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Invariant manifolds in singular perturbation problems for ordinary differential equations

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Dedicated to Professor Jack K. Hale on the occasion of his 60th birthday

Synopsis

Based on Fenichel's geometric idea, invariant manifold theory is applied to singular perturbation problems. This approach clarifies the nature of outer and inner solutions. A specific condition is given to ensure the existence of heteroclinic connections between normally hyperbolic invariant manifolds. A method to approximate the connections is also presented.

1. Introduction

This is the first of a series of papers in which a geometric approach to singular perturbation problems for ordinary differential equations is presented. Some of the concepts from dynamical system theory, such as centre manifold (in a generalised sense), stable and unstable manifolds, will find natural positions and play important roles in clarifying the behaviour of solutions on an infinite time interval. Standard techniques of asymptotic expansions, inner and outer solutions will reveal a renewed meaning in this context.

Specifically, we are concerned with the study of differential equations of the form

$$(\mathrm{S})_\varepsilon \quad x' = f_0(x, y, \varepsilon), \quad \varepsilon y' = g_0(x, y, \varepsilon)$$

for a positive parameter ε near zero. Here $x, f_0 \in R^m$ and $y, g_0 \in R^n$ and they satisfy the following conditions:

(I) *For a positive integer r, f_0 and g_0 are C^r-bounded as a function of (x, y, ε), and there is a function $h(x)$ whose derivatives up through order r are bounded (except for the function itself), such that $g_0(x, h(x), \varepsilon) = 0$.*

(II) *For some integer k, $0 \leqq k \leqq n$, the matrices $D_y g_0(x, y, 0)$, $x \in R^m$, have k eigenvalues with real part smaller than -2μ and $(n-k)$ eigenvalues with real part larger than 2μ. Here μ is a fixed positive number.*

Under these conditions the first theorem (Theorem 2.1) says: There is an invariant manifold $\mathbf{C}_\varepsilon$ for $(\mathrm{S})_\varepsilon$ which behaves like a centre manifold relative to the flow in its neighbourhood and is a perturbation of $\mathbf{C}_0 = \{(x, h(x)); x \in R^m\}$. Theorem 2.9 gives a method of approximating $\mathbf{C}_\varepsilon$ as a power series of ε. Outer solutions can be viewed as an approximation to solutions whose orbits lie on $\mathbf{C}_\varepsilon$. In Theorem 3.1, we show that $\mathbf{C}_\varepsilon$ has stable and unstable manifolds, $W^s(\mathbf{C}_\varepsilon)$ and

$W^u(\mathbf{C}_\varepsilon)$, which are characterised in terms of an exponential decay rate of solutions as $t\to\infty$ and $t\to-\infty$, respectively. Theorem 3.1 also says that $W^s(\mathbf{C}_\varepsilon)$ and $W^u(\mathbf{C}_\varepsilon)$ constitute fibre bundles

$$W^s(\mathbf{C}_\varepsilon)=\bigcup_{\xi\in R^m} W^s(h_\varepsilon(\xi)),\quad W^u(\mathbf{C}_\varepsilon)=\bigcup_{\xi\in R^m} W^u(h_\varepsilon(\xi)),$$

in which each fibre $W^s(h_\varepsilon(\xi))(W^u(h_\varepsilon(\xi)))$ is the set of initial values of solutions which converge with an exponential rate, as $t\to\infty$ (respectively, $t\to-\infty$), to the solution on $\mathbf{C}_\varepsilon$ passing through $(\xi, h_\varepsilon(\xi))$ at the initial time. Theorem 3.11 then shows how to approximate $W^s(\mathbf{C}_\varepsilon)$ and $W^u(\mathbf{C}_\varepsilon)$ by means of inner solutions. It should be noted that Theorem 3.11 enables us to approximate the stable and unstable manifolds "semi-globally". To explain what is meant here by "semi-global", let $W^s_{\mathrm{loc}}(\mathbf{C}_\varepsilon)$ be the local stable manifold of $\mathbf{C}_\varepsilon$. Then global stable manifold is expressed as $W^s(\mathbf{C}_\varepsilon)=\bigcup_{t\leqq 0} W^s_{\mathrm{loc}}(\mathbf{C}_\varepsilon)\,.\,t$, where $p\to p\,.\,t$ represents the flow associated with $(\mathrm{S})_\varepsilon$. Then Theorem 3.11 says: For any $T>0$, there is an $\varepsilon^*>0$ such that $\bigcup_{-T\leqq t\leqq 0} W^s_{\mathrm{loc}}(\mathbf{C}_\varepsilon)\,.\,t$ can be approximated by a power series of inner solutions for $\varepsilon\in(0,\varepsilon^*]$. Under additional conditions, Theorem 3.11 implies Theorem 4.2, in which we present a situation where transverse heteroclinic connections exist between two normally hyperbolic invariant manifolds. If there are two functions $h^\pm$ satisfying the conditions (I), (II), then we know from Theorems 2.1 and 3.1, that there are normally hyperbolic invariant sets $\mathbf{C}^\pm_\varepsilon$ near $\mathbf{C}^\pm_0=\{(x, h^\pm(x)); x\in R^m\}$. The conditions (III), (IV) and (V) in Section 4 imply that $W^u(\mathbf{C}^-_\varepsilon)$ and $W^s(\mathbf{C}^+_\varepsilon)$ intersect transversely along an m-dimensional manifold. The idea and technique of the proof of the last theorem strongly resemble those of heteroclinic bifurcation theory.

Interest in the study of the system $(\mathrm{S})_\varepsilon$ is not of recent origin. Consult Levin **[11]** and Hoppensteadt **[7]** and references therein for the historical background of the problem. The basic difference between **[7, 11]** and the present work is that those authors are mainly concerned with individual solutions on a finite time interval and their perturbations, while our focus in this work is to treat families of solutions on an infinite time interval all at once. It should be also pointed out that the problem $(\mathrm{S})_\varepsilon$ contains a seemingly more general situation

$$x'=f(t,x,y,\varepsilon),\quad \varepsilon y'=g(t,x,y,\varepsilon),$$

which **[7]** and **[11]** dwell on. To see this, we consider t as a dependent variable and append the trivial equation $t'=1$.

The geometric ideas employed in this paper are by no means the present author's inventions, although he was naturally led to the same ideas while tackling an existence question for transition layer solutions. N. Fenichel's work **[4]** seems to be the first in which basic geometric pictures of the problem were made clear. In fact, the existence of $\mathbf{C}_\varepsilon$ (Theorem 2.1) and $W^s_{\mathrm{loc}}(\mathbf{C}_\varepsilon)$, $W^u_{\mathrm{loc}}(\mathbf{C}_\varepsilon)$ (Theorem 3.1) were already proved in **[4]**. Fenichel also touched upon the approximation of the manifolds $\mathbf{C}_\varepsilon$, $W^s_{\mathrm{loc}}(\mathbf{C}_\varepsilon)$ and $W^u_{\mathrm{loc}}(\mathbf{C}_\varepsilon)$, but did not pursue it thoroughly. Henry's work **[6]** also contains many geometric ideas and the technical machinery to implement those ideas. Recent works by Knobloch and Aulbach **[9]** and Knobloch **[10]** also present results similar to those of **[4]**.

The proof of Theorems 2.1 and 3.1 is by now fairly standard due to the works of Henry [6], Vanderbauwhede and Van Gils [14], and Chow and Lu [1, 2]. Although this is the case, a reasonably detailed proof will be given in the subsequent sections, because the generalisation of the results in [1, 2, 6, 14] to the present situation does not seem to be transparent. The proof is given in such a way that we have available the analytical expressions of the conditions and the results of the theorems for application to practical problems. Fenichel's proof in [4] depends heavily upon his previous work [3]. Unfortunately, the ideas developed in [4] have not been appreciated until recently. One of the reasons is, probably, that there is a gap between the geometric ideas in [4] and their analytical implementation. It is hoped that the present paper may serve to fill the gap. Now the time is ripe to pursue thoroughly the study of singular perturbation problems along the line of Fenichel's geometric ideas. It seems nontrivial to obtain Theorem 4.2 of the present work from the results in [4]. In particular, some of the computations to detect approximate heteroclinic connections seem to be unknown, although such computations were previously carried out in [5] for a simpler problem.

Applications of Theorems 2.1, 2.9, 3.1, 3.11 and 4.2 to boundary value problems, such as those treated in Lin's work [12], and to the existence of travelling front (pulse) waves for reaction-diffusion systems on the real line, will appear in later publications.

Throughout this paper, prime "$'$" and dot "$\cdot$" are used to indicate the differentiations with respect to slow time t and fast time τ, respectively. Differentiations with respect to variables other than t and τ are denoted by D_x, D_y, D_ε, etc. For example, $D_x f(x, y)\langle\phi\rangle$ stands for the Jacobian matrix of the function $f(x, y)$ with respect to x, operated on a vector ϕ, and $D_y^l g(y)\langle\psi_1, \dots, \psi_l\rangle$ designates the l-linear operator associated with the l-th order derivatives of $g(x)$ with respect to y, operated on l-tuple of vectors $(\psi_1, \dots, \psi_l)$. We also denote by $L^l(X^l, Y)$ the space of l-linear operators from X to Y. Throughout this paper, N_i, $i = 0, \dots, r$, stands for the supremum norm of i-th derivatives of f_0, and M_i, $i = 0, \dots, r$, that of g_0,

$$N_i = \sup\{|D_x^\alpha D_y^\beta D_\varepsilon^\gamma f_0(x, y, \varepsilon)|;\ \alpha+\beta+\gamma = i,\ x \in R^m,\ y \in R^n,\ \varepsilon \in R_+\},$$
$$M_i = \sup\{|D_x^\alpha D_y^\beta D_\varepsilon^\gamma g_0(x, y, \varepsilon)|;\ \alpha+\beta+\gamma = i,\ x \in R^m,\ y \in R^n,\ \varepsilon \in R_+\}.$$

2. Existence and smoothness of centre-like manifolds outer solutions

Consider the system $(S)_\varepsilon$ under the conditions (I), (II). The "time" t appearing in $(S)_\varepsilon$ is called the slow time. The fast time τ is introduced by $dt/d\tau = \varepsilon$, i.e. $t = t_0 + \varepsilon\tau$ for some $t_0 \in R$. In terms of the fast time scale τ, $(S)_\varepsilon$ can be written as

$$(\mathrm{F})_\varepsilon \quad \dot{x} = \varepsilon f_0(x, y, \varepsilon), \quad \dot{y} = g_0(x, y, \varepsilon) \quad \cdot = \frac{d}{d\tau}.$$

2.1. Centre-like manifolds

When $\varepsilon = 0$, $(S)_\varepsilon$ reduces to a system of differential equations coupled with a system of nondifferential equations. On the other hand, the system $(F)_\varepsilon$ has a manifold of equilibria when $\varepsilon = 0$, namely, $\mathbf{C}_0 = \{(x, h(x)); x \in R^m\}$. For $\varepsilon > 0$, $\mathbf{C}_0$ will survive as a manifold which is invariant relative to both $(\mathrm{S})_\varepsilon$ and $(\mathrm{F})_\varepsilon$. More precisely, we have the following theorem:

THEOREM 2.1. (i) *There exist* $\varepsilon_0 > 0$ *and a* C^{r-1}*-bounded function* $h: R^m \times [0, \varepsilon_0] \to R^n$ *such that the set* $\mathbf{C}_\varepsilon$ *defined by* $\mathbf{C}_\varepsilon = \{(x, h(x, \varepsilon)); x \in R^m\}$, $\varepsilon \in (0, \varepsilon_0]$, *is invariant under the flow generated by* $(\mathrm{S})_\varepsilon$ *(and* $(\mathrm{F})_\varepsilon$*) and, moreover,* $\sup\{|h(x, \varepsilon) - h(x)|; x \in R^m\} = O(\varepsilon)$ *as* $\varepsilon \to 0$.

(ii) *There is a constant* $\delta > 0$ *independent of* $\varepsilon \in (0, \varepsilon_0]$ *such that any solution* $(x(t), y(t))$ *of* $(\mathrm{S})_\varepsilon$ *which stays in a* δ*-neighbourhood of* $\mathbf{C}_0$ *lies on* $\mathbf{C}_\varepsilon$.

Change variables in $(\mathrm{S})_\varepsilon$ by $x \to x$, $y \to y + h(x)$ to obtain the equations for the new unknown (x, y):

$$(\mathrm{S})'_\varepsilon \quad x' = f_1(x, y, \varepsilon), \quad \varepsilon y' = A(x)y + G(x, y, \varepsilon),$$

where

$$f_1(x, y, \varepsilon) = f_0(x, h(x) + y, \varepsilon), \quad A(x) = D_y g_0(x, h(x), 0),$$
$$G(x, y, \varepsilon) = g_1(x, y, \varepsilon) - g_1(x, 0, 0) - D_y g_1(x, 0, 0)y = O(\varepsilon + |y|^2) \quad \text{as} \quad \varepsilon \to 0,$$
$$g_1(x, y, \varepsilon) = g_0(x, y + h(x), \varepsilon) - \varepsilon D_x h(x) f_0(x, y + h(x), \varepsilon).$$

Notice that f_1 is C^r-bounded and g_1 and G are C^{r-1}-bounded.

Remark 2.2. The reason why $h(x, \varepsilon)$ in Theorem 2.1 is C^{r-1} instead of being C^r comes from the fact that G is C^{r-1}.

The first key for the proof is the following Lemma 2.3 cited from Henry [**6**].

LEMMA 2.3 (Uniform exponential dichotomy lemma). *Under the condition* (II) *on* $A(x)$ *and for* $N_0 > 0$, *there exist* $\varepsilon_0 > 0$ *and* $K \geqq 1$ *such that the linear system*

$$\varepsilon y' = A(x(t))y \tag{2.1}$$

has an exponential dichotomy on R *uniformly with respect to* C^1 *function* $x(t) \in C^1(R, R^m)$, *with* $\sup_{t \in R} |x'(t)| \leqq N_0$, *i.e.*

$$\begin{cases} |T^\varepsilon(t, t'; x)P^\varepsilon(t; x)| \leqq K e^{-\mu(t-t')/\varepsilon}, & t \geqq t', \\ |T^\varepsilon(t, t'; x)Q^\varepsilon(t'; x)| \leqq K e^{\mu(t-t')/\varepsilon}, & t \leqq t', \end{cases} \tag{2.2}$$

where $T^\varepsilon(t, t'; x)$ *is the principal solution operator for* (2.1) *and* P^ε, Q^ε *stand for the stable and unstable projections. Moreover,* $\dim.\mathrm{Range}\, P^\varepsilon(t; x) = k$, *and* $\dim.\mathrm{Range}\, Q^\varepsilon(t; x) = n - k$.

Solutions of $(\mathrm{S})'_\varepsilon$ whose y-component stays bounded for $t \in R$ have to satisfy the following

$$(\mathrm{ID}) \quad \begin{cases} x' = f_1(x, y, \varepsilon), \\ y(t) = \dfrac{1}{\varepsilon} \displaystyle\int_{-\infty}^{\infty} U^\varepsilon(t, t'; x) G(x(t'), y(t'), \varepsilon)\, dt', \end{cases}$$

where $U^\varepsilon(t, t'; x)$ is the Green's function for the equation (2.1) multiplied by ε, i.e.

$$U^\varepsilon(t, t'; x) = T^\varepsilon(t, t'; x)P^\varepsilon(t'; x) \text{ for } t \geqq t', = T^\varepsilon(t, t'; x)Q^\varepsilon(t'; x) \text{ for } t < t'.$$

In order to study solutions of (ID), we introduce function spaces: for $\rho \in R$,

$$X_\rho = \{\phi \in C^1(R, R^m); |\phi|_\rho + |\phi'|_\rho < \infty\},$$
$$Y_\rho = \{\psi \in C^0(R, R^n); |\psi|_\rho < \infty\},$$

where $|\phi|_\rho = \sup_{t\in R} |\phi(t)|\, e^{-\rho|t|}$ is a weighted norm. We also denote $X_\rho^N = \{\phi \in X_\rho; |\phi'| \leqq N\}$. In terms of this notation, the linear equation (2.1) has an exponential dichotomy uniformly with respect to $x \in X_\rho^{N_0}$. Notice that $X_\rho \subset X_{\rho'}$, $Y_\rho \subset Y_{\rho'}$ if $\rho < \rho'$. For each $\xi \in R^m$ and $y \in Y_\rho$, we denote by $H_\varepsilon(\xi, y)(t)$ a unique solution of $x' = f_1(x, y, \varepsilon)$, $x(0) = \xi$. Lemma 2.4 below follows from a standard argument using Gronwall's inequality and variational equations.

LEMMA 2.4. (i) *For each $(\xi, y) \in R^m \times Y_\rho$, $\rho \in R$, the function $H_\varepsilon(\xi, y)$ belongs to $X_{N_0}^{N_0}$:*

$$|H_\varepsilon(\xi, y)(t)| \leqq |\xi| + N_0\,|t| \leqq |\xi| + e^{N_0|t|}, \quad \left|\frac{d}{dt} H_\varepsilon(\xi, y)(t)\right| \leqq N_0, \quad t \in R.$$

(ii) *For each $(\xi_i, y_i) \in R^m \times Y_{\rho_0}$, $\rho_0 > N_1$, $i = 1, 2$,*

$$|H_\varepsilon(\xi_2, y_2)(t) - H_\varepsilon(\xi_1, y_1)(t)| \leqq |\xi_2 - \xi_1|\, e^{N_1|t|} + \frac{N_1}{\rho_0 - N_1}\,|y_2 - y_1|_{\rho_0}, \quad t \in R.$$

(iii) *The function $H_\varepsilon(\xi, y)$ is C^r-bounded in $(\xi, y) \in R^m \times Y_{\rho_0}$, $\rho_0 > N_1$, in the following sense: for nonnegative integers α, β with $\alpha + \beta = l \leqq r$, the operator*

$$D_\xi^\alpha D_y^\beta H_\varepsilon : R^m \times Y_{\rho_0} \to L^l((R^m)^\alpha \times (Y_{\rho_0})^\beta, X_{\rho_1})$$

is well-defined for $\rho_1 \geqq l\rho_0$ and continuous for $\rho_1 > l\rho_0$.

From Lemma 2.3 and Lemma 2.4(i), the linear equation $\varepsilon\dot\psi = A(H_\varepsilon(\xi, y)(t)\psi$ has an exponential dichotomy for each $(\xi, y) \in R^m \times Y_\rho$, $\rho \in R$, and the corresponding nonhomogeneous equation $\varepsilon\dot\psi = A(H_\varepsilon(\xi, y)(t))\psi + \psi_1$, $\psi_1 \in Y_{\rho'}$, $|\rho'| < \mu/\varepsilon$, has a unique solution in $Y_{\rho''}$ for $\rho'' \in [\rho', \mu/\varepsilon)$ given by

$$\psi(t) = \frac{1}{\varepsilon}\int_{-\infty}^{\infty} U^\varepsilon(t, t'; H_\varepsilon(\xi, y))\psi_1(t')\, dt'.$$

We denote the correspondence $\psi_1 \to \psi$ by $\psi = K^\varepsilon(\xi, y)\psi_1$. For the linear operator K^ε, we have:

LEMMA 2.5. (i) *For each $(\xi, y) \in R^m \times Y_\rho$, $\rho \in R$, and a pair $\rho_1 \leqq \rho_2$, $|\rho_i| < \mu/\varepsilon$, $i = 1, 2$,*

$$K^\varepsilon(\xi, y) \in L(Y_{\rho_1}, Y_{\rho_2}), \quad |K^\varepsilon(\xi, y)|_{L(Y_{\rho_1}, Y_{\rho_2})} \leqq \frac{2K}{\mu - \varepsilon\rho_1}.$$

(ii) *For each $\xi \in R^m$, the mapping $K^\varepsilon(\xi, \cdot): Y_{\rho_0} \to L(Y_{\rho_1}, Y_{\rho_2})$, $\rho_0 > N_1$ is Lipschitzian for any $\rho_2 \geqq \rho_1 + \rho_0$ with $|\rho_i| < \mu/\varepsilon$, $i = 0, 1, 2$ with the Lipschitz constant*

$$\frac{4K^2 M_2}{(\mu - \varepsilon\rho_1)(\mu - \varepsilon\rho_1 - \varepsilon\rho_0)}\,\frac{N_1}{\rho_0 - N_1}.$$

(iii) *For each* $y \in Y_{\rho_0}$, $\rho_0 > N_1$, *the mapping*

$$K^\varepsilon(\cdot, y): R^m \to L(Y_{\rho_1}, Y_{\rho_2})$$

is Lipschitzian for $\rho_2 \geqq \rho_1 + \rho_0$, $|\rho_i| < \mu/\varepsilon$, $i = 1, 2$ *with Lipschitz constant*

$$\frac{4K^2M_2}{(\mu - \varepsilon\rho_1)(\mu - \varepsilon\rho_1 - \varepsilon\rho_0)}.$$

(iv) *The mapping* $K^\varepsilon: R^m \times Y_{\rho_0} \to L(Y_{\rho_1}, Y_{\rho_2})$, $\rho_0 > N_1$, *is* C^{r-1}*-bounded in the following sense: for non-negative integers* α, β *with* $\alpha + \beta = l \leqq r - 1$,

$$D_\xi^\alpha D_y^\beta K^\varepsilon: R^m \times Y_{\rho_0} \to L^l((R^m)^\alpha \times (Y_{\rho_0})^\beta, L(Y_{\rho_1}, Y_{\rho_2}))$$

is well-defined for $\rho_2 \geqq l\rho_0 + \rho_1$ *and continuous for* $\rho_2 > l\rho_0 + \rho_1$, $|\rho_i| < \mu/\varepsilon$, $i = 0, 1, 2$, $|l\rho_0 + \rho_1| < \mu/\varepsilon$.

The proof of Lemma 2.5 is given in Appendix A.

Next, we introduce a nonlinear operator associated with $G(x, y, \varepsilon)$. As noted before, $G(x, y, \varepsilon)$ is a C^{r-1}-bounded function. Assuming $r \geqq 3$, it is easy to see that

$$|D_y^\alpha G(x, y, \varepsilon)| = O(\varepsilon + |y|^{2-\alpha}), \quad |D_x^\alpha G(x, y, \varepsilon)| = O(\varepsilon + |y|^2), \quad \alpha = 1, 2, \tag{2.3}$$

from the definition of $G(x, y, \varepsilon)$. Let $G^\varepsilon: R^m \times Y_\rho \to Y_0$, $\rho \in R$, be defined by: $G^\varepsilon(\xi, y)(t) = G(H_\varepsilon(\xi, y)(t), y(t), \varepsilon)$.

LEMMA 2.6. (i) $G^\varepsilon: R^m \times Y_{\rho_0} \to Y_{\rho_1}$, $\rho_0 > N_1$ *is* C^{r-1}*-bounded in the following sense: for non-negative integers* α, β *with* $\alpha + \beta = l \leqq r - 1$, *the operator* $D_\xi^\alpha D_y^\beta G^\varepsilon: R^m \times Y_{\rho_0} \to L^l((R^m)^\alpha \times (Y_{\rho_0})^\beta, Y_{\rho_1})$ *is well defined for* $\rho_1 \geqq l\rho_0$ *and continuous for* $\rho_1 > l\rho_0$.

(ii) *For each* $\xi \in R^m$, *the mapping* $G^\varepsilon(\xi, \cdot): Y_{\rho_0} \to Y_{\rho_0}$, $\rho_0 > N_1$, *is Lipschitzian with a Lipschitz constant:* $M_1(1 + (N_1)/(\rho_0 - N_1))$.

This lemma can be proved from Lemma 2.4 by using the argument in **[14]**.

Employing the operators introduced above, the solution of $(\mathrm{S})'_\varepsilon$ with $x(0) = \xi$ and $\sup_{t \in R} |y(t)| < \infty$ can be given by a pair $(x(t), y(t)) = (H_\varepsilon(\xi, y^\varepsilon(\xi)), y^\varepsilon(\xi))$, where $y^\varepsilon(\xi)$ is a fixed point of the operator

$$\Gamma^\varepsilon(\xi, y) = K^\varepsilon(\xi, y)G^\varepsilon(\xi, y), \quad \Gamma^\varepsilon: R^m \times Y_\rho \to Y_0.$$

We shall show that the operator Γ^ε has a unique fixed point on appropriate function spaces.

It is possible, because of the properties in (2.3), to choose $\delta > 0$ and $\varepsilon_0 > 0$ so small that the following conditions are satisfied:

$$\sup\{|G(x, y, \varepsilon)|; x \in R^m, |y| < 2\delta, 0 \leqq \varepsilon \leqq \varepsilon_0\} < \frac{\mu\delta}{2K}, \tag{2.4}$$

$$\sup\{|D_xG| + |D_yG|; x \in R^m, |y| < 2\delta, 0 \leqq \varepsilon \leqq \varepsilon_0\} < M\delta, \tag{2.5}$$

where $M \geqq 1$ is a constant independent of (ε, δ), and

$$\left(\frac{2KM_2 + 4KM}{\mu - 2\varepsilon N_1}\right)\delta < 1. \tag{2.6}$$

We modify G outside the set $\{(x, y, \varepsilon); x \in R^m, |y| < \delta, 0 \leqq \varepsilon \leqq \varepsilon_0\}$ so that (2.4) and (2.5) remain true when the supremum is taken over $R^m \times R^n \times [0, \varepsilon_0]$. We denote the modified G by the same symbol G. After the modification, consider the fixed point equation: $y = \Gamma^\varepsilon(\xi, y)$, where Γ^ε is considered as: $\Gamma^\varepsilon: R^m \times Y_{2N_1} \to Y_{2N_1}$. It is easy, by using (2.2), Lemma 2.5(i)(ii), Lemma 2.4(i) and Lemma 2.6(ii), to verify the following inequalities:

$$|\Gamma^\varepsilon(\xi, y)|_{2N_1} \leqq |\Gamma^\varepsilon(\xi, y)|_0 \leqq \frac{2K}{\mu} |G^\varepsilon(\xi, y)|_0 < \frac{2K}{\mu} \frac{\mu\delta}{2K} = \delta,$$

$$\begin{aligned} |\Gamma^\varepsilon(\xi, y_2) - \Gamma^\varepsilon(\xi, y_1)|_{2N_1} &\leqq |K^\varepsilon(\xi, y_2) - K^\varepsilon(\xi, y_1)|_{L(Y_0, Y_{2N_1})} |G^\varepsilon(\xi, y_1)|_0 \\ &\quad + |K^\varepsilon(\xi, y_2)|_{L(Y_{2N_1}, Y_{2N_1})} |G^\varepsilon(\xi, y_2) - G^\varepsilon(\xi, y_1)|_{2N_1} \\ &\leqq \frac{4K^2M_2}{\mu(\mu - 2\varepsilon N_1)} |y_1 - y_2|_{2N_1} \frac{\mu\delta}{2K} + \frac{2K}{(\mu - 2\varepsilon N_1)} 2M\, |y_1 - y_2|_{2N_1} \\ &\leqq \frac{2KM_2 + 4KM}{\mu - 2\varepsilon N_1} \delta\, |y_1 - y_2|_{2N_1} < |y_1 - y_2|_{2N_1}. \end{aligned}$$

Therefore, the operator $\Gamma^\varepsilon(\xi, \cdot): Y_{2N_1} \to Y_{2N_1}$ has a unique fixed point $y^\varepsilon(\xi)$ in the δ-neighbourhood of the origin in Y_{2N_1}. Moreover, $y^\varepsilon(\xi) \in Y_0$ and $|y^\varepsilon(\xi)|_0 < \delta$. Since $G(x, y, \varepsilon) = O(\varepsilon + |y|^2)$ as $\varepsilon + |y| \to 0$, we can also conclude that $|y^\varepsilon(\xi)|_0 = O(\varepsilon)$ as $\varepsilon \to 0$. As for the smoothness of the fixed point, we have the following lemma:

LEMMA 2.7. *The mapping $y^\varepsilon: R^m \to Y_\rho$ is C^{r-1}-bounded for $\rho \in [2rN_1, \mu/\varepsilon)$.*

Proof. Since $\Gamma^\varepsilon: R^m \times Y_{N_1} \to Y_\rho$ is C^{r-1}-bounded for $\rho \in [2rN_1, \mu/\varepsilon)$, which follows from Lemma 2.5(iv) and Lemma 2.6(i), the argument in [**2**, **14**] proves the lemma. □

Now let us define $h(\xi, \varepsilon) = y^\varepsilon(\xi)(0) + h(\xi)$. The function $h(\xi, \varepsilon)$ defined in this way is C^{r-1}-bounded in ξ for each fixed $\varepsilon > 0$. In order to prove that the same function is C^{r-1}-bounded in (ξ, ε) on $R^m \times [0, \varepsilon]$, we use the system $(\mathrm{F})_\varepsilon$ in which the parameter ε enters the problem in a regular manner. The method of proof employed above works equally for this case, giving the same function $h(\xi, \varepsilon)$ which is now C^{r-1}-bounded jointly in (ξ, ε). We claim that Theorem 2.1 is true for the function $h(\xi, \varepsilon)$ obtained above. Before we complete the proof, we prepare the following lemma:

LEMMA 2.8. (i) *A function $y^\varepsilon(t)$ is a unique fixed point of $y = \Gamma^\varepsilon(\xi, y)$ in the δ-neighbourhood of zero in Y_0 if and only if the pair $(H_\varepsilon(\xi, y^\varepsilon)(t), y^\varepsilon(t)) = (x(t), y(t))$ is a solution of $(\mathrm{S})'_\varepsilon$ with $x(0) = \xi$.*

(ii) *If $y^\varepsilon(\xi)$ is the unique fixed point of $y = \Gamma^\varepsilon(\xi, y)$, then for each $s \in R$, the following holds true: $y^\varepsilon(H_\varepsilon(\xi, y^\varepsilon(\xi))(s))(0) = y^\varepsilon(\xi)(s)$.*

Proof. (i) This follows immediately from the way we constructed the operator Γ^ε.

(ii) For each $s \in R$, consider the translate

$$x_s(t) = H_\varepsilon(\xi, y^\varepsilon(\xi))(s + t), \quad y_s(t) = y^\varepsilon(\xi)(s + t).$$

Then $(x_s(t), y_s(t))$ solves $(S)'_\varepsilon$, and $|y_s|_0 = |y^\varepsilon(\xi)|_0 \leqq \delta$. Hence part (i) of the lemma implies that y_s is a fixed point of $y = \Gamma^\varepsilon(x_s(0), y)$. From the uniqueness of the fixed point, we have: $y_s(t) = y^\varepsilon(x_s(0))(t)$; in particular, $y_s(0) = y^\varepsilon(x_s(0))(0)$, or $y^\varepsilon(\xi)(s) = y^\varepsilon(H_\varepsilon(\xi, y^\varepsilon(\xi)(s))(0)$, proving statement (ii). □

Proof of Theorem 2.1. (i) It only remains to prove the invariance of the manifold $\mathbf{C}_\varepsilon$. Let $(x_0, y_0) \in \mathbf{C}_\varepsilon$, and $x(t) = H_\varepsilon(x_0, y^\varepsilon(x_0))(t)$, $y(t) = h(x(t)) + y^\varepsilon(x_0)(t)$; then the pair $(x(t), y(t))$ solves $(S)_\varepsilon$ with $x(0) = x_0$, $y(0) = h(x(0), \varepsilon)$. On the other hand, Lemma 2.8(ii) implies:

$$\begin{aligned} y(t) &= h(x(t)) + y^\varepsilon(x_0)(t) \\ &= h(x(t)) + y^\varepsilon(H_\varepsilon(x_0, y^\varepsilon(x_0))(t))(0) \\ &= h(x(t)) + y^\varepsilon(x(t))(0) = h(x(t), \varepsilon), \quad t \in R, \end{aligned}$$

which proves the invariance of $\mathbf{C}_\varepsilon$.

(ii) To prove Theorem 2.1(ii), let $(x(t), y(t))$ be a solution of $(S)_\varepsilon$ with $\sup_{t\in R} |y(t) - h(x(t))| \leqq \delta$. Then the pair $(x(t), y(t) - h(x(t)))$ solves $(S)'_\varepsilon$ and hence $y(t) - h(x(t))$ is a fixed point of $y = \Gamma^\varepsilon(x(0), y)$, or $y(t) = h(x(t)) + y^\varepsilon(x(0))(t)$. In particular, $y(0) = h(x(0), \varepsilon)$ and the invariance of the manifold $\mathbf{C}_\varepsilon$ implies $(x(t), y(t)) \in \mathbf{C}_\varepsilon$. This completes the proof of Theorem 2.1. □

2.2. Outer expansions

In this subsection, we will exhibit a method of approximating the function $h(x, \varepsilon)$ as a power series of ε. This corresponds to classical outer expansions.

We assume the formal expansion $h(x, \varepsilon) = \sum_{j\geqq 0} \varepsilon^j h_j(x)$. Since $\mathbf{C}_\varepsilon$ is invariant relative to $(S)_\varepsilon$, the function $h(x, \varepsilon)$ satisfies the following partial differential equation:

$$\varepsilon D_x h(x, \varepsilon)\langle f_0(x, h(x, \varepsilon), \varepsilon)\rangle = g_0(x, h(x, \varepsilon), \varepsilon). \tag{2.7}$$

Expanding formally both sides of (2.7), we obtain

$$\sum_{j\geqq 1} \varepsilon^j \sum_{l=1}^{j} D_x h_{l-1}(x)\bar{f}_{j-l}(x) = g_0(x, h_0(x), 0) + \sum_{j\geqq 1} \varepsilon^j \bar{g}_j(x), \tag{2.8}$$

where $\bar{f}_0(x) = f_0(x, h_0(x), 0)$ and for $j \geqq 1$,

$$\bar{f}_j = \sum_{l=1}^{j} \sum{}^* \frac{1}{\alpha!\,\beta!} D_y^\alpha D_\varepsilon^\beta f_0(x, h_0(x), 0)\langle h_{j_1}, \ldots, h_{j_\alpha}\rangle$$

$$\bar{g}_j = \sum_{l=1}^{j} \sum{}^* \frac{1}{\alpha!\,\beta!} D_y^\alpha D_\varepsilon^\beta g_0(x, h_0(x), 0)\langle h_{j_1}, \ldots, h_{j_\alpha}\rangle.$$

The summation $\sum^*$ is taken over the indices $\alpha \geqq 0$, $\beta \geqq 0$, $j_i \geqq 1$, $i = 1, \ldots, \alpha$, satisfying $\alpha + \beta = l$, $j_1 + \ldots + j_\alpha + \beta = j$. The relation (2.8) gives rise to a series of equations

$$0 = g_0(x, h(x), 0) \tag{2.9-0}$$

$$D_x h_0(x) f_0(x, h_0(x), 0) = D_y g_0(x, h_0(x), 0) h_1(x) + D_\varepsilon g_0(x, h_0(x), 0) \tag{2.9-1}$$

$$\sum_{l=1}^{j} D_x h_{l-1}(x)\bar{f}_{j-l}(x) = \bar{g}_j(x). \tag{2.9-j}$$

Because of the hypothesis (I), (2.9-0) has a solution $h_0(x) = h(x)$. Once $h_0(x)$ is chosen, all the $h_j(x)$, $j \geqq 1$ can be determined recursively by means of (2.9-j). To see this, notice that in (2.9-j) the left-hand side involves only known functions and $h_0, \ldots, h_{j-1}$, while the right-hand side is of the form: $D_y g_0(x, h(x), 0)h_j(x) + \hat{g}_j(x)$, where the term $\hat{g}_j$ involves only known functions and $h_0, \ldots, h_{j-1}$. Therefore, once $h_0, \ldots, h_{j-1}$ are known, the j-th term h_j can be identified uniquely as

$$h_j(x) = [D_y g_0(x, h(x), 0)]^{-1}\left[\sum_{l=1}^{j} D_x h_{l-1}(x)\bar{f}_{j-l}(x) - \hat{g}_j(x)\right]$$

due to the condition (II). Notice that the function h_j is C^{r-j}-bounded.

Now changing the variables in $(\mathrm{S})_\varepsilon$ by $y \to y + \sum_0^p \varepsilon^j h_j(x)$, $p < r$, the equations for the new unknown (x, y) are:

$$x' = f_p(x, y, \varepsilon), \quad \varepsilon y' = A_p(x, \varepsilon)y + G_p(x, y, \varepsilon), \tag{2.10}$$

where

$$f_p(x, y, \varepsilon) = f_0\left(x, \sum_0^p \varepsilon^j h_j(x) + y, \varepsilon\right), \quad A_p(x, \varepsilon) = D_y g_0\left(x, \sum_0^p \varepsilon^j h_j(x), \varepsilon\right)$$

$$|G_p(x, y, \varepsilon)| \leqq M_2 |y|^2 + \varepsilon |y| + O(\varepsilon^{p+1}), \quad \text{as} \quad |y| + \varepsilon \to 0.$$

By virtue of Lemma 2.3, the equation $\varepsilon y' = A_p(x(t), \varepsilon)y$ has an exponential dichtomy with constant K an exponent μ/ε uniformly with respect to $x \in X_p^{N_0}$, $\varepsilon \in (0, \varepsilon_0]$ for sufficiently small $\varepsilon_0 > 0$. By applying the same method employed in the previous subsection, we obtain a manifold invariant relative to (2.10), which is expressed as a graph of a C^{r-p}-bounded function $h_p(x, \varepsilon)$, with $\sup_{x \in R^m} |h_p(x, \varepsilon)| = O(\varepsilon^{p+1})$. From the uniqueness of the invariant manifold $\mathbf{C}_\varepsilon$ for $(\mathrm{S})_\varepsilon$ in the δ-neighbourhood of $\mathbf{C}_0$, we obtain, for $\varepsilon_0 > 0$ small,

$$h(x, \varepsilon) = \sum_0^p \varepsilon^j h_j(x) + h_p(x, \varepsilon), \quad p < r,$$

after transforming (2.10) back to $(\mathrm{S})_\varepsilon$. We thus have proved the following theorem:

THEOREM 2.9. *The invariant manifold $\mathbf{C}_\varepsilon$ is approximate in the following sense:*

$$|h(x, \varepsilon) - \sum_0^p \varepsilon^j h_j(x)| = O(\varepsilon^{p+1}), \quad \text{as} \quad \varepsilon \to 0$$

for each integer $p < r$, where $h_0, \ldots, h_p$ are computed recursively by (2.9-j).

An implication of Theorem 2.9 is that the classical outer solutions are an approximation to solutions whose orbits lie on the invariant manifold $\mathbf{C}_\varepsilon$.

3. Existence and smoothness of stable and unstable manifolds

In this section, we show that the manifold $\mathbf{C}_\varepsilon$ has stable and unstable manifolds. We also present a method of approximating these manifolds in an explicit manner. The fast time system $(\mathrm{F})_\varepsilon$ will be exploited in this and the next sections.

3.1. Stable and unstable manifolds

THEOREM 3.1. (i) *These are C^{r-1}-immersed submanifolds $W^u(\mathbf{C}_\varepsilon)$ and $W^u(\mathbf{C}_\varepsilon)$, in $R^m \times R^n$, of dimension $k+m$ and $n-k+m$ respectively. These are characterised by*

$$W^s(\mathbf{C}_\varepsilon) = \left\{(x_0, y_0); \sup_{\tau \geqq 0} |y(\tau; x_0, y_0) - h_\varepsilon(x(\tau; x_0, y_0))|\, e^{\mu\tau/2} < \infty\right\},$$

$$W^u(\mathbf{C}_\varepsilon) = \left\{(x_{0,0}); \sup_{\tau \leqq 0} |y(\tau; x_0, y_0) - h_\varepsilon(x(\tau; x_0, y_0))|\, e^{-\mu\tau/2} < \infty\right\},$$

where $(x(\tau; x_0, y_0), y(\tau; x_0, y_0))$ is the solution of $(\mathrm{F})_\varepsilon$ passing through (x_0, y_0) and $h_\varepsilon(x) = h(x, \varepsilon)$ is the function defining $\mathbf{C}_\varepsilon$.

(ii) *The manifolds $W^s(\mathbf{C}_\varepsilon)$ and $W^u(\mathbf{C}_\varepsilon)$ are a disjoint union of C^{r-1}-immersed manifolds $W^s(h_\varepsilon(\xi))$ and $W^u(h_\varepsilon(\xi))$ of dimension k and $n-k$, respectively:*

$$W^s(\mathbf{C}_\varepsilon) = \bigcup_{\xi \in R^m} W^s(h_\varepsilon(\xi)), \quad W^u(\mathbf{C}_\varepsilon) = \bigcup_{\xi \in R^m} W^u(h_\varepsilon(\xi)).$$

Moreover, these manifolds are characterised as:

$$W^s(h_\varepsilon(\xi)) = \left\{(x_0, y_0); \sup_{\tau \geqq 0} |\tilde{x}(\tau)|\, e^{\mu\tau/2} < \infty,\ \sup_{\tau \geqq 0} |\tilde{y}(\tau)|\, e^{\mu\tau/2} < \infty\right\},$$

$$W^u(h_\varepsilon(\xi)) = \left\{(x_0, y_0); \sup_{\tau \leqq 0} |\tilde{x}(\tau)|\, e^{-\mu\tau/2} < \infty,\ \sup_{\tau \leqq 0} |\tilde{y}(\tau)|\, e^{-\mu\tau/2} < \infty\right\},$$

where $\tilde{x}(\tau) = x(\tau; x_0, y_0) - H_\varepsilon(\xi)(\tau)$, $\tilde{y}(\tau) = y(\tau; x_0, y_0) - h_\varepsilon(H_\varepsilon(\xi)(\tau))$, *and $H_\varepsilon(\xi)(\tau)$ stands for a unique solution of $\dot{x} = f_0(x, h_\varepsilon(x), \varepsilon)$, $x(0) = \xi \in R^m$.*

(iii) *There is a constant $\delta_0 > 0$ such that if a solution $(x(\tau), y(\tau))$ of $(\mathrm{F})_\varepsilon$ satisfies*

$$\sup_{\tau \geqq 0} |y(\tau) - h_\varepsilon(x(\tau))| < \delta_0 \quad \left(or \sup_{\tau \leqq 0} |y(\tau) - h_\varepsilon(x(\tau))| < \delta_0\right)$$

then $(x(0), y(0)) \in W^s(\mathbf{C}_\varepsilon)$ (respectively $(x(0), y(0)) \in W^u(\mathbf{C}_\varepsilon)$).

In what follows, we give a proof for W^s. The proof for W^u is almost identical with obvious changes. Changing variables by $x \to x$, $y \to y + h_\varepsilon(x)$ in $(\mathrm{F})_\varepsilon$ and using the identity

$$\varepsilon D_x h_\varepsilon(x) f_0(x, h_\varepsilon(x), \varepsilon) = g_0(x, h_\varepsilon(x), \varepsilon)$$

the equations for the new unknowns (x, y) are

$$\dot{x} = \varepsilon f_1(x, y, \varepsilon), \quad \dot{y} = g_1(x, y, \varepsilon), \tag{3.1}$$

where

$$f_1(x, y, \varepsilon) = f_0(x, h_\varepsilon(x) + y, \varepsilon),$$
$$g_1(x, y, \varepsilon) = g_0(x, h_\varepsilon(x) + y, \varepsilon) - g_0(x, h_\varepsilon(x), \varepsilon).$$

As before, let us denote by $H_\varepsilon(\xi)(\tau)$ a unique solution of $\dot{x} = \varepsilon f_1(x, 0, \varepsilon)$, $x(0) = \xi$, and transform (3.1) further by $x \to x + H_\varepsilon(\xi)(\tau)$ to obtain

$$(\mathrm{F})'_\varepsilon \quad \dot{x} = \varepsilon f_2(H_\varepsilon(\xi)(\tau), x, y, \varepsilon), \quad \dot{y} = A(H_\varepsilon(\xi)(\tau), \varepsilon)y + g_2(H_\varepsilon(\xi)(\tau), x, y, \varepsilon),$$

where

$$f_2(H, x, y, \varepsilon) = f_1(H + x, y, \varepsilon) - f_1(H, 0, \varepsilon),$$
$$g_2(H, x, \varepsilon) = g_1(x + H, y, \varepsilon) - D_y g_1(H, 0, \varepsilon)y,$$
$$A(H, \varepsilon) = D_y g_1(H, 0, \varepsilon) = D_y g_0(H, h_\varepsilon(H), \varepsilon).$$

Notice that f_2, g_2 and $A(H, \varepsilon)$ are C^{r-1}-bounded and

$$f_2(H, 0, 0, \varepsilon) \equiv 0, \quad g_2(H, 0, 0, \varepsilon) \equiv 0, \quad D_y g_2(H, 0,0, \varepsilon) \equiv 0.$$

For each $\xi \in R^m$, the linear equation

$$\dot{y} = A(H_\varepsilon(\xi)(\tau), \varepsilon)y \tag{3.2}$$

has an exponential dichotomy on R uniformly with respect to $\xi \in R^m$, due to Lemma 2.3. More precisely, we have

$$|T^\varepsilon(\tau, \tau'; \xi)P^\varepsilon(\tau'; \xi)| \leqq Ke^{-\mu(\tau-\tau')}, \quad \tau \geqq \tau',$$
$$|T^\varepsilon(\tau, \tau'; \xi)Q^\varepsilon(\tau'; \xi)| \leqq Ke^{\mu(\tau-\tau')}, \quad \tau \leqq \tau',$$

where $T^\varepsilon(\tau, \tau'; \xi)$ is the fundamental solution operator for (3.2) and $P^\varepsilon(\tau; \xi)$ and $Q^\varepsilon(\tau; \xi)$ are stable and unstable projections associated with the dichotomy. For the sake of proof, we introduce function spaces X^+_ρ and Y^+_ρ for $\rho \in R$ defined by

$$X^+_\rho = \left\{\phi \in C^0(R_+, R^m); \sup_{\tau \geqq 0} |\phi(\tau)|\, e^{-\rho|\tau|} < \infty\right\},$$

$$Y^+_\rho = \left\{\psi \in C^0(R_+, R^n); \sup_{\tau \geqq 0} |\psi(\tau)|\, e^{-\rho|\tau|} < \infty\right\}.$$

Solutions $(x(\tau), y(\tau))$ of $(\mathrm{F})'_\varepsilon$ which are bounded on R_+ and satisfying $\lim_{\tau \to \infty} x(\tau) = 0$ solve the integral equations

$$x(\tau) = \varepsilon \int_\infty^\tau f_2(H_\varepsilon(\xi)(\tau'), x(\tau'), y(\tau'), \varepsilon)\, d\tau' \tag{3.3-1}$$

$$y(\tau) = S^\varepsilon(\tau; \xi)y(0) + \int_0^\infty U^\varepsilon(\tau, \tau'; \xi)g_2(H_\varepsilon(\xi)(\tau'), x(\tau'), y(\tau'), \varepsilon)\, d\tau', \tag{3.3-2}$$

where U^ε is the Green's function for (3.2) on R;

$$U^\varepsilon(\tau, \tau'; \xi) = T^\varepsilon(\tau, \tau'; \xi)P^\varepsilon(\tau'; \xi) \text{ if } \tau \geqq \tau', \quad = T^\varepsilon(\tau, \tau'; \xi)Q^\varepsilon(\tau'; \xi) \text{ if } \tau < \tau'$$

and $S^\varepsilon(\tau; \xi) = T^\varepsilon(\tau, 0; \xi)P^\varepsilon(0; \xi)$. Notice that $U^\varepsilon(\tau, \tau', \xi)$ and $S^\varepsilon(\tau; \xi)$ are defined for $\tau, \tau' \in R$, although we are concerned with $\tau, \tau' \in R_+$ below.

LEMMA 3.2. (i) *For each τ, τ', and $s \in R$, the following identities hold true:*

$$U^\varepsilon(\tau, \tau; H_\varepsilon(\xi)(s)) = U^\varepsilon(\tau + s, \tau' + s; \xi), \quad S^\varepsilon(\tau; H_\varepsilon(\xi)(s))S^\varepsilon(s; \xi) = S^\varepsilon(\tau + s; \xi).$$

(ii) *The unions* $\bigcup_{\xi\in R^m}$ Range $(P^\varepsilon(0;\xi))$ *and* $\bigcup_{\xi\in R^m}$ Range $(Q^\varepsilon(0;\xi))$ *constitute* C^{r-1}*-vector bundles with fibre dimension* k *and* $n-k$*, respectively.*

Proof. Statement (i) follows from the uniqueness property of solutions for initial value problems for (3.2). Statement (ii) will be proved in Appendix B. □

Let us define linear operators I^ε and $K^\varepsilon(\xi)$, $\xi\in R^m$, by

$$I^\varepsilon\phi(\tau)=\varepsilon\int_\infty^\tau \phi(\tau')\,d\tau',\quad K^\varepsilon(\xi)\psi(\tau)=\int_0^\infty U^\varepsilon(\tau,\tau';\xi)\psi(\tau')\,d\tau'.$$

It is easy to verify that $I^\varepsilon\in L(X_\rho^+, X_\rho^+)$ for $\rho<0$ and $K^\varepsilon(\xi)\in L(Y_\rho^+, Y_\rho^+)$ for $|\rho|<\mu$, and

$$|I^\varepsilon|_\rho\leqq\frac{\varepsilon}{|\rho|},\quad\text{for}\quad\rho<0,\quad |K^\varepsilon(\xi)|_\rho\leqq K\left[\frac{1}{\mu+\rho}+\frac{1}{\mu-\rho}\right],\quad\text{for}\quad|\rho|<\mu.\tag{3.4}$$

We also introduce nonlinear operators G^ε and F^ε defined by

$$F^\varepsilon(\xi,x,y)(\tau)=f_2(H_\varepsilon(\xi)(\tau),x(\tau),y(\tau),\varepsilon),$$
$$G^\varepsilon(\xi,x,y)(\tau)=g_2(H_\varepsilon(\xi)(\tau),x(\tau),y(\tau),\varepsilon).$$

Lemma 3.3. $H_\varepsilon\colon R^m\to X_{\varepsilon N_0}^+$ *is* C^{r-1}*-bounded in the following sense: for each integer* $l\leqq r-1$*, the mapping* $D_\xi^l H_\varepsilon\colon R^m\to L^l((R^m)^l, X_\rho^+)$ *is well-defined for* $\rho\geqq\varepsilon l N_1$ *and continuous for* $\rho>\varepsilon l N_1$.

This lemma is essentially a restatement of smoothness properties of solutions for initial value problems and hence its proof may be omitted.

Lemma 3.4. (i) $F^\varepsilon\colon R^m\times X_\rho^+\times Y_\rho^+\to X_\rho^+$ *is* C^{r-1}*-bounded in the following sense: for non-negative integers* α, β *and* γ *satisfying* $\alpha+\beta+\gamma=l\leqq r-1$*, the operator*

$$D_\xi^\alpha D_x^\beta D_y^\gamma F^\varepsilon\colon R^m\times X_\rho^+\times Y_\rho^+\to L^l((R^m)^\alpha\times(X_\rho^+)^\beta\times(Y_\rho^+)^\gamma, X_{\rho_1}^+)$$

is well-defined for $\rho_1\geqq\rho+\varepsilon l N_1$ *and continuous for* $\rho_1>\rho+\varepsilon l N_1$.

(ii) *A similar statement holds true for* G^ε.

We refer readers to **[14]** for a proof.

Lemma 3.5. (i) *The operator* $K^\varepsilon\colon R^m\to L(Y_\rho^+, Y_\rho^+)$, $|\rho|<\mu$ *is* C^{r-1}*-bounded in the following sense: for each non-negative integer* $l\leqq r-1$*, the mapping* $D_\xi^l K^\varepsilon\colon R^m\to L^l((R^m)^l, L(Y_\rho^+, Y_{\rho_1}^+))$ *is well-defined for* $\rho_1\geqq\rho+\varepsilon l N_1$ *and continuous for* $\rho_1>\rho+\varepsilon l N_1$, $\rho_1<\mu$.

(ii) *For each pair* $(\xi,\eta)\in E_\varepsilon^s:=\bigcup_{\xi\in R^m}$ Range $(P^\varepsilon(0;\xi))$ *and* $\rho\geqq-\mu$*, the estimate* $|S^\varepsilon(\cdot,\xi)\eta|_\rho\leqq K|\eta|$ *holds true.*

(iii) $S^\varepsilon(\tau)\colon E_\varepsilon^s\to Y_\rho^+$ *is* C^{r-1}*-bounded in the following sense: for non-negative integers* α, β, $\alpha+\beta=l\leqq r-1$*, the mapping* $D_\xi^\alpha D_\eta^\beta S^\varepsilon(\tau)\colon E_\varepsilon^s\to Y_\rho^+$ *is well-defined for* $\rho\geqq-\mu+\varepsilon l N_1$*, and continuous for* $\rho>-\mu+\varepsilon l N_1$*, where the derivatives are taken with respect to a local coordinate system for the bundle* E_ε^s.

The proof of this lemma is similar to, and easier than, those of Lemma 2.5 and Lemma 3.2.

We are now ready to prove Theorem 3.1. For $\rho \in (-\mu, 0)$, consider the map $\Pi^\varepsilon: E^s_\varepsilon \times X^+_\rho \times Y^+_\rho \to X^+_\rho \times Y^+_\rho$ defined by

$$\Pi^\varepsilon(\xi, \eta, x, y) = \begin{bmatrix} I^\varepsilon F^\varepsilon(\xi, x, y) \\ S^\varepsilon(\cdot, \xi)\eta + K^\varepsilon(\xi) G^\varepsilon(\xi, x, y) \end{bmatrix}.$$

From Lemma 3.3 through to Lemma 3.5 and the estimates in (3.4), it is easy to verify the following inequalities:

$$|I^\varepsilon F^\varepsilon(\xi, x, y)|_\rho \leqq \frac{\varepsilon N_1}{|\rho|}(|x|_\rho + |y|_\rho) \tag{3.5}$$

$$|K^\varepsilon(\xi) G^\varepsilon(\xi, x, y)|_\rho \leqq KM_2\left[\frac{1}{\mu+\rho} + \frac{1}{\mu-\rho}\right](|x|_\rho + |y|_\rho)\,|y|_\rho \tag{3.6}$$

$$|S^\varepsilon(\cdot, \xi)\eta|_\rho \leqq K\,|\eta| \tag{3.7}$$

$$|I^\varepsilon F^\varepsilon(\xi, x_1, y_1) - I^\varepsilon F^\varepsilon(\xi, x_2, y_2)|_\rho \leqq \frac{\varepsilon N_1}{|\rho|}(|x_1 - x_2|_\rho + |y_1 - y_2|_\rho) \tag{3.8}$$

$$\begin{aligned} |K^\varepsilon(\xi)[G^\varepsilon(\xi, x_1, y_1) - G^\varepsilon(\xi, x_2, y_2)]|_\rho \leqq KM_2\left[\frac{1}{\mu-\rho} + \frac{1}{\mu+\rho}\right]\{&|y_1|_\rho\,|x_1 - x_2|_\rho \\ &+ (|y_1|_\rho + |y_2|_\rho)\,|y_1 - y_2|_\rho\} \end{aligned} \tag{3.9}$$

Let us denote by $[X^+_\rho \times Y^+_\rho]_\delta$ the closed δ-ball around the origin in $X^+_\rho \times Y^+_\rho$ equipped with the product norm $|(x, y)|_\rho = |x|_\rho + |y|_\rho$. We also write as: $[E^s_\varepsilon]_\delta = \{(\xi, \eta) \in E^s_\varepsilon \colon |\eta| \leqq \delta\}$.

Now setting $\rho = -\mu/2$ and choosing $\delta > 0$ and $\varepsilon_0 > 0$ so small that

$$\frac{4\varepsilon_0 N_1}{\mu} < 1, \quad \frac{16KM_2}{\mu}\delta < 1, \quad \frac{2\varepsilon_0 r N_1}{\mu} < 1 \tag{3.10}$$

are satisfied, we can verify from (3.5) through to (3.9) that the operator $\Pi^\varepsilon(\xi, \eta, \cdot, \cdot)$ is a contraction on $[X^+_{-\mu/2} \times Y^+_{-\mu/2}]_\delta$ uniformly with respect to $(\xi, \eta) \in [E^s_\varepsilon]_{\delta/2K}$ and $\varepsilon \in (0, \varepsilon_0]$. Therefore it has a unique fixed point, denoted by $(x^*(\xi, \eta, \varepsilon), y^*(\xi, \eta, \varepsilon))$. From Lemma 3.3 through to Lemma 3.5 and the third inequality in (3.10), it follows that the map

$$\Pi^\varepsilon: E^s_\varepsilon \times X^+_{-\mu/2} \times Y^+_{-\mu/2} \to X^+_0 \times Y^+_0$$

is C^{r-1}-bounded for $\varepsilon \in (0, \varepsilon_0]$. Therefore we can conclude from the proof of **[14]** that

$$x^*(\cdot, \cdot, \varepsilon) \times xy^*(\cdot, \cdot, \varepsilon) \colon [E^s_\varepsilon]_{\delta/2K} \to X^+_0 \times Y^+_0$$

is C^{r-1}-bounded. It is also easy to compute the Lipschitz constant of $x^*(\cdot, \cdot, \varepsilon)$ in ξ, which is $(2\varepsilon N_1/\mu^2)(\mu - 2\varepsilon N_1)$. So we set

$$U^\varepsilon(\xi, \eta) = x^*(\xi, \eta, \varepsilon)(0), \quad V^\varepsilon(\xi, \eta) = y^*(\xi, \eta, \varepsilon)(0)$$

and define the set

$$W^s_{\delta/2K} := \{(\xi + U^\varepsilon(\xi, \eta), V^\varepsilon(\xi, \eta)); (\xi, \eta) \in [E^s_\varepsilon]_{\delta/2K}\},$$

which is a C^{r-1}-manifold with boundaries.

CLAIM 3.6. *The set $W_{\delta/2K}$ is uniformly "future invariant" with respect to* (3.1) *in the following sense: if $(x_0, y_0) \in W^s_{\delta/2K}$, then $(x(\tau; x_0, y_0), y(\tau; x_0, y_0)) \in W^s_{\delta/2K}$ for $\tau \geqq 1/\mu \ln K$.*

Proof. Since $(x_0, y_0) \in W_{\delta/2K}$, there exists $(\xi_0, \eta_0) \in [E^s_\varepsilon]_{\delta/2K}$ such that $x_0 = \xi_0 + U^\varepsilon(\xi_0, \eta_0)$, $y_0 = V^\varepsilon(\xi_0, \eta_0)$. The pair of functions $(H_\varepsilon(\xi_0) + x^*(\xi_0, \eta_0; \varepsilon)$, $y^*(\xi_0, \eta_0; \varepsilon))$ is a solution of (3.1) passing through (x_0, y_0) at $\tau = 0$, and hence coincident with $(x(\tau; x_0, y_0), y(\tau; x_0, y_0))$. Now for $s \geqq 1/\mu \ln K$, consider the translates

$$x^*_s(\tau) := x^*(\tau + s) = x^*(\xi_0, \eta_0, \varepsilon)(\tau + s),$$
$$y^*_s(\tau) := y^*(\tau + s) = y^*(\xi_0, \eta_0, \varepsilon)(\tau + s).$$

From the fixed point equation for (x^*, y^*), we have

$$x^*(\tau + s) = \varepsilon \int_\infty^{\tau+s} f_2(H_\varepsilon(\xi_0)(\tau'), x^*(\tau'), y^*(\tau'), \varepsilon)\, d\tau \tag{3.11-1}$$

$$y^*(\tau + s) = S^\varepsilon(\tau + s, \xi_0)\eta_0 + \int_0^\infty U^\varepsilon(\tau + s, \tau'; \xi_0) g_2(H_\varepsilon(\xi_0)(\tau'), x(\tau'), y(\tau'), \varepsilon)\, d\tau'. \tag{3.11-2}$$

Changing the integration variable in (3.11) by $\tau' \to \tau' + s$, and using Lemma 3.2(i) and the fact that $H_\varepsilon(\xi)(\tau + s) = H_\varepsilon(H_\varepsilon(\xi)(s))(\tau)$, (3.11) can be written as

$$x^*_s(\tau) = \varepsilon \int_\infty^{\tau} f_2(H_\varepsilon(\xi)(\tau'), x^*_s(\tau'), y^*_s(\tau'), \varepsilon)\, d\tau' \tag{3.12-1}$$

$$y^*_s(\tau) = S^\varepsilon(\tau, \xi)\eta + \int_0^\infty U^\varepsilon(\tau, \tau'; \xi) g_2(H_\varepsilon(\xi)(\tau'), x^*_s(\tau'), y^*_s(\tau'), \varepsilon)\, d\tau', \tag{3.12-2}$$

where $\xi = H_\varepsilon(\xi_0)(s)$ and $\eta = S^\varepsilon(s, \xi_0)\eta_0$. Notice also that $|\eta| \leqq \delta/2K$ because of the condition $s \geqq (1/\mu) \ln K$. Therefore (3.12) says that (x^*_s, y^*_s) is a fixed point of $\Pi^\varepsilon(\xi, \eta, \cdot, \cdot)$ and the uniqueness of the fixed point implies

$$x^*(\tau + s) = x^*(H_\varepsilon(\xi_0)(\tau), S^\varepsilon(s, \xi_0)\eta_0; \varepsilon)(\tau),$$
$$y^*(\tau + s) = y^*(H_\varepsilon(\xi_0)(\tau), S^\varepsilon(s, \xi_0)\eta_0; \varepsilon)(\tau).$$

Therefore it follows that

$$x(s; x_0, y_0) = H_\varepsilon(\xi_0)(s) + x^*(s) = \xi + U^\varepsilon(\xi, \eta),$$
$$y(s; x_0, y_0) = y^*(\xi, \eta, \varepsilon)(0) = V^\varepsilon(\xi, \eta),$$

which proves $(x(s; x_0, y_0), y(s; x_0, y_0)) \in W^s_{\delta/2K}$, $s \geqq (1/\mu) \ln K$. □

CLAIM 3.7. *If $(x(\tau), y(\tau))$ is a solution of* (3.1) *with $\sup_{\tau \geqq 0} |y(\tau)| \leqq \delta/2K$ and $|y(0)| \leqq \delta/2K^2$, then $(x(0), y(0)) \in W^s_{\delta/2K}$.*

Before we prove Claim 3.7, we prepare two lemmas.

LEMMA 3.8. *For $x(\tau) \in C^1(R_+, R^m)$ with $|\dot{x}| \leqq \varepsilon N_0$, if $y(\tau)$ is a solution of $\dot{y} = g_1(x(\tau), y, \varepsilon)$ satisfying $\sup_{\tau \geqq 0} |y(\tau)| \leqq \delta/2K$, then $\sup_{\tau \geqq 0} |y(\tau)|\, e^{\mu\tau/2} \leqq \delta$.*

Proof. The equation for $y(\tau)$ can be written as $\dot{y} = A(x(\tau), \varepsilon)y + G_1(\tau, y, \varepsilon)$, where

$$G_1(\tau, y, \varepsilon) = g_0(x(\tau), h_\varepsilon(x(\tau)) + y, \varepsilon) - g_0(x(\tau), h_\varepsilon(x(\tau)), \varepsilon) - D_y g_0(x(\tau), h_\varepsilon(x(\tau)), \varepsilon)y.$$

Since the linear equation $\dot{y} = A(x(\tau), \varepsilon)y$ has an exponential dichotomy due to Lemma 2.3, the boundedness of y on R_+ implies $y(\tau) = S^\varepsilon(\tau)y(0) + K^\varepsilon G_1^\varepsilon(y)$, where S^ε, K^ε, and G_1^ε are operators constructed in a manner analogous to those for $S^\varepsilon(\tau, \xi)$, $K^\varepsilon(\xi)$ and G^ε earlier. It is easy to see that the operator $\tilde{\Pi}^\varepsilon: R^n \times Y_\rho^+ \to Y_\rho^+$, $\rho \in [-\mu/2, 0]$ defined by $\tilde{\Pi}^\varepsilon = S^\varepsilon(\tau)\eta + K^\varepsilon G_1^\varepsilon(y)$ satisfies $|\tilde{\Pi}^\varepsilon(\eta, y)|_\rho \leqq \delta$ if $|\eta| \leqq \delta/2K$ and $|y| \leqq \delta$. Moreover, $\tilde{\Pi}^\varepsilon(\eta, \cdot): [Y_\rho^+]_\delta \to [Y_\rho^+]_\delta$ is a contraction uniformly in η, $|\eta| \leqq \delta/2K$ and $\rho \in [-\mu/2, 0]$. Therefore $\tilde{\Pi}^\varepsilon$ has a unique fixed point $y_{\varepsilon,\rho}(\eta)(\tau) \in [Y_\rho^+]_\delta$ for each $\rho \in [-\mu/2, 0]$. Notice that $|S^\varepsilon(0)y(0)| \leqq K\,|y(0)| \leqq \delta/2K$ and hence y is a fixed point of $\tilde{\Pi}^\varepsilon(y(0), \cdot)$, namely, $y(\tau) = y_{\varepsilon,0}(y(0))(\tau)$. The uniqueness of the fixed point $y_{\varepsilon,\rho}(\eta)$ in $[Y_\rho^+]_\delta$, $\rho \in [-\mu/2, 0]$ and the inclusions $[Y_\rho^+]_\delta \subset [Y_0^+]_\delta$, $\rho \in [-\mu/2, 0]$ imply $y(\cdot) = y_{\varepsilon,0}(y(0)) = y_{\varepsilon,-\mu/2}(y(0)) \in [Y_{-\mu/2}^+]_\delta$. This completes the proof of Lemma 3.8. □

LEMMA 3.9. *If $y(\tau) \in [Y_{-\mu/2}^+]_\delta$, then for each solution of $\dot{x}(\tau) = \varepsilon f_1(x(\tau), y(\tau), \varepsilon)$, there exists a unique $\xi \in R^m$ such that*

$$\sup_{\tau \geqq 0} |x(\tau) - H_\varepsilon(\xi)(\tau)|\, e^{\mu\tau/2} < \frac{4\varepsilon N_1 \delta}{\mu}.$$

Proof. If we set $\phi(\tau) = x(\tau) - H_\varepsilon(\xi)(\tau)$ for some $\xi \in R^m$ to be determined, the requirement $\lim_{\tau\to\infty} \phi(\tau) = 0$ gives rise to an integral equation

$$\phi(\tau) = \varepsilon \int_\infty^\tau [f_1(x(\tau'), y(\tau'), \varepsilon) - f_1(x(\tau') - \phi(\tau'), 0, \varepsilon)]\, d\tau'.$$

The right-hand side of this integral equation defines an operator $\tilde{F}^\varepsilon: X_{-\mu/2}^+ \to X_{-\mu/2}^+$ satisfying $|\tilde{F}^\varepsilon(\phi)|_{-\mu/2} \leqq 2\varepsilon N_1/\mu(\delta + |\phi|_{-\mu/2})$. Hence $\tilde{F}^\varepsilon$ maps $[X_{-\mu/2}^+]_\delta$ onto itself because $4\varepsilon N_1/\mu \leqq 1$ (see (3.10)). Moreover, $\tilde{F}^\varepsilon$ is a contraction with Lipschitz constant $2\varepsilon N_1/\mu < 1/2$ (cf. (3.10)) and has a fixed point, which we denote by $\phi^*(\tau)$. It is now easy to verify that the difference $H(\tau) = x(\tau) - \phi^*(\tau)$ satisfies the equation $\dot{H} = f_1(H, 0, \varepsilon)$. By choosing ξ as $\xi = x(0) - \phi^*(0)$, we obtain the desired solution. □

Proof of Claim 3.7. By Lemma 3.8, we have $|y(\tau)| \leqq \delta e^{-\mu\tau/2}$, $\tau \geqq 0$. Hence, Lemma 3.9 gives a unique $\xi \in R^m$ such that $\phi(\tau) = x(\tau) - H_\varepsilon(\xi)(\tau) \in X_{-\mu/2}^+$. Therefore the pair (ϕ, y) satisfies a fixed point equation for Π^ε, and the uniqueness of the fixed point implies

$$\phi(\tau) = x^*(\xi, P^\varepsilon(0; \xi)y(0); \varepsilon)(\tau), \quad y(\tau) = y^*(\xi, P^\varepsilon(0; \xi)y(0); \varepsilon)(\tau).$$

This, in turn, means

$$x(0) = \xi + U^\varepsilon(\xi, P^\varepsilon(0; \varepsilon)y(0)), \quad y(0) = V^\varepsilon(\xi, P^\varepsilon(0; \varepsilon)y(0)),$$

proving $(x(0), y(0)) \in W_{\delta/2K}^s$. □

Now we can conclude the proof of Theorem 3.1. Let

$$\tilde{W}^s_{\delta/2K} = \{(x, y); x = \xi + U^\varepsilon(\xi, \eta), y = h_\varepsilon(x) + V^\varepsilon(\xi, \eta), (\xi, \eta) \in [E^s_\varepsilon]_{\delta/2K}\}$$

and

$$\tilde{W}^s_{\delta/2K}(h_\varepsilon(\xi)) = \{(x, y); x = \xi + U^\varepsilon(\xi, \eta), y = h_\varepsilon(x) + V^\varepsilon(\xi, \eta), (\xi, \eta) \in [E^s_\varepsilon]_{\delta/2K}\}$$

for each fixed $\xi \in R^m$. We also let $(x, y) \,.\, \tau$ denote the flow generated by $(\mathrm{F})_\varepsilon$. By setting

$$W^s(\mathbf{C}_\varepsilon) = \bigcup_{\tau \leqq 0} [\tilde{W}^s_{\delta/2K}] \,.\, \tau$$

$$W^s(h_\varepsilon(\xi) = \bigcup_{\tau \leqq 0} [\tilde{W}^s_{\delta/2K}(h_\varepsilon(H_\varepsilon(\xi)(-\tau)))] \,.\, \tau$$

and $\delta_0 = \delta/2K$, all the assertions in Theorem 3.1 are verified. □

COROLLARY 3.10. *For each $\tau \in R$, $W^s(h_\varepsilon(\xi)) \,.\, \tau = W^s(h_\varepsilon(H_\varepsilon(\xi)(\tau)))$ holds true.*

3.2. Inner expansions

We will present a method of approximating $W^s(h_\varepsilon(\xi))$ in the sequel. Let us come back to $(\mathrm{F})'_\varepsilon$, which can be recast as

$$(\mathrm{F})''_\varepsilon \quad \dot{\phi} = \varepsilon f_3(\tau, \phi, \psi, \varepsilon), \quad \dot{\psi} = g_3(\tau, \phi, \psi, \varepsilon),$$

where

$$f_3(\tau, \phi, \psi, \varepsilon) = f_2(H_\varepsilon(\xi_\varepsilon)(\tau), \phi, \psi, \varepsilon)$$
$$g_3(\tau, \phi, \phi, \psi, \varepsilon) = A(H_\varepsilon(\xi_\varepsilon)(\tau), \varepsilon)\psi + g_2(H_\varepsilon(\xi_\varepsilon)(\tau), \phi, \psi, \varepsilon)$$

and

$$\xi_\varepsilon = \sum_{j=0}^{p} \varepsilon^j \xi_j, \quad \xi_j \in R^m, \quad p < r - 1.$$

For the purpose of this section, we only need results for the case in which $\xi_1 = \xi_2 = \ldots = \xi_p = 0$, but we work in the general situation, which will be used in the next section. Notice that $f_3(\tau, 0, 0, \varepsilon) \equiv 0$ and $g_3(\tau, 0, 0, \varepsilon) \equiv 0$. One should also recall from the proof of Theorem 3.1 that $W^s(h_\varepsilon(\xi_\varepsilon))$ is characterised as the set of initial values of solutions $(\phi_\varepsilon(\tau), \psi_\varepsilon(\tau))$ of $(\mathrm{F})''_\varepsilon$ satisfying

$$\lim_{\tau \to \infty} \phi_\varepsilon(\tau) = 0, \quad \lim_{\tau \to \infty} \psi_\varepsilon(\tau) = 0, \tag{3.13}$$

or, more precisely,

$$W^s(h_\varepsilon(\xi_\varepsilon)) = h_\varepsilon(\xi_\varepsilon)$$
$$+ \{(\phi_\varepsilon(0), \psi_\varepsilon(0)); (\phi_\varepsilon, \psi_\varepsilon) \text{ solves } (\mathrm{F})''_\varepsilon \text{ and } |\phi_\varepsilon|_{-\mu/2} < \infty, |\psi_\varepsilon|_{-\mu/2} < \infty\}.$$

Assuming the formal expansions

$$\phi_\varepsilon(\tau) = \sum_{j \geqq 0} \varepsilon^j \phi_j(\tau), \quad \psi_\varepsilon(\tau) = \sum_{j \geqq 0} \varepsilon^j \psi_j(\tau),$$

the relation $(\mathrm{F})''_\varepsilon$ gives rise to a series of equations for (ϕ_j, ψ_j), $j=0, \ldots, p$.

$(\mathrm{F})''_0$ $\quad \dot{\phi}_0 = 0, \quad \dot{\psi}_0 = g_3(\tau, \phi_0, \psi_0, 0),$

$(\mathrm{F})''_1$ $\quad \dot{\phi}_1 = f_3(\tau, \phi_0, \psi_0, 0), \quad \dot{\psi}_1 = D_\psi g_3(\tau, \phi_1, \psi_1, 0)\psi_1 + D_\phi g_3(\tau, \phi_0, \psi_0, 0)\phi_1 + D_\varepsilon g_3(\tau, \phi_0, \psi_0, 0),$

and for $j \geqq 2$,

$$(\mathrm{F})''_j \begin{cases} \dot{\phi}_j = \sum_{l=1}^{j-1} \sum^{\#} \dfrac{1}{\alpha!\,\beta!\,\gamma!} D^\alpha_\phi D^\beta_\psi D^\gamma_\varepsilon f_3(\tau, \phi_0, \psi_0, 0)\langle \phi_{j_1}, \ldots, \phi_{j_\alpha}\rangle \langle \psi_{l_1}, \ldots, \psi_{l_\beta}\rangle, \\ \dot{\psi}_j = \sum_{l=1}^{j-1} \sum^{\#} \dfrac{1}{\alpha!\,\beta!\,\gamma!} D^\alpha_\phi D^\beta_\psi D^\gamma_\varepsilon g_3(\tau, \phi_0, \psi_0, 0)\langle \phi_{j_1}, \ldots, \phi_{j_\alpha}\rangle \langle \psi_{l_1}, \ldots, \psi_{l_\beta}\rangle, \end{cases}$$

where the summation $\sum^{\#}$ is taken over indices $\alpha \geqq 0$, $\beta \geqq 0$, $\gamma \geqq 0$, $j_i \geqq 1$, $l_i \geqq 1$ satisfying $\alpha + \beta + \gamma = l$, $\sum_{i=1}^{\alpha} j_i + \sum_{i=1}^{\beta} l_i + \gamma = j$. The condition in (3.13) can be interpreted as

$$\lim_{\tau \to \infty} \phi_j(\tau) = 0, \quad \lim_{\tau \to -\infty} \psi_j(\tau) = 0, \quad j = 0, \ldots, p. \tag{3.14-j}$$

The problem $(\mathrm{F}''_j + (3.14\text{-j})$ can be solved recursively. For $j=0$, $\dot{\phi}_0 = 0$ and the condition (3.14-0) gives $\phi_0 \equiv 0$. Then the equation for ψ_0 reads:

$$\dot{\psi}_0 = g_0(\xi_0, h(\xi_0) + \psi_0, 0). \tag{3.15}$$

For each $\xi_0 \in R^m$, (3.15) can be regarded as an autonomous equation in ψ_0 and the conditions (I), (II) imply that $\psi_0 = 0$ is a hyperbolic fixed point. Let us denote by $W^s(h(\xi_0))$ the stable manifold of $\psi_0 = 0$ for the flow of (3.15). For each $\eta \in W^s(h(\xi_0))$, the equation (3.15) has a unique solution $\psi_0(\tau; \eta)$ with $\psi_0(0; \eta) = \eta$ which decays exponentially, or $\sup_{\tau \geqq 0} |\psi_0(\tau)|\, e^{2\mu\tau} < \infty$, because of condition (II). With the choice, $\phi_0 \equiv 0$, and $\psi_0(\tau) = \psi_0(\tau; \eta)$, the problem $(\mathrm{F})''_j + (3.14\text{-}1)$ can be solved as follows. It is easy to see that ϕ_1 is given by

$$\phi_1(\tau) = \int_\infty^\tau f_3(s, 0, \psi_0(s), 0)\, ds, \quad \sup_{\tau \geqq 0} |\phi_1(\tau)|\, e^{2\mu\tau} \leqq \frac{N_1}{2\mu} |\psi_0|_{-2\mu}.$$

Now the equation for ψ_1 reads: $\dot{\psi}_1 = B(\tau)\psi_1 + q_1(\tau)$, where $B(\tau) = D_y g_0(\xi_0, h(\xi_0) + \psi_0(\tau), 0)$ and $q_1(\tau)$ is a function satisfying $\sup_{\tau \geqq 0} |q_1(\tau)|\, e^{2\mu\tau}(1+\tau)^{-1} < \infty$. The last inequality follows from the fact that $g_3(\tau, 0, 0, \varepsilon) = 0$, $|\phi_1|_{-2\mu} < \infty$, $|\psi_0|_{-2\mu} < \infty$, and the fact that $H_j(\tau)$ is a polynomial in τ of degree j where $H_\varepsilon(\xi_\varepsilon)(\tau) = \sum_{j \geqq 0} \varepsilon^j H_j(\tau)$. Condition (II) implies that the linear equation $\dot{y} = B(\tau)y$ has an exponential dichotomy on R_+, namely,

$$\begin{cases} |T(\tau, \tau')P(\tau) \leqq K e^{-2\mu(\tau - \tau')}, & \tau \geqq \tau' \geqq 0, \\ |T(\tau, \tau')Q(\tau) \leqq K e^{2\mu(\tau - \tau')}, & \tau' \geqq \tau \geqq 0. \end{cases} \tag{3.16}$$

Projections P and Q are not unique, but we can choose them uniquely by requiring them to be orthogonal at $\tau = 0$. For each $\eta_1 \in T_\eta W^s(h(\xi_0))$, we can uniquely determine the solution $\psi_1(\tau)$ satisfying $P(0)\psi_1(0) = \eta_1$, by

$$\psi_1(\tau) = T(\tau, 0)\eta_1 + \int_0^\infty U(\tau, \tau')q_1(\tau')\, d\tau',$$

where $U(\tau, \tau')$ is the Green's function for $\dot{y} = B(\tau)y$ on R_+.

Proceeding in a way similar to that described above for (ϕ_1, ψ_1), we can determine the solutions (ϕ_j, ψ_j), $j=2, \ldots, p$, with $\psi_0(0)=\eta$, $P(0)\psi_j(0)=\eta_j$, $j=1, \ldots, p$, for given $(\eta, \eta_1, \ldots, \eta_p)$. Moreover, the following inequalities hold true:

$$|\phi_j|_{-3\mu/2}<\infty, \quad |\psi_j|_{-3\mu/2}<\infty, \quad j=0, \ldots, p.$$

By changing variables in $(\mathrm{F})''_\varepsilon$ by $\phi \to \phi + \Phi^\varepsilon_p$, $\psi \to \psi + \Psi^\varepsilon_p$, where $\Phi^\varepsilon_p(\tau, \xi_\varepsilon, \eta, \eta_1, \ldots, \eta_p) = \sum_{j=1}^p \varepsilon^j \phi_j(\tau)$ and $\Psi^\varepsilon_p(\tau, \xi_\varepsilon, \eta, \eta_1, \ldots, \eta_p) = \sum_{j=1}^p \varepsilon^j \psi_j(\tau)$, we obtain the equations for the new variables (ϕ, ψ)

$$\begin{cases} \dot{\phi} = \varepsilon \tilde{f}_3(\tau, \phi, \psi, \varepsilon) + p_1(\tau, \varepsilon), \\ \dot{\psi} = B(\tau, \varepsilon)\psi + \tilde{g}_3(\tau, \phi, \psi, \varepsilon) + p_2(\tau, \varepsilon), \end{cases} \tag{3.17}$$

where

$$\begin{aligned}
\tilde{f}_3 &= f_3(\tau, \phi + \Phi^\varepsilon_p, \psi + \Psi^\varepsilon_p, \varepsilon) - f_3(\tau, \Phi^\varepsilon_p, \Psi^\varepsilon_p, \varepsilon), \\
p_1(\tau, \varepsilon) &= \varepsilon f_3(\tau, \Phi^\varepsilon_p, \Psi^\varepsilon_p, \varepsilon) - \dot{\Phi}^\varepsilon_p(\tau), \\
B(\tau, \varepsilon) &= D_y g_3(\tau, \Phi^\varepsilon_p, \Psi^\varepsilon_p, \varepsilon), \\
\tilde{g}_3 &= g_3(\tau, \phi + \Phi^\varepsilon_p, \psi + \Psi^\varepsilon_p, \varepsilon) - g_3(\tau, \Phi^\varepsilon_p, \Psi^\varepsilon_p, \varepsilon) - B(\tau, \varepsilon)\psi, \\
p_2(\tau, \varepsilon) &= g_3(\tau, \Phi^\varepsilon_p, \Psi^\varepsilon_p, \varepsilon) - \dot{\Psi}^\varepsilon_p(\tau).
\end{aligned}$$

From the construction of Φ^ε_p and Ψ^ε_p and the Taylor expansion, one can verify that $|p_i(\cdot)|_{-3\mu/2} = O(\varepsilon^{p+1})$, $i=1, 2$. The linear equation

$$\dot{\psi} = B(\tau, \varepsilon)\psi \tag{3.18}$$

has an exponential dichotomy with constant L and exponent μ, namely,

$$\begin{cases} |T^\varepsilon(\tau, \tau')P^\varepsilon(\tau') \leqq Le^{-\mu(\tau-\tau')}, & \tau \geqq \tau' \geqq 0, \\ |T^\varepsilon(\tau, \tau')Q^\varepsilon(\tau') \leqq Le^{\mu(\tau-\tau')}, & \tau' \geqq \tau \geqq 0, \end{cases} \tag{3.19}$$

where the stable and unstable projections are normalised to be orthogonal. We emphasise that the constant L appearing in (3.19) depends on $(\eta, \eta_1, \ldots, \eta_p)$. To be more precise, let us introduce

$$W^s_{\delta,T}(h(\xi_0)) = \{\eta \in W^s(h(\xi_0)); |\psi_0(\tau; \eta)| \leqq \delta, \text{ for } \tau \geqq T\},$$

where δ is the same constant as that which appeared in (3.10). For $\eta \in W^s_{\delta,T}(h(\xi_0))$, $\eta_i \in T_\eta W^s(h(\xi_0))$, $|\eta_i| \leqq K_1$, $i=1, \ldots, p$, the constant L in (3.19) depends on T and K_1, i.e. $L = L(T, K_1)$.

Now, on looking for solutions of (3.17) with $\lim_{\tau\to\infty} \phi(\tau) = 0$, $\lim_{\tau\to\infty} \psi(\tau) = 0$, we arrive, after an additional normalisation $P^\varepsilon(0)y(0) = 0$, at the integral equations

$$\begin{cases} \phi(\tau) = \displaystyle\int_\infty^\tau [\varepsilon \tilde{f}_3(\tau', \phi(\tau'), \psi(\tau'), \varepsilon) + p_1(\tau', \varepsilon)]\, d\tau', \\ \psi(\tau) = \displaystyle\int_0^\infty U^\varepsilon(\tau, \tau')[\tilde{g}_3(\tau', \phi(\tau'), \psi(\tau'), \varepsilon) + p_2(\tau', \varepsilon)]\, d\tau', \end{cases} \tag{3.20}$$

where U^ε is the Green's function for (3.18) on $[0, \infty)$. By using (3.19), we can verify that the relations in (3.20) define a fixed point equation on $X^+_{-\mu} \times Y^+_{-\mu}$. For sufficiently small $\varepsilon_0 = \varepsilon_0(T, K_1)$, the right-hand side of (3.20) is a contraction on

$[X^{+}_{-\mu}\times Y^{+}_{-\mu}]_{\delta_1}$ for some $\delta_1>0$, uniformly with respect to $(\xi_\varepsilon, \eta, \eta_1, \ldots, \eta_p, \varepsilon)$, $\xi_\varepsilon \in R^m$, $\eta \in W^s_{\delta,T}(h(\xi_0))$, $\eta_i \in T_\eta W^s(h(\xi_0))$, $|\eta_i| \leqq K_1$, $i=1,\ldots,p$ and $\varepsilon \in (0, \varepsilon_0]$. Now one can apply the proof of Theorem 3.1 to obtain a unique fixed point

$$(\phi^\varepsilon(\xi_\varepsilon, \eta, \eta_1, \ldots, \eta_p)(\tau), \psi^\varepsilon(\xi_\varepsilon, \eta, \eta_1, \ldots, \eta_p)(\tau))$$

of (3.20), which is C^{r-p-1}-bounded as a $[X^{+}_{-\mu+\rho}\times Y^{+}_{-\mu+\rho}]$-valued function for $\rho \geqq \varepsilon(r-p)N_1$, and

$$|\phi^\varepsilon(\xi, \eta, \eta_1, \ldots, \eta_p)|_{-\mu} = O(\varepsilon^{p+1}),$$
$$|\psi^\varepsilon(\xi, \eta, \eta_1, \ldots, \eta_p)|_{-\mu} = O(\varepsilon^{p+1}), \quad \text{as} \quad \varepsilon \to 0.$$

From the unique characterisation of $W^s(h_\varepsilon(\xi))$ in Theorem 3.1(ii), and the exponentially decaying properties of Φ^ε_p and Ψ^ε_p, we have the following theorem:

THEOREM 3.11. (i) *For $K_1>0$ and $T>0$, there exists $\varepsilon_0=\varepsilon_0(K_1, T)$, such that for $\varepsilon \in (0, \varepsilon_0]$, $W^s(h_\varepsilon(\xi_\varepsilon))$ can be approximated by a power series of ε of order $p \leqq r-1$, in the following sense:*

$$\begin{aligned} W^s(h_\varepsilon(\xi_\varepsilon)) \supset \{(x,y); x &= \xi_\varepsilon + \Phi^\varepsilon_p(\xi_\varepsilon, \eta, \eta_1, \ldots, \eta_p)(0) + \phi^\varepsilon(\xi_\varepsilon, \eta, \eta_1, \ldots, \eta_p)(0), \\ y &= h_\varepsilon(x) + \Psi^\varepsilon_p(\xi_\varepsilon, \eta, \eta_1, \ldots, \eta_p)(0) + \psi^\varepsilon(\xi_\varepsilon, \eta, \eta_1, \ldots, \eta_p)(0), \\ \eta &\in W^s_{\delta,T}(h(\xi_0)), |\eta_i| \leqq K_1, \eta_i \in T_\eta W^s(h(\xi_0)), i=1,\ldots,p\} \supset W^s(h_\varepsilon(\xi_\varepsilon))_{\delta,T/2},\end{aligned}$$

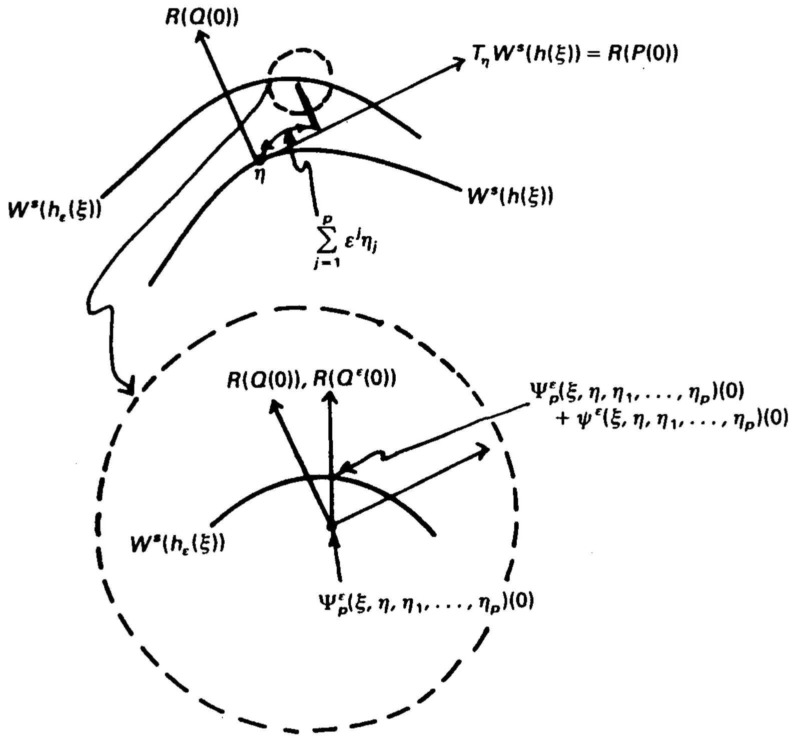

Figure 3.1.

where

$$W^s(h_\varepsilon(\xi))_{\delta,T} = \{(x_0, y_0) \in W^s(h_\varepsilon(\xi));\, |y(\tau; x_0, y_0) - h_\varepsilon(H_\varepsilon(\xi)(\tau))| \leqq \delta, \text{ for } \tau \geqq T\},$$

$\xi_\varepsilon = \xi_0 + \varepsilon\xi_1 + \ldots + \varepsilon^p\xi_p$, *and* Φ_p^ε *and* Ψ_p^ε *are obtained by solving* $(\mathrm{F})_j'' + (3.14\text{-j})$, $j \geqq 0$, *successively.*

(ii) *An analogous result to part* (i) *holds true for the unstable manifold* $W^u(h_\varepsilon(\xi_\varepsilon))$.

Remark 3.12. For each $\xi_\varepsilon \in R^m$, the dimension of $W^s(h_\varepsilon(\xi_\varepsilon))$ is k. But the variables $(\eta, \eta_1, \ldots, \eta_p)$ range over a $k(p+1)$-dimensional space. So different values of $(\eta, \eta_1, \ldots, \eta_p)$ may correspond to a one point of $W^s(h_\varepsilon(\xi_\varepsilon))$. Geometrically, the situation can be explained as follows. For each $\xi \in R^m$ and $\eta \in W^s(h(\xi))$, the manifold $W^s(h_\varepsilon(\xi))$ is, locally, expressed as a graph over $T_\eta W^s(h(\xi)) = \mathrm{Range}\,(P(0))$ with the essential variable on $T_\eta W^s(h(\xi))$ being $\sum_{j=1}^p \varepsilon^j \eta_j$. Figure 3.1 may help the reader to understand the situation.

4. Heteroclinic connections

In this section, we present a situation in which there is a heteroclinic connection between two invariant sets for $(\mathrm{F})_\varepsilon$. Suppose that there are two functions $h^+(x)$ and $h^-(x)$ satisfying the conditions (I), (II) imposed in Section 1. Applying the results of the previous two sections, we know that there are normally hyperbolic invariant manifolds for $(\mathrm{F})_\varepsilon$, denoted by $\mathbf{C}_\varepsilon^\pm$, and their stable and unstable manifolds $W^s(\mathbf{C}_\varepsilon^\pm)$ and $W^u(\mathbf{C}_\varepsilon^\pm)$. Under certain additional conditions to be exhibited below, we show that $W^u(\mathbf{C}_\varepsilon^-)$ and $W^s(\mathbf{C}_\varepsilon^+)$ intersect along an m-dimensional submanifold.

When $\varepsilon = 0$, the equation $(\mathrm{F})_\varepsilon$ reduces to

$$(\mathrm{F})_0 \quad \begin{cases} \dot{x} = 0 \\ \dot{y} = g_0(x, y, 0) \end{cases}$$

One can regard this as a family of autonomous equations in R^n with $x \in R^m$ being a parameter. The conditions (I), (II) for $h^\pm(x)$ imply that $y = h^\pm(x)$ are hyperbolic equilibrium points of $\dot{y} = g_0(x, y, 0)$ for each $x \in R^m$. Our third condition is:

(III) *There exists a* $\bar{\xi} \in R^m$ *such that the equation* $\dot{y} = g_0(\bar{\xi}, y, 0)$ *has a solution* $z(\tau)$ *satisfying* $\lim_{\tau\to\infty} z(\pm\tau) = h^\pm(\bar{\xi})$ *and the linear variational equation*

$$\dot{y} = D_y g_0(\bar{\xi}, z(\tau), 0)y \tag{4.1}$$

has a unique (up to scalar multiplications) bounded solution $\dot{z}(\tau)$.

Condition (III) implies that the equation adjoint to (4.1):

$$\dot{\zeta} = -\zeta D_y g_0(\bar{\xi}, z(\tau), 0), \quad \zeta\text{: an } n\text{-dimensional row vector} \tag{4.2}$$

also has a unique (up to scalar multiplications) bounded solution $\zeta(\tau)$ which decays exponentially as $\tau \to \pm\infty$. We also impose the following conditions:

(IV) $$\int_{-\infty}^{\infty} \zeta(\tau) D_x g_0(\bar{\xi}, z(\tau), 0)\, d\tau \neq 0$$

and

(V.a) $$\int_{-\infty}^{\infty} \zeta(\tau) D_x g_0(\bar{\xi}, z(\tau), 0)\, d\tau \langle f_0(\bar{\xi}, h^+(\bar{\xi}), 0)\rangle \neq 0,$$

or

(V.b) $$\int_{-\infty}^{\infty} \zeta(\tau) D_x g_0(\bar{\xi}, z(\tau), 0)\, d\tau \langle f_0(\bar{\xi}, h^-(\bar{\xi}), 0)\rangle \neq 0.$$

Let $z(\tau; \bar{\xi})$ be the solution appearing in condition (III) normalised so that $z(0; \bar{\xi}) \in \Sigma$, where Σ is an $(n-1)$-dimensional plane transversal to the vector $g_0(\bar{\xi}, z(0; \bar{\xi}), 0)$. We also normalise $\zeta(\tau; \bar{\xi})$ so that $|\zeta(0; \bar{\xi})| = 1$ and one of the nonzero components of $\zeta(0; \bar{\xi})$, say $\zeta_1(0; \bar{\xi})$, is positive.

It is a well known fact (cf. [**8, 13**]) that under the conditions (III), (IV) there is an $(m-1)$-dimensional manifold $\mathbf{S}$ of C^{r-1}-class passing through $\bar{\xi}$ in R^m such that for each $\xi \in \mathbf{S}$, the equation $\dot{y} = g_0(\xi, y, 0)$ has a heteroclinic orbit $z(\tau; \xi)$ with $z(0; \xi) \in \Sigma$, $\lim_{\tau\to\infty} z(\pm\tau) = h^{\pm}(\xi)$. Moreover, the adjoint equation (4.2) around $z(\tau; \xi)$ also has a unique (up to scalar multiplications) bounded solution $\zeta(\tau; \xi)$ which decays exponentially as $|\tau| \to \infty$ and normalised so that $|\zeta(0; \xi)| = 1$, $\zeta_1(0; \xi) > 0$. The conditions (III), (IV) are satisfied for $z(\tau; \xi)$, $\zeta(\tau; \xi)$ for each $\xi \in \mathbf{S}$, and the vector

$$\int_{-\infty}^{\infty} \zeta(\tau; \xi) D_x g_0(\xi, z(\tau; \xi), 0)\, d\tau, \quad \xi \in \mathbf{S},$$

is orthogonal to the tangent space $T_{\xi}\mathbf{S}$ in R^m.

Condition (V) is also satisfied for $\xi \in \mathbf{S}$ by virtue of continuity.

Remark 4.1. Condition (V.a) means that the flow on $\mathbf{C}_\varepsilon^+$ is transversal to $(\mathbf{S}, h_\varepsilon^+(\mathbf{S}))$ and condition (V.b) means that the flow on $\mathbf{C}_\varepsilon^-$ is transversal to $(\mathbf{S}, h_\varepsilon^-(\mathbf{S}))$, where the function $h_\varepsilon^{\pm}$ defines the set $\mathbf{C}_\varepsilon^{\pm}$.

THEOREM 4.2. *Suppose the conditions* (I), (II), (III), (IV) *and* (V.a) (*or* (V.b)) *are satisfied. For each $\xi \in R^m$, there exists ξ_ε with $|\xi - \xi_\varepsilon| = O(\varepsilon)$ such that $W^u(h_\varepsilon^-(\xi_\varepsilon))$ (respectively $W^u(h_\varepsilon^-(\xi))$) and $W^s(h_\varepsilon^+(\xi))$ (respectively $W^s(h_\varepsilon^+(\xi_\varepsilon))$) intersect transversely at a uniquely defined point $(x(\xi, \varepsilon), y(\xi, \varepsilon))$. Moreover, the set $\mathbf{S}_\varepsilon = \{\xi_\varepsilon; \xi \in \mathbf{S}\}$, is an $(m-1)$-dimensional manifold of class C^{r-1} in an $O(\varepsilon)$-neighbourhood of* $\mathbf{S}$.

The remaining part of this section is devoted to the proof of Theorem 4.2, assuming (V.b) out of two alternatives.

We express the right-hand side of $(F)_j''$, $j = 0, \ldots, p$ in terms of f_0, g_0, ϕ_j, ψ_j, and H_j.

$$(F)_j'' \quad \begin{cases} \dot{\phi}_j(\tau) = \varepsilon \hat{f}_{j-1}(\phi_0 + H_0, \ldots, \phi_{j-1} + H_{j-1}; \psi_0 + \Psi_0, \ldots, \psi_{j-1} + \Psi_{j-1}) \\ \qquad - \dot{H}_j(\tau), \\ \dot{\psi}_j(\tau) = \hat{g}_j(\phi_0 + H_0, \ldots, \phi_j + H_j; \psi_0 + \Psi_0, \ldots, \psi_j + \Psi_j) - D_\tau^* \Psi_j(\tau), \end{cases}$$

where $H_\varepsilon(\xi_\varepsilon)(\tau)=\sum_{j\geqq 0}\varepsilon^j H_j(\tau)$ is the outer expansion and $h_\varepsilon(\sum_{j\geqq 0}\varepsilon^j(\phi_j+H_j))=\sum_{j\geqq 0}\varepsilon^j\Psi_j(\phi_0+H_0,\ldots,\phi_j+H_j)(\tau)$,

$$\hat{f}_{-1}=0,\quad \hat{f}_0=f_0(\phi_0+H_0,\,\psi_0+\Psi_0,\,0),\quad \hat{g}_0=g_0(\phi_0+H_0,\,\psi_0+\Psi_0,\,0)$$

and for $j\geqq 1$,

$$\begin{aligned}\hat{f}_j=\sum_{l=1}^{j}\sum{}^{\#\#}\frac{1}{\alpha!\,\beta!\,\gamma!}D_x^\alpha D_y^\beta D_\varepsilon^\gamma f_0(\phi_0+H_0,\,\psi_0+\Psi_0,\,0)\\ \times\langle\phi_{p_1}+H_{p_1},\ldots,\phi_{p_\alpha}+H_{p_\alpha}\rangle\langle\psi_{q_1}+\Psi_{q_1},\ldots,\psi_{q_\beta}+\Psi_{q_\beta}\rangle,\\ \hat{g}_j=\sum_{l=1}^{j}\sum{}^{\#\#}\frac{1}{\alpha!\,\beta!\,\gamma!}D_x^\alpha D_y^\beta D_\varepsilon^\gamma g_0(\phi_0+H_0,\,\psi_0+\Psi_0,\,0)\\ \times\langle\phi_{p_1}+H_{p_1},\ldots,\phi_{p_\alpha}+H_{p_\alpha}\rangle\langle\psi_{q_1}+\Psi_{q_1},\ldots,\psi_{q_\beta}+\Psi_{q_\beta}\rangle.\end{aligned}$$

In the above, the symbol $D_\tau^*\Psi_j$ stands for the differentiation of Ψ_j along the solutions ϕ_l+H_l of the equations $\dot{\phi}_l+\dot{H}_l=\hat{f}_{l-1}$, $l\leqq j$, and the summation $\sum^{\#\#}$ is taken over indices $\alpha\geqq 0$, $\beta\geqq 0$, $\gamma\geqq 0$, $p_i\geqq 1$, $q_i\geqq 1$ satisfying $\alpha+\beta+\gamma=l$, $\sum_{i=1}^{\alpha}p_i+\sum_{i=1}^{\beta}q_i+\gamma=j$. We apply the expressions above to both of the cases

$$\phi_\varepsilon=\phi_\varepsilon^\pm,\quad \psi_\varepsilon=\psi_\varepsilon^\pm,\quad H_\varepsilon=H_\varepsilon^\pm,\quad h_\varepsilon=h_\varepsilon^\pm,$$

where $H_\varepsilon^\pm(\tau)$ is the outer solutions on $\mathbf{C}_\varepsilon^\pm$, i.e. the solutions of

$$\begin{cases}\dot{H}_\varepsilon^\pm=\varepsilon f_0(H_\varepsilon^\pm,\,h_\varepsilon^\pm(H_\varepsilon^\pm),\,\varepsilon), & \tau\in R,\\ H_\varepsilon^\pm(0)=\sum_{j=0}^{p}\varepsilon^j\xi_j^\pm, & \xi_j^\pm\in R^m.\end{cases}$$

In order to have a heteroclinic connection between $\mathbf{C}_\varepsilon^-$ and $\mathbf{C}_\varepsilon^+$, we impose the conditions

$$\begin{cases}\phi_\varepsilon^-(0)+H_\varepsilon^-(0)=\phi_\varepsilon^+(0)+H_\varepsilon^+(0),\\ \psi_\varepsilon^-(0)+h_\varepsilon^-(\phi_\varepsilon^-(0)+H_\varepsilon^-(0))=\psi_\varepsilon^+(0)+h_\varepsilon^+(\phi_\varepsilon^+(0)+H_\varepsilon^+(0)),\end{cases}\tag{4.3}$$

$$\lim_{\tau\to\infty}\phi_\varepsilon^\pm(\pm(\tau))=0,\quad \lim_{\tau\to\infty}\psi_\varepsilon^\pm(\tau)=0.\tag{4.4}$$

In terms of $\phi_j^\pm$, $\psi_j^\pm$, $H_j^\pm$, and $\Psi_j^\pm$, these conditions are identified as:

$$\phi_j^-(0)+H_j^-(0)=\phi_j^+(0)+H_j^+(0),\quad \psi_j^-(0)+\Psi_j^-(0)=\psi_j^+(0)+\Psi_j^+(0),\tag{4.3-j}$$

and

$$\lim_{\tau\to\infty}\phi_j^\pm=0,\quad \lim_{\tau\to\infty}\psi_j^\pm=0.\tag{4.4-j}$$

Let us start solving $(\mathrm{F})_j''$ successively under the conditions (4.3-j) and (4.4-j). For $j=0$, the equation reads

$$\dot{\phi}_0^\pm=0,\quad \dot{\psi}_0^\pm=g_0(\xi_0^\pm+\phi_0^\pm,\,\psi_0^\pm+h^\pm(\xi_0^\pm),\,0).$$

We can immediately see that $\phi_0^\pm\equiv 0$. Then the first condition in (4.3-0) implies $\xi_0^-=H_0^-(0)=H_0^+(0)=\xi_0^+$. We choose the common value $\xi_0^\pm=\xi\in\mathbf{S}$ from the manifold $\mathbf{S}\subset R^m$. The equation for $\psi_0^\pm$ is

$$\dot{\psi}_0^\pm=g_0(\xi,\,\psi_0^\pm+h^\pm(\xi),\,0).$$

Since we have, from conditions (III) and (IV), that

$$\{z(\tau;\xi);\tau\in R\}\subset W^s(h^+(\xi))\cap W^u(h^-(\xi)),\quad\text{for}\quad \xi\in \mathbf{S},$$

we choose

$$\psi_0^+(\tau)=z(\tau+\theta;\xi)-h^+(\xi)\quad\text{for}\quad \tau\geqq 0,$$
$$\psi_0^-(\tau)=z(\tau+\theta;\xi)-h^-(\xi)\quad\text{for}\quad \tau\leqq 0,$$

for some constant $\theta\in R$, which will be determined below. Let us denote by $x_0(\tau)$ and $y_0(\tau)$ the sums

$$x_0(\tau)=\phi_0^\pm(\tau)+H_0^\pm(\tau)=\xi,\quad y_0(\tau)=\psi_0^\pm(\tau)+h^\pm(\xi)=z(\tau+\theta;\xi)\quad\text{for}\quad \pm\tau\geqq 0,$$

respectively. The equation for $\phi_1^\pm(\tau)$ and the first condition in (4.4-1) determine $\phi_1^\pm$ uniquely as

$$\phi_1^\pm(\tau)=\int_{\pm\infty}^{\tau}[f_0(\xi,y_0(s),0)-f_0(\xi,h^\pm(\xi),0)]\,ds.$$

The first condition in (4.3-1) then becomes

$$\xi_1^-+\int_{-\infty}^{0}[f_0(\xi,y_0(s),0)-f_0(\xi,h^-(\xi),0)]\,ds$$
$$=\xi_1^++\int_{\infty}^{0}[f_0(\xi,y_0(s),0)-f_0(\xi,h^+(\xi),0)]\,ds.$$

The last relation determines only the difference $\xi_1^+-\xi_1^-$ uniquely, so we set $\xi_1^-=0$. Hereafter, we always set $\xi_j^-=0$ in accordance with the statement of Theorem 4.2 and condition (V.b). Notice that $x_1(\tau):=\phi_1^\pm(\tau)+H_1^\pm(\tau)$ is a solution of $\dot{x}_1(\tau)=\hat{f}_0=f_0(\xi,y_0(\tau),0)$ on R with $x_1(0)=\phi_1^\pm(0)+H_1^\pm(0)$. The equation for $\psi_1^\pm$ now reads

$$\dot{\psi}_1^\pm=D_yg_0(\xi,y_0(\tau),0)\langle\psi_1^\pm+\Psi_1^\pm\rangle+D_xg_0(\xi,y_0(\tau),0)\langle x_1(\tau)\rangle$$
$$+D_\varepsilon g_0(\xi,y_0(\tau),0)-\dot{\Psi}_1^\pm(\tau).$$

One should notice that $x_j=\phi_j^\pm+H_j^\pm$, $j=0,1$, are chosen so that $D_\tau^*\Psi_1^\pm=\dot{\Psi}_1^\pm$. The linear equation $\dot{\psi}=D_yg_0(\xi,z(\tau;\xi),0)\psi$ has an exponential dichotomy on $(-\infty,\theta]$ and on $[\theta,\infty)$, for each $\theta\in R$. We denote, suppressing the reference to ξ, the solution operator by $T(\tau,\tau')$ and the stable and unstable projections by $P^+(\tau)$, $Q^+(\tau)$ for $\tau\geqq\theta$, and by $P^-(\tau)$, $Q^-(\tau)$ for $\tau\leqq\theta$. The projections $P^\pm$, $Q^\pm$ are normalised so that they are orthogonal projections at $\tau=0$. The second condition in (4.4-1) implies

$$\psi_1^+(\tau)=T(\tau+\theta,\theta)P^+(\theta)\psi_1^+(0)+\int_0^\tau T(\tau+\theta,s+\theta)P^+(s+\theta)q_1^+(s)\,ds$$
$$+\int_\infty^\tau T(\tau+\theta,s+\theta)Q^+(\tau+\theta)q_1^+(s)\,ds,$$

$$\psi_1^-(\tau)=T(\tau+\theta,\theta)Q^-(\theta)\psi_1^-(0)+\int_0^\tau T(\tau+\theta,s+\theta)Q^-(s+\theta)q_1^-(s)\,ds$$
$$+\int_{-\infty}^\tau T(\tau+\theta,s+\theta)P^-(s+\theta)q_1^-(s)\,ds,$$

where

$$q_1^{\pm}(\tau) = D_y g_0(\xi, z(\tau+\theta;\xi), 0)\langle \Psi_1^{\pm}(\tau)\rangle + D_x g_0(\xi, z(\tau+\theta;\xi), 0)\langle x_1(\tau)\rangle$$
$$+ D_\varepsilon g_0(\xi, z(\tau+\theta;\xi), 0) - \dot{\Psi}_1^{\pm}(\tau).$$

In what follows, we let Ra denote the range of operators. From condition (III) for $\xi \in \mathbf{S}$, we have

$$R^n = [\dot{z}(\theta)] \oplus [v^*(\theta)] \oplus \{Ra(Q^-(\theta)) \cap Ra(Q^+(\theta))\}$$
$$\oplus \{Ra(P^-(\theta)) \cap Ra(P^+(\theta))\},$$

where $v^*(\theta)$ is a unit vector such that $[v^*(\theta)] = \{Ra(Q^+(\theta)) \cap Ra(P^-(\theta))\}$. Therefore, the second condition (4.3-1) is equivalent to

$$\dot{z}(\theta)[\psi_1^-(0) + \Psi_1^-(0)] = \dot{z}(\theta)[\psi_1^+(0) + \Psi_1^+(0)] \tag{4.5}$$

$$P^+(\theta)P^-(\theta)\Psi_1^-(0) + P^+(\theta)\int_{-\infty}^{0} T(\theta, s+\theta)P^-(s+\theta)q_1^-(s)\,ds$$
$$= P^-(\theta)P^+(\theta)[\psi_1^+(0) + \Psi_1^+(0)] \tag{4.6}$$

$$Q^+(\theta)Q^-(\theta)[\psi_1^-(0) + \Psi_1^-(0)]$$
$$= Q^-(\theta)Q^+(\theta)\Psi_1^+(0) + Q^-(\theta)\int_{\infty}^{0} T(\theta, s+\theta)Q^+(s+\theta)q_1^+(s)\,ds \tag{4.7}$$

$$\zeta(\theta)\Psi_1^-(0) + \int_{-\infty}^{0} \zeta(s+\theta)q_1^-(s)\,ds = \zeta(\theta)\Psi_1^+(0) + \int_{\infty}^{0} \zeta(s+\theta)q_1^+(s)\,ds. \tag{4.8}$$

Here, in order to obtain (4.8), we used the fact that the range of the operators

$$Q^+(\theta)T(\theta, s+\theta)P^-(s+\theta) = P^-(\theta)T(\theta, s+\theta)Q^+(s+\theta)$$

is spanned by $\zeta(s+\theta)$. By using integration by parts and the equation $\dot{\zeta} = -\zeta D_y g_0$, we obtain

$$\int_{\pm\infty}^{0} \zeta(s+\theta)q_1^{\pm}(s)\,ds = \int_{\pm\infty}^{0} \zeta(s+\theta)\{D_x g_0(\xi, z(s+\theta), 0)\langle x_1(s)\rangle$$
$$+ D_\varepsilon g_0(\xi, z(s+\theta), 0)\}\,ds - \zeta(\theta)\Psi_1^{\pm}(\theta).$$

The condition (4.8) is thus equivalent to

$$\int_{-\infty}^{\infty} \zeta(\tau)[D_x g_0(\xi, z(\tau), 0)\langle x_1(\tau-\theta)\rangle + D_\varepsilon g_0(\xi, z(\tau), 0)]\,d\tau = 0. \tag{4.8$'$}$$

By differentiating the left-hand side of (4.8)$'$ with respect to θ, we obtain

$$\int_{-\infty}^{\infty} \zeta(\tau)D_x g_0(\xi, z(\tau), 0)\,d\tau \langle f_0(\xi, h^-(\xi), 0)\rangle, \tag{4.9}$$

where we used the following expression for $x_1(\tau)$:

$$x_1(\tau) = \int_{-\infty}^{\tau+\theta} [f_0(\xi, z(s), 0) - f_0(\xi, h^-(\xi), 0)]\,d\tau + f_0(\xi, h^-(\xi), 0)\tau,$$

which follows from the choice of $\xi_1^\pm$. Notice that the quantity in (4.9) is independent of θ and nonzero by virtue of the condition (V.b). Therefore, we can choose $\theta = \theta_0(\xi)$ uniquely so that (4.8) is fulfilled. As soon as θ is determined in this way, we can choose $P^-(\theta)P^+(\theta)\psi_1^+(0)$ and $Q^+(\theta)Q^-(\theta)\psi_1^-(0)$ uniquely so that (4.6) and (4.7) are satisfied. On the other hand, the condition (4.5) determined only the difference $\dot{z}(\theta)[\psi_1^+(0) - \psi_1^-(0)]$ uniquely. Let $a_1^\pm = \dot{z}(\theta)\psi_1^\pm(0)/|\dot{z}(\theta)|^2$ be chosen so that (4.5) is satisfied. Then the function $y_1(\tau) := \psi_1^\pm(\tau) + \Psi_1^\pm(\tau)$, for $\pm\tau \geqq 0$, solves the equation

$$\dot{y}_1(\tau) = \hat{g}_1(x_0(\tau), x_1(\tau); y_0(\tau), y_1(\tau)), \quad \tau \in R,$$

and, moreover, $y_1(\tau) = a_1\dot{z}(\tau + \theta) + \bar{y}_1(\tau)$, where $\bar{y}_1$ is a solution of

$$\dot{\bar{y}}_1(\tau) = \hat{g}_1(x_0(\tau), x_1(\tau); y_0(\tau), \bar{y}_1(\tau)), \quad \text{with} \quad \bar{y}_1(0) = \psi_1^\pm(0) + \Psi_1^\pm(0) - a_1^\pm\dot{z}(\theta)$$

and a_1 is an arbitrary constant. The function $\bar{y}_1$ also satisfies $\dot{z}(\theta)\bar{y}_1(0) = 0$. At this stage, we cannot determine the constant a_1 uniquely in a natural way. As a matter of fact, the free parameter a_1 plays an important role when we determine $\psi_2^\pm$ satisfying (4.3-2). We will come back to this point later on. For now, fix a_1, say $a_1 = 0$, in order to complete the proof of Theorem 4.2.

Observe that the tangent spaces at $(x_0(0) + \varepsilon x_1(0), y_0(0) + \varepsilon y_1(0))$ of the ε-approximations of stable and unstable fibres $W^u(h_\varepsilon^+(\xi + \varepsilon\xi_1^+))$ and $W^s(h_\varepsilon^-(\xi))$ are spanned, respectively, by

$$[\dot{z}(\theta)] \oplus Ra(P^-(\theta)P^+(\theta)) \oplus [v_+(\theta)]$$

and

$$[\dot{z}(\theta)] \oplus Ra(Q^+(\theta)Q^-(\theta)) \oplus [v_-(\theta)],$$

where

$$v_\pm(\theta) = \left[\frac{d}{d\theta}\int_{\pm\infty}^{0} \zeta(\tau + \theta)q_1^\pm(\tau)\, d\tau\right] v^*(\theta).$$

Since we have

$$R_n = [\dot{z}(\theta)] \oplus Ra(P^-(\theta)P^+(\theta)) \oplus Ra(Q^+(\theta)Q^-(\theta)) \oplus [v^*(\theta)]$$

and $v_+(\theta) - v_-(\theta) \neq 0$ (which follows from the fact that the quantity in (4.9) is nonzero), the ε-approximate stable and unstable fibres intersect transversely at the point $(x_0(0) + \varepsilon x_1(0), y_0(0) + \varepsilon y_1(0))$. On the other hand, Theorem 3.11 in the previous section says that $W^s(h_\varepsilon^+(\xi + \varepsilon\xi_1^+))$ and $W^u(h_\varepsilon^-(\xi))$ are an ε^2-perturbation of the ε-approximate stable and unstable fibres. Therefore, $W^s(h_\varepsilon^+(\xi + \varepsilon\xi_1^+))$ and $W^u(h_\varepsilon^-(\xi))$ intersect transversely at a point $(x(\xi, \varepsilon), y(\xi, \varepsilon))$ near the point $(x_0(0) + \varepsilon x_1(0), y_0(0) + \varepsilon y_1(0))$ for $\varepsilon > 0$ small. The function $\theta_0(\xi)$ is of C^{r-1} in $\xi \in \mathbf{S}$ and hence the function $\xi_1^+ = \xi_1^+(\xi)$ is of C^{r-1} in $\xi \in \mathbf{C}$. Therefore the set $\mathbf{S}_\varepsilon = \{\xi + \varepsilon\xi_1^+(\xi) + O(\varepsilon^2); \xi \in \mathbf{S}\}$ is C^{r-1}-diffeomorphic to $\mathbf{S}$ for $\varepsilon \in (0, \varepsilon_0]$,, for $\varepsilon_0 > 0$ sufficiently small. This complete the proof of Theorem 4.2. □

Let us now come back to the question of how the free parameter a_1 in y_1 can be determined in the best possible way.

The solutions $\phi_2^\pm$ are determined as

$$\phi_2^\pm(\tau) = \int_{\pm\infty}^{\tau} [\hat{f}_1(x_0(s), x_1(s); y_0(s)) - \dot{H}_2^\pm(s)]\, ds$$

and the condition (4.3-2) together with $\xi_2^- = 0$ determines ξ_2^+ uniquely. The function $x_2 := \phi_2^\pm(\tau) + H_2^\pm(\tau)$, $\pm\tau \geqq 0$, satisfies $\dot{x}_2 = \hat{f}_1(x_0, x_1; y_0)$ and hence we have $D_\tau^*\Psi_2^\pm = \dot{\Psi}_2^\pm$. The equation for $\psi_2^\pm$ then reads

$$\dot{\psi}_2^\pm = D_y g_0(\xi, z(\tau+\theta; \xi), 0)\langle \psi_2^\pm + \Psi_2^\pm\rangle + \bar{g}_2(x_0, x_1, x_2; y_0, y_1) - \dot{\Psi}_2^\pm,$$

where

$$\bar{g}_2 = \hat{g}_2 - D_y g_0(\xi, z(\tau+\theta; \xi), 0)\langle \psi_2^\pm + \Psi_2^\pm\rangle.$$

The conditions (4.3-2) and (4.4-2) further dictate:

$$\dot{z}(\theta)[\psi_j^-(0) + \Psi_j^-(0)] = \dot{z}(\theta)[\psi_j^+(0) + \Psi_j^+(0)] \tag{4.10-j}$$

$$P^+(\theta)P^-(\theta)\Psi_j^-(0) + P^+(\theta)\int_{-\infty}^{0} T(\theta, s+\theta)P^-(s+\theta)q_j^-(s)\, ds = P^-(\theta)P^+(\theta)[\psi_j^+(0) + \Psi_j^+(0)] \tag{4.11-j}$$

$$Q^+(\theta)Q^-(\theta)[\psi_j^-(0) + \Psi_j^-(0)] = Q^-(\theta)Q^+(\theta)\Psi_j^+(0) + Q^-\int_{\infty}^{0} T(\theta, s+\theta)Q^+(s+\theta)q_j^+(s)\, ds \tag{4.12-j}$$

$$\zeta(\theta)\Psi_j^-(0) + \int_{-\infty}^{0} \zeta(s+\theta)q_j^-(s)\, ds = \zeta(\theta)\Psi_j^+(0) + \int_{\infty}^{0} \zeta(s+\theta)q_j^+(s)\, ds, \tag{4.13-j}$$

in which $j = 2$, $\theta = \theta_0(\xi)$ and $q_j^\pm$ are given by

$$q_j^\pm = D_y g_0(\xi, z(\tau+\theta; \xi), 0)\langle \Psi_j^\pm\rangle - \dot{\Psi}_j^\pm + \bar{g}_j(x_0, \ldots, x_j; y_0, \ldots, y_{j-1}).$$

By using integration by parts and the relation $\dot{\zeta} = -\zeta D_y g_0(x_0, y_0, 0)$, we can see that the condition (4.13-2) is equivalent to

$$\int_{-\infty}^{\infty} \zeta(\tau+\theta)\bar{g}_2(x_0(\tau), x_1(\tau), x_2(\tau); y_0(\tau), y_1(\tau))\, d\tau = 0. \tag{4.14}$$

The parameter a_1 is contained in y_1 and x_2, so, pulling out the terms involving y_1, the coefficients of a_1 and a_1^2 are, respectively, given by

$$I_1 = \int_{-\infty}^{\infty} \zeta(\tau+\theta)[D_x D_y g_0(\#)\langle x_1(\tau)\rangle\langle \dot{z}(\tau+\theta)\rangle + D_y D_\varepsilon g_0(\#)\langle \dot{z}(\tau+\theta)\rangle + D_y^2 g_0(\#)\langle \dot{z}(\tau+\theta), \bar{y}_1(\tau)\rangle]\, d\tau,$$

$$I_2 = \frac{1}{2}\int_{-\infty}^{\infty} \zeta(\tau+\theta)D_y^2 g_0(\#)\langle \dot{z}(\tau+\theta), \dot{z}(\tau+\theta)\rangle\, d\tau,$$

in which $g_0(\#)$ is evaluated at $(\xi, z(\tau+\theta; \xi), 0)$. Then the condition (4.14) is written as $I_2 a_1^2 + (I_1 + I_1')a_1 + I_0 = 0$, where I_0 does not involve a_1 and I_1' is the contribution from x_2. It then follows from the equation for ζ and the relation

$\ddot{z} = D_y g_0(\#)\langle \dot{z} \rangle$ that

$$2I_2 = -\int_{-\infty}^{\infty} \dot{\zeta}(\tau+\theta) D_y g_0(\#)\langle \dot{z}(\tau+\theta)\rangle \, d\tau$$
$$- \int_{-\infty}^{\infty} \zeta(\tau+\theta) D_y g_0(\#)\langle \ddot{z}(\tau+\theta)\rangle \, d\tau = 0.$$

Similarly, we obtain

$$\begin{aligned} I_1 &= -\int_{-\infty}^{\infty} \dot{\zeta}(\tau+\theta) D_x g_0(\#)\langle x_1(\tau)\rangle \, d\tau - \int_{-\infty}^{\infty} \zeta(\tau+\theta) D_x g_0(\#)\langle \dot{x}_1(\tau)\rangle \, d\tau \\ &\quad - \int_{-\infty}^{\infty} \dot{\zeta}(\tau+\theta) D_y g_0(\#)\langle \bar{y}_1(\tau)\rangle \, d\tau - \int_{-\infty}^{\infty} \zeta(\tau+\theta) D_y g_0(\#)\langle \dot{\bar{y}}_1(\tau)\rangle \, d\tau \\ &\quad - \int_{-\infty}^{\infty} \dot{\zeta}(\tau+\theta) D_\varepsilon g_0(\#) \, d\tau \\ &= \int_{-\infty}^{\infty} \dot{\zeta}(\tau+\theta)\langle \dot{\bar{y}}_1 - \hat{g}_1(x_0, x_1; y_0, \bar{y}_1)\rangle \, d\tau - \int_{-\infty}^{\infty} \zeta(\tau+\theta) D_x g_0(\#)\langle \hat{f}_0 \rangle \, d\tau \\ &= -\int_{-\infty}^{\infty} \zeta(\tau) D_x g_0(\xi, z(\tau;\xi), 0)\langle f_0(\xi, z(\tau;\xi), 0)\rangle \, d\tau, \end{aligned}$$

where we used the facts that $\dot{\bar{y}}_1 = \hat{g}_1(x_0, x_1; y_0, \bar{y}_1)$, $\dot{x}_1 = \hat{f}_0(x_0; y_0)$. On the other hand, the coefficient of a_1 in x_2 is given by $f_0(\xi, z(\tau+\theta), 0) - f_0(\xi, h^-(\xi), 0)$ and the contribution from x_2 to the coefficient of a_1 in (4.14) is given by

$$I_1' = -I_1 - \int_{-\infty}^{\infty} \zeta(\tau) D_x g_0(\xi, z(\tau), 0) \, d\tau \langle f_0(\xi, h^-(\xi), 0)\rangle$$

and therefore the coefficient of a_1 in (4.14) is nonzero due to the condition (V.b). Therefore, the parameter a_1 is determined so that (4.13-2) is satisfied. Then $P^-(\theta)P^+(\theta)\psi_2^+(0)$ and $Q^+(\theta)Q^-(\theta)\psi_2^-(0)$ are chosen so that (4.11-2) and (4.12-2) are fulfilled. The condition (4.10-2) determines only the difference $\dot{z}(\theta)[\psi_2^-(0) - \psi_2^+(0)]$. This leaves us another free parameter a_2 as $y_2(\tau) = a_2\dot{z}(\tau+\theta) + \bar{y}_2(\tau)$, where $y_2(\tau) := \psi_2^\pm(\tau) + \Psi_2^\pm(\tau)$ for $\pm\tau \geqq 0$, and $\dot{z}(\theta)\bar{y}_2(0) = 0$. Both y_2 and $\bar{y}_2$ satisfy $\dot{y} = \hat{g}_2(x_0, x_1, x_2; y_0, y_1, y)$. The parameter a_2 is uniquely chosen so that (4.13-3) is satisfied. We show this process for general cases.

Let us assume, for $j \geqq 2$, that $(\phi_i^\pm, \psi_i^\pm)$, $i \leqq j$ are determined so that (4.3-i) and (4.4-i) are satisfied. Let us set

$$x_i(\tau) = \phi_i^\pm(\tau) + H_i^\pm(\tau), \quad y_i^\pm(\tau) = \psi_i^\pm(\tau) + \Psi_i^\pm(\tau), \quad \text{for} \quad \pm\tau \geqq 0, \quad i \leqq j.$$

Then y_i contains a free parameter a_j as: $y_j(\tau) = a_j\dot{z}(\tau+\theta) + \bar{y}_j(\tau)$. It is easy to determine $\phi_{j+1}^\pm$, ξ_{j+1}^+ satisfying (4.3-j + 1), (4.4-j + 1). The conditions (4.3-j + 1), (4.4-j + 1) for $\psi_{j+1}^\pm$ are equivalent to (4.10-j + 1) through to (4.13-j + 1). The parameter a_j is determined as follows. By using integration by parts, the equation for ζ and $D_\tau^* \Psi_{j+1}^\pm = \dot{\Psi}_{j+1}^\pm$, we can verify, as before, that (4.13-j + 1) is equivalent to

$$\int_{-\infty}^{\infty} \zeta(\tau+\theta)\bar{g}_{j+1}(x_0, \ldots, x_{j+1}; y_0, \ldots, y_j) \, d\tau = 0. \tag{4.15}$$

Since $j \geqq 2$, the terms involving y_j are:

$$\int_{-\infty}^{\infty} \zeta(\tau+\theta)[D_y D_x g_0(\#)\langle x_1(\tau), y_j(\tau)\rangle + D_y D_\varepsilon g_0(\#)\langle y_i(\tau)\rangle + D_y^2 g_0(\#)\langle y_1(\tau), y_j(\tau)\rangle]\, d\tau$$

where $g_0(\#)$ is evaluated at $(\xi, z(\tau+\theta; \xi), 0)$. The coefficient of a_j is given by the above formula, with y_j being replaced by $\dot{z}(\tau+\theta)$. Therefore the coefficient of a_j contributed from y_j is expressed as

$$\begin{aligned}
&-\int_{-\infty}^{\infty} \dot{\zeta}(\tau+\theta) D_y g_0(\#)\langle y_1(\tau)\rangle\, d\tau - \int_{-\infty}^{\infty} \zeta(\tau+\theta) D_y g_0(\#)\langle \dot{y}_1(\tau)\rangle\, d\tau \\
&-\int_{-\infty}^{\infty} \dot{\zeta}(\tau+\theta) D_x g_0(\#)\langle x_1(\tau)\rangle\, d\tau - \int_{-\infty}^{\infty} \zeta(\tau+\theta) D_x g_0(\#)\langle \dot{x}_1(\tau)\rangle\, d\tau \\
&-\int_{-\infty}^{\infty} \dot{\zeta}(\tau+\theta) D_\varepsilon g_0(\#)\, d\tau \\
={}&-\int_{-\infty}^{\infty} \zeta(\tau) D_x g_0(\xi, z(\tau; \xi), 0)\langle f_0(\xi, z(\tau; \xi), 0)\rangle\, d\tau \\
&+\int_{-\infty}^{\infty} \dot{\zeta}(\tau+\theta)\langle \dot{y}_1 - \hat{g}_1(x_0, x_1; y_0, y_1)\rangle\, d\tau \\
={}&-\int_{-\infty}^{\infty} \zeta(\tau) D_x g_0(\xi, z(\tau; \xi), 0)\langle f_0(\xi, z(\tau; \xi), 0)\rangle\, d\tau.
\end{aligned}$$

Here we used integration by parts together with the relations

$$\dot{\zeta} = -\zeta D_y g_0(\#), \quad \dot{x}_1 = f_0(\#), \quad \dot{y}_1 = \hat{g}_1(x_0, x_1; y_0, y_1).$$

On the other hand, the coefficient of a_j in x_{j+1} is again given by $f_0(\xi, z(\tau+\theta), 0) - f_0(\xi, h^{-1}(\xi), 0)$ and the coefficient of a_j in (4.15) is given by

$$-\int_{-\infty}^{\infty} \zeta(\tau) D_x g_0(\xi, z(\tau), 0)\, d\tau \langle f_0(\xi, h^-(\xi), 0)\rangle$$

which is nonzero due to the condition (V.b), determining the a_j uniquely.

Appendix A. Proof of Lemma 2.5

For $(\xi, y) \in R^m \times Y_\rho$, $\rho \in R$, the operator $K^\varepsilon(\xi, y)$ is defined by

$$K^\varepsilon(\xi, y)\psi(t) = \frac{1}{\varepsilon}\int_{-\infty}^{\infty} U^\varepsilon(t, s; H_\varepsilon(\xi, y))\psi(s)\, ds$$

for $\psi \in Y_{\rho_1}$, $|\rho_1| < \mu/\varepsilon$. By using (2.2), we have

$$\begin{aligned}
|K^\varepsilon(\xi, y)\psi(t)| &\leqq \frac{1}{\varepsilon}\int_{-\infty}^{t} K e^{-(\mu/\varepsilon)(t-t')}|\psi(t')|\, dt' + \frac{1}{\varepsilon}\int_{t}^{\infty} K e^{(\mu/\varepsilon)(t-t')}|\psi(t')|\, dt' \\
&< \frac{K}{\varepsilon}\left[\frac{1}{\mu/\varepsilon + \rho_1} + \frac{1}{\mu/\varepsilon - \rho_1}\right]|\psi|_{\rho_1} e^{\rho_1|t|} \\
&< \frac{2k}{\mu - \varepsilon\rho_1}|\psi|_{\rho_1} e^{\rho_1|t|}.
\end{aligned}$$

Therefore, $|K^\varepsilon(\xi, y)\psi|_{\rho_2} \leqq |K^\varepsilon(\xi, y)\psi|_{\rho_1} \leqq (2K/\mu - \varepsilon\rho_1)\,|\psi|_{\rho_1}$ for $\rho_1 \leqq \rho_2$, which proves (i).

For $\xi_1, \xi_2 \in R^m$, $y_1, y_2 \in Y_{\rho_0}$, $\rho_0 > N_1$ and $\psi \in Y_{\rho_1}$, let $\psi_i = K^\varepsilon(\xi_i, y_i)\psi$, $x_i(t) = H_\varepsilon(\xi_i, y_i)(t)$, $i = 1, 2$. Then the difference $v = \psi_2 - \psi_1$ satisfies

$$\varepsilon v' = A(x_1(t))v + \int_0^1 D_xA(x_1(t) + su(t))\langle u(t)\rangle\, ds\psi_2(t), \tag{A.1}$$

where $u(t) = x_2(t) - x_1(t)$. Notice that the non-homogeneous term in (A.1) belongs to $Y_{\rho_0+\rho_1}$, since $N_1 < \rho_0$. Moreover, we know that $v \in Y_{\rho_1} \subset Y_{\rho_0+\rho_1}$, and hence

$$v = K^\varepsilon(\xi_1, y_1)\left[\int_0^1 D_xA(x_1 + su)\langle u\rangle\, ds\psi_2\right].$$

Therefore by using part (i) of the lemma, and Lemma 2.4, we obtain:

$$\begin{aligned}|\psi_2 - \psi_1|_{\rho_0+\rho_1} &\leqq \frac{2K}{\mu - \varepsilon\rho_0 - \varepsilon\rho_1} M_2\left[|\xi_1 - \xi_2| + \frac{N_1}{\rho_0 - N_1}|y_1 - y_2|_{\rho_0}\right]|\psi_2|_{\rho_1} \\ &\leqq \frac{4K^2M_2}{(\mu - \varepsilon\rho_0 - \varepsilon\rho_1)(\mu - \varepsilon\rho_1)}\left[|\xi_1 - \xi_2| + \frac{N_1}{\rho_0 - N_1}|y_1 - y_2|_{\rho_0}\right]|\psi|_{\rho_1},\end{aligned}$$

which proves (ii), (iii) of Lemma 2.5.

With the same notation as above, a unique solution $\bar{v}$ in $Y_{\rho_0+\rho_1}$ of the following equation

$$\varepsilon\bar{v}' = A(x_1(t))\bar{v} + D_xA(x_1)\langle D_\xi H_\varepsilon(\xi_1, y_1)\langle \xi_2 - \xi_1\rangle + D_yH_\varepsilon(\xi_1, y_1)\langle y_2 - y_1\rangle\rangle\psi_1$$

is expressed as:

$$\begin{aligned}\bar{v} = K^\varepsilon(\xi_1, y_1)[&D_xA(x_1)\langle D_\xi H_\varepsilon(\xi_1, y_1)\langle \xi_2 - \xi_1\rangle \\ &+ D_yH_\varepsilon(\xi_1, y_1)\langle y_2 - y_1\rangle\rangle K^\varepsilon(\xi_1, y_1)\psi].\end{aligned} \tag{A.2}$$

The difference $v - \bar{v}$ belongs to $Y_{\rho_0+\rho_1}$ and satisfies

$$\begin{aligned}\varepsilon(v - \bar{v})' = A(x_1)(v - \bar{v}) &+ \int_0^1 [D_xA(x_1 + su) - D_xA(x_1)]\, ds\langle u\rangle\psi_1 \\ &+ D_xA(x_1)\langle u - D_\xi H_\varepsilon(\xi_1, y_1)\langle \xi_2 - \xi_1\rangle - D_yH_\varepsilon(\xi_1, y_1)\langle y_2 - y_1\rangle\rangle\psi_1 \\ &+ \int_0^1 [D_xA(x_1 + su)\langle u\rangle\, ds[\psi_2 - \psi_1].\end{aligned}$$

Therefore for $\rho_2 > \rho_0 + \rho_1$, we have

$$\begin{aligned}|v - \bar{v}|_{\rho_2} \leqq \frac{2K}{\mu - \varepsilon\rho_2}\Bigg\{&\left|\int_0^1 [D_xA(x_1 + su) - D_xA(x_1)]\, ds\right|_{\rho_2-\rho_1-\rho_0} |u|_{\rho_0}|\psi_1|_{\rho_1} \\ &+ |D_xA(x_1)|_0\,|u - D_\xi H_\varepsilon\langle \xi_2 - \xi_1\rangle - D_yH_\varepsilon\langle y_2 - y_1\rangle|_{\rho_2-\rho_1}|\psi_1|_{\rho_1} \\ &+ \left|\int_0^1 [D_xA(x_1 + su)\, ds\right|_0 |u|_{\rho_0}|\psi_2 - \psi_1|_{\rho_2-\rho_0}\Bigg\}.\end{aligned}$$

One can show that $|\int_0^1 [D_xA(x_1 + su) - D_xA(x_1)]\, ds|_{\rho_2-\rho_1-\rho_0} \to 0$ as $(\xi_2, y_2) \to (\xi_1, y_1)$ (see, e.g. [**14**, Lemma 3]. On the other hand, Lemma 2.4 implies that

$$\frac{|u - D_\xi H_\varepsilon\langle \xi_2 - \xi_1\rangle - D_yH_\varepsilon\langle y_2 - y_1\rangle|_{\rho_2-\rho_1}}{(|\xi_1 - \xi_2| + |y_1 - y_2|_\rho)} \to 0,$$

as $(\xi_2, y_2) \to (\xi_1, y_1)$, on account of $\rho_2 - \rho_1 > \rho_0$, and that

$$\frac{|u|_{\rho_0}}{(|\xi_1 - \xi_2| + |y_1 - y_2|_{\rho_0})} \leqq 1 + \frac{N_1}{\rho_1 - N_1}.$$

One can also show that $|\psi_2 - \psi_1|_{\rho_2 - \rho_0} \to 0$ as $(\xi_2, y_2) \to (\xi_1, y_1)$ as follows. The equation (A.1) could also be written as:

$$\varepsilon v' = A(x_1(t))v + [A(x_1 + su) - A(x_1)]\psi_2$$

and hence

$$|\psi_2 - \psi_1|_{\rho_2 - \rho_0} = |v|_{\rho_2 - \rho_0} \leqq \frac{2K}{\mu - \varepsilon\rho_2 + \varepsilon\rho_0} |A(x_1 + su) - A(x_1)|_{\rho_2 - \rho_1 - \rho_0} |\psi_2|_{\rho_1}.$$

Since $\rho_2 - \rho_1 - \rho_0 > 0$, $|A(x_1 + su) - A(x_1)|_{\rho_2 - \rho_1 - \rho_0} \to 0$ as $(\xi_2, y_2) \to (\xi_1, y_1)$ (see [**14**, Lemma 3]). Therefore, $|v - \bar{v}|_{\rho_2}/(|\xi_1 - \xi_2| + |y_1 - y_2|_{\rho_0}) \to 0$ as $(\xi_2, y_2) \to (\xi_1, y_1)$, proving the differentiability of K^ε: $R^m \times Y_{\rho_0} \to L(R^m \times Y_{\rho_0}, L(Y_{\rho_1}, Y_{\rho_2}))$ at (ξ_1, y_1). Moreover, the operator $D_{(\xi,y)}K^\varepsilon(\xi_1, y_1)\langle(\xi, y)\rangle\psi$ is represented as in (A.2), i.e. for $(\xi, y) \in R^m \times Y_{\rho_0}$, $\psi \in Y_{\rho_1}$.

$$\begin{aligned} D_{(\xi,y)}K^\varepsilon(\xi_1, y_1)\langle(\xi, y)\rangle\psi = K^\varepsilon(\xi_1, y_1)[&D_xA(x_1)\langle D_\xi H_\varepsilon(\xi_1, y_1)\langle\xi\rangle \\ &+ D_yH_\varepsilon(\xi_1, y_1)\langle y\rangle\rangle K^\varepsilon(\xi_1, y_1)\psi]. \end{aligned} \tag{A.3}$$

Notice that the operator $D_{(\xi,y)}K^\varepsilon(\xi_1, y_1)$ defined by (A.3) is well-defined in $L(R^m \times Y_{\rho_0}, L(Y_{\rho_1}, Y_{\rho_2}))$ even for $\rho_2 = \rho_1 + \rho_0$.

The continuity of $D_{(\xi,y)}K^\varepsilon(\xi, y)$ in (ξ, y) can be verified by using (A.3) and following the argument presented above, provided that $\rho_2 > \rho_1 + \rho_0$ is satisfied. This completes the proof of Lemma 2.5(iii) for $l = 1$. For $l \geqq 2$, the proof goes along the same line of argument as above, although computation is extremely tedious. The formula for $D^l_{(\xi,y)}K^\varepsilon(\xi, y)$ involves the higher order Leibnitz rule and would not be given.

Appendix B. Proof of Lemma 3.2(ii)

Let $T^\varepsilon(\tau, \tau'; H_\varepsilon(\xi))$ be the solution operator of (B.1):

$$\dot{y} = A(H_\varepsilon(\xi), \varepsilon)y, \tag{B.1}$$

where $A(H, \varepsilon) = D_yg_0(H, h_\varepsilon(H), \varepsilon)$, and $H_\varepsilon(\xi)$ is the solution of $\dot{H} = \varepsilon f_0(H, h_\varepsilon(H), \varepsilon)$, $H(0) = \xi$. Equation (B.1) has an exponential dichotomy on R uniformly with respect to $\xi \in R^m$. Namely,

$$\begin{cases} |T^\varepsilon(\tau, \tau'; H_\varepsilon(\xi))P^\varepsilon(\tau'; \xi)| \leqq Ke^{-\mu(\tau-\tau')}, & \tau \geqq \tau', \\ |T^\varepsilon(\tau, \tau'; H_\varepsilon(\xi))Q^\varepsilon(\tau'; \xi)| \leqq Ke^{\mu(\tau-\tau')}, & \tau \leqq \tau'. \end{cases} \tag{B.2}$$

First of all, we prove the following lemma:

LEMMA B.1. *The $L(R^n, R^n)$-valued functions $P^\varepsilon(0; \xi)$ and $Q^\varepsilon(0; \xi)$ are C^{r-1}-bounded functions of $\xi \in R^m$.*

Proof. For $\xi_i \in R^m$, $i = 1, 2$, and $\eta \in R^n$, let $y_i(\tau) = T_1(\tau, 0)P_1(0)\eta$, $i = 1, 2$, where $T_i(\tau, s) = T^\varepsilon(\tau, s; H_\varepsilon(\xi_i))$, $P_i(\tau) = P^\varepsilon(\tau; \xi_i)$. In this proof, we suppress the dependence of functions on ε from our notation. So we set

$$Q_i(\tau) = Q^\varepsilon(\tau: \xi_i), \quad H^i(\tau) = H_\varepsilon(\xi_i)(\tau), \quad A(H) = A(H, \varepsilon).$$

Since $y_1(\tau)$ is a solution of $\dot{y}_1 = A(H^2(\tau))y_1 + [A(H^1(\tau)) - A(H^2(\tau))]y_1$ bounded on R_+, there exists $\eta' \in R^n$ such that

$$\begin{aligned} y_1(\tau) = T_2(\tau, 0)P_2(0)\eta' &+ \int_0^\tau T_2(\tau, s)P_2(s)[A(H^1(s)) - A(H^2(s))]y_1(s)\,ds \\ &+ \int_\infty^\tau T_2(\tau, s)Q_2(s)[A(H^1(s)) - A(H^2(s))]y_1(s)\,ds. \end{aligned} \tag{B.3}$$

Setting $\tau = 0$ in (B.3), we have

$$P_1(0)\eta = P_2(0)\eta' + \int_\infty^0 T_2(0, s)Q_2(s)[A(H^1(s)) - A(H^2(s))]T_1(s, 0)P_1(0)\eta\,ds.$$

Since the last integrand belongs to $\ker P_2(0)$, we have $P_2(0)P_1(0)\eta = P_2(0)\eta'$ and hence obtain

$$\begin{aligned} P_1(0)\eta - P_2(0)P_1(0)\eta = \int_\infty^0 T_2(0, s)Q_2(s)[A(H^1(s)) \\ - A(H^2(s))]T_1(s, 0)P_1(0)\eta\,ds \end{aligned} \tag{B.4}$$

and

$$\begin{aligned} |P_1(0)\eta - P_2(0)P_1(0)\eta| &\leqq K^2 \int_0^\infty e^{-2\mu s}\,|A(H^1(s)) - A(H^2(s))|\,|\eta|\,ds \\ &\leqq \frac{K^2 M_2 |\eta|}{2\mu - \varepsilon N_1}\,|\xi_1 - \xi_2|, \end{aligned}$$

where we used (B.2) and $|H^1(s) - H^2(s)|_{\varepsilon N_1} \leqq |\xi_1 - \xi_2|$, which follows from the same proof as that of Lemma 2.4. Let us set $C = K^2 M_2/(2\mu - \varepsilon N_1)$. Hence we have obtained

$$|P_1(0)\eta - P_2(0)P_1(0)\eta| \leqq C\,|\eta|\,|\xi_1 - \xi_2|. \tag{B.5}$$

The identity (B.3), together with $P_2(0)P_1(0)\eta = P_2(0)\eta'$ and (B.2), gives

$$\begin{aligned} &|T_1(\tau, 0)P_1(0)\eta - T_2(\tau, 0)P_2(0)P_1(0)\eta| \\ &\qquad \leqq \frac{2\mu}{\varepsilon N_1}\,C\,|\eta|\,|\xi_1 - \xi_2|\,e^{(-\mu + \varepsilon N_1)\tau}, \quad \text{for} \quad \tau \geqq 0. \end{aligned} \tag{B.6}$$

On the other hand, arguing as above, $T_1(\tau, 0)Q_1(0)\eta$ can be represented, for some $\eta'' \in R^n$, as

$$\begin{aligned} T_1(\tau, 0)Q_1(0)\eta = T_2(\tau, 0)Q_2(0)\eta'' \\ &+ \int_0^\tau T_2(\tau, s)Q_2(s)[A(H^1(s)) - A(H^2(s))]T_1(\tau, 0)Q_1(0)\eta\,ds \\ &+ \int_{-\infty}^\tau T_2(\tau, s)P_2(s)[A(H^1(s)) - A(H^2(s))]T_1(\tau, 0)Q_1(0)\eta\,ds. \end{aligned} \tag{B.7}$$

This immediately implies

$$P_2(0)Q_1(0)\eta = \int_{-\infty}^{0} T_2(0, s)P_2(s)[A(H^1(s)) - A(H^2(s))]T_1(\tau, 0)Q_1(0)\eta \, ds \tag{B.8}$$

and hence

$$|P_2(0)Q_1(0)\eta| \leqq C\,|\eta|\,|\xi_1 - \xi_2|. \tag{B.9}$$

Moreover, the identity (B.7) implies

$$|T_2(\tau, 0)P_2(0)Q_1(0)\eta| \leqq C\,|\eta|\,|\xi_1 - \xi_2|\,e^{-\mu\tau}, \quad \tau \geqq 0. \tag{B.10}$$

The inequalities (B.5) and (B.9) imply the Lipschitz continuity of $P^\varepsilon(0; \xi)$ in ξ as follows. For $\eta \in R^n$,

$$\begin{aligned} |P_1(0)\eta - P_2(0)\eta| &= |P_1(0)\eta - P_2(0)P_1(0)\eta| + |-P_2(0)Q_1(0)\eta| \\ &\leqq 2C\,|\eta|\,|\xi_1 - \xi_2|. \end{aligned}$$

Therefore $|P_1(0) - P_2(0)| \leqq 2C\,|\xi_1 - \xi_2|$ follows.

Motivated by the computations above, define

$$\begin{aligned} D_\xi P_2(0)\langle \xi \rangle \eta = &\int_{\infty}^{0} T_2(0, s)Q_2(s)D_xA(H^2(s))\langle D_\xi H^2(\tau)\langle \xi \rangle \rangle T_2(s, 0)P_2(0)\eta \, ds \\ &- \int_{-\infty}^{0} T_2(0, s)P_2(s)D_xA(H^2(s))\langle D_\xi H^2(\tau)\langle \xi \rangle \rangle T_2(s, 0)Q_2(0)\eta \, ds. \end{aligned}$$

We will show below that this operator defines the derivative of $P^\varepsilon(0; \xi)$ at $\xi = \xi_2$. By using (B.4) and (B.8), we have

$$\begin{aligned} |[P_1(0) - P_2(0) - D_\xi P_2(0)\langle \xi \rangle]\eta| = |[P_1(0) - P_2(0)]P_1(0)\eta - P_2(0)Q_1(0)\eta \\ - D_\xi P_2(0)\langle \xi \rangle \eta| \leqq \sum_{i=1}^{6} |I_i|, \end{aligned}$$

where $\xi = \xi_1 - \xi_2$ and I_i, $i = 1, \ldots, 6$ are given by

$$I_1 = \int_{\infty}^{0} T_2(0, s)Q_2(s)[A(H^1) - A(H^2) - D_xA(H^2)\langle D_\xi H^2\langle \xi \rangle \rangle T_1(s, 0)P_1(0)\eta] \, ds,$$

$$I_2 = \int_{\infty}^{0} T_2(0, s)Q_2(s)D_xA(H^2)\langle D_\xi H^2\langle \xi \rangle \rangle [T_1(s, 0)P_1(0)\eta - T_2(s, 0)P_2(0)P_1(0)\eta] \, ds,$$

$$I_3 = \int_{\infty}^{0} T_2(0, s)Q_2(s)D_xA(H^2)\langle D_\xi H^2\langle \xi \rangle \rangle [-T_2(s, 0)P_2(0)Q_1(0)\eta] \, ds,$$

$$I_4 = \int_{-\infty}^{0} T_2(0, s)P_2(s)[A(H^1) - A(H^2) - D_xA(H^2)\langle D_\xi H^2\langle \xi \rangle \rangle T_1(s, 0)Q_1(0)\eta] \, ds,$$

$$\begin{aligned} I_5 = &\int_{-\infty}^{0} T_2(0, s)P_2(s)D_xA(H^2)\langle D_\xi H^2\langle \xi \rangle \rangle [T_1(s, 0)Q_1(0)\eta \\ &- T_2(s, 0)Q_2(0)Q_1(0)\eta] \, ds, \end{aligned}$$

$$I_6 = \int_{-\infty}^{0} T_2(0, s)P_2(s)D_xA(H^2)\langle D_\xi H^2\langle \xi \rangle \rangle [-T_2(s, 0)Q_2(0)P_1(0)\eta] \, ds.$$

Now corresponding to (B.6) and (B.10), we have the following inequalities available:

$$|T_1(z,0)Q_1(0)\eta - T_2(s,0)Q_2(0)Q_1(0)\eta| \leqq \frac{2\mu}{\varepsilon N_1} C\,|\eta|\,|\xi_1-\xi_2|\,e^{(\mu-\varepsilon N_1)\tau}, \quad \tau \leqq 0, \qquad \text{(B.6)}'$$

$$|T_2(\tau,0)Q_2(0)P_1(0)\eta| \leqq C\,|\eta|\,|\xi_1-\xi_2|\,e^{\mu\tau}, \quad \tau \leqq 0. \qquad \text{(B.10)}'$$

By using (B.2) and $|D_\xi H^2\langle \xi\rangle|_{\varepsilon N_1} \leqq |\xi|$, we obtain the following estimates:

$$\frac{|I_j|}{|\eta|} \leqq \frac{K^2}{2\mu - 2\varepsilon N_1} |A(H^1) - A(H^2) - D_x A(H^2)\langle D_\xi H^2\langle \xi\rangle\rangle|_{2\varepsilon N_1}, \quad j = 1, 4, \qquad \text{(B.11)}$$

$$\frac{|I_j|}{|\eta|} \leqq \frac{2\mu C\,|\xi_1-\xi_2|^2}{\varepsilon N_1(2\mu - 2\varepsilon N_1)}, \quad j = 2, 5, \qquad \text{(B.12)}$$

$$\frac{|I_j|}{|\eta|} \leqq \frac{C\,|\xi_1-\xi_2|^2}{2\mu - 2\varepsilon N_1}, \quad j = 3, 6. \qquad \text{(B.13)}$$

Moreover, we know (essentially from Lemma 2.4) that $|A(H^1) - A(H^2) - D_x A(H^2)\langle D_\xi H^2\langle\xi\rangle\rangle|_{2\varepsilon N_1} = o(|\xi_1-\xi_2|)$ as $\xi_1 \to \xi_2$. This fact, together with (B.11)–(B–13), proves $|P_1(0) - P_2(0) - D_\xi P_2(0)\langle\xi\rangle| = o(|\xi|)$ as $\xi \to 0$. This completes the proof of the differentiability of $P^\varepsilon(0;\xi)$ in ξ. The differentiability of $Q^\varepsilon(0;\xi)$ can be proved in the same way as above. Higher order differentiability follows along the same line of argument, although computation gets complicated and tedious. The formula of higher order derivative is, as was the case for $K^\varepsilon(\xi, y)$ in Appendix A, very complicated and will not be given here. □

Lemma B.1 implies Lemma 3.2(ii) as follows. To prove that $E^s_\varepsilon = \bigcup_{\xi\in R^m} \mathrm{Range}\,(P^\varepsilon(0;\xi))$ is a C^{r-1}-vector bundle, it suffices to show that we can choose k linearly independent vectors $p_1(\xi), \ldots, p_k(\xi)$ such that $\mathrm{span}\,[p_1(\xi), \ldots, p_k(\xi)] = \mathrm{Range}\,(P^\varepsilon(0;\xi))$ and $p_i(\xi)$, $i = 1, \ldots, k$, are C^{r-1}-functions of ξ in a neighbourhood of each point $\xi_0 \in R^m$. Since $P^\varepsilon(0;\xi)$ is a C^{r-1}-function, each column vector of the matrix representation of $P^\varepsilon(0;\xi)$ relative to the standard basis of R^n is also a C^{r-1}-function. Now, for each $\xi_0 \in R^m$, we can choose k column vectors $p_1(\xi_0), \ldots, p_k(\xi_0)$ so that they are linearly independent because $\mathrm{Rank}\, P^\varepsilon(0;\xi_0) = k$. Then for a neighbourhood $U(\xi_0)$ of ξ_0, $p_1(\xi), \ldots, p_k(\xi)$ are linearly independent and C^{r-1}-functions of $\xi \in U(\xi_0)$, completing the proof of Lemma 3.2(ii).

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