An Empirical Evaluation of the Black-Litterman Approach to Portfolio Choice*

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Abstract

We evaluate the Black-Litterman equilibrium model approach to portfolio choice. We quantify the improvement in portfolio performance of a privately informed investor who learns from market prices over an equally informed, but dogmatic investor who only uses private information. We extend the approach to any linear multi-factor asset pricing model (e.g. ICAPM) to examine how learning from prices using different equilibrium models affects portfolio performance. We find that even a misspecified asset-pricing model can improve portfolio performance when private signals are not extremely precise. As we increase the noise in private information, learning from prices is initially harmful and gradually becomes more beneficial.

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1 Introduction

Prices in financial markets aggregate dispersed information from millions of investors. A vast literature considers portfolio choice problems under symmetric information, but ignores this fundamental function of financial markets. It is no coincidence that the overwhelming majority of empirically successful equilibrium asset pricing models that embed portfolio problems assume that information is symmetric, or equivalently, that a representative agent exists. Equilibrium asset pricing models with dispersed information are complicated, informationally demanding and suffer from testability issues that limit their usefulness. However, a realistic though challenging portfolio problem must recognize that investors allocate capital across multiple assets without much knowledge of other investors, their tastes or the precision of their information. Money managers and retail investors alike must therefore ask themselves two questions: First, what do current prices tell them about the private information of others? Second, is the information in prices valuable given their private information?

To answer the first question, Black and Litterman (1992) extract market-implied expected returns using a symmetric information equilibrium model, the CAPM, as an elegant approximation.[1] They combine this information from prices with the private information of the investor in a Bayesian fashion. The approach is popular among active money managers who believe they hold information superior to that of other market participants, but wish to update their beliefs using market prices. However, the benefits of using a misspecified asset pricing model to learn from market prices about other investors' views on expected returns remains an empirical question.[2]

[1]Sharpe (1974) introduced the possibility to extract expected returns from the weights of a portfolio held by an institution rather than from equilibrium weights of the market portfolio.

[2]We are aware of no other study that empirically evaluates the Black-Litterman despite its popularity. The Black-Litterman model and its extensions are used by money management firms such as Goldman Sachs and JPMorgan; implemented in popular investors services such as Bloomberg and Ibbotson; taught at universities and schools around the globe including Chicago, Princeton, NYU, and Columbia; covered in standard investments textbooks such as Benninga (2000) and Bodie, Kane, and Marcus (2009); and are required material for CFA exams. It is hard to name another methodology in finance that is this popular yet academically untested.

To answer the second question, we quantify the improvement in portfolio performance of a privately informed investor who updates his beliefs using market prices with those of an equally informed "dogmatic" investor who only uses his private information. We do this by providing the investor with a noisy signal of next period's actual returns, which we, the econometricians, can correctly observe. Our simple approach allows us to overcome the fundamental problem of testing a model in which agents have private information unobservable to the econometrician.

Our empirical investigation leads us to extend the Black-Litterman model along two dimensions. Since Black and Litterman first introduced the approach, one of the tenets of their work, the CAPM, has been rejected empirically (Fama and French, 1992, 1996). Following this realization, several multi-factor asset pricing models have been suggested, the most popular of which is the Fama and French (1993) three-factor model. These multi-factor models can be thought of as special case of the Merton (1973) Intertemporal Capital Asset Pricing Model (ICAPM). Below, we extend the Black-Litterman approach to any linear multi-factor asset pricing model, carefully formulating it in a natural Bayesian framework.[3] This extension allows us to examine how the use of different equilibrium models for learning from prices can affect portfolio performance and establish a framework to test asset pricing models for their informational content about expected returns.

We treat a full information equilibrium model as an approximation for the behavior of an average investor in a dispersed information model. Fully specified equilibrium models with dispersed information, beginning with Grossman (1976), Hellwig (1980) and Diamond and Verrecchia (1981), and more recently Qiu and Wang (2010), provide a theoretical justification why agents with private signals should learn from equilibrium prices. In these models, agents observe private signals about payoffs and learn from prices about other participants' signals. The multiple-asset Admati (1985) model can almost rational-

[3] Strictly speaking the original Black-Litterman formulation is not Bayesian, though it combine the information from prices with the private information of the investor in a Bayesian fashion. After we formulate a Bayesian framework for learning from prices, we show how it can incorporate equilibrium model uncertainty into the inference problem in the Appendix.

ize the Black and Litterman approach, though in that model, the unconditional version of the CAPM as assumed by Black and Litterman does not hold. In order for investors to learn from prices in such models, they require a full and correct specification of preferences and all parameters that govern the distribution of other agents' private signals. By contrast, the informational requirement in the Black and Litterman model is much smaller. Since in equilibrium, the average investor holds the market portfolio, prices aggregate information about all market participants' beliefs. The degree to which prices are informative depends on the distance between the assumed full information model and the dispersed information model that governs prices in reality. In section 3 we develop a framework to evaluate this approximation empirically, which is a new way to test asset pricing models.

The Black and Litterman (1992) model can be interpreted as an approach for estimation of risk loadings (βs) using current market prices instead of regression slopes from time series regressions. Our extension of their model to the ICAPM prescribes the use of asset weights in the market portfolio as well as in factor-mimicking portfolios of concern to investors, to estimate risk loadings in a multi-factor asset pricing models. We perform cross-sectional regressions using risk loadings estimated intra-temporally for single-factor and three-factor models. The covariance matrix of the pricing errors then provides us with a new natural measure of model misspecification that we use to determine the weights placed on the model and on the private signals when the investor forms his forecast of expected returns. This is potentially an important advance as Black and Litterman simply left the choice of a model misspecification meta-parameter (τ) to implementor discretion.

We evaluate the out-of-sample performance of the equilibrium approach along two dimensions. First, We quantify the improvement in portfolio performance of a privately informed investor who learns from market prices over an equally informed, but "dogmatic" investor who only uses private information. Second, we examine how the use of various equilibrium models to learn from prices affects portfolio performance. Since the CAPM has been shown to perform poorly compared with the ICAPM-motivated Fama and French three-factor model, it is plausible that market-implied expected returns which use such models can improve investors' learning and portfolio selection.

We find that as we increase the noise in the private signal, learning from prices is initially harmful and gradually becomes more beneficial. On one extreme, when private information is perfect, ignoring market prices and following the private information alone is clearly optimal. However, on the other extreme, when the investor's private information is pure noise, his portfolio should underperform a simple strategy of holding the market portfolio. The fact that we find a benefit of learning from equilibrium prices for an investor whose private information is reasonably precise, is evidence that even misspecified models have useful predictive power.

We further find that an informed investor whose asset universe is one hundred size and book-to-market portfolios, and who achieves an annualized Sharpe ratio between 0.5 and 1.0, improves his Sharpe ratio by 7 to 35 percent using the Black-Litterman approach and by 6 to 41 percent using our extension of the model to the three-factor model. The implementation of the ICAPM-motivated three-factor model depends however on the ability to determine the weights of each asset in the factor mimicking portfolios. Whereas these weights can be easily constructed for the one hundred size and book-to-market portfolios, they can not be constructed for industry portfolios. For this reason, learning using the CAPM outperforms the three-factor model in tests based on industry portfolios.

Our paper is related to a portfolio choice literature that extends the classical portfolio choice problem pioneered by Markowitz (1952) to a setting in which an investor learns about model parameters from historical data (Pastor, 2000; Pastor and Stambaugh, 2000; Garlappi, Uppal, and Wang, 2007; Johannes, Korteweg, and Polson, 2011). Recent contributions that study this type of *inter-temporal* learning, over time, in a portfolio choice setting include Avramov (2004) and Zhou (2009). Avramov and Zhou (2010) provides a recent review of this literature and Pastor and Veronesi (2009) survey other finance applications of inter-temporal learning. By contrast, we follow the Black and Litterman (1992) approach and focus on *intra-temporal* learning about expected returns, at a point in time, from a cross-section of current prices. Furthermore, unlike most of the above papers, we allow investors to include private information in addition to historical returns and current prices in their portfolio choice.[4]

In the next section we explain the theoretical basis for the equilibrium approach and generalize it along several dimensions. The results of the empirical investigation appear in Section 3. Section 4 concludes.

2 Theory

In speculative markets where agents hold different information, equilibrium prices plausibly contain information beyond that held originally by each agent. This information can be formally incorporated into an investor's portfolio choice using a Bayesian framework. This approach allows for the incorporation of both model uncertainty and private information in a natural and flexible fashion. Below, we extend the approach of Black and Litterman (1992) that relies on the CAPM as the equilibrium model to any linear multi-factor asset pricing model.

2.1 Learning from Prices

Classical statistic methods are much more frequently used to forecast expected returns than Bayesian ones. The few exceptions that do use Bayesian methods in portfolio choice

[4]Zhou (2009) argues that the Black and Litterman (1992) approach disregards the information in historical returns. Theoretically, given that historical returns are public information, prices already incorporate this information and historical data does not provide any additional value. However, if the asset pricing model is misspecified then historical returns can still be informative. In section 3.3, we compare the out-of-sample performance of intra-temporal learning models relative to the historical means and find that the latter can perform better when the number of assets is large.

assume that returns are i.i.d. and drawn from a distribution with constant parameters.[5] These papers pose the investor's learning problem as one of estimating the constant parameters of the return distribution using historical information. Under such conditions, current prices might be less useful than past returns for predictive purposes.

We focus on *intra-temporal* learning, at a point in time, and abstract from the *inter-temporal* learning problem. Every period, investors use information about the current state to form their expectations about next period returns. To this end we can envision no better statistic than current market prices. At the very least, since prices can be effortlessly observed in most security markets, they can be used by investors to update their expectations. Additional correct information that can be costlessly observed and processed without altering the feasible set of actions cannot make the agent worse off (see Blackwell, 1953; Marschak and Miyasawa, 1968; Sulganik and Zilcha, 1997). The reason is that more information increases the opportunity set of the investor. It allows him to condition his choices on more variables. The investor has the *option* but not the *obligation* to use the added information in his decision. Such an option must have non-negative value to the optimizing agent when he knows its precision.

Information incorporated in market prices can be used for learning of expected returns in the following way. Suppose that θ is a parameter vector that fully describes the distribution of asset returns. Let $\ell(\omega|\theta)$ denote the likelihood of equilibrium prices (weights) implied by an equilibrium model, given the state of the economy that governs expected returns. The investor does not necessarily need to agree with all the assumptions maintained under the equilibrium model. Instead, our investor only uses the equilibrium model as an approximation for the distribution of prices given other traders' beliefs. We assume that this likelihood is unaffected by prior information. Next, we require an assumption about the investor's prior distribution about future states which we denote $\pi(\theta|\psi)$. With these two objects, we can form a posterior distribution over θ using Bayes'

[5] These include Pastor (2000), Pastor and Stambaugh (2000) and others.

rule:

$$\pi^*(\theta|\psi,\omega) \propto \ell(\omega|\theta)\pi(\theta|\psi), \tag{1}$$

where we omit a constant term which guarantees that this conditional distribution integrates to one. In the terminology of Bayesian statistics, the investor "shrinks" his estimates of the state towards those implied by market prices. While a dogmatic investor would simply use his prior, a more humble investor would likely use the Bayesian posterior in (1). On the other hand, an uninformed investor who has a diffuse prior about future returns would use only market-implied parameters. Thus, the equilibrium approach can benefit all investors, not just superiorly informed investors.

Finally, we require a specification of the personal preferences, $u(\cdot)$, of our investor and his opportunity set. These likely differ as well from those of the representative agent commonly assumed for the simplification of equilibrium analysis. With these ingredients specified, the investor can use the predictive distribution of returns implied by the posterior distribution to rank portfolios and choose an optimal one. We emphasize that the common investment advice to simply hold the market, not only assumes that the investor has no private information, but also implicitly assumes that his preferences are identical to those of the average investor. This approach would leave the investor with a portfolio that fails to account for his heterogeneity in preferences or opportunities. Instead, the investor should learn from market valuations about the average investor's expectations, rather than use the average investor's portfolio weights.

2.2 A Multi-factor Asset Pricing Model

We next focus our attention on linear, possibly multi-factor, asset pricing models. We show that the Black and Litterman (1992) model can be interpreted as an alternative approach for estimation of risk loadings (βs) using current market prices instead of regression slopes from historical data regressions. The Black and Litterman (1992) model uses the CAPM as its equilibrium model. Since the CAPM is a special case of the ICAPM, we present only our more general learning process that uses a multi-factor model and point out that our development collapses to the Black and Litterman model when the only risk factor of concern to investors is market risk.

Following Fama (1996) we assume that a risk-averse investor maximizes the expected utility of his lifetime consumption. Investors agree about the joint distribution of asset returns, $r_t \in \mathcal{R}^N$, and the state variables, $K_t \in \mathcal{R}^S$. Most importantly, this distribution is multivariate normal:

$$r_{t+1} \sim \mathcal{N}(\mu_t, \Sigma). \tag{2}$$

In what follows we mostly omit time subscripts, but note that this is a conditional asset pricing model, and all expectations are conditional on the state at time t. In this world, investors choose multifactor-efficient portfolios, p, which maximize expected return, $E(r^p)$, given their variance, $\sigma^2(r^p)$, and covariance with the state variables, $cov(r_p, K_s)$. Joint normality implies that

$$r_p = E(r_p) + \sum_{s=1}^{S} b_{ps} K_{st} + \epsilon_{pt}, \tag{3}$$

where $E(\epsilon_{pt}) = 0$, $cov(\epsilon_{pt}, K_{st}) = 0, s = 1, \ldots, S$, and without loss of generality, we assume that the state variables are rescaled to have zero means.

As Fama (1996) shows, in equilibrium, expected excess returns satisfy the following linear relationship with expected excess returns on the market portfolio and the mimicking portfolios:

$$E(r) - r^f = \beta \begin{bmatrix} E(r^{em}) \\ E(r^{es}) \end{bmatrix}, \tag{4}$$

where r is an N vector of asset returns, r^f is the risk free rate, r^{em} is the excess return on the value-weighted market portfolio and r^{es} is a vector of excess returns on the remaining factors.

The ICAPM, as summarized by the familiar linear multifactor asset pricing equation

(4) can be used to deduce the market's expectation about future returns from current period weights of the market portfolio and the factors of each asset. We decompose β into covariance matrices, the weights vector on the market portfolio, ω_m and the weights matrix on the S mimicking portfolios, ω_s, as follows:

$$\beta(\omega) = \Sigma[\omega_m \, \omega_s]\Sigma_f^{-1}, \tag{5}$$

where Σ is an $N \times N$ nonsingular covariance matrix of asset returns, Σ_f^{-1} is an $(S+1) \times (S+1)$ covariance matrix of factor returns. Then, under the assumption that the covariance matrices as well as the risk premia on the factors are known, the expected next-period returns vector implied by market weights at time t is:

$$\mu_{\omega,t} = E_t(r_{t+1}) = r_{t+1}^f + \beta(\omega_t) \begin{bmatrix} E_t(r_{t+1}^{em}) \\ E_t(r_{t+1}^{es}) \end{bmatrix}. \tag{6}$$

Thus, the Black and Litterman (1992) model can be interpreted as an alternative approach for estimation of risk loadings (βs) using current market prices (ω_t). This approach can be superior to estimation using regression slopes from historical data regressions if β is time varying. There is ample evidence that this is indeed the case (Fama and French, 1997; Hrdlicka, 2010).

We allow for model uncertainty by adding a pricing error to the conditional expected return:

$$\mu = \mu_{\omega,t} + \alpha, \tag{7}$$

where $\alpha \sim \mathcal{N}(0, V_\omega)$ parameterizes model misspecification. The original Black and Litterman model implicitly assumed such a discrepancy term giving rise to a randomly distributed μ conditional on the risk premia and their betas. A similar assumption is made explicitly in Pastor (2000). The likelihood $\ell(\omega|\theta) = \ell([\omega_m \, \omega_s]|\mu, \Sigma)$ for expected returns based on a model M which satisfies the ICAPM relation (4) is then

$$\mu|\omega, M \sim \mathcal{N}(\mu_\omega, V_\omega). \tag{8}$$

This formulation suggests an intuitive empirical counterpart for V_ω, namely the covariance matrix of the pricing errors from a cross-sectional regression of expected returns on risk loadings. We pursue this approach in the empirical section below.

At this point we have calculated a vector of expected returns that make the market portfolio multifactor-efficient. This vector summarizes the beliefs of the average investor implied by the ICAPM. In the following section we describe how to incorporate these market views with those of the investor.

2.3 Incorporating Private Information

Investors can have several sources of information that they believe are not 100% reflected in current prices. This information can be some type of inside information or the result of a complicated forecasting algorithm. We follow the Black and Litterman approach and model investors' private information in a reduced form. Investors are asked to provide their expected returns on portfolios of assets based on all of their private sources. In addition, they are required to provide their level of confidence in their expectations and correlations between their views. The limiting case where their prior variance is very large corresponds to the case of an investor that has no additional information (diffuse prior). The portfolio formulation allows the investor to provide expectations for returns of any single asset, but this is an option rather than a requirement. He can instead provide expected returns for a sub group of traded assets or returns on long-short zero cost portfolios.

Formally, the investor provides a $P = K \times N$ matrix of portfolio weights and $Q =$

$K \times 1$ vector of expected returns such that his views are described by

$$P\mu = Q + \epsilon, \tag{9}$$

where $\epsilon \sim \mathcal{N}(0, \Omega)$ and independent of error in market prices.[6] For example, an investor who has an independent view for each traded asset will have a specification where P is an identity matrix and Ω is a diagonal matrix. The inputs on the diagonal of the Ω matrix are inversely related to the precision of his views.[7] The investor's prior distribution about the parameters governing returns is then conditionally normal and denoted $\pi(\theta|\psi) = \pi(\mu|P, Q, \Omega)$.

The use of a conjugate normal prior parameterized by P, Q and Ω with a normal likelihood parameterized by μ_ω and V_ω leads to the following normal posterior:

$$\mu|P, Q, \Omega, \omega \sim \mathcal{N}(\mu^*, V^*), \tag{10}$$

where

$$\begin{aligned} V^* &= [P'\Omega^{-1}P + V_\omega^{-1}]^{-1} \\ \mu^* &= V^*[P'\Omega^{-1}Q + V_\omega^{-1}\mu_\omega]. \end{aligned}$$

Finally, we must update the returns covariance matrix for the posterior mean,

$$\Sigma^* = \Sigma + V^*, \tag{11}$$

so that the updated distribution of returns, which combines the investor's views with

[6]It is a straightforward exercise to allow for correlation between the private information of the investor and that of the market. In that case, the reduction in uncertainty is smaller since we have fewer independent observations.

[7]Explicit learning over time about the precision of the views and correlations between them is beyond the scope of this paper and is left for future research.

those of the average investor are distributed:

$$r|P, Q, \Omega, \omega \sim \mathcal{N}(\mu^*, \Sigma^*). \tag{12}$$

One could use this distribution of returns and solve for the weights on the multifactor-efficient portfolio given these beliefs. However, we emphasize that this is not necessarily optimal from the perspective of the informed investor. Instead, he should use his personal preferences and his subjective opportunity set to pose an optimization problem. This problem can be different from that of the average investor assumed by the equilibrium model.

3 Empirical Analysis

We wish to evaluate the performance of the equilibrium approach to portfolio choice along several dimensions. First, we ask whether the use of current market prices for estimation of risk loadings results in an improvement over estimation using time-series regression slopes. Second, we attempt to quantify the improvement in portfolio performance of a privately informed investor who updates his beliefs using market prices with that of a "dogmatic" informed investor who only uses his private information. Third, we examine how the use of various equilibrium models to learn from prices affects portfolio performance. Since the CAPM has been shown to perform poorly compared with its ICAPM motivated, multifactor counterparts, it is plausible that market-implied expected returns that use such models can improve investors portfolio choice.

3.1 Data

The data used for the empirical evaluation of the Bayesian approach includes monthly returns on 100 portfolios formed on size and book-to-market, industry portfolios and the

Fama-French factors all available from Ken French's data library.[8] We use monthly data to get a larger sample for our tests. In addition, many informed investors, such as active money managers, trade more frequently than on an annual basis, and therefore monthly trading frequency is an interesting benchmark to look at. Monthly returns data for these assets is available starting in July 1926, but only sparsely. We omit from our tests months in which any of the assets considered has no return information. We use several portfolios of assets rather than the basic securities to avoid the need to estimate the full covariance matrix of these returns. Estimation of large covariance matrices is possible[9], but is not the main focus of this paper.

We first consider an investor faced with a choice between 100 value-weighted portfolios formed on size (market equity, ME) and the ratio of book equity to market equity (BE/ME). This asset space is particularly convenient for our purposes since we wish to use the Fama and French three-factor model as the equilibrium model and compare it with using the CAPM as in Black and Litterman (1992). The likelihood implied by the three-factor model, which appears in (6) requires in addition to current market weights ($\omega_{m,t}$) an observation of the weights of each asset in SMB and HML portfolios ($\omega_{s,t}$). While these weights can always be calculated when the asset space is the full set of basic securities used to form the factor mimicking portfolios, this is not generally the case when the assets are portfolios of the basic securities.

To see why recall the definition of the weighting vectors:

$$r^s = \omega_s r, \tag{13}$$

where r^s is the return on factor s and r is a vector of the N basic asset returns. This ω_s can always be calculated based on the mimicking portfolios' definition. Next consider an I

[8] http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html

[9] See Carvalho and West (2007) for a recent treatment.

vector of portfolios r^I formed from the same set of N assets:

$$r^I = Wr, \tag{14}$$

where W is an $I \times N$ matrix of portfolio weights. If we restrict the portfolio problem to the set of I assets, then the weights matrix required is such that:

$$r^s = x^s r^I = x^s W r, \tag{15}$$

where x^s is an $1 \times I$ vector of the weights that each portfolio has in the factor s. But, this implies that we require $x^s W = \omega_s$ for every factor s. While this equation has a solution for value-weighted portfolios when the factor is the market, it does not have a general solution for every set of portfolios I and for any factor s. We therefore calculate the weights of the value-weighted 100 portfolios of securities sorted by size and book-to-market using their weights in the six value-weighted portfolios used to calculate Fama-French factors.

The second set of assets we consider are the 49 industry portfolios. A small number of industry portfolios decreases the dimension of the covariance matrix substantially and decreases its estimation error. This can be important when the history of available historical returns is limited. The first set of 100 portfolios is an appropriate asset space for money managers who specialize in small versus big stocks and growth stocks versus value stocks. Industry portfolios is the asset universe for money managers who collect signals about future performance of different industries. There is no exact solution for the weights of the industries in SMB and HML factor-mimicking portfolios. Therefore industry portfolios are used primary to evaluate the CAPM-based Black-Litterman approach for the industry portfolios, because the weights of the industry portfolios in the market portfolio are computed exactly. We nonetheless proceed with evaluation of the ICAPM-based approach by estimating the weights of the industry portfolios in the Fama-French factors by solving (5) for ω_s, where we estimate β from a time-series regression, but this is a join test of the benefits of the ICAPM-based model and of the approach to estimate asset weights in the factor-mimicking portfolios when exact solution is not available.

3.2 Estimating Model Misspecification using Cross-Sectional Regressions

Black and Litterman and the literature that followed have usually estimated the covariance matrix of expected returns V_ω using an unbiased estimator of the sample mean's covariance $(\frac{1}{T-1})\Sigma$ with T the length of the time-series used to estimate the covariance matrix Σ (see He 1999; Walters 2010). We instead suggest a more intuitive estimator given this formulation. The variance of μ conditional on risk premia and covariances, comes entirely from α. We use the sample covariance estimate of the pricing errors vector $Cov[\alpha]$ as an estimator for model misspecification V_ω. The intuition for this choice is clear. When the model under consideration explains only a small part of the variation in expected returns, the variance of the pricing errors is large. As a result, we are less confident about the model and assign a large confidence interval around its implied expected returns.

We calculate β in two ways. The first uses current weights in the risk factor according the decomposition in (5).[10] Time variation in market prices results in variation in risk loadings. The second assumes that risk loading specific to each asset are constant and therefore uses the slopes from a time-series regression of excess returns on the excess returns of the risk factors. We compare the fit of the CAPM to that of the Fama and French (1993) three factor version of the ICAPM.

Fama and MacBeth (1973) cross-sectional regressions of average returns of each portfolio on its β are reported in Table 1 for the 100 Size and B/M sorted portfolios. The χ^2 statistic tests whether the pricing errors are jointly different from zero. All four specifica-

[10] We use a full sample estimate of the covariance matrix. We avoid shrinkage approaches for optimal estimation of the covariance matrix to isolate the benefits of the Black-Litterman approach from the benefits of different estimation techniques. The interested reader is referred to Disatnik and Benninga 2007 for comparison of different shrinkage techniques.

tions reject the model at the 1% significance level. Comparing specification (i) to (ii) and specification (iii) to (iv) we find that regression-based betas explain more of the variation in expected returns than those using market prices. This small difference can be due to estimation error of a large returns covariance matrix Σ using a relatively short time series. Use of shrinkage estimators for the Σ can improve on this result. As expected, the three-factor model does better than the CAPM. We also report the average risk premia on the factors as well. Recall that in a cross-sectional regression, the estimated λ should theoretically correspond to the risk premia (Cochrane, 2005). On this dimension the results are very close. None of the estimated premia is statistically different from its average.

Table 2 reports the results of a similar regressions for the 49 industry portfolios. For these assets, both models can hardly be rejected at usual significance levels. Once again, constant betas using time series regression slopes explain better the variation in expected returns. The slope on market beta is again very close to the average premium, but the slopes on SMB and HML are quite different.

Based on these results it is reasonable to expect that learning about expected returns using the multifactor model can improve portfolio performance of investor who invests in 49 industry portfolios. The χ^2 statistics summarize the extent to which each model of expected returns is misspecified. It is not clear ex-ante if using the CAPM and the Fama and French (1993) three factor version of the ICAPM will improve performance of an investor who invests in 100 Size and B/M sorted portfolios because both models are rejected based on the classical tests. However, a Bayesian approach allows to account for model misspecification and investor can optimally decide to put less weight on the information implied by the models. Below, we use the sample covariance estimate of the pricing errors vector $Cov[\alpha]$ as an estimator for model misspecification V_ω. We next turn to out-of-sample tests of the predictive ability of the two models models.

3.3 Out-of-Sample Predictability Tests

An alternative measure of model fit is its ability to predict future returns out-of-sample. We calculate mean-squared-error (MSE) between realized and expected returns for each model M_j using the market-implied betas:

$$MSE(M_j) \equiv \frac{1}{T}\left(r^{ei}_{t+1} - \beta_{i,f}(\omega_t)E_t[r^{ef}]\right)^2. \tag{16}$$

We then compare the MSE of the model to that of simple benchmark which uses each asset's historical average returns as a prediction of its future return. This predictor should theoretically perform well if expected returns are constant over time. An intuitive measure of relative performance is the *Relative* R^2 defined as

$$R^2 \equiv 1 - MSE(M_j)/MSE(hist). \tag{17}$$

A positive relative R^2 means that model M_j is a better predictor of future returns compared with the historical returns benchmark.

Table 3 reports out-of-sample relative R^2 for each of the 100 portfolios formed on Size and B/M in ascending order as well as an equal-weighted average. The first twenty years are omitted because a long enough history is required to estimate the average risk premia and loadings. We find that the three-factor model perform better than the historical average according to this metric, while the CAPM does not. The results for 49 industry portfolios reported in Table 4 show that the models do better in predicting future returns compared with the historical average. The CAPM performs slightly better than the three-factor model. However, the comparison between the two models for this universe of assets is biased against the three-factor model because it is impossible to compute weights of industry portfolios in the HML and SMB risk factors (see section 3.1).

These results reveal that the CAPM is a poor predictor of future returns for portfolios formed on Size and B/M. However, for 49 industry portfolios the CAPM's predictive power is comparable to that of the three-factor model. This predictive ability will play a key role in the out-of-sample portfolio performance tests to which we turn next.

3.4 Portfolio Choice with Noisy Private Information

We compare three alternative approaches for constructing portfolios that utilize this private information of the investor. The first strategy creates a portfolio with the maximum Sharpe ratio, based only on the private information ("dogmatic informed investor strategy"). The second strategy combines the private signals with the expected returns implied by the CAPM model as proposed by Black and Litterman (1992). The third strategy combines the private signals with the expected returns implied by the ICAPM model as proposed in this paper. For practical purposes, we assume that Fama-French HML and SMB portfolios together with the market portfolio and the risk-free security span the multifactor minimum variance (MMV) frontier.

Investors are allowed to form long-short portfolios that they re-balance monthly. In general, investors can combine model-implied expected returns with any optimization approach for constructing portfolios. For the purposes of the quantitative comparison between the performance of the three strategies, we consider an investor who invests in the maximum Sharpe ratio (tangency) portfolio every period given his beliefs. Weights of the tangency portfolio are given by:

$$\omega_t^* = \frac{\widehat{\Sigma}_t^{-1} \mu_t}{\mathbf{1}|\widehat{\Sigma}_t^{-1} \mu_t|}. \tag{18}$$

Here, μ_t is the vector of the expected excess returns at time t, $\widehat{\Sigma}_t$ is the estimate of the covariance matrix based on the history of realized returns available at time t and the covariance matrix for the signals, and $\mathbf{1}$ is a row vector of ones. That is, we normalize the weights by the sum of the weights' absolute values.[11] We use a minimal period of 20 years for the investors to estimate the covariance matrix before they begin to form portfolios.

The focus of this paper is the vector of expected excess returns μ_t. The strategy of the dogmatic informed investor is to use only his private signals as expectations. The strategy of the informed investor, who believes that the CAPM is the equilibrium model, combines private signals with the implied expected returns using the CAPM version of (6). The third strategy is similar to the second, but the informed investor believes in the ICAPM and uses the implied expected returns that appear in (6) with the three factors of Fama and French. To implement the first strategy, the informed investor needs to know only his signals and the variance of the noise of the signals. For the second strategy, the informed investor also needs to know the valuations of the assets in order to calculate their weights in the market portfolio. For the third strategy the informed investor needs also to know what is the weight of each asset in the three Fama-French factors.

3.5 An Empirical Counterpart to Private Information

Empirically evaluating the performance of a Bayesian portfolio choice model requires an empirical counterpart to the prior distribution. The Black and Litterman model was designed for investors who posses information superior to that of the average investor. In their model, the investor shrinks his private views toward those of the average investor as implied by market prices. Since private information is by definition unobservable, we let each hypothetical investor observe a noisy signal of next-period returns:

$$Q_t = r^e_{t+1} + \sigma\tilde{\epsilon}_t, \tag{19}$$

[11]This normalization avoids the usual problem that investor is required to take extreme long-short positions in his portfolio. This problem occurs when the sum of the unnormalized weights is close to zero and as a result constraining weights to sum up to one is equivalent to deciding each unnormalized weight by a number close to zero. In our normalization we assume that investor can bet at most one dollar each period but the net position can be positive or negative.

where ϵ_t is randomly drawn from a zero mean normal distribution with covariance matrix of excess returns Σ. That is, investors get the ultimate private information, a peak at next period's returns, albeit augmented by varying amounts of noise.

We report performance estimates for five levels of signal noise factor (σ) ranging from 10 to 160 in our main specifications. A factor of one means that the covariance matrix of the signals is the same as the one of the historical returns. Our specification for the noise preserves the correlation structure between the assets and allows us to control the precision of the signal with a single parameter. As $\sigma \to \infty$ the signals become uninformative and the investor only uses market-implied expected returns to form his posterior. To get consistent estimates we use the same random vector ϵ_t to generate the signals with different precisions. Therefore, the only difference across different values of σ is in the magnitude of the noise but not the realization of the noise. We repeat the procedure 300 times and report the means and standard errors for all the reported statistics.

3.6 Results

The results for the strategy performance for 100 size and book-to-market portfolios are presented in Table 5. We report mean percent excess returns for each strategy and three performance evaluation metrics. The monthly Sharpe ratio of the strategy is important because the objective function of our investor is to maximize the Sharpe ratio. These estimates allow us to compare different investment approaches. In addition, we want to learn whether the returns of the strategies come from alpha or from factor loadings. For that purpose, we report the monthly market alpha and the three-factor alpha computed from monthly time-series regressions of returns to the strategy on the risk factors. All of the reported means across simulations are annualized.

Table 5 reports our main results. We report the mean difference between Sharpe ratios relative to the dogmatic strategy and their standard errors. We find that both the Black-Litterman approach and its extension to the ICAPM have Sharpe ratios lower than those of the dogmatic informed investor when the signal noise factor (σ) is 20 or lower. For a low noise factor of 10 the Sharpe ratio of the dogmatic investor is statistically higher. However, for higher values of the signal noise factor both equilibrium based strategies exhibit higher Sharpe ratios than those of the dogmatic investor. Comparing the CAPM and ICAPM models for learning from prices, we can see that the FF3F model performs better than the Black-Litterman model that uses the CAPM when the signal noise factor is 80 or higher. Thus, an informed investor with noisy private information whose asset universe is one hundred size and book-to-market portfolios, can significantly improve his portfolio performance by using the Black-Litterman approach and do even better using our extension of the model to the three-factor model.

Interestingly, even when the signal is very noisy ($\sigma = 80$), intra-temporal learning from prices is beneficial compared with holding the market, forming the maximum Sharpe ratio portfolio based on historical means, or holding an equal-weighted portfolio. There is wide-spread belief that using historical averages as estimates for expected returns results in portfolios with poor performance out-of-sample. By contrast, we find that when the weights are normalized as we advocate according to (18), the Sharpe ratio of this portfolio, reported at the bottom of the table, is slightly better than that of holding the market.

An investor with a high signal noise factor ($\sigma = 160$) who assumes the FF3F version of the ICAPM is the equilibrium model receives an annualized alpha of 1.43 percent but a negative three-factor alpha of 7 basis point. This result might seem surprising due to the common misconception that the Black-Litterman approach achieves balanced portfolios by shrinking portfolio weights towards the market portfolio. But as we explained above, learning and optimization can be separated in this framework. The objective of our investor is to maximize the Sharpe ratio of his portfolio, meaning that he does not consider HML and SMB as risk factors and loads on them, even though he believes those are risk factors for an average investor and learns from them. Since the portfolio problem in our tests is different from the portfolio problem of an average ICAPM investor, our optimal weights differ from ICAPM market weights.

Table 6 repeats this experiment for the 49 industry portfolios. Again, as we increase the amount of noise in the private signals, Sharpe ratios decrease monotonically in all strategies. As the signal becomes noisy, learning from prices again becomes more and more beneficial. Unlike the case of the Size and B/M sorted portfolios, this time the three-factor model does poorly compared with the CAPM and not better than the dogmatic strategy even for very noisy private information. Recall that for the industry portfolios, there is no exact weight of an industry in the factor mimicking portfolios. It is quite possible that our procedure to estimate this weight using regression slopes, as explained above, introduces an estimation error to such a degree that learning from prices using the three-factor model is no longer beneficial.

Both tests reveal that as we increase the noise in the private signal, learning from prices becomes more and more beneficial. This is quite intuitive. One extreme, when private information is perfect, is obvious. In this case, ignoring market prices and following the private information alone is optimal. However, in the other extreme, when private information is very noisy, or when the investor has a diffuse prior, it is less obvious that there is value in learning from prices compared with, for example, passively holding the market. That we find such a benefit is evidence of the predictive power of the asset pricing models.

For each investment strategy one can use our results to calculate the precision of the private signals that achieves a certain level of Sharpe ratio. For example, to achieve a monthly Sharpe ratio close to 0.75, an informed investor, who invests in 49 industry portfolios and uses only his private information, requires an unbiased signal for next-month returns with a covariance matrix equal to one hundred times that of the historical returns.

As a robustness test for our results we test the out-of-sample performance of these strategies for 5 and 10 industry portfolios. The results reported in Tables 7 and 8 reveal that once again, as the signal becomes noisier, both equilibrium based strategies exhibit higher Sharpe ratios than those of the dogmatic investor. Similarly to the 49 industry portfolios tests, CAPM-based learning from prices performs better than ICAPM-based learning. It seems that this pattern is robust as long as proper weights for the assets tested are not available.

When we compare the results reported across tables 5 through 8, we learn that Sharpe ratios for all strategies increase with the number of investment assets. This result should not be surprising since a larger number of assets implies a larger number of private signals that translate into better performance. The specification of one signal per one asset was chosen here in order to test the benefit of the equilibrium approach to portfolio choice that assumes that market prices reveal additional information. However, the investment approach itself is flexible enough to accommodate views about any subset of assets. Though our results are silent about the benefit of using the strategy when an investor has less signals than the number of assets, we expect it still depends on the noisiness of the signal.

4 Conclusion

The Black and Litterman approach to portfolio choice is simple, intuitive and results in portfolio weights which incorporate both private information of an investor and information aggregated in prices. We quantify the benefits of the approach using simulations based on actual returns data. Our analysis exposes the benefits as well as the limits of this approach. An investor whose private information is sufficiently noisy can benefit from the approach. On the other hand, an informed investor whose information is highly accurate stands to lose by adjusting his estimates of expected returns too much toward implied market expectations. Thus, the answer to the question whether an investor should use the equilibrium approach depends on the precision of his information and is certainly not categorical.

Our results can be of interest to academic researchers and to practitioners. First, we provide a novel approach to assess the informativeness of different asset pricing models. Even a model which is statistically rejected based on classical statistics tests can enhance portfolio performance. Second, our approach can provide the foundations for fully specified equilibrium models with dispersed information in which agents learn from equilibrium prices even when they do not know the precision of other investors' signals. Third, active money managers ought to know whether they can benefit from adjusting their private expectations toward those of other market participants. Since financial markets aggregate participants' private information and beliefs into prices, intra-temporal learning from prices can improve their decisions. In addition, while multi-factor models are a popular benchmark for performance evaluation, portfolio choice in many financial institutions still relies on the single factor Black and Litterman model. Such institutions might wish to first decide which asset pricing model they trust most, and then use it consistently for both portfolio choice and performance evaluation of risky strategies.

Our empirical approach focuses on the evaluation of the Black-Litterman model and its extension to multi-factor models. For this purpose, we abstract from parameter learning over time, model learning over time, and shrinkage approaches for estimation of the covariance matrix. Incorporating these approaches in future work can provide additional benefits of the equilibrium approach to portfolio choice or further expose its limitations.

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A Appendix

A.1 Equilibrium Model Uncertainty

In this section we discuss how to incorporate model uncertainty into the portfolio choice of an investor who has private information. Let us assume that the investor considers J potential asset pricing models $\{M_j\}_{j=1}^{J}$ and has a belief in each model described by a discrete density function $f(M_j|\psi)$ where ψ represents all available information other than current market weights. Without loss of generality he can consider only models with positive density. The described procedure is valid and optimal from the point of view of the investor even if the true model generating the data is not considered by him.

The investor incorporates model uncertainty in the following way. First, he needs to extract the implied expected returns conditional on each equilibrium model. Second, he incorporates his own views with the implied expected returns separately for each model. Then he calculates an unconditional expected return which is a weighted-average of conditional expected returns weighted by the probabilities of each model.[12] The resulting expected returns incorporate the views of the investor, the views of the average investor for each of the models under consideration, and the investor's belief about the probability of each model. This vector of expected returns is used by the investor in his optimization procedure to calculate optimal investment policy as usual.

The unconditional expected return distribution is the expected return distribution conditional on a model times the probability of each model:

$$p(\mu|\psi,\omega) = \sum_{j=1}^{J} p(\mu|\psi,\omega,M_j) f(M_j|\psi,\omega). \tag{20}$$

With normal conditional distributions, the unconditional distribution is also normal and as shown by Pastor and Stambaugh (2000) is given by:

$$\mu|\psi \sim \mathcal{N}(\mu_M^*, V_M^*), \tag{21}$$

[12]The model probabilities are based on all available information at time t so they are posterior probabilities. We abstract here from the procedure used by the investor to update his priors about models at time t with a new information at time t as in Johannes, Korteweg, and Polson (2011). Although it is crucial for learning problems over time it is less important for decision problem of investor at any specific point at time which is the focus of this paper. Moreover, the information used by the investor to build his beliefs about models can be based not only on a realization of returns but also on soft information such as new research papers that he reads or opinions of people who he talks to.

where

$$\mu_M^* = \sum_{j=1}^{J} \mu_j^* f(M_j|\psi)$$

$$V_M^* = \sum_{j=1}^{J} V_j^* f(M_j|\psi) + \sum_{j=1}^{J} (\mu_j^* - \mu_M^*)(\mu_j^* - \mu_M^*)' f(M_j|\psi).$$

The unconditional means of the expected returns (μ_M^*) is a weighted average of means of expected returns of each model (μ_j^*) weighted by the probability of each model ($f(M_j|\psi)$). The unconditional covariance matrix of expected returns (V_M^*) is a weighted average of covariance matrices of expected returns of each model (V_j^*) weighted by the probability of each model plus a covariance matrix of means across models. Finally, we must update the returns covariance matrix for the posterior mean:

$$\Sigma_M^* = \Sigma + V_M^*. \tag{22}$$

The updated predictive return distribution is therefore

$$r \sim \mathcal{N}(\mu_M^*, \Sigma_M^*). \tag{23}$$

This distribution incorporates three sources of information: investor's beliefs about different equilibrium asset pricing models, investor's private views and equilibrium beliefs of the average investor calculated from market weights. This distribution of returns enters the optimization problem of the investor.

Table 1: Fama-Macbeth regressions: 100 Size and Book-to-Market Portfolios

$r^{ei}_{t+1} = \boldsymbol{\lambda}\boldsymbol{\beta}^{i}_{f,t} + \alpha^{i}_{t+1}$				
	(i)	(ii)	(iii)	(iv)
β^i_M	0.59	0.58	0.51	0.52
	(0.20)	(0.20)	(0.19)	(0.19)
β^i_{SMB}			0.09	0.09
			(0.11)	(0.11)
β^i_{HML}			0.36	0.34
			(0.11)	(0.11)
χ^2_{N-K}	203.11	197.57	198.48	184.39
p-value	(0.00)	(0.00)	(0.00)	(0.00)
$E_T[\boldsymbol{r}^{ef}_t] = (0.49, 0.14, 0.32)$				

Reported are average slopes λ from cross-sectional regressions of monthly percent excess returns r^{ei}_{t+1} on risk loadings $\beta^{i,f}_t$. The sample period is 7/1937 to 6/2011. We omit months in which any of the assets has an invalid return. In specifications (i) and (iii), βs are calculated using the full sample covariance matrices Σ and Σ_f, and the time t factor weights using equation (5). By contrast, specifications (ii) and (iv) use the slopes from a full sample time series regression of asset excess returns on a constant and factor excess returns. In parenthesis are standard errors for the average slopes. χ^2_{N-K} is a test of the null hypothesis that the pricing errors α are zero. The last row reports the average risk premia on the factors.

Table 2: Fama-Macbeth Regressions: 49 Industry Portfolios

$r^{ei}_{t+1} = \boldsymbol{\lambda}\boldsymbol{\beta}^i_{f,t} + \alpha^i_{t+1}$				
	(i)	(ii)	(iii)	(iv)
β^i_M	0.49	0.49	0.58	0.58
	(0.21)	(0.22)	(0.21)	(0.21)
β^i_{SMB}			-0.49	-0.38
			(0.19)	(0.19)
β^i_{HML}			0.05	-0.05
			(0.18)	(0.18)
χ^2_{N-K}	62.51	54.68	55.39	48.12
p-value	(0.08)	(0.24)	(0.16)	(0.39)
$E_T[\boldsymbol{r}^{ef}_t] = (0.44, 0.19, 0.39)$				

Reported are average slopes λ from cross-sectional regressions of monthly percent excess returns r^{ei}_{t+1} on risk loadings $\beta^{i,f}_t$. The sample period is 7/1969 to 6/2011. We omit months in which any of the assets has an invalid return. In specifications (i) and (iii), βs are calculated using the full sample covariance matrices Σ and Σ_f, and the time t factor weights using equation (5). By contrast, specifications (ii) and (iv) use the slopes from a full sample time series regression of asset excess returns on a constant and factor excess returns. In parenthesis are standard errors for the average slopes. χ^2_{N-K} is a test of the null hypothesis that the pricing errors α are zero. The last row reports the average risk premia on the factors.

Table 3: Out-of-Sample Relative R^2: 100 Size and Book-to-Market Portfolios

$$R^2 = 1 - MSE(model)/MSE(hist)$$

(a) $MSE(model) = \frac{1}{T}\sum_{t=1}^{T}\left(r_{t+1}^{ei} - \beta_{m,t}^{i} E_t[r^{em}]\right)^2$

-1.50	-1.47	-1.20	-1.15	-1.12	-1.05	-1.04	-1.01	-0.96	-0.95
-0.90	-0.85	-0.81	-0.81	-0.78	-0.77	-0.75	-0.72	-0.71	-0.71
-0.69	-0.68	-0.65	-0.61	-0.60	-0.60	-0.60	-0.59	-0.56	-0.54
-0.47	-0.46	-0.46	-0.44	-0.42	-0.39	-0.38	-0.34	-0.34	-0.34
-0.34	-0.33	-0.33	-0.32	-0.32	-0.32	-0.29	-0.29	-0.23	-0.22
-0.21	-0.15	-0.12	-0.10	-0.10	-0.07	-0.07	-0.05	-0.05	-0.04
-0.04	-0.03	-0.02	-0.02	-0.00	0.01	0.02	0.02	0.05	0.05
0.08	0.10	0.12	0.14	0.15	0.15	0.16	0.17	0.17	0.18
0.18	0.25	0.28	0.29	0.29	0.30	0.31	0.32	0.32	0.35
0.36	0.37	0.44	0.46	0.49	0.51	0.53	0.67	0.78	1.03

Average R^2 = -0.21

(b) $MSE(model) = \frac{1}{T}\sum_{t=1}^{T}\left(r_{t+1}^{ei} - \boldsymbol{\beta}_{f,t}^{i} E_t[\boldsymbol{r}^{ef}]\right)^2$

-0.61	-0.58	-0.57	-0.55	-0.50	-0.48	-0.46	-0.44	-0.42	-0.39
-0.38	-0.37	-0.34	-0.33	-0.33	-0.31	-0.30	-0.30	-0.29	-0.27
-0.26	-0.25	-0.21	-0.20	-0.19	-0.18	-0.18	-0.18	-0.17	-0.15
-0.15	-0.14	-0.12	-0.12	-0.12	-0.12	-0.11	-0.10	-0.08	-0.08
-0.07	-0.07	-0.07	-0.06	-0.06	-0.05	-0.04	-0.03	-0.03	-0.02
-0.01	0.00	0.01	0.01	0.02	0.03	0.04	0.05	0.06	0.08
0.08	0.11	0.11	0.12	0.12	0.13	0.13	0.14	0.15	0.15
0.15	0.17	0.18	0.18	0.21	0.21	0.22	0.23	0.23	0.25
0.28	0.29	0.30	0.31	0.31	0.34	0.35	0.37	0.39	0.40
0.40	0.41	0.43	0.48	0.49	0.69	0.71	0.87	0.99	1.12

Average R^2 = 0.03

Reported are the percent out-of-sample R^2 of monthly percent excess returns r_{t+1}^{ei} predicted using the expected return given time t information. The R^2 statistic compares the forecasting ability of the model with a forecast using the historical average return on the asset. A positive R^2 means that the model forecast is better than historical averages. The first panel uses the Black-Litterman model while the second panel uses our extension with the Fama-French three factor model. Risk loadings β are calculated using the time t sample covariance matrices Σ and Σ_f, and the time t factor weights using equation (5). The sample period is 7/1937 to 6/2011. We omit the first 240 months to reliably estimate the covariance matrices, as well as months in which any of the assets has an invalid return. Results are sorted in ascending order.

Table 4: Out-of-Sample Relative R^2: 49 Industry Portfolios

$$R^2 = 1 - MSE(model)/MSE(hist)$$

(a) $MSE(model) = \frac{1}{T}\sum_{t=1}^{T}\left(r_{t+1}^{ei} - \beta_{m,t}^{i} E_t[r^{em}]\right)^2$

-0.80	-0.63	-0.60	-0.45	-0.35	-0.28	-0.26
-0.25	-0.24	-0.16	-0.14	-0.10	-0.10	-0.05
-0.04	-0.03	-0.03	-0.03	-0.01	0.01	0.02
0.03	0.04	0.05	0.05	0.05	0.07	0.07
0.07	0.08	0.13	0.14	0.15	0.15	0.16
0.18	0.19	0.20	0.23	0.23	0.25	0.27
0.27	0.28	0.32	0.33	0.34	0.43	0.87

Average R^2 = 0.05

(b) $MSE(model) = \frac{1}{T}\sum_{t=1}^{T}\left(r_{t+1}^{ei} - \boldsymbol{\beta}_{f,t}^{i} E_t[\boldsymbol{r}^{ef}]\right)^2$

-1.62	-1.04	-1.03	-0.75	-0.72	-0.44	-0.42
-0.32	-0.25	-0.24	-0.16	-0.08	-0.07	-0.07
-0.04	-0.01	-0.01	-0.01	0.01	0.01	0.01
0.02	0.05	0.05	0.06	0.06	0.07	0.07
0.07	0.07	0.08	0.13	0.15	0.15	0.17
0.18	0.21	0.22	0.25	0.26	0.29	0.29
0.30	0.32	0.32	0.40	0.46	0.50	0.69

Average R^2 = 0.03

Reported are the percent out-of-sample R^2 of monthly percent excess returns r_{t+1}^{ei} predicted using the expected return given time t information. The R^2 statistic compares the forecasting ability of the model with a forecast using the historical average return on the asset. A positive R^2 means that the model forecast is better than historical averages. The first panel uses the Black-Litterman model while the second panel uses our extension with the Fama-French three factor model. Risk loadings β are calculated using the time t sample covariance matrices Σ and Σ_f, and the time t factor weights using equation (5). The sample period is 7/1969 to 6/2011. We omit the first 240 months to reliably estimate the covariance matrices, as well as months in which any of the assets has an invalid return. Results are sorted in ascending order.

Table 5: Simulated Strategies Performance: 100 Size and Book-to-Market Portfolios

Max Sharpe Ratio Portfolio	$E[r^e]$		Sharpe Ratio		SR Diff		Alpha		Alpha (FF3F)	
Signal noise factor: $\sigma = 10$										
Dogmatic informed investor	3.81	(0.18)	3.21	(0.19)			3.81	(0.18)	3.82	(0.19)
CAPM and signals	4.34	(0.20)	2.98	(0.19)	-0.23	(0.07)	4.28	(0.21)	4.28	(0.22)
ICAPM (FF3F) and signals	4.35	(0.20)	3.00	(0.19)	-0.21	(0.08)	4.30	(0.21)	4.17	(0.21)
Signal noise factor: $\sigma = 20$										
Dogmatic informed investor	1.93	(0.18)	1.84	(0.18)			1.93	(0.19)	1.93	(0.19)
CAPM and signals	2.30	(0.21)	1.81	(0.17)	-0.04	(0.06)	2.17	(0.22)	2.15	(0.23)
ICAPM (FF3F) and signals	2.43	(0.21)	1.80	(0.17)	-0.04	(0.08)	2.32	(0.21)	2.05	(0.22)
Signal noise factor: $\sigma = 40$										
Dogmatic informed investor	0.97	(0.18)	0.96	(0.18)			0.97	(0.19)	0.97	(0.19)
CAPM and signals	1.35	(0.21)	1.03	(0.17)	0.07	(0.06)	1.08	(0.22)	1.05	(0.23)
ICAPM (FF3F) and signals	1.65	(0.20)	1.02	(0.13)	0.06	(0.08)	1.43	(0.21)	0.91	(0.22)
Signal noise factor: $\sigma = 80$										
Dogmatic informed investor	0.49	(0.18)	0.49	(0.18)			0.49	(0.19)	0.49	(0.19)
CAPM and signals	1.06	(0.21)	0.66	(0.14)	0.17	(0.07)	0.52	(0.22)	0.45	(0.23)
ICAPM (FF3F) and signals	1.62	(0.19)	0.69	(0.08)	0.20	(0.11)	1.22	(0.19)	0.27	(0.20)
Signal noise factor: $\sigma = 160$										
Dogmatic informed investor	0.24	(0.18)	0.25	(0.18)			0.25	(0.19)	0.25	(0.19)
CAPM and signals	1.28	(0.22)	0.52	(0.09)	0.27	(0.11)	0.22	(0.22)	0.08	(0.23)
ICAPM (FF3F) and signals	2.03	(0.15)	0.60	(0.05)	0.35	(0.14)	1.43	(0.15)	-0.07	(0.16)
Reference Portfolios:										
Market	8.22		0.52							
SMB	2.46		0.27							
HML	3.38		0.37							
1/N	10.37		0.57							
Max SR - Historical Means	0.76		0.61							

Reported are annualized monthly portfolio performance measures for five different levels of private signal noise σ, where the signal is actual next month asset returns augmented by noise as in (19). For each σ, we compare three strategies which aim to maximize portfolio Sharpe ratios: (1) *Dogmatic informed investor* only uses the private signals, (2) *CAPM and signals* uses the Black-Litterman approach with learning from market weights at time t about returns at $t+1$, and (3) *ICAPM (FF3F) and signals* uses our extension with learning from Fama-French factor weights. *SR Diff* is the Sharpe Ratio of the strategy less that of the Dogmatic one. We use the sample covariance estimate of the pricing errors vector $Cov[\alpha]$ from the cross-sectional regressions as an estimator for model misspecification V_ω in (12). The sample period is 7/1937 to 6/2011. For reference, we report the performance of the three factors portfolio, a strategy that equally weighs each asset every period (*1/N*), and *Max SR - Historical Means*, which uses the historical average returns of each asset at time t to form the maximum Sharpe ratio portfolio held to $t+1$. We omit the first 240 months to reliably estimate the covariance matrices, as well as months in which any of the assets has an invalid return. In parenthesis are standard errors for the reported means across the 300 simulated draws of private signal noise.

Table 6: Simulated Strategies Performance: 49 Industry Portfolios

Max Sharpe Ratio Portfolio	$E[r^e]$		Sharpe Ratio		SR Diff		Alpha		Alpha (FF3F)	
Signal noise factor: $\sigma = 10$										
Dogmatic informed investor	7.11	(0.45)	2.60	(0.18)			7.11	(0.46)	7.14	(0.47)
CAPM and signals	7.94	(0.50)	2.51	(0.17)	-0.09	(0.06)	7.80	(0.51)	7.81	(0.52)
ICAPM (FF3F) and signals	7.60	(0.49)	2.41	(0.17)	-0.19	(0.08)	7.40	(0.50)	7.19	(0.51)
Signal noise factor: $\sigma = 20$										
Dogmatic informed investor	3.59	(0.46)	1.49	(0.19)			3.60	(0.47)	3.61	(0.48)
CAPM and signals	4.22	(0.52)	1.49	(0.18)	-0.00	(0.06)	3.95	(0.53)	3.92	(0.55)
ICAPM (FF3F) and signals	3.87	(0.49)	1.23	(0.16)	-0.26	(0.09)	3.50	(0.50)	3.05	(0.52)
Signal noise factor: $\sigma = 40$										
Dogmatic informed investor	1.80	(0.47)	0.77	(0.20)			1.80	(0.47)	1.81	(0.49)
CAPM and signals	2.49	(0.53)	0.84	(0.18)	0.07	(0.06)	1.95	(0.54)	1.86	(0.55)
ICAPM (FF3F) and signals	1.99	(0.44)	0.51	(0.11)	-0.26	(0.11)	1.36	(0.45)	0.54	(0.46)
Signal noise factor: $\sigma = 80$										
Dogmatic informed investor	0.90	(0.47)	0.39	(0.20)			0.90	(0.48)	0.91	(0.49)
CAPM and signals	1.98	(0.52)	0.53	(0.14)	0.14	(0.09)	0.91	(0.53)	0.73	(0.55)
ICAPM (FF3F) and signals	1.17	(0.33)	0.23	(0.07)	-0.15	(0.15)	0.25	(0.33)	-0.95	(0.34)
Signal noise factor: $\sigma = 160$										
Dogmatic informed investor	0.44	(0.47)	0.19	(0.21)			0.45	(0.48)	0.45	(0.49)
CAPM and signals	2.40	(0.51)	0.42	(0.09)	0.22	(0.13)	0.37	(0.52)	0.02	(0.54)
ICAPM (FF3F) and signals	0.90	(0.20)	0.16	(0.04)	-0.04	(0.17)	-0.19	(0.20)	-1.62	(0.21)
Reference Portfolios:										
Market	6.35		0.41							
SMB	2.25		0.19							
HML	2.75		0.24							
1/N	7.56		0.48							
Max SR - Historical Means	0.56		0.22							

Reported are annualized monthly portfolio performance measures for five different levels of private signal noise σ, where the signal is actual next month asset returns augmented by noise as in (19). For each σ, we compare three strategies which aim to maximize portfolio Sharpe ratios: (1) *Dogmatic informed investor* only uses the private signals, (2) *CAPM and signals* uses the Black-Litterman approach with learning from market weights at time t about returns at $t+1$, and (3) *ICAPM (FF3F) and signals* uses our extension with learning from Fama-French factor weights. *SR Diff* is the Sharpe Ratio of the strategy less that of the Dogmatic one. We use the sample covariance estimate of the pricing errors vector $Cov[\alpha]$ from the cross-sectional regressions as an estimator for model misspecification V_ω in (12). The sample period is 7/1969 to 6/2011. For reference, we report the performance of the three factors portfolio, a strategy that equally weighs each asset every period (*1/N*), and *Max SR - Historical Means*, which uses the historical average returns of each asset at time t to form the maximum Sharpe ratio portfolio held to $t+1$. We omit the first 240 months to reliably estimate the covariance matrices, as well as months in which any of the assets has an invalid return. In parenthesis are standard errors for the reported means across the 300 simulated draws of private signal noise.

Table 7: Simulated Strategies Performance: 5 Industry Portfolios (Robustness)

Max Sharpe Ratio Portfolio	$E[r^e]$		Sharpe Ratio		SR Diff		Alpha		Alpha (FF3F)	
Signal noise factor: $\sigma = 1$										
Dogmatic informed investor	20.58	(0.28)	3.66	(0.09)			20.63	(0.29)	20.59	(0.31)
CAPM and signals	17.77	(0.23)	3.47	(0.07)	-0.19	(0.06)	17.55	(0.23)	17.45	(0.24)
ICAPM (FF3F) and signals	15.44	(0.21)	3.30	(0.08)	-0.36	(0.07)	15.17	(0.22)	14.99	(0.22)
Signal noise factor: $\sigma = 5$										
Dogmatic informed investor	6.26	(0.52)	1.27	(0.11)			6.27	(0.54)	6.25	(0.56)
CAPM and signals	6.45	(0.49)	1.24	(0.10)	-0.02	(0.06)	5.49	(0.51)	5.45	(0.53)
ICAPM (FF3F) and signals	5.52	(0.46)	1.09	(0.10)	-0.17	(0.08)	4.44	(0.48)	3.94	(0.49)
Signal noise factor: $\sigma = 10$										
Dogmatic informed investor	3.20	(0.56)	0.67	(0.12)			3.20	(0.58)	3.19	(0.60)
CAPM and signals	4.64	(0.56)	0.74	(0.09)	0.07	(0.08)	2.79	(0.59)	2.76	(0.61)
ICAPM (FF3F) and signals	3.76	(0.46)	0.63	(0.08)	-0.04	(0.09)	1.95	(0.48)	1.18	(0.49)
Signal noise factor: $\sigma = 20$										
Dogmatic informed investor	1.59	(0.56)	0.33	(0.12)			1.58	(0.59)	1.58	(0.61)
CAPM and signals	4.69	(0.58)	0.53	(0.07)	0.20	(0.09)	1.36	(0.62)	1.34	(0.64)
ICAPM (FF3F) and signals	3.41	(0.40)	0.45	(0.06)	0.12	(0.10)	0.72	(0.42)	-0.37	(0.43)
Signal noise factor: $\sigma = 40$										
Dogmatic informed investor	0.77	(0.57)	0.16	(0.12)			0.76	(0.59)	0.76	(0.61)
CAPM and signals	5.85	(0.50)	0.48	(0.04)	0.32	(0.10)	0.72	(0.53)	0.70	(0.55)
ICAPM (FF3F) and signals	3.72	(0.30)	0.41	(0.03)	0.25	(0.10)	0.25	(0.31)	-1.12	(0.32)
Reference Portfolios:										
Market	6.80		0.45							
SMB	1.83		0.19							
HML	4.44		0.47							
1/N	7.40		0.50							
Max SR - Historical Means	2.90		0.48							

Reported are annualized monthly portfolio performance measures for five different levels of private signal noise σ, where the signal is actual next month asset returns augmented by noise as in (19). For each σ, we compare three strategies which aim to maximize portfolio Sharpe ratios: (1) *Dogmatic informed investor* only uses the private signals, (2) *CAPM and signals* uses the Black-Litterman approach with learning from market weights at time t about returns at $t+1$, and (3) *ICAPM (FF3F) and signals* uses our extension with learning from Fama-French factor weights. *SR Diff* is the Sharpe Ratio of the strategy less that of the Dogmatic one. We use the sample covariance estimate of the pricing errors vector $Cov[\alpha]$ from the cross-sectional regressions as an estimator for model misspecification V_ω in (12). The sample period is 7/1926 to 6/2011. For reference, we report the performance of the three factors portfolio, a strategy that equally weighs each asset every period (*1/N*), and *Max SR - Historical Means*, which uses the historical average returns of each asset at time t to form the maximum Sharpe ratio portfolio held to $t+1$. We omit the first 240 months to reliably estimate the covariance matrices, as well as months in which any of the assets has an invalid return. In parenthesis are standard errors for the reported means across the 300 simulated draws of private signal noise.

Table 8: Simulated Strategies Performance: 10 Industry Portfolios (Robustness)

Max Sharpe Ratio Portfolio	$E[r^e]$		Sharpe Ratio		SR Diff		Alpha		Alpha (FF3F)	
Signal noise factor: $\sigma = 1$										
Dogmatic informed investor	23.90	(0.23)	4.66	(0.09)			23.84	(0.23)	23.74	(0.24)
CAPM and signals	22.74	(0.22)	4.59	(0.08)	-0.06	(0.04)	22.55	(0.22)	22.42	(0.22)
ICAPM (FF3F) and signals	20.45	(0.21)	4.58	(0.08)	-0.07	(0.06)	20.25	(0.21)	20.09	(0.21)
Signal noise factor: $\sigma = 5$										
Dogmatic informed investor	7.03	(0.44)	1.81	(0.12)			7.01	(0.45)	6.98	(0.45)
CAPM and signals	7.34	(0.47)	1.78	(0.12)	-0.03	(0.04)	6.86	(0.47)	6.81	(0.47)
ICAPM (FF3F) and signals	6.37	(0.43)	1.66	(0.12)	-0.15	(0.06)	5.87	(0.43)	5.62	(0.43)
Signal noise factor: $\sigma = 10$										
Dogmatic informed investor	3.59	(0.46)	0.98	(0.13)			3.58	(0.47)	3.56	(0.47)
CAPM and signals	4.44	(0.49)	1.03	(0.12)	0.05	(0.05)	3.52	(0.50)	3.47	(0.49)
ICAPM (FF3F) and signals	3.73	(0.43)	0.92	(0.11)	-0.07	(0.07)	2.82	(0.44)	2.39	(0.44)
Signal noise factor: $\sigma = 20$										
Dogmatic informed investor	1.80	(0.46)	0.50	(0.13)			1.79	(0.47)	1.79	(0.48)
CAPM and signals	3.55	(0.50)	0.65	(0.09)	0.15	(0.06)	1.78	(0.51)	1.71	(0.51)
ICAPM (FF3F) and signals	2.73	(0.40)	0.55	(0.08)	0.04	(0.08)	1.15	(0.41)	0.45	(0.41)
Signal noise factor: $\sigma = 40$										
Dogmatic informed investor	0.90	(0.46)	0.25	(0.13)			0.89	(0.47)	0.89	(0.48)
CAPM and signals	4.16	(0.50)	0.51	(0.06)	0.26	(0.09)	0.92	(0.51)	0.80	(0.51)
ICAPM (FF3F) and signals	2.76	(0.34)	0.42	(0.05)	0.17	(0.09)	0.33	(0.35)	-0.70	(0.34)
Reference Portfolios:										
Market	6.80		0.45							
SMB	1.83		0.19							
HML	4.44		0.47							
1/N	7.50		0.53							
Max SR - Historical Means	2.10		0.54							

Reported are annualized monthly portfolio performance measures for five different levels of private signal noise σ, where the signal is actual next month asset returns augmented by noise as in (19). For each σ, we compare three strategies which aim to maximize portfolio Sharpe ratios: (1) *Dogmatic informed investor* only uses the private signals, (2) *CAPM and signals* uses the Black-Litterman approach with learning from market weights at time t about returns at $t+1$, and (3) *ICAPM (FF3F) and signals* uses our extension with learning from Fama-French factor weights. *SR Diff* is the Sharpe Ratio of the strategy less that of the Dogmatic one. We use the sample covariance estimate of the pricing errors vector $Cov[\alpha]$ from the cross-sectional regressions as an estimator for model misspecification V_ω in (12). The sample period is 7/1926 to 6/2011. For reference, we report the performance of the three factors portfolio, a strategy that equally weighs each asset every period (*1/N*), and *Max SR - Historical Means*, which uses the historical average returns of each asset at time t to form the maximum Sharpe ratio portfolio held to $t+1$. We omit the first 240 months to reliably estimate the covariance matrices, as well as months in which any of the assets has an invalid return. In parenthesis are standard errors for the reported means across the 300 simulated draws of private signal noise.