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Weighted progressive iteration approximation and convergence analysis

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ABSTRACT

We present a new and efficient method for weighted progressive iteration approximations of data points by using normalized totally positive bases. Compared to the usual progressive iteration approximation, our method has a faster convergence rate for any normalized totally positive basis, which is achieved by choosing an optimal value for the weight. For weighted progressive iteration approximations, we prove that the normalized B-basis of a space provides the fastest convergence rate among all normalized totally positive bases of the space. These results are also valid for tensor product surfaces.

1. Introduction

Approximation by polynomial curves and surfaces is one of the fundamental problems in CAGD. For a given set of data points one has to find a parametric curve or surface to approximate them. The literature on this topic is extensive and we refer the interested reader to Wang et al. (2006), Weiss et al. (2002) and the references therein.

A classical approach for solving it is the least-square approximation. The approximating curve or surface is obtained by minimizing an error criterion, which is generally defined as the weighted sum of a distance function to data points and some smooth functions. The active contour model for parametric curve and surface approximation was introduced in Pottmann et al. (2002) and Wang et al. (2006). The key idea is to iteratively change the control points of the active curve (or surface) so that it deforms towards the target shape represented by the point data. Recently, Feichtinger et al. (2008) formulated a novel framework of *dual evolution*, by simultaneously considering evolution processes for parametric spline curves and implicitly defined curves.

Another related idea is the use of the *progressive iteration approximation* property of some polynomial bases. Qi et al. (1975) and de Boor (1979) showed that the uniform cubic B-spline basis shares this property. This result was extended to the nonuniform cubic B-spline basis and the nonuniform cubic tensor product B-spline basis for surfaces (Lin et al., 2004). We outline their methods as follows. Let $(u_0(t),\dots,u_n(t))$ be a *blending basis*, i.e., these functions are nonnegative and satisfy $\sum_{i=0}^n u_i(t)=1$. Given a sequence of points $\{\mathbf{p}_i\}_{i=0}^n$, whose ith point is assigned to a parameter value t_i, $i=0,1,\dots,n$. The initial curve is generated by $\mathcal{C}^0(t)=\sum_{i=0}^n \mathbf{p}_i^0 u_i(t)$ with $\mathbf{p}_i^0=\mathbf{p}_i$ for all $i=0,1,\dots,n$. Then, the approximating curve after $k+1$ ($k\geqslant 0$) iterations is defined by $\mathcal{C}^{k+1}(t)=\sum_{i=0}^n \mathbf{p}_i^{k+1}u_i(t)$, where the control points are iteratively updated by $\mathbf{p}_i^{k+1}=\mathbf{p}_i^k+\Delta_i^k$, with $\Delta_i^k=\mathbf{p}_i-\mathcal{C}^k(t_i)$ denoting adjusting vectors. We say that the initial curve has the *progressive iteration approximation* property if $\mathcal{C}^k(t_i)$ converges to $\mathbf{p}_i$ for all i.

Recall that the collocation matrix of a system $(u_0(t),\dots,u_n(t))$ at the parameters $\{t_i\}_{i=0}^m$ in $\mathbb{R}$ is given by

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$$M\begin{pmatrix} u_0, \dots, u_n \\ t_0, \dots, t_m \end{pmatrix} := \left(u_j(t_i)\right)_{i=0,\dots,m}^{j=0,\dots,n}.$$

A matrix is *totally positive* (TP) if all its minors are nonnegative and a basis is *totally positive* (TP) when all its collocation matrices are TP. A blending basis that is TP is said to be a *normalized totally positive* (NTP) basis. It is well known that (cf. Peña, 1999) the bases providing shape preserving representations are precisely the NTP bases and that a space with an NTP basis always has a unique normalized B-basis, which is the basis with optimal shape preserving properties. For instance, both the Bernstein basis and the B-spline basis are normalized B-bases of the polynomial space (see Carnicer and Peña, 1993, 1994).

It was proved in Lin et al. (2005) that curves and tensor product surfaces generated by NTP bases satisfy the progressive iteration approximation property. Furthermore, Delgado and Peña (2007) proved that the normalized B-basis is the NTP basis with the fastest convergence rates and that the tensor product of normalized B-bases also present the fastest convergence rates. Recently, Cheng et al. (2009) used the progressive iteration approximation to construct interpolating Loop subdivision surfaces.

In this paper, we present a new and efficient method for weighted progressive iteration approximations of data points by using NTP bases. Different from previous methods (Delgado and Peña, 2007; Lin et al., 2004, 2005), we multiply all the adjusting vectors by a common weight for the iteration approximation. By choosing an optimal value for the weight, we prove that the weighted progressive iteration approximation shares the progressive iteration approximation property and has a faster convergence rate than the usual progressive iteration approximation, and that the normalized B-basis of a space provides the fastest convergence rate among all NTP bases of the space. These theoretical results are derived in Section 3. In Section 4, we show some numerical experiments which verify our theoretical analysis and demonstrate that, in comparison with previous methods, faster convergence has been obtained by our method.

2. Weighted progressive iteration approximation

2.1. The case of curves

Let $(u_0, \dots, u_n)$ be an NTP basis. Given a sequence of control points $\{\mathbf{p}_i\}_{i=0}^n$ in $\mathbb{R}^2$ or $\mathbb{R}^3$, we can generate the initial curve

$$\mathcal{C}^0(t) = \sum_{i=0}^{n} \mathbf{p}_i^0 u_i(t)$$

with $\mathbf{p}_i^0 = \mathbf{p}_i$ for all $i = 0, 1, \dots, n$.

We assign the control points $\mathbf{p}_i$ with the parameters t_i for all $i = 0, 1, \dots, n$, where $\{t_i\}_{i=0}^n$ is a real increasing sequence, i.e., $t_0 < t_1 < \dots < t_n$. Then, the remaining curves of the sequence, $\mathcal{C}^{k+1}(t)$ for $k \geqslant 0$, can be calculated as follows

$$\mathcal{C}^{k+1}(t) = \sum_{i=0}^{n} \mathbf{p}_i^{k+1} u_i(t),$$

where

$$\mathbf{p}_i^{k+1} = \mathbf{p}_i^k + \omega \Delta_i^k, \quad \Delta_i^k = \mathbf{p}_i - \mathcal{C}^k(t_i), \; i = 0, 1, \dots, n. \tag{1}$$

By (1), we can obtain

$$\Delta_j^k = \Delta_j^{k-1} - \omega \sum_{i=0}^{n} \Delta_i^{k-1} u_i(t_j), \quad j = 0, 1, \dots, n. \tag{2}$$

Then, the iterative process can be written in matrix form as follows:

$$\left[\Delta_0^k, \Delta_1^k, \dots, \Delta_n^k\right]^{\mathrm{T}} = (I - \omega B)\left[\Delta_0^{k-1}, \Delta_1^{k-1}, \dots, \Delta_n^{k-1}\right]^{\mathrm{T}} = (I - \omega B)^k \left[\Delta_0^0, \Delta_1^0, \dots, \Delta_n^0\right]^{\mathrm{T}}, \tag{3}$$

where I is the identity matrix of order $n+1$ and $B = (u_j(t_i))_{i=0,\dots,n}^{j=0,\dots,n}$ is the collocation matrix of the basis $(u_0, \dots, u_n)$ at $\{t_i\}_{i=0}^n$.

Thus, we get a curve sequence $\mathcal{C}^k(t), k = 0, 1, \dots$. If

$$\lim_{k \to \infty} \mathcal{C}^k(t_i) = \mathbf{p}_i, \quad i = 0, 1, \dots, n,$$

then the initial curve has the *weighted progressive iteration approximation* property. Moreover, it is equivalent to showing that $(I - \omega B)^k$ converges to the zero matrix of the same size.

Given an $n \times n$ matrix M, we denote by $\lambda_i(M)$, $i = 0, 1, \ldots, n$ its eigenvalues, which are sorted in nonincreasing order, i.e., $\lambda_0 \geqslant \lambda_1 \geqslant \cdots \geqslant \lambda_n$, and denote by $\rho(M)$ its spectral radius, which is the maximal absolute value of its eigenvalues, i.e., $\rho(M) = \max\{|\lambda_0|, |\lambda_n|\}$. Then the iterative process (3) converges when $\rho(I - \omega B) < 1$, and its speed depends on $\rho(I - \omega B)$: the smaller the value is, the faster the speed converges.

Remark 1. The weight ω in (1) can be taken as any positive value, as long as it can guarantee the convergence of the iterative process. We will show how to determine its value in Section 3 so that we can obtain the best approximation with the fastest convergence rate. On the contrary, a fixed value of $\omega = 1$ is implied in the earlier work (Delgado and Peña, 2007; Lin et al., 2004, 2005).

2.2. The case of surfaces

Let $(u_0, \ldots, u_m)$ and $(v_0, \ldots, v_n)$ be two NTP bases. Then, given a sequence of control points $\{\mathbf{p}_{ij}\}_{i=0,\ldots,m}^{j=0,\ldots,n}$ in $\mathbb{R}^3$, we can generate the initial surface

$$\mathcal{S}^0(x, y) = \sum_{i=0}^{m}\sum_{j=0}^{n} \mathbf{p}_{ij}^0 u_i(x) v_j(y)$$

with $\mathbf{p}_{ij}^0 = \mathbf{p}_{ij}$ for all $i = 0, 1, \ldots, m$, $j = 0, 1, \ldots, n$.

Analogous to the curve case, we assign the control points $\mathbf{p}_{ij}$ with the parameters (x_i, y_j), where $\{x_i\}_{i=0}^m$ and $\{y_j\}_{j=0}^n$ are two real increasing sequences, respectively. Then, the remaining surfaces of the sequence, $\mathcal{S}^{k+1}(x, y)$ for $k \geqslant 0$, can be calculated as follows

$$\mathcal{S}^{k+1}(x, y) = \sum_{i=0}^{m}\sum_{j=0}^{n} \mathbf{p}_{ij}^{k+1} u_i(x) v_j(y),$$

where

$$\mathbf{p}_{ij}^{k+1} = \mathbf{p}_{ij}^k + \omega \Delta_{ij}^k, \quad \Delta_{ij}^k = \mathbf{p}_{ij} - \mathcal{S}^k(x_i, y_j),\ i = 0, 1, \ldots, m,\ j = 0, 1, \ldots, n. \tag{4}$$

Similarly, by (4), we can obtain

$$\Delta_{rs}^k = \Delta_{rs}^{k-1} - \omega \sum_{i=0}^{m}\sum_{j=0}^{n} \Delta_{ij}^{k-1} u_i(x_r) v_j(y_s), \quad r = 0, 1, \ldots, m,\ s = 0, 1, \ldots, n. \tag{5}$$

Then, the iterative process can be written in matrix form

$$\Delta^k = (I - \omega B)\Delta^{k-1} = (I - \omega B)^k \Delta^0, \tag{6}$$

where

$$\Delta^j = \left[\Delta_{00}^j, \Delta_{01}^j, \ldots, \Delta_{0n}^j, \Delta_{10}^j, \Delta_{11}^j, \ldots, \Delta_{1n}^j, \ldots, \Delta_{m0}^j, \Delta_{m1}^j, \ldots, \Delta_{mn}^j\right]^{\mathrm{T}}, \quad j = 0, 1, \ldots, k,$$

I is the identity matrix of order $(m+1)(n+1)$, and $B = B_1 \otimes B_2$ is the Kronecker product of two collocation matrices $B_1 = (u_j(x_i))_{i=0,\ldots,m}^{j=0,\ldots,m}$ and $B_2 = (v_j(y_i))_{i=0,\ldots,n}^{j=0,\ldots,n}$, see e.g. Delgado and Peña (2007), Stephen (1979).

Thus, we get a surface sequence $\mathcal{S}^k(x, y), k = 0, 1, \ldots.$ If

$$\lim_{k\to\infty} \mathcal{S}^k(x_i, y_j) = \mathbf{p}_{ij}, \quad i = 0, 1, \ldots, m,\ j = 0, 1, \ldots, n,$$

then the initial surface has the *weighted progressive iteration approximation* property. Moreover, it is equivalent to showing that $(I - \omega B)^k$ converges to the zero matrix of the same size.

3. Main results

We first introduce the following lemma.

Lemma 1. *Let $B = (u_j(t_i))_{i=0,\ldots,n}^{j=0,\ldots,n}$ be any nonsingular collocation matrix of an NTP basis, and let $\lambda_i(B)$, $i = 0, 1, \ldots, n$ be its eigenvalues sorted in nonincreasing order. Then,*

(a) $\lambda_0(B) = 1$, *and* $0 < \lambda_i(B) \leqslant 1$ *for all* $i = 1, 2, \ldots, n$;
(b) $0 \leqslant \rho(I - B) < 1$;
(c) $\rho(I - \omega B) = \max\{|1 - \omega|, |1 - \omega\lambda_n(B)|\}$.

Proof. For (a) and (b), see the proof of Theorem 2.1 in Lin et al. (2005). Then, we can easily deduce (c) from (a). $\square$

The main results of the paper are the following.

Theorem 1. *Let $(u_0(t), \ldots, u_n(t))$ be an NTP basis of the space U, and let $B = (u_j(t_i))_{i=0,\ldots,n}^{j=0,\ldots,n}$ be any nonsingular collocation matrix of the basis at $\{t_i\}_{i=0}^n$. Then, the weighted progressive iteration approximation based on $\{t_i\}_{i=0}^n$ has the fastest convergence rate when*

$$\omega = \frac{2}{1+\lambda_n(B)}, \tag{7}$$

where $\lambda_n(B)$ is the smallest eigenvalue of B, and in such case,

$$\rho(I-\omega B) = \frac{1-\lambda_n(B)}{1+\lambda_n(B)}. \tag{8}$$

Proof. It follows from Lemma 1 that $\lambda_n \in (0,1]$. When $\omega = 1$, we have

$$\rho(I-\omega B) = 1-\lambda_n.$$

Firstly, we show that $0<\omega<2$ must hold. Otherwise,

$$\rho(I-\omega B) = \max\bigl\{|1-\omega|, |1-\omega\lambda_n|\bigr\} \geqslant |1-\omega| \geqslant 1,$$

which will violate the convergence of the iterative process.

Secondly, for any $\omega \in (0,1)$, we can obtain

$$\rho(I-\omega B) = \max\bigl\{|1-\omega|, |1-\omega\lambda_n|\bigr\} = 1-\omega\lambda_n > 1-\lambda_n.$$

Therefore, when $\omega \in (0,1)$, the convergence rate is always slower than that of $\omega = 1$.

Finally, we assume $\omega \in (1,2)$. If $\omega\lambda_n > 1$, then

$$\rho(I-\omega B) = \max\bigl\{|1-\omega|, |1-\omega\lambda_n|\bigr\} = \max\{\omega-1, \omega\lambda_n-1\} \geqslant \omega-1 > \frac{1}{\lambda_n}-1 \geqslant 1-\lambda_n,$$

which means that the convergence rate in such case is always slower than that of $\omega = 1$. On the other hand, if $\omega\lambda_n \leqslant 1$, then

$$\rho(I-\omega B) = \max\bigl\{|1-\omega|, |1-\omega\lambda_n|\bigr\} = \max\{\omega-1, 1-\omega\lambda_n\} = \begin{cases} 1-\omega\lambda_n, & 1<\omega<\frac{2}{1+\lambda_n}, \\ \omega-1, & \frac{2}{1+\lambda_n} \leqslant \omega < 2. \end{cases}$$

Obviously, $\rho(I-\omega B)$ will reach the minimum $\frac{1-\lambda_n}{1+\lambda_n}$ when $\omega = \frac{2}{1+\lambda_n}$. And such value of ω can be chosen since $\frac{2}{1+\lambda_n} \cdot \lambda_n \leqslant 1$ is always satisfied due to $\lambda_n \in (0,1]$. This completes the proof. $\square$

Theorem 2. *Given a space U with an NTP basis, let $\{t_i\}_{i=0}^n$ in the parameter domain of U be a real increasing sequence. Then, for the weighted progressive iteration approximation based on $\{t_i\}_{i=0}^n$, the normalized B-basis of U provides the fastest convergence rate among all NTP bases of U.*

Proof. By Theorem 4.2(i) of Carnicer and Peña (1994), U has a unique normalized B-basis $(b_0(t), \ldots, b_n(t))$. Let $(v_0(t), \ldots, v_n(t))$ be any other NTP basis of U. Denote by $B := (b_j(t_i))_{i=0,\ldots,n}^{j=0,\ldots,n}$ and $V := (v_j(t_i))_{i=0,\ldots,n}^{j=0,\ldots,n}$ their collocation matrices at $\{t_i\}_{i=0}^n$, respectively.

From Theorem 1, we obtain the spectral radii of the two kinds of iteration approximations when having the fastest convergence rates as follows:

$$\rho(I-\omega_B B) = \frac{1-\lambda_n(B)}{1+\lambda_n(B)}, \qquad \rho(I-\omega_V V) = \frac{1-\lambda_n(V)}{1+\lambda_n(V)}.$$

By Theorem 4 of Delgado and Peña (2007), we get $1-\lambda_n(B) < 1-\lambda_n(V)$, which implies $\lambda_n(B) > \lambda_n(V)$. Then, it is easy to verify that

$$\rho(I-\omega_B B) < \rho(I-\omega_V V),$$

which completes the proof. $\square$

Theorem 3. *Let $(u_0(t),\dots,u_m(t))$ and $(v_0(t),\dots,v_n(t))$ be the NTP bases of the spaces U and V respectively, and let $B = B_1 \otimes B_2$ be the Kronecker product of two collocation matrices $B_1 = (u_j(x_i))_{i=0,\dots,m}^{j=0,\dots,m}$ and $B_2 = (v_j(y_i))_{i=0,\dots,n}^{j=0,\dots,n}$. Then, the weighted progressive iteration approximation based on $\{x_i\}_{i=0}^m$ and $\{y_i\}_{i=0}^n$ has the fastest convergence rate when*

$$\omega = \frac{2}{1+\lambda_m(B_1)\mu_n(B_2)}, \tag{9}$$

where $\lambda_m(B_1)$ and $\mu_n(B_2)$ are the smallest eigenvalues of B_1 and B_2 respectively, and in such case,

$$\rho(I-\omega B) = \frac{1-\lambda_m(B_1)\mu_n(B_2)}{1+\lambda_m(B_1)\mu_n(B_2)}. \tag{10}$$

Proof. We denote the eigenvalues of the matrices B_1 and B_2 by $\lambda_i(B_1)$, $i = 0,1,\dots,m$ and by $\mu_j(B_2)$, $j = 0,1,\dots,n$, respectively, and sort them as follows:

$$\lambda_0 \geqslant \lambda_1 \geqslant \cdots \geqslant \lambda_m, \qquad \mu_0 \geqslant \mu_1 \geqslant \cdots \geqslant \mu_n.$$

It is well known that the $(m+1)(n+1)$ eigenvalues of the Kronecker product $B = B_1 \otimes B_2$ are given by the products of λ_i and μ_j (cf. Theorem 4.2.12 of Stephen, 1979),

$$\lambda(B) = \{\lambda_i\mu_j \mid i = 0,1,\dots,m,\ j = 0,1,\dots,n\}.$$

Therefore, similar to Lemma 1, we get

$$\rho(I-\omega B) = \max\{|1-\omega|, |1-\omega\lambda_m\mu_n|\}.$$

Then, we can derive the conclusion in the same way as in the proof of Theorem 1. □

Theorem 4. *Given spaces U, V with NTP bases, let $\{x_i\}_{i=0}^m$ and $\{y_i\}_{i=0}^n$ in the parameter domains of U and V be two real increasing sequences, respectively. Then, for the weighted progressive iteration approximation based on $\{x_i\}_{i=0}^m$ and $\{y_i\}_{i=0}^n$, the tensor product of the normalized B-bases of U, V provides the fastest convergence rate among all bases which are tensor product of NTP bases of U, V.*

Proof. The proof is similar to that of Theorem 2 for the curve case, and is therefore left to the reader. □

4. Numerical experiments

In this section, we will apply weighted progressive iteration approximations to the following two examples proposed in Delgado and Peña (2007) and make a comparison with their method.

Example 1. Consider the Lemniscate of Gerono given by

$$\big(x(t), y(t)\big) = (\cos t, \sin t \cos t), \quad t \in [0, 2\pi].$$

A sequence of 11 points $\{\mathbf{p}_i\}_{i=0}^{10}$ is sampled from the parameter curve in the following way

$$\mathbf{p}_i = \big(x(s_i), y(s_i)\big), \quad s_i = -\frac{\pi}{2} + i\cdot\frac{2\pi}{10},\ i = 0,1,\dots,10.$$

Example 2. Consider a helix of radius 5 given by

$$\big(x(t), y(t), z(t)\big) = (5\cos t, 5\sin t, t), \quad t \in [0, 6\pi].$$

A sequence of 19 points $\{\mathbf{p}_i\}_{i=0}^{18}$ is sampled from the helix in the following way

$$\mathbf{p}_i = \big(x(s_i), y(s_i), z(s_i)\big), \quad s_i = i\cdot\frac{\pi}{3},\ i = 0,1,\dots,18.$$

Given the control points $\{\mathbf{p}_i\}_{i=0}^n$, a Bézier curve of degree n is defined by

$$\mathbf{b}(t) = \sum_{i=0}^{n} \mathbf{p}_i b_i^n(t), \quad t \in [0,1],$$

where $b_i^n(t) = \binom{n}{i}(1-t)^{n-i}t^i$, $i = 0,1,\dots,n$ are the Bernstein polynomials of degree n. The Bernstein basis $(b_0^n(t),\dots,b_n^n(t))$ is the normalized B-basis of the space of polynomials of degree at most n (Π_n). On the other hand, a Said–Ball curve of degree n can be defined by

Table 1
The smallest eigenvalue λ_n of the collocation matrix and the weight ω when the iteration approximation has the fastest convergence rate. The value of $\omega = 2.000000$ is rounded in their last digit and cannot be reached.

n	Bernstein basis		Said–Ball basis	
	λ_n	ω	λ_n	ω
1	1	1	1	1
2	0.5	1.333333	0.5	1.333333
3	2.222222e−01	1.636364	1.481481e−01	1.741935
4	9.375000e−02	1.828571	8.178722e−02	1.848792
5	3.840000e−02	1.926040	1.966332e−02	1.961432
6	1.543210e−02	1.969605	1.113068e−02	1.977984
7	6.119899e−03	1.987835	2.493639e−03	1.995025
8	2.403259e−03	1.995205	1.432656e−03	1.997139
9	9.366567e−04	1.998128	3.097357e−04	1.999381
10	3.628800e−04	1.999275	1.795492e−04	1.999641
11	1.399059e−04	1.999720	3.799744e−05	1.999924
12	5.372322e−05	1.999893	2.215383e−05	1.999956
13	2.055970e−05	1.999959	4.622286e−06	1.999991
14	7.845414e−06	1.999984	2.705427e−06	1.999995
15	2.986281e−06	1.999994	5.588150e−07	1.999999
16	1.134227e−06	1.999998	3.279531e−07	1.999999
17	4.299687e−07	1.999999	6.723373e−08	2.000000
18	1.627181e−07	2.000000	3.953194e−08	2.000000
19	6.148599e−08	2.000000	8.057731e−09	2.000000
20	2.320196e−08	2.000000	4.744060e−09	2.000000

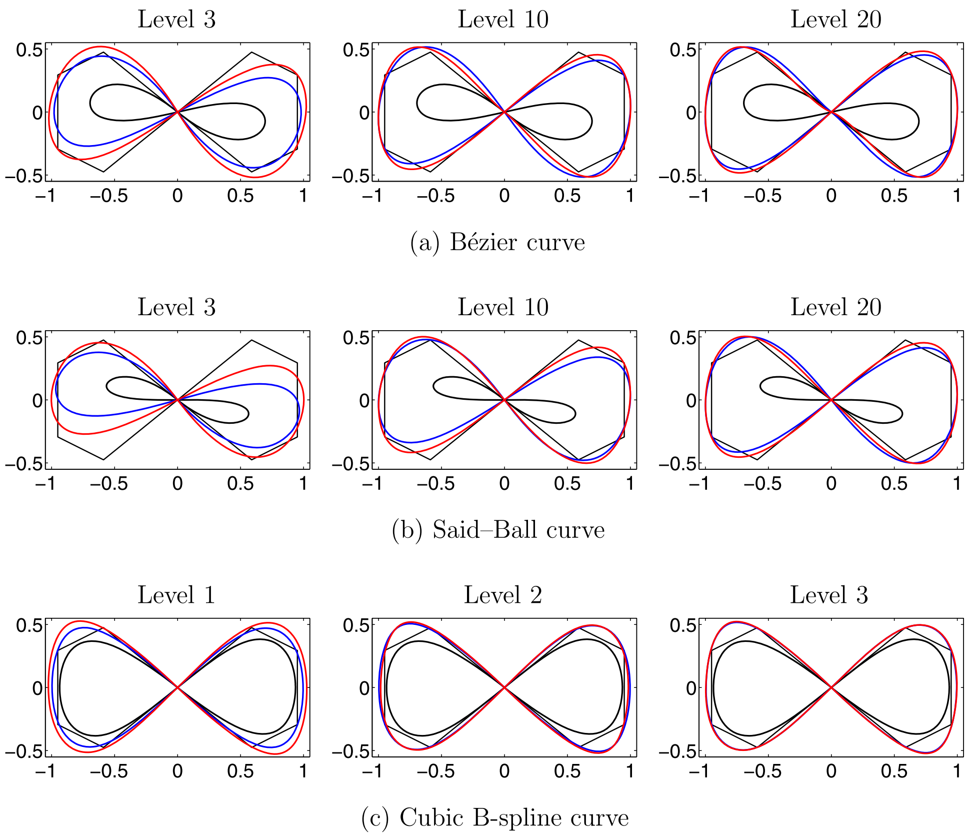
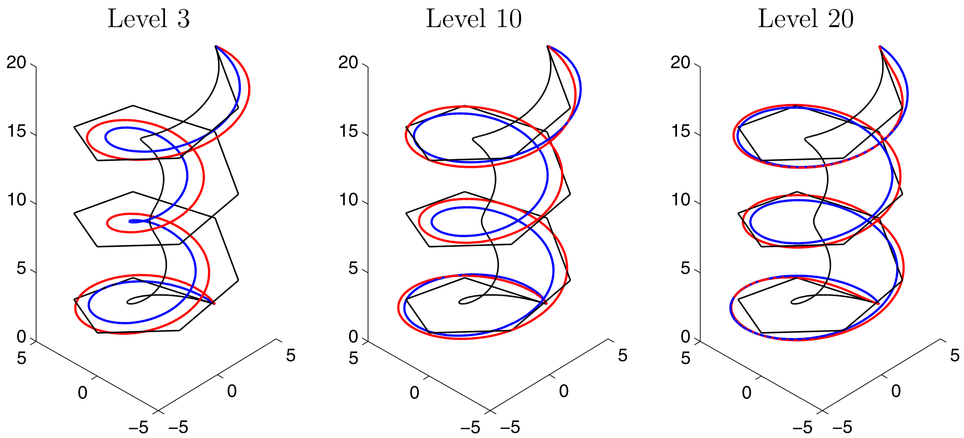


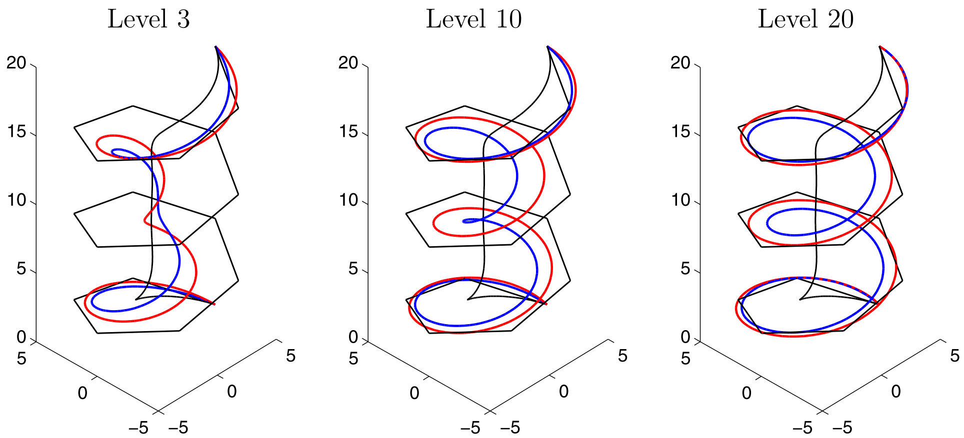
Fig. 1. Approximations of the Lemniscate of Gerono.

$$\mathbf{s}(t) = \sum_{i=0}^{n} \mathbf{p}_i s_i^n(t), \quad t \in [0,1],$$

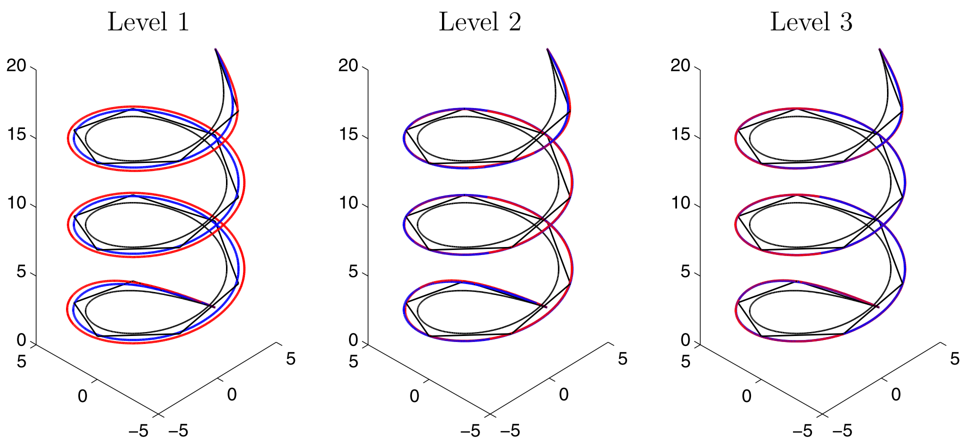
where the Said–Ball basis $(s_0^n(t), \ldots, s_n^n(t))$ forms an NTP basis of Π_n (cf. Proposition 3 of Delgado and Peña, 2006), whose definition can be found e.g. in Delgado and Peña (2006).



(a) Bézier curve



(b) Said–Ball curve



(c) Cubic B-spline curve

Fig. 2. Approximations of the helix.

Table 2
Maximum Euclidean norm when fitting the Lemniscate of Gerono.

Level	Bézier curve		Said–Ball curve		Cubic B-spline curve	
	WPIA	PIA	WPIA	PIA	WPIA	PIA
0	4.620135e–01	4.620135e–01	5.470408e–01	5.470408e–01	1.723725e–01	1.723725e–01
1	2.631793e–01	3.490831e–01	4.142717e–01	4.597077e–01	4.090235e–02	5.708451e–02
2	1.721945e–01	2.801053e–01	2.899616e–01	4.013677e–01	1.237301e–02	2.281479e–02
3	1.340257e–01	2.312598e–01	2.224117e–01	3.516669e–01	5.329585e–03	1.048108e–02
4	1.092036e–01	1.954023e–01	1.778551e–01	3.086119e–01	2.036615e–03	5.304174e–03
5	9.512785e–02	1.685992e–01	1.486947e–01	2.719752e–01	8.263994e–04	2.855718e–03
6	8.350914e–02	1.482074e–01	1.288604e–01	2.412384e–01	3.261151e–04	1.598827e–03
7	7.463709e–02	1.323846e–01	1.148307e–01	2.156355e–01	1.330285e–04	9.265572e–04
8	6.687703e–02	1.198404e–01	1.044383e–01	1.943528e–01	5.267752e–05	5.665645e–04
9	6.027894e–02	1.096699e–01	9.636226e–02	1.766392e–01	2.194962e–05	3.469003e–04
10	5.446468e–02	1.012377e–01	8.979356e–02	1.618460e–01	8.665911e–06	2.127948e–04
15	3.441901e–02	7.333693e–02	6.743287e–02	1.158150e–01	1.115186e–07	1.887495e–05
20	2.369717e–02	5.637195e–02	5.251825e–02	9.293990e–02	1.380113e–09	1.700982e–06
25	1.786766e–02	4.447245e–02	4.151614e–02	7.883642e–02	2.174935e–11	1.539271e–07
30	1.461343e–02	3.579771e–02	3.329604e–02	6.853643e–02	3.017558e–13	1.394385e–08
35	1.309044e–02	2.940473e–02	2.713955e–02	6.025712e–02	4.986434e–15	1.263466e–09
40	1.271751e–02	2.466991e–02	2.252130e–02	5.330010e–02	2.288783e–16	1.144914e–10

Table 3
Maximum Euclidean norm when fitting the helix.

Level	Bézier curve		Said–Ball curve		Cubic B-spline curve	
	WPIA	PIA	WPIA	PIA	WPIA	PIA
0	4.624577e+00	4.624577e+00	5.698051e+00	5.698051e+00	1.105845e+00	1.105845e+00
1	3.868353e+00	4.246465e+00	6.613912e+00	5.965671e+00	3.097317e–01	3.676239e–01
2	3.036176e+00	3.849364e+00	6.510416e+00	5.668207e+00	8.375265e–02	1.471971e–01
3	2.244280e+00	3.428212e+00	4.892650e+00	5.546793e+00	2.975002e–02	6.779905e–02
4	2.117219e+00	3.048900e+00	4.137706e+00	5.339926e+00	1.166351e–02	3.438666e–02
5	1.622221e+00	2.723716e+00	3.395129e+00	5.065304e+00	4.533284e–03	1.852054e–02
6	1.558006e+00	2.447822e+00	2.970326e+00	4.768933e+00	1.848516e–03	1.035501e–02
7	1.256018e+00	2.213283e+00	2.566844e+00	4.453045e+00	8.056670e–04	5.938120e–03
8	1.211104e+00	2.012801e+00	2.270679e+00	4.141841e+00	3.220072e–04	3.626373e–03
9	1.010787e+00	1.840375e+00	2.017167e+00	3.847403e+00	1.532987e–04	2.221317e–03
10	9.820949e–01	1.691200e+00	1.812416e+00	3.574839e+00	6.254915e–05	1.365647e–03
15	6.042277e–01	1.181968e+00	1.164660e+00	2.539184e+00	1.334829e–06	1.264084e–04
20	4.652951e–01	8.971303e–01	8.342646e–01	1.901447e+00	3.205500e–08	1.275990e–05
25	3.338619e–01	7.190871e–01	6.308106e–01	1.490377e+00	8.089789e–10	1.390944e–06
30	2.786617e–01	5.968398e–01	4.945052e–01	1.209260e+00	2.130513e–11	1.542153e–07
35	2.322219e–01	5.068952e–01	3.997958e–01	1.006389e+00	5.854147e–13	1.733723e–08
40	2.013669e–01	4.376396e–01	3.318458e–01	8.576443e–01	1.670845e–14	2.003222e–09

Firstly, let us consider the collocation matrices $B = (b_j^n(t_i))_{i=0,\ldots,n}^{j=0,\ldots,n}$ and $S = (s_j^n(t_i))_{i=0,\ldots,n}^{j=0,\ldots,n}$ of the Bernstein and Said–Ball bases at $\{t_i\}_{i=0}^n$, respectively. In all the examples, we have used a simple choice for the parameter values by setting $t_i = i/n$, $i = 0, 1, \ldots, n$. In Table 1, we have listed the smallest eigenvalue λ_n of every collocation matrix for degree n from 1 to 20. Together, for every case, the value of the weight ω calculated by (7) is also shown, which implies that the weighted progressive iteration approximation will have the fastest convergence rate. We can see from the results that the value of ω is close to (but less than) 2 when $n \geqslant 5$.

In Fig. 1, we have compared the approximations of the Lemniscate of Gerono in Example 1 by Bézier curves and Said–Ball curves of degree 10, and uniform cubic B-spline curves. And in Fig. 2, we have compared the approximations of the helix in Example 2 by Bézier curves and Said–Ball curves of degree 18, and uniform cubic B-spline curves. For simplicity, we only use uniform cubic B-spline curves associated with the knot vector $\mathbf{t} = \{0, 0, 0, 0, 1, 2, \ldots, n-1, n, n, n, n\}$. In all figures, the initial curves are demonstrated in the innermost position, and the approximating curves after specific iteration levels by our method are displayed in red and those by the method in Delgado and Peña (2007) and Lin et al. (2005) are displayed in blue.[1] Clearly, the weighted progressive iteration approximations in this paper provide better results.

In Tables 2 and 3, we have listed the fitting errors of the curve sequences after specific iteration levels. The fitting error at each level $k \geqslant 0$ is taken as the maximum Euclidean norm of the adjusting vectors,

$$\varepsilon_k := \max\{\|\Delta_i^k\|: i = 0, 1, \ldots, n\}.$$

[1] For interpretation of the references to colour in these figures, the reader is referred to the web version of this article.

From the tables, we can see that weighted progressive iteration approximations have faster convergence rates than progressive iteration approximations, which confirms to Theorem 1. And the convergence rates can be greatly improved (about two times) by weighted approximations. For example, by weighted approximations after 5 iterations, we can obtain the errors which will be reached by previous method after 10 iterations. Furthermore, weighted progressive iteration approximations by Bézier curves have faster convergence rates than those by Said–Ball curves, which has been theoretically proved in Theorem 2. Finally, we have shown experimentally that the B-spline form exhibits faster convergence than the Bézier form.

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