7-1-1. $x_1[n] = 0.5^n u[n]$ $\quad x_2[n] = u[n]$

$$X_1(z) = \sum_{-\infty}^{+\infty} 0.5^n u[n] z^{-n} = \sum_{n=0}^{+\infty} \left(\frac{1}{2}\right)^n z^{-n} = \sum_{0}^{+\infty} \left(\frac{1}{2} z^{-1}\right)^n$$

$$= \frac{1 - \lim\limits_{n\to\infty} \left(\frac{1}{2} z^{-1}\right)^n}{1 - \frac{1}{2} z^{-1}} \qquad \left|\frac{1}{2} z^{-1}\right| < 1 \qquad \left|\frac{1}{z}\right| < 2$$

$$|z| > \frac{1}{2}$$

$$= \frac{1}{1 - \frac{1}{2} z^{-1}} = \frac{z}{z - \frac{1}{2}}$$

$$X_2(z) = \sum_{-\infty}^{+\infty} u[n] z^{-n} = \sum_{0}^{+\infty} z^{-n} = \frac{1}{1 - z^{-1}} \qquad |z| > 1$$

$$= \frac{z}{z-1}$$

$$\therefore \quad X(z) = \frac{z^2}{(z-\frac{1}{2})(z-1)}$$

为什么 $\dfrac{X(z)}{z} = \dfrac{z}{(z-\frac{1}{2})(z-1)} = \dfrac{-1}{z-\frac{1}{2}} + \dfrac{2}{z-1}$.

但 $X(z) = \dfrac{-z}{z-\frac{1}{2}} + \dfrac{2z}{z-1} = \dfrac{-1}{1-\frac{1}{2}z^{-1}} + \dfrac{2}{1-z^{-1}}$

$$x(t) = -\left(\frac{1}{2}\right)^n u[n] + 2u[n]$$

其实没什么必要？不过 $\dfrac{z^2}{(z-\frac{1}{2})(z-1)}$ 好像得先分离一次常数，所以可以去用一下

7-1-2 $x_1[n] = 2^n u[n-1]$ $x_2[n] = 2^n u[n+1]$

$$X_1(z) = \sum_{n=0}^{+\infty} x_1[n] z^{-n} = \sum_{n=1}^{+\infty} 2^n z^{-n} = \sum_{n=1}^{+\infty} (2z^{-1})^n \cdot ✗$$

最好别这样. $x_1[n] = 2\cdot 2^{n-1} u[n-1] = 2\, 2^n u[n] * \delta[n-1]$

$\Rightarrow$ $\delta[n-1] \xleftrightarrow{z} \sum_{-\infty}^{+\infty} \delta[n-1] z^{-n} = \frac{1}{z}$ $|z|>0$

$\Rightarrow$. $2^n u[n] \xleftrightarrow{z} \sum_{-\infty}^{+\infty} (2z^{-1})^n u[n] = \frac{1}{1-2z^{-1}}$ $|z|>2$

$$= \frac{z}{z-2}.$$

$x_1[n] \xleftrightarrow{z} 2 \frac{z}{(z-2)} \cdot \frac{1}{z}$

其实z变换,光做题很容易,无非是常用变换对,但希望可以从实用角度了解z变换

$x_2[n] = \frac{1}{2} 2^{n+1} u[n+1] = \frac{1}{2} 2^n u[n] * \delta[n+1]$

$\delta[n+1] \xleftrightarrow{z} z.$

$\therefore$ $X_2(z) = \frac{1}{2} \frac{z}{z-2} \cdot z = \frac{z^2}{2(z-2)}$

$\therefore$ $X(z) = 2 \cdot \frac{1}{z-2} \frac{z^2}{2(z-2)} = \frac{z^2}{(z-2)^2} = \left(\frac{1}{1-2z^{-1}}\right)^2$

$\frac{1}{(1-z^{-1})^2} = (k+1) u[k]$

$\therefore y[n] = (n+1) 2^n u[n]$

7-1-3 $F_1(z) = z^{-1} \cdot \cancel{+}\ z^{-3}$

$F_2(z) = z \cdot$

有时别急着用z变换

$\therefore F(z) = F_1(z)F_2(z) = 1 - z^{-2}$

$f[n] = \delta[n] - u[n-2]$ ✗ 而且都变错了.

$f_1[n] = \delta[n-1] + \delta[n-2]$

$f_2[n] = u[n+1]$

$\therefore f_1[n] * f_2[n] = u[n] + u[n-1] = \delta[n] + 2u[n-1]$

$\therefore F(z) = 1 + 2\frac{1}{z-1}$

$\sum_{-\infty}^{+\infty} u[n-1] z^{-1} = \sum_{n=1}^{+\infty} z^{-n} = \frac{z^{-1}}{1-z^{-1}} = \frac{1}{z-1}$

不如 $\sum_{0}^{+\infty} z^{-n} = \frac{1}{1-z^{-1}} \cdot z^{-1}$ $|z| \geq 1$

7-1-4 $x[n] = a^n u[-n]$

$x[-n] = a^{-n} u[n]$

$a^{-n} u[n] = \frac{z}{z-\frac{1}{a}}$ $|z| > \frac{1}{a}$

7-1-4 主要得熟

① 反因果信号 $u[-n-1]$

② $a^n u[n] \leftrightarrow \frac{z}{z-a}$

$x[n] = a^n u[-n] = a^n u[-n-1] + \delta[n]$

$X(z) = \frac{-z}{z-a} + 1$ $|z| < a$

$X[-n] = a^{-n}u[n] = \dfrac{z}{z-\frac{1}{a}}$

$\therefore\ X(z)\ X(-z) = \left(\dfrac{-z}{z-a}+1\right)\dfrac{z}{z-\frac{1}{a}}$

$= \dfrac{-za}{(z-a)(z-\frac{1}{a})}$ $\quad \frac{1}{a}<|z|<a$

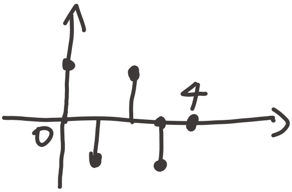
把目标式转Z变换 $\frac{4}{3}\left(\dfrac{z}{z-\frac{1}{2}}+\dfrac{-z}{z-2}\right)$

$= \dfrac{-2z}{(z-0.5)(z-2)}$ $\quad a=2$

7-2-1 $\quad u[n]-u[n-8] \xleftrightarrow{z} \dfrac{z}{z-1} - z^{-8}\dfrac{z}{z-1}$

$|z|>1$

7-2-2 (1) $\quad x[n] = (-1)^n\{u[n]-u[n-4]\}$



(2) $\quad u[n] \xleftrightarrow{z} \dfrac{z}{z-1}$

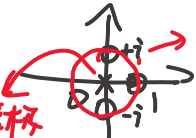
$u[n-4] \xleftrightarrow{z} z^{-4}\dfrac{z}{z-1}$

或 $= \delta[n]-\delta[n-1]+\delta[n-2]-\delta[n-3]$

$\xleftrightarrow{z} 1-z^{-1}+z^{-2}-z^{-3} = \dfrac{(z-1)(z^2+1)}{z^3}$

$|z|>0.$

(3)



单位圆画出

三重极

7-2-3 $2^n u[n+1] = \frac{1}{2} 2^{n+1} u[n+1]$

$$= \frac{1}{2} 2^n u[n] * \delta[n+1]$$

$\delta[n+1] \xleftrightarrow{z} z^1$

$2^n u[n] \leftrightarrow \frac{1}{1-2z^{-1}}$ $\quad |z|>2$

$\therefore 2^n u[n+1] = \frac{1}{2} 2^n u[n] * \delta[n+1] \xleftrightarrow{z} \frac{1}{2} \frac{1}{1-2z^{-1}} \cdot z$

$$= \frac{z}{2-4z^{-1}} = \frac{z^2}{2z-4}$$

$2^n u[n-2] = 4\, 2^{n-2} u[n-2] = 4\, \delta[n-2] * 2^n u[n]$

$\xleftrightarrow{z} 4 z^{-2} \frac{1}{1-2z^{-1}}$ $\quad |z|>2$

$\therefore X(z) = \frac{z^2}{2z-4} - \frac{4}{z^2-2z}$

$\cancel{X}$, $\frac{\frac{1}{2}z^2}{z-2} - \frac{4}{z(z-2)} = \frac{\frac{1}{2}z^3 - 4}{z(z-2)}$

为什么这样不好

u[n+1]-u[n-2] 是个时限信号，不如转化冲激序列.

7-2-4 $f[n] = 3^n u[-n-1] + 2^{n+1} u[n]$ 双边!

$2 \cdot 2^{n} u[n] \leftrightarrow 2 \frac{1}{1-2z^{-1}} = 2 \cdot \frac{z}{z-2}$ $\quad |z|>2$

$3^n u[-n-1] \leftrightarrow \frac{-1}{1-3z^{-1}} = \frac{-z}{z-3}$ $\quad |z|<3$

解. $3^n u[-n-1] \xleftrightarrow{z} \sum_{-\infty}^{+\infty} 3^n u[-n-1] z^{-n} = \sum_{-\infty}^{-1} 3^n z^{-n} = \sum_{n=1}^{+\infty} 3^{-n} z^n = \frac{\frac{1}{3}z}{1-\frac{1}{3}z}$

$$= \frac{z}{3-z}$$

$$= \frac{-z}{z-3} \quad |z|<3.$$

$\because F(z) = \frac{-z}{z-3} + 2\frac{z}{z-2} = \frac{-z(z-2)+2z(z-3)}{(z-3)(z-2)}$

$= \frac{-z^2+2z+2z^2-6z}{(\quad)(\quad)}$

$= \frac{z^2-4z}{(\quad)(\quad)}, \quad 2<|z|<3.$

7-2-9 $f[n] = (n+1)^2 u[n]$

$n u[n] \overset{z}{\longleftrightarrow} \frac{d\left(\frac{z}{z-1}\right)}{dz}(-z)$ 性质:微分

$= \frac{z-1-z}{(z-1)^2}(-z). = \frac{z}{(z-1)^2}$

$n^2 u[n] \overset{z}{\longleftrightarrow} \frac{d^2\left(\frac{z}{z-1}\right)}{dz^2} z^2.$ 不用这样.

把 $n x[n] \overset{z}{\longleftrightarrow} (-z)\cdot\frac{d(X(z))}{dz}$

中 $x[n]$ 当作 $n u[n]$

$n^2 x[n] \overset{z}{\longleftrightarrow} (-z)\cdot\frac{d\left(\frac{z}{(z-1)^2}\right)}{dz} = \frac{z(z+1)}{(z-1)^3}$

$(n^2+2n+1)u[n] \overset{z}{\longleftrightarrow} \frac{z(z+1)}{(z-1)^3} + 2\frac{z}{(z-1)^2} + 1\cdot\frac{z}{z-1}$

$= \frac{z^2+z+2z(z-1)+z(z-1)^2}{(z-1)^3}$ ROC $\boxed{|z|>1}$

7-2.10 $f[n] = \left\{\frac{1}{5}2^n - \frac{(-3)^n}{5}\right\}u[n]$

$2^n u[n] \overset{z}{\longleftrightarrow} \frac{1}{1-2z^{-1}} = \frac{z}{z-2}$

$(-3)^n u[n] \overset{z}{\longleftrightarrow} \frac{1}{1+3z^{-1}} = \frac{z}{z+3} \Rightarrow$ 不用𝑧. 直接收敛

$$F(z) = \frac{1}{5}\frac{z}{z-2} - \frac{1}{5}\frac{z}{z+3}, \quad |z|>2$$

$$= \frac{1}{5}\frac{z(z+3)-z(z-2)}{(z-2)(z+3)} = \frac{1}{5}\frac{3z+2z}{(z-2)(z+3)}$$

$$= \frac{z}{(z-2)(z+3)} \quad |z|>2$$

7-2-20 $\quad x[n] \leftrightarrow X(z)$

$$\left(\frac{1}{a}\right)^n x[n] \overset{Z}{\longleftrightarrow} \sum_{-\infty}^{+\infty}\left(\frac{1}{a}\right)^n x[n] z^{-n}$$

$$= \sum_{-\infty}^{+\infty} x[n]\left(\frac{1}{a}z^{-1}\right)^n = \sum_{-\infty}^{+\infty} x[n](az)^{-n} = X(az)$$

$$\lim_{T} \int_0^{+\infty} e^{-2\alpha t}\,dt$$

$$-\frac{1}{2\alpha} - \frac{1}{2b} \qquad \frac{2}{\alpha+\beta}$$

$$e^{-\alpha t}u(t)\,dt.$$

7-3-1 $y[n]-ay[n-1]=x[n]$

$$Y(z)-az^{-1}Y(z)=X(z) \quad X(z)=\frac{z}{z-1}$$

$$(1-az^{-1})Y(z)=\frac{z}{z-1}$$

$$Y(z)=\frac{z}{(1-az^{-1})(z-1)}=\frac{z^2}{(z-a)(z-1)}$$

① $a=1 \Rightarrow z=1$ 为二阶极点

$\therefore$ ROC: $|z|>1$ X 其实并不需要 ROC

把 $Y(z)$ 反变回去即可

此外，除分式展开，熟练掌握长除、泰勒等方法

$a=1 \quad Y(z)=\frac{z^2}{(z-1)^2}=?$

$$nu[n]=\frac{z}{(z-1)^2} \quad u[n]=\frac{z}{z-1}$$

只能说记 $(n+1)u[n] \Leftrightarrow Y(z)$

② $a\neq1$ 拆吧

7-3-2 $G(z)=\frac{H(z)}{1-z^{-1}}$

7-3-3. $x[n]=-2\delta[n-1]+4\delta[n-2]$

$$y[n]=-2\delta[n]+6\delta[n-2]+4\delta[n-3]$$

$$Y(z)=-2+6z^{-2}+4z^{-3}$$

$$X(z)=-2z^{-1}+4z^{-2}.$$

$\therefore H(z) = \frac{Y(z)}{X(z)} = \frac{-2+6z^{-2}+4z^{-3}}{-2z^{-1}+4z^{-2}} = \frac{-2z^2+6+4z^{-1}}{-2z+4}$

$= \frac{-2z^3+6z+4}{-2z^2+4z} = \frac{z^3-3z-2}{z(z-2)} = \frac{z^3-2z^2+2z^2-3z-2}{z(z-2)}$

$= z + \frac{2z^2-3z-2}{z(z-2)} = z + \frac{2z^2-4z+z-2}{z(z-2)}$

$= z+2+\frac{1}{z} \overset{z^{-1}}{\longleftrightarrow}$.

7-3-19 $\quad Y(z) - \frac{5}{2}z^{-1}Y(z) + z^{-2}Y(z) = -\frac{3}{2}z^{-1} X(z)$

$(1-\frac{5}{2}z^{-1}+z^{-2})\,Y(z) = -\frac{3}{2}z^{-1}X(z)$

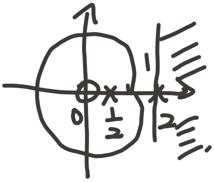
$$H(z) = \frac{-\frac{3}{2}z^{-1}}{1-\frac{5}{2}z^{-1}+z^{-2}}$$

零点: $z_{01}=0$

$$= \frac{-\frac{3}{2}z}{z^2-\frac{5}{2}z+1}$$

极点 $z_1 = 2$

$z_2 = \frac{1}{2}$



因果. ROC $\Rightarrow |z|>2$

(2). 单位样值. $\frac{H(z)}{z} = \frac{-1}{z-2} + \frac{1}{z-\frac{1}{2}}$

$\therefore H(z) = \frac{-z}{z-2} + \frac{z}{z-\frac{1}{2}} = \frac{-1}{1-2z^{-1}} + \frac{1}{1-\frac{1}{2}z^{-1}}$

$u[n] \leftarrow$

$a^n u[n] \overset{z}{\leftarrow}$ $= -2^n u[n] + (\frac{1}{2})^n u[n]$.

重做

$$H(z)=\frac{-\frac{3}{2}z}{(z-2)(z-\frac{1}{2})}$$ 求其反变换

$$\frac{H(z)}{z}=\frac{-\frac{3}{2}}{(z-2)(z-\frac{1}{2})}=\frac{-1}{z-2}+\frac{1}{z-\frac{1}{2}}$$

$$\therefore H(z)=\frac{-z}{z-2}+\frac{z}{z-\frac{1}{2}}=\frac{-1}{1-2z^{-1}}+\frac{1}{1-\frac{1}{2}z^{-1}}$$

$$\overset{z}{\longleftrightarrow}\ \boxed{-2^n u[n]+(\tfrac{1}{2})^n u[n]}$$

or

$$H(z)=\frac{-\frac{3}{2}z}{(z-2)(z-\frac{1}{2})}$$ 留数

$$=\left(\frac{-2}{z-2}+\frac{\frac{1}{2}}{z-\frac{1}{2}}\right)$$

$$=\frac{-2z^{-1}}{1-2z^{-1}}+\frac{\frac{1}{2}z^{-1}}{1-\frac{1}{2}z^{-1}}$$

$$=-2z^{-1}\frac{1}{1-2z^{-1}}+\frac{1}{2}z^{-1}\frac{1}{1-\frac{1}{2}z^{-1}}$$

$$\overset{z}{\longleftrightarrow}\ -2\delta[n-1]*2^n u[n]+\frac{1}{2}\delta[n-1]*(\tfrac{1}{2})^n u[n-1]$$

$$-2\,2^{n-1}u[n-1]+(\tfrac{1}{2})^n u[n-1]$$

$$-2^n u[n-1]+(\tfrac{1}{2})^n u[n-1]$$

关于 7-3-19 这道题，拓展出几种方法，并化简成同样形式

对于 $\dfrac{z+1}{z+\frac{1}{2}}$ 的几种反Z变换求法。

7-4-1 对于分母多项式 $A(z)=z^2+z+k$

要稳定 则 $A(1)=2+k>0$ $A(-1)=k>0$

$1>k$ ∴ $0<k<1$

至于朱里判稳是为何,可以不管 且只适用二阶.

$$z=\frac{-1\pm\sqrt{1-4k}}{2}$$

7-4-2. $A(z)=z^2+\frac{1}{2}z+\underline{k+1}$

$A(1)\cancel{<}>0$ $A(-1)\cancel{<}>0$ $a_2>|a_0|$

$\frac{3}{2}+1+k>0 \Rightarrow k>-\frac{5}{2}$ $-2<k<0$

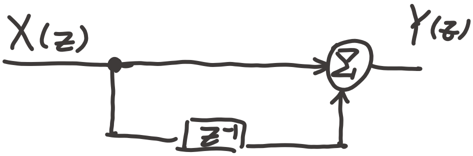
$$\begin{cases} 1-\frac{1}{2}+k+1<0 \Rightarrow \underline{k\cancel{<}>-\frac{3}{2}} \\ 1>\underline{|k+1|} \Rightarrow 1>-k-1 \Rightarrow k>-2 \end{cases}$$

$|k+1|<1$

∴ $-\frac{3}{2}<\cancel{k}<-2$ $-\frac{3}{2}<k<0$

7-5-1 什么是梅森公式? 其实没啥用，不如自己推

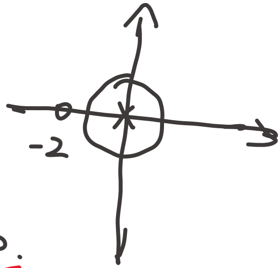
(1) $Y(z) = X(z)\left(z^{-1}+\frac{1}{2}\right)$



$H(z) = z^{-1}+\frac{1}{2}$

$= \frac{1}{z}+\frac{1}{2} = \frac{2+z}{2z}$

$= \frac{1+\frac{1}{2}z}{z}$ $|z|>0$.



默认在右边.

(2) 前提是稳定，才可分析幅频与相频.

$H(e^{j\omega}) = H(z)\big|_{z=e^{j\omega}} = \frac{1}{2}+e^{-j\omega} = \sqrt{\frac{5}{4}+\cos\omega}$

幅频 求 $|H(e^{j\omega})|$ ⇒ 代入 $\omega = 0, \frac{\pi}{2}, \pi$ 可以确定幅频曲线.

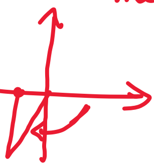
相频 求 $e^{\angle e^{j\omega}}$

① $\omega=0$ $H(e^{j0}) = \frac{1}{2}+1 = \frac{3}{2}$ $\varphi(0)=0$

($\varphi(0)=0$ 是因为 $H(e^{j0})$ 被视作复数 $H(e^{j0}) = |H(e^{j0})| e^{\angle H(e^{j0})}$

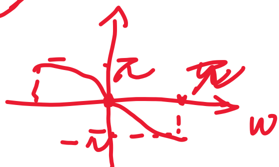
② $\omega=\frac{\pi}{2}$ $|H(e^{j\frac{\pi}{2}})| = |\frac{1}{2}-j| = \frac{\sqrt{5}}{2}$

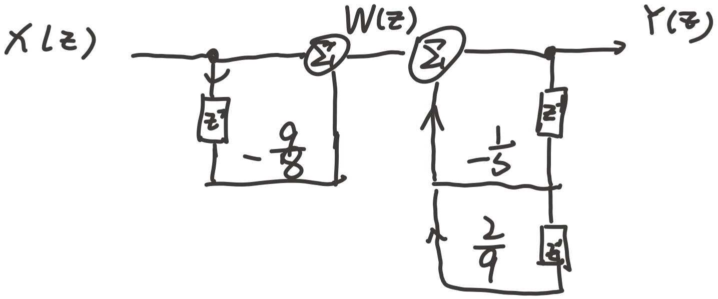
$\varphi(\frac{\pi}{2}) = \arg(\frac{1}{2}-j) = \arctan(-2)$



③ $\omega=\pi$ $H(e^{j\pi}) = -\frac{1}{2}$

$\varphi(\pi) = -\pi$





$$W(z) = X(z)\left(1 + \left(-\frac{9}{8}\right)z^{-1}\right)$$

$$Y(z) = W(z) + \left(-\frac{1}{3}\right)z^{-1}Y(z) + \frac{2}{9}z^{-2}Y(z)$$

$$\therefore \left(-\frac{2}{9}z^{-2} + \frac{1}{3}z^{-1} + 1\right)Y(z) = X(z)\left(1 - \frac{9}{8}z^{-1}\right)$$

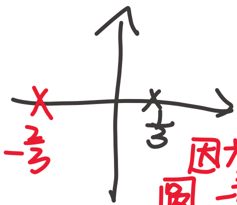
$$\therefore H(z) = \frac{1 - \frac{9}{8}z^{-1}}{-\frac{2}{9}z^{-2} + \frac{1}{3}z^{-1} + 1} = \frac{z^2 - \frac{9}{8}z}{z^2 + \frac{1}{3}z - \frac{2}{9}}$$

$$y[n] + \frac{1}{3}y[n-1] - \frac{2}{9}y[n-2] = x[n] - \frac{9}{8}x[n-1]$$

$$z^2 + \frac{1}{3}z - \frac{2}{9} = 0$$

$$\Delta = \frac{1}{9} + \frac{8}{9} = 1 \qquad \therefore z = \frac{-\frac{1}{3} \pm 1}{2}$$

$$z_1 = \frac{1}{3} \qquad z_2 = -\frac{2}{3}$$



ROC $|z| > \cancel{\frac{1}{3}}$ $\frac{2}{3}$ 含 $|z|=1$

因为z域是圆 $-\frac{2}{3}$ 看绝对值

不稳定

7-6-1 极点 $z_{11}=+0.5j$ $z_{12}=-0.5j$

对全通系统 一组极点、零点互为共轭倒数

$$z_{01}=\frac{1}{0.5j}=-2j \qquad z_{02}=2j.$$

$$H(z)=\frac{A(z-z_{01})(z-z_{02})}{(z-z_1)(z-z_2)}$$

$$=\frac{A(z^2+4)}{z^2+0.25}$$ 令 $z=e^{jw}$ $w=0$

比较好求

$$\left|H(e^{jw})\right|=\frac{|A|(5)}{1.25}=1 \qquad |A|=\frac{1}{4}$$

梳理 ① 前提．系统要稳定 ROC含 e^{jw} $z=$ ($|z|=1$)

②为方便，取 $e^{jw}|_{w=0}$ 模是 $H(e^{jw})$ 才有。

③ $H(z)$ 没有模一说，因为 FT $\begin{cases}\text{DTFT}\\ \text{CFT}\end{cases}$ 才有实际意义

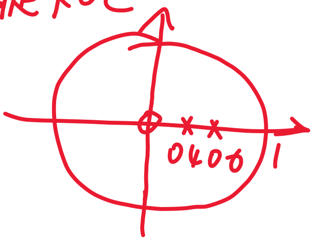
7-9-1 $f(0)=\lim_{z\to\infty} F(z)=0$ （高阶无穷小）

7-9-2 初值 $x(0)=\lim_{z\to\infty} F(z)=1$

$$x(\infty)=\lim_{z\to 0}\frac{z^2}{z^2-z+0.24}=0 \quad ✗.$$

系统讲一下. initial-theory / final-theory

$F(z)=\frac{z^2}{(z-0.4)(z-0.6)}$ 不确定ROC



① $|z|>0.6$ 时. ROC含单位圆

∴ 稳定 ⇒ 存在终值.

⇒ $f(\infty)=\lim_{z\to 1}(z-1)F(z)=0$

② $0.4<|z|<0.6$ 或 $|z|<0.4$

不稳定, 无终值.

因为终值实际是直流分量, 因果信号时 其他项 $z^{-1}, z^{-2}\ldots$ 均收敛成0 ⇔ 稳定 ⇔ ROC含单位圆

$F(z)$.