Elena Rubei
Algebraic Geometry

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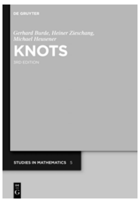

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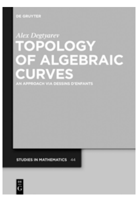

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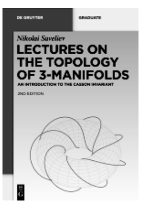

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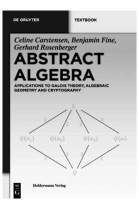

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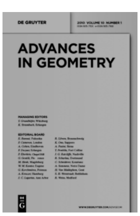

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Algebraic Geometry

A Concise Dictionary

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Preface

This little dictionary of algebraic geometry is intended to be useful mainly to undergraduate and Ph.D. students. For each word listed in the dictionary, we have given the definition, some references, and the main theorems about that term (without the proofs). Some terms from other subjects close to algebraic geometry have also been included. The dictionary has been conceived to help beginners who know some basic facts on algebraic geometry, but not every basic fact, to follow seminars and read papers by giving them basic definitions and statements and providing them with a glimpse of what is nowadays considered to be the basic notions of algebraic geometry. For the sake of simplicity, in some items (for instance algebraic surfaces and Abelian varieties), we have considered only the case of the varieties over the field of complex numbers.

I warmly thank Giorgio Ottaviani for many helpful discussions on algebraic geometry and his invaluable support during all the years I have been in Firenze and, in particular, during the writing of this book. I thank also Roberto Pignatelli for a helpful suggestion.

Firenze, December 2013 Elena Rubei

Notation

$\mathbb{A}^n_K$	We denote by $\mathbb{A}^n_K$ the affine space of dimension n over K.
$A^p(X)$	For any C^∞ manifold X, $A^p(X)$ denotes the vector space of the C^∞ p-forms on X.
$A^{p,q}(X)$	For any almost complex manifold X (see "Almost complex manifolds, holomorphic maps, holomorphic tangent bundles"), $A^{p,q}(X)$ denotes the vector space of the C^∞ (p,q)-forms on X.
$b_i(X,R)$	For any topological space X and ring R, $b_i(X,R)$ (Betti number) denotes the rank of $H_i(X,R)$; it is also denoted by $h_i(X,R)$; see "Singular homology and cohomology".
b.p.f.	The abbreviation b.p.f. stands for "base point free", see "Bundles, fibre -".
$CH^p(X)$	For any algebraic variety X, $CH^p(X)$ denotes the p-th Chow group of X; see "Equivalence, algebraic, rational, linear -, Chow, Neron–Severi and Picard groups".
$Cl(X)$	For any algebraic variety X, $Cl(X)$ denotes the divisor class group of X; see "Divisors".
$C^\infty(X,E)$	For any C^∞ vector bundle E on a C^∞ manifold X, $C^\infty(X,E)$ denotes the set of the C^∞ sections of E; see "Bundles, fibre -".
$\delta_{\alpha,\beta}$	$\delta_{\alpha,\beta}$ stands (as usual) for the Kronecker delta.
(D)	For any Cartier divisor D, (D) denotes the line bundle associated to D; see "Divisors".
f_*, f^*, f^{-1}	See "Pull-back and push-forward of cycles", "Direct and inverse image sheaves", "Singular homology and cohomology".
$\lvert D\rvert, \mathcal{L}(D)$	For any divisor D, we denote by $\lvert D\rvert$ the complete linear system associated to D, see "Linear systems"; see also "Linear systems" for the definition of $\mathcal{L}(D)$.
φ_L	For any line bundle L on a variety X, φ_L denotes the map associated to L; see "Bundles, fibre -".
g^r_d	g^r_d denotes a linear system on a Riemann surface of degree d and projective dimension r; see "Linear systems".
$G(k,V), G(k,\mathbb{P})$	For any vector space V, $G(k,V)$ denotes the Grassmannian of k-planes in V, see "Grassmannians"; analogously, for any projective space $\mathbb{P}$, $G(k,\mathbb{P})$ denotes the Grassmannian of projective k-planes in $\mathbb{P}$.
$H_i(X,R), h_i(X,R)$	For any topological space X and any ring R, $H_i(X,R)$ denotes the i-th homology module of X with coefficients in R (see "Singular homology and cohomology"); $h_i(X,R)$ denotes its rank; $h_i(X,R)$ is also denoted by $b_i(X,R)$ (Betti number)
$H^i(X,R), h^i(X,R)$	For any topological space X and any ring R, $H^i(X,R)$ denotes the i-th cohomology module of X with coefficients in R (see "Singular homology and cohomology"); $h^i(X,R)$ denotes its rank.

$H^i(X, \mathcal{E})$	For any sheaf of Abelian groups $\mathcal{E}$ on a topological space X, $H^i(X, \mathcal{E})$ denotes the i-th cohomology group of $\mathcal{E}$, see "Sheaves"; $h^i(X, \mathcal{E})$ denotes its rank; if E is a holomorphic vector bundle on a complex manifold X or an algebraic vector bundle on an algebraic variety X, we sometimes write $H^i(X, E)$ instead of $H^i(X, \mathcal{O}(E))$.
$\chi(X)$	For any topological space X, $\chi(X)$ denotes the Euler–Poincaré characteristic of X, i.e., the sum $\sum_{i=1,\dots,\infty}(-1)^i b_i(X, \mathbb{Z})$, when it is defined.
$\chi(X, \mathcal{E})$	For any sheaf of Abelian groups $\mathcal{E}$ on a topological space X, $\chi(X, \mathcal{E})$ denotes the Euler–Poincaré characteristic of $\mathcal{E}$, i.e., $\sum_{i=1,\dots,\infty}(-1)^i h^i(X, \mathcal{E})$, when it is defined.
K_X, ω_X	For any complex manifold or smooth algebraic variety X, K_X denotes the canonical bundle and ω_X the canonical sheaf, i.e., $\mathcal{O}(K_X)$; see "Canonical bundle, canonical sheaf".
$M_{\mathcal{F}}, \mathcal{F}_M$	See "Serre correspondence".
$M(m \times n, K)$	We denote by $M(m \times n, K)$ the space of the $m \times n$ matrices with entries in K.
nef	The abbreviation nef stands for "numerically effective"; see "Bundles, fibre -".
$NS^p(X)$	For any algebraic variety X, $NS^p(X)$ denotes the p-th Neron–Severi group of X; see "Equivalence, algebraic, rational, linear -, Chow, Neron–Severi and Picard groups".
$\mathcal{O}_X, \mathcal{O}_X(s)$	If X is a complex manifold, $\mathcal{O}_X$ (or simply $\mathcal{O}$) denotes the sheaf of holomorphic functions; if X is an algebraic variety, it denotes the sheaf of the regular functions; more generally, it denotes the structure sheaf of a ringed space; see "Space, ringed -"; for the definition of $\mathcal{O}_X(s)$, see "Hyperplane bundles, twisting sheaves".
$\mathcal{O}(E), \Omega^p(E)$	Let E be an algebraic vector bundle on an algebraic variety or a holomorphic vector bundle on a complex manifold; then $\mathcal{O}(E)$ denotes the sheaf of the (regular, resp. holomorphic) sections of E; see "Bundles, fibre -". We denote $\Omega^p \otimes \mathcal{O}(E)$ by $\Omega^p(E)$.
Ω^p	For any complex manifold X, Ω^p denotes the sheaf of the holomorphic p-forms; for any algebraic variety, Ω^p denotes the sheaf of the regular p-forms; see "Zariski tangent space, differential forms, tangent bundle, normal bundle".
$\mathbb{P}^n_K$	We denote by $\mathbb{P}^n_K$ the projective space of dimension n over K.
$p_a(X), p_g(X), P_i(X)$	The symbols $p_a(X)$, $p_g(X)$ and $P_i(X)$ denote respectively the arithmetic genus, the geometric genus and the i-th plurigenus of X (for X variety or manifold); see "Genus, arithmetic, geometric, real, virtual -", "Plurigenera".
$Pic(X)$	For any algebraic variety X, $Pic(X)$ denotes the Picard group of X; see "Equivalence, algebraic, rational, linear -, Chow, Neron–Severi and Picard groups".
PID	PID stands for "principal ideal domain", i.e., for an integral domain such that every ideal is principal.
$\pi_i(X, x)$	For any topological space X and any $x \in X$, $\pi_i(X, x)$ denotes the i-th fundamental group of X at the basepoint x; see "Fundamental group".

$q(X)$	For any complex manifold or algebraic variety X, $q(X)$ denotes the irregularity of X; see "Irregularity".
$R^q f_*$, R^q_f	See "Direct and inverse image sheaves"
$sat(I)$	For any ideal I, $sat(I)$ denotes the saturation of I; see "Saturation".
Σ_d	For any $d \in \mathbb{N}$, we denote by Σ_d the group of the permutations on d elements.
$Spec(R)$, $Proj(S)$	For any ring R, $Spec(R)$ denotes its spectrum, see "Schemes". See "Schemes" also for the definition of $Proj(S)$ for S graded ring.
$S \times_B S'$	The symbol $S \times_B S'$ denotes the fibred product of S and S' over B; see "Fibred product".
$\cdot$	The symbol $\cdot$ denotes the intersection of cycles; see "Intersection of cycles". Sometimes it is omitted.
$\sqrt{J}$	For any ideal J in a ring R, we denote by $\sqrt{J}$ the radical of J, i.e., $$\sqrt{J} = \{x \in R \mid \exists n \in \mathbb{N} - \{0\} \text{ s.t. } x^n \in J\}.$$
UV, $U+V$, $(U:V)$	Let R be a commutative ring and U and V be two ideals in R. We define UV to be the ideal $\left\{\sum_{i=1,\dots,k} u_i v_i \mid k \in \mathbb{N},\ u_i \in U,\ v_i \in V\right\}$. Moreover, we define $U+V = \{u+v \mid u \in U\ v \in V\}$, and $(U:V) := \{x \in R \mid xV \subset U\}$.
$V^\vee$	For any vector space V, we denote by $V^\vee$ its dual space.
$\sqcup$	$\sqcup$ denotes the disjoint union.
$f:(X,S) \to (Y,T)$	If $S \subset X$ and $T \subset Y$, the notation $f:(X,S) \to (Y,T)$ stands for a map $f: X \to Y$ such that $f(S) \subset T$.

Note. The end of the definitions, theorems, and propositions is indicated by the symbol $\square$.

A

Abelian varieties. See “Tori, complex - and Abelian varieties”.

Adjunction formula. ([72], [93], [107], [129], [140]). Let X be a complex manifold or a smooth algebraic variety and let Z be a submanifold, respectively a smooth closed subvariety. We have

$$K_Z = K_X|_Z \otimes det\, N_{Z,X},$$

where $N_{Z,X}$ is the normal bundle and K_X and K_Z are the canonical bundles respectively of X and Z (see “Canonical bundle, canonical sheaf”, “Zariski tangent space, differential forms, tangent bundle, normal bundle”).
If, in addition, Z is a hypersurface, the formula becomes

$$K_Z = K_X|_Z \otimes (Z)|_Z,$$

where (Z) is the bundle associated to the divisor Z, since, in this case the bundle $N_{Z,X}$ is the bundle given by Z (see “Bundles, fibre -” for the definition of bundles associated to divisors).

Albanese varieties. ([93], [163], [166]). The Albanese variety is a generalization of the Jacobian of a compact Riemann surface (see “Jacobians of compact Riemann surfaces”) for manifolds of higher dimension.
Let X be a compact Kähler manifold of dimension n (see “Hermitian and Kählerian metrics” and “Hodge theory”). The Albanese variety of X is the complex torus (see “Tori, complex - and Abelian varieties”)

$$Alb(X) := \frac{H^0(X,\Omega^1)^\vee}{j(H_1(X,\mathbb{Z}))},$$

where $j : H_1(X,\mathbb{Z}) \to H^0(X,\Omega^1)^\vee$ is defined by $j(\alpha) = \int_\alpha$ for any $\alpha \in H_1(X,\mathbb{Z})$.
The Albanese map

$$\mu : X \to Alb(X)$$

is defined in the following way: we fix a point P_0 of X (base point) and we define

$$\mu(P) = \int_{P_0}^{P}$$

for any $P \in X$, where $\int_{P_0}^{P}$ is the integral along a path joining P_0 and P (thus, obviously, it defines a linear function on $H^0(X,\Omega^1)$ only up to elements of $j(H_1(X,\mathbb{Z}))$). If we choose a basis $\omega_1,\dots,\omega_k$ of $H^0(X,\Omega^1)$ we can describe μ in the following way:

$$\mu(P) = \left(\int_{P_0}^{P}\omega_1,\dots,\int_{P_0}^{P}\omega_k\right).$$

For any compact Kähler manifold X of dimension n, the Albanese variety of X is isomorphic to the n-th intermediate Griffiths' Jacobian of X. The Albanese variety of a smooth complex projective algebraic variety is an Abelian variety, that is, it can be embedded in a projective space. See "Jacobians, Weil and Griffiths intermediate -", "Tori, complex - and Abelian varieties".

Algebras. We say that A is an algebra over a ring R if it is an R-module and a ring with unity (with the same sum) and, for all $a, b \in A$ and $r \in R$, we have

$$r(ab) = (ra)b = a(rb).$$

Algebraic groups. ([27], [126], [228], [235], and references in "Tori, complex - and Abelian varieties"). An algebraic group is a set A that is both an algebraic variety (see "Varieties, algebraic -, Zariski topology, regular and rational functions, morphisms and rational maps") and a group and the two structures are compatible, that is, the map

$$\begin{aligned} A \times A &\to A, \\ (x, y) &\mapsto xy^{-1} \end{aligned}$$

is a morphism between algebraic varieties.

Structure theorem for algebraic groups (Chevalley's theorem). Let A be an algebraic group over an algebraically closed field. Then there exists a (unique) normal affine subgroup N such that A/N is an Abelian variety. □

Definition. We say that an algebraic group is **reductive** if all its representations are completely reducible (see "Representations"). □

Almost complex manifolds, holomorphic maps, holomorphic tangent bundles. ([121], [147], [192], [251]). An **almost complex manifold** is the data of

- a C^∞ manifold M;
- a C^∞ section J of the bundle $TM_{\mathbb{R}} \otimes TM_{\mathbb{R}}^\vee$ (where $TM_{\mathbb{R}}$ is the real tangent bundle) such that, if we see J as a map

  $$J : C^\infty(M, TM_{\mathbb{R}}) \to C^\infty(M, TM_{\mathbb{R}})$$

  (where $C^\infty(M, TM_{\mathbb{R}})$ is the vector space of the C^∞ sections of $TM_{\mathbb{R}}$), we have that

  $$J^2 = -I,$$

  where I is the identity map; in other words, for every $P \in M$, the linear map $J_P : T_P M_{\mathbb{R}} \to T_P M_{\mathbb{R}}$ induced on the real tangent space of M at P is such that

  $$J_P^2 = -I.$$

We can extend J_P to $T_PM_{\mathbb{C}} := T_PM_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ by $\mathbb{C}$-linearity. We define the **holomorphic tangent space** of M at P to be the i-eigenspace of J_P; we denote it by $T_P^{1,0}M$ and we denote the holomorphic tangent bundle, i.e., the bundle whose fibre in P is $T_P^{1,0}M$, by $T^{1,0}M$. We define the **antiholomorphic tangent space** of M at P to be the $-i$-eigenspace of J_P; we denote it by $T_P^{0,1}M$ and we denote the antiholomorphic tangent bundle, i.e., the bundle whose fibre in P is $T_P^{0,1}M$, by $T^{0,1}M$. Thus

$$T_PM_{\mathbb{C}} = T_P^{1,0}M \oplus T_P^{0,1}M.$$

Obviously a **complex manifold** (see "Manifolds") is an almost complex manifold: if $\{z_\alpha = x_\alpha + iy_\alpha\}_\alpha$ are the coordinates on a coordinate open subset, we can define J by

$$J\left(\frac{\partial}{\partial x_\alpha}\right) = \frac{\partial}{\partial y_\alpha}, \qquad J\left(\frac{\partial}{\partial y_\alpha}\right) = -\frac{\partial}{\partial x_\alpha}$$

for any α. Thus, in the case of a complex manifold,

$$T_P^{1,0}M = \left\langle \left(\frac{\partial}{\partial z_\alpha}\right)_P \right\rangle_\alpha \qquad T_P^{0,1}M = \left\langle \left(\frac{\partial}{\partial \overline{z}_\alpha}\right)_P \right\rangle_\alpha,$$

where

$$\frac{\partial}{\partial z_\alpha} := \frac{\partial}{\partial x_\alpha} - i\frac{\partial}{\partial y_\alpha}, \qquad \frac{\partial}{\partial \overline{z}_\alpha} := \frac{\partial}{\partial x_\alpha} + i\frac{\partial}{\partial y_\alpha}.$$

We define $\{dx_\alpha, dy_\alpha\}$ to be the dual basis of $\frac{\partial}{\partial x_\alpha}, \frac{\partial}{\partial y_\alpha}$. Let

$$dz_\alpha := dx_\alpha + idy_\alpha$$

and

$$d\overline{z}_\alpha := dx_\alpha - idy_\alpha.$$

Observe that $dz_\alpha(\frac{\partial}{\partial z_\beta}) = 2\delta_{\alpha,\beta}$, where $\delta_{\alpha,\beta}$ is the Kronecker delta.

Remark. There is an isomorphism

$$T_PM_{\mathbb{R}} \cong T_P^{1,0}M$$

given by $x \mapsto \frac{1}{2}(x - iJ(x))$ (observe that, through such isomorphism, $\frac{\partial}{\partial x_\alpha}$ goes to $\frac{1}{2}\frac{\partial}{\partial z_\alpha}$). Analogously, there is an isomorphism

$$T_PM_{\mathbb{R}} \cong T_P^{0,1}M$$

given by $x \mapsto \frac{1}{2}(x + iJ(x))$. □

For any almost complex manifold M, let $A^{p,q}(M)$ be the space of the C^∞ (p,q)-forms on M.

Newlander–Nirenberg theorem. Let (M, J) be an almost complex manifold of (real) dimension $2n$. The almost complex structure is induced by a complex structure if and only if one of the following conditions holds:

(1) for all $A, B \in C^\infty(M, T^{1,0}M)$, we have $[A, B] \in C^\infty(M, T^{1,0}M)$ (where $[A, B]$ stands for $A \circ B - B \circ A$);
(2) for all $A, B \in C^\infty(M, T^{0,1}M)$, we have $[A, B] \in C^\infty(M, T^{0,1}M)$;
(3) for all $\alpha \in A^{0,1}(M)$, we have $d\alpha \in A^{1,1}(M) \oplus A^{0,2}(M)$ and for all $\alpha \in A^{1,0}(M)$, we have $d\alpha \in A^{1,1}(M) \oplus A^{2,0}(M)$;
(4) for all $\alpha \in A^{p,q}(M)$, we have $d\alpha \in A^{p+1,q}(M) \oplus A^{p,q+1}(M)$ for every $p, q \in \{0, \dots, n\}$;
(5) for all $X, Y \in C^\infty(M, TM_{\mathbb{R}})$, we have

$$[X, Y] + J[X, JY] + J[JX, Y] - [JX, JY] = 0$$

(the first member of the equality is called Nijenhuis tensor). □

Definition. Let M be a complex manifold. We can decompose

$$d : A^{p,q}(M) \longrightarrow A^{p+1,q}(M) \oplus A^{p,q+1}(M)$$

as

$$d = \partial + \bar{\partial},$$

where

$$\partial : A^{p,q}(M) \longrightarrow A^{p+1,q}(M),$$

$$\bar{\partial} : A^{p,q}(M) \longrightarrow A^{p,q+1}(M)$$

are the compositions of d respectively with the projections

$$A^{p+1,q}(M) \oplus A^{p,q+1}(M) \to A^{p+1,q}(M),$$

$$A^{p+1,q}(M) \oplus A^{p,q+1}(M) \to A^{p,q+1}(M).$$ □

Let $f : M \to N$ be a C^∞ map between two complex manifolds. By extending the differential

$$df_P^{\mathbb{R}} : T_P M_{\mathbb{R}} \to T_P N_{\mathbb{R}}$$

by $\mathbb{C}$-linearity, we get a map

$$df_P^{\mathbb{C}} : T_P M_{\mathbb{C}} \to T_P N_{\mathbb{C}}.$$

We say that f is **holomorphic** if one of the following equivalent conditions holds:
(i) for every component $f^j = f_1^j + i f_2^j$ of f in local coordinates of M and N, we have $\frac{\partial f_1^j}{\partial x_\alpha} = \frac{\partial f_2^j}{\partial y_\alpha}$ and $\frac{\partial f_2^j}{\partial x_\alpha} = -\frac{\partial f_1^j}{\partial y_\alpha}$;
(ii) $\frac{\partial f}{\partial \bar{z}_\alpha} = 0$ for every $\alpha = 1, \dots, dim(M)$, where $\{z_\alpha\}_\alpha$ are local coordinates of M;
(iii) $df^{\mathbb{R}} \circ J^M = J^N \circ df^{\mathbb{R}}$ (where J^M and J^N denote the operators J on M and N respectively), i.e., the differential of f is "$\mathbb{C}$-linear" for the complex structures given by J;

(iv) $df^{\mathbb{C}}(T^{1,0}M) \subseteq T^{1,0}N$:
(v) $df^{\mathbb{C}}(T^{0,1}M) \subseteq T^{0,1}N$.

Ample and very ample. See "Bundles, fibre -" (or "Divisors").

Anticanonical. See "Fano manifolds".

Arithmetically Cohen–Macaulay or arithmetically Gorenstein. See "Cohen–Macaulay, Gorenstein, (arithmetically -,-)".

Artinian. See "Noetherian (and Artinian)".

B

Base point free (b.p.f.) See "Bundles, fibre -".

Beilinson's complex. ([5], [23], [63], [207], [209], [137]).

Beilinson's theorem I. Let $\mathcal{F}$ be a coherent sheaf on $\mathbb{P}^n_{\mathbb{C}}$ (see "Coherent sheaves"). Then there exists a complex of sheaves

$$0 \to L^{-n} \xrightarrow{d_{-n}} L^{-n+1} \xrightarrow{d_{-n+1}} \cdots\cdots\cdots \xrightarrow{d_{n-2}} L^{n-1} \xrightarrow{d_{n-1}} L^n \to 0$$

with $L^k = \oplus_{i,j \text{ s.t. } i-j=k} \Omega^j(j)^{h^i(\mathcal{F}(-j))}$ such that

$$\frac{Ker\ d_k}{Im\ d_{k-1}} = \begin{cases} \mathcal{F} & \text{if} \quad k = 0, \\ 0 & \text{if} \quad k \neq 0. \end{cases}$$

Every morphism $\Omega^p(p) \to \Omega^p(p)$ induced by one of the morphisms d_k is zero. □

Beilinson's theorem II. Let $\mathcal{F}$ be a coherent sheaf on $\mathbb{P}^n_{\mathbb{C}}$. Then there exists a complex of sheaves

$$0 \to L^{-n} \xrightarrow{d_{-n}} L^{-n+1} \xrightarrow{d_{-n+1}} \cdots\cdots\cdots \xrightarrow{d_{n-2}} L^{n-1} \xrightarrow{d_{n-1}} L^n \to 0$$

with $L^k = \oplus_{i,j \text{ s.t. } i-j=k} \mathcal{O}(-j)^{h^i(\mathcal{F}\otimes\Omega^j(j))}$ such that

$$\frac{Ker\ d_k}{Im\ d_{k-1}} = \begin{cases} \mathcal{F} & \text{if} \quad k = 0, \\ 0 & \text{if} \quad k \neq 0. \end{cases}$$

Every morphism $\mathcal{O}(-p) \to \mathcal{O}(-p)$ induced by one of the morphisms d_k is zero. □

Bertini's theorem. ([93], [104], [107], [129], [136], [228]). On a smooth quasi-projective algebraic variety X over an algebraically closed field of characteristic 0, the general element of a finite-dimensional linear system (see "Linear systems") is smooth away from the base locus of the system.

Bezout's theorem.
([72], [93], [104], [107], [196]). Let K be an algebraic closed field.

Bezout's theorem. Let V and V' be two projective algebraic varieties of respective dimension m and m' in $\mathbb{P}^n_K$. Suppose that $m + m' \geq n$ and that, for every irreducible component C of $V \cap V'$ and for P general point of C, V and V' are smooth at P and $T_P(\mathbb{P}^n_K) = T_P(V) + T_P(V')$, where T_P denotes the tangent space at P. Then

$$\deg(V \cap V') = \deg(V)\ \deg(V') . \qquad \square$$

(See "Degree of an algebraic subset" for the definition of degree).

By using intersection multiplicities (see "Intersection of cycles"), we can state a stronger result: suppose that V and V' are two projective algebraic varieties of dimension m and m' in $\mathbb{P}^n_K$ with $m + m' \geq n$ and suppose they intersect properly, i.e., the codimension of every irreducible component C of $V \cap V'$ is the sum of the codimension of V and the codimension of V'; then, by using an appropriate definition of intersection multiplicity of V and V' along C, which we denote by $i_C(V, V')$, we have that

$$deg(V)\, deg(V') = \sum_C i_C(V, V')\, deg(C),$$

where the sum runs over all irreducible components C of $V \cap V'$ (see, e.g., [72]).

Bielliptic surfaces.
See "Surfaces, algebraic -".

Big.
See "Bundles, fibre -" or "Divisors".

Birational.
See "Varieties, algebraic -, Zariski topology, regular and rational functions, morphisms and rational maps".

Blowing-up (or σ-process).
([22], [93], [104], [107], [196], [228]). We follow mainly [93] and [104].

Roughly speaking, the blow-up of a manifold along a subvariety is a geometric transformation replacing the subvariety with all the directions pointing out from it. For instance the blow-up of a manifold in a point replaces the point with all the directions pointing out from it.

We define the blow-up of $\mathbb{C}^n$ in a point $P \in \mathbb{C}^n$ as follows. By changing coordinates we can suppose $P = 0$; we define the blow-up of $\mathbb{C}^n$ in 0 as the set

$$Bl_0(\mathbb{C}^n) := \{(x, l) \in \mathbb{C}^n \times \mathbb{P}^{n-1}_{\mathbb{C}} \mid x \in l\}$$

(recall that $\mathbb{P}^{n-1}_{\mathbb{C}} = \mathbb{P}(\mathbb{C}^n)$ is the set of lines of $\mathbb{C}^n$ passing through 0), with the projection

$$\pi : Bl_0(\mathbb{C}^n) \longrightarrow \mathbb{C}^n, \qquad (x, l) \longmapsto x.$$

Observe that

$$\pi^{-1}(x) = \begin{cases} \text{a point} & \text{if} \quad x \neq 0, \\ \{0\} \times \mathbb{P}^{n-1}_{\mathbb{C}} & \text{if} \quad x = 0. \end{cases}$$

Precisely, if $x \neq 0$, the set $\pi^{-1}(x)$ is $\{(x, l)\}$ where l is the unique line through x and 0. Thus we can say that $Bl_0(\mathbb{C}^n)$ is obtained from $\mathbb{C}^n$ by replacing 0 with $\mathbb{P}^{n-1}_{\mathbb{C}}$, which represents the set of the lines of $\mathbb{C}^n$ passing through 0.
We call $\pi^{-1}(0)$ the **exceptional divisor** of the blow-up, and we denote it by E.

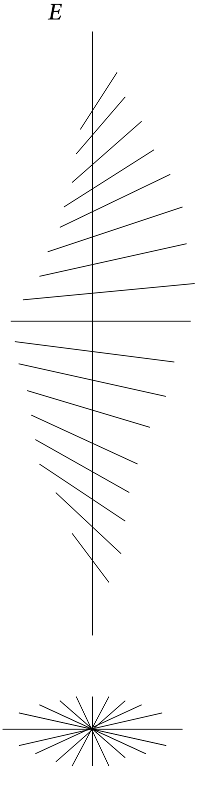


Fig. 1. Blowing up.

By using the definition of blow-up of $\mathbb{C}^n$ in a point, we can define the blow-up of any manifold in a point. Let M be a **complex manifold** and let $P \in M$. We define the blow-up of M in P as follows: let $U \cong \mathbb{C}^n$ be a neighborhood of P; the blow-up of M in P is the set

$$\tilde{M} := Bl_P(M) := \big((M - \{P\}) \sqcup Bl_P(U)\big)/\sim,$$

where $\sim$ is the relation that identifies the points in $U - \{P\}$ in $M - \{P\}$ with the points of $U - \{P\}$ in $Bl_P(U)$, with the obvious projection

$$\pi : Bl_P(M) \to M.$$

Observe that π restricted to $Bl_P(M) - \pi^{-1}(P)$ is an isomorphism onto $M - \{P\}$. Again, $E := \pi^{-1}(P)$ is called the exceptional divisor of the blow-up in P. We say that M is obtained from $\tilde{M}$ by blowing down E.

Let V be an analytic subvariety of M. The **proper or strict transform** of V in $\tilde{M}$, denoted by $\tilde{V}$, is defined by

$$\tilde{V} := \overline{\pi^{-1}(V - \{P\})} = \overline{\pi^{-1}(V) - E},$$

that is, it is the closure of $\pi^{-1}(V)$ minus the exceptional divisor.

We can define also the blow-up of a manifold along a submanifold. Let $k < n$; the blow-up of $\mathbb{C}^n$ along $\mathbb{C}^k = \{x \in \mathbb{C}^n |\ x_{k+1} = \cdots = x_n = 0\}$ is defined to be

$$Bl_{\mathbb{C}^k}(\mathbb{C}^n) = \{(x, l) \in \mathbb{C}^n \times \mathbb{P}^{n-k-1} |\ [x_{k+1} : \cdots : x_n] \in l\}$$

with the projection map

$$\pi : Bl_{\mathbb{C}^k}(\mathbb{C}^n) \longrightarrow \mathbb{C}^n,$$
$$(x, l) \longmapsto x.$$

Observe that the map π is an isomorphism from $Bl_{\mathbb{C}^k}(\mathbb{C}^n) - \pi^{-1}(\mathbb{C}^k)$ onto $\mathbb{C}^n - \mathbb{C}^k$. Let M be a complex manifold of dimension n and let V be a submanifold of dimension k. Let $\{U_\alpha\}_\alpha$ be a family of open subsets such that $U_\alpha \cong \mathbb{C}^n$ for every α, V is contained in $U := \cup_\alpha U_\alpha$ and, for every α, there exist local coordinates $x_1, \dots, x_n$ on U_α such that $V \cap U_\alpha = \{x_{k+1} = \cdots = x_n = 0\}$. Let

$$\pi_\alpha : \tilde{U}_\alpha \to U_\alpha$$

be the blow-up of U_α along $U_\alpha \cap V$. We have that $\pi_\alpha^{-1}(U_\alpha \cap U_\beta)$ and $\pi_\beta^{-1}(U_\alpha \cap U_\beta)$ are isomorphic, so we can glue the blow-ups $\pi_\alpha : \tilde{U}_\alpha \to U_\alpha$ to get a manifold $\tilde{U}$ and a map $\pi' : \tilde{U} \to U$. Let

$$\tilde{M} := Bl_V(M) := \big(\tilde{U} \sqcup (M - V)\big)/\sim$$

where $\sim$ is the equivalence relation given by the identification of the points of $U - V$ in $M - V$ and in $\tilde{U}$. We define the blow-up of M along V to be $Bl_V(M)$ with the map $\pi : Bl_V(M) \longrightarrow M$ equal to π' on $\tilde{U}$ and equal to the identity on $M - V$.

Let K be an algebraically closed field. The blow-up can be defined for **algebraic varieties** over K in the following way. Let M be an affine algebraic variety over K and V be a closed subvariety. Let $\{f_0, \dots, f_s\}$ be a set of generators of the ideal of V in M. Let

$$F : M \longrightarrow \mathbb{P}^s$$

be the rational map

$$x \longmapsto [f_0(x) : \dots : f_s(x)]$$

We define the blow-up of M along V as the graph Γ_F of F with the obvious projection $\pi : \Gamma_F \to M$ (the graph of a rational map F is defined to be the closure of the graph of $F|_U$, where U is any open subset on which F is defined). The divisor $\pi^{-1}(V)$ is called the exceptional divisor of the blow-up. If M is a projective variety, we take as $f_0, \dots, f_s$ homogeneous polynomials of the same degree generating an ideal whose saturation (see "Saturation") is the ideal of V in M, and we define the blow-up analogously.
We can prove that the definition of blow-up doesn't depend on the choice of $f_0, \dots, f_s$ and that, for any open affine subset U in M, we have that the blow-up of U along $V \cap U$ is equal to the inverse image of U in the blow-up of M along V. The map $\pi : \Gamma_F \to M$ is birational.
For example, the blow-up of $\mathbb{A}^n_K$ in 0 is

$$Bl_0(\mathbb{A}^n_K) = \{(x, l) \in \mathbb{A}^n_K \times \mathbb{P}^{n-1}_K | \ x \in l\}$$

with the projection

$$\pi : Bl_0(\mathbb{A}^n_K) \longrightarrow \mathbb{A}^n_K, \qquad (x, l) \longmapsto x.$$

The proper transform in $Bl_0(\mathbb{A}^n_K)$ of an affine algebraic variety $M \subset \mathbb{A}^n_K$ containing 0, i.e., $\overline{\pi^{-1}(M - \{0\})}$, is isomorphic to the blow-up of M in 0.

Proposition. Let M be a nonsingular algebraic variety of dimension n over K and let $P \in M$; let $\pi : \tilde{M} \to M$ be the blow-up in P and call E the exceptional divisor. Then we have

- $K_{\tilde{M}} = \pi^* K_M + (n-1)(E)$,
- $Pic(\tilde{M}) = \pi^* Pic(M) \oplus \mathbb{Z}E$,

where (E) is the line bundle associated to E (see "Canonical bundle, canonical sheaf" and "Equivalence, algebraic, rational, linear -, Chow, Neron–Severi and Picard groups" for the definitions of K_M and Pic; see "Bundles, fibre -" for the definition of the line bundle associated to a divisor; see "Pull-back and push-forward of cycles" for the definition of π^*),

- the bundle induced by E on E is the dual of the hyperplane bundle on E (see "Hyperplane bundles, twisting sheaves").

Suppose now that M is a projective surface; then we have

- $E^2 = -1$ (which is an obvious consequence of the fact that the bundle induced by E on E is the dual of the hyperplane bundle on E);
- $\pi^*(D) \cdot \pi^*(D') = D \cdot D'$ for any D, D' divisors of M;
- $E \cdot \pi^* D = 0$ for any D divisor of M. □

Blowing-ups are the fundamental building blocks in birational geometry (see "Hironaka's decomposition of birational maps", "Varieties, algebraic -, Zariski topology, regular and rational functions, morphisms and rational maps"). Furthermore, they are useful in the study of singularities (see "Regular rings, smooth points, singular points" and "Genus, arithmetic, geometric, real, virtual -").

Buchberger's algorithm. See "Groebner bases".

Bundles, fibre -. ([93], [110], [128], [135], [169], [228], [230], [237]).

Definition. A fibre bundle (bundle for short in the following) is a quadruple (E, p, X, F), where E, X, F are topological spaces,

$$p : E \to X$$

is a continuous surjective map, and there exists an open covering $\{U_\alpha\}_{\alpha \in A}$ of X and homeomorphisms

$$\varphi_{U_\alpha} : p^{-1}U_\alpha \to U_\alpha \times F$$

such that the following diagram commutes:

$$\begin{tikzcd} p^{-1}(U_\alpha) \arrow[rr, "\varphi_{U_\alpha}"] \arrow[dr, "p|_{p^{-1}(U_\alpha)}"'] & & U_\alpha \times F \arrow[dl, "\pi_1"] \\ & U_\alpha & \end{tikzcd},$$

where $\pi_1 : U_\alpha \times F \to U_\alpha$ is the projection onto the first component. We call the φ_{U_α} "trivializing homeomorphisms". □

Observe that, for all $x \in X$, the fibre $p^{-1}(x)$ is homeomorphic to F. It is generally denoted by E_x.
We say that F is the **fibre** of the bundle, X is the **base** and E is the **total space**. For the sake of brevity, we sometimes say that E is the bundle (on X) or that $p : E \to X$ is the bundle.

We say that two bundles on X, (E, p, X, F) and (E', p', X, F'), are **isomorphic** if and only if there is a homeomorphism $h : E \to E'$ such that $p' \circ h = p$.
More generally, a **morphism from a bundle on** X**,** (E, p, X, F)**, to another bundle on** X**,** (E', p', X, F'), is a continuous map $f : E \to E'$ such that $p' \circ f = p$.
We say that a bundle (E, p, X, F) is **trivial** if it is isomorphic to $(X \times F, \pi_1, X, F)$ where $\pi_1 : X \times F \to X$ is the projection onto the first component.
If (E, p, X, F) is a bundle, we say that an open subset U of X is **trivializing** if $p^{-1}(U)$ is trivial, i.e., there is a homeomorphism $\varphi_U : p^{-1}(U) \to U \times F$ such that $\pi_1 \circ \varphi_U = p$.

Example. One of the simplest examples of a nontrivial bundle is the Möbius strip, which is a bundle on the circle with a segment as fibre.

Let (E, p, X, F) be a bundle and let $\{U_\alpha\}_{\alpha \in A}$ be a trivializing open covering of X and $\varphi_{U_\alpha} : p^{-1}U_\alpha \to U_\alpha \times F$ for $\alpha \in A$ be trivializing homeomorphisms. For any $\alpha, \beta \in A$, let us consider the composition

$$(U_\alpha \cap U_\beta) \times F \xrightarrow{\varphi_{U_\beta}|^{-1}_{(U_\alpha \cap U_\beta)\times F}} p^{-1}(U_\alpha \cap U_\beta) \xrightarrow{\varphi_{U_\alpha}|_{p^{-1}(U_\alpha \cap U_\beta)}} (U_\alpha \cap U_\beta) \times F;$$

it sends (x, z) to (x, z') for some z'; we define

$$\boxed{f_{\alpha,\beta} : U_\alpha \cap U_\beta \to Homeo(F),}$$

(where $Homeo(F)$ is the set of the homeomorphisms of F) to be the map such that

$$(x, z') = (x, f_{\alpha,\beta}(x)(z))$$

for any x, z. The $f_{\alpha,\beta}$ are called the **transition functions** of the bundle E. They satisfy

$$(*) \qquad f_{\alpha,\beta}(x) \circ f_{\beta,\gamma}(x) = f_{\alpha,\gamma}(x) \quad \forall x \in U_\alpha \cap U_\beta \cap U_\gamma, \ \forall\, \alpha, \beta, \gamma \in A.$$

Conversely, given topological spaces X, F, an open covering $\{U_\alpha\}_{\alpha\in A}$ of X and functions $f_{\alpha,\beta} : U_\alpha \cap U_\beta \to Homeo(F)$ satisfying $(*)$ and such that the maps $(x, w) \mapsto (x, f_{\alpha,\beta}(x)(w))$ are homeomorphisms on $(U_\alpha \cap U_\beta) \times F$, it is easy to construct a bundle on X with fibre F and with the $f_{\alpha,\beta}$ as transition functions: define

$$E = (\sqcup_{\alpha\in A} U_\alpha \times F)/ \sim$$

where, if $(x, v) \in U_\alpha \times F$ and $(y, w) \in U_\beta \times F$, we say that $(x, v) \sim (y, w)$ if and only if $x = y$ and $v = f_{\alpha,\beta}(x)(w)$. The transition functions determine the bundle up to isomorphism.

Let (E, p, X, F) be a bundle. Let $C \subset E$ and $D \subset F$ and suppose there exist a trivializing open covering $\{U_\alpha\}_{\alpha\in A}$ for E and trivializing homeomorphisms $\varphi_{U_\alpha} : p^{-1}(U_\alpha) \longrightarrow U_\alpha \times F$ such that

$$\varphi_{U_\alpha}(p^{-1}(U_\alpha) \cap C) = U_\alpha \times D;$$

in this case we say that $(C, p|_C, X, D)$ is a **subbundle** of (E, p, X, F).

A **section** σ of a bundle (E, p, X, F) is a continuous map

$$\boxed{\sigma : X \to E}$$

such that $\sigma(x) \in E_x$ for all $x \in X$.
If $\{U_\alpha\}_{\alpha\in A}$ is a trivializing open covering for the bundle and $\varphi_{U_\alpha} : p^{-1}U_\alpha \to U_\alpha \times F$ is a trivializing homeomorphism, we can consider

$$\left(\varphi_{U_\alpha} \circ \sigma\right)|_{U_\alpha} : U_\alpha \to U_\alpha \times F.$$

It sends a point $x \in U_\alpha$ to a point whose first coordinate is x, let us say $(x, \sigma_\alpha(x))$. The maps

$$\sigma_\alpha : U_\alpha \to F$$

which we have just defined have the property that

$$\sigma_\alpha(x) = f_{\alpha,\beta}(x)(\sigma_\beta(x))$$

for any $x \in U_\alpha \cap U_\beta$. Giving σ is equivalent to giving the maps σ_α.

If (E, p, X, F) is a bundle, X' is a topological space and $f : X' \to X$ a continuous map, then the **pull-back** of E through f is (E', p', X', F) where

$$E' := X' \times_X E := \{(x', e) \in X' \times E |\ f(x') = p(e)\}$$

and $p' : E' \to X'$ is the projection onto the first factor. The pull-back bundle E' is denoted by f^*E. Observe that

$$(f^*E)_{x'} = \{(x', e)|\ p(e) = f(x')\} = E_{f(x')}.$$

We say that a bundle (E, p, X, F) is a **differentiable**, respectively **holomorphic**, if X, E, and F are differentiable manifolds, respectively complex manifolds, and the trivializing functions are diffeomorphisms, respectively biholomorphisms. In this case, we have obviously that the functions $(x, z) \mapsto f_{\alpha,\beta}(x)(z)$ are also differentiable, respectively holomorphic.

By a **section** of a differentiable, respectively holomorphic, bundle (E, p, X, F), we mean (unless otherwise specified) a differentiable, respectively holomorphic, map $\sigma : X \to E$ such that $\sigma(x) \in E_x$ for any $x \in X$.

We say that a bundle (E, p, X, F) is a **vector bundle** over a field K if F is a topological K-vector space, the fibres E_x have a structure of K-vector space and there exist $\{U_\alpha\}_{\alpha\in A}$ trivializing open covering and φ_{U_α} trivializing homeomorphisms such that the maps $\varphi_{U_\alpha}|_{E_x} : E_x \to \{x\} \times F$ are vector spaces isomorphisms.
Thus the images of the transition functions $f_{\alpha,\beta}$ are in $GL(F)$.
In this case, the dimension of F is called the **rank** of the bundle. A vector bundle of rank 1 is called a **line bundle**. A complex vector bundle is a vector bundle over the field $\mathbb{C}$.

Let (E, p, X, F) and (E', p', X, F') be two K-vector bundles on X and let $\{U_\alpha\}_\alpha$ be an open covering of X trivializing for both bundles. Let the transition functions be $f_{\alpha,\beta}$ and $f'_{\alpha,\beta}$ respectively.
The **direct sum** $E \oplus E'$ is the bundle whose fibre is $F \oplus F'$ and whose transition functions are

$$\begin{pmatrix} f_{\alpha,\beta} & 0 \\ 0 & f'_{\alpha,,\beta} \end{pmatrix}.$$

The **tensor product** $E \otimes E'$ is the bundle whose fibre is $F \otimes F'$ and whose transition functions are $f_{\alpha,\beta} \otimes f'_{\alpha,,\beta}$. The tensor product $E \otimes \cdots \cdots \otimes E$ (E repeated k times) is denoted by $E^{\otimes k}$, or, if E is a line bundle, also by E^k or kE.

The **wedge product** $\wedge^k E$ is the bundle whose fibre is $\wedge^k F$ and whose transition functions are $\wedge^k f_{\alpha,\beta}$. If $k = r$, where r is the rank of E, then $\wedge^r E$ is called **determinant bundle** and denoted by $det(E)$ (and the transition functions are obviously $det(f_{\alpha,\beta})$).
The **dual** bundle $E^\vee$ is the bundle with fibre on $x \in X$ is $E_x^\vee$ and whose transition functions are

$$x \longmapsto {}^t f_{\alpha,\beta}(x)^{-1}.$$

If E has rank 1, the bundle $E^\vee$ is denoted also by E^{-1}, since $E \otimes E^\vee$ is trivial.

Let (E, p, X, F) be a vector bundle. Let $C \subset E$ and D be a vector subspace of F and suppose there exist a trivializing open covering $\{U_\alpha\}_{\alpha \in A}$ for E and trivializing homeomorphisms $\varphi_{U_\alpha} : p^{-1}(U_\alpha) \longrightarrow U_\alpha \times F$ such that

$$\varphi_{U_\alpha}(p^{-1}(U_\alpha) \cap C) = U_\alpha \times D.$$

We say that $(C, p|_C, X, D)$ is a **(vector) subbundle** of (E, p, X, F).
Let $c_{\alpha,\beta}$ be the transition functions of C. Then the transition functions of E are

$$\begin{pmatrix} c_{\alpha,\beta} & s_{\alpha,\beta} \\ 0 & q_{\alpha,\beta} \end{pmatrix}$$

for some functions $s_{\alpha,\beta}, q_{\alpha,\beta}$. The **quotient bundle** E/C is the bundle whose fibre on x is E_x/C_x and whose transition functions are $q_{\alpha,\beta}$. We have that $E = C \oplus Q$ if and only if $s_{\alpha,\beta} = 0$ for all α, β.

A **morphism between vector bundles on X**, precisely from (E, p, X, F) to (E', p', X, F'), is a continuous map $f : E \to E'$ such that $p' \circ f = p$ and such that $f_x : E_x \to E'_x$ is linear.
Let $Ker(f) := \cup_x Ker(f_x)$ and $Im(f) := \cup_x Im(f_x)$. They determine subbundles if and only if the rank of f_x does not depend on $x \in X$.

A **frame** of a vector bundle (E, p, X, F) on an open subset U of X is a set of sections $\{\sigma_1, \dots, \sigma_k\}$ of E on U such that for all $x \in U$ the set $\{\sigma_1(x), \dots, \sigma_k(x)\}$ is a basis of E_x. Obviously, giving a frame of E on U is the same as giving a trivializing of E on U.

The **projectivized of a vector bundle** E on X, which we denote by $\mathbb{P}(E)$, is the bundle on X whose fibre on $x \in X$ is $\mathbb{P}(E_x)$ with the obvious trivializing maps.

Let G be a topological group. A bundle (E, p, X, F) is said to be a **principal G-bundle** if there is an action of G on E that preserves the fibres and acts freely and transitively on every fibre.
Let (E, p, X, G) be a principal bundle with transition functions $f_{\alpha,\beta}$. Let V be a topological space. A homomorphism $\rho : G \to Homeo(V)$ determines a bundle on X with fibre V: the bundle

$$E \times_\rho V := E \times V / \sim,$$

where
$$(e, v) \sim (eg, \rho(g^{-1})v)$$
for any $g \in G$.
Obviously, if V is a topological vector space and $\rho : G \to GL(V)$, then the determined bundle is a vector bundle on X with fibre V.

Definition. An **(algebraic) vector bundle of rank r on an algebraic variety X** over a field K is a scheme E with a surjective morphism $p : E \to X$ such that every fibre has a structure of K-vector space and there is an open cover $\{U_\alpha\}_{\alpha \in A}$ and trivializing maps $\varphi_\alpha : p^{-1}(U_\alpha) \to U_\alpha \times K^r$ such that the φ_α are isomorphisms, we have $\pi_1 \circ \varphi_\alpha = p$, where $\pi_1 : U_\alpha \times K^r \to U_\alpha$ is the projection onto the first factor, and the maps induced by the φ_α from $p^{-1}(x)$ to $\{x\} \times K^r$ are vector spaces isomorphisms. □

By a **section** of an algebraic bundle $p : E \to X$ on an algebraic variety X, we mean (unless otherwise specified) a morphism $\sigma : X \to E$ such that $\sigma(x) \in E_x$ for any $x \in X$, where E_x is the fibre $p^{-1}(x)$. In an obvious way we can define the morphisms and the isomorphisms between algebraic vector bundles.

When we speak of a vector bundle on an algebraic variety, we always mean an algebraic vector bundle.

Let X be an algebraic variety. We can **associate a sheaf to any vector bundle** E on X in the following way: let $\mathcal{O}(E)$ be the sheaf associating to any open subset U of X the set of the sections of E on U. Analogously for holomorphic vector bundles on complex manifolds. We call $\mathcal{O}(E)$ the **sheaf of sections of** E.
The map sending a vector bundle E to the sheaf $\mathcal{O}(E)$ gives a bijection between the set of vector bundles on a smooth algebraic variety X up to isomorphisms and the set of locally free sheaves of $\mathcal{O}_X$-modules of finite rank up to isomorphisms (see [228, Chapter 6] or [107, Chapter 2, Exercise 5.18]).

We recall that the group of Cartier divisors of an algebraic variety X up to linear equivalence is called **Picard group** of X and denoted by $Pic(X)$ (see "Divisors" and "Equivalence, algebraic, rational, linear -, Chow, Neron–Severi and Picard groups").

Definition. We can **associate to any Cartier divisor D of an algebraic variety X a line bundle** we denote by (D) in the following way: if the Cartier divisor D is given by local data (U_i, f_i), we define (D) to be the line bundle whose transition functions $f_{i,j}$ from U_j to U_i are f_i/f_j. □

Theorem. The map associating to a Cartier divisor D of an algebraic variety X the line bundle (D) induces an isomorphism from the Picard group $Pic(X)$ to the group of the isomorphism classes of line bundles on X, where the multiplication is the tensor product and the inverse is the dual. The group of the isomorphism classes of line bundles on X is isomorphic also to $H^1(X, \mathcal{O}^*)$, via the isomorphism induced by the map sending a bundle to its transition functions. $\square$

If X is a smooth projective algebraic variety over $\mathbb{C}$, one can prove that, for any divisor D, the first Chern class (see "Chern classes") of the line bundle associated to D is the Poincaré dual of the class of D.

Definition. Let X be a projective algebraic variety over an algebraically closed field (or a compact complex manifold) of dimension n. Let L be an algebraic line bundle on X (resp. a holomorphic line bundle on the complex manifold X); in this case we have that $H^0(X, \mathcal{O}(L))$ is finite dimensional, see for instance [107, Chapter 2, § 5] and [93, Chapter 0, § 6]. Let

$$\boxed{\varphi_L : X \longrightarrow \mathbb{P}(H^0(X, \mathcal{O}(L))^\vee),}$$

be the rational map

$$x \longmapsto \{s \in H^0(X, \mathcal{O}(L)) | \ \ s(x) = 0\};$$

defined only on the set of the points x of X such that there exists $s \in H^0(X, \mathcal{O}(L))$ with $s(x) \neq 0$.

We say that L is **very ample** if the associated map φ_L is well defined on the whole X and is an embedding. In this case we have that $\varphi_L^*(\mathcal{O}(1)) = L$, where $\mathcal{O}(1)$ is the hyperplane bundle on $\mathbb{P}(H^0(X, \mathcal{O}(L))^\vee)$ (see "Hyperplane bundles, twisting sheaves").
We say that L is **ample** if there exists $k \in \mathbb{N}$ such that $L^{\otimes k}$ is very ample.
We say that L is **numerically effective (nef)** if $deg(L|_C) \geq 0$ for any C (irreducible) curve in X, that is the intersection number $D \cdot C$ is nonnegative for any divisor D such that the associated line bundle is L and any C (irreducible) curve (see "Intersection of cycles").
Let

$$N(L) = \{k \in \mathbb{N} | \ H^0(X, \mathcal{O}(L^{\otimes k})) \neq 0\};$$

we define the **Iitaka dimension** of L to be $-\infty$ if $N(L) = \{0\}$, to be

$$\max_{k \in N(L)} \{dim \, \varphi_{L^{\otimes k}}(X)\}$$

if $N(L) \neq \{0\}$ and X is normal, to be the Iitaka dimension of $\nu^* L$ on $\tilde{X}$ if X is not normal and $\nu : \tilde{X} \to X$ is the normalization of X (see "Normal").
The line bundle L is said to be **big** if its Iitaka dimension is equal to n.

We say that a vector bundle E on X is ample if $\mathcal{O}_{\mathbb{P}(E)}(1)$ is ample on $\mathbb{P}(E)$, where $\mathcal{O}_{\mathbb{P}(E)}(1)$ is the line bundle that restricted to any fibre of $\mathbb{P}(E) \to X$ is $\mathcal{O}(1)$, i.e., the dual of the universal bundle (see "Tautological (or universal) bundle").
Let E be a vector bundle on X. We say that $V \subset H^0(X, \mathcal{O}(E))$ is **base point free (b.p.f.)** if there does not exist $x \in X$ such that $\sigma(x) = 0$ for all $\sigma \in V$. We say that E is base point free if there does not exist $x \in X$ such that $\sigma(x) = 0$ for all $\sigma \in H^0(X, \mathcal{O}(E))$.

Let E be a vector bundle on X. We say that E is **globally generated** if for all $x \in X$ we have that E_x is generated by $\cup_{\sigma \in H^0(X,\mathcal{O}(E))} \sigma(x)$.
Obviously, for E line bundle, being globally generated is equivalent to being base point free.

We say that a Cartier divisor D is ample, very ample, big, nef, b.p.f., if and only if the corresponding line bundle (D) is.

We say that a holomorphic line bundle L on a complex manifold X is **positive** if in the first Chern class $c_1(L) \in H^2(X, \mathbb{Z})$ there is a positive $(1,1)$-form (see "Positive"). □

If D is a nef Cartier divisor on a complex projective algebraic variety of dimension n, we can prove that D is big if and only if $D^n > 0$ (see "Intersection of cycles" for the definition of the intersection number D^n); see, e.g., [169].

Nakai–Moishezon theorem. ([107], [142], [190], [203]). Let X be a projective algebraic variety of dimension n over an algebraically closed field. Let D be a Cartier divisor on X. Then D is ample if and only if $D^s C > 0$ for all C irreducible subvariety in X with $dim\ C = s$ and for all $s \leq n$ (see "Intersection of cycles" for the definition of the intersection number $D^s C$). □

Thus ample implies nef. We have also that, under nice assumptions, for instance if X is a smooth projective algebraic variety over an algebraically closed field, the interior part of the cone generated by the nef divisors in the space of the divisors up to numerical equivalence is the cone generated by the ample divisors; see [142] (we say that two divisors are numerically equivalent if their intersection number with any curve is the same).

By Kodaira embedding theorem (see "Kodaira embedding theorem") a holomorphic line bundle on a compact complex manifold is positive if and only if it is ample.
Observe that, if L is a line bundle associated to a divisor D, then the positivity of L is not equivalent to the effectivity of D (and no implication is true: to show that the implication $\Rightarrow$ is false one can take a noneffective divisor of positive degree on a Riemann surface; to show that that the implication $\Leftarrow$ is false one can take the exceptional divisor E of the blow-up of a surface in a point: this is effective, but the associated line bundle (E) is not positive, because, by Nakai–Moishezon criterion, it is not ample, since E^2 is equal to -1.)

Finally one can easily prove that b.p.f. implies nef.

Summarizing, for a line bundle on a projective algebraic variety over an algebraically closed field, the following implications hold:

$$\begin{tikzcd}
& \text{very ample} \arrow[dl] \arrow[dr] & & \\
\text{b.p.f.} \arrow[dr] & & \text{ample} \arrow[dl] \arrow[dr] & \\
& \text{nef} & & \text{big}
\end{tikzcd}$$

See also “Sheaves”, “Chern classes”, “Equivalence, algebraic, rational, linear -, Chow, Neron–Severi and Picard groups”, “Cartan–Serre theorem”.

C

Calabi–Yau manifolds. ([94], [252]). There are several possible (also nonequivalent) definitions. One of the most common definitions is the following (see “Canonical bundle, canonical sheaf” and “Hermitian and Kählerian metrics” for the definitions of canonical bundle and Kähler manifold).

Definition. A Calabi–Yau manifold is a compact Kähler manifold with trivial canonical bundle. □

Another (nonequivalent, precisely weaker) definition is: a Calabi–Yau manifold is a compact Kähler manifold X such that there exists a finite (holomorphically) covering space (see “Covering projections”) of X with trivial canonical bundle.

Examples. Elliptic Riemann surfaces and K3 surfaces satisfy both definitions, while Enriques surfaces satisfy the second definition, but not the first (see “Riemann surfaces (compact -) and algebraic curves”, “Surfaces, algebraic -” for the definitions).

Canonical bundle, canonical sheaf. ([93], [107], [228]). Let X be a complex manifold. The canonical bundle K_X is the determinant bundle of the dual of the holomorphic tangent bundle $T^{1,0}X$ (see “Almost complex manifolds, holomorphic maps, holomorphic tangent bundles”). The canonical sheaf, denoted by ω_X, is the sheaf of the holomorphic sections of the canonical bundle, i.e., $\mathcal{O}(K_X)$.

If X is a smooth algebraic variety of dimension n over an algebraically closed field, we define the canonical bundle to be the determinant of dual of the tangent bundle Θ_X, i.e., to be the bundle determined by the sheaf Ω^n (see “Zariski tangent space, differential forms, tangent bundle, normal bundle”).
The associated sheaf of sections, i.e., $\mathcal{O}(K_X)$, coincides obviously with Ω^n and is called **canonical sheaf**; it is denoted also by ω_X.

Example.

$$K_{\mathbb{P}^n} = (-n-1)H,$$

where H is the hyperplane bundle on $\mathbb{P}^n$ (see "Hyperplane bundles, twisting sheaves").

See also "Dualizing sheaf".

Cap product. See "Singular homology and cohomology".

Cartan–Serre theorems. ([42], [93], [103], [107], [223]).

Serre's theorem. Let $\mathcal{F}$ be a coherent sheaf (see "Coherent sheaves") on a projective algebraic variety X over a field K. For any $k \in \mathbb{Z}$, let $\mathcal{F}(k)$ be $\mathcal{F} \otimes \mathcal{O}_X(k)$ (see "Hyperplane bundles, twisting sheaves" for the definition of $\mathcal{O}_X(k)$). Then

(i) the K-vector space $H^q(X, \mathcal{F})$ has finite dimension for any $q \in \mathbb{N}$ and has dimension 0 for every $q > dim(X)$;
(ii) there exists $l \in \mathbb{Z}$ such that, for any $k \geq l$ and for any $x \in X$, the stalk $\mathcal{F}(k)_x$ is spanned, as $\mathcal{O}_{X,x}$-module, by the elements of $H^0(X, \mathcal{F}(k))$;
(iii) there exists $l \in \mathbb{Z}$ such that $H^q(X, \mathcal{F}(k)) = 0$ for all $k \geq l$, $q > 0$. □

Statements (ii) and (iii) are often called Serre's theorem A and B, respectively.

Furthermore, in [42], Cartan and Serre proved that, for any coherent sheaf $\mathcal{F}$ on a compact complex manifold X, the complex vector space $H^q(X, \mathcal{F})$ is finite dimensional for any $q \in \mathbb{N}$.

Castelnuovo–Enriques Criterion. See "Surfaces, algebraic -".

Castelnuovo–Enriques theorem. See "Surfaces, algebraic -".

Castelnuovo–De Franchis theorem. See "Surfaces, algebraic -".

Categories. ([26], [79], [116], [168], [179]). We follow mainly the exposition in [79].

A category $\mathcal{C}$ consists of

(i) a set of **objects**;
(ii) for every ordered pair of objects, X, Y, a set denoted $Hom_{\mathcal{C}}(X, Y)$ (or simply $Hom(X, Y)$), whose elements are called **morphisms** or **arrows**;
(iii) for each triple of objects X, Y, Z, a map

$$Hom(X, Y) \times Hom(Y, Z) \to Hom(X, Z)$$

(the image of $(\varphi, \psi) \in Hom(X, Y) \times Hom(Y, Z)$ by this map is called the **composition** of φ and ψ and is denoted by $\varphi \circ \psi$ or by $\varphi\psi$) such that

(a) $Hom(X_1, Y_1)$ and $Hom(X_2, Y_2)$ are disjoint unless $X_1 = X_2$ and $Y_1 = Y_2$;
(b) the associativity of the composition holds;
(c) for each object X, there exists an arrow $1_X \in Hom(X, X)$ such that $1_X \circ f = f$ and $g \circ 1_X = g$ for any f, g arrows.

We write an arrow $f \in Hom(X, Y)$ as $f : X \to Y$.

Definition. Let $\mathcal{C}$, $\mathcal{D}$ be two categories; a **(covariant) functor** F from $\mathcal{C}$ to $\mathcal{D}$ (we will write $F : \mathcal{C} \to \mathcal{D}$) is given by a map, which we denote again by F, from the set of the objects of $\mathcal{C}$ to the set of objects of $\mathcal{D}$ and, for each ordered pair of objects X, Y of $\mathcal{C}$, a map, which we denote again by F, from $Hom_{\mathcal{C}}(X, Y)$ to $Hom_{\mathcal{D}}(F(X), F(Y))$ such that
(i) $F(\varphi\psi) = F(\varphi)F(\psi)$,
(ii) $F(1_X) = 1_{F(X)}$. □

We can define the composition of two functors in the obvious way.

Definition. Let F be a functor from a category $\mathcal{C}$ to a category $\mathcal{D}$.
We say that F is **faithful** if, for any X, Y objects of $\mathcal{C}$, we have that

$$F : Hom_{\mathcal{C}}(X, Y) \to Hom_{\mathcal{D}}(F(X), F(Y))$$

is injective.
We say that F is **full** if, for any X, Y objects of $\mathcal{C}$, we have that

$$F : Hom_{\mathcal{C}}(X, Y) \to Hom_{\mathcal{D}}(F(X), F(Y))$$

is surjective. □

Definition. Let $\mathcal{C}$ be a category. The **dual category** $\mathcal{C}^o$ is the category defined in the following way: the objects are the objects of $\mathcal{C}$ (an object X in $\mathcal{C}$ will be denoted by X^o as an object of $\mathcal{C}^o$); we define $Hom_{\mathcal{C}^o}(X^o, Y^o) := Hom_{\mathcal{C}}(Y, X)$ (an arrow $\varphi : Y \to X$ in $Hom_{\mathcal{C}}(Y, X)$ will be denoted $\varphi^o : X^o \to Y^o$ as an element of $Hom_{\mathcal{C}^o}(X^o, Y^o)$) and we define $I_{X^o} = (I_X)^o$ and $\psi^o\varphi^o = (\varphi\psi)^o$.
A **controvariant functor** from $\mathcal{C} \to \mathcal{D}$ is a covariant functor $\mathcal{C}^o \to \mathcal{D}$, thus it is given by a map from the set of the objects of $\mathcal{C}$ to the set of objects of $\mathcal{D}$ and, for each ordered pair of objects X, Y of $\mathcal{C}$, a map

$$Hom_{\mathcal{C}}(X, Y) \to Hom_{\mathcal{D}}(F(Y), F(X))\,. \qquad \square$$

Definition. We say that a category $\mathcal{C}'$ is a **subcategory** of a category $\mathcal{C}$ if
- the set of the objects of $\mathcal{C}'$ is a subset of the set of the objects of $\mathcal{C}$;
- for any X, Y objects of $\mathcal{C}'$, we have that $Hom_{\mathcal{C}'}(X, Y)$ is a subset of $Hom_{\mathcal{C}}(X, Y)$:
- the composition of the morphisms in $\mathcal{C}'$ coincides with the composition of the morphisms in $\mathcal{C}$ and, for any X object of $\mathcal{C}'$, the identity morphism I_X in $\mathcal{C}'$ coincides with the identity morphism I_X in $\mathcal{C}$. □

Definition. We say that two objects X, Y in a category $\mathcal{C}$ are **isomorphic** if and only if there are two arrows $f : X \to Y$ and $g : Y \to X$ such that $f \circ g = 1_Y$ and $g \circ f = 1_X$. □

Definition. Let $\mathcal{C}$, $\mathcal{D}$ be two categories and F and G be two functors from $\mathcal{C}$ to $\mathcal{D}$. A **morphism of functors** (called also a **natural transformation**) from F to G

$$m : F \to G$$

is a family of arrows in $\mathcal{D}$, $m(X) : F(X) \to G(X)$, one for each X object of $\mathcal{C}$, such that, for any arrow $\phi : X \to Y$ in $\mathcal{C}$, the diagram

$$\begin{tikzcd} F(X) \arrow[r, "m(X)"] \arrow[d, "F(\phi)"'] & G(X) \arrow[d, "G(\phi)"] \\ F(Y) \arrow[r, "m(Y)"] & G(Y) \end{tikzcd}$$

is commutative, i.e., $m(Y) \circ F(\phi) = G(\phi) \circ m(X)$. □

Taking the natural transformations as morphisms, we get that the functors from $\mathcal{C}$ to $\mathcal{D}$ form a new category, which we denote by $Funct(\mathcal{C}, \mathcal{D})$.
In particular two functors F, G from $\mathcal{C}$ to $\mathcal{D}$ are isomorphic if there exist a natural transformation φ from F to G and a natural transformation ψ from G to F such that $\varphi \circ \psi = I_G$ and $\psi \circ \varphi = I_F$. Equivalently, F and G are isomorphic if there exists a natural transformation φ from F to G such that $\varphi(X) : F(X) \to G(X)$ is an isomorphism for any X object of $\mathcal{C}$.

Definition. We say that two categories $\mathcal{C}$ and $\mathcal{D}$ are **isomorphic** if and only if there are functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ such that $F \circ G = I_{\mathcal{D}}$ and $G \circ F = I_{\mathcal{C}}$. □

Definition. We say that two categories $\mathcal{C}$ and $\mathcal{D}$ are **equivalent** if and only if there are functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ such that $F \circ G$ is isomorphic to $I_{\mathcal{D}}$ and $G \circ F$ is isomorphic to $I_{\mathcal{C}}$. □

Proposition. Two categories $\mathcal{C}$ and $\mathcal{D}$ are equivalent if and only if there exists a functor $F : \mathcal{C} \to \mathcal{D}$ such that

$$F : Hom_{\mathcal{C}}(X, Y) \to Hom_{\mathcal{D}}(F(X), F(Y))$$

is a bijection for any X, Y objects of $\mathcal{C}$ (that is, F is full and faithful) and, for any object Z of $\mathcal{D}$, there exists an object X of $\mathcal{C}$ such that Z and $F(X)$ are isomorphic. □

Let $\mathcal{C}$ be a category and $\mathcal{S}$ be the category of the sets (that is, the category whose objects are the sets and, for any sets S_1, S_2, $Hom(S_1, S_2)$ is the set of the functions from S_1 to S_2). Let X be an object of $\mathcal{C}$. Let $h_X : \mathcal{C} \to \mathcal{S}$ be the covariant functor defined by

$$h_X(Y) = Hom_{\mathcal{C}}(X, Y)$$

for any Y object of $\mathcal{C}$ and sending an arrow $f : Y \to Z$ to the arrow

$$Hom_{\mathcal{C}}(X, Y) \to Hom_{\mathcal{C}}(X, Z)$$

given by the composition with f. Let $h^X : \mathcal{C} \to \mathcal{S}$ be the controvariant functor defined by

$$h^X(Y) = Hom_{\mathcal{C}}(Y, X)$$

for any Y object of $\mathcal{C}$ and sending an arrow $f : Y \to Z$ to the arrow

$$Hom_{\mathcal{C}}(Z, X) \to Hom_{\mathcal{C}}(Y, X)$$

given by the composition with f. Sometimes h_X and h^X are denoted respectively by $Hom(X, -)$ and $Hom(-, X)$.

Definition. We say that a covariant functor $F : \mathcal{C} \to \mathcal{S}$ is **representable** if it is isomorphic to h_X for some X object of $\mathcal{C}$ (in this case, we say that X represents F).
We say that a controvariant functor $F : \mathcal{C} \to \mathcal{S}$ is **representable** if it is isomorphic to h^X for some X object of $\mathcal{C}$ (in this case, we say that X represents F). □

We can prove that an object representing a functor is unique up to isomorphism.

Definition. We say that a category $\mathcal{C}$ is **additive** if it satisfies the following three axioms:
- $Hom_{\mathcal{C}}(X, Y)$ is an Abelian group for any objects X and Y of $\mathcal{C}$ and the composition of arrows is bi-additive, i.e.,

$$(\varphi + \varphi') \circ \psi = \varphi \circ \psi + \varphi' \circ \psi, \qquad \varphi \circ (\psi + \psi') = \varphi \circ \psi + \varphi \circ \psi'$$

  for all morphisms $\psi, \psi', \varphi, \varphi'$.
- "Zero object". There exists an object 0 of $\mathcal{C}$ such that $Hom_{\mathcal{C}}(0, 0)$ is the zero group.
- "Direct sum". For any two objects X_1, X_2 in $\mathcal{C}$ there is an object P in $\mathcal{C}$ and arrows

$$i_i : X_i \to P, \quad p_i : P \to X_i \quad \text{for} \quad i = 1, 2$$

  such that $p_i \circ i_i = I_{X_i}$ for $i = 1, 2$, $p_2 \circ i_1 = p_1 \circ i_2 = 0$ and

$$i_1 \circ p_1 + i_2 \circ p_2 = I_P\,.$$

□

Definition. Let $\mathcal{C}$ and $\mathcal{D}$ be two additive categories. We say that a functor F from $\mathcal{C}$ to $\mathcal{D}$ is **additive** if, for any X, Y, objects of $\mathcal{C}$, the map

$$F : Hom_{\mathcal{C}}(X, Y) \to Hom_{\mathcal{D}}(F(X), F(Y))$$

is a homomorphism of Abelian groups. □

Let Ab be the category of Abelian groups, that is, the category whose objects are the Abelian groups and, for any Abelian groups G_1, G_2, $Hom(G_1, G_2)$ is the set of the homomorphisms from G_1 to G_2. Let $\mathcal{C}$ be an additive category and let $\varphi : X \to Y$ be an arrow in $\mathcal{C}$. We define a controvariant functor

$$Ker\,\varphi : \mathcal{C} \to Ab,$$

in the following way: for any Z object of $\mathcal{C}$, we define

$$(Ker\ \varphi)(Z) = Ker(Hom_{\mathcal{C}}(Z, X) \to Hom_{\mathcal{C}}(Z, Y)),$$

where $Hom_{\mathcal{C}}(Z, X) \to Hom_{\mathcal{C}}(Z, Y)$ is the map given by the composition with φ, and, for any $f : Z \to R$ arrow in $\mathcal{C}$, let $(Ker\ \varphi)(f)$ be the morphism $(Ker\ \varphi)(R) \to (Ker\ \varphi)(Z)$, i.e.,

$$Ker(Hom_{\mathcal{C}}(R, X) \to Hom_{\mathcal{C}}(R, Y)) \longrightarrow Ker(Hom_{\mathcal{C}}(Z, X) \to Hom_{\mathcal{C}}(Z, Y)),$$

given by the composition with f. If the functor $Ker\ \varphi$ is represented by an object K, one can prove easily that there is a morphism $k : K \to X$ such that $\varphi \circ k = 0$. The morphism k, or the pair (K, k), or K, is called the **kernel** of φ.
The **cokernel** of a morphism $\varphi : X \to Y$ is a morphism $c : Y \to C$ such that, for all Z object in $\mathcal{C}$, the following sequence (where the maps are induced by c and φ) is an exact sequence of groups:

$$0 \to Hom(C, Z) \to Hom(Y, Z) \to Hom(X, Z).$$

Again, sometimes we call a cokernel c or C.

Definition. We say that a category $\mathcal{C}$ is **Abelian** if it is additive and it satisfies the following axiom:
"Ker and Coker". For any arrow $\varphi : X \to Y$ in $\mathcal{C}$, there exists a sequence

$$K \xrightarrow{k} X \xrightarrow{i} I \xrightarrow{j} Y \xrightarrow{c} C$$

such that
(i) $j \circ i = \varphi$;
(ii) K is the kernel of φ, C is the cokernel of φ;
(iii) I is the kernel of c and the cokernel of k. □

Chern classes. ([45], [93], [96], [107], [135], [146], [147], [181],[188]). Let E be a holomorphic vector bundle of rank r on a complex compact manifold X. The Chern classes

$$c_i(E) \in H^{2i}(X, \mathbb{Z})$$

for $i = 1, \dots, r$ and the total Chern class

$$c(E) = 1 + c_1(E) + \cdots + c_r(E) \in \oplus_{i=0,\dots,r} H^{2i}(X, \mathbb{Z})$$

(where $1 \in H^0(X, \mathbb{Z}) \cong \mathbb{Z}$) are defined by the following three axioms:

Axiom 1: Normalization. if $r = 1$, c_1 is the map $H^1(X, \mathcal{O}^*) \to H^2(X, \mathbb{Z})$ induced by the exponential sequence on X,

$$0 \to \mathbb{Z} \to \mathcal{O} \to \mathcal{O}^* \to 0$$

(the first map is given by the inclusion and the second by $f \mapsto e^{2\pi i f}$; see “Exponential sequence”).

Axiom 2: Multiplicativity. if $0 \to F \to E \to G \to 0$ is an exact sequence of vector bundles on X, then

$$c(E) = c(G)c(F),$$

where the product at the second member is the cup product, see “Singular homology and cohomology”.

Axiom 3: Functoriality. if $f : X \to Y$ is a continuous map and E a vector bundle on Y, then

$$c(f^* E) = f^* c(E).$$

We define the **Chern polynomial** of E to be

$$c(E)(t) = 1 + c_1(E)t + \cdots + c_r(E)t^r.$$

Leray–Hirsch theorem. Let $\pi : \mathbb{P}(E) \to X$ be the projectivized bundle of E. Let U be the tautological bundle on $\mathbb{P}(E)$, i.e., the subbundle of $\pi^* E$ whose fibre on a point of $\mathbb{P}(E)$ is the line represented by this point (see “Tautological (or universal) bundle”)

$$\begin{tikzcd} U \subset \pi^*(E) \arrow[d] & E \arrow[d] \\ \mathbb{P}(E) \arrow[r, "\pi"] & X \end{tikzcd}$$

We have that $H^*(\mathbb{P}(E), \mathbb{Z})$ is a free $H^*(X, \mathbb{Z})$-module generated by $1, \zeta, \ldots, \zeta^{r-1}$, where $\zeta = c_1(U^{-1})$. □

Theorem. If $\pi : \mathbb{P}(E) \to X$ is the projectivized bundle of E and $\zeta = c_1(U^{-1})$, where U is the tautological bundle on $\mathbb{P}(E)$, we have

$$\zeta^r + \sum_i \pi^*(c_i(E))\zeta^{r-i} = 0\,.$$ □

The theorem above is sometimes used (once c_0 and c_1 are defined) to define the Chern classes (to be the unique elements satisfying the equation above).

Some useful formulas about Chern classes are:

(i) For any i

$$c_i(E^\vee) = (-1)^i c_i(E).$$

(ii) Let E and F be holomorphic vector bundles on X. Let

$$c(E)(t) = \prod_i (1 + x_i t), \qquad c(F)(t) = \prod_i (1 + y_i t)$$

be the “formal factorizations” of the Chern polynomials of E and F. Then

$$c(E \otimes F)(t) = \prod_{i,j}(1 + (x_i + y_j)t).$$

From (ii) we easily get that

$$c_1(E \otimes F) = rk(F)c_1(E) + rk(E)c_1(F).$$

Moreover, if $rk(E) = r$ and $rk(L) = 1$,

$$c_i(E \otimes L) = \sum_{j=0,\dots,i} \binom{r-j}{i-j} c_j(E)c_1(L)^{i-j}.$$

In particular, if E is a bundle on the projective space and $rk(E) = r$, then

$$c_1(E(k)) = c_1(E) + k\,r,$$

where $E(k) = E \otimes H^k$, where H is the hyperplane bundle (see “Hyperplane bundles, twisting sheaves”).

The definition of Chern classes can be given, more generally, for complex bundles on compact C^∞ manifolds in a way analogous to the definition above (the only difference being in the normalization axiom). We skip it, but we mention other ways to define Chern classes.
If E is a complex vector bundle on a compact C^∞-manifold and ∇ is a connection on E (see “Connections”), we can define for any k

$$c_k(E) = tr\left(\wedge^k\left(\frac{R}{2\pi i}\right)\right),$$

where R is the curvature of ∇, tr is the trace, and i is the imaginary unit.

Another way is the following: if the rank of E is r, we define
– $c_r(E)$ to be the Poincaré dual of the zero locus of a general C^∞ section s of E;
– $c_{r-1}(E)$ to be the Poincaré dual of the zero locus of $s_1 \wedge s_2$ where s_1, s_2 are general C^∞ section of E;
– more generally, $c_i(E)$ to be the Poincaré dual of the zero locus of $s_1 \wedge \cdots \wedge s_{r-i+1}$, where $s_1, \dots, s_{r-i+1}$ are general C^∞ sections of E.

Finally we want to mention the definition of Chern character.
Let

$$c(E)(t) = \prod_i (1 + x_i t)$$

be the “formal factorization” of the Chern polynomial $c(E)(t)$ of a complex bundle E; the x_i are called the Chern roots. Observe that, if E splits, i.e.,

$$E = L_1 \oplus \cdots \oplus L_r$$

with the L_i line bundles, the x_i are the $c_1(L_i)$. We define the **Chern character** $ch(E) = \sum_k ch_k(E)$ of E by

$$ch(E) = \sum_{i=1,\dots,r} \exp(x_i) = \sum_{i=1,\dots,r} \left(1 + x_i + \frac{x_i^2}{2!} + \frac{x_i^3}{3!} + \cdots\cdots\right).$$

One can easily see that

$$ch(E) = r + c_1(E) + \frac{1}{2}\left[c_1(E)^2 - 2c_2(E)\right] + \frac{1}{6}\left[c_1(E)^3 - 3c_1(E)c_2(E) + 3c_3(E)\right] + \cdots .$$

We have that

$$ch(E \oplus F) = ch(E) + ch(F),$$
$$ch(E \otimes F) = ch(E)ch(F).$$

Chow's group. See "Equivalence, algebraic, rational, linear -, Chow, Neron–Severi and Picard groups".

Chow's theorem. ([93], [103], [196]). Any analytic subvariety in $\mathbb{P}^n_{\mathbb{C}}$ is algebraic, i.e., it is the zero locus of a finite number of homogeneous polynomials (see "Varieties and subvarieties, analytic -", "Varieties, algebraic -, Zariski topology, regular and rational functions, morphisms and rational maps"). □
See "G.A.G.A."

Class group, divisor -. See "Equivalence, algebraic, rational, linear -, Chow, Neron–Severi and Picard groups".

Clifford's Index and Clifford's theorem. See "Riemann surfaces (compact -) and algebraic curves".

Cohen–Macaulay, Gorenstein, (arithmetically -,-). ([20], [39], [62], [159], [186], [187]). Let (R, m) be a local Noetherian ring. We say that a finitely generated R-module M is Cohen–Macaulay if

$$dim(M) = depth(M)$$

(see "Local", "Noetherian, Artinian", "Depth" and "Dimension").
A ring R is Cohen-Macaulay if it is Cohen-Macaulay as module on itself.

Let K be an algebraically closed field.

(1) We say that an algebraic set X over K is **Cohen–Macaulay** if, for any $P \in X$, the local ring $\mathcal{O}_{X,P}$ (the stalk in P of the sheaf of the regular functions on X) is Cohen–Macaulay.

One can prove that a Cohen–Macaulay algebraic set is equidimensional: for instance, the union of a line and a plane meeting in a point is not Cohen–Macaulay.
Hartshorne's connectedness theorem states that a Cohen–Macaulay algebraic set must be locally connected in codimension 1, i.e., removing a subvariety of codimension ≥ 2 cannot disconnect it.
For instance two surfaces meeting in a point in a space of dimension 4 cannot be Cohen–Macaulay.

Example of Cohen–Macaulay algebraic sets: (locally) complete intersections.

(2) An equidimensional projective algebraic set X of codimension c in $\mathbb{P}^n_K$ is said **arithmetically Cohen–Macaulay** if the minimal free resolution (see "Minimal free resolutions") of the sheaf $\mathcal{O}_X$ has length c (which is the minimal possible length):

$$0 \to \mathcal{F}_c \to \mathcal{F}_{c-1} \to \cdots \to \mathcal{F}_1 \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_X \to 0.$$

Equivalently, if we define $S = K[x_0, \ldots, x_n]$ and $I = I(X)$ is the ideal associated to X, the minimal free resolution of the projective coordinate ring S/I, as module over S, has length c:

$$0 \to F_c \to F_{c-1} \to \cdots \to F_1 \to S \to S/I \to 0.$$

This is equivalent to the condition

$$pd(S/I) = c,$$

where the pd stands for "projective dimension" and the projective dimension of an R-module M is the minimal length of a projective resolution of M (a projective resolution of M is an exact sequence

$$0 \to P_k \to \cdots \to P_0 \to M \to 0$$

with P_i projective R-modules; see "Injective and projective modules" and "Injective and projective resolutions"; k is the length of the resolution).
The condition $pd(S/I) = c$ is obviously equivalent to the condition

$$pd(S/I) = n - dim(X) = depth(S) - dim(S/I),$$

that is,

$$depth(S) - pd(S/I) = dim(S/I),$$

which is equivalent, by the Auslander–Buchsbaum theorem (see "Depth"), to

$$depth(S/I) = dim(S/I).$$

Thus X is arithmetically Cohen–Macaulay if and only if the coordinate ring S/I, is Cohen–Macaulay as S-module.

The rank of $\mathcal{F}_c$ is called the Cohen–Macaulay type of X.

One can show that, for X of dimension $m \geq 1$, being arithmetically Cohen–Macaulay is also equivalent to the condition

$$H^i(\mathcal{I}(t)) = 0$$

for $1 \leq i \leq m$ and for all $t \in \mathbb{Z}$, where $\mathcal{I}$ is the ideal sheaf of X.

Proposition. Arithmetically Cohen–Macaulay ⇒ Cohen–Macaulay. □

(3) An equidimensional projective algebraic set X of codimension c in $\mathbb{P}^n_K$ is said to be **arithmetically Gorenstein** if it is arithmetically Cohen–Macaulay of type 1, i.e., the minimal free resolution of the sheaf $\mathcal{O}_X$ has length c

$$0 \to \mathcal{F}_c \to \mathcal{F}_{c-1} \to \cdots \to \mathcal{F}_1 \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_X \to 0$$

and the rank of $\mathcal{F}_c$ is 1.

Remark. (See [186].) If X is arithmetically Cohen–Macaulay of codimension c in $\mathbb{P}^n_K$ and

$$0 \to F_c \to F_{c-1} \to \cdots \to F_1 \to S \to S/I \to 0$$

is a minimal free resolution of S/I, then

$$0 \to S \to F_1^\vee \to \cdots \to F_c^\vee \to Ext^c_S(S/I, S) \to 0$$

is a minimal free resolution of $Ext^c_S(S/I, S)$.
The module $K_X := Ext^c_S(S/I, S)(-n-1)$ is called the canonical module. □

Proposition. Let X be an a.C.M. algebraic subset of $\mathbb{P}^n_K$. We have: X is arithmetically Gorenstein ⇔ $S/I = K_X(l)$ for some l ⇔ the minimal free resolution of S/I is selfdual up to twist by $n+1$. □

(4) We say that an equidimensional projective algebraic subset X of $\mathbb{P}^n_K$ of codimension c is **Gorestein** if it is Cohen–Macaulay and its dualizing sheaf, that is $\mathcal{EXT}^c(\mathcal{O}_X, \mathcal{O}_{\mathbb{P}^n_K})(-n-1)$ (see "Dualizing sheaf", "*Ext*, $\mathcal{EXT}$") is locally free of rank 1.

Remark. Strict complete intersection ⇒ arithmetically Gorenstein ⇒ arithmetically Cohen–Macaulay (the first implication is due to the fact that, if X is a strict complete intersection, then its minimal resolution is the Koszul complex). □

Coherent sheaves. ([93], [107], [129], [146], [223]).

The most common definition of coherent sheaf is the following.

Definition. Let $(X, \mathcal{O}_X)$ be a ringed space (see "Space , ringed -").

– We say that a sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules (see "Sheaves") is **of finite type** if for every point $x \in X$ there is an open neighborhood U of x and a surjective morphism of sheaves of $\mathcal{O}_X$-modules $\mathcal{O}^n_X|_U \to \mathcal{F}|_U$ for some $n \in \mathbb{N}$.

– We say that a sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules is **coherent** if the following two properties hold:
  (i) $\mathcal{F}$ is of finite type,
  (ii) for any open subset U of X, any $n \in \mathbb{N}$, and any morphism of sheaves of $\mathcal{O}_X$-modules $\varphi : \mathcal{O}_X^n|_U \to \mathcal{F}|_U$, we have that $Ker(\varphi)$ is of finite type.
– We say that a sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ is **quasi-coherent** if, for every point $x \in X$, there is an open neighborhood U of x and an exact sequence

$$\mathcal{O}_X^I|_U \to \mathcal{O}_X^J|_U \to \mathcal{F}|_U \to 0,$$

where $\mathcal{O}_X^I$ and $\mathcal{O}_X^J$ are direct sum of (possibly infinite) copies of $\mathcal{O}_X$. □

Theorem. Let $(X, \mathcal{O}_X)$ be a ringed space.
(i) Let $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ be an exact sequence of sheaves of $\mathcal{O}_X$-modules. If two of the $\mathcal{F}_i$ are coherent, so is the third.
(ii) The kernel, the image and the cokernel of a morphism of sheaves of $\mathcal{O}_X$-modules between two coherent sheaves are coherent.
(iii) Let $\mathcal{F}$ and $\mathcal{G}$ be two coherent sheaves on X. Then $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ and $\mathcal{HOM}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ are coherent. □

In some texts (for instance [107]), the definition of coherent sheaf on a scheme (see "Schemes") is the following.

Definition. Let R be a commutative ring with unity and M an R-module. We define $\mathcal{F}_M$ to be the following sheaf on $Spec(R)$: for any U open subset of $Spec(R)$, let $\mathcal{F}_M(U)$ be the set of the functions

$$s : U \to \sqcup_{p \in U} M_p$$

(where M_p is the localization of M in p, see "Localization, quotient ring, quotient field") such that $s(p) \in M_p$ for every $p \in U$ and s is locally a fraction, that is, for all $p \in U$, there exists a neighborhood V of p in U and $m \in M$, $r \in R$ such that for all $p' \in V$, we have $r \notin p'$ and $s(p') = \frac{m}{r}$. Consider the obvious restriction maps.
Let $(X, \mathcal{O}_X)$ be a scheme and $\mathcal{F}$ a sheaf of $\mathcal{O}_X$-modules. We say $\mathcal{F}$ is a **coherent** (respectively **quasi-coherent**) sheaf if there is an open affine covering of X, $\{U_i = Spec(R_i)\}_i$, and finitely generated (respectively not necessarily finitely generated) R_i-modules M_i such that $\mathcal{F}|_{U_i} = \mathcal{F}_{M_i}$. □

The two definitions of coherent sheaf agree on Noetherian schemes (see, e.g., [129, Theorem 1.13] for a proof).

By Serre's G.A.G.A. theorem (see "G.A.G.A.") there is a functor from the category of schemes of finite type over $\mathbb{C}$ to the category of analytic spaces (see "Spaces analytic -") and, given a projective scheme X over $\mathbb{C}$, the functor induces an equivalence of categories from the category of coherent sheaves on X to the category of coherent sheaves on the analytic space associated to X and this equivalence maintains the cohomology.

See also "Cartan–Serre theorems".

Cohomology of a complex. See "Complexes".

Cohomology, singular -. See "Singular homology and cohomology".

Complete intersections. ([104], [107], [228]). We say that a projective algebraic variety X of dimension m in $\mathbb{P}^n$ is a **strict complete intersection** if there exist $n-m$ elements of the ideal of X that generate the ideal of X; we say that X is **set-theoretically complete intersection** if it is the intersections of $n-m$ hypersurfaces.

Complete varieties. ([107], [129], [140], [197], [201], [202], [228]).

Definition. An algebraic variety X is said to be complete if, for all algebraic varieties Y, the projection morphism $\pi : X \times Y \to Y$ is a closed map, i.e., the image through π of any closed subset (closed in the Zariski topology) is a closed subset. □

Proposition. An algebraic variety X over $\mathbb{C}$ is complete if and only if it is compact with the usual topology. □

Proposition. A projective algebraic variety X over an algebraically closed field is complete. □

There exist complete nonprojective algebraic varieties; see [201].

Chow's lemma. For any complete variety X over an algebraically closed field, there exists a projective algebraic variety Y and a surjective birational morphism from Y to X. □

Completion. ([12], [62], [107], [185], [256]). Let $(G, +)$ be a topological Abelian group with a sequence of subgroups, $\{G_i\}_{i\in\mathbb{N}}$, such that

$$G = G_0 \supseteq G_1 \supseteq \cdots \supseteq G_n \supseteq \cdots$$

and $\{G_i\}_{i\in\mathbb{N}}$ is a fundamental system of neighborhoods of 0.
We say that a sequence $(x_n)_{n\in\mathbb{N}}$ in G is a Cauchy sequence if, for any U neighborhood of 0, there exists $k \in \mathbb{N}$ such that

$$x_n - x_m \in U \quad \forall\, n, m \geq k.$$

We say that two Cauchy sequences, $(x_n)_{n\in\mathbb{N}}$ and $(x'_n)_{n\in\mathbb{N}}$, are equivalent if

$$\lim_{n\to\infty} (x_n - x'_n) = 0.$$

The completion of G, denoted by $\hat{G}$, is defined to be the set of the equivalence classes of the Cauchy sequences in G.

The map $G \to \hat{G}$ sending $x \in G$ to the constant sequence $x_n = x$ for all n is injective if and only if G is Hausdorff. If the map $G \to \hat{G}$ is an isomorphism, we say that G is complete. We can prove that a completion is complete.

The completion of G can be also defined by using the inverse limit (see "Limits, Direct and inverse"): we can define

$$\hat{G} := \varprojlim G/G_n.$$

Example. Let $G = \mathbb{Z}$ and $G_n = (p^n)$ for every $n \in \mathbb{N}$, where p is a prime number. Take $\{G_n\}_{n\in\mathbb{N}}$ as fundamental system of neighborhoods of 0; the topology we get is called p-adic and the ring $\hat{G}$ is called ring of the p-adic integers.
More generally, let G be a ring, I be an ideal and define $G_n = I^n$ for any $n \in \mathbb{N}$ (define $I^0 = G$) . The topology we get on G by taking $\{G_n\}_{n\in\mathbb{N}}$ as a fundamental system of neighborhoods of 0 is called I-adic.

In algebraic geometry one often meets completions of local rings (R, m) (see "Local"); in this case we consider $\{m^n\}_{n\in\mathbb{N}}$ as fundamental system of neighborhoods of 0. Precisely, for any algebraic variety X and any $P \in X$, the completion of the local ring $(\mathcal{O}_{X,P}, m_P)$ gives information about the local behavior of X around P and the study of the completions of the local rings $(\mathcal{O}_{X,P}, m_P)$ for $P \in X$ is linked to the study of singularities of X.
If X and Y are two algebraic varieties over a field K and $P \in X$ and $Q \in Y$, we say that P in X and Q in Y are **analytically isomorphic** if and only if the completions of $\mathcal{O}_{X,P}$ and $\mathcal{O}_{Y,Q}$ are isomorphic as K-algebras.

Cohen's theorem. Let (R, m) be a complete regular local Noetherian ring of dimension d (see "Noetherian, Artinian" and "Dimension") containing some field and let K be its residue field, i.e., R/m. Then R is isomorphic to the ring of the formal powers series in d variables over K, usually denoted by $K[[x_1, \dots, x_d]]$.
More generally, if (R, m) is a complete local Noetherian ring containing some field and K is its residue field, then R is isomorphic to $K[[x_1, \dots, x_d]]/I$ for some d and some ideal I. □

In particular, if P is a smooth point of an algebraic variety X of dimension d over a field K, then the completion of $\mathcal{O}_{X,P}$ is isomorphic to $K[[x_1, \dots, x_d]]$.

See "Regular rings, smooth points, singular points".

Complexes. Let R be a ring. A complex of R-modules, which is usually written

$$\cdots \xrightarrow{f_{i-2}} M_{i-1} \xrightarrow{f_{i-1}} M_i \xrightarrow{f_i} M_{i+1} \xrightarrow{f_{i+1}} \cdots,$$

is the datum of a sequence of R-modules M_i and R-homomorphisms $f_i : M_i \to M_{i+1}$ such that $f_{i+1} \circ f_i = 0$ for any i.

The **cohomology** of the complex M^* above in degree i, usually denoted by $H^i(M^*)$, is defined as

$$H^i(M^*) := \frac{Ker\ (f_i : M_i \to M_{i+1})}{Im\ (f_{i-1} : M_{i-1} \to M_i)}.$$

We call it "**homology**", instead of cohomology, if the indices of the R-modules are decreasing (instead of increasing).

A **morphism** from a complex M_* of R-modules

$$\cdots \xrightarrow{f_{i-2}} M_{i-1} \xrightarrow{f_{i-1}} M_i \xrightarrow{f_i} M_{i+1} \xrightarrow{f_{i+1}} \cdots$$

to another complex N_* of R-modules

$$\cdots \xrightarrow{g_{i-2}} N_{i-1} \xrightarrow{g_{i-1}} N_i \xrightarrow{g_i} N_{i+1} \xrightarrow{g_{i+1}} \cdots$$

is the datum of a sequence of R-homomorphism $r_i : M_i \to N_i$ such that the following diagram commutes

$$\begin{tikzcd}
\cdots \arrow[r, "f_{i-2}"] & M_{i-1} \arrow[r, "f_{i-1}"] \arrow[d, "r_{i-1}"] & M_i \arrow[r, "f_i"] \arrow[d, "r_i"] & M_{i+1} \arrow[r, "f_{i+1}"] \arrow[d, "r_{i+1}"] & \cdots \\
\cdots \arrow[r, "g_{i-2}"] & N_{i-1} \arrow[r, "g_{i-1}"] & N_i \arrow[r, "g_i"] & N_{i+1} \arrow[r, "g_{i+1}"] & \cdots
\end{tikzcd}.$$

We can see easily that it induces homomorphisms from $H^i(M^*)$ to $H^i(N^*)$ for any i.
We say that two morphisms $r = (r_i)$ and $s = (s_i)$ from M_* to N_* are **homotopically equivalent** if there are R-homomorphisms $k_i : M_i \to N_{i-1}$ such that

$$r_i - s_i = k_{i+1} \circ f_i + g_{i-1} \circ k_i.$$

$$\begin{tikzcd}
\cdots \arrow[r, "f_{i-2}"] & M_{i-1} \arrow[r, "f_{i-1}"] \arrow[d] & M_i \arrow[r, "f_i"] \arrow[d, "r_i - s_i"] \arrow[dl, "k_i"] & M_{i+1} \arrow[r, "f_{i+1}"] \arrow[d] \arrow[dl, "k_{i+1}"] & \cdots \\
\cdots \arrow[r, "g_{i-2}"'] & N_{i-1} \arrow[r, "g_{i-1}"'] & N_i \arrow[r, "g_i"'] & N_{i+1} \arrow[r, "g_{i+1}"] & \cdots
\end{tikzcd}.$$

Homotopically equivalent morphisms induce the same homomorphisms in cohomology.
In an analogous way we can define complexes of sheaves, etc.
See also "Exact sequences".

Cone, tangent -. ([104], [228]). Let X be an affine algebraic variety over an algebraically closed field and let $I = I(X)$ be the ideal of X (see "Varieties, algebraic -, Zariski topology, regular and rational functions, morphisms and rational maps"). Let

$P \in X$. Choose coordinates such that $P = 0$. For every $f \in I$, let $In(f)$ be the sum of the monomials of f of the lowest degree. Let

$$In(I) := \{In(f) | \, f \in I\}.$$

The tangent cone to X at $P = 0$ is defined to be the zero locus of $In(I)$.

Example. Let I be generated by $x^2 - y^2 + x^3$. The ideal $In(I)$ is generated by $x^2 - y^2$; see Figure 2.

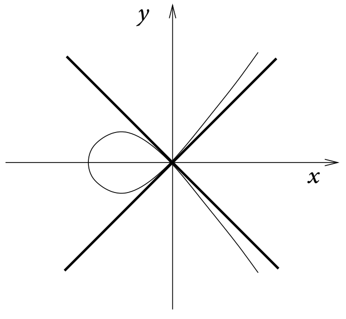


Fig. 2. The tangent cone.

Connections. ([46], [93], [135], [146], [188], [251]). Let E be a C^∞ vector bundle on a C^∞ manifold X. A connection on E is a map

$$\boxed{\nabla : A(E) \to A(E \otimes TX^\vee)}$$

(where, for any bundle F, $A(F)$ denotes the set of C^∞ sections of F and $TX^\vee$ denotes the dual of the tangent bundle) such that

$$\begin{aligned} \nabla(s_1 + s_2) &= \nabla s_1 + \nabla s_2 \quad && \forall s_1, s_2 \in A(E), \\ \nabla(fs) &= s \otimes df + f\nabla s \quad && \forall s \in A(E), \; \forall f : X \to X \;\; C^\infty. \end{aligned}$$

Remark. Let $e_1, \dots, e_r$ be a frame for E on an open subset U of X, i.e., let $e_1, \dots, e_r$ be sections of $E|_U$ such that, for all $x \in U$, $\{e_1(x), \dots, e_r(x)\}$ is a basis of the fibre E_x. We define the **connection matrix** θ of ∇ with respect to $e_1, \dots, e_r$ in the following way:

$$\nabla e_i = \sum_{j=1,\dots,r} e_j \otimes \theta_{i,j}$$

(the entries of θ are 1-forms). Then

$$\begin{aligned} \nabla(\textstyle\sum_i s_i e_i) &= \textstyle\sum_j e_j \otimes ds_j + \sum_i s_i \nabla e_i \\ &= \textstyle\sum_j e_j \otimes ds_j + \sum_i s_i \sum_j e_j \otimes \theta_{i,j} = \sum_j e_j \otimes (ds_j + \sum_i s_i \theta_{i,j}), \end{aligned}$$

which can be written for short as

$$ds + \theta s,$$

where

$$s = \begin{pmatrix} s_1 \\ \vdots \\ s_r \end{pmatrix}. \qquad \square$$

Let $A^p(E) := A(E \otimes \wedge^p TX^\vee)$. Given a connection on E, $\nabla : A^0(E) \to A^1(E)$, we can define a map, which is usually called again ∇,

$$\boxed{\nabla : A^p(E) \to A^{p+1}(E)}$$

in the following way:

$$\nabla(s \otimes \omega) = s \otimes d\omega + \nabla s \wedge \omega$$

for any $s \in A(E)$, $\omega \in A(\wedge^p TX^\vee)$. By composing $\nabla : A^0(E) \to A^1(E)$ with $\nabla : A^1(E) \to A^2(E)$ we get a map

$$\boxed{D : A^0(E) \to A^2(E),}$$

called **curvature** of the connection. It satisfies

$$D(fs) = f\ D(s)$$

for all $s \in A(E)$ and for all C^∞ maps $f : X \to X$. If θ is the connection matrix of ∇ with respect to a frame on an open subset U of X, we have that D is given by the matrix of 2-forms

$$R = d\theta + \theta \wedge \theta,$$

where $\theta \wedge \theta$ denotes the usual matrix product where the entries are multiplied by the wedge product.

Remark. Suppose $\{e_1, \dots, e_r\}$ and $\{e'_1, \dots, e'_r\}$ are two frames on an open subset U of X for the vector bundle E and, if $\sum s_i e_i = \sum s'_i e'_i$, then

$$\begin{pmatrix} s'_1 \\ \vdots \\ s'_r \end{pmatrix} = g \begin{pmatrix} s_1 \\ \vdots \\ s_r \end{pmatrix}.$$

Then, if θ and θ' are the connection matrices of ∇ with respect the two frames and R and R' are the matrices of the curvatures, we have that

$$\theta = g^{-1} dg + g^{-1} \theta' g,$$
$$R = g^{-1} R' g. \qquad \square$$

Proposition. Let ∇ be a connection on a vector bundle E on X and D be its curvature. Let $v, w \in TX$. Then

$$D_{v,w} = \nabla_v \nabla_w - \nabla_w \nabla_v - \nabla_{[v,w]},$$

where the subscript in an operator means that, after applying the operator, we evaluate the result in the vector indicated in the subscript and $[v, w]$ means $vw - wv$. □

Connections on tensor products and dual bundles

Remark. If E and E' are two vector bundles and ∇ and ∇' are two connections respectively on E and E', we can define a connection $\nabla \otimes \nabla'$ on the **tensor product** $E \otimes E'$ in the following way:

$$(\nabla \otimes \nabla')(e \otimes e') = \nabla e \otimes e' + e \otimes \nabla e'$$

for any $e \in A(E)$, $e' \in A(E')$. □

Remark. If ∇ is a connection on a vector bundle E, there exists a connection $\nabla^\vee$ on the **dual bundle** $E^\vee$ such that

$$d(e, e') = (\nabla e, e') + (e, \nabla^\vee e')$$

for any $e \in A(E)$, $e' \in A(E^\vee)$ (where $(\cdot, \cdot)$ means that we are applying the element of $A(E^\vee)$ to the element of $A(E)$). □

Bianchi's identity. Let ∇ be a connection on a vector bundle E. Let D be its curvature; it can be seen as an element of $A^2(End\, E)$. Then

$$\nabla D = 0$$

(by the remarks above, ∇ defines a connection on $E^\vee$ and thus also on $E \otimes E^\vee = End(E)$ and thus we have a map, we call again ∇, from $A^2(End\, E)$ to $A^3(End\, E)$). □

Compatibility with holomorphic structures and metrics

Definition. Let $(E, (\cdot, \cdot))$ be a complex vector bundle with a Hermitian metric. A connection ∇ on E is said to be **compatible with the metric** if

$$d(s_1, ds_2) = (\nabla s_1, s_2) + (s_1, \nabla s_2)$$

for all $s_1, s_2 \in A(E)$. □

Definition. Let X be a complex manifold and E be a holomorphic vector bundle. A connection ∇ on E is said to be **compatible with the holomorphic structure** if $p \circ \nabla = \overline{\partial}$ where p is the projection

$$A(E \otimes T^{1,0} X^\vee) \oplus A(E \otimes T^{0,1} X^\vee) \longrightarrow A(E \otimes (T_X^{0,1})^\vee)$$

(i.e., the entries of the connection matrix are of type $(1, 0)$). □

Proposition. Let X be a complex manifold and E be a holomorphic vector bundle with a Hermitian metric. There is a unique connection on E compatible both with the holomorphic structure and with the metric. □

Correspondences.

([72], [93], [196]). Let X and Y be two algebraic varieties; a correspondence from X to Y is an algebraic cycle in $X \times Y$ (see "Cycles").

Let X, Y, Z be three smooth algebraic varieties and let $p_{X,Y}$ be the projection from $X \times Y \times Z$ onto $X \times Y$ and analogously $p_{X,Z}$ and $p_{Y,Z}$. If T_1 is a correspondence from X to Y and T_2 is a correspondence from Y to Z, we define

$$T_2 \circ T_1 = p_{X,Z*}(p_{X,Y}^* T_1 \cdot p_{Y,Z}^* T_2),$$

where $\cdot$ is the intersection of cycles (see "Pull-back and push-forward of cycles", "Intersection of cycles").

One can prove that, if X is smooth, then the set of the correspondences from X to X with the product $\circ$ and the usual sum of cycles is an associative ring.

Covering projections.

([33], [91], [112], [158], [184], [215], [234], [247]).

Definition. Let X' and X be two topological spaces. We say that a map

$$p : X' \to X$$

is a **topological covering projection** (covering projection for short) of X if, for all $x \in X$, there exists an open subset U of X such that $p^{-1}(U)$ is a disjoint union of open subsets U_i of X' such that $p|_{U_i} : U_i \to U$ is a homeomorphism for all i. The space X' is said to be **covering space**. □

Definitions.

- We say that a map between two topological spaces, $p : X' \to X$, is a **local homeomorphism** if, for all $x' \in X'$, there exists an open subset U of X' containing x' such that $p(U)$ is an open subset of X and $p : U \to p(U)$ is a homeomorphism.
- Let X', X, Z be topological spaces. Let $p : X' \to X$ and $f : Z \to X$ be continuous maps. A **lifting** of f (for p) is a continuous map $\tilde{f} : Z \to X'$ such that $p \circ \tilde{f} = f$.

$$\begin{tikzcd} & X' \arrow[d, "p"] \\ Z \arrow[ur, "\tilde{f}"] \arrow[r, "f"'] & X \end{tikzcd}$$

- We say that a continuous map $p : X' \to X$ is **complete with respect to a topological space** Y, if, for every continuous map $f : Y \times [0,1] \to X$, every lifting of $f|_{Y \times \{0\}}$ can be extended to a lifting of f.
- We say that a continuous map $p : X' \to X$ has **unique path lifting** if, given paths σ and $\tilde{\sigma}$ in X' (that is two continuous maps from $[0,1]$ to X') such that $\sigma(0) = \tilde{\sigma}(0)$ and $p \circ \sigma = p \circ \tilde{\sigma}$, then $\sigma = \tilde{\sigma}$.
- We say that a topological space is **locally path-connected** if the path-connected components of open subsets are open.

– We say that a topological space X is **semi-locally simply connected** if every $x \in X$ has a neighborhood U such that all loops in U are homotopically trivial in X. □

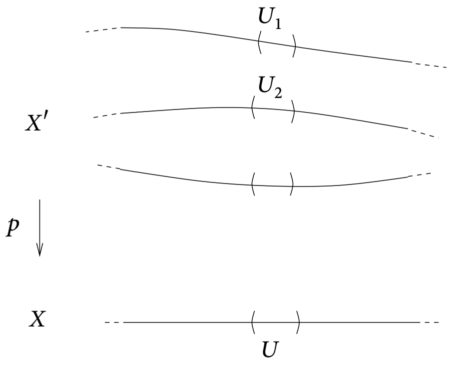


Fig. 3. A covering projection.

Theorem. Let X and X' be topological spaces. A covering projection $p : X' \to X$ is a local homeomorphism and is complete with respect to any topological space. □

Theorem. Let X be a locally path-connected and semi-local simply connected topological space and let X' be a locally path-connected topological space. Let $p : X' \to X$ be a continuous map complete with respect to any topological space and with unique path lifting. Then $p : X' \to X$ is a covering projection. □

Definition. Let X and X' be two topological spaces and let x and x' be points respectively of X and X'. A **pointed covering projection** $p : (X', x') \to (X, x)$ is a covering projection $p : X' \to X$ such that $p(x') = x$. □

Proposition $*$. Let $p : (X', x') \to (X, x)$ be a pointed covering projection. Let Y be a connected topological space, $y \in Y$ and $f : (Y, y) \to (X, x)$ be a continuous map.
(a) If there exists a lifting $f' : (Y, y) \to (X', x')$ of f, it is unique.
(b) If Y is locally path-connected, there exists a lifting $f' : (Y, y) \to (X', x')$ of f if and only if

$$f_*(\pi_1(Y, y)) \subset p_*(\pi_1(X', x')),$$

where we denote by π_1 the first fundamental group (see "Fundamental group", also for the definition of f_* and p_*). □

Proposition. Let $p : (X', x') \to (X, x)$ be a pointed covering projection. The maps

$$p_* : \pi_n(X', x') \to \pi_n(X, x)$$

are injective for every n and are isomorphisms for $n \geq 2$. □

Theorem. Let X be a path-connected, locally path-connected, semi-locally simply connected topological space. Let $x \in X$. There is a bijection between the following sets:

$$\{\text{pointed covering maps on } (X, x)\}/ \text{ pointed covering homeomorphisms}$$

and

$$\{\text{subgroups of } \pi_1(X, x)\},$$

where a pointed covering homeomorphism between two pointed covering projections on (X, x), $p' : (X', x') \to (X, x)$ and $p'' : (X'', x'') \to (X, x)$, is a homeomorphism $\phi : X' \to X''$ such that $\phi(x') = x''$ and $p'' \circ \phi = p'$. The bijection is given by associating to a pointed covering projection on (X, x), $p' : (X', x') \to (X, x)$, the subgroup $p_*(\pi_1(X', x'))$ of $\pi_1(X, x)$. □

Definition. We say that a covering projection $p : X' \to X$ with X' path-connected, is **universal** if the first fundamental group of X' is trivial. □

Let $p : (X', x') \to (X, x)$ be a pointed covering projection with X' path-connected. Let

$$H = p_*(\pi_1(X', x')).$$

Obviously the fibre $p^{-1}(x)$ is in bijection with the set of lateral classes of H in $\pi_1(X, x)$:

$$\{H\alpha |\ \alpha \in \pi_1(X, x)\}.$$

In fact, if $x'' \in p^{-1}(x)$, we can send x'' to $H\alpha$, where α is the image through p of a path from x'' to x'.
We have an action, called **monodromy action**, of $\pi_1(X, x)$ on $p^{-1}(x)$: if c is a point of $p^{-1}(x)$ and $\alpha \in \pi_1(X, x)$, let $\tilde{\alpha}$ be the lifting of α such that $\tilde{\alpha}(0) = c$; we define $\alpha \cdot c$ to be $\tilde{\alpha}(1)$.
Observe that, if x'' is a point of X' such that $p(x'') = x$ and α is the image through p of a path from x'' to x', then $p_*(\pi_1(X', x'')) = \alpha^{-1}H\alpha$. Thus, in the assumptions of the theorem above, there exists a pointed covering homeomorphism between (X', x') and (X', x'') if and only if $\alpha H \alpha^{-1} = H$.
In particular, if H is normal, the group of covering homeomorphisms of X' on X is transitive on $p^{-1}(x)$ and is isomorphic to $\pi_1(X, x)/H$.

Note. In the case of two complex manifolds, "(ramified) covering projections" sometimes stands for holomorphic surjective maps between complex manifolds of the same dimension. When it is a true topological covering projection, i.e., it is not ramified, we say that it is an étale covering projection. □

See also "Riemann's existence theorem"

Cremona transformations. See "Quadratic transformations, Cremona transformations".

Cross ratio. Let K be a field and let $\mathbb{P}^1 = \mathbb{P}^1_K$. The cross ratio of an ordered set of four points of $\mathbb{P}^1$, (P_1, P_2, P_3, P_4), with P_1, P_2, P_3 distinct, is the element $\begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} \in \mathbb{P}^1$ such that there exists an automorphism $f : \mathbb{P}^1 \to \mathbb{P}^1$ such that

$$(f(P_1), f(P_2), f(P_3), f(P_4)) = \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} \right),$$

or, using not homogeneous coordinates,

$$(f(P_1), f(P_2), f(P_3), f(P_4)) = (\infty, 0, 1, \lambda),$$

where $\lambda = \lambda_2/\lambda_1$. It is equal to

$$\frac{\text{simple ratio}(z_1, z_2, z_3)}{\text{simple ratio}(z_1, z_2, z_4)},$$

where, if $P_i = \begin{bmatrix} x_i \\ y_i \end{bmatrix}$ and $z_i := y_i/x_i$, we define simple ratio$(z_1, z_2, z_3) = \frac{z_3 - z_1}{z_3 - z_2}$. More precisely, the cross ratio is equal to

$$\frac{\det \begin{pmatrix} x_1 & x_3 \\ y_1 & y_3 \end{pmatrix} \det \begin{pmatrix} x_2 & x_4 \\ y_2 & y_4 \end{pmatrix}}{\det \begin{pmatrix} x_1 & x_4 \\ y_1 & y_4 \end{pmatrix} \det \begin{pmatrix} x_2 & x_3 \\ y_2 & y_3 \end{pmatrix}}.$$

Since the cross ratio of (P_1, P_2, P_3, P_4) and the cross ratio of $(P_{\sigma(1)}, P_{\sigma(2)}, P_{\sigma(3)}, P_{\sigma(4)})$ are the same for any σ composition of two disjoint transpositions, we have that the possible values of the cross ratio of $(P_{\sigma(1)}, P_{\sigma(2)}, P_{\sigma(3)}, P_{\sigma(4)})$ for $\sigma \in \Sigma_4$ are at most 6, precisely, if λ is the cross ratio of (P_1, P_2, P_3, P_4), the 6 possible values are

$$\lambda, \quad \frac{1}{\lambda}, \quad 1 - \lambda, \quad \frac{1}{1-\lambda}, \quad 1 - \frac{1}{\lambda}, \quad 1 - \frac{1}{1-\lambda}.$$

Cup product. See "Singular homology and cohomology".

Curves. See "Riemann surfaces (compact -) and algebraic curves".

Cusps. See "Regular rings, smooth points, singular points".

Cycles. Let X be an algebraic variety. An algebraic cycle of codimension p in X is an element of the free Abelian group generated by the closed (for the Zariski topology) irreducible subsets of X of codimension p (see "Varieties, algebraic -, Zariski topology, regular and rational functions, morphisms and rational maps"); in other words an algebraic cycle of codimension p in X is an element of the form

$$\sum_i n_i Z_i$$

with Z_i irreducible algebraic subset of X of codimension p and $n_i \in \mathbb{Z}$.
Analogously for analytic cycles.
See "Equivalence, algebraic, rational, linear -, Chow, Neron–Severi and Picard groups", "Pull-back and push-forward of cycles".

D

Deformations. ([43], [151], [153], [154], [160], [161], [218], [221]).

Deformation theory is strictly connected with the theory of moduli spaces (see "Moduli spaces"), i.e., varieties parametrizing geometric objects of a certain kind. It was created to parametrize the possible complex structures on a fixed differentiable manifold. We recall that if $\pi : \chi \to B$ is a holomorphic surjective map between complex manifolds such that the differential of π at every point has maximal rank and the fibres of π are compact complex manifolds, then the fibres are diffeomorphic by Ehresmann's theorem (see [58] or [149, Theorem. 2.3]), and so B parametrizes some complex structures on the same differentiable manifold.

Definition. Let X be a compact complex manifold. A **deformation** of X is the datum of a proper flat morphism $\pi : \chi \to B$ of complex analytic spaces, a point $P \in B$ and an isomorphism $\pi^{-1}(P) \cong X$. We denote this by $\pi : \chi \to (B, P)$. □

(See "Proper", "Flat (module, morphism)", "Spaces, analytic -" for the definitions of these terms.)

Let $\pi : \chi \to (B, P)$ be a deformation and let $\varphi : (B', P') \to (B, P)$ a morphism of analytic spaces. The **pull-back** $\pi' : \chi' \to (B', P')$ of the deformation $\pi : \chi \to (B, P)$ through φ is defined in the following way: let

$$\chi' = \chi \times_B B' = \{(x, b') \in \chi \times B' |\ \ \pi(x) = \varphi(b')\}$$

and let $\pi' : \chi' \to B'$ be the projection onto the second factor.

We say that two deformations of X, $\pi : \chi \to (B, P)$ and $\pi' : \chi' \to (B', P')$, are **isomorphic** if there exist isomorphisms of analytic spaces $\phi : \chi' \to \chi$ and $\varphi : (B', P') \to (B, P)$ such that $\varphi \circ \pi' = \pi \circ \phi$ and the composition of ϕ from $\pi'^{-1}(P')$ to $\pi^{-1}(P)$ with the isomorphisms with X is the identity.

Definition. We say that a deformation $\pi : \chi \to (B, P)$ of X is **complete** if any other deformation of X, $\pi' : \chi' \to (B', P')$, is locally the pull-back of $\pi : \chi \to (B, P)$, i.e., there exist U neighborhood of P' in B' and $\varphi : U \to B$ such that the deformation $\chi'|_U$ (in the obvious sense) is isomorphic to the pull-back of $\pi : \chi \to (B, P)$ through φ.
We say that $\pi : \chi \to (B, P)$ is **universal** if it is complete and the map φ is locally unique.
We say that $\pi : \chi \to (B, P)$ is **semiuniversal** if it is complete and the map $d\varphi_{P'}$ (the differential of φ in P') is unique. □

Observe that all the universal deformations of a complex manifold X are locally canonically isomorphic, and all the semiuniversal deformations of X are locally isomorphic. Obviously, from the "moduli problem" viewpoint, the best situation is the one of universal deformations.

Definition. Let $\pi : \chi \to (B, P)$ be a deformation of X. Let $T_P^{1,0}B$ be the holomorphic tangent space of B at P and let $\Theta_X = \mathcal{O}(T^{1,0}X)$, where $T^{1,0}X$ is the holomorphic tangent bundle of X. The **Kodaira–Spencer map** associated to $\pi : \chi \to (B, P)$ is the map

$$\rho : T_P^{1,0}B \to H^1(\Theta_X)$$

defined as follows: let U be a neighborhood of P and let $\{U_\alpha\}_\alpha$ be a finite open covering of $\pi^{-1}U$ such that, defined $X_\alpha = X \cap U_\alpha$, there are isomorphisms $s_\alpha : U_\alpha \cong X_\alpha \times U$ such that the composition of s_α with the projection onto the second factor is π; let $\eta \in T_P^{1,0}B$; for all $x \in X_\alpha$ there exists a unique $\eta_\alpha(x) \in T_x^{1,0}U_\alpha$ orthogonal ("through s_α") to $T_x^{1,0}X_\alpha$ and such that $d\pi(\eta_\alpha(x)) = \eta$. The family

$$\eta_{\alpha,\beta} := \eta_\alpha - \eta_\beta$$

defines an element of $H^1(\Theta_X)$, which we call $\rho(\eta)$. □

Observe that the trivial deformation (i.e., $\chi \cong X \times B$) has zero Kodaira–Spencer map.

Remark. If $\chi \to (B, P)$ and $\chi' \to (B', P')$ are two deformations of X with Kodaira–Spencer maps respectively ρ and ρ' and the second deformation is the pull-back of the first through a map $\varphi : (B', P') \to (B, P)$, then

$$\rho \circ d\varphi = \rho' .$$ □

Theorem. For every complex compact manifold X, there exists a deformation $\chi \to (B, P)$, called **Kuranishi family**, with the following properties:
(1) its Kodaira–Spencer map is bijective;
(2) it is a semiuniversal deformation of X; furthermore, and it is complete in every point Q of $B - P$, if we consider it as a deformation of the fibre on Q;
(3) if $H^0(\Theta_X) = 0$, then it is a universal deformation of X;
(4) B is the zero locus of a holomorphic map from a neighborhood of 0 in $H^1(\Theta_X)$ to $H^2(\Theta_X)$; in particular $dim(B) \geq h^1(\Theta_X) - h^2(\Theta_X)$ and, if $H^2(\Theta_X) = 0$, then B is smooth. □

From the remark and from statement (1) of the theorem we have that a complete deformation must have surjective Kodaira–Spencer map. Furthermore, if a complete deformation has bijective Kodaira–Spencer map, then it is semiuniversal.

We considered deformations of compact complex manifolds, but in an analogous way we can consider deformations of other objects: schemes, bundles . . . . In [218], Schlessinger developed a theory to which many deformation theories can be related.

Let R be a local Noetherian complete ring with residue field k and let Art be the category of the local Artinian R-algebras with residue field k. Schlessinger's theory studies the functors

$$F : Art \to Sets$$

(see "Categories" for the definition of functor) such that $F(k)$ is a set with only one element.

The link between this theory and deformation theory is the following: consider for instance deformations of schemes; consider the functor $F : Art \to Sets$ such that $F(A)$ is the set of isomorphisms classes of deformations on $Spec(A)$ of a scheme X and, if $\varphi : A \to B$ is a homomorphism, $F(\varphi)$ is the map from the set of isomorphisms classes of deformations on $Spec(A)$ of X to the set of isomorphisms classes of deformations on $Spec(B)$ of X associating to a deformation on $Spec(A)$ its pull-back through $\varphi : Spec(B) \to Spec(A)$. If we fix $A \in Art$ and $\chi \in F(A)$, the pull-back gives a morphism of functors $Hom(A, \cdot) \to F$; obviously its surjectivity, that is the surjectivity of the maps $Hom(A, B) \to F(B)$ for $B \in Art$, corresponds to the completeness of the deformation $\chi \to Spec(A)$ and the bijectivity to the universality; so the universality is related with representability of the functor F.

Degeneracy locus of a morphism of vector bundles. See "Determinantal varieties".

Degree of an algebraic subset. ([93], [104], [107], [196]). Let K be an algebraically closed field.

Let X be an algebraic subset of dimension n in $\mathbb{P}^N_K$. The degree of X is defined to be $n!$ times the leading coefficient of the Hilbert polynomial p_X (see "Hilbert function and Hilbert polynomial").

Equivalently, one can define the degree of an algebraic variety X of dimension n in $\mathbb{P}^N_K$ to be the number of intersection points of X with a generic $(N - n)$-dimensional subspace (i.e., a subspace of complementary dimension) and the degree of an algebraic subset of dimension n in $\mathbb{P}^N_K$ to be the sum of the degrees of its irreducible components of dimension n.

If X is the zero locus of a homogeneous polynomial F, we have that the degree of X is the degree of F.

If $K = \mathbb{C}$, the degree of a n-dimensional smooth algebraic variety X in $\mathbb{P}^N_{\mathbb{C}}$ is its fundamental class in $H_{2n}(\mathbb{P}^N_{\mathbb{C}}, \mathbb{Z}) \cong \mathbb{Z}$ (see "Singular homology and cohomology"). More precisely $H_{2n}(\mathbb{P}^N_{\mathbb{C}}, \mathbb{Z})$ is $\mathbb{Z}S$, where S is the fundamental class of a n-subspace of $\mathbb{P}^N_{\mathbb{C}}$; so the fundamental class of X is dS for some $d \in \mathbb{N}$; d is the degree of X.

The degree of X is also $\int_X \omega^n$, where ω is the Fubini-Study form on $\mathbb{P}^N_{\mathbb{C}}$ (see "Fubini-Study metric").

See also "Bezout's Theorem" and "Minimal degree".

Depth. ([62], [159], [185]).
Let R be a Noetherian ring and let M be a finitely generated R-module. If I is an ideal of R such that $IM \neq M$, we define the I-depth (sometimes called I-grade) of M to be the number of the elements of a maximal M-regular sequence in I (see "Regular sequences"). If R is also local and m is its maximal ideal, we call "depth of M" the m-depth of M.

Auslander–Buchsbaum theorem. Let R be a Noetherian local ring. For any finitely generated R-module M with finite projective dimension, we have that

$$pd(M) + depth(M) = depth(R),$$

where pd is the projective dimension (see "Dimension"). □

See also "Cohen–Macaulay, Gorenstein, (arithmetically -,-)".

Del Pezzo surfaces. See "Surfaces, algebraic -".

De Rham's theorem. ([85], [93], [251]).

De Rham's theorem. Let M be a C^∞ real manifold. Let $H^k_{DR}(M)$ denote the so-called de Rham cohomology, which is defined to be the quotient

$$\frac{\{\text{closed } C^\infty \ k\text{-forms on } M\}}{\{\text{exact } C^\infty \ k\text{-forms on } M\}},$$

where a k-form ω is said to be closed if $d\omega = 0$ and is said to be exact if there exists a $(k-1)$-form η such that $\omega = d\eta$. Let $H^k(M, \mathbb{R})$ denote the singular cohomology (see "Singular homology and cohomology"). Then we have

$$H^k(M, \mathbb{R}) \cong H^k_{DR}(M)\,.$$ □

For de Rham's abstract theorem, see "Sheaves".

Derived categories and derived functors. ([5], [79], [108], [242]).

Definition. Let $\mathcal{A}$ be an Abelian category (see "Categories"). The **homotopy category** of $\mathcal{A}$, denoted by $K(\mathcal{A})$, is the following category:
- the objects are the complexes of objects of $\mathcal{A}$;
- the morphisms are homotopy equivalence classes of morphisms of complexes (See "Complexes" for the definition of homotopically equivalent morphisms of complexes).

We denote by $K^b(\mathcal{A})$ the subcategory of $K(\mathcal{A})$ whose objects are the bounded complexes, by $K^+(\mathcal{A})$ the subcategory whose objects are the complexes bounded below, and by $K^-(\mathcal{A})$ the subcategory whose objects are the complexes bounded above. □

We say that a morphism in $K^*(\mathcal{A})$, where $*$ is one among $\emptyset, b, +, -$, is a **quasi-isomorphism** if it induces an isomorphism in cohomology.
Let X be a complex

$$\cdots \longrightarrow X^{p-1} \longrightarrow X^p \longrightarrow X^{p+1} \longrightarrow \cdots ;$$

we denote by $T(X)$ the complex such that $T(X)^p = X^{p+1}$ for any p and $d_{T(X)} = -d_X$. We call T "shift operator".
Let $h : X \to Y$ be a morphism in $K^*(\mathcal{A})$, where $*$ is one among $\emptyset, b, +, -$. The **cone** of h, denoted by $C(h)$, is the complex $T(X) \oplus Y$ with the differential

$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} d_{T(X)} & 0 \\ h & d_Y \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Proposition. Let $\mathcal{A}$ be an Abelian category. If $\alpha : Z \to Y$ and $\beta : U \to Y$ are two morphisms in $K(\mathcal{A})$ and β is a quasi-isomorphism, then there exist a complex R in $\mathcal{A}$ and two morphisms $t : R \to Z$ and $g : R \to U$ with t quasi-isomorphism such that the following diagram commutes:

$$\begin{tikzcd} & Y & \\ Z \arrow[ur, "\alpha"] & & U . \arrow[ul, "\beta"'] \\ & R \arrow[ul, "t"] \arrow[ur, "g"'] & \end{tikzcd}$$

□

Definition. Let $\mathcal{A}$ be an Abelian category. The **derived category** of $\mathcal{A}$, denoted by $D(\mathcal{A})$ is the following category:
- the objects of $D(\mathcal{A})$ are the complexes of objects of $\mathcal{A}$;
- a morphism $f : X \to Y$ in $D(\mathcal{A})$ is a triplet (Z, g, h), where Z is a third complex, $g : Z \to X$ and $h : Z \to Y$ are homotopy equivalence classes of morphisms of complexes and g is a quasi-isomorphism. We write it in the following way:

$$\begin{tikzcd} X & & Y . \\ & Z \arrow[ul, "g"] \arrow[ur, "h"'] & \end{tikzcd}$$

By the proposition above one can define the composition of two morphisms

$$\begin{tikzcd} X & & Y \\ & Z \arrow[ul, "\delta"] \arrow[ur, "\alpha"'] & \end{tikzcd}$$

and

$$\begin{tikzcd} Y & & W \\ & U \arrow[ul, "\beta"'] \arrow[ur, "\gamma"] & \end{tikzcd}$$

in $D(\mathcal{A})$ in the following way: by the proposition above there exist R, t, g with t quasi-isomorphism such that the following diagram is commutative:

$$\begin{tikzcd} X & & Y & & W; \\ & Z \arrow[ul, "\delta"'] \arrow[ur, "\alpha"] & & U \arrow[ul, "\beta"'] \arrow[ur, "\gamma"] & \\ & & R \arrow[ul, "t"'] \arrow[ur, "g"] & & \end{tikzcd}$$

let the composition be given by the triplet $R, \delta \circ t, \gamma \circ g$. We denote by $D^b(\mathcal{A})$ the subcategory of $D(\mathcal{A})$ whose objects are the bounded complexes, and analogously $D^+(\mathcal{A})$ and $D^-(\mathcal{A})$. □

Let $f : X \to Y$ be the following morphism in $D^*(\mathcal{A})$ (where $*$ is one among $\emptyset, b, +, -$ and so throughout the item):

$$\begin{tikzcd} X & & Y. \\ & R \arrow[ul, "g"'] \arrow[ur, "h"] & \end{tikzcd}$$

We define the **cone** $C(f)$ to be the cone of h.

Definition. We define the **localizing functor from the homotopy category to the derived category**

$$Q_{\mathcal{A}} : K^*(\mathcal{A}) \to D^*(\mathcal{A})$$

in the following way: it is the identity on the set of the objects and, for any f morphism in $K^*(\mathcal{A})$, we define $Q_{\mathcal{A}}(f)$ to be

$$\begin{tikzcd} X & & Y \\ & X \arrow[ul, "I"'] \arrow[ur, "f"] & \end{tikzcd} .$$

□

Observe that $Q_{\mathcal{A}}(f)$ is an isomorphism in $D^*(\mathcal{A})$ for every f quasi-isomorphism in $K^*(\mathcal{A})$.

In fact, the idea of derived category is to identify an object of an Abelian category $\mathcal{A}$ with all its resolutions; to do this we consider a category, $D(\mathcal{A})$, whose objects are all the complexes of objects in $\mathcal{A}$ and the morphisms are defined in such way that two quasi-isomorphic complexes are isomorphic in $D(\mathcal{A})$. In such a way, any object X of $\mathcal{A}$, considered as an element of $D(\mathcal{A})$ (that is, $0 \to X \to 0$) is isomorphic to all its resolutions.

Definition. Let $\mathcal{C}$ be an additive category and let T be an additive automorphism of $\mathcal{C}$ (we call T shift operator). A **triangle** in $\mathcal{C}$ is a sextuple (X, Y, Z, u, v, w) of objects X, Y, Z in $\mathcal{C}$ and morphisms $u : X \to Y$, $v : Y \to Z$, $w : Z \to T(X)$. It is often denoted

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} T(X).$$

A morphism of triangles is a commutative diagram

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"] & Y \arrow[r, "v"] \arrow[d, "g"] & Z \arrow[r, "w"] \arrow[d, "h"] & T(X) \arrow[d, "T(f)"] \\ X' \arrow[r, "u'"] & Y' \arrow[r, "v'"] & Z' \arrow[r, "w'"] & T(X') . \end{tikzcd}$$ □

Definition. We say that an additive category $\mathcal{C}$ equipped with an additive automorphism T and with a family of triangles, called distinguished triangles, is a **triangulated category** if the following axioms hold:

(1) Every triangle isomorphic to a distinguished triangle is a distinguished triangle. For every morphism $u : X \to Y$, there is a distinguished triangle (X, Y, Z, u, v, w). The triangle $(X, X, 0, I_X, 0, 0)$ is a distinguished triangle.

(2) A triangle (X, Y, Z, u, v, w) is distinguished if and only if the triangle

$$(Y, Z, T(X), v, w, \ -T(u))$$

is distinguished.

(3) Given two distinguished triangles (X, Y, Z, u, v, w) and (X', Y', Z', u', v', w') and morphism $f : X \to X'$ and $g : Y \to Y'$ commuting with u and u', there exists a morphism $h : Z \to Z'$ such that (f, g, h) is a morphism from the first triangle to the second.

(4) Given distinguished triangles

$$(X, Y, Z', u, j, k),$$
$$(Y, Z, X', v, l, i),$$
$$(X, Z, Y', vu, m, n),$$

there exist morphisms $f : Z' \to Y'$, $g : Y' \to X'$, such that

$$(Z', Y', X', f, g, T(j)i)$$

is a distinguished triangle and (I_X, v, f) and (u, I_Z, g) are morphisms of triangles (i.e., $l = gm$, $k = nf$, $ig = un$, $fj = mv$). □

Definition. We say that a functor between two triangulated categories is a **∂-functor** if it is additive, if it commutes with the shift operators, and if it takes distinguished triangles to distinguished triangles. □

A **distinguished triangle** in $K^*(\mathcal{A})$ is defined to be a triangle isomorphic to a triangle of the form

$$X \xrightarrow{u} Y \xrightarrow{v} C(u) \xrightarrow{w} T(X),$$

where v and w are the natural maps $Y \to C(u)$ and $C(u) \to T(X)$.
Analogously we define the distinguished triangles in $D^*(\mathcal{A})$. With these families of distinguished triangles, $K^*(\mathcal{A})$ and $D^*(\mathcal{A})$ are triangulated categories.

Definition. Let $\mathcal{A}$ and $\mathcal{B}$ be Abelian categories and let

$$F : K^*(\mathcal{A}) \to K(\mathcal{B})$$

be a ∂-functor. A **right derived functor** of F is a ∂-functor

$$R^*F : D^*(\mathcal{A}) \to D(\mathcal{B})$$

together with a morphism of functors from $K(\mathcal{A})$ to $D(\mathcal{B})$

$$\mu : Q_{\mathcal{B}} \circ F \to R^*F \circ Q_{\mathcal{A}}$$

with the following universal property: if

$$G : D^*(\mathcal{A}) \to D(\mathcal{B})$$

is a ∂-functor and

$$\lambda : Q_{\mathcal{B}} \circ F \to G \circ Q_{\mathcal{A}}$$

is a morphism of functors, then there exists a unique morphism $\eta : R^*F \to G$ such that

$$\lambda = (\eta \circ Q_{\mathcal{A}}) \circ \mu.$$

Analogously, a **left derived functor** of F is a ∂-functor

$$L^*F : D^*(\mathcal{A}) \to D(\mathcal{B})$$

together with a morphism of functors from $K(\mathcal{A})$ to $D(\mathcal{B})$

$$\mu : LF \circ Q_{\mathcal{A}} \to Q_{\mathcal{B}} \circ F$$

with an analogous universal property. □

If R^*F exists, it is unique up to isomorphism of functors.

Theorem. Let $\mathcal{A}$ and $\mathcal{B}$ be two Abelian categories and $F : K^*(\mathcal{A}) \to K(\mathcal{B})$ a ∂-functor. Suppose there exists a triangulated subcategory S of $K^*(\mathcal{A})$ such that

(i) for every object of $K^*(\mathcal{A})$, there exists a quasi-isomorphism from it to an object of S;
(ii) for every exact object X of S (i.e., $H^i(X) = 0$ for any i), we have that $F(X)$ is also exact.

Then there exists a right derived functor R^*F of F and, if X is an object of $D^*(\mathcal{A})$ and X' an object of S and they are quasi-isomorphic, then $R^*F(X)$ and $F(X')$ are isomorphic in $D(\mathcal{B})$. □

Let F be an additive functor between two Abelian categories. If it is a left exact functor (that is, it takes any exact sequence $0 \to A \to B \to C$ to an exact sequence $0 \to F(A) \to F(B) \to F(C)$), and RF exists, then we define

$$R^iF := H^i(RF)$$

and we call it the **classical i-th right derived functor for F**. Analogously, if F is a right exact functor, we can define the classical i-th left derived functor for F, L^iF.
For instance, let $\mathcal{A}$ be an Abelian category such that every bounded complex of objects in $\mathcal{A}$ admits a quasi-isomorphism to a bounded complex of injective objects (we say that an object I in an Abelian category is injective if, for any morphism $g : M \to I$ and any monomorhism $f : M \to N$, there exists a morphism $h : N \to I$ such that $hf = g$), e.g., we can take $\mathcal{A}$ equal to the category of R-modules for some commutative ring with unity R. Let Y be an object of $\mathcal{A}$. Let S be the subcategory of $K^b(\mathcal{A})$ given by the complexes of of injective objects. Consider the left exact functor $F = Hom(Y, \cdot)$ from $K(\mathcal{A})$ to $K(Ab)$ (where Ab is the category of Abelian groups). Then there exists the derived functor $R\,Hom(Y, \cdot)$ and

$$R^iHom(Y, \cdot) \cong Ext^i(Y, \cdot).$$

Let $\mathcal{A}$ be the category of R-modules for some commutative ring with unity R and let N be an R-module. The classical i-th left derived functor of the right exact functor $\cdot \otimes_R N$ is $Tor_i(\cdot, N)$.
See "Ext, $\mathcal{EXT}$" and "Tor, $\mathcal{TOR}$".

Determinantal varieties. ([15], [77], [104], [106], [209]). Let X be an algebraic variety (or a manifold) and E and F be two vector bundles on X and let $\varphi : E \to F$ be a morphism of vector bundles. For any $k \in \mathbb{N}$, the set

$$D_k(\varphi) = \{x \in X |\ rk(\varphi_x : E_x \to F_x) \leq k\}$$

is said to be a determinantal variety (or the k-degeneracy locus of φ).

Example. Take $X = \mathbb{P}^N_{\mathbb{C}}$ and $E = \oplus_{j=1,\dots,r}\mathcal{O}(a_j)$, $F = \oplus_{i=1,\dots,l}\mathcal{O}(b_i)$. Then φ is given by a matrix $l \times r$ whose entry i, j is a polynomial of degree $b_i - a_j$ if $b_i \geq a_j$ and 0 if $b_i < a_j$ and $D_k(\varphi)$ is the zero locus in $\mathbb{P}^N$ of the minors $(k + 1) \times (k + 1)$ of φ.

For instance, take $N = rl - 1$ for some r and l. Let $E = \mathcal{O}^r$ and $F = \mathcal{O}(1)^l$. Call the coordinates in $\mathbb{P}^N$ $x_{i,j}$ for $i = 1, \dots, r$, $j = 1, \dots, l$. Let φ be the matrix such that $\varphi_{i,j} = x_{i,j}$. We have that $D_1(\varphi)$ is the image S of the Segre embedding (see "Segre embedding")

$$\mathbb{P}^{r-1} \times \mathbb{P}^{l-1} \to \mathbb{P}^N,$$
$$(\dots, t_i, \dots), (\dots, s_j, \dots) \mapsto (\dots, t_i s_j, \dots).$$

Furthermore, $D_k(\varphi)$ is the k-secant variety to S (see [104]).

Dimension. ([12], [104], [107], [159], [164], [185], [228]). We define the **dimension of a topological space** X to be the supremum of the set of all integers n such that there exists a chain $X_0 \subset X_1 \subset \cdots \subset X_n$ of distinct irreducible closed subsets of X.
The **(Krull) dimension of a ring** R is defined to be the supremum of the set of all integers n such that there exists a chain $p_0 \subset p_1 \subset \cdots \subset p_n$ of distinct prime ideals of R.
The **dimension of an algebraic variety** is its dimension as topological space (with the Zariski topology).

Let K be an algebraically closed field. By using Hilbert's Nullstellensatz (see "Hilbert's Nullstellensatz"), we can prove easily that the dimension of an affine algebraic variety over K is the dimension of the affine coordinate ring and the dimension of a projective algebraic variety over K is the dimension of the homogeneous coordinate ring minus 1.

Theorem. The dimension of an integral domain R that is a finitely generated algebra over a field K is the transcendence degree over K of the quotient field of R (see "Transcendence degree" and "Localization, quotient ring, quotient field"). □

In particular, the dimension of the affine coordinate ring of an irreducible affine algebraic variety over a field K is equal to the transcendence degree over K of its quotient field.

Proposition. The degree of the Hilbert polynomial (see "Hilbert function and Hilbert polynomial") of a projective algebraic variety is the dimension of the projective variety. □

The **(Krull) dimension of a nonzero R-module** M is defined to be the Krull dimension of

$$R/Ann(M),$$

where $Ann(M) = \{x \in R | \ xM = 0\}$.

We say that an R-module M has finite projective dimension if there exists a projective resolution of M (see "Injective and projective modules","Injective and projective resolutions") of the form

$$0 \to P_n \to \cdots \to P_0 \to M \to 0$$

(the number n is said to be the length of the resolution). In this case, the **projective dimension of the R-module M**, denoted by $pd(M)$, is the minimum of the lengths of such resolutions.

If $R = K[x_0, \dots, x_n]$ for some field K, then $pd(M)$ is the minimum of the length of a free resolution (which is the length of a minimal free resolution), since any finitely generated projective $K[x_0, \dots, x_n]$-module is free (see [159, Chapter IV, Theorem 3.15]).

See also "Length of a module".

Direct and inverse image sheaves. ([93], [107]). Let $f : X \to Y$ be a continuous map between two topological spaces. Let $\mathcal{F}$ be a sheaf on X (see "Sheaves").
The **direct image sheaf** (push-forward) $f_*\mathcal{F}$ is the sheaf on Y

$$U \longmapsto \mathcal{F}(f^{-1}(U)),$$

for any open subset U of Y.
The **q-direct image sheaf** $R^q f_*(\mathcal{F})$ (or $R^q_f(\mathcal{F})$) is the sheaf on Y associated to the presheaf

$$U \longmapsto H^q(f^{-1}(U), \mathcal{F}|_{f^{-1}(U)}),$$

for any open subset U of Y.
We can prove that it is the q-th right derived functor of the left exact functor f_* (see "Derived categories and derived functors").
The sheaf $f^{-1}\mathcal{G}$ of a sheaf $\mathcal{G}$ on Y is the sheaf on X associated to the presheaf

$$U \longmapsto \lim_{V \text{ open},\, f(U) \subset V \subset Y} \mathcal{G}(V)$$

for any open subset U of X (see "Limits, direct and inverse -").
Moreover, if $(X, \mathcal{O}_X)$, $(Y, \mathcal{O}_Y)$ are ringed spaces (see "Spaces, ringed -") and $\mathcal{G}$ is a sheaf of $\mathcal{O}_Y$-modules on Y, then we define

$$f^*\mathcal{G} = f^{-1}\mathcal{G} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X;$$

this is called **inverse image sheaf** by f. One can prove easily that, if $\mathcal{F}$ is a sheaf of $\mathcal{O}_X$-modules, then

$$Hom_{\mathcal{O}_X}(f^*\mathcal{G}, \mathcal{F}) \cong Hom_{\mathcal{O}_Y}(\mathcal{G}, f_*\mathcal{F}).$$

Discrete valuation rings. ([12], [73], [107]). Let R be an integral domain that is not a field. We say that it is a discrete valuation ring if it is Noetherian (see "Noetherian, Artinian"), local (see "Local") and the maximal ideal m is principal.
We can prove that these three conditions together are equivalent to the following condition: there exists an irreducible element $t \in R$ such that every $x \in R - \{0\}$ can be written uniquely in the form

$$x = u\, t^n, \tag{1}$$

where u is a unit in R and $n \in \mathbb{N}$. The element t is a generator of m and it is called **uniformizing parameter**.
If R is a discrete valuation ring, we can define a map

$$v : R - \{0\} \to \mathbb{N}$$

by sending x to n, with n as in (1); we can prove that the map v does not depend on the choice of the uniformizing parameter t; the map v is called **valuation map** of R. If we denote by K the quotient field of R (see "Localization, quotient ring, quotient field"), we can extend v to a map

$$V : K - \{0\} \to \mathbb{Z}$$

by sending $x \in K - \{0\}$ to the unique $n \in \mathbb{Z}$ such that $x = ut^n$, where u is a unit in R; we have that

$$R = \{0\} \cup \{x \in K - \{0\} |\ V(x) \geq 0\}, \quad m = \{0\} \cup \{x \in K - \{0\} |\ V(x) > 0\}.$$

We can prove that a point P in an algebraic curve C over an algebraically closed field is smooth (see "Regular rings, smooth points, singular points") if and only if $\mathcal{O}_{C,P}$ is a discrete valuation ring.

Divisors. ([72], [74], [93], [107], [129], [228], [241]). Let K be an algebraically closed field and let X be an algebraic variety over K.

A **Cartier divisor** on X is given by an open covering $\{U_i\}_{i \in I}$ of X and not identically zero rational functions f_i on U_i such that, for any i and j, the function f_i/f_j on $U_i \cap U_j$ is a nowhere vanishing regular function. Other data $(U'_l, f'_l)_{l \in L}$ give the same Cartier divisor if f_i/f'_l is a nowhere vanishing regular function on $U_i \cap U'_l$ for any i, l. In other words, a Cartier divisor is a global section of the sheaf that is the quotient of the sheaf of the not identically vanishing rational functions and the sheaf of the nowhere vanishing regular functions.
(Analogously a Cartier divisor on a complex manifold X is a global section of the sheaf $\mathcal{M}^*/\mathcal{O}^*$, where $\mathcal{M}^*$ is the sheaf of the not identically vanishing meromorphic functions on X and $\mathcal{O}^*$ is the sheaf of nowhere vanishing holomorphic functions on X).
In some books Cartier divisors are called "locally principal divisors". The **support** of a Cartier divisor on X given by $\{(U_i, f_i)\}_i$ is the set of the points $x \in X$ such that, if $x \in U_i$, then f_i in x is either zero or not regular.
We can define the sum of two Cartier divisors on X by multiplying their local equations; so we get an Abelian group structure on the set of Cartier divisors on X.

A **Weil divisor** on X a finite formal sum $\sum_{i=1,\dots,N} n_i V_i$ where V_i is a subvariety of X of codimension 1 and $n_i \in \mathbb{Z}$ for any i. Its **support** is the set $\cup_{i \text{ s.t. } n_i \neq 0} V_i$.
Also the set of the Weil divisors on X forms an Abelian group in an obvious way.

For any subvariety V of X of codimension 1, let $\mathcal{O}_{X,V}$ be the set of the pairs (U, f), where U is an open subset of X such that $X \cap V \neq \emptyset$ and f is a regular function on U, up to the following equivalence relation: (U, f) is equivalent to (U', f') if $f = f'$ on $U \cap U'$.

Definition. For any subvariety V on X of codimension 1 and any element g of $\mathcal{O}_{X,V}$ we define

$$ord_V(g) = length_{\mathcal{O}_{X,V}}(\mathcal{O}_{X,V}/(g)),$$

where $length_{\mathcal{O}_{X,V}}(\mathcal{O}_{X,V}/(g))$ denotes the length of the $\mathcal{O}_{X,V}$-module $\mathcal{O}_{X,V}/(g)$, (see "Length of a module").

If X is smooth along V, we can write g as ut^d for some unit u of $\mathcal{O}_{X,V}$, t a generator of the maximal ideal of $\mathcal{O}_{X,V}$, d an integer and, equivalently, we can define $ord_V(g)$ to be d.

For any not identically zero rational function f on X, we define

$$ord_V(f) = ord_V(g) - ord_V(h),$$

where $g, h \in \mathcal{O}_{X,V}$ and $f = g/h$. □

Theorem. There is a group homomorphism

$$h : \{\text{Cartier divisors on } X\} \to \{\text{Weil divisors on } X\}$$

such that
(1) if X is normal, then h is injective;
(2) if X is smooth, then h is bijective.

The map h can be described as follows. For any Cartier divisor D on X and any subvariety V on X of codimension 1, we define $ord_V(D)$ to be $ord_V(f_i)$, where f_i is a local equation of D on any affine open subset U_i such that $U_i \cap V \neq \emptyset$. For any Cartier divisor D, we define $h(D)$ to be the Weil divisor

$$\sum ord_V(D)\, V,$$

where the sum is over all the subvarieties V of X of codimension 1 with $ord_V(D) \neq 0$ (we can prove that only a finite number of subvarieties V of codimension 1 are such that $ord_V(D) \neq 0$). □

Definition. A Weil divisor $\sum_{i=1,\dots,N} n_i V_i$ is said to be **effective** if and only if $n_i \geq 0$ for any i.

A Cartier divisor is said to be effective if it is defined by regular local equations f_i. □

Definition. Let f be a not identically vanishing rational function on X. Obviously it defines a Cartier divisor, we denote by $(f)_C$ or simply by (f), by taking any open covering and the restriction of f to any open set of the covering.

The function f defines also a Weil divisor, we denote by $(f)_W$ or simply by (f): we define

$$(f)_W = \sum_V ord_V(f)V,$$

where the sum is over all the codimension 1 subvarieties V with $ord_V(f) \neq 0$.
A divisor of the form (f) for some rational function f is called **principal**. □

Obviously $h((f)_C) = (f)_W$.

Definition. Two Cartier divisors, respectively Weil divisors, D_1 and D_2 on X are said **linearly equivalent** if there exists a rational function f on X such that $D_1 - D_2 = (f)_C$, respectively $D_1 - D_2 = (f)_W$.
The group of Weil divisors of X up to linear equivalence is called **divisor class group** of X and denoted by $Cl(X)$.
The group of Cartier divisors of X up to linear equivalence is called **Picard group** of X and denoted by $Pic(X)$. □

Theorem. The map h induces a homomorphism $\tilde{h} : Pic(X) \to Cl(X)$ and $\tilde{h}$ is injective if X is normal, it is bijective if X is smooth. □

Definition. We can **associate to any Cartier divisor D a line bundle** we denote by (D) in the following way: if the Cartier divisor D is given by local data (U_i, f_i), we define (D) to be the line bundle whose transition functions $f_{i,j}$ from U_j to U_i are f_i/f_j, see "Bundles, fibre -". □

Theorem. The map $D \mapsto (D)$ induces an isomorphism from $Pic(X)$ to the group of the isomorphism classes of line bundles on X. □

Definition. We say that a Cartier divisor D is **ample**, or **very ample**, or **big**, or **nef**, or **b.p.f.** if the corresponding line bundle (D) is. See "Bundles, fibre -". □

See "Linear systems" and "Equivalence, algebraic, rational, linear -, Chow, Neron–Severi and Picard groups".

Dolbeault's theorem. ([93], [251]). Let M be a complex manifold. Let Ω^p be the sheaf of the holomorphic p-forms and let $H^{p,q}_{\overline{\partial}}(M)$ be

$$\frac{\{\overline{\partial}\text{-closed } C^\infty \ (p,q)\text{-forms on } M\}}{\{\overline{\partial}\text{-exact } C^\infty \ (p,q)\text{-forms on } M\}},$$

where a (p,q)-form ω is said to be $\overline{\partial}$-closed if $\overline{\partial}\omega = 0$ and is said to be $\overline{\partial}$-exact if there exists a $(p, q-1)$-form η such that $\omega = \overline{\partial}\eta$. Then

$$H^q(M, \Omega^p) \cong H^{p,q}_{\overline{\partial}}(M),$$

Dominant. See "Varieties, algebraic -, Zariski topology, regular and rational functions, morphisms and rational maps".

Dual variety and biduality theorem. ([59], [60], [78], [143], [162]). Let X be a projective algebraic variety in $\mathbb{P}^n := \mathbb{P}^n_{\mathbb{C}}$. Let $X^\vee \subseteq (\mathbb{P}^n)^\vee$ be the closure of the set of all hyperplanes tangent to X (we say that a hyperplane H is tangent to X if there exists a smooth point P in X such that H contains the tangent space of X at P). It is called the dual variety of X. In most cases it is a hypersurface.

Theorem. (Biduality theorem) Let X be a projective algebraic variety in $\mathbb{P}^n$. Then $(X^\vee)^\vee = X$. □

See [143], [60] for the case of fields different from $\mathbb{C}$.

Dualizing sheaf. ([107], [224]). We follow strictly [107].
The dualizing sheaf on a singular algebraic variety is a sheaf that has, in some sense, the same role of the canonical sheaf on a nonsingular algebraic variety (see "Canonical bundle, canonical sheaf").
Let K be an algebraically closed field. For the canonical sheaf on the projective space we have the following theorem.

Theorem. (Serre duality for $\mathbb{P}^n$). Let ω be the canonical sheaf on $\mathbb{P}^n := \mathbb{P}^n_K$.
We have that $H^n(\mathbb{P}^n, \omega) \cong K$ and, for any coherent sheaf $\mathcal{F}$ on $\mathbb{P}^n$ (see "Coherent sheaves"), the natural bilinear form

$$Hom(\mathcal{F}, \omega) \times H^n(\mathbb{P}^n, \mathcal{F}) \longrightarrow H^n(\mathbb{P}^n, \omega) \cong K$$

is a perfect pairing from finite-dimensional vector spaces over K to K (let V, W be vector spaces over a field K; a bilinear form $b : V \times W \to K$ is said to be a perfect pairing if the induced maps $V \to W^\vee$ and $W \to V^\vee$ are isomorphisms). Furthermore, for all $i \geq 0$, there is a natural isomorphism

$$Ext^i(\mathcal{F}, \omega) = H^{n-i}(\mathbb{P}^n, \mathcal{F})^\vee,$$

which, for $i = 0$, is the one induced by the bilinear form above (see "Ext, $\mathcal{EXT}$" for the definition of Ext^i). □

Definition. Let X be a proper scheme over K of dimension n. A dualizing sheaf for X is defined to be a coherent sheaf ω_X^0 on X and a linear map $l : H^n(X, \omega_X^0) \to K$ such that the composition

$$Hom(\mathcal{F}, \omega_X^0) \times H^n(X, \mathcal{F}) \longrightarrow H^n(X, \omega_X^0) \xrightarrow{l} K,$$

where the first map is the natural bilinear form, induces an isomorphism

$$Hom(\mathcal{F}, \omega_X^0) \cong H^n(X, \mathcal{F})^\vee . \qquad \square$$

One can prove that, if a dualizing sheaf exists, then it is unique.
If X is projective scheme, then a dualizing sheaf exists: if $X \subset \mathbb{P}^n_K$ and its codimension is r, we can prove that the dualizing sheaf is the sheaf

$$\mathcal{EXT}^r_{\mathbb{P}^n_K}(\mathcal{O}_X, \omega_{\mathbb{P}^n_K}).$$

Theorem. If X is a Cohen–Macaulay and equidimensional projective scheme of dimension n over K, we have that the natural maps

$$Ext^i(\mathcal{F}, \omega^0_X) \to H^{n-i}(X, \mathcal{F})^\vee$$

are isomorphisms for any coherent sheaf $\mathcal{F}$ and any $i \geq 0$. □

Corollary. (Serre duality). If X is a Cohen–Macaulay and equidimensional projective scheme of dimension n over K, we have a natural isomorphism

$$H^i(X, \mathcal{F}) \cong H^{n-i}(X, \mathcal{F}^\vee \otimes \omega^0_X)^\vee$$

for any locally free sheaf $\mathcal{F}$ of finite rank on X. □

In fact,
- for any sheaf $\mathcal{G}$ of $\mathcal{O}_X$-modules, $Ext^i(\mathcal{O}_X, \mathcal{G}) \cong H^i(X, \mathcal{G})$ for any $i \geq 0$;
- for any sheaves of $\mathcal{O}_X$-modules, $\mathcal{G}$, $\mathcal{G}'$, $\mathcal{L}$ with $\mathcal{L}$ locally free of finite rank, we have that $Ext^i(\mathcal{G} \otimes \mathcal{L}, \mathcal{G}') \cong Ext^i(\mathcal{G}, \mathcal{L}^\vee \otimes \mathcal{G}')$;

(see “Ext, $\mathcal{EXT}$”), so

$$H^{n-i}(X, \mathcal{F}^\vee \otimes \omega^0_X)^\vee \cong Ext^{n-i}(\mathcal{O}_X, \mathcal{F}^\vee \otimes \omega^0_X)^\vee \cong Ext^{n-i}(\mathcal{F}, \omega^0_X)^\vee \cong H^i(X, \mathcal{F}).$$

Corollary. (Serre duality). If X is a smooth projective algebraic variety of dimension n over K, we have a natural isomorphism

$$H^p(X, \Omega^q \otimes \mathcal{F}) \cong H^{n-p}(X, \Omega^{n-q} \otimes \mathcal{F}^\vee)^\vee$$

for any locally free sheaf $\mathcal{F}$ of finite rank on X (see “Zariski tangent space, differential forms, tangent bundle, normal bundle” for the definition of Ω^p). □

(It follows from the previous corollary and from the fact that $(\Omega^p)^\vee \otimes \omega \cong \Omega^{n-p}$). See “Serre duality”.

If X is a locally complete intersection of codimension r in a projective space $\mathbb{P} := \mathbb{P}^n_K$ we can prove that

$$\omega^0_X \cong \omega_{\mathbb{P}} \otimes \wedge^r(\mathcal{I}/\mathcal{I}^2))^\vee,$$

where $\mathcal{I}$ is the ideal sheaf of X. Finally, if X is a smooth projective algebraic variety, then the dualizing sheaf ω^0_X coincides with the canonical sheaf ω_X.

Dynkin diagrams.

See “Lie algebras”.

E

Effective. See “Divisors”.

Elliptic Riemann surfaces, elliptic curves. ([93], [107], [127], [144], [148], [165]). A compact Riemann surface X (see “Riemann surfaces (compact -) and algebraic curves”) is said to be elliptic if its genus is 1. One can easily prove that it is isomorphic to its Jacobian (see “Jacobians of compact Riemann surfaces”), thus it is isomorphic to a torus

$$\mathbb{C}/\Lambda,$$

where Λ is a lattice of rank 2 in $\mathbb{C}$, that is, there exists a basis $\{\lambda_1, \lambda_2\}$ of $\mathbb{C}$ over $\mathbb{R}$ such that $\Lambda = \mathbb{Z}\lambda_1 + \mathbb{Z}\lambda_2$ (see Figure 4).

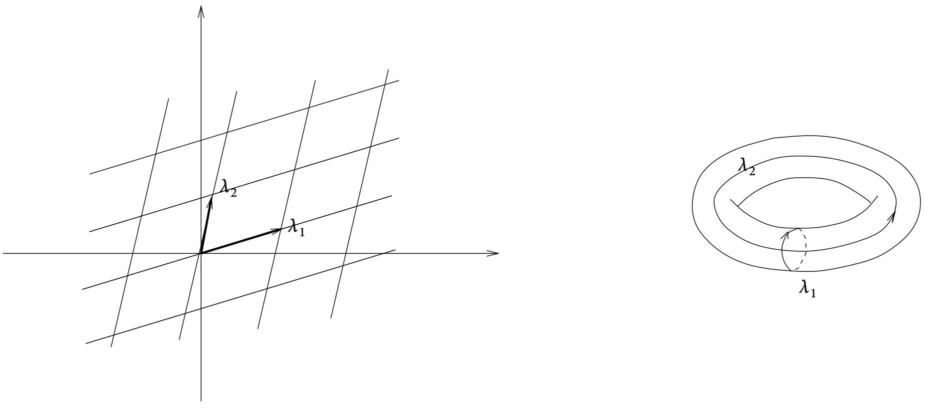


Fig. 4. Elliptic Riemann surface.

Up to isomorphisms, we can suppose that

$$\Lambda = \langle 1, \tau \rangle$$

with the imaginary part of τ, $Im(\tau)$, greater than 0 , where $\langle 1, \tau \rangle$ means $\mathbb{Z}1 + \mathbb{Z}\tau$ (see Figure 5).

Observe that a holomorphic map $f : \mathbb{C}/\Lambda_1 \to \mathbb{C}/\Lambda_2$ is, up to translation, equal to a map $z \mapsto az$ for some $a \in \mathbb{C}$ such that $a\Lambda_1 \subset \Lambda_2$, in fact: f can be lifted to a map $F : \mathbb{C} \to \mathbb{C}$ (see “Covering projections”) and $\frac{\partial F}{\partial z}$ is Λ_1-invariant, thus it defines a holomorphic function $\mathbb{C}/\Lambda_1 \to \mathbb{C}$, which must be constant (since $\mathbb{C}/\Lambda_1$ is compact); therefore f is given by the multiplication by a constant.

Hence two tori $\mathbb{C}/\langle 1, \tau_1 \rangle$ and $\mathbb{C}/\langle 1, \tau_2 \rangle$ with $Im(\tau_1), Im(\tau_2) > 0$ are isomorphic if and only if

$$\tau_1 = \frac{\alpha\tau_2 + \beta}{\gamma\tau_2 + \delta}$$

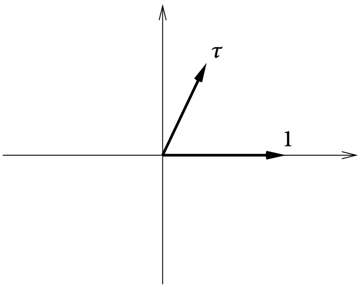


Fig. 5. Generators of the lattice.

with $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL(2,\mathbb{Z})$. In fact, let $f : \mathbb{C}/\langle 1,\tau_1\rangle \to \mathbb{C}/\langle 1,\tau_2\rangle$ be an isomorphism; then f is the multiplication by a constant $a \in \mathbb{C}$ with $a\langle 1,\tau_1\rangle \subset \langle 1,\tau_2\rangle$; so $a1 = \gamma\tau_2 + \delta$, $a\tau_1 = \alpha\tau_2 + \beta$ for some $\alpha,\beta,\gamma,\delta \in \mathbb{Z}$; since f is an isomorphism, the matrix $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ must be invertible over $\mathbb{Z}$ and, since f preserves the orientation, the determinant must be 1.

Therefore the set of the elliptic Riemann surfaces up to isomorphisms is **parametrized** by

$$\mathcal{H}/SL(2,\mathbb{Z}),$$

where $\mathcal{H} = \{\tau \in \mathbb{C} | \; Im(\tau) > 0\}$ (Siegel upper half-plane) and $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL(2,\mathbb{Z})$ acts on $\mathcal{H}$ by

$$\tau \longmapsto \frac{\alpha\tau + \beta}{\gamma\tau + \delta}.$$

Let K be an algebraic closed field. A complete (see "Complete varieties") smooth algebraic curve X is said to be elliptic if its genus is 1.

Observe that, if $P \in X$, then the line bundle $(3P)$ embeds X into $\mathbb{P}^2_K$ and the image is a **smooth cubic curve** (see again "Riemann surfaces (compact -) and algebraic curves"). Moreover, any smooth cubic curve in $\mathbb{P}^2_K$ is an elliptic curve (see "Genus, arithmetic, geometric, real, virtual -"). If the characteristic of K is 0, a smooth cubic curve can be written in the **Weierstrass form**

$$y^2 z = x(x-z)(x-\lambda z)$$

for some $\lambda \in K - \{0,1\}$ (see "Weierstrass form of cubic curves") and two curves,

$$y^2 z = x(x-z)(x-\lambda z),$$
$$y^2 z = x(x-z)(x-\lambda' z),$$

are isomorphic if and only if there is an automorphism f of $\mathbb{P}^1_K$ such that

$$f(\{0,1,\infty,\lambda\}) = \{0,1,\infty,\lambda'\},$$

thus if and only if $\lambda' \in \{\frac{1}{\lambda}, 1-\lambda, \frac{1}{1-\lambda}, \frac{\lambda}{\lambda-1}, \frac{\lambda-1}{\lambda}\}$ (see "Cross ratio"). We can prove that this is true if and only if $j(\lambda) = j(\lambda')$, where

$$j(\lambda) = 2^8 \frac{(\lambda^2-\lambda+1)^3}{\lambda^2(\lambda-1)^2}.$$

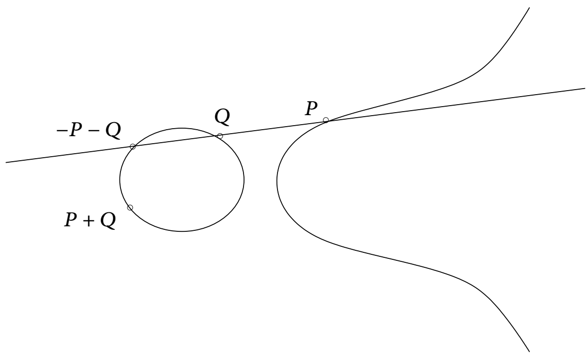


Fig. 6. The group law on an elliptic curve.

Thus, through the function j, the set of the elliptic curves over K, up to isomorphisms, is parametrized by K.
Let X be an elliptic complete smooth curve over K and let $P_0 \in X$. Observe that the map

$$X \longrightarrow Pic^0(X)$$

sending $P \in X$ to the class of the divisor $P - P_0$ is a bijection (see "Equivalence, algebraic, rational, linear -, Chow, Neron–Severi and Picard groups" for the definition of $Pic^0(X)$); in fact, by using Riemann–Roch theorem, we can prove that, for any divisor D on X of degree 0, there exists $P \in X$ such that D is linearly equivalent to $P - P_0$ (apply Riemann–Roch theorem to $D+P_0$). So X inherits a group structure from $Pic^0(X)$. In particular it is an Abelian variety (of dimension 1), see "Tori, complex - and Abelian varieties".
As we have already said, the line bundle $(3P_0)$ embeds X into $\mathbb{P}^2_K$ and the image is a smooth cubic curve, we call again X. The smooth cubic curve inherits a group structure from $Pic^0(X)$. We want to illustrate how we can "read" **the group structure inherited from $Pic^0(X)$ on the cubic curve**. Observe that

$$\sum_{i=1,\dots,m} P_i = \sum_{j=1,\dots,n} Q_j$$

(where $\sum$ is the sum on X inherited from $Pic^0(X)$) if and only if there is the following linear equivalence of divisors:

$$\sum_{i=1,\dots,m} P_i - mP_0 \sim \sum_{j=1,\dots,n} Q_j - nP_0,$$

(where $\sim$ denotes linear equivalence), and obviously this holds if and only if

$$\sum_{i=1,\dots,m} P_i \sim \sum_{j=1,\dots,n} Q_j + (m-n)P_0.$$

In particular P_0 is the zero element of the group. Observe also that three points P, Q, S in X embedded in $\mathbb{P}^2_k$ are collinear if and only if the divisor $P+Q+S$ is linearly equivalent

to $3P_0$. So, if P, Q, S are collinear, then for the sum we have defined on X we have that $P + Q + S = 0$.
Now take coordinates on $\mathbb{P}^2_K$ such that $P_0 = [0 : 1 : 0]$. Observe that any vertical line $x = kz$ meets the cubic in P_0 and in two points, call them A, B; then, for the sum we have defined on X, we have that $A + B = 0$.
Thus, if P, Q are two points on the cubic, then $-P-Q$ is the third point of intersection of the cubic with the line PQ and $P+Q$ is the third point of intersection of the cubic with the line passing through $-P - Q$ and P_0 (which will be a vertical line); see Figure 6.

Elliptic surfaces. See "Surfaces, algebraic -".

Embedded components. See "Primary ideals, primary decompositions, embedded ideals".

Embedding. An embedding (between groups, rings, varieties ...) is a injective map that is an isomorphism onto its image.
We can prove (see [228, Chapter 2, § 5]) that a morphism between two algebraic varieties $f : X \to Y$ is an embedding if and only if it is injective and the differential in any point $x \in X$ is an isomorphic embedding of the Zariski tangent space of X at x into the Zariski tangent space of Y at $f(x)$ (see "Zariski tangent space, differential forms, tangent bundle, normal bundle").

Enriques surfaces. See "Surfaces, algebraic -".

Equivalence, algebraic, rational, linear -, Chow, Neron–Severi and Picard groups. ([16], [72], [74], [93], [107], [166], [217]). Let X be an algebraic variety of dimension n over an algebraically closed field K. Throughout this item, with the word cycle we will mean an algebraic cycle (see "Cycles").

Notation. Let T be a smooth algebraic curve. If C is a cycle in $X \times T$ and $t \in T$, we denote by C_t the push-forward through the projection from $X \times T$ onto X of the intersection cycle $C \cdot (X \times t)$ when this is defined (see "Pull-back and push-forward of cycles" and "Intersection of cycles"). □

Definition. Two Weil divisors D_1 and D_2 on X are said **linearly equivalent** if there exists a rational map f on X such that $D_1 - D_2 = (f)$, where (f) is the Weil divisor on X associated to f, and analogously for Cartier divisors (see "Divisors").
Two cycles of codimension p in X, Z_1 and Z_2, are said **rational equivalent** if there are a finite number of $(p-1)$-codimensional subvarieties in X, $V_1, \dots, V_k$, and, for any $i = 1, \dots, k$, a not identically vanishing rational function f_i on V_i such that

$$Z_1 - Z_2 = \sum_{i=1,\dots,k} (f_i).$$

Another (equivalent) definition of rational equivalence is the following: two cycles of codimension p in X, Z_1 and Z_2, are said **rational equivalent** if there is a p-codimensional cycle C of $X \times \mathbb{P}^1_K$ such that

$$Z_1 - Z_2 = C_0 - C_\infty.$$

Two cycles of codimension p in X, Z_1 and Z_2, are said **algebraic equivalent** if there are a smooth algebraic curve T, a cycle C of codimension p in $X \times T$ and two points in T, t_1 and t_2, such that

$$Z_1 - Z_2 = C_{t_1} - C_{t_2}. \qquad \square$$

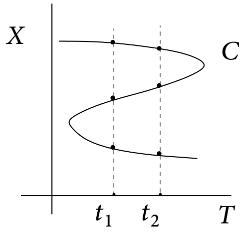


Fig. 7. Algebraic equivalence.

Remark. Obviously rational equivalence implies algebraic equivalence.
For divisors, linear equivalence and rational equivalence coincide (it is clear from the first definition of rational equivalence). □

Definition. The p-th **Chow group** $CH^p(X)$ of X is defined to be the group of the cycles of codimension p in X up to rational equivalence.
The p-th **Neron–Severi group** $NS^p(X)$ of X is defined to be the group of the cycles of codimension p in X up to algebraic equivalence.
We write $CH_q(X)$, respectively $NS_q(X)$, to denote the group of the cycles of dimension q in X up to rational equivalence, respectively algebraic equivalence.
The group of Weil divisors of X up to linear equivalence is called **divisor class group** of X and denoted by $Cl(X)$.
The group of Cartier divisors of X up to linear equivalence is called **Picard group** of X and denoted by $Pic(X)$. □

Theorem. There is a homomorphism $Pic(X) \to Cl(X)$ that is injective if X is normal and that is bijective if X is smooth (see “Divisors” for the description of this homomorphism). □

Definition. We can **associate to any Cartier divisor D a line bundle** we denote by (D) in the following way: if the Cartier divisor D is given by local data (U_i, f_i), we define (D) to be the line bundle whose transition functions $f_{i,j}$ from U_j to U_i are f_i/f_j, see “Bundles, fibre -”. □

Theorem. The map associating to a Cartier divisor D the line bundle (D) induces an isomorphism from the Picard group $Pic(X)$ to the group of the isomorphism classes of line bundles on X. The group of the isomorphism classes of line bundles on X is isomorphic also to $H^1(X, \mathcal{O}^*)$, via the isomorphism induced by the map sending a bundle to its transition functions. □

Suppose from now on that X **is a smooth projective algebraic variety over** $\mathbb{C}$.
In this case, one can prove that, for any divisor D, the first Chern class (see "Chern classes") of the line bundle associated to D is the Poincaré dual of the class of D (see "Singular homology and cohomology" for the definition of Poincaré duality).

Let $Pic^0(X)$ denote the group of the isomorphism classes of the holomorphic line bundles L on X with first Chern class $c_1(L)$ equal to 0. Since the first Chern class of a line bundle associated to a divisor D is the Poincaré dual of the class of D, the group $Pic^0(X)$ can be seen also as the group of divisors on X that are homologous to 0, up to linear equivalence.

Consider the exponential sequence

$$0 \to \mathbb{Z} \to \mathcal{O} \to \mathcal{O}^* \to 0$$

(see "Exponential sequence"); by taking the cohomology we get

$$0 \to H^1(X, \mathcal{O})/H^1(X, \mathbb{Z}) \to H^1(X, \mathcal{O}^*) \xrightarrow{c_1} H^2(X, \mathbb{Z}) \xrightarrow{r} H^2(X, \mathcal{O}) \to \cdots \tag{2}$$

where c_1 is the first Chern class. Thus

$$Pic^0(X) = \frac{H^1(X, \mathcal{O})}{H^1(X, \mathbb{Z})}$$

(since $Pic^0(X) = Ker(c_1)$). So the group $Pic^0(X)$ is the first intermediate Jacobian J_1 of X (see "Jacobians, Weil and Griffiths intermediate -").

We can prove that $Pic^0(X)$ is an Abelian variety and thus an algebraic variety, and, by using this result, one can easily prove:

Proposition. Two divisors on a smooth projective algebraic variety over $\mathbb{C}$ are algebraic equivalent if and only if they are homologous. □

Therefore, $NS^1(X)$, which is defined to be the set of divisors up to algebraic equivalence, is also the set of divisors up to homological equivalence and thus it is equal to $H^1(X, \mathcal{O}^*)/Ker(c_1) \cong Im(c_1)$ and the sequence (2) becomes

$$0 \to Pic^0(X) \to Pic(X) \to NS^1(X) \to 0.$$

Observe that $NS^1(X) \cong Im(c_1) = Ker(r)$ and that $Ker(r)$ is $H^2(X, \mathbb{Z}) \cap H^{1,1}(X)$ (in fact $H^2(X, \mathcal{O}) \cong H^{0,2}(X)$ and the $(2,0)$-part of a real form, and in particular an integer form, is conjugate of the $(0,2)$-part); thus we get:

Lefschetz theorem on $(1,1)$-classes. Let X be a smooth projective algebraic variety over $\mathbb{C}$. Every element of $H^2(X,\mathbb{Z}) \cap H^{1,1}(X)$ is the class of a divisor on X. □

See “Albanese varieties”, “Jacobians of compact Riemann surfaces”, “Jacobians, Weil and Griffiths intermediate -”.

Euler sequence. ([93], [207]). Let V be a complex vector space of dimension $n+1$ and let $\mathbb{P}^n = \mathbb{P}(V)$. Consider the trivial bundle on $\mathbb{P}^n$ whose total space is $\mathbb{P}^n \times V$; by a slight abuse of notation (that is, by using for the bundle the notation of the associated sheaf) we denote it $\mathcal{O}_{\mathbb{P}^n} \otimes V$ or simply $\mathcal{O} \otimes V$. Let U be the subbundle of $\mathcal{O} \otimes V$ whose total space is

$$\{(x,v) \in \mathbb{P}^n \times V |\ \ v \in x\}$$

with the obvious map to $\mathbb{P}^n$ (U is called universal bundle, see “Tautological (or universal) bundle”).
The Euler sequence on $\mathbb{P}^n$ is the following exact sequence:

$$0 \longrightarrow U \xrightarrow{i} \mathcal{O} \otimes V \xrightarrow{p} Q \longrightarrow 0,$$

where i is the inclusion map, Q is the quotient bundle and p the projection.
We can easily prove that U is $\mathcal{O}(-1)$ (see “Hyperplane bundles, twisting sheaves”) and Q is $T^{1,0}\mathbb{P}^n(-1)$, where $T^{1,0}\mathbb{P}^n$ is the holomorphic tangent bundle of $\mathbb{P}^n$. So the Euler sequence is

$$0 \longrightarrow \mathcal{O}(-1) \xrightarrow{i} \mathcal{O} \otimes V \xrightarrow{p} T^{1,0}\mathbb{P}^n(-1) \longrightarrow 0.$$

Exact sequences. We say that a complex of modules (see “Complexes”)

$$\dots \xrightarrow{f_{i-2}} M_{i-1} \xrightarrow{f_{i-1}} M_i \xrightarrow{f_i} M_{i+1} \xrightarrow{f_{i+1}} \dots$$

is an exact sequence if $Ker\ f_{i+1} = Im\ f_i$ for all i.
We have that a complex of the kind $0 \to M \xrightarrow{f} M'$ is exact if and only if f is injective.
We have that a complex of the kind $M \xrightarrow{g} M' \to 0$ is exact if and only if g is surjective.
A resolution of a module U is an exact sequence of modules of the kind

$$\dots \longrightarrow M_1 \longrightarrow M_0 \longrightarrow U \longrightarrow 0$$

or of the kind

$$0 \longrightarrow U \longrightarrow M_0 \longrightarrow M_1 \longrightarrow \cdots$$

(see “Sheaves” for the definition of exact sequences of sheaves).

Exponential sequence. ([93], [107]). Let M be a complex manifold. The exponential sequence is the exact sequence of sheaves on M

$$0 \longrightarrow \mathbb{Z} \xrightarrow{j} \mathcal{O} \xrightarrow{exp} \mathcal{O}^* \longrightarrow 0,$$

where
- $\mathbb{Z}$ is the sheaf associating to any open subset U of M the set of the continuous functions from U to the group $\mathbb{Z}$ endowed with the discrete topology;
- $\mathcal{O}$ is the sheaf associating to any open subset U the group of the holomorphic functions on U (with the sum as group operation) ;
- $\mathcal{O}^*$ is the sheaf associating to any open subset U the group of the holomorphic functions on U that vanish nowhere (with the product as group operation);
- the map j is given by the inclusion and the map exp is the map sending any $f \in \mathcal{O}(U)$ to $e^{2\pi i f} \in \mathcal{O}^*(U)$.

Ext, $\mathcal{EXT}$. ([62], [79], [84], [93], [107], [116]). Let R be a commutative ring with unity. Let M and N be R-modules and let

$$\longrightarrow P_n \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

be a projective resolution of M (see "Injective and projective resolutions"); we denote by P_* the complex

$$\longrightarrow P_n \xrightarrow{f_{n-1}} P_{n-1} \longrightarrow \cdots \xrightarrow{f_0} P_0 \longrightarrow 0.$$

We can consider the complex $Hom_R(P_*, N)$:

$$0 \longrightarrow Hom_R(P_0, N) \xrightarrow{d^0} \cdots \longrightarrow Hom_R(P_{n-1}, N) \xrightarrow{d^{n-1}} Hom_R(P_n, N) \xrightarrow{d^n} \cdots,$$

where, for every k, the map d^k is given by the composition with the map f_k. For every k, we define

$$Ext^k_R(M, N) = H^k(Hom_R(P_*, N)),$$

i.e.,

$$Ext^k_R(M, N) = Ker(d^k)/Im(d^{k-1}).$$

One can prove that, if

$$0 \longrightarrow N \longrightarrow I^0 \longrightarrow \cdots \longrightarrow I^{n-1} \longrightarrow I^n \longrightarrow \cdots$$

is an injective resolution of N and I^* is the complex

$$0 \to I^0 \to \cdots \to I^{n-1} \to I^n \to \cdots,$$

then $Ext^k_R(M, N)$ is equal to $H^k(Hom_R(M, I^*))$. We can easily prove that $Ext^k_R(M, N)$ does not depend on the choice of the resolutions and that

$$Ext^0_R(M, N) = Hom_R(M, N).$$

We will write $Ext^k(M, N)$ instead of $Ext^k_R(M, N)$ for simplicity.

Proposition. For any exact sequence of R-modules

$$0 \to A \to B \to C \to 0,$$

there are exact sequences

$$\begin{aligned} 0 &\to Hom(C,N) \to Hom(B,N) \to Hom(A,N) \\ &\to Ext^1(C,N) \to Ext^1(B,N) \to Ext^1(A,N) \\ &\to Ext^2(C,N) \to Ext^2(B,N) \to Ext^2(A,N) \\ &\to \cdots\cdots\cdots , \end{aligned}$$

$$\begin{aligned} 0 &\to Hom(M,A) \to Hom(M,B) \to Hom(M,C) \\ &\to Ext^1(M,A) \to Ext^1(M,B) \to Ext^1(M,C) \\ &\to Ext^2(M,A) \to Ext^2(M,B) \to Ext^2(M,C) \\ &\to \cdots\cdots\cdots . \end{aligned}$$

□

More synthetically, $Ext^i(M,\cdot)$ can be defined as the i-th classical right derived functor of the left exact functor $Hom(M,\cdot)$ (see "Derived categories and derived functors").

Let $(X, \mathcal{O}_X)$ be a ringed space (see "Spaces, ringed -"). Let $\mathcal{M}(X)$ be the category of sheaves of $\mathcal{O}_X$-modules (see "Sheaves").
Let $\mathcal{F}$ and $\mathcal{G}$ be in $\mathcal{M}(X)$. Define $Hom(\mathcal{F},\mathcal{G})$ to be the group of $\mathcal{O}_X$-module homomorphisms and $\mathcal{HOM}(\mathcal{F},\mathcal{G})$ to be the sheaf associated to the presheaf

$$U \mapsto Hom_{\mathcal{O}_{X|U}}(\mathcal{F}|_U, \mathcal{G}|_U).$$

For $\mathcal{F} \in \mathcal{M}(X)$ we define $Ext^i(\mathcal{F},\cdot)$ to be the i-th right derived functor of the functor $Hom(\mathcal{F},\cdot)$ and we define $\mathcal{EXT}^i(\mathcal{F},\cdot)$ to be the i-th right derived functor of the functor $\mathcal{HOM}(\mathcal{F},\cdot)$.

Properties. Let $\mathcal{F}$ be in $\mathcal{M}(X)$. We have

(a) $\mathcal{EXT}^i(\mathcal{O}_X,\mathcal{F}) = \begin{cases} \mathcal{F} & \text{if} \quad i = 0, \\ 0 & \text{if} \quad i > 0; \end{cases}$

(b) $Ext^i(\mathcal{O}_X,\mathcal{F}) \cong H^i(X,\mathcal{F}) \ \ \forall i \geq 0$.

(c) If $0 \to \mathcal{H} \to \mathcal{G} \to \mathcal{F} \to 0$ is an exact sequence in $\mathcal{M}(X)$, then, for any $\mathcal{T}$ in $\mathcal{M}(X)$, we have an exact sequence

$$\begin{aligned} 0 &\to Hom(\mathcal{F},\mathcal{T}) \to Hom(\mathcal{G},\mathcal{T}) \to Hom(\mathcal{H},\mathcal{T}) \to \\ &\to Ext^1(\mathcal{F},\mathcal{T}) \to Ext^1(\mathcal{G},\mathcal{T}) \to Ext^1(\mathcal{H},\mathcal{T}) \to \cdots \end{aligned}$$

and analogously for $\mathcal{EXT}$.

(d) Let $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{L}$ be in $\mathcal{M}(X)$. Let $\mathcal{L}$ be locally free of finite rank and let $\mathcal{L}^\vee = \mathcal{HOM}(\mathcal{L},\mathcal{O}_X)$. Then

$$Ext^i(\mathcal{F}\otimes\mathcal{L},\mathcal{G}) \cong Ext^i(\mathcal{F},\mathcal{L}^\vee\otimes\mathcal{G})$$

and

$$\mathcal{EXT}^i(\mathcal{F}\otimes\mathcal{L},\mathcal{G}) \cong \mathcal{EXT}^i(\mathcal{F},\mathcal{L}^\vee\otimes\mathcal{G}) \cong \mathcal{EXT}^i(\mathcal{F},\mathcal{G})\otimes\mathcal{L}^\vee.$$

(f) If X is a Noetherian scheme and $\mathcal{F},\mathcal{G} \in \mathcal{M}(X)$ with $\mathcal{F}$ coherent (see "Schemes" and "Coherent sheaves"), then, for any $x \in X$ and any $i \geq 0$, we have:

$$\mathcal{EXT}^i(\mathcal{F},\mathcal{G})_x \cong Ext^i_{\mathcal{O}_x}(\mathcal{F}_x,\mathcal{G}_x).$$

(g) Let X be a projective scheme over a Noetherian ring A and let $\mathcal{L}$, $\mathcal{F}$ and $\mathcal{G}$ be in $\mathcal{M}(X)$; let $\mathcal{L}$ be a very ample locally free sheaf of rank 1 and $\mathcal{F}$ and $\mathcal{G}$ be coherent. Then there is an integer $n_0 > 0$ depending on $\mathcal{F}$, $\mathcal{G}$ and i such that for all $n \geq n_0$ we have
$$Ext^i(\mathcal{F}, \mathcal{G} \otimes \mathcal{L}^n) \cong H^0(X, \mathcal{EXT}^i(\mathcal{F}, \mathcal{G} \otimes \mathcal{L}^n)) . \qquad \square$$

More generally the relation between Ext and $\mathcal{EXT}$ is given by a spectral sequence, see [84].

See also "Tor, $\mathcal{TOR}$".

F

Fano varieties. ([68], [131], [191]). A Fano (or anticanonical) variety X is a smooth complete algebraic variety such that K_X^{-1} is ample (where K_X is the canonical bundle, see "Complete varieties", "Canonical bundle, canonical sheaf").
Examples of Fano varieties are the projective spaces and the Del Pezzo surfaces (see "Surfaces, algebraic -").
An instrument to classify Fano varieties is the so-called "index": the index of a Fano variety X is defined to be

$$r_X := \max\{m \in \mathbb{N} |\ K_X^{-1} = mL \ \text{ for some line bundle } L\}.$$

Fibred product. If $f : S \to B$ and $f' : S' \to B$ are two maps, then the fibred product of the sets S and S' with respect to f and f' is defined to be the set

$$S \times_B S' := \{(s, s') \in S \times S' |\ f(s) = f'(s')\}.$$

Five Lemma. ([62], [79], [91], [208], [234]). Let $A_1, \dots, A_5, B_1, \dots, B_5$ be modules over a ring R and let

$$\begin{array}{ccccccccc} A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 & \longrightarrow & A_4 & \longrightarrow & A_5 \\ \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} & & \downarrow{\scriptstyle f_4} & & \downarrow{\scriptstyle f_5} \\ B_1 & \longrightarrow & B_2 & \longrightarrow & B_3 & \longrightarrow & B_4 & \longrightarrow & B_5 \end{array}$$

be a commutative diagram of morphisms of R-modules. If the rows are exact sequences, f_1 is surjective, f_5 is injective and f_2 and f_4 are isomorphisms, then f_3 is an isomorphism.

Flag varieties. ([36], [126], [181]). Flag varieties parametrize the chains of projective subspaces in a projective space. Precisely, for any field K and $k_1, \dots, k_s, n \in \mathbb{N}$, we define the flag variety $F(k_1, \dots, k_s, n, K)$ in the following way:

$$F(k_1, \dots, k_s, n, K) = \{(T_1, \dots, T_s) \in G(k_1, \mathbb{P}^n_K) \times \dots \times G(k_s, \mathbb{P}^n_K) | \, T_1 \subset \dots \subset T_s\},$$

where $G(k, \mathbb{P}^n_K)$ denotes the Grassmannian of projective k-planes in $\mathbb{P}^n_K$.

Theorem. $F(k_1, \dots, k_s, n, \mathbb{C})$ is a rational, homogeneous (hence smooth) variety (see "Homogeneous varieties", "Rational varieties") of dimension

$$\sum_{j=1,\dots,s} (k_{j+1} - k_j)(k_j + 1)$$

where we set $k_{s+1} = n$. □

Flat (module, morphism). ([12], [62], [107], [116], [185]). Let R be a ring. We say that an R-module N is flat if the functor

$$M \mapsto M \otimes_R N$$

is exact, that is, if it changes every exact sequence into an exact sequence.

Proposition. We have that N is flat if and only if every exact sequence of R-modules

$$0 \longrightarrow M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3 \longrightarrow 0 \tag{3}$$

implies the exact sequence

$$0 \longrightarrow M_1 \otimes N \xrightarrow{f \otimes I} M_2 \otimes N \xrightarrow{g \otimes I} M_3 \otimes N \longrightarrow 0\,.$$ □

Observe that, in general, the exact sequence (3) implies only the exact sequence

$$M_1 \otimes N \xrightarrow{f \otimes I} M_2 \otimes N \xrightarrow{g \otimes I} M_3 \otimes N \longrightarrow 0,$$

thus the functor $M \mapsto M \otimes_R N$ is only right-exact. For instance, let m and k be positive natural numbers and consider the exact sequence of $\mathbb{Z}$-modules

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\cdot m} \mathbb{Z} \longrightarrow \mathbb{Z}/m\mathbb{Z} \longrightarrow 0,$$

where the map we have indicated by $\cdot m$ is the multiplication by the number m; by tensoring by $\mathbb{Z}/k$, we get

$$0 \longrightarrow \mathbb{Z}/GCD(k, m) \longrightarrow \mathbb{Z}/k \xrightarrow{\cdot m} \mathbb{Z}/k \longrightarrow \mathbb{Z}/GCD(k, m) \longrightarrow 0.$$

We say that a morphism of varieties (or schemes) $f : X \to Y$ is flat if, for all $x \in X$, $y \in Y$ such that $f(x) = y$, we have that $\mathcal{O}_{X,x}$ is a flat $\mathcal{O}_{Y,y}$-module via the map induced by f, $f_x : \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$, where $\mathcal{O}_{X,x}$ is the stalk in x of the sheaf of the regular functions on X and analogously $\mathcal{O}_{Y,y}$.
See "Tor", "Grauert's semicontinuity theorem".

Flexes. ([93], [107], [228], [246]). Let K be an algebraic closed field of characteristic 0.

Definition. A point P of a plane algebraic curve C over K is said to be a flex if it is a smooth point and $mult_P(C, T_P(C)) \geq 3$, where $T_P(C)$ is the tangent line of C at P and $mult_P(C, T_P(C))$ is defined as follows:
In an affine neighborhood of P, write $T_P(C) = \{P + tv | \ t \in K\}$ for some vector v; let f be a polynomial whose zero locus is C; consider the polynomial, in t, $f|_{T_P(C)}$; define $mult_P(C, T_P(C))$ to be the multiplicity of 0 as zero of $f|_{T_P(C)}$. □

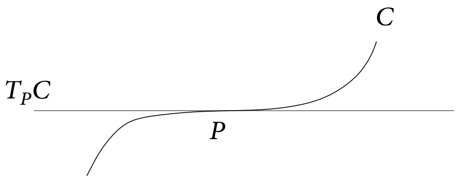


Fig. 8. Flex.

Proposition. Let C be a projective algebraic curve in $\mathbb{P}^2_K$ and let F be a homogeneous polynomial in z_0, z_1, z_2 such that C is the zero locus of F. A smooth point of C is a flex if and only if it is in the zero locus of the hessian of C, where the hessian of C is the determinant of the matrix

$$\left(\frac{\partial^2 F}{\partial z_i \partial z_j} \right)_{i,j \in \{0,1,2\}}$$

(observe that the partial derivatives of a polynomial can be defined over any field by applying the usual rules for derivatives).
In particular, if C is smooth and of degree ≥ 3, then C has at least one flex (see "Bezout's theorem"). □

Fubini-Study metric. ([93]). Let $\pi : \mathbb{C}^{n+1} - \{0\} \longrightarrow \mathbb{P}^n := \mathbb{P}^n_{\mathbb{C}}$ be the standard projection. Let $\sigma : U \longrightarrow \mathbb{C}^{n+1} - \{0\}$ be a holomorphic map on an open subset U of $\mathbb{P}^n$ such that $\pi \circ \sigma$ is the identity; define

$$\omega_U := \frac{i}{2\pi} \partial\bar{\partial} \log(|\sigma|^2).$$

One can show that ω_U does not depend on the choice of the map σ and in this way we define a form ω on all $\mathbb{P}^n$. We can prove that ω is closed, of type $(1, 1)$, positive and its class in cohomology is the Poincaré dual of a hyperplane (see "Positive" and "Singular homology and cohomology"). The form ω is called the Fubini-Study form. We define the Fubini-Study metric to be the Kälerian metric whose Kähler form is ω (see "Hermitian and Kählerian metrics").

Functors. See "Categories".

Fundamental group. ([37], [38], [91], [112], [158], [215], [234]). Let X be a topological space.
A **path** in X is a continuous map $\gamma : [0, 1] \to X$.
A **loop** in X is a path $\gamma : [0, 1] \to X$ such that $\gamma(0) = \gamma(1)$.
Given two paths α and β in X, we define their composition $\alpha\beta$ to be the path in X such that

$$(\alpha\beta)(s) = \begin{cases} \alpha(2s) & \text{if } s \in [0, 1/2], \\ \beta(2s-1) & \text{if } s \in [1/2, 1]. \end{cases}$$

Let $x_0, x_1 \in X$ and let α and β be two paths in X with $\alpha(0) = \beta(0) = x_0$ and $\alpha(1) = \beta(1) = x_1$. We say that α and β are **homotopic with end points held fixed** if there exists a continuous map $F : [0, 1] \times [0, 1] \to X$ such that

a) $F(t, 0) = \alpha(t)$ for any t;
b) $F(t, 1) = \beta(t)$ for any t;
c) $F(0, s) = x_0$ for any s;
d) $F(1, s) = x_1$ for any s.

Obviously, for each s, the map $t \mapsto F(t, s)$ is a path from x_0 to x_1.

Let $x_0 \in X$. Let Ω_{X,x_0} be the set of loops γ in X such that $\gamma(0) = \gamma(1) = x_0$. The **first fundamental group** of X, denoted by $\pi_1(X, x_0)$, is defined as follows:

$$\pi_1(X, x_0) := \Omega_{X,x_0} / \text{ homotopy with end points held fixed.}$$

It is a group with the composition given by the composition of paths defined above.

Definition. We say that a path-connected topological space is **simply connected** if its first fundamental group is trivial. □

Definition. Let X and Y be two topological spaces and let $x_0 \in X$ and $y_0 \in Y$. If $f : X \to Y$ is a continuous map such that $f(x_0) = y_0$, then we define

$$f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$$

to be the homomorphism such that $f_*([\gamma]) = [f \circ \gamma]$ for any $\gamma \in \Omega_{X,x_0}$. □

Before stating the main theorems about the fundamental group, we need to recall the definition of the free product of groups.

Let G_i, for $i \in I$, be groups. Their **free product** $*_{i\in I} G_i$ is defined to be

$$W(\sqcup_{i\in I} G_i)/\sim,$$

where $W(\sqcup_{i\in I} G_i)$ is the set of words with letters in $\sqcup_{i\in I} G_i$ (i.e., the finite sequences of elements of the union of the G_i) and $\sim$ is the equivalence relation satisfying

$$(g_1 \dots g_k \, e_{G_j} \, g_{k+1} \dots g_{k+s}) \sim (g_1 \dots g_k \, g_{k+1} \dots g_{k+s}) \tag{1}$$

for any $k, s \in \mathbb{N}$, for any $g_1, \ldots, g_{k+s} \in \sqcup_{i \in I} G_i$, for any $j \in I$ and where e_{G_j} is the identity element of G_j, and

(2) $$(g_1 \, \ldots \, g_k \, g' \, g'' \, g_{k+1} \, \ldots \, g_{k+s}) \sim (g_1 \, \ldots \, g_k \, (g' \cdot g'') \, g_{k+1} \, \ldots \, g_{k+s})$$

for any $k, s \in \mathbb{N}$, for any $g_1, \ldots, g_{k+s}, g', g'' \in \sqcup_{i \in I} G_i$, if g' and g'' are elements of the same group G_j for some j.
Let us denote by $[g_1 \, \ldots \, g_s]$ the equivalence class of the word $(g_1 \, \ldots \, g_s)$.
We define the product in $*_{i \in I} G_i$ to be the product given by the concatenation, i.e., let

$$[h_1 \, \ldots \, h_k] \cdot [l_1 \, \ldots \, l_s] = [h_1 \, \ldots \, h_k l_1 \, \ldots \, l_s].$$

With this product, the free product $*_{i \in I} G_i$ is a group; it contains all the G_i as subgroups.

If $\mathcal{R} = \{R_\alpha\}_{\alpha \in A}$ is a subset of $W(\sqcup_{i \in I} G_i)$, then the **free product of the G_i with the relations $\mathcal{R}$**, denoted by $*G_i/\mathcal{R}$, is defined to be

$$W(\sqcup_{i \in I} G_i)/ \sim',$$

where $\sim'$ is the equivalence relation satisfying (1), (2), and

(3) $$(g_1 \, \ldots \, g_k \, R_\alpha \, g_{k+1} \, \ldots \, g_{k+s}) \sim (g_1 \, \ldots \, g_k \, g_{k+1} \, \ldots \, g_{k+s})$$

for any $k, s \in \mathbb{N}$, for any $g_1, \ldots, g_{k+s} \in \sqcup_{i \in I} G_i$ and for any $\alpha \in A$. If we endow it with the product given by concatenation, it is a group.

The following theorem (due to Seifert and Van Kampen, but generally called Van Kampen's theorem I) computes, under some assumptions, the fundamental group of a space X from the fundamental groups of two subspaces whose union is X.

Van Kampen's theorem I. Let $X = X_1 \cup X_2$ be a topological space and let $X_0 = X_1 \cap X_2$ with X_j open, path-connected for $j = 0, 1, 2$. Let $x_0 \in X_0$. We have

$$\pi_1(X, x_0) = \big(\pi_1(X_1, x_0) * \pi_1(X_2, x_0)\big)/\langle i_{1\,*}(\gamma) \; i_{2\,*}(\gamma)^{-1} \rangle_{\gamma \in \pi_1(X_0, x_0)},$$

where $i_j : X_j \to X$ are the inclusions and the second member can be interpreted both as the free product of $\pi_1(X_1, x_0)$ and $\pi_1(X_2, x_0)$ with the relations $i_{1\,*}(\gamma) \; i_{2\,*}(\gamma)^{-1}$ for $\gamma \in \pi_1(X_0, x_0)$ and as the quotient of $\pi_1(X_1, x_0) * \pi_1(X_2, x_0)$ by the normal subgroup generated by $i_{1\,*}(\gamma) \; i_{2\,*}(\gamma)^{-1}$ for $\gamma \in \pi_1(X_0, x_0)$. □

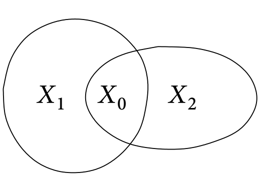


Fig. 9. Van Kampen's theorem I.

For the following theorem, which is often called Van Kampen's theorem II, see [37]. The main instrument for the proof is the so-called "fundamental groupoid": instead of considering the loops based at a fixed point up to homotopy, one can consider all the paths up to homotopy fixing the initial and final point; this yields not a group but a groupoid, a category where all arrows are isomorphisms; the groupoid obtained in this way is called the fundamental groupoid of the space.

Van Kampen's theorem II. Let $X = X_1 \cup X_2$ be a topological space such that
- X_i is open and path-connected for $i = 1, 2$;
- $X_1 \cap X_2 = A \cup B$ with A and B disjoint path-connected open subsets;
- X_2, A and B are simply connected.

Then
$$\pi_1(X) = \pi_1(X_1) * \mathbb{Z}. \qquad \square$$

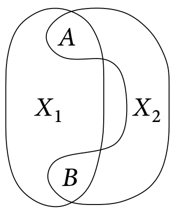


Fig. 10. Van Kampen's theorem II.

Attaching cells. Let $B^n = \{x \in \mathbb{R}^n \mid |x| \le 1\}$ and $S^{n-1} = \{x \in \mathbb{R}^n \mid |x| = 1\}$.
Let X be a path-connected Hausdorff topological space. Let $\tilde{X}$ be a topological space obtained by attaching an n-cell (B^n, S^{n-1}) to X by a continuous map
$$f : S^{n-1} \longrightarrow X,$$
i.e., let $\tilde{X}$ be the quotient of $X \sqcup B^n$ by the equivalence relation determined by the identification of x with $f(x)$ for every $x \in S^{n-1}$. Let $x_0 \in X$.
If $n = 1$, then $\pi_1(\tilde{X}, x_0) = \pi_1(X, x_0) * \mathbb{Z}$.
If $n > 1$, then $\pi_1(\tilde{X}, x_0) = \pi_1(X, x_0)/i_*(\pi_1(f(S^{n-1})))$, where $i : f(S^{n-1}) \hookrightarrow X$ is the inclusion.

Theorem. Let X be a topological space, and let $x_0 \in X$. Let $H_1(X, \mathbb{Z})$ denotes the first singular homology module of X over $\mathbb{Z}$ (see "Singular homology and cohomology"). There is a group homomorphism
$$p : \pi_1(X, x_0) \to H_1(X, \mathbb{Z})$$
which sends the homotopy class of γ to the homology class of γ for any loop γ with $\gamma(0) = x_0$. If X is path connected, then p is surjective and its kernel is the commutator

subgroup

$$\langle \alpha\beta\alpha^{-1}\beta^{-1} | \ \alpha, \beta \in \pi_1(X, x_0)\rangle \,. \qquad \square$$

Definition. For any topological space X, we denote by $\pi_0(X)$ the set of the path-connected components of X. $\square$

Let X be a topological space. Let $X^{[0,1]}$ be the set of the paths in X; endow it with the compact-open topology, i.e., the topology whose prebase is the set of the subsets

$$[K, U] := \{\sigma : [0,1] \to X | \ \sigma(K) \subset U\}$$

for K compact subset of $[0, 1]$ and U open subset of X. For any $x_0 \in X$, the subset Ω_{X,x_0} is closed in $X^{[0,1]}$ if x_0 is closed in X. Endow the subset Ω_{X,x_0} with the induced topology.

Remark. Let X be a topological space and $x_0 \in X$. Two elements of Ω_{X,x_0} are in the same path-connected component of Ω_{X,x_0} if and only if they are homotopic with end points held fixed, i.e., $\pi_1(X, x_0) = \pi_0(\Omega_{X,x_0})$. $\square$

Definition. We define

$$\pi_n(X, x_0) = \pi_{n-1}(\Omega_{X,x_0}, C)$$

for all $n \geq 2$, where C is the constant path in x_0. $\square$

Theorem. $\pi_n(X, x_0)$ is commutative for all $n \geq 2$. $\square$

Remark. $\pi_n(X\times Y, (x_0, y_0)) = \pi_n(X, x_0)\times\pi_n(Y, y_0)$ for any X, Y topological spaces, $x_0 \in X$, $y_0 \in Y$ and for all $n \geq 1$. $\square$

Remark. If a topological space X is path-connected, then, for every n, the group $\pi_n(X, x_0)$ does not depend on x_0; thus we can speak of n-fundamental group $\pi_n(X)$ without specifying the point x_0. $\square$

Remark. Let X and Y be two topological spaces and let $x_0 \in X$ and $y_0 \in Y$. If $f : X \to Y$ is a continuous map such that $f(x_0) = y_0$, then there is an obvious induced map from Ω_{X,x_0} to Ω_{Y,y_0}, so, by induction, we can define a homomorphism

$$f_* : \pi_n(X, x_0) \to \pi_n(Y, y_0)$$

for every n. $\square$

Theorem. Let $f : X \to Y$ and $g : X \to Y$ be two continuous maps between two topological spaces such that $f(x_0) = g(x_0) = y_0$. If f and g are homotopic, then, for every n, the maps $f_* : \pi_n(X, x_0) \to \pi_n(Y, y_0)$ and $g_* : \pi_n(X, x_0) \to \pi_n(Y, y_0)$ are equal. $\square$

In particular the fundamental groups of two homotopic topological spaces are isomorphic.

Hurewicz theorem. If $n \geq 2$ and $\pi_q(X)$ is trivial for any $q \in \{0, \dots, n-1\}$, then $H_q(X) = 0$ for any $q \in \{1, \dots, n-1\}$ and $\pi_n(X) = H_n(X)$. □

Theorem. Let $\pi : E \to B$ be a topological bundle and let B be path-connected. Let $f \in E$, $b = \pi(f)$ and let F be the fibre on b. Then there is an exact sequence

$$\cdots \to \pi_q(F, f) \to \pi_q(E, f) \to \pi_q(B, b) \to \pi_{q-1}(F, f) \to \cdots \to \pi_0(E) \to 1.$$

In particular, if the fibre is discrete, i.e., the bundle is a covering space, we get that $\pi_q(E, f)$ and $\pi_q(B, b)$ are isomorphic for any $q \geq 2$. □

Examples.
- $\pi_1(S^1) = \mathbb{Z}$, $\pi_1(S^n) = 0$ if $n \geq 2$.
- If B_q is the bouquet of q circles, then $\pi_1(B_q) = \mathbb{Z} * \cdots * \mathbb{Z}$ (q times).
- $\pi_1(\mathbb{P}^1_{\mathbb{R}}) = \mathbb{Z}$.
- $\pi_1(\mathbb{P}^n_{\mathbb{R}}) = \mathbb{Z}/2$ if $n \geq 2$.
- $\pi_1(\mathbb{P}^n_{\mathbb{C}})$ is trivial for all n.
- Let T_g be the topological torus with g holes, i.e., the topological space obtained by attaching to a bouquet of $2g$ circles ($\lambda_1, \dots, \lambda_{2g}$) a $4g$-agon with the law $\lambda_1, \lambda_{g+1}, \lambda_1^{-1}, \lambda_{g+1}^{-1}, \dots$. We have

$$\pi_1(T_g) = \mathbb{Z}\lambda_1 * \cdots * \mathbb{Z}\lambda_{2g} / \lambda_1 \lambda_{g+1} \lambda_1^{-1} \lambda_{g+1}^{-1} \cdots \lambda_g \lambda_{2g} \lambda_g^{-1} \lambda_{2g}^{-1}.$$

- $\pi_m(S^n) = 0$ if $m < n$, $\pi_m(S^m) = \mathbb{Z}$.

G

G.A.G.A. ([93], [107], [222], [228], [241]). G.A.G.A. is the abbreviation of the title of Serre's paper [222] "Géométrie algébrique et géométrie analytique". Roughly speaking, the G.A.G.A. principle can be formulated as follows. Global analytic "things" on a projective algebraic variety X over $\mathbb{C}$ are algebraic; for instance an analytic subvariety of X is an algebraic subvariety (see "Chow's theorem"), a meromorphic function on X is rational, a holomorphic vector bundle is an algebraic vector bundle. Precisely, there is a functor h from the category of schemes of finite type over $\mathbb{C}$ to the category of analytic spaces (see "Spaces, analytic -") and Serre proved the following theorem and obtained some of the results above (some of them were already known) as consequences of it.

Theorem. For any projective scheme X over $\mathbb{C}$, the functor h induces an equivalence of categories from the category of coherent sheaves on X (see "Coherent sheaves") to the category of coherent sheaves on the analytic space associated to X and this equivalence maintains the cohomology. □

Gauss–Bonnet–Hopf theorem. ([93]). Let M be a compact complex manifold of dimension n. We have

$$c_n(M) = \chi(M),$$

where $\chi(M)$ is the Euler characteristic $\sum_i (-1)^i b_i(M, \mathbb{R})$ and $c_n(M)$ is the n-th Chern class of the holomorphic tangent bundle $T^{1,0}M$. More precisely, here, by $c_n(M)$, we mean $c_n(M)$ evaluated in the fundamental class of M, i.e., in the element of $H_{2n}(M, \mathbb{Z})$ giving the orientation of M (see "Chern classes", "Singular homology and cohomology").

General type, of -. See "Kodaira dimension (or Kodaira number)".

Genus, arithmetic, geometric, real, virtual -. ([93], [107], [118], [196], [241]).
Let K be an algebraically closed field.
Let X be a compact complex manifold or a projective algebraic variety over K. The **arithmetic genus** $p_a(X)$ is defined by

$$\boxed{p_a(X) = (-1)^n(\chi(X, \mathcal{O}_X) - 1).}$$

where $\mathcal{O}_X$ is the sheaf of the holomorphic, respectively regular, functions on X.
Let X be a compact complex manifold or a *smooth* projective algebraic variety over K. The **geometric genus** $p_g(X)$ is defined by

$$\boxed{p_g(X) = h^0(X, \omega_X),}$$

where ω_X is the canonical sheaf (see "Canonical bundle, canonical sheaf").

Theorem. If X and X' are birationally equivalent smooth projective algebraic varieties over K (see "Varieties, algebraic -, Zariski topology, regular and rational functions, morphisms and rational maps" for the definition of birational equivalent), then $p_g(X) = p_g(X')$ and, if the characteristic of K is 0, $p_a(X) = p_a(X')$ (see [118], [107]). □

If X is a singular projective algebraic variety, its geometric genus is defined to be the geometric genus of a desingularization of X, that is, the geometric genus of a smooth projective algebraic variety that is birational equivalent to X, when it exists.

Proposition. If C is a projective algebraic curve, then $p_a(C) = h^1(C, \mathcal{O}_C)$.
Moreover, if C is smooth, we have $p_g(C) = p_a(C) = h^1(C, \mathcal{O}_C)$ (by Serre duality; see "Serre duality").
If $K = \mathbb{C}$ and C is smooth, and then, from the topological viewpoint, C is a topological torus with a certain number of holes, we have that the number of holes is equal to $p_g(C)$ (and $p_a(C)$) (Riemann's Theorem; see "Riemann surfaces (compact -) and algebraic curves"). □

Let S be a smooth projective algebraic surface and C be an algebraic curve in S (we recall that for us a curve is irreducible). By applying the Riemann–Roch theorem for surfaces (see "Surfaces, algebraic -") to calculate $\chi(\mathcal{O}_S(-C))$ and by using the exact sequence $0 \to \mathcal{O}_S(-C) \to \mathcal{O}_S \to \mathcal{O}_C \to 0$, we get that

$$h^1(C, \mathcal{O}_C) = \frac{C \cdot C + K_S \cdot C}{2} + 1,$$

and thus

$$p_a(C) = \frac{C \cdot C + K_S \cdot C}{2} + 1$$

(observe that, if C is smooth, we can be prove that this number is also equal to the geometric genus by adjunction formula).
Sometimes, the number

$$\frac{C \cdot C + K_S \cdot C}{2} + 1$$

is called the **virtual genus** of a curve C on a smooth algebraic projective surface S, or, more generally, of a divisor C on a smooth projective algebraic surface S. If C is a curve in a smooth projective algebraic surface over $\mathbb{C}$, we have that the virtual genus is the genus (the geometric one, the arithmetic one and the number of the holes) of every smooth curve C' homologous to C (in fact, since C' is homologous to C we have that $C \cdot C = C' \cdot C'$ and $K_S \cdot C = K_S \cdot C'$, so $\frac{C \cdot C + K_S \cdot C}{2} + 1 = \frac{C' \cdot C' + K_S \cdot C'}{2} + 1$, which is the genus of C').

Observe that, if $S = \mathbb{P}^2$ and C is an algebraic curve of degree d, then the virtual genus (and thus the arithmetic genus) is

$$\frac{(d-1)(d-2)}{2}$$

(in fact $C \cdot C = d^2$ and $C \cdot K_S = -3d$).

In some books, e.g., [93], the **real genus** of a projective algebraic curve C is defined to be the genus of $\tilde{C}$, where $\tilde{C}$ is the desingularization (which is the normalization; see "Normal") of C. So it is another name for the geometric genus.

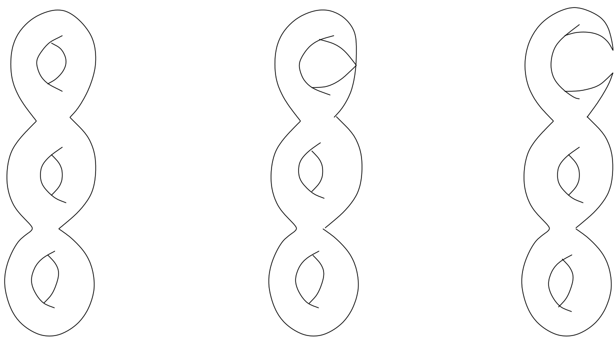
Let S be a smooth projective algebraic surface over K, and let C be an algebraic curve in S. It holds (see, e.g., [107, Chapter V, § 3])

$$\boxed{p_g(C) = p_a(C) - \sum_P \frac{m_P(m_P - 1)}{2},}$$

where m_P is the multiplicity at the point P (see "Multiplicity of a curve in a surface at a point") and the sum is over all points P of C, including the so-called "infinitely near points", i.e., all the points on the strict transforms obtained by blowing up the curve in the singular points until we get a smooth curve (see "Blowing-up (or σ-process)").

Example 1. Let $C = \{[x : y : z] \in \mathbb{P}^2_{\mathbb{C}} | \ z^2(x^2 - y^2) + x^4 + y^4 = 0\}$; we have that

$$p_a(C) = \frac{(4-1)(4-2)}{2} = 3.$$



C', *smooth curve homolgous to* C $\qquad$ C $\qquad$ $\tilde{C}$ *desingularization of* C

Fig. 11. Real genus and virtual genus.

Let us calculate $p_g(C)$. The unique singular point of C is the point $P = [0:0:1]$; its multiplicity is 2. Let $U = \{[x:y:z] \in \mathbb{P}^2_{\mathbb{C}} |\ z \neq 0\} = \mathbb{C}^2_{x,y}$:

$$C \cap U = \{(x,y) |\ x^2 - y^2 + x^4 + y^4 = 0\};$$

the blowing up of $U = \mathbb{C}^2_{x,y}$ in $P = (0,0)$ is

$$\{((x,y),[t_0:t_1]) |\ t_0 y = t_1 x\} \xrightarrow{\pi} U = \mathbb{C}^2_{x,y};$$

let $t = t_1/t_0$; the strict transform C_1 of $C \cap U$ is

$$\{(x,t) |\ 1 - t^2 + x^2 + t^4 x^2 = 0\}$$

and $\pi^{-1}(P) \cap C_1 = \{x = 0\} \cap C_1$ is the union of the points $Q = (0,1)$ and $R = (0,-1)$, both smooth (thus P is a node; see "Regular rings, smooth points, singular points"). Hence

$$p_g(C) = 3 - \frac{1}{2} m_P(m_P - 1) = 2$$

(see Figure 11).

Example 2. Let $C = \{[x:y:z] \in \mathbb{P}^2_{\mathbb{C}} |\ z^4 y^2 - x^6 + z^6 = 0\}$; we have

$$p_a(C) = \frac{(6-1)(6-2)}{2} = 10.$$

Let us calculate $p_g(C)$. The unique singular point of C is the point $P = [0:1:0]$; its multiplicity is 4. Let $U = \{[x:y:z] \in \mathbb{P}^2_{\mathbb{C}} |\ y \neq 0\} = \mathbb{C}^2_{x,z}$:

$$C \cap U = \{(x,z) |\ z^4 - x^6 + z^6 = 0\};$$

the blowing up of $U = \mathbb{C}^2_{x,z}$ in $P = (0,0)$ is

$$\{((x,z),[t_0:t_1]) |\ t_0 z = t_1 x\} \xrightarrow{\pi_1} U = \mathbb{C}^2_{x,z};$$

let $t = t_1/t_0$; the strict transform C_1 of $C \cap U$ is

$$\{(x,t)| \ -x^2 + t^4 + x^2t^6 = 0\}$$

and $\pi_1^{-1}(P) \cap C_1 = \{x = 0\} \cap C_1$ is the point $Q = (0,0)$, which has multiplicity 2; the blowing up of $\mathbb{C}^2_{x,t}$ in $Q = (0,0)$ is

$$\{((x,t),[l_0:l_1])| \ l_0 t = l_1 x\} \xrightarrow{\pi_2} \mathbb{C}^2_{x,t};$$

let $l = l_0/l_1$; the strict transform C_2 of C_1 is

$$\{(l,t)| \ -l^2 + t^2 + l^2t^6 = 0\},$$

and $\pi_2^{-1}(Q) \cap C_2 = \{t = 0\} \cap C_2$ is the point $R = (0,0)$, which has multiplicity 2; the blowing up of $\mathbb{C}^2_{l,t}$ in $R = (0,0)$ is

$$\{((l,t),[s_0:s_1])| \ s_0 t = s_1 l\} \xrightarrow{\pi_3} \mathbb{C}^2_{l,t};$$

let $s = s_1/s_0$; the strict transform C_3 of C_2 is

$$\{(s,l)| \ -1 + s^2 + s^6l^6 = 0\}$$

and $\pi_3^{-1}(R) \cap C_3 = \{l = 0\} \cap C_3$ is the set of the smooth points $G = (1,0)$ and $H = (-1,0)$. Thus

$$p_g(C) = 10 - \frac{1}{2}[m_P(m_P - 1) + m_Q(m_Q - 1) + m_R(m_R - 1)] = 10 - 8 = 2.$$

Geometric invariant theory (G.I.T.). ([81], [82], [194], [199], [204]). Let K be an algebraic closed field. Let G be a reductive algebraic group over K (see "Representations", "Algebraic groups", "Lie groups"). Suppose G acts on a scheme X of finite type over $Spec(K)$ (see "Schemes"). Geometric invariant theory is the theory investigating whether the quotient X/G is a "good" quotient. More precisely:

Definition. A **good quotient** of X by G is a surjective affine G-invariant morphism

$$f : X \to Y,$$

with Y scheme of finite type over $Spec(K)$, such that
(i) $f_*(\mathcal{O}_X)^G = \mathcal{O}_Y$, where $f_*(\mathcal{O}_X)^G$ denotes the G-invariant part of $f_*(\mathcal{O}_X)$ (see "Direct and inverse image sheaves" for the definition of $f_*(\mathcal{O}_X)$);
(ii) the image of a G-invariant closed subset is closed and the images of two disjoint G-invariant closed subsets are disjoint.

If, in addition, for all $y \in Y$, the set $f^{-1}(y)$ is exactly one orbit, then we say that f is a **geometric quotient**. □

Remark. When a good quotient exists, it is unique (up to isomorphism). □

From now on, suppose that $X \subset \mathbb{P}^n_K$ and that the action of G on X is the restriction of a representation $\rho : G \to GL(n+1, K)$.

Definition. We say that $x \in X$ is **semistable** if there exists a G-invariant homogeneous nonconstant polynomial F in $n+1$ variables such that $F(x) \neq 0$.
We say that $x \in X$ is **stable** if
(i) the dimension of the orbit of x is equal to the dimension of G;
(ii) there exists a G-invariant homogeneous nonconstant polynomial F such that $F(x) \neq 0$, i.e., $x \in X - Z(F)$, where $Z(F)$ is the zero locus of F in X, and, for every $x' \in X - Z(F)$, the orbit of x' is closed in $X - Z(F)$.

We denote the set of semistable points by X^{ss} and the set of stable points by X^s. □

Theorem. There exists a projective scheme Y and a morphism $f : X^{ss} \to Y$ which is a good quotient of X^{ss} by G. Moreover, there exists an open subset U of Y such that $f^{-1}(U) = X^s$ and $f|_{X^s} : X^s \to U$ is a geometric quotient. □

Definition. An **1-parameter subgroup** of an algebraic group H is a nontrivial homomorphism of algebraic groups

$$\lambda : K^* \to H,$$

where K^* is the multiplicative group of the nonzero elements of the field K. □

Definition. Fix a 1-parameter subgroup $\lambda : K^* \to G$; we can prove that there exists a basis $\{v_0, \dots, v_n\}$ of K^{n+1} and integers r_i for $i = 0, \dots, n$ such that

$$(\rho \circ \lambda)(t)(v_i) = t^{r_i} v_i,$$

for any $t \in K^*$, $i = 0, \dots, n$. Let $x_0, \dots, x_n$ be the coordinates with respect to $\{v_0, \dots, v_n\}$. Let $x = [x_0 : \cdots : x_n] \in X \subset \mathbb{P}^n_K$. We say that x is **λ-semistable** (respectively **λ-stable**) if $\max\{r_i | \; x_i \neq 0\} \geq 0$ (respectively > 0). □

Theorem. A point in X is semistable (respectively stable) if and only if it is λ-semistable (respectively λ-stable) for every λ 1-parameter subgroup of G. □

Globally generated.
See "Bundles, fibre -".

Gorenstein.
See "Cohen-Macaulay, Gorenstein, (arithmetically -,-)".

Grassmannians.
([75], [93], [104], [120], [181], [188]). Let V be a vector space of dimension n over a field K. For any $k = 1, \dots, n$, let $G(k, V)$ denote the Grassmannian of the k-subspaces of V.
If $K = \mathbb{R}$ or $\mathbb{C}$, the set $G(k, V)$ can be endowed with a topology which makes it a compact manifold of dimension $k(n-k)$ respectively over $\mathbb{R}$ and over $\mathbb{C}$ (for instance, if

$K = \mathbb{R}$; see $G(k, V)$ as $O(n)/O(k) \times O(n-k)$, where $O(n)$ is the group of the orthogonal matrices $n \times n$).
The map

$$\boxed{\varphi_k : G(k, V) \longrightarrow \mathbb{P}(\wedge^k V)},$$
$$\langle v_1, \dots, v_k \rangle \longmapsto [v_1 \wedge \dots \wedge v_k]$$

is injective and is called the **Plücker embedding**.
Let us fix a basis $\{e_1, \dots, e_n\}$ of V. It induces a basis on $\wedge^k V$ and thus homogeneous coordinates on $\mathbb{P}(\wedge^k V)$:

$$p_{j_1,\dots,j_k}, \qquad j_i \in \{1, \dots, n\}, \qquad j_1 < \dots < j_k.$$

For any $W \in G(k, V)$, the coordinates $p_{j_1,\dots,j_k}$ of $\varphi_k(W)$ are called **Plucker coordinates** of W. We can define $p_{j_1,\dots,j_k}$ for *all* $j_1, \dots, j_k \in \{1, \dots, n\}$: let $p_{j_1,\dots,j_k} = 0$ if $j_1, \dots, j_k$ are not distinct, and let

$$p_{j_1,\dots,j_k} = \varepsilon(\sigma) p_{j_{\sigma(1)},\dots,j_{\sigma(k)}}$$

for every σ permutation on $\{1, \dots, k\}$ (where $\varepsilon(\sigma)$ is the sign of σ).

Theorem. Let the characteristic of K be 0. The image of the Grassmannian $G(k, V)$ by the Plücker embedding is a smooth algebraic variety defined by quadrics, precisely by the following quadrics:

$$\boxed{\textstyle\sum_{i=0,\dots,k} (-1)^i p_{j_0,\dots,\hat{j_i},\dots,j_k} p_{j_i, r_1,\dots,r_{k-1}} = 0}$$

for all $j_0, \dots, j_k, r_1, \dots, r_{k-1}$. □

Let $K = \mathbb{C}$ (but an analogous theory can be developed also over $\mathbb{R}$; see [188]). Let

$$V = V_n \supseteq V_{n-1} \supseteq \dots \supseteq V_1 \supseteq V_0$$

be a flag of subspaces in V with $dim(V_i) = i$. Let $W \in G(k, V)$. For any $i = 1, \dots, k$ we define

$$\boxed{\sigma_i(W) = \min\{t \in \mathbb{N} |\ dim W \cap V_t = i\}}.$$

Observe that $1 \le \sigma_1(W) < \dots < \sigma_k(W) \le n$. We define the **Schubert symbol of W with respect to the given flag** by

$$\sigma(W) = (\sigma_1(W), \dots, \sigma_k(W)).$$

Now let $\sigma_1, \dots, \sigma_k$ be natural numbers with $1 \le \sigma_1 < \dots < \sigma_k \le n$. We define the **Schubert cell** with Schubert symbol $\sigma = (\sigma_1, \dots, \sigma_k)$, and we denote it by C_σ, as follows:

$$\boxed{C_\sigma = \{W \in G(k, V) |\ \sigma(W) = \sigma\}}.$$

Thus,

$$C_\sigma = \{W \in G(k,V) |\ dim(W \cap V_{\sigma_i}) = i \text{ and } dim(W \cap V_{\sigma_i - 1}) < i \ \ \forall i = 1, \dots, k\}.$$

Proposition. Let $\sigma_1, \dots, \sigma_k$ be natural numbers with $1 \le \sigma_1 < \dots < \sigma_k \le n$. We have that $C_\sigma \cong \mathbb{C}^{\sum_{i=1,\dots,k}(\sigma_i - i)}$. Moreover, if we denote the closure of C_σ by $\overline{C}_\sigma$, we have that

$$\overline{C}_\sigma = \cup_{\tau \le \sigma} C_\tau = \{W \in G(k,V) |\ dim(W \cap V_{\sigma_i}) \ge i \ \ \forall i = 1, \dots, k\}$$

(where $\tau \le \sigma$ means $\tau_i \le \sigma_i$ for any i) and $\overline{C}_\sigma$ is an algebraic subvariety of the Plücker embedding of $G(k,V)$. The Grassmannian $G(k,V)$ is a CW-complex, the cells of which are the Schubert cells. □

The subset $\overline{C}_\sigma$ is called the Schubert cycle, with the Schubert symbol σ.

Please note that in some works (for instance [75], [93], [181]) another notation is used: let

$$\lambda_i = n - k + i - \sigma_i$$

for $i = 1, \dots, k$; then

$$0 \le \lambda_k \le \dots \le \lambda_1 \le n - k;$$

the Schubert cells $C_{(\sigma_1,\dots,\sigma_k)}$ are denoted by $X_{(\lambda_1,\dots,\lambda_k)}$ (or something else instead of X) and analogously the Schubert cycles, so

$$\begin{aligned} C_{(\sigma_1,\dots,\sigma_k)} &= X_{(\lambda_1,\dots,\lambda_k)} \\ &= \left\{ W \in G(k,n) \,\middle|\, dim(W \cap V_j) = i \quad \begin{array}{ll} \text{if} & n-k+i-\lambda_i \le j \le n-k+i-\lambda_{i+1} \\ \text{for} & 1 \le i \le k \end{array} \right\} \end{aligned}$$

and

$$\begin{aligned} \overline{C}_{(\sigma_1,\dots,\sigma_k)} &= \overline{X}_{(\lambda_1,\dots,\lambda_k)} \\ &= \{W \in G(k,n) |\ dim(W \cap V_{n-k+i-\lambda_i}) \ge i \quad \text{for} \quad 1 \le i \le k\} \end{aligned}.$$

The dimension of $X_{(\lambda_1,\dots,\lambda_k)}$ in terms of $\lambda_1, \dots, \lambda_k$ is $k(n-k) - \sum_{i=1,\dots,k} \lambda_i$.

Theorem. The algebra $H_*(G(k,V), \mathbb{Z})$ (with the intersection as product) is torsion free, and it is freely generated by the Schubert cycles. □

Finally, we mention that the algebra $H_*(G(k,V), \mathbb{Z})$ can be seen as a quotient of the algebra of symmetric polynomials in k variables; see [75] and[181] (see "Symmetric polynomials" for the definition of the symmetric polynomials).

Grauert's semicontinuity theorem. ([86], [107], [241]). Let T be a topological space. We recall that a map $h : T \to \mathbb{Z}$ is said to be upper semicontinuous if and only if for every $a \in \mathbb{Z}$ the subset $\{t \in T |\ h(t) \ge a\}$ is closed. This is equivalent to the following condition: for all $t_0 \in T$, there exists an open subset $U \ni t_0$ such that $h(t) \le h(t_0)$ for all $t \in U$.

Grauert's semicontinuity theorem. Let $f : X \to Y$ be a proper morphism of analytic spaces (respectively a projective morphism of Noetherian schemes), and let $\mathcal{F}$ be a coherent sheaf on X that is flat over Y, i.e., $\mathcal{F}_x$ is a flat $\mathcal{O}_{Y,f(x)}$-module for all $x \in X$, where $\mathcal{O}_{Y,f(x)}$ is the stalk in $f(x)$ of the structure sheaf on Y (see "Spaces, analytic -", "Coherent sheaves", "Schemes", "Flat (module, morphism)").

(a) Then the map $h^i : Y \to \mathbb{Z}$

$$y \longmapsto dim\ H^i(X_y, \mathcal{F}|_{X_y})$$

(where X_y is the fibre of f on y) is upper semicontinuous for all i.

(b) If Y is connected and reduced, where reduced means that $\mathcal{O}(U)$ has no nilpotent elements for any open subset U (respectively Y is an integral scheme), and for some i the map h^i is constant, then the sheaf $R^i f_*(\mathcal{F})$ (see "Direct and inverse image sheaves") is locally free and the fibre on y of the corresponding bundle is $H^i(X_y, \mathcal{F}|_{X_y})$. □

Groebner bases. ([50], [51], [62]). We follow mainly the exposition in [62].
Let K be a field and let $R = K[x_1, \dots, x_r]$. A monomial in R is an element of R of the form $x_1^{n_1} \dots x_r^{n_r}$ for some $n_1, \dots, n_r \in \mathbb{N}$.
Let F be a free finitely generated R-module, and let us fix a basis $\mathcal{E} = \{e_1, \dots, e_k\}$ of F.

A **monomial in F with basis** $\mathcal{E}$ is an element of F equal to me_i for some monomial $m \in R$ and some i.
A **term in F with basis** $\mathcal{E}$ is the product of an element of K and of a monomial in F with basis $\mathcal{E}$. Obviously any element of F can be written as sum of terms.

A **monomial order on F with basis** $\mathcal{E}$ is a total order on the monomials of F such that if m is a monomial in $R - \{1\}$ and f_1, f_2 are monomials in F, then

$$f_1 > f_2 \implies m f_1 > m f_2 > f_2.$$

If f_1, f_2 are monomials in F, with $f_1 > f_2$, and $a, b \in K - \{0\}$, then we say that $af_1 > bf_2$, and analogously replacing $>$ with $\geq$.
Let $f \in F$. We denote the greatest (with respect to $>$) term of f by $in_>(f)$.

Let M be a submodule of F. Let $in_> M$ be the submodule of F generated by the elements $in_>(f)$ for f in M. We say that $g_1, \dots, g_n$ form a **Groebner basis** of M if they generate M and $in_>(g_1), \dots, in_>(g_n)$ generate $in_> M$.

Let $g_1, \dots, g_n \in F - \{0\}$. For every $i, j \in \{1, \dots, n\}$ with $i \neq j$, we define
- $h_{i,j} = 0$ if $in_>(g_i)$ and $in_>(g_j)$ involve different elements of the basis of F (i.e., $in_>(g_i) = me_l$ and $in_>(g_j) = m' e_s$ with $l \neq s$);
- otherwise (i.e., if $in_>(g_i)$ and $in_>(g_j)$ involve the same element of the basis of F), we define
$$m_{i,j} = \frac{in_>(g_i)}{GCD(in_>(g_i), in_>(g_j))} \in R,$$

where GCD stands for "the greatest common divisor") in the obvious sense; we define $h_{i,j}$ in the following way: one can prove that we can write $m_{j,i}g_i - m_{i,j}g_j$ as

$$m_{j,i}g_i - m_{i,j}g_j = \sum_{u=1,\dots,n} f_u^{i,j} g_u + h_{i,j}$$

for some $f_u^{i,j} \in R$ and $h_{i,j} \in F$, with

$$in_>(m_{j,i}g_i - m_{i,j}g_j) \geq in_>(f_u^{i,j} g_u)$$

for every u, and there is no monomial of $h_{i,j}$ in $\{in_>(g_1), \dots, in_>(g_n)\}$.

Buchberger's theorem. The elements $g_1, \dots, g_n$ form a Groebner basis for the submodule they generate if and only if $h_{i,j} = 0$ for all i, j. □

Buchberger's algorithm. Buchberger's algorithm is useful for finding a Groebner basis for a submodule M of F: let $g_1, \dots, g_n$ be a set of generators of M; if they are a Groebner basis, we have finished; if not, add to them the $h_{i,j}$ computed as above; if $g_1, \dots, g_n$ and the $h_{i,j}$ are a Groebner basis, then we have finished; if not, repeat the operation, and so on. □

Groebner basis give us a method to compute syzygies (see "Syzygies"). Precisely the following theorem tells us how to compute the syzygies of a Groebner basis.

Schreyer's theorem. Let $g_1, \dots, g_n$ be a Groebner basis of the submodule of F they generate. Then, with the notation above,

$$m_{j,i}g_i - m_{i,j}g_j - \sum_u f_u^{i,j} g_u$$

generate all the syzygies among the g_i.
More precisely, consider on the free R-module $\oplus_{j=1,\dots,n} R\varepsilon_j$ the following monomial order: $q\,\varepsilon_a > p\,\varepsilon_b$ if and only if

$$in_>(q\,g_a) > in_>(p\,g_b) \qquad \text{or} \qquad in_>(q\,g_a) = in_>(p\,g_b) \quad \text{and} \quad a < b.$$

Then the elements

$$m_{j,i}\varepsilon_i - m_{i,j}\varepsilon_j - \sum_u f_u^{i,j} \varepsilon_u$$

form a Groebner basis of the submodule of the syzygies among the g_i. □

Obviously computing syzygies among a Groebner basis of a submodule M of F allows us to compute syzygies among any set of generators of M.

Grothendieck group.

([11], [12], [21] [72]). Let R be a Noetherian ring. Let $\mathcal{M}$ be the free Abelian group generated by the isomorphism classes of the finitely generated R-modules. For any finitely generated R-module M, denote the isomorphism class of

M by $[M]$. Let $\mathcal{S}$ be the subgroup of $\mathcal{M}$ generated by the elements $[M_1] - [M_2] + [M_3]$ for any exact sequence of finitely generated R-modules $0 \to M_1 \to M_2 \to M_3 \to 0$. We define the **Grothendieck group of the ring R** to be the quotient

$$\mathcal{M}/\mathcal{S}.$$

Let X be an algebraic variety. The **Grothendieck group of vector bundles on X**, denoted by $K^0(X)$, is the quotient of the free Abelian group generated by the set of isomorphism classes of vector bundles on X by the subgroup generated by the elements $[E_1] - [E_2] + [E_3]$ for any exact sequence $0 \to E_1 \to E_2 \to E_3 \to 0$ (where $[E]$ is the isomorphism class of E). The group $K^0(X)$ with the tensor product is a ring.
The **Grothendieck group of coherent sheaves on X** (see "Coherent sheaves"), denoted by $K_0(X)$, is the quotient of the free Abelian group generated by the set of isomorphism classes of coherent sheaves on X by the subgroup generated by the elements $[\mathcal{E}_1] - [\mathcal{E}_2] + [\mathcal{E}_3]$ for any exact sequence $0 \to \mathcal{E}_1 \to \mathcal{E}_2 \to \mathcal{E}_3 \to 0$, where $[\mathcal{E}]$ is the isomorphism class of $\mathcal{E}$. We have that $K_0(X)$ is a $K^0(X)$-module with the tensor product as product.
We say that a function f from the set of isomorphism classes of coherent sheaves on X to an Abelian group G is additive if $f(\mathcal{A}) + f(\mathcal{C}) - f(\mathcal{B}) = 0$ for any exact sequence $0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$. Obviously, for any additive function f from the set of isomorphism classes of coherent sheaves on X to an Abelian group G (for instance for the Euler–Poincaré characteristic), we have that f factorizes through a map from $K_0(X)$ to G; analogously for bundles and $K^0(X)$. The Grothendieck groups of bundles and of coherent sheaves were introduced by Grothendieck to prove the Hirzebruch–Riemann–Roch theorem (see "Hirzebruch–Riemann–Roch theorem").

Grothendieck–Segre theorem. ([80], [93, Chapter IV, §3], [97], [107, Chapter V], [207], [219]). Let K be an algebraically closed field. Let $\mathcal{E}$ be a locally free sheaf (see "Sheaves") on $\mathbb{P}^1_K$ of rank $r \in \mathbb{N}$. Then there exists a unique (up to permutation) sequence of integer numbers $a_1, \dots, a_r$ such that

$$\mathcal{E} \cong \bigoplus_{i=1,\dots,r} \mathcal{O}(a_i)$$

(see "Hyperplane bundles, twisting sheaves" for the definition of $\mathcal{O}(a_i)$).

Grothendieck's vanishing theorem. See "Vanishing theorems".

Group algebra. Let G be a group. The group algebra of G over a field K, generally denoted by $K[G]$, is the algebra whose underlying vector space is

$$\oplus_{g \in G} K e_g$$

endowed with the product given by

$$e_g e_h = e_{gh}.$$

H

Hartogs' theorem. ([40], [93], [121]). Let $n > 1$. Let U be an open set in $\mathbb{C}^n$ and let V be a compact subset of U such that $U - V$ is connected. Then any holomorphic function on $U - V$ extends to a holomorphic function on U.

Hartshorne's conjecture. ([19], [109], [209], [240]). Hartshorne's conjecture (formulated in 1974) states that if X is a smooth algebraic variety of dimension m in $\mathbb{P}^n_{\mathbb{C}}$ (or, more generally, in $\mathbb{P}^n_K$, where K is an algebraically closed field) and $m > \frac{2}{3}n$, then X is a complete intersection.
In particular, the conjecture says that if $n \geq 7$, every smooth algebraic subvariety of $\mathbb{P}^n_{\mathbb{C}}$ of codimension 2 is a complete intersection. One can prove that another formulation of the last statement is: if $n \geq 7$, every (algebraic) vector bundle of rank 2 on $\mathbb{P}^n_{\mathbb{C}}$ is a direct sum of line bundles. This can also be conjectured for $n \geq 5$, while we know that there exists an indecomposable algebraic bundle of rank 2 on $\mathbb{P}^4_{\mathbb{C}}$: the Horrocks–Mumford bundle (see "Horrocks–Mumford bundle"). Tango found an indecomposable algebraic vector bundle of rank 2 over $\mathbb{P}^5_K$, with K a field of characteristic 2.

Hartshorne–Serre theorem (correspondence). ([9], [87], [109], [209]). Let X be a smooth projective algebraic variety over $\mathbb{C}$ and Z be a subscheme locally complete intersection of codimension 2. Suppose that there exists a line bundle L on X such that $\det(N_{Z,X}) = L|_Z$ (where $N_{Z,X}$ is the normal bundle, which in this case can be defined to be the vector bundle associated to the sheaf $(\mathcal{I} \otimes \mathcal{O}_Z)^\vee$, where $\mathcal{I}$ is the ideal sheaf of Z in X).
If $H^2(\mathcal{O}(L^\vee)) = 0$, then there exists a vector bundle E on X of rank 2 such that $E|_Z = N_{Z,X}$, and there exists $s \in H^0(\mathcal{O}(E))$ such that Z is the zero locus of s.
Furthermore, if $H^1(\mathcal{O}(L^\vee)) = 0$ and Z is connected and reduced, then the vector bundle E is unique up to isomorphism.

Hermitian and Kählerian metrics. ([93], [147], [192]). We use the notation introduced in "Almost complex manifolds, holomorphic maps, holomorphic tangent bundles".
A **Riemannian metric** g on a C^∞ manifold X is given by a positive definite symmetric bilinear form

$$g_P : (T_P X)_{\mathbb{R}} \times (T_P X)_{\mathbb{R}} \longrightarrow \mathbb{R}$$

for every $P \in X$, depending in a C^∞ way on P.
A **Hermitian metric** on a complex manifold X is a R iemannian metric g such that

$$g_P(J_P v, J_P w) = g_P(v, w)$$

for all $v, w \in (T_P X)_{\mathbb{R}}$ and for all $P \in X$.

Let g be a Hermitian metric on X. The skew-symmetric form ω given by

$$\omega_P : (T_P X)_{\mathbb{R}} \times (T_P X)_{\mathbb{R}} \to \mathbb{R},$$
$$\omega_P(v, w) := g_P(J_P v, w),$$

for any v, w, is called **Kähler form** of the Hermitian metric g.
Let $g_{\mathbb{C}}$ be the $\mathbb{C}$-bilinear extension of g to $(TX)_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$. Observe that it is zero on $T^{1,0}X \times T^{1,0}X$ and on $T^{0,1}X \times T^{0,1}X$. For any $P \in X$, define
- $\alpha_P : (T_P X)_{\mathbb{R}} \to T_P^{1,0}X$ to be the isomorphism given by $\alpha(x) = x - iJ_P x$;
- $\beta_P : (T_P X)_{\mathbb{R}} \to T_P^{0,1}X$ to be the isomorphism given by $\beta(x) = x + iJ_P x$.

Remark. Obviously $g_{\mathbb{C}}(\cdot, \bar{\cdot}) = g_{\overline{\mathbb{C}}}$, where $g_{\overline{\mathbb{C}}}$ denotes g extended by $\mathbb{C}$-linearity in the first variable and by $\overline{\mathbb{C}}$-linearity in the second variable. On $(T_P X)_{\mathbb{R}} \times (T_P X)_{\mathbb{R}}$ we have that

$$g_{\overline{\mathbb{C}},P}(\alpha_P \cdot, \alpha_P \cdot) = g_{\mathbb{C},P}(\alpha_P \cdot, \beta_P \cdot) = 2[g_P(\cdot, \cdot) - i\, \omega_P(\cdot, \cdot)]\,. \qquad \square$$

We recall that a Hermitian form h on a complex vector space V is a map $h : V \times V \to \mathbb{C}$, $\mathbb{C}$-linear in the first variable and such that $h(v, w) = \overline{h(w, v)}$ for any $v, w \in V$.

By the remark above, a Hermitian metric g gives, for every $P \in X$, a Hermitian form $h_P : T_P^{1,0}X \times T_P^{1,0}X \to \mathbb{C}$ defined to be $g_{\mathbb{C},P}(\cdot, \bar{\cdot})$ restricted to $T_P^{1,0}X \times T_P^{1,0}X$.
Vice versa, given, for any $P \in X$, a positive definite Hermitian form

$$h_P : T_P^{1,0}X \times T_P^{1,0}X \to \mathbb{C},$$

we can define $g_P = Re\, h_P(\alpha_P \cdot, \alpha_P \cdot)$ on $(T_P X)_{\mathbb{R}} \times (T_P X)_{\mathbb{R}}$; one can easily check that g is Riemannian and that $g(J\cdot, J\cdot) = g(\cdot, \cdot)$.

In local coordinates, given a Hermitian metric g, define $g_{i,\bar{j}} := g_{\mathbb{C}}(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \overline{z_j}})$. We write the Hermitian metric locally as

$$\sum g_{i,\bar{j}} dz_i d\overline{z_j},$$

and we have

$$\omega = i \sum g_{i,\bar{j}} dz_i \wedge d\overline{z_j}.$$

We say that a Hermitian metric g is a **Kähler metric** if and only if the associated Kähler form ω is closed.

Hilbert Basis theorem. ([12], [164] [256]).

Theorem "Hilbert basis theorem". If R is a Noetherian ring, then $R[x]$ is Noetherian. $\square$

Corollary. If R is a Noetherian ring, then $R[x_1, \ldots, x_n]$ is Noetherian. $\square$

See "Noetherian, Artinian" for the definition of Noetherian ring. Since every field is Noetherian, from the Hilbert basis theorem we get the following corollary:

Corollary. If K is a field, then $K[x_1, \dots, x_n]$ is Noetherian. □

Hilbert's Nullstellensatz. ([12], [104], [256]).

Notation. For any ideal J in a ring R, let $\sqrt{J}$ denote the radical ideal of J, i.e.,

$$\sqrt{J} := \{f \in R |\ \exists\, r \in \mathbb{N} - \{0\} \ \text{ such that } \ f^r \in J\}\,. \qquad \square$$

Notation. Let K be a field.

– For any $S \subset K[x_1, \dots, x_n]$, let $Z(S)$ denote the zero locus of S in the affine space $\mathbb{A}^n_K$, i.e.,
$$Z(S) := \{P \in \mathbb{A}^n_K |\ f(P) = 0\ \forall f \in S\}.$$
– For any $X \subset \mathbb{A}^n_K$, let $I(X)$ denote the ideal of X, i.e.,
$$I(X) := \{f \in K[x_1, \dots, x_n] |\ f(P) = 0\ \forall P \in X\}\,. \qquad \square$$

Theorem "Hilbert's Nullstellensatz". Let K be an algebraically closed field. We have that
$$I(Z(J)) = \sqrt{J}$$
for any ideal J of $K[x_1, \dots, x_n]$.
In particular, if $Z(J) = \emptyset$, then $\sqrt{J} = K[x_1, \dots, x_n]$ and then $J = K[x_1, \dots, x_n]$. □

Corollary. Let K be an algebraically closed field. There is a bijection between the set of the algebraic subsets of $\mathbb{A}^n_K$ and the set of the radical ideals of $K[x_1, \dots, x_n]$. □

Notation. Let K be a field.

– For any subset S of homogeneous elements of $K[x_0, \dots, x_n]$, we denote
$$Z(S) = \{P \in \mathbb{P}^n_K |\ f(P) = 0\ \forall f \in S\}.$$
For any homogeneous ideal I (see "Homogeneous ideals"), we define $Z(I)$ to be the zero locus of the set of the homogeneous elements of I.
– For any $X \subset \mathbb{P}^n_K$, we denote by $I(X)$ the ideal generated by
$$\{f \in K[x_0, \dots, x_n] |\ f \ \text{ homogeneous and } \ f(P) = 0\ \forall P \in X\};$$
obviously it is a homogeneous ideal. □

Theorem "Projective Hilbert's Nullstellensatz". Let K be an algebraically closed field. Let J be a homogeneous ideal of $K[x_0, \dots, x_n]$.
We have that $Z(J) = \emptyset$ if and only if there exists $\overline{d} \in \mathbb{N}$ such that the part of degree d of $K[x_0, \dots, x_n]$ is contained in J for any $d \geq \overline{d}$.
If $Z(J) \neq \emptyset$, then $I(Z(J)) = \sqrt{J}$. □

See also "Weierstrass preparation theorem and Weierstrass division theorem".

Hilbert function and Hilbert polynomial. ([12], [104], [107], [164]).

Definition. We say that a polynomial $P(x) \in \mathbb{Q}[x]$ is numerical if $P(s) \in \mathbb{Z}$ for any $s \in \mathbb{Z}$ with $s \gg 0$. □

Proposition. If $P(x) \in \mathbb{Q}[x]$ is a numerical polynomial of degree d, then it is a linear combination with integer coefficients of the polynomials $\binom{x}{i} := \frac{x(x-1)\cdots(x-i+1)}{i!}$ for $i = 0, \ldots, d$ (where $\binom{x}{0} = 1$). In particular $P(s) \in \mathbb{Z}$ for every $s \in \mathbb{Z}$. □

Let K be an algebraically closed field.

Definition. Let $M = \oplus_{s \in \mathbb{Z}} M_s$ be a graded module over $K[x_0, \ldots, x_r]$, where M_s is its part of degree s. The Hilbert function of M, we denote f_M, is defined in the following way:

$$f_M(s) := dim_K M_s \quad \forall s \in \mathbb{Z}.$$

□

Theorem (Hilbert–Serre). Let M be a graded finitely generated module on $K[x_0, \ldots, x_r]$. There exists a unique polynomial, called Hilbert polynomial of M, $P_M(x) \in \mathbb{Q}[x]$ such that

$$P_M(s) = f_M(s)$$

for $s \gg 0$, $s \in \mathbb{N}$. □

Let X be a projective algebraic variety in $\mathbb{P}^N := \mathbb{P}^N_K$. Its Hilbert function and its Hilbert polynomial are defined to be the Hilbert function and the Hilbert polynomial of its homogeneous coordinates ring. Thus the Hilbert function is

$$f_X(s) = dim_K \frac{H^0(\mathcal{O}_{\mathbb{P}^N}(s))}{H^0(\mathcal{I}_X(s))}$$

and the Hilbert polynomial is

$$P_X(s) = \chi(\mathcal{O}_X(s)),$$

where $\mathcal{O}_X$ is the sheaf of the regular functions on X, $\mathcal{I}_X$ is the ideal sheaf of X and χ is the Euler–Poincaré characteristic. In fact, by Serre's theorem (see "Cartan–Serre theorems", for $s \gg 0$, we have $h^i(\mathcal{I}_X(s)) = h^i(\mathcal{O}_X(s)) = 0$ for all $i > 0$, thus, for $s \gg 0$, we have

$$dim_K \frac{H^0(\mathcal{O}_{\mathbb{P}^N}(s))}{H^0(\mathcal{I}_X(s))} = h^0(\mathcal{O}_X(s)) = \chi(\mathcal{O}_X(s))$$

by the sequence

$$0 \to \mathcal{I}_X \to \mathcal{O}_{\mathbb{P}^N} \to \mathcal{O}_X \to 0.$$

We can prove that the degree of P_X is the dimension n of X, and we may define the degree (see "Degree of an algebraic subset") of X to be $n!$ times the leading coefficient of P_X.

Examples.

– Let $\nu_{n,d} : \mathbb{P}^n \to \mathbb{P}^N$, where $N = \binom{n+d}{d} - 1$, be the Veronese embedding (see "Veronese embedding"). Let $V_{n,d}$ be the Veronese variety $\nu_{n,d}(\mathbb{P}^n)$. Its Hilbert function is

$$f_{V_{n,d}}(s) = h^0(\mathcal{O}_{\mathbb{P}^N}(s))/h^0(\mathcal{I}_{V_{n,d}}(s)) = h^0(\mathcal{O}_{V_{n,d}}(s)) = h^0(\mathcal{O}_{\mathbb{P}^n})(ds)) = \binom{sd+n}{n},$$

where the second equality holds since the Veronese variety is projectively normal (see "Normal, projectively -, k-normal, linearly normal"). Since $f_{V_{n,d}}(s)$ is a polynomial we have that

$$p_{V_{n,d}}(s) = f_{V_{n,d}}(s).$$

In particular for $n = 1$, i.e., for the rational normal curve, we get

$$p_{V_{1,d}}(s) = sd + 1.$$

– Let C be a smooth projective curve of degree d and genus g. For $s \gg 0$

$$h^0(\mathcal{O}_C(s)) = sd - g + 1,$$

by the Riemann–Roch theorem (see "Riemann surfaces (compact -) and algebraic curves"), since $\mathcal{O}_C(s)$ is a line bundle of degree sd on C (the restriction of s times the hyperplane bundle to C, i.e., s times the line bundle given the embedding of C in the projective space). Thus

$$p_C(s) = sd - g + 1.$$

Hilbert schemes. See "Moduli spaces"

Hilbert syzygy theorem. ([62], [159], [164]).

Hilbert syzygy theorem. Let K be a field and $R = K[x_0, \dots, x_n]$. Every finitely generated graded R-module M has a finite graded free resolution of length $\leq n$ by finitely generated free R-modules, that is, there exists a graded exact sequence (i.e., an exact sequence where the maps preserve the degree)

$$0 \to F_n \to \cdots \to F_1 \to F_0 \to M \to 0,$$

with F_i finitely generated free R-modules. □

See also "Horrocks' theorem".

Hironaka's decomposition of birational maps. ([107], [117], [118]).

Let K be a field of characteristic zero and let V and V' be two smooth projective algebraic varieties over K. Let $f : V \to V'$ be a birational map. Then there exists a morphism $g : \tilde{V} \to V$ such that g is the composition of a finite sequence of blow-ups along smooth subvarieties and the birational map $f \circ g$ is a morphism. □

In the case where V and V' are surfaces, we have a stronger statement; see "Structure of birational maps on surfaces" in "Surfaces, algebraic -".
Finally, we want to mention that it is known that in dimension ≥ 3 not every birational morphism is the composition of blow-ups along smooth subvarieties (see [229]).
See also [2].

Hirzebruch surfaces. See "Surfaces, algebraic -".

Hirzebruch–Riemann–Roch theorem. ([13], [29], [107], [119], [146]). Let X be a compact complex manifold and let E be a vector bundle on X of rank r. Write the Chern polynomial (see "Chern classes")

$$c(E)(t) = c_0(E) + c_1(E)t + \cdots + c_r(E)t^r$$

as

$$c(E)(t) = \Pi_{i=1,\dots,r}(1 + a_i t).$$

We define the Chern character to be

$$ch(E) := \sum_{i=1,\dots,r} e^{a_i}$$

(where $e^x = 1 + x + \frac{1}{2}x^2 + \cdots$). Furthermore, we define the Todd class by

$$td(E) = \Pi_{i=1,\dots,r} \frac{a_i}{1 - e^{-a_i}},$$

where $\frac{x}{1-e^{-x}}$ is the series $1 + \frac{1}{2}x + \frac{1}{12}x^2 - \frac{1}{720}x^4 \dots$.
We recall also that the symbol $\cup$ denotes the cup product; see "Singular homology and cohomology" for the definition.

Hirzebruch–Riemann–Roch theorem. Let X be a compact complex manifold of dimension n and let E be a holomorphic vector bundle on X of rank r. We have that

$$\chi(\mathcal{O}(E)) = \int_X ch(E) \cup td(T_X^{1,0}),$$

that is, $\chi(\mathcal{O}(E))$ (the Euler–Poincaré characteristic of $\mathcal{O}(E)$) is equal to the component in $H^{2n}(X, \mathbb{R})$ of $ch(E) \cup td(T_X^{1,0})$ evaluated in the fundamental class of X, i.e., in the class of the $2n$-cycle determined by the natural orientation of X. □

In the case where X is a compact Riemann surface of genus g (see "Riemann surfaces (compact -) and algebraic curves") and E is a line bundle, we get the usual Riemann–Roch theorem:

$$h^0(X, \mathcal{O}(E)) - h^1(X, \mathcal{O}(E)) = deg(E) - g + 1;$$

in fact, the Chern polynomial is $c(E)(t) = 1 + c_1(E)t$ and the Chern character is $ch(E) = e^{c_1(E)} = 1 + c_1(E)$; in addition, $T_X^{1,0} = -K_X$, and thus $td(T_X^{1,0}) = 1 - \frac{1}{2}c_1(K_X)$.

The component of $(1 + c_1(E)) \cup (1 - \frac{1}{2}c_1(K_X))$ in $H^2(X)$ is $c_1(E) - \frac{1}{2}c_1(K_X)$; thus we get

$$\chi(\mathcal{O}(E)) = \deg(E) - \frac{1}{2}deg(K_X).$$

Taking E to be trivial, we get that the degree of K_X is $2g - 2$; hence, by substituting, we get the Riemann–Roch formula for any holomorphic line bundle E on a compact Riemann surface.
Analogously, in the case where X is a surface and E is a holomorphic line bundle, we get the usual Riemann–Roch theorem for surfaces (see "Surfaces, algebraic -").
The theorem also holds for nonsingular projective algebraic varieties over algebraically closed fields; we will not state it in this context, since it requires the generalization of some concepts, such as Chern classes and intersection theory, for such varieties. The statement for complex nonsingular projective varieties is due to Hirzebruch, the one for compact complex manifolds to Atiyah–Singer, the one for nonsingular projective varieties over algebraically closed fields to Grothendieck (see [29]).

Hodge theory. ([44], [90], [93], [245]). Let X be a compact complex manifold of (complex) dimension n. Let h be a Hermitian metric on X and let ω be the associated $(1,1)$-form (see "Hermitian and Kählerian metrics").

Let us denote by $T^{\vee\,(p,q)}X$ the bundle $\wedge^p T^{1,0}X^\vee \otimes \wedge^q T^{0,1}X^\vee$, where $T^{1,0}X$ and $T^{0,1}X$ are the holomorphic and the antiholomorphic tangent bundles (see "Almost complex manifolds, holomorphic maps, holomorphic tangent bundles").
For any $x \in X$, the metric h induces a Hermitian metric on $T_x^{\vee\,(p,q)}X$, we call again h, defined in the following way: let $\omega = \frac{i}{2}\sum_{j=1,\dots,n} z_j \wedge \overline{z_j}$ in local coordinates $z_1, \dots, z_n$ around x; let h be such that the $z_I \wedge \overline{z}_J$ for $|I| = p$ and $|J| = q$ form an orthogonal basis in $T_x^{\vee\,(p,q)}X$ and the norm of every $z_I \wedge \overline{z}_J$ is 2^{p+q}. For any $p, q \in \mathbb{N}$, let $(\ ,\ )$ be the following positive definite product on the set $A^{p,q}(X)$ of the C^∞ (p,q)-forms on X:

$$(\eta, \gamma) = \int_X h(\eta(z), \gamma(z)) \frac{\omega^n(z)}{n!}.$$

Let $\overline{\partial}^*$ be the adjoint operator of $\overline{\partial} : A^{p,q}(X) \to A^{p,q+1}(X)$ with respect to $(\ ,\ )$, i.e., let $\overline{\partial}^* : A^{p,q}(X) \to A^{p,q-1}(X)$ be the operator such that

$$(\overline{\partial}^* \eta, \theta) = (\eta, \overline{\partial}\theta)$$

for all $\eta \in A^{p,q}(X)$, $\theta \in A^{p,q-1}(X)$.
Let $\Delta_{\overline{\partial}} : A^{p,q}(X) \to A^{p,q}(X)$ be the so-called **Laplacian operator**:

$$\Delta_{\overline{\partial}} := \overline{\partial}^* \overline{\partial} + \overline{\partial}\,\overline{\partial}^*.$$

The forms η s.t $\Delta_{\overline{\partial}}\eta = 0$ are called **$\overline{\partial}$-harmonic**. Let $\mathcal{H}^{p,q}_{\overline{\partial}}(X)$ be the space of the $\overline{\partial}$-harmonic (p,q)-forms. Observe that $\Delta_{\overline{\partial}}\eta = 0$ if and only if $\overline{\partial}\eta = 0$ and $\overline{\partial}^*\eta = 0$, in fact $(\cdot,\cdot)$ is positive definite and

$$(\Delta_{\overline{\partial}}\eta, \eta) = ((\overline{\partial}\,\overline{\partial}^* + \overline{\partial}^*\overline{\partial})\eta, \eta) = (\overline{\partial}^*\eta, \overline{\partial}^*\eta) + (\overline{\partial}\eta, \overline{\partial}\eta).$$

Theorem.

(i) The space $\mathcal{H}^{p,q}_{\bar\partial}(X)$ is finite dimensional. Thus, we can define an orthogonal projection $H : A^{p,q}(X) \to \mathcal{H}^{p,q}_{\bar\partial}(X)$.

(ii) There exists an operator (called Green's operator) $G : A^{p,q}(X) \to A^{p,q}(X)$ such that $G(\mathcal{H}^{p,q}_{\bar\partial}(X)) = 0$, G commutes with $\bar\partial$ and $\bar\partial^*$ and, for all $\eta \in A^{p,q}(X)$,

$$\eta = H\eta + \Delta_{\bar\partial} G\eta.$$

The above decomposition is called Hodge decomposition.

(iii) Let $\eta \in A^{p,q}(X)$; there exists $\gamma \in A^{p,q}(X)$ such that $\Delta_{\bar\partial}\gamma = \eta$ if and only if η is orthogonal to $\mathcal{H}^{p,q}_{\bar\partial}(X)$. □

Observe that the implication $\Rightarrow$ of (iii) is obvious and the other implication follows from (ii); in fact, let $\eta \perp \mathcal{H}^{p,q}_{\bar\partial}(X)$; by (ii) we have $\eta = H\eta + \Delta_{\bar\partial} G\eta = \Delta_{\bar\partial} G\eta$.

Corollary. Every $\bar\partial$-closed form η is $\bar\partial$-homologous to a $\bar\partial$-harmonic form. Thus

$$\mathcal{H}^{p,q}_{\bar\partial}(X) \cong H^{p,q}_{\bar\partial}(X)\,.$$ □

In fact, let η be such that $\bar\partial\eta = 0$; by Hodge decomposition, we have

$$\eta = H\eta + \bar\partial\,\bar\partial^* G\eta + \bar\partial^*\bar\partial G\eta = H\eta + \bar\partial\,\bar\partial^* G\eta + G\bar\partial^*\bar\partial\eta = H\eta + \bar\partial\,\bar\partial^* G\eta,$$

so we have found a harmonic form, $H\eta$, that is $\bar\partial$-homologous to η.

Thus, by Dolbeault's theorem, $\mathcal{H}^{p,q}_{\bar\partial}(X) \cong H^q(X, \Omega^p)$ (see "Dolbeault's theorem").

An analogous theory can be developed for a Riemannian manifold and the operator d instead of the operator $\bar\partial$ (and $\Delta_d := d\,d^* + d^* d$ instead of $\Delta_{\bar\partial}$, where d^* is the adjoint operator of d with respect to $\int \cdot \wedge * \cdot$, where $*$ is the star operator; see "Star operator").
If X is a Kähler manifold, we have that

$$2\Delta_{\bar\partial} = \Delta_d = 2\Delta_\partial.$$

In particular, $\mathcal{H}^{p,q}_{\bar\partial}(X) = \mathcal{H}^{p,q}_{\partial}(X) = \mathcal{H}^{p,q}_{d}(X)$ (where $\mathcal{H}^{p,q}_{d}(X)$ is the set of the (p,q)-forms η such that $\Delta_d\,\eta = 0$).
Since $\Delta_d = 2\Delta_{\bar\partial}$, the operator Δ_d preserves the bidegree; hence

$$\mathcal{H}^r_d(X) = \oplus_{p+q=r}\mathcal{H}^{p,q}_d(X);$$

moreover Δ_d is "real", thus

$$\overline{\mathcal{H}^{p,q}_d(X)} = \mathcal{H}^{q,p}_d(X).$$

By Hodge decomposition for d, we have $H^r_{DR}(X) = \mathcal{H}^r(X)$ and $H^{p,q}(X) = \mathcal{H}^{p,q}_d(X)$, where $H^r_{DR}(X)$ is the set the d-closed r-forms (over $\mathbb{C}$) modulo the set of the d-exact r-forms (over $\mathbb{C}$) and $H^{p,q}_d(X)$ is the set of the d-closed (p,q)-forms modulo the d-exact (p,q)-forms. Thus we get the following theorem:

Theorem. If X is a compact Kähler manifold, then

$$H^r(X,\mathbb{C}) = \oplus_{p+q=r} H^{p,q}(X),$$
$$\overline{H^{p,q}(X)} = H^{q,p}(X)\,. \qquad \square$$

Holomorphic.

See “Almost complex manifolds, holomorphic maps, holomorphic tangent bundles”.

Homogeneous bundles.

([30], [210]). Let X be a G-homogeneous complex algebraic variety (see “Homogeneous varieties”) and E be a vector bundle on X. We say that E is homogeneous if there is an action of G on E (that is, a homomorphism from G to the group of automorphisms of E) such that

$$gE_x \subset E_{gx}$$

for all $g \in G$ and for all $x \in X$ (where E_x denotes the fibre on x).
If we write X as G/P where P is the isotropy subgroup of a point of X, we can easily prove that E is homogeneous if and only if it comes from the principal bundle $G \to G/P$ and a representation $\rho : P \to GL(r,\mathbb{C})$, where r is the rank of E (see “Bundles, fibre -” and precisely principal bundles), i.e., E is homogeneous if and only if

$$E \cong G \times_\rho \mathbb{C}^r := G \times \mathbb{C}^r / \sim,$$

where $\sim$ is the equivalence relation such that

$$(g,v) \sim (gp, \rho(p^{-1})g)$$

for any $p \in P$.
For a vector bundle E on a homogeneous rational variety G/P, with G simply connected semisimple group, P parabolic subgroup (see “Lie groups”), we have that E is homogeneous if and only if $l_g^* E \cong E$ for every $g \in G$, where $l_g : G \to G$ is the left multiplication by g.

Homogeneous ideals.

([73], [185], [256]). Let K be a field and I be an ideal of $K[x_0,\dots,x_n]$. We say that I is a homogeneous ideal if and only if the following property holds: if we write an element F of I as sum of homogeneous polynomials, $F = \sum_{i=1,\dots,k} F_i$, we have that $F_i \in I$ for $i = 1,\dots,k$.

Proposition.
- An ideal in $K[x_0,\dots,x_n]$ is homogeneous if and only if it is generated by homogeneous polynomials.
- The sum, product, intersection of homogeneous ideals are homogeneous, the radical of a homogeneous ideal is homogeneous.
- A homogeneous ideal I is prime if and only if, for any $f,g \in K[x_0,\dots,x_n]$ with f,g homogeneous and such that $fg \in I$, we have either $f \in I$ or $g \in I$. $\square$

Homogeneous varieties. ([28], [210]). Let G be an algebraic group, respectively a topological group. Let X be an algebraic variety, respectively a manifold. We say that G acts on X if there is a morphism $G \times X \to X$, $(g, x) \mapsto gx$ such that $1x = x$ for any $x \in X$, $g_1(g_2 x) = (g_1 g_2)x$ for any $x \in X$ and for any $g_1, g_2 \in G$; we say that the action is transitive if, for any $x, x' \in X$, there exists $g \in G$ such that $gx = x'$. We say that X is G-homogeneous if G acts transitively on it.

Remark. Every homogeneous variety is smooth. □

Theorem (Borel–Remmert). A homogeneous compact Kähler manifold is isomorphic to the product of a complex torus and a rational homogeneous projective algebraic variety (see "Tori, complex - and Abelian varieties", "Rational varieties").
Furthermore, a rational homogeneous projective algebraic variety is isomorphic to

$$G_1/P_1 \times \cdots \times G_k/P_k$$

for some simple Lie groups G_i and P_i parabolic subgroups (see "Lie groups"). □

Homology, Singular -. See "Singular homology and cohomology".

Homology of a complex. See "Complexes".

Horrocks' Criterion. ([17], [122], [207]). An algebraic vector bundle E on $\mathbb{P}^n_{\mathbb{C}}$ is direct sum of algebraic line bundles if and only if the intermediate cohomology of all its twists is zero, i.e.,

$$H^i(\mathbb{P}^n_{\mathbb{C}}, \mathcal{O}(E)(t)) = 0$$

for all $t \in \mathbb{Z}$ and for all i with $0 < i < n$ (see "Hyperplane bundles, twisting sheaves" for the defintion of $\mathcal{O}(E)(t)$).

Horrocks–Mumford bundle. ([122], [123], [124]). The Horrocks–Mumford bundle is an indecomposable algebraic vector bundle of rank 2 on $\mathbb{P}^4_{\mathbb{C}}$ (indecomposable means that is it not the direct sum of two algebraic line bundles). It is the only bundle with these features we know so far.
The zero locus of a generic section of the Horrocks–Mumford bundle is an Abelian surface (see "Tori, complex - and Abelian varieties") of degree 10.
See also "Hartshorne's conjecture".

Horrocks' theorem. ([122], [209]). Let E be an algebraic vector bundle (or more generally a torsion-free sheaf of $\mathcal{O}$-modules) on $\mathbb{P}^n_{\mathbb{C}}$. Then E has a free resolution of length $\leq n-1$, that is, there exists an exact sequence

$$0 \longrightarrow F_{n-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow E \longrightarrow 0$$

with all the F_i direct sum of algebraic line bundles on $\mathbb{P}^n_{\mathbb{C}}$, thus direct sum of hyperplanes bundles (see "Hyperplane bundles, twisting sheaves").

Horseshoe lemma. ([208]).

Horseshoe lemma. Let R be a ring and let

$$0 \longrightarrow M_1 \xrightarrow{\alpha} M_2 \xrightarrow{\beta} M_3 \longrightarrow 0$$

be an exact sequence of R-modules. Let

$$\cdots \longrightarrow F_t \xrightarrow{f_t} \cdots\cdots \xrightarrow{f_1} F_0 \xrightarrow{f_0} M_1 \longrightarrow 0$$

and

$$\cdots \longrightarrow G_t \xrightarrow{g_t} \cdots\cdots \xrightarrow{g_1} G_0 \xrightarrow{g_0} M_3 \longrightarrow 0$$

be two projective resolutions of M_1 and M_3 respectively (see "Injective and projective resolutions"). Then there is a projective resolution of M_2

$$\cdots \longrightarrow F_t \oplus G_t \xrightarrow{d_t} \cdots\cdots \xrightarrow{d_1} F_0 \oplus G_0 \xrightarrow{d_0} M_2 \longrightarrow 0$$

such that the diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & & & 0 \arrow[d] & 0 \arrow[d] & \\
\cdots \arrow[r] & F_t \arrow[r, "f_t"] \arrow[d, "i_t"] & \cdots\cdots \arrow[r, "f_1"] & & F_0 \arrow[r, "f_0"] \arrow[d, "i_0"] & M_1 \arrow[r] \arrow[d, "\alpha"] & 0 \\
\cdots \arrow[r] & F_t \oplus G_t \arrow[r, "d_t"] \arrow[d, "p_t"] & \cdots\cdots \arrow[r, "d_1"] & & F_0 \oplus G_0 \arrow[r, "d_0"] \arrow[d, "p_0"] & M_2 \arrow[r] \arrow[d, "\beta"] & 0 \\
\cdots \arrow[r] & G_t \arrow[r, "g_t"] \arrow[d] & \cdots\cdots \arrow[r, "g_1"] & & G_0 \arrow[r, "g_0"] \arrow[d] & M_3 \arrow[r] \arrow[d] & 0 \\
 & 0 & & & 0 & 0 &
\end{tikzcd}$$

is commutative with exact rows and columns (where, for any t, the map i_t is the obvious injection and the map p_t is the obvious projection). □

The lemma takes its name from the shape of the given part of the diagram (the short exact sequence, and the given projective resolutions).

An analogous statement holds in case we have an injective resolution of M_1 and an injective resolution of M_3.

Hurwitz's theorem. See "Riemann surfaces (compact -) and algebraic curves".

Hypercohomology of a complex of sheaves. ([79], [93]). We follow strictly [93].

Let

$$\cdots \longrightarrow \mathcal{F}^p \xrightarrow{d} \mathcal{F}^{p+1} \longrightarrow \cdots\cdots$$

be a **complex of sheaves** of Abelian groups on a topological space X (see "Sheaves"). We denote it by $\mathcal{F}^*$.
The **cohomology sheaf** $\mathcal{H}^q(\mathcal{F}^*)$ is the sheaf on X associated to the presheaf

$$U \longmapsto \frac{Ker\left(\mathcal{F}^q(U) \xrightarrow{d} \mathcal{F}^{q+1}(U)\right)}{Im\left(\mathcal{F}^{q-1}(U) \xrightarrow{d} \mathcal{F}^q(U)\right)}$$

for any U open subset of X. Let $\mathcal{U} = \{U_\alpha\}_{\alpha\in A}$ be an open covering of X. Consider the bigraded complex $C^p(\mathcal{U}, \mathcal{F}^q)$ (see "Sheaves" and in particular the cohomology of the sheaves for the notation) with the differentials

$$\delta : C^p(\mathcal{U}, \mathcal{F}^q) \longrightarrow C^{p+1}(\mathcal{U}, \mathcal{F}^q),$$
$$d : C^p(\mathcal{U}, \mathcal{F}^q) \longrightarrow C^p(\mathcal{U}, \mathcal{F}^{q+1}),$$

where d is induced by the differential of $\mathcal{F}^*$. Let $C^*(\mathcal{U})$ be the associated single complex given by $C^n(\mathcal{U}) = \oplus_{p+q=n} C^p(\mathcal{U}, \mathcal{F}^q)$ with differential $D = d + \delta$.
The **hypercohomology** of the complex of sheaves $\mathcal{F}^*$ is defined to be the limit on $\mathcal{U}$ of the cohomology of the complex $C^*(\mathcal{U})$, where the set of the open coverings is partially ordered by refinement (see "Limits, direct and inverse -"):

$$\mathbb{H}^q(X, \mathcal{F}^*) = \lim_{\substack{\mathcal{U}\\ \rightarrow}} H^q(C^*(\mathcal{U}))$$

for any q.
By considering the two filtrations of $C^*\mathcal{U}$ and by passing to the limit for $\mathcal{U}$, we get two spectral sequences (see "Spectral sequences") E_r and E'_r with $E_2^{p,q} \cong H^p(X, \mathcal{H}^q(\mathcal{F}^*))$ and $E_2^{'p,q} \cong H^q_d(H^p(X, \mathcal{F}^*))$.

Proposition. If a map between complexes of sheaves induces an isomorphism on cohomology sheaves, then it also induces an isomorphism on hypercohomology. □

Hyperelliptic Riemann surfaces. See "Riemann surfaces (compact -) and algebraic curves".

Hyperelliptic or bielliptic surfaces. See "Surfaces, algebraic -".

Hyperplane bundles, twisting sheaves. ([93], [107], [129]). Let K be a field and $s \in \mathbb{Z}$. The line bundle H_s on $\mathbb{P}^n_K$ is defined in the following way: let the total space be

$$((K^{n+1} - \{0\}) \times K)/\sim,$$

where $\sim$ is the following equivalence relation:

$$(x_0, \ldots, x_n, t) \sim (\lambda x_0, \ldots, \lambda x_n, \lambda^s t) \quad \forall \lambda \in K - \{0\};$$

the projection

$$\pi : H_s \to \mathbb{P}^n_K$$

is given by

$$\pi([(x_0, \dots, x_n, t)]) = [x_0 : \dots : x_n].$$

We can take as trivializing subsets for H_s the sets

$$U_i = \{[x_0 : \dots : x_n] \in \mathbb{P}^n_K \mid x_i \neq 0\}$$

for $i = 0, \dots, n$, and the trivializing function $\phi_i : U_i \times K \to H_s|_{U_i}$ is

$$\phi_i([x_0 : \dots : x_n], t) = [(x_0, \dots, x_n, x_i^s t)].$$

Thus the transition functions are

$$f_{i,j}([x_0 : \dots : x_n]) = x_i^{-s} x_j^s.$$

The bundle H_1 is called a hyperplane bundle (since it is associated to the divisor given by a hyperplane) and is the dual of the universal bundle (see "Tautological (or universal) bundle"). Obviously $H_s = H^{\otimes s}$. The sheaf $\mathcal{O}(s)$ (or, more precisely, $\mathcal{O}_{\mathbb{P}^n_K}(s)$) is defined to be the sheaf of the sections of H_s. Sometimes $\mathcal{O}(s)$ denotes also the line bundle itself. The sheaves $\mathcal{O}(s)$ are called Serre's twisting sheaves.
For any sheaf $\mathcal{E}$ on $\mathbb{P}^n_K$, we denote $\mathcal{E} \otimes \mathcal{O}(s)$ by $\mathcal{E}(s)$.

A more general way to define the hyperplane bundles and the twisting sheaves is the following one.
Let R be a graded commutative ring with unity. For any graded R-module M, let $\mathcal{F}_M$ be the following sheaf on the scheme $Proj(R)$ (see "Schemes"): for every $p \in Proj(R)$, let $M_{(p)}$ be the group of the elements of degree 0 in the localization of M with respect to the multiplicative system of the homogeneous elements in $R - p$ (see "Localization, quotient ring, quotient field"); for every open subset U of $Proj(R)$, let $\mathcal{F}_M(U)$ be the set of the functions

$$\sigma : U \to \sqcup_{p \in U} M_{(p)}$$

such that $\sigma(p) \in M_{(p)}$ for every $p \in U$ and σ is locally a fraction, i.e., for all $p \in U$, there exists a neighborhood V of p in U and homogeneous elements $m \in M$, $r \in R$ of the same degree such that, for all $p' \in V$, we have $r \notin p'$ and $\sigma(p') = \frac{m}{r}$.
For any graded R-module M and $s \in \mathbb{Z}$, let $M(s)$ be the graded module whose part of degree d is M_{d+s} (the part of degree $d + s$ of M). For any $s \in \mathbb{Z}$, we define the twisting sheaf $\mathcal{O}_X(s)$ on $X := Proj(R)$ in the following way:

$$\mathcal{O}_X(s) = \mathcal{F}_{R(s)},$$

i.e., $\mathcal{O}_X(s)$ is defined to be the sheaf associated to the R-module $R(s)$. We can prove that $\mathcal{O}_X(s)$ is locally free (see "Sheaves") of rank 1 and that, for any $s, t \in \mathbb{Z}$,

$$\mathcal{O}_X(s) \otimes \mathcal{O}_X(t) = \mathcal{O}_X(t + s).$$

If we take $X = Proj(R)$ where $R = K[x_0, \dots, x_n]$ for some algebraically closed field K (thus X is the scheme corresponding to the projective space of dimension n over K), we get the sheaves we have described before.

Theorem.

- Let A be a Noetherian graded commutative ring with unity and let $R = A[x_0, \dots, x_n]$. Let $X = Proj(R)$. Denote $\mathcal{O}_X(s)$ by $\mathcal{O}(s)$ for the sake of brevity.
- For any $s \in \mathbb{Z}$, we have

  $$H^0(X, \mathcal{O}(s)) = A[x_0, \dots, x_n]_s,$$

  where $A[x_0, \dots, x_n]_s$ is the part of degree s of $A[x_0, \dots, x_n]$, i.e., the one given by homogeneous polynomials of degree s. Therefore

  $$h^0(X, \mathcal{O}(s)) = \binom{n+s}{s} \qquad \text{for } \ s \geq 0$$

  and

  $$h^0(X, \mathcal{O}(s)) = 0 \qquad \text{for } \ s < 0.$$

- Furthermore,

  $$H^k(X, \mathcal{O}(s)) = 0 \qquad \forall s \in \mathbb{Z}, \quad \text{for } k > 0, k \neq n.$$

- By Serre duality (see “Serre duality”)

  $$h^0(X, \mathcal{O}(s)) = h^n(X, \mathcal{O}(-s-n-1)) . \qquad \square$$

Theorem. Let K be an algebraically closed field. Then any locally free sheaf of rank 1 on the scheme $\mathbb{P}^n_K$ is isomorphic to $\mathcal{O}(s)$ for some $s \in \mathbb{Z}$. $\square$

I

Injective and projective modules. ([41], [62], [116], [159], [208]). Let R be a ring. An R-module I is said to be **injective** if for every injective R-morphism of R-modules $f : M \to N$ and every R-morphism $g : M \to I$ there exists an R-morphism $h : N \to I$ such that $g = h \circ f$:

$$\begin{tikzcd} M \arrow[r, hook, "f"] \arrow[d, "g"] & N \\ I & \end{tikzcd} \qquad \begin{tikzcd} M \arrow[r, hook, "f"] \arrow[d, "g"] & N \arrow[dl, "h"] \\ I & \end{tikzcd} .$$

An R-module P is said to be **projective** if for every surjective R-morphism of R-modules $f : M \to N$ and every R-morphism $g : P \to N$ there exists an R-morphism $h : P \to M$

such that $g = f \circ h$:

$$\begin{tikzcd} & P \arrow[d, "g"] \\ M \arrow[r, two heads, "f"] & N \end{tikzcd} \qquad \begin{tikzcd} & P \arrow[d, "g"] \arrow[dl, "h"'] \\ M \arrow[r, two heads, "f"] & N \end{tikzcd} .$$

Proposition. Let R be a ring with unity and consider only unital modules.
Free R-modules (i.e., direct sums of copies of R) are projective.
Any projective module is a direct summand of a free module.
Let K be a field. Any finitely generated projective $K[x_0, \dots, x_n]$-module is free (see [159, Chapter IV, Theorem 3.15]). □

Injective and projective resolutions. ([41], [62], [116], [208]). Let R be a ring.
An **injective resolution** of an R-module M is an exact sequence of R-modules

$$0 \to M \to I_0 \to \cdots \to I_k \to \cdots$$

with I_j injective R-modules.
A **projective resolution** of an R-module M is an exact sequence of R-modules

$$\cdots \to P_k \to P_{k-1} \to \cdots \to P_0 \to M \to 0$$

with P_j projective R-modules.

Integrally closed. ([12], [62], [164], [256]). Let B be a commutative ring with unity.
Let A be a subring of B (so the unity belongs to A). We say that $x \in B$ is **integral over** A if there exists a monic polynomial p with coefficients in A such that $p(x) = 0$.
The **integral closure** of A in B is defined to be the set

$$C := \{x \in B \mid x \text{ integral over } A\}.$$

If $A = C$, then we say that A is **integrally closed in** B. If $B = C$, then we say that B is **integral over** A.
We say that a domain A is **integrally closed** if it is integrally closed in its quotient field (see "Localization, quotient ring, quotient field").

Proposition.
(i) Let B be a commutative ring with unity. Let A be a subring of B.
- An element $x \in B$ is integral over A if and only if $A[x]$ is a finitely generated A-module.
- The integral closure of A in B is a subring of B.
- If B is integral over A and b is an ideal of B, then B/b is integral over $A/(b \cap A)$.
- Let C be the integral closure of A in B and let S be a multiplicative part of A, then $S^{-1}C$ is the integral closure of $S^{-1}A$ in $S^{-1}B$ (see "Localization, quotient ring, quotient field" for the notation $S^{-1}C$).

(ii) Let R be a commutative ring with unity and let R', R'' be two subrings of R with $R'' \subset R' \subset R$. If R is integral over R' and R' is integral over R'', then R is integral over R''.
(iii) Let R be a domain. Then R is integrally closed if and only if R_p is integrally closed for every prime ideal p of R and this is true if and only if R_m is integrally closed for every maximal ideal m of R (see "Localization, quotient ring, quotient field" for the notation R_p and R_m).
(iv) A unique factorization domain is integrally closed, in particular, for any field K, the ring $K[x_1, \dots, x_n]$ is integrally closed. □

Intersection of cycles. ([72], [74], [93], [107], [214], [227], [228]).

Intersection in topology. We follow mainly [93].
Let X be an oriented C^∞ manifold of (real) dimension n. Let c_1 and c_2 be two singular homology classes of **cycles of complementary dimension**, precisely,

$$c_1 \in H_k(X, \mathbb{Z}), \qquad c_2 \in H_{n-k}(X, \mathbb{Z})$$

(see "Singular homology and cohomology" for the notation). Let γ_1 and γ_2 be two piecewise smooth cycles such that $[\gamma_1] = c_1$, $[\gamma_2] = c_2$, and γ_1 and γ_2 intersect transversely (i.e., every intersection point P is smooth in γ_1 and γ_2 and $T_P X = T_P\gamma_1 \oplus T_P\gamma_2$, where T_P denotes the tangent space at P). We define

$$c_1 \cdot c_2 = \sum_{P \in \gamma_1 \cap \gamma_2} i_P(\gamma_1, \gamma_2),$$

where $i_P(\gamma_1, \gamma_2)$ is defined as follows: take a positively oriented basis $\mathcal{B}_1$ of $T_P\gamma_1$ and a positively oriented basis $\mathcal{B}_2$ of $T_P\gamma_2$ (observe that γ_1 and γ_2 have natural orientations given respectively by c_1 and c_2); if $\mathcal{B}_1, \mathcal{B}_2$ is a positively oriented basis of $T_P X$, then we define $i_P(\gamma_1, \gamma_2)$ to be 1, otherwise we set $i_P(\gamma_1, \gamma_2) = -1$. One can show that we can find γ_1 and γ_2 as above and that $c_1 \cdot c_2$ does not depend on the choice of γ_1 and γ_2.
We can define also the intersection of two singular homology classes of **cycles of non-complementary dimension**. Precisely, let

$$c_1 \in H_{n-k_1}(X, \mathbb{Z}), \qquad c_2 \in H_{n-k_2}(X, \mathbb{Z}),$$

with $k_1 + k_2 < n$. Let γ_1 and γ_2 be two piecewise smooth cycles such that $[\gamma_1] = c_1$, $[\gamma_2] = c_2$ and γ_1 and γ_2 intersect transversely almost everywhere (we say that they intersect transversely in a point P if P is smooth in γ_1 and γ_2 and $T_P\gamma_1 + T_P\gamma_2 = T_P X$).
We define $c_1 \cdot c_2$ to be the element of $H_{n-k_1-k_2}(X, \mathbb{Z})$ given by $\gamma_1 \cap \gamma_2$ with the following orientation: if $\mathcal{B}$ is a positively oriented basis of $T_P(\gamma_1 \cap \gamma_2)$, where P is a smooth point of $\gamma_1 \cap \gamma_2$, and $\mathcal{B}_1, \mathcal{B}$ is a positively oriented basis of $T_P\gamma_1$ and $\mathcal{B}, \mathcal{B}_2$ is a positively oriented basis of $T_P\gamma_2$, then $\mathcal{B}_1, \mathcal{B}, \mathcal{B}_2$ is a positively oriented basis of $T_P X$.

One can prove that the intersection of cycles corresponds through the Poincaré duality to the cup product; see "Singular homology and cohomology".

Intersection in algebraic geometry. Let X be a smooth algebraic variety of dimension n over an algebraic closed field K and let V_1 and V_2 be two subvarieties of X of codimension k_1 and k_2, respectively. Suppose $k_1 + k_2 \leq n$. We say that V_1 and V_2 intersect properly if the codimension of every irreducible component of $V_1 \cap V_2$ is equal to $k_1 + k_2$. Suppose that V_1 and V_2 intersect properly. We define the intersection cycle of V_1 and V_2, which we denote by $V_1 \cdot V_2$ or simply by $V_1 V_2$, to be the algebraic cycle (see "Cycles")

$$\sum_W i_W(V_1, V_2)\ W,$$

where the sum runs over all the irreducible components of $V_1 \cap V_2$ and the number $i_W(V_1, V_2)$ is the so-called "intersection multiplicity" of V_1 and V_2 along W. There are several definitions of intersection multiplicity; we report Serre's definition (see [227]): if W is an irreducible component of $V_1 \cap V_2$,

$$i_W(V_1, V_2) := \sum_i (-1)^i\, length\left(Tor_i^A(A/I_1, A/I_2)\right),$$

where A is the local ring $\mathcal{O}_{X,x}$ of X at a generic point $x \in W$ and I_i is the ideal of V_i in A for $i = 1, 2$ (see "Length of a module", "Tor, $\mathcal{TOR}$" for the definitions of *length* and *Tor*). Serre showed that the numbers $i_W(V_1, V_2)$ are nonnegative.
By linearity we can define the intersection cycle of any two cycles V_1 and V_2 when they intersect properly, i.e., when the subvarieties of which V_1 is a linear combination intersect properly the subvarieties of which V_2 is a linear combination.
Chow's moving lemma states that if X is a smooth quasi-projective algebraic variety and V_1 and V_2 are cycles of X of codimension k_1 and k_2, respectively, we can find a cycle V_1' rationally equivalent to V_1 and such that V_1' and V_2 intersect properly (see "Equivalence, algebraic, rational, linear -, Chow, Neron–Severi and Picard groups" for the definition of rational equivalence). Moreover, one can show that if V_1 and V_1' are rationally equivalent and intersect properly V_2, then $V_1 \cdot V_2$ and $V_1' \cdot V_2$ are rationally equivalent. This allows us to define the intersection of algebraic cycles (also intersecting not properly) up to rational equivalence.

Inverse image sheaf.
See "Direct and inverse image sheaves".

Irreducible topological space.
We say that a topological space is irreducible if it is not the union of two proper closed subsets.

Irregularity.
([93], [107]). Let X be a complex manifold or an algebraic variety over a field K. The irregularity of X is

$$h^1(X, \mathcal{O}_X),$$

where $\mathcal{O}_X$ is the sheaf of the holomorphic functions, respectively of the regular functions, on X. The irregularity of X is generally denoted by $q(X)$.

If X is a compact Kähler manifold (see "Hermitian and Kählerian metrics"), then, by Dolbeault's theorem and by Hodge's theorem (see "Dolbeault's theorem", "Hodge theory"), we have that $h^1(X, \mathcal{O}_X) = h^{0,1}(X) = h^{1,0}(X) = h^0(X, \Omega^1_X)$ (and somewhere the irregularity is defined to be $h^{1,0}(X)$).
If X is a smooth projective algebraic surface over an algebraic closed field, then the irregularity is equal to $p_g(X) - p_a(X)$ (where $p_a(X)$ and $p_g(X)$ are respectively the arithmetic and the geometric genus of X; see "Genus, arithmetic, geometric, real, virtual -" and "Dualizing sheaf").

J

Jacobians of compact Riemann surfaces. ([93], [101], [102], [163], [165], [195]). To every compact Riemann surface we can associate a principally polarized Abelian variety, called its Jacobian. Jacobians were the first Abelian varieties to be studied.
Let X be a **compact Riemann surface** of genus g (see "Riemann surfaces (compact -) and algebraic curves"). By Riemann's theorem, the complex vector space $H^0(X, \mathcal{O}(K_X))$, where K_X is the canonical bundle of X (see "Canonical bundle, canonical sheaf"), has dimension g. Consider the map

$$p : H_1(X, \mathbb{Z}) \longrightarrow H^0(X, \mathcal{O}(K_X))^\vee$$

defined by

$$p(\gamma) = \int_\gamma \cdot$$

It is injective, and the quotient

$$\frac{H^0(X, \mathcal{O}(K_X))^\vee}{p(H_1(X, \mathbb{Z}))}$$

is a complex torus of dimension g (see "Tori, complex - and Abelian varieties"). The Jacobian of X is the complex torus above endowed with the following polarization: let E be the alternating form on $H^0(X, \mathcal{O}(K_X))^\vee$ obtained extending on $\mathbb{R}$ the form on $H_1(X, \mathbb{Z})$ given by the intersection of 1-cycles; let

$$H : H^0(X, \mathcal{O}(K_X))^\vee \times H^0(X, \mathcal{O}(K_X))^\vee \longrightarrow \mathbb{C},$$

defined by

$$H(v, w) = E(iv, w) + iE(v, w),$$

for any $v, w \in H^0(X, \mathcal{O}(K_X))^\vee$. We endow the complex torus above with the polarization given by H. One can prove that it is a principal polarization. Thus the Jacobian of X is a principally polarized Abelian variety.

The Jacobian of X coincides with the Albanese variety of X (see "Albanese varieties").

Definition. Let $Div^d(X)$ be the set of divisors of degree d on X. The map

$$Div^0(X) \longrightarrow J(X)$$

defined by

$$\sum_{a=1,\dots,d} (P_a - Q_a) \longmapsto \sum_{a=1,\dots,d} \int_{Q_a}^{P_a},$$

for any $d \in \mathbb{N}$, $P_a, Q_a \in X$, is called the **Abel–Jacobi map** of X and denoted by μ. Obviously, if we fix a point P on X, we can also define a map

$$Div^d(X) \longrightarrow J(X)$$

(again called the Abel–Jacobi map and denoted by μ) by composing the map

$$Div^d(X) \longrightarrow Div^0(X),$$
$$D \longmapsto D - dP$$

with the Abel–Jacobi map $Div^0(X) \longrightarrow J(X)$. □

Abel–Jacobi theorem. The Abel–Jacobi map defines an isomorphism (again called the Abel–Jacobi map)

$$Pic^0(X) \longrightarrow J(X).$$ □

(See "Equivalence, algebraic, rational, linear -, Chow, Neron–Severi and Picard groups" for the definition of the Picard group $Pic^0(X)$; in case X is a compact Riemann surface, $Pic^0(X)$ is the set of the divisors of degree 0 up to linear equivalence.)

Notation. Define $V^d := \mu(X^{(d)})$, where $X^{(d)}$ is the symmetric d-product of X, i.e., the set of effective divisors of degree d on X. □

Proposition. The dimension of V^d is d if $d \leq g$, while it is g if $d \geq g$. If $D \in X^{(d)}$, then

$$\mu^{-1}(\mu(D)) = |D| = \mathbb{P}(H^0(X, \mathcal{O}(D)),$$

where $|D|$ denotes the linear system of D (see "Linear systems") Let $g \geq 1$; then the map $\mu : X \longrightarrow J(X)$ is injective and the map $\mu : X^{(d)} \longrightarrow J(X)$ is generically injective for $d \leq g$. □

Theorem. Let Θ be the divisor in $J(X)$ associated to a section of a holomorphic line bundle on $J(X)$ defining the polarization. We have that the intersection number of V^1 and Θ is g:

$$V^1 \cdot \Theta = g.$$

In addition, Θ is a translate of the image through the Abel–Jacobi map of the set of the effective divisors of degree $g-1$ on X, i.e.,

$$\Theta = V^{g-1} + K,$$

for some $K \in J(X)$. □

Torelli's theorem. The map from the set of compact Riemann surfaces up to isomorphisms to the set of the principally polarized Abelian varieties up to isomorphisms, associating to every Riemann surface X its Jacobian $J(X)$ is injective. □

Poincaré's formula. Let $[V^d]$ be the class of V^d in the singular homology group $H_{2g-2d}(J(X), \mathbb{Z})$. We have that

$$[V^d] = \frac{[\Theta]^{g-d}}{(g-d)!},$$

where Θ is the divisor associated to a section of a holomorphic line bundle defining the polarization on $J(X)$. □

Jacobians, Weil and Griffiths intermediate -.

([92], [93], [166], [250]). We follow mainly [166].

Let X be a compact Kähler manifold of dimension n (see "Hermitian and Kählerian metrics"). Let $1 \le q \le n$. The **Griffiths q-th intermediate Jacobian** of X is the complex torus (see "Tori, complex - and Abelian varieties")

$$J_q(X) = \frac{\oplus_{i=q,\dots,2q-1} H^{2q-1-i,i}(X, \mathbb{C})}{p(H^{2q-1}(X, \mathbb{Z}))},$$

where $p : H^{2q-1}(X, \mathbb{Z}) \to \oplus_{i=q,\dots,2q-1} H^{2q-1-i,i}(X, \mathbb{C})$ is given by the composition of the canonical map $u : H^{2q-1}(X, \mathbb{Z}) \to H^{2q-1}(X, \mathbb{C})$ with the projection $H^{2q-1}(X, \mathbb{C}) \to \oplus_{i=q,\dots,2q-1} H^{2q-1-i,i}(X, \mathbb{C})$ induced by the Hodge decomposition (see "Hodge theory" and "Singular homology and cohomology", in particular the Universal Coefficient Theorem). If $q = 1$ we have the Picard variety of X (see "Equivalence, algebraic, rational, linear -, Chow, Neron–Severi and Picard groups"):

$$J_1(X) = \frac{H^1(X, \mathcal{O})}{p(H^1(X, \mathbb{Z}))} = Pic^0(X).$$

If $q = n$ we have the Albanese variety of X (see "Albanese varieties"):

$$J_n(X) = \frac{H^{n-1,n}(X, \mathbb{C})}{p(H^{2n-1}(X, \mathbb{Z}))} \cong \frac{H^{1,0}(X, \mathbb{C})^\vee}{j(H_1(X, \mathbb{Z}))} \cong \frac{H^0(X, \Omega^1)^\vee}{j(H_1(X, \mathbb{Z}))} = Alb(X),$$

where j is the map $\gamma \mapsto \int_\gamma$ (see "Dolbeault's theorem", "Hodge theory", "Serre duality" to understand the isomorphisms above).

We have

$$\begin{aligned} Pic^0(Alb(X)) &= \frac{H^{0,1}(Alb(X), \mathbb{C})}{p(H^1(Alb(X), \mathbb{Z}))} \cong \frac{Hom_{\overline{\mathbb{C}}}(H^{1,0}(X, \mathbb{C}), \mathbb{C})}{p(H^1(X, \mathbb{Z}))} \\ &\cong \frac{H^1(X, \mathcal{O})}{p(H^1(X, \mathbb{Z}))} = Pic^0(X), \end{aligned}$$

and analogously we also have an isomorphism $Alb(Pic^0(X)) \cong Alb(X)$. So $Pic^0(X)$ and $Alb(X)$ are dual complex tori (see "Tori, complex - and Abelian varieties"). More generally, we can prove that $J_q(X)$ and $J_{n-q+1}(X)$ are dual complex tori.
Now let X be a smooth complex projective algebraic variety, and let ω be the Fubini-Study form restricted to X (thus ω is a closed positive integer $(1,1)$-form).
If $2q-1 \leq n$, we can consider on $J_q(X)$ the polarization with index defined by the following Hermitian form (if $2q-1 > n$, we define the polarization with index as the dual polarization with index of the one on $J_{n-q+1}(X)$):

$$H(\varphi,\psi) = 2i(-1)^q \int_X \omega^{n-2q+1} \wedge \varphi \wedge \overline{\psi};$$

the form H is Hermitian; in fact it is $\mathbb{C}$-linear in the first variable and

$$\begin{aligned} H(\psi,\varphi) &= 2i(-1)^q \int_X \omega^{n-2q+1} \wedge \psi \wedge \overline{\varphi} \\ &= -2i(-1)^q \int_X \omega^{n-2q+1} \wedge \overline{\varphi} \wedge \psi = \overline{2i(-1)^q \int_X \omega^{n-2q+1} \wedge \varphi \wedge \overline{\psi}} = \overline{H(\varphi,\psi)}. \end{aligned}$$

In general it is not positive definite, so in general the Griffiths intermediate Jacobian is not an Abelian variety; the Albanese variety and the Picard variety (of a smooth complex projective algebraic variety) are Abelian varieties.

Let $1 \leq q \leq n$. Let $C : H^q(X,\mathbb{C}) \to H^q(X,\mathbb{C})$ be the linear operator defined to be the multiplication by i^{a-b} on $H^{a,b}(X)$; it takes $H^q(X,\mathbb{R})$ to $H^q(X,\mathbb{R})$ and, if q is odd, we have $C^2 = -1$. The operator C is called Weil's operator. The **q-th Weil intermediate Jacobian** of X is the following torus:

$$W_q(X) = \frac{H^{2q-1}(X,\mathbb{R})}{u(H^{2q-1}(X,\mathbb{Z}))},$$

with the complex structure given by $-C$; here $u : H^{2q-1}(X,\mathbb{Z}) \to H^{2q-1}(X,\mathbb{R})$ is the canonical map; its image is isomorphic to the free part of $H^{2q-1}(X,\mathbb{Z})$. We can prove that $W_q(X)$ and $W_{n-q+1}(X)$ are dual complex tori.
If $2q-1 \leq n$, we endow $W_q(X)$ with the polarization defined by the Hermitian form

$$\begin{aligned} &H : H^{2q-1}(M,\mathbb{R}) \times H^{2q-1}(M,\mathbb{R}) \to \mathbb{C}, \\ &H(\varphi,\psi) := -E(C\varphi,\psi) + iE(\varphi,\psi), \end{aligned}$$

where, if $\varphi = \sum_j L^j\varphi_j$ and $\psi = \sum_j L^j\psi_j$ are the Lefschetz decomposition of φ and ψ (see "Lefschetz decomposition and Hard Lefschetz theorem"),

$$E(\varphi,\psi) := \sum_j (-1)^{q+j} \int_X \omega^{n-2q+1+2j} \wedge \varphi_j \wedge \psi_j.$$

For $2q - 1 > n$, we define the polarization on $W_q(X)$ to be the dual polarization of the one on $W_{n-q+1}(X)$.
One can prove that the Weil intermediate Jacobian is an Abelian variety (with the polarization we have just defined).
As we have already said, Griffiths intermediate Jacobian is not an Abelian variety in general. The advantage of Griffiths Jacobian with respect to Weil Jacobian is that Griffiths Jacobian varies holomorphically in a family of smooth projective algebraic varieties (while Weil Jacobian does not).

See also "Jacobians of compact Riemann surfaces".

Jumping lines and splitting type of a vector bundle on $\mathbb{P}^n$. ([61], [207]).

Let E be a holomorphic vector bundle of rank r on $\mathbb{P}^n = \mathbb{P}^n_{\mathbb{C}}$. Let l be a line in $\mathbb{P}^n$. By the Grothendieck–Segre theorem (see "Grothendieck–Segre theorem"), we have that

$$E|_l = \mathcal{O}_l(a_1) \oplus \cdots \oplus \mathcal{O}_l(a_r)$$

for some $a_1, \ldots, a_r \in \mathbb{Z}$ (see "Hyperplane bundles, twisting sheaves" for the definition of $\mathcal{O}_l(a_i)$). Suppose $a_1 \geq \cdots \geq a_r$. We say that

$$(a_1, \ldots, a_r)$$

is **the splitting type** of E on l. We denote it by $a_E(l)$.
We order $\mathbb{Z}^r$ by the lexicographic order. By using Grauert's semicontinuity theorem (see "Grauert's semicontinuity theorem"), one can prove the following lemma:

Lemma. Let $(a_1, \ldots, a_r) \in \mathbb{Z}^r$. The set $\{l \in G(1, \mathbb{P}^n) |\ a_E(l) > (a_1, \ldots, a_r)\}$ is a Zarisky closed subset of $G(1, \mathbb{P}^n)$ (the Grassmannian of the lines in $\mathbb{P}^n$). □

Let

$$a_E := inf\{a_E(l) |\ l \in G(1, \mathbb{P}^n)\}.$$

By Lemma above, there is a Zariski open subset U of $G(1, \mathbb{P}^n)$ such that $a_E(l) = a_E$ for $l \in U$. We say that a_E is the **generic splitting type of** E. If $l \in G(1, \mathbb{P}^n) - U$, we say that l is a **jumping line** for E.
About the generic splitting type we can recall the following theorem (see "Stable sheaves" for the definition of semistable bundle):

Grauert–Mülich–Spindler theorem. Let E be a semistable holomorphic r-vector bundle on $\mathbb{P}^n$ with generic splitting type $(a_1, \ldots, a_r)$, $a_1 \geq \cdots \geq a_r$. We have $a_i - a_{i+1} \leq 1$ for $1 \leq i \leq r - 1$. □

K

Kähler. See “Hermitian and Kählerian metrics”.

Kodaira dimension (or Kodaira number).

([22], [93], [107], [241]). Let X be a compact complex manifold or a smooth projective algebraic variety over a field K.
Let I be the set of positive natural numbers i such that $h^0(X, \omega_X^i) \neq 0$, where ω_X is the canonical sheaf (see “Canonical bundle, canonical sheaf”).
If $I = \emptyset$, we say that the **Kodaira dimension** of X is -1 (or $-\infty$).
If $I \neq \emptyset$, the Kodaira dimension of X is the maximum of the following set:

$$\{dim(\varphi_{K_X^i}(X)) | \ i \in I\},$$

where K_X is the canonical bundle and $\varphi_{K_X^i}$ is the map associated to the line bundle K_X^i (see “Bundles, fibre -”).

For instance, the Kodaira dimension of a compact Riemann surface of genus g is -1 if and only if $g = 0$; it is 0 if and only if $g = 1$; and it is 1 if and only if $g \geq 2$.

Let us denote $h^0(X, \omega_X^i)$ by $P_i(X)$ (plurigenera). If X is a complex smooth algebraic surface, one can prove the following statement: if $(P_i(X))_i$ is a bounded nonzero sequence, the Kodaira dimension of X is 0; if $(P_i(X))_i$ is an unbounded sequence, but there exists c such that $P_i(X) \leq ci$ for all i, the Kodaira dimension of X is 1; if $(P_i(X)/i)_i$ is an unbounded sequence, the Kodaira dimension of X is 2.

When the Kodaira dimension of X is equal to dimension of X, we say that X is **of the general type**.

Kodaira embedding theorem.

([93], [107], [150]).

Kodaira embedding theorem. Let X be a compact complex manifold and L be a positive holomorphic line bundle on X (see “Positive”).
Then there exists a natural number $\overline{k}$ such that, for all $k \geq \overline{k}$, the map associated to L^k, $\varphi_{L^k} : X \to \mathbb{P}(H^0(X, \mathcal{O}(L^k))^\vee)$, is well defined, and an embedding (see “Bundles, fibre -” for the definition of the map associated to L^k).
In other words, a positive holomorphic line bundle is ample. □

Observe that the converse is also true: an ample holomorphic line bundle is positive. In fact, if L^k embeds X into $\mathbb{P}^n_{\mathbb{C}}$ and H is the hyperplane bundle of $\mathbb{P}^n_{\mathbb{C}}$, then $\varphi^*_{L^k}(H) = L^k$ and so $c_1(L^k) = \varphi^*_{L^k}(c_1(H))$ and, since the hyperplane bundle is positive (see “Fubini-Study metric”), we conclude.

Another statement of the Kodaira embedding theorem is the following.

Kodaira embedding theorem. Let X be a compact complex manifold. It is embedable in a projective space if and only if it has a closed positive $(1,1)$-form η such that the cohomology class of η is rational, that is, such that $[\eta] \in H^2(X, \mathbb{Q})$. □

Kodaira–Nakano vanishing theorem.

See "Vanishing theorems".

Koszul complex.

([62], [89], [93], [107], [164], [185], [209]). Let R be a commutative ring with unity and $f_1, \dots, f_r \in R$; let R^r be a free R-module of rank r with basis $\{e_1, \dots, e_r\}$; the Koszul complex is the following complex:

$$0 \longrightarrow \wedge^r R^r \xrightarrow{\varphi_r} \wedge^{r-1} R^r \xrightarrow{\varphi_{r-1}} \cdots\cdots \longrightarrow R^r \xrightarrow{\varphi_1} R \xrightarrow{\varphi_0} R/(f_1, \dots, f_r)R \longrightarrow 0, \tag{4}$$

where φ_0 is the projection and, for every $p = 1, \dots, r$, $\varphi_p : \wedge^p R^r \to \wedge^{p-1} R^r$ is the module morphism such that

$$m(e_{i_1} \wedge \cdots \wedge e_{i_p}) \longmapsto \sum_{k=1,\dots,p} (-1)^{k+1} (m f_{i_k})(e_{i_1} \wedge \cdots \wedge \widehat{e_{i_k}} \wedge \cdots \wedge e_{i_p})$$

for any $m \in R$ and for any distinct $i_1, \dots, i_p \in \{1, \dots, r\}$. If M is an R-module, by tensoring (4) by M we get the complex

$$0 \to \wedge^r M^r \xrightarrow{\varphi_r} \wedge^{r-1} M^r \xrightarrow{\varphi_{r-1}} \cdots\cdots \to M^r \to M \to M/(f_1, \dots, f_r)M \to 0, \tag{5}$$

Proposition. If $f_1, \dots, f_r$ is an M-regular sequence (see "Regular sequences"), the Koszul complex (5) is exact. □

K3 surfaces.

See "Surfaces, algebraic -".

Kummer surfaces.

See "Surfaces, algebraic -".

L

Lefschetz decomposition and hard Lefschetz theorem.

([44], [90], [93], [245], [250], [251]). Let (X, g) be a compact Hermitian manifold of complex dimension n (see "Hermitian and Kählerian metrics"). Let ω be the $(1,1)$-form associated to g. Let L be the operator on the set of forms on X defined by

$$L(\eta) = \omega \wedge \eta$$

and let Λ be the adjoint operator of L with respect to the product $(\eta, \gamma) = \int \eta \wedge * \overline{\gamma}$, where the star operator $*$ is defined as usual on real forms and extended by $\mathbb{C}$-linearity on complex forms (see "Star operator"). We say that a q-form η is primitive if

$$\Lambda(\eta) = 0.$$

We can prove that, if $q \leq n$, then a q-form η is primitive if and only if $L^{n-q+1}(\eta) = 0$ and that any primitive form of degree greater than n must be zero. Let us denote by $H^q_{DR}(X, \mathbb{C})$ the q-th De Rham cohomology of X, i.e., the vector space of the closed complex q-forms on X modulo the vector space of the exact complex q-forms on X. If (X, g) is Kähler, i.e., ω is closed, then the operators L and Λ induce operators, called again L and Λ, on the De Rham cohomology $H^*_{DR}(X, \mathbb{C})$ and $\alpha \in H^q_{DR}(X, \mathbb{C})$ is said to be primitive if $\Lambda(\alpha) = 0$.
Let

$$P^q(X, \mathbb{C}) = \{\alpha \in H^q_{DR}(X, \mathbb{C}) | \ \Lambda(\alpha) = 0\}.$$

Lefschetz decomposition.

$$H^q_{DR}(X, \mathbb{C}) = \oplus_k L^k P^{q-2k}(X, \mathbb{C}) . \qquad \square$$

Hard Lefschetz. For any k with $1 \leq k \leq n$, the map

$$L^k : H^{n-k}_{DR}(X, \mathbb{C}) \to H^{n+k}_{DR}(X, \mathbb{C})$$

is an isomorphism. □

Lefschetz theorem on (1, 1)-classes. ([16], [93], [245]). Let X be a smooth complex projective algebraic variety. Every element of the intersection $H^2(X, \mathbb{Z}) \cap H^{1,1}(X)$ is the class of a divisor on X (see "Equivalence, algebraic, rational, linear -, Chow, Neron–Severi and Picard groups").

Lefschetz hyperplane theorem. ([6], [31], [93], [172], [245]). Let X be an n-dimensional compact complex manifold and $D \subset X$ a smooth hypersurface such that the associated line bundle (D) (see "Bundles, fibre -") is positive. The map

$$H^i(X, \mathbb{Q}) \longrightarrow H^i(D, \mathbb{Q})$$

induced by the inclusion of D into X is an isomorphism for $i \leq n-2$ and injective for $i = n-1$ (see "Singular homology and cohomology" for the definition of $H^i(X, \mathbb{Q})$).

Length of a module. ([12], [62], [256]). Let R be a commutative ring with unity. Let M be an R-module. A chain of length n of submodules of M is a sequence of submodules M_i, $i = 0, \ldots, n$, such that $M = M_0 \supset M_1 \supset \cdots \supset M_n = 0$ and all the inclusions are strict.
We say that a chain is maximal if we cannot insert other submodules. One can prove that two maximal chains of a module M have the same length.
If in a module M there exists a maximal chain of submodules, we say that M is of finite length and we call "length of M" the length of any maximal chain; we denote it by $l(M)$. If in M there is not a maximal chain of submodules, we say that the length of M is $+\infty$.

Proposition.

(i) A module M is of finite length if and only if it is Artinian and Noetherian (see "Noetherian, Artinian").

(ii) If $0 \to M_1 \to M_2 \to M_3 \to 0$ is an exact sequence of R-modules of finite length, then $l(M_2) = l(M_1) + l(M_3)$.

(iii) Let K be a field. A K-vector space has finite dimension if and only if has finite length and, in this case, the dimension is equal to the length. □

Leray spectral sequence. ([34], [53], [84, II 4.17], [93], [129], [175], [176]). Let $f : X \to Y$ be a continuous map between two topological spaces. Let $\mathcal{F}$ be a sheaf on X. The q-direct image sheaf $\mathcal{R}^q f_*(\mathcal{F})$ is the sheaf on Y associated to the presheaf

$$U \mapsto H^q(f^{-1}(U), \mathcal{F}|_{f^{-1}(U)}),$$

where U is any open subset of Y (see "Sheaves", "Direct and inverse image sheaves"). Under certain assumptions, there exists a spectral sequence $\{E_r\}_r$ (see "Spectral sequences"), called Leray spectral sequence, such that

$$E_2^{p,q} = H^p(Y, \mathcal{R}^q f_*(\mathcal{F})),$$
$$E_\infty \Rightarrow H^*(X, \mathcal{F}).$$

Let X and Y be compact Kähler manifolds (see "Hermitian and Kählerian metrics") and $f : X \to Y$ be a surjective holomorphic of maximal rank. One can prove (see [53] or [93]) that in this case Leray spectral sequence degenerates at E_2, i.e.,

$$E_2 = E_\infty.$$

Liaison or linkage. ([186], [187], [212], [213]). Let K be an algebraically closed field and let $\mathbb{P}^n$ be the scheme $Proj(K[x_0, \dots, x_n])$. Let Y_1 and Y_2 be two subschemes of $\mathbb{P}^n$ such that no component of Y_1 is contained in any component of Y_2 and conversely. We say that Y_1 and Y_2 are **geometrically G (respectively CI) linked** if $Y_1 \cup Y_2$ is arithmetically Gorenstein (respectively complete intersection).

Let Y_1 and Y_2 be two subschemes of $\mathbb{P}^n$. We say that Y_1 and Y_2 are **algebraically G- (resp. CI-) linked** if there exists X arithmetically Gorenstein (resp. complete intersection) such that $I_X \subset I_{Y_1} \cap I_{Y_2}$, $(I_X : I_{Y_1}) = I_{Y_2}$ and $(I_X : I_{Y_2}) = I_{Y_1}$, where I_X denotes the saturated ideal of X (see Notation for the definition of $(I_X : I_{Y_i})$).

One can prove that geometric linkage implies algebraic linkage.

Let Y be a subscheme of $\mathbb{P}^n$. The G-linkage (respectively CI-linkage) class of Y is the set of the subschemes of $\mathbb{P}^n$ that can be obtained from Y by a finite number of G (respectively CI) algebraic linkages.

Lie algebras. ([76], [115], [125], [133]). A Lie algebra is a vector space g over a field K with a map,

$$[\ ,\ ]: g \times g \to g,$$

called a bracket, such that
(i) $[\ ,\ ]$ is bilinear;
(ii) $[x, x] = 0$ for all $x \in g$;
(iii) $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ for all $x, y, z \in g$ (Jacobi identity).

Examples. Let $gl(n, K)$ be the set of the $n \times n$ matrices with entries in K. It is a Lie algebra with the bracket:

$$[A, B] = AB - BA.$$

Let $sl(n, K)$ be the set of the $n \times n$ matrices with entries in K and with null trace. It is a Lie algebra with the bracket:

$$[A, B] = AB - BA.$$

The Lie algebras $gl(n, K)$, $sl(n, K)$ take their names from the fact that they can be identified respectively with the tangent space in the identity of $GL(n, K)$ (the set of invertible $n \times n$ matrices with entries in K) and the tangent space in the identity of $SL(n, K)$ (the set of $n \times n$ matrices with entries in K and with determinant equal to 1).

Definition. A **Lie algebras morphism** from a Lie algebra $(g, [\ ,\ ])$ to a Lie algebra $(g', [\ ,\ ]')$ is a linear map

$$f : g \to g'$$

such that $[f(x), f(y)]' = f([x, y])$ for any $x, y \in g$. □

Definition. We say that a Lie algebra $(g, [\ ,\ ])$ is **Abelian** if $[x, y] = 0$ for any $x, y \in g$. □

Definition.
(i) A subspace a of a Lie algebra $(g, [\ ,\ ])$ is called a **Lie subalgebra** of g if $[a, a] \subset a$.
(ii) A subspace a of a Lie algebra $(g, [\ ,\ ])$ is called an **ideal** of g if $[a, g] \subset a$.
(iii) Let a be a Lie subalgebra of a Lie algebra $(g, [\ ,\ ])$. The **normalizer** of a is the set

$$\{x \in g |\ [x, a] \subset a\}.$$ □

From now on, we will consider only Lie algebras of finite dimension.

Definition.
– We say that a Lie subalgebra a of a Lie algebra $(g, [\ ,\ ])$ is **solvable** if the sequence of subalgebras

$$a^1 := [a, a], \quad a^2 := [a^1, a^1], \quad \cdots \quad a^i := [a^{i-1}, a^{i-1}], \ldots$$

terminates to zero.

– We say that a Lie subalgebra a of a Lie algebra $(g, [\ ,\ ])$ is **nilpotent** if the sequence of subalgebras

$$a_1 := [a, a], \quad a_2 := [a, a_1], \quad \cdots \quad a_i := [a, a_{i-1}], \ldots$$

terminates to zero. □

From now on, we will consider only Lie algebras over an algebraically closed field K of characteristic 0.

Notation. Let $(g, [\ ,\ ])$ be a Lie algebra and let $x \in g$; we denote by

$$ad(x) : g \longrightarrow g$$

the map

$$y \longmapsto [x, y]$$

for any $y \in g$. □

Notation. The **Killing form** on a Lie algebra $(g, [\ ,\ ])$ is the symmetric bilinear form B defined in the following way:

$$B(x, y) := tr(ad(x) \circ ad(y))$$

for any $x, y \in g$ (where tr denotes the trace). □

Definition.
(a) We say that a nonzero Lie algebra $(g, [\ ,\ ])$ is **simple** if there is no nontrivial ideals (in some works, the condition $dim\ \ g > 1$ is also required; in other works, the (stronger) condition that g is not Abelian is also required).
(b) We say that a nonzero Lie algebra $(g, [\ ,\ ])$ is **semisimple** if there are no nonzero solvable ideals (equivalently if the Killing form is nondegenerate). □

Theorem. A semisimple Lie algebra is the direct sum of all its simple ideals. □

Definition. We say that a Lie subalgebra c of a Lie algebra $(g, [\ ,\ ])$ is a **Cartan** subalgebra if it is nilpotent and is equal to its normalizer. □

The Cartan subalgebras are useful for classifying semisimple Lie algebras.

Definition. Let $(g, [\ ,\ ])$ be a semisimple Lie algebra and let c be a Cartan subalgebra (we can prove that it exists and that, in the case of semisimple Lie algebras, it is Abelian and its elements are semisimple, i.e., diagonalizable).
We say that a linear map $\alpha : c \to K$ is a **root** if it is nonzero, and if we define

$$g^\alpha := \{x \in g \mid [h, x] = \alpha(h)x \quad \forall\, h \in c\}$$

we have that $g^\alpha \neq \{0\}$. We call the subspace g^α the **root space of** α. Let R be the set of the roots. □

Definition. Let $(E, (\ ,\ ))$ be a finite dimensional euclidean space, i.e., a finite dimensional vector space over $\mathbb{R}$ endowed with a positive definite symmetric bilinear form $(\ ,\ )$. Define

$$\langle \beta, \alpha \rangle = \frac{2(\beta, \alpha)}{(\alpha, \alpha)}$$

for any $\alpha, \beta \in E$ with $\alpha \neq 0$. We say that a subset R of E is a **root system** if
(1) R is finite, it generates E and does not contain 0;
(2) if $\alpha \in R$, the unique multiples of α in R are α and $-\alpha$;
(3) if $\alpha \in R$, $\sigma_\alpha(R) = R$, where σ_α is defined by $\sigma_\alpha(\beta) = \beta - \langle \beta, \alpha \rangle \alpha$;
4) if $\alpha, \beta \in R$, $\langle \alpha, \beta \rangle \in \mathbb{Z}$.

We say that a root system is reducible if there exist $R_1, R_2 \subset R$ such that R_1 and R_2 are perpendicular for $(\ ,\ )$ and $R = R_1 \cup R_2$. □

Definition. Let $(E, (\ ,\ ))$ be a finite dimensional euclidean space, and let R be a root system. Let $\Delta = \{\alpha_1, \dots, \alpha_r\}$ be a subset of R. We say that Δ is a **basis** of R if
(1) Δ is a basis of E;
(2) we can write every $\beta \in R$ as $\beta = \sum_{i=1,\dots,r} n_i \alpha_i$ with $\alpha_i \in \Delta$, $n_i \in \mathbb{Z}$ and the n_i all nonnegative or all nonpositive. □

Definition. Let $(E, (\ ,\ ))$ be a finite dimensional euclidean space and let R be a root system. Let $\Delta = \{\alpha_1, \dots, \alpha_r\}$ be a basis of R. The associated **Cartan matrix** A is defined by

$$A_{i,j} = \langle \alpha_i, \alpha_j \rangle = \frac{2(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)}$$

for any $i, j \in \{1, \dots, r\}$.
The associated **Dynkin diagram** is defined in the following way: we consider a vertex for every element of the basis and we join the two vertices corresponding to α_i and α_j by $\langle \alpha_i, \alpha_j \rangle \langle \alpha_j, \alpha_i \rangle$ lines (we can prove that $\langle \alpha_i, \alpha_j \rangle \langle \alpha_j, \alpha_i \rangle$ is a number between 0 and 3); if between the two vertices corresponding to α_i and α_j there are 2 or 3 lines, we also put an arrow toward the vertex corresponding to α_i if $|\langle \alpha_i, \alpha_j \rangle| < |\langle \alpha_j, \alpha_i \rangle|$. □

Theorem. The Dynkin diagram associated to a roots system does not depend on the choice of a basis.
The Dynkin diagram of an irreducible root system is one of the diagrams in Figure 12.
The map

$$\{\text{irreducible roots systems}\}/\text{isomorphisms} \longrightarrow \{\text{Dynkin diagrams of kind } A, \dots, G\},$$

associating to every class of an irreducible root system its Dynkin diagram, is a bijection. □

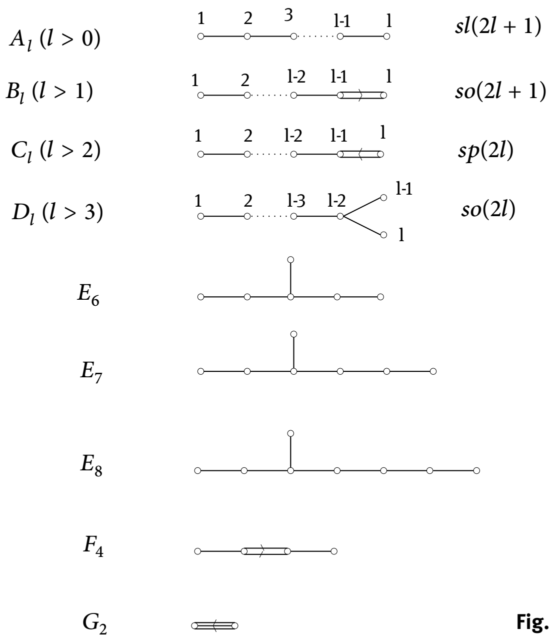


Fig. 12. Dynkin diagrams and Lie algebras.

Theorem. Let g be a semisimple Lie algebra and c one of its Cartan subalgebras. On $c^\vee$ (the dual space of c) we define the following form $(\, , \,)$:
For any $\alpha \in c^\vee$ there exists a unique $t_\alpha \in c$ such that $B(t_\alpha, h) = \alpha(h)$ for all $h \in c$, where B is the Killing form; we set $(\lambda, \mu) = B(t_\lambda, t_\mu)$ for any $\lambda, \mu \in c^\vee$.
Let E be the $\mathbb{R}$-subspace of $c^\vee$ spanned by the set R of the roots; the form just defined gives a positive definite bilinear form on it, so it is an euclidean space.

(1) The set R of roots is a root system of E.
(2) We have
$$g = c \oplus \oplus_{\alpha \in R} \, g^\alpha$$
(Cartan decomposition).
(3) For any $\alpha \in R$, the subalgebra g^α has dimension 1. For any $\alpha \in R$, the subalgebra $[g^\alpha, g^{-\alpha}]$ has dimension 1 and it is generated by t_α.
(4) If $\alpha \in R$, $x_\alpha \in g^\alpha$, $x_\alpha \neq 0$, then there exists $y_\alpha \in g^{-\alpha}$ such that, if we define $h_\alpha = [x_\alpha, y_\alpha]$, we have $x_\alpha, y_\alpha, h_\alpha$ generate a subalgebra of g isomorphic to $sl(2, K)$ via
$$x_\alpha \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad y_\alpha \mapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad h_\alpha \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \qquad \square$$

Theorem. The root system associated to a semisimple Lie algebra, as described in the theorem above, does not depend on the choice of the Cartan subalgebra. The map
$$\{\text{semisimple Lie algebras over } K\}/\text{isomorphisms} \to \{\text{root systems}\}/\text{isomorphisms}$$
is a bijection. $\square$

Therefore the classification of the Dynkin diagrams associated to the irreducible root systems allows us to classify simple Lie algebras and then semisimple Lie algebras.

See “Lie groups” and “Representations”.

Lie groups. ([3], [76], [115], [126], [210], [226]). A **Lie group** is a group G endowed with a structure of C^∞ manifold such that the product and the map associating to any element of G its inverse are C^∞ maps.
A morphism of Lie groups is a C^∞ homomorphism of groups.
Let G be a Lie group, and let $Aut(G)$ be the group of the automorphisms of G. Let

$$a : G \longrightarrow Aut(G)$$

be defined in the following way:

$$a(g)(h) = ghg^{-1}$$

for any $g, h \in G$. Observe that, if e is the identity element, $a(g)(e) = e$ for all $g \in G$; thus the differential of $a(g)$ in e is a map from the tangent space of G at e to itself:

$$d(a(g))_e : T_eG \longrightarrow T_eG.$$

We define the **adjoint representation** (see “Representations”) of G to be

$$Ad : G \to GL(T_eG),$$
$$g \mapsto d(a(g))_e.$$

Its differential in e is denoted by ad:

$$ad : T_eG \longrightarrow End(T_eG),$$

where $End(T_e(G))$ is the vector space of the endomorphisms of $T_e(G)$. We can define a bracket on T_eG by

$$[x, y] = ad(x)(y).$$

With this bracket, the vector space T_eG is a Lie algebra (see “Lie algebras”), which is denoted $Lie(G)$, and ad is a representation of the Lie algebra $Lie(G)$ on $Lie(G)$.

Remark. If $G \subset M(n \times n, \mathbb{R})$, then the bracket in $Lie(G)$ is

$$[x, y] = xy - yx.$$

In fact, for any $g \in G$, the map $da(g)_e$ is the map $t \mapsto gtg^{-1}$, then $Ad(g)$ is the map $t \mapsto gtg^{-1}$ and, for any $l \in \mathbb{R}, k \in G$, the map $Ad(g+lk)$ is the map $t \mapsto (g+lk)t(g+lk)^{-1}$. So $d(Ad)_e(t)$ is the map $k \mapsto tk - kt$. □

Definition. We say that a subset H of a Lie group G is a **Lie subgroup** if it is a subgroup and a closed submanifold. □

Remark. Let G and G' be two Lie groups and let $\varphi : G \to G'$ be a morphism of Lie groups. The map $d\varphi_e : T_eG \to T_eG'$ is a Lie algebra morphism. □

Proposition. Let G and G' be two Lie groups with G connected. Then a morphism of Lie groups $\varphi : G \to G'$ is uniquely determined by $d\varphi_e$.
Furthermore, if G is simply connected, the map sending a morphism of Lie groups $\varphi : G \to G'$ to its differential at e, $d\varphi_e : T_eG \to T_eG'$, is a bijection between the set of the morphisms of Lie groups from G to G' and the set of the morphisms of Lie algebras from $Lie(G)$ to $Lie(G')$. □

Proposition. Let G be a Lie group. Let $v \in T_eG$. There exists a unique Lie group morphism $\varphi_v : \mathbb{R} \to G$ such that $(d\varphi_v)_0(1) = v$. □

Definition. We define the **exponential map**

$$\exp_G : T_eG \to G$$

by

$$\exp_G(tv) = \varphi_v(t),$$

for any $t \in \mathbb{R}$. □

Proposition. For every Lie group G, the map exp_G is smooth.
Furthermore, if $\varphi : G \to G'$ is a morphism of Lie groups, we have $\exp_{G'} \circ d\varphi_e = \varphi \circ \exp_G$. □

Theorem.
(a) Every finite dimensional Lie algebra is the Lie algebra of a simply connected Lie group.
(b) Let G be a connected Lie group. The subalgebras of $Lie(G)$ are in bijection with the connected Lie subgroups of G. The bijection sends a Lie subalgebra h to the subgroup generated by $\exp(h)$ (its tangent space is h).
In addition, if H is a connected Lie subgroup of G and h the corresponding subalgebra, then H is normal if and only if h is an ideal. □

Proposition. Let G be a connected Lie group. We have that G is a solvable group if and only if $Lie(G)$ is solvable (see “Lie algebras” for the definition of solvable Lie algebra). □

Definition. We say that a connected Lie group G is **simple** if there are not in G nontrivial normal connected Lie subgroups.
We say that a connected Lie group G is **semisimple** if there are not in G nontrivial normal solvable connected Lie subgroups. □

By the proposition above, G is (semi)simple if $Lie(G)$ is (semi)simple.

Example. $SL(n, \mathbb{C})$ is simple.

Definition. We say that a subalgebra of a semisimple Lie algebra g is **Borel** if it is a maximal solvable Lie subalgebra.
We say that a Lie subgroup of a semisimple Lie group G is **Borel** if the corresponding subalgebra is Borel (or equivalently if it is a maximal solvable connected subgroup).
We say that a Lie subgroup of a semisimple Lie group G is **parabolic** if it contains a Borel subgroup. □

Proposition. Let G be a semisimple Lie group. A Lie subgroup P is parabolic if and only if G/P is a projective algebraic variety. □

Example. Let $G = SL(3)$; the subgroup $P = \{A \in SL(3) |\ a_{2,1} = a_{3,1} = 0\}$ is parabolic; it contains the Borel subgroup of the upper triangular matrices. The quotient $SL(3)/P$ is $\mathbb{P}^2$.

Definition. We say that a Lie group is **reductive** if all its representations are completely reducible (see "Representations"). □

Definition. A **complex Lie group** is a group G endowed with a structure of complex manifold such that the product and the map associating to any element of G its inverse are holomorphic. □

Theorem. (Unitary trick). Let G be a complex Lie group. If there is a real compact Lie group G' such that $Lie(G) = Lie(G') \otimes_{\mathbb{R}} \mathbb{C}$, then G is reductive. □

Limits, direct and inverse -.

([12], [91], [208]). Let $(F, \leq)$ be a partially ordered set. We say that it is a **filtering set** if, for all $i_1, i_2 \in F$, there exists $j \in F$ such that $i_1 \leq j$, $i_2 \leq j$.

Let $(F, \leq)$ be a filtering set; a family of R-modules $\{M_i\}_{i \in F}$ is said to be a **direct system of modules** if and only if, for all $i_1, i_2 \in F$ with $i_1 \leq i_2$, we have a homomorphism

$$\phi_{i_2,i_1} : M_{i_1} \to M_{i_2}$$

such that
(i) if $i_1 \leq i_2 \leq i_3$, then $\phi_{i_3,i_2} \circ \phi_{i_2,i_1} = \phi_{i_3,i_1}$;
(ii) $\phi_{i,i}$ is the identity for all $i \in F$.

Let $\{M_i\}_{i \in F}$ be a direct system of modules on a filtering set $(F, \leq)$ (with homomorphisms $\phi_{i,j}$); a **direct (or inductive) limit** of $\{M_i\}_{i \in F}$ is an R-module M with a homomorphism

$$\phi_i : M_i \to M$$

for all $i \in F$, such that, for all $i_1, i_2 \in F$ with $i_1 \leq i_2$, we have $\phi_{i_2} \circ \phi_{i_2,i_1} = \phi_{i_1}$ and the following universal property holds: if N is an R-module and, for all $i \in F$, we have homomorphisms $\varphi_i : M_i \to N$ such that for all $i_1, i_2 \in F$ with $i_1 \leq i_2$ we have

$\varphi_{i_2} \circ \phi_{i_2,i_1} = \varphi_{i_1}$, then there exists a homomorphism $\varphi : M \to N$ such that $\varphi \circ \phi_i = \varphi_i$ for all $i \in F$.

By the universal property, the direct limit must be unique; in addition we can prove that the direct limit always exists.

We denote the direct limit of a direct system of modules $\{M_i\}_{i\in F}$ by

$$\varinjlim M_i$$

Let $\{M_i\}_{i\in F}$ and $\{N_i\}_{i\in F}$ be two direct systems of R-modules on a filtering set $(F, \leq)$, and suppose that for any $i \in F$ there exists a homomorphism $h_i : M_i \to N_i$; suppose that such homomorphisms commute with the homomorphisms of the direct systems. Then for every i we have a homomorphism $M_i \to \lim_{\to} N_i$ (the composition of the maps $M_i \to N_i$ and $N_i \to \lim_{\to} N_i$) and then a homomorphism

$$H : \varinjlim M_i \longrightarrow \varinjlim N_i$$

by the universal property.

Proposition $*$**.** Let $\{N_i\}_{i\in F}$, $\{M_i\}_{i\in F}$ and $\{P_i\}_{i\in F}$ be three direct systems of R-modules on a filtering set $(F, \leq)$, and suppose that for all $i \in F$ there is an exact sequence of modules $M_i \xrightarrow{h_i} N_i \xrightarrow{g_i} P_i$ such that for all $i_1, i_2 \in F$ with $i_1 \leq i_2$ we have the commutative diagram

$$\begin{tikzcd}
M_{i_1} \arrow[r, "h_{i_1}"] \arrow[d] & N_{i_1} \arrow[r, "g_{i_1}"] \arrow[d] & P_{i_1} \arrow[d] \\
M_{i_2} \arrow[r, "h_{i_2}"] & N_{i_2} \arrow[r, "g_{i_2}"] & P_{i_2}
\end{tikzcd}$$

where the vertical arrows are the maps of the direct systems. Then we have an exact sequence

$$\varinjlim M_i \xrightarrow{H} \varinjlim N_i \xrightarrow{G} \varinjlim P_i\,. \qquad \square$$

Let $(F, \leq)$ be a filtering set; we say that a subset F' of F is **cofinal** if F' is a filtering set with the order induced by F and if for all $i \in F$ there exists $i' \in F'$ such that $i \leq i'$.

Proposition. Let $\{M_i\}_{i\in F}$ be a **direct system of modules** on a filtering set $(F, \leq)$ and let F' be a cofinal subset of F; then the limit of $\{M_i\}_{i\in F}$ is isomorphic to the limit of $\{M_{i'}\}_{i'\in F'}$. $\square$

Reverting the arrows in the definition of direct system and direct limit, we get the notion of **inverse (or projective) system** and **inverse (or projective) limit**, which we denote by

$$\varprojlim M_i.$$

For the inverse limit, we have analogous propositions except Proposition $*$.

Linear systems. ([93], [107], [129], [228]). Let K be an algebraically closed field. Let X be a complex manifold, respectively an algebraic variety over K that is smooth in codimension 1 (i.e., the codimension of the set of the singular points is greater than or equal to 2), e.g., a normal variety (see "Normal").
Let D_1 and D_2 be two Weil divisors on X. We say that D_1 and D_2, are linearly equivalent if there exists a meromorphic, respectively rational, function f on X such that $D_1 - D_2 = (f)$, where (f) is the Weil divisor associated to f (see "Divisors", "Equivalence, algebraic, rational, linear -, Chow, Neron–Severi and Picard groups"). If D is a Weil divisor on X, the **complete linear system** $|D|$ is defined in the following way:

$$|D| = \{D' \mid D' \text{ effective Weil divisor on } X \text{ and } D' \sim D\},$$

where $\sim$ denotes the linear equivalence. Furthermore, we define

$$\mathcal{L}(D) = \{f \mid f = 0 \text{ or } f \text{ merom., resp. rat., function on } X \text{ s.t. } (f) + D \text{ effective}\}.$$

One can easily prove that $\mathcal{L}(D)$ is a vector space over K. Analogous definitions can be given for Cartier divisors.
Let X be a compact complex manifold or a complete normal algebraic variety over K. In this case, if f and g are two meromorphic, respectively rational, functions such that $(f) = (g)$, then f and g are multiple by a constant. Therefore, for any divisor D there is an obvious bijection

$$|D| \cong \mathbb{P}(\mathcal{L}(D)).$$

A **linear system** on a X is a family of divisors corresponding to a projective subspace of $\mathbb{P}(\mathcal{L}(D))$; its dimension is its dimension as projective subspace; we say that it is a **pencil** if its dimension is 1, a **net** if its dimension is 2, and a **web** if its dimension is 3.
The **base locus** of a linear system is the intersection of the supports of the divisors in the linear system.
An often used notation about linear systems on Riemann surfaces is "g^r_d": it denotes a linear system of degree d and projective dimension r.
Let X be a complex manifold or a smooth algebraic variety over K. For any divisor D, we have that

$$\mathcal{L}(D) \cong H^0(X, \mathcal{O}(D)),$$

where (D) is the line bundle associated to D. The bijection can be described as follows. If $\{U_\alpha\}_\alpha$ is an open covering of X and D on U_α is given by $f_\alpha = 0$ (f_α meromorphic, respectively rational, function), the transition functions of the bundle associated to D are $f_{\alpha,\beta} := \frac{f_\alpha}{f_\beta}$.
We can associate to $g \in \mathcal{L}(D)$ the section of $\mathcal{O}(D)$ given by gf_α on U_α.

Vice versa, let $\sigma \in H^0(X, \mathcal{O}(D))$ be given by $\sigma_\alpha : U_\alpha \to \mathbb{C}$, respectively $\sigma_\alpha : U_\alpha \to K$, such that $\sigma_\alpha = f_{\alpha,\beta}\sigma_\beta = \frac{f_\alpha}{f_\beta}\sigma_\beta$; we can associate to σ the meromorphic, respectively rational, function in $\mathcal{L}(D)$ given by $\frac{\sigma_\alpha}{f_\alpha}$ on U_α.

See also “Bertini’s theorem”.

Linkage. See “Liaison or linkage”.

Local. ([12], [62]). We say that a ring is local if it has a unique maximal ideal.

Localization, quotient ring, quotient field. ([12], [62]). Let R be a commutative ring with unity 1. We say that S is a multiplicative part of R if $1 \in S$ and S is closed under multiplication. Let

$$S^{-1}R := R \times S/\sim,$$

where $\sim$ is the equivalence relation defined in the following way:

$$(a, s) \sim (a', s') \iff \exists\, t \in S \,|\, t(s'a - sa') = 0.$$

We denote by a/s the equivalence class of (a, s). With the usual definition of sum and product of fractions,

$$a/s + a'/s' := (s'a + sa')/ss', \qquad (a/s)(a'/s') := aa'/ss',$$

we have that $S^{-1}R$ is a commutative ring with unity. We call $S^{-1}R$ the **quotient ring** of R with respect to S.
Let p be a prime ideal of R; then $R - p$ is a multiplicative part of R; we define the **localization** of R in p to be $S^{-1}R$, where $S = R - p$, and we denote this by R_p. It is a local ring: the unique maximal ideal is $\{a/s \,|\, a \in p, s \in S\}$.

If R is an integral domain and $S = R - \{0\}$, then $S^{-1}R$ is equal to

$$R \times (R - \{0\})/\sim,$$

where

$$(a, s) \sim (a', s') \Leftrightarrow s'a - sa' = 0;$$

this is a field (with the usual definition of sum and product of fractions) and is called the **quotient field** of R and denoted by $K(R)$.

Let R be a commutative ring with unity. If M is an R-module and S is a multiplicative part of R, we define

$$S^{-1}M = M \times S/\sim,$$

where $(m, s) \sim (m', s')$ if and only if there exists $t \in S$ such that $t(s'm - sm') = 0$. The set $S^{-1}M$, with the obvious sum and product, is an $S^{-1}R$-module. A homomorphism of

R-modules $\varphi : M_1 \to M_2$ induces a homomorphism of $S^{-1}R$-modules $S^{-1}\varphi : S^{-1}M_1 \to S^{-1}M_2$. If p is a prime ideal of R and $S = R - p$, we denote $S^{-1}M$ by M_p.

Proposition. Let R be a commutative ring with unity and S be a multiplicative part of R.

(i) If $M_1 \to M_2 \to M_3$ is an exact sequence of R-modules, then the induced sequence

$$S^{-1}M_1 \longrightarrow S^{-1}M_2 \longrightarrow S^{-1}M_3$$

is exact.

(ii) Let M be an R-module. Then $S^{-1}R \otimes_R M \cong S^{-1}M$ as $S^{-1}R$-modules.

(iii) Let $\varphi : M \to N$ be a homomorphism of R-modules. Then φ is injective (respectively surjective) $\Longleftrightarrow \varphi_p : M_p \to N_p$ is injective (respectively surjective) for every prime ideal $p \Longleftrightarrow \varphi_m : M_m \to N_m$ is injective (respectively surjective) for every maximal ideal m.

(iv) Let M be an R-module. We have: M is a flat R-module $\Longleftrightarrow M_p$ is a flat R_p-module for every prime ideal $p \Longleftrightarrow M_m$ is a flat R_m-module for every maximal ideal m (see "Flat (module, morphism)"). □

Lüroth problem. See "Unirational, Lüroth problem".

M

Manifolds. ([171], [200]) A topological manifold of dimension n is a connected Hausdorff topological space M such that there exists a countable open covering $\{U_\alpha\}_{\alpha \in A}$ and, for any $\alpha \in A$, a homeomorphism

$$\varphi_\alpha : U_\alpha \to \mathbb{R}^n.$$

We say that M is a differentiable (respectively C^k) manifold if, for any $\alpha, \beta \in A$, the map $\varphi_\beta \circ \varphi_\alpha^{-1}$ (where it is defined) is differentiable (respectively C^k). The maps φ_α are called local coordinates. A complex manifold of complex dimension n is a manifold whose local coordinates are

$$\varphi_\alpha : U_\alpha \to \mathbb{C}^n$$

and such that, for any $\alpha, \beta \in A$, the map $\varphi_\beta \circ \varphi_\alpha^{-1}$ (where is defined) is holomorphic.

Proposition. Every manifold is paracompact. □

Mapping cone lemma. ([186], [180]). Let

$$0 \longrightarrow M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3 \longrightarrow 0$$

be an exact sequence of R-modules for some commutative ring R with unity. Let

$$0 \longrightarrow F_t \xrightarrow{a_t} \cdots\cdots \xrightarrow{a_1} F_0 \xrightarrow{a_0} M_1 \longrightarrow 0$$

and

$$0 \longrightarrow G_t \xrightarrow{b_t} \cdots\cdots \xrightarrow{b_1} G_0 \xrightarrow{b_0} M_2 \longrightarrow 0$$

be two free resolutions of M_1 and M_2 respectively (that is, two exact sequences with F_i and G_i free R-modules for any i). Then we have a free resolution of M_3

$$\cdots \longrightarrow F_1 \oplus G_2 \xrightarrow{c_2} F_0 \oplus G_1 \xrightarrow{c_1} G_0 \xrightarrow{c_0} M_3 \longrightarrow 0$$

and a commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & F_2 \arrow[r, "a_2"] \arrow[d, "f_2"] & F_1 \arrow[r, "a_1"] \arrow[d, "f_1"] & F_0 \arrow[r, "a_0"] \arrow[d, "f_0"] & M_1 \arrow[r] \arrow[d, "f"] & 0 \\
\cdots \arrow[r] & G_2 \arrow[r, "b_2"] \arrow[d, "g_2"] & G_1 \arrow[r, "b_1"] \arrow[d, "g_1"] & G_0 \arrow[r, "b_0"] \arrow[d, "g_0"] & M_2 \arrow[r] \arrow[d, "g"] & 0 \\
\cdots \arrow[r] & F_1 \oplus G_2 \arrow[r, "c_2"] & F_0 \oplus G_1 \arrow[r, "c_1"] & G_0 \arrow[r, "c_0"] & M_3 \arrow[r] & 0,
\end{tikzcd}$$

where the maps g_i are the maps $x \mapsto (0, x)$ for $i \geq 1$ and the identity for $i = 0$ and the maps $c_i : F_{i-1} \oplus G_i \to F_{i-2} \oplus G_{i-1}$ are defined by

$$c_i(x, y) = (-a_{i-1}(x), f_{i-1}(x) + b_i(y))$$

for $i \geq 2$, $c_0 := g \circ b_0$, and c_1 is defined by $c_1(x, y) = f_0(x) + b_1(y)$.

Minimal set of generators. ([51]). We say that a set of generators of a module is minimal if no proper subset generates the module.

Minimal free resolutions. ([51], [89], [164], [209]). For any graded module M over a ring R and for any $s \in \mathbb{Z}$, we denote by $M(s)$ the graded module whose part of degree d is M_{d+s} (the part of degree $d+s$ of M). Let $R = K[x_0, \ldots, x_n]$, where K is a field and let M be a finitely generated graded R-module. Let

$$\cdots \longrightarrow E_r \xrightarrow{\varphi_r} E_{r-1} \to \cdots \longrightarrow E_1 \xrightarrow{\varphi_1} E_0 \xrightarrow{\varphi_0} M \longrightarrow 0$$

be a free graded resolution of M, that is, an exact sequence where the E_i are twisted free graded modules (i.e., every E_i is equal to $R(-s_1) \oplus \cdots \oplus R(-s_r)$ for some $r \in \mathbb{N}$, $s_1, \ldots, s_r \in \mathbb{Z}$) and the homomorphisms φ_i are graded homomorphisms of degree 0. We say that the resolution above is minimal if for every $i \geq 1$ the constant entries of the matrix of $\varphi_i : E_i \to E_{i-1}$ are zero.

Proposition. The resolution above is minimal if and only if for every $i \geq 0$ the map φ_i takes the standard basis of E_i to a minimal set of generators of the image of φ_i (see "Minimal set of generators"). □

Thus, to construct a minimal free resolution we can take as E_0 the free module generated by a minimal set of generators of M, then as E_1 the free module generated by a minimal set of generators of the kernel of $\varphi_0 : E_0 \to M$, and so on.
If we write E_i as $\oplus_j B_{i,j} \otimes R(-j)$ for some vector spaces $B_{i,j}$, we can prove that the dimensions of the $B_{i,j}$ depend only on M.
Furthermore one can prove that every finitely generated graded module over R has a minimal free resolution of length $\leq n$ (see "Hilbert syzygy theorem").

A minimal free resolution of a coherent sheaf $\mathcal{F}$ on $\mathbb{P}^n_K$ is defined in the following way (see "Hyperplane bundles, twisting sheaves" for the defintion of $\mathcal{O}_{\mathbb{P}^n}(s)$):
Let $R = \oplus_{s\in\mathbb{Z}} H^0(\mathcal{O}_{\mathbb{P}^n}(s)) = K[x_0, \dots, x_n]$ and let $M_{\mathcal{F}}$ be the graded R-module associated to $\mathcal{F}$ (see "Serre correspondence"), i.e., the R-module

$$\oplus_{s\in\mathbb{Z}} H^0(\mathcal{F}(s));$$

a minimal free resolution of $\mathcal{F}$ is the sheafified of a minimal free resolution of $M_{\mathcal{F}}$.
Observe that R is $M_{\mathcal{O}}$, i.e., it is the module associated to $\mathcal{O}$, and it is true in general that $\mathcal{F}_{M_{\mathcal{F}}} = \mathcal{F}$, and thus the terms of the sheafification of a minimal free resolution of $M_{\mathcal{F}}$ are of the kind $\oplus_i \mathcal{O}(a_i)$ for some $a_i \in \mathbb{Z}$.

Minimal degree. ([93], [104]). Let K be an algebraically closed field.

Theorem. The degree of a nondegenerate algebraic variety of dimension m in $\mathbb{P}^n_K$ is greater than or equal to $n - m + 1$.
A rational normal scroll of dimension m in in $\mathbb{P}^n_K$ has degree $n - m + 1$.
If X is a nondegenerate algebraic variety of dimension m in $\mathbb{P}^n_K$ of degree $n - m + 1$, then it is one of the following varieties:
(a) a quadric hypersurface;
(b) a cone over the surface $\nu_{2,2}(\mathbb{P}^2_K) \subset \mathbb{P}^5_K$;
(c) a rational normal scroll. □

(See "Degree of an algebraic subset", "Scrolls, rational normal -" and "Veronese embedding" for the relative definitions. We recall that for us any alegbraic variety is irreducible.)

Modules.

Definition. Let R be a ring. We say that a set M is a (left) R-module if it is an Abelian group for an operation $+ : M \times M \to M$ and there is an operation $\cdot : R \times M \to M$ such that

$$r \cdot (a + b) = r \cdot a + r \cdot b,$$
$$(r + s) \cdot a = r \cdot a + s \cdot a,$$
$$r \cdot (s \cdot a) = (r \cdot s) \cdot a,$$

for all $r, s \in R$, $a, b \in M$. (In general the symbol $\cdot$ is omitted). □

In some works, if R is a ring with unity, denoted by 1_R, it is also required that $1_R \cdot a = a$ for any $a \in M$. In other works it is not required, and an R-module is said to be unital (or unitary) if $1_R \cdot a = a$ for any $a \in M$.

Moduli spaces. ([7], [99], [100], [105], [111], [194], [199], [244]). Roughly speaking, the expression "moduli space" means "variety parametrizing", i.e., a moduli space is a geometric space whose points represent geometric objects of some fixed kind or isomorphism classes of such geometric objects. For example, if V is a vector space, the Grassmannian $G(k, V)$ (see "Grassmannians") is the moduli space of the k-planes in V.
Let

$$F : \text{(Schemes)} \to \text{(Sets)}$$

be a controvariant functor from the category of schemes to the category of sets that associates to a scheme S the set of (the equivalence classes of) the families of objects over S of a certain kind (see "Categories", "Schemes").

We say that a scheme M is a **fine moduli space** for F if F is representable by M, i.e., if F is isomorphic to $Hom(\cdot, M)$.
In this case, there exists a universal family U over M, i.e., a family over M, such that every other family over a scheme S is the pull-back of U by a certain map $f : S \to M$. To get the universal family, take the element of $F(M)$ corresponding to the identity in $Hom(M, M)$.

Fine moduli spaces seldom exist. Therefore, we consider a weaker definition.

We say that a scheme M is a **coarse moduli space** for F if there is a natural transformation from F to $Hom(\cdot, M)$ such that
(1) the induced map from $F(Spec\ \mathbb{C})$ to $Hom(Spec\ \mathbb{C}, M) = M$ is a bijection (i.e., the set of the points of M is in bijection with the set of the equivalence classes of families over a point);
(2) if M' is another scheme with the properties above, then there is a unique morphism $\pi : M \to M'$ such that the following diagram of natural transformations commutes

$$\begin{tikzcd} F \arrow[rr] \arrow[dr] & & Hom(\cdot, M') \arrow[dl] \\ & Hom(\cdot, M) & \end{tikzcd}$$

where $Hom(\cdot, M) \to Hom(\cdot, M')$ is induced by $M \to M'$.

For example, let $P \in K[x]$, where K is a field. A flat family over a scheme S of subschemes of $\mathbb{P}^n_K$ with Hilbert polynomial P is a closed subscheme $V \subset \mathbb{P}^n_K \times S$ such that, if we denote the projection $\mathbb{P}^n_K \times S \to S$ by π, we have
(1) $\pi|_V : V \to S$ is flat;
(2) $\pi|_V^{-1}(s)$ is a subscheme with Hilbert polynomial P for any $s \in S$

(see “Flat (module, morphism)” and “Hilbert function and Hilbert polynomial”).
We can consider the functor

$$Hilb_{P,\mathbb{P}^n_K} : \text{(Schemes)} \longrightarrow \text{(Sets)},$$

that associates to S the set of the flat families over S of subschemes of $\mathbb{P}^n_K$ with Hilbert polynomial P. In 1961 Grothendieck proved that there exists a projective scheme which is a fine moduli space for such a functor; it is called **Hilbert scheme** $H_{P,\mathbb{P}^n_K}$. In 1966 Hartshorne proved that it is connected.

One of the most studied moduli spaces is the one of smooth **curves with a fixed genus**. There is not a fine moduli space for them, but only a coarse one.

Theorem (Baily, Mumford, Deligne, Knudsen). ([7], [14], [54], [145]). The set $\mathcal{M}_g$ of the isomorphism classes of smooth projective algebraic curves of genus g over $\mathbb{C}$ has a structure of quasi-projective normal algebraic variety. If $g > 1$, its dimension is $3g - 3$. There is a natural compactification $\overline{\mathcal{M}}_g$, consisting of the isomorphism classes of the so-called stable curves of arithmetic genus g (a stable curve is a curve such that the only singularities are nodes and whose smooth rational components contain at least three singular points of the curve; see “Genus, arithmetic, geometric, real, virtual -” for the definition of arithmetic genus). $\overline{\mathcal{M}}_g$ is a projective algebraic variety. □

In general $\mathcal{M}_g$ and $\overline{\mathcal{M}}_g$ are singular; the singularities come from the curves with non-trivial group of automorphisms.

A subject obviously linked to the one of moduli spaces is the one of deformations (see “Deformations”). Another one is the geometric invariant theory (see “Geometric invariant theory (G.I.T.)”); in fact, the existence of nontrivial automorphisms of the objects which we want to classify causes trouble for the construction of moduli spaces, so one can look for the moduli space of these objects with some additional structure in such a way that the only automorphism is the identity; the moduli space of the original objects will be the quotient of the moduli space of the objects with the additional structure by a group, and so one can reduce the original problem to a problem of geometric invariant theory.

Monoidal transformations. In some books the term “monoidal transformation” is a synonym of “blowing-up” (see “Blowing-up (or σ-process)”). In other books, it denotes the blowing-up of a point in a surface.

Morphisms. See “Varieties, algebraic -, Zariski topology, regular and rational functions, morphisms and rational maps”.

Multiplicity of a curve in a surface at a point. ([93], [107], [196], [228]). Let K be an algebraic closed field and C be an algebraic curve in $\mathbb{A}^2_K$; let F be a generator of the ideal of C. Let $P \in C$. If we take coordinates such that $P = (0,0)$, and we write

$F = \sum_i F_i$ with F_i homogeneous polynomial of degree i, we define the multiplicity of C at P, which we denote by $mult_P(C)$, to be the minimum of the i such that $F_i \neq 0$. An equivalent definition is the following: $mult_P(C)$ is the maximum r such that $F \in m_P^r$, where m_p is the maximal ideal of the local ring $\mathcal{O}_{\mathbb{A}^2_K,P}$. Analogously, we can define the multiplicity at a point of a curve (or, more generally, of an effective Cartier divisor) in a smooth surface.
See [93, Chapter 0, § 2.2] and [107, Chapter V, Example 3.4] for the general definition of multiplicity of a subvariety at a point.

Multiplicity of intersection. See “Intersection of cycles”.

N

Nakai–Moishezon theorem. See “Bundles, fibre -”.

Nef. See “Bundles, fibre -” or “Divisors”.

Neron–Severi group. See “Equivalence, algebraic, rational, linear -, Chow, Neron–Severi and Picard groups”.

Net. A net is a linear system (see “Linear systems”) of dimension 2.

Node. See “Regular rings, smooth points, singular points”.

Noetherian, Artinian. ([12], [62], [164], [256]). Let R be a commutative ring with unity and let M be an R-**module**. We say that M is **Noetherian** (respectively **Artinian**) if for every chain of submodules $M_1, M_2, \dots$ such that

$$M_1 \subset M_2 \subset \cdots$$

(respectively $M_1 \supset M_2 \supset \cdots$) of M there exists n such that

$$M_n = M_{n+1} = \cdots ,$$

or, equivalently, if for every nonempty set of submodules of M there exists a maximal element for $\subset$ (respectively $\supset$).

Proposition. A module M is Noetherian if and only if every submodule of M is finitely generated. □

We say that the **ring** R is **Noetherian** (respectively **Artinian**) if it is Noetherian (respectively Artinian) as R-module, and thus if, for every chain of ideals $I_1, I_2 \dots$ such that $I_1 \subset I_2 \subset \cdots$ (respectively $I_1 \supset I_2 \supset \cdots$) of R, there exists n such that $I_n = I_{n+1} = \cdots$.

Proposition.
(i) The direct sum of a finite number of Noetherian (respectively Artinian) R-modules is Noetherian (respectively Artinian).
(ii) If R is a Noetherian (respectively Artinian) ring, every finitely generated R-module is Noetherian (respectively Artinian).
(iii) The quotient of a Noetherian (respectively Artinian) ring by an ideal is Noetherian (respectively Artinian).
(iv) If R is a Noetherian ring and S is a multiplicative part of R, then $S^{-1}R$ is Noetherian (see "Localization, quotient ring, quotient field"). □

Hilbert basis theorem. If R is a Noetherian ring, then $R[x]$ is Noetherian. □

Corollary. If R is a Noetherian ring, then $R[x_1, \dots, x_n]$ is Noetherian. □

Corollary. If K is a field, then $K[x_1, \dots, x_n]$ is Noetherian (since every field is Noetherian). □

Definition. We say that an ideal I in a ring R is irreducible if it satisfies the following condition: if I is the intersection of two ideals, J and K, then or $I = J$ either $I = K$. □

Theorem. In a Noetherian ring every ideal is the intersection of a finite number of irreducible ideals and every irreducible ideal is primary. So, in a Noetherian ring, every ideal has a primary decomposition (see "Primary ideals, primary decompositions"). □

Theorem.
(i) In an Artinian ring every prime ideal is maximal.
(ii) Every Artinian ring is Noetherian.
(iii) Every Artinian ring is the direct sum of a finite number of local Artinian rings. □

Proposition. Let R be a Noetherian ring. Then it is Artinian if and only if $Spec(R)$ is finite and discrete (see "Schemes" for the definition of $Spec$). □

Noether's formula.
See "Surfaces, algebraic -".

Nondegenerate.
We say that an affine or a projective algebraic variety is nondegenerate if it is not contained in any hyperplane.

Normal.
([107], [129], [228]). Let K be an algebraically closed field. We say that an algebraic variety X over K is normal if and only if for any $P \in X$ the stalk $\mathcal{O}_{X,P}$ of $\mathcal{O}_X$ in P is integrally closed, where $\mathcal{O}_X$ is the sheaf of the regular functions on X (see

"Integrally closed" and "Varieties, algebraic -, Zariski topology, regular and rational functions, morphisms and rational maps").

Proposition. An affine algebraic variety is normal if and only if its coordinate ring is integrally closed.
A quasi-projective algebraic variety is normal if and only if every point has a normal affine neighborhood. □

Theorem. Smooth algebraic varieties are normal. □

Theorem. The codimension of the set of singular points of a normal algebraic variety is ≥ 2. In particular, an algebraic curve is normal if and only if it is smooth. □

A normalization of an algebraic variety X is a normal algebraic variety X^n together with a finite birational morphism $X^n \to X$.

Theorem. An affine algebraic variety has a (unique) affine normalization.
A quasi-projective algebraic curve has a quasi-projective normalization. If the curve is projective, the normalization is projective. □

(We recall that for us all algebraic varieties, in particular all algebraic curves, are irreducible).

Normal, projectively -, k-normal, linearly normal. ([107]). Let K be an algebraically closed field and let X be a projective algebraic variety in $\mathbb{P}^n_K$.

– We say that X is **k-normal** if and only if the map

$$H^0(X, \mathcal{O}_{\mathbb{P}^n}(k)) \to H^0(X, \mathcal{O}_X(k))$$

given by the restriction is surjective, that is, if and only if the hypersurfaces of degree k cut on X a complete linear system.
– We say that X is **linearly normal** if and only if it is 1-normal.
– We say that X is **projectively normal** if and only if the projective coordinate ring is integrally closed (see "Integrally closed").

We can prove that X is projectively normal if and only if it is normal (see "Normal") and it is k-normal for any k. Thus a smooth projective algebraic variety is projectively normal if and only if it is k-normal for any k.

Normal crossing and log complex. ([66], [93]). Let X be a complex manifold of dimension n and D a divisor on X. We say that D has normal crossings if the irreducible components of D are smooth and meet transversally (i.e., for every point P in the intersection of some components and for any s, any s of the tangent spaces at P of these components intersect in a subspace of codimension s).

Let

$$\Omega^p(*D) = \cup_{k\in\mathbb{N}}\Omega^p(kD)$$

be the sheaf on X of the meromorphic p–forms that are holomorphic on $X - D$ and have poles on D.
Let P be a point such that exactly k of the components of D contain P. Let $z_1, \dots, z_n$ be local coordinates in a neighborhood of P such that $P = (0, \dots, 0)$ and D locally is given by $z_1 \dots z_k = 0$. Let $\Omega^p(logD)$ be the subsheaf of $\Omega^p(*D)$ generated by holomorphic forms and logarithmic differentials $\frac{dz_i}{z_i}$ for $i = 1, \dots, k$. The complexes $(\Omega^*(D), d)$ and $(\Omega^*(logD), d)$ are used to study the singular cohomology of $X - D$. The complex $(\Omega^*(logD), d)$ is called log complex.

N_p, Property -. See “Syzygies”.

Null correlation bundle. ([207]). Let n be an odd natural number. A null correlation bundle E on $\mathbb{P}^n_{\mathbb{C}}$ is a bundle on $\mathbb{P}^n_{\mathbb{C}}$ that is the kernel of a surjective bundle morphism

$$T^{1,0}(-1) \longrightarrow \mathcal{O}(1),$$

where $T^{1,0}$ is the holomorphic tangent bundle on $\mathbb{P}^n_{\mathbb{C}}$ and $\mathcal{O}(1)$ is the hyperplane bundle (see “Hyperplane bundles, twisting sheaves”). So there is an exact sequence

$$0 \longrightarrow E \longrightarrow T^{1,0}(-1) \longrightarrow \mathcal{O}(1) \longrightarrow 0.$$

By using Euler sequence, one can easily see that the Chern classes of a null correlation bundle E are the following: $c_i(E) = 1$ for i even, $i \leq n-1$, $c_i(E) = 0$ otherwise (see “Euler sequence” and “Chern classes”).

Proposition. Every null correlation bundle is simple (see “Simple bundles”). □

Theorem. For every n odd, there exists a null correlation bundle on $\mathbb{P}^n_{\mathbb{C}}$. □

Observe that to find a surjective bundle morphism $T^{1,0}(-1) \longrightarrow \mathcal{O}(1)$ it is sufficient to construct a section without zeroes of $\Omega^1(2)$ and then dualize.

O

$\mathcal{O}(s)$. See “Hyperplane bundles, twisting sheaves”.

Orbit Lemma, Closed -. ([27], [126], [210], [235] §4.3). Let K be an algebraically closed field and G be an algebraic group over K (see “Algebraic groups”). Suppose that

G acts on an algebraic variety X over K, that is, there is a morphism

$$\begin{aligned} G \times X &\longrightarrow X \\ (g, x) &\longmapsto gx \end{aligned}$$

such that $ex = x$ and $g(g'x) = (gg')x$ for any $x \in X$, $g, g' \in G$, where e is the identity element of G. Then every orbit is a smooth variety and it is open in its closure. Moreover, the boundary of any orbit is the disjoint union of orbits of lower dimension. In particular, the orbits of minimum dimension are closed; therefore, there exists a closed orbit.

Orientation. Let M be a differentiable manifold. We say that M is orientable if there exists a local system of coordinates $(U_\alpha, \varphi_\alpha)_{\alpha \in A}$ (see "Manifolds" for the notation) such that, for any $\alpha, \beta \in A$, the determinant of the Jacobian of $\varphi_\alpha^{-1} \circ \varphi_\beta$ is positive in every point where $\varphi_\alpha^{-1} \circ \varphi_\beta$ is defined.
See "Singular homology and cohomology" for a topological approach.

P

Pencil. A pencil is a linear system (see "Linear systems") of dimension 1.

Pfaffian. ([188, Appendix C, Lemma 9], [211]). Let $n \in \mathbb{N} - \{0\}$. One can prove that, if A is a $2n \times 2n$ antisymmetric matrix, then the determinant of A can be written as the square of a polynomial in the entries of A. One of such polynomials is called the Pfaffian of A.
Precisely, if A is a $2n \times 2n$ antisymmetric matrix, we define the Pfaffian of A, written $Pf(A)$, in the following way:

$$Pf(A) := \sum_p \varepsilon(p) a_{i_1, j_1} \dots a_{i_n, j_n},$$

where the sum is over all the partitions p of $\{1, \dots, 2n\}$ into n disjoint 2-subsets $\{i_1, j_1\}, \dots, \{i_n, j_n\}$ and $\varepsilon(p)$ is defined as follows: we can suppose $i_k < j_k$ for $k = 1, \dots, n$; we define $\varepsilon(p)$ to be the sign of the permutation sending $1, \dots, 2n$ respectively to $i_1, j_1, \dots, i_n, j_n$. It holds that

$$Pf(A)^2 = det(A).$$

Examples.

$$Pf\begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix} = a, \qquad Pf\begin{pmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{pmatrix} = af - be + dc.$$

Remark. The determinant of an antisymmetric square matrix A with an odd number of rows is 0. Let $A_{\hat{i},\hat{j}}$ be the matrix obtained from A by taking off the i-th row and the j-th column; the determinant of $A_{\hat{i},\hat{j}}$ is the product of the Pfaffian of $A_{\hat{i},\hat{i}}$ and of the Pfaffian of $A_{\hat{j},\hat{j}}$. □

Picard groups. See "Equivalence, algebraic, rational, linear -, Chow, Neron–Severi and Picard groups".

Plurigenera. ([93], [107], [241]). Let X be a compact complex manifold or a smooth projective algebraic variety over a field. We define the plurigenera to be the numbers

$$P_i(X) := h^0(X, \omega_X^i),$$

for $i = 1, 2, \dots$, where ω_X is the canonical sheaf (see "Canonical bundle, canonical sheaf").
See also "Kodaira dimension (or Kodaira number)".

Positive. ([93], [169]). Let X be a complex manifold.

- We say that a $(1,1)$ **real form** ω on X is positive if $i\,\omega(x)(v \wedge \overline{v}) > 0$ for any $x \in X$ and for any $v \in T_x^{1,0}X$ (where $T_x^{1,0}X$ is the holomorphic tangent space of X at x). In other words, if we write
$$\omega = i \sum_{r,l} h_{r,l}(x) dz_r \wedge d\overline{z}_l$$
in local holomorphic coordinates, the matrix $(h_{r,l}(x))_{r,l}$ (which is a Hermitian matrix since ω is real) is positive definite for any $x \in X$.
- We say that a holomorphic **line bundle** L on X is positive if in its first Chern class $c_1(L) \in H^2(X, \mathbb{Z})$ there is a positive $(1,1)$-form (see "Bundles, fibre -" and "Chern classes").

By Kodaira Embedding theorem (see "Kodaira Embedding theorem"), a holomorphic line bundle on a compact complex manifold is positive if and only if it is ample.

Observe that if L is a line bundle associated to a divisor D (see "Bundles, fibre -" or "Divisors" for the definition), then the positivity of L is not equivalent to the effectivity of D and no implication is true:
To show $\Rightarrow$ is false, one can take on a Riemann surface a divisor of positive degree but not effective;
to show $\Leftarrow$ is false, one can take the exceptional divisor E of the blow-up of a point in a surface: it is effective but the associated line bundle is not positive because, by Nakai-Moishezon criterion (see "Bundles, fibre -"), it is not ample since $E^2 = -1$.

Primary ideals, primary decompositions, embedded ideals. ([12], [62], [107]). Let R be a commutative ring with unity.

Definition. We say that an ideal I in R is **primary** if $xy \in I$ implies $x \in I$ or $y \in \sqrt{I}$ (where $\sqrt{I}$ is the radical ideal of I). □

Proposition.
(a) Let I be an ideal in R.
- If I is primary, then $\sqrt{I}$ is prime, more precisely $\sqrt{I}$ is the smallest prime ideal containing I.
- If $\sqrt{I}$ is maximal, then I is primary.

(b) Let $J_1, \ldots, J_k$ be primary ideals such that $\sqrt{J_1} = \cdots = \sqrt{J_k} = P$; then the intersection $J_1 \cap \cdots \cap J_k$ is a primary ideal whose radical is P. □

Definition. A **primary decomposition** of an ideal I is an expression of I as a finite intersection of primary ideals.
We say that an ideal is **decomposable** if it has a primary decomposition.
We say that a primary decomposition of an ideal I,

$$I = \cap_{i=1,\ldots,k} J_i$$

is **minimal** if the radical ideals of the J_i are distinct and $\cap_{i=1,\ldots,k, i \neq r} J_i \not\subset J_r$ for any $r = 1, \ldots, k$. □

From a primary decomposition of an ideal we can easily get a minimal primary decomposition.

Theorem. Let I be a decomposable ideal and let $I = \cap_{i=1,\ldots,k} J_i$ be a minimal primary decomposition. Let $P_i = \sqrt{J_i}$. The prime ideals $P_1, \ldots, P_k$ do not depend on the particular decomposition of I. □

Definition. The prime ideals $P_1, \ldots, P_k$ in the theorem above are said the **prime ideals associated** to I. The minimal ones (minimal for the inclusion) are called **minimal prime ideals**, the others are called **embedded prime ideals**. □

Proposition. In a Noetherian ring (see Noetherian, Artinian) every ideal has a primary decomposition. □

Let $R = K[x_1, \ldots, x_n]$ for some algebraically closed field K. Then the minimal prime ideals associated to an ideal I correspond to the irreducible components of the scheme given by I and the embedded prime ideals correspond to the subschemes of the irreducible components, the so-called "embedded components". If a component is given by only one point, we speak of "embedded point".

Example. Let $I = (x^2, xy)$ in $K[x, y]$. Then $I = P_1 \cap P_2^2$ where $P_1 = (x)$ and $P_2 = (x, y)$. Both P_1 and P_2^2 are primary, in fact: P_1 is prime and thus primary, besides $\sqrt{P_2^2} = P_2$ is maximal so P_2^2 is primary. Thus the prime ideals associated to I are P_1 and P_2. Since

$P_1 \subset P_2$, we have that P_1 is minimal and P_2 is embedded. The variety defined by I is the line $x = 0$ and the embedded ideal P_2 corresponds to the point $(0, 0)$.
See "Schemes".

Principal bundles. See "Bundles, fibre -".

Process, σ-. See "Blowing-up (or σ-process)".

Product, Semidirect -. ([164]). Let H and K be two groups and let

$$\varphi : K \to Aut(H)$$

be a homomorphism, where $Aut(H)$ denotes the group of the automorphisms of H. The semidirect product of H and K with respect to φ, denoted by $H \times_\varphi K$, is the following group: the Cartesian product $H \times K$ with the product

$$(h_1, k_1) \cdot (h_2, k_2) := \Big(h_1\varphi(k_1)(h_2)\,,\; k_1k_2\Big).$$

Observe that the subgroup

$$\{(h, e_K)|\; h \in H\}$$

(where e_K is the identity element of K) is a normal subgroup of $H \times_\varphi K$.

Remark. Let G be a group and H and K two subgroups such that H is normal, $G = HK$ and $H \cap K = \{e_G\}$. Then

$$G \cong H \times_\varphi K,$$

where $\varphi : K \to Aut(H)$ is defined by

$$k \longmapsto (h \mapsto k^{-1}hk)\,. \qquad \square$$

Proper. A map from a topological space to another is proper if the inverse image of any compact subset is compact. See "Schemes" for the definition of proper morphism of schemes.

Proper mapping theorem. See "Remmert's proper mapping theorem".

Projective modules. See "Injective and projective modules".

Projective resolutions. See "Injective and projective resolutions".

Pull-back and push-forward of cycles. ([72],[74]). Let X and Y be two varieties over an algebraically closed field K and let $f : X \to Y$ be a proper morphism. For any subvariety V of X we define

$$\deg(V/f(V)) = \begin{cases} 0 & \text{if} \quad dim(V) > dim(f(V)), \\ [K(V) : K(f(V))] & \text{if} \quad dim(V) = dim(f(V)), \end{cases}$$

where $K(V)$ is the rational function field of V, $K(f(V))$ is the rational function field of $f(V)$ and $[K(V) : K(f(V))]$ is the degree of $K(V)$ as field extension of $K(f(V))$ (see "Varieties, algebraic -, Zariski topology, regular and rational functions, morphisms and rational maps" for the definition of rational function field). In the complex case, if $dim(V) = dim(f(V))$, the map $V \to f(V)$ is generically a finite sheeted covering and we can also define $\deg(V/f(V))$ to be the number of the sheets of the covering.
We define the **push-forward** of the cycle V by f, denoted by $f_*(V)$, to be $\deg(V/f(V))$ times the cycle $f(V)$, i.e.,

$$f_*(V) = \deg(V/f(V))\ f(V)$$

(see "Cycles"); we extend the definition of f_* to all the cycles by linearity. One can prove that the image through f_* of any cycle that is rationally equivalent to 0 is rationally equivalent to 0. So f_* induces a map, called again f_*, between the Chow groups:

$$f_* : CH_q(X) \to CH_q(Y)$$

for any q (see "Equivalence, algebraic, rational, linear -, Chow, Neron–Severi and Picard groups" for the definition of Chow group).

Let X and Y be two varieties and $f : X \to Y$ be a flat morphism (see "Flat (module, morphism)"). Suppose that there exists an integer k such that, for any subvariety W of Y, all the irreducible components of $f^{-1}(W)$ have dimension $dim(W)+k$ (for instance a projection from a vector bundle to its base or an open embedding).
For any subvariety W of Y, we define the **pull-back** of the cycle W, which we denote by $f^*(W)$, to be the cycle given by the inverse image scheme $f^{-1}(W)$ (see "Schemes"). We extend the definition of f^* to all the cycles by linearity. This map induces a map between the Chow groups:

$$f^* : CH_q(Y) \to CH_{q+k}(X).$$

Pull-back and push-forward of sheaves. See "Direct and inverse image sheaves".

Q

Quadratic transformations, Cremona transformations. ([93], [107], [220], [228]). Let K be an algebraically closed field. A **quadratic transformation** of $\mathbb{P}^2_K$ is a birational map $b : \mathbb{P}^2_K \to \mathbb{P}^2_K$ defined in the following way:
Let P, Q, R be three noncollinear points of $\mathbb{P}^2_K$; let $\tilde{\mathbb{P}}^2_K$ be the blow-up of $\mathbb{P}^2_K$ at P, Q, R and let $\widetilde{PQ}, \widetilde{QR}, \widetilde{RP}$ be the strict transforms of the lines PQ, QR, RP; denote by $\varphi : \tilde{\mathbb{P}}^2_K \to \mathbb{P}^2_K$ the induced morphism and let φ^{-1} be the inverse birational map from $\mathbb{P}^2_K$ to $\tilde{\mathbb{P}}^2_K$; let

$$\psi : \tilde{\mathbb{P}}^2_K \to \mathbb{P}^2_K$$

be the morphism given by the blow-down of $\widetilde{PQ}, \widetilde{QR}, \widetilde{RP}$; define

$$b = \psi \circ \varphi^{-1}.$$

If we take $P = [1:0:0]$, $Q = [0:1:0]$, $R = [0:0:1]$, the map b is

$$[x_0 : x_1 : x_2] \longmapsto [x_1 x_2 : x_0 x_2 : x_0 x_1].$$

A **Cremona transformation** of $\mathbb{P}^2_K$ is a birational map of $\mathbb{P}^2_K$.
A classical theorem by Noether asserts that a Cremona transformation of $\mathbb{P}^2_K$ is the composition of a projective automorphism and of quadratic transformations (see [220, Chapter V], for a proof).

Quotient field. See "Localization, quotient ring, quotient field".

R

Rank of finitely generated Abelian groups. ([164]). Let A be a finitely generated Abelian group. The elements of A of finite order form a group T, called torsion group, and one can easily prove that A/T is a free Abelian group. The minimal number of generators of A/T, i.e., the cardinality of a basis of A/T, is called rank of A.
The rank of a module is its rank as Abelian group.

Rational normal curves. ([104], [107]). Let K be an algebraically closed field. We say that a curve in $\mathbb{P}^n_K$ is a rational normal curve if it is projectively equivalent to the image of $\mathbb{P}^1_K$ embedded into $\mathbb{P}^n_K$ by the map

$$[t_0 : t_1] \mapsto [t_0^n : t_0^{n-1} t_1 : \cdots : t_0 t_1^{n-1} : t_1^n],$$

i.e., of $\mathbb{P}^1_K$ embedded into $\mathbb{P}^n_K$ by the n-uple Veronese embedding (see "Veronese embedding"). One can prove easily that it is projectively normal (see "Normal, projectively -, k-normal, linearly normal"), and that it is the zero locus of polynomials of degree 2, in fact $\mathbb{P}^1_K$ embedded into $\mathbb{P}^n_K$ by the n-uple Veronese embedding is the zero locus of the polynomials $z_i z_j - z_{i-1} z_{j+1}$ for $1 \le i \le j \le n-1$, where $z_0, \ldots, z_n$ are the projective coordinates on $\mathbb{P}^n_K$.

Any rational normal curve has degree n. One can show that any nondegenerate algebraic curve in $\mathbb{P}^n_K$ has degree greater than or equal to n and that any nondegenerate algebraic curve in $\mathbb{P}^n_K$ of degree n is a rational normal curve (we recall that, for us, an algebraic curve is irreducible).

Rational normal scrolls. See "Scrolls, rational normal -".

Rational functions, rational maps. See "Varieties, algebraic -, Zariski topology, regular and rational functions, morphisms and rational maps".

Rational varieties. We say that an algebraic variety is rational if it is birational to a projective space. See "Varieties, algebraic -,Zariski topology, regular and rational functions, morphisms and rational maps", "Unirational, Lüroth problem".

Reduced. See "Schemes".

Regular functions. See "Varieties, algebraic -,Zariski topology, regular and rational functions, morphisms and rational maps".

Regular rings, smooth points, singular points. ([12], [62], [73], [104], [107], [113], [114], [117], [156], [228], [246]).

Definition. Let (R, m) be a **Noetherian local ring** of dimension d (see "Dimension", "Local", "Noetherian, Artinian"). We say that R is **regular** if

$$dim_K(m/m^2) = d,$$

where K is the residue field R/m. □

Now let K be an algebraically closed field.

Definition. Let X be an **algebraic variety** over K. For any $P \in X$, denote by $\mathcal{O}_{X,P}$ the stalk in P of the sheaf of the regular functions on X and by m_P the maximal ideal of $\mathcal{O}_{X,P}$. We say that a point P of X is a **smooth (or nonsingular or regular) point** of X if $\mathcal{O}_{X,P}$ is a regular ring, i.e., the dimension of m_P/m_P^2 over K is equal to the dimension of $\mathcal{O}_{X,P}$ (which is equal to the dimension of X). We say that a point is a **singular point (or a singularity)** of X if it is not a smooth point of X. We say that X is smooth if all its points are smooth points of X. □

Proposition. If X is an affine algebraic variety in $\mathbb{A}^n_K$ and the ideal $I(X)$ of X is generated by $f_1, \ldots, f_k$, we have that a point P of X is a smooth point of X if and only if the rank of the matrix $(\frac{\partial f_i}{\partial x_j})_{i=1,\ldots,k,j=1,\ldots,n}$ in P is equal to the codimension of X. □

Proposition. The set of the singular points of an algebraic variety is a proper closed subset. □

Definition. If X and Y are two algebraic varieties over K and $P \in X$ and $Q \in Y$, we say that P in X and Q in Y are **analytically isomorphic** if and only if the completions of $\mathcal{O}_{X,P}$ and $\mathcal{O}_{Y,Q}$ are isomorphic as K-algebras (see "Completion"). □

Let C be an affine plane algebraic curve and let F be a generator of the ideal of C. Let $P \in C$. By changing coordinates, we can suppose $P = (0, 0)$. If we write $F = \sum_{i=0,\ldots,d} F_i$, with F_i homogeneous polynomial of degree i, we define $mult_P(C)$ to be the minimum of the i such that $F_i \neq 0$ (see "Multiplicity of a curve in a surface at a point"). Obviously P is singular if and only if $mult_P(C) > 1$.

We now give some examples of singularities of multiplicity 2 on plane curves.

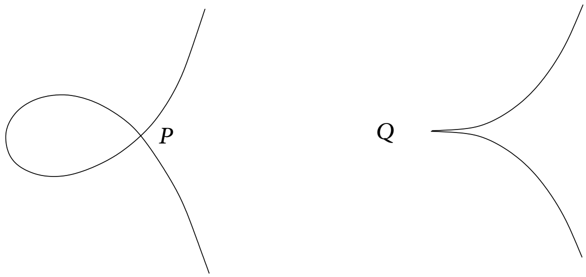


Fig. 13. A node (on the left) and a cusp (on the right).

A **node** is a singularity analytically equivalent to the point $(0,0)$ in the curve $xy = 0$ (or, equivalently, in the curve $x^2 = y^2$), i.e., a point of multiplicity 2 with the tangent cone given by two distinct lines (see "Cone, tangent -"). In other words, it is a point of multiplicity 2 such that, by blowing it up, we have that its fibre in the strict transform are two distinct smooth points; see "Blowing-up (or σ-process)".

A **cusp** is a singularity analytically equivalent to the point $(0,0)$ in the curve $y^2 = x^3$, thus it is a point of multiplicity 2 with the tangent cone given by a line. If we blow it up, we have that its fibre in the strict transform is a single smooth point.

A **tacnode** is a singularity analytically equivalent to the point $(0,0)$ in the curve $y^2 = x^4$, thus it is given by two arcs meeting one other with multiplicity two.

Theorem. (See [35]). Let the characteristic of K be different from 2. A singular point of multiplicity 2 on a plane algebraic curve over K is analytically equivalent the point $(0,0)$ in the curve $y^2 = x^n$ for some (unique) $n \in \mathbb{N} - \{0,1\}$. □

We say that an algebraic variety X has a **resolution of singularities** if there exists a nonsingular algebraic variety X' and a proper birational map from X' to X. In [117], Hironaka proved that every algebraic variety in characteristic 0 has a resolution of singularities. For curves it is unique, while in higher dimension this is not true. There are several methods to resolve singularities. See [113], [114], [156], [1].
One of the main tools to resolve singularities is the blow-up (see "Blowing-up (or σ-process)", "Genus, arithmetic, geometric, real, virtual -"; see also "Normal").

Regular sequences. ([62], [159], [185]). Let R be a commutative ring with unity and let M be an R-module.
An M-regular sequence is a sequence of elements $(x_1, \dots, x_n)$ in R such that $(x_1, \dots, x_n)M \neq M$ and x_{i+1} is not a zerodivisor in $M/(x_1, \dots, x_i)M$ for $i = 0, \dots, n-1$.

Proposition. If (R, m) is a Noetherian local ring (see "Noetherian, Artinian" and "Local"), and $(x_1, \dots, x_n)$ is an R-regular sequence contained in m, then also $(x_{\sigma(1)}, \dots, x_{\sigma(n)})$ is an R-regular sequence for any permutation σ of $\{1, \dots, n\}$. □

Proposition. Let M be a finitely generated module over a Noetherian ring R. Let I be an ideal in R such that $IM \neq M$. Then any two maximal M-regular sequences in I have the same number of elements (where we say that an M-regular sequence $(x_1, \dots, x_n)$ in I is maximal in I if there does not exist y in I such that $(x_1, \dots, x_n, y)$ is M-regular). □

See "Depth".

Regularity.

([62], [89], [198]). We say that a sheaf $\mathcal{F}$ on a projective space $\mathbb{P}^n$ is m-regular if

$$H^i(\mathcal{F}(m-i)) = 0 \quad \forall i > 0.$$

Castelnuovo–Mumford theorem. Let $\mathcal{F}$ be a coherent sheaf on $\mathbb{P}^n_{\mathbb{C}}$ (see "Coherent sheaves"). If $\mathcal{F}$ is m-regular, then it is $(m+1)$-regular. □

We define the regularity of $\mathcal{F}$ to be $\min\{m|\ \mathcal{F} \text{ is } m\text{-regular}\}$.

Green's theorem. Let $\mathcal{F}$ be a coherent sheaf on $\mathbb{P}^n_{\mathbb{C}}$. If

$$0 \to \mathcal{E}_r \to \cdots \to \mathcal{E}_1 \to \mathcal{E}_0 \to \mathcal{F} \to 0$$

is a minimal free resolution of $\mathcal{F}$ (see "Minimal free resolutions") and we write $\mathcal{E}_p = \oplus_q \mathcal{O}_{\mathbb{P}^n_{\mathbb{C}}}(-q) \otimes_{\mathbb{C}} B_{p,q}$ for some complex vector spaces $B_{p,q}$, then the regularity of $\mathcal{F}$ is

$$\max\{q - p|\ B_{p,q} \neq 0\}\,.$$ □

Remmert's proper mapping theorem.

([93], [241]). Let X and Y be two complex manifolds, and let $f : X \to Y$ be a holomorphic map. Let S be an analytic subvariety of X such that $f|_S$ is proper (see "Varieties and subvarieties, analytic -" and "Proper"). Then $f(S)$ is an analytic subvariety of Y.

Representations.

([76], [115], [130], [164], [225], [248]). A **representation of a group** G on a vector space V is a homomorphism

$$\rho : G \to GL(V).$$

Sometimes we say that V is a G-representation, and we write gv instead of $\rho(g)v$.

A **representation of a Lie algebra** a (see "Lie algebras") on a vector space V is a Lie algebra morphism

$$r : a \to gl(V),$$

where $gl(V)$ is the Lie algebra defined to be the vector space of the endomorphisms of V, endowed with the bracket $[A, B] = AB - BA$.

A **representation of a Lie group** G (see “Lie groups”) on a finite-dimensional real vector space V is a morphism of Lie groups

$$\rho : G \to GL(V).$$

We say that a representation V of a group G is **irreducible** if any G-invariant subspace W of V is equal either to $\{0\}$ or to V (where we say that a subspace W is G-invariant if $gw \in W$ for any $w \in W$ and for any $g \in G$).
We say that a representation V of a group G is **completely reducible** if it is the direct sum of irreducible representations.
Let V and W be two vector spaces over the same field, and let $\varphi : V \to W$ be a linear map. If V and W are representations of a group G, we say that φ is a ***G*-homomorphism** or is ***G*-equivariant** if $\varphi(gv) = g\varphi(v)$ for all $g \in G$ and for all $v \in V$.

Schur’s lemma. Let G be a group. Let φ be a G-homomorphism between two irreducible G-representations, V and W, vector spaces over the field K. Then, either φ is an isomorphism or $\varphi = 0$. Furthermore, if K is algebraically closed, $V = W$ and the dimension of V is finite, then $\varphi = \lambda I$ for some $\lambda \in K$. □

Maschke’s theorem. Let K be a field of characteristic 0. If G is a finite group, any G-representation over K is completely reducible. □

Definition. Let V be a vector space of finite dimension over a field K and let G be a finite group. We define the **character** of a representation $\rho : G \to GL(V)$ to be

$$\chi_V : G \longrightarrow K,$$
$$g \longmapsto tr(\rho(g) : V \to V),$$

where tr denotes the trace. □

Theorem. Let K be a field of characteristic 0 and let G be a finite group. Then any irreducible finite-dimensional G-representation over K is determined by its character. □

Theorem. All the representations of the complex Lie groups $SL(\mathbb{C}^n)$ and of $GL(\mathbb{C}^n)$ are completely reducible. □

See “Schur functors” for a description of the representations of the complex Lie groups $SL(\mathbb{C}^n)$ and of $GL(\mathbb{C}^n)$. See also “Lie algebras” and “Lie groups”.

Residue field. Let R be a local ring (that is, a ring with a unique maximal ideal) and let m be its maximal ideal. The field R/m is called the residue field of R.

Resolutions. See “Exact sequences”.

Riemann's existence theorem. ([71], [189]).

Riemann's existence theorem. Let X be a compact Riemann surface. Let F be a finite subset of X, $x \in X - F$ and $n \in \mathbb{N} - \{0\}$. The following sets are in a natural bijective correspondence:

(i) the set of equivalences classes of connected n-sheeted branched covering spaces of X with branch locus contained in F;

(ii) the set of connected equivalence classes of n-sheeted topological covering spaces of $X - F$;

(iii) the set of the elements of $Hom(\pi_1(X - F, x), \Sigma_n)$ whose images are transitive subgroups of Σ_n, where π_1 denotes the first fundamental group and Σ_n is the group of permutations on n elements. □

(See "Riemann surfaces (compact -) and algebraic curves", "Covering projections" and "Fundamental group" for the definitions of such terms).

The equivalence between (ii) and (iii) can be described in the following way:
Let $f : Y \to X - F$ be a topological covering; then $\pi_1(X - F, x)$ acts transitively on $f^{-1}(x)$, so we get a homomorphism from $\pi_1(X - F, x)$ to Σ_n whose image is a transitive subgroup of Σ_n. Conversely, let $\alpha \in Hom(\pi_1(X - F, x), \Sigma_n)$ such that its image is a transitive subgroup of Σ_n; consider the topological covering space given by the following subgroup of $\pi_1(X - F, x)$:

$$H = \{g \in \pi_1(X - F, x) | \ \alpha(g)(1) = 1\}.$$

See "Riemann surfaces (compact -) and algebraic curves" for another Riemann's existence theorem.

Riemann–Roch theorem. See "Hirzebruch–Riemann–Roch theorem".

Riemann surfaces (compact -) and algebraic curves. ([8], [35], [69], [73], [93], [101], [102], [107], [129] [189], [195], [196], [246]).

A Riemann surface is a complex manifold of (complex) dimension 1. An algebraic curve is an algebraic variety of dimension 1 (see "Varieties, algebraic -, Zariski topology, regular and rational functions, morphisms and rational maps").
From the topological viewpoint, a compact Riemann surface is a topological torus with g holes, for some unique $g \geq 0$, since it is an orientable, compact, real manifold of real dimension 2 (see [184] for instance). The number g is called the genus of the Riemann surface.

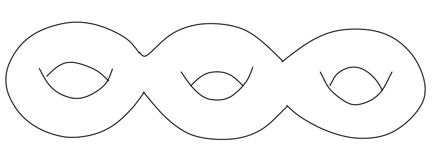


Fig. 14. A topological torus with three holes.

Hurwitz formula. Let X and Y be two compact Riemann surfaces of genus respectively g and j, and let $f : X \to Y$ a nonconstant holomorphic map; we have that

$$2g - 2 = \deg(f)(2j - 2) + \sum_{P \in X} (i_P(f) - 1),$$

where the numbers $i_P(f)$ are defined in the following way:
If we choose a local coordinate z around a point P and a local coordinate around $f(P)$ such that $P = \{z = 0\}$ and locally $f(z) = z^n$, we define the ramification index $i_P(f)$ to be n (and we say that P is a ramification point if $i_p(f) > 1$). □

We recall that, for any holomorphic line bundle L, we denote by $\mathcal{O}(L)$ the sheaf of the (holomorphic) sections of L (see "Bundles, fibre -").

Riemann's theorem. Let X be a compact Riemann surface of genus g. Let K_X be the canonical bundle (see "Canonical bundle, canonical sheaf"). We have

$$h^0(X, \mathcal{O}(K_X)) = g\,. \qquad \square$$

Observe that for any holomorphic line bundle L on a compact Riemann surface X we have that $H^i(X, \mathcal{O}(L)) = 0$ for $i \geq 2$; to prove this, for instance we can apply the abstract de Rham's theorem (see "Sheaves") to the exact sequence

$$0 \longrightarrow \mathcal{O}(L) \longrightarrow C^\infty(L) \xrightarrow{\bar{\partial}} A^{0,1}(L) \longrightarrow 0,$$

where $C^\infty(L)$ is the sheaf of the C^∞ sections of L, $A^{p,q}(L)$ is the sheaf of the C^∞ (p, q)-forms with values in L and the map $\mathcal{O}(L) \longrightarrow C^\infty(L)$ is given by the inclusion.

A divisor D on a compact Riemann surface is a finite formal sum

$$\sum_{i=1,\dots,k} n_i P_i$$

with $P_i \in X$ and $n_i \in \mathbb{Z}$ (see "Divisors"). We define the degree of D, written $\deg(D)$, in the following way:

$$\deg(D) := \sum_{i=1,\dots,k} n_i.$$

Let L be a holomorphic line bundle on a compact Riemann surface X. Any meromorphic section σ of L gives a divisor on X: the linear combination of the set of the zeroes of σ with coefficients their multiplicities as zeroes minus the linear combination of the set of the poles of σ with coefficients their multiplicities as poles. We define $\deg(L)$ to be the degree of any divisor whose associated line bundle is L (see "Bundles, fibre -") or, equivalently, to be the degree of the divisor given by any meromorphic section of L.

Riemann–Roch theorem. Let X be a compact Riemann surface of genus g and L be a holomorphic line bundle on X. We have

$$h^0(X, \mathcal{O}(L)) - h^1(X, \mathcal{O}(L)) = deg(L) - g + 1\,. \qquad \square$$

Corollary 1. If X is a compact Riemann surface of genus g, then

$$\deg(K_X) = 2g - 2.$$ □

In fact, by Riemann's theorem, we have that $h^0(X, \mathcal{O}(K_X)) = g$ and, by Serre duality (see "Serre duality"), we have that $h^1(\mathcal{O}(K_X)) = h^0(\mathcal{O})$, which is equal to 1, since any holomorphic map from X to $\mathbb{C}$ is constant.

Remark 2. Let L be a holomorphic line bundle of degree d on a compact Riemann surface X of genus g.
(i) If $d < 0$, then obviously $h^0(\mathcal{O}(L)) = 0$.
(ii) If $d = 0$, then either $h^0(\mathcal{O}(L)) = 0$ or $\mathcal{O}(L) = \mathcal{O}$.
(iii) If $d > 2g - 2$, then obviously $h^1(\mathcal{O}(L))$, which is equal to $h^0(\mathcal{O}(K_X L^{-1}))$ by Serre duality, is equal to 0.
(iv) If $d = 2g - 2$, then either $h^1(\mathcal{O}(L)) = 0$ or $L = K_X$. □

Remark 3. Let X be a compact Riemann surface and L be a holomorphic line bundle on X. Let

$$\varphi_L : X \to \mathbb{P}(H^0(X, \mathcal{O}(L))^\vee$$

be the map associated to L (see "Bundles, fibre -"). We have that
- the map φ_L is holomorphic, i.e., L is base point free (see again "Bundles, fibre -" for the definition), if and only if $h^0(\mathcal{O}(L - P)) = h^0(\mathcal{O}(L)) - 1$ for any $P \in X$, where $\mathcal{O}(L - P)$ is $\mathcal{O}(L) \otimes \mathcal{O}(-P)$ and $\mathcal{O}(-P)$ is the sheaf of the sections of the holomorphic bundle $(-P)$ associated to the divisor $-P$ and analogously in the sequel;
- the map φ_L is injective if and only if $h^0(\mathcal{O}(L-P-Q)) = h^0(\mathcal{O}(L))-2$ for any $P, Q \in X$ with $P \neq Q$;
- the differential of φ_L is injective if and only if $h^0(\mathcal{O}(L - 2P)) = h^0(\mathcal{O}(L)) - 2$ for any $P \in X$. □

By using the previous remark and the Riemann–Roch theorem, we can prove

Theorem 4. Let X be a compact Riemann surface of genus g and L be a holomorphic line bundle on X of degree d.
(i) If $d \geq 2g$, then φ_L is holomorphic.
(ii) If $d \geq 2g + 1$, then φ_L is an embedding. □

Riemann's existence theorem. The image of a compact Riemann surface by an embedding in a projective space is an algebraic curve. □

(See also "Chow's theorem" and "Remmert's proper mapping theorem").

Therefore, we can embed every compact Riemann surface in a projective space (by Theorem 4) and the image is an algebraic curve. Vice versa we can associate to any algebraic curve in a projective space over $\mathbb{C}$ a Riemann surface: the desingularization of C, which is equal to the normalization of C (see "Normal" and "Regular rings, smooth points, singular points"). We have seen that every compact Riemann surface can embedded in a projective space, but one can prove a stronger result:

Proposition 5. Any compact Riemann surface can be embedded in $\mathbb{P}^3_{\mathbb{C}}$. □

Definition 6. Let X be a compact Riemann surface of genus $g \geq 2$. We say that X is **hyperelliptic** if and only if there exists a holomorphic map $X \to \mathbb{P}^1_{\mathbb{C}}$ of degree 2. □

Remark 7. A compact Riemann surface of genus $g \geq 2$ is hyperelliptic if and only if there exist $P, Q \in X$ such that

$$h^0(\mathcal{O}(P+Q)) = 2,$$

i.e., if and only if there is a g^1_2, i.e., a linear system of degree 2 and projective dimension 1 (see "Linear systems"). □

Let L be a holomorphic line bundle of degree d on a compact Riemann surface of genus g. By Remark 2 and by the Riemann–Roch theorem, in the case $d < 0$ and in the case $d > 2g-2$ we can calculate $h^0(\mathcal{O}(L))$ and $h^1(\mathcal{O}(L))$. In the case $0 \leq d \leq 2g-2$ the following theorem can help.

Clifford's theorem. Let L be a holomorphic line bundle of degree d on a compact Riemann surface of genus g such that $h^0(\mathcal{O}(L)) > 0$ and $h^1(\mathcal{O}(L)) > 0$. Then

$$h^0(\mathcal{O}(L)) - 1 \leq \frac{d}{2},$$

If we have the equality $h^0(\mathcal{O}(L)) - 1 = \frac{d}{2}$, then we are in one of the following cases:
(i) L is the trivial bundle;
(ii) $L = K_X$;
(iii) X is hyperelliptic and the linear system given by L is a g^1_2. □

Clifford's index. Let X be a compact Riemann surface and L be a holomorphic line bundle on X; we define

$$\mathit{Cliff}(L) = \deg(L) - 2h^0(\mathcal{O}(L)) + 2.$$

In addition we define

$$\mathit{Cliff}(X) = \min\{\mathit{Cliff}(L) |\ L \ \text{ s.t. } \ h^0(\mathcal{O}(L)) \geq 2, \ \ h^1(\mathcal{O}(L)) \geq 2\}. \qquad \square$$

Hence, by Clifford's theorem, we can say that the Clifford's index of L, when $h^0(\mathcal{O}(L)) > 0$ and $h^1(\mathcal{O}(L)) > 0$, is a measure of how special L is; precisely it is greater than or equal to 0 and it is equal to 0 only in special cases.

We want now to investigate the map φ_{K_X}.

Remark 8. Let X be a compact Riemann surface of genus g and $P \in X$. We have

$$1 \le h^0(\mathcal{O}(P)) \le 2$$

and, if $h^0(\mathcal{O}(P)) = 2$, then $g = 0$ and X is biholomorphic to $\mathbb{P}^1_{\mathbb{C}}$. In other words, if $g \ge 1$, then $h^0(\mathcal{O}(P)) = 1$. By using this, Remark 3, and the Riemann–Roch theorem, one can prove easily the following statements:

(i) if $g \ge 1$, then K_X is base point free, i.e., φ_{K_X} is holomorphic;
(ii) if $g \ge 2$ and X is not hyperelliptic, then φ_{K_X} is an embedding. □

Geometric Riemann–Roch theorem. Let X be a compact Riemann surface of genus $g \ge 2$ and D be an effective divisor on X. We denote by $\langle D \rangle$ "the subspace of $\mathbb{P}(H^0(X, \mathcal{O}(K_X))^\vee)$ generated by D", precisely the intersection of the hyperplanes H in $\mathbb{P}(H^0(X, \mathcal{O}(K_X))^\vee)$ such that $\varphi_{K_X}(X) \subset H$ or the divisor $\varphi^*_{K_X}(H) - D$ is effective. Then we have

$$dim\langle D \rangle = \deg D - h^0(\mathcal{O}(D)) .$$ □

Corollary. Let X be a Riemann surface of genus $g \ge 1$. We have

$$h^0(\mathcal{O}(D)) = \begin{cases} 1 & \text{if } \quad 0 \le d \le g, \\ d - g + 1 & \text{if } \quad d \ge g, \end{cases}$$

for a general effective divisor D of degree d. □

In particular, for general $P \in X$, we have

$$h^0(\mathcal{O}(dP)) = \begin{cases} 1 & \text{if } \quad 0 \le d \le g, \\ d - g + 1 & \text{if } \quad d \ge g. \end{cases}$$

The points that do not have the general behaviour are called Weierstrass points (see "Weierstrass points").

Castelnuovo's theorem. Let X be a nondegenerate smooth projective algebraic curve in $\mathbb{P}^n_{\mathbb{C}}$ of genus g and degree d (see "Degree of an algebraic subset"). Let m be the integer part of $\frac{d-1}{n-1}$ and $\epsilon = d - 1 - m(n-1)$. Then

$$g \le (n-1)\frac{(m-1)m}{2} + m\epsilon .$$ □

Now let us consider algebraic curves over algebraic closed fields. The notion of genus of a Riemann surface is replaced by the ones of geometric genus p_g and arithmetic genus p_a (see "Genus, arithmetic, geometric, real, virtual -"). We have:

Proposition. Let C be a smooth complete algebraic curve over an algebraic closed field. Then $p_a(C) = p_g(C) = h^1(C, \mathcal{O}_C)$. □

We will call this number simply "genus of C".

For smooth complete algebraic curves over an algebraic closed field, the theory is very similar to the one of Riemann surfaces. Precisely, by replacing holomorphic line bundles with algebraic line bundles and holomorphic maps with morphisms in the Riemann–Roch theorem, in Clifford's Theorem, and in the corollaries, remarks, propositions. and definitions above from 1 to 8, we get the analogous statements for any smooth complete curve over an algebraic closed field.
As to the Hurwitz formula we have the following theorem (see "Varieties, algebraic -, Zariski topology, regular and rational functions, morphisms and rational maps" for the definition of rational functions field):

Hurwitz formula. Let X and Y be two smooth complete algebraic curves over an algebraic closed field of characteristic 0 and let g and j be their respective genera.
Let $f : X \to Y$ be a finite morphism such that $K(X)$ is a separable field extension of $K(Y)$, where $K(X)$ and $K(Y)$ are the rational functions fields respectively of X and Y. We define $\deg(f)$ to be the degree of $K(X)$ as field extension of $K(Y)$. Then

$$2g - 2 = \deg(f)(2j - 2) + \sum_{P \in X} (i_P(f) - 1),$$

where the numbers $i_P(f)$ are defined as follows: let t be a uniformizing parameter of the discrete valuation ring $\mathcal{O}_{f(P)}$ (see "Discrete valuation rings"), let s be the image of t through the map from $\mathcal{O}_{f(P)}$ to $\mathcal{O}_P$ induced by f; let $v_P : \mathcal{O}_P \to \mathbb{N}$ the valuation map of the discrete valuation ring $\mathcal{O}_P$; we define $i_P(f) = v_P(s)$. □

See also "Genus, arithmetic, geometric, real, virtual -", "Jacobians of compact Riemann surfaces".

S

Saturation. ([104], [107]). Let $R = k[x_0, \dots, x_n]$ for some algebraically closed field k and let I be a homogeneous ideal of R; the saturation of I, we denote by $sat(I)$, is defined as follows:

$$sat(I) := \{f \in R \mid (x_0, \dots, x_n)^p f \subset I \ \text{ for some } p \in \mathbb{N}\},$$

where $(x_0, \dots, x_n)$ is the ideal generated by $x_0, \dots, x_n$.
We say that I is saturated if $I = sat(I)$.

Remark. For any homogeneous ideal I of R, we have
(i) $sat(I)$ is homogeneous;
(ii) I and $sat(I)$ coincide from a certain degree on;
(iii) in $\mathbb{P}^n_k$ the zero locus of I and the zero locus of $sat(I)$ coincide; moreover, I and $sat(I)$ determine the same ideal locally, precisely, for any $i = 0, \dots, n$ and for any homogeneous polynomial F, let $\hat{F}^i$ be the polynomial $\frac{F}{x_i^{\deg(F)}}$ in the variables $z_j :=$

x_j/x_i for $j = 0, \ldots, n$, $j \neq i$; we have that, for any $i = 0, \ldots, n$, the ideal generated by the $\hat{F}^i$ for F homogeneous polynomial of I is equal to the ideal generated by the $\hat{F}^i$ for F homogeneous polynomial of $sat(I)$;

iv) two homogeneous ideals give the same closed subscheme of $Proj(R)$ if and only if they have the same saturation and there is a bijection between saturated ideals of R and closed subschemes of $Proj(R)$ (see "Schemes"). □

Thus, if two ideals have the same saturation, not only do they have the same zero locus (that is, they cut out the same variety "set-theoretically"), but they also determine the same ideals locally, and they determine the same subscheme (i.e., they cut out the same variety "scheme-theoretically").

Schemes.

([12], [64], [95], [107], [129], [228]). We strictly follow the exposition in [107].

Definition. Let R be a commutative ring with unity.

- We denote by $Spec(R)$ the set of the prime ideals of R, endowed with following topology:
  For any ideal a of R, we define
  $$V(a) = \{p \mid p \text{ prime ideal of } R,\ p \supset a\};$$
  the closed subsets of $Spec(R)$ are the $V(a)$ for a varying in the ideals of R.
- Let $\mathcal{O}_{Spec(R)}$ ($\mathcal{O}$ for short) be the sheaf of rings on $Spec(R)$ defined in the following way:
  For any open subset U of $Spec(R)$, let
  $$\mathcal{O}(U) = \{\sigma : U \to \sqcup_{p \in U} R_p \mid \sigma \text{ section, } \sigma \text{ locally quotient of elements of } R\},$$
  where
  - R_p denotes the localization of R in p (see "Localization, quotient ring, quotient field");
  - "σ section" means that $\sigma(p) \in R_p$ for any $p \in U$;
  - "σ locally quotient of elements of R" means that, for all $p \in U$, there is a neighborhood V of p in U and $a_1, a_2 \in R$ such that for each $p' \in V$
    $$\sigma(p') = \frac{a_1}{a_2}$$
    and $a_2 \notin p'$.

The **spectrum** of R is the pair $(Spec(R), \mathcal{O}_{Spec(R)})$. Sometimes we use the notation $Spec(R)$ to denote the spectrum. □

Definition.
- A **ringed space** $(X, \mathcal{O}_X)$ is a topological space X with a sheaf of rings $\mathcal{O}_X$. We say that it is a **locally ringed space** if the stalk of the sheaf in every point is a local ring (see "Local").
- A **morphism of ringed spaces** from a ringed space $(M, \mathcal{O}_M)$ to a ringed space $(N, \mathcal{O}_N)$ is given by a continuous map $f : M \to N$ and a morphism of sheaves $\mathcal{O}_N \to f_*\mathcal{O}_M$ (see "Direct and inverse image sheaves" for the definition of $f_*\mathcal{O}_M$).
- A **morphism of locally ringed spaces** is a morphism of ringed spaces such that the maps induced on the stalks are local homomorphisms of local rings (i.e., homomorphisms such that the inverse image of the maximal ideal of the second ring is the maximal ideal of the first ring). □

Proposition.
(i) Let R be a commutative ring with unity and let $(Spec(R), \mathcal{O})$ be its spectrum. Then
  (a) for every $p \in Spec(R)$, the subset $\{p\}$ in $Spec(R)$ is closed if and only if p is maximal;
  (b) for every $p \in Spec(R)$, the closure of the subset $\{p\}$ is $V(p)$;
  (c) $\mathcal{O}_p \cong R_p$ for every $p \in Spec(R)$;
  (d) $\mathcal{O}(Spec(R)) = R$.
  In particular, the spectrum of a ring is a locally ringed space.
(ii) Let A and B be two commutative rings with unity. A homomorphism of rings $A \to B$ induces a morphism of locally ringed spaces

$$(Spec(B), \mathcal{O}_{Spec(B)}) \to (Spec(A), \mathcal{O}_{Spec(A)}).$$

  Any morphism of locally ringed spaces $(Spec(B), \mathcal{O}_{Spec(B)}) \to (Spec(A), \mathcal{O}_{Spec(A)})$ is induced by a homomorphism $A \to B$. □

Definition. An **affine scheme** is a locally ringed space isomorphic to the spectrum of a ring. A **scheme** is a locally ringed space $(X, \mathcal{O}_X)$ locally isomorphic to an affine scheme, i.e., such that for every point of X there exists a neighborhood U such that $(U, \mathcal{O}_X|_U)$ is an affine scheme. The sheaf $\mathcal{O}_X$ is called **structure sheaf**. A **morphism of schemes** is a morphism of locally ringed spaces. □

Example. Let K be an algebraically closed field. Then $Spec(K[x_1, \dots, x_n])$ is denoted by $\mathbb{A}^n_K$ (as the affine space of dimension n over K) and called **affine space of dimension** n **over** K **(scheme)**: in fact the closed points of $Spec(K[x_1, \dots, x_n])$ are the maximal ideals of $K[x_1, \dots, x_n]$, which are the ideals of $K[x_1, \dots, x_n]$ of the kind $(x_1 - a_1, \dots, x_n - a_n)$ for some $a_1, \dots, a_n \in K$; thus they correspond to the points of the affine space of dimension n over K. Obviously $Spec(K[x_1, \dots, x_n])$ also contains the points corresponding to the prime ideals of $K[x_1, \dots, x_n]$ that are not maximal; the closure of such a point p is all $V(p)$ (which contains all the points that are in the zero locus of the ideal p in the affine space); p is said to be a generic point for $V(p)$. In particular the closure of the zero ideal is all $Spec(K[x_1, \dots, x_n])$, and so it is called the generic point of $Spec(K[x_1, \dots, x_n])$.

Definition. Let $R = \oplus_{k \geq 0} R_k$ be a graded commutative ring with unity.

– We define $Proj(R)$ to be the set

$$\{p |\ p \text{ homogeneous prime ideal of } R,\ \ \oplus_{k>0} R_k \not\subset p\}.$$

endowed with the following topology: the closed subsets are the $V(a)$ for a homogeneous ideal of R, where

$$V(a) := \{p \in Proj(R) |\ p \supset a\}.$$

– Let $\mathcal{O}_{Proj(R)}$ ($\mathcal{O}$ for short) be the sheaf of rings on $Proj(R)$ defined in the following way: for any open subset U of $Proj(R)$ let

$$\mathcal{O}(U) = \{\sigma : U \to \sqcup_{p \in U} R_{(p)} |\ \sigma \text{ section, } \sigma \text{ locally quotient of elements of } R\},$$

where

  – $R_{(p)}$ is the set of the elements of degree 0 in the localization $M^{-1}R$ where $M = \{f \in R |\ f \text{ homogeneous, } f \notin p\}$;
  – "σ section" means $\sigma(p) \in R_{(p)}$ for all $p \in U$;
  – "σ locally quotient of two elements of R" means that, for all $p \in U$, there exists a neighborhood V of p in U and there exist $r_1, r_2 \in R$ homogeneous with $deg(r_1) = deg(r_2)$ such that for all $p' \in V$

$$\sigma(p') = \frac{r_1}{r_2}$$

  in $R_{(p')}$ and $r_2 \notin p'$. □

Proposition. Let R be a graded commutative ring with unity. We have

(a) $\mathcal{O}_p \cong R_{(p)}$ for every $p \in Proj(R)$;

(b) let $f \in \oplus_{k>0} R_k$; then the set

$$A_f := \{p \in Proj(R) |\ f \notin p\}$$

is an open subset of $Proj(R)$ and $(A_f, \mathcal{O}_{Proj(R)}|_{A_f})$ is isomorphic, as locally ringed space, to the spectrum of $R_{(f)}$ (the subring of the elements of degree 0 of the localized ring R_f). The A_f for $f \in \oplus_{k>0} R_k$ cover $Proj(R)$. Hence $(Proj(R), \mathcal{O}_{Proj(R)})$ is a scheme. □

Sometimes we use the notation $Proj(R)$ to denote this scheme.

Example. Let K be an algebraically closed field. Then $Proj(K[x_0, \dots, x_n])$ is denoted by $\mathbb{P}^n_K$ (as the projective space of dimension n over K) and called **projective space of**

dimension n over K (scheme): in fact, the closed points of $Proj(K[x_0, \dots, x_n])$ correspond to the points of the projective space of dimension n over K.

Definition. Let S be a scheme. A **scheme over** S is another scheme X and a morphism of schemes $X \to S$. If R is a ring, a scheme over R is a scheme over $Spec(R)$. If X and Y are two schemes over S, an S-morphism from X to Y is a morphism $X \to Y$ such that the morphism $X \to S$ is the composition of the morphism $X \to Y$ with the morphism $Y \to S$. □

Let K be an algebraically closed field. **One can associate to any algebraic variety V over K** (see "Varieties, algebraic -, Zariski topology, regular and rational functions, morphisms and rational maps") **a scheme $s(V)$ over K** such that V is homeomorphic to the subset of the closed points of $s(V)$ and the sheaf of the regular functions on V is obtained by restricting the structure sheaf $\mathcal{O}_{s(V)}$ to the subset of closed points identified with V. The definition of s is as follows:
- define $s(V)$ to be the set of the nonempty irreducible closed subsets of V;
- define $s(Z)$ to be the set of the nonempty irreducible closed subsets of Z for any closed subset Z of V; endow $s(V)$ with the topology such that a subset of $s(V)$ is closed if and only if it is equal to $s(Z)$ for some closed subset Z of V;
- define $\alpha : V \to s(V)$ to be $\alpha(P) := \overline{\{P\}}$ for all $P \in V$; let $\mathcal{O}_V$ be the sheaf of regular functions on V.

One can prove that $s(V)$ with the sheaf $\alpha_*(\mathcal{O}_V)$ is a scheme with the properties we want. Moreover, we can prove that the map s is a full faithful functor (see "Categories") from the category of algebraic varieties over K to the category of schemes over K.

Definition. We say that a scheme is **irreducible** (respectively **connected**) if and only if the corresponding topological space is irreducible (respectively connected).
Let S be a scheme and let $\mathcal{O}$ be its structure sheaf. We say that S is **reduced** if $\mathcal{O}(U)$ has no nilpotent elements for any U open subset of S. This is equivalent to the condition that the stalk $\mathcal{O}_P$ has no nilpotent elements for any $P \in S$.
We say that S is **integral** if $\mathcal{O}(U)$ is an integral domain for any U open subset of S; one can prove that this holds if and only if S is reduced and irreducible.
We say that S is **locally Noetherian** if it can be covered by open affine subsets $Spec(R_i)$ with R_i Noetherian rings.
We say that S is **Noetherian** if if it can be covered by a finite number of open affine subsets $Spec(R_i)$ with R_i Noetherian rings.
Let $f : M \to N$ be a morphism of schemes. We say that f is respectively
- **locally of finite type**,
- **of finite type**,
- **finite**,

if there exists an open affine covering $\{Spec(B_i)\}_i$ of N such that, respectively,
- there exists an open affine covering of $f^{-1}(Spec(B_i))$, $\{Spec(A_{i,j})\}_j$ with $A_{i,j}$ finitely generated B_i-algebra,

– there exists a finite open affine covering of $f^{-1}(Spec(B_i))$, $\{Spec(A_{i,j})\}_j$ with $A_{i,j}$ finitely generated B_i-algebra,
– $f^{-1}(Spec(B_i)) = Spec(A_i)$ for some A_i B_i-algebra that is a finitely generated B_i-module.

We say that a morphism of schemes $f : M \to N$ is **affine** if there is an open affine covering $\{U_\alpha\}$ of N such that $f^{-1}(U_\alpha)$ is affine for every α.

An **open subscheme** of a scheme X is an open subset U of the topological space of X with the induced topology and with the sheaf given by the restriction of the structure sheaf of X to U. We can prove that it is a scheme.
An **open embedding** of schemes is a morphism of schemes $f : X \to Y$ that induces an isomorphism of X with an open subscheme of Y.
A **closed embedding** of schemes is a morphism of schemes $f : X \to Y$ inducing a homeomorphism between the topological space of X and a closed subset of the topological space of Y and the induced map of sheaves $\mathcal{O}_X \to f_*\mathcal{O}_Y$ is surjective.
A **closed subscheme** is an equivalence class of closed embeddings, where two closed embeddings $f : X \to Y$ and $g : Z \to Y$ are equivalent if there is an isomorphism $h : X \to Z$ such that $g \circ h = f$.
We say that a scheme over K is **projective** if it is a closed subscheme of the scheme $\mathbb{P}^n_K = Proj(K[x_0, \dots, x_n])$ for some n. □

Remark. Let R be a ring and let I be an ideal of R. The homomorphism $R \to R/I$ induces a closed embedding from the spectrum of R/I to the spectrum of R and the image of the map between the topological spaces is $V(I)$. Thus any ideal I induces a structure of closed subscheme on $V(I)$.
For instance, the ideals (x), (x^2), (x^2, xy) in $K[x, y]$ have the same zero locus (the y-axis) in the affine plane over K, but induce different structures of closed subscheme on the y-axis. Observe that the ideal (x^2) gives a subscheme such that for every point p of y-axis the stalk $\mathcal{O}_p$ has nilpotent elements, while the ideal (x^2, xy) gives a subscheme such that only the stalk of the sheaf in the origin has nilpotent elements (in this case we say that the origin is an embedded point; see “Primary ideals, primary decompositions, embedded ideals”). □

Definition.
– Let X_1 and X_2 be two schemes over another scheme S. The **fibred product** of X_1 and X_2 over S, denoted by $X_1 \times_S X_2$, is a scheme with morphisms $\pi_i : X_1 \times_S X_2 \to X_i$ for $i = 1, 2$ such that, if $f_i : X_i \to S$ are the given morphisms, we have $f_1 \circ \pi_1 = f_2 \circ \pi_2$ and, given another scheme Z over S and morphisms $h_i : Z \to X_i$ for $i = 1, 2$ such that $f_1 \circ h_1 = f_2 \circ h_2$, there exists a unique morphism $\alpha : Z \to X_1 \times_S X_2$ such that $h_i = \pi_i \circ \alpha$ for $i = 1, 2$. We can prove that the fibred product exists and that it is unique up to isomorphisms.

- Given a morphism of schemes $f : X \to Y$ and a subscheme W of Y, we define the **inverse image scheme** of W through f to be the fibred product $X \times_Y W$. Its underlying topological space is homeomorphic to $f^{-1}(W)$. Sometimes the inverse image scheme of W by f is denoted by $f^{-1}(W)$.
- A morphism of schemes $f : X \to Y$ is said to be **separated** (and X is said to be separated over Y) if the diagonal morphism $X \to X \times_Y X$ is a closed embedding, where the diagonal morphism $X \to X \times_Y X$ is the unique morphism whose composition with both the projections $X \times_Y X \to X$ is the identity. We say that a scheme X is separated if it is separated over $Spec(\mathbb{Z})$.
- A morphism of schemes $f : X \to Y$ is said to be **proper** if it is separated, of finite type, closed and, for any morphism $Y' \to Y$, the induced morphism $X \times_Y Y' \to Y'$ is closed (closed means that the image of any closed subset is a closed subset).
- A morphism of schemes $f : X \to Y$ is said to be **projective** if it is the composition of a closed embedding $X \to \mathbb{P}^n_{\mathbb{Z}} \times_{Spec(\mathbb{Z})} Y$ and the projection $\mathbb{P}^n_{\mathbb{Z}} \times_{Spec(\mathbb{Z})} Y \to Y$ (where $\mathbb{P}^n_{\mathbb{Z}} = Proj(\mathbb{Z}[x_0, \dots, x_n])$). □

We can prove that the properties of separatedness and properness for schemes, the definitions of which may sound a bit "strange", correspond respectively to the Hausdorff property and to the usual notion of properness through the functor from the category of the schemes of finite type over $\mathbb{C}$ to the category of complex analytic spaces (see "G.A.G.A.").

Theorem. The image of the map s described above from the category of algebraic varieties over an algebraically closed field K to the category of schemes over K is the set of the quasi-projective integral schemes and the image of the set the projective algebraic varieties is the set of the projective integral schemes. □

Schur functors. ([75], [76]). Let d be a natural number and let $\lambda = (\lambda_1, \dots, \lambda_k)$ be a partition of d, that is, let $\lambda_i \in \mathbb{N}$ for $i = 1, \dots, k$ with $\lambda_1 + \cdots + \lambda_k = d$ and $\lambda_1 \geq \cdots \geq \lambda_k$. We consider $(\lambda_1, \dots, \lambda_k)$ and $(\lambda_1, \dots, \lambda_k, 0, \dots, 0)$ the same partition and usually we don't write the zeroes.
We can associate to λ a diagram, which we call the **Young diagram** of λ, with λ_i boxes in the i-th row for $i = 1, \dots, k$ and the rows lined up on the left.

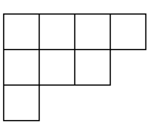


Fig. 15. Young diagram of $(4, 3, 1)$.

The **conjugate** partition λ' is the partition of d whose Young diagram is obtained from the Young diagram of λ by interchanging rows and columns.

A **tableau** with entries in $\{1, \dots, n\}$ on the Young diagram of a partition $\lambda = (\lambda_1, \dots, \lambda_k)$ of d is a numbering of the boxes by the integers $1, \dots, n$, allowing repetitions (we also say that it is a tableau on λ).

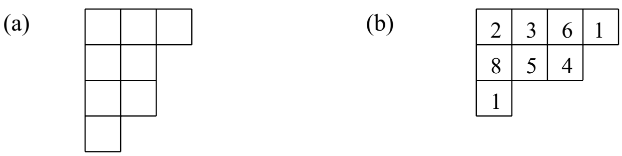


Fig. 16. (a) The Young diagram of $(3, 2, 2, 1)$, conjugate of $(4, 3, 1)$; (b) a tableau on $(4, 3, 1)$.

Definition. Let V be a complex vector space of dimension n. Let $d \in \mathbb{N}$, and let $\lambda = (\lambda_1, \dots, \lambda_k)$ be a partition of d. Number the boxes of the Young diagram of λ with the numbers $1, \dots, d$ from left to right, beginning from the top row; let Σ_d be the group of permutations on d elements. Let R be the subgroup of Σ_d given by the permutations preserving the rows, and let C be the subgroup of Σ_d given by the permutations preserving the columns. We define

$$S^\lambda V := Im\left(\sum_{a \in C,\ s \in R} sign(a)\ s \circ a : \otimes^d V \to \otimes^d V \right),$$

where Im stands for "image" and $sign(a)$ is the sign of the permutation a. The $S^\lambda V$ are called **Schur representations**. □

In particular, $S^d V$ denotes the symmetric product of V, and $S^{(1,\dots,1)} V$ denotes the alternating product. We can prove that $S^\lambda V$ is isomorphic to

$$Im\left(\sum_{a \in C,\ s \in R} sign(a)\ a \circ s : \otimes^d V \to \otimes^d V \right).$$

Theorem. Let V be a complex vector space of dimension n. For every $\lambda = (\lambda_1, \dots, \lambda_k)$ with $\lambda_i \in \mathbb{N}$ for $i = 1, \dots, k$ and $\lambda_1 \geq \dots \geq \lambda_k$, $S^\lambda V$ is an $SL(V)$-irreducible representation and all the (complex) **irreducible $SL(V)$-representations** are of this form. □

Proposition. Let V be a complex vector space of dimension n. Let $\lambda = (\lambda_1, \dots, \lambda_k)$ with $\lambda_i \in \mathbb{N}$ for $i = 1, \dots, k$ and $\lambda_1 \geq \dots \geq \lambda_k$. The dimension of $S^\lambda V$ is 0 if $\lambda_{n+1} \neq 0$, and is

$$\prod_{1 \leq i < j \leq n} \frac{\lambda_i - \lambda_j + j - i}{j - i}$$

if $\lambda_i = 0$ for $i \geq n + 1$. □

Observe that, for any $\lambda = (\lambda_1, \dots, \lambda_k)$ with $\lambda_i \in \mathbb{N}$ for $i = 1, \dots, k$ and $\lambda_1 \geq \dots \geq \lambda_k$ and for any $s \in \mathbb{N}$,

$$S^{(\lambda_1 + s, \dots, \lambda_n + s)} V \cong S^{(\lambda_1, \dots, \lambda_n)} V$$

as $SL(V)$-representations. In particular, as $SL(V)$-representation,

$$S^{(\lambda_1,\dots,\lambda_n)}V \cong S^{(\lambda_1-\lambda_n,\dots,\lambda_{n-1}-\lambda_n)}V.$$

Furthermore, for every $\lambda = (\lambda_1, \dots, \lambda_k)$ with $\lambda_i \in \mathbb{N}$ for $i = 1, \dots, k$ and $\lambda_1 \geq \cdots \geq \lambda_k$, the space $S^\lambda V$ is also a $GL(V)$-irreducible representation. Observe that, for all $s \in \mathbb{N}$,

$$S^{(\lambda_1+s,\dots,\lambda_n+s)}V \cong S^{(\lambda_1,\dots,\lambda_n)}V \otimes (\wedge^n V)^s$$

as $GL(V)$-representations. By defining $(\wedge^n V)^s = ((\wedge^n V)^{-s})^\vee$ if s negative, this allows us to define $S^{(\lambda_1,\dots,\lambda_n)}V$ for every $\lambda = (\lambda_1, \dots, \lambda_n)$ with $\lambda_1 \geq \cdots \geq \lambda_n$, not necessarily greater than or equal to 0.
Every (complex) irreducible $GL(V)$-representation is equal to $S^{(\lambda_1,\dots,\lambda_n)}V$ with $\lambda_1 \geq \cdots \geq \lambda_n$, for some $\lambda_1, \dots, \lambda_n \in \mathbb{Z}$.

The following theorem, called the Littlewood–Richardson rule, shows the decomposition in irreducible components of the tensor product of two irreducible representations of $GL(V)$. To state it, we need the following definition:

A $\mu = (\mu_1, \dots, \mu_k)$-expansion of a Young diagram is obtained by first adding μ_1 boxes, not two in the same column, with a 1 in each box, then adding μ_2 boxes, not two in the same column, with a 2 in each box, and so on. We say that the expansion is strict if the list of the numbers in the boxes we add, read from the top row to lowest one and every row from right to left, has the following property: for every r, in the first r entries of the list, the number of the p is greater than or equal to the number of the $p+1$ for every $p = 1, \dots, k-1$.

Littlewood–Richardson rule. Let $d, f \in \mathbb{N}$. For every $\lambda = (\lambda_1, \dots, \lambda_s)$ partition of d and $\mu = (\mu_1, \dots, \mu_k)$ partition of f, we have

$$S^\lambda V \otimes S^\mu V = \oplus_{\nu \text{ partition of } d+f}\, m_{\lambda,\mu,\nu} S^\nu V,$$

as $GL(V)$-representation, where $m_{\lambda,\mu,\nu}$ is the number of ways in which the Young diagram of ν can be obtained from the Young diagram of λ by a strict μ-expansion. □

For instance, if λ is a partition of a natural number d and t is a natural number, the Young diagrams of the irreducible components of $S^\lambda V \otimes S^t V$ are obtained from the Young diagram of S^λ adding t boxes not two in the same column and all the multiplicity are 1 (**Pieri's formula**).

Finally we observe that, if $dim(V) = n$, $(S^{\lambda_1,\dots,\lambda_n}V)^\vee$ is isomorphic as $SL(V)$-representation to $S^{\lambda_1-\lambda_n,\lambda_1-\lambda_{n-1},\dots,\lambda_1-\lambda_2}V$. Moreover $(S^\lambda V)^\vee \cong S^\lambda V^\vee$.

Now we want to describe the irreducible representations of Σ_d (the group of permutations on d elements).
Let $d, k \in \mathbb{N}$ and let $\lambda = (\lambda_1, \dots, \lambda_k)$ be a partition of d. As before, number the boxes of the Young diagram of λ with the numbers $1, \dots, d$ from left to right beginning from the

top row. Let R be the subgroup of Σ_d given by the permutations preserving the rows and let C be the subgroup of Σ_d given by the permutations preserving the columns. We define

$$c_\lambda := \sum_{a\in C,\ s\in R} sign(a)\ e_s \circ e_a \in \mathbb{C}\Sigma_d,$$

where $\mathbb{C}\Sigma_d$ is the group algebra associated to Σ_d (i.e., the algebra whose underlying vector space is $\oplus_{g\in\Sigma_d}\mathbb{C}e_g$, endowed with the product $e_g \cdot e_h = e_{gh}$). Let

$$S_\lambda = \mathbb{C}\Sigma_d \cdot c_\lambda.$$

Theorem. The only **irreducible representations of** Σ_d are the S_λ for λ partition of d. □

Scrolls.

A scroll is a projective bundle (i.e., a bundle whose fibres are projective spaces) embedded into a projective space $\mathbb{P}$ in such a way that the fibres are projective subspaces of $\mathbb{P}$.

Scrolls, rational normal -.

([93], [104]). Let K be a field and let $\mathbb{P}^r$ denote $\mathbb{P}^r_K$ for any $r \in \mathbb{N}$.
Let $n, k, r_0, \dots, r_k \in \mathbb{N} - \{0\}$ with

$$r_0 \le \dots \le r_k \quad \text{and} \quad \sum_{i=0,\dots,k} r_i = n - k + 1.$$

Let $\Lambda_i \cong \mathbb{P}^{r_i} \subset \mathbb{P}^n$, for $i = 0, \dots, k$, be complementary subspaces and C_i be (nondegenerate) rational normal curves in Λ_i (see "Rational normal curves"); let $\varphi_i : C_0 \to C_i$, for $i = 1, \dots, k$, be isomorphisms. Define

$$S_{r_0,\dots,r_k} = \cup_{P\in C_0} \langle P, \varphi_1(P), \dots, \varphi_k(P)\rangle,$$

where $\langle P, \varphi_1(P), \dots, \varphi_k(P)\rangle$ is the minimal subspace containing $P, \varphi_1(P), \dots, \varphi_k(P)$. The variety $S_{r_0,\dots,r_k}$ is called a rational normal scroll. Its dimension is obviously $k+1$. It is determined up to projective equivalence by the integers r_i.

Caution! The name may be somewhat misleading: it is not true that if a scroll is rational and normal, then it is a rational normal scroll.

Example of a rational normal scroll. $\mathbb{P}^1 \times \mathbb{P}^t$ embedded into $\mathbb{P}^{rt+r+t} = \mathbb{P}(K^{(t+1)(r+1)})$ by the bundle $\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^t}(r, 1)$, i.e., by the map

$$([x_0 : x_1], [y_0 : \dots : y_t]) \longmapsto$$
$$[x_0^r y_0 : x_0^{r-1} x_1 y_0 : \dots : x_1^r y_0 : \dots\dots : x_0^r y_t : x_0^{r-1} x_1 y_t : \dots : x_1^r y_t].$$

The k above is t, $r = r_i$ for any i, and the C_i are:

$$C_0 = \{[x_0^r y_0 : x_0^{r-1} x_1 y_0 : \cdots : x_1^r y_0 : 0 : \cdots\cdots : 0]\},$$
$$\cdots\cdots$$
$$C_t = \{[0 : \cdots\cdots 0 : x_0^r y_t : x_0^{r-1} x_1 y_t : \cdots : x_1^r y_t].$$

See “Minimal degree”.

Segre classes. ([72], [74]). Let E be a holomorphic vector bundle of rank r on a complex smooth projective algebraic variety X of dimension n; let $c(E)(t)$ be the Chern polynomial (see “Chern classes”)

$$1 + c_1(E)t + c_2(E)t^2 + \cdots .$$

We define the Segre polynomial

$$s(E)(t) = 1 + s_1(E)t + s_2(E)t^2 + \cdots$$

to be the polynomial such that

$$c(E)(t) \cdot s(E)(t) = 1.$$

Thus $s_1(E) = -c_1(E)$, $s_2(E) = c_1(E)^2 - c_2(E), \dots$. The element $s_i(E) \in H^{2i}(X, \mathbb{Z})$ is called the i-th Segre class of E.
We have that

$$PD(s_i(E)) = \pi_*(PD(c_1(H)^{r-1+i})),$$

where

- PD is the Poincaré duality (see “Singular homology and cohomology”);
- π is the projection $\mathbb{P}(E) \to X$ and π_* is the map induced by π in homology;
- $H = \mathcal{O}_{\mathbb{P}(E)}(1)$, i.e., H is the line bundle on $\mathbb{P}(E)$ whose restriction to $\mathbb{P}(E_x)$ (E_x fibre on x) is $\mathcal{O}(1)$ for all $x \in X$ (i.e., H is the dual of the tautological bundle; see “Tautological (or universal) bundle”).

The Segre classes can be defined also for cones and subvarieties, see [72], [74].

Segre embedding. ([104], [107]). Let K be an algebraically closed field and, for every n, let $\mathbb{P}^n = \mathbb{P}^n_K$ be the projective space over K of dimension n.
Let $n, m \in \mathbb{N}$. The Segre map on $\mathbb{P}^n \times \mathbb{P}^m$ is the map

$$s_{n,m} : \mathbb{P}^n \times \mathbb{P}^m \longrightarrow \mathbb{P}^{(n+1)(m+1)-1},$$
$$([x_0 : \cdots : x_n], [y_0 : \cdots : y_m]) \longmapsto [\cdots : x_i y_j : \cdots],$$

i.e., the map sending $([x_0 : \cdots : x_n], [y_0 : \cdots : y_m])$ to the point of $\mathbb{P}^{(n+1)(m+1)-1}$ whose coordinates are all the products $x_i y_j$ for $i = 0, \dots, n$, $j = 0, \dots, m$. Obviously, the map $s_{n,m}$ is an embedding (see “Embedding”).

Let $z_{i,j}$ for $i = 0, \dots, n$, $j = 0, \dots, m$ be the coordinates on $\mathbb{P}^{(n+1)(m+1)-1}$; the image of the Segre map $s_{n,m}$ is the zero locus of the polynomials of degree 2:

$$z_{i,j}z_{k,l} - z_{i,l}z_{k,j},$$

with $i, k \in \{0, \dots, n\}$, $j, l \in \{0, \dots, m\}$. It is a smooth variety of dimension $m + n$ and degree $\binom{m+n}{m}$ (see "Degree of an algebraic subset"), called Segre variety. It is a determinantal variety (see "Determinantal varieties"), since it is the zero locus of the determinants of the (2×2)-submatrices of the matrix $(z_{i,j})_{i,j}$.
A coordinate-free way to describe the Segre map $s_{n,m}$ is the following: Let V be a vector space over K of dimension $n + 1$ and W be a vector space over K of dimension $m + 1$; the map $s_{n,m}$ is the map

$$\begin{aligned} \mathbb{P}(V) \times \mathbb{P}(W) &\longrightarrow \mathbb{P}(V \otimes W), \\ (\langle v \rangle, \langle w \rangle) &\longmapsto \langle v \otimes w \rangle \end{aligned}$$

for $v \in V$, $w \in W$, where $\langle v \rangle \in \mathbb{P}(V)$ is the line generated by v.
One can define also the Segre map on the product of more than two projective spaces in the obvious way: let $n_1, \dots, n_k \in \mathbb{N}$; the map $s_{n_1,\dots,n_k}$ is the map

$$\mathbb{P}^{n_1} \times \dots \times \mathbb{P}^{n_k} \longrightarrow \mathbb{P}^{(n_1+1)\dots(n_k+1)-1},$$

$$([x^1_0 : \dots : x^1_{n_1}], \dots, [x^k_0 : \dots : x^k_{n_k}]) \longmapsto [\dots\dots : x^1_{i_1} \cdot \ \cdots \ \cdot x^k_{i_k} : \dots\dots],$$

where $i_j \in \{0, \dots, n_j\}$ for $j = 1, \dots, k$ (the point on the right is the point whose coordinates are all the products $x^1_{i_1} \cdot \ \cdots \ \cdot x^k_{i_k}$, where $i_j \in \{0, \dots, n_j\}$ for $j = 1, \dots, k$).

Semicontinuity theorem. See "Grauert's semicontinuity theorem".

Serre correspondence. ([64], [107], [223]). Let R be a graded commutative ring with unity and let X be the scheme $Proj(R)$ (see "Schemes").

– For any graded R-module M, let $\mathcal{F}_M$ be the following sheaf on X: for every $p \in X$, let $M_{(p)}$ be the group of the elements of degree 0 in the localization of M with respect to the multiplicative system of the homogeneous elements in $R - p$ (see "Localization, quotient ring, quotient field"); for every open subset U of X, let $\mathcal{F}_M(U)$ be the set of the functions

$$\sigma : U \to \sqcup_{p \in U} M_{(p)}$$

such that $\sigma(p) \in M_{(p)}$ for every $p \in U$ and σ is locally a fraction, that is, for all $p \in U$, there exists a neighborhood V of p in U and homogeneous elements $m \in M$, $r \in R$ of the same degree such that, for all $p' \in V$, we have $r \notin p'$ and $\sigma(p') = \frac{m}{r}$. The functor $M \mapsto \mathcal{F}_M$ is called sheafification.

– For any sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules on X, let

$$M_{\mathcal{F}} := \oplus_{n \in \mathbb{Z}} H^0(X, \mathcal{F}(n)).$$

We can make this a graded R-module, in fact, given $f \in H^0(X, \mathcal{F}(n))$ and $r \in R_k$, which gives an element in $H^0(X, \mathcal{O}_X(k))$ (see "Hyperplane bundles, twisting sheaves"), we can easily define their product $r \cdot f \in H^0(X, \mathcal{F}(n+k))$ by using the isomorphism $\mathcal{F}(n) \otimes \mathcal{O}_X(k) \cong \mathcal{F}(n+k)$.

Proposition. Let K be a field and let $R = K[x_0, \dots, x_n]$. For any quasi-coherent sheaf $\mathcal{F}$ on $\mathbb{P}^n_K = Proj(R)$ (see "Coherent sheaves"), we have

$$\mathcal{F}_{M_{\mathcal{F}}} \cong \mathcal{F}. \qquad \square$$

It is not true that for any finitely generated graded R-module M we have that $M_{\mathcal{F}_M} \cong M$, it is only true that $M_{\mathcal{F}_M}$ and M are isomorphic from a certain degree on. We say that two graded R-modules are equivalent if they are isomorphic from a certain degree on. We say that a graded R-module is quasi-finitely generated if it is equivalent to a finitely generated graded R-module. The functors $M \mapsto \mathcal{F}_M$, $\mathcal{F} \mapsto M_{\mathcal{F}}$ induce an equivalence of categories between the following categories:

- the category of the graded quasi-finitely generated R-modules modulo the equivalence above, where $R = K[x_0, \dots, x_n]$;
- the category of coherent sheaves of $\mathcal{O}_{\mathbb{P}^n_K}$-modules.

See [107, Chapter II, § 5 and Example 5.9] for more general statements.

Serre duality. ([93], [107], [224]).

Theorem. Let X be a compact complex manifold of dimension n and E a holomorphic vector bundle on X. The following isomorphism holds:

$$H^q(X, \Omega^p(E)) \cong H^{n-q}(X, \Omega^{n-p}(E^\vee))^\vee. \qquad \square$$

The isomorphism above is called Serre duality. A completely analogous statement holds for a vector bundle on a smooth projective algebraic variety of dimension n over an algebraically closed field. See "Dualizing sheaf".

Serre's theorems A and B. See "Cartan–Serre theorems".

Sheafify. See "Serre correspondence".

Sheaves. ([34], [84], [93], [103], [107], [119], [146], [152], [223], [228], [234]).

Definition. A **presheaf** of Abelian groups $\mathcal{F}$ on a topological space X is the datum of a map associating to any open subset U of X an Abelian group $\mathcal{F}(U)$ and, for any V, U open subsets of X with $V \subset U$, a homomorphism (called restriction map) $\rho^U_V : \mathcal{F}(U) \longrightarrow$

$\mathcal{F}(V)$ such that

(a) $\mathcal{F}(\emptyset) = \{0\}$;

(b) $\rho_U^U = I$ for every open subset U of X;

(c) if $W \subset V \subset U$, then $\rho_W^V \circ \rho_V^U = \rho_W^U$. □

Thus a presheaf of Abelian groups is a controvariant functor from the category of the open subsets of X with the inclusion maps to the category of Abelian groups.
Obviously we can consider also presheaves "with values" in an other category, say $\mathcal{C}$, i.e., such that $\mathcal{F}(U)$ is an object of $\mathcal{C}$ for any U open subset of X (for instance presheaves of modules over some ring).

For any open subsets U and V of X with $V \subset U$ and any $s \in \mathcal{F}(U)$, we often denote $\rho_V^U(s)$ by $s|_V$.
For any open subset U of X, the group $\mathcal{F}(U)$ is often denoted also by $\Gamma(U, \mathcal{F})$. Its elements are called **sections** of $\mathcal{F}$ over U. The elements of $\mathcal{F}(X)$ are called global sections.

Definition. We say that a presheaf $\mathcal{F}$ on a topological space X is a **sheaf** if and only if the two following conditions hold:

(1) (Locality) Let U be an open subset of X and $\{U_\alpha\}_\alpha$ be an open covering of U. If $s, t \in \mathcal{F}(U)$ and $s|_{U_\alpha} = t|_{U_\alpha}$ for all α, then $s = t$.

(2) (Gluing) Let U be an open subset of X and $\{U_\alpha\}_\alpha$ be an open covering of U. Let $s_\alpha \in \mathcal{F}(U_\alpha)$ for all α such that $s_\alpha|_{U_\alpha \cap U_\beta} = s_\beta|_{U_\alpha \cap U_\beta}$; then there exists $s \in \mathcal{F}(U)$ such that $s_\alpha = s|_{U_\alpha}$. □

Let $x \in X$. The **stalk** of a presheaf $\mathcal{F}$ on X is

$$\mathcal{F}_x = \lim_{\substack{\longrightarrow \\ U \text{ open subset},\, U \ni x}} \mathcal{F}(U)$$

(see "Limits, direct and inverse -"). We denote the map from $\mathcal{F}(U)$ to $\mathcal{F}_x$ by ρ_x^U.
Let

$$\hat{\mathcal{F}} = \sqcup_{x \in X} \mathcal{F}_x.$$

We can associate to any presheaf $\mathcal{F}$ on a topological space X, a sheaf $\mathcal{F}'$ on X in the following way: let $\mathcal{F}'$ be the sheaf

$$U \longmapsto \left\{ s : U \to \hat{\mathcal{F}} \;\middle|\; \begin{array}{l} s(y) \in \mathcal{F}_y \quad \forall y \in U; \\ \forall\, x \in U,\ \exists\, V \subset U,\ V \text{ neighborhood of } x, \text{ and} \\ \qquad\qquad \exists\, \sigma \in \mathcal{F}(V) \ \text{ s.t. } s(y) = \rho_y^V(\sigma)\ \forall y \in V \end{array} \right\}$$

for any open subset U.
Equivalently we can define $\mathcal{F}'$ in the following way: endow $\hat{\mathcal{F}}$ with the strongest topology such that, for any open subset U and for any $s \in \mathcal{F}(U)$, the map $U \to \hat{\mathcal{F}}$, $x \mapsto \rho_x^U(s)$ is continuous (with this topology, $\hat{\mathcal{F}}$ is called "espace étalé of $\mathcal{F}$"); define $\mathcal{F}'$ to be the

following sheaf:

$$U \mapsto \{s : U \to \hat{\mathcal{F}} |\ \ s(y) \in \mathcal{F}_y \ \ \forall y \in U, \ \ s \text{ continuous}\}.$$

If $\mathcal{F}$ is a sheaf, then $\mathcal{F} = \mathcal{F}'$.

Examples.

- Let X be a topological space and let G be an Abelian group. Endow G with the discrete topology.
  - We define a sheaf G_X (or simply G) on X in the following way: For any open subset U, let $G_X(U)$ be the group of the continuous functions from U to G and let the restriction maps be the usual restriction maps. We call G_X a **constant sheaf**. For instance, we can take $G = \mathbb{Z}, \mathbb{R}, \mathbb{C} \ldots$.
  - If P is a point of X, the **skyscraper sheaf** G_P on X is defined to be the sheaf such that, for any open subset U, $G_P(U)$ is G if $P \in U$, while it is 0 if $p \notin U$.
- Let X be a C^∞ manifold. We define the sheaves C^∞, $A^{p,q}$ and, for any C^∞ bundle E on X, the sheaves $C^\infty(E)$ and $A^{p,q}(E)$ in the following way:
  - $C^\infty(U)$ is the group of the C^∞ functions from U to $\mathbb{R}$;
  - $A^{p,q}(U)$ is the group of the C^∞ (p,q)-forms on U;
  - $C^\infty(E)(U)$ is the group of the C^∞ sections of E on U;
  - $A^{p,q}(E)(U)$ is the group of the C^∞ (p,q)-forms on U with values in E,

  for any open subset U. We should write respectively C^∞_X, $A^{p,q}_X$, $C^\infty(E)_X$, $A^{p,q}(E)_X$ when the manifold X is not clear from the context.
- Let X be a complex manifold. We define the sheaf $\mathcal{O}$, the sheaf Ω^p, and for any holomorphic bundle E on X, the sheaf $\mathcal{O}(E)$ in the following way:
  - $\mathcal{O}(U)$ is the group of the holomorphic functions from U to $\mathbb{C}$;
  - $\Omega^p(U)$ is the group of the holomorphic p-forms on U;
  - $\mathcal{O}(E)(U)$ is the group of the (holomorphic) sections of E on U;

  for any open subset U. We should write respectively $\mathcal{O}_X$, Ω^p_X, $\mathcal{O}(E)_X$ when the manifold is not clear from the context.
- If X is an algebraic variety, $\mathcal{O}_X$ denotes the sheaf of the regular functions (see “Varieties, algebraic -, Zariski topology, regular and rational functions, morphisms and rational maps”). If E is an (algebraic) vector bundle on X, we denote by $\mathcal{O}(E)$ the sheaf on X associating to any open subset U of X the group of the sections of E on U.

A **subsheaf** of a sheaf $\mathcal{F}$ is a sheaf $\mathcal{G}$ such that, for any U open subset of X, $\mathcal{G}(U)$ is a subgroup of $\mathcal{F}(U)$ and the restriction maps of $\mathcal{G}$ are induced by the ones of $\mathcal{F}$.

Definition. Let $\mathcal{F}$, $\mathcal{G}$ be two sheaves on a topological space X. A **morphism of sheaves** $f : \mathcal{F} \to \mathcal{G}$ is the datum of a homomorphism

$$\alpha_U : \mathcal{F}(U) \to \mathcal{G}(U)$$

for any U open subset of X such that, if V and U are open subsets with $V \subset U$ and ρ_V^U is the restriction map of $\mathcal{F}$ and τ_V^U is the restriction map of $\mathcal{G}$, the diagram

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "\alpha_U"] \arrow[d, "\rho_V^U"] & \mathcal{G}(U) \arrow[d, "\tau_V^U"] \\ \mathcal{F}(V) \arrow[r, "\alpha_V"] & \mathcal{G}(V) \end{tikzcd}$$

commutes. Denote by $Hom(\mathcal{F}, \mathcal{G})$ the set of the morphisms from $\mathcal{F}$ to $\mathcal{G}$. □

Obviously, a morphism of sheaves $f : \mathcal{F} \to \mathcal{G}$ induces a map $f_x : \mathcal{F}_x \to \mathcal{G}_x$ for any $x \in X$.
We say that f is injective if and only if $f_x : \mathcal{F}_x \to \mathcal{G}_x$ is injective for all x and this is true if and only if the maps $f_U : \mathcal{F}(U) \to \mathcal{G}(U)$ are injective for any U open subset of X.
We say that f is surjective if and only if $f_x : \mathcal{F}_x \to \mathcal{G}_x$ is surjective for all x. This is not equivalent to requiring that the maps $f_U : \mathcal{F}(U) \to \mathcal{G}(U)$ are surjective for any U open subset of X. For instance consider the morphism (exponential morphism) $exp : \mathcal{O} \to \mathcal{O}^*$ on $\mathbb{C} - \{0\}$ that sends $g \in \mathcal{O}(U)$ to $e^{2\pi i g} \in \mathcal{O}^*(U)$ for any open subset U (where $\mathcal{O}^*$ is the sheaf associating to any open subset U the group of the holomorphic functions on U that are never zero); it is surjective but, if we take $U = \mathbb{C} - \{0\}$, the map $\exp_U : \mathcal{O}(U) \to \mathcal{O}^*(U)$ is not surjective; in fact $z \in \mathcal{O}^*(U)$, but z is not in the image of $\exp_U$.

We define $Ker\ f$ to be the sheaf

$$U \longmapsto Ker\ f_U.$$

We define $Im\ f$ to be the sheaf associated to the presheaf

$$U \longmapsto Im\ f_U$$

and $Coker\ f$ to be the sheaf associated to the presheaf

$$U \longmapsto Coker\ f_U.$$

We say that

$$\cdots\cdots \mathcal{F}_{i-1} \xrightarrow{f_{i-1}} \mathcal{F}_i \xrightarrow{f_i} \mathcal{F}_{i+1} \cdots\cdots$$

is a complex if $f_{i+1} \circ f_i = 0$ and is exact if

$$\cdots\cdots (\mathcal{F}_{i-1})_x \xrightarrow{(f_{i-1})_x} (\mathcal{F}_i)_x \xrightarrow{(f_i)_x} (\mathcal{F}_{i+1})_x \cdots\cdots$$

is exact for all $x \in X$.

For any subsheaf $\mathcal{G}$ of a sheaf $\mathcal{F}$, we define the quotient sheaf $\mathcal{F}/\mathcal{G}$ to be the sheaf associated to the presheaf $U \mapsto \mathcal{F}(U)/\mathcal{G}(U)$; so we get an exact sequence $0 \to \mathcal{G} \to \mathcal{F} \to \mathcal{F}/\mathcal{G} \to 0$. Obviously we can define the quotient sheaf only for sheaves with values in categories where the quotient is defined.

Cohomology

Let X be a topological space. Let $\mathcal{F}$ be a sheaf on X and $\mathcal{U} = \{U_i\}_{i\in I}$ be an open covering of X. For any $p \geq 0$, let

$$\mathcal{C}^p(\mathcal{U}, \mathcal{F}) = \prod_{\substack{(i_0,\dots,i_p) \text{ s.t.} \\ i_0,\dots,i_p \in I \text{ and } U_{i_0}\cap\cdots\cap U_{i_p} \neq \emptyset}} \mathcal{F}(U_{i_0} \cap \cdots \cap U_{i_p}).$$

Let

$$\delta_p : \mathcal{C}^p(\mathcal{U}, \mathcal{F}) \longrightarrow \mathcal{C}^{p+1}(\mathcal{U}, \mathcal{F})$$

be the following map: for $s = (s_{i_0,\dots,i_p})_{i_0,\dots,i_p} \in \mathcal{C}^p(\mathcal{U}, \mathcal{F})$ we define

$$(\delta_p s)_{i_0,\dots,i_{p+1}} = \sum_{j=0,\dots,p+1} (-1)^j s_{i_0,\dots,\hat{i_j},\dots,i_{p+1}}|_{U_{i_0}\cap\cdots\cap U_{i_{p+1}}},$$

where $\hat{i_j}$ means that we omit i_j.
We have that $\delta_p \circ \delta_{p-1} = 0$. Define

$$\check{H}^p(\mathcal{U}, \mathcal{F}) = \frac{Ker\ \delta_p}{Im\ \delta_{p-1}}.$$

Let $\mathcal{U} = \{U_a\}_{a\in A}$ and $\mathcal{V} = \{V_b\}_{b\in B}$ be two open coverings of X. We say that $\mathcal{V}$ is finer than $\mathcal{U}$ if and only if there exists a map $j : B \to A$ such that $V_b \subset U_{j(b)}$ for any $b \in B$. The map j induces a map

$$\mathcal{C}^p(\mathcal{U}, \mathcal{F}) \longrightarrow \mathcal{C}^p(\mathcal{V}, \mathcal{F}),$$

$$(s_{a_0,\dots,a_p})_{a_0,\dots,a_p} \longmapsto (s_{j(b_0),\dots,j(b_p)}|_{V_{b_0}\cap\cdots\cap V_{b_p}})_{b_0,\dots,b_p}$$

and, since this map commutes with δ, it induces a map

$$\check{H}^p(\mathcal{U}, \mathcal{F}) \to \check{H}^p(\mathcal{V}, \mathcal{F}).$$

One can show that this map does not depend on j; therefore we can define the **Čech cohomology** of $\mathcal{F}$ to be

$$\check{H}^p(X, \mathcal{F}) := \lim_{\substack{\rightarrow \\ \mathcal{U}}} \check{H}^p(\mathcal{U}, \mathcal{F})$$

(see “Limits, direct and inverse -”); sometimes it is denoted simply by $H^p(X, \mathcal{F})$.

Remark. $\check{H}^0(\mathcal{U}, \mathcal{F}) = \mathcal{F}(X)$ for any $\mathcal{U}$ open covering of X. □

Leray's theorem. Let $\mathcal{F}$ be a sheaf on a topological space X.
Let $\mathcal{U} = \{U_i\}_i$ be an acyclic open covering of X, i.e., an open covering such that $\check{H}^p(U_{i_1} \cap \cdots \cap U_{i_r}, \mathcal{F}|_{U_{i_1} \cap \cdots \cap U_{i_r}}) = 0$ for all $p \geq 1$, for all r and for all $i_1, \ldots, i_r$. Then

$$\check{H}^p(\mathcal{U}, \mathcal{F}) \cong \check{H}^p(X, \mathcal{F})$$

for all p. □

Proposition. Let

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$$

be an exact sequence of sheaves on a paracompact topological space X. Then there is a long exact sequence

$$\cdots \to \check{H}^p(X, \mathcal{F}) \to \check{H}^p(X, \mathcal{G}) \to \check{H}^p(X, \mathcal{H}) \to \check{H}^{p+1}(X, \mathcal{F}) \to \cdots .$$ □

Definition. We say that a sheaf $\mathcal{F}$ on a topological space X is **acyclic** if, for any $p \geq 1$, we have $\check{H}^p(X, \mathcal{F}) = 0$. □

Abstract De Rham's theorem. Let

$$0 \to \mathcal{F} \to \mathcal{A}_0 \to \mathcal{A}_1 \to \cdots\cdots \to \mathcal{A}_k \to 0$$

be an exact sequence of sheaves on a paracompact topological space X and suppose that the sheaves $\mathcal{A}_i$ are acyclic. Then, for any natural number p,

$$\check{H}^p(X, \mathcal{F}) \cong \frac{Ker\ \Gamma(X, \mathcal{A}_p) \to \Gamma(X, \mathcal{A}_{p+1})}{Im\ \Gamma(X, \mathcal{A}_{p-1}) \to \Gamma(X, \mathcal{A}_p)},$$

where we set $\mathcal{A}_{-1} = 0$. □

Definition. We say that a sheaf $\mathcal{F}$ on a topological space X is **fine** if, for any open subset U of X and any locally finite open covering $\{U_\alpha\}_\alpha$ of U, there exist maps $\eta_\alpha : \mathcal{F}(U_\alpha) \to \mathcal{F}(U)$ such that
(1) if $s \in \mathcal{F}(U_\alpha)$, then $\eta_\alpha(s)$ has support in U_α, that is, $\rho_x^U(\eta_\alpha(s)) = 0$ for any $x \notin U_\alpha$;
(2) $\sum_\alpha \eta_\alpha(s|_{U_\alpha}) = s$ for any $s \in \mathcal{F}(U)$. □

Theorem. Let X be a paracompact topological space. A fine sheaf on X is acyclic. □

Definition. We say that a sheaf $\mathcal{F}$ on a topological space X is **flasque** (or flabby) if, for all V, U open subsets of X with $V \subset U$, the map $\rho_V^U : \mathcal{F}(U) \to \mathcal{F}(V)$ is surjective. □

Theorem. Let X be a paracompact topological space. A flasque sheaf on X is acyclic. □

There are several sheaves cohomology theories besides Čech cohomology.

One can prove that Čech cohomology of constant sheaves and singular cohomology (see "Singular homology and cohomology") coincide on manifolds (see [234, Chapter 6]).
Another cohomology is the so-called **derived functor cohomology**. Let X be a topological space; we define the cohomology functors $H^i(X, \cdot)$ to be the classical right derived functors of the global section functor $\Gamma(X, \cdot)$ from the category of sheaves of Abelian groups on X to the category of Abelian groups (see "Derived categories and derived functors"):
For any sheaf of Abelian groups $\mathcal{F}$, take an injective resolution $\mathcal{I}^*$ (we can prove that it exists), i.e, a complex of injective sheaves $\mathcal{I}^0 \to \mathcal{I}^1 \to \cdots$ with a morphism $\mathcal{F} \to \mathcal{I}^0$ such that

$$0 \to \mathcal{F} \to \mathcal{I}^0 \to \mathcal{I}^1 \to \cdots$$

is an exact sequence (we say that a sheaf $\mathcal{N}$ is injective if $Hom(\cdot, \mathcal{N})$ is an exact functor); define $H^i(X, \mathcal{F})$ to be the cohomology in i of the complex $\Gamma(X, \mathcal{I}^*)$.
For any exact sequence of sheaves on a topological space X

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0,$$

there is a long exact sequence

$$\cdots \to H^p(X, \mathcal{F}) \to H^p(X, \mathcal{G}) \to H^p(X, \mathcal{H}) \to H^{p+1}(X, \mathcal{F}) \to \cdots .$$

Čech cohomology and derived functor cohomology generally coincide (but on pathological spaces they an be different). In particular we have the following theorems.

Theorem $*$. (See [107, Example 4.11 and Theorem 3.5, Chapter III)]).
(a) Let $\mathcal{F}$ be a sheaf (of Abelian groups) on a topological space X. Let $\mathcal{U} = \{U_\alpha\}_\alpha$ be an open covering of X acyclic for the derived functor cohomology, that is, such that $H^p(U_{i_1} \cap \cdots \cap U_{i_r}, \mathcal{F}|_{U_{i_1} \cap \cdots \cap U_{i_r}}) = 0$ for all $p \geq 1$, for all r and for all $i_1, \dots, i_r$, where H^p denotes the derived functor cohomology. Then

$$\check{H}^p(\mathcal{U}, \mathcal{F}) \cong H^p(X, \mathcal{F})$$

for all p.
(b) Let $\mathcal{F}$ be a quasi-coherent sheaf on a Noetherian affine scheme X (see "Coherent sheaves" and "Schemes"). Then the derived functor cohomology $H^p(X, \mathcal{F})$ is zero for all $p \geq 1$. □

Serre's theorem. (See [223, part 46]) Let $\mathcal{F}$ be a coherent sheaf on an affine algebraic variety. Then the Čech cohomolgy $\check{H}^p(X, \mathcal{F})$ is zero for all $p \geq 1$. □

Thus affine open subsets of an algebraic variety give an acyclic open covering both for Čech cohomology of coherent sheaves (by Serre's Theorem) and derived cohomology of coherent sheaves (by Theorem $*$, part (b)). Thus, by part (a) of Theorem $*$ and by

Leray's Theorem, the two cohomologies coincide for coherent sheaves on algebraic varieties.
Furthermore, derived cohomology and Čech cohomology coincide on paracompact topological spaces (see [84, Theorem. 5.10.1, p. 228, Example 7.2.1, p. 263]):

Sheaves of $\mathcal{O}_X$-Modules

Definition. Let $(X, \mathcal{O}_X)$ be a ringed space (see "Spaces, ringed -").
We say that a sheaf of groups $\mathcal{F}$ on X is **a sheaf of $\mathcal{O}_X$-modules** if, for any U open subset of X, $\mathcal{F}(U)$ is an $\mathcal{O}_X(U)$-module and, for any V open subset of U, the diagram

$$\begin{tikzcd} \mathcal{O}_X(V) \times \mathcal{F}(V) \arrow[r] \arrow[d] & \mathcal{F}(V) \arrow[d] \\ \mathcal{O}_X(U) \times \mathcal{F}(U) \arrow[r] & \mathcal{F}(U) \end{tikzcd}$$

commutes (where the vertical maps are the maps induced by the restriction maps).
We say that a sheaf of $\mathcal{O}_X$-modules is **free** if it is isomorphic to a direct sum of copies of $\mathcal{O}_X$. The number of the copies is called the rank.
We say that a sheaf of $\mathcal{O}_X$-modules is **locally free of finite rank** if there is an open covering $\{U_\alpha\}_\alpha$ of X and there exist $r_\alpha \in \mathbb{N}$ such that $\mathcal{F}|_{U_\alpha}$ is isomorphic to $(\mathcal{O}_X|_{U_\alpha})^{r_\alpha}$. If X is connected all the r_α are equal.
We say that a sheaf of $\mathcal{O}_X$-modules is **invertible** if it is locally free of rank 1.
Let $\mathcal{F}$ and $\mathcal{G}$ be sheaves of $\mathcal{O}_X$-modules.
Define the **tensor product** $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ (or simply $\mathcal{F} \otimes \mathcal{G}$) to be the sheaf associated to the presheaf

$$U \longmapsto \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U)$$

for any open subset U.
Define $Hom_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ to be the group of $\mathcal{O}_X$-module homomorphisms and define $\mathcal{HOM}(\mathcal{F}, \mathcal{G})$ to be the sheaf associated to the presheaf

$$U \longmapsto Hom_{\mathcal{O}_{X|U}}(\mathcal{F}|_U, \mathcal{G}|_U).$$

For any sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules, let $\mathcal{F}^\vee = \mathcal{HOM}(\mathcal{F}, \mathcal{O})$ (**dual sheaf**).
We say that a sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules is **torsion-free** if every stalk is torsion-free.
We say that $\mathcal{F}$ is **reflexive** if the natural map $\mathcal{F} \to (\mathcal{F}^\vee)^\vee$ is bijective. □

Proposition. (see [146]). Let X be a complex manifold. Let $\mathcal{F}$ be a coherent sheaf of $\mathcal{O}_X$-modules (see "Coherent sheaves").
(1) The sheaf $\mathcal{F}$ is torsion-free if and only if the natural map $\mathcal{F} \to (\mathcal{F}^\vee)^\vee$ is injective.
(2) The sheaf $\mathcal{F}^\vee$ is reflexive.
(3) We have

$$\text{locally free} \;\Rightarrow\; \text{reflexive} \;\Rightarrow\; \text{torsion-free}\,.$$ □

Let X be a smooth algebraic variety and let $\mathcal{O}_X$ be the sheaf of the regular functions. There is a bijection between the set of vector bundles on X up to isomorphisms and the set of locally free sheaves of $\mathcal{O}_X$-modules of finite rank up to isomorphisms: the bijection is given by the map sending a vector bundle E to the sheaf $\mathcal{O}(E)$ associating to any open subset U of X the group of the sections of E on U (see [228, Chapter 6], or [107, Chapter 2, Exercise 5.18]).
The fibre of the vector bundle can be recovered from the stalk of the sheaf in the following way:

$$E_x = (\mathcal{O}(E))_x / m_x (\mathcal{O}(E))_x,$$

where m_x is the maximal ideal of $\mathcal{O}_x$ for any $x \in X$.

A **sheaf of ideals** on a ringed space $(X, \mathcal{O}_X)$ is a sheaf of $\mathcal{O}_X$-modules $\mathcal{I}$ such that for any U open subset of X, $\mathcal{I}(U)$ is an ideal of $\mathcal{O}_X(U)$.
If Y is a subvariety of an algebraic variety X, we define

$$\mathcal{I}_{Y,X}(U) = \{f \in \mathcal{O}_X(U) |\ f|_{Y \cap U} = 0\}.$$

Siegel half-space.

The Siegel half-space of degree g is

$$\mathcal{H}_g = \{Z \in M(g \times g, \mathbb{C}) |\ Z = {}^t Z, \quad Im(Z) > 0\},$$

where $Im(Z) > 0$ means that the imaginary part of Z is positive definite. It is the moduli space of polarized Abelian varieties of fixed type with symplectic basis. See "Tori, complex - and Abelian varieties".

Siegel's theorem.

([228], [232]).

Siegel's theorem. The transcendence degree over $\mathbb{C}$ (see "Transcendence degree") of the field of meromorphic functions on a compact complex manifold M is less than or equal to the dimension of M. □

See also "Dimension".

Simple bundles.

([146], [207]). We say that a holomorphic vector bundle E on a compact complex manifold X is simple if any holomorphic morphism of bundles from E to E is a scalar multiple of the identity, that is, if $H^0(X, \mathcal{O}(End\ E)) = \mathbb{C}$, where $End\ E = E^\vee \otimes E$.

Proposition. Every stable vector bundle (see "Stable sheaves") on a compact Kähler manifold is simple. □

Singular homology and cohomology.

([33], [91], [112], [183], [215], [234], [247]). Throughout the item, R will denote a commutative ring with unity.

Homology

For any $k \in \mathbb{N}$, let $\{e_1, \ldots, e_k\}$ be the canonical basis of $\mathbb{R}^k$ and let e_0 be the zero element of $\mathbb{R}^k$. We define Δ_k to be $\langle e_0, \ldots, e_k \rangle$, i.e., the simplex spanned by $e_0, \ldots, e_k$. Thus Δ_0 is a point, Δ_1 is a segment, Δ_2 a triangle and so on.
Let X be a topological space. A (singular) ***k*-simplex** in X is a continuous map $\sigma : \Delta_k \to X$. We define

$$\boxed{C_k(X,R) = \left\{ \sum_{\sigma \text{ } k\text{-simplex in } X} n_\sigma \sigma \;\middle|\; \begin{array}{l} n_\sigma \in R, \\ \text{all } n_\sigma \text{ are zero apart from a finite number} \end{array} \right\}}.$$

Thus $C_k(X,R)$ is the set of the finite formal linear combinations of the k-simplexes in X with coefficients in R. It is called the set of the ***k*-chains** in X with coefficients in R.
We define $\varepsilon_i : \Delta_{k-1} \to \Delta_k$ to be the affine map sending e_j to e_j for any $j \in \{0, \ldots, i-1\}$ and e_j to e_{j+1} for any $j \in \{i, \ldots, k-1\}$, so sending Δ_{k-1} onto the face of Δ_k spanned by $e_0, \ldots, \hat{e_i}, \ldots, e_k$.
For any $k \geq 1$, we define the **border operator**

$$\partial_k : C_k(X,R) \longrightarrow C_{k-1}(X,R)$$

to be the linear map such that, if σ is a singular k-simplex in X,

$$\boxed{\partial_k \sigma = \sum_{i=0,\ldots,k} (-1)^i \sigma \circ \varepsilon_i}$$

and we define $\partial_0 : C_0(X,R) \to \{0\}$ to be the zero map. We should write $\partial_k(\sigma)$ instead of $\partial_k \sigma$, but the brackets are generally omitted.

Lemma. $\partial_k \circ \partial_{k+1} = 0$. □

Thus

$$\cdots \longrightarrow C_{k+1}(X,R) \xrightarrow{\partial_{k+1}} C_k(X,R) \xrightarrow{\partial_k} C_{k-1}(X,R) \longrightarrow \cdots$$

is a complex; we denote it by $(C_*(X,R), \partial_*)$. We define the **singular homology module** of X of degree k with coefficients in R, which we denote by $H_k(X,R)$, to be the homology of the complex $(C_*(X,R), \partial_*)$ in $C_k(X,R)$, i.e.,

$$\boxed{H_k(X,R) := \frac{Ker\ \partial_k}{Im\ \partial_{k+1}}}.$$

The elements of $Ker\ \partial_k$ are called ***k*-cycles**, the elements of $Im\ \partial_{k+1}$ are called ***k*-borders**. Two k-cycles are said **homologous** if their difference is a k-border. The rank of $H_i(X,R)$ is denoted by $h_i(X,R)$ or by $b_i(X,R)$ (Betti numbers).

Remark. If X is a topological space with r path-connected components, then $H_0(X,R) \cong R^r$. □

A slight variant of the homology is the so-called "reduced homology": for $k \geq 1$, we define

$$\partial_k^\sharp : C_k(X,R) \longrightarrow C_{k-1}(X,R)$$

to be equal to ∂_k and we define $\partial_0^\sharp : C_0(X,R) \to R$ to be the map

$$\sum_\sigma n_\sigma \sigma \longmapsto \sum_\sigma n_\sigma .$$

We define the reduced homology to be

$$\boxed{H_k^\sharp(X,R) = \frac{Ker\ \partial_k^\sharp}{Im\ \partial_{k+1}^\sharp}.}$$

So we have $H_k^\sharp(X,R) = H_k(X,R)$ for any $k \geq 1$; moreover, if X is path-connected, then $H_0^\sharp(X,R) = 0$, if X has r path-connected components with $r \geq 2$, then $H_0^\sharp(X,R) \cong R^{r-1}$.

Definition. Let $f : X \to Y$ be a continuous map between two topological spaces. For any $k \in \mathbb{N}$, we can define

$$f_* : C_k(X,R) \longrightarrow C_k(Y,R)$$

to be the linear map such that

$$f_*(\sigma) = f \circ \sigma$$

for every σ singular k-simplex (we should write $f_{*,k}$ since there is one such map for every k, but we omit the subscript k for simplicity). □

Remark. If $f : X \to Y$ and $g : Y \to Z$ are two continuous maps, then

$$(g \circ f)_* = g_* \circ f_* .$$ □

Remark. Let $f : X \to Y$ be a continuous map between two topological spaces. The maps f_* commute with the border maps, therefore they define operators

$$\boxed{H_k(f) : H_k(X,R) \longrightarrow H_k(Y,R).}$$

These operators are more often denoted by f_*. □

Homotopy theorem. If $f : X \to Y$ and $g : X \to Y$ are homotopic continuous maps, then

$$H_k(f) = H_k(g)$$

for any k. In particular homotopic topological spaces have the same homology. □

Definition. Let X be a topological space and let $Y \subset X$. Obviously the inclusion of Y into X induces an injection from $C_k(Y,R)$ into $C_k(X,R)$. For any $k \in \mathbb{N}$, we define

$H_k(X, Y, R)$ to be the homology in the place k of the complex

$$\cdots \longrightarrow \frac{C_{k+1}(X,R)}{C_{k+1}(Y,R)} \longrightarrow \frac{C_k(X,R)}{C_k(Y,R)} \longrightarrow \frac{C_{k-1}(X,R)}{C_{k-1}(Y,R)} \longrightarrow \cdots,$$

where the maps are the maps induced by the border maps ∂_k. □

Couple exact sequence. Let X be a topological space and let $Y \subset X$. Then there is an exact sequence

$$\cdots \longrightarrow H_k(Y,R) \longrightarrow H_k(X,R) \longrightarrow H_k(X,Y,R) \longrightarrow H_{k-1}(Y,R) \longrightarrow \cdots,$$

where the first map is induced by the inclusion of Y into X and the second is induced by the projection $C_k(X,R) \to \frac{C_k(X,R)}{C_k(Y,R)}$. □

Triple exact sequence. Let X be a topological space and let $Z \subset Y \subset X$. Then there is an exact sequence

$$\cdots \longrightarrow H_k(Y,Z,R) \longrightarrow H_k(X,Z,R) \longrightarrow H_k(X,Y,R) \longrightarrow H_{k-1}(Y,Z,R) \longrightarrow \cdots,$$

where the first map is induced by the map $\frac{C_k(Y,R)}{C_k(Z,R)} \to \frac{C_k(X,R)}{C_k(Z,R)}$ given by the inclusion $C_k(Y,R) \to C_k(X,R)$ and the second map is induced by the projection $\frac{C_k(X,R)}{C_k(Z,R)} \to \frac{C_k(X,R)}{C_k(Y,R)}$. □

Excision theorem. Let X be a topological space and let $Y \subset X$. Let F be a closed subset included in the interior part of Y. Then

$$H_k(X-F, Y-F, R) \cong H_k(X,Y,R)$$

for any $k \in \mathbb{N}$ (the isomorphism is induced by the inclusion of $(X-F, Y-F)$ into (X,F)). □

Mayer–Vietoris theorem. Let X be a topological space and let X_1 and X_2 be two open subsets such that $X = X_1 \cup X_2$. Let $j_1 : X_1 \to X$, $j_2 : X_2 \to X$, $i_1 : X_1 \cap X_2 \to X_1$, $i_2 : X_1 \cap X_2 \to X_2$ be the inclusions.
Then there is an exact sequence

$$\cdots \longrightarrow H_k(X_1 \cap X_2, R) \xrightarrow{\alpha} H_k(X_1,R) \oplus H_k(X_2,R) \xrightarrow{\beta} H_k(X,R) \longrightarrow$$
$$\xrightarrow{\hat{\partial}} H_{k-1}(X_1 \cap X_2, R) \longrightarrow \cdots,$$

where
- α is the map $x \longmapsto \big(H_k(i_1)(x), H_k(i_2)(x)\big)$;
- β is the map $(y,z) \longmapsto H_k(-j_1)(y) + H_k(j_2)(z)$;

and, if a k-cycle w in X is homologous to $c_1 + c_2$ with $c_i \in C_k(X_i, R)$ for $i = 1,2$, then $\hat{\partial}([w]) := [\partial c_1]$ (we can prove that, given w, we can always find c_1 and c_2 as above). □

Attaching cells. Define

$$B^n = \{x \in \mathbb{R}^n | \, |x| \leq 1\}, \qquad S^{n-1} = \{x \in \mathbb{R}^n | \, |x| = 1\}.$$

Let X be a Hausdorff topological space. Let $\tilde{X}$ be the topological space obtained by attaching an n-cell (B^n, S^{n-1}) to X by a continuous map

$$f : S^{n-1} \longrightarrow X,$$

i.e., let $\tilde{X}$ be the quotient of $X \sqcup B^n$ by the equivalence relation determined by the identification of x with $f(x)$ for every $x \in S^{n-1}$.
If $k \neq n-1, n$, then

$$H_k^\sharp(\tilde{X}, R) = H_k^\sharp(X, R);$$

moreover,

$$H_{n-1}^\sharp(\tilde{X}, R) \cong H_{n-1}^\sharp(X, R)/Im(H_{n-1}^\sharp(f))$$

and there is an exact sequence

$$0 \longrightarrow H_n^\sharp(X, R) \longrightarrow H_n^\sharp(\tilde{X}, R) \longrightarrow Ker(H_{n-1}^\sharp(f)) \longrightarrow 0,$$

where $H_{n-1}^\sharp(f) : H_{n-1}^\sharp(S^{n-1}, R) \to H_{n-1}^\sharp(X, R)$.

Orientation

Let X be a manifold of dimension n. Here, a manifold will not be considered necessarily connected. By the Excision Theorem we can easily prove that

$$H_n(X, X - x, R) = R$$

for all $x \in X$ (where we write $H_n(X, X - x, R)$ instead of $H_n(X, X - \{x\}, R)$ for simplicity).
An ***R*-orientation system** of X is given by
(a) a family of open subsets, $\{U_\alpha\}_\alpha$, that covers X;
(b) for any α, an element $\psi_\alpha \in H_n(X, X - U_\alpha, R)$ such that, for all $x \in U_\alpha$, the element $i_x^{U_\alpha}(\psi_\alpha)$ is a generator of $H_n(X, X - x, R)$ and

$$i_x^{U_\alpha}(\psi_\alpha) = i_x^{U_\beta}(\psi_\beta)$$

for all $x \in U_\alpha \cap U_\beta$, where

$$i_x^{U_\alpha} : H_n(X, X - U_\alpha, R) \longrightarrow H_n(X, X - x, R)$$

is the map induced by the inclusion.

We say that two R-orientation systems are equivalent if they induce the same elements in $H_n(X, X - x, R)$ for all $x \in X$.
An ***R*-orientation** of X is the equivalence class of an R-orientation system.
We say that X is R-orientable if there exists an R-orientation on X.

Proposition. An open submanifold of an R-orientable manifold is R-orientable.
A manifold is R-orientable if and only if all its connected components are R-orientable.
If a manifold is connected and two R-orientations coincide in a point, then they are equal.
All manifolds are $\mathbb{Z}/2$-orientable (and the $\mathbb{Z}/2$-orientation is unique).
All connected manifolds have at most two $\mathbb{Z}$-orientations. □

Theorem. Let X be a connected manifold of dimension n. Then $H_k(X, R) = 0$ for all $k > n$; moreover, if X is not compact, $H_n(X, R) = 0$.
If X is compact and R an integral domain, then

$$H_n(X, R) = \begin{cases} R & \text{if } X \text{ is } R\text{-orientable,} \\ 0 & \text{if } X \text{ is not } R\text{-orientable,} \end{cases}$$

and, if X is R-orientable, the natural map $H_n(X, R) \to H_n(X, X - x, R)$ is an isomorphism for any $x \in X$. □

Definition. If X is a compact connected $\mathbb{Z}$-oriented manifold of dimension n, we define the **fundamental class** of X to be the element of $H_n(X, \mathbb{Z})$ inducing the orientation in $H_n(X, X - x, \mathbb{Z})$ for every $x \in X$. □

Examples.

$$H_k(\mathbb{P}^n_{\mathbb{C}}, R) = \begin{cases} R & \text{if } k \text{ is even and } 0 \le k \le 2n, \\ 0 & \text{otherwise.} \end{cases}$$

$$H_k(\mathbb{P}^n_{\mathbb{R}}, R) = \begin{cases} 0 & \text{if } k > n, \\ R & \text{if } k = 0 \text{ or } k = n \text{ if } n \text{ odd,} \\ R/2R & \text{if } 1 \le k < n \text{ and } k \text{ is odd,} \\ R_2 & \text{if } 1 < k \le n \text{ and } k \text{ is even,} \end{cases}$$

where R_2 is $\{x \in R \mid 2x = 0\}$.

Let T_g be the topological torus with g holes (i.e., the topological space obtained by attaching to a bouquet of $2g$ circles, $\lambda_1, \dots, \lambda_{2g}$, a $4g$-agon with the law $\lambda_1, \lambda_{g+1}, \lambda_1^{-1}, \lambda_{g+1}^{-1}, \dots$). We have

$$H_k(T_g, R) = \begin{cases} 0 & \text{if } k > 2, \\ R & \text{if } k = 0 \text{ or } k = 2, \\ R^{2g} & \text{if } k = 1. \end{cases}$$

Let U_h be the nonorientable compact surface obtained by attaching to a bouquet of h circles $\lambda_1, \dots, \lambda_h$ a $2h$-agon with the law $\lambda_1, \lambda_1, \lambda_2, \lambda_2, \dots$. We can see easily that U_1 is $\mathbb{P}^2_{\mathbb{R}}$ and U_2 is the Klein's bottle. We have

$$H_k(U_h, R) = \begin{cases} 0 & \text{if } k > 2, \\ R & \text{if } k = 0, \\ R_2 & \text{if } k = 2, \\ R/2R \times R^{h-1} & \text{if } k = 1. \end{cases}$$

Cohomology

Let X be a topological space. We define

$$\boxed{C^k(X,R) = Hom_R(C_k(X,R),R);}$$

its elements are called ***k*-cochains**. Let

$$\boxed{\delta_k : C^k(X,R) \longrightarrow C^{k+1}(X,R)}$$

be the cobordism operator defined by

$$(\delta_k\, t)(z) = t(\partial_{k+1}\, z)$$

for any $t \in C^k(X,R)$, $z \in C_{k+1}(X,R)$. Obviously $\delta_{k+1} \circ \delta_k = 0$, thus

$$\cdots \longrightarrow C^{k-1}(X,R) \xrightarrow{\delta_{k-1}} C^k(X,R) \xrightarrow{\delta_k} C^{k+1}(X,R) \xrightarrow{\delta_{k+1}} C^{k+2}(X,R) \longrightarrow \cdots$$

is a complex. We define the **singular cohomology module** of X of degree k with coefficients in R, which we denote by $H^k(X,R)$, to be the cohomology of such a complex in $C^k(X,R)$, i.e.,

$$\boxed{H^k(X,R) = \frac{Ker\ \delta_k}{Im\ \delta_{k-1}}.}$$

The elements of $Ker\ \ \delta_k$ are called ***k*-cocycles**, the elements of $Im\ \ \delta_{k-1}$ are called ***k*-coborders**. The rank of $H^i(X,R)$ is denoted by $h^i(X,R)$.

Definition. Let $f : X \to Y$ be a continuous map between two topological spaces. We can define

$$f^* : C^k(Y,R) \longrightarrow C^k(X,R)$$

to be the linear map such that

$$f^*(t) = t \circ f_*$$

for every $t \in C^k(Y,R)$, where f_* is the map $f_* : C_k(X,R) \to C_k(Y,R)$ we have already defined. □

Remark. If $f : X \to Y$ and $g : Y \to Z$ are two continuous maps, then

$$(g \circ f)^* = f^* \circ g^* . \qquad \square$$

Remark. If $f : X \to Y$ is a continuous map between two topological spaces, then f^* commutes with the coborder maps. Thus we can define operators

$$\boxed{H^k(f) : H^k(Y,R) \longrightarrow H^k(X,R)}\,. \qquad \square$$

Definition. Let X be a topological space and let $Y \subset X$. We define $H^k(X, Y, R)$ to be the cohomology in the place k of the complex

$$\cdots \longrightarrow \left(\frac{C_{k-1}(X,R)}{C_{k-1}(Y,R)}\right)^{\vee} \longrightarrow \left(\frac{C_{k}(X,R)}{C_{k}(Y,R)}\right)^{\vee} \longrightarrow \left(\frac{C_{k+1}(X,R)}{C_{k+1}(Y,R)}\right)^{\vee} \longrightarrow \cdots,$$

where $\cdot^{\vee}$ denotes $Hom_R(\cdot, R)$ and the maps are induced by the coborder maps δ_k. □

Homotopy theorem. If $f : X \to Y$ and $g : X \to Y$ are homotopic continuous maps, then

$$H^k(f) = H^k(g)$$

for any k. In particular, homotopic topological spaces have the same cohomology. □

Couple exact sequence. Let X be a topological space and let $Y \subset X$. There is an exact sequence

$$\cdots \longrightarrow H^k(X, Y, R) \longrightarrow H^k(X, R) \longrightarrow H^k(Y, R) \longrightarrow H^{k+1}(X, Y, R) \longrightarrow \cdots,$$

where the first map is induced by the inclusion $\left(\frac{C_k(X,R)}{C_k(Y,R)}\right)^{\vee} \to C_k(X,R)^{\vee}$ and the second map is induced by the inclusion of Y into X. □

Triple exact sequence. Let X be a topological space and let $Z \subset Y \subset X$. There is an exact sequence

$$\cdots \longrightarrow H^k(X, Y, R) \longrightarrow H^k(X, Z, R) \longrightarrow H^k(Y, Z, R) \longrightarrow H^{k-1}(X, Y, R) \longrightarrow \cdots,$$

where the first map is induced by the injection $\left(\frac{C_k(X,R)}{C_k(Y,R)}\right)^{\vee} \to \left(\frac{C_k(X,R)}{C_k(Z,R)}\right)^{\vee}$ and the second is induced by the map $\left(\frac{C_k(X,R)}{C_k(Z,R)}\right)^{\vee} \to \left(\frac{C_k(Y,R)}{C_k(Z,R)}\right)^{\vee}$ given by the restriction. □

Excision theorem. Let X be a topological space and let $Y \subset X$. If F is a closed subset included in the interior part of Y, then

$$H^k(X, Y, R) \cong H^k(X - F, Y - F, R)$$

for any $k \in \mathbb{N}$ (the isomorphism is induced by the inclusion of $(X - F, Y - F)$ into (X, F)). □

Mayer–Vietoris theorem. Let X be a topological space and let X_1 and X_2 be two open subsets such that $X = X_1 \cup X_2$. Let $j_1 : X_1 \to X$, $j_2 : X_2 \to X$, $i_1 : X_1 \cap X_2 \to X_1$, $i_2 : X_1 \cap X_2 \to X_2$ be the inclusions.
Then there is an exact sequence

$$\begin{aligned}\cdots \longrightarrow H^k(X, R) \xrightarrow{\alpha} H^k(X_1, R) \oplus H^k(X_2, R) \xrightarrow{\beta} H^k(X_1 \cap X_2, R) \longrightarrow \\ \longrightarrow H^{k+1}(X, R) \longrightarrow \cdots,\end{aligned}$$

where
- α is the map $x \longmapsto \big(H^k(j_1)(x), H^k(j_2)(x)\big)$;
- β is the map $(y,z) \longmapsto H^k(-i_1)(y) + H^k(i_2)(z)$. □

Theorem $*$**.** Let X be a topological space and let $Y \subset X$. If R is a PID, i.e., an integral domain such that every ideal is principal, and $H_{k-1}(X,Y,R)$ is free (for instance if R is a field), then there is a canonical isomorphism

$$H_k^{\vee}(X,Y,R) \cong H^k(X,Y,R)$$

for every k.
If R is a PID and $H_{k-1}(X,Y,R)$ and $H_k(X,Y,R)$ are finitely generated, then

$$H^k(X,R) \cong F_k \oplus T_{k-1},$$

for every k, where F_k is the free part of $H_k(X,R)$ and T_{k-1} is the torsion part of $H_{k-1}(X,R)$. □

For any $p \in \mathbb{N}$, let $\lambda_p : \langle e_0,\dots,e_p\rangle \longrightarrow \langle e_0,\dots,e_{p+q}\rangle$ be the affine inclusion sending e_i to e_i for $i = 0,\dots,p$ and, for any $q \in \mathbb{N}$, let $\rho_q : \langle e_0,\dots,e_q\rangle \longrightarrow \langle e_0,\dots,e_{p+q}\rangle$ be the affine inclusion sending e_i to e_{i+p} for $i = 0,\dots,q$.

Definition. Let X be a topological space. For every p, q, we define the **cup product** to be the map

$$C^p(X,R) \times C^q(X,R) \longrightarrow C^{p+q}(X,R),$$
$$(c,d) \mapsto c \cup d,$$

where $c \cup d$ is the element of $C^{p+q}(X,R)$ such that, for any σ $(p+q)$-simplex in X,

$$(c \cup d)(\sigma) = c(\sigma \circ \lambda_p)\, d(\sigma \circ \rho_q)..$$ □

The cup product in $C^*(X,R)$ is bilinear, associative and has the identity element (the 0-cochain sending every point P in X to 1).

Proposition. Let X be a topological space. For any $a \in C^p(X,R)$, $b \in C^q(X,R)$, we have

$$a \cup b = (-1)^{pq}\, b \cup a\,.$$ □

Proposition. If $f : X \to Y$ is a continuous map, then

$$f^*(c \cup d) = f^*(c) \cup f^*(d)$$

for any $c \in C^p(Y,R), d \in C^q(Y,R)$. □

Remark. Let X be a topological space. Then

$$\delta(c \cup d) = \delta c \cup d + (-1)^p c \cup \delta d$$

for all $c \in C^p(X,R)$, $d \in C^q(X,R)$. Therefore, the cup product induces a product, called again cup product,

$$\boxed{H^p(X,R) \times H^q(X,R) \longrightarrow H^{p+q}(X,R)}\,. \qquad \square$$

Definition. Let X be a topological space. We define the **cap product** to be the map

$$C^p(X,R) \times C_{p+q}(X,R) \longrightarrow C_q(X,R),$$
$$(c,\sigma) \longmapsto c \cap \sigma,$$

where

$$c \cap \sigma := c(\sigma \circ \lambda_p)\sigma \circ \rho_q$$

for any σ $(p+q)$-simplex in X. $\square$

Remark.

(1) For any $c \in C^p(X,R)$, $d \in C^q(X,R)$, $\sigma \in C_{p+q}(X,R)$,

$$d(c \cap \sigma) = (c \cup d)(\sigma).$$

(2) For any $c \in C^p(X,R)$, $\sigma \in C_{p+q}(X,R)$,

$$\partial(c \cap \sigma) = (-1)^p[(c \cap \partial\sigma) - (\delta c \cap \sigma)].$$

Thus the cap product induces a product, called again cap product,

$$\boxed{H^p(X,R) \times H_{p+q}(X,R) \longrightarrow H_q(X,R)}\,. \qquad \square$$

Projection formula. If $f : X \to Y$ is a continuous map, then

$$H_q(f)[H^p(f)(b) \cap a] = b \cap H_{p+q}(f)(a)$$

for any $a \in H_{p+q}(X,R)$, $b \in H^p(Y,R)$. $\square$

In the relative case, for X topological space and Y subset of X, we have maps

$$H^p(X,Y,R) \times H^q(X,Y,R) \xrightarrow{\cup} H^{p+q}(X,Y,R),$$
$$H^p(X,Y,R) \times H_{p+q}(X,Y,R) \xrightarrow{\cap} H_q(X,R),$$
$$H^p(X,R) \times H_{p+q}(X,Y,R) \xrightarrow{\cap} H_q(X,Y,R).$$

The following theorem describes the homology and the cohomology of the product of two topological spaces.

Künneth formula. Let R be a PID. Let X and Y be two topological spaces. There is an exact sequence (see "Tor, $\mathcal{TOR}$" for the definition of Tor_1^R)

$$0 \longrightarrow \oplus_{p+q=k} H_p(X,R) \otimes_R H_q(Y,R) \longrightarrow H_k(X \times Y, R) \longrightarrow$$
$$\longrightarrow \oplus_{p+q=k-1} Tor_1^R(H_p(X,R), H_q(Y,R)) \longrightarrow 0$$

and, if all the homology modules are finitely generated, an exact sequence

$$0 \longrightarrow \oplus_{p+q=k} H^p(X,R) \otimes_R H^q(Y,R) \longrightarrow H^k(X \times Y, R) \longrightarrow$$
$$\longrightarrow \oplus_{p+q=k+1} Tor_1^R(H^p(X,R), H^q(Y,R)) \longrightarrow 0\,.$$

Universal coefficient theorem. For any X topological space (and R commutative ring with unity), there is an exact sequence

$$0 \longrightarrow H_k(X,\mathbb{Z}) \otimes_{\mathbb{Z}} R \longrightarrow H_k(X,R) \longrightarrow Tor_1^{\mathbb{Z}}(H_{k-1}(X,\mathbb{Z}),R) \longrightarrow 0\,.$$ □

Let X be a manifold. For every $k \in \mathbb{N}$, we define the **cohomology with compact supports** $H^k_C(X,R)$ in the following way:

$$H^k_C(X,R) = \varinjlim_{P \subset X,\ P \text{ compact}} H^k(X, X-P, R)$$

(see "Limits, direct and inverse -"). If X is compact, then $H^k_C(X,R) = H^k(X,R)$.
If $f: X \to Y$ is a proper continuous map (see "Proper"), then we have an induced map

$$f^*: H^k_C(Y,R) \longrightarrow H^k_C(X,R).$$

If X is an R-oriented manifold of dimension n, then, for every P compact subset, we have a map

$$H^k(X, X-P, R) \longrightarrow H_{n-k}(X,R)$$

given by the cap product with the element of $H_n(X, X-P, R)$ induced by the orientation and these maps induce a map

$$\boxed{PD: H^k_C(X,R) \longrightarrow H_{n-k}(X,R),}$$

called **Poincaré duality**.

Theorem. Let X be an R-oriented manifold of dimension n. Then the Poincaré duality

$$PD: H^k_C(X,R) \longrightarrow H_{n-k}(X,R)$$

is an isomorphism for any k. □

Corollary. Let X be an R-orientable connected manifold of dimension n, then

$$H^n_C(X,R) \cong R.$$

If X is a compact R-orientable manifold of dimension n, then

$$h^k(X,R) = h_{n-k}(X,R)$$

for any k. □

By the de Rham's theorem (see "De Rham's theorem")

$$H^k(X,\mathbb{R}) = H^k_{DR}(X,\mathbb{R})$$

for any C^∞ manifold X and the isomorphism $H^k_{DR}(X,\mathbb{R}) \cong (H_k(X,\mathbb{R}))^\vee$ (given by the De Rham's theorem and Theorem $*$) is induced by the integration.
Moreover, the cup product in $H^*(X,\mathbb{R})$ corresponds to the wedge product in $H^*_{DR}(X,\mathbb{R})$ and to the intersection of cycles in $H_*(X,\mathbb{R})$ through the Poincaré duality, i.e.,

$$PD(PD^{-1}(c_1) \cup PD^{-1}(c_2)) = c_1 \cdot c_2$$

for any $c_1, c_2 \in H_*(X,\mathbb{R})$ (see "Intersection of cycles" for the definition of $c_1 \cdot c_2$).

Let X be an R-oriented manifold. Let A be a closed subset in X. We define the **Alexander cohomology** of A, which we denote by $\overline{H}^q(A,R)$, in the following way:

$$\overline{H}^q(A,R) = \varinjlim_{V \text{ open subset } \supset A} H^q(V,R).$$

Theorem. If X is an R-oriented compact manifold of dimension n, then there is an isomorphism, called **Alexander duality**,

$$\boxed{\overline{H}^q(A,R) \cong H_{n-q}(X, X - A, R)}.$$ □

Corollary.
(i) Let A be a compact submanifold in $\mathbb{R}^n$. Then, for any k,

$$H^k(A,R) \cong H^\sharp_{n-k-1}(\mathbb{R}^n - A, R).$$

(ii) If A is a compact submanifold in $\mathbb{R}^n$ with dimension $n-1$ and r connected components, then $\mathbb{R}^n - A$ has $r+1$ connected components.
(iii) If A is a nonorientable compact manifold of dimension $n-1$, then it cannot be embedded in $\mathbb{R}^n$. □

Theorem. Let X be a compact manifold with boundary. Denote the boundary of X by ∂X. Suppose the interior part of X is R-oriented. Then, for any k, there are isomorphisms (called **Lefschetz dualities**):

$$\boxed{H^k(X,\partial X,R) \cong H_{n-k}(X,R),}$$
$$\boxed{H^k(X,R) \cong H_{n-k}(X,\partial X,R)}.$$ □

Lefschetz fixed points theorem. Let K be a field and let X be a compact K-oriented manifold. Let $f: X \to X$ be a continuous map. We have

$$\Delta \cdot \Gamma_f = \sum_{q\in\mathbb{N}} (-1)^q tr\, H^q(f),$$

where Δ is the diagonal in $X \times X$, Γ_f is the graph of f, $\Delta \cdot \Gamma_f$ is the intersection number (see "Intersection of cycles") and tr is the trace. □

Singularities. See "Regular rings, smooth points, singular points".

Smooth. See "Regular rings, smooth points, singular points".

Snake lemma. ([12], [79], [116], [164]). Let

$$\begin{tikzcd} & A_1 \arrow[r, "f_1"] \arrow[d, "r_1"] & A_2 \arrow[r, "f_2"] \arrow[d, "r_2"] & A_3 \arrow[r] \arrow[d, "r_3"] & 0 \\ 0 \arrow[r] & B_1 \arrow[r, "g_1"] & B_2 \arrow[r, "g_2"] & B_3 & \end{tikzcd}$$

be a commutative diagram of Abelian groups (or of R-modules for some ring R). If the rows are exact, then there is an exact sequence

$$Ker(r_1) \longrightarrow Ker(r_2) \longrightarrow Ker(r_3) \longrightarrow Coker(r_1) \longrightarrow Coker(r_2) \longrightarrow Coker(r_3).$$

Moreover, if f_1 is injective, then the map $Ker(r_1) \to Ker(r_2)$ is injective; if g_2 is surjective, then the map $Coker(r_2) \to Coker(r_3)$ is surjective.

Spaces, analytic -. ([103], [107]). Let D_n be the polydisc

$$\{z \in \mathbb{C}^n | \ |z_i| < 1 \ \forall i \in \{1, \dots, n\}\}.$$

Definition.

– An analytic space is a locally ringed space $(X, \mathcal{O}_X)$ (see "Spaces, ringed -") such that, for any $x \in X$, there exist an open neighborhood U of x, natural numbers k, n and holomorphic functions $f_1, \dots, f_k$ on D_n such that

$$(U, \mathcal{O}_X|_U)$$

is isomorphic, as locally ringed space, to

$$(D_n, \mathcal{O}/(f_1, \dots, f_k)),$$

where $\mathcal{O}$ is the sheaf of the holomorphic functions on D_n and $(f_1, \dots, f_k)$ is the ideal sheaf generated by $f_1, \dots, f_k$ (see "Sheaves").

– Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be analytic spaces. A map from X to Y is said to be a morphism of analytic spaces if it is a morphism of locally ringed spaces. □

Spaces, ringed -. ([103], [107]). A **ringed space** $(X, \mathcal{O}_X)$ is a topological space X with a sheaf of rings $\mathcal{O}_X$ (see "Sheaves"). The sheaf $\mathcal{O}_X$ is called "structure sheaf".
We say that a ringed space $(X, \mathcal{O}_X)$ is **reduced** if, for any open subset U of X, $\mathcal{O}_X(U)$ has no nilpotent elements. We say that a ringed space is a **locally ringed space** if the stalk of the structure sheaf in every point is a local ring.

A **morphism of ringed spaces** from a ringed space $(X, \mathcal{O}_X)$ to a ringed space $(Y, \mathcal{O}_Y)$ is given by a continuous map $f : X \to Y$ and a morphism of sheaves $\mathcal{O}_Y \to f_*\mathcal{O}_X$ (see "Direct and inverse image sheaves" for the definition of $f_*\mathcal{O}_X$). A **morphism of locally ringed spaces** is a morphism of ringed spaces such that the maps induced on the stalks are local homomorphisms of local rings (i.e., homomorphisms such that the inverse image of the maximal ideal of the second ring is the maximal ideal of the first ring).

Spectral sequences. ([32], [41], [62], [84], [93], [116], [180]). A spectral sequence is a sequence $\{E_r, d_r\}_{r\in\mathbb{N}}$ of bigraded groups

$$E_r = \oplus_{p,q\in\mathbb{N}} E_r^{p,q}$$

together with differentials (see Figure 17)

$$d_r : E_r^{p,q} \longrightarrow E_r^{p+r,q-r+1}$$

such that $d_r^2 = 0$ and

$$H^*(E_r) = E_{r+1},$$

i.e.,

$$\frac{Ker(d_r : E_r^{p,q} \longrightarrow E_r^{p+r,q-r+1})}{Im(d_r : E_r^{p-r,q+r-1} \longrightarrow E_r^{p,q})} = E_{r+1}^{p,q}$$

for any $p, q, r \in \mathbb{N}$.

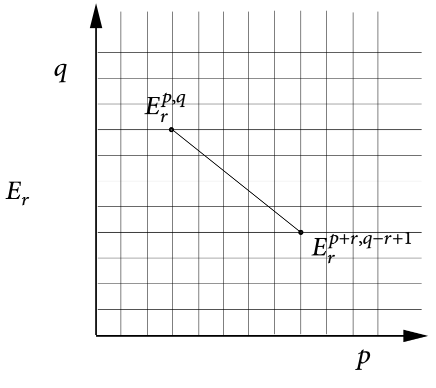


Fig. 17. Spectral sequence.

If $E_r = E_{r+n}$ for all $n \in \mathbb{N}$, we denote E_r by E_∞; in this case we say that the spectral sequence converges to E_∞ and we write $E_r \Rightarrow E_\infty$.

Let H be a group. Given a filtration in subgroups on H

$$H = F^0H \supset F^1H \supset \cdots \supset F^kH \supset F^{k+1}H = 0,$$

we set, for any p,

$$Gr^pH := F^pH/F^{p+1}H.$$

A filtered complex

$$C^* = F^0C^* \supset F^1C^* \supset \cdots \supset F^kC^* \supset F^{k+1}C^* = 0$$

is defined to be a finite sequence of complexes F^pC^* such that F^pC^* is a subcomplex of $F^{p+1}C^*$ for any p.

Theorem. Given a **filtered complex** $C^* = F^0C^* \supset F^1C^* \supset \cdots \supset F^kC^* \supset F^{k+1}C^* = 0$, there is a spectral sequence E_r, called **associated spectral sequence**, such that

$$E_0^{p,q} = Gr^pC^{p+q},$$
$$E_1^{p,q} = H^{p+q}(Gr^pC^*),$$
$$E_\infty^{p,q} = Gr^p(H^{p+q}(C^*)).$$

Precisely, E_r is the spectral sequence defined in the following way: for any p, q, r,

$$E_r^{p,q} := \frac{\{x \in F^pC^{p+q} |\ dx \in F^{p+r}C^{p+q+1}\}}{d(F^{p-r+1}C^{p+q-1}) + F^{p+1}C^{p+q}}. \qquad \square$$

Given a **bigraded complex** $(C^{*,*}, d, d')$, i.e., a bigraded group

$$C = \oplus_{p,q\in\mathbb{N}}C^{p,q}$$

with differentials

$$d : C^{p,q} \longrightarrow C^{p+1,q}, \quad d' : C^{p,q} \longrightarrow C^{p,q+1}$$

such that $d^2 = 0$, $d'^2 = 0$ and $dd' + d'd = 0$, we can define the **associated single complex** C^* in the following way (see Figure 18):

$$C^n = \oplus_{p+q=n}C^{p,q}$$

with the differential

$$D = d + d' : C^n \longrightarrow C^{n+1}.$$

It has two obvious filtrations:

$$F^pC^n = \oplus_{s\geq p, s+q=n}C^{s,q}, \qquad F'^qC^n = \oplus_{s\geq q, p+s=n}C^{p,s}.$$

By the theorem above, there are two spectral sequences E_r and E'_r associated to C^* endowed respectively with the two filtrations F and F'. We have:

$$E_2^{p,q} \cong H_d^p(H_{d'}^q(C^{*,*})), \qquad E_2'^{p,q} \cong H_{d'}^q(H_d^p(C^{*,*})).$$

See also “Hypercohomology of a complex of sheaves”, “Leray spectral sequence”.

Spin groups.

([47], [76], [167]). Let $SO(n)$ be the group of the orthogonal real $n \times n$ matrices with determinant 1. Its fundamental group is

$$\pi_1(SO(n)) = \begin{cases} \mathbb{Z} & \text{if} \quad n = 2, \\ \mathbb{Z}/2 & \text{if} \quad n \geq 3. \end{cases}$$

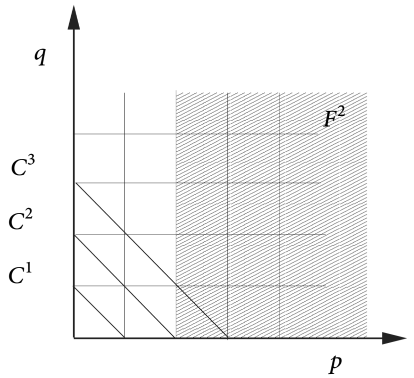


Fig. 18. Bigraded complex.

Thus there is a double covering space of $SO(n)$, which is universal if $n \geq 3$ (see "Covering projections"); it is denoted by $Spin_{\mathbb{R}}(n)$.
To describe it and to define a group structure on it, we must define Clifford algebras.

Definition. Let V be a vector space over $\mathbb{R}$ or over $\mathbb{C}$ and endow $\oplus_{n\in\mathbb{N}} V^{\otimes n}$ with the bilinear product defined by

$$(v_1 \otimes \cdots \otimes v_r) \cdot (v_1' \otimes \cdots \otimes v_s') = v_1 \otimes \cdots \otimes v_r \otimes v_1' \otimes \cdots \otimes v_s'.$$

We will omit the symbol $\cdot$ in the sequel. With such a product, $\oplus_{n\in\mathbb{N}} V^{\otimes n}$ is a ring. Let b be a non degenerate bilinear symmetric form on V. Let

$$Cliff\,(V, b) = \oplus_{n\geq 0} V^{\otimes n}/I,$$

where I is the two-sides ideal generated by the elements of the form $v \otimes v - b(v, v)$. □

For every $x = [v_1 \otimes \cdots \otimes v_r] \in Cliff\,(V, b)$ ($v_i \in V$), we denote by x^* the element

$$(-1)^r [v_r \otimes \cdots \otimes v_1];$$

we extend the definition of x^* to every element $x \in Cliff\,(V, b)$ by linearity.

Definition. Let

$$Spin(V, b) = \{x \in Cliff\,(V, b)^{\text{even}} \mid xx^* = 1 \quad \text{and} \quad xVx^* \subset V\},$$

where $Cliff(V, b)^{\text{even}} = \oplus_{n\geq 0,\, n \text{ even}} V^{\otimes n}/I$. We endow $Spin(V, b)$ with the product induced by the product $\cdot$. □

Now take $V = \mathbb{R}^n$ and b the opposite of the standard positive definite bilinear form on $\mathbb{R}^n$. In this case $Spin(V, b)$ is denoted $Spin_{\mathbb{R}}(n)$ and we can prove that:

– if we define, for any $x \in Spin(V, b)$, a map $f(x) : V \to V$ to be the map

$$v \longmapsto xvx^*,$$

  we have that $f(x)$ is an element of $SO(V, b)$;

– the map

$$f : Spin(V, b) \longrightarrow SO(V, b),$$
$$x \longmapsto f(x)$$

is a two-sheeted covering projection and a group homomorphism.

An analogous statement holds for V complex vector space and b non degenerate bilinear symmetric form on V.

Splitting type of a vector bundle. See "Jumping lines and splitting type of a vector bundle on $\mathbb{P}^n$".

Stable sheaves. ([83], [146], [182], [199], [207], [239]). Let (M, g) be a compact Kähler manifold of dimension n and let ω be the Kähler form. Let $\mathcal{F}$ be a coherent sheaf on M (see "Hermitian and Kählerian metrics", "Sheaves" and "Coherent sheaves"). The determinant bundle of $\mathcal{F}$, denoted by $det\ \mathcal{F}$, is defined in the following way: let

$$0 \to \mathcal{E}_n \to \cdots\cdots \to \mathcal{E}_1 \to \mathcal{E}_0 \to \mathcal{F}|_U \to 0$$

be a resolution of $\mathcal{F}|_U$ with $\mathcal{E}_i$ locally free coherent sheaves and U small open subset of M; let E_i be the bundle corresponding to $\mathcal{E}_i$. We define

$$det\ \mathcal{F}|_U = \otimes_{i=0,\dots,n}(det\ E_i)^{(-1)^i}.$$

If $\mathcal{F}$ is also torsion-free and its rank is r, then we can give the following equivalent definition: $det\ \mathcal{F} := (\wedge^r \mathcal{F})^{\vee\vee}$.
We define the first Chern class of $\mathcal{F}$ by

$$c_1(\mathcal{F}) = c_1(det\ \mathcal{F})$$

(see "Chern classes"). The ω-degree of $\mathcal{F}$ is

$$\deg(\mathcal{F}) = \int_M c_1(\mathcal{F}) \wedge \omega^{n-1}.$$

The degree/rank ratio of $\mathcal{F}$ is

$$\mu(\mathcal{F}) = \deg(\mathcal{F})/rank(\mathcal{F}).$$

According to Takemoto's definition, we say that $\mathcal{F}$ is ω-semistable if, for every coherent subsheaf $\mathcal{F}'$ with $0 < rank(\mathcal{F}')$, we have

$$\mu(\mathcal{F}') \leq \mu(\mathcal{F}).$$

If we have also that, for every coherent subsheaf $\mathcal{F}'$ with $0 < rank(\mathcal{F}') < rank(\mathcal{F})$,

$$\mu(\mathcal{F}') < \mu(\mathcal{F}),$$

we say that $\mathcal{F}$ is ω-stable.
We say that a holomorphic vector bundle E on M is ω-(semi)stable if and only if the sheaf $\mathcal{O}(E)$ of the sections of E is ω-(semi)stable.

There are other definitions of stability. Gieseker's one is the following. Let M be a smooth projective algebraic variety over $\mathbb{C}$ and let H be an ample line bundle on M. Let $\mathcal{F}$ be a torsion-free coherent sheaf on M. Let

$$p(\mathcal{F}(k)) = \frac{\sum_i (-1)^i h^i(M, \mathcal{F} \otimes \mathcal{O}(H^k))}{rank(\mathcal{F})}.$$

We say that $\mathcal{F}$ is Gieseker H-stable (respectively Gieseker H-semistable) if, for every coherent subsheaf $\mathcal{F}'$ of $\mathcal{F}$ with $0 < rank(\mathcal{F}') < rank(\mathcal{F})$, there exist $k_0 \in \mathbb{N}$ such that

$$p(\mathcal{F}'(k)) < p(\mathcal{F}(k)), \qquad \text{(respectively} \quad p(\mathcal{F}'(k)) \leq p(\mathcal{F}(k)))$$

for any $k \geq k_0$.
We can prove that Takemoto ω-stability implies Gieseker H-stability, which implies Gieseker H-semistability, which implies Takemoto ω-semistability, where $\omega \in c_1(H)$.

Any (Takemoto or Gieseker) stable bundle is simple; see "Simple bundles".

By using geometric invariant theory (see "Geometric invariant theory (G.I.T.)"), Gieseker and Maruyama constructed moduli spaces of Gieseker stable sheaves.

Star operator. ([44], [90], [93], [135], [250]). Let (M, g) be a riemannian manifold of dimension n and let $A^p(M)$ denote the set of the C^∞ p-forms on M. We define the star operator

$$* : A^p(M) \longrightarrow A^{n-p}(M)$$

in the following way: let $dx_1, \dots, dx_n$ be an orthonormal frame on an open subset, i.e., for every point P of the open subset, let it give a basis of $T_P^\vee(M)$ (the dual of the tangent space at P) orthonormal for g; locally $*$ is defined by

$$* \left(f(x) dx_{i_1} \wedge \cdots \wedge dx_{i_p} \right) = f(x) dx_{j_1} \wedge \cdots \wedge dx_{j_{n-p}},$$

where $dx_{i_1} \wedge \cdots \wedge dx_{i_p} \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_{n-p}} = dx_1 \wedge \cdots \wedge dx_n$; we extend this definition to any p-form by linearity.
We have that

$$** = (-1)^{p(n-p)}.$$

If (M, h) is a Hermitian complex manifold of dimension n (see "Hermitian and Kählerian metrics"), the star operator is defined on complex forms by extending the star operator defined above, by $\overline{\mathbb{C}}$-linearity in some books, by $\mathbb{C}$-linearity in other books.

Stein factorization. ([107], [241]).

Theorem. ("algebraic version"; see [107]) Let $f : X \to Y$ be a projective morphism of Noetherian schemes (see "Schemes"). Then there exists a scheme Z, a finite morphism $g : Z \to Y$ and a projective morphism with connected fibres $h : X \to Z$ such that

$$f = g \circ h. \qquad \square$$

Theorem. ("analytic version"); (see [241]). Let $f : X \to Y$ be a proper surjective morphism of reduced complex analytic spaces (see "Spaces, analytic -"). Then there exists a reduced complex analytic space Z, a surjective morphism $g : Z \to Y$ such that the fibres consist of a finite number of points and a surjective morphism with connected fibres $h : X \to Z$ such that

$$f = g \circ h. \qquad \square$$

Subcanonical.

We say that a smooth subvariety Z of a smooth algebraic variety X is subcanonical if there exists a line bundle L on X such that $L|_Z = K_Z$, where K_Z is the canonical bundle of Z (see "Canonical bundle, canonical sheaf").

Surfaces, algebraic -.

([18], [22], [25], [65], [93], [107], [253]). In the sequel the word surface will denote a smooth projective algebraic variety of dimension 2 over $\mathbb{C}$. We will denote the projective space $\mathbb{P}^n_{\mathbb{C}}$ by $\mathbb{P}^n$.

We start by stating some basic theorems and notation.
Let S be a surface; then $H_4(S, \mathbb{Z})$ and $H^4(S, \mathbb{Z})$ are isomorphic to $\mathbb{Z}$, since S is oriented (as it is a complex manifold), compact and of real dimension 4 (see "Singular homology and cohomology"). We denote by $[S]$ the fundamental class of S, that is, the generator of $H_4(S, \mathbb{Z})$ giving the orientation of S. If L and L' are two holomorphic line bundles on S, we denote by $L \cdot L'$ the number obtained by evaluating $c_1(L) \cup c_1(L') \in H^4(S, \mathbb{Z})$ in $[S]$ (where $\cup$ is the cup product; see again "Singular homology and cohomology"). For any divisors D and D' on S, let $D \cdot D'$ their intersection number (see "Intersection of cycles").
Observe that, for any holomorphic line bundle L on S, we have that $H^i(\mathcal{O}(L)) = 0$ for $i \geq 3$; to prove this, we can apply the Abstract de Rham's theorem (see "Sheaves") to the exact sequence

$$0 \longrightarrow \mathcal{O}(L) \longrightarrow C^\infty(L) \xrightarrow{\bar\partial} A^{0,1}(L) \xrightarrow{\bar\partial} A^{0,2}(L) \longrightarrow 0,$$

where $C^\infty(L)$ is the sheaf of the C^∞ sections of L, $A^{p,q}(L)$ is the sheaf of the C^∞ (p, q)-forms with values in L and the map $\mathcal{O}(L) \longrightarrow C^\infty(L)$ is given by inclusion.
So, if we denote by $\chi(\mathcal{O}(L))$ the Euler characteristic of $\mathcal{O}(L)$, i.e., the number $\sum_{i=0,\dots,\infty}(-1)^i h^i(\mathcal{O}(L))$, we have that, in this case (i.e., in the case of surfaces),

$$\chi(\mathcal{O}(L)) = h^0(\mathcal{O}(L)) - h^1(\mathcal{O}(L)) + h^2(\mathcal{O}(L)),$$

and so $\chi(\mathcal{O}_S) = h^0(\mathcal{O}_S) - h^1(\mathcal{O}_S) + h^2(\mathcal{O}_S)$. Furthermore, we recall that $\chi(S)$ denotes the Euler-Poincaré characteristic of S, i.e., $\sum_{i=0,\dots,\infty}(-1)^i b_i(S)$, where $b_i(S)$ is the Betti number $dim\, H_i(S,\mathbb{R})$; in this case,

$$\chi(S) = b_0(S) - b_1(S) + b_2(S) - b_3(S) + b_4(S) = 2b_0(S) - 2b_1(S) + b_2(S)$$

by Poincaré duality (see "Singular homology and cohomology").

Index theorem. Let S be a surface. The intersection form is negative definite on a subspace of codimension 1 in $H^{1,1}(S,\mathbb{C})$, precisely is negative definite on the subspace of the primitive forms (see "Lefschetz decomposition and Hard Lefschetz theorem" for the definition of primitive). □

Riemann–Roch theorem for surfaces. Let L be a holomorphic line bundle on a surface S. We have:

$$\chi(\mathcal{O}(L)) = \chi(\mathcal{O}_S) + \frac{L\cdot L - L\cdot K_S}{2},$$

where K_S is the canonical bundle of S (see "Canonical bundle, canonical sheaf"). □

Noether's theorem. Let S be a surface. We denote by $c_i(S)$ the i-th Chern class of the holomorphic tangent bundle $T^{1,0}S$ and we denote by $c_1(S)^2$ the cup product $c_1(S)\cup c_1(S)$. We have

$$\chi(\mathcal{O}_S) = \frac{1}{12}\left(K_S\cdot K_S + \chi(S)\right)$$

or, in other words,

$$\chi(\mathcal{O}_S) = \frac{1}{12}\left(c_1(S)^2 + c_2(S)\right),$$

where by $c_1(S)^2$ and $c_2(S)$, we mean them evaluated in $[S]$. □

The two formulas are equivalent because $c_1(K_S) = c_1(T^{1,0}S)$ and, by the Gauss–Bonnet–Hopf theorem (see "Gauss–Bonnet–Hopf theorem") $c_2(T^{1,0}S) = \chi(S)$.

Castelnuovo–Enriques criterion. Let S be a surface and C a smooth rational curve in S with $C\cdot C = -1$. Then there exists a surface S' and a map $p : S \to S'$ such that p is the blow-up of S' in a point P and $p^{-1}(P) = C$. □

Structure of birational maps on surfaces. Let S_1 and S_2 be two surfaces and $f : S_1 \to S_2$ be a birational map. Then there exists another surface $\tilde{S}$ and two morphisms, g_1 and g_2, both given by a sequence of blowing-up maps, such that the following diagram commutes:

$$\begin{tikzcd} & \tilde{S} \arrow[dl, "g_1"'] \arrow[dr, "g_2"] & \\ S_1 \arrow[rr, "f"] & & S_2 \end{tikzcd}.$$

In other words, a birational map between surfaces is given by a sequence of blowing-ups followed by a sequence of blowing-downs. □

Definition. We say that a surface S is **minimal** if there does not exist another surface $\tilde{S}$ and a blowing-up $S \to \tilde{S}$. A minimal model for a surface S is a minimal surface birational to S. □

We recall that if M is a compact complex manifold of dimension n, we define
- $q(M) = h^{1,0}(M)$ (irregularity);
- $P_r(M) = h^0(\mathcal{O}(K_M^r))$ for any $r \in \mathbb{N}$ (plurigenera);
- $p_g(M) = h^{n,0}(M) = P_1(M)$ (geometric genus).

One can prove that they are birational invariants. With the notation above, we have

$$\begin{aligned}\chi(\mathcal{O}_S) &= h^0(\mathcal{O}_S) - h^1(\mathcal{O}_S) + h^2(\mathcal{O}_S)\\ &= h^{0,0}(S) - h^{0,1}(S) + h^{0,2}(S) = 1 - q(S) + p_g(S)\end{aligned}$$

by Dolbeault's theorem and the Hodge theorem (see "Hodge theory").

The most important tool to classify surfaces is Kodaira dimension K (see "Kodaira dimension (or Kodaira number)"). For surfaces, it can be obviously only $-\infty, 0, 1, 2$.

First, we will deal with rational and ruled surfaces.

Definition. We say that a surface is **rational** if it is birational to $\mathbb{P}^2$. □

Definition. We say that a surface is **ruled** over a compact Riemann surface C if it is birational to $C \times \mathbb{P}^1$. □

Remark. Let S be a surface. Then S is ruled on $\mathbb{P}^1 \Leftrightarrow S$ is rational (in fact it is birational to $\mathbb{P}^1 \times \mathbb{P}^1$ if and only if it is birational to $\mathbb{P}^2$). □

Definition. We say that a surface S is **geometrically ruled** over a compact Riemann surface C if there exists a smooth morphism $S \to C$ such that the fibres are isomorphic to $\mathbb{P}^1$. □

(Please note that in some works the term "ruled" means "geometrically ruled".)

Noether–Enriques theorem. A geometrically ruled surface is equal to the projectivized $\mathbb{P}(E)$ of a vector bundle E of rank 2. □

Proposition 1. Let S be a ruled surface on a compact Riemann surface C. Let $g(C)$ be the genus of C. Then

$$q(S) = g(C), \qquad p_g(S) = 0, \qquad P_n(S) = 0 \quad \forall n \geq 2.$$

Moreover, if S is geometrically ruled, then

$$K_S^2 = 8(-g(C) + 1), \qquad b_2(S) = 2\,.$$ □

Definition. Let $n \in \mathbb{N}$. The n-th **Hirzebruch surface** is

$$S_n = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(n) \oplus \mathcal{O}_{\mathbb{P}^1}),$$

where $\mathcal{O}_{\mathbb{P}^1}$ is the trivial bundle on $\mathbb{P}^1$ and $\mathcal{O}_{\mathbb{P}^1}(n)$ is the n-th power of the hyperplane bundle on $\mathbb{P}^1$; see "Hyperplane bundles, twisting sheaves" (we are making a slight abuse of the notation: $\mathcal{O}_{\mathbb{P}^1}(n)$ generally denotes the sheaf of the holomorphic sections of the n-th power of the hyperplane bundle, but sometimes, as here, it denotes just the n-th power of the hyperplane bundle). □

Remark. The Hirzebruch surfaces are exactly the surfaces that are geometrically ruled on $\mathbb{P}^1$. In fact, by the Grothendieck–Segre theorem (see "Grothendieck–Segre theorem") and by the Noether–Enriques theorem, we have that a surface that is geometrically ruled on $\mathbb{P}^1$ is equal to $\mathbb{P}(L_1 \oplus L_2) = \mathbb{P}((L_1 \otimes L_2^{-1}) \oplus \mathcal{O}_{\mathbb{P}^1})$ for some L_1 and L_2 holomorphic line bundles on $\mathbb{P}^1$ and $\mathbb{P}((L_1 \otimes L_2^{-1}) \oplus \mathcal{O}_{\mathbb{P}^1}) = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(n) \oplus \mathcal{O}_{\mathbb{P}^1})$ for some n. Obviously we can suppose $n \in \mathbb{N}$. □

We can prove that for all $n \neq 0$ the surface S_n is the unique $\mathbb{P}^1$-bundle on $\mathbb{P}^1$ with an irreducible curve E with $E \cdot E = -n$ and that the blow-up of $\mathbb{P}^2$ in a point is S_1. We can also prove that S_{n-1} can be obtained from S_n by a blowing-up followed by a blowing-down.

Let F be a fibre of $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(n) \oplus \mathcal{O}_{\mathbb{P}^1})$ and Z be the image of the section given by the zero section of $\mathcal{O}_{\mathbb{P}^1}(n)$ and by the section 1 of $\mathcal{O}_{\mathbb{P}^1}$. Let $S_{k,n}$ be the image of S_n by the map $\varphi_{(Z+kF)}$ associated to the linear system $|Z + kF|$ (see "Bundles fibre -"). We can prove that it is a surface in $\mathbb{P}^{n+2k+1}$ of degree $n+2k$, which is the minimal achievable degree for a surface in $\mathbb{P}^{n+2k+1}$ (see "Minimal degree"). The surfaces $S_{k,n}$ are the rational normal scrolls of dimension 2 (see "Scrolls, rational normal -").

Proposition 2. Every nondegenerate surface of minimal degree in $\mathbb{P}^d$, i.e., every nondegenerate surface of degree $d-1$ in $\mathbb{P}^d$, is either a rational normal scroll or the Veronese surface in $\mathbb{P}^5$, i.e., the image of $\mathbb{P}^2$ embedded in $\mathbb{P}^5$ by $\mathcal{O}_{\mathbb{P}^2}(2)$ (see "Veronese embedding"). □

Another example of rational surfaces are **Del Pezzo surfaces**. Let $r \leq 6$. Let $P_1, \dots, P_r$ be r distinct points in $\mathbb{P}^2$ in general position (i.e., no 3 of them are collinear, and no 6 of them lie on a conic). Let $\pi : S \to \mathbb{P}^2$ be the blowing-up of $\mathbb{P}^2$ in $P_1, \dots, P_r$. Then $-K_S$ defines an embedding $i : S \hookrightarrow \mathbb{P}^{9-r}$, whose image has degree $9-r$ and is called a Del Pezzo surface.

If $r = 6$, $i(S)$ is a smooth cubic in $\mathbb{P}^3$; if $r = 5$, $i(S)$ is a complete intersection of two quadrics in $\mathbb{P}^4$.

One can show that $i(S)$ contains only a finite number of lines, precisely the images under i of

(a) the exceptional curves;
(b) the strict transforms of the lines P_iP_j for $i \neq j$;

(c) the strict transforms of the conics through 5 of the P_i.

In the case $r = 6$ we have exactly 27 lines.

Theorem.
(i) The minimal ruled surfaces on $\mathbb{P}^1$ are isomorphic either to the Hirzebruch surfaces or to $\mathbb{P}^2$.
(ii) The minimal ruled surfaces on a Riemann surface of genus ≥ 1 are the geometrically ruled ones and the minimal model is not unique. □

Theorem. The nonruled surfaces have a unique minimal model (up to isomorphisms). □

Remark. If S is a rational surface, then, for every $n \geq 1$, we have $P_n(S) = q(S) = p_g(S) = 0$. □

The following theorem tells us that also the converse is true.

Castelnuovo–Enriques theorem. Let S be a surface such that $q(S) = P_2(S) = 0$. Then S is rational. □

Theorem. Any unirational surface (see "Unirational, Lüroth problem") is rational. □

Theorem (Castelnuovo–De Franchis).
- Let S be a minimal surface with $\chi(S) < 0$; then S is irrational ruled.
- Let S be a minimal surface with $c_1(S)^2 < 0$; then S is irrational ruled.
- Let S be a minimal surface with $\chi((\mathcal{O}_S)) < 0$; then S is irrational ruled. □

(Observe that the third statement follows trivially from the first two and Noether's theorem.)

Enriques' theorem. A surface S is ruled if and only if $P_{12}(S) = 0$. □

As we have already said, the most important tool to classify surfaces is Kodaira dimension.

Enriques' theorem and Proposition 1 tell us that $K(S) = -\infty$ if and only if S is ruled.

Definition. We say that a surface S is **hyperelliptic** or **bielliptic** if it is equal to

$$(E \times F)/G,$$

where E and F are elliptic curves and G is a finite group of translations of E acting on F in such a way that $F/G = \mathbb{P}^1$. □

Definition. Let C be a smooth curve. An **elliptic** surface S with base C is a surface such that there exists a surjective morphism $S \to C$ such that the generic fibre is an elliptic (irreducible) curve. □

Classification theorem (Enriques–Kodaira).
- A minimal surface S with $K(S) = 0$ is one of the following:
  (a) a surface with $q = 0$ and $p_g = 1$ (this implies $K_S = \mathcal{O}$); such surfaces are called **K3 surfaces**;
  (b) a surface with $q = 0$ and $p_g = 0$ (this implies $K_S^{\otimes 2} = \mathcal{O}$); such surfaces are called **Enriques surfaces**;
  (c) a **hyperelliptic surface** if $q = 1$;
  (d) an **Abelian variety** if $q = 2$ (see "Tori, complex - and Abelian varieties").
- A surface S with $K(S) = 1$ is **elliptic**. □

A particular case of K3 surfaces are the **Kummer surfaces**. A Kummer surface X is a surface obtained in the following way: let A be an Abelian surface and let $\hat{A}$ be the surface obtained from A by blowing up the 16 points of order 2; define

$$X = \hat{A}/\langle I, \sigma \rangle,$$

where σ is the map induced on $\hat{A}$ by the map $a \mapsto -a$ on A.

Proposition. The quotient of a K3 surface by a fixed-point free involution is an Enriques surface.
Conversely, let S be an Enriques surface. As we have already said, $K_S^{\otimes 2}$ is the trivial bundle. Let $\tilde{S} = \{u \in K_S | \ i(u \otimes u) = 1\}$ where i is the isomorphism $K_S^{\otimes 2} \cong S \times \mathbb{C}$. The surface $\tilde{S}$ is a (nonramified) double covering space of S, where the covering map is induced by the projection of K_S to S, and it is a K3 surface. □

Definition. We say that a surface S is **of general type** if $K(S) = 2$. □

Bogomolov–Miyaoka–Yau inequality. For a surface S of general type, it holds that

$$c_1^2(S) \leq 3\, c_2(S),$$

where $c_i(S) = c_i(T^{1,0}S)$. □

Noether inequality. For a minimal surface of general type, S, it holds that

$$p_g(S) \leq \frac{1}{2} c_1^2(S) + 2,$$

where $c_i(S) = c_i(T^{1,0}S)$. □

Bombieri and Mumford extended the classification of surfaces to arbitrary algebraically closed fields (see [25]). The classification in the case where the characteristic is different from 2, 3 is analogous to the one over the complex numbers.

Symmetric polynomials ([76], [164], [178], [181], [236], [238]). Let $n \in \mathbb{N}$. Let $P \in \mathbb{Z}[x_1,\dots,x_n]$; we say that P is **symmetric** if it is invariant for the action of the symmetric group Σ_n (in other words, if interchanging any of the variables does not modify the polynomial).
We will denote by $\mathbb{Z}[x_1,\dots,x_n]^{\Sigma_n}$ the set of the symmetric polynomials in $x_1,\dots,x_n$ with coefficients in $\mathbb{Z}$.
Let $E_j(x_1,\dots,x_n)$ be the sum of the squarefree monomials of degree j in $x_1,\dots,x_n$ (squarefree means not divisible by the square of a variable). The polynomials E_j are symmetric and they are called **elementary symmetric polynomials**.
For example,

- $E_0(x_1,\dots,x_n) = 1$;
- $E_1(x_1,\dots,x_n) = x_1 + \dots + x_n$;
- $E_2(x_1,\dots,x_n) = \sum_{1\le i<j\le n} x_i x_j$.

Observe that $E_i(x_1,\dots,x_n) = 0$ for any $i \ge n+1$.

The elementary symmetric polynomials can be defined also by the following formula:

$$\Pi_{i=1,\dots,n}(1 + x_i t) = \sum_{j\in\mathbb{N}} E_j(x_1,\dots,x_n)t^j.$$

Gauss' theorem. If $P \in \mathbb{Z}[x_1,\dots,x_n]^{\Sigma_n}$, then there exists a polynomial $F \in \mathbb{Z}[x_1,\dots,x_n]$ such that $P = F(E_1,\dots,E_n)$. □

Let $C_j(x_1,\dots,x_n)$ be the sum of the monomials of degree j in $x_1,\dots,x_n$. The C_j are called **complete symmetric polynomials**.
For example:

- $C_0(x_1,\dots,x_n) = 1$;
- $C_1(x_1,\dots,x_n) = x_1 + \dots + x_n$;
- $C_2(x_1,\dots,x_n) = \sum_{1\le i\le j\le n} x_i x_j$.

The complete symmetric polynomials can be defined also by the following formula:

$$\Pi_{i=1,\dots,n}\frac{1}{1 - x_i t} = \sum_{j\in\mathbb{N}} C_j(x_1,\dots,x_n)t^j.$$

Remark. The following relations hold:

$$\sum_{j\in\mathbb{N}} x_k^{n-j}(-1)^j E_j(x_1,\dots,x_n) = 0 \qquad \forall k \in \{1,\dots,n\},$$
$$\sum_{j\in\mathbb{N}} (-1)^j E_j C_{p-j} = 0 \qquad \forall p \ge 1.$$

The first follows from the second definition of E_j taking $t = -1/x_k$. The second follows from the second definitions of E_j and C_j:

$$1 = \Pi_{i=1,\dots,n}(1 - x_i t)\ \Pi_{i=1,\dots,n}\frac{1}{1 - x_i t}$$
$$= \Big(\sum_{j\in\mathbb{N}}(-1)^j E_j(x_1,\dots,x_n)t^j\Big)\Big(\sum_{k\in\mathbb{N}} C_k(x_1,\dots,x_n)t^k\Big).$$

From the second relation we have $\mathbb{Z}[C_1,\dots,C_n] = \mathbb{Z}[E_1,\dots,E_n]$, which, by Gauss' theorem, is $\mathbb{Z}[x_1,\dots,x_n]^{\Sigma_n}$. Thus both the elementary symmetric polynomials and the complete symmetric polynomials generate the algebra of the symmetric polynomials. □

Now we will define another class of symmetric polynomials such that they generate $\mathbb{Z}[x_1,\dots,x_n]^{\Sigma_n}$ as $\mathbb{Z}$-module: the **Schur polynomials**.

Let $\lambda = (\lambda_1,\dots,\lambda_n)$ with $\lambda_i \in \mathbb{N}$ and $\lambda_1 \geq \cdots \geq \lambda_n$ (we call it a partition of $\lambda_1 + \cdots + \lambda_n$); we can associate to λ a diagram, called a Young diagram, with λ_i boxes in the i-th row for any $i \in \{1,\dots,n\}$ and the rows lined up on the left; see Figure 19.

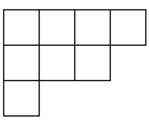


Fig. 19. Young diagram of $(4,3,1)$.

Let A be the matrix $n \times \infty$:

$$\begin{pmatrix} 1 & x_1 & x_1^2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & x_n & x_n^2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}.$$

Number the columns of A beginning from 0. For distinct $t_1,\dots,t_n \in \mathbb{N}$, define $a_{t_1,\dots,t_n}(x_1,\dots,x_n)$ to be the determinant of the matrix obtained by taking the columns $t_1,\dots,t_n$ of A.

For $\lambda = (\lambda_1,\dots,\lambda_n)$ with $\lambda_i \in \mathbb{N}$ and $\lambda_1 \geq \cdots \geq \lambda_n$, define

$$S_\lambda(x_1,\dots,x_n) = \frac{a_{\lambda+\delta}(x_1,\dots,x_n)}{a_\delta(x_1,\dots,x_n)},$$

where $\delta = (n-1, n-2,\dots,0)$. The S_λ are symmetric polynomials and they are called Schur polynomials.

For example: let $n = 3$ and consider $\lambda = (1,1,0)$, which we write $(1,1)$ (in general the zeroes at the end of a partition are omitted); we have

$$S_{(1,1)} = \frac{x_3^2x_2^2(x_3 - x_2) - x_1^2x_3^2(x_3 - x_1) + x_2^2x_1^2(x_2 - x_1)}{\prod_{i>j}(x_i - x_j)} = x_1x_2 + x_1x_3 + x_2x_3.$$

Theorem. The Schur polynomials are a basis of $\mathbb{Z}[x_1, \dots, x_n]^{\Sigma_n}$ as $\mathbb{Z}$-module. □

The following formulas express the Schur polynomials in terms of elementary symmetric polynomials and in terms of complete symmetric polynomials.

Jacobi–Trudi–Giambelli formulas. For any $\lambda = (\lambda_1, \dots, \lambda_n)$ with $\lambda_i \in \mathbb{N}$ and $\lambda_1 \geq \dots \geq \lambda_n$, we have

$$S_\lambda = \det\left((C_{\lambda_i + j - i})_{i,j \in \{1,\dots,n\}}\right) = \det\begin{pmatrix} C_{\lambda_1} & C_{\lambda_1+1} & \cdot & \cdot \\ C_{\lambda_2 - 1} & C_{\lambda_2} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix},$$

$$S_\lambda = \det\left((E_{\gamma_i + j - i})_{i,j \in \{1,\dots,n\}}\right) = \det\begin{pmatrix} E_{\gamma_1} & E_{\gamma_1+1} & \cdot & \cdot \\ E_{\gamma_2 - 1} & E_{\gamma_2} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix},$$

where $\gamma = (\gamma_1, \dots, \gamma_s)$ is the conjugate of λ, i.e., γ_j is the number of the boxes of the j-th column of the Young diagram of λ (in other words, the Young diagram of γ is obtained from the Young diagram of λ by interchanging rows and columns). □

Littlewood–Richardson rule. For any $\lambda = (\lambda_1, \dots, \lambda_n)$ with $\lambda_i \in \mathbb{N}$ and $\lambda_1 \geq \dots \geq \lambda_n$ and for any $\mu = (\mu_1, \dots, \mu_n)$ with $\mu_i \in \mathbb{N}$ and $\mu_1 \geq \dots \geq \mu_n$, we have

$$S_\lambda S_\mu = \sum_\nu N_{\lambda,\mu,\nu} S_\nu,$$

where $N_{\lambda,\mu,\nu}$ is the number of the ways the Young diagram of λ can be expanded to the Young diagram of ν by a strict μ-expansion, where

- a $\mu = (\mu_1, \dots, \mu_n)$-expansion of the Young diagram of λ is a Young diagram obtained from the Young diagram of λ by adding μ_1 boxes not two in the same column, then μ_2 boxes not two in the same column, and so on;
- a μ-expansion is called strict if the following condition hold: put a 1 in each of the μ_1 boxes, a 2 in each of the μ_2 boxes, and so on; form a list reading the numbers in the boxes, reading from right to left and beginning from the top row; we must have that for every r with $1 \leq r \leq \mu_1 + \dots + \mu_n$, and for every p with $1 \leq p \leq n-1$, in the first r entries of the list, the number of the p's is greater than or equal to the number of the $(p+1)$'s. □

See "Schur functors".

Syzygies. Let K be a field. A **syzygy among some polynomials**, $P_1, \dots, P_r \in K[x_1, \dots, x_n]$, is a r-uple of polynomials $(Q_1, \dots, Q_r)$ with $Q_1, \dots, Q_r \in K[x_1, \dots, x_n]$ such that

$$\sum_{i=1,\dots,r} Q_i P_i = 0.$$

More generally, a **syzygy among some m-uples of polynomials in $K[x_1, \dots, x_n]$,**

$$\overline{P}_1 = \begin{pmatrix} P_1^1 \\ \vdots \\ P_1^m \end{pmatrix}, \dots\dots\dots, \overline{P}_r = \begin{pmatrix} P_r^1 \\ \vdots \\ P_r^m \end{pmatrix},$$

is a r-uple of polynomials $(Q_1, \dots, Q_r)$ with $Q_1, \dots, Q_r \in K[x_1, \dots, x_n]$ such that

$$\sum_{i=1,\dots,r} Q_i \overline{P}_i = 0.$$

Given a projective algebraic variety, one can study the syzygies among generators of the ideal of the variety and then the syzygies among these syzygies, and so on. In particular, one can study the degree of such syzygies. A definition which is often used is the following one; it is due to Green and Lazarsfeld (see [88], [89], [170]):

Let X be a smooth complex projective algebraic variety of dimension n and let L be a holomorphic line bundle on X defining an embedding $\varphi_L : X \to \mathbb{P}$, where $\mathbb{P} = \mathbb{P}(H^0(X, L)^\vee)$ (see "Bundles, fibre -"). Set $S = \oplus_{d \in \mathbb{N}} Sym^d H^0(\mathcal{O}(L))$, the homogeneous coordinate ring of the projective space $\mathbb{P}$, and consider the graded S-module $G = \oplus_{d \in \mathbb{N}} H^0(X, \mathcal{O}(L^d))$. Let E_*

$$\cdots \to E_1 \to E_0 \to G \to 0$$

be a minimal graded free resolution of G (see "Minimal free resolutions"). For any $p \in \mathbb{N}$, we say that the line bundle L satisfies **Property** N_p if the two following conditions hold:

$$E_0 = S,$$
$$E_i = \oplus S(-i-1) \quad \text{for} \quad 1 \le i \le p,$$

where the second condition means that E_i is the direct sum of some copies of $S(-i-1)$. Observe that the kernel of the map $S \to G$ is the ideal of $\varphi_L(X)$ (take the cohomology of the exact sequence $0 \to \mathcal{I}_{\varphi_L(X)}(d) \to \mathcal{O}_{\mathbb{P}}(d) \to \mathcal{O}_{\varphi_L(X)}(d) \to 0$, where $\mathcal{I}_{\varphi_L(X)}$ is the ideal sheaf $\varphi_L(X)$).
Thus, L satisfies Property N_0 if and only if L is normally generated, L satisfies Property N_1 if and only if it satisfies Property N_0 and the ideal of $\varphi_L(X)$ is generated by quadrics, and L satisfies Property N_2 if and only if it satisfies Property N_1 and the syzygies among these quadrics are linear, and so on.
See also "Groebner bases", "Hilbert syzygy theorem".

T

Tautological (or universal) bundle. ([93], [188]).

- Let V be a vector space of dimension n and let $r < n$. The **tautological (or universal bundle) on the Grassmannian** of r-subspaces in V, $G(r, V)$ (see "Grassmannians"), is the bundle whose fibre on $W \in G(r, V)$ is the r-subspace W. In particular the tautological bundle on the projective space $\mathbb{P}(V)$ is the line bundle whose fibre on l is the line l; it is the dual of the hyperplane bundle (see "Hyperplane bundles, twisting sheaves").
- Let E be a vector bundle on a manifold (or an algebraic variety) X and let $\pi : \mathbb{P}(E) \to X$ be the projectivized bundle, i.e., the bundle whose fibre on $x \in X$ is the projectivized of E_x. The **tautological (or universal) bundle U on the projectivized bundle** $\mathbb{P}(E)$ is the following bundle: the subbundle of $\pi^* E$ whose fibre on a point l of $\mathbb{P}(E)$ is the line represented by l.
  Its dual is the line bundle $\mathcal{O}_{\mathbb{P}(E)}(1)$, i.e., the line bundle whose restriction on $\mathbb{P}(E_x)$ is $\mathcal{O}(1)$ for all $x \in X$.
- More generally: let E be a vector bundle on a manifold (or an algebraic variety) X, and let $\pi : G(r, E) \to X$ be the bundle whose fibre on $x \in X$ is $G(r, E_x)$; the **tautological (or universal) bundle U on $G(r, E)$** is the following bundle on $G(r, E)$: the subbundle of $\pi^* E$ whose fibre on a point W of $G(r, E)$ is the r-subspace of $E_{\pi(W)} = (\pi^* E)_W$ given by W:

$$\begin{tikzcd} U \subset \pi^*(E) \arrow[d] & E \arrow[d] \\ G(r,E) \arrow[r, "\pi"] & X \end{tikzcd}$$

  Obviously taking X equal to a point, we get the notion of universal bundle on the Grassmannian and taking $r = 1$ we get the notion of universal bundle on the projectivized of a bundle.

Tor, $\mathfrak{TOR}$. ([41], [62], [93], [116]). Let R be a commutative ring with unity. Let M be an R-module. Let

$$\cdots \longrightarrow P_n \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

be a projective resolution of M (see "Injective and projective resolutions"); we denote by P_* the complex

$$\cdots \longrightarrow P_n \longrightarrow P_{n-1} \longrightarrow \cdots . \longrightarrow P_0 \longrightarrow 0.$$

Let N be another R-module. We can consider the complex $P_* \otimes_R N$:

$$\cdots \longrightarrow P_n \otimes_R N \longrightarrow P_{n-1} \otimes_R N \longrightarrow \cdots \longrightarrow P_0 \otimes_R N \longrightarrow 0.$$

We define

$$Tor_i^R(M,N) := H_i(P_* \otimes_R N).$$

One can prove that if

$$\cdots \longrightarrow Q_n \longrightarrow Q_{n-1} \longrightarrow \cdots \longrightarrow Q_0 \longrightarrow N \longrightarrow 0$$

is a projective resolution of N and Q_* is the complex

$$\cdots \longrightarrow Q_n \longrightarrow Q_{n-1} \longrightarrow \cdots \longrightarrow Q_0 \longrightarrow 0,$$

then $Tor_i^R(M,N)$ is equal to $H_i(M \otimes_R Q_*)$. Moreover, one can demonstrate that the R-module $Tor_i^R(M,N)$ does not depend on the choice of the resolutions. Sometimes we will omit the superscript R in $Tor_i^R(M,N)$.

Proposition.

(1) $Tor_i(M,N) \cong Tor_i(N,M)$.

(2) For any short exact sequence of R-modules

$$0 \to A \to B \to C \to 0,$$

we have an exact sequence:

$$\begin{aligned} \cdots\cdots \to Tor_2(A,M) \to Tor_2(B,M) \to Tor_2(C,M) \to \\ \to Tor_1(A,M) \to Tor_1(B,M) \to Tor_1(C,M) \to \\ \to A \otimes_R M \to B \otimes_R M \to C \otimes_R M \to \quad 0. \end{aligned}$$

(3) Let M be an R-module. It is flat (see "Flat (module, morphism)") if and only if $Tor_i(M,N) = 0$ for every R-module N and for every $i \geq 1$ and this is true if and only if $Tor_1(M,N) = 0$ for every R-module N.

(4) Free implies flat and also projective implies flat (see "Injective and projective modules").

(5) If $R = \mathbb{Z}$, or more generally R is a principal ideal domain, then M is flat if and only if it is torsion-free. □

More synthetically, we can define $Tor_i^R(M,\cdot)$ to be the i-th classical left derived functor of the right exact functor $M \otimes_R \cdot$ (see "Derived categories and derived functors").

Let $(X,\mathcal{O}_X)$ be a ringed space (see "Spaces, ringed -"). Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules on X (see "Sheaves"). We define $\mathcal{TOR}^{\mathcal{O}_X}(\mathcal{F},\cdot)$ to be the left derived functor of $\mathcal{F} \otimes_{\mathcal{O}_X} \cdot$.

Proposition. If $\mathcal{F}$ is locally free, then $\mathcal{TOR}_i^{\mathcal{O}_X}(\mathcal{F},\mathcal{G}) = 0$ for all $i > 0$ and for all $\mathcal{G}$ sheaf of $\mathcal{O}_X$-modules. □

Observe that the Proposition above can deduced from properties (a) and (d) of $\mathcal{EXT}^i$ and the fact that, if $\mathcal{F}$ is locally free, then $\mathcal{HOM}(\mathcal{F}^\vee,\cdot) = \mathcal{F} \otimes \cdot$ (see "*Ext*, $\mathcal{EXT}$").

Torelli's theorem. See "Jacobians of compact Riemann surfaces" for Torelli's theorem on Riemann surfaces. More generally we say that a theorem is of the "Torelli" kind if it states the injectivity of a map associating a certain kind of varieties to another kind of varieties.

Tori, complex - and Abelian varieties. ([93], [139], [163], [165], [166], [193]). We follow mainly the exposition in [165].

A **complex torus** X of dimension g is defined to be the quotient of a complex vector space V of dimension g by a lattice of maximal rank Λ in V (we say that Λ is a lattice of maximal rank in V if there exists a basis of V over $\mathbb{R}$, $\{\lambda_1, \dots, \lambda_{2g}\}$, such that $\Lambda = \mathbb{Z}\lambda_1 + \cdots + \mathbb{Z}\lambda_{2g}$):

$$X = V/\Lambda.$$

Example. Let $V = \mathbb{C}$, $\Lambda = \mathbb{Z}\lambda_1 + \mathbb{Z}\lambda_2$ with $\lambda_1, \lambda_2 \in V$ indipendent over $\mathbb{R}$. The quotient $X = V/\Lambda$ is a complex torus of dimension 1; see Figure 20.

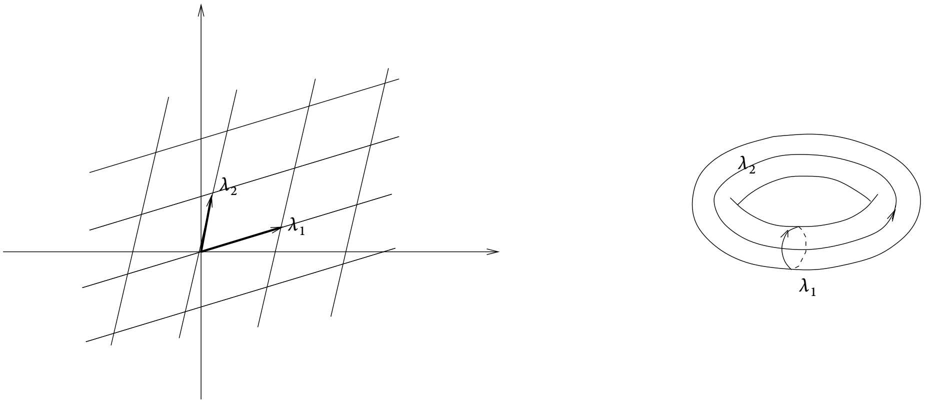


Fig. 20. A complex torus.

A **complex Abelian variety** is defined to be a complex torus embeddable into a projective space.

Remark. Let $X_1 = V_1/\Lambda_1$ and $X_2 = V_2/\Lambda_2$ be complex tori. Let $f : X_1 \longrightarrow X_2$ be a holomorphic map such that $f(0) = 0$. Then f is a group homomorphism and there is a unique $\mathbb{C}$-linear map $F : V_1 \longrightarrow V_2$ sending Λ_1 into Λ_2 and inducing f. □

Definition. Let $X_1 = V_1/\Lambda_1$ and $X_2 = V_2/\Lambda_2$ be two complex tori. An **isogeny** from X_1 to X_2 is a surjective holomorphic homomorphism $f : X_1 \longrightarrow X_2$ with a finite kernel. □

Remark. The isogeny is an equivalence relation. □

One can easily prove that any holomorphic line bundle on a complex vector space is trivial; thus, if $X = V/\Lambda$ is a complex torus, $\pi : V \longrightarrow X$ is the projection and L is a holomorphic **line bundle** on X, then $\pi^* L$ is trivial. Therefore for any $\lambda \in \Lambda$ and for any $v \in V$ there is an isomorphism from $(\pi^* L)_v$ onto $(\pi^* L)_{v+\lambda}$ given by an element of $\mathbb{C}^*$, we call $a_{\lambda,v}$. The $a_{\lambda,v}$, $\lambda \in \Lambda, v \in V$ are called **factors of automorphy** of L and satisfy the following relation:

$$a_{\mu,v+\lambda} a_{\lambda,v} = a_{\lambda+\mu,v} \qquad \forall \lambda, \mu \in \Lambda, \quad \forall v \in V. \tag{6}$$

On the other hand, given a set $\{a_{\lambda,v}\}_{v \in V, \lambda \in \Lambda}$ such that $a_{\lambda,v} \in \mathbb{C}^*$ for any λ and v, they are holomorphic functions in v and they satisfy (6), one can define a holomorphic line bundle L on X as follows:

$$L = (V \times \mathbb{C}) / \sim,$$

where $\sim$ is the relation $(v, z) \sim (v + \lambda, a_{\lambda,v} z)$.

Remark $\star$. There is a bijection between the set $\mathcal{T}$ of the Hermitian forms H on a complex vector space V and the set $\mathcal{E}$ of the real valued alternating forms E on V such that $E(iv, iu) = E(v, u)$ for any $v, u \in V$.
The bijection can be described as follows. Send $H \in \mathcal{T}$ to $E := Im(H)$, where Im denotes the imaginary part, and, conversely, send $E \in \mathcal{E}$ to the form H defined by $H(v, u) = E(iv, u) + iE(v, u)$ for any $v, u \in V$. □

Theorem (Appell–Humbert). Let $X = V/\Lambda$ be a complex torus. There is a canonical isomorphism of exact sequences

$$\begin{array}{ccccccccc}
1 & \longrightarrow & Hom(\Lambda, \mathbb{C}_1) & \longrightarrow & P(\Lambda) & \longrightarrow & N(\Lambda) & \longrightarrow & 0 \\
 & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \\
1 & \longrightarrow & Pic^0(X) & \longrightarrow & Pic(X) & \xrightarrow{c_1} & NS(X) & \longrightarrow & 0
\end{array}$$

where

- $\mathbb{C}_1$ is the group $\{z \in \mathbb{C} \mid |z| = 1\}$;
- $N(\Lambda) = \{H : V \times V \longrightarrow \mathbb{C} \text{ Hermitian s.t. } Im\, H(\Lambda, \Lambda) \subset \mathbb{Z}\}$;
- $P(\Lambda) = \left\{ (H, \chi) \,\middle|\, \begin{array}{l} H \in N(\Lambda), \quad \chi : \Lambda \longrightarrow \mathbb{C}_1, \\ \chi(\lambda + \mu) = \chi(\lambda)\chi(\mu) e^{\pi i Im H(\lambda,\mu)} \quad \forall \lambda, \mu \in \Lambda \end{array} \right\}$;
- the map $Hom(\Lambda, \mathbb{C}_1) \longrightarrow \mathcal{P}(\Lambda)$ is the map $\chi \mapsto (0, \chi)$;
- the map $P(\Lambda) \longrightarrow N(\Lambda)$ is the map $(H, \chi) \mapsto H$;
- the map $P(\Lambda) \longrightarrow Pic(X)$ is the following map: let $(H, \chi) \in P(\Lambda)$; define

  $$a_{\lambda,v} = \chi(\lambda) e^{\pi H(v,\lambda) + \frac{\pi}{2} H(\lambda,\lambda)}$$

  for $\lambda \in \Lambda, v \in V$; they satisfy (6), and thus they define a holomorphic line bundle on X, which we call $L(H, \chi)$; we associate to (H, χ) the line bundle $L(H, \chi)$.

See "Chern classes", "Equivalence, algebraic, rational, linear -, Chow, Neron–Severi and Picard groups" for the definition of c_1, $Pic^0(X)$, $Pic(X)$, $NS(X)$ and of the exact sequence in the second line of the diagram. □

Definition. Let $X = V/\Lambda$ be a complex torus. We define the **dual complex torus**

$$\hat{X} = \frac{Hom_{\overline{\mathbb{C}}}(V, \mathbb{C})}{\{m \in Hom_{\overline{\mathbb{C}}}(V, \mathbb{C}) |\ m(\Lambda) \subset \mathbb{Z}\}},$$

where $Hom_{\overline{\mathbb{C}}}(V, \mathbb{C})$ is the vector space of the additive functions $f : V \to \mathbb{C}$ such that $f(av) = \overline{a} f(v)$ for any $a \in \mathbb{C}, v \in V$.

Proposition. The map

$$Hom_{\overline{\mathbb{C}}}(V, \mathbb{C}) \longrightarrow Hom(\Lambda, \mathbb{C}_1),$$
$$m \mapsto e^{2\pi i m(\cdot)}$$

induces an isomorphism from $\hat{X}$ to $Pic^0(X)$. □

Definition. Let $X = V/\Lambda$ a complex torus. A **Poincaré bundle** for X is a holomorphic line bundle $\mathcal{P}$ on $X \times \hat{X}$ such that
(1) $\mathcal{P}|_{X \times \{L\}} \cong L$ for any $L \in \hat{X}$ (where we are identifying $\hat{X}$ with $Pic^0(X)$);
(2) $\mathcal{P}|_{\{0\} \times \hat{X}}$ is trivial. □

Proposition. There exists a unique, up to isomorphisms, Poincaré bundle on $X \times \hat{X}$. □

Definition. Let L be a holomorphic line bundle on X. Let $\phi_L : X \to \hat{X}$ be the map

$$x \longmapsto t_x^*(L) \otimes L^{-1},$$

where we identify $\hat{X}$ with $Pic^0(X)$, and, for any $x \in X$, the map $t_x : X \to X$ is defined to be the map $y \mapsto x + y$. We define $K(L) = Ker\ \phi_L$. □

We can prove that ϕ_L is induced by the map $V \longrightarrow Hom_{\overline{\mathbb{C}}}(V, \mathbb{C})$, $v \mapsto ImH(v, \cdot)$, where H is the element of $N(\Lambda)$ representing $c_1(L)$; thus ϕ_L depends only on H and then it can be denoted also by ϕ_H. Thus

$$K(L) = \{v \in V |\ \ ImH(v, \Lambda) \subset \mathbb{Z}\}/\Lambda.$$

Let $X = V/\Lambda$ be a complex torus. Let $H \in N(\Lambda)$. One can prove that there is a symplectic basis $\mathcal{B}$ of Λ, that is, there exist $d_i \in \mathbb{N}$ for $i = 1, \dots, g$ with $d_1 | d_2 | \cdots | d_g$ and a basis $\mathcal{B}$ of Λ such that E is expressed in the basis $\mathcal{B}$ by the matrix

$$\begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix},$$

where D is the diagonal matrix with diagonal $(d_1, \dots, d_g)$. If L is a holomorphic line bundle on X and H is the element of $N(\Lambda)$ representing $c_1(L)$, the g-uple $(d_1, \dots, d_g)$ is called the **type** of L. We say that L is **nondegenerate** if and only if H is nondegenerate.

Proposition. If L is nondegenerate, then

$$K(L) = \oplus_{j=1,\dots,g}(\mathbb{Z}/d_j\mathbb{Z})^2,$$

where $(d_1, \dots, d_g)$ is the type of L. □

Let $\pi : V \longrightarrow V/\Lambda = X$ be the canonical projection. Let L be a holomorphic line bundle on X. Since $\pi^* L$ is trivial, having chosen a system of factors of automorphy $a_{\lambda,v}$, one can identify $H^0(\mathcal{O}(L))$ with the set of holomorphic functions θ on V such that, for any λ and v,

$$\theta(v + \lambda) = a_{\lambda,v}\theta(v).$$

Such functions are called **theta functions**.

Theorem. Let X be a complex torus of dimension g and L a holomorphic line bundle on X such that $c_1(L)$ has r positive eigenvalues and s negative eigenvalues. Then

$$h^q(\mathcal{O}(L)) = \begin{cases} \binom{g-r-s}{q-s} d_1 \cdot \dots \cdot d_{s+r} & \text{if } s \le q \le g - r \text{ and } L|_{K(L)_0} \text{ is trivial,} \\ 0 & \text{otherwise,} \end{cases}$$

where $(d_1, \dots, d_{r+s}, 0, \dots, 0)$ is the type of L and $K(L)_0$ is the connected component of $K(L)$ containing 0. In particular, if $c_1(L)$ is positive definite, then

$$h^0(\mathcal{O}(L)) = d_1 \cdot \dots \cdot d_g,$$

where $(d_1, \dots, d_g)$ is the type of L. □

Corollary (Riemann–Roch theorem). If L is a holomorphic line bundle on a complex torus X of dimension g, then

$$\chi(\mathcal{O}(L)) = \frac{1}{g!}L^g,$$

where $\chi(\mathcal{O}(L))$ is the Euler–Poincaré characteristic of $\mathcal{O}(L)$, i.e., the number $\sum_{i=0,\dots,\infty}(-1)^i h^i(X, \mathcal{O}(L))$, and L^g is the cup product $c_1(L) \cup \dots \cup c_1(L)$ (g times) evaluated in the fundamental class of X, that is, in the element of $H_{2g}(X, \mathbb{Z})$ giving the orientation of X (see "Singular homology and cohomology"). □

By using Kodaira embedding theorem (see "Kodaira Embedding theorem") or directly, one can prove the following proposition.

Proposition. A complex torus is embeddable in a projective space, i.e., it is an Abelian variety, if and only if there is a $(1, 1)$ closed positive integer translation-invariant form. This is equivalent to requiring that there be a positive holomorphic line bundle L (the first Chern class $c_1(L)$ gives the required form). □

The form above is called **polarization**. The polarization is said to be **principal** if L is of type $(1, \dots, 1)$. We say that two polarized Abelian varieties are isomorphic if there is a biholomorphism such that the pull-back of the polarization of the second Abelian variety is the polarization of the first.
More generally, following [166], we can give the following definition: Let L be a nondegenerate holomorphic line bundle on a complex torus X; let its first Chern class be given by a Hermitian form with exactly k negative eigenvalues; then we say that the first Chern class of L is a **polarization of index k** on X.

Definition. Let X be a complex torus and L a nondegenerate holomorphic line bundle on X defining a polarization of type $(d_1, \dots, d_g)$ with an index. One can prove that the map

$$d_1 d_g \phi_L^{-1} : \hat{X} \to X$$

is an isogeny and that there is a line bundle $\hat{L}$ on $\hat{X}$ such that $\phi_{\hat{L}} = d_1 d_g \phi_L^{-1}$; the polarization with an index on $\hat{X}$ given by $\hat{L}$ is called **dual polarization** (with index) (see [166]). □

Theorem (Riemann relations). Let $X = V/\Lambda$ be a complex torus of dimension g; let $\{e_1, \dots, e_g\}$ be a complex basis of V, $\{\lambda_1, \dots, \lambda_{2g}\}$ be a basis of Λ and Π be the matrix expressing $\{\lambda_1, \dots, \lambda_{2g}\}$ in function of $\{e_1, \dots, e_g\}$ (thus $X \cong \mathbb{C}^g/\Pi\mathbb{Z}^{2g}$). We have that X is an Abelian variety if and only if there is a nondegenerate alternating matrix $A \in M(2g \times 2g, \mathbb{Z})$ such that

(i) $\Pi A^{-1t}\Pi = 0$;

(ii) $i\Pi A^{-1t}\overline{\Pi}$ is positive definite;

A is the matrix expressing the alternating form E defining the polarization in the basis $\{\lambda_1, \dots, \lambda_{2g}\}$. □

Theorem. Let X be an Abelian variety with an ample holomorphic line bundle L of type $(d_1, \dots, d_g)$. Let φ_L be the rational map on X associated to L (see "Bundles, fibre -").

(1) If $d_1 \geq 2$ the map φ_L is holomorphic.

(2) (Lefschetz) If $d_1 \geq 3$, then φ_L is an embedding. □

Definition. We define the **Siegel upper-half space**

$$\mathcal{H}_g := \{Z \in M(g \times g, \mathbb{C}) | \ Z = {}^tZ \ \text{ and } \ Im\, Z > 0\},$$

where $Im\, Z > 0$ means that the imaginary part of Z is positive definite. □

Suppose that $X = V/\Lambda$ is an Abelian variety of dimension g and let H be a Hermitian form on V defining a polarization of type $(d_1, \dots, d_g)$; let $\{\lambda_1, \dots, \lambda_g, \lambda_{g+1}, \dots, \lambda_{2g}\}$ be a symplectic basis of Λ, i.e., a basis in which $E = Im\, H$ is expressed by the matrix

$$\begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix},$$

where D is the diagonal matrix with diagonal $(d_1, \dots, d_g)$; let $e_i := \frac{1}{d_i}\lambda_{i+g}$ for $i = 1, \dots, g$; one can prove that $\{e_1, \dots, e_g\}$ is a complex basis of V; the matrix Π expressing $\{\lambda_1, \dots, \lambda_{2g}\}$ in function of $\{e_1, \dots, e_g\}$ is of the form $\Pi = (Z, D)$ for some $Z \in M(g \times g, \mathbb{C})$. We can prove that $Z \in \mathcal{H}_g$.
Conversely, an element $Z \in \mathcal{H}_g$ determines a polarized Abelian variety with polarization of type $(d_1, \dots, d_g)$ with a symplectic basis: the torus $X = \mathbb{C}^g/(Z, D)\mathbb{Z}^{2g}$, with the polarization whose imaginary part is expressed by the matrix $\begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix}$ in the basis of the lattice given by the columns of the matrix (Z, D) (the imaginary part of the polarization determines the whole polarization by Remark $\star$).

Theorem. Fix a type $(d_1, \dots, d_g)$. The space $\mathcal{H}_g$ is the **moduli space** of polarized Abelian varieties of type $(d_1, \dots, d_g)$ with symplectic basis. Moreover, if $Z_1, Z_2 \in \mathcal{H}_g$, the polarized Abelian varieties of type $(d_1, \dots, d_g)$ determined by Z_1 and Z_2 are isomorphic if and only if

$$Z_2 = (\alpha Z_1 + \beta)(\gamma Z_1 + \delta)^{-1}$$

for some $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in M(2g \times 2g, \mathbb{Q})$ such that

$${}^t\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\begin{pmatrix} I & 0 \\ 0 & D \end{pmatrix}\mathbb{Z}^{2g} \subset \begin{pmatrix} I & 0 \\ 0 & D \end{pmatrix}\mathbb{Z}^{2g},$$

where D is the diagonal matrix with diagonal $(d_1, \dots, d_g)$, and

$${}^t\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}.$$ □

More generally, we can define an **Abelian variety over an algebraic closed field K** to be a complete algebraic variety X which is also a group and such that the group operation and the map associating to any element its inverse are morphisms.
We can prove that the group operation, which is usually called a "sum", is necessarily Abelian. Many properties we have seen for complex Abelian varieties also hold for Abelian varieties over K. We refer to [163] and [193].

See also "Albanese varieties", "Elliptic Riemann surfaces, elliptic curves" "Equivalence, algebraic, rational, linear -, Chow, Neron–Severi and Picard groups", "Jacobians of compact Riemann surfaces", "Jacobians, Weil and Griffiths intermediate -".

Sometimes (but not in this book) the term "complex torus" is used also to denote $(\mathbb{C}^*)^k$ for some $k \in \mathbb{N}$, where $\mathbb{C}^*$ is $\mathbb{C} - \{0\}$.

Toric varieties. ([52], [56], [67], [70], [78], [141], [205], [206]). A (complex) toric variety X is an algebraic variety over $\mathbb{C}$ such that there is an action of the group $(\mathbb{C}^*)^n$ on X for some n and such an action has a dense orbit.
A normal toric variety can be constructed as follows.

Let N be a $\mathbb{Z}$-module isomorphic to $\mathbb{Z}^n$ and let $M = Hom_{\mathbb{Z}}(N, \mathbb{Z})$. If $u \in N \otimes_{\mathbb{Z}} \mathbb{R}$ and $w \in M \otimes_{\mathbb{Z}} \mathbb{R}$, we say that $u \perp w$ if $w(u) = 0$.

Let σ be a **rational strongly convex polyhedral cone** in $N \otimes_{\mathbb{Z}} \mathbb{R}$, i.e., a subset of $N \otimes_{\mathbb{Z}} \mathbb{R}$ such that there exists a finite number of elements of N, $n_1, \dots, n_s$, such that

$$\sigma = \mathbb{R}^+ n_1 + \dots + \mathbb{R}^+ n_s$$

and the unique linear subspace contained in σ is $\{0\}$ (here $\mathbb{R}^+ = \{x \in \mathbb{R} | \ x \geq 0\}$). We define

$$\sigma^\vee := \{m \in M \otimes_{\mathbb{Z}} \mathbb{R} | \ \ m(n) \geq 0 \ \forall n \in \sigma\},$$
$$S_\sigma := M \cap \sigma^\vee.$$

We can prove that the semigroup S_σ is finitely generated (Gordon's Lemma). Let $u_1, \dots, u_t$ be a set of generators of the semigroup S_σ.

The **affine toric variety** V_σ associated to σ can be defined in the following way. Consider $\mathbb{C}[S_\sigma]$, the $\mathbb{C}$-algebra generated by χ^u for all $u \in S_\sigma$ with the property

$$\chi^u \chi^{u'} = \chi^{u+u'},$$

for any $u, u' \in S_\sigma$. Thus

$$\mathbb{C}[S_\sigma] = \mathbb{C}[\chi^{u_1}, \dots, \chi^{u_t}]/I,$$

where I is the ideal generated by the binomials:

$$(\chi^{u_1})^{a_1} \cdots (\chi^{u_t})^{a_t} - (\chi^{u_1})^{b_1} \cdots (\chi^{u_t})^{b_t}$$

for $a_i, b_j \in \mathbb{N}$ such that

$$\sum_{i=1,\dots,t} a_i u_i = \sum_{j=1,\dots,t} b_j u_j.$$

Define V_σ to be the zero locus in $\mathbb{C}^t$ of the ideal I; in the language of the schemes (see "Schemes")

$$V_\sigma = Spec\,(\mathbb{C}[S_\sigma]) = Spec\,(\mathbb{C}[\chi^{u_1}, \dots, \chi^{u_t}]/I).$$

The set of the closed points of V_σ correspond to the set of the maximal ideals of $\mathbb{C}[S_\sigma]$, $Specm(\mathbb{C}[S_\sigma])$, and there is a bijection from $Specm(\mathbb{C}[S_\sigma])$ to $Hom(S_\sigma, \mathbb{C})$ (the set of the semigroup homomorphisms from S_σ to $\mathbb{C}$) defined by

$$P \mapsto f,$$

where $f(u) := \chi^u(P)$ for all $u \in S_\sigma$.

Definition. Let σ be a rational strongly convex polyhedral cone. We say that $\tau \subset \sigma$ is a face of σ if there exists $m \in \sigma^\vee$ such that $\tau = \sigma \cap \{m\}^\perp$.

Thus we have

$$S_\tau = S_\sigma + \mathbb{N}(-m).$$

Then

$$\mathbb{C}[S_\sigma] \hookrightarrow \mathbb{C}[S_\tau]$$

and V_τ is an open subset of V_σ complementary of the zero locus of the equation $\chi^m = 0$. □

Definition. A **fan** F in N is a set of rational strongly convex polyhedral cones in $N \otimes_{\mathbb{Z}} \mathbb{R}$ such that
(1) every face of an element of F is an element of F;
(2) the intersection of two elements of F is a face of each of them.

The **toric variety $T(F)$ associated to the fan F** is obtained by considering the union of V_σ for $\sigma \in F$ and gluing them by identifying the images of $V_{\sigma\cap\tau}$ in V_σ and V_τ. □

One can prove that in this way we obtain a normal toric variety of dimension n and every normal toric variety is equal to $T(F)$ for some finite fan F (see [141, §1.2, Theorem 6]).
The action of $(\mathbb{C}^*)^n$ is the following: identify $(\mathbb{C}^*)^n$ with $Hom(M, \mathbb{C}^*)$; identify every closed point x of V_σ with a semigroup homomorphism $S_\sigma \to \mathbb{C}$. Let $t \in Hom(M, \mathbb{C}^*)$; tx is the semigroup homomorphism $S_\sigma \to \mathbb{C}$ such that $(tx)(s) = t(s)x(s)$.

Example. Let F be the following fan: $\sigma_1 = \langle e_1, e_2\rangle$, $\sigma_2 = \langle -e_1, e_2\rangle$ (see Figure 21).

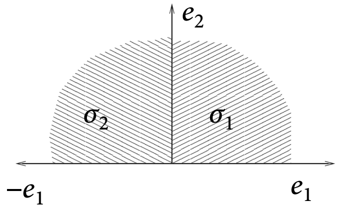


Fig. 21. A fan.

Then $S_{\sigma_1} = \langle e_1^\vee, e_2^\vee\rangle$, $S_{\sigma_2} = \langle -e_1^\vee, e_2^\vee\rangle$ and $S_{\sigma_1\cap\sigma_2} = \langle e_1^\vee, -e_1^\vee, e_2^\vee\rangle$.
Thus $\mathbb{C}[S_{\sigma_1}] = \mathbb{C}[\chi^{e_1}, \chi^{e_2}]$ and $V_{\sigma_1} = \mathbb{C}^2$, $\mathbb{C}[S_{\sigma_2}] = \mathbb{C}[\chi^{-e_1}, \chi^{e_2}]$ and $V_{\sigma_2} = \mathbb{C}^2$; finally, $\mathbb{C}[S_{\sigma_1\cap\sigma_2}] = \mathbb{C}[\chi^{e_1}, \chi^{e_2}, \chi^{-e_1}]/\chi^{e_1}\chi^{-e_1} = 1$ and then $V_{\sigma_1\cap\sigma_2} = \mathbb{C} \times \mathbb{C}^*$. Therefore $T(F) = \mathbb{C} \times \mathbb{P}^1$.

Theorem. Let σ be a rational strongly convex polyhedral cone in $N \otimes_{\mathbb{Z}} \mathbb{R}$. Then V_σ is smooth if and only if σ is generated by a part of a $\mathbb{Z}$-basis of N. If V_σ is smooth, then $V_\sigma = \mathbb{C}^k \times (\mathbb{C}^*)^{n-k}$ where $k = dim\sigma$. □

Notation. For every fan F, let $|F|$ be $\cup_{\sigma\in F}\sigma$. □

Theorem. Let F be fan. Then $T(F)$ is compact if and only if F is finite and $|F| = N \otimes_{\mathbb{Z}} \mathbb{R}$. □

Theorem. Let F be a finite fan in N. Then

$$\pi_1(T(F)) \cong N/\langle |F| \cap N \rangle,$$

where π_1 denotes the first fundamental group (see "Fundamental group"). In particular, if F contains a cone of dimension n, then $T(F)$ is simply connected. Moreover, for any σ cone in N,

$$H^i(V_\sigma, \mathbb{Z}) \cong \wedge^i(\sigma^\perp \cap M),$$

where H^i denotes the singular cohomology (see "Singular homology and cohomology"). □

Theorem. Let F be a finite fan. The set of the Weil divisors (see "Divisors") in $T(F)$ that are invariant for the action of $(\mathbb{C}^*)^n$ is

$$\oplus_{\tau \text{ edges of } F} \; T(F(\tau)),$$

where $F(\tau)$ is the fan given by the cones of F containing τ.
A $(\mathbb{C}^*)^n$-invariant Cartier divisor is the zero locus of χ^u for some $u \in M$.
If F is a fan such that $|F|$ is not contained in any proper subspace of $N \otimes_{\mathbb{Z}} \mathbb{R}$, then there is a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "\varphi"] \arrow[d, "I"] & \{(\mathbb{C}^*)^n\text{-invar. C-divisors}\} \arrow[r] \arrow[d] & Pic(T(F)) = \{\text{C-divisors}\}/\sim \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M \arrow[r, "\varphi"] & \{(\mathbb{C}^*)^n\text{-invar. W-divisors}\} \arrow[r] & \{\text{W-divisors}\}/\sim \arrow[r] & 0
\end{tikzcd}$$

whose rows are exact, the vertical maps are injective and φ is defined by $\varphi(u) = \{\chi^u = 0\}$ and where "C-divisors" stands for Cartier divisors, "W-divisors" stands for Weil divisors, $\sim$ stands for linear equivalence and $Pic(T(F))$ is the Picard group of $T(F)$ (see "Equivalence, algebraic, rational, linear -, Chow, Neron–Severi and Picard groups"). In particular

$$rk(Pic(T(F))) \leq \sharp(\text{ edges in } F) - n.$$

Moreover, $Pic(T(F))$ is torsion free. □

For any $(\mathbb{C}^*)^n$-invariant Cartier divisor D on $T(F)$, define

$$l_D : |F| \to \mathbb{R}$$

in the following way: for any cone σ of F, let

$$l_D|_\sigma(v) = u(\sigma)(v),$$

where $u(\sigma)$ is the element such that $D \cap V_\sigma = \{\chi^{-u(\sigma)} = 0\}$.

Theorem. Let D be a $(\mathbb{C}^*)^n$-invariant Cartier divisor on $T(F)$. Let $\mathcal{O}(D)$ be the sheaf of the sections of the bundle associated to D (see "Divisors").
If $T(F)$ is compact, then D is ample if and only if l_D is strictly upper convex (i.e., for all σ, the graph of l_D on the complementary of σ is strictly under the graph of $u(\sigma)$). Moreover,

$$H^0(T(F), \mathcal{O}(D)) = \oplus_{u \in M,\, u \geq l_D \text{ on } |F|} \mathbb{C}\chi^u.$$

More generally, for all $i \geq 0$,

$$H^i(T(F), \mathcal{O}(D)) \cong \oplus_{u \in M} H^i(|F|, |F|_{D,u}, \mathbb{C}),$$

where $|F|_{D,u} := \{v \in |F| \mid u(v) < l_D(v)\}$ and the H^i at the second member denotes the singular relative cohomology. □

Now we are describing a way to **construct toric varieties from convex polytopes.** A subset of $M \otimes \mathbb{R}$ is said to be a rational polytope if it is the convex hull of a finite set of points of M. Let P be an n-dimensional rational polytope in $M \otimes \mathbb{R}$.
For every face L of P, let σ_L be the cone defined by

$$\sigma_L = \{v \in N \otimes_{\mathbb{Z}} \mathbb{R} |\ u(v) \leq u'(v)\ \forall u \in L \text{ and } u' \in P\}.$$

We can easily prove that the cones σ_L, for L varying among the faces of P, form a fan F_P. We define the toric variety associated to P to be $T(F_P)$.

Finally we sketch **another approach to toric varieties** (see [78]).
For any $x \in (\mathbb{C}^*)^n$ and any $a \in \mathbb{Z}^n$, let x^a denote $x_1^{a_1} \dots x_n^{a_n}$. Let $A = \{a^1, \dots, a^N\}$ be a finite subset of $\mathbb{Z}^n$; we define X_A to be

$$X_A := \overline{\{[x^{a^1} : \cdots : x^{a^N}] \mid x \in (\mathbb{C}^*)^n\}}$$

(the closure in $\mathbb{P}^{N-1}$) and we define Y_A to be

$$Y_A := \overline{\{(x_{n+1}x^{a^1}, \dots, x_{n+1}x^{a^N}) \mid x = (x_1, \dots, x_n) \in (\mathbb{C}^*)^n, \quad x_{n+1} \in \mathbb{C}^*\}}$$

(the closure in $\mathbb{C}^N$).
Obviously X_A is the closure of the orbit of $[1 : \cdots : 1]$ under the action of $(\mathbb{C}^*)^n$ on $\mathbb{P}^{N-1}$ given by the formula

$$x \cdot [z_1 : \cdots : z_N] = [x^{a^1} z_1 : \cdots : x^{a^N} z_N]$$

for $x \in (\mathbb{C}^*)^n, [z_1 : \cdots : z_N] \in \mathbb{P}^{N-1}$, so it is a toric variety. Vice versa one can easily prove the following proposition.

Proposition. Let X be a projective toric variety in $\mathbb{P}^{M-1}$ such that the action of the torus $(\mathbb{C}^*)^n$ extends to the whole $\mathbb{P}^{M-1}$. Let $N-1$ be the dimension of the minimal projective space in $\mathbb{P}^{M-1}$ containing X. Then there exists a subset A of $\mathbb{Z}^n$ containing exactly N

elements such that there exists an isomorphism from X_A to X equivariant under the torus action and extending to an equivariant projective isomorphism from $\mathbb{P}^{N-1}$ to the minimal projective space in $\mathbb{P}^{M-1}$ containing X. □

The link between this approach and the first description we have given of normal toric varieties is the following:
let $A = \{a^1, \dots, a^N\}$ a finite subset of $\mathbb{Z}^n$; let S_A be the semigroup generated by the elements $(a, 1)$ for $a \in A$; we can prove that $Y_A = \mathit{Spec}\,\mathbb{C}[S_A]$; besides we can prove that, if S is a finitely generated semigroup of $\mathbb{Z}^{n+1}$ containing 0, then $\mathit{Spec}\,\mathbb{C}[S]$ is an affine toric variety and it is normal if and only if S is the intersection of the convex hull of S and the Abelian group generated by S.

The theory of toric varieties has been developed more generally over algebraically closed fields, see [141] and [206].

Transcendence degree. ([12], [62], [164], [256]). Let L be a field and let K be a subfield of L. A subset $\{\alpha_1, \dots, \alpha_k\}$ of L is said to be algebraically independent over K if there doesn't exist a nonzero polynomial P in k variables with coefficients in K such that $P(\alpha_1, \dots, \alpha_k) = 0$. The transcendence degree of L over K is the largest cardinality of a subset of L algebraically independent over K.

Theorem. If R is an integral domain and a finitely generated K-algebra for some field K, then the dimension of R is the transcendence degree over K of the quotient field of R. □

See "Dimension", "Localization, quotient ring, quotient field", "Siegel's theorem".

Transcendental. The word "transcendental" in algebraic geometry means "concerning complex analysis".

U

Unirational, Lüroth problem. ([10], [48], [107], [132], [228], [254]) We say that an algebraic variety X of dimension n over a field K is unirational if there exists a dominant rational map $\mathbb{P}^n_K \to X$ (see "Varieties, algebraic -, Zariski topology, regular and rational functions, morphisms and rational maps" for the definition of dominant). The Lüroth problem can be formulated as follows: is any unirational variety X rational? (see "Rational varieties").
If the dimension of X is 1, the answer is yes for any field K (Lüroth's theorem).
If the dimension of X is 2, the answer is yes if $K = \mathbb{C}$ (Castelnuovo's theorem).

In higher dimension, the answer is no, also over $\mathbb{C}$: the first counterexamples were given by Iskovskikh and Manin, Artin and Mumford, and Clemens and Griffiths. In particular, a smooth cubic threefold in $\mathbb{P}^4_{\mathbb{C}}$ is unirational but not rational, see [48].

Universal bundle. See "Tautological (or universal) bundle".

V

Vanishing theorems. ([4], [66], [93], [98], [107], [138], [149], [169], [173], [174], [231], [243]). (See "Bundles, fibre -", "Positive", "Dimension", "Canonical bundle, canonical sheaf" and "Sheaves" for the definitions of nef, big, ample, positive, dimension, canonical bundle, cohomology of sheaves.)

Grothendieck's theorem. Let $\mathcal{F}$ be a sheaf of Abelian groups on a Noetherian topological space X of dimension n (we say that a topological space is Noetherian if for every chain of closed subset $C_1 \supset C_2 \supset \cdots$, there exists k such that $C_k = C_{k+1} = \cdots$). Then

$$H^i(X, \mathcal{F}) = 0 \quad \text{for } i > n,$$

where H^i denotes the derived functor cohomology. □

Akizuki–Kodaira–Nakano theorem. Let L be a positive (thus ample) holomorphic line bundle on a compact complex manifold X of dimension n (thus X is projective algebraic, see "Kodaira Embedding theorem", "Remmert's proper mapping theorem", "Chow's theorem"). Then

$$H^p(X, \Omega^q(L)) = 0 \qquad \text{for } p + q > n\,. \qquad \square$$

Observe that in particular $H^p(X, \mathcal{O}(K_X \otimes L)) = 0$ for $p > 0$, where K_X is the canonical bundle. The last statement is generalized by Kawamata-Viehweg theorem (the case $n = 2$ was proved first by Ramanujam):

Kawamata–Viehweg theorem. Let L be a nef and big holomorphic line bundle on a smooth complex projective algebraic variety X of dimension n. Then

$$H^p(X, \mathcal{O}(K_X \otimes L)) = 0 \qquad \text{for } p > 0\,. \qquad \square$$

Le Potier's theorem generalizes the Akizuki–Kodaira–Nakano theorem to a bundle of any rank.

Le Potier's theorem. Let E be an ample holomorphic vector bundle of rank r on a compact complex manifold X of dimension n. Then

$$H^p(X, \Omega^q(E)) = 0 \qquad \text{for } p + q \geq n + r.$$

More generally, for $1 \leq s \leq r$,

$$H^p(X, \Omega^q(\wedge^s E)) = 0 \quad \text{for } p + q > n + s(r - s)\,. \qquad \square$$

See also “Cartan–Serre theorems”.

Varieties, algebraic -, Zariski topology, regular and rational functions, morphisms and rational maps. ([73], [104], [107], [140], [159], [228]). Let K be a field. Let $\mathbb{A}_K^n$ be the affine n-space over K. For any $S \subset K[x_1, \ldots, x_n]$, we denote the zero locus of S by $Z(S)$, i.e.,

$$Z(S) = \{P \in \mathbb{A}_K^n |\ f(P) = 0\ \forall f \in S\}.$$

Furthermore, for any $X \subset \mathbb{A}_K^n$, we define

$$I(X) = \{f \in K[x_1, \ldots, x_n] |\ f(P) = 0\ \forall P \in X\}$$

We can prove easily that $I(X)$ is an ideal; we call it the ideal of X.

A subset X of $\mathbb{A}_K^n$ is said to be an **(affine) algebraic set** if it is equal to $Z(S)$ for some $S \subset K[x_1, \ldots, x_n]$.

The **Zariski topology** on $\mathbb{A}_K^n$ is the topology such that the closed subsets are the algebraic sets (it is a topology by items 3 and 4 of the following theorem).

For any ideal J in a ring R, we define the radical ideal of J, which we denote by $\sqrt{J}$, as follows:

$$\sqrt{J} := \{f \in R |\ \exists r \in \mathbb{N} - \{0\} \quad \text{s.t.} \quad f^r \in J\}.$$

We say that an ideal J is radical if and only if $J = \sqrt{J}$.

Theorem.
(1) For any S subset of $K[x_1, \ldots, x_n]$, we have that $Z(S) = Z(I)$, where I is the ideal generated by S.
(2) If $S_1 \subset S_2 \subset K[x_1, \ldots, x_n]$, then $Z(S_1) \supset Z(S_2)$.
(3) If I_α is a collection of ideals in $K[x_1, \ldots, x_n]$, then $Z(\cup_\alpha I_\alpha) = \cap_\alpha Z(I_\alpha)$, so the intersection of algebraic sets is an algebraic set.
(4) $Z(I) \cup Z(J) = Z(IJ)$ for any I, J ideals in $K[x_1, \ldots, x_n]$, so the finite union of algebraic sets is an algebraic set.
(5) If $X_1 \subset X_2 \subset \mathbb{A}_K^n$, then $I(X_1) \supset I(X_2)$.
(6) $I(X_1 \cup X_2) = I(X_1) \cap I(X_2)$ for any $X_1, X_2 \subset \mathbb{A}_K^n$.
(7) $I(Z(S)) \supset S$ for any $S \subset K[x_1, \ldots, x_n]$.
(8) $Z(I(X)) \supset X$ for any $X \subset \mathbb{A}_K^n$, more precisely $Z(I(X)) = \overline{X}$ (the closure of X for the Zariski topology).
(9) $I(X)$ is a radical ideal for any $X \subset \mathbb{A}_K^n$.

(10) (Hilbert basis theorem) If R is a Noetherian ring, then $R[x_1, \dots, x_n]$ is Noetherian; in particular for any field K, $K[x_1, \dots, x_n]$ is Noetherian, thus any ideal of $K[x_1, \dots, x_n]$ is finitely generated.

(11) (Hilbert's Nullstellensatz) Let K be an algebraically closed field. For any ideal J in $K[x_1, \dots, x_n]$ we have:
$$I(Z(J)) = \sqrt{J};$$
therefore, if $Z(J) = \emptyset$, then $\sqrt{J} = K[x_1, \dots, x_n]$ and then $J = K[x_1, \dots, x_n]$.
In particular there is a bijection between the set of the algebraic subsets of $\mathbb{A}^n_K$ and the set of the radical ideals of $K[x_1, \dots, x_n]$.

(12) An algebraic subset Z of $\mathbb{A}^n_K$ is irreducible in the Zariski topology (see "Irreducible topological space") if and only if $I(Z)$ is prime. □

Let $\mathbb{P}^n_K$ be the n-projective space over K. For any subset S of homogeneous elements of $K[x_0, \dots, x_n]$, we denote the zero locus of S by $Z(S)$, i.e.,
$$Z(S) = \{P \in \mathbb{P}^n_K |\ f(P) = 0\ \forall f \in S\}.$$
For any homogeneous ideal I (see "Homogeneous ideals"), we define $Z(I)$ to be the zero locus of the set of the homogeneous elements of I. Moreover, for any $X \subset \mathbb{P}^n_K$, we denote by $I(X)$ the ideal generated by
$$\{f \in K[x_0, \dots, x_n] |\ f \text{ homogeneous and } f(P) = 0\ \forall P \in X\}.$$
Obviously it is a homogeneous ideal. We call it the ideal of X.

A subset X of $\mathbb{P}^n_K$ is said to be a **(projective) algebraic set** if it is equal to $Z(S)$ for some S subset of homogeneous elements of $K[x_0, \dots, x_n]$.
For the projective algebraic sets, we have properties analogous to the ones for the affine algebraic sets. See "Hilbert's Nullstellensatz" for the projective version of Hilbert's Nullstellensatz.
The **Zariski topology** on $\mathbb{P}^n_K$ is the topology such that the closed subsets are the algebraic sets.

From now on, let K be an algebraically closed field.
A subset of $\mathbb{A}^n_K$ is said to be an **affine algebraic variety** if it is an irreducible algebraic set.
A **quasi-affine algebraic variety** in $\mathbb{A}^n_K$ is an open subset of an affine algebraic variety.
A **projective algebraic variety** is an irreducible algebraic subset of $\mathbb{P}^n_K$.
A **quasi-projective algebraic variety** is an open subset of a projective algebraic variety in $\mathbb{P}^n_K$.

The quasi-affine algebraic varieties in $\mathbb{A}^n_K$ and the quasi-projective algebraic varieties in $\mathbb{P}^n_K$ are called **algebraic varieties** (obviously affine and projective algebraic varieties are included). More generally, an algebraic variety in $\mathbb{A}^n_K \times \mathbb{P}^{n_1}_K \times \cdots \times \mathbb{P}^{n_k}_K$ is an open subset of an irreducible algebraic subset.

The Zariski topology on an algebraic variety in $\mathbb{A}^n_K \times \mathbb{P}^{n_1}_K \times \cdots \times \mathbb{P}^{n_k}_K$ is the topology induced by the Zariski topology of $\mathbb{A}^n_K \times \mathbb{P}^{n_1}_K \times \cdots \times \mathbb{P}^{n_k}_K$ (i.e., by the topology where the closed subsets are the algebraic subsets).

A topological space is said to be Noetherian if for any sequence $Z_1 \supset Z_2 \supset \cdots$ of closed subsets, there is an integer such that $Z_r = Z_{r+1} = \cdots$.

Proposition. In a Noetherian topological space, every nonempty closed subset F can be written uniquely as a finite union of irreducible closed subsets, one not contained in the other. These irreducible closed subsets are called the irreducible components of F. □

The spaces $\mathbb{A}^n_K$ and $\mathbb{P}^n_K$ and more generally $\mathbb{A}^n_K \times \mathbb{P}^{n_1}_K \times \cdots \times \mathbb{P}^{n_k}_K$ with the Zariski topology are Noetherian topological spaces, so that the proposition above can be applied to them with the Zariski topology, and so every algebraic set can be expressed as a finite union of irreducible closed subsets (i.e., varieties), each not contained in another.

If X is an algebraic set in $\mathbb{A}^n_K$, then we define the **affine coordinate ring** $\Gamma(X)$ to be

$$K[x_1, \ldots, x_n]/I(X).$$

Observe that any element of $\Gamma(X)$ defines a function from X to K.
If X is an algebraic set in $\mathbb{P}^n_K$, then we define the **projective coordinate ring** $\Gamma(X)$ to be

$$K[x_0, \ldots, x_n]/I(X).$$

We say that a function $f : X \to K$ on a quasi-affine algebraic variety X is **regular** in a point $P \in X$ if there is an open neighborhood U of P in X such that $f = g/h$ for some g and h polynomials with h nowhere zero on U.

We say that a function $f : X \to K$ on a quasi-projective algebraic variety X is regular in a point $P \in X$ if there is an open neighborhood U of P in X such that $f = g/h$ for some g and h homogeneous polynomials of the same degree with h nowhere zero on U.

We say that a function $f : X \to K$ is regular on X if it is regular in any point of X.
The sheaf of the regular functions on a variety X is denoted by $\mathcal{O}_X$.
The stalk $\mathcal{O}_{X,P}$ of $\mathcal{O}_X$ in a point $P \in X$ is a local ring (see "Local") and the unique maximal ideal, denoted by m_P, is given by the elements of $\mathcal{O}_{X,P}$ that vanish in P (it is the unique maximal ideal because every element of $\mathcal{O}_{X,P} - m_P$ is invertible). Obviously $\mathcal{O}_{X,P}/m_P \cong K$.

A **rational function** on an algebraic variety is an equivalence class of couples (U, f) where U is an open subset and f a regular function on U and (U, f) and (U', f') are equivalent if and only if $f = f'$ on $U \cap U'$.
The set of the rational functions on a variety X form a field, which is called the **rational functions field** of X and denoted by $K(X)$.

Theorem.

(a) If X is an affine algebraic variety in $\mathbb{A}^n_K$, we have
   - $\mathcal{O}(X) = \Gamma(X)$ (which is a domain since $I(X)$ is prime);
   - $K(X)$ is the quotient field of $\Gamma(X)$;
   - for any $P \in X$, the stalk $\mathcal{O}_{X,P}$ is the localization of $\Gamma(X)$ with respect to $M_P(X)$, where $M_P(X)$ is the ideal in $\Gamma(X)$ given by the elements that are 0 in P (see "Localization, quotient ring, quotient field"). Thus $\mathcal{O}_{X,P}$ is a Noetherian (see "Noetherian, Artinian") local domain. The map $P \mapsto M_P$ is a bijection between the points of X and the maximal ideals of $\Gamma(X)$.

(b) If X is a projective algebraic variety in $\mathbb{P}^n_K$, we have
   - $\mathcal{O}(X) = K$;
   - $K(X)$ is the subring of the elements of degree 0 in the localization of $\Gamma(X)$ with respect the multiplicative system given by the nonzero homogeneous elements of $\Gamma(X)$;
   - for any $P \in X$, $\mathcal{O}_{X,P}$ is the subring of the elements of degree 0 in the localization of $\Gamma(X)$ with respect the multiplicative system given by the homogeneous elements f of $\Gamma(X)$ such that $f(P) \neq 0$. □

Let X and Y be two algebraic varieties. A **morphism** $f : X \to Y$ is a continuous map such that, for any open subset U of Y and for any regular function g on U, the map $g \circ f : f^{-1}(U) \to K$ is regular.
A **rational map** $\varphi : X \to Y$ is an equivalence class of couples (U, f) where U is an open subset and f a morphism on U and (U, f) and (U', f') are equivalent if $f = f'$ on $U \cap U'$. We say that it is **dominant** if there exists one of these couples, (U, f), such that the image of f is dense in Y.
We say that a rational map $f : X \to Y$ is **birational** if there exists a rational map $g : Y \to X$ such that $g \circ f = I_X$ as rational map.
We say that two varieties X and Y are **birational equivalent** (or birational) if there is a birational map from X to Y.

Example. The blowing-up is the typical example of birational map (see "Blowing-up (or σ-process)").

Theorem. Two algebraic varieties (over K) are birational equivalent if and only if their rational functions fields are isomorphic as K-algebras. □

See "Regular rings, smooth points, singular points", "Zariski tangent space, differential forms, tangent bundle, normal bundle", "Schemes", "Primary ideals, primary decompositions, embedded ideals", "G.A.G.A.", "Hironaka's decomposition of birational maps", "Unirational, Lüroth problem".

Varieties and subvarieties, analytic -. ([93], [103], [121]). We say that a subset V of $\mathbb{C}^n$ is an analytic variety if locally it is the zero locus of a finite number holomorphic functions, precisely, if, for all $P \in V$, there exists an open subset U of $\mathbb{C}^n$ containing P and a finite number of holomorphic functions on U, $f_1, \dots, f_k$, such that $V \cap U = \{z \in U| \quad f_1(z) = \cdots = f_k(z) = 0\}$. More generally, an analytic subvariety of a complex manifold is a subset that is locally the zero locus of a finite number of holomorphic functions.

Veronese embedding. ([104], [107]). Let K be an algebraically closed field and, for every n, let $\mathbb{P}^n = \mathbb{P}^n_K$ be the projective space of dimension n over K.
Let $n, d \in \mathbb{N}$ and $N = \binom{n+d}{d} - 1$. The Veronese map of degree d on $\mathbb{P}^n$ is the map

$$\nu_{n,d} : \mathbb{P}^n \longrightarrow \mathbb{P}^N,$$
$$[x_0 : \cdots : x_n] \longmapsto [\cdots : x^I : \cdots]_{I \in \mathcal{I}},$$

where $\mathcal{I} := \{(i_0, \dots, i_n)| \; i_0, \dots, i_n \in \mathbb{N} \quad i_0 + \cdots + i_n = d\}$, that is, $\nu_{n,d}$ is the map sending $[x_0 : \cdots : x_n]$ to the point whose coordinates are all the monomials of degree d in $x_0, \dots, x_n$. Obviously, the map $\nu_{n,d}$ is an embedding (see "Embedding").
Let z_I, for $I \in \mathcal{I}$, be the coordinates on $\mathbb{P}^N$; the image of the Veronese map is the zero locus of the set of the following polynomials of degree 2:

$$z_I z_J - z_K z_L$$

with $I, J, K, L \in \mathcal{I}$ such that $I + J = K + L$.
The image of the Veronese map $\nu_{n,d}$ is a smooth variety of dimension n and degree d^n (see "Degree of an algebraic subset"), called Veronese variety.
If K has characteristic 0, a coordinate-free way to describe the Veronese map $\nu_{n,d}$ is the following: let V be a vector space over K of dimension $n + 1$; $\nu_{n,d}$ is the map

$$\mathbb{P}(V) \longrightarrow \mathbb{P}(Sym^d V),$$
$$\langle v \rangle \longmapsto \langle v^d \rangle$$

for any $v \in V$, where $\langle v \rangle \in \mathbb{P}(V)$ is the line generated by v and v^d is the image of $v \otimes \cdots \otimes v$ (d times) through the natural map $V \otimes \cdots \otimes V \to Sym^d V$.

W

Web. A web is a linear system (see "Linear systems") of dimension 3.

Weighted projective spaces. ([24], [55], [57], ([104]). Let K be an algebraically closed field of characteristic 0. Let $a_0, \dots, a_n \in \mathbb{N} - \{0\}$. The weighted projective space of weights $a_0, \dots, a_n$ over K, which we denote by $\mathbb{P}_K(a_0, \dots, a_n)$, or simply by $\mathbb{P}(a_0, \dots, a_n)$, is defined to be the quotient of $K^{n+1} - \{0\}$ by the following equivalence relation: for any $\lambda \in K - \{0\}$,

$$(x_0, \dots, x_n) \sim (\lambda^{a_0} x_0, \dots, \lambda^{a_n} x_n).$$

We can define $\mathbb{P}_K(a_0, \dots, a_n)$ also to be the quotient of $\mathbb{P}^n_K$ by the action of the group $\mathbb{Z}/a_0 \times \cdots \times \mathbb{Z}/a_n$ generated by the automorphisms

$$[x_0 : \cdots : x_n] \mapsto [x_0 : \cdots : \psi x_i : \cdots : x_n],$$

where ψ is a primitive a_i-th root of the unity.

The isomorphism from $\mathbb{P}^n_K/(\mathbb{Z}/a_0 \times \cdots \times \mathbb{Z}/a_n)$ to $(K^{n+1} - \{0\})/\sim$ is given by sending the class of $[x_0 : \cdots : x_n]$ to the class of $(x_0^{a_0}, \dots, x_n^{a_n})$.

Theorem.

(i) Let $a, a_0, \dots, a_n \in \mathbb{N} - \{0\}$. Then the weighted projective spaces $\mathbb{P}(a_0, \dots, a_n)$ and $\mathbb{P}(aa_0, \dots, aa_n)$ are isomorphic.

(ii) Let $a_0, \dots, a_n \in \mathbb{N} - \{0\}$ with greatest common divisor equal to 1. For any $i = 0, \dots, n$, let
  - $d_i = GCD(a_0, \dots, a_{i-1}, a_{i+1}, \dots, a_n)$;
  - $r_i = lcm(d_0, \dots, d_{i-1}, d_{i+1}, \dots, d_n)$;

  where GCD stands for greatest common divisor and lcm stands for least common multiple. Then $\mathbb{P}(a_0, \dots, a_n)$ and $\mathbb{P}(a_0/r_0, \dots, a_n/r_n)$ are isomorphic.

(iii) For any $a_0, \dots, a_n \in \mathbb{N} - \{0\}$, the weighted projective space $\mathbb{P}(a_0, \dots, a_n)$ is a normal projective algebraic variety (see "Normal"). □

Weierstrass form of cubic curves. ([93], [107], [228], [246]). Let K be an algebraically closed field of characteristic 0.

Let C be a smooth cubic algebraic curve in $\mathbb{P}^2_K$ and F be a homogeneous polynomial in x, y, z whose zero locus is C. By Bezout's theorem applied to F and to the Hessian of C, the curve C must have a flex P (see "Bezout's theorem", "Flexes"). By changing coordinates in $\mathbb{P}^2_K$, we can suppose that $P = [0 : 1 : 0]$ and that the tangent of C at P, $T_P(C)$, is the line $z = 0$, thus that the coefficients of y^3 and xy^2 are zero. Since C is irreducible, one of the coefficients of x^3 and x^2y must be nonzero. Furthermore, the coefficient of x^2y must be zero (otherwise P would not be a flex). Thus we can suppose that

$$F(x, y, z) = x^3 + z\,G(x, y, z),$$

where $G(x, y, z)$ is a polynomial of degree 2 with nonzero coefficient of y^2 (otherwise P would be a singular point); we can suppose the coefficient of y^2 equal to -1; thus

$$G(x, y, z) = -y^2 + y(a\,z + b\,x) + c\,x^2 + d\,z^2 + e\,xz$$

for some $a, b, c, d, e \in K$. With a change of coordinates (the new y equal to $y - \frac{1}{2}(az+bx)$), we can suppose

$$F(x, y, z) = x^3 + z(-y^2 + \gamma\,x^2 + \delta\,z^2 + \epsilon\,xz)$$

for some $\gamma, \delta, \epsilon \in K$. With a new change of coordinates (the new x equal to $x + \frac{\gamma}{3}z$) we can suppose

$$F(x, y, z) = x^3 - y^2z + \alpha\,z^3 + \beta\,xz^2$$

for some $\alpha, \beta \in K$. One can easily see that C is nonsingular if and only if the roots of $x^3 + \alpha z^3 + \beta xz^2$ are distinct. We can suppose they are $0, 1, \lambda$ for some $\lambda \neq 0, 1$. Thus

$$F(x, y, z) = -y^2 z + x(x - z)(x - \lambda z).$$

Therefore $C \cap \{z = 1\}$ is the set of the (x, y) such that

$$y^2 = x(x - 1)(x - \lambda),$$

which is called Weierstrass form.
See "Elliptic Riemann surfaces, elliptic curves".

Weierstrass points. ([8], [69], [93], [101], [189]). By the geometric Riemann–Roch theorem (see "Riemann surfaces (compact -) and algebraic curves"), for a general divisor D of degree d on a compact Riemann surface X of genus $g \geq 1$, we have

$$h^0(\mathcal{O}(D)) = \begin{cases} 1 & \text{if } 0 \leq d \leq g, \\ d - g + 1 & \text{if } d \geq g. \end{cases}$$

In particular, for general $P \in X$, we have

$$h^0(\mathcal{O}(dP)) = \begin{cases} 1 & \text{if } 0 \leq d \leq g, \\ d - g + 1 & \text{if } d \geq g. \end{cases}$$

The points P that have *not* this behaviour are called Weierstrass points.

For every $P \in X$ let

$$G_P = \{d \in \mathbb{N} | \ h^0(\mathcal{O}(dP)) = h^0(\mathcal{O}((d-1)P))\}$$

The set G_p ordered by increasing order is called Weierstrass gap sequence.

Remark. By using the Riemann–Roch theorem one can easily prove that, for every $P \in X$, the cardinality of G_P is g (in fact $h^0(\mathcal{O}(dP)) = d - g + 1$ for every $d \geq 2g - 1$ and $h^0(\mathcal{O}(0P)) = 1$). □

Obviously a point P is a Weiersrtrass point if and only if $G_P \neq (1, \dots, g)$.

Let P be a Weierstrass point, and let $G_P = (a_1, \dots, a_g)$ (with $a_1 < \dots < a_g$). We define the weight of P to be

$$w(P) = \sum_{i=1,\dots,g} (a_i - i).$$

Proposition. Let X be a Riemann surface of genus $g \geq 1$.
If $g = 1$, there are no Weierstrass points on X.
If $g \geq 2$, there exist at least $2g + 2$ Weierstrass points on X (and the equality holds if and only if X is hyperelliptic). Besides the sum of the weights of the Weierstrass points on X is $(g-1)g(g+1)$; therefore the maximum number of the Weierstrass points on a Riemann surface of genus g is $(g-1)g(g+1)$. □

Weierstrass preparation theorem and Weierstrass division theorem. ([93], [121], [151]).

Weierstrass preparation theorem. Let U be a neighborhood of 0 in $\mathbb{C}^n$; call $z_1, \dots, z_n$ the coordinates in $\mathbb{C}^n$ and let $z = (z_1, \dots, z_{n-1})$. Let $f : U \to \mathbb{C}$ be holomorphic, not identically zero on the z_n-axis and such that $f(0) = 0$. Then, in a neighborhood of 0 in $\mathbb{C}^n$, the function f can be written uniquely as

$$f(z, z_n) = h(z, z_n)(z_n^d + \alpha_1(z) z_n^{d-1} + \dots + \alpha_d(z)),$$

for some $d \geq 1$, h, α_i holomorphic functions such that $h(0) \neq 0$ and $\alpha_i(0) = 0$ for all i. □

Corollary. The stalk of the sheaf of the holomorphic functions in a point of $\mathbb{C}^n$ is a unique factorization domain. □

Weierstrass division theorem. Let U be a neighborhood of 0 in $\mathbb{C}^n$ and call $z_1, \dots, z_n$ the coordinates in $\mathbb{C}^n$ and let $z = (z_1, \dots, z_{n-1})$. Let $f : U \to \mathbb{C}$ be holomorphic. Let

$$g(z, z_n) = z_n^d + \alpha_1(z) z_n^{d-1} + \dots + \alpha_d(z),$$

where α_i are holomorphic functions such that $\alpha_i(0) = 0$ for all i. Then, in a neighborhood of 0, the function f can be written as

$$f = gh + r,$$

where h and r are holomorphic and r can be written as a polynomial of degree $\leq d - 1$ in z_n with some holomorphic functions in z as coefficients. □

Corollary: Weak Nullstellensatz. Let f and h be holomorphic maps from a neighborhood of 0 in $\mathbb{C}^n$ to $\mathbb{C}$. Let f be irreducible (in the ring of the holomorphic functions defined in some neighborhood of 0) and let h vanish on the zero locus of f. Then $h = fg$ for some holomorphic function g on a neighborhood of 0. □

See "Hilbert's Nullstellensatz".

Z

Zariski's main theorem. ([107], [241], [255]). There are several statements that are called Zariski's main theorem. We report the versions of Zariski's main theorem in [107] and [241] respectively.

Theorem. Let X and Y be Noetherian integral schemes (see "Schemes"). Suppose that Y is normal, that is, all the stalks of the structure sheaf are integrally closed domains (see "Normal", "Integrally closed"). Let $f : X \to Y$ be a birational projective morphism. Then $f^{-1}(y)$ is connected for any $y \in Y$. □

Theorem. Let X and Y be reduced irreducible complex analytic spaces (see "Spaces, analytic -") and suppose Y is normal (reduced means that all the stalks of the structure sheaf have no nilpotent elements and normal that they are integrally closed domains; irreducible means that the underlying topological space is irreducible; see "Irreducible topological space"). Let $f : X \to Y$ be a proper surjective morphism with finite fibres and suppose there is an open subset U of Y such that $f^{-1}(u)$ is a point for all $u \in U$. Then f is an isomorphism. □

Zariski tangent space, differential forms, tangent bundle, normal bundle. ([104], [107], [140], [228]). Let K be an algebraically closed field. The Zariski tangent space of an algebraic variety X at a point P, denoted by $\Theta_{X,P}$, is defined to be the dual of m_P/m_P^2, where m_P is the maximal ideal of $\mathcal{O}_{X,P}$ and $\mathcal{O}_{X,P}$ is the stalk in P of the sheaf of the regular functions on X (see "Varieties, algebraic -, Zariski topology, regular and rational functions, morphisms and rational maps"):

$$\Theta_{X,P} = (m_P/m_P^2)^\vee.$$

If X is an affine algebraic variety in $\mathbb{A}^n_K$ and $f_1, \ldots, f_k$ generate $I(X)$, we can define the tangent space T_PX of X at $P \in X$ as

$$T_PX = \left\{ (x_1, \ldots, x_n) \,\middle|\, \sum_{i=1,\ldots,n} \frac{\partial f_j}{\partial x_i}(P)(x_i - P_i) = 0 \;\text{ for }\; j = 1, \ldots, k \right\}$$

(observe that the partial derivatives and the differential of a polynomial can be defined in an obvious way over any field by applying the usual rules for derivatives).
We can prove that the differential at P gives an isomorphism from m_P/m_P^2 to $T_PX^\vee$, so the Zariski tangent space $\Theta_{X,P}$ is isomorphic to T_PX. In particular for any quasi-projective algebraic variety X the Zariski tangent space at $P \in X$ is isomorphic to the tangent space at P of any of its affine neighborhoods. If X is a projective variety contained in $\mathbb{P}^n_K$ (with homogeneous coordinates $x_0, \ldots, x_n$), the closure in $\mathbb{P}^n_K$ of the tangent space at P of $X \cap \{x_i \neq 0\}$ does not depend on i, for i such that $P_i \neq 0$, and it too is often called the "tangent space of X at P".

Let X be an algebraic variety over K. We say that a function η associating to every point $P \in X$ an element of $\wedge^k \Theta^\vee_{X,P}$ is a **regular differential *k*-form** if every point $P \in X$ has an open neighborhood U such that η restricted to U can be written as linear combination with coefficients in the coordinate ring $\Gamma(U)$ of elements of the kind $df_1 \wedge \cdots \wedge df_k$ with $f_1, \ldots, f_k \in \Gamma(U)$, where d is the differential (see "Varieties, algebraic -, Zariski topology, regular and rational functions, morphisms and rational maps" for the definition of coordinate ring). For any open subset V, denote by Ω^k_V the set of the regular differential k-forms on V. Let Ω^k be the sheaf of modules over $\mathcal{O}_X$ assigning Ω^k_V to any open subset V of X.

Furthermore, consider the set of the pairs (U, η) where U is an open subset of X and η is a regular differential k-form on U. We say that (U, η) and (U', η') are equivalent if $\eta = \eta'$ on $U \cap U'$. A **rational differential *k*-form** on X is an equivalence class of a pair.

Theorem. If X is smooth, the sheaf Ω^k is a locally free sheaf. □

Thus, if X is smooth, for any $k \in \mathbb{N}$, the sheaf Ω^k determines a vector bundle. The bundle determined by Ω^1 is called cotangent bundle. Obviously the bundle determined by Ω^k is the k-wedge product of the cotangent bundle. The dual of the cotangent bundle is called **tangent bundle**; we denote it by Θ_X; its fibre in every point $P \in X$ is the Zariski tangent space at P.
Moreover, if X is a smooth variety and Y a smooth closed subvariety, we define the **normal bundle** $N_{X,Y}$ to Y in X to be the quotient of the restriction of Θ_X to Y by Θ_Y:

$$N_{X|Y} = \Theta_X|_Y / \Theta_Y.$$

Zariski topology. See "Varieties, algebraic -, Zariski topology, regular and rational functions, morphisms and rational maps".

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