Fourier another general signals

complex exponentials 复指数本征.

$\boxed{\text{Eigenfunction}}$

$e^{jw_k t} \to H(jw)e^{jw_k t} \nearrow$ eigenfunction

$H(jw) = \int_{-\infty}^{+\infty} h(\tau)e^{-jw_k\tau}d\tau$

eigenvalue

e-function won't change its form, while it's put into a system

we hope these could demonstrate signals by linear combination

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Fourier Series / transform

period Aperiod

Harmonically Related 谐波相关复指数.

$e^{jkw_0 t}$ $\frac{T_0}{k} = \frac{2\pi}{kw_0}$

$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jkw_0 t}$ F-Series 复指形式

polar form ⇒ Euler formula.

$x(t) = a_0 + 2\sum A_k \cos(kw_0 t + \theta_k)$ .. (cos form)

(not convenient)

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3 How to determine a_k's value

Consider Fourier Series(复指形式)

$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jkw_0 t}$. (synthesis equation)

$x(t)e^{-jnw_0 t} = e^{-jnw_0 t}\sum_{k=-\infty}^{+\infty} a_k e^{jkw_0 t}$

integral to a period signal we can do it during a T_0

$$\int_{T_0} x(t)e^{-jn\omega_0 t} = \sum_{k=-\infty}^{+\infty} a_k \cdot \int_{T_0} e^{-j(n-k)\omega_0 t}dt$$

we also know that $\int_{T_0} e^{-jm\omega t}dt = \begin{cases} T_0 & (m=0) \\ 0 & (m\neq 0) \end{cases}$

when $n=k$, $a_n T_0 = \int_{T_0} x(t)e^{-jn\omega_0 t}$

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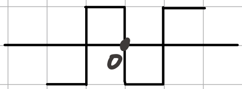
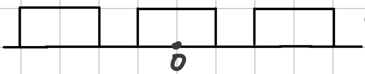
4. odd/even harmonic wave 谐波



only sine series (antisymmetric)



only cosine series (symmetric)

when signal isn't anti or sy-, it would involves both in Series (general case)

吉布斯现象：用 Fourier Series 表方波出现不连续点的"尖刺"

高频：产生不连续点

Series 收敛与狄利赫特条件　不存在能量差距

error $= |x_N(t) - x_\infty(t)|$

as $N\uparrow$, error$\Downarrow$

① Square integratable

② 狄里赫利 condition:

1) absolutely integratable

2). finite ups and downs

3) finite inconstant points

} 充分条件 都不必要

权系数.

$x^{(N)}(t)$ 来逼近 $x(t)$,

$x^{(N)}(t)=\sum_{-N}^{+N}\tilde{a}_k e^{+jN\omega_0 t}$

误差越小

但有"尖刺"

工程上的 converge $k=\forall$.

能量上的收敛 见何，

Fourier is an engineering tool, not a math tool

傅里叶级数系数的物理含义

$$\min_{\{\widetilde{a}_i\}_{i=1}^{N}} \left\{ \|x^{(N)}(t) - x(t)\|^2 \right\}, \qquad x^{(N)}(t) = \sum_{k=-N}^{N} \widetilde{a}_k e^{jk\omega_0 t}$$

❖ **根据投影定理**

$$\left[x(t) - x^{(N)}\right] \perp e^{jk\omega_0 t}, \quad k = -N, -N+1, \ldots, N-1, N$$

$$\int_T \left[x(t) - x^{(N)}\right] e^{-jk\omega_0 t} dt = \int_T \left[x(t) - \sum_{m=-N}^{N} \widetilde{a}_k e^{jm\omega_0 t}\right] e^{-jk\omega_0 t} dt = 0$$

$$\int_T x(t) e^{-jk\omega_0 t} dt = \sum_{k=-N}^{N} \widetilde{a}_k \int_T e^{j(m-k)\omega_0 t} dt$$

仅有 $m=k$ 时非0

$$\int_T x(t) e^{-jk\omega_0 t} dt = T\widetilde{a}_k \qquad \textbf{VS.} \qquad a_n = \frac{1}{T} \int_T x(t) e^{-jn\omega_0 t} dt$$

Fourier Transform when period → Aperiod

$T \to \infty$

For a Fourier Series:

Synthesis Equation. $x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$

Analysis Equation $a_k = \frac{1}{T_0}\int_{T_0} x(t) e^{-jk\omega_0 t} dt$

when $T_0 \to \infty$

$$a_k = \frac{1}{T_0}\int_{-\infty}^{+\infty} x(t) e^{-jk\omega_0 t} dt$$ (analysis equation)

Define
$$\begin{cases} X(\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt & (\omega = k\omega_0) \\ T_0 a_k = X(\omega)\big|_{\omega = k\omega_0} \end{cases}$$

at first apply $X(\omega)$ as a_k.

$$\tilde{x}(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{+\infty} \frac{1}{T_0} X(k\omega_0) e^{jk\omega_0 t}$$

$$= \frac{1}{2\pi}\sum_{k=-\infty}^{+\infty} X(k\omega_0) e^{jk\omega_0 t}\omega_0$$

As $T_0 \to \infty$ $\omega_0 \to d\omega_0$ $\tilde{x}(t) \Rightarrow x(t)$ $\sum \to \int$

$\Rightarrow \mathcal{F}^{-1}$: $\frac{1}{2\pi}\int_{-\infty}^{+\infty} X(\omega) e^{j\omega t} d\omega$ (synthesis equation)

原本周期⇒包络 $\xrightarrow{T变大}$ 间隔更紧的包络样本

$x(t)$ real → $X(\omega)$ complex

Let's look at an example.

$e^{-at}u(t) \xrightarrow{\mathcal{F}} \frac{1}{a+j\omega}$ (only when $a>0$, or it doesn't converge 不收敛)

考研书会有更多这些结论

Some signals can't be analyzed by Fourier Transform

all happen in a period (but infinite)

$\tilde{x}(t)$: period signal

$x(t)$: Aperiod signal

Bode Plot (波特图) $\begin{cases} 20\log_{10}|X(w)| \triangleq dB \text{ (分贝)} \\ X(w) \end{cases}$

$$\tilde{X}(w) \triangleq \sum_{k=-\infty}^{+\infty} 2\pi a_k \delta(w-kw_0)$$

离散样本点。

Duality(对偶性) $\begin{cases} x(t) \xrightarrow{\mathcal{F}} X(jw) \\ X(t) \xrightarrow{\mathcal{F}} 2\pi x(-w) \end{cases}$

(Basically identical)

no need to develop it. If needed, there will be chances

帕塞瓦尔定理.

$$\underbrace{\int_{-\infty}^{+\infty} |x(t)|^2 dt}_{\text{energy in time domain}} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |X(jw)|^2 dw$$

energy in time domain　　energy in frequency domain

单位时间の能量　　单位频率の能量

其它广泛运算性质

① time delay

$$x(t-t_0) \xrightarrow{\mathcal{F}} e^{-jwt_0}F(jw)$$

② differentiation

$$\frac{dx(t)}{dt} \xrightarrow{\mathcal{F}} jwF(jw)$$

③ Integration · a bit complicated

$$\int_{-\infty}^{t} x(\tau)d\tau \xrightarrow{\mathcal{F}} \frac{1}{jw}X(jw) + \pi X(0)\delta(w)$$

④ linear combination

also with scale property

$$x(at) \to \frac{1}{|a|}X\left(\frac{jw}{a}\right)$$

$$x(-t) \to X(-jw)$$

⑤ Convolution property $\begin{cases} \text{滤波} \\ \text{调制} \end{cases}$

$$x(t) \xrightarrow{} \boxed{h(t)} \rightarrow x(t) * h(t)$$

$$X(j\omega) \xrightarrow{} \boxed{H(j\omega)} \rightarrow X(j\omega) \cdot H(j\omega)$$

general response

$$\delta(t) \xrightarrow{} \boxed{h(t)} \rightarrow h(t)$$

$$1 \xrightarrow{} \boxed{H(\omega)} \rightarrow H(j\omega)$$

called "impulse response."

eiger-function : $e^{j\omega_0 t}$

$$e^{j\omega_0 t} \rightarrow \boxed{H(j\omega)} \rightarrow e^{j\omega_0 t} H(j\omega)$$

frequency response.

This is the base of 滤波(笑)

modulation 调制 $\quad s(t)p(t) \xrightarrow{\mathcal{F}} \frac{1}{2\pi}[S(j\omega) * P(j\omega)]$

convolution: $s(t) * p(t) \xrightarrow{\mathcal{F}} S(j\omega)\ P(j\omega)$

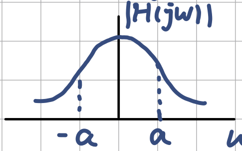
Now you can apply those properties to solve a linear differential equation

$$\frac{dy(t)}{dt} + a\,y(t) = x(t)$$

$$\xrightarrow{\mathcal{F}} \quad j\omega Y(j\omega) + aY(j\omega) = X(j\omega)$$

$$Y(j\omega) = \frac{1}{j\omega + a} X(j\omega)$$

$$H(j\omega) = \frac{1}{j\omega + a} \longleftrightarrow h(t) = e^{-at}u(t)$$



when you look at that frequency response, you find it attenuate high fs and retain low fs

响应の背景 保低频 减高频.

In other words 卷积背景即滤波 放大/削弱某一频率波的权重

lowpass filter eliminate f outside the specific range

Laplace Transform

F: $X(\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt$

L: $X(s) = \int_{-\infty}^{+\infty} x(t) e^{-st} dt \quad s = \sigma + j\omega$

if $\sigma = 0$, $F \Leftrightarrow L$ (纯复数)

$\left. X(s) \right|_{s=j\omega} = \mathcal{F}\{x(t)\} = X(\omega)$

$\downarrow$

$X(j\omega)$

So, F-Transform is a function of ω. but also a function of $j\omega$ in essense

会推翻表示，让傅氏也变成 $X(j\omega)$

* how does σ come from?

$X(s) = \mathcal{F}\{x(t) e^{-\sigma t}\}$

Fourier Transform doesn't force $x(t)$ to converge, which requires a absolute integratable function

this is why σ exists — to restruct $x(t)$ $\uparrow$