An Empirical Evaluation of the Black-Litterman Approach to Portfolio Choice

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Motivation

- The Black-Litterman approach is applied to allocate billions of dollars.
  - Applications: Goldman Sachs Asset Management, Ambrose Capital Management, etc.
  - Software: Ibbotson Associates, Zephyr Associates, etc.
  - Bob Litterman: 2008 Financial Engineer of the Year Award, CFA Institute Nicholas Molodovsky Award.
- Useful for active portfolio managers
  - Allows to incorporate private signals
  - Views on portfolios of assets
  - Allows to combine private views with market views
- Uses equilibrium weights to learn about “market views”
  - Addresses the problem of extreme portfolio weights
  - Account for information available to other investors

What do we do?

- Extend the Black-Litterman approach to linear multi-factor models.
  - The CAPM has been rejected empirically.
  - Consistency between the equilibrium model and benchmarking (CAPM, FF3F).
- Extend their approach to account for asset pricing model uncertainty.
- Evaluate empirically.
  - A novel approach to simulate private signals.
  - Industry portfolios and portfolios formed on book-to-market and size.
  - Long time-series of data.

One-period Portfolio Choice Framework

- Input to portfolio problem: distribution of returns
- Use historical covariance matrix
- Prior about expected returns
- Define the set of models and provide probability for each model
- For each model:
    - Learn from prices about expected excess returns
    - Posterior about expected excess returns
- Bayesian Model Averaging
- Portfolio decision

Learning from weights about expected excess returns

- In equilibrium:

$$\begin{aligned}
\mu_{\omega,t}^{CAPM} &= E_t(R_{t+1}^{e}) = \underbrace{\Sigma[\omega_{m,t}]Var(R^{em})^{-1}}_{\beta_t^{market}} E_t(R_{t+1}^{em}) \\
\mu_{\omega,t}^{ICAPM} &= E_t(R_{t+1}^{e}) = \underbrace{\Sigma[\omega_{m,t}\ \omega_{s,t}]\Sigma_f^{-1}}_{\beta_t^{factors}} \begin{bmatrix} E_t(R_{t+1}^{em}) \\ E_t(R_{t+1}^{es}) \end{bmatrix}
\end{aligned} \tag{1}$$

The likelihood $\ell(\omega|\theta) = \ell([\omega_m\ \omega_s]|\mu,\Sigma)$ for expected returns is then:

$$\mu|\omega \sim \mathcal{N}(\mu_\omega, V_\omega) \tag{2}$$

where V_ω is an unbiased estimator of the sample mean's covariance $(\frac{1}{T-1})\Sigma$ and T is the length of the time-series used to estimate the covariance matrix Σ.

- The Black-Litterman approach assumes there is only market factor.
- Approximation: ICAPM and CAPM hold only for homogeneous information.
- Admati (1985) models heterogeneously informed agents and CAPM does not hold in her setting.

Calculating weights

To calculate weights:

$$R^s = \omega_s R \tag{3}$$

where R^s is the return on factor s and R is a vector of the N basic assets' returns. This ω_s can always be calculated based on the mimicking portfolios' definition. Next consider an I vector of portfolios R^I formed from the same set of N assets:

$$R^I = MR \tag{4}$$

where M is an $I \times N$ matrix of portfolio weights. If we restrict the portfolio problem to the set of I assets, then the weights matrix required is such that:

$$R^s = X^s R^I = X^s MR \tag{5}$$

where X^s is an $1 \times I$ vector of the weights that each portfolio has in the factor s. But, this implies that we require $X^s M = \omega_s$ for every factor s. While this equation has a solution for value-weighted portfolios when the factor is the market, it does not have a general solution for every set of portfolios I and for any factor s.

Investor's Prior and Posterior Distribution

An investor provides a $P = K \times N$ matrix of portfolio weights and $Q = K \times 1$ vector of expected returns such that his views are described by:

$$P\mu = Q + \varepsilon,\ \varepsilon \sim \mathcal{N}(0, \Omega) \tag{6}$$

The use of a conjugate normal prior parameterized by P, Q and Ω with a normal likelihood parameterized by μ_ω and V_ω leads to the following normal posterior:

$$\mu | P, Q, \Omega, \omega \sim \mathcal{N}(\mu^*, V^*) \tag{7}$$

where

$$\begin{aligned} \mu^* &= [P'\Omega^{-1}P + V_\omega^{-1}]^{-1}[P'\Omega^{-1}Q + V_\omega^{-1}\mu_\omega] \\ V^* &= [P'\Omega^{-1}P + V_\omega^{-1}]^{-1} \end{aligned}$$

Finally, we must update the returns covariance matrix:

$$\Sigma^* = \Sigma + V^* \tag{8}$$

So that the posterior distribution of returns, which updates the investor's views with those of the average investor are distributed:

$$R | P, Q, \Omega, \omega \sim \mathcal{N}(\mu^*, \Sigma^*) \tag{9}$$

Bayesian Model Averaging

The unconditional expected return distribution is the expected return distribution conditional on a model times the probability of each model:

$$p(\mu|\psi,\omega) = \sum_{j=1}^{J} p(\mu|\psi,\omega,M_j) f(M_j|\psi,\omega) \tag{10}$$

With normal conditional distributions, the unconditional distribution is also normal and as shown by Pastor and Stambaugh (2000) is given by:

$$\mu|\psi \sim \mathcal{N}(\mu_M^*, V_M^*) \tag{11}$$

$$\mu_M^* = \sum_{j=1}^{J} \mu_j^* f(M_j|\psi)$$

$$V_M^* = \sum_{j=1}^{J} V_j^* f(M_j|\psi) + \sum_{j=1}^{J} (\mu_j^* - \mu_M^*)(\mu_j^* - \mu_M^*)' f(M_j|\psi)$$

Finally, we must update the returns covariance matrix for the posterior mean:

$$\Sigma_M^* = \Sigma + V_M^* \tag{12}$$

Bayesian Model Averaging (2)

The resulting posterior distribution of excess returns is:

$$R \sim \mathscr{N}(\mu_M^*, \Sigma_M^*) \tag{13}$$

This distribution incorporates three sources of information:

- Investor's beliefs about different equilibrium asset pricing models
- Investor's private views
- Views of the average investor calculated from prices

This distribution of returns enters the optimization problem of the investor.

Modeling private signals

- Private information is by definition unobservable.
- We let our investor observe a noisy signal of next-period returns:

$$Q_t = R^e_{t+1} + \sigma\tilde{\varepsilon}_t \tag{14}$$

  where ε_t is randomly drawn from a zero mean normal distribution with covariance matrix of excess returns Σ.
- As $\sigma \longrightarrow \infty$ the signals become uninformative and the investor only uses market implied expected returns to form his posterior.
- Our specification for the noise preserves the correlation structure between the assets and allows us to control the precision of the signal with a single parameter.
- We report performance estimates for six levels of signal noise factor (σ) ranging from 0.1 to 10.

Empirical Evaluation

- Equilibrium Models
  - No model (Dogmatic informed investor)
  - CAPM (Black-Litterman)
  - FF3F model (ICAPM)
- Investment Opportunity Sets
  - 5,10 and 49 industry portfolios and 100 book-to-market and size portfolios from Ken French's website
  - Commonly used, bias-free, long time-series
  - B/M and Size decile portfolios allow for exact solution for the weights in HML and SMB.
- Track performance of the Maximum Sharpe ratio portfolio
  - Sensitive to the inputs of expected returns
  - Simple approach to evaluate informativeness of equilibrium prices
  - Not effected by leverage
  - Normalization eliminates variation in the dollar size of the bet

$$\omega_t^* = \frac{\widehat{\Sigma}_t^{-1} \mu_t}{\mathbf{1}\left|\widehat{\Sigma}_t^{-1} \mu_t\right|} \tag{15}$$

Description of the Strategies

Periods	**Scenarios**			
	1		…	D
	Optimal weight	Realized return		
Data First Month				
	w(1)			
Strategy First Month	w(2)	R(1)=w(1)*R(assets)		
	w(3)	R(2)=w(2)*R(assets)		
	…	R(3)=w(3)*R(assets)		
		…		
Last Month		R(T)		
	Mean return	Mean 1		Mean D
	Sharpe Ratio	SR 1		SR D

Table 1

Assets: 100 Portfolios Formed on Size and Book-to-Market

Period: (7/1937-9/2009)

Strategy Frequency: Monthly

Minimum History Length: 240

Number of Simulations: 100

Max Sharpe Ratio portfolio	Mean Excess Return		Sharpe Ratio		Alpha		Alpha (FF3F)	
Signal noise factor: $\sigma = 0.10$								
Dogmatic informed investor	2.36	(0.00)	3.04	(0.00)	2.36	(0.00)	2.37	(0.00)
CAPM and signals	2.36	(0.00)	3.05	(0.00)	2.36	(0.00)	2.37	(0.00)
ICAPM (FF3F) and signals	2.36	(0.00)	3.06	(0.00)	2.37	(0.00)	2.37	(0.00)
Signal noise factor: $\sigma = 0.25$								
Dogmatic informed investor	2.33	(0.00)	3.00	(0.00)	2.33	(0.00)	2.34	(0.00)
CAPM and signals	2.38	(0.00)	3.27	(0.00)	2.37	(0.00)	2.38	(0.00)
ICAPM (FF3F) and signals	2.38	(0.00)	3.27	(0.00)	2.37	(0.00)	2.35	(0.00)
Signal noise factor: $\sigma = 0.50$								
Dogmatic informed investor	2.19	(0.00)	2.85	(0.00)	2.19	(0.00)	2.20	(0.00)
CAPM and signals	2.18	(0.00)	2.98	(0.00)	2.15	(0.00)	2.15	(0.00)
ICAPM (FF3F) and signals	2.04	(0.00)	2.67	(0.00)	2.01	(0.00)	1.95	(0.00)
Signal noise factor: $\sigma = 1.00$								
Dogmatic informed investor	1.76	(0.01)	2.54	(0.01)	1.76	(0.01)	1.77	(0.01)
CAPM and signals	1.69	(0.01)	1.74	(0.00)	1.59	(0.01)	1.59	(0.01)
ICAPM (FF3F) and signals	1.20	(0.00)	1.19	(0.00)	1.15	(0.00)	1.00	(0.00)
Signal noise factor: $\sigma = 2.00$								
Dogmatic informed investor	1.09	(0.01)	2.19	(0.01)	1.09	(0.01)	1.10	(0.01)
CAPM and signals	1.07	(0.01)	0.61	(0.00)	0.85	(0.00)	0.84	(0.00)
ICAPM (FF3F) and signals	0.58	(0.00)	0.47	(0.00)	0.52	(0.00)	0.32	(0.00)
Signal noise factor: $\sigma = 10.00$								
Dogmatic informed investor	0.23	(0.00)	0.88	(0.01)	0.24	(0.00)	0.24	(0.00)
CAPM and signals	0.58	(0.00)	0.15	(0.00)	0.09	(0.00)	0.08	(0.00)
ICAPM (FF3F) and signals	0.29	(0.00)	0.21	(0.00)	0.22	(0.00)	-0.00	(0.00)
Reference Portfolios:								
Market	0.59		0.13					
SMB	0.14		0.05					
HML	0.35		0.13					
Max Sharpe Ratio portfolio (historical mean)	0.06		0.19					

In parenthesis are standard errors for the mean values across the simulations.

Only months in which all assets have valid returns are included.

Table 2

Assets: 5 Industry Portfolios

Period: (7/1926-12/2007)

Strategy Frequency: Monthly

Minimum History Length: 240

Number of Simulations: 100

Max Sharpe Ratio portfolio	Mean Excess Return		Sharpe Ratio		Alpha		Alpha (FF3F)	
Signal noise factor: $\sigma = 0.10$								
Dogmatic informed investor	2.38	(0.00)	1.65	(0.00)	2.38	(0.00)	2.36	(0.00)
CAPM and signals	2.43	(0.00)	1.63	(0.00)	2.38	(0.00)	2.36	(0.00)
ICAPM (FF3F) and signals	2.40	(0.00)	1.61	(0.00)	2.35	(0.00)	2.33	(0.00)
Signal noise factor: $\sigma = 0.25$								
Dogmatic informed investor	2.31	(0.01)	1.57	(0.01)	2.31	(0.01)	2.30	(0.01)
CAPM and signals	2.41	(0.01)	0.96	(0.00)	2.15	(0.01)	2.13	(0.01)
ICAPM (FF3F) and signals	1.99	(0.01)	0.97	(0.00)	1.79	(0.01)	1.73	(0.01)
Signal noise factor: $\sigma = 0.50$								
Dogmatic informed investor	2.13	(0.02)	1.39	(0.01)	2.12	(0.02)	2.11	(0.02)
CAPM and signals	1.51	(0.00)	0.39	(0.00)	0.99	(0.00)	0.99	(0.00)
ICAPM (FF3F) and signals	1.06	(0.01)	0.40	(0.00)	0.76	(0.00)	0.62	(0.00)
Signal noise factor: $\sigma = 1.00$								
Dogmatic informed investor	1.71	(0.03)	1.08	(0.02)	1.70	(0.03)	1.69	(0.03)
CAPM and signals	0.88	(0.00)	0.21	(0.00)	0.30	(0.00)	0.29	(0.00)
ICAPM (FF3F) and signals	0.57	(0.00)	0.20	(0.00)	0.24	(0.00)	0.08	(0.00)
Signal noise factor: $\sigma = 2.00$								
Dogmatic informed investor	1.12	(0.03)	0.73	(0.02)	1.12	(0.03)	1.11	(0.03)
CAPM and signals	0.68	(0.00)	0.16	(0.00)	0.09	(0.00)	0.08	(0.00)
ICAPM (FF3F) and signals	0.42	(0.00)	0.15	(0.00)	0.09	(0.00)	-0.07	(0.00)
Signal noise factor: $\sigma = 10.00$								
Dogmatic informed investor	0.25	(0.01)	0.19	(0.01)	0.26	(0.01)	0.25	(0.02)
CAPM and signals	0.61	(0.00)	0.15	(0.00)	0.02	(0.00)	0.01	(0.00)
ICAPM (FF3F) and signals	0.37	(0.00)	0.13	(0.00)	0.04	(0.00)	-0.12	(0.00)
Reference Portfolios:								
Market	0.59		0.14					
SMB	0.12		0.04					
HML	0.40		0.15					
Max Sharpe Ratio portfolio (historical mean)	0.23		0.14					

In parenthesis are standard errors for the mean values across the simulations.

Table 3

Assets: 10 Industry Portfolios
Period: (7/1926-12/2007)
Strategy Frequency: Monthly
Minimum History Length: 240
Number of Simulations: 100

Max Sharpe Ratio portfolio	Mean Excess Return		Sharpe Ratio		Alpha		Alpha (FF3F)	
Signal noise factor: $\sigma = 0.10$								
Dogmatic informed investor	2.80	(0.00)	2.05	(0.00)	2.79	(0.00)	2.78	(0.00)
CAPM and signals	2.83	(0.00)	2.03	(0.00)	2.79	(0.00)	2.78	(0.00)
ICAPM (FF3F) and signals	2.81	(0.00)	2.02	(0.00)	2.78	(0.00)	2.76	(0.00)
Signal noise factor: $\sigma = 0.25$								
Dogmatic informed investor	2.72	(0.01)	1.96	(0.01)	2.70	(0.01)	2.69	(0.01)
CAPM and signals	2.79	(0.01)	1.52	(0.00)	2.63	(0.01)	2.62	(0.01)
ICAPM (FF3F) and signals	2.45	(0.01)	1.52	(0.00)	2.32	(0.01)	2.28	(0.01)
Signal noise factor: $\sigma = 0.50$								
Dogmatic informed investor	2.48	(0.02)	1.74	(0.01)	2.46	(0.02)	2.45	(0.02)
CAPM and signals	2.14	(0.01)	0.67	(0.00)	1.72	(0.01)	1.73	(0.01)
ICAPM (FF3F) and signals	1.48	(0.01)	0.67	(0.00)	1.23	(0.01)	1.13	(0.01)
Signal noise factor: $\sigma = 1.00$								
Dogmatic informed investor	1.95	(0.03)	1.39	(0.02)	1.94	(0.03)	1.92	(0.03)
CAPM and signals	1.17	(0.00)	0.29	(0.00)	0.60	(0.00)	0.61	(0.00)
ICAPM (FF3F) and signals	0.69	(0.00)	0.28	(0.00)	0.40	(0.00)	0.27	(0.00)
Signal noise factor: $\sigma = 2.00$								
Dogmatic informed investor	1.26	(0.03)	1.00	(0.02)	1.24	(0.03)	1.23	(0.03)
CAPM and signals	0.76	(0.00)	0.18	(0.00)	0.17	(0.00)	0.17	(0.00)
ICAPM (FF3F) and signals	0.41	(0.00)	0.16	(0.00)	0.11	(0.00)	-0.03	(0.00)
Signal noise factor: $\sigma = 10.00$								
Dogmatic informed investor	0.29	(0.01)	0.29	(0.01)	0.28	(0.01)	0.28	(0.01)
CAPM and signals	0.61	(0.00)	0.15	(0.00)	0.02	(0.00)	0.01	(0.00)
ICAPM (FF3F) and signals	0.31	(0.00)	0.12	(0.00)	0.01	(0.00)	-0.13	(0.00)
Reference Portfolios:								
Market	0.59		0.14					
SMB	0.12		0.04					
HML	0.40		0.15					
Max Sharpe Ratio portfolio (historical mean)	0.17		0.15					

In parenthesis are standard errors for the mean values across the simulations.

Table 4

Assets: 49 Industry Portfolios

Period: (7/1969-12/2007)

Strategy Frequency: Monthly

Minimum History Length: 240

Number of Simulations: 100

Max Sharpe Ratio portfolio	Mean Excess Return		Sharpe Ratio		Alpha		Alpha (FF3F)	
Signal noise factor: $\sigma = 0.10$								
Dogmatic informed investor	3.86	(0.00)	2.67	(0.00)	3.86	(0.00)	3.87	(0.00)
CAPM and signals	3.86	(0.00)	2.68	(0.00)	3.86	(0.00)	3.87	(0.00)
ICAPM (FF3F) and signals	3.86	(0.00)	2.68	(0.00)	3.86	(0.00)	3.86	(0.00)
Signal noise factor: $\sigma = 0.25$								
Dogmatic informed investor	3.80	(0.00)	2.61	(0.00)	3.80	(0.00)	3.81	(0.00)
CAPM and signals	3.85	(0.00)	2.72	(0.00)	3.83	(0.00)	3.84	(0.00)
ICAPM (FF3F) and signals	3.66	(0.00)	2.57	(0.00)	3.64	(0.00)	3.62	(0.00)
Signal noise factor: $\sigma = 0.50$								
Dogmatic informed investor	3.59	(0.01)	2.43	(0.01)	3.59	(0.01)	3.60	(0.01)
CAPM and signals	3.56	(0.01)	2.43	(0.01)	3.49	(0.01)	3.50	(0.01)
ICAPM (FF3F) and signals	2.31	(0.01)	1.76	(0.00)	2.27	(0.01)	2.19	(0.01)
Signal noise factor: $\sigma = 1.00$								
Dogmatic informed investor	2.94	(0.02)	2.07	(0.01)	2.94	(0.02)	2.94	(0.02)
CAPM and signals	2.74	(0.01)	1.50	(0.00)	2.53	(0.01)	2.54	(0.01)
ICAPM (FF3F) and signals	0.81	(0.00)	0.79	(0.00)	0.76	(0.00)	0.64	(0.00)
Signal noise factor: $\sigma = 2.00$								
Dogmatic informed investor	1.90	(0.02)	1.69	(0.01)	1.90	(0.02)	1.91	(0.02)
CAPM and signals	1.59	(0.01)	0.59	(0.00)	1.20	(0.01)	1.21	(0.01)
ICAPM (FF3F) and signals	0.28	(0.00)	0.29	(0.00)	0.23	(0.00)	0.11	(0.00)
Signal noise factor: $\sigma = 10.00$								
Dogmatic informed investor	0.43	(0.01)	0.69	(0.01)	0.43	(0.01)	0.43	(0.01)
CAPM and signals	0.67	(0.00)	0.17	(0.00)	0.09	(0.00)	0.08	(0.00)
ICAPM (FF3F) and signals	0.11	(0.00)	0.11	(0.00)	0.06	(0.00)	-0.06	(0.00)
Reference Portfolios:								
Market	0.61		0.15					
SMB	0.09		0.03					
HML	0.29		0.09					
Max Sharpe Ratio portfolio (historical mean)	0.08		0.11					

In parenthesis are standard errors for the mean values across the simulations.

Summary of the Empirical Part

- Informativeness of prices relative to private information
- CAPM vs. ICAPM
- Sharpe ratios and number of assets

Conclusion

- Black-Litterman approach: Information (Overall No), Wacky-weights (Maybe), Partial Views (Yes)
- If decide to use: consider model uncertainty
- Consistency between equilibrium model and benchmarking (CAPM, FF3F)
- Caveats for the results