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Totally positive bases for shape preserving curve design and optimality of B-splines

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Abstract

Normalized totally positive (NTP) bases present good shape preserving properties when they are used in Computer Aided Geometric Design. Here we characterize all the NTP bases of a space and obtain a test to know if they exist. Furthermore, we construct the NTP basis with optimal shape preserving properties in the sense of (Goodman and Said, 1991), that is, the shape of the control polygon of a curve with respect to the optimal basis resembles with the highest fidelity the shape of the curve among all the control polygons of the same curve corresponding to NTP bases. In particular, this is the case of the B-spline basis in the space of polynomial splines. Further examples are given.

Keywords: B-splines; Totally positive; Shape preserving; Normalized bases

1. Introduction

In (Goodman, 1989), (Goodman and Said, 1991) and (Goodman, 1991) it is shown that totally positive bases present good shape preserving properties due to the variation diminishing properties of totally positive matrices. Roughly speaking, the shape of a parametrically defined curve mimics the shape of its control polygon when the corresponding blending functions form a totally positive system. Thus, we can predict or manipulate the shape of the curves by choosing or changing the control polygon. One of the aims of this paper is to characterize all the totally positive bases of a space. On the other hand, in (Goodman and Said, 1991) it was conjectured that the Bernstein basis has optimal shape preserving properties among all normalized totally positive bases of

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the space of polynomials of degree less than or equal to n. An affirmative answer was given in (Carnicer and Peña, 1993).

Here we construct and characterize the normalized totally positive basis with optimal shape preserving properties for any finite dimensional vector space of real valued functions which has a totally positive basis. Such a basis is called B-basis because, as we shall see, the B-spline basis is the normalized totally positive basis with optimal shape preserving properties in the space of polynomial splines. Another examples of B-bases are the Bernstein basis, the basis of β-splines, and the basis used for generating nonuniform rational B-spline curves.

Let us introduce now the main definitions. A matrix is said *totally positive* (TP) if all its minors are nonnegative. The following well-known property of TP matrices (cf. Theorem 3.1 of (Ando, 1987)) will be used throughout this paper: the product of TP matrices is a TP matrix. Some other basic properties of TP matrices can be found in (Ando, 1987; Gasca and Peña, 1992, 1993).

We say that a sequence of functions $(u_0,\dots,u_n)$ defined on a subset I of $\mathbb{R}$ is a totally positive (TP) system if for any $t_0 < t_1 < \cdots < t_m$ in I the collocation matrix

$$M\begin{pmatrix} u_0,\dots u_n \\ t_0,\dots,t_m \end{pmatrix} := (u_j(t_i))_{i=0,\dots,m;j=0,\dots,n} \tag{1.1}$$

is TP.

In Section 2, some auxiliary results for TP systems of functions are proved. In Section 3 we transform a TP basis of a space into another special basis which will be called B-basis (Theorem 3.6). From a B-basis we shall be able to obtain all the TP bases of the space (Corollary 3.10). Proposition 3.12 gives an interesting characterization of a B-basis. In Section 4 we start defining normalized totally positive (NTP) bases, which are the bases used to generate curves from their control polygons in the framework of Computer Aided Geometric Design. In Theorem 4.2, we shall characterize all the NTP bases of a given space from the unique NTP B-basis. In the mentioned characterization, the matrix of change of basis will be TP and stochastic (the sum of each row is one), which provides (if we factorize this matrix as a product of nonnegative bidiagonal and stochastic matrices) a corner cutting algorithm (see p. 240 of (Goodman and Micchelli, 1988) and (Goodman and Said, 1991)) from the control polygon of the given NTP basis to the control polygon of the NTP B-basis.

In Remark 4.3, we give a test to know if a space which has a TP basis is adequate for design purposes, that is, if it has a NTP basis. This test provides a method to obtain the NTP B-basis. In Corollary 4.4 we point out that, in a space which has a NTP basis, the unique optimal NTP basis is the NTP B-basis. Finally we include some examples of B-bases for several spline spaces. In particular we prove (Theorem 4.6) that the B-spline basis is a B-basis and so it has optimal shape preserving properties among all NTP bases of its space.

We thank the referees for pointing out to us the thesis of Schmeltz (1992), which was not referenced in an earlier version of the manuscript. This thesis contains some results of optimality of bases for spaces with C^n functions which have bases satisfying special properties of the derivatives at the end points. It also provides some interesting properties of variation diminishing algorithms for generating curves through this kind of bases.

2. Auxiliary results

Let us start with some ancillary results about TP systems.

Lemma 2.1. *Let $(u_0,\dots,u_n)$ be a TP system of functions defined on I and let $C_i := \{t \in I \mid u_i(t) \neq 0\}$. Then the function $u_j(t)/u_i(t)$, $j > i$ defined on C_i is monotone increasing.*

Proof. Let $s < t$ in C_i. Since $u_0,\dots,u_n$ is TP

$$0 \leqslant \begin{vmatrix} u_i(s) & u_j(s) \\ u_i(t) & u_j(t) \end{vmatrix} = u_i(s)u_j(t) - u_j(s)u_i(t),$$

and so $u_j(t)/u_i(t) \geqslant u_j(s)/u_i(s)$. $\square$

We shall make use of the following auxiliary result, which is a generalization for TP systems of functions of the well-known shadow's lemma for TP matrices (Lemma A of (de Boor and Pinkus, 1982)).

Lemma 2.2. *Let $(u_0,\dots,u_n)$ be a TP system of functions on I and let $t_0 \in I$. If $u_i(t_0) = 0$ for some $i \in \{0,\dots,n\}$ then*
(i) *$u_i(t) = 0$ for all $t \geqslant t_0$ or $u_j(t_0) = 0$ for all $j \geqslant i$.*
(ii) *$u_i(t) = 0$ for all $t \leqslant t_0$ or $u_j(t_0) = 0$ for all $j \leqslant i$.*

Proof. Let $t \in I$ with $t > t_0$ and $j > i$. Since $u_0,\dots,u_n$ is TP, then

$$0 \leqslant \begin{vmatrix} u_i(t_0) & u_j(t_0) \\ u_i(t) & u_j(t) \end{vmatrix} = -u_j(t_0)u_i(t)$$

and $u_j(t_0), u_i(t) \geqslant 0$. Thus $u_j(t_0)u_i(t) = 0$, for all $t > t_0$, $j > i$. If there exists some $j > i$ such that $u_j(t_0) \neq 0$, then $u_i(t) = 0$ for all $t > t_0$ and then (i) follows. Using analogous arguments (ii) can be derived. $\square$

The next auxiliary results will be a useful tool in the following.

Lemma 2.3. *Let $(u_0,\dots,u_n)$ be a system of linearly independent functions on I. Then*
(i) *There exists a collocation matrix of the system which is nonsingular.*
(ii) *$(u_0,\dots,u_n)$ is TP if each nonsingular collocation matrix is TP.*

Proof. (i) Let us show the existence of nonsingular collocation matrices by induction on n. If $n = 0$ it is obvious. Suppose now that the result holds for $n-1$. Since $u_1,\dots,u_n$ are linearly independent, by the induction hypothesis there exist $t_1,\dots,t_n$ in I such that $M\binom{u_1,\dots,u_n}{t_1,\dots,t_n}$ is nonsingular. Let $\alpha = (\alpha_0,\dots,\alpha_n)^{\mathrm{T}}$ be a nonzero vector such that

$M\binom{u_0,u_1,\ldots u_n}{t_1,\ldots,t_n}\alpha = 0$. Since the functions $u_0,\ldots,u_n$ are linearly independent on I, there exists $s \in I$ such that $\sum_{i=0}^{n} \alpha_i u_i(s) \neq 0$. Clearly $s \neq t_i$, $i = 1,\ldots,n$, and the collocation matrix of $u_0,\ldots,u_n$ with the values $s, t_1, \ldots t_n$ (conveniently ordered) is nonsingular and (i) follows.

(ii) Consider $d = \det M\binom{u_{i_0}\ldots u_{i_r}}{t_0,\ldots,t_r}$. Let us prove that d is nonnegative. By (i) there exist $s_0,\ldots,s_n$ such that $\det M\binom{u_0,\ldots u_n}{s_0,\ldots,s_n} \neq 0$. If $d \neq 0$ then $(u_0(t_i),\ldots,u_n(t_i))$, $i = 0,\ldots,r$, are linearly independent row vectors. Now we can complete this set of linearly independent vectors to a basis of $\mathbb{R}^{n+1}$ with some vectors from $(u_0(s_i),\ldots,u_n(s_i))$, $i = 0,\ldots,n$. Let A be the matrix whose rows are the vectors conveniently ordered, so that A is a nonsingular collocation matrix of $(u_0,\ldots,u_n)$, which will be TP. Since d is a minor of this matrix $d > 0$. □

Lemma 2.4. *Let $(u_0,\ldots,u_n)$ be a TP system of linearly independent functions on I and let $(v_0,\ldots,v_n)$ be a system such that there exists an upper triangular TP matrix U with unit diagonal such that*

$$(u_0,\ldots,u_n) = (v_0,\ldots,v_n)U$$

If $(v_1,\ldots,v_n)$ is TP on I, then $(v_0,v_1,\ldots,v_n)$ is also TP.

Proof. By Lemma 2.3(ii) it is sufficient to show that any nonsingular collocation matrix of $(v_0,\ldots,v_n)$ is TP. Let $t_0 < \cdots < t_n$ in I such that $M\binom{v_0,\ldots v_n}{t_0,\ldots,t_n}$ is nonsingular. From $M\binom{u_0,\ldots u_n}{t_0,\ldots,t_n} = M\binom{v_0,\ldots v_n}{t_0,\ldots,t_n}U$ we obtain that all the minors involving initial (consecutive) columns of both collocation matrices coincide. On the other hand, the submatrix $M\binom{v_1,\ldots v_n}{t_0,\ldots,t_n}$ is TP. Therefore all the minors of $M\binom{v_0,\ldots v_n}{t_0,\ldots,t_n}$ with consecutive columns are nonnegative. It is well-known that a nonsingular matrix satisfying this property is TP (cf. Theorem 1.3 of (Cryer, 1976)). □

In the next result we shall perform an elementary step to transform a given TP system of functions $(u_0,\ldots,u_n)$ on I into a TP system $(v_0,\ldots,v_n)$ satisfying the property $\inf\{v_i(t)/v_0(t) \mid t \in I, v_0(t) \neq 0\} = 0$, $i = 1,\ldots,n$. Let $a = \inf\{t \in I \mid v_0(t) \neq 0\}$. Taking into account Lemma 2.1, this condition for the TP system $(v_0,\ldots,v_n)$ is equivalent to the following

$$\begin{aligned}
&\text{if } a \in I \text{ and } v_0(a) \neq 0, \text{ then } v_i(a) = 0, \quad i = 1,\ldots,n;\\
&\text{if } a \notin I \text{ or } v_0(a) = 0, \text{ then } \lim_{t\to a^+} \frac{v_i(t)}{v_0(t)} = 0, \quad i = 1,\ldots,n.
\end{aligned} \tag{2.1}$$

Roughly speaking, we construct a TP system $(v_0,\ldots,v_n)$ with the following property: the functions $v_1,\ldots,v_n$ have a zero of higher order than v_0 to the left of I.

Proposition 2.5. *Let $(u_0,\ldots,u_n)$ be a TP-system of linearly independent functions on I. Then there exists a TP system $(v_0,\ldots,v_n)$ of the form*

$$(u_0,\ldots,u_n) = (v_0,\ldots,v_n)U, \tag{2.2}$$

where U is an upper triangular TP matrix with unit diagonal, such that

$$\inf\{v_i(t)/v_0(t) \mid t \in I, v_0(t) \neq 0\} = 0, \quad i = 1, \ldots, n. \tag{2.3}$$

Proof. Let us begin with the construction of the system of functions $(v_0, \ldots, v_n)$. Let $C_0 := \{t \in I \mid u_0(t) \neq 0\}$ and let $a := \inf C_0 \in \mathbb{R} \cup \{-\infty\}$. Let $s_0 < s_1 < \cdots < s_n$ in I such that $\det M\binom{u_0,\ldots,u_n}{s_0,\ldots,s_n} \neq 0$. The existence of $s_0, \ldots, s_n$ is guaranteed by Lemma 2.3(i). Since $M\binom{u_0,\ldots,u_n}{s_0,\ldots,s_n}$ is a nonsingular TP matrix, by Corollary 3.8 of (Ando, 1987)

$$u_i(s_i) \neq 0, \quad i = 0, \ldots, n \tag{2.4}$$

and in particular $s_0 \in C_0$. So, $a \leqslant s_0$.

For any $\tau \in C_0$ such that $\tau \leqslant s_0$, let us consider the collocation matrix $M\binom{u_0,u_1,\ldots,u_n}{\tau,s_1,\ldots,s_n}$ which is TP. Dividing the first row of the matrix by $u_0(\tau) > 0$ we deduce that the matrix

$$\begin{pmatrix} 1 & \dfrac{u_1(\tau)}{u_0(\tau)} & \cdots & \dfrac{u_n(\tau)}{u_0(\tau)} \\ u_0(s_1) & u_1(s_1) & \cdots & u_n(s_1) \\ \cdots & \cdots & \cdots & \cdots \\ u_0(s_n) & u_1(s_n) & \cdots & u_n(s_n) \end{pmatrix}$$

is TP.

Let $(\tau_m)_{m\in\mathbb{N}}$ a monotone decreasing sequence with $\tau_m \in C_0$, $\tau_m \leqslant s_0$, $\lim_{m\to\infty} \tau_m = a$ and if $a \in C_0$, $\tau_m = a$ for all m. From Lemma 2.1, it follows that

$$\lim_{m\to\infty} \frac{u_i(\tau_m)}{u_0(\tau_m)} = \inf_{\tau\in C_0} \frac{u_i(\tau)}{u_0(\tau)}, \quad i = 1, \ldots, n.$$

Therefore

$$\begin{pmatrix} 1 & \inf\limits_{\tau\in C_0} \dfrac{u_1(\tau)}{u_0(\tau)} & \cdots & \inf\limits_{\tau\in C_0} \dfrac{u_n(\tau)}{u_0(\tau)} \\ u_0(s_1) & u_1(s_1) & \cdots & u_n(s_1) \\ \cdots & \cdots & \cdots & \cdots \\ u_0(s_n) & u_1(s_n) & \cdots & u_n(s_n) \end{pmatrix} = \lim_{m\to\infty} \begin{pmatrix} 1 & \dfrac{u_1(\tau_m)}{u_0(\tau_m)} & \cdots & \dfrac{u_n(\tau_m)}{u_0(\tau_m)} \\ u_0(s_1) & u_1(s_1) & \cdots & u_n(s_1) \\ \cdots & \cdots & \cdots & \cdots \\ u_0(s_n) & u_1(s_n) & \cdots & u_n(s_n) \end{pmatrix} \tag{2.5}$$

is a TP matrix since the set of TP matrices is closed.

Lemma A of (de Boor and Pinkus, 1982) implies that if a TP matrix with nonzero diagonal elements has zero in the first row and jth column then every entry in the first row and rth column $(r > j)$ is also zero. Therefore, using (2.4), we can deduce that there exists $k \in \{0, \ldots, n\}$ such that

$$\inf_{\tau\in C_0} \frac{u_i(\tau)}{u_0(\tau)} > 0, \quad i = 0, \ldots, k,$$
$$\inf_{\tau\in C_0} \frac{u_i(\tau)}{u_0(\tau)} = 0, \quad i = k+1, \ldots, n.$$

Let us define

$$\alpha_i := \begin{cases} \inf_{\tau\in C_0} \dfrac{u_i(\tau)}{u_0(\tau)} \Big/ \inf_{\tau\in C_0} \dfrac{u_{i-1}(\tau)}{u_0(\tau)}, & i=1,\dots,k, \\ 0, & i=k+1,\dots,n, \end{cases}$$

$$v_0 := u_0, \qquad v_i := u_i - \alpha_i u_{i-1}, \quad i=1,\dots,n,$$

and

$$U := \begin{pmatrix} 1 & -\alpha_1 & 0 & \cdots & 0 \\ 0 & 1 & -\alpha_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & 0 & 1 & -\alpha_n \\ 0 & \cdots & \cdots & 0 & 1 \end{pmatrix}^{-1}.$$

Then formula (2.2) holds. The matrix U is upper triangular with unit diagonal. Since U^{-1} is a bidiagonal matrix with positive elements on and only on the main diagonal, by Proposition 5.6 of (Gasca and Peña, 1992), U is TP.

Let us show that

$$v_1(t) = \cdots = v_n(t) = 0, \quad \text{for all } t \leqslant a, t \in I. \tag{2.6}$$

If $t \leqslant a$ and $t \notin C_0$ then $u_0(t) = 0$ and by Lemma 2.2(i)

$$u_i(t) = 0, \quad i=0,\dots,n \tag{2.7}$$

and so (2.6) holds.

If $t = a \in C_0$ one has by Lemma 2.1

$$\inf_{\tau\in C_0} \frac{u_i(\tau)}{u_0(\tau)} = \frac{u_i(a)}{u_0(a)}. \tag{2.8}$$

Besides, we have $v_i(a) = u_i(a) - \alpha_i u_{i-1}(a)$, $i=1,\dots,n$. If $u_{i-1}(a) = 0$, then by Lemma 2.2(i) either $u_i(a) = 0$ or $u_{i-1}(t) = 0$, $\forall t \geqslant a$. Let us assume that $u_{i-1}(t) = 0$, $\forall t \geqslant a$. As we have shown above (see (2.7)), $u_{i-1}(t) = 0$, $\forall t < a$ and taking into account that u_{i-1} is a nonzero function we obtain a contradiction. Thus $u_i(a) = 0$ and then $v_i(a) = 0$. If $u_{i-1}(a) \neq 0$, from (2.8) we derive that $\alpha_i = u_i(a)/u_{i-1}(a)$ and then $v_i(a) = 0$. Thus formula (2.6) is confirmed.

In view of Lemma 2.4, in order to prove that $(v_0, v_1, \dots, v_n)$ is TP, it is sufficient to show that $(v_1, \dots, v_n)$ is a TP system.

By Lemma 2.3(ii) we only need to show the total positivity of nonsingular collocation matrices

$$H = M\begin{pmatrix} v_1, \dots v_n \\ t_1, \dots, t_n \end{pmatrix}.$$

Since $\det H \neq 0$, we deduce that $t_1 > a$, because otherwise by (2.6) the first row of H would be zero.

For $\tau \leqslant t_1$ and $\tau \in C_0$ we may consider collocation matrices of the form $M\binom{u_0,u_1\dots u_n}{\tau,t_1,\dots,t_n}$. Dividing the first row by $u_0(\tau)$ and taking limits analogously to (2.5) we may deduce that the matrix

$$A = \begin{pmatrix} 1 & \inf\limits_{\tau\in C_0} \dfrac{u_1(\tau)}{u_0(\tau)} & \cdots & \inf\limits_{\tau\in C_0} \dfrac{u_n(\tau)}{u_0(\tau)} \\ u_0(t_1) & u_1(t_1) & \cdots & u_n(t_1) \\ \cdots & \cdots & \cdots & \cdots \\ u_0(t_n) & u_1(t_n) & \cdots & u_n(t_n) \end{pmatrix}$$

is TP.

In order to prove the total positivity of H, by Theorem 1.3 of (Cryer, 1976) it is sufficient to show the nonnegativity of the minors with consecutive columns $d = \det M\binom{v_i,v_{i+1}\dots v_{i+r}}{t_{j_0},t_{j_1},\dots,t_{j_r}}$ of the matrix H. If $i > k$, then $v_{i+l} = u_{i+l}$, $l = 0,\dots,r$ and so $d \geqslant 0$ because it coincides with a minor of A, which is a TP matrix. If $i \leqslant k$, then $d = \det B$, where B is the $(r+2)\times(r+2)$ matrix with $(1,0,\dots,0)$ in the first row and $M\binom{u_{i-1},v_i,v_{i+1}\dots v_{i+r}}{t_{j_0},t_{j_1},\dots,t_{j_r}}$ in the rest of the rows. Let us transform B by performing some elementary column operations which do not change the determinant. First, we add to the second column of B α_i times the previous column. Second we add to the third column of the transformed matrix α_{i+1} times the previous one. We continue the process and finally we obtain

$$d = \frac{1}{\inf\limits_{\tau\in C_0}(u_{i-1}(\tau)/u_0(\tau))} \begin{vmatrix} \inf\limits_{\tau\in C_0} \dfrac{u_{i-1}(\tau)}{u_0(\tau)} & \inf\limits_{\tau\in C_0} \dfrac{u_i(\tau)}{u_0(\tau)} & \cdots & \inf\limits_{\tau\in C_0} \dfrac{u_{i+r}(\tau)}{u_0(\tau)} \\ u_{i-1}(t_{j_0}) & u_i(t_{j_0}) & \cdots & u_{i+r}(t_{j_0}) \\ \cdots & \cdots & \cdots & \cdots \\ u_{i-1}(t_{j_r}) & u_i(t_{j_r}) & \cdots & u_{i+r}(t_{j_r}) \end{vmatrix},$$

that is, d is the product of a positive number by a minor of A. Therefore $d \geqslant 0$.

It only remains to show that formula (2.3) holds. Let $(\tau_m)_{m\in\mathbb{N}}$ be any decreasing sequence of values in C_0 which converges to a. In the particular case of $a \in C_0$ we choose $\tau_m := a$ for all $m \in \mathbb{N}$. Thus, taking into account Lemma 2.1,

$$\begin{aligned} \inf_{\tau\in C_0} \frac{v_i(\tau)}{v_0(\tau)} &= \lim_{m\to\infty} \frac{v_i(\tau_m)}{v_0(\tau_m)} = \lim_{m\to\infty} \frac{u_i(\tau_m) - \alpha_i u_{i-1}(\tau_m)}{u_0(\tau_m)} = \\ &= \lim_{m\to\infty} \frac{u_i(\tau_m)}{u_0(\tau_m)} - \alpha_i \lim_{m\to\infty} \frac{u_{i-1}(\tau_m)}{u_0(\tau_m)} = \inf_{\tau\in C_0} \frac{u_i(\tau)}{u_0(\tau)} - \alpha_i \inf_{\tau\in C_0} \frac{u_{i-1}(\tau)}{u_0(\tau)}, \end{aligned}$$

which is zero by the definition of α_i. □

Given $(u_0,\dots,u_n)$ a system of functions on I, we define the *conversion* $(u_0,\dots,u_n)^\#$ as the system of functions on $-I$ given by

$$(u_0,\dots,u_n)^\#(s) := (u_n,\dots,u_0)(-s), \quad s \in -I. \tag{2.9}$$

Following (Ando, 1987, p. 171), we introduce now the definition of *conversion* of a matrix. Given any matrix $K = (k_{ij})_{i,j=0,\dots,n}$, we define its conversion $K^\# :=$

$(k_{n-i,n-j})_{i,j=0,\dots,n}$. From the definition, it is clear that K is TP if and only if $K^\#$ is TP.

Given two systems of functions $(u_0,\dots,u_n)$, $(v_0,\dots,v_n)$ related by a TP matrix K

$$(u_0,\dots,u_n) = (v_0,\dots,v_n)K,$$

the conversion of the systems will be related in the following way

$$(u_0,\dots,u_n)^\# = (v_0,\dots,v_n)^\# K^\#.$$

In the next proposition we shall state an analogous result to Proposition 2.5, which can be readily derived applying Proposition 2.5 to $(u_0,\dots,u_n)^\#$.

Proposition 2.6. *Let $(u_0,\dots,u_n)$ be a TP-system of linearly independent functions on I. Then there exists a TP system $(v_0,\dots,v_n)$ of the form*

$$(u_0,\dots,u_n) = (v_0,\dots,v_n)L, \tag{2.10}$$

where L is a lower triangular TP matrix with unit diagonal, such that

$$\inf\{v_i(t)/v_n(t) \mid t \in I, v_n(t) \neq 0\} = 0, \quad i = 0,\dots,n-1. \tag{2.11}$$

3. B-bases

Any TP system of linearly independent functions $(u_0,\dots,u_n)$, can be considered as a basis of the vector space $\mathcal{U}$ generated by these functions. Any other system of functions $(v_0,\dots,v_n)$ of the form

$$(u_0,\dots,u_n) = (v_0,\dots,v_n)K,$$

where K is a nonsingular $(n+1)\times(n+1)$ matrix, is also a basis of the same space. In this section we shall construct some special bases which we shall call B-bases. The B-bases will be useful to characterize the set of all TP bases of a given space of functions.

For the construction of B-bases we shall define, for any space $\mathcal{U}$ generated by a TP system of linearly independent functions $(u_0,\dots,u_n)$, some particular subspaces which do not depend on the particular given system $(u_0,\dots,u_n)$.

Let $C := \{t \in I \mid (u_0+\dots+u_n)(t) \neq 0\}$. Since the functions $u_i(t)$ are nonnegative, it is clear that

$$C = \bigcup_{u\in\mathcal{U}} \{t \in I \mid u(t) \neq 0\}$$

and then C is independent of the given TP basis $(u_0,\dots,u_n)$ of $\mathcal{U}$. Following the terminology in (Schumaker, 1981) C is the set of essential points with respect to the space of functions $\mathcal{U}$. On the other hand $C_0 := \{t \in I \mid u_0(t) \neq 0\} \subseteq C$ depends on u_0.

However it coincides with C in a neigbourhood of the left end, as the following lemma shows.

Lemma 3.1. *Given $s \in C_0$, one has*

$$C_0 \cap (-\infty, s) = C \cap (-\infty, s).$$

Proof. Since $u_0(s) \neq 0$, it follows from Lemma 2.2(i) that, for any $t < s$ with $u_0(t) = 0$, $u_i(t) = 0$, for all i and so $(u_0 + \cdots + u_n)(t) = 0$. This means that if $t < s$ and $t \notin C_0$ then $t \notin C$ and taking into account that $C_0 \subseteq C$ the result follows. □

Let $a := \inf C \in \mathbb{R} \cup \{-\infty\}$. Now let us define $L(\mathcal{U})$ as the set

$$L(\mathcal{U}) := \begin{cases} \{u \in \mathcal{U} \mid u(a) = 0\} & \text{if } a \in C, \\ \left\{ u \in \mathcal{U} \;\middle|\; \lim\limits_{\substack{t \to a^+ \\ t \in C}} \dfrac{u(t)}{u_0(t) + \cdots + u_n(t)} = 0 \right\} & \text{if } a \notin C, \end{cases}$$

where $(u_0, \ldots, u_n)$ is a TP basis of $\mathcal{U}$. Clearly $L(\mathcal{U})$ is a vector subspace of $\mathcal{U}$. Besides, $L(\mathcal{U}) \neq \mathcal{U}$ because $u_0 + \cdots + u_n \notin L(\mathcal{U})$.

Remark 3.2. From the nonnegativity of the basis $(u_0, \ldots, u_n)$ it easily follows that, for any $u \in \mathcal{U}$, the function $u/(u_0 + \cdots + u_n)$ defined on C is bounded. Furthermore, for any other TP basis of $\mathcal{U}(v_0, \ldots, v_n)$, we deduce that the functions $(u_0 + \cdots + u_n)/(v_0 + \cdots + v_n)$ and $(v_0 + \cdots + v_n)/(u_0 + \cdots + u_n)$, defined on C, are bounded. This fact implies that the definition of $L(\mathcal{U})$ does not depend on the choice of the basis of $\mathcal{U}$.

The following result gives a sufficient condition to find functions in $L(\mathcal{U})$.

Proposition 3.3. *Let $(v_0, \ldots, v_n)$ be a TP basis of $\mathcal{U}$. Then*

$$\inf\{v_i(t)/v_0(t) \mid t \in I, v_0(t) \neq 0\} = 0$$

if and only if $v_i \in L(\mathcal{U})$.

Proof. Let $C_0 := \{t \in I \mid v_0(t) \neq 0\}$ By Lemma 3.1, we know that $a = \inf C = \inf C_0$ and $a \in C$ if and only if $a \in C_0$. We shall distinguish two cases. If $a \in C$, then from Lemma 2.1 it follows that $\inf_{t \in C_0} v_i(t)/v_0(t) = v_i(a)/v_0(a)$ and the left-hand side of the previous formula vanishes if and only if $v_i(a) = 0$. By the definition of $L(\mathcal{U})$, the result follows.

If $a \notin C$ and $\inf_{t \in C_0} v_i(t)/v_0(t) = 0$, then from Lemma 2.1 it follows that

$$\lim_{t \to a^+, t \in C_0} v_i(t)/v_0(t) = 0.$$

Taking limits when $t \to a^+$ in

$$\frac{v_i(t)}{v_0(t) + \cdots + v_n(t)} = \frac{v_i(t)}{v_0(t)} \cdot \frac{v_0(t)}{v_0(t) + \cdots + v_n(t)}, \quad t \in C_0,$$

we derive from Remark 3.2 that

$$\lim_{\substack{t\to a^+\\ t\in C_0}} \frac{v_i(t)}{v_0(t)+\cdots+v_n(t)} = 0.$$

By Lemma 3.1, taking limits when $t \to a^+$ and $t \in C$ yield the same result as taking limits when $t \to a^+$ and $t \in C_0$. Consequently $v_i \in L(\mathcal{U})$.

Conversely, if $a \notin C$ and $v_i \in L(\mathcal{U})$ we know by Lemma 3.1 that

$$\lim_{\substack{t\to a^+\\ t\in C_0}} \frac{v_i(t)}{v_0(t)+\cdots+v_n(t)} = 0.$$

Since

$$\frac{v_i(t)}{v_0(t)} = \frac{v_i(t)}{v_0(t)+\cdots+v_n(t)} \cdot \frac{v_0(t)+\cdots+v_n(t)}{v_0(t)}, \quad t \in C_0,$$

we obtain $\lim_{t\to a^+, t\in C_0} v_i(t)/v_0(t) = 0$ because by Lemma 2.1, $(v_0+\cdots+v_n)/v_0$ is an increasing function and thus is bounded when $t \to a^+$. The result follows from Lemma 2.1. □

Corollary 3.4. *$L(\mathcal{U})$ has a TP basis and* $\dim L(\mathcal{U}) = \dim \mathcal{U} - 1$.

Proof. By Proposition 2.5, there exists a TP basis of $\mathcal{U}$ $(v_0,\ldots,v_n)$ which satisfies $\inf\{v_i(t)/v_0(t) \mid t \in I, v_0(t) \neq 0\} = 0$, $i = 1,\ldots,n$. Then by Proposition 3.3, $v_i \in L(\mathcal{U})$, $i = 1,\ldots,n$, and since $v_0+\cdots+v_n \notin L(\mathcal{U})$ we deduce that $(v_1,\ldots,v_n)$ is a TP basis of $L(\mathcal{U})$. □

We have shown that, given an $n+1$-dimensional vector space $\mathcal{U}$ with a TP basis, we may construct an n-dimensional subspace $L(\mathcal{U})$. Since $L(\mathcal{U})$ has also a TP basis by Corollary 3.4, we can iterate this process and obtain a decreasing chain of subspaces

$$\mathcal{U} = L^0(\mathcal{U}) \supset L(\mathcal{U}) \supset L^2(\mathcal{U}) \supset \cdots \supset L^n(\mathcal{U}),$$

such that each of them has a TP basis and

$$\dim L^i(\mathcal{U}) = \dim(\mathcal{U}) - i = n+1-i. \tag{3.1}$$

We may also define the vector space $\mathcal{U}^\#$ generated by the conversion of any TP basis of $\mathcal{U}$ (see (2.9)) and then define $R(\mathcal{U}) := (L(\mathcal{U}^\#))^\#$. Then we have that $R(\mathcal{U})$ has a TP basis and $\dim R(\mathcal{U}) = n$. So we can iterate this construction and obtain the following decreasing chain of subspaces

$$\mathcal{U} = R^0(\mathcal{U}) \supset R(\mathcal{U}) \supset R^2(\mathcal{U}) \supset \cdots \supset R^n(\mathcal{U}),$$

each of them with a TP basis and

$$\dim R^i(\mathcal{U}) = \dim(\mathcal{U}) - i = n+1-i. \tag{3.2}$$

In order to give a sufficient condition to find functions in $R(\mathcal{U})$, we derive the following result.

Proposition 3.5. *Let $(v_0, \ldots, v_n)$ be a TP basis of $\mathcal{U}$. Then*

$$\inf\{v_i(t)/v_n(t) \mid t \in I, v_n(t) \neq 0\} = 0$$

if and only if $v_i \in R(\mathcal{U})$.

Proof. Apply Proposition 3.3 to the basis $(v_0, \ldots, v_n)^\#$ of $\mathcal{U}^\#$. □

Now we shall transform any given TP basis into another special basis which will be called a B-basis.

Theorem 3.6. *Let $(u_0, \ldots, u_n)$ be a TP system of linearly independent functions on I and $\mathcal{U}$ the space generated by these functions. Then there exists a TP basis $(b_0, \ldots, b_n)$ of $\mathcal{U}$ satisfying $b_i \in L^i(\mathcal{U}) \cap R^{n-i}(\mathcal{U})$, for all $i = 0, \ldots, n$, and such that*

$$(u_0, \ldots, u_n) = (b_0, \ldots, b_n)K, \tag{3.3}$$

where K is a TP matrix.

Proof. First, let us show that there exists a TP basis $(w_0, \ldots, w_n)$ given by

$$(u_0, \ldots, u_n) = (w_0, \ldots, w_n)U,$$

where U is an upper triangular matrix with unit diagonal and $w_i \in L^i(\mathcal{U})$. We shall prove it by induction on n. If $n = 0$, it is trivial taking $U = 1$. If $n = 1$ the result follows from Proposition 2.5 and Proposition 3.3. Let us assume now that the result holds for $n - 1$ and let $(u_0, \ldots, u_n)$ be a TP system of linearly independent functions. Again by Propositions 2.5 and 3.3, there exists a TP system $(v_0, \ldots, v_n)$ such that

$$(u_0, \ldots, u_n) = (v_0, \ldots, v_n)U_1,$$

where U_1 is an upper triangular TP matrix with unit diagonal and $v_1, \ldots, v_n \in L(\mathcal{U})$. Since $v_1, \ldots, v_n$ are linearly independent, by Corollary 3.4 $v_1, \ldots, v_n$ form a TP basis of $L(\mathcal{U})$. By the induction hypothesis, there exists a TP basis of $L(\mathcal{U})$ and an upper triangular TP matrix U_2 with unit diagonal such that

$$(v_1, \ldots, v_n) = (w_1, \ldots, w_n)U_2$$

and $w_i \in L^{i-1}(L(\mathcal{U})) = L^i(\mathcal{U})$. Now defining $w_0 := v_0 = u_0 \in L^0(\mathcal{U})$ we have

$$(u_0, \ldots, u_n) = (w_0, \ldots, w_n)U,$$

where

$$U := \begin{pmatrix} 1 & 0 \\ 0 & U_2 \end{pmatrix} U_1.$$

Let us observe that U is an upper triangular matrix with unit diagonal and, since it is product of TP matrices, U is TP. Finally, by Lemma 2.4 $(w_0, \ldots, w_n)$ is TP.

Now, it can be shown that there exists a TP basis $(b_0, \ldots, b_n)$ given by

$$(w_0, \ldots, w_n) = (b_0, \ldots, b_n)L,$$

where L is a lower triangular TP matrix with unit diagonal and $b_i \in R^{n-i}(\mathcal{U})$. This can be proved by induction on n using Corollary 2.6 and Proposition 3.5 instead of Proposition 2.5 and Proposition 3.3, analogously to the first part of this proof.

The matrix $K := LU$ is TP and satisfies (3.3). It remains to show that $b_i \in L^i(\mathcal{U})$. Since

$$(b_0, \ldots, b_n) = (w_0, \ldots, w_n)L^{-1}, \tag{3.4}$$

we know that

$$b_i = w_i + \sum_{j=i+1}^{n} \alpha_{ij} w_j \quad \forall i.$$

Since $w_i, w_{i+1}, \ldots, w_n$ are in $L^i(\mathcal{U})$, it follows that $b_i \in L^i(\mathcal{U})$. □

Definition 3.7. Let $\mathcal{U}$ be a finite dimensional vector space of functions defined on I. A B-basis $(b_0, \ldots, b_n)$ of $\mathcal{U}$ is a TP basis such that $b_i \in L^i(\mathcal{U}) \cap R^{n-i}(\mathcal{U})$.

Remark 3.8. Observe that if $\mathcal{U}$ is a finite dimensional vector space of functions with a TP basis, then it has a B-basis as stated in Theorem 3.6.

Corollary 3.9. *Let $(u_0, \ldots, u_n)$ be any TP system of linearly independent functions on I and $\mathcal{U}$ the vector space generated by these functions. Let $(c_0, \ldots, c_n)$ be any given B-basis of $\mathcal{U}$. Then*

(i) $\dim(L^k(\mathcal{U}) \cap R^i(\mathcal{U})) = n + 1 - i - k$, $k = 0, \ldots, n$, $i = 0, \ldots, n - k$.

(ii) *If $(v_0, \ldots, v_n)$ is a system of nonnegative functions and $v_k \in L^k(\mathcal{U}) \cap R^{n-k}(\mathcal{U}) \setminus \{0\}$ for all k, then there exist $d_0, \ldots, d_n > 0$ such that $v_k = d_k c_k$ for all k, and so $(v_0, \ldots, v_n)$ is a B-basis of $\mathcal{U}$.*

(iii) *A system of functions is a B-basis if and only if it is of the form $(d_0 c_0, \ldots, d_n c_n)$, where $d_k > 0$ for all k.*

(iv) *There exists a TP matrix K such that*

$$(u_0, \ldots, u_n) = (c_0, \ldots, c_n)K.$$

Proof. (i) We shall prove it by induction on i. If $i = 0$, then by (3.1) $\dim(L^k(\mathcal{U}) \cap R^0(\mathcal{U})) = \dim L^k(\mathcal{U}) = n + 1 - k$. Suppose that the result holds for $i - 1$. Then $\dim(L^k(\mathcal{U}) \cap R^{i-1}(\mathcal{U})) = n + 1 - (i - 1) - k$. We know that the $n + 2 - i - k$ linearly independent functions $c_k, c_{k+1}, \ldots, c_{n-i}, c_{n-i+1}$ are in $L^k(\mathcal{U}) \cap R^{i-1}(\mathcal{U})$ and so they form a basis. From

$$L^k(\mathcal{U}) \cap R^i(\mathcal{U}) \subseteq L^k(\mathcal{U}) \cap R^{i-1}(\mathcal{U}),$$

$c_k, c_{k+1}, \ldots, c_{n-i} \in L^k(\mathcal{U}) \cap R^{i-1}(\mathcal{U})$ and $c_{n-i+1} \notin R^i(\mathcal{U})$ (because $c_0, c_1, \ldots, c_{n-i}$ form a basis of $R^i(\mathcal{U})$ by (3.2)), it follows that $\dim(L^k(\mathcal{U}) \cap R^i(\mathcal{U})) = n + 1 - i - k$.

(ii) By (i), the following formula holds: $\dim L^k(\mathcal{U}) \cap R^{n-k}(\mathcal{U}) = 1$. We know that $v_k, c_k \in L^k(\mathcal{U}) \cap R^{n-k}(\mathcal{U})\backslash\{0\}$ and therefore $v_k = d_k c_k$, where $d_k \neq 0$, for all $k = 0, \ldots, n$. Furthermore, v_k and c_k are nonnegative functions which implies $d_k > 0$ for all k.

(iii) This is an immediate consequence of (ii).

(iv) By Theorem 3.6, there exists a TP matrix K_1 such that

$$(u_0, \ldots, u_n) = (b_0, \ldots, b_n) K_1,$$

where $(b_0, \ldots, b_n)$ is a B-basis. By (iii), we know that

$$(b_0, \ldots, b_n) = (c_0, \ldots, c_n) D,$$

where $D = \operatorname{diag}(d_0, \ldots, d_n)$ is obviously a TP matrix. Then

$$(u_0, \ldots, u_n) = (c_0, \ldots, c_n) D K_1,$$

and taking $K = DK_1$, (iv) follows. $\square$

The following result will allow us to provide a characterization of all TP bases of a given space in terms of any B-basis.

Corollary 3.10. *Let $\mathcal{U}$ be a vector space generated by a TP system of linearly independent functions on I and $(b_0, \ldots, b_n)$ be a B-basis of $\mathcal{U}$. A basis $(u_0, \ldots, u_n)$ of $\mathcal{U}$ is TP if and only if the matrix K of change of basis*

$$(u_0, \ldots, u_n) = (b_0, \ldots, b_n) K$$

is TP.

Proof. If $(u_0, \ldots, u_n)$ is TP, by Corollary 3.9(iv), K is TP. For the converse, any collocation matrix of $(u_0, \ldots, u_n)$ can be written

$$M\begin{pmatrix} u_0, \ldots u_n \\ t_0, \ldots, t_n \end{pmatrix} = M\begin{pmatrix} b_0, \ldots b_n \\ t_0, \ldots, t_n \end{pmatrix} K$$

and so is the product of TP matrices. $\square$

Given a TP basis $(u_0, \ldots, u_n)$ and a nonsingular TP matrix K, $(u_0, \ldots, u_n)K$ is also a TP basis of the same space. Thus we may construct a family of TP bases from a given one, but not necessarily all TP bases. If we obtain by this process all the TP bases (as it happens with the B-bases by the previous corollary) we say that $(u_0, \ldots, u_n)$ is an *optimal* TP basis. In the next section, we shall derive from the optimality property some interesting consequences in Computer Aided Geometric Design. The next result (together with Corollary 3.10) will show that the optimal TP bases coincide with the B-bases.

Proposition 3.11. *If $(c_0, \ldots, c_n)$ is a TP basis of a space $\mathcal{U}$ such that, for any other TP basis $(u_0, \ldots, u_n)$, the matrix K of the change of basis*

$$(u_0, \dots, u_n) = (c_0, \dots, c_n)K$$

is TP, then $(c_0, \dots, c_n)$ *is a B-basis.*

Proof. Taking a TP B-basis $(b_0, \dots, b_n)$ of $\mathcal{U}$, we can write

$$(b_0, \dots, b_n) = (c_0, \dots, c_n)K,$$

where K is TP. Now

$$(c_0, \dots, c_n) = (b_0, \dots, b_n)K^{-1},$$

and by Corollary 3.9(iv), K^{-1} must be also TP. It can be easily proved (see for instance the proof of Proposition 4.4 of (Carnicer and Peña, 1993)) that if K and K^{-1} are both TP, then K is a diagonal matrix. By Corollary 3.9(iii), $(c_0, \dots, c_n)$ is a B-basis. □

In general, it is not easy to check whether a given function u is in $L^i(\mathcal{U})$, $R^{n-i}(\mathcal{U})$ because the spaces were defined recursively. So this problem arises when using Corollary 3.9(ii) as a test to recognize B-bases. However, the following result provides a general characterization of B-bases, which can be applied directly.

Proposition 3.12. *Let* $(b_0, \dots, b_n)$ *be a TP basis of a space* $\mathcal{U}$. *Then* $(b_0, \dots, b_n)$ *is a B-basis if and only if the following conditions hold*

$$\inf\{b_i(t)/b_j(t) \mid t \in I, b_j(t) \neq 0\} = 0, \tag{3.5}$$

for all $i \neq j$.

Proof. Let $(b_0, \dots, b_n)$ be a B-basis. Now we shall prove (3.5) with $i > j$. By (3.1), the linearly independent functions $b_j, b_{j+1}, \dots, b_n$ form a basis of $L^j(\mathcal{U})$. Besides, $b_{j+1}, \dots, b_n \in L^{j+1}(\mathcal{U}) \subseteq L(L^j(\mathcal{U}))$. By Proposition 3.3 formula (3.5) holds for $i > j$. The case $i < j$ can be dealt analogously using (3.2) and Proposition 3.5 instead of (3.1) and Proposition 3.3.

For the converse, it will be proved by induction on n that $b_i \in L^i(\mathcal{U})$, $i = 0, \dots, n$. If $n = 1$, it follows from Proposition 3.3. Let us assume now that the result holds for $n - 1$. Let $(b_0, \dots, b_n)$ be a TP basis satisfying (3.5). By Proposition 3.3, $b_1, \dots, b_n \in L(\mathcal{U})$ and by (3.1) they form a basis of $L(\mathcal{U})$. By the induction hypothesis $b_j \in L^{j-1}(L(\mathcal{U})) = L^j(\mathcal{U})$ for $j > 1$. Analogously it can be shown by induction on n that $b_i \in R^{n-i}(\mathcal{U})$, using (3.2) and Proposition 3.5 instead (3.1) and Proposition 3.3. □

Finally let us show some elementary examples of B-bases.

Example 3.13. (i) In the interval $[0, 1]$, the Bernstein basis of the space Π_n of polynomials of degree less than or equal to n is a B-basis. It is well-known that it is a TP basis and clearly satisfies (3.5).

(ii) In the interval $[0, \infty)$, the monomial basis $(1, t, \dots, t^n)$ of Π_n is a B-basis. This basis is TP (see for instance Theorem 3.7 of (Schumaker, 1981)) and again satisfies (3.5).

(iii) Given $\lambda_0 < \lambda_1 < \cdots < \lambda_n$ in $\mathbb{R}$ the system $(\exp(\lambda_0 t), \exp(\lambda_1 t), \ldots, \exp(\lambda_n t))$ of functions defined on $\mathbb{R}$ is a B-basis of the space generated by these functions. The system is TP (see pp. 15–16 of (Karlin, 1968)) and the condition (3.5) can be readily checked.

In the next section we shall give other examples of B-bases for some spaces which are useful in Computer Aided Geometric Design, such as the B-spline basis or β-spline basis.

4. Shape preserving properties and optimality of B-splines

Let $\mathcal{U}$ be a vector space of functions defined on I. If $(u_0, \ldots, u_n)$ is a TP basis of $\mathcal{U}$ such that $\sum_{i=0}^n u_i(t) = 1$, for all $t \in I$, we say that the basis $(u_0, \ldots, u_n)$ is *normalized totally positive* (NTP). Thus, the collocation matrix $M\binom{u_0,\ldots,u_n}{t_0,\ldots,t_n}$, $t_0 < \cdots < t_n$ in I, is TP and stochastic (each row has sum 1). We note that if $(u_0, \ldots, u_n)$ is a TP basis of functions defined on I with $\sum_{i=0}^n u_i(t) > 0$, for all $t \in I$, then the system $(w_0, \ldots, w_n)$ defined by

$$w_i := \frac{u_i}{\sum_{i=0}^n u_i}, \quad i = 0, \ldots, n,$$

is clearly NTP. However, the space $\mathcal{W}$ generated by $w_0, \ldots, w_n$ does not necessarily coincide with the space $\mathcal{U}$ generated by $u_0, \ldots, u_n$. If $\sum_{i=0}^n u_i(t) > 0$ for all $t \in I$, we say that the TP basis $(u_0, \ldots, u_n)$ of $\mathcal{U}$ is *normalizable.* Obviously, every normalized totally positive (NTP) basis is normalizable. As it was shown at the beginning of Section 3, if $(v_0, \ldots, v_n)$ is any TP basis of a space $\mathcal{U}$ which has a normalizable basis $(u_0, \ldots, u_n)$, then the set of essential points C satisfies

$$C = \{t \in I \mid (v_0 + \cdots + v_n)(t) > 0\} = \{t \in I \mid (u_0 + \cdots + u_n)(t) > 0\} = I,$$

which means that every TP basis of $\mathcal{U}$ is normalizable. Therefore a normalizable TP basis can be seen as a TP basis of a space whose set of essential points coincides with the domain of definition I.

In Computer Aided Geometric Design the curves are often generated by a system of functions defined on an interval. By this reason, we shall only consider throughout this section the case of $I = C$ being an *interval.*

The following result generalizes well-known properties of the basis of B-splines.

Proposition 4.1. *Let $(u_0, \ldots, u_n)$ be a normalizable TP basis of $\mathcal{U}$. Then*

(i) *$I_i = \{t \in I \mid u_i(t) \neq 0\}$ is an interval for all $i = 0, \ldots, n$.*

(ii) *If $u_i(t_0) = 0$, then either $u_j(t) = 0$, $\forall t \leqslant t_0$, $\forall j \geqslant i$, or $u_j(t) = 0$, $\forall t \geqslant t_0$, $\forall j \leqslant i$.*

Proof. (i) Let us assume that $I_i \neq \emptyset$ is not an interval. Then there exist $t_0 < t_1 < t_2$ such that $u_i(t_0) > 0$, $u_i(t_1) = 0$, $u_i(t_2) > 0$. By Lemma 2.2(i), $u_j(t_0) = 0$ for all $j \geqslant i$

and by Lemma 2.2(ii), $u_j(t_0) = 0$ for all $j \leqslant i$. Thus $(u_0 + \cdots + u_n)(t_0) = 0$, which is a contradiction.

(ii) By (i), if $u_i(t_0) = 0$, then either $u_i(t) = 0$ for all $t \leqslant t_0$ or $u_i(t) = 0$ for all $t \geqslant t_0$. If $u_i(t) = 0$ for all $t \leqslant t_0$, then Lemma 2.2(i) implies $u_j(t) = 0$ for all $j \geqslant i$, for each $t \leqslant t_0$. If $u_i(t) = 0$ for all $\geqslant t_0$, then Lemma 2.2(ii) implies $u_j(t) = 0$ for all $j \leqslant i$ for each $t \geqslant t_0$. $\square$

The NTP bases $(u_0, \ldots, u_n)$ of a given space $\mathcal{U}$ can be used in Computer Aided Geometric Design to obtain curves.

Given a sequence $U_0, \ldots, U_n$ of points in $\mathbb{R}^k$, we may define a curve

$$\gamma(t) = \sum_{i=0}^{n} U_i u_i(t), \quad t \in I.$$

We shall denote by $U_0 \cdots U_n$ the polygonal arc with vertices $U_0, \ldots, U_n$. This is usually called the *control polygon* of γ and the points U_i, $i = 0, \ldots, n$, are called control points.

In Computer Aided Geometric Design it is only required that the functions u_i, $i = 0, \ldots, n$, are nonnegative and $\sum_{i=0}^{n} u_i(t) = 1$ for all $t \in I$. However, it is convenient that the curves generated from their control polygon satisfy the variation diminishing property because this implies that some shape properties, such as convexity, should be preserved (see p. 48 of (Farin, 1988)). It is well known (cf. chapter 5 of (Karlin, 1968), Section 5 of (Ando, 1987)) that sign-regular transformations characterize those transformations with the variation diminishing property. Let us recall that a Descartes system is a system of functions $(u_0, \ldots, u_n)$ such that the subdeterminants of order k of the collocation matrices (1.1) all have the same sign, but this sign may vary with k; in other words, their collocation matrices are sign-regular. In particular every TP system is a Descartes system. In Bemerkung II.4 of (Schmeltz, 1992) it is shown that a normalized Descartes system which generates curves such that the first control point always coincides with the start-point of the curve and the last control point always coincides with the end-point of the curve (which is a usual requirement in Computer Aided Geometric Design) is necessarily totally positive. Further shape preserving properties of NTP bases have been shown in (Goodman, 1989), (Goodman and Said, 1991), (Goodman, 1991). In fact, the most interesting systems used for curve design (Bernstein polynomials, B-spline basis, β-spline basis, ...) are NTP bases.

The next theorem characterizes all the NTP bases of a space.

Theorem 4.2. *Let $\mathcal{U}$ be a vector space of functions defined on an interval I with an NTP basis. Then*

(i) *There exists a unique NTP B-basis $(b_0, \ldots, b_n)$.*

(ii) *A basis $(u_0, \ldots, u_n)$ of $\mathcal{U}$ is NTP if and only if the matrix K of change of basis*

$$(u_0, \ldots, u_n) = (b_0, \ldots, b_n)K$$

is TP and stochastic.

Proof. (i) Let $(u_0,\ldots,u_n)$ be an NTP basis of $\mathcal{U}$. By Remark 3.8, there exists a B-basis $(c_0,\ldots,c_n)$ of $\mathcal{U}$. From Corollary 3.9(iv) there exists a (nonsingular) TP matrix $H=(h_{ij})_{i,j=0,\ldots,n}$ such that

$$(u_0,\ldots,u_n)=(c_0,\ldots,c_n)H.$$

Let $b_i := \left(\sum_{j=0}^n h_{ij}\right)c_i$, which is a B-basis by Corollary 3.9(iii). Then $(b_0,\ldots,b_n)$ is normalized because

$$\sum_{i=0}^n b_i=\sum_{i=0}^n\left(\sum_{j=0}^n h_{ij}\right)c_i=\sum_{j=0}^n\left(\sum_{i=0}^n h_{ij}c_i\right)=\sum_{j=0}^n u_j=1.$$

If $(\bar{b}_0,\ldots,\bar{b}_n)$ is another NTP B-basis, then by Corollary 3.9(iii) $\bar{b}_i=d_ib_i$, for some $d_i>0$, $i=0,\ldots,n$, and so $\sum_{i=0}^n b_i=1=\sum_{i=0}^n \bar{b}_i=\sum_{i=0}^n d_ib_i$. Since $(b_0,\ldots,b_n)$ is a basis then $d_i=1$ for all i.

(ii) If $(u_0,\ldots,u_n)$ is NTP, then K is TP by Corollary 3.9(iv) and

$$\sum_{i=0}^n b_i=1=\sum_{j=0}^n u_j=\sum_{j=0}^n\left(\sum_{i=0}^n k_{ij}b_i\right)=\sum_{i=0}^n\left(\sum_{j=0}^n k_{ij}\right)b_i.$$

Since $(b_0,\ldots,b_n)$ is a basis $\sum_{j=0}^n k_{ij}=1$ for all i.

If K is TP and stochastic, then $(u_0,\ldots,u_n)$ is TP and

$$\sum_{j=0}^n u_j=\sum_{j=0}^n\left(\sum_{i=0}^n k_{ij}b_i\right)=\sum_{i=0}^n\left(\sum_{j=0}^n k_{ij}\right)b_i=\sum_{i=0}^n b_i=1. \qquad \square$$

Remark 4.3. We have mentioned that we need NTP bases for the generation of curves from their control polygon. Thus, a space is suitable for design purposes if it has an NTP basis. From Theorem 4.2(i) we know that in a space with a TP basis, there exists a NTP basis if and only if each element of a B-basis can be conveniently normalized in such a way that the sum of the normalized functions is 1. An important example of the situation described above is the B-spline basis of the space of polynomial splines: it is well-known that the B-splines can be normalized so that they add up to 1 and, as we shall see later, the B-spline basis is a B-basis. An "algorithm" to know if a space $\mathcal{U}$ with a TP basis has an NTP basis and to obtain it (if it exists) is:

(I) Check if $1\in\mathcal{U}$.

(II) Take a TP basis $(u_0,\ldots,u_n)$ of $\mathcal{U}$ and transform it into a B-basis $(b_0,\ldots,b_n)$ following the steps suggested by Theorem 3.6. More precisely, let us define

$$u_i^0 := u_i, \quad i=0,\ldots,n,$$

and for $j=0,\ldots,n-1$ define iteratively

$$u_i^{j+1} := \begin{cases} u_i^j-\inf(u_i^j/u_{i-1}^j)u_{i-1}^j, & i=n,n-1,\ldots,j+1,\\ := u_i^j, & i=j,j-1,\ldots,0.\end{cases}$$

Now let

$$v_i^0 := u_i^{n-1}, \quad i = 0, \dots, n,$$

and for $j = 0, \dots, n-1$ define

$$v_i^{j+1} := \begin{cases} v_i^j - \inf(v_i^j / v_{i+1}^j) v_{i+1}^j, & i = 0, 1, \dots, n-j-1, \\ v_i^j, & i = n-j, \dots, n. \end{cases}$$

Finally $b_i := v_i^{n-1}$, $i = 0, \dots, n$ form a B-basis.

(III) Write down 1 as a linear combination of $(b_0, \dots, b_n)$:

$$1 = d_0 b_0 + \cdots + d_n b_n$$

and check that the coefficients d_i are positive (this condition is necessary by Corollary 3.9(iii)). Obtain the NTP basis $(d_0 b_0, \dots, d_n b_n)$.

Finally, Theorem 4.2(ii) allows us to construct all the NTP bases of the space, that is, all the convenient bases for design purposes.

Now, let us turn our attention to the idea of optimal NTP basis.

Corollary 4.4. *Let $\mathcal{U}$ be a vector space of functions defined on an interval I with an NTP basis. Then there exists a unique optimal NTP basis, which is the NTP B-basis.*

Proof. From Corollary 3.9(iv) and Proposition 3.11 the optimal TP bases coincide with B-bases. The result follows from Proposition 4.2(i). □

Now we know that the optimal NTP bases coincide with the NTP B-bases. The idea of NTP B-basis puts the stress on its construction (Theorem 3.6) and its recognition by functional properties (Proposition 3.12). However, the idea of optimal NTP basis comes from the geometric point of view. In (Goodman and Said, 1991) appeared the problem of knowing if an NTP basis $(w_0, \dots, w_n)$ has optimal shape preserving properties among all NTP bases. That is, if there exists a corner cutting algorithm (see (Goodman, 1989), (Goodman and Micchelli, 1988)) from the control polygon $U_0 \cdots U_n$ of the curve γ with respect to any NTP basis $(u_0, \dots, u_n)$ to the control polygon $W_0 \cdots W_n$ of the same curve respect to the NTP basis $(w_0, \dots, w_n)$. The existence of a corner cutting algorithm from the control polygon $U_0 \cdots U_n$ to the control polygon $W_0 \cdots W_n$ implies that $W_0 \cdots W_n$ lies between $U_0 \cdots U_n$ and γ. Furthermore $W_0 \cdots W_n$ is more similar to γ than $U_0 \cdots U_n$ in many geometrical properties such as the length, angular variation, number of inflections, ... (see (Goodman and Said, 1991) and (Goodman, 1991)).

Specifically, if K is the matrix of change of basis

$$(u_0, \dots, u_n) = (w_0, \dots, w_n) K,$$

the control polygons of γ are related by

$$(W_0, \dots, W_n)^{\mathrm{T}} = K (U_0, \dots, U_n)^{\mathrm{T}}.$$

The relationship between both control polygons corresponds to a corner cutting algorithm precisely when K is TP and stochastic (cf. (Goodman and Micchelli, 1988, p. 240)). Thus, the idea of basis with optimal shape preserving properties of (Goodman and Said, 1991) coincides with our previously defined concept of optimal NTP basis. We know that the NTP B-basis is the optimal NTP basis (Corollary 4.4) and Remark 4.3 provides a procedure to obtain it.

We have considered until now NTP bases in a general framework. But in Computer Aided Geometric Design, the basic functions are usually continuous in order to generate continuous curves. This assumption allows us to simplify the characterization provided in Proposition 3.12 for B-bases.

Proposition 4.5. *Let $(b_0,\dots,b_n)$ be a NTP basis of a space $\mathcal{U} \subseteq C(I)$. $(b_0,\dots,b_n)$ is a B-basis if and only if the following conditions hold*

$$\lim_{t\to\alpha_j^+} \frac{b_i(t)}{b_j(t)} = 0, \quad i > j \quad \textit{and} \quad \lim_{t\to\beta_j^-} \frac{b_i(t)}{b_j(t)} = 0, \quad i < j, \tag{4.1}$$

where $\alpha_j = \inf I_j \in \mathbb{R} \cup \{-\infty\}$, $\beta_j = \sup I_j \in \mathbb{R} \cup \{\infty\}$ and I_j is the interval $I_j := \{t \in I \mid b_j(t) \neq 0\}$.

Proof. By Proposition 4.1(i), I_j is an interval for all j. The continuous function b_i/b_j defined on the interval is increasing if $i > j$ and decreasing if $i < j$ by Lemma 2.1. Thus,

$$\inf_{t\in I_j} \frac{b_i(t)}{b_j(t)} = \lim_{t\to\alpha_j^+} \frac{b_i(t)}{b_j(t)}, \quad \text{if } i > j,$$
$$\inf_{t\in I_j} \frac{b_i(t)}{b_j(t)} = \lim_{t\to\beta_j^-} \frac{b_i(t)}{b_j(t)}, \quad \text{if } i < j.$$

Now the result follows from Proposition 3.12. □

Now we shall show that the B-spline basis is the optimal NTP basis of the space of polynomial splines. Let Π_{k-1} be the space of polynomials of degree less than or equal to $k-1$. Let $\Delta = \{\theta_1,\dots,\theta_l\}$ be a partition of the interval $I = [\alpha,\beta]$, where

$$\alpha = \theta_0 < \theta_1 < \cdots < \theta_l < \theta_{l+1} = \beta,$$

and let $I_i = [\theta_i,\theta_{i+1}]$, $i = 0,\dots,l$. For any multiplicity vector $m = (m_1,\dots,m_l)$ of positive integers such that $m_i \leqslant k-1$, $i = 1,\dots,l$, let us define

$$\begin{aligned} \mathcal{S}_{k,\Delta,m} = \{ s : I \to \mathbb{R} \mid s|_{I_i} \in \Pi_{k-1},\ i = 0,\dots,l; \\ s^{(j)}(\theta_i^-) = s^{(j)}(\theta_i^+),\ i = 1,\dots,l,\ j = 0,1,\dots,k-m_i-1\} \end{aligned}$$

the space of polynomial splines of order k with knots $\theta_1,\dots,\theta_l$ of multiplicities $m_1,\dots,m_l$.

Now let us define the extended sequence of knots (that is, each knot θ_i is repeated m_i times)

$$\alpha = \tau_0 = \cdots = \tau_{k-1} < \tau_k \leqslant \cdots \leqslant \tau_n < \tau_{n+1} = \cdots = \tau_{n+k} = \beta,$$

where $n = k + m_1 + \cdots + m_l - 1$. It is well-known that the functions

$$N_{i,k}(t) := (\tau_{i+k} - \tau_i)[\tau_i, \ldots, \tau_{i+k}](\cdot - t)_+^{k-1}, \quad i = 0, \ldots, n$$

form a basis of $\mathcal{S}_{k,\Delta,m}$ called the B-spline basis. Besides $\operatorname{supp} N_{i,k} = [\tau_i, \tau_{i+k}]$, $N_{i,k}(t) > 0$ for all $t \in [\tau_i, \tau_{i+k}]$ and $\sum_{i=0}^n N_{i,k} = 1$.

Furthermore $N_{i,k}$ satisfies

$$N_{i,k}^{(j)}(\tau_i^+) = 0, \quad j = 0, 1, \ldots, k - \nu_i - 2 \text{ and } N_{i,k}^{(k-\nu_i-1)}(\tau_i^+) \neq 0, \tag{4.2}$$

and

$$N_{i,k}^{(j)}(\tau_{i+k}^-) = 0, \quad j = 0, 1, \ldots, k - \mu_i - 2 \text{ and } N_{i,k}^{(k-\mu_i-1)}(\tau_{i+k}^-) \neq 0, \tag{4.3}$$

where ν_i and μ_i are given by

$$\nu_i = \max\{j \mid \tau_i = \tau_{i+1} = \cdots = \tau_{i+j-1}\},$$
$$\mu_i = \max\{j \mid \tau_{i+k} = \tau_{i+k-1} = \cdots = \tau_{i+k-j+1}\}$$

(see Theorem 4.17 of (Schumaker, 1981)).

The B-spline basis is TP (see (Karlin, 1968; de Boor, 1976)) and consequently a NTP basis of $\mathcal{S}_{k,\Delta,m}$.

Theorem 4.6. *The B-spline basis is a B-basis and so the optimal NTP basis of the space $\mathcal{S}_{k,\Delta,m}$.*

Proof. By Corollary 4.4, it is sufficient to see that the B-spline basis is a B-basis. Let $i > j$ and let us show that

$$\lim_{t \to \tau_j^+} \frac{N_{i,k}(t)}{N_{j,k}(t)} = 0. \tag{4.4}$$

If $i \geqslant j + \nu_j$, then $\tau_i > \tau_j$ and so $N_{i,k}(t) = 0$, $N_{j,k}(t) > 0$ for all $t \in [\tau_j, \tau_{j+\nu_j}]$, which implies (4.4).

If $i < j + \nu_j$, we know from (4.2) that the restriction of $N_{i,k}$ to the interval $(\tau_j, \tau_{j+\nu_j})$ is a polynomial which has a zero of higher order at τ_j than the restriction of $N_{j,k}$ to the same interval, which implies that (4.4) holds.

Analogously we may show that, if $i < j$, $\lim_{t \to \tau_{j+k}^-} N_{i,k}(t)/N_{j,k}(t) = 0$ using (4.3) instead of (4.2). Then from Proposition 4.5, the result follows. □

Remark 4.7. For any given $w_0, \ldots, w_n > 0$ positive weights and a control polygon $R_0 \cdots R_n$, the nonuniform rational B-spline (NURBS) curve is defined by

$$\gamma(t) = \frac{\sum_{j=0}^n w_j R_j N_{j,k}(t)}{\sum_{j=0}^n w_j N_{j,k}(t)}$$

(see (Farin, 1988), (Farin, 1989)). The corresponding basic functions

$$r_i(t) := \frac{w_i N_{i,k}(t)}{\sum_{j=0}^n w_j N_{j,k}(t)}$$

form the optimal NTP basis of the space $\mathcal{R}$ generated by these functions. In fact, by Corollary 3.9 (iii), the basis of $\mathcal{S}_{k,\Delta,m}$ $(w_0N_{0,k},\dots,w_nN_{n,k})$ is a B-basis. Then, if we divide by the fixed positive function $\sum_{j=0}^{n} w_jN_{j,k}$, the obtained system $(r_0,\dots,r_n)$ is again a TP basis of the space $\mathcal{R}$ and still satisfies (3.5). By Proposition 3.12, $(r_0,\dots,r_n)$ is a B-basis of $\mathcal{R}$ and since $\sum_{i=0}^{n} r_i = 1$, by Corollary 4.4, it is the optimal NTP basis.

With the same proof as Theorem 4.6, the following result can be shown:

Proposition 4.8. *Let $\mathcal{S}$ be a space with an NTP basis $(N_0,\dots,N_n)$ of continuous functions such that* $\operatorname{supp} N_i = [\alpha_i,\beta_i]$ *and $N_i(t) > 0$ for all $t \in [\alpha_i,\beta_i]$, $i = 0,\dots,n$. Let r_i, s_i, $i = 0,\dots,n$, be nonnegative integers such that if $\alpha_i = \alpha_{i+1}$ then $r_i < r_{i+1}$ and if $\beta_i = \beta_{i+1}$ then $s_{i+1} < s_i$. If the functions N_i satisfy the following conditions*

$$N_i^{(j)}(\alpha_i^+) = 0, \quad j = 0,\dots,r_i - 1 \quad \text{and} \quad N_i^{(r_i)}(\alpha_i^+) \neq 0,$$
$$N_i^{(j)}(\beta_i^-) = 0, \quad j = 0,\dots,s_i - 1 \quad \text{and} \quad N_i^{(s_i)}(\beta_i^-) \neq 0,$$

for all $i = 0,\dots,n$, then $(N_0,\dots,N_n)$ is a B-basis and so the optimal NTP basis of $\mathcal{S}$.

Now, let us turn our attention to the generalized spline spaces used in (Dyn and Micchelli, 1988) to model space curves which have geometric continuity. In general, conditions of geometric continuity at some knot τ give rise to the following conditions for the basic functions

$$(u_i,u_i',\dots,u_i^{(r)})^{\mathrm{T}}(\tau^+) = A_\tau(u_i,u_i',\dots,u_i^{(r)})^{\mathrm{T}}(\tau^-),$$

where A_τ is the connection matrix at τ. If all the connection matrices are lower triangular, TP and their first column is $(1,0,\dots,0)^{\mathrm{T}}$, then the basis mentioned in Theorem 4.6 of (Dyn and Micchelli, 1988) satisfies the conditions of Proposition 4.8, in view of the results of Section 4 of the same paper. Thus, such a basis is a B-basis and so the optimal NTP basis of the corresponding space of generalized splines. In particular this is true for the β-spline basis.

Summarizing, we have seen how to obtain bases whose control polygons represent the shape of the curve with the highest fidelity. Furthermore, for some classical spline spaces we have identified these bases with the B-spline, rational B-spline and β-spline bases. However, our construction allows us to obtain the optimal NTP basis (NTP B-basis) for any other space which has an NTP basis. Moreover, our results allow us to obtain all the NTP bases which can be also used in Computer Aided Geometric Design to obtain curves from a control polygon preserving the shape properties.

References

Ando, T. (1987), Totally positive matrices, Linear Algebra Appl. 90, 165–219.

Carnicer, J.M. and Peña, J.M. (1993), Shape preserving representations and optimality of the Bernstein basis, Adv. Comput. Math. 1, 173–196.

Cryer, C.W. (1976), Some properties of totally positive matrices, Linear Algebra Appl. 15, 1–25.

de Boor, C. and Pinkus, A. (1982), The approximation of a banded totally positive matrix by a strictly banded totally positive one, Linear Algebra Appl. 42, 81–98.

Dyn, N. and Micchelli, C.A. (1988), Piecewise polynomial spaces and geometric continuity of curves, Numer. Math. 54, 319–337.

Farin, G. (1988), *Curves and Surfaces for Computer Aided Geometric Design*, Academic Press, Boston, MA.

Farin, G. (1989), Rational curves and surfaces, in: Lyche, T. and Schumaker, L.L., eds., *Mathematical Methods in CAGD*, Academic Press, Boston, MA, 215–238.

Gasca, M. and Peña, J.M. (1992), Total positivity and Neville elimination, Linear Algebra Appl. 165, 25–44.

Gasca, M. and Peña, J.M. (1993), Total positivity, QR factorization and Neville elimination, SIAM J. Matrix Anal. Appl. 14, 1132–1140.

Goodman, T.N.T. (1989), Shape preserving representations, in: Lyche, T. and Schumaker, L.L., eds., *Mathematical Methods in CAGD*, Academic Press, Boston, MA, 333–357.

Goodman, T.N.T. (1991), Inflections on curves in two and three dimensions, Computer Aided Geometric Design 8, 37–50.

Goodman, T.N.T. and Micchelli, C.A. (1988), Corner cutting algorithms for the Bézier representation of free form curves, Linear Algebra Appl. 99, 225–252.

Goodman, T.N.T. and Said, H.B. (1991), Shape preserving properties of the generalized Ball basis, Computer Aided Geometric Design 8, 115–121.

Karlin, S. (1968), *Total Positivity*, Stanford University Press, Stanford, CA.

Schmeltz, G. (1992), *Variationsreduzierende Kurvendarstellungen und Krümmungskriterien für Bézierflächen*, Thesis, Fachbereich Mathematik, Technische Hochschule Darmstadt.

Schumaker, L.L. (1981), *Spline Functions: Basic Theory*, Wiley, New York.