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Perverse Sheaves and Applications to Representation Theory

Pramod N. Achar

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Preface

Perverse sheaves were discovered in the fall of 1980 by Beilinson–Bernstein–Deligne–Gabber [**24**], sitting at the confluence of two major developments of the 1970s: the intersection homology theory of Goresky–MacPherson [**85**, **86**], and the Riemann–Hilbert correspondence, due to Kashiwara [**121**] and Mebkhout [**175**]. Those same two ingredients had already been combined a few months prior for a breakthrough in representation theory: the proof of the Kazhdan–Lusztig conjecture on Lie algebra representations [**26**, **50**, **129**, **130**]. From today's perspective, the Kazhdan–Lusztig conjecture may be seen as the spectacular first application of perverse sheaves. Ever since, perverse sheaves have been a powerful tool of fundamental importance in geometric representation theory.

This is partly due to the diversity of perspectives from which one may approach this subject. Perverse sheaves have close connections (especially in their computational aspects) to topics in classical algebraic topology, including fundamental groups, covering spaces, and singular cohomology. On the other hand, perverse sheaves (at least with field coefficients) have algebraic features reminiscent of modules over an artinian ring: every perverse sheaf has a composition series, and one can classify the simple perverse sheaves.

But in my opinion, the most significant reason for the usefulness of perverse sheaves is the following secret known to experts: perverse sheaves are *easy*, in the sense that most arguments come down to a rather short list of tools, such as proper base change, smooth pullback, and open–closed distinguished triangles. In practice, one can reason and compute with perverse sheaves just using a list of these tools, much as calculus students might use a table of integrals. One does not have to dig into the details of flabby resolutions or sheafification any more than a calculus student needs to revisit Riemann sums to integrate a polynomial. In this book, I have tried to emphasize this perspective with computational exercises and with the **Quick Reference** pages near the end of the book.

Organization and prerequisites. This book is divided into two parts: the first six chapters develop the general theory of constructible sheaves on complex algebraic varieties, and the last four chapters give brief introductions to selected applications of perverse sheaves in representation theory. The prerequisites for the first six chapters are: familiarity with the language of derived and triangulated categories; familiarity with introductory algebraic topology; and (starting from Chapter 2) some minimal familiarity with complex algebraic varieties. For the applications in Chapters 7–10, some knowledge of Lie theory is required.

Chapter 1 covers the foundations of sheaf theory on topological spaces, including the definitions of the six basic sheaf operations, and a number of natural compatibilities between them, such as the proper base change theorem and the projection formula. This chapter also contains material on local systems and fundamental groups. Much of this material can be found in many other textbook-level sources, so a number of proofs in this chapter are merely sketched, or sometimes omitted entirely.

In **Chapter 2** we begin the study of constructible sheaves on complex algebraic varieties. Some highlights of results proved in this chapter include Artin's vanishing theorem, the Verdier duality theorem, and the "constructibility theorem" (which says that in the algebraic setting, all six sheaf operations preserve constructibility). We also show that in the setting of constructible sheaves, the external tensor product functor and the extension-of-scalars functor commute with all sheaf operations. The chapter ends with a selection of other topics related to constructible sheaves, including hyperbolic localization, Borel–Moore homology, and fundamental classes.

In **Chapter 3**, we begin the study of perverse sheaves, including the important special case of intersection cohomology complexes. Key results in this chapter describe the behavior of perverse sheaves with respect to push-forward along affine morphisms, and pullback along smooth morphisms. The former is very closely related to Artin's vanishing theorem. In the context of the latter, we prove that perverse sheaves satisfy "smooth descent"—that is, a perverse sheaf can be recovered from its pullback along a smooth surjective morphism. The chapter concludes with a discussion of two of the deepest results about perverse sheaves with coefficients in $\mathbb{Q}$: the decomposition theorem and the hard Lefschetz theorem.

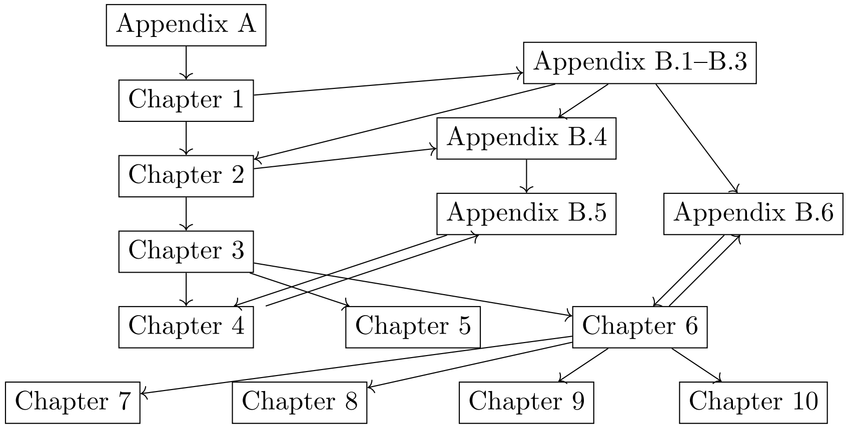
Chapter 4 discusses the nearby cycles functor. The definition of this functor requires leaving the algebraic setting (it involves the exponential map $\exp : \mathbb{C} \to \mathbb{C}^\times$), so most results from Chapter 2 cannot be applied directly. Nevertheless, we prove that this functor preserves constructibility; that it takes perverse sheaves to perverse sheaves; and that it is compatible with Verdier duality and the extension of scalars. As an application, we prove Beilinson's theorem, which says that the derived category of perverse sheaves (with coefficients in a field) is equivalent to the constructible derived category.

Chapter 5 gives an overview of two separate (but conceptually related) topics: mixed ℓ-adic sheaves in the étale topology, and mixed Hodge modules. Both of these theories provide a kind of "enrichment" of perverse sheaves: the objects carry additional structure, most notably the weight filtration. This chapter includes discussions of some related side topics, including the sheaf–function correspondence and the Riemann–Hilbert correspondence. Most theorems in this chapter are stated without proof.

The final chapter in the first part of the book, **Chapter 6**, is devoted to the study of equivariant sheaves. It is straightforward to define the abelian category of equivariant sheaves (or equivariant perverse sheaves), but it is rather nontrivial to define the correct triangulated analogue. (The derived category of the abelian category of equivariant sheaves is usually the "wrong" answer.) We present a solution to this problem following Bernstein–Lunts. We also study compatibilities of sheaf functors with various ways of modifying the group action, such as forgetting, inflation, and averaging. Perhaps the two most useful results from this chapter are the quotient equivalence and the induction equivalence.

The remaining chapters deal with applications in representation theory.

Chapter 7 deals with the study of Borel-equivariant perverse sheaves on the flag variety of a reductive group. This chapter contains a proof that these perverse



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sheaves give a categorification of the Hecke algebra. This fact, which essentially goes back to a 1980 paper of Kazhdan–Lusztig, is one of the ingredients in the Kazhdan–Lusztig conjectures for Lie algebra representations. This chapter also discusses some more recent developments around sheaves on flag varieties, including parity sheaves and Soergel bimodules.

Chapter 8 studies perverse sheaves on the nilpotent cone of the Lie algebra of a reductive group. The starting point for this topic was Springer's discovery in the late 1970s that the stalks of some of these perverse sheaves (with coefficients in $\mathbb{Q}$) carry a natural action of the Weyl group. By the mid-1980s, Lusztig had extended Springer's work to cover all perverse sheaves (still with coefficients in $\mathbb{Q}$); this became the starting point for his theory of character sheaves. This chapter also discusses recent developments on Springer theory for perverse sheaves with coefficients in a field of positive characteristic.

In **Chapter 9**, we study perverse sheaves on the affine Grassmannian of a reductive group G. Results of Lusztig going back to 1983 indicated that these perverse sheaves contained a great deal of information about representations of the Langlands dual group $\check{G}$. In a landmark 2007 paper, Mirković and Vilonen, following an idea of Drinfeld, proved that this can be upgraded to an equivalence of tensor categories, known as the geometric Satake equivalence. We give proofs of the more sheaf-theoretic steps in this theorem, but we will not prove it in full.

Lastly, in **Chapter 10**, we use perverse sheaves on the space of representations of a quiver to construct the canonical basis for a quantum group. The fact that quantum groups are related to (functions on the space of) quiver representations is due to Ringel. The project of upgrading this by replacing functions by sheaves is due to Lusztig.

The book concludes with two appendices. **Appendix A** contains background (mostly without proofs) on category theory and homological algebra. One fact that is proved is the duality theorem for rings of finite global dimension. This result can be seen as a precursor to Verdier duality. **Appendix B** contains a number of calculations involving sheaves on $\mathbb{C}^n$. The results in this appendix are enlightening examples in their own right, but they are also needed for the proofs of a number of theorems in the main body of the book.

Acknowledgments. This book grew out of notes for a mini-course I gave at East China Normal University in July 2015 on the topic of "Perverse sheaves in representation theory." This mini-course was part of a workshop organized by Bin Shu and Weiqiang Wang. I am grateful to them for the opportunity to participate, and especially to Weiqiang Wang for strongly encouraging me to expand my lecture notes into a book.

Over the years, I have had the opportunity to teach a number of graduate courses at Louisiana State University on topics related to this book, including homological algebra, sheaf theory, and Lie theory. In the 2017–2018 academic year, I co-taught a two-semester sequence with my colleague Daniel Sage on sheaves and geometric representation theory. I am grateful to him and to all the students in these courses for the feedback they have given me. These experiences have shaped the presentation of a number of topics in this book.

I learned this subject myself largely from my collaborators. In particular, I have learned a number of explicit examples from Anthony Henderson, Daniel Juteau, Carl Mautner, and Simon Riche. Some of these examples appear as exercises in this book.

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Some notation and conventions. The 1-point topological space is denoted by pt. For any topological space X, the unique continuous map from X to pt is denoted by $\mathrm{a}_X : X \to \mathrm{pt}$.

All rings in this book are unital. Sheaves will almost always have coefficients in a commutative ring, usually denoted by $\Bbbk$. Starting from Chapter 2, the ring $\Bbbk$ is almost always assumed to be noetherian and of finite global dimension. The category of $\Bbbk$-modules is denoted by $\Bbbk$-mod, and the category of finitely generated $\Bbbk$-modules by $\Bbbk\text{-mod}^{\mathrm{fg}}$. However, if π is a group and $\Bbbk[\pi]$ is its group ring, the notation $\Bbbk[\pi]\text{-mod}^{\mathrm{fg}}$ means the category of $\Bbbk[\pi]$-modules that are finitely generated over $\Bbbk$ (and not merely over $\Bbbk[\pi]$).

We write $\mathsf{H}^i(A)$ for the the ith cohomology object of a chain complex A. We write $\mathbf{H}^i(X, \mathcal{F})$ for the ith sheaf (hyper)cohomology of a topological space X with coefficients in a sheaf (or chain complex of sheaves) $\mathcal{F}$.

Sheaf functors such as $f_!$ and f_* are always derived; a separate notation (${}^\circ\! f_!$, ${}^\circ\! f_*$) is used for their non-derived counterparts.

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Baton Rouge
March 2021

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CHAPTER 1

Sheaf theory

This chapter covers the foundations of sheaves and their (derived) functors. Much of the content of this chapter is well covered in other sources, so many proofs are only sketched or entirely omitted.

Basic definitions and some introductory results are given in Section 1.1. Following this, Sections 1.2–1.6 are devoted to the study of the six main functors on (chain complexes of) sheaves, often called **Grothendieck's six operations**. These sections include the definitions of the functors, their basic properties, and statements of a number of natural transformations or natural isomorphisms among them. Perhaps the most useful for explicit calculations are the proper base change theorem (Theorem 1.2.13) and the natural distinguished triangles associated to complementary open and closed subsets (Section 1.3).

In Section 1.7 we study locally constant sheaves, which are the building blocks from which constructible sheaves will be built in Chapter 2. Section 1.8 contains an assortment of results on complexes of sheaves with locally constant cohomology. Some of these results are needed in Section 1.9, which discusses various additional base change theorems.

1.1. Sheaves

Definition 1.1.1. Let X be a topological space, and let $\Bbbk$ be a commutative ring. A **presheaf of $\Bbbk$-modules** on X (also called a **presheaf with coefficients in $\Bbbk$**) is a rule $\mathcal{F}$ that assigns:

- to each open set $U \subset X$ a $\Bbbk$-module $\mathcal{F}(U)$, called the module of **sections** of $\mathcal{F}$ over U,
- to each pair of open sets $U \subset V$ a homomorphism of $\Bbbk$-modules $\mathrm{res}^{\mathcal{F}}_{V,U} = \mathrm{res}_{V,U} : \mathcal{F}(V) \to \mathcal{F}(U)$, called the **restriction map**, that satisfies

$$\begin{aligned} \mathrm{res}_{U,U} &= \mathrm{id}_{\mathcal{F}(U)}, \\ \mathrm{res}_{W,U} &= \mathrm{res}_{V,U} \circ \mathrm{res}_{W,V} \qquad\qquad \text{if } U \subset V \subset W. \end{aligned}$$

A **morphism** of presheaves $\phi : \mathcal{F} \to \mathcal{G}$ consists of a collection of homomorphisms of $\Bbbk$-modules $\phi_U : \mathcal{F}(U) \to \mathcal{G}(U)$ that "commute with restriction"; that is, whenever $U \subset V$, the following diagram commutes:

$$\begin{tikzcd} \mathcal{F}(V) \arrow[r, "\phi_V"] \arrow[d, "\mathrm{res}^{\mathcal{F}}_{V,U}"'] & \mathcal{G}(V) \arrow[d, "\mathrm{res}^{\mathcal{G}}_{V,U}"] \\ \mathcal{F}(U) \arrow[r, "\phi_U"] & \mathcal{G}(U) \end{tikzcd}$$

The category of presehaves of $\Bbbk$-modules on X is denoted by $\mathrm{Presh}(X, \Bbbk)$.

We often use a more compact notation for restriction of sections: if $s \in \mathcal{F}(V)$ and $U \subset V$, we write

$$s|_U = \mathrm{res}_{V,U}(s).$$

REMARK 1.1.2. A more concise way to define presheaves is as follows. Let $\mathrm{Op}(X)$ be the category whose objects are open subsets of X and whose morphisms are inclusion maps. Then a presheaf $\mathcal{F}$ is a functor $\mathcal{F} : \mathrm{Op}(X)^{\mathrm{op}} \to \Bbbk\text{-mod}$, and a morphism of presheaves is a natural transformation of functors.

DEFINITION 1.1.3. Let $\mathcal{F}$ be a presheaf on X, and let $x \in X$. The **stalk** of $\mathcal{F}$ at x, denoted by $\mathcal{F}_x$, is the $\Bbbk$-module

$$\mathcal{F}_x = \varinjlim_{U \ni x} \mathcal{F}(U).$$

If $x \in U$, the image of a section $s \in \mathcal{F}(U)$ under the canonical map $\mathcal{F}(U) \to \mathcal{F}_x$ is denoted by s_x and is called the **germ** of s at x. The **support** of a section $s \in \mathcal{F}(U)$ is the closed set (see Exercise 1.1.1) given by

$$\mathrm{supp}\, s = \{x \in U \mid s_x \neq 0\}.$$

The **support** of $\mathcal{F}$ is the closed set

$$\mathrm{supp}\, \mathcal{F} = \overline{\{x \in X \mid \mathcal{F}_x \neq 0\}}.$$

We sometimes say that $\mathcal{F}$ is **supported on** $Z \subset X$ to mean that $\mathrm{supp}\, \mathcal{F} \subset Z$.

Here is a more concrete description of $\mathcal{F}_x$: it is the set of equivalence classes

$$\mathcal{F}_x = \{(U, s) \mid U \ni x \text{ open in } X,\ s \in \mathcal{F}(U)\}/\sim \tag{1.1.1}$$

where $\sim$ is the following equivalence relation: $(U_1, s_1) \sim (U_2, s_2)$ if there is an open set $V \subset U_1 \cap U_2$ containing x such that $s_1|_V = s_2|_V$.

For any morphism of presheaves $\phi : \mathcal{F} \to \mathcal{G}$ and any point $x \in X$, one can consider the induced map of stalks $\phi_x : \mathcal{F}_x \to \mathcal{G}_x$. Many properties of ϕ can be detected at the level of stalks (see Lemma 1.1.6 and Proposition 1.1.12 below).

DEFINITION 1.1.4. A presheaf $\mathcal{F}$ on X is called a **sheaf** if it satisfies the following two axioms:

(1) (Gluing) Given an open covering $(U_\alpha)_{\alpha \in I}$ of an open set $U \subset X$ and a collection of sections $(s_\alpha \in \mathcal{F}(U_\alpha))_{\alpha \in I}$ such that for all $\alpha, \beta \in I$ we have

$$s_\alpha|_{U_\alpha \cap U_\beta} = s_\beta|_{U_\alpha \cap U_\beta},$$

then there exists a section $s \in \mathcal{F}(U)$ such that $s|_{U_\alpha} = s_\alpha$ for all $\alpha \in I$.

(2) (Local identity) Given an open covering $(U_\alpha)_{\alpha \in I}$ of an open set $U \subset X$ and sections $s, t \in \mathcal{F}(U)$, if $s|_{U_\alpha} = t|_{U_\alpha}$ for all $\alpha \in I$, then $s = t$.

A **morphism** of sheaves is the same as a morphism of presheaves. The full subcategory of $\mathrm{Presh}(X, \Bbbk)$ consisting of sheaves is denoted by $\mathrm{Sh}(X, \Bbbk)$.

The local identity axiom implies that for a sheaf $\mathcal{F}$, we have $\mathcal{F}(\varnothing) = 0$. The following lemma is a "stalkwise" restatement of this axiom. We omit its proof.

LEMMA 1.1.5. *Let $\mathcal{F} \in \mathrm{Sh}(X, \Bbbk)$. Given an open set $U \subset X$ and two sections $s, t \in \mathcal{F}(U)$, we have $s_x = t_x$ for all $x \in U$ if and only if $s = t$. As a consequence, $\mathcal{F} = 0$ if and only if $\mathcal{F}_x = 0$ for all $x \in X$.*

LEMMA 1.1.6. *A morphism $\phi : \mathcal{F} \to \mathcal{G}$ in $\mathrm{Sh}(X, \Bbbk)$ is an isomorphism if and only if $\phi_x : \mathcal{F}_x \to \mathcal{G}_x$ is an isomorphism for all $x \in X$.*

Proof sketch. The "only if" direction is obvious. For the "if" direction, assume that $\phi_x : \mathcal{F}_x \to \mathcal{G}_x$ is an isomorphism for all $x \in X$. It is enough to show that $\phi_U : \mathcal{F}(U) \to \mathcal{G}(U)$ is an isomorphism for all open sets $U \subset X$. The fact that ϕ_U is injective can be deduced from the local identity axiom and from the fact that it is surjective from the gluing axiom. □

Lemma 1.1.7. *The inclusion functor* $\mathrm{Sh}(X, \Bbbk) \hookrightarrow \mathrm{Presh}(X, \Bbbk)$ *admits a left adjoint* $\mathrm{Presh}(X, \Bbbk) \to \mathrm{Sh}(X, \Bbbk)$, *denoted by* $\mathcal{F} \mapsto \mathcal{F}^+$ *and called the* ***sheafification functor***. *The canonical map* $\iota : \mathcal{F} \to \mathcal{F}^+$ *induces an isomorphism of stalks* $\iota_x : \mathcal{F}_x \to \mathcal{F}_x^+$ *for all* $x \in X$.

Proof sketch. Define $\mathcal{F}^+$ to be the sheaf given by

$$\mathcal{F}^+(U) = \left\{ \begin{array}{c} \text{functions} \\ s : U \to \displaystyle\bigsqcup_{x \in U} \mathcal{F}_x \end{array} \,\middle|\, \begin{array}{c} \text{for all } x \in U, \text{ there is an open subset} \\ V \subset U \text{ containing } x \text{ and a section } t \in \mathcal{F}(V) \\ \text{such that } s(y) = t_y \in \mathcal{F}_y \text{ for all } y \in V \end{array} \right\}.$$

One can show that the obvious map $\iota : \mathcal{F} \to \mathcal{F}^+$ induces isomorphisms of stalks, and that for any sheaf $\mathcal{G}$, we get a natural isomorphism $\mathrm{Hom}(\mathcal{F}^+, \mathcal{G}) \cong \mathrm{Hom}(\mathcal{F}, \mathcal{G})$. □

In particular, if $\mathcal{F}$ is already a sheaf, then Lemma 1.1.6 implies that $\iota : \mathcal{F} \xrightarrow{\sim} \mathcal{F}^+$ is an isomorphism.

The following example is of fundamental importance.

Example 1.1.8. Let M be a $\Bbbk$-module. The **constant presheaf** on X with value M, denoted by $\underline{M}_{\mathrm{pre}}$ or $\underline{M}_{\mathrm{pre},X}$, is given by

$$\begin{aligned} \underline{M}_{\mathrm{pre}}(U) &= M && \text{for all open subsets } U \subset X, \\ \mathrm{res}_{V,U} &= \mathrm{id}_M : M \to M && \text{for all inclusions } U \hookrightarrow V. \end{aligned}$$

The **constant sheaf** on X with value M, denoted by $\underline{M}_X$ or $\underline{M}$, is given by

$$\begin{aligned} \underline{M}(U) &= \{\text{locally constant functions } s : U \to M\}, \\ \mathrm{res}_{V,U}(s) &= (s|_U : U \to M). \end{aligned}$$

One can show that $\underline{M}$ can be identified with the sheafification of $\underline{M}_{\mathrm{pre}}$. If X is locally connected, then all stalks of $\underline{M}$ (or of $\underline{M}_{\mathrm{pre}}$) are canonically identified with M. In particular, if X is locally connected, and $U \subset X$ is a connected open subset, then for any point $x \in U$, the natural map $\underline{M}_X(U) \to \underline{M}_{X,x}$ is an isomorphism.

Example 1.1.9. Suppose X is Hausdorff (or at least T_1), and let $x \in X$. The **skyscraper sheaf** at x with value M is sheaf $\underline{M}_{x \in X}$ (or just $\underline{M}_x$) given by

$$\underline{M}_{x \in X}(U) = \begin{cases} M & \text{if } x \in U, \\ 0 & \text{otherwise.} \end{cases}$$

Remark 1.1.10. Consider the 1-point topological space pt. The functor $\mathcal{F} \mapsto \mathcal{F}(\mathrm{pt})$ gives an equivalence of categories

$$\mathrm{Sh}(\mathrm{pt}, \Bbbk) \cong \Bbbk\text{-mod}.$$

The inverse functor is the constant sheaf functor $M \mapsto \underline{M}_{\mathrm{pt}}$.

Theorem 1.1.11. *The category* $\mathrm{Sh}(X, \Bbbk)$ *is an abelian category.*

Proof sketch. One first proves the following two claims:

(1) Morphisms in $\mathrm{Sh}(X, \Bbbk)$ admit kernels and cokernels.

(2) Kernels and cokernels commute with taking stalks at any point x.

For part (1), one can give an explicit construction: the kernel of a morphism $\phi : \mathcal{F} \to \mathcal{G}$ is given by $(\ker \phi)(U) = \ker(\phi_U : \mathcal{F}(U) \to \mathcal{G}(U))$, and its cokernel is the sheafification of the presheaf $\operatorname{cok}_{\mathrm{pre}} \phi$ given by

$$(\operatorname{cok}_{\mathrm{pre}} \phi)(U) = \operatorname{cok}(\phi_U : \mathcal{F}(U) \to \mathcal{G}(U)).$$

Part (2) says that

$$(\ker \phi)_x \cong \ker(\phi_x : \mathcal{F}_x \to \mathcal{G}_x), \qquad (\operatorname{cok} \phi)_x \cong \operatorname{cok}(\phi_x : \mathcal{F}_x \to \mathcal{G}_x).$$

In view of Lemma 1.1.5, this implies that ϕ is a monomorphism (resp. epimorphism) if and only if ϕ_x is injective (resp. surjective) for all $x \in X$. Using Lemma 1.1.6, one can then show that every monomorphism is the kernel of its cokernel, and every epimorphism is the cokernel of its kernel. By Theorem A.3.7, we are done. □

As a consequence of the description of kernels and cokernels in the proof of Theorem 1.1.11, we obtain the following characterization of short exact sequences:

PROPOSITION 1.1.12. *A sequence* $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ *in* $\mathrm{Sh}(X, \Bbbk)$ *is a short exact sequence if and only if for every* $x \in X$*, the sequence* $0 \to \mathcal{F}_x \to \mathcal{G}_x \to \mathcal{H}_x \to 0$ *is exact. In particular, for any* $x \in X$*, the stalk functor* $\mathrm{Sh}(X, \Bbbk) \to \Bbbk\text{-mod}$ *given by* $\mathcal{F} \mapsto \mathcal{F}_x$ *is exact.*

Thus, a morphism of sheaves $\phi : \mathcal{F} \to \mathcal{G}$ is injective (resp. surjective) if and only if each stalk morphism $\phi_x : \mathcal{F}_x \to \mathcal{G}_x$ is injective (resp. surjective).

In view of Theorem 1.1.11, it makes sense to consider the derived category $D(\mathrm{Sh}(X, \Bbbk))$. Derived categories of sheaves will be the main setting for all of the work in this book, so it is convenient to introduce a slightly abbreviated notation: from now on, we write

$$D(X, \Bbbk) = D(\mathrm{Sh}(X, \Bbbk)),$$

and likewise $D^+(X, \Bbbk)$, $D^-(X, \Bbbk)$, and $D^{\mathrm{b}}(X, \Bbbk)$. The **support** of an object $\mathcal{F} \in D(X, \Bbbk)$ is defined by

$$\operatorname{supp} \mathcal{F} = \overline{\bigcup_{i \in \mathbb{Z}} \operatorname{supp} \mathsf{H}^i(\mathcal{F})}.$$

Of course, if $\mathcal{F} \in D^{\mathrm{b}}(X, \Bbbk)$, then the union on the right-hand side is already closed.

PROPOSITION 1.1.13. *The category* $\mathrm{Sh}(X, \Bbbk)$ *has enough injectives.*

PROOF SKETCH. Let $\mathcal{F}$ be a sheaf. For each point $x \in X$, consider the $\Bbbk$-module $\mathcal{F}_x$. Choose some injective map to an injective $\Bbbk$-module $\iota_x : \mathcal{F}_x \to I_x$. Define a new sheaf $\mathcal{I}$ by

$$\mathcal{I}(U) = \prod_{x \in U} I_x.$$

There is an obvious map $\theta : \mathcal{F} \to \mathcal{I}$. One can show that θ is an injective map and that $\mathcal{I}$ is an injective object. □

In contrast, $\mathrm{Sh}(X, \Bbbk)$ may not have enough projectives! (See Exercise 1.3.10.) We conclude this section with a few more basic operations on sheaves.

DEFINITION 1.1.14. Let $\mathcal{F}$ be a sheaf on X, and let $Y \subset X$. The **restriction** of $\mathcal{F}$ to Y, denoted by $\mathcal{F}|_Y$, is the sheafification of the presheaf on Y given by

$$U \mapsto \varinjlim_{\substack{V \subset X \text{ open} \\ V \supset U}} \mathcal{F}(V). \tag{1.1.2}$$

The module of **global sections**, denoted by $\Gamma(\mathcal{F})$, is the $\Bbbk$-module $\Gamma(\mathcal{F}) = \mathcal{F}(X)$.

The module of **global sections with compact support**, denoted by $\Gamma_c(\mathcal{F})$, is the $\Bbbk$-module $\Gamma_c(\mathcal{F}) = \{s \in \mathcal{F}(X) \mid \operatorname{supp} s \text{ is compact}\}$.

The following statement is a special case of Lemma 1.2.2. We omit its proof.

LEMMA 1.1.15. (1) *Let $Y \subset X$. The restriction functor $\mathrm{Sh}(X, \Bbbk) \to \mathrm{Sh}(Y, \Bbbk)$ given by $\mathcal{F} \mapsto \mathcal{F}|_Y$ is exact.*
(2) *The functors $\Gamma : \mathrm{Sh}(X, \Bbbk) \to \Bbbk\text{-mod}$ and $\Gamma_c : \mathrm{Sh}(X, \Bbbk) \to \Bbbk\text{-mod}$ are left exact.*

EXAMPLE 1.1.16. Let X be a discrete topological space. Then

$$\Gamma(\mathcal{F}) \cong \prod_{x \in X} \mathcal{F}_x \qquad \text{and} \qquad \Gamma_c(\mathcal{F}) \cong \bigoplus_{x \in X} \mathcal{F}_x.$$

It follows that on a discrete topological space, both Γ and Γ_c are exact functors.

Thanks to Proposition 1.1.13, Γ and Γ_c admit right derived functors

$$R\Gamma, R\Gamma_c : D^+(X, \Bbbk) \to D^+(\Bbbk\text{-mod}).$$

For $\mathcal{F} \in D^+(X, \Bbbk)$, the cohomology objects $\mathsf{H}^k(R\Gamma(\mathcal{F}))$ and $\mathsf{H}^k(R\Gamma_c(\mathcal{F}))$ are important enough to merit special notation.

DEFINITION 1.1.17. Let X be a topological space, and let $\mathcal{F} \in D^+(X, \Bbbk)$. The kth **hypercohomology** of $\mathcal{F}$, denoted by $\mathbf{H}^k(X, \mathcal{F})$, is the $\Bbbk$-module given by

$$\mathbf{H}^k(X, \mathcal{F}) = \mathsf{H}^k(R\Gamma(\mathcal{F})).$$

Similarly, the kth **hypercohomology with compact support** of $\mathcal{F}$, denoted by $\mathbf{H}^k_c(\mathcal{F})$, is the $\Bbbk$-module given by

$$\mathbf{H}^k_c(X, \mathcal{F}) = \mathsf{H}^k(R\Gamma_c(\mathcal{F})).$$

In the special case of a constant sheaf, we often use slightly different notation: for a $\Bbbk$-module M, we write

$$\mathbf{H}^k(X; M) = \mathbf{H}^k(X, \underline{M}_X) \qquad \text{and} \qquad \mathbf{H}^k_c(X; M) = \mathbf{H}^k_c(X, \underline{M}_X).$$

See Remark 1.2.5 for another description of hypercohomology.

Finally, let us mention a connection to algebraic topology. Given a space X and a $\Bbbk$-module M, let $\mathbf{H}^k_{\mathrm{sing}}(X; M)$ denote its kth singular cohomology group with coefficients in M. Similarly, let $\mathbf{H}^k_{\mathrm{sing,c}}(X; M)$ denote its kth singular cohomology group with compact supports and coefficients in M.

THEOREM 1.1.18. *If X is a locally contractible and hereditarily paracompact topological space, then there is a natural isomorphism*

$$\mathbf{H}^k(X; M) \cong \mathbf{H}^k_{\mathrm{sing}}(X; M).$$

Similarly, if X is locally contractible, hereditarily paracompact, and locally compact, then there is a natural isomorphism

$$\mathbf{H}^k_c(X; M) \cong \mathbf{H}^k_{\mathrm{sing,c}}(X; M).$$

For a proof, see [**44**, Theorem III.1.1] or [**189**, Theorem 4.14]. (See also [**207**] for a version that drops the hereditary paracompactness assumption.) This theorem is not strictly logically necessary for the main content of the book. However, it motivates the way many computations of explicit examples are structured: such computations can often be reduced to computing hypercohomology of a constant sheaf on some familiar topological space—for instance, a projective space or a Grassmannian—whose singular cohomology can be found in an algebraic topology textbook, such as [**98**, **185**, **225**].

Exercises.

1.1.1. Show that the support of section of a presheaf is automatically closed.

1.1.2. Show that $(\underline{M}_{\mathrm{pre},X})^+ \cong \underline{M}_X$.

1.1.3. Show that $\mathrm{Presh}(X, \Bbbk)$ is an abelian category, and that the inclusion functor $\mathrm{Sh}(X, \Bbbk) \to \mathrm{Presh}(X, \Bbbk)$ is left exact. Is it exact?

1.1.4. Give an example of a presheaf that satisfies gluing but not local identity. Then give an example of a presheaf that satisfies local identity but not gluing.

1.1.5. Give an example showing that the presheaf in (1.1.2) need not be a sheaf.

1.1.6. Let $\phi : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves with the property that for every open set $U \subset X$, $\phi_U : \mathcal{F}(U) \to \mathcal{G}(U)$ is surjective. Show that ϕ is surjective.

1.1.7. Let $(\mathcal{F}_\alpha)_{\alpha\in I}$ be a collection of sheaves on X. Show that the direct sum $\bigoplus_{\alpha\in I} \mathcal{F}_\alpha$ exists. Specifically, show that it is the sheafification of the presheaf

$$\left(\bigoplus_{\alpha\in I}^{\mathrm{pre}} \mathcal{F}_\alpha\right)(U) = \bigoplus_{\alpha\in I} \mathcal{F}_\alpha(U).$$

Also show by example that $\bigoplus^{\mathrm{pre}}_{\alpha\in I} \mathcal{F}_\alpha$ may fail to be a sheaf. Then show that stalks commute with arbitrary direct sums, i.e., that there is a natural isomorphism

$$\left(\bigoplus_{\alpha\in I} \mathcal{F}_\alpha\right)_x \cong \bigoplus_{\alpha_I} (\mathcal{F}_\alpha)_x.$$

(For another construction of the direct sum, see Exercise 1.2.5.) What about direct products of sheaves?

1.1.8. Let $X = \mathbb{C} \smallsetminus \{0\}$, and let $\mathcal{Q} \in \mathrm{Sh}(X, \mathbb{C})$ be the sheaf given by

$$\mathcal{Q}(U) = \{\text{solutions } g : U \to \mathbb{C} \text{ to the differential equation } z\tfrac{dg}{dz} - \tfrac{1}{2}g = 0\}.$$

We will call this the **square-root sheaf**, because its sections are scalar multiples of a function g satisfying $g(z)^2 = z$. Show that every point $x \in X$ has a neighborhood U such that $\mathcal{Q}|_U \cong \underline{\mathbb{C}}_U$. (A sheaf with this property is said to be **locally constant**.) On the other hand, show that $\mathcal{Q}$ is not itself a constant sheaf.

1.1.9. Let $X = \mathbb{C} \smallsetminus \{0\}$. Let $\mathcal{F}$ and $\mathcal{G}$ be the sheaves in $\mathrm{Sh}(X, \mathbb{Z})$ given by

$$\mathcal{F}(U) = \{\text{analytic functions } U \to \mathbb{C}\},$$
$$\mathcal{G}(U) = \{\text{analytic functions } U \to \mathbb{C} \smallsetminus \{0\}\}.$$

Here, $\mathcal{F}(U)$ is an abelian group under addition, and by $\mathcal{G}(U)$ is an abelian group under multiplication. Let $\exp : \mathcal{F} \to \mathcal{G}$ be the morphism given by $\exp_U(f(z)) = e^{f(z)}$.

(a) Show that $\exp : \mathcal{F} \to \mathcal{G}$ is a surjective morphism of sheaves. Then show that its kernel is isomorphic to the constant sheaf $\underline{\mathbb{Z}}_X$.
(b) Show that $\Gamma(\mathcal{F}) \to \Gamma(\mathcal{G})$ is *not* surjective. For which open sets $U \subset X$ is it true that $\mathcal{F}(U) \to \mathcal{G}(U)$ is surjective?

1.1.10 (Sheaves of rings). Let $\mathcal{R}$ be a **sheaf of rings**, i.e., a sheaf such that for each open set $U \subset X$, $\mathcal{R}(U)$ is a (not necessarily commutative) ring, and such that each map $\mathrm{res}_{V,U} : \mathcal{R}(V) \to \mathcal{R}(U)$ is a ring homomorphism. Define an **$\mathcal{R}$-module** to be a sheaf $\mathcal{M}$ such that for each open set $U \subset X$, $\mathcal{M}(U)$ is an $\mathcal{R}(U)$-module, and the restriction maps $\mathrm{res}_{V,U} : \mathcal{M}(V) \to \mathcal{M}(U)$ are "module homomorphisms" in the following sense: for $r \in \mathcal{R}(V)$ and $m \in \mathcal{M}(V)$, we require

$$\mathrm{res}_{V,U}(r \cdot m) = \mathrm{res}_{V,U}(r) \cdot \mathrm{res}_{V,U}(m).$$

A **morphism** of $\mathcal{R}$-modules $\phi : \mathcal{M} \to \mathcal{M}'$ is a morphism of sheaves such that each $\phi_U : \mathcal{M}(U) \to \mathcal{M}'(U)$ is an $\mathcal{R}(U)$-module homomorphism. Show that the category $\mathcal{R}$-mod of $\mathcal{R}$-modules is an abelian category, and that it has enough injectives.

1.1.11 (Gluing morphisms). Let $\mathcal{F}$ and $\mathcal{G}$ be sheaves on X. Let $(U_\alpha)_{\alpha\in I}$ be an open covering of X, and let $(f_\alpha : \mathcal{F}|_{U_\alpha} \to \mathcal{G}|_{U_\alpha})_{\alpha\in I}$ be a collection of morphisms such that for any two elements $\alpha, \beta \in I$, the two morphisms

$$f_\alpha|_{U_\alpha\cap U_\beta}, f_\beta|_{U_\alpha\cap U_\beta} : \mathcal{F}|_{U_\alpha\cap U_\beta} \to \mathcal{G}|_{U_\alpha\cap U_\beta}$$

are equal. Prove that there exists a unique morphism of sheaves $f : \mathcal{F} \to \mathcal{G}$ such that $f|_{U_\alpha} = f_\alpha$ for all $\alpha \in I$.

1.1.12 (Gluing sheaves). Let $(U_\alpha)_{\alpha\in I}$ be an open covering of X. Suppose that for each $\alpha \in I$, we have a sheaf $\mathcal{F}_\alpha$ on U_α, and for each pair of elements $\alpha, \beta \in I$, we have an isomorphism $\phi_{\alpha\beta} : \mathcal{F}_\alpha|_{U_\alpha\cap U_\beta} \xrightarrow{\sim} \mathcal{F}_\beta|_{U_\alpha\cap U_\beta}$, such that for $\alpha, \beta, \gamma \in I$, the diagram

$$\begin{tikzcd}
\mathcal{F}_\alpha|_{U_\alpha\cap U_\beta\cap U_\gamma} \arrow[rr, "\phi_{\alpha\gamma}|_{U_\alpha\cap U_\beta\cap U_\gamma}"] \arrow[dr, swap, "\phi_{\alpha\beta}|_{U_\alpha\cap U_\beta\cap U_\gamma}"] & & \mathcal{F}_\gamma|_{U_\alpha\cap U_\beta\cap U_\gamma} \\
& \mathcal{F}_\beta|_{U_\alpha\cap U_\beta\cap U_\gamma} \arrow[ur, swap, "\phi_{\beta\gamma}|_{U_\alpha\cap U_\beta\cap U_\gamma}"] &
\end{tikzcd}$$

commutes. Show that there exists a sheaf $\mathcal{F}$ on X together with isomorphisms $\psi_\alpha : \mathcal{F}|_{U_\alpha} \xrightarrow{\sim} \mathcal{F}_\alpha$ such that the diagram

$$\begin{tikzcd}
\mathcal{F}|_{U_\alpha\cap U_\beta} \arrow[rr, "\psi_\beta|_{U_\alpha\cap U_\beta}"] \arrow[dr, swap, "\psi_\alpha|_{U_\alpha\cap U_\beta}"] & & \mathcal{F}_\beta|_{U_\alpha\cap U_\beta} \\
& \mathcal{F}_\alpha|_{U_\alpha\cap U_\beta} \arrow[ur, swap, "\phi_{\alpha\beta}"] &
\end{tikzcd}$$

commutes. Moreover, the datum $(\mathcal{F}, (\psi_\alpha)_{\alpha\in I})$ is unique up to isomorphism.

1.2. Pullback, push-forward, and base change

In this section, we introduce three operations on sheaves associated to a continuous map $f : X \to Y$, and we study some relations among them. Some statements will involve locally compact spaces and proper maps. We adopt the following conventions, following [**42**]:

- A space X is said to be **locally compact** if it is Hausdorff and if each point $x \in X$ is contained in a pair of subsets $U \subset K \subset X$ with U open and K compact.

- A continuous map $f : X \to Y$ is said to be **proper** if it is universally closed, i.e., if for any other space Z, the map $f \times \mathrm{id}_Z : X \times Z \to Y \times Z$ is a closed map.

When X and Y are both locally compact, the following three conditions on a continuous map $f : X \to Y$ are all equivalent:

(1) f is proper.
(2) For every compact subset $K \subset Y$, the set $f^{-1}(K)$ is also compact.
(3) The map f is closed, and for every point $y \in Y$, the set $f^{-1}(y)$ is compact.

DEFINITION 1.2.1. Let $f : X \to Y$ be a continuous map.

(1) For $\mathcal{F} \in \mathrm{Sh}(Y, \Bbbk)$, the **pullback** of $\mathcal{F}$, denoted by $f^*\mathcal{F}$, is the sheafification of the presheaf $f^*_{\mathrm{pre}}(\mathcal{F}) \in \mathrm{Presh}(X, \Bbbk)$ given by
$$f^*_{\mathrm{pre}}(\mathcal{F})(U) = \varinjlim_{\substack{V \subset Y \text{ open} \\ V \supset f(U)}} \mathcal{F}(V).$$
(2) For $\mathcal{F} \in \mathrm{Sh}(X, \Bbbk)$, its **push-forward** is the sheaf ${}^\circ f_*\mathcal{F} \in \mathrm{Sh}(Y, \Bbbk)$ given by
$${}^\circ f_*(\mathcal{F})(U) = \mathcal{F}(f^{-1}(U)).$$
(3) For $\mathcal{F} \in \mathrm{Sh}(X, \Bbbk)$, its **proper push-forward** is the sheaf ${}^\circ f_!\mathcal{F} \in \mathrm{Sh}(Y, \Bbbk)$ given by
$${}^\circ f_!(\mathcal{F})(U) = \{s \in \mathcal{F}(f^{-1}(U)) \mid f|_{\mathrm{supp}\, s} : \mathrm{supp}\, s \to U \text{ is proper}\}.$$

LEMMA 1.2.2. *Let $f : X \to Y$ be a continuous map. The functor $f^* : \mathrm{Sh}(Y, \Bbbk) \to \mathrm{Sh}(X, \Bbbk)$ is exact, and the functors ${}^\circ f_*, {}^\circ f_! : \mathrm{Sh}(X, \Bbbk) \to \mathrm{Sh}(Y, \Bbbk)$ are left exact.*

The proof is left as an exercise. As a consequence, we have
$$f^* : D(Y, \Bbbk) \to D(X, \Bbbk)$$
as well as the the derived functors
$$f_* = R({}^\circ f_*) : D^+(X, \Bbbk) \to D^+(Y, \Bbbk),$$
$$f_! = R({}^\circ f_!) : D^+(X, \Bbbk) \to D^+(Y, \Bbbk).$$
The functor $f_!$ is usually only used on locally compact spaces, as the proofs of some basic properties (such as Proposition 1.2.8(3) below) require local compactness.

REMARK 1.2.3. Traditionally, the push-forward functors in Definition 1.2.1 would have been denoted by f_* and $f_!$, and their derived functors by Rf_* and $Rf_!$. However, in the geometric representation theory literature, it has become common practice (going back at least to [**24**]) to omit the "R," and to use f_* and $f_!$ to mean the derived versions. The underived versions are rarely needed in applications—they do not appear at all in Chapters 7–10 of this book—so this is a convenient way to reduce clutter.

In this book, I follow the practice of using f_* and $f_!$ for the derived functors. But I cannot avoid mentioning the underived versions in the early chapters, so I have chosen new notation for them. The symbols ${}^\circ f_*$ and ${}^\circ f_!$ can be read as abbreviated forms of $\mathsf{H}^0(f_*)$ and $\mathsf{H}^0(f_!)$, for which they are synonyms.

By construction, for $\mathcal{F} \in \mathrm{Sh}(X, \Bbbk)$, the sheaf ${}^\circ f_! \mathcal{F}$ is a subsheaf of ${}^\circ f_* \mathcal{F}$. Therefore, for any $\mathcal{F} \in D^+(X, \Bbbk)$, we have a natural transformation

$$f_! \mathcal{F} \to f_* \mathcal{F}. \tag{1.2.1}$$

Note that stalks and restriction of sheaves (Definition 1.1.14) are both special cases of pullback. Some other notable special cases involve the constant map $\mathrm{a}_X : X \to \mathrm{pt}$ to a point. Identify $\mathrm{Sh}(\mathrm{pt}, \Bbbk)$ with $\Bbbk$-mod as in Remark 1.1.10. Then we have a natural isomorphism

$$\mathrm{a}_X^*(M) \cong \underline{M}_X. \tag{1.2.2}$$

On the other hand, we have

$$\begin{aligned} {}^\circ\mathrm{a}_{X*}(\mathcal{F}) &\cong \Gamma(\mathcal{F}), & {}^\circ\mathrm{a}_{X!}(\mathcal{F}) &\cong \Gamma_{\mathrm{c}}(\mathcal{F}), \\ \mathrm{a}_{X*}(\mathcal{F}) &\cong R\Gamma(\mathcal{F}), & \mathrm{a}_{X!}(\mathcal{F}) &\cong R\Gamma_{\mathrm{c}}(\mathcal{F}). \end{aligned}$$

THEOREM 1.2.4. *Let $f : X \to Y$ be a continuous map. For $\mathcal{F} \in D^-(Y, \Bbbk)$ and $\mathcal{G} \in D^+(X, \Bbbk)$, there are natural isomorphisms*

$$R\mathrm{Hom}(f^*\mathcal{F}, \mathcal{G}) \cong R\mathrm{Hom}(\mathcal{F}, f_*\mathcal{G}),$$
$$\mathrm{Hom}_{D(X,\Bbbk)}(f^*\mathcal{F}, \mathcal{G}) \cong \mathrm{Hom}_{D(Y,\Bbbk)}(\mathcal{F}, f_*\mathcal{G}).$$

PROOF SKETCH. Observe that the formula for ${}^\circ f_*$ also makes sense for presheaves. The proof involves the following steps.

Step 1. For presheaves $\mathcal{F} \in \mathrm{Presh}(Y, \Bbbk)$ and $\mathcal{G} \in \mathrm{Presh}(X, \Bbbk)$, there is a natural isomorphism $\mathrm{Hom}(f^*_{\mathrm{pre}}\mathcal{F}, \mathcal{G}) \cong \mathrm{Hom}(\mathcal{F}, {}^\circ f_* \mathcal{G})$. One can write down the unit and counit maps for the putative adjunction explicitly: the unit map $\mathcal{F} \to {}^\circ f_* f^*_{\mathrm{pre}} \mathcal{F}$ is given over an open set $V \subset Y$ by the obvious map

$$\mathcal{F}(V) \to {}^\circ f_* f^*_{\mathrm{pre}} \mathcal{F}(V) = \varinjlim_{\substack{W \subset Y \text{ open} \\ W \supset f(f^{-1}(V))}} \mathcal{F}(W),$$

and the counit map $f^*_{\mathrm{pre}} {}^\circ f_* \mathcal{G} \to \mathcal{G}$ is given by

$$f^*_{\mathrm{pre}} {}^\circ f_* \mathcal{G}(U) = \varinjlim_{\substack{W \subset Y \text{ open} \\ W \supset f(U)}} \mathcal{G}(f^{-1}(W)) \to \mathcal{G}(U).$$

It is straightforward to check the zig-zag equations.

Step 2. For $\mathcal{F} \in \mathrm{Sh}(Y, \Bbbk)$ and $\mathcal{G} \in \mathrm{Sh}(X, \Bbbk)$, there is a natural isomorphism $\mathrm{Hom}(f^*\mathcal{F}, \mathcal{G}) \cong \mathrm{Hom}(\mathcal{F}, {}^\circ f_* \mathcal{G})$. This follows from Step 1 using Lemma 1.1.7.

Step 3. Conclusion of the proof. Since f^* is exact, Step 2 implies that ${}^\circ f_*$ takes injective sheaves to injective sheaves. Thus, by Proposition A.6.6, both $R\mathrm{Hom}(f^*\mathcal{F}, \mathcal{G})$ and $R\mathrm{Hom}(\mathcal{F}, f_*\mathcal{G})$ can be computed by replacing $\mathcal{G}$ by an injective resolution. The first assertion in the theorem thus follows from Step 2, and then the second assertion follows by Proposition A.6.13(2). □

REMARK 1.2.5. Recall that $\mathbf{H}^k(X, \mathcal{F}) = \mathsf{H}^k(R\Gamma(\mathcal{F})) \cong \mathrm{Hom}(\Bbbk, R\Gamma(\mathcal{F})[k])$. Applying Theorem 1.2.4 to $\mathrm{a}_X : X \to \mathrm{pt}$ and using (1.2.2), we find that $\mathbf{H}^k(X, \mathcal{F}) \cong \mathrm{Hom}(\underline{\Bbbk}_X, \mathcal{F}[k])$. In particular, we have

$$\mathbf{H}^k(X; \Bbbk) \cong \mathrm{Hom}_{D(X,\Bbbk)}(\underline{\Bbbk}_X, \underline{\Bbbk}_X[k]) \cong \mathrm{Ext}^k_{\mathrm{Sh}(X,\Bbbk)}(\underline{\Bbbk}_X, \underline{\Bbbk}_X).$$

This description shows that $\mathbf{H}^\bullet(X; \Bbbk)$ has the structure of a graded ring, given by the composition of morphisms in $D(X, \Bbbk)$. It can be shown that this ring structure agrees with the cup product on singular cohomology via Theorem 1.1.18.

Adapted classes. In the following several sections, we will see a large number of natural isomorphisms between compositions of derived sheaf functors. The proofs of all of these follow the same general pattern as that of Theorem 1.2.4, summarized in the following remark.

REMARK 1.2.6. Suppose we wish to prove that for all $\mathcal{F} \in D^+(X, \Bbbk)$, there is a natural isomorphism of the form

$$RF(RG(\mathcal{F})) \cong RF'(RG'(\mathcal{F})), \tag{1.2.3}$$

where F, G, F', and G' are various left exact functors. The usual plan goes in two steps. First, prove an analogous statement at the level of abelian categories, namely, for $\mathcal{F} \in \mathrm{Sh}(X, \Bbbk)$, there is a natural isomorphism

$$F(G(\mathcal{F})) \cong F'(G'(\mathcal{F})).$$

This implies immediately that for $\mathcal{F} \in D^+(X, \Bbbk)$, there is a natural isomorphism

$$R(F \circ G)(\mathcal{F}) \cong R(F' \circ G')(\mathcal{F}). \tag{1.2.4}$$

The second step is to exhibit some adapted class $\mathscr{R} \subset \mathrm{Sh}(X, \Bbbk)$ for G and G' that matches the hypotheses of Proposition A.6.6. That proposition then tells us that (1.2.4) implies (1.2.3).

The main technical difficulty is that not all sheaf functors take injective sheaves to injective sheaves, so injective sheaves are not always adequate for the second step of this plan. Here are two additional classes of sheaves that we will use. For others, see Definitions 1.2.14 and 1.4.3.

DEFINITION 1.2.7. A sheaf $\mathcal{F}$ is said to be **flabby** if for every open set $U \subset X$, the restriction map $\mathcal{F}(X) \to \mathcal{F}(U)$ (or equivalently $\Gamma(\mathcal{F}) \to \Gamma(\mathcal{F}|_U)$) is surjective.

Assume now that X is a locally compact topological space. A sheaf $\mathcal{F}$ on X is said to be ***c*-soft** if for every compact subset $K \subset X$, the natural map $\Gamma(\mathcal{F}) \to \Gamma(\mathcal{F}|_K)$ is surjective.

The main facts we need about these kinds of sheaves at the moment are as follows. Let $f : X \to Y$ be a continuous map.

- The class of flabby sheaves on X is an adapted class for ${}^\circ f_*$, and ${}^\circ f_*$ takes flabby sheaves to flabby sheaves.
- Assume that X and Y are locally compact. The class of c-soft sheaves on X is an adapted class for ${}^\circ f_!$, and ${}^\circ f_!$ takes c-soft sheaves to c-soft sheaves.

Proofs of these assertions can be found in [**125**, Sections 2.4–2.5]. Another quick observation, immediate from the definitions, is that restriction to an open (resp. locally closed) subset sends flabby (resp. c-soft) sheaves to flabby (resp. c-soft) sheaves. For a summary of these and other facts about sheaf functors and adapted classes, see Table 1.4.1 in Section 1.4.

PROPOSITION 1.2.8. *Let $f : X \to Y$ and $g : Y \to Z$ be continuous maps.*

(1) *For $\mathcal{F} \in D(Z, \Bbbk)$, there is a natural isomorphism $(g \circ f)^*\mathcal{F} \cong f^*g^*\mathcal{F}$.*
(2) *For $\mathcal{F} \in D^+(X, \Bbbk)$, there is a natural isomorphism $g_*f_*\mathcal{F} \cong (g \circ f)_*\mathcal{F}$.*
(3) *Assume that X, Y, and Z are locally compact. Then, for $\mathcal{F} \in D^+(X, \Bbbk)$, there is a natural isomorphism $g_!f_!\mathcal{F} \cong (g \circ f)_!\mathcal{F}$.*

PROOF SKETCH. We follow the plan of Remark 1.2.6. For all three parts, the analogous statement at the abelian-category level follows easily from the definitions. Since pullback is exact, part (1) is immediate. For part (2), use either a flabby or an injective resolution (see Table 1.4.1(iii) or (iv)); for part (3), use a c-soft resolution (see Table 1.4.1(vi)). □

As a special case of Proposition 1.2.8(1), for any continuous map $f : X \to Y$ and any point $x \in X$, we have an isomorphism

$$(f^*\mathcal{F})_x \cong \mathcal{F}_{f(x)}. \tag{1.2.5}$$

Another special case is the following generalization of (1.2.2): for any $\Bbbk$-module M, we have

$$f^*\underline{M}_Y \cong \underline{M}_X. \tag{1.2.6}$$

REMARK 1.2.9 (Induced maps in hypercohomology). Let $f : X \to Y$ be a continuous map, and let M be a $\Bbbk$-module. Combining (1.2.6) with the unit of the adjunction from Theorem 1.2.4, we obtain a natural map

$$\underline{M}_Y \to f_*\underline{M}_X.$$

Now apply $R\Gamma$. By Proposition 1.2.8(2), we obtain a natural map $R\Gamma(\underline{M}_Y) \to R\Gamma(\underline{M}_X)$. Finally, apply H^i, and denote the resulting map by

$$f^\sharp : \mathbf{H}^i(Y; M) \to \mathbf{H}^i(X; M).$$

This is called the **induced map in hypercohomology**. When $M = \Bbbk$, one can show that this map is a ring homomorphism with respect to the ring structure from Remark 1.2.5. In the setting of Theorem 1.1.18, it agrees with the induced map in singular cohomology; see the discussion in [**44**, Section III.1].

As another application of flabby resolutions, we give the following formula for the cohomology of a stalk. (This result can also be proved using injective resolutions; see Exercise 1.3.3.)

LEMMA 1.2.10. *Let X be a topological space, and let $x \in X$. For $\mathcal{F} \in D^+(X, \Bbbk)$, there is a natural isomorphism*

$$\mathsf{H}^k(\mathcal{F}_x) \cong \varinjlim_{U \ni x} \mathbf{H}^k(U, \mathcal{F}|_U).$$

PROOF. Assume without loss of generality that $\mathcal{F}$ is a bounded below complex of flabby sheaves $\cdots \to \mathcal{F}^{k-1} \to \mathcal{F}^k \to \mathcal{F}^{k+1} \to \cdots$. It is clear from the definition that the restriction of a flabby sheaf to an open subset is still flabby. Thus $R\Gamma(\mathcal{F}|_U)$ is given by the chain complex of $\Bbbk$-modules

$$\cdots \to \mathcal{F}^{k-1}(U) \to \mathcal{F}^k(U) \to \mathcal{F}^{k+1}(U) \to \cdots.$$

Because direct limits of $\Bbbk$-modules are exact, the limit of cohomology modules $\lim_\to \mathsf{H}^k(R\Gamma(\mathcal{F}|_U))$ can be identified with the kth cohomology of the chain complex

$$\cdots \to \varinjlim_{U \ni x} \mathcal{F}^{k-1}(U) \to \varinjlim_{U \ni x} \mathcal{F}^k(U) \to \varinjlim_{U \ni x} \mathcal{F}^{k+1}(U) \to \cdots.$$

But this chain complex is precisely the stalk chain complex $\mathcal{F}_x$. □

Cartesian squares and base change. Suppose we have a commutative square of continuous maps

$$
\begin{tikzcd}
X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\
Y' \arrow[r, "g"] & Y
\end{tikzcd}
\tag{1.2.7}
$$

Recall that such a square is said to be **cartesian** if X', f', and g' have the following universal property: for any other space X'' and maps $f'' : X'' \to Y'$, $g'' : X'' \to X$ such that $g \circ f'' = f \circ g''$, there is a unique map $h : X'' \to X'$ making the following diagram commute:

$$
\begin{tikzcd}
X'' \arrow[dr, dashed, "\exists! h"] \arrow[drr, bend left, "g''"] \arrow[ddr, bend right, "f''"'] & & \\
& X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\
& Y' \arrow[r, "g"] & Y
\end{tikzcd}
$$

In the cartesian square (1.2.7), the space Y is sometimes called the **base** of f, and the map f' is said to be obtained from f by **base change**. There are many properties of continuous maps that are preserved by base change: for example, if f is proper, or surjective, or a covering map, or a topological submersion (see Definition 1.5.10), or a locally trivial fibration, then f' has the same property.

Given maps $f : X \to Y$ and $g : Y' \to Y$, one can always construct a cartesian square as above, by setting

$$X' = X \times_Y Y' = \{(x, y') \in X \times Y' \mid f(x) = g(y')\}.$$

We wish to compare various sheaf functors coming from the diagram (1.2.7).

LEMMA 1.2.11. *In the cartesian square* (1.2.7)*, for* $\mathcal{F} \in D^+(X', \Bbbk)$*, there is a natural transformation* $f_! g'_* \mathcal{F} \to g_* f'_! \mathcal{F}$ *that fits into a commutative diagram*

$$
\begin{tikzcd}
f_! g'_* \mathcal{F} \arrow[r] \arrow[d, "(1.2.1)"'] & g_* f'_! \mathcal{F} \arrow[d, "(1.2.1)"] \\
f_* g'_* \mathcal{F} \arrow[r, "Prop.\ 1.2.8(2)", "\sim"'] & g_* f'_* \mathcal{F}
\end{tikzcd}
$$

PROOF SKETCH. Let $h = f \circ g' = g \circ f' : X' \to Y$. For $\mathcal{F} \in \mathrm{Sh}(X', \Bbbk)$, consider the diagram

$$
\begin{tikzcd}
{}^\circ f_! {}^\circ g'_* \mathcal{F} \arrow[rr, dashed] \arrow[d, "(1.2.1)"'] & & {}^\circ g_* {}^\circ f'_! \mathcal{F} \arrow[d, "(1.2.1)"] \\
{}^\circ f_* {}^\circ g'_* \mathcal{F} \arrow[r, equal] & {}^\circ h_* \mathcal{F} \arrow[r, equal] & {}^\circ g_* {}^\circ f'_* \mathcal{F}
\end{tikzcd}
$$

The vertical maps are inclusions of subsheaves coming from (1.2.1). One can check by unwinding the definitions that the dotted arrow along the top (necessarily unique) exists. Now pass to derived functors, and take $\mathcal{F} \in D^+(X', \Bbbk)$. At each corner of the diagram, we have an instance of the natural map (A.6.3) for a composition of derived functors, labelled $m_1, \dots, m_4$ below:

$$\begin{tikzcd}
f_! g'_* \mathcal{F} \arrow[d, "(1.2.1)"'] & R({}^\circ f_! {}^\circ g'_*)(\mathcal{F}) \arrow[l, "m_1"'] \arrow[r] \arrow[d, "(1.2.1)"'] & R({}^\circ g_* {}^\circ f'_!)(\mathcal{F}) \arrow[r, "m_2"] \arrow[d, "(1.2.1)"] & g_* f'_! \mathcal{F} \arrow[d, "(1.2.1)"] \\
f_* g'_* \mathcal{F} & R({}^\circ f_* {}^\circ g'_*)(\mathcal{F}) \arrow[l, "m_3"'] \arrow[r, equal] & R({}^\circ g_* {}^\circ f'_*)\mathcal{F} \arrow[r, "m_4"] & g_* f'_* \mathcal{F}
\end{tikzcd}$$

The maps m_3 and m_4 are isomorphisms by Proposition 1.2.8(2). Since ${}^\circ g'_*$ takes injective sheaves to injective sheaves (see Table 1.4.1(iii)), m_1 is also an isomorphism by Proposition A.6.6, and we obtain the desired diagram. (The map m_2 need not be an isomorphism.) $\square$

Lemma 1.2.12. *In the cartesian square* (1.2.7)*, for $\mathcal{F} \in D^+(X, \Bbbk)$, there is a natural commutative diagram*

$$\begin{tikzcd}
g^* f_! \mathcal{F} \arrow[r] \arrow[d, "(1.2.1)"'] & f'_!(g')^* \mathcal{F} \arrow[d, "(1.2.1)"] \\
g^* f_* \mathcal{F} \arrow[r] & f'_*(g')^* \mathcal{F}
\end{tikzcd}$$

Proof sketch. Apply Lemma 1.2.11 to $(g')^*\mathcal{F}$. Then apply g^* to the whole diagram, and use the adjunction maps $\mathrm{id} \to g'_*(g')^*$ and $g^* g_* \to \mathrm{id}$ to obtain the commutative diagram

$$\begin{tikzcd}
g^* f_! \mathcal{F} \arrow[r] \arrow[d] & g^* f_! g'_*(g')^* \mathcal{F} \arrow[r] \arrow[d] & g^* g_* f'_!(g')^* \mathcal{F} \arrow[r] \arrow[d] & f'_!(g')^* \mathcal{F} \arrow[d] \\
g^* f_* \mathcal{F} \arrow[r] & g^* f_* g'_*(g')^* \mathcal{F} \arrow[r, "\sim"] & g^* g_* f'_*(g')^* \mathcal{F} \arrow[r] & f'_*(g')^* \mathcal{F}
\end{tikzcd}$$

as desired. $\square$

Both horizontal maps in Lemma 1.2.12 are called **base change morphisms**. For the upper horizontal map, we have the following fundamental result.

Theorem 1.2.13 (Proper base change). *Consider the cartesian square* (1.2.7)*, and assume that all the spaces are locally compact. For any $\mathcal{F} \in D^+(X, \Bbbk)$, the base change map $g^* f_! \mathcal{F} \xrightarrow{\sim} f'_!(g')^*\mathcal{F}$ is an isomorphism.*

Proof sketch. Consider the special case where g is the inclusion of a single point $\{y\}$ into Y, and X' is identified with $f^{-1}(y) \subset X$. In this case the base change map can be rewritten as

$$(f_! \mathcal{F})_y \to R\Gamma_c(\mathcal{F}|_{f^{-1}(y)}). \tag{1.2.8}$$

The first step is to prove that this map is an isomorphism, following the plan of Remark 1.2.6. Most of the work is in establishing the abelian-category version; see [**125**, Proposition 2.5.2] for details. To finish the proof that (1.2.8) is an isomorphism, replace $\mathcal{F}$ by a c-soft resolution.

Now return to the case of a general cartesian square. It is enough to show that the induced map on stalks

$$(g^* f_! \mathcal{F})_y \to (f'_!(g')^* \mathcal{F})_y \tag{1.2.9}$$

is an isomorphism for every point $y \in Y'$. Consider the following diagram, which contains three cartesian squares:

$$\begin{tikzcd}
(f')^{-1}(y) \arrow[r, "i'"] \arrow[d, "\mathrm{a}_{(f')^{-1}(y)}"'] \arrow[rr, bend left, "h'=g'\circ i'"] & X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\
\{y\} \arrow[r, "i"] \arrow[rr, bend right, "h=g\circ i"'] & Y' \arrow[r, "g"] & Y
\end{tikzcd}$$

This gives rise to a diagram

$$\begin{tikzcd}
(g^* f_! \mathcal{F})_y \arrow[r, "(1.2.9)"] \arrow[d, "\wr"'] & (f'_!(g')^* \mathcal{F})_y \arrow[r, "b_1"] & R\Gamma_c(((g')^*\mathcal{F})|_{(f')^{-1}(y)}) \arrow[d, "\wr"] \\
h^* f_! \mathcal{F} \arrow[rr, "b_2"] & & R\Gamma_c((h')^*\mathcal{F})
\end{tikzcd}$$

in which b_1 and b_2 are both base change maps. One can check from the construction in Lemma 1.2.12 that this diagram commutes. The maps b_1 and b_2 are isomorphisms by the special case in (1.2.8), so (1.2.9) is as well. $\square$

The special case (1.2.8) of Theorem 1.2.13 implies that the following generalization of c-soft sheaves is also an adapted class for ${}^\circ f_!$.

DEFINITION 1.2.14. Let $f : X \to Y$ be a continuous map of locally compact spaces. A sheaf $\mathcal{F}$ on X is said to be **relatively c-soft for f**, or **f-soft**, if for each point $y \in Y$, the sheaf $\mathcal{F}|_{f^{-1}(y)}$ is c-soft.

The second base change map from Lemma 1.2.12, $(g^* f_* \mathcal{F} \to f'_*(g')^*\mathcal{F})$, is not an isomorphism in general, although it is an isomorphism in some important special cases; see Section 1.9. It is also an isomorphism when f is proper or when g is an open embedding. These cases are spelled out in the following two propositions.

PROPOSITION 1.2.15 (Proper base change). *In the cartesian square* (1.2.7), *assume that f is proper. For any $\mathcal{F} \in D^+(X, \Bbbk)$, the base change map $g^* f_* \mathcal{F} \xrightarrow{\sim} f'_*(g')^*\mathcal{F}$ is an isomorphism.*

If the spaces are locally compact, this follows trivially from Theorem 1.2.13, but local compactness is not required for this statement.

PROOF SKETCH. The proof follows the pattern of that of Theorem 1.2.13 very closely, but in place of (1.2.8), the first step is to prove that

$$(f_*\mathcal{F})_y \to R\Gamma(\mathcal{F}|_{f^{-1}(y)})$$

is an isomorphism. The proof of the abelian-category version is explained in [**125**, Remark 2.5.3], and then the derived version is obtained by replacing $\mathcal{F}$ by a flabby resolution. $\square$

PROPOSITION 1.2.16 (Open base change). *Let $f : X \to Y$ be a continuous map. Let $V \subset Y$ be an open subset, and consider the cartesian square*

$$\begin{tikzcd} f^{-1}(V) \arrow[r, hook] \arrow[d, "f|_{f^{-1}(V)}"'] & X \arrow[d, "f"] \\ V \arrow[r, hook] & Y \end{tikzcd}$$

For any $\mathcal{F} \in D^+(X, \Bbbk)$, the base change map $(f_*\mathcal{F})|_V \xrightarrow{\sim} (f|_{f^{-1}(V)})_*(\mathcal{F}|_{f^{-1}(V)})$ is an isomorphism.

PROOF SKETCH. The abelian-category version is obvious: for a sheaf $\mathcal{F}$ on X and any open set $W \subset V$, the sections of both $(f_*\mathcal{F})|_V$ and $(f|_{f^{-1}(V)})_*(\mathcal{F}|_{f^{-1}(V)})$ are given by $\mathcal{F}(f^{-1}(W))$. For the derived version, replace $\mathcal{F}$ by a flabby resolution. (This works because the restriction functor from X to $f^{-1}(V)$ sends flabby sheaves to flabby sheaves; see Table 1.4.1(ii).) □

Like Lemma 1.2.10, the preceding result can also be proved with injective resolutions, using Exercise 1.3.3.

Exercises.

1.2.1. Let $f : X \to Y$ be a continuous map. Prove that for any $\mathcal{F} \in \mathrm{Sh}(X, \Bbbk)$, the presheaf $^\circ f_!(\mathcal{F})$ is a sheaf.

1.2.2. Let X be a topological space, and let X_{disc} be the same set equipped with the discrete topology. There is an obvious continuous map $i : X_{\mathrm{disc}} \to X$.

(a) Show that for any sheaf $\mathcal{F}$ on X, the sheaf $^\circ i_* i^* \mathcal{F}$ is flabby, and that the adjunction map $\eta : \mathcal{F} \to {}^\circ i_* i^* \mathcal{F}$ is injective.

(b) By iterating this construction, one obtains a sequence of sheaves and morphisms as follows:

$$0 \to \mathcal{F} \xrightarrow{\eta} {}^\circ i_* i^* \mathcal{F} \xrightarrow{d^0} {}^\circ i_* i^* (\operatorname{cok} \eta) \xrightarrow{d^1} {}^\circ i_* i^* (\operatorname{cok} d^0) \xrightarrow{d^2} {}^\circ i_* i^* (\operatorname{cok} d^1) \to \cdots .$$

Show that this sequence is exact.

The construction above gives a functorial flabby resolution of $\mathcal{F}$, known as the **Godement resolution**.

(c) Now let $\mathcal{F} \in Ch^+(\mathrm{Sh}(X, \Bbbk))$. Roughly speaking, one can define the "Godement resolution of $\mathcal{F}$" to be the total complex of a double complex whose rows are the Godement resolutions of the terms of $\mathcal{F}$. Explain how to make this definition precise, and then show that the natural map from $\mathcal{F}$ to its Godement resolution is a quasi-isomorphism.

1.2.3. Let $X = \mathbb{C} \smallsetminus \{0\}$, and let $f : X \to X$ be the map $f(z) = z^2$. Let $\mathcal{Q}$ be the "square-root sheaf" from Exercise 1.1.8. Show that $f_*\underline{\mathbb{C}}_X \cong \underline{\mathbb{C}}_X \oplus \mathcal{Q}$ and that $f^*\mathcal{Q} \cong \underline{\mathbb{C}}_X$. (*Hint*: First show "by hand" that $^\circ f_*\underline{\mathbb{C}}_X \cong \underline{\mathbb{C}}_X \oplus \mathcal{Q}$. Then use the proper base change theorem to show that $^\circ f_*\underline{\mathbb{C}}_X \cong f_*\underline{\mathbb{C}}_X$.)

1.2.4. Generalize the preceding exercise as follows: Let $n \geq 1$, and let $f : X \to X$ be the map $f(z) = z^n$. Find a collection of sheaves $\mathcal{Q}_1, \ldots, \mathcal{Q}_{n-1}$ with the following properties:

(a) Each $\mathcal{Q}_i$ is locally constant but not constant. Moreover, $\mathcal{Q}_i \ncong \mathcal{Q}_j$ if $i \neq j$.

(b) We have $f_*\underline{\mathbb{C}}_X \cong \underline{\mathbb{C}}_X \oplus \mathcal{Q}_1 \oplus \ldots \oplus \mathcal{Q}_{n-1}$.

(c) For each i, $f^*\mathcal{Q}_i \cong \underline{\mathbb{C}}_X$.

For further generalizations, see Exercise 1.7.3.

1.2.5. Let $(\mathcal{F}_\alpha)_{\alpha\in I}$ be a collection of sheaves on X. Consider the space $I \times X$, where I is equipped with the discrete topology. Define a sheaf $\hat{\mathcal{F}}$ on $I \times X$ by requiring that $\hat{\mathcal{F}}|_{\{\alpha\}\times X} = \mathcal{F}_\alpha$. (This defines a sheaf by Exercise 1.1.12.) Let $\mathrm{pr}_2 : I \times X \to X$ be the obvious map. Show that ${}^\circ\mathrm{pr}_{2!}$ is an exact functor and that there is a natural isomorphism

$$\mathrm{pr}_{2!}\hat{\mathcal{F}} \cong \bigoplus_{\alpha\in I} \mathcal{F}_\alpha.$$

(See Exercise 1.1.7 for a discussion of $\bigoplus_{\alpha\in I}\mathcal{F}_\alpha$.) Deduce that for any continuous map of locally compact spaces $f : X \to Y$, the functor $f_!$ commutes with arbitrary direct sums.

1.2.6. Suppose that X is the disjoint union of a collection of subspaces $(X_\alpha)_{\alpha\in I}$. For $\mathcal{F} \in D^+(X, \Bbbk)$, show that there is a natural isomorphism

$$\mathbf{H}^i(X, \mathcal{F}) \cong \prod_{\alpha\in I} \mathbf{H}^i(X_\alpha, \mathcal{F}|_{X_\alpha}).$$

1.3. Open and closed embeddings

When $h : Y \hookrightarrow X$ is the inclusion map of an open or closed subset (or, more generally, a locally closed subset), the functors h^*, $h_!$, and h_* have additional nice properties. These properties are the focus of this section.

Lemma 1.3.1. *Let $h : Y \hookrightarrow X$ be the inclusion of a locally closed subset. Then, for $\mathcal{F} \in \mathrm{Sh}(Y, \Bbbk)$, the sheaf ${}^\circ h_!(\mathcal{F})$ is naturally isomorphic to the sheafification of the presheaf ${}^\circ h_{!,\mathrm{pre}}(\mathcal{F})$ given by*

$${}^\circ h_{!,\mathrm{pre}}(\mathcal{F})(U) = \begin{cases} \mathcal{F}(U \cap Y) & \text{if } U \cap \overline{Y} \subset Y, \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, ${}^\circ h_!$ is exact, and its stalks are given by

$${}^\circ h_!(\mathcal{F})_x \cong \begin{cases} \mathcal{F}_x & \text{if } x \in Y, \\ 0 & \text{if } x \notin Y. \end{cases}$$

Proof sketch. Recall that an inclusion map of a subset is proper if and only if the subset is closed [**42**, Proposition I.10.2]. If $U \subset X$ is an open subset such that $U \cap \overline{Y} \subset Y$, then any closed subset of $U \cap Y$ remains closed as a subset of U. It follows that

$$\{s \in \mathcal{F}(U \cap Y) \mid h|_{\mathrm{supp}\, s} : \mathrm{supp}\, s \to U \text{ is proper}\} = \mathcal{F}(U \cap Y).$$

Thanks to this observation, we have an obvious presheaf morphism ${}^\circ h_{!,\mathrm{pre}}(\mathcal{F}) \to {}^\circ h_!(\mathcal{F})$, and hence a sheaf morphism

$$({}^\circ h_{!,\mathrm{pre}}(\mathcal{F}))^+ \to {}^\circ h_!(\mathcal{F}). \tag{1.3.1}$$

Next, one can show that both ${}^\circ h_{!,\mathrm{pre}}(\mathcal{F})$ and ${}^\circ h_!(\mathcal{F})$ have stalks as described in the statement of the lemma. This implies that (1.3.1) is an isomorphism. Finally, the stalk formula also implies that ${}^\circ h_!$ is exact. $\square$

Because ${}^\circ h_!$ is exact, we will henceforth omit the symbol "$\circ$" and simply write

$$h_! : \mathrm{Sh}(Y, \Bbbk) \to \mathrm{Sh}(X, \Bbbk).$$

Because this functor preserves stalks over Y and has zero stalks outside of Y, it is often called the **extension by zero** functor.

One remarkable property of $h_!$ for a locally closed embedding is that it has a rather down-to-earth right adjoint. (For more general maps f, the existence of a right adjoint to $f_!$ will be discussed in Section 1.5.)

Definition 1.3.2. Let $h : Y \hookrightarrow X$ be the inclusion of a locally closed subset. The functor of **restriction with supports** in Y is the functor ${}^\circ h^! : \mathrm{Sh}(X, \Bbbk) \to \mathrm{Sh}(Y, \Bbbk)$ given by

$${}^\circ h^!(\mathcal{F})(U) = \varinjlim_{\substack{V \subset X \text{ open} \\ V \cap \overline{Y} = U}} \{s \in \mathcal{F}(V) \mid \operatorname{supp} s \subset U\}.$$

It is left as an exercise to show that this defines a sheaf, and not just a presheaf. The following two lemmas are also left as exercises.

Lemma 1.3.3. *If $j : U \hookrightarrow X$ is an open embedding, then for $\mathcal{F} \in D^+(X, \Bbbk)$, there is a natural isomorphism $j^!\mathcal{F} \cong j^*\mathcal{F}$.*

Lemma 1.3.4. *Let $h : Y \hookrightarrow X$ be a locally closed embedding. The functor ${}^\circ h^! : \mathrm{Sh}(X, \Bbbk) \hookrightarrow \mathrm{Sh}(Y, \Bbbk)$ is left exact.*

As a consequence, we can form its right derived functor

$$h^! = R({}^\circ h^!) : D^+(X, \Bbbk) \to D^+(Y, \Bbbk).$$

If $i_x : \{x\} \hookrightarrow X$ is the inclusion of a point, the object $i_x^!\mathcal{F} \in D^+(\mathrm{pt}, \Bbbk)$ is sometimes called the **costalk** of $\mathcal{F}$ at x, as a counterpart to the stalk $\mathcal{F}_x = i_x^*\mathcal{F}$.

Remark 1.3.5. Given a locally closed embedding $h : Y \hookrightarrow X$, some sources define a functor $\Gamma_Y : \mathrm{Sh}(X, \Bbbk) \to \mathrm{Sh}(X, \Bbbk)$. It is related to ${}^\circ h^!$ by the formulas

$${}^\circ h^!\mathcal{F} \cong h^*\Gamma_Y(\mathcal{F}) \qquad \text{and} \qquad \Gamma_Y(\mathcal{F}) \cong {}^\circ h_* {}^\circ h^!(\mathcal{F}).$$

We will not use the notation Γ_Y in this book, but it may be useful for readers who wish to compare the statements below to those in other sources such as [**125**].

Theorem 1.3.6. *Let $h : Y \hookrightarrow X$ be a locally closed embedding. For $\mathcal{F} \in D^+(Y, \Bbbk)$ and $\mathcal{G} \in D^+(X, \Bbbk)$, there is a natural isomorphism*

$$\mathrm{Hom}_{D^+(X,\Bbbk)}(h_!\mathcal{F}, \mathcal{G}) \cong \mathrm{Hom}_{D^+(Y,\Bbbk)}(\mathcal{F}, h^!\mathcal{G}).$$

The proof follows the same pattern as that of Theorem 1.2.4 and will be omitted. Note that for the abelian-category version, one may instead construct a natural isomorphism $\mathrm{Hom}({}^\circ h_{!,\mathrm{pre}}\mathcal{F}, \mathcal{G}) \cong \mathrm{Hom}(\mathcal{F}, {}^\circ h^!\mathcal{G})$.

Since $h_!$ is an exact functor, Theorem 1.3.6 implies that ${}^\circ h^!$ sends injective sheaves to injective sheaves. In the special case where h is an open embedding, this observation has some consequences for the behavior of injective sheaves on open subsets; see Exercise 1.3.3.

Proposition 1.3.7. *Let $k : W \hookrightarrow Y$ and $h : Y \hookrightarrow X$ be inclusion maps of locally closed subsets.*

(1) *For $\mathcal{F} \in D^+(W, \Bbbk)$, there is a natural isomorphism $h_!k_!\mathcal{F} \cong (h \circ k)_!\mathcal{F}$.*
(2) *For $\mathcal{F} \in D^+(X, \Bbbk)$, there is a natural isomorphism $k^!h^!\mathcal{G} \cong (h \circ k)^!\mathcal{F}$.*

Proof sketch. Part (1) closely resembles Proposition 1.2.8(3), but the spaces here are not assumed to be locally compact. But since ${}^\circ h_!$ and ${}^\circ k_!$ are exact, the desired isomorphism follows immediately from the abelian-category version, without using c-soft (or any other adapted class of) sheaves.

Next, Theorem 1.3.6 implies that $k^! \circ h^!$ is right adjoint to $h_! \circ k_!$ and that $(h \circ k)^!$ is right adjoint to $(h \circ k)_!$. Part (2) thus follows from part (1) by uniqueness of right adjoints. □

Remark 1.3.8 (Induced maps in hypercohomology with compact support). In contrast with Remark 1.2.9, arbitrary continuous maps do not give rise to an induced map in hypercohomology with compact support. However, for an open embedding $j : U \hookrightarrow X$, we have $j^! \underline{M}_X \cong j^* \underline{M}_X \cong \underline{M}_U$, and hence an adjunction map $j_! \underline{M}_U \to \underline{M}_X$. Applying $R\Gamma_c$, we get a map

$$j_\sharp : \mathbf{H}^k_c(U; M) \to \mathbf{H}^k_c(X; M). \tag{1.3.2}$$

More generally, suppose we have a commutative (not necessarily cartesian) square

$$\begin{tikzcd} U \arrow[r, "j"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ V \arrow[r, "h"] & Y \end{tikzcd}$$

in which the horizontal maps are open embeddings. Let us define

$$j_\sharp : f'_! \underline{\Bbbk}_U \to h^! f_! \underline{\Bbbk}_X$$

to be the natural transformation obtained as the following composition, using the adjunction maps $\mathrm{id} \to h^! h_!$ and $j_! j^! \to \mathrm{id}$:

$$f'_! \underline{\Bbbk}_U \cong f'_! j^! \underline{\Bbbk}_X \to h^! h_! f'_! j^! \underline{\Bbbk}_X \cong h^! f_! j_! j^! \underline{\Bbbk}_X \to h^! f_! \underline{\Bbbk}_X.$$

In the special case where $V = Y = \mathrm{pt}$, this agrees with (1.3.2). See Exercise 1.3.6 and Section B.1 for more on this construction.

From the definition, we see that for a sheaf $\mathcal{F}$, there is a natural injective map ${}^\circ h^!(\mathcal{F}) \hookrightarrow h^*(\mathcal{F})$. This gives rise to a natural transformation of derived functors

$$h^! \mathcal{F} \to h^* \mathcal{F}$$

for any $\mathcal{F} \in D^+(X, \Bbbk)$.

Proposition 1.3.9. *Let $h : Y \hookrightarrow X$ be a locally closed embedding. For any $\mathcal{F} \in D^+(Y, \Bbbk)$, the natural maps*

$$\mathcal{F} \to h^! h_! \mathcal{F} \to h^* h_! \mathcal{F} \qquad \text{and} \qquad h^! h_* \mathcal{F} \to h^* h_* \mathcal{F} \to \mathcal{F}$$

are all isomorphisms.

Proof. As in Remark 1.2.6, it is enough to prove the abelian-category version; the derived version then follows by taking injective resolutions. For a sheaf $\mathcal{F} \in \mathrm{Sh}(Y, \Bbbk)$, we proceed as follows.

Let $\mathcal{F} \in \mathrm{Sh}(Y, \Bbbk)$. The description of stalks in Lemma 1.3.1 implies that for any section $s \in h_! \mathcal{F}(U)$, we have $\operatorname{supp} s \subset U \cap Y$. From the definition of ${}^\circ h^!$, we deduce that the natural map

$$ {}^\circ h^! h_! \mathcal{F} \to h^* h_! \mathcal{F} \tag{1.3.3}$$

is an isomorphism. By Lemma 1.3.1 again, the composition

$$\mathcal{F} \to {}^\circ h^! h_! \mathcal{F} \xrightarrow{\sim} h^* h_! \mathcal{F}$$

induces an isomorphism on all stalks, so it is an isomorphism, as is $\mathcal{F} \to {}^\circ h^! h_! \mathcal{F}$.

Next, for an open subset $U \subset Y$, we have

$$h^*_{\mathrm{pre}}{}^\circ h_*(\mathcal{F})(U) = \varinjlim_{\substack{V \subset X \text{ open} \\ V \supset U}} {}^\circ h_*(\mathcal{F})(V) = \varinjlim_{\substack{V \subset X \text{ open} \\ V \supset U}} \mathcal{F}(V \cap Y).$$

The last term has a natural restriction map to $\mathcal{F}(U)$. But for V sufficiently small, we have $V \cap Y = U$, so in fact the natural map $h^*_{\mathrm{pre}}{}^\circ h_*(\mathcal{F})(U) \to \mathcal{F}(U)$ is an isomorphism. It follows that $h^*{}^\circ h_*(\mathcal{F}) \to \mathcal{F}$ is an isomorphism as well.

Finally, it remains to show that ${}^\circ h^! h_* \mathcal{F} \to h^* h_* \mathcal{F}$ is an isomorphism. In the special case where h is an open embedding, this follows from Lemma 1.3.3. In the special case where h is a closed embedding, we have $h_*\mathcal{F} \cong h_!\mathcal{F}$, and the claim follows from (1.3.3). Since any locally closed embedding can be factored as the composition of the open embedding $Y \hookrightarrow \overline{Y}$ followed by the closed embedding $\overline{Y} \hookrightarrow X$, our claim follows from the two special cases and Propositions 1.2.8 and 1.3.7. □

The following theorem is the most important result in this section. The distinguished triangles in this theorem are incredibly useful for computations and will be one of the most frequently used tools throughout the book.

THEOREM 1.3.10. *Let $i : Z \hookrightarrow X$ be a closed embedding, and let $j : U \hookrightarrow X$ be the complementary open embedding.*

(1) *We have $i^* \circ j_! = 0$, $i^! \circ j_* = 0$, and $j^* \circ i_* = 0$.*

(2) *For any $\mathcal{F} \in D^+(X, \Bbbk)$, there is a natural distinguished triangle*

$$j_! j^* \mathcal{F} \to \mathcal{F} \to i_* i^* \mathcal{F} \to . \tag{1.3.4}$$

Moreover, if $\mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to$ is a distinguished triangle in which $i^*\mathcal{F}' = 0$ and $j^*\mathcal{F}'' = 0$, then it is canonically isomorphic to that in (1.3.4).

(3) *For any $\mathcal{F} \in D^+(X, \Bbbk)$, there is a natural distinguished triangle*

$$i_* i^! \mathcal{F} \to \mathcal{F} \to j_* j^* \mathcal{F} \to . \tag{1.3.5}$$

Moreover, if $\mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to$ is a distinguished triangle in which $i^!\mathcal{F}'' = 0$ and $j^*\mathcal{F}' = 0$, then it is canonically isomorphic to that in (1.3.5).

In each of these triangles, the first two maps are adjunction maps.

PROOF. (1) Since i^*, $j_!$, j^*, and i_* come from exact functors, the claims that $i^* \circ j_! = 0$ and $j^* \circ i_* = 0$ can be proved at the level of abelian categories. This is left as an exercise. By Theorem 1.2.4 and Theorem 1.3.6, $i^! \circ j_*$ is right adjoint to $j^* \circ i_! \cong j^* \circ i_*$, so it vanishes as well.

(2) Given $\mathcal{F} \in \mathrm{Sh}(X, \Bbbk)$ and $x \in X$, consider the maps on stalks induced by the adjunction maps

$$(j_! j^* \mathcal{F})_x \to \mathcal{F}_x \to (i_* i^* \mathcal{F})_x.$$

If $x \in U$, the first map is an isomorphism, and $(i_* i^* \mathcal{F})_x = 0$. If $x \in Z$, then $(j_! j^* \mathcal{F})_x = 0$, and the second map is an isomorphism. In either case, the sequence above is (trivially) a short exact sequence. Therefore, by Proposition 1.1.12,

$$0 \to j_! j^* \mathcal{F} \to \mathcal{F} \to i_* i^* \mathcal{F} \to 0 \tag{1.3.6}$$

is a short exact sequence of sheaves.

Next, for $\mathcal{F} \in D^+(X, \Bbbk)$, apply (1.3.6) to get a short exact sequence of chain complexes. By the fancy snake lemma (Lemma A.5.12), we get a distinguished triangle $j_! j^* \mathcal{F} \to \mathcal{F} \to i_* i^* \mathcal{F} \to$. Note that

$$\mathrm{Hom}(j_! j^* \mathcal{F}, i_* i^* \mathcal{F}[-1]) \cong \mathrm{Hom}(i^* j_! j^* \mathcal{F}, i^* \mathcal{F}[-1]) = 0,$$

so the third morphism in our triangle is unique by Corollary A.4.11. Any other triangle $\mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to$ as in the statement of the proposition is canonically isomorphic to (1.3.4) by Lemma A.4.10.

(3) For $\mathcal{F} \in \mathrm{Sh}(X, \Bbbk)$, it follows from the definitions that $({}^\circ j_* j^* \mathcal{F})(V) = \mathcal{F}(V \cap U)$. Therefore, the kernel $\mathcal{K}$ of the adjunction map $\mathcal{F} \to {}^\circ j_* j^* \mathcal{F}$ is given by

$$\mathcal{K}(V) = \{s \in \mathcal{F}(V) \mid s|_{V \cap U} = 0\} = \{s \in \mathcal{F}(V) \mid \mathrm{supp}\, s \subset V \cap Z\}.$$

It is straightforward to check that $\mathcal{K}$ can be identified with $i_* {}^\circ i^! \mathcal{F}$, and the inclusion map $\mathcal{K} \to \mathcal{F}$ with the adjunction map $i_* {}^\circ i^! \mathcal{F} \to \mathcal{F}$. We at least have a left exact sequence

$$0 \to i_* {}^\circ i^! \mathcal{F} \to \mathcal{F} \to {}^\circ j_* j^* \mathcal{F}. \tag{1.3.7}$$

For any open set $V \subset X$, the map $\mathcal{F}(V) \to ({}^\circ j_* j^* \mathcal{F})(V)$ can be identified with the restriction map $\mathcal{F}(V) \to \mathcal{F}(U \cap V)$. If $\mathcal{F}$ is flabby (for instance, if $\mathcal{F}$ is injective), this is surjective, so by Exercise 1.1.6, (1.3.7) is a short exact sequence.

The rest of the proof proceeds as in part (2), by applying the short exact sequence (1.3.7) to a chain complex of injective sheaves. ☐

Corollary 1.3.11. *Let $i : Z \hookrightarrow X$ be a closed embedding. The functor $i_* : D^+(Z, \Bbbk) \to D^+(X, \Bbbk)$ is fully faithful, and it induces an equivalence of categories*

$$D^+(Z, \Bbbk) \xrightarrow{\sim} \{\mathcal{F} \in D^+(X, \Bbbk) \mid \mathrm{supp}\, \mathcal{F} \subset Z\}.$$

Corollary 1.3.12. *Let $i : Z \hookrightarrow X$ be a closed embedding, and let $j : U \hookrightarrow X$ be the complementary open embedding. For any $\mathcal{F} \in D^+(U, \Bbbk)$, there is a canonical isomorphism $i^* j_* \mathcal{F} \cong i^! j_! \mathcal{F}[1]$.*

Proof. Apply (1.3.4) to the object $j_* \mathcal{F}$ to obtain a distinguished triangle $j_! \mathcal{F} \to j_* \mathcal{F} \to i_* i^* j_* \mathcal{F} \to$. Then apply (1.3.5) to the object $j_! \mathcal{F}$ to obtain a distinguished triangle $i_* i^! j_! \mathcal{F} \to j_! \mathcal{F} \to j_* \mathcal{F} \to$. The uniqueness assertions in Theorem 1.3.10 imply that these two triangles are (after rotation) canonically isomorphic to one another. In particular, there is a natural isomorphism $i_* i^* j_* \mathcal{F} \cong i_* i^! j_! \mathcal{F}[1]$. The result then follows from Corollary 1.3.11. ☐

Very similar ideas to those in the proof Theorem 1.3.10 yield the following.

Proposition 1.3.13 (Mayer–Vietoris distinguished triangles). *Let Y_1 and Y_2 be subsets of X that are either both open or both closed, and such that $Y_1 \cup Y_2 = X$. Let $h_1 : Y_1 \hookrightarrow X$, $h_2 : Y_2 \hookrightarrow X$, and $h : Y_1 \cap Y_2 \hookrightarrow X$ be the inclusion maps. For any $\mathcal{F} \in D^+(X, \Bbbk)$, there are distinguished triangles*

$$\mathcal{F} \to h_{1*} h_1^* \mathcal{F} \oplus h_{2*} h_2^* \mathcal{F} \to h_* h^* \mathcal{F} \to,$$
$$h_! h^! \mathcal{F} \to h_{1!} h_1^! \mathcal{F} \oplus h_{2!} h_2^! \mathcal{F} \to \mathcal{F} \to .$$

Proof sketch. Both distinguished triangles can be obtained from suitable short exact sequences in $\mathrm{Sh}(X, \Bbbk)$, following the pattern of the proof of Theorem 1.3.10. Let us briefly discuss how to obtain a short exact sequence corresponding to the first distinguished triangle above; the second one is similar.

Let $\mathcal{F} \in \mathrm{Sh}(X, \Bbbk)$, and let $\eta_1 : \mathcal{F} \to {}^\circ h_{1*}h_1^*\mathcal{F}$ and $\eta_2 : \mathcal{F} \to {}^\circ h_{2*}h_2^*\mathcal{F}$ be the adjunction maps. We also have adjunction maps $\eta_1' : {}^\circ h_{1*}h_1^*\mathcal{F} \to {}^\circ h_* h^*\mathcal{F}$ and $\eta_2' : {}^\circ h_{2*}h_2^*\mathcal{F} \to {}^\circ h_* h^*\mathcal{F}$. Consider the sequence

$$0 \to \mathcal{F} \xrightarrow{\left[\begin{smallmatrix}\eta_1\\ -\eta_2\end{smallmatrix}\right]} {}^\circ h_{1*}h_1^*\mathcal{F} \oplus {}^\circ h_{2*}h_2^*\mathcal{F} \xrightarrow{[\,\eta_1'\ \eta_2'\,]} {}^\circ h_* h^*\mathcal{F} \to 0.$$

If Y_1 and Y_2 are closed, a stalk calculation shows that this is a short exact sequence.

If Y_1 and Y_2 are open, a little more work is needed. Assume from now on that $\mathcal{F}$ is an injective sheaf. Let $W_1 = X \smallsetminus Y_1$ and $W_2 = X \smallsetminus Y_2$, and let $v_1 : W_1 \hookrightarrow X$, $v_2 : W_2 \hookrightarrow X$, and $v : W_1 \cup W_2 \hookrightarrow X$ be the inclusion maps. Since W_1 and W_2 are disjoint closed sets, it is easy to see that $v_*{}^\circ v^!\mathcal{F} \cong v_{1*}{}^\circ v_1^!\mathcal{F} \oplus v_{2*}{}^\circ v_2^!\mathcal{F}$. We then consider the diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & v_*{}^\circ v^!\mathcal{F} \arrow[r] \arrow[d, equal] & \mathcal{F} \arrow[r] \arrow[d, "\left[\begin{smallmatrix}\eta_1\\ -\eta_2\end{smallmatrix}\right]"] & h_*h^*\mathcal{F} \arrow[r] \arrow[d, "\left[\begin{smallmatrix}\mathrm{id}\\ -\mathrm{id}\end{smallmatrix}\right]"] & 0 \\
0 \arrow[r] & v_{2*}{}^\circ v_2^!\mathcal{F} \oplus v_{1*}{}^\circ v_1^!\mathcal{F} \arrow[r] & {}^\circ h_{1*}h_1^*\mathcal{F} \oplus {}^\circ h_{2*}h_2^*\mathcal{F} \arrow[r] \arrow[d, "{[\,\eta_1'\ \eta_2'\,]}"] & {}^\circ h_*h^*\mathcal{F} \oplus {}^\circ h_*h^*\mathcal{F} \arrow[r] \arrow[d, "{[\,\mathrm{id}\ \mathrm{id}\,]}"] & 0 \\
 & & {}^\circ h_*h^*\mathcal{F} \arrow[r, equal] \arrow[d] & {}^\circ h_*h^*\mathcal{F} \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

The top two rows of this diagram are instances of (1.3.7). A diagram chase then shows that the middle column is exact. $\square$

Remark 1.3.14. The proof above shows that the map $h_{1*}h_1^*\mathcal{F} \oplus h_{2*}h_2^*\mathcal{F} \to h_*h^*\mathcal{F}$ in Proposition 1.3.13 is the sum of the adjunction maps $h_{1*}h_1^*\mathcal{F} \to h_*h^*\mathcal{F}$ and $h_{2*}h_2^*\mathcal{F} \to h_*h^*\mathcal{F}$.

Exercises.

1.3.1. Let $h : Y \hookrightarrow X$ be a locally closed embedding. Show that the presheaf ${}^\circ h^!(\mathcal{F})$ is a sheaf.

1.3.2. Let $j : U \hookrightarrow X$ be the inclusion of an open subset. Show that $j_!\underline{\Bbbk}_U$ is isomorphic to the sheaf given by

$$V \mapsto \{\text{locally constant functions } s : V \to \Bbbk \text{ such that } s|_{V \smallsetminus U} = 0\}.$$

1.3.3. Using Theorem 1.3.6, show that the restriction of an injective sheaf to an open subset is injective, and that every injective sheaf is flabby.

1.3.4. Let $i : Z \hookrightarrow X$ be a closed embedding, and let $j : U \hookrightarrow X$ be the complementary open embedding. Show that the diagram of categories and functors

$$\begin{tikzcd}
D^+(Z,\Bbbk) \arrow[r, "i_*"] & D^+(X,\Bbbk) \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "i^!"] \arrow[r, "j^*"] & D^+(U,\Bbbk) \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "j_*"]
\end{tikzcd}$$

is a **recollement diagram**, as in Exercise A.7.4.

1.3.5. Suppose X has a sequence of closed subsets $\varnothing = X_0 \subset X_1 \subset X_2 \subset \cdots \subset X_n = X$ such that $X_i \smallsetminus X_{i-1}$ is homeomorphic to some even-dimensional real vector space $\mathbb{R}^{2k_i}$. Use Theorem 1.3.10 to show that

$$\mathbf{H}^k_{\mathrm{c}}(X; \Bbbk) \cong \begin{cases} \Bbbk^{|\{i \mid 2k_i = k\}|} & \text{if } k \text{ is even,} \\ 0 & \text{if } k \text{ is odd.} \end{cases}$$

(This is a variant of a well-known fact from algebraic topology; see, for instance, the discussion in [**98**, Section 2.2].)

1.3.6. Suppose we have a commutative diagram

$$\begin{tikzcd} U' \arrow[r, hook, "j'"'] \arrow[rr, bend left, hook, "j''=jj'"] \arrow[d, "f''"'] & U \arrow[r, hook, "j"'] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ V' \arrow[r, hook, "h'"] \arrow[rr, bend right, hook, "h''=hh'"'] & V \arrow[r, hook, "h"] & Y \end{tikzcd}$$

in which the horizontal maps are open embeddings. Show that the following diagram commutes (see Remark 1.3.8 for the definition of $j_\sharp$):

$$\begin{tikzcd} f''_! \underline{\Bbbk}_{U'} \arrow[r, "j''_\sharp"] \arrow[d, "j'_\sharp"'] & (h'')^! f_! \underline{\Bbbk}_X \arrow[d, "\wr"] \\ (h')^! f'_! \underline{\Bbbk}_U \arrow[r, "(h')^! j_\sharp"] & (h')^! h^! f_! \underline{\Bbbk}_X \end{tikzcd}$$

1.3.7. Let $i : Z \hookrightarrow X$ be a closed embedding, and let $j : U \hookrightarrow X$ be the complementary open embedding. Show that for $\mathcal{F} \in D^-(X, \Bbbk)$ and $\mathcal{G} \in D^+(X, \Bbbk)$, we have the following natural distinguished triangle (see also Exercise 1.5.2):

$$R\mathrm{Hom}(i^*\mathcal{F}, i^!\mathcal{G}) \to R\mathrm{Hom}(\mathcal{F}, \mathcal{G}) \to R\mathrm{Hom}(j^*\mathcal{F}, j^*\mathcal{G}) \to .$$

1.3.8 (Relative cohomology). Let $Z \subset X$ be a closed subset, and let $j : (X \smallsetminus Z) \hookrightarrow X$ be the inclusion map of its complement. Define the **relative cohomology** of the pair (X, Z) by

$$\mathbf{H}^k(X, Z; M) = \mathbf{H}^k(X, j_! \underline{M}_{X \smallsetminus Z}).$$

Show that there is a natural long exact sequence

$$\cdots \to \mathbf{H}^k(X, Z; M) \to \mathbf{H}^k(X; M) \to \mathbf{H}^k(Z; M) \to \mathbf{H}^{k+1}(X, Z; M) \to \cdots .$$

1.3.9 (Mayer–Vietoris sequence). Let $Y_1, Y_2 \subset X$ be either two open subsets or two closed subsets such that $Y_1 \cup Y_2 = X$. Let $h_1 : Y_1 \hookrightarrow X$, $h_2 : Y_2 \hookrightarrow X$, and $h : Y_1 \cap Y_2 \hookrightarrow X$ be the inclusion maps. Using Proposition 1.3.13, show that there is a canonical long exact sequence

$$\cdots \to \mathbf{H}^k(X; M) \to \mathbf{H}^k(Y_1; M) \oplus \mathbf{H}^k(Y_2; M) \to \mathbf{H}^k(Y_1 \cap Y_2; M) \\ \to \mathbf{H}^{k+1}(X; M) \to \cdots .$$

1.3.10. Let X be a locally connected topological space. Let $x \in X$ be a point with the following property: for every open set containing x, there is another strictly smaller open set that also contains x. (This holds, for example, on $\mathbb{R}^n$.) In this problem, you will show that $\mathrm{Sh}(X, \Bbbk)$ does not have enough projectives.

(a) Let $\mathcal{F}$ be the skyscraper sheaf with value $\Bbbk$ at x. Show that if $j : V \hookrightarrow X$ is the inclusion of an open set containing x, then $\mathcal{F}$ is a quotient of $j_!\underline{\Bbbk}_V$.

(b) Suppose we have a projective sheaf $\mathcal{P}$ and a morphism $\phi : \mathcal{P} \twoheadrightarrow \mathcal{F}$. Show that if U is a connected open set containing x, then $\phi_U : \mathcal{P}(U) \to \mathcal{F}(U)$ is zero, because it factors through $(j_!\underline{\Bbbk}_V)(U)$, where $j : V \hookrightarrow X$ is the inclusion of a strictly smaller open set. Conclude that $\phi = 0$, and hence that $\mathrm{Sh}(X, \Bbbk)$ does not have enough projectives.

1.4. Tensor product and sheaf Hom

In this section, we introduce a few more fundamental operations on sheaves, and we list a large number of additional natural isomorphisms between compositions of sheaf functors.

Definition 1.4.1. For $\mathcal{F}, \mathcal{G} \in \mathrm{Sh}(X, \Bbbk)$, their **tensor product** $\mathcal{F} \otimes \mathcal{G}$ is the sheafification of the presheaf $\mathcal{F} \otimes_{\mathrm{pre}} \mathcal{G}$ given by

$$(\mathcal{F} \otimes_{\mathrm{pre}} \mathcal{G})(U) = \mathcal{F}(U) \otimes \mathcal{G}(U).$$

Their **sheaf Hom** is the sheaf $\mathscr{H}om(\mathcal{F}, \mathcal{G})$ given by

$$\mathscr{H}om(\mathcal{F}, \mathcal{G})(U) = \mathrm{Hom}_{\mathrm{Sh}(U,\Bbbk)}(\mathcal{F}|_U, \mathcal{G}|_U).$$

Both of these functors can be generalized to chain complexes. The chain complex tensor product $\otimes^{\mathrm{ch}}$ and the chain complex sheaf Hom functor $\mathrm{ch}\mathscr{H}om$ are defined like their analogues in Section A.6.

Lemma 1.4.2. *The functor $\otimes : \mathrm{Sh}(X, \Bbbk) \times \mathrm{Sh}(X, \Bbbk) \to \mathrm{Sh}(X, \Bbbk)$ is right exact in both variables, while $\mathscr{H}om : \mathrm{Sh}(X, \Bbbk)^{\mathrm{op}} \times \mathrm{Sh}(X, \Bbbk) \to \mathrm{Sh}(X, \Bbbk)$ is left exact in both variables.*

Since $\mathrm{Sh}(X, \Bbbk)$ has enough injectives, the latter has a derived functor

$$R\mathscr{H}om : D^-(X, \Bbbk)^{\mathrm{op}} \times D^+(X, \Bbbk) \to D^+(X, \Bbbk).$$

But $\mathrm{Sh}(X, \Bbbk)$ does *not* have enough projectives in general (see Exercise 1.3.10), so it takes a bit more work to construct the derived functor of $\otimes$.

Definition 1.4.3. A sheaf $\mathcal{F} \in \mathrm{Sh}(X, \Bbbk)$ is said to be **flat** if $\mathcal{F} \otimes (-) : \mathrm{Sh}(X, \Bbbk) \to \mathrm{Sh}(X, \Bbbk)$ is an exact functor.

For a proof that every sheaf is a quotient of a flat sheaf, see [**125**, Proposition 2.4.12]. Using flat resolutions, one can define the derived functor

$$\overset{L}{\otimes} : D^-(X, \Bbbk) \times D^-(X, \Bbbk) \to D^-(X, \Bbbk).$$

If $\Bbbk$ has finite global dimension, then the method of Theorem A.6.10 or Proposition A.6.16 shows that we also have a functor

$$\overset{L}{\otimes} : D^+(X, \Bbbk) \times D^+(X, \Bbbk) \to D^+(X, \Bbbk).$$

The basic facts we now need about how $\otimes$ and $\mathscr{H}om$ interact with various classes of sheaves are given in Table 1.4.1(v), (viii), and (ix). (Proofs can be found in [**125**, Section 2.2–2.4].) Using these, one can prove a large number of natural isomorphisms among various compositions of derived sheaf functors. The proofs of all of the following statements follow the general pattern of Remark 1.2.6. We omit most details, but we include remarks on the proof in a few cases.

TABLE 1.4.1. Quick reference for adapted classes for sheaf functors

Functor	*Exactness*	*Adapted classes*
f^*	exact	—
f_*, $R\Gamma$	left	injective, flabby
$f_!$	left	injective, flabby, c-soft, relatively c-soft
$R\Gamma_{\mathrm{c}}$	left	injective, flabby, c-soft
$\otimes^L$, $\Bbbk' \otimes^L_{\Bbbk} (-)$	right	flat
$R\mathscr{H}om$	left	injective

Implications among right adapted classes:

$$\text{injective} \implies \text{flabby} \implies c\text{-soft} \implies \text{relatively } c\text{-soft (for any } f). \tag{1.4.1}$$

Behavior of sheaf functors on adapted classes: For items (vi), (vii), (x), and (xi), assume the spaces are locally compact.

(i) $(\text{injective})|_{\text{open}} = \text{injective}$
(ii) $(\text{flabby})|_{\text{open}} = \text{flabby}$
(iii) ${}^\circ f_*(\text{injective}) = \text{injective}$
(iv) ${}^\circ f_*(\text{flabby}) = \text{flabby}$
(v) $f^*(\text{flat}) = \text{flat}$
(vi) ${}^\circ f_!(c\text{-soft}) = c\text{-soft}$
(vii) $(c\text{-soft})|_{\text{locally closed}} = c\text{-soft}$
(viii) $\mathscr{H}om(\text{any}, \text{injective}) = \text{flabby}$
(ix) $\mathscr{H}om(\text{flat}, \text{injective}) = \text{injective}$
(x) ${}^\circ f_!(\text{flat and } c\text{-soft}) = \text{flat and } c\text{-soft}$
(xi) $(\text{any}) \otimes (\text{flat and } c\text{-soft}) = c\text{-soft}$

We begin with several easy statements. The abelian-category versions of the following three propositions are almost immediate from the definitions.

PROPOSITION 1.4.4. (1) *For $\mathcal{F} \in D^-(X, \Bbbk)$, there is a natural isomorphism $\underline{\Bbbk}_X \otimes^L \mathcal{F} \xrightarrow{\sim} \mathcal{F}$. If $\Bbbk$ has finite global dimension, the same result holds for $\mathcal{F} \in D^+(X, \Bbbk)$.*
(2) *For $\mathcal{F} \in D^+(X, \Bbbk)$, there is a natural isomorphism $R\mathscr{H}om(\underline{\Bbbk}_X, \mathcal{F}) \cong \mathcal{F}$.*
(3) *For $\mathcal{F} \in D^-(X, \Bbbk)$ and $\mathcal{G} \in D^+(X, \Bbbk)$, and for any open subset $U \subset X$, there is a natural isomorphism $R\mathscr{H}om(\mathcal{F}, \mathcal{G})|_U \cong R\mathscr{H}om(\mathcal{F}|_U, \mathcal{G}|_U)$.*

PROPOSITION 1.4.5. *Let $f : X \to Y$ be a continuous map. For $\mathcal{F}, \mathcal{G} \in D^-(Y, \Bbbk)$, there is a natural isomorphism $f^*(\mathcal{F} \otimes^L \mathcal{G}) \cong f^*\mathcal{F} \otimes^L f^*\mathcal{G}$. If $\Bbbk$ has finite global dimension, the same result holds for $\mathcal{F}, \mathcal{G} \in D^+(Y, \Bbbk)$.*

PROPOSITION 1.4.6. *For $\mathcal{F} \in D^-(X, \Bbbk)$ and $\mathcal{G} \in D^+(X, \Bbbk)$, there is a natural isomorphism $R\Gamma(R\mathscr{H}om(\mathcal{F}, \mathcal{G})) \xrightarrow{\sim} R\mathrm{Hom}(\mathcal{F}, \mathcal{G})$.*

Next, we have an upgrade of Theorem 1.2.4.

PROPOSITION 1.4.7. *Let $f : X \to Y$ be a continuous map. For $\mathcal{F} \in D^-(Y, \Bbbk)$ and $\mathcal{G} \in D^+(X, \Bbbk)$, there are natural isomorphisms*

$$f_* R\mathscr{H}om(f^*\mathcal{F}, \mathcal{G}) \cong R\mathscr{H}om(\mathcal{F}, f_*\mathcal{G}),$$
$$R\mathrm{Hom}(f^*\mathcal{F}, \mathcal{G}) \cong R\mathrm{Hom}(\mathcal{F}, f_*\mathcal{G}),$$
$$\mathrm{Hom}(f^*\mathcal{F}, \mathcal{G}) \cong \mathrm{Hom}(\mathcal{F}, f_*\mathcal{G}).$$

THEOREM 1.4.8. *For $\mathcal{F}, \mathcal{G} \in D^-(X, \Bbbk)$ and $\mathcal{H} \in D^+(X, \Bbbk)$, there are natural isomorphisms*

$$R\mathscr{H}om(\mathcal{F} \overset{L}{\otimes} \mathcal{G}, \mathcal{H}) \cong R\mathscr{H}om(\mathcal{F}, R\mathscr{H}om(\mathcal{G}, \mathcal{H})),$$

$$R\mathrm{Hom}(\mathcal{F} \overset{L}{\otimes} \mathcal{G}, \mathcal{H}) \cong R\mathrm{Hom}(\mathcal{F}, R\mathscr{H}om(\mathcal{G}, \mathcal{H})),$$

$$\mathrm{Hom}(\mathcal{F} \overset{L}{\otimes} \mathcal{G}, \mathcal{H}) \cong \mathrm{Hom}(\mathcal{F}, R\mathscr{H}om(\mathcal{G}, \mathcal{H})).$$

Finally, perhaps the most significant result of this section is the following.

THEOREM 1.4.9 (Projection formula). *Let $f : X \to Y$ be a continuous map of locally compact spaces, and assume that $\Bbbk$ has finite global dimension. For $\mathcal{F} \in D^+(X, \Bbbk)$ and $\mathcal{G} \in D^+(Y, \Bbbk)$, there is a natural isomorphism $f_! \mathcal{F} \otimes^L \mathcal{G} \xrightarrow{\sim} f_!(\mathcal{F} \otimes^L f^* \mathcal{G})$.*

There is a subtlety in the proof of this theorem: the abelian-category analogue isn't true! For sheaves $\mathcal{F} \in \mathrm{Sh}(X, \Bbbk)$ and $\mathcal{G} \in \mathrm{Sh}(Y, \Bbbk)$, there is a natural map

$$ {}^\circ f_! \mathcal{F} \otimes \mathcal{G} \to {}^\circ f_!(\mathcal{F} \otimes f^* \mathcal{G}), \tag{1.4.2}$$

but it is not an isomorphism in general. However, it is an isomorphism if $\mathcal{G}$ is a flat sheaf (see [**125**, Proposition 2.5.13]), and this is enough to carry out the rest of the plan from Remark 1.2.6.

REMARK 1.4.10. The projection formula has a number of consequences for sheaves that are both flat and c-soft. If $\mathcal{F}$ is a flat and c-soft on X, then:

- the map (1.4.2) is an isomorphism for any sheaf $\mathcal{G} \in \mathrm{Sh}(Y, \Bbbk)$;
- the sheaf ${}^\circ f_! \mathcal{F} \in \mathrm{Sh}(Y, \Bbbk)$ is flat (and, of course, also c-soft);
- for any sheaf $\mathcal{H} \in \mathrm{Sh}(X, \Bbbk)$, $\mathcal{F} \otimes \mathcal{H}$ is c-soft.

The latter two items are recorded in Table 1.4.1(x) and (xi).

REMARK 1.4.11. Under some boundedness conditions, Theorem 1.4.8 can be used to define a number of other natural transformations involving $R\mathscr{H}om$ and $\otimes^L$, most of which are not isomorphisms. For example, suppose that $R\mathscr{H}om(\mathcal{F}, \mathcal{G}) \in D^{\mathrm{b}}(X, \Bbbk)$. Then one instance of Theorem 1.4.8 gives us a natural isomorphism

$$\mathrm{Hom}(R\mathscr{H}om(\mathcal{F}, \mathcal{G}) \overset{L}{\otimes} \mathcal{F}, \mathcal{G}) \cong \mathrm{Hom}(R\mathscr{H}om(\mathcal{F}, \mathcal{G}), R\mathscr{H}om(\mathcal{F}, \mathcal{G})).$$

The identity map on the right-hand side corresponds to a canonical map

$$R\mathscr{H}om(\mathcal{F}, \mathcal{G}) \overset{L}{\otimes} \mathcal{F} \to \mathcal{G}. \tag{1.4.3}$$

Next, assume in addition that $\Bbbk$ has finite global dimension, and take another object $\mathcal{H} \in D^{\mathrm{b}}(X, \Bbbk)$. Another instance of Theorem 1.4.8 gives us

$$\mathrm{Hom}(R\mathscr{H}om(\mathcal{F}, \mathcal{G}) \overset{L}{\otimes} \mathcal{H} \overset{L}{\otimes} \mathcal{F}, \mathcal{G} \overset{L}{\otimes} \mathcal{H}) \cong \mathrm{Hom}(R\mathscr{H}om(\mathcal{F}, \mathcal{G}) \overset{L}{\otimes} \mathcal{H}, R\mathscr{H}om(\mathcal{F}, \mathcal{G} \overset{L}{\otimes} \mathcal{H})).$$

There is an element of the left-hand side here given by tensoring (1.4.3) by $\mathcal{H}$. The corresponding map on the right-hand side is a canonical map

$$R\mathscr{H}om(\mathcal{F}, \mathcal{G}) \overset{L}{\otimes} \mathcal{H} \to R\mathscr{H}om(\mathcal{F}, \mathcal{G} \overset{L}{\otimes} \mathcal{H}). \tag{1.4.4}$$

Again, this is not an isomorphism in general. But if, for example, $\mathcal{H} = \underline{\Bbbk}_X$, then Proposition 1.4.4(1) can be used to show that it is an isomorphism.

Finally, if $R\mathscr{H}om(\mathcal{G},\mathcal{H})$ lies in $D^{\mathrm{b}}(X,\Bbbk)$, we can construct a natural "composition" map

$$R\mathscr{H}om(\mathcal{F},\mathcal{G}) \overset{L}{\otimes} R\mathscr{H}om(\mathcal{G},\mathcal{H}) \to R\mathscr{H}om(\mathcal{F},\mathcal{H})$$

by combining some instances of the maps defined above:

$$R\mathscr{H}om(\mathcal{F},\mathcal{G}) \overset{L}{\otimes} R\mathscr{H}om(\mathcal{G},\mathcal{H}) \xrightarrow{(1.4.4)} R\mathscr{H}om(\mathcal{F}, R\mathscr{H}om(\mathcal{G},\mathcal{H}) \overset{L}{\otimes} \mathcal{G})$$
$$\xrightarrow{(1.4.3)} R\mathscr{H}om(\mathcal{F},\mathcal{H}).$$

Remark 1.4.12. Here is another variant of the ideas in Remark 1.4.11. Let $f : X \to Y$ be a continuous map, and suppose $\mathcal{F},\mathcal{G} \in D^{\mathrm{b}}(X,\Bbbk)$ are objects such that $f_*\mathcal{F}$ and $f_*\mathcal{G}$ belong to $D^{\mathrm{b}}(Y,\Bbbk)$. Then $f^*f_*\mathcal{F}$ and $f^*f_*\mathcal{G}$ belong to $D^{\mathrm{b}}(X,\Bbbk)$, and it makes sense to form their tensor product $f^*f_*\mathcal{F} \otimes^L f^*f_*\mathcal{G}$. The tensor product of the adjunction maps $f^* \circ f_* \to \mathrm{id}$ gives us a natural map $f^*f_*\mathcal{F} \otimes^L f^*f_*\mathcal{G} \to \mathcal{F} \otimes^L \mathcal{G}$. Following this map through the sequence of isomorphisms

$$\mathrm{Hom}(f^*f_*\mathcal{F} \overset{L}{\otimes} f^*f_*\mathcal{G}, \mathcal{F} \overset{L}{\otimes} \mathcal{G}) \cong \mathrm{Hom}(f^*(f_*\mathcal{F} \overset{L}{\otimes} f_*\mathcal{G}), \mathcal{F} \overset{L}{\otimes} \mathcal{G})$$
$$\cong \mathrm{Hom}(f_*\mathcal{F} \overset{L}{\otimes} f_*\mathcal{G}, f_*(\mathcal{F} \overset{L}{\otimes} \mathcal{G}))$$

gives us a natural map

$$f_*\mathcal{F} \overset{L}{\otimes} f_*\mathcal{G} \to f_*(\mathcal{F} \overset{L}{\otimes} \mathcal{G}),$$

which is not an isomorphism in general. Similar considerations show that for $\mathcal{F} \in D^-(Y,\Bbbk)$ and $\mathcal{G} \in D^+(Y,\Bbbk)$, there is a natural map

$$f^*R\mathscr{H}om(\mathcal{F},\mathcal{G}) \to R\mathscr{H}om(f^*\mathcal{F}, f^*\mathcal{G}).$$

Change of scalars. We now define two ways to change the ring $\Bbbk$ over which we are working.

Definition 1.4.13. Let $\varphi : \Bbbk \to \Bbbk'$ be a homomorphism of commutative rings. Any sheaf $\mathcal{F} \in \mathrm{Sh}(X,\Bbbk')$ can be regarded as a sheaf of $\Bbbk$-modules via φ. This defines a functor

$$\mathrm{res}^{\Bbbk'}_{\Bbbk} : \mathrm{Sh}(X,\Bbbk') \to \mathrm{Sh}(X,\Bbbk),$$

called the **restriction of scalars** functor. Next, for $\mathcal{F} \in \mathrm{Sh}(X,\Bbbk)$, let $\Bbbk' \otimes \mathcal{F}$ be the sheafification of the presheaf $\Bbbk' \otimes_{\mathrm{pre}} \mathcal{F}$ given by

$$(\Bbbk' \otimes_{\mathrm{pre}} \mathcal{F})(U) = \Bbbk' \otimes \mathcal{F}(U).$$

This sheaf is said to be obtained from $\mathcal{F}$ by **extension of scalars** from $\Bbbk$ to $\Bbbk'$.

Restriction of scalars is in some sense the "easier" of these two operations, while extension of scalars is closely related to the sheaf tensor product defined earlier. We omit the proof of the following basic fact.

Lemma 1.4.14. *Let $\varphi : \Bbbk \to \Bbbk'$ be a ring homomorphism. The functor $\mathrm{res}^{\Bbbk'}_{\Bbbk} : \mathrm{Sh}(X,\Bbbk') \to \mathrm{Sh}(X,\Bbbk)$ is exact, and the functor $\Bbbk' \otimes (-) : \mathrm{Sh}(X,\Bbbk) \to \mathrm{Sh}(X,\Bbbk')$ is right exact. Flat sheaves form an adapted class for $\Bbbk' \otimes (-)$.*

We thus have derived functors

$$\mathrm{res}^{\Bbbk'}_{\Bbbk} : D(X,\Bbbk') \to D(X,\Bbbk),$$
$$\Bbbk' \overset{L}{\otimes} (-) : D^-(X,\Bbbk) \to D^-(X,\Bbbk').$$

If $\Bbbk$ has finite global dimension, then by Theorem A.6.10 there is also a derived functor

$$\Bbbk' \overset{L}{\otimes} (-) : D^+(X, \Bbbk) \to D^+(X, \Bbbk').$$

It is obvious that $\mathrm{res}_{\Bbbk}^{\Bbbk'}$ sends flabby (resp. c-soft) sheaves to flabby (resp. c-soft) sheaves, and from this it is easily deduced that $\mathrm{res}_{\Bbbk}^{\Bbbk'}$ commutes with f_* and $f_!$. (It also commutes with f^*.)

The question of whether extension of scalars commutes with various sheaf operations is more difficult. We will prove some results in this direction below, and we will study this question further in Chapter 2. The following is a minor variation on Proposition 1.4.5, with the same proof.

PROPOSITION 1.4.15. *Let $f : X \to Y$ be a continuous map, and let $\varphi : \Bbbk \to \Bbbk'$ be a ring homomorphism. For $\mathcal{F} \in D^-(X, \Bbbk)$, there is a natural isomorphism $f^*(\Bbbk' \otimes^L \mathcal{F}) \cong \Bbbk' \otimes^L f^*\mathcal{F}$. If $\Bbbk$ has finite global dimension, the same result holds for $\mathcal{F} \in D^+(X, \Bbbk)$.*

The next statement follows easily from the definitions.

PROPOSITION 1.4.16. *Let $\varphi : \Bbbk \to \Bbbk'$ be a ring homomorphism. For $\mathcal{F}, \mathcal{G} \in D^-(X, \Bbbk)$, there is a natural isomorphism $\Bbbk' \otimes^L (\mathcal{F} \otimes^L \mathcal{G}) \cong (\Bbbk' \otimes^L \mathcal{F}) \otimes^L_{\Bbbk'} (\Bbbk' \otimes^L \mathcal{G})$. If both $\Bbbk$ and $\Bbbk'$ have finite global dimension, the same result holds for $\mathcal{F}, \mathcal{G} \in D^+(X, \Bbbk)$.*

Finally, the following statement bears some resemblance to the projection formula (Theorem 1.4.9), but its proof requires some additional care.

PROPOSITION 1.4.17. *Let $f : X \to Y$ be a continuous map of locally compact spaces, and let $\varphi : \Bbbk \to \Bbbk'$ be a ring homomorphism. Assume that $\Bbbk$ is noetherian and has finite global dimension. Then, for $\mathcal{F} \in D^+(X, \Bbbk)$, there is a natural isomorphism $\Bbbk' \otimes^L f_!\mathcal{F} \xrightarrow{\sim} f_!(\Bbbk' \otimes^L \mathcal{F})$.*

PROOF SKETCH. One can construct a natural map $\Bbbk' \otimes {}^\circ f_!\mathcal{F} \to {}^\circ f_!(\Bbbk' \otimes \mathcal{F})$ the same way as in (1.4.2). This map is not an isomorphism in general, but we claim that it is an isomorphism when $\mathcal{F}$ is flat and c-soft. It is enough to check that it becomes an isomorphism after applying $\mathrm{res}_{\Bbbk}^{\Bbbk'}$, which gives us a map

$$\underline{\Bbbk}'_Y \otimes {}^\circ f_!\mathcal{F} \to {}^\circ f_!(\underline{\Bbbk}'_X \otimes \mathcal{F})$$

in $\mathrm{Sh}(Y, \Bbbk)$. This is an isomorphism by Remark 1.4.10.

Next, we claim that if $\mathcal{F}$ is flat and c-soft, then $\Bbbk' \otimes \mathcal{F}$ is c-soft. After applying $\mathrm{res}_{\Bbbk}^{\Bbbk'}$, this follows from Table 1.4.1(xi).

According to Lemma 1.4.19 below, every object in $D^+(X, \Bbbk)$ is quasi-isomorphic to a complex whose terms are flat and flabby, and hence flat and c-soft. To finish the proof according the plan of Remark 1.2.6, we must show that for any $\mathcal{F} \in D^+(X, \Bbbk)$, both $\Bbbk' \otimes^L f_!\mathcal{F}$ and $f_!(\Bbbk' \otimes^L \mathcal{F})$ can be computed by replacing $\mathcal{F}$ by a complex of flat and c-soft sheaves. For the former, this is true by Table 1.4.1(x); for the latter, it is true by (the variant in the previous paragraph of) Table 1.4.1(xi). □

As a special case of Proposition 1.4.17, we have the following.

COROLLARY 1.4.18 (Universal coefficient theorem for cohomology with compact support). *Let X be a locally compact space. There is a natural isomorphism $\Bbbk \otimes^L R\Gamma_c(\underline{\mathbb{Z}}_X) \cong R\Gamma_c(\underline{\Bbbk}_X)$.*

Lemma 1.4.19. *Assume that* $\Bbbk$ *is noetherian and has finite global dimension. Let* $\mathcal{F} \in Ch^+\mathrm{Sh}(X, \Bbbk)$ *be a chain complex of flat sheaves. Then there exists a quasi-isomorphism* $\mathcal{F} \to \mathcal{I}$*, where* $\mathcal{I} \in Ch^+\mathrm{Sh}(X, \Bbbk)$ *is a bounded-below complex of flat and flabby sheaves.*

Proof. We will show that the Godement resolution of $\mathcal{F}$ (see Exercise 1.2.2) has the desired property. Of course, it is enough to consider the case where $\mathcal{F}$ is a single sheaf (rather than a chain complex).

Let $i : X_{\mathrm{disc}} \to X$ be as in Exercise 1.2.2. We claim that if $\mathcal{G}$ is flat, then ${}^{\circ}i_*i^*\mathcal{G}$ is flat, and the cokernel of $\mathcal{G} \to {}^{\circ}i_*i^*\mathcal{G}$ is also flat. From the definition of the Godement resolution, one sees that these two claims imply the lemma.

The stalks of ${}^{\circ}i_*i^*\mathcal{G}$ are given by

$$({}^{\circ}i_*i^*\mathcal{G})_x = \varinjlim_{U \ni x} ({}^{\circ}i_*i^*\mathcal{G})(U) \cong \varinjlim_{U \ni x} (i^*\mathcal{G})(U_{\mathrm{disc}}) \cong \varinjlim_{U \ni x} \prod_{y \in U} \mathcal{G}_y.$$

Because $\Bbbk$ is noetherian, Chase's theorem (see [**145**, Theorem 4.47]) tells us that an arbitrary direct product of flat $\Bbbk$-modules is flat. Since a direct limit of flat modules is also flat, we conclude that ${}^{\circ}i_*i^*\mathcal{G}$ has flat stalks, and then that it is a flat sheaf.

The stalk of the cokernel of $\mathcal{G} \to i_*i^*\mathcal{G}$ at $x \in X$ is given by

$$\operatorname{cok}\left(\mathcal{G}_x \to \varinjlim_{U \ni x} \prod_{y \in U} \mathcal{G}_y\right) \cong \varinjlim_{U \ni x} \prod_{y \in U \smallsetminus \{x\}} \mathcal{G}_y,$$

and then the reasoning above shows that this cokernel is also a flat sheaf. □

External tensor product. We conclude this section with one more variant of the tensor product construction.

Definition 1.4.20. Let $\mathrm{pr}_1 : X \times Y \to X$ and $\mathrm{pr}_2 : X \times Y \to Y$ be the projection maps. For $\mathcal{F} \in D^-(X, \Bbbk)$ and $Y \in D^-(Y, \Bbbk)$, their **external tensor product** $\mathcal{F} \boxtimes \mathcal{G}$ is the object

$$\mathcal{F} \boxtimes \mathcal{G} = \mathrm{pr}_1^*\mathcal{F} \overset{L}{\otimes} \mathrm{pr}_2^*\mathcal{G} \in D^-(X \times Y, \Bbbk).$$

This defines a functor $\boxtimes : D^-(X, \Bbbk) \times D^-(Y, \Bbbk) \to D^-(X \times Y, \Bbbk)$. If $\Bbbk$ has finite global dimension, the same formula defines a functor $\boxtimes : D^+(X, \Bbbk) \times D^+(Y, \Bbbk) \to D^+(X \times Y, \Bbbk)$.

The following proposition is easily deduced from results above and is left as an exercise. As with the extension of scalars, we will see further properties of external tensor product in Chapter 2.

Proposition 1.4.21. *Let* $f : X \to X'$ *and* $g : Y \to Y'$ *be continuous maps.*

(1) *For* $\mathcal{F} \in D^-(X', \Bbbk)$ *and* $\mathcal{G} \in D^-(Y', \Bbbk)$*, there is a natural isomorphism*

$$f^*\mathcal{F} \boxtimes g^*\mathcal{G} \cong (f \times g)^*(\mathcal{F} \boxtimes \mathcal{G}).$$

If $\Bbbk$ *has finite global dimension, then the same result also holds for* $\mathcal{F} \in D^+(X', \Bbbk)$ *and* $\mathcal{G} \in D^+(Y', \Bbbk)$*.*

(2) *For* $\mathcal{F}, \mathcal{F}' \in D^-(X, \Bbbk)$ *and* $\mathcal{G}, \mathcal{G}' \in D^-(Y, \Bbbk)$*, there is a natural isomorphism*

$$(\mathcal{F} \overset{L}{\otimes} \mathcal{F}') \boxtimes (\mathcal{G} \overset{L}{\otimes} \mathcal{G}') \cong (\mathcal{F} \boxtimes \mathcal{G}) \overset{L}{\otimes} (\mathcal{F}' \boxtimes \mathcal{G}').$$

If $\Bbbk$ has finite global dimension, then the same result also holds for $\mathcal{F}, \mathcal{F}' \in D^+(X, \Bbbk)$ and $\mathcal{G}, \mathcal{G}' \in D^+(Y, \Bbbk)$.

(3) *Assume that $\Bbbk$ has finite global dimension and that our spaces are locally compact. For $\mathcal{F} \in D^+(X, \Bbbk)$ and $\mathcal{G} \in D^+(Y, \Bbbk)$, there is a natural isomorphism*

$$f_!\mathcal{F} \boxtimes g_!\mathcal{G} \cong (f \times g)_!(\mathcal{F} \boxtimes \mathcal{G}).$$

As an application of Proposition 1.4.21(3), we have the following result.

COROLLARY 1.4.22 (Künneth formula for cohomology with compact support). *Let X and Y be locally compact spaces, and assume that $\Bbbk$ has finite global dimension. Then there is a natural isomorphism $R\Gamma_c(\underline{\Bbbk}_X) \otimes^L R\Gamma_c(\underline{\Bbbk}_Y) \cong R\Gamma_c(\underline{\Bbbk}_{X\times Y})$.*

Exercises.

1.4.1. Let $h : Y \hookrightarrow X$ be a locally closed embedding. Prove that for $\mathcal{F} \in D^-(X, \Bbbk)$ and $\mathcal{G} \in D^+(X, \Bbbk)$, there are natural isomorphisms

$$h_!\underline{\Bbbk}_Y \overset{L}{\otimes} \mathcal{F} \cong h_!h^*\mathcal{F}, \qquad R\mathscr{H}om(h_!\underline{\Bbbk}_Y, \mathcal{G}) \cong h_*h^!\mathcal{G}.$$

1.4.2. In this exercise, $\mathbb{Q}$ will occur both as a topological space and as a ring of coefficients. Let $i : \mathbb{Q}_{\mathrm{disc}} \to \mathbb{Q}$ be the natural map, and let $\mathcal{F} \in \mathrm{Sh}(\mathbb{Q}_{\mathrm{disc}}, \mathbb{Z})$ be the sheaf whose stalk at a rational number $\frac{p}{q} \in \mathbb{Q}_{\mathrm{disc}}$ (where $\gcd(p, q) = 1$) is $\mathbb{Z}/q\mathbb{Z}$. Show that $\mathbb{Q} \otimes \mathcal{F} = 0$, but that $\mathbb{Q} \otimes i_*\mathcal{F}$ has nonzero stalks everywhere. In particular, $\mathbb{Q} \otimes i_*\mathcal{F}$ and $i_*(\mathbb{Q} \otimes \mathcal{F})$ are not isomorphic.

1.4.3. Let $\mathcal{R}$ be a sheaf of $\Bbbk$-algebras on X, and let $\mathcal{R}$-mod be the category of (left) $\mathcal{R}$-modules (see Exercise 1.1.10).

(a) For $\mathcal{M}_1, \mathcal{M}_2 \in \mathcal{R}$-mod, let $\mathscr{H}om_{\mathcal{R}}(\mathcal{M}_1, \mathcal{M}_2)$ be the presheaf given by

$$\mathscr{H}om_{\mathcal{R}}(\mathcal{M}_1, \mathcal{M}_2)(U) = \mathrm{Hom}_{\mathcal{R}|_U\text{-mod}}(\mathcal{M}_1|_U, \mathcal{M}_2|_U).$$

Show that this indeed a sheaf, and that this construction is left exact in both variables. Then explain how to form the derived functor

$$R\mathscr{H}om_{\mathcal{R}} : D^-(\mathcal{R}\text{-mod})^{\mathrm{op}} \times D^+(\mathcal{R}\text{-mod}) \to D^+(X, \Bbbk).$$

(b) Let mod-$\mathcal{R}$ denote the category of *right* $\mathcal{R}$-modules. Explain how to define the functor $\otimes_{\mathcal{R}} : \text{mod-}\mathcal{R} \times \mathcal{R}\text{-mod} \to \mathrm{Sh}(X, \Bbbk)$. Explain how to define flat (left or right) $\mathcal{R}$-modules, and show that flat $\mathcal{R}$-modules form an adapted class for $\otimes_{\mathcal{R}}$ in both variables. As a result, one has the derived functor

$$\overset{L}{\otimes}_{\mathcal{R}} : D^-(\text{mod-}\mathcal{R}) \times D^-(\mathcal{R}\text{-mod}) \to D^-(X, \Bbbk).$$

(c) Let $\mathcal{S}$ be another sheaf of $\Bbbk$-algebras, and let $\mathcal{R}$-mod-$\mathcal{S}$ be the category of $(\mathcal{R}, \mathcal{S})$-bimodules. Prove the following generalization of Theorem 1.4.8: For $\mathcal{F} \in D^-(\mathcal{R}\text{-mod-}\mathcal{S})$, $\mathcal{G} \in D^-(\mathcal{S}\text{-mod})$, and $\mathcal{H} \in D^+(\mathcal{R}\text{-mod})$, there is a natural isomorphism

$$R\mathrm{Hom}_{\mathcal{R}}(\mathcal{F} \overset{L}{\otimes}_{\mathcal{S}} \mathcal{G}, \mathcal{H}) \cong R\mathrm{Hom}_{\mathcal{S}}(\mathcal{G}, R\mathscr{H}om_{\mathcal{R}}(\mathcal{F}, \mathcal{H})).$$

(You will have to first explain how to define suitably generalized versions of $\otimes^L$ and $R\mathscr{H}om$.)

1.4.4. Let $(\mathcal{F}_\alpha)_{\alpha\in I}$ be a collection of sheaves on X. Show that for any sheaf $\mathcal{G}$ on X, there is a natural isomorphism

$$\bigoplus_{\alpha\in I}(\mathcal{F}_\alpha \otimes \mathcal{G}) \xrightarrow{\sim} \left(\bigoplus_{\alpha\in I}\mathcal{F}_\alpha\right) \otimes \mathcal{G}.$$

1.5. The right adjoint to proper push-forward

Construction and general properties. In this section, we discuss the right adjoint to $f_!$ for more general maps $f : X \to Y$. In order for this adjoint to exist, we must assume that ${}^\circ f_!$ has finite cohomological dimension. The following few statements are useful for checking this condition.

LEMMA 1.5.1. *Let $f : X \to Y$ be a continuous map of locally compact spaces. Then ${}^\circ f_!$ has cohomological dimension $\le n$ if and only if every sheaf $\mathcal{F} \in \mathrm{Sh}(X, \Bbbk)$ admits a relatively c-soft resolution of length at most n.*

For a proof, see the discussion following [**125**, Eq. (3.1.3)].

DEFINITION 1.5.2. Let X be a locally compact topological space. We say that X **has c-soft dimension** $\le n$ if the functor ${}^\circ\mathrm{a}_{X!}$ has cohomological dimension $\le n$.

By Lemma 1.5.1, X has c-soft dimension $\le n$ if and only if every sheaf admits a c-soft resolution of length $\le n$. Since c-soft sheaves are also relatively c-soft for any f, we see that if X is locally compact of finite c-soft dimension, then for any continuous map $f : X \to Y$, ${}^\circ f_!$ has finite cohomological dimension. The following proposition thus gives us a concrete class of spaces to which Lemma 1.5.1 can be applied.

PROPOSITION 1.5.3. *If X is a locally closed subset of a real n-dimensional manifold, then X has c-soft dimension $\le n$.*

This proposition follows from [**38**, Proposition V.1.15 and Section V.6.3], or else from [**125**, Proposition 3.2.2]. (The latter statement assumes that X actually is a manifold, but since restriction to a locally closed subset is exact and takes c-soft sheaves to c-soft sheaves, the proposition also holds for locally closed subsets of manifolds.)

When ${}^\circ f_!$ has finite cohomological dimension, Theorem A.6.10 tells us that it makes sense to speak of $f_! : D^-(X, \Bbbk) \to D^-(Y, \Bbbk)$. We are now ready to state the main theorem of this section, due to Verdier [**238**].

THEOREM 1.5.4. *Let $f : X \to Y$ be a continuous map of locally compact spaces, and let $\Bbbk$ be a noetherian ring of finite global dimension. Assume that ${}^\circ f_!$ has finite cohomological dimension. Then there exists a triangulated functor*

$$f^! : D^+(Y, \Bbbk) \to D^+(X, \Bbbk)$$

such that for $\mathcal{F} \in D^-(X, \Bbbk)$ and $\mathcal{G} \in D^+(Y, \Bbbk)$, there are natural isomorphisms

$$R\mathscr{H}om(f_!\mathcal{F}, \mathcal{G}) \cong f_*R\mathscr{H}om(\mathcal{F}, f^!\mathcal{G}),$$
$$R\mathrm{Hom}(f_!\mathcal{F}, \mathcal{G}) \cong R\mathrm{Hom}(\mathcal{F}, f^!\mathcal{G}),$$
$$\mathrm{Hom}(f_!\mathcal{F}, \mathcal{G}) \cong \mathrm{Hom}(\mathcal{F}, f^!\mathcal{G}).$$

Before discussing the proof, let us carry out a calculation that motivates the construction. Suppose that the theorem is already proved, and that for some sheaf $\mathcal{G} \in \mathrm{Sh}(Y, \Bbbk)$, the object $f^!\mathcal{G}$ is also a sheaf (rather than some more general object of the derived category). To describe this sheaf, we would need to determine the $\Bbbk$-module $(f^!\mathcal{G})(U)$ for each open set $U \subset X$. Let $j_U : U \hookrightarrow X$ be the inclusion map. Then

$$(f^!\mathcal{G})(U) \cong \Gamma(j_U^! f^!\mathcal{G}) \cong \mathrm{Hom}(\underline{\Bbbk}_U, j_U^! f^!\mathcal{G}) \cong \mathrm{Hom}(f_! j_{U!}\underline{\Bbbk}_U, \mathcal{G}).$$

More generally, given a sheaf $\mathcal{F}$ on X, the sheaf $\mathscr{H}om(\mathcal{F}, f^!\mathcal{G})$ is described by

$$\begin{aligned}\mathscr{H}om(\mathcal{F}, f^!\mathcal{G})(U) &= \mathrm{Hom}(\mathcal{F}|_U, (f^!\mathcal{G})|_U) \\ &\cong \mathrm{Hom}(j_{U!}j_U^*\mathcal{F}, f^!\mathcal{G}) \cong \mathrm{Hom}(f_!(\mathcal{F} \otimes j_{U!}\underline{\Bbbk}_U), \mathcal{G}).\end{aligned}$$

Thus, $\mathscr{H}om(\mathcal{F}, f^!\mathcal{G})$ can be described by a formula that makes no mention of $f^!$.

The proof of Theorem 1.5.4 consists of defining and studying a chain complex $\mathcal{E}(\mathcal{F}, \mathcal{G})$ whose definition is inspired by the calculation above. We will very briefly indicate the main ideas of the proof; for full details, see [**38**, Sections V.7.14–V.7.19] or [**125**, Theorem 3.1.5 and Proposition 3.1.10].

Proof sketch. *Step 1.* Show that the constant sheaf $\underline{\Bbbk}_X$ admits a flat and c-soft resolution with finitely many terms. (This step relies on Lemma 1.4.19, and hence on the assumption that $\Bbbk$ is noetherian and of finite global dimension.)

For the rest of the proof, we work with a fixed finite resolution $\mathcal{K}$ of $\underline{\Bbbk}_X$ by flat and c-soft sheaves. For $\mathcal{F} \in Ch^-(\mathrm{Sh}(X, \Bbbk))$ and $\mathcal{G} \in Ch^+(\mathrm{Sh}(Y, \Bbbk))$, let $\mathcal{E}(\mathcal{F}, \mathcal{G})$ be the chain complex of presheaves on X whose sections over an open subset $U \subset X$ are given by

$$\mathcal{E}(\mathcal{F}, \mathcal{G})(U) = \mathrm{chHom}({}^\circ f_!(\mathcal{F} \overset{\mathrm{ch}}{\otimes} j_{U!}(\mathcal{K}|_U)), \mathcal{G}).$$

Step 2. Show that this is actually a chain complex of sheaves, so that we have a functor

$$\mathcal{E} : K^-(\mathrm{Sh}(X, \Bbbk))^{\mathrm{op}} \times K^+(\mathrm{Sh}(Y, \Bbbk)) \to K^+(\mathrm{Sh}(X, \Bbbk)).$$

Moreover, if $\mathcal{G}$ is a complex of injective sheaves, then $\mathcal{E}(\mathcal{F}, \mathcal{G})$ is also a complex of injective sheaves. There are natural isomorphisms

$$\mathcal{E}(\mathcal{F}, \mathcal{G}) \cong \mathrm{ch}\mathscr{H}om(\mathcal{F}, \mathcal{E}(\underline{\Bbbk}_X, \mathcal{G})),$$

$$\mathrm{ch}\mathscr{H}om({}^\circ f_!(\mathcal{F} \overset{\mathrm{ch}}{\otimes} \mathcal{K}), \mathcal{G}) \cong {}^\circ f_*\mathcal{E}(\mathcal{F}, \mathcal{G}).$$

Step 3. The functor $\mathcal{E}$ admits a "derived" functor

$$R\mathcal{E} : D^-(X, \Bbbk)^{\mathrm{op}} \times D^+(Y, \Bbbk) \to D^+(Y, \Bbbk),$$

related to $\mathcal{E}$ by a suitable universal property, and computed by replacing $\mathcal{G}$ by an injective resolution. For $\mathcal{F} \in D^-(X, \Bbbk)$ and $\mathcal{G} \in D^+(Y, \Bbbk)$, there are natural isomorphisms

$$\begin{aligned}R\mathcal{E}(\mathcal{F}, \mathcal{G}) &\cong R\mathscr{H}om(\mathcal{F}, R\mathcal{E}(\underline{\Bbbk}_X, \mathcal{G})), \\ R\mathscr{H}om(f_!\mathcal{F}, \mathcal{G}) &\cong f_*R\mathcal{E}(\mathcal{F}, \mathcal{G}).\end{aligned}$$

Step 4. Define $f^! : D^+(Y, \Bbbk) \to D^+(X, \Bbbk)$ by setting

$$f^!\mathcal{G} = R\mathcal{E}(\underline{\Bbbk}_X, \mathcal{G}).$$

The isomorphisms above imply that $R\mathscr{H}om(f_!\mathcal{F}, \mathcal{G}) \cong f_*R\mathscr{H}om(\mathcal{F}, f^!\mathcal{G})$, and this implies the other isomorphisms in the statement of the theorem. ☐

Remark 1.5.5. Suppose $h : Y \hookrightarrow X$ is a locally closed embedding of locally compact spaces. Since ${}^\circ h_!$ is exact, it is trivially of finite cohomological dimension, and Theorem 1.5.4 gives us a right adjoint $h^! : D^+(X, \Bbbk) \to D^+(Y, \Bbbk)$.

In Section 1.3, we gave a different definition of $h^!$, but by Theorem 1.3.6, that functor was also a right adjoint to $h_!$, so in fact the two versions of $h^!$ are canonically isomorphic.

The next three statements can be deduced from Proposition 1.2.8(3), Theorem 1.2.13, and Theorem 1.4.9, respectively, by adjunction and Yoneda's lemma.

Proposition 1.5.6. *Let $f : X \to Y$ and $g : Y \to Z$ be continuous maps of locally compact spaces. Assume that $\Bbbk$ is noetherian and of finite global dimension, and that ${}^\circ f_!$ and ${}^\circ g_!$ have finite cohomological dimension. For $\mathcal{F} \in D^+(Z, \Bbbk)$, there is a natural isomorphism $f^! g^! \mathcal{F} \cong (g \circ f)^! \mathcal{F}$.*

Proposition 1.5.7. *Suppose we have a cartesian square of continuous maps between locally compact spaces:*

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

Assume that $\Bbbk$ is noetherian and of finite global dimension, and that ${}^\circ f_!$ has finite cohomological dimension.

1. *For $\mathcal{F} \in D^+(Y, \Bbbk)$, there is a natural map $(g')^* f^! \mathcal{F} \to (f')^! g^* \mathcal{F}$.*
2. *For $\mathcal{F} \in D^+(Y', \Bbbk)$, there is a natural commutative diagram*

$$\begin{tikzcd} g'_!(f')^! \mathcal{F} \arrow[r] \arrow[d] & f^! g_! \mathcal{F} \arrow[d] \\ g'_*(f')^! \mathcal{F} \arrow[r, "\sim"] & f^! g_* \mathcal{F} \end{tikzcd}$$

in which the map $g'_*(f')^! \mathcal{F} \xrightarrow{\sim} f^! g_* \mathcal{F}$ is an isomorphism.

Proposition 1.5.8. *Let $f : X \to Y$ be a continuous map of locally compact spaces. Assume that $\Bbbk$ is noetherian and of finite global dimension, and that ${}^\circ f_!$ has finite cohomological dimension. For $\mathcal{F} \in D^{\mathrm{b}}(Y, \Bbbk)$ and $\mathcal{G} \in D^+(Y, \Bbbk)$, there is a natural isomorphism $f^! R\mathscr{H}om(\mathcal{F}, \mathcal{G}) \cong R\mathscr{H}om(f^* \mathcal{F}, f^! \mathcal{G})$.*

In view of Lemma 1.3.3, this is a generalization of Proposition 1.4.4(3).

Proposition 1.5.9. *Let $f : X \to Y$ be a continuous map of locally compact spaces of finite c-soft dimension, and suppose $\Bbbk$ is noetherian and of finite global dimension. For $\mathcal{F}, \mathcal{G} \in D^+(Y, \Bbbk)$, there is a natural map*

$$f^! \mathcal{F} \overset{L}{\otimes} f^* \mathcal{G} \to f^! (\mathcal{F} \overset{L}{\otimes} \mathcal{G}). \tag{1.5.1}$$

Moreover, if $\mathcal{G}$ is a local system of finite type, then this map is an isomorphism.

See Definition 1.7.1 for the notion of a "local system of finite type." (No other statements from Section 1.7 are needed for the proof.) For some additional cases in which (1.5.1) is an isomorphism, see Theorem 1.5.11, Theorem 2.2.9, and Theorem 2.2.13.

Proof. Using the projection formula (Theorem 1.4.9) and the adjunction map $f_! f^! \mathcal{F} \to \mathcal{F}$, we obtain a natural map

$$f_!(f^!\mathcal{F} \overset{L}{\otimes} f^*\mathcal{G}) \cong f_! f^! \mathcal{F} \overset{L}{\otimes} \mathcal{G} \to \mathcal{F} \overset{L}{\otimes} \mathcal{G}.$$

This map is an element of the left-hand side of the isomorphism

$$\mathrm{Hom}(f_!(f^!\mathcal{F} \overset{L}{\otimes} f^*\mathcal{G}), \mathcal{F} \overset{L}{\otimes} \mathcal{G}) \cong \mathrm{Hom}(f^!\mathcal{F} \overset{L}{\otimes} f^*\mathcal{G}, f^!(\mathcal{F} \overset{L}{\otimes} \mathcal{G})).$$

The corresponding element on the right-hand side is our desired natural map

$$f^!\mathcal{F} \overset{L}{\otimes} f^*\mathcal{G} \to f^!(\mathcal{F} \overset{L}{\otimes} \mathcal{G}). \tag{1.5.2}$$

For the remainder of the proof, assume that $\mathcal{G}$ is a local system of finite type. We wish to prove that (1.5.2) is an isomorphism. It is enough to prove that every point $x \in X$ has a neighborhood U over which it is an isomorphism. Choose a neighborhood $V \subset Y$ of $f(x)$ such that $\mathcal{G}|_V$ is a constant sheaf; that is, $\mathcal{G}|_V \cong \underline{M}_V$ for some finitely generated $\Bbbk$-module M. Let $U = f^{-1}(V)$, and let $f_U = f|_U : U \to V$. We have $f_U^*(\mathcal{G}|_V) \cong \underline{M}_U$. The restriction of (1.5.2) to U can be rewritten as

$$f_U^!(\mathcal{F}|_V) \overset{L}{\otimes} \underline{M}_U \to f_U^!(\mathcal{F}|_V \overset{L}{\otimes} \underline{M}_V). \tag{1.5.3}$$

If $M = \Bbbk$, then both sides are identified with $f_U^!(\mathcal{F}|_V)$, and this map is an isomorphism. More generally, if M is a finite-rank free module, then (1.5.3) is an isomorphism.

To prove that (1.5.3) is an isomorphism in general, we proceed by induction on the projective dimension of M. If M is projective, then it is a direct summand of a finite-rank free module, so the claim follows from the case considered above. Otherwise, there exists a short exact sequence

$$0 \to M' \to P \to M \to 0$$

of finitely generated $\Bbbk$-modules, where P is projective, and M' has smaller projective dimension than M. Applying (1.5.3) to this sequence yields the commutative diagram

$$\begin{tikzcd}
f_U^!(\mathcal{F}|_V) \otimes^L \underline{M'}_U \arrow[r] \arrow[d, "\wr"] & f_U^!(\mathcal{F}|_V) \otimes^L \underline{P}_U \arrow[r] \arrow[d, "\wr"] & f_U^!(\mathcal{F}|_V) \otimes^L \underline{M}_U \arrow[r] \arrow[d] & {} \\
f_U^!(\mathcal{F}|_V \otimes^L \underline{M'}_V) \arrow[r] & f_U^!(\mathcal{F}|_V \otimes^L \underline{P}_V) \arrow[r] & f_U^!(\mathcal{F}|_V \otimes^L \underline{M}_V) \arrow[r] & {}
\end{tikzcd}$$

The first two vertical maps are isomorphisms by induction, so the last one is as well. ☐

Topological submersions. We will now study a class of maps f for which the functor $f^!$ can be given a more concrete description.

Definition 1.5.10. A continuous map $f : X \to Y$ is said to be a **topological submersion** of relative dimension n if every point $x \in X$ has a neighborhood U such that $f(U)$ is open in Y, and such that there is a homeomorphism $b : U \xrightarrow{\sim}$

$f(U) \times \mathbb{R}^n$ that makes the following diagram commute:

$$\begin{tikzcd} U \arrow[r, "b", "\sim"'] \arrow[d, "f|_U"'] & f(U) \times \mathbb{R}^n \arrow[dl, "\mathrm{pr}_1"] \\ f(U) & \end{tikzcd}$$

The **relative orientation sheaf** of a topological submersion $f : X \to Y$ of relative dimension n, denoted by or_f, is defined to be the sheafification of the presheaf

$$\mathit{or}_{f,\mathrm{pre}}(U) = \mathrm{Hom}(\mathsf{H}^n((f|_U)_!\underline{\Bbbk}_U), \underline{\Bbbk}_{f(U)}).$$

(The restriction maps in $\mathit{or}_{f,\mathrm{pre}}$ are defined as follows: for a pair of open subsets $U \subset V$ of X, let $j : U \hookrightarrow V$ be the inclusion map. Then $\mathrm{res}_{V,U} : \mathit{or}_{f,\mathrm{pre}}(V) \to \mathit{or}_{f,\mathrm{pre}}(U)$ is induced by the map $j_\sharp : \mathsf{H}^n((f|_U)_!\underline{\Bbbk}_U) \to \mathsf{H}^n((f|_V)_!\underline{\Bbbk}_V)|_{f(U)}$ from Remark 1.3.8.)

THEOREM 1.5.11. *Let $f : X \to Y$ be a topological submersion of relative dimension n. Assume that $\Bbbk$ is noetherian and of finite global dimension. For $\mathcal{F} \in D^+(Y, \Bbbk)$, there is a natural isomorphism $\mathit{or}_f \otimes^L f^*\mathcal{F}[n] \xrightarrow{\sim} f^!\mathcal{F}$.*

Briefly, the proof consists of first showing that the natural map $f^!\underline{\Bbbk}_Y \otimes^L f^*\mathcal{F} \to f^!\mathcal{F}$ from Proposition 1.5.9 is an isomorphism, and then showing that

$$f^!\underline{\Bbbk}_Y \cong \mathit{or}_f[n].$$

For details, see [**125**, Propositions 3.3.2(ii) and 3.3.6(i)].

For the following statement, note that a **local homeomorphism** is the same as a topological submersion of relative dimension 0.

PROPOSITION 1.5.12. *Let $f : X \to Y$ be a local homeomorphism of locally compact topological spaces. Assume that $\Bbbk$ is noetherian and of finite global dimension. There is a canonical isomorphism $\mathit{or}_f \cong \underline{\Bbbk}_X$. As a consequence, for $\mathcal{F} \in D^-(X, \Bbbk)$ and $\mathcal{G} \in D^+(Y, \Bbbk)$, there are natural isomorphisms $f^*\mathcal{G} \cong f^!\mathcal{G}$ and*

$$\mathrm{Hom}(f_!\mathcal{F}, \mathcal{G}) \cong \mathrm{Hom}(\mathcal{F}, f^*\mathcal{G}).$$

We will sketch a proof that depends on Theorem 1.5.11, and thus requires $\Bbbk$ to be noetherian and of finite global dimension. A direct proof without these assumptions on $\Bbbk$ is also possible; see Exercise 1.5.1.

PROOF SKETCH. For an open set $U \subset X$, let us describe the sheaf $\mathsf{H}^0(f_{U!}\underline{\Bbbk}_U) = {}^\circ f_{U!}\underline{\Bbbk}_U$. Unwinding the definitions, we find that for an open set $V \subset f(U)$, we have

$$({}^\circ f_{U!}\underline{\Bbbk}_U)(V) = \left\{ \begin{array}{c} \text{locally constant functions} \\ s : U \cap f^{-1}(V) \to \Bbbk \end{array} \;\middle|\; f_U|_{\mathrm{supp}\, s} : \mathrm{supp}\, s \to V \text{ is proper} \right\}.$$

Since the fibers are f are discrete, the fibers of $f_U|_{\mathrm{supp}\, s} : \mathrm{supp}\, s \to V$ must be finite, so for any section $s \in ({}^\circ f_{U!}\underline{\Bbbk}_U)(V)$, there is a well-defined function

$$\chi^U_V(s) : V \to \Bbbk \qquad \text{given by} \qquad \chi^U_V(s)(v) = \sum_{u \in f^{-1}(v) \cap \mathrm{supp}\, s} s(u).$$

Of course, $\mathrm{supp}\, s$ is closed as a subset of $U \cap f^{-1}(V)$, but since s is locally constant, its support is also open, and so $f_U|_{\mathrm{supp}\, s} : \mathrm{supp}\, s \to V$ is still a local homeomorphism. It can be shown that a proper local homeomorphism between locally compact spaces is in fact a covering map over its image (see [**102**, Lemma 2] for a proof). From this fact, one can deduce that $\chi^U_V(s) : V \to \Bbbk$ is locally constant.

Thus, the assignment $s \mapsto \chi^U_V(s)$ defines a map $\chi^U_V : ({}^\circ f_{U!}\underline{\Bbbk}_U)(V) \to \underline{\Bbbk}_{f(U)}(V)$. Letting V vary, we get a morphism $\chi^U : {}^\circ f_{U!}\underline{\Bbbk}_U \to \underline{\Bbbk}_{f(U)}$.

Now define a map of presheaves $\phi : \underline{\Bbbk}_{X,\mathrm{pre}} \to \mathscr{or}_{f,\mathrm{pre}}$ as follows: over an open set $U \subset X$, ϕ_U sends any scalar $a \in \Bbbk = \underline{\Bbbk}_{X,\mathrm{pre}}(U)$ to $a\chi^U \in \mathscr{or}_{f,\mathrm{pre}}(U)$. One can show that this map induces an isomorphism of stalks, so that after sheafifying, it gives us an isomorphism $\underline{\Bbbk}_X \xrightarrow{\sim} \mathscr{or}_f$. $\square$

The dualizing complex. The following definitions were introduced in [**20**], by translating an idea of Borel–Moore [**31**] into the language of derived categories. These notions will be further studied in Section 2.8.

DEFINITION 1.5.13. Let X be a locally compact space of finite c-soft dimension. The **dualizing complex** of X, denoted by ω_X, is the object of $D^+(X,\Bbbk)$ given by

$$\omega_X = \mathrm{a}^!_X\underline{\Bbbk}_{\mathrm{pt}}.$$

The **Verdier** (or **Borel–Moore–Verdier**) **duality functor** is the functor $\mathbb{D} : D^-(X,\Bbbk)^{\mathrm{op}} \to D^+(X,\Bbbk)$ given by $\mathbb{D}(\mathcal{F}) = R\mathscr{H}om(\mathcal{F},\omega_X)$.

As an immediate consequence of Proposition 1.5.6, we obtain the following.

LEMMA 1.5.14. *Let $f : X \to Y$ be a continuous map of locally compact spaces of finite c-soft dimension. There is a canonical isomorphism $f^!\omega_Y \cong \omega_X$.*

The next statement follows from Theorem 1.5.4 and Proposition 1.5.8.

LEMMA 1.5.15. *Let $f : X \to Y$ be a continuous map of locally compact spaces of finite c-soft dimension. For any $\mathcal{F} \in D^-(X,\Bbbk)$, there is a natural isomorphism $f_*\mathbb{D}\mathcal{F} \cong \mathbb{D}(f_!\mathcal{F})$. For any $\mathcal{G} \in D^-(Y,\Bbbk)$, there is a natural isomorphism $f^!\mathbb{D}\mathcal{F} \cong \mathbb{D}(f^*\mathcal{F})$.*

REMARK 1.5.16. Let $\mathcal{F} \in D^-(X,\Bbbk)$, and suppose $\mathbb{D}(\mathcal{F})$ lies in $D^{\mathrm{b}}(X,\Bbbk)$. As an application of Remark 1.4.11, we have a natural map

$$\mathbb{D}(\mathcal{F}) \overset{L}{\otimes} \mathcal{F} \to \omega_X,$$

called the **pairing map**.

Exercises.

1.5.1. Let $f : X \to Y$ be a local homeomorphism of locally compact spaces.

(a) For $\mathcal{G} \in \mathrm{Sh}(Y,\Bbbk)$, let $f^\flat_{\mathrm{pre}}\mathcal{G}$ be the presheaf on X given by $(f^\flat_{\mathrm{pre}}\mathcal{G})(U) = \mathrm{Hom}({}^\circ f_! j_{U!}\underline{\Bbbk}_U, \mathcal{G})$, and let $f^\flat\mathcal{G} = (f^\flat_{\mathrm{pre}}\mathcal{G})^+$ be its sheafification. Show that there is a natural isomorphism $f^*\mathcal{G} \xrightarrow{\sim} f^\flat\mathcal{G}$.

(b) Show that for $\mathcal{F} \in D^+(X,\Bbbk)$ and $\mathcal{G} \in D^+(Y,\Bbbk)$, there is a natural isomorphism $\mathrm{Hom}(f_!\mathcal{F},\mathcal{G}) \cong \mathrm{Hom}(\mathcal{F}, f^*\mathcal{G})$.

1.5.2. Upgrade Exercise 1.3.7 as follows: let $i : Z \hookrightarrow X$ and $j : U \hookrightarrow X$ be complementary closed and open embeddings in a locally compact space, and show that for $\mathcal{F} \in D^{\mathrm{b}}(X,\Bbbk)$ and $\mathcal{G} \in D^+(X,\Bbbk)$, there is a natural distinguished triangle

$$i_*R\mathscr{H}om(i^*\mathcal{F}, i^!\mathcal{G}) \to R\mathscr{H}om(\mathcal{F},\mathcal{G}) \to j_*R\mathscr{H}om(j^*\mathcal{F}, j^*\mathcal{G}) \to .$$

1.6. Relations among natural transformations

In the preceding sections, we have constructed a large number of natural morphisms or isomorphisms between sheaf functors, summarized in Table 1.6.1. These maps may be combined in various ways, and it is natural to ask whether the resulting diagrams commute. Very often, the answer is "yes." In this section, we collect a number of statements of this nature. The proofs are straightforward, and we will omit most details. For more statements of this nature, see [**11**, Appendix B] and [**192**]. See also the comments in [**96**, Section II.6].

To avoid clutter, the statements in this section do not always include the assumptions that are in effect. Throughout this section, we assume:

- In any statement involving the functor $f_!$, the spaces should be assumed to be locally compact.
- In any statement involving the functor $\otimes^L$, the ring $\Bbbk$ should be assumed to be of finite global dimension.
- In any statement involving the functor $f^!$, the functor ${}^\circ f_!$ should be assumed to have finite cohomological dimension, and the ring $\Bbbk$ should be assumed to be noetherian and of finite global dimension.

Most arrows in commutative diagrams in this section carry labels such as "adj.," "comp.," "b.ch.," etc., to indicate the entries of Table 1.6.1 from which they are obtained. The results of this section can be broadly grouped into three (overlapping) categories:

- Commutativity of units (or counits) of adjunctions with other maps: Propositions 1.6.1, 1.6.4, and 1.6.6.
- Commutativity of composition isomorphisms with other maps: Propositions 1.6.1, 1.6.2, 1.6.3, and 1.6.7.
- Commutativity of maps involving only left (or only right) adjoints: Propositions 1.6.7 and 1.6.8.

PROPOSITION 1.6.1. *In the setting of Table* 1.6.1(ii)*, for* $\mathcal{F} \in D^+(X_3, \Bbbk)$ *and* $\mathcal{G} \in D^+(X_1, \Bbbk)$*, the following diagrams commute:*

$$\begin{tikzcd}
\mathcal{F} \arrow[r, "\text{adj.}"] \arrow[d, "\text{adj.}"'] & f_{2*}f_2^*\mathcal{F} \arrow[d, "\text{adj.}"] \\
(f_2f_1)_*(f_2f_1)^*\mathcal{F} \arrow[r, "\text{comp.}", "\sim"'] & f_{2*}f_{1*}f_1^*f_2^*\mathcal{F}
\end{tikzcd}
\qquad
\begin{tikzcd}
f_1^*f_2^*f_{2*}f_{1*}\mathcal{G} \arrow[r, "\text{comp.}", "\sim"'] \arrow[d, "\text{adj.}"'] & (f_2f_1)^*(f_2f_1)_*\mathcal{G} \arrow[d, "\text{adj.}"] \\
f_1^*f_{1*}\mathcal{G} \arrow[r, "\text{adj.}"] & \mathcal{G}
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{G} \arrow[r, "\text{adj.}"] \arrow[d, "\text{adj.}"'] & f_1^!f_{1!}\mathcal{G} \arrow[d, "\text{adj.}"] \\
(f_2f_1)^!(f_2f_1)_!\mathcal{G} \arrow[r, "\text{comp.}", "\sim"'] & f_1^!f_2^!f_{2!}f_{1!}\mathcal{G}
\end{tikzcd}
\qquad
\begin{tikzcd}
f_{2!}f_{1!}f_1^!f_2^!\mathcal{F} \arrow[r, "\text{comp.}", "\sim"'] \arrow[d, "\text{adj.}"'] & (f_2f_1)_!(f_2f_1)^!\mathcal{F} \arrow[d, "\text{adj.}"] \\
f_{2!}f_2^!\mathcal{F} \arrow[r, "\text{adj.}"] & \mathcal{F}
\end{tikzcd}$$

PROOF SKETCH. For the diagrams involving the adjoint pair (f^*, f_*), it is enough to prove the commutativity of the analogous diagrams in the abelian category of sheaves, for the pair $(f^*, {}^\circ f_*)$. One can then further reduce to considering presheaves and the pair $(f^*_{\mathrm{pre}}, {}^\circ f_*)$. In this setting, the proof can be completed by examining the explicit construction of the unit and counit maps in the proof of Theorem 1.2.4, as well as the construction of the composition isomorphisms in Proposition 1.2.8.

TABLE 1.6.1. Natural morphisms of sheaf functors

Diagrams of spaces:

(i) $X_1 \xrightarrow{f} X_2$

(ii) $X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3$

(iii)
$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

Adjunction maps:

$$\mathcal{F} \xrightarrow{\text{adj.}} f_* f^* \mathcal{F} \qquad f^* f_* \mathcal{F} \xrightarrow{\text{adj.}} \mathcal{F}$$

$$\mathcal{F} \xrightarrow{\text{adj.}} f^! f_! \mathcal{F} \qquad f_! f^! \mathcal{F} \xrightarrow{\text{adj.}} \mathcal{F}$$

$$\mathcal{F} \xrightarrow{\text{adj.}} R\mathscr{H}om(\mathcal{G}, \mathcal{F} \otimes^L \mathcal{G}) \qquad \mathcal{G} \otimes^L R\mathscr{H}om(\mathcal{G}, \mathcal{F}) \xrightarrow{\text{adj.}} \mathcal{F}$$

Composition isomorphisms:

$$(f_2 \circ f_1)^* \mathcal{F} \xrightarrow[\sim]{\text{comp.}} f_1^* f_2^* \mathcal{F} \qquad f_{2*} f_{1*} \mathcal{F} \xrightarrow[\sim]{\text{comp.}} (f_2 \circ f_1)_* \mathcal{F}$$

$$f_{2!} f_{1!} \mathcal{F} \xrightarrow[\sim]{\text{comp.}} (f_2 \circ f_1)_! \mathcal{F} \qquad (f_2 \circ f_1)^! \mathcal{F} \xrightarrow[\sim]{\text{comp.}} f_1^! f_2^! \mathcal{F}$$

Natural maps from cartesian squares:

$$f_! g'_* \mathcal{F} \xrightarrow{\text{cart.}} g_* f'_! \mathcal{F} \qquad (g')^* f^! \mathcal{F} \xrightarrow{\text{cart.}} (f')^! g^* \mathcal{F}$$

$$g^* f_* \mathcal{F} \xrightarrow{\text{b.ch.}} f'_* (g')^* \mathcal{F} \qquad g'_! (f')^! \mathcal{F} \xrightarrow{\text{b.ch.}} f^! g_! \mathcal{F}$$

Natural isomorphisms involving only left adjoints or only right adjoints:

$$g^* f_! \mathcal{F} \xrightarrow[\sim]{\text{b.ch.}} f'_! (g')^* \mathcal{F} \qquad g'_* (f')^! \mathcal{F} \xrightarrow{\sim} f^! g_* \mathcal{F}$$

$$f^* \mathcal{F} \otimes^L f^* \mathcal{G} \xrightarrow[\sim]{\text{tens.}} f^* (\mathcal{F} \otimes^L \mathcal{G}) \qquad R\mathscr{H}om(\mathcal{G}, f_* \mathcal{F}) \xrightarrow{\sim} f_* R\mathscr{H}om(f^* \mathcal{G}, \mathcal{F})$$

$$f_! \mathcal{F} \otimes^L \mathcal{G} \xrightarrow[\sim]{\text{proj.}} f_! (\mathcal{F} \otimes^L f^* \mathcal{G}) \qquad R\mathscr{H}om(f^* \mathcal{G}, f^! \mathcal{F}) \xrightarrow{\sim} f^! R\mathscr{H}om(\mathcal{G}, \mathcal{F})$$

TABLE 1.6.2. More diagrams of spaces

(i) $X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3 \xrightarrow{f_3} X_4$

(ii)
$$\begin{tikzcd} X'' \arrow[r, "g'_1"] \arrow[d, "f''"'] & X' \arrow[r, "g'_2"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y'' \arrow[r, "g_1"] & Y' \arrow[r, "g_2"] & Y \end{tikzcd}$$

(iii)
$$\begin{tikzcd} X' \arrow[r, "g''"] \arrow[d, "f'_1"'] & X \arrow[d, "f_1"] \\ Y' \arrow[r, "g'"] \arrow[d, "f'_2"'] & Y \arrow[d, "f_2"] \\ Z' \arrow[r, "g"] & Z \end{tikzcd}$$

For the diagrams involving the pair $(f_!, f^!)$, the proof is different. The isomorphism $(f_2f_1)^!\mathcal{F} \cong f_1^! f_2^! \mathcal{F}$ from Proposition 1.5.6 is defined to be the composition

$$(f_2f_1)^!\mathcal{F} \to f_1^! f_2^! f_{2!} f_{1!} (f_2f_1)^!\mathcal{F} \xrightarrow{\sim} f_1^! f_2^! (f_2f_1)_! (f_2f_1)^!\mathcal{F} \to f_1^! f_2^! \mathcal{F}. \tag{1.6.1}$$

The desired commutative diagrams then follow from the zig-zag equations. □

As a consequence of Proposition 1.6.1, each of the composition isomorphisms can be expressed in terms of that of its adjoint. For instance, the isomorphism $f_1^* f_2^* \mathcal{F} \cong (f_2f_1)^*\mathcal{F}$ is equal to the composition

$$\begin{aligned} f_1^* f_2^* \mathcal{F} \to f_1^* f_2^* (f_2f_1)_* (f_2f_1)^*\mathcal{F} &\cong f_1^* f_2^* f_{2*} f_{1*} (f_2f_1)^*\mathcal{F} \\ &\to f_1^* f_{1*} (f_2f_1)^*\mathcal{F} \to (f_2f_1)^*\mathcal{F}. \end{aligned} \tag{1.6.2}$$

Proposition 1.6.2. *In the setting of Table* 1.6.2(i)*, for $\mathcal{F} \in D^+(X_4, \Bbbk)$ and $\mathcal{G} \in D^+(X_1, \Bbbk)$, the following diagrams commute:*

$$\begin{tikzcd} (f_3f_2f_1)^!\mathcal{F} \arrow[r, "\text{comp.}", "\sim"'] \arrow[d, "\text{comp.}"', "\wr"] & f_1^!(f_3f_2)^!\mathcal{F} \arrow[d, "\text{comp.}", "\wr"'] \\ (f_2f_1)^! f_3^! \mathcal{F} \arrow[r, "\text{comp.}", "\sim"'] & f_1^! f_2^! f_3^! \mathcal{F} \end{tikzcd} \qquad \begin{tikzcd} f_{3!} f_{2!} f_{1!} \mathcal{G} \arrow[r, "\text{comp.}", "\sim"'] \arrow[d, "\text{comp.}"', "\wr"] & (f_3f_2)_! f_{1!}\mathcal{G} \arrow[d, "\text{comp.}", "\wr"'] \\ f_{3!}(f_2f_1)_!\mathcal{G} \arrow[r, "\text{comp.}", "\sim"'] & (f_3f_2f_1)_!\mathcal{G} \end{tikzcd}$$

$$\begin{tikzcd} (f_3f_2f_1)_*\mathcal{G} \arrow[r, "\text{comp.}", "\sim"'] \arrow[d, "\text{comp.}"', "\wr"] & (f_3f_2)_* f_{1*}\mathcal{G} \arrow[d, "\text{comp.}", "\wr"'] \\ f_{3*}(f_2f_1)_*\mathcal{G} \arrow[r, "\text{comp.}", "\sim"'] & f_{3*} f_{2*} f_{1*} \mathcal{G} \end{tikzcd} \qquad \begin{tikzcd} f_1^* f_2^* f_3^* \mathcal{F} \arrow[r, "\text{comp.}", "\sim"'] \arrow[d, "\text{comp.}"', "\wr"] & f_1^*(f_3f_2)^*\mathcal{F} \arrow[d, "\text{comp.}", "\wr"'] \\ (f_2f_1)^* f_3^* \mathcal{F} \arrow[r, "\text{comp.}", "\sim"'] & (f_3f_2f_1)^*\mathcal{F} \end{tikzcd}$$

Proof sketch. The diagrams for f_* and $f_!$ are easily seen to commute at the level of abelian categories from the construction of the composition isomorphisms in Proposition 1.2.8. The case of f^* can likewise be proved directly (with a bit more work); alternatively, it can be deduced from the case of f_* using (1.6.2). Finally, the case of $f^!$ follows from $f_!$ by (1.6.1). □

Proposition 1.6.3. *In the setting of Table* 1.6.2(ii) *or* 1.6.2(iii)*, let $\mathcal{F} \in D^+(X, \Bbbk)$. Then the following diagrams commute:*

$$\begin{tikzcd} (g_2g_1)^* f_* \mathcal{F} \arrow[r, "\text{comp.}", "\sim"'] \arrow[dd, "\text{b.ch.}"'] & g_1^* g_2^* f_* \mathcal{F} \arrow[d, "\text{b.ch.}"] \\ & g_1^* f'_* (g'_2)^* \mathcal{F} \arrow[d, "\text{b.ch.}"] \\ f''_* (g'_2 g'_1)^* \mathcal{F} \arrow[r, "\text{comp.}", "\sim"'] & f''_* (g'_1)^* (g'_2)^* \mathcal{F} \end{tikzcd} \qquad \begin{tikzcd} g^* f_{2*} f_{1*} \mathcal{F} \arrow[r, "\text{comp.}", "\sim"'] \arrow[d, "\text{b.ch.}"'] & g^*(f_2f_1)_*\mathcal{F} \arrow[dd, "\text{b.ch.}"] \\ f'_{2*}(g')^* f_{1*}\mathcal{F} \arrow[d, "\text{b.ch.}"'] & \\ f'_{2*} f'_{1*} (g'')^* \mathcal{F} \arrow[r, "\text{comp.}", "\sim"'] & (f'_2 f'_1)_* (g'')^* \mathcal{F} \end{tikzcd}$$

Proof sketch. Recall that the base change map $g^* f_* \mathcal{F} \to f'_*(g')^*\mathcal{F}$ is defined to be the composition

$$g^* f_* \mathcal{F} \xrightarrow{\text{adj.}} g^* f_* g'_* (g')^* \mathcal{F} \xrightarrow[\sim]{\text{comp.}} g^* g_* f'_* (g')^* \mathcal{F} \xrightarrow{\text{adj.}} f'_* (g')^* \mathcal{F}.$$

The proposition can then be deduced from Propositions 1.6.1 and 1.6.2. □

PROPOSITION 1.6.4. *In the setting of Table* 1.6.1(iii), *for* $\mathcal{F} \in D^+(X, \Bbbk)$, *the following diagrams commute:*

$$\begin{tikzcd} (f')^* g^* f_* \mathcal{F} \arrow[r, "\text{comp.}", "\sim"'] \arrow[d, "\text{b.ch.}"'] & (g')^* f^* f_* \mathcal{F} \arrow[d, "\text{adj.}"] \\ (f')^* f'_* (g')^* \mathcal{F} \arrow[r, "\text{adj.}"] & (g')^* \mathcal{F} \end{tikzcd} \qquad \begin{tikzcd} f_* \mathcal{F} \arrow[r, "\text{adj.}"] \arrow[d, "\text{adj.}"'] & g_* g^* f_* \mathcal{F} \arrow[d, "\text{b.ch.}"] \\ f_* g'_* (g')^* \mathcal{F} \arrow[r, "\text{comp.}", "\sim"'] & g_* f'_* (g')^* \mathcal{F} \end{tikzcd}$$

For $\mathcal{G} \in D^+(Y, \Bbbk)$ *and* $\mathcal{H} \in D^+(X', \Bbbk)$, *the following diagrams commute:*

$$\begin{tikzcd} g^* \mathcal{G} \arrow[r, "\text{adj.}"] \arrow[d, "\text{adj.}"'] & g^* f_* f^* \mathcal{G} \arrow[d, "\text{b.ch.}"] \\ f'_* (f')^* g^* \mathcal{G} \arrow[r, "\text{comp.}", "\sim"'] & f'_* (g')^* f^* \mathcal{G} \end{tikzcd} \qquad \begin{tikzcd} g^* f_* g'_* \mathcal{H} \arrow[r, "\text{comp.}", "\sim"'] \arrow[d, "\text{b.ch.}"'] & g^* g_* f'_* \mathcal{H} \arrow[d, "\text{adj.}"] \\ f'_* (g')^* g'_* \mathcal{H} \arrow[r, "\text{adj.}"] & f'_* \mathcal{H} \end{tikzcd}$$

PROOF SKETCH. The two paths around the first diagram give a pair of maps $\phi_1, \phi_2 : (f')^* g^* f_* \mathcal{F} \to (g')^* \mathcal{F}$. By adjunction, these maps correspond to a pair of maps $\psi_1, \psi_2 : g^* f_* \mathcal{F} \to (f')_* (g')^* \mathcal{F}$. By unwinding the definitions, one can check that both ψ_1 and ψ_2 are equal to the base change map, so $\phi_1 = \phi_2$. In other words, the diagram commutes. Similar remarks apply to the other diagrams. $\square$

The next statement is very similar to Proposition 1.6.4, but involves $g^* f_! \mathcal{F} \xrightarrow{\sim} f'_! (g')^* \mathcal{F}$ in place of $g^* f_* \mathcal{F} \to f'_* (g')^* \mathcal{F}$. We omit its proof.

PROPOSITION 1.6.5. *In the setting of Table* 1.6.1(iii), *for* $\mathcal{F} \in D^+(X, \Bbbk)$, *the following diagrams commute:*

$$\begin{tikzcd} (g')^* \mathcal{F} \arrow[r, "\text{adj.}"] \arrow[d, "\text{adj.}"'] & (f')^! f'_! (g')^* \mathcal{F} \\ (g')^* f^! f_! \mathcal{F} \arrow[r, "\text{cart.}"] & (f')^! g^* f_! \mathcal{F} \arrow[u, "\wr"', "\text{b.ch.}"] \end{tikzcd} \qquad \begin{tikzcd} f_! \mathcal{F} \arrow[r, "\text{adj.}"] \arrow[d, "\text{adj.}"'] & g_* g^* f_! \mathcal{F} \arrow[d, "\wr"', "\text{b.ch.}"] \\ f_! g'_* (g')^* \mathcal{F} \arrow[r, "\text{cart.}"] & g_* f'_! (g')^* \mathcal{F} \end{tikzcd}$$

For $\mathcal{G} \in D^+(Y, \Bbbk)$ *and* $\mathcal{H} \in D^+(X', \Bbbk)$, *the following diagrams commute:*

$$\begin{tikzcd} g^* f_! f^! \mathcal{G} \arrow[r, "\text{adj.}"] \arrow[d, "\text{b.ch.}"'] & g^* \mathcal{G} \\ f'_! (g')^* f^! \mathcal{G} \arrow[r, "\text{cart.}"] & f'_! (f')^! g^* \mathcal{G} \arrow[u, "\text{adj.}"'] \end{tikzcd} \qquad \begin{tikzcd} g^* f_! g'_* \mathcal{H} \arrow[r, "\text{cart.}"] \arrow[d, "\text{b.ch.}"'] & g^* g_* f'_! \mathcal{H} \arrow[d, "\text{adj.}"] \\ f'_! (g')^* g'_* \mathcal{H} \arrow[r, "\text{adj.}"] & f'_! \mathcal{H} \end{tikzcd}$$

The proofs of the remaining three propositions in this section follow the pattern of Proposition 1.6.1. We omit the details. The first statement below involves units of adjunctions, while the second and third involve only left adjoints. It is left to the reader to formulate the analogous statements for counits and for right adjoints.

PROPOSITION 1.6.6. (1) *In the setting of Table* 1.6.1(iii), *for any* $\mathcal{F} \in D^+(X, \Bbbk)$, *the following diagram commutes:*

$$\begin{tikzcd} \mathcal{F} \arrow[rr, "\text{adj.}"] \arrow[d, "\text{adj.}"'] & & f^! f_! \mathcal{F} \arrow[d, "\text{adj.}"] \\ g'_* (g')^* \mathcal{F} \arrow[d, "\text{adj.}"'] & & f^! g_* g^* f_! \mathcal{F} \arrow[d, "\wr"', "\text{b.ch.}"] \\ g'_* (f')^! f'_! (g')^* \mathcal{F} \arrow[rr, "\text{b.ch.}", "\sim"'] & & f^! g_* f'_! (g')^* \mathcal{F} \end{tikzcd}$$

(2) *In the setting of Table* 1.6.1(i), *for* $\mathcal{F} \in D^+(X_2, \mathbb{k})$ *and* $\mathcal{G} \in D^+(X_1, \mathbb{k})$, *the following diagrams commute:*

$$\begin{tikzcd}
\mathcal{F} \arrow[rr, "\text{adj.}"] \arrow[d, "\text{adj.}"'] & & f_* f^* \mathcal{F} \arrow[d, "\text{adj.}"] \\
R\mathscr{H}om(\mathcal{G}, \mathcal{F} \otimes^L \mathcal{G}) \arrow[d, "\text{adj.}"'] & & f_* R\mathscr{H}om(f^*\mathcal{G}, f^*\mathcal{F} \otimes^L f^*\mathcal{G}) \arrow[d, "\text{tens.}", "\wr"'] \\
R\mathscr{H}om(\mathcal{G}, f_* f^*(\mathcal{F} \otimes^L \mathcal{G})) \arrow[rr, "\sim"] & & f_* R\mathscr{H}om(f^*\mathcal{G}, f^*(\mathcal{F} \otimes^L \mathcal{G}))
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{G} \arrow[rr, "\text{adj.}"] \arrow[d, "\text{adj.}"'] & & f^! f_! \mathcal{G} \arrow[d, "\text{adj.}"] \\
R\mathscr{H}om(f^*\mathcal{F}, \mathcal{G} \otimes^L f^*\mathcal{F}) \arrow[d, "\text{adj.}"'] & & f^! R\mathscr{H}om(\mathcal{F}, f_!\mathcal{G} \otimes^L \mathcal{F}) \arrow[d, "\text{proj.}", "\wr"'] \\
R\mathscr{H}om(f^*\mathcal{F}, f^! f_!(\mathcal{G} \otimes^L f^*\mathcal{F})) \arrow[rr, "\sim"] & & f^! R\mathscr{H}om(\mathcal{F}, f_!(\mathcal{G} \otimes^L f^*\mathcal{F}))
\end{tikzcd}$$

PROPOSITION 1.6.7. (1) *In the setting of Table* 1.6.2(ii) *or* 1.6.2(iii), *let* $\mathcal{F} \in D^+(X, \mathbb{k})$. *Then the following diagrams commute:*

$$\begin{tikzcd}
(g_2 g_1)^* f_! \mathcal{F} \arrow[r, "\text{comp.}", "\sim"'] \arrow[dd, "\text{b.ch.}"', "\wr"] & g_1^* g_2^* f_! \mathcal{F} \arrow[d, "\text{b.ch.}", "\wr"'] \\
& g_1^* f'_!(g'_2)^* \mathcal{F} \arrow[d, "\text{b.ch.}", "\wr"'] \\
f''_!(g'_2 g'_1)^* \mathcal{F} \arrow[r, "\text{comp.}", "\sim"'] & f''_!(g'_1)^*(g'_2)^* \mathcal{F}
\end{tikzcd}
\qquad
\begin{tikzcd}
g^* f_{2!} f_{1!} \mathcal{F} \arrow[r, "\text{comp.}", "\sim"'] \arrow[d, "\text{b.ch.}"', "\wr"] & g^*(f_2 f_1)_! \mathcal{F} \arrow[dd, "\text{b.ch.}", "\wr"'] \\
f'_{2!}(g')^* f_{1!} \mathcal{F} \arrow[d, "\text{b.ch.}"', "\wr"] & \\
f'_{2!} f'_{1!}(g'')^* \mathcal{F} \arrow[r, "\text{comp.}", "\sim"'] & (f'_2 f'_1)_!(g'')^* \mathcal{F}
\end{tikzcd}$$

(2) *In the setting of Table* 1.6.1(ii), *let* $\mathcal{F} \in D^+(X_1, \mathbb{k})$, *and let* $\mathcal{G}, \mathcal{H} \in D^+(X_3, \mathbb{k})$. *Then the following diagrams commute:*

$$\begin{tikzcd}
(f_2 f_1)^* \mathcal{G} \otimes^L (f_2 f_1)^* \mathcal{H} \arrow[r, "\text{comp.}", "\sim"'] \arrow[dd, "\text{tens.}"', "\wr"] & f_1^* f_2^* \mathcal{G} \otimes^L f_1^* f_2^* \mathcal{H} \arrow[d, "\text{tens.}", "\wr"'] \\
& f_1^*(f_2^* \mathcal{G} \otimes^L f_2^* \mathcal{H}) \arrow[d, "\text{tens.}", "\wr"'] \\
(f_2 f_1)^*(\mathcal{G} \otimes^L \mathcal{H}) \arrow[r, "\text{comp.}", "\sim"'] & f_1^* f_2^*(\mathcal{G} \otimes^L \mathcal{H})
\end{tikzcd}
\qquad
\begin{tikzcd}
(f_2 f_1)_! \mathcal{F} \otimes^L \mathcal{G} \arrow[r, "\text{comp.}", "\sim"'] \arrow[dd, "\text{proj.}"', "\wr"] & f_{2!} f_{1!} \mathcal{F} \otimes^L \mathcal{G} \arrow[d, "\text{proj.}", "\wr"'] \\
& f_{2!}(f_{1!} \mathcal{F} \otimes^L f_2^* \mathcal{G}) \arrow[d, "\text{proj.}", "\wr"'] \\
(f_2 f_1)_!(\mathcal{F} \otimes^L (f_2 f_1)^* \mathcal{G}) \arrow[r, "\text{comp.}", "\sim"'] & f_{2!} f_{1!}(\mathcal{F} \otimes^L f_1^* f_2^* \mathcal{G})
\end{tikzcd}$$

PROPOSITION 1.6.8. *In the setting of Table* 1.6.1(iii), *for* $\mathcal{F} \in D^+(X, \mathbb{k})$ *and* $\mathcal{G} \in D^+(Y, \mathbb{k})$, *the following diagram commutes:*

$$\begin{tikzcd}
g^* f_! \mathcal{F} \otimes^L g^* \mathcal{G} \arrow[r, "\text{tens.}", "\sim"'] \arrow[d, "\text{b.ch.}"', "\wr"] & g^*(f_! \mathcal{F} \otimes^L \mathcal{G}) \arrow[rr, "\text{proj.}", "\sim"'] & & g^* f_!(\mathcal{F} \otimes^L f^* \mathcal{G}) \arrow[d, "\text{b.ch.}", "\wr"'] \\
f'_!(g')^* \mathcal{F} \otimes^L g^* \mathcal{G} \arrow[r, "\text{proj.}", "\sim"'] & f'_!((g')^* \mathcal{F} \otimes^L (f')^* g^* \mathcal{G}) \arrow[r, "\text{comp.}", "\sim"'] & f'_!((g')^* \mathcal{F} \otimes^L (g')^* f^* \mathcal{G}) \arrow[r, "\text{tens.}", "\sim"'] & f'_!(g')^*(\mathcal{F} \otimes^L f^* \mathcal{G})
\end{tikzcd}$$

Exercises.

1.6.1. In the setting of Table 1.6.1(iii), assume that g and g' are local homeomorphisms. Show that for $\mathcal{F} \in D^+(Y, \mathbb{k})$, the natural map $(g')^* f^! \mathcal{F} \to (f')^! g^* \mathcal{F}$ is an isomorphism.

1.6.2. In the setting of Table 1.6.1(iii), show that for $\mathcal{F}, \mathcal{G} \in D^+(Y', \Bbbk)$, there is a commutative diagram

$$\begin{tikzcd}[column sep=small]
& g'_*(f')^!\mathcal{F} \otimes^L f^*g_*\mathcal{G} \arrow[dl, "\text{b.ch.}"'] \arrow[dr, "\text{b.ch.}", "\sim"'] & \\
g'_*(f')^!\mathcal{F} \otimes^L g'_*(f')^*\mathcal{G} \arrow[d, "\text{Rmk. 1.4.12}"'] & & f^!g_*\mathcal{F} \otimes^L f^*g_*\mathcal{G} \arrow[d, "\text{Prop. 1.5.9}"] \\
g'_*((f')^!\mathcal{F} \otimes^L (f')^*\mathcal{G}) \arrow[d, "\text{Prop. 1.5.9}"'] & & f^!(g_*\mathcal{F} \otimes^L g_*\mathcal{G}) \arrow[d, "\text{Rmk. 1.4.12}"] \\
g'_*(f')^!(\mathcal{F} \otimes^L \mathcal{G}) \arrow[rr, "\text{b.ch.}", "\sim"'] & & f^!g_*(\mathcal{F} \otimes^L \mathcal{G})
\end{tikzcd}$$

1.7. Local systems

The notion of a **constant sheaf** was introduced in Example 1.1.8. In this section, we will study the following generalization of this notion.

DEFINITION 1.7.1. A sheaf $\mathcal{L}$ on a space X is said to be **locally constant**, or a **local system**, if there is an open covering $(U_\alpha)_{\alpha \in I}$ of X such that $\mathcal{L}|_{U_\alpha}$ is a constant sheaf for all $\alpha \in I$. The full subcategory of $\mathrm{Sh}(X, \Bbbk)$ consisting of local systems is denoted by $\mathrm{Loc}(X, \Bbbk)$.

A local system $\mathcal{L}$ on a space X is said to be of **finite type** if its stalks are finitely generated $\Bbbk$-modules. The full subcategory of $\mathrm{Loc}(X, \Bbbk)$ consisting of local systems of finite type is denoted by $\mathrm{Loc}^{\mathrm{ft}}(X, \Bbbk)$.

A local system is said to be **locally free** if its stalks are free $\Bbbk$-modules.

Here are a few easy observations about local systems:

- If $\Bbbk$ is a field, every local system is automatically locally free.
- For any continuous map $f : X \to Y$, the functor f^* takes local systems on Y to local systems on X. (This follows from (1.2.6).)
- If X is connected, then any two stalks of a local system are isomorphic. (See Exercise 1.7.1.)

DEFINITION 1.7.2. Suppose X is connected.

(1) Let $\mathcal{L}$ be a locally free local system. The **rank** of $\mathcal{L}$, denoted by $\operatorname{rank} \mathcal{L}$, is defined to be the rank of any stalk of $\mathcal{L}$ as a $\Bbbk$-module.
(2) Suppose $\Bbbk$ is noetherian and of finite global dimension, and let $\mathcal{L}$ be a local system of finite type on X. The **grade** of $\mathcal{L}$, denoted by $\operatorname{grade} \mathcal{L}$, is defined to be the grade of any stalk of $\mathcal{L}$ (see Definition A.10.3).

If $\Bbbk$ is a field, then every nonzero local system of finite type has grade 0.

We saw in Example 1.1.8 that if X is connected and locally connected, then for any point $x \in X$, the natural map

$$\Gamma(\underline{M}_X) \to \underline{M}_{X,x} \tag{1.7.1}$$

is an isomorphism, and both modules are canonically identified with M. Below are a few more basic results on constant sheaves.

LEMMA 1.7.3. *Let X be a connected and locally connected topological space. The constant sheaf functor* $\Bbbk\text{-mod} \to \mathrm{Sh}(X, \Bbbk)$ *given by* $M \mapsto \underline{M}_X$ *is fully faithful.*

Proof. The constant sheaf functor is identified with a_X^* (see (1.2.2)). By Theorem 1.2.4 and (1.7.1), we have $\mathrm{Hom}_{\mathrm{Sh}(X,\Bbbk)}(\underline{M}_X, \underline{N}_X) \cong \mathrm{Hom}_{\Bbbk}(M, \Gamma(\underline{N}_X)) \cong \mathrm{Hom}_{\Bbbk}(M, N)$, as desired. □

Proposition 1.7.4. *Suppose X is locally connected. The category $\mathrm{Loc}(X,\Bbbk)$ is closed under kernels and cokernels. In particular, $\mathrm{Loc}(X,\Bbbk)$ is an abelian subcategory of $\mathrm{Sh}(X,\Bbbk)$.*

Later, we will establish a slightly stronger version of this statement under the assumption that X is locally contractible; see Lemma 1.8.6. Note that in general, $\mathrm{Loc}(X,\Bbbk)$ is *not* closed under taking arbitrary subobjects and quotients. (Find counterexamples.)

Proof. Let $\phi : \mathcal{L} \to \mathcal{M}$ be a morphism in $\mathrm{Loc}(X,\Bbbk)$. Let $x \in X$, and choose a connected open set U containing x such that both $\mathcal{L}|_U$ and $\mathcal{M}|_U$ are constant sheaves. Lemma 1.7.3 implies that the kernel and cokernel of $\phi|_U : \mathcal{L}|_U \to \mathcal{M}|_U$ are constant sheaves, so $\ker\phi$ and $\operatorname{cok}\phi$ are locally constant. □

The following lemma, a kind of converse to the last part of Lemma 1.7.3, will be useful as a criterion for identifying (locally) constant sheaves.

Lemma 1.7.5. *Let $\mathcal{F}$ be a presheaf on X, and let $U \subset X$ be an open set. Consider the following two conditions on $\mathcal{F}$:*

(1) *Every point $x \in U$ has a basis of neighborhoods $V \subset U$ such that $\mathrm{res}_{U,V} : \mathcal{F}(U) \xrightarrow{\sim} \mathcal{F}(V)$ is an isomorphism.*
(2) *The sheaf $\mathcal{F}^+|_U$ is isomorphic to the constant sheaf with value $\mathcal{F}(U)$.*

Condition (1) implies condition (2). If $\mathcal{F}$ is a sheaf, and if U is connected and locally connected, then the opposite implication also holds.

Proof. Let $M = \mathcal{F}(U)$. By the presheaf version of Theorem 1.2.4, we have

$$\mathrm{Hom}(\underline{M}_{U,\mathrm{pre}}, \mathcal{F}|_U) \cong \mathrm{Hom}(M, \Gamma(\mathcal{F}|_U)) = \mathrm{Hom}(\mathcal{F}(U), \mathcal{F}(U)),$$

and so we obtain a map $\phi : \underline{M}_{U,\mathrm{pre}} \to \mathcal{F}|_U$ that induces an isomorphism on global sections. Condition (1) (together with (1.7.1)) implies that ϕ induces isomorphisms of all stalks, so it becomes an isomorphism after sheafifying.

When $\mathcal{F}$ is a sheaf and U is connected and locally connected, the opposite implication is just a restatement of (1.7.1). □

Example 1.7.6. Let X be a locally connected space, and let $j : U \hookrightarrow X$ be the inclusion of a connected, dense, open subset. Suppose every point $x \in X \smallsetminus U$ has a basis of neighborhoods V such that $V \cap U$ is connected. Then, for any $\Bbbk$-module M, we have

$$^\circ j_* \underline{M}_U \cong \underline{M}_X.$$

To see this, note first that our assumptions imply that X is connected and that every point in X has a basis of neighborhoods V such that $V \cap U$ is connected. (For points in U, this is just a restatement of the fact that U is locally connected.) For any such V, the map $\mathrm{res}_{X,V} : (^\circ j_* \underline{M}_U)(X) \to {}^\circ j_* \underline{M}_U(V)$ is identified with $\mathrm{res}_{U,V\cap U} : \underline{M}_U(U) \to \underline{M}_U(V \cap U)$, which is an isomorphism. We are done by Lemma 1.7.5.

The monodromy representation. Given a local system $\mathcal{L}$ on X and a point $x_0 \in X$, we will now construct an action of the fundamental group $\pi_1(X, x_0)$ on the stalk $\mathcal{L}_{x_0}$, called the **monodromy representation**.

LEMMA 1.7.7. *Any local system on* $[0,1]$ *or on* $[0,1] \times [0,1]$ *is a constant sheaf.*

PROOF. Let $\mathcal{L}$ be a local system on $[0,1]$. Since $[0,1]$ is compact, we can find a finite collection of connected open subsets (i.e., intervals) $U_1, \ldots, U_n$ such that each $\mathcal{L}|_{U_i}$ is a constant sheaf. We may assume that these sets are numbered in such a way that $0 \in U_1$, and for $i > 1$, U_i contains the right endpoint of the closed interval $\overline{U_{i-1}}$. Let $V_i = U_1 \cup U_2 \cup \cdots \cup U_i$. Our assumptions imply that V_i is a connected open set containing 0. Of course, $V_n = [0,1]$.

We will show by induction on i that $\mathcal{L}|_{V_i}$ is a constant sheaf. For $i = 1$, there is nothing to prove. If $i > 1$, then V_{i-1} and U_i are both intervals, so $V_{i-1} \cap U_i$ is as well. In particular, $V_{i-1} \cap U_i$ is connected, and hence both the restriction maps

$$\mathcal{L}(V_{i-1}) \to \mathcal{L}(V_{i-1} \cap U_i) \leftarrow \mathcal{L}(U_i)$$

are isomorphisms. From this it is easily deduced that the restriction maps $\mathcal{L}(V_i) \to \mathcal{L}(V_{i-1})$ and $\mathcal{L}(V_i) \to \mathcal{L}(U_i)$ are isomorphisms, and then Lemma 1.7.5 implies that $\mathcal{L}|_{V_i}$ is a constant sheaf. In particular, $\mathcal{L} = \mathcal{L}|_{V_n}$ is a constant sheaf.

The proof for $[0,1] \times [0,1]$ is similar in spirit, although it is somewhat more cumbersome to arrange a list of open subsets of $[0,1] \times [0,1]$ in a suitable order. The details are left to the reader. □

Given a local system $\mathcal{L}$ on X and a path $\gamma : [0,1] \to X$, we define a map

$$\rho(\gamma) : \mathcal{L}_{\gamma(1)} \to \mathcal{L}_{\gamma(0)}$$

to be the composition of the following sequence of natural isomorphisms:

$$\mathcal{L}_{\gamma(1)} \cong (\gamma^*\mathcal{L})_1 \xleftarrow{\sim} \Gamma(\gamma^*\mathcal{L}) \xrightarrow{\sim} (\gamma^*\mathcal{L})_0 \cong \mathcal{L}_{\gamma(0)}. \tag{1.7.2}$$

Here, the first and last isomorphisms come from (1.2.5), and the middle two from (1.7.1), using the fact that $\gamma^*\mathcal{L}$ is a constant sheaf.

If $\gamma, \gamma' : [0,1] \to X$ are two paths such that $\gamma(1) = \gamma'(0)$, let $\gamma * \gamma'$ be the path given by

$$(\gamma * \gamma')(t) = \begin{cases} \gamma(2t) & \text{if } 0 \le t \le 1/2, \\ \gamma'(2t-1) & \text{if } 1/2 \le t \le 1. \end{cases}$$

LEMMA 1.7.8. *Let* $\mathcal{L}$ *be a local system on* X.

(1) *If* $\gamma : [0,1] \to X$ *is a constant path, then* $\rho(\gamma) = \mathrm{id} : \mathcal{L}_{\gamma(0)} \to \mathcal{L}_{\gamma(0)}$.
(2) *If* $\gamma, \gamma' : [0,1] \to X$ *satisfy* $\gamma(1) = \gamma'(0)$, *then* $\rho(\gamma * \gamma') = \rho(\gamma) \circ \rho(\gamma')$.
(3) *If* $\gamma, \gamma' : [0,1] \to X$ *are homotopic paths (with* $\gamma(0) = \gamma'(0)$ *and* $\gamma(1) = \gamma'(1)$*), then* $\rho(\gamma) = \rho(\gamma') : \mathcal{L}_{\gamma(1)} \to \mathcal{L}_{\gamma(0)}$.

PROOF. Parts (1) and (2) follow easily from the definition. For part (3), let $x_0 = \gamma(0) = \gamma'(0)$ and $x_1 = \gamma(1) = \gamma'(1)$, and let $H : [0,1] \times [0,1] \to X$ be a homotopy between γ and γ'. That is, we have

$$H(t,0) = \gamma(t), \qquad H(t,1) = \gamma'(t), \qquad H(0,s) = x_0, \qquad H(1,s) = x_1.$$

By Lemma 1.7.7, $H^*\mathcal{L}$ is a constant sheaf. The claim follows from the commutativity of the following diagram:

$$\begin{tikzcd}
\mathcal{L}_{x_1} \arrow[d, equal] & \Gamma(\gamma^*\mathcal{L}) = \Gamma((H^*\mathcal{L})|_{[0,1]\times\{0\}}) \arrow[l, "\sim"'] \arrow[r, "\sim"] & \mathcal{L}_{x_0} \arrow[d, equal] \\
\Gamma((H^*\mathcal{L})|_{\{1\}\times[0,1]}) \arrow[d, equal] & \Gamma(H^*\mathcal{L}) \arrow[l, "\sim"'] \arrow[r, "\sim"] \arrow[u, "\wr"] \arrow[d, "\wr"'] & \Gamma((H^*\mathcal{L})|_{\{0\}\times[0,1]}) \arrow[d, equal] \\
\mathcal{L}_{x_1} & \Gamma((\gamma')^*\mathcal{L}) = \Gamma((H^*\mathcal{L})|_{[0,1]\times\{1\}}) \arrow[l, "\sim"'] \arrow[r, "\sim"] & \mathcal{L}_{x_0}
\end{tikzcd}$$

□

Let $\mathcal{L}$ be a local system on X. Lemma 1.7.8 gives us a well-defined action of the fundamental group $\pi_1(X, x_0)$ on the stalk $\mathcal{L}_{x_0}$. This action is called the **monodromy representation**. Equivalently, this construction lets us regard $\mathcal{L}_{x_0}$ as a module for the group ring $\Bbbk[\pi_1(X, x_0)]$. Because the maps in (1.7.2) are all natural, this construction actually defines a functor

(1.7.3) $$\mathrm{Mon}_{x_0} : \mathrm{Loc}(X, \Bbbk) \to \Bbbk[\pi_1(X, x_0)]\text{-mod}.$$

This restricts to a functor

(1.7.4) $$\mathrm{Mon}_{x_0} : \mathrm{Loc}^{\mathrm{ft}}(X, \Bbbk) \to \Bbbk[\pi_1(X, x_0)]\text{-mod}^{\mathrm{fg}}$$

where the right-hand side is the category of $\Bbbk[\pi_1(X, x_0)]$-modules that are finitely generated over $\Bbbk$. The main result of this section is the following.

THEOREM 1.7.9. *Suppose X is connected, locally path-connected, and semilocally simply connected, and let $x_0 \in X$. Then* (1.7.3) *is an equivalence of categories.*

Of course, this immediately implies that (1.7.4) is an equivalence as well.

PROOF SKETCH. Let us explain how to construct the inverse functor. Our assumptions imply that X is path-connected. For each point $x \in X$, let us choose, once and for all, a path $\alpha_x : [0, 1] \to X$ such that $\alpha_x(0) = x_0$ and $\alpha_x(1) = x$. In particular, let us take α_{x_0} to be the constant path at x_0. Given a $\Bbbk[\pi_1(X, x_0)]$-module M, define a presheaf $Q(M)$ by

$$Q(M)(U) = \left\{ \begin{matrix} \text{functions} \\ s : U \to M \end{matrix} \;\middle|\; \begin{matrix} \text{for any path } \gamma : [0,1] \to U, \text{ we have} \\ s(\gamma(0)) = (\alpha_{\gamma(0)} * \gamma * \alpha_{\gamma(1)}^{-1}) \cdot s(\gamma(1)) \end{matrix} \right\}.$$

We briefly indicate the remaining steps of the proof below.

Step 1. Show that $Q(M)$ is a sheaf (and not just a presheaf). The local identity axiom is obvious, but the gluing axiom requires a little bit of work.

Step 2. Show that $Q(M)$ is a local system. By our assumptions on X, any point $x \in X$ has a path-connected neighborhood U with the additional property that any loop in U is null-homotopic in X. For such an open set U, one can show that $Q(M)|_U \cong \underline{M}_U$. This step also shows that all stalks of $Q(M)$ are isomorphic to M.

Step 3. Show that there is a natural isomorphism $\mathrm{Mon}_{x_0}(Q(M)) \cong M$. In fact, one can show more generally that for any path $\gamma : [0, 1] \to X$, the map $\rho(\gamma) : Q(M)_{\gamma(1)} \to Q(M)_{\gamma(0)}$ is identified with the action of $\alpha_{\gamma(0)} * \gamma * \alpha_{\gamma(1)}^{-1} \in \pi_1(X, x_0)$.

Step 4. Show that there is a natural isomorphism $\mathcal{L} \cong Q(\mathrm{Mon}_{x_0}(\mathcal{L}))$. The map $\mathcal{L} \to Q(\mathrm{Mon}_{x_0}(\mathcal{L}))$ can be defined as follows: given an open set $U \subset X$ and a section $s \in \mathcal{L}(U)$, the corresponding section of $Q(\mathrm{Mon}_{x_0}(\mathcal{L}))$ is the function $\tilde{s} : U \to \mathcal{L}_{x_0}$ given by $\tilde{s}(x) = \rho(\alpha_x)(s_x)$. □

Sheaf functors and monodromy. We will now investigate the behavior of the monodromy representation under various sheaf functors. We sometimes write $f : (X, x_0) \to (Y, y_0)$ to indicate that f is a continuous map $X \to Y$ with the property that $f(x_0) = y_0$. Of course, such a map induces a group homomorphism

$$\pi_1(f) : \pi_1(X, x_0) \to \pi_1(Y, y_0).$$

Via this homomorphism, any $\Bbbk[\pi_1(Y, y_0)]$-module M can be regarded as a module over $\Bbbk[\pi_1(X, x_0)]$. The resulting $\Bbbk[\pi_1(X, x_0)]$-module is denoted by

$$\operatorname{Res}^{\Bbbk[\pi_1(Y,y_0)]}_{\Bbbk[\pi_1(X,x_0)]} M.$$

PROPOSITION 1.7.10. *Let X and Y be connected, locally path-connected, and semilocally simply connected topological spaces, and let $f : (X, x_0) \to (Y, y_0)$ be a continuous map. For $\mathcal{F} \in \mathrm{Loc}(Y, \Bbbk)$, there is a natural isomorphism*

$$\mathrm{Mon}_{x_0}(f^*\mathcal{F}) \cong \operatorname{Res}^{\Bbbk[\pi_1(Y,y_0)]}_{\Bbbk[\pi_1(X,x_0)]} \mathrm{Mon}_{y_0}(\mathcal{F}).$$

This follows easily from the construction of the monodromy representation. We omit the details.

PROPOSITION 1.7.11. *Suppose X is connected, locally path-connected, and semilocally simply connected. For any $\mathcal{L}, \mathcal{M} \in \mathrm{Loc}(X, \Bbbk)$, the sheaves $\mathcal{L} \otimes \mathcal{M}$ and $\mathscr{H}\!om(\mathcal{L}, \mathcal{M})$ are also local systems. Moreover, given $x_0 \in X$, there are natural isomorphisms*

$$\mathrm{Mon}_{x_0}(\mathcal{L}) \otimes \mathrm{Mon}_{x_0}(\mathcal{M}) \xrightarrow{\sim} \mathrm{Mon}_{x_0}(\mathcal{L} \otimes \mathcal{M}),$$
$$\mathrm{Mon}_{x_0}(\mathscr{H}\!om(\mathcal{L}, \mathcal{M})) \cong \mathrm{Hom}_{\Bbbk}(\mathrm{Mon}_{x_0}(\mathcal{L}), \mathrm{Mon}_{x_0}(\mathcal{M})).$$

PROOF SKETCH. See Exercise 1.7.2 for the claim that $\otimes$ and $\mathscr{H}\!om$ send local systems to local systems. The tensor product isomorphism follows from the observation that for any path $\gamma : [0, 1] \to X$, we have $\gamma^*(\mathcal{L} \otimes \mathcal{M}) \cong (\gamma^*\mathcal{L}) \otimes (\gamma^*\mathcal{M})$. Next, using the tensor product isomorphism, for any $N \in \Bbbk[\pi_1(X, x_0)]$-mod, we have

$$\begin{aligned} &\mathrm{Hom}(N, \mathrm{Mon}_{x_0}(\mathscr{H}\!om(\mathcal{L}, \mathcal{M}))) \cong \mathrm{Hom}(\mathrm{Mon}^{-1}_{x_0}(N) \otimes \mathcal{L}, \mathcal{M}) \\ &\quad \cong \mathrm{Hom}(N \otimes \mathrm{Mon}_{x_0}(\mathcal{L}), \mathrm{Mon}_{x_0}(\mathcal{G})) \cong \mathrm{Hom}(N, \mathrm{Hom}_{\Bbbk}(\mathrm{Mon}_{x_0}(\mathcal{L}), \mathrm{Mon}_{x_0}(\mathcal{M}))). \end{aligned}$$

By Yoneda's lemma, we obtain the second isomorphism in the proposition. $\square$

Under some conditions, we can replace the functors in Proposition 1.7.11 by their derived versions. The following statement is particularly useful when working with field coefficients.

PROPOSITION 1.7.12. *Let $\mathcal{L}, \mathcal{M} \in \mathrm{Loc}(X, \Bbbk)$. If $\mathcal{L}$ is locally free of finite rank, then $\mathcal{L} \otimes^L \mathcal{M} \cong \mathcal{L} \otimes \mathcal{M}$ and $R\mathscr{H}\!om(\mathcal{L}, \mathcal{M}) \cong \mathscr{H}\!om(\mathcal{L}, \mathcal{M})$. If both $\mathcal{L}$ and $\mathcal{M}$ are locally free of finite rank, then $\mathcal{L} \otimes^L \mathcal{M}$ and $R\mathscr{H}\!om(\mathcal{L}, \mathcal{M})$ are also locally free local systems of finite rank.*

PROOF. For the first assertion, it is enough to prove that any point admits a neighborhood U such that $(\mathcal{L} \otimes^L \mathcal{M})|_U \cong (\mathcal{L} \otimes \mathcal{M})|_U$ and $R\mathscr{H}\!om(\mathcal{L}, \mathcal{M})|_U \cong \mathscr{H}\!om(\mathcal{L}, \mathcal{M})|_U$. Choose U so that $\mathcal{L}|_U$ is a constant sheaf, and hence a finite direct sum of copies of $\underline{\Bbbk}_U$. Then the claim follows from Propositions 1.4.4 and 1.4.5. The same reasoning shows that if $\mathcal{M}$ is also locally free of finite rank, then $(\mathcal{L} \otimes^L \mathcal{M})|_U$ and $R\mathscr{H}\!om(\mathcal{L}, \mathcal{M})|_U$ are as well. $\square$

DEFINITION 1.7.13. Let $\mathcal{L}$ be a locally free local system of finite rank on X. The **dual local system** to $\mathcal{L}$ is the local system $\mathcal{L}^\vee$ given by

$$\mathcal{L}^\vee = \mathscr{H}om(\mathcal{L}, \underline{\Bbbk}_X) \cong R\mathscr{H}om(\mathcal{L}, \underline{\Bbbk}_X).$$

Proposition 1.7.11 tells us that the assignment $\mathcal{L} \mapsto \mathcal{L}^\vee$ corresponds to taking the $\Bbbk$-linear dual of a $\Bbbk[\pi_1(X, x_0)]$-module. There is a canonical isomorphism

$$\mathcal{L} \xrightarrow{\sim} (\mathcal{L}^\vee)^\vee.$$

The next few statements study pullbacks and push-forwards of local systems along covering maps.

LEMMA 1.7.14. *Let $f : X \to Y$ be a covering map. Assume that Y is locally path-connected and locally simply connected.*

(1) *The functor ${}^\circ f_* : \mathrm{Sh}(X, \Bbbk) \to \mathrm{Sh}(Y, \Bbbk)$ restricts to an exact functor ${}^\circ f_* : \mathrm{Loc}(X, \Bbbk) \to \mathrm{Loc}(Y, \Bbbk)$.*
(2) *If X and Y are locally compact, then ${}^\circ f_! : \mathrm{Sh}(X, \Bbbk) \to \mathrm{Sh}(Y, \Bbbk)$ restricts to an exact functor ${}^\circ f_! : \mathrm{Loc}(X, \Bbbk) \to \mathrm{Loc}(Y, \Bbbk)$.*

PROOF. To show that ${}^\circ f_!$ is exact, it is enough to prove exactness after taking stalks. In other words, we must prove that for any $y \in Y$, the functor $\mathcal{F} \mapsto {}^\circ f_!(\mathcal{F})_y \cong \Gamma_c(\mathcal{F}|_{f^{-1}(y)})$ is exact. Because f is a covering map, $f^{-1}(y)$ is a discrete topological space, so the desired exactness follows from Example 1.1.16.

Next, we will show that both ${}^\circ f_!$ and ${}^\circ f_*$ send local systems to local systems. Let $y \in Y$, and let $U \subset Y$ be an evenly covered open set. That is, $f^{-1}(U)$ is the disjoint union of a collection of open sets $(V_\alpha)_{\alpha \in I}$, each of which is homeomorphic to U. By replacing U by a smaller open set if necessary, we may assume that it is simply connected, and then each V_α is also simply connected. Theorem 1.7.9 implies that any local system on U or on V_α is constant.

Let $\mathcal{L} \in \mathrm{Loc}(X, \Bbbk)$. For each $\alpha \in I$, let $M_\alpha = \mathcal{L}(V_\alpha)$, so that $\mathcal{L}|_{V_\alpha} \cong \underline{M_\alpha}_{V_\alpha}$. One can check from the definitions that

$${}^\circ f_!(\mathcal{L})(U) \cong \bigoplus_{\alpha \in I} M_\alpha, \qquad {}^\circ f_*(\mathcal{L})(U) \cong \prod_{\alpha \in I} M_\alpha.$$

Moreover, these same formulas hold if U is replaced by some smaller connected open set $U' \subset U$, so Lemma 1.7.5 implies that ${}^\circ f_!(\mathcal{L})|_U$ and ${}^\circ f_*(\mathcal{L})|_U$ are constant sheaves. Thus, ${}^\circ f_! \mathcal{L}$ and ${}^\circ f_* \mathcal{L}$ are locally constant.

It remains to show that ${}^\circ f_*$ is exact on local systems. It is at least left exact; we just need to show that it sends surjective maps to surjective maps. Let $\phi : \mathcal{L} \to \mathcal{M}$ be a surjective map of local systems on X. Let U and $(V_\alpha)_{\alpha \in I}$ be as above. Since $\mathcal{L}|_{V_\alpha}$ and $\mathcal{M}|_{V_\alpha}$ are constant sheaves, Lemma 1.7.3 implies that the map $\phi_{V_\alpha} : \mathcal{L}(V_\alpha) \to \mathcal{M}(V_\alpha)$ is surjective. Then the map $({}^\circ f_* \phi)_U$ can be identified with

$$\prod_{\alpha \in I} \phi_{V_\alpha} : \prod_{\alpha \in I} \mathcal{L}(V_\alpha) \to \prod_{\alpha \in I} \mathcal{M}(V_\alpha),$$

and this is clearly surjective. $\square$

REMARK 1.7.15. In the setting of Lemma 1.7.14, given $\mathcal{L} \in \mathrm{Loc}(X, \Bbbk)$, it is natural to ask whether $f_* \mathcal{L}$ and $f_! \mathcal{L}$ also lie in $\mathrm{Loc}(Y, \Bbbk)$, i.e., whether $\mathsf{H}^i(f_* \mathcal{L})$ and $\mathsf{H}^i(f_! \mathcal{L})$ vanish for $i > 0$. For $f_! \mathcal{L}$, this is easy, by the same reasoning as in the proof of Lemma 1.7.14. For $f_* \mathcal{L}$, see Corollary 1.9.6.

PROPOSITION 1.7.16. *Let X and Y be connected, locally path-connected, and locally simply connected topological spaces, and let $f : (X, x_0) \to (Y, y_0)$ be a covering map. For $\mathcal{L} \in \mathrm{Loc}(X, \Bbbk)$, there is a natural isomorphism*

$$\mathrm{Mon}_{y_0}({}^\circ f_* \mathcal{L}) \cong \mathrm{Hom}_{\Bbbk[\pi_1(X,x_0)]}(\Bbbk[\pi_1(Y, y_0)], \mathrm{Mon}_{x_0}(\mathcal{L})).$$

If X and Y are locally compact, then there is also a natural isomorphism

$$\mathrm{Mon}_{y_0}({}^\circ f_! \mathcal{L}) \cong \Bbbk[\pi_1(Y, y_0)] \otimes_{\Bbbk[\pi_1(X,x_0)]} \mathrm{Mon}_{x_0}(\mathcal{L}).$$

PROOF. By Theorem 1.2.4 and Proposition 1.5.12, the functors ${}^\circ f_*$ and ${}^\circ f_!$ are right and left adjoint, respectively, to f^*. In view of Proposition 1.7.10, the functors $\mathrm{Mon}_{y_0} \circ {}^\circ f_*$ and $\mathrm{Mon}_{y_0} \circ {}^\circ f_!$ must be right and left adjoint, respectively, to $\mathrm{Mon}_{x_0}^{-1} \circ \mathrm{Res}^{\Bbbk[\pi_1(Y,y_0)]}_{\Bbbk[\pi_1(X,x_0)]}$, and the result follows. $\square$

REMARK 1.7.17. Because the isomorphisms in Proposition 1.7.16 were established by adjunction, it follows that the image under Mon_{y_0} of the adjunction map

$${}^\circ f_! f^* \mathcal{L} \to \mathcal{L}$$

can be identified with the adjunction map

$$\Bbbk[\pi_1(Y, y_0)] \otimes_{\Bbbk[\pi_1(X,x_0)]} \mathrm{Res}^{\Bbbk[\pi_1(Y,y_0)]}_{\Bbbk[\pi_1(X,x_0)]} \mathrm{Mon}_{y_0}(\mathcal{L}) \to \mathrm{Mon}_{y_0}(\mathcal{L})$$

given by $r \otimes m \mapsto rm$. Similarly, the image of

$$\mathcal{L} \to {}^\circ f_* f^* \mathcal{L}$$

can be identified with the adjunction map

$$\mathrm{Mon}_{y_0}(\mathcal{L}) \to \mathrm{Hom}_{\Bbbk[\pi_1(X,x_0)]}(\Bbbk[\pi_1(Y, y_0)], \mathrm{Res}^{\Bbbk[\pi_1(Y,y_0)]}_{\Bbbk[\pi_1(X,x_0)]} \mathrm{Mon}_{y_0}(\mathcal{L}))$$

given by $m \mapsto (r \mapsto rm)$.

Deck transformations. Let X and Y be connected, locally path-connected, and locally simply connected topological spaces, and let $f : (X, x_0) \to (Y, y_0)$ be a covering map. Recall that the **Galois group** of f, also known as the group of **deck transformations**, is the group

$$\mathrm{Gal}(X/Y) = \{\text{homeomorphisms } h : X \to X \mid f \circ h = f\}.$$

The group of deck transformations has the following group-theoretic interpretation. Recall that the induced map $\pi_1(f) : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ is injective (see [**98**, Proposition 1.31]), so that we may regard $\pi_1(X, x_0)$ as a subgroup of $\pi_1(Y, y_0)$. Consider its normalizer

$$\mathrm{N}_{\pi_1(Y,y_0)}\pi_1(X, x_0) = \{g \in \pi_1(Y, y_0) \mid g\pi_1(X, x_0)g^{-1} = \pi_1(X, x_0)\}.$$

Given $h \in \mathrm{Gal}(X/Y)$, choose a path $\gamma : [0, 1] \to X$ such that $\gamma(0) = x_0$ and $\gamma(1) = h(x_0)$. Then $f \circ \gamma$ is a loop based at y_0, so it determines an element $g \in \pi_1(Y, y_0)$. In fact, it can be shown that g belongs to the normalizer of $\pi_1(X, x_0)$. In general, g depends on the choice of γ, but it turns out that the coset $g\pi_1(X, x_0)$ depends only on h. In this way, we obtain a map

$$\theta : \mathrm{Gal}(X/Y) \to \mathrm{N}_{\pi_1(Y,y_0)}\pi_1(X, x_0)/\pi_1(X, x_0). \tag{1.7.5}$$

According to [**225**, Theorem 2.6.2] or [**98**, Proposition 1.3.9], this map is an isomorphism of groups.

Let M be a $\Bbbk[\pi_1(X, x_0)]$-module, and let $g \in \mathrm{N}_{\pi_1(Y,y_0)}\pi_1(X, x_0)$. Let M^g be the $\Bbbk[\pi_1(X, x_0)]$-module with the same underlying $\Bbbk$-module as M, but where the action of $r \in \Bbbk[\pi_1(X, x_0)]$ on $m \in M$ is given by

$$r \cdot^g m = (grg^{-1}) \cdot m. \tag{1.7.6}$$

If g and g' are in the same coset of $\pi_1(X, x_0)$, then there is a canonical isomorphism

$$M^g \xrightarrow{\sim} M^{g'} \qquad \text{given by} \qquad m \mapsto g'g^{-1} \cdot m. \tag{1.7.7}$$

It therefore makes sense to consider the module $M^{\bar{g}}$, where $\bar{g}$ is an element of $\mathrm{N}_{\pi_1(Y,y_0)}\pi_1(X, x_0)/\pi_1(X, x_0)$.

Note that if $M = \mathrm{Res}^{\Bbbk[\pi_1(Y,y_0)]}_{\Bbbk[\pi_1(X,x_0)]} M'$ for some $\Bbbk[\pi_1(Y, y_0)]$-module M', then the map (1.7.7) makes sense for arbitrary g and g' in $\mathrm{N}_{\pi_1(Y,y_0)}\pi_1(X, x_0)$, not necessarily in the same coset of $\pi_1(X, x_0)$. In other words, in this situation, we have

$$M \xrightarrow{\sim} M^g \qquad \text{for all } g \in \mathrm{N}_{\pi_1(Y,y_0)}\pi_1(X, x_0). \tag{1.7.8}$$

The following lemma explains how to interpret pullback along a deck transformation in terms of the constructions with $\Bbbk[\pi_1(X, x_0)]$-modules described above.

LEMMA 1.7.18. *Let X and Y be connected, locally path-connected, and locally simply connected topological spaces, and let $f : (X, x_0) \to (Y, y_0)$ be a covering map. Let $h : X \to X$ be a deck transformation. For $\mathcal{L} \in \mathrm{Loc}(Y, \Bbbk)$, there is a natural isomorphism*

$$\mathrm{Mon}_{x_0}(h^*\mathcal{L}) \cong \mathrm{Mon}_{x_0}(\mathcal{L})^{\theta(h)}.$$

Moreover, for $\mathcal{M} \in \mathrm{Loc}(Y, \Bbbk)$, we have a commutative diagram

$$\begin{tikzcd} \mathrm{Mon}_{x_0}(f^*\mathcal{M}) \arrow[r, "\sim"] \arrow[d, "\wr"', "\text{Prop. 1.2.8(1)}"'] & \mathrm{Res}^{\Bbbk[\pi_1(Y,y_0)]}_{\Bbbk[\pi_1(X,x_0)]} \mathrm{Mon}_{y_0}(\mathcal{M}) \arrow[d, "\wr", "(1.7.8)"] \\ \mathrm{Mon}_{x_0}(h^*f^*\mathcal{M}) \arrow[r, "\sim"] & (\mathrm{Res}^{\Bbbk[\pi_1(Y,y_0)]}_{\Bbbk[\pi_1(X,x_0)]} \mathrm{Mon}_{y_0}(\mathcal{M}))^{\theta(h)} \end{tikzcd} \tag{1.7.9}$$

PROOF. Let $\gamma : [0, 1] \to X$ be a loop based at x_0, and let r be its class in $\pi_1(X, x_0)$. We wish to study the map $\rho(\gamma) : (h^*\mathcal{L})_{x_0} \to (h^*\mathcal{L})_{x_0}$.

Choose a path $\gamma_0 : [0, 1] \to X$ such that $\gamma_0(0) = x_0$ and $\gamma_0(1) = h(x_0)$. Then $f \circ \gamma_0$ is a loop in Y, and it determines an element $g \in \mathrm{N}_{\pi_1(Y,y_0)}\pi_1(X, x_0)$ whose coset $g\pi_1(X, x_0)$ is precisely $\theta(h)$. From the definition of ρ (see (1.7.2)), one can check that the following diagram commutes:

$$\begin{tikzcd} (h^*\mathcal{L})_{x_0} \arrow[r, "(1.2.5)", "\sim"'] \arrow[d, "\rho(\gamma)"'] & \mathcal{L}_{h(x_0)} \arrow[r, "\rho(\gamma_0)"] & \mathcal{L}_{x_0} \arrow[d, "\rho(\gamma_0 * (h\circ\gamma) * \gamma_0^{-1})"] \\ (h^*\mathcal{L})_{x_0} \arrow[r, "(1.2.5)", "\sim"'] & \mathcal{L}_{h(x_0)} \arrow[r, "\rho(\gamma_0)"] & \mathcal{L}_{x_0} \end{tikzcd} \tag{1.7.10}$$

The image in $\pi_1(Y, y_0)$ of the loop $\gamma_0 * (h \circ \gamma) * \gamma_0^{-1}$ appearing on the right-hand side of this diagram is the class of $(f \circ \gamma_0) * (f \circ h \circ \gamma) * (f * \gamma_0^{-1})$, which simplifies to grg^{-1}. In other words, the commutative diagram above shows that the action of $\pi_1(X, x_0)$ on $\mathrm{Mon}_{x_0}(h^*\mathcal{L})$ is given by (1.7.6), as desired.

Now suppose $\mathcal{L} = f^*\mathcal{M}$. By (1.2.5), we have isomorphisms

$$(f^*\mathcal{M})_{x_0} \cong \mathcal{M}_{y_0} \cong (h^*f^*\mathcal{M})_{x_0}.$$

This composition can be rewritten as

$$(f^*\mathcal{M})_{x_0} \cong \mathcal{M}_{y_0} \xrightarrow{\rho(f\circ\gamma_0)} \mathcal{M}_{y_0} \cong (f^*\mathcal{M})_{x_0} \xrightarrow{\sim} (h^*f^*\mathcal{M})_{x_0}$$

where the last isomorphism is inverse to that coming from the top (or bottom) horizontal arrows in (1.7.10). Thus, the left-hand vertical map in (1.7.9) corresponds to the action of (the class of) $f \circ \gamma_0$, i.e., to the action of $g \in \pi_1(Y, y_0)$. $\square$

Since a deck transformation $h : X \to X$ is a homeomorphism, the functors $h_! = h_*$ can be identified with the pullback functor $(h^{-1})^*$. In particular, for any $\mathcal{L} \in \mathrm{Loc}(X, \Bbbk)$, Lemma 1.7.18 gives us natural isomorphisms

$$\mathrm{Mon}_{x_0}(h_!\mathcal{L}) = \mathrm{Mon}_{x_0}(h_*\mathcal{L}) \cong \mathrm{Mon}_{x_0}(\mathcal{L})^{\theta(h)^{-1}}.$$

Proposition 1.2.8 also gives us natural isomorphisms $f_!\mathcal{L} \xrightarrow{\sim} f_!h_!\mathcal{L}$ and $f_*h_*\mathcal{L} \xrightarrow{\sim} f_*\mathcal{L}$. Our next task is to interpret these in terms of $\Bbbk[\pi_1(Y, y_0)]$-modules. Let $g \in \mathrm{N}_{\pi_1(Y,y_0)}\pi_1(X, x_0)$, and let $\bar{g}$ denote its coset in $\mathrm{N}_{\pi_1(Y,y_0)}\pi_1(X, x_0)/\pi_1(X, x_0)$. Given a $\Bbbk[\pi_1(X, x_0)]$-module M, we define two maps

$$a_{\bar{g}} : \Bbbk[\pi_1(Y, y_0)] \otimes_{\Bbbk[\pi_1(X,x_0)]} M \to \Bbbk[\pi_1(Y, y_0)] \otimes_{\Bbbk[\pi_1(X,x_0)]} M^{\bar{g}^{-1}},$$

$$b_{\bar{g}} : \mathrm{Hom}_{\Bbbk[\pi_1(X,x_0)]}(\Bbbk[\pi_1(Y, y_0)], M^{\bar{g}^{-1}}) \to \mathrm{Hom}_{\Bbbk[\pi_1(X,x_0)]}(\Bbbk[\pi_1(Y, y_0)], M)$$

by the following formulas: for $r \in \Bbbk[\pi_1(Y, y_0)]$, $s \in \mathrm{Hom}_{\Bbbk[\pi_1(X,x_0)]}(\Bbbk[\pi_1(Y, y_0)], M)$, and $m \in M$, we put

$$a_{\bar{g}}(r \otimes m) = rg^{-1} \otimes m, \qquad b_{\bar{g}}(s)(r) = s(gr).$$

Similar considerations to those in Lemma 1.7.18 yield the following result, whose proof we omit.

LEMMA 1.7.19. *Let X and Y be connected, locally path-connected, and locally simply connected topological spaces, and let $f : (X, x_0) \to (Y, y_0)$ be a covering map. Let $h : X \to X$ be a deck transformation. For $\mathcal{L} \in \mathrm{Loc}(X, \Bbbk)$, the following diagrams commute:*

$$\begin{tikzcd}
\mathrm{Mon}_{y_0}(f_!\mathcal{L}) \arrow[r, "\sim"] \arrow[d, "\wr"'] & \Bbbk[\pi_1(Y, y_0)] \otimes_{\Bbbk[\pi_1(X,x_0)]} \mathrm{Mon}_{x_0}(\mathcal{L}) \arrow[d, "a_{\theta(h)}"] \\
\mathrm{Mon}_{y_0}(f_!h_!\mathcal{L}) \arrow[r, "\sim"] & \Bbbk[\pi_1(Y, y_0)] \otimes_{\Bbbk[\pi_1(X,x_0)]} \mathrm{Mon}_{x_0}(\mathcal{L})^{\theta(h)^{-1}}
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{Mon}_{y_0}(f_*h_*\mathcal{L}) \arrow[r, "\sim"] \arrow[d, "\wr"'] & \mathrm{Hom}_{\Bbbk[\pi_1(X,x_0)]}(\Bbbk[\pi_1(Y, y_0)], \mathrm{Mon}_{x_0}(\mathcal{L})^{\theta(h)^{-1}}) \arrow[d, "b_{\theta(h)}"] \\
\mathrm{Mon}_{y_0}(f_*\mathcal{L}) \arrow[r, "\sim"] & \mathrm{Hom}_{\Bbbk[\pi_1(X,x_0)]}(\Bbbk[\pi_1(Y, y_0)], \mathrm{Mon}_{x_0}(\mathcal{L}))
\end{tikzcd}$$

We are now ready to combine the various constructions described above. Suppose $M = \mathrm{Res}^{\Bbbk[\pi_1(Y,y_0)]}_{\Bbbk[\pi_1(X,x_0)]} M'$ for some $\Bbbk[\pi_1(Y, y_0)]$-module M'. Then the maps $a_{\bar{g}}$ and $b_{\bar{g}}$ for M can be combined with the isomorphism from (1.7.8) to obtain two endomorphisms

$$\mathrm{Int}_{\mathrm{l}}(\bar{g}) : \Bbbk[\pi_1(Y, y_0)] \otimes_{\Bbbk[\pi_1(X,x_0)]} M \to \Bbbk[\pi_1(Y, y_0)] \otimes_{\Bbbk[\pi_1(X,x_0)]} M,$$

$$\mathrm{Int}_{\mathrm{r}}(\bar{g}) : \mathrm{Hom}_{\Bbbk[\pi_1(X,x_0)]}(\Bbbk[\pi_1(Y, y_0)], M) \to \mathrm{Hom}_{\Bbbk[\pi_1(X,x_0)]}(\Bbbk[\pi_1(Y, y_0)], M)$$

given by

$$\mathrm{Int}_{\mathrm{l}}(\bar{g})(r \otimes m) = rg^{-1} \otimes gm, \qquad \mathrm{Int}_{\mathrm{r}}(\bar{g})(s)(r) = g^{-1}s(gr).$$

It follows immediately that $\mathrm{Int}_{\mathrm{l}}$ gives a group homomorphism

$$\mathrm{Int}_{\mathrm{l}} : \mathrm{N}_{\pi_1(Y,y_0)}\pi_1(X,x_0)/\pi_1(X,x_0) \to \mathrm{Aut}(\Bbbk[\pi_1(Y,y_0)] \otimes_{\Bbbk[\pi_1(X,x_0)]} M). \tag{1.7.11}$$

On the other hand, $\mathrm{Int}_{\mathrm{r}}$ defines an antihomomorphism, or a homomorphism

$$\begin{aligned}\mathrm{Int}_{\mathrm{r}} : (\mathrm{N}_{\pi_1(Y,y_0)}\pi_1(X,x_0)/\pi_1(X,x_0))^{\mathrm{op}}& \\ \to \mathrm{Aut}(\mathrm{Hom}_{\Bbbk[\pi_1(X,x_0)]}(\Bbbk[\pi_1(Y,y_0)], M)).&\end{aligned} \tag{1.7.12}$$

Next, let $\mathcal{L} \in \mathrm{Loc}(Y, \Bbbk)$, and let $h : X \to X$ be a deck transformation. We define the maps

$$\mathrm{mon}_!(h) : f_!f^*\mathcal{L} \to f_!f^*\mathcal{L} \qquad \text{and} \qquad \mathrm{mon}_*(h) : f_*f^*\mathcal{L} \to f_*f^*\mathcal{L}$$

to be the following compositions, where every map is defined using either adjunction or the isomorphisms of sheaf functors coming from $f \circ h = f$:

$$\mathrm{mon}_!(h) : f_!f^*\mathcal{L} = (f \circ h)_!(f \circ h)^*\mathcal{L} \xrightarrow{\sim} f_!h_!h^*f^*\mathcal{L} \to f_!f^*\mathcal{L},$$
$$\mathrm{mon}_*(h) : f_*f^*\mathcal{L} \to f_*h_*h^*f^*\mathcal{L} \xrightarrow{\sim} (f \circ h)_*(f \circ h)^*\mathcal{L} = f_*f^*\mathcal{L}.$$

Proposition 1.7.20. *Let X and Y be connected, locally path-connected, and locally simply connected topological spaces, and let $f : (X, x_0) \to (Y, y_0)$ be a covering map. Let $h : X \to X$ be a deck transformation. For $\mathcal{L} \in \mathrm{Loc}(Y, \Bbbk)$, the following diagrams commute:*

$$\begin{tikzcd}
\mathrm{Mon}_{y_0}(f_!f^*\mathcal{L}) \arrow[r, "\sim"] \arrow[d, "\mathrm{mon}_!(h)"'] & \Bbbk[\pi_1(Y,y_0)] \otimes_{\Bbbk[\pi_1(X,x_0)]} \mathrm{Mon}_{y_0}(\mathcal{L}) \arrow[d, "\mathrm{Int}_{\mathrm{l}}(\theta(h))"] \\
\mathrm{Mon}_{y_0}(f_!f^*\mathcal{L}) \arrow[r, "\sim"] & \Bbbk[\pi_1(Y,y_0)] \otimes_{\Bbbk[\pi_1(X,x_0)]} \mathrm{Mon}_{y_0}(\mathcal{L})
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{Mon}_{y_0}(f_*f^*\mathcal{L}) \arrow[r, "\sim"] \arrow[d, "\mathrm{mon}_*(h)"'] & \mathrm{Hom}_{\Bbbk[\pi_1(X,x_0)]}(\Bbbk[\pi_1(Y,y_0)], \mathrm{Mon}_{y_0}(\mathcal{L})) \arrow[d, "\mathrm{Int}_{\mathrm{r}}(\theta(h))"] \\
\mathrm{Mon}_{y_0}(f_*f^*\mathcal{L}) \arrow[r, "\sim"] & \mathrm{Hom}_{\Bbbk[\pi_1(X,x_0)]}(\Bbbk[\pi_1(Y,y_0)], \mathrm{Mon}_{y_0}(\mathcal{L}))
\end{tikzcd}$$

It follows from (1.7.11) and (1.7.12) that $\mathrm{mon}_!$ and mon_* define group homomorphisms as shown below.

Definition 1.7.21. Let X and Y be connected, locally path-connected, and locally simply connected topological spaces, and let $f : (X, x_0) \to (Y, y_0)$ be a covering map. For $\mathcal{L} \in \mathrm{Loc}(Y, \Bbbk)$, the group homomorphisms

$$\mathrm{mon}_! : \mathrm{Gal}(X/Y) \to \mathrm{Aut}(f_!f^*\mathcal{L}), \qquad \mathrm{mon}_* : \mathrm{Gal}(X/Y)^{\mathrm{op}} \to \mathrm{Aut}(f_*f^*\mathcal{L})$$

are called the **internal monodromy actions** on $f_!f^*\mathcal{L}$ and $f_*f^*\mathcal{L}$, respectively.

Exercises.

1.7.1. Let $\mathcal{L}$ be a local system on X. Show that if X is connected, then all stalks of $\mathcal{L}$ are isomorphic.

1.7.2. Let X be a topological space, and let $\mathcal{L}$ and $\mathcal{M}$ be local systems on X.
(a) Show that $\mathcal{L} \otimes \mathcal{M}$ is a local system.
(b) Show that if X is locally connected, then $\mathscr{H}om(\mathcal{L}, \mathcal{M})$ is a local system.

1.7.3. This exercise gives a concrete realization of local systems on $\mathbb{C} \smallsetminus \{0\}$, using a generalization of the sheaves from Exercises 1.1.8 and 1.2.4. Let $\lambda \in \mathbb{C}$, and let $\mathcal{Q}_\lambda$ be the sheaf given by

$$\mathcal{Q}_\lambda(U) = \{\text{solutions } g : U \to \mathbb{C} \text{ to the differential equation } z\tfrac{dg}{dz} - \lambda g = 0\}.$$

(a) Show that $\mathcal{Q}_\lambda$ is a local system of rank 1. Show that $\mathcal{Q}_\lambda \otimes \mathcal{Q}_\mu \cong \mathcal{Q}_{\lambda+\mu}$.
(b) Show that $\mathcal{Q}_\lambda \cong \mathcal{Q}_\mu$ if and only if $\lambda - \mu \in \mathbb{Z}$.
(c) Let $\gamma : [0,1] \to \mathbb{C} \smallsetminus \{0\}$ be the loop given by $\gamma(t) = e^{2\pi i t}$. This is a generator of $\pi_1(\mathbb{C} \smallsetminus \{0\}, 1)$. Show that in the monodromy representation $\mathrm{Mon}_1(\mathcal{Q}_\lambda)$, γ acts by multiplication by $e^{2\pi i\lambda}$. Deduce that every rank-1 local system in $\mathrm{Sh}(\mathbb{C} \smallsetminus \{0\}, \mathbb{C})$ is isomorphic to some $\mathcal{Q}_\lambda$.

For more on this example from the perspective of $\mathscr{D}$-modules and integrable connections, see Examples 5.5.8(2) and 5.5.23.

1.7.4. For $\lambda \in \mathbb{C}$, define a sheaf $\mathcal{Q}'_\lambda \in \mathrm{Sh}(\mathbb{C}, \mathbb{C})$ by the same formula as above.
(a) Compute the stalk of $\mathcal{Q}'_\lambda$ at the point $0 \in \mathbb{C}$.
(b) Show that $\mathcal{Q}'_\lambda \cong {}^\circ j_* \mathcal{Q}_\lambda$, where $j : \mathbb{C} \smallsetminus \{0\} \hookrightarrow \mathbb{C}$ is the inclusion map.
(c) For which λ is $\mathcal{Q}'_\lambda$ locally constant?

1.7.5. This exercise deals with the n-sphere S^n. You may use the following fact from algebraic topology without proof:

$$\mathbf{H}^k(S^n; \Bbbk) \cong \begin{cases} \Bbbk & \text{if } k = 0 \text{ or } k = n, \\ 0 & \text{otherwise.} \end{cases}$$

Let $f : S^3 \to S^2$ be the **Hopf fibration** (see, for instance, [**98**, Example 4.45]).
(a) Prove that

$$\mathsf{H}^k(f_* \underline{\Bbbk}_{S^3}) \cong \begin{cases} \underline{\Bbbk}_{S^2} & \text{if } k = 0 \text{ or } k = 1, \\ 0 & \text{otherwise.} \end{cases}$$

(b) Consider the truncation distinguished triangle $\tau^{\leq 0} f_* \underline{\Bbbk}_{S^3} \to f_* \underline{\Bbbk}_{S^3} \to \tau^{\geq 1} \underline{\Bbbk}_{S^3} \to$. By the previous part, this triangle can be rewritten as

$$\underline{\Bbbk}_{S^2} \to f_* \underline{\Bbbk}_{S^3} \to \underline{\Bbbk}_{S^2}[-1] \to .$$

Show that this distinguished triangle does not split.

1.8. Homotopy

Let X be a locally contractible space. We define two full subcategories of $D^+(X, \Bbbk)$ as follows:

$$D^+_{\mathrm{loc}}(X, \Bbbk) = \{\mathcal{F} \in D^+(X, \Bbbk) \mid \mathsf{H}^k(\mathcal{F}) \in \mathrm{Loc}(X, \Bbbk) \text{ for all } k \in \mathbb{Z}\},$$
$$D^+_{\mathrm{locf}}(X, \Bbbk) = \{\mathcal{F} \in D^+(X, \Bbbk) \mid \mathsf{H}^k(\mathcal{F}) \in \mathrm{Loc}^{\mathrm{ft}}(X, \Bbbk) \text{ for all } k \in \mathbb{Z}\}.$$

This section is devoted to the study of $D^+_{\mathrm{loc}}(X, \Bbbk)$ (but most results apply to $D^+_{\mathrm{locf}}(X, \Bbbk)$ as well). In particular, we will prove that this category is a homotopy invariant of X. To get started, we need some preliminaries on homotopy from a sheaf-theoretic perspective.

Homotopy. In this section, we will make use of the following consequence of Theorem 1.1.18.

LEMMA 1.8.1. *Let M be a $\Bbbk$-module. For the unit interval $[0,1] \subset \mathbb{R}$, there is a canonical isomorphism $M \xrightarrow{\sim} R\Gamma(\underline{M}_{[0,1]})$. In particular, $\mathbf{H}^i([0,1];M) = 0$ for $i > 0$.*

For a direct sheaf-theoretic proof that avoids Theorem 1.1.18, see Exercise 1.8.1.

LEMMA 1.8.2. *Let X be a locally contractible space, and let $\mathrm{pr}_2 : [0,1]\times X \to X$ be the projection map. For $t \in [0,1]$, let $i_t : X \to [0,1] \times X$ be the map given by $i_t(x) = (t,x)$.*

(1) *The functor pr_2^* restricts to an equivalence of categories $\mathrm{pr}_2^* : \mathrm{Loc}(X,\Bbbk) \to \mathrm{Loc}([0,1] \times X, \Bbbk)$. For any $t \in [0,1]$, the functor $i_t^* : \mathrm{Loc}([0,1] \times X, \Bbbk) \to \mathrm{Loc}(X,\Bbbk)$ is an inverse to pr_2^*.*
(2) *For $\mathcal{L} \in \mathrm{Loc}(X,\Bbbk)$, the adjunction map $\mathcal{L} \to \mathrm{pr}_{2*}\mathrm{pr}_2^*\mathcal{L}$ is an isomorphism.*
(3) *For $\mathcal{L} \in \mathrm{Loc}([0,1] \times X,\Bbbk)$, we have $\mathrm{pr}_{2*}\mathcal{L} \in \mathrm{Loc}(X)$. Moreover, the functor $\mathrm{pr}_{2*} : \mathrm{Loc}([0,1] \times X,\Bbbk) \to \mathrm{Loc}(X,\Bbbk)$ is an equivalence of categories, inverse to pr_2^*.*

PROOF. (1) Choose a point $x_0 \in X$. It is clear that the maps

$$\pi_1(X,x_0) \xrightarrow{\pi_1(i_t)} \pi_1([0,1] \times X, (t,x_0)) \xrightarrow{\pi_1(\mathrm{pr}_2)} \pi_1(X,x_0)$$

are both isomorphisms and are inverse to one another. It then follows from Theorem 1.7.9 and Proposition 1.7.10 that pr_2^* and i_t^* are equivalences of categories, inverse to one another.

(2) It is enough to show that for any point $x \in X$, the induced map on stalks

$$\mathcal{L}_x \to (\mathrm{pr}_{2*}\mathrm{pr}_2^*\mathcal{L})_x \tag{1.8.1}$$

is an isomorphism. Consider the cartesian square

$$\begin{tikzcd} {[0,1]} \arrow[r, "g"] \arrow[d, "\mathrm{a}_{[0,1]}"'] & {[0,1] \times X} \arrow[d, "\mathrm{pr}_2"] \\ \{x\} \arrow[r] & X \end{tikzcd}$$

where $g(t) = (t,x)$. Since $[0,1]$ is compact, the vertical maps are proper, so by Proposition 1.2.15, we have $(\mathrm{pr}_{2*}\mathrm{pr}_2^*\mathcal{L})_x \cong R\Gamma(g^*\mathrm{pr}_2^*\mathcal{L}) \cong R\Gamma(\mathrm{a}_{[0,1]}^*\mathcal{L}_x) \cong R\Gamma(\underline{\mathcal{L}_x}_{[0,1]})$. Combining these with (1.8.1), we obtain a map

$$\mathcal{L}_x \to R\Gamma(\underline{\mathcal{L}_x}_{[0,1]}).$$

By Proposition 1.6.4, this is the adjunction map $\mathrm{id} \to \mathrm{a}_{[0,1]*}\mathrm{a}_{[0,1]}^*$, so it is an isomorphism by Lemma 1.8.1.

(3) By part (1), any local system $\mathcal{L} \in \mathrm{Loc}([0,1] \times X,\Bbbk)$ can be written as $\mathcal{L} \cong \mathrm{pr}_2^*\mathcal{L}'$ for some $\mathcal{L}' \in \mathrm{Loc}(X,\Bbbk)$. Then part (2) tells us that $\mathrm{pr}_{2*}\mathcal{L} \cong \mathrm{pr}_{2*}\mathrm{pr}_2^*\mathcal{L}' \cong \mathcal{L}'$, so pr_{2*} does indeed restrict to a functor $\mathrm{Loc}([0,1] \times X,\Bbbk) \to \mathrm{Loc}(X,\Bbbk)$. Since $\mathrm{pr}_2^* : \mathrm{Loc}(X,\Bbbk) \to \mathrm{Loc}([0,1]\times X,\Bbbk)$ is an equivalence of categories, its right adjoint pr_{2*} is also an equivalence, inverse to pr_2^*. $\square$

LEMMA 1.8.3. *Let X and Y be locally contractible spaces, and let $f, g : X \to Y$ be homotopic maps. Then the induced maps*

$$f^\sharp, g^\sharp : \mathbf{H}^k(Y;M) \to \mathbf{H}^k(X;M)$$

are equal.

The analogous statement in singular cohomology is, of course, well known; see, for instance, [**98**, Section 3.1].

PROOF. Consider the projection map $\mathrm{pr}_2 : [0,1] \times X \to X$. By Lemma 1.8.2(2), the adjunction map $\underline{M}_X \to \mathrm{pr}_{2*}\underline{M}_{[0,1]\times X}$ is an isomorphism, so by construction,

$$\mathrm{pr}_2^\sharp : \mathbf{H}^k(X; M) \to \mathbf{H}^k([0,1] \times X; M)$$

is an isomorphism. Next, let $i_t : X \to [0,1] \times X$ be as in Lemma 1.8.2. We have $\mathrm{pr}_2 \circ i_t = \mathrm{id}_X$ for any t, so $i_t^\sharp \circ \mathrm{pr}_2^\sharp$ is the identity map on $\mathbf{H}^k(X; M)$. Thus, $i_t^\sharp$ is an isomorphism, inverse to $\mathrm{pr}_2^\sharp$. In particular, $i_t^\sharp$ is independent of t.

Now let $H : [0,1] \times X \to Y$ be a homotopy from f to g. In other words, we have $H(0,x) = f(x)$ and $H(1,x) = g(x)$ for all $x \in X$. The composition

$$\mathbf{H}(Y; M) \xrightarrow{H^\sharp} \mathbf{H}([0,1] \times X; M) \xrightarrow{i_t^\sharp} \mathbf{H}(X; M)$$

agrees with $f^\sharp$ when $t = 0$, and with $g^\sharp$ when $t = 1$. But the preceding paragraph tells us that it is independent of t, so $f^\sharp = g^\sharp$. $\square$

COROLLARY 1.8.4. *Let X and Y be locally contractible spaces. If $f : X \to Y$ is a homotopy equivalence, then for any $\Bbbk$-module M, the induced maps $f^\sharp : \mathbf{H}^k(Y; M) \to \mathbf{H}^k(X; M)$ are isomorphisms.*

As a special case, we have the following.

COROLLARY 1.8.5. *Let X be a contractible and locally contractible space. For any $\Bbbk$-module M, there is a canonical isomorphism $M \xrightarrow{\sim} R\Gamma(\underline{M}_X)$.*

Complexes of sheaves with locally constant cohomology. We are now ready to upgrade Proposition 1.7.4 as folllows.

LEMMA 1.8.6. *Let X be a locally contractible space. Then* $\mathrm{Loc}(X, \Bbbk)$ *is closed under kernels, cokernels, and extensions.*

PROOF. In view of Proposition 1.7.4, we just need to show that it is closed under extensions. Let $0 \to \mathcal{L}' \to \mathcal{L} \to \mathcal{L}'' \to 0$ be a short exact sequence of sheaves, where $\mathcal{L}'$ and $\mathcal{L}''$ are local systems. Let $x \in X$, and choose a contractible neighborhood U of x. Then $\mathcal{L}'|_U$ is a constant sheaf, so by Corollary 1.8.5, we have $\mathsf{H}^1(R\Gamma(\mathcal{L}'|_U)) = 0$. The long exact sequence in hypercohomology then shows that

$$0 \to \Gamma(\mathcal{L}'|_U) \to \Gamma(\mathcal{L}|_U) \to \Gamma(\mathcal{L}''|_U) \to 0$$

is in fact a short exact sequence. The same holds for any smaller contractible open set $V \subset U$ containing x. One can then consider the diagram

$$\begin{tikzcd}
0 \arrow[r] & \Gamma(\mathcal{L}'|_U) \arrow[r] \arrow[d] & \Gamma(\mathcal{L}|_U) \arrow[r] \arrow[d] & \Gamma(\mathcal{L}''|_U) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Gamma(\mathcal{L}'|_V) \arrow[r] & \Gamma(\mathcal{L}|_V) \arrow[r] & \Gamma(\mathcal{L}''|_V) \arrow[r] & 0
\end{tikzcd}$$

where the vertical maps are given by restriction. Since $\mathcal{L}'|_U$ and $\mathcal{L}''|_U$ are both constant sheaves, the first and third vertical maps are isomorphisms, by Lemma 1.7.5. Then the middle vertical map must be an isomorphism as well, so by Lemma 1.7.5 again, $\mathcal{L}|_U$ is a constant sheaf. Since this holds in a neighborhood of every point, $\mathcal{L}$ is a local system. $\square$

The following lemma is a consequence of Lemma 1.8.6. (See Section A.7 for a review of the notion of a t-structure.)

LEMMA 1.8.7. *Let X be a locally contractible space. The category $D^+_{\mathrm{loc}}(X,\Bbbk)$ is a full triangulated subcategory of $D^+(X,\Bbbk)$. Moreover, it is stable under the truncation functors $\tau^{\leq n}$ and $\tau^{\geq n}$, so the natural t-structure on $D^+(X,\Bbbk)$ induces a t-structure on $D^+_{\mathrm{loc}}(X,\Bbbk)$ whose heart is identified with* $\mathrm{Loc}(X,\Bbbk)$.

Thanks to this lemma, $D^+_{\mathrm{loc}}(X,\Bbbk)$ is a reasonable notion to work with. Note that if $f : X \to Y$ is a continuous map of locally contractible spaces, then because f^* is exact and takes local systems to local systems, it induces a functor

$$f^* : D^+_{\mathrm{loc}}(Y,\Bbbk) \to D^+_{\mathrm{loc}}(X,\Bbbk).$$

LEMMA 1.8.8. *Let X be a locally contractible space, and consider the projection map* $\mathrm{pr}_2 : [0,1] \times X \to X$.

(1) *The functor pr_{2*} restricts to a functor $D^+_{\mathrm{loc}}([0,1] \times X,\Bbbk) \to D^+_{\mathrm{loc}}(X,\Bbbk)$. Moreover, this functor is t-exact with respect to the natural t-structure.*
(2) *The functor $\mathrm{pr}_2^* : D^+_{\mathrm{loc}}(X,\Bbbk) \to D^+_{\mathrm{loc}}([0,1] \times X,\Bbbk)$ is an equivalence of categories, and pr_{2*} is its inverse.*

PROOF. Part (1) follows easily from Lemma 1.8.2(3). To prove part (2), we will show that for $\mathcal{F} \in D^+_{\mathrm{loc}}(X,\Bbbk)$ and $\mathcal{G} \in D^+_{\mathrm{loc}}([0,1] \times X,\Bbbk)$, the adjunction maps

$$\mathcal{F} \to \mathrm{pr}_{2*}\mathrm{pr}_2^*\mathcal{F}, \tag{1.8.2}$$

$$\mathrm{pr}_2^*\mathrm{pr}_{2*}\mathcal{G} \to \mathcal{G} \tag{1.8.3}$$

are isomorphisms. Let us first consider the special case where $\mathcal{F} \in \mathrm{Loc}(X,\Bbbk)$ and $\mathcal{G} \in \mathrm{Loc}([0,1] \times X,\Bbbk)$. Then (1.8.2) is an isomorphism by Lemma 1.8.2(2). For (1.8.3), we may assume that $\mathcal{G} = \mathrm{pr}_2^*\mathcal{F}$ by Lemma 1.8.2(1). The claim then follows from (1.8.2) and the zig-zag equation.

Now consider general $\mathcal{F} \in D^+_{\mathrm{loc}}(X,\Bbbk)$ and $\mathcal{G} \in D^+_{\mathrm{loc}}([0,1] \times X,\Bbbk)$. It is enough to prove that (1.8.2) and (1.8.3) become isomorphisms after applying any H^i. Since pr_{2*} and pr_2^* are both t-exact, this reduces the problem to the special case considered above. □

LEMMA 1.8.9. *Let X be a locally contractible space. Let $t \in [0,1]$, and let $i_t : X \to [0,1] \times X$ be the map given by $i_t(x) = (t,x)$. Then there is an isomorphism of functors*

$$i_t^* \cong \mathrm{pr}_{2*} : D^+_{\mathrm{loc}}([0,1] \times X,\Bbbk) \to D^+_{\mathrm{loc}}(X,\Bbbk).$$

In particular, i_t^* is an equivalence of categories, and the isomorphism class of the functor i_t^* is independent of t.

PROOF. Since $\mathrm{pr}_2 \circ i_t$ is the identity map on X, we see from Proposition 1.2.8(1) that $i_t^* \circ \mathrm{pr}_2^*$ is isomorphic to the identity functor. The rest of the lemma then follows from Lemma 1.8.8. □

The main result of this section is the following.

THEOREM 1.8.10. *Let X and Y be locally contractible spaces.*

(1) *If $f, g : X \to Y$ are homotopic maps, then the pullback functors $f^*, g^* : D^+_{\mathrm{loc}}(Y,\Bbbk) \to D^+_{\mathrm{loc}}(X,\Bbbk)$ are isomorphic.*

(2) *If $f : X \to Y$ is a homotopy equivalence, with homotopy inverse $g : Y \to X$, then $f^* : D^+_{\mathrm{loc}}(Y, \Bbbk) \to D^+_{\mathrm{loc}}(X, \Bbbk)$ is an equivalence of categories, and $g^* : D^+_{\mathrm{loc}}(X, \Bbbk) \to D^+_{\mathrm{loc}}(Y, \Bbbk)$ is an inverse to f^*.*

Proof. Let $f, g : X \to Y$ be homotopic maps, and let $H : [0,1] \times X \to Y$ be a homotopy between them, so that $H \circ i_0 = f$ and $H \circ i_1 = g$. Therefore, $f^* \cong i_0^* \circ H^*$ and $g^* \cong i_1^* \circ H^*$. By Lemma 1.8.9, f^* and g^* are isomorphic.

The second assertion in the theorem follows easily. □

As an application, we can upgrade Lemma 1.7.3 as follows.

Corollary 1.8.11. *Let X be a contractible and locally contractible space, and let $x \in X$. For $\mathcal{F} \in D^+_{\mathrm{loc}}(X, \Bbbk)$, the natural map $R\Gamma(\mathcal{F}) \to \mathcal{F}_x$ is an isomorphism.*

Induced maps in cohomology with compact support. We conclude this section with the following variant of Lemma 1.8.3.

Lemma 1.8.12. *Let U and X be locally compact and locally contractible spaces. Let $f : [0,1] \times U \to X$ be a continuous map, and for $t \in [0,1]$, let $f_t = f(t, -) : U \to X$. Assume that the map $\tilde{f} : [0,1] \times U \to [0,1] \times X$ given by $\tilde{f}(t, u) = (t, f(t, u))$ is an open embedding. Then, for any $\Bbbk$-module M, the induced map*

$$(f_t)_\sharp : \mathbf{H}^k_{\mathrm{c}}(U; M) \to \mathbf{H}^k_{\mathrm{c}}(X; M)$$

is independent of t.

Note that the assumption that $\tilde{f}$ is an open embedding implies that each f_t is an open embedding as well, so $(f_t)_\sharp$ makes sense.

Proof. Let $i_t : X \to [0,1] \times X$ be the map $i_t(x) = (t, x)$. We use the same notation for the analogous maps $U \to [0,1] \times U$ and $\mathrm{pt} \to [0,1]$. Consider the following diagram, which contains three cartesian squares:

$$\begin{tikzcd}
& {[0,1]\times U} \arrow[rr, "\tilde f"] \arrow[ddrr, "\mathrm{pr}_1" near start] & & {[0,1]\times X} \arrow[dd, "\mathrm{pr}_1"] \\
U \arrow[ur, "i_t"] \arrow[rr, "f_t"] \arrow[dr, "\mathrm{a}_U"'] & & X \arrow[ur, "i_t"] \arrow[dl, "\mathrm{a}_X"'] & \\
& \mathrm{pt} \arrow[rr, "i_t"'] & & {[0,1]}
\end{tikzcd}$$

Now consider the adjunction map

$$\tilde{f}_! \tilde{f}^* \underline{M}_{[0,1]\times X} \to \underline{M}_{[0,1]\times X}. \tag{1.8.4}$$

If we apply i_t^* to this map, then by the proper base change theorem, the result can be identified with $f_{t!} f_t^* \underline{M}_X \to \underline{M}_X$. Thus, the map $(f_t)_\sharp : \mathbf{H}^k_{\mathrm{c}}(U; M) \to \mathbf{H}^k_{\mathrm{c}}(X; M)$ can be computed by applying $\mathrm{a}_{X!} i_t^*$ to (1.8.4). By the proper base change theorem again, we can instead compute $(f_t)_\sharp$ by applying $i_t^* \mathrm{pr}_{1!}$ to (1.8.4).

Since $\underline{M}_{[0,1]\times U} \cong \underline{\Bbbk}_{[0,1]} \boxtimes \underline{M}_U$, we have

$$\mathrm{pr}_{1!}(f_! f^* \underline{M}_{[0,1]\times X}) \cong \mathrm{pr}_{1!}(\underline{\Bbbk}_{[0,1]} \boxtimes \underline{M}_U) \cong \underline{\Bbbk}_{[0,1]} \boxtimes \mathrm{a}_{U!} \underline{M}_U \cong \mathrm{a}^*_{[0,1]} \mathrm{a}_{U!} \underline{M}_U.$$

(For the second isomorphism, we have used Proposition 1.4.21.) In particular, the cohomology sheaves of this object are constant sheaves:

$$\mathsf{H}^k(\mathrm{pr}_{1!}(f_! f^* \underline{M}_{[0,1]\times X})) \cong \underline{\mathbf{H}^k_{\mathrm{c}}(U; M)}_{[0,1]}.$$

The same reasoning shows that $\mathsf{H}^k(\mathrm{pr}_{1!}\underline{M}_{[0,1]\times X}) \cong \underline{\mathbf{H}^k_{\mathrm{c}}(X; M)}$. Now, the adjunction map (1.8.4) induces a map of constant sheaves

$$\mathsf{H}^k(\mathrm{pr}_{1!}(f_! f^* \underline{M}_{[0,1]\times X})) \to \mathsf{H}^k(\mathrm{pr}_{1!}\underline{M}_{[0,1]\times X}). \tag{1.8.5}$$

At the beginning of the proof, we saw that $(f_t)_\sharp$ is given by taking the stalk at $t \in [0,1]$ of (1.8.5). But because these are constant sheaves, taking the stalk at any point is isomorphic to taking global sections (see (1.7.1)). In particular, the map $(f_t)_\sharp$ is independent of t, as desired. $\square$

Exercises.

1.8.1. Let M be a $\Bbbk$-module. In this exercise, you will show that if $i > 0$, then $\mathbf{H}^i([0,1]; M) = 0$. Let $s \in \mathbf{H}^i([0,1]; M)$.

(a) Show that for any point $x \in [0,1]$, there is an interval $[a,b] \subset [0,1]$ containing x such that the image of s under the restriction map $\mathbf{H}^i([0,1]; M) \to \mathbf{H}^i([a,b]; M)$ is 0. (*Hint*: Use Lemma 1.2.10.)

(b) Show that there are numbers $0 = a_0 < a_1 < \cdots < a_k = 1$ such that the image of s under $\mathbf{H}^i([0,1]; M) \to \mathbf{H}^i([a_j, a_{j+1}]; M)$ is 0 for all j.

(c) Show that the image of s under $\mathbf{H}^i([0,1]; M) \to \mathbf{H}^i([0, a_j]; M)$ is 0 for all j. (*Hint*: Use a Mayer–Vietoris sequence and an induction on j.)

Conclude that $s = 0$, and hence that $\mathbf{H}^i([0,1]; M) = 0$.

1.8.2. Using Exercise 1.8.1, show that $\mathbf{H}^i_{\mathrm{c}}([0,1); M) = 0$ for all i, and that

$$\mathbf{H}^i_{\mathrm{c}}((0,1); M) \cong \begin{cases} M & \text{if } i = 1, \\ 0 & \text{otherwise.} \end{cases}$$

1.8.3. Using the preceding exercises and the Künneth formula, prove the following well-known fact from algebraic topology (see, for instance, [**98**, Example 3.34]):

$$\mathbf{H}^i_{\mathrm{c}}(\mathbb{R}^n; \Bbbk) \cong \begin{cases} \Bbbk & \text{if } i = n, \\ 0 & \text{otherwise.} \end{cases}$$

1.9. More base change theorems

In this final section, we consider a few special cases in which the second base change map from Lemma 1.2.12 is an isomorphism.

Base change for products. The first case we consider is that of a cartesian square in which the horizontal maps are projection maps from a product. (See Proposition 1.9.2 for the statement.) We first need a lemma about pullbacks along such projection maps.

LEMMA 1.9.1. *Let $\mathrm{pr}_1 : X \times Y \to X$ be the projection map onto the first factor, and let $\mathcal{F} \in \mathrm{Sh}(X, \Bbbk)$. For any open set $U \subset X$ and any connected open set $V \subset Y$, we have* $(\mathrm{pr}_1^*\mathcal{F})(U \times V) \cong \mathcal{F}(U)$.

PROOF. Since pr_1 is an open map, the presheaf pullback $\mathrm{pr}_{1,\mathrm{pre}}^*(\mathcal{F})$ is particularly easy to describe: for any open set $W \subset X \times Y$, we have

$$\mathrm{pr}_{1,\mathrm{pre}}^*(\mathcal{F})(W) = \mathcal{F}(\mathrm{pr}_1(W)).$$

The germ at $(x, y) \in W$ of any section $s \in \mathrm{pr}^*_{1,\mathrm{pre}}(\mathcal{F})(W)$ can be identified with the germ of the corresponding section of $\mathcal{F}(\mathrm{pr}_1(W))$ at x under the isomorphisms

$$(1.9.1) \qquad (\mathrm{pr}^*_{1,\mathrm{pre}}(\mathcal{F}))_{(x,y)} \cong (\mathrm{pr}_1^*\mathcal{F})_{(x,y)} \cong \mathcal{F}_x$$

from (1.2.5). In particular, every nonzero section of $\mathrm{pr}^*_{1,\mathrm{pre}}(\mathcal{F})$ has a nonzero germ. Since the sheafification map $\mathrm{pr}^*_{1,\mathrm{pre}}(\mathcal{F}) \to \mathrm{pr}_1^*\mathcal{F}$ induces an isomorphism on stalks, it cannot kill any section of $\mathrm{pr}^*_{1,\mathrm{pre}}(\mathcal{F})$. In other words, we have a natural injective map

$$(1.9.2) \qquad \mathcal{F}(U) = \mathrm{pr}^*_{1,\mathrm{pre}}(\mathcal{F})(U \times V) \hookrightarrow (\mathrm{pr}_1^*\mathcal{F})(U \times V).$$

Our aim is to show that this map is an isomorphism.

Given a section $s \in (\mathrm{pr}_1^*\mathcal{F})(U \times V)$ and a point $x \in U$, we define a function $p_{s,x} : V \to \mathcal{F}_x$ as follows: $p_{s,x}(y)$ is the image of the germ $s_{(x,y)} \in (\mathrm{pr}_1^*\mathcal{F})_{(x,y)}$ under the isomorphism (1.9.1).

We claim that $p_{s,x}$ is locally constant. Let $y \in V$. Recall from the construction of the sheafification that the germs of s are given locally near (x, y) by a section of $\mathrm{pr}^*_{1,\mathrm{pre}}\mathcal{F}$. That is, there is a small open set $U' \times V' \subset U \times V$ containing (x, y) and a section $t \in (\mathrm{pre}^*_{1,\mathrm{pre}}(\mathcal{F}))(U' \times V') = \mathcal{F}(U')$ such that $s_{(x',y')} = t_{(x',y')}$ for all $(x', y') \in U' \times V'$. Regarding t as a section of $\mathcal{F}$, this equation becomes $s_{(x',y')} = t_{x'}$. In particular, $p_{s,x}(y') = t_x$ for all $y' \in V'$. Thus, $p_{s,x}$ is locally constant. Since V is connected, $p_{s,x}$ is in fact constant.

Choose an open covering $(W_\alpha)_{\alpha \in I}$ of $U \times V$ such that for each α, $s|_{W_\alpha}$ is given by a section $t_\alpha \in (\mathrm{pr}^*_{1,\mathrm{pre}}\mathcal{F}(W)) = \mathcal{F}(\mathrm{pr}_1(W_\alpha))$. The open sets $(\mathrm{pr}_1(W_\alpha))_{\alpha \in I}$ form an open covering of U. We claim that for all α and β, we have $t_\alpha|_{\mathrm{pr}_1(W_\alpha) \cap \mathrm{pr}_1(W_\beta)} = t_\beta|_{\mathrm{pr}_1(W_\alpha) \cap \mathrm{pr}_1(W_\beta)}$. It is enough to compare the germs of these two sections at any point $x \in \mathrm{pr}_1(W_\alpha) \cap \mathrm{pr}_1(W_\beta)$. They are given by

$$\begin{aligned} (t_\alpha)_x &= p_{s,x}(y_1) && \text{for any } y_1 \in V \text{ such that } (x, y_1) \in W_\alpha, \\ (t_\beta)_x &= p_{s,x}(y_2) && \text{for any } y_2 \in V \text{ such that } (x, y_2) \in W_\beta. \end{aligned}$$

Since $p_{s,x}$ is constant, these germs are equal. We conclude that the collection of sections $(t_\alpha)_{\alpha \in I}$ can be glued to give a section $t \in \mathcal{F}(U)$. By construction, the corresponding section in $\mathrm{pr}^*_{1,\mathrm{pre}}(\mathcal{F})(U \times V)$ has the same germs as s at every point of $U \times V$. It follows that s lies in the image of the map (1.9.2), as desired. □

Proposition 1.9.2. *Let $f : X \to X'$ be a continuous map, and let Y be a locally contractible space. Let $f' = f \times \mathrm{id} : X \times Y \to X' \times Y$, and consider the cartesian square*

$$\begin{tikzcd} X \times Y \arrow[r, "\mathrm{pr}_1"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ X' \times Y \arrow[r, "\mathrm{pr}_1"] & X' \end{tikzcd}$$

For $\mathcal{F} \in D^+(X, \Bbbk)$, there is a natural isomorphism $\mathrm{pr}_1^* f_*\mathcal{F} \xrightarrow{\sim} f'_*\mathrm{pr}_1^*\mathcal{F}$.

Another way to express the conclusion of this proposition is that there is a natural isomorphism $f_*\mathcal{F} \boxtimes \underline{\Bbbk}_Y \cong (f \times \mathrm{id}_Y)_*(\mathcal{F} \boxtimes \underline{\Bbbk}_Y)$.

Proof. *Step 1. Proof of the abelian category version.* Let $U \subset X'$ be an open set, and let $V \subset Y$ be a connected open set. By Lemma 1.9.1, for $\mathcal{F} \in \mathrm{Sh}(X, \Bbbk)$, we have

$$(\mathrm{pr}_1^* {}^\circ f_* \mathcal{F})(U \times V) \cong ({}^\circ f_* \mathcal{F})(U) = \mathcal{F}(f^{-1}(U)),$$
$$({}^\circ f'_* \mathrm{pr}_1^* \mathcal{F})(U \times V) \cong (\mathrm{pr}_1^* \mathcal{F})(f^{-1}(U) \times V) \cong \mathcal{F}(f^{-1}(U)).$$

This calculation shows that the map $\mathrm{pr}_1^* {}^\circ f_* \mathcal{F} \to {}^\circ f'_* \mathrm{pr}_1^* \mathcal{F}$ that was defined in Lemma 1.2.12 induces an isomorphism of sections over any open set of the form $U \times V \subset X \times Y$ with V connected. Since every point in $X \times Y$ has a basis of neighborhoods of this form, our map induces an isomorphism of stalks at every point, so it is an isomorphism.

Step 2. Sheaves on X_{disc}. Let X_{disc} be the set X equipped with the discrete topology. Let $i : X_{\mathrm{disc}} \to X$ be the obvious map, and let $i' = i \times \mathrm{id} : X_{\mathrm{disc}} \times Y \to X \times Y$. We have a cartesian square

$$\begin{array}{ccc} X_{\mathrm{disc}} \times Y & \xrightarrow{\mathrm{pr}_1} & X_{\mathrm{disc}} \\ {\scriptstyle i'}\downarrow & & \downarrow{\scriptstyle i} \\ X \times Y & \xrightarrow{\mathrm{pr}_1} & X \end{array}$$

Observe that as a special case of Step 1, we have a natural isomorphism $\mathrm{pr}_1^* {}^\circ i_* \mathcal{F} \xrightarrow{\sim} {}^\circ i'_* \mathrm{pr}_1^* \mathcal{F}$ for any $\mathcal{F} \in \mathrm{Sh}(X_{\mathrm{disc}}, \Bbbk)$.

Step 3. Locally constant sheaves on $X_{\mathrm{disc}} \times Y$ are acyclic for ${}^\circ(f' \circ i')_*$. Let $\mathcal{F}$ be a locally constant sheaf on $X_{\mathrm{disc}} \times Y$, and choose a flabby resolution $0 \to \mathcal{F} \to \mathcal{I}^0 \to \mathcal{I}^1 \to \cdots$. We wish to prove that $\mathsf{H}^k((f' \circ i')_* \mathcal{F}) = 0$ for $k > 0$, or, equivalently, that the sequence

$$0 \to {}^\circ(f' \circ i')_* \mathcal{F} \to {}^\circ(f' \circ i')_* \mathcal{I}^0 \to {}^\circ(f' \circ i')_* \mathcal{I}^1 \to \cdots$$

is exact. It is enough to show exactness after taking sections over a small open set $U' \times V$, where $U' \subset X'$ is open and $V \subset Y$ is a contractible open set. Let $U = f^{-1}(U') \subset X$, and let $U_{\mathrm{disc}} = i^{-1}(U)$. It is enough to show that the sequence

$$0 \to \mathcal{F}(U_{\mathrm{disc}} \times V) \to \mathcal{I}^0(U_{\mathrm{disc}} \times V) \to \mathcal{I}^1(U_{\mathrm{disc}} \times V) \to \cdots$$

is exact. Now, $U_{\mathrm{disc}} \times V$ is the disjoint union of open subsets of the form $\{x\} \times V$, as x ranges over points of U_{disc}, so the sequence above becomes

$$0 \to \prod_{x \in U_{\mathrm{disc}}} \mathcal{F}(\{x\} \times V) \to \prod_{x \in U_{\mathrm{disc}}} \mathcal{I}^0(\{x\} \times V) \to \prod_{x \in U_{\mathrm{disc}}} \mathcal{I}^1(\{x\} \times V) \to \cdots.$$

To check the exactness of this sequence, we can check it componentwise: it is enough to prove that for a fixed $x \in U_{\mathrm{disc}}$, the sequence

$$0 \to \mathcal{F}(\{x\} \times V) \to \mathcal{I}^0(\{x\} \times V) \to \mathcal{I}^1(\{x\} \times V) \to \cdots \tag{1.9.3}$$

is exact. This sequence, in turn, is obtained by applying Γ to the exact sequence

$$0 \to \mathcal{F}|_{\{x\} \times V} \to \mathcal{I}^0|_{\{x\} \times V} \to \mathcal{I}^1|_{\{x\} \times V} \to \cdots.$$

This is an injective resolution of the locally constant sheaf $\mathcal{F}|_{\{x\} \times V}$. Because V is contractible and locally contractible, $\mathcal{F}|_{\{x\} \times V}$ is in fact a constant sheaf, and then Corollary 1.8.5 tells us that it is acyclic for Γ. In other words, (1.9.3) is exact.

Step 4. Locally constant sheaves on $X_{\mathrm{disc}} \times Y$ are acyclic for ${}^\circ i'_*$. This is just the result of Step 3 in the special case where $X' = X$ and f is the identity map.

Step 5. Pseudoflabby sheaves on $X \times Y$. Let us say that a sheaf $\mathcal{F}$ on $X \times Y$ is *pseudoflabby* if it is isomorphic to ${}^\circ i'_* \mathcal{G}$ for some locally constant sheaf $\mathcal{G}$ on $X_{\mathrm{disc}} \times Y$. We claim that pseudoflabby sheaves are acyclic for ${}^\circ f'_*$. Indeed, if $\mathcal{F} \cong {}^\circ i'_* \mathcal{G}$ is pseudoflabby (with $\mathcal{G}$ locally constant), then Step 4 tells us that $\mathcal{F} \cong i'_*\mathcal{G}$. Then, for $k > 0$, we have $\mathsf{H}^k(f'_*\mathcal{F}) \cong \mathsf{H}^k((f' \circ i')_*\mathcal{G}) = 0$ by Step 3, as desired.

Step 6. For any $\mathcal{F} \in D^+(X, \Bbbk)$, pr_1^* takes the Godement resolution of $\mathcal{F}$ to a pseudoflabby resolution of $\mathrm{pr}_1^*\mathcal{F}$. Let $\mathcal{F} \to \mathcal{I}$ be the Godement resolution of $\mathcal{F}$. Recall from Exercise 1.2.2 that each term $\mathcal{I}^i$ is defined by applying ${}^\circ i_* i^*$ to some other sheaf $\mathcal{H}$. By Step 2, we have

$$\mathrm{pr}_1^* {}^\circ i_* i^* \mathcal{H} \cong {}^\circ i'_* \mathrm{pr}_1^* i^* \mathcal{H}.$$

It is clear that $\mathrm{pr}_1^* : \mathrm{Sh}(X_{\mathrm{disc}}, \Bbbk) \to \mathrm{Sh}(X_{\mathrm{disc}} \times Y, \Bbbk)$ takes any sheaf to a locally constant sheaf on $X_{\mathrm{disc}} \times Y$. By definition, ${}^\circ i'_*$ takes the latter to a pseudoflabby sheaf on $X \times Y$. Thus, each $\mathrm{pr}_1^*\mathcal{I}^i$ is pseudoflabby.

Step 7. Conclusion of the proof. We can compute $\mathrm{pr}_1^* f_*\mathcal{F}$ by applying $\mathrm{pr}_1^* {}^\circ f_*$ to the Godement resolution of $\mathcal{F}$. Separately, Steps 5 and 6 imply that we can compute $f'_*\mathrm{pr}_1^*\mathcal{F}$ by applying ${}^\circ f'_*\mathrm{pr}_1^*$ to its Godement resolution. The claim then follows by Step 1. □

Base change for topological submersions. Proposition 1.9.2 applies, in particular, to projection maps of the form $\mathrm{pr}_1 : X \times \mathbb{R}^n \to X$. The following theorem generalizes this special case of Proposition 1.9.2.

THEOREM 1.9.3. *Let $g : Y' \to Y$ be a topological submersion, and suppose we have a cartesian square*

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

For any $\mathcal{F} \in D^+(X, \Bbbk)$, there is a natural isomorphism $g^* f_*\mathcal{F} \xrightarrow{\sim} f'_*(g')^*\mathcal{F}$.

PROOF. Let $p \in Y'$, and choose a neighborhood $U \subset Y'$ of p such that $g(U)$ is open in Y, and such that there is a homeomorphism $U \xrightarrow{\sim} g(U) \times \mathbb{R}^n$ as in Definition 1.5.10. It is enough to show that the base change map $g^* f_*\mathcal{F} \to f'_*(g')^*\mathcal{F}$ is an isomorphism after restriction to U. Let $V = g(U)$, and consider the cube

$$\begin{tikzcd} (f')^{-1}(U) \arrow[rr] \arrow[dd] \arrow[dr] & & f^{-1}(V) \arrow[dd] \arrow[dr] & \\ & X' \arrow[rr, "g'"] \arrow[dd, "f'"] & & X \arrow[dd, "f"] \\ U \arrow[rr] \arrow[dr] & & V = g(U) \arrow[dr] & \\ & Y' \arrow[rr, "g"] & & Y \end{tikzcd}$$

Here, all the vertical faces are cartesian. The base change maps for the left and right vertical faces are isomorphisms by Proposition 1.2.16. To prove the result for the front face, Proposition 1.6.3 lets us reduce the problem to studying the back face of the cube. By Definition 1.5.10, this face can be replaced by

$$\begin{tikzcd} f^{-1}(V) \times \mathbb{R}^n \arrow[r, "\mathrm{pr}_1"] \arrow[d] & f^{-1}(V) \arrow[d] \\ V \times \mathbb{R}^n \arrow[r, "\mathrm{pr}_1"] & V \end{tikzcd}$$

The base change map for this square is an isomorphism by Proposition 1.9.2. □

Base change for locally trivial fibrations. Recall that a map $f : X \to Y$ is said to be a **locally trivial fibration** (with fiber F) if every point $y \in Y$ has a neighborhood U such that there is a homeomorphism $b : f^{-1}(U) \to F \times U$ that makes the following diagram commute:

$$\begin{tikzcd} f^{-1}(U) \arrow[r, "b", "\sim"'] \arrow[d, "f|_{f^{-1}(U)}"'] & F \times U \arrow[dl, "\mathrm{pr}_2"] \\ U & \end{tikzcd} \tag{1.9.4}$$

In particular, $f^{-1}(y)$ is homeomorphic to F for all $y \in Y$.

Proposition 1.9.4. *Let $f : X \to X'$ be a continuous map of locally contractible spaces, and let Y be a contractible and locally contractible space. Let $y_0 \in Y$. Let $f' = f \times \mathrm{id} : X \times Y \to X' \times Y$, and consider the cartesian square*

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "f"'] & X \times Y \arrow[d, "f'"] \\ X' \arrow[r, "i'"] & X' \times Y \end{tikzcd}$$

where i and i' are defined by $x \mapsto (x, y_0)$. For $\mathcal{F} \in D^+_{\mathrm{loc}}(X \times Y, \Bbbk)$, there is a natural isomorphism $(i')^* f'_* \mathcal{F} \xrightarrow{\sim} f_* i^* \mathcal{F}$.

Proof. Consider the larger diagram

$$\begin{tikzcd} X \arrow[r, "i"'] \arrow[d, "f"'] \arrow[rr, bend left, "\mathrm{id}"] & X \times Y \arrow[r, "\mathrm{pr}_1"'] \arrow[d, "f'"] & X \arrow[d, "f"] \\ X' \arrow[r, "i'"] \arrow[rr, bend right, "\mathrm{id}"'] & X' \times Y \arrow[r, "\mathrm{pr}_1"] & X' \end{tikzcd}$$

Our assumptions imply that $\mathrm{pr}_1 : X \times Y \to X$ is a homotopy equivalence, so by Theorem 1.8.10, there is an object $\mathcal{G} \in D^+_{\mathrm{loc}}(X, \Bbbk)$ such that $\mathcal{F} \cong \mathrm{pr}_1^* \mathcal{G}$. Now consider the following sequence of base change maps:

$$\begin{tikzcd} \mathrm{id}_X^* f_* \mathcal{G} \cong (i')^* \mathrm{pr}_1^* f_* \mathcal{G} \arrow[r, "\sim"] \arrow[rr, bend left, "\sim"] & (i')^* f'_*(\mathrm{pr}_1^* \mathcal{G}) \arrow[r] & f_* i^* (\mathrm{pr}_1^* \mathcal{G}) \cong f_* \mathrm{id}_X^* \mathcal{G} \end{tikzcd}$$

The first map is an isomorphism by Proposition 1.9.2, and the composition is clearly an isomorphism because it comes from base change along an identity map, so the second map is an isomorphism as well. $\square$

Theorem 1.9.5. *Let $f : X \to Y$ be a locally trivial fibration of locally contractible spaces.*

(1) *For $\mathcal{F} \in D^+_{\mathrm{loc}}(X, \Bbbk)$, we have $f_* \mathcal{F} \in D^+_{\mathrm{loc}}(Y, \Bbbk)$.*
(2) *Suppose we have a cartesian square of locally contractible spaces:*

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

For any $\mathcal{F} \in D^+_{\mathrm{loc}}(X)$, there is a natural isomorphism $g^* f_* \mathcal{F} \xrightarrow{\sim} f'_*(g')^*\mathcal{F}$.

Proof. Let $y \in Y$, and let U be a contractible neighborhood of y such that $f^{-1}(U)$ is homeomorphic to $f^{-1}(y) \times U$, as in (1.9.4). In the following diagram of cartesian squares, the maps along the bottom are inclusion maps:

$$\begin{tikzcd} f^{-1}(y) \arrow[r, "i'"] \arrow[d] & f^{-1}(y) \times U \arrow[r, "j'"] \arrow[d, "\mathrm{pr}_2"'] & X \arrow[d, "f"] \\ \{y\} \arrow[r, "i"] & U \arrow[r, "j"] & Y \end{tikzcd}$$

The base change map for the right-hand square is an isomorphism by Proposition 1.2.16, and the base change map for the left-hand square is an isomorphism by Proposition 1.9.4. Combining these, we see that the base change map

$$(f_*\mathcal{F})_y \xrightarrow{\sim} R\Gamma(\mathcal{F}|_{f^{-1}(y)})$$

is an isomorphism. This is a special case of part (2) of the theorem. The general case then follows by the same reasoning as in the proof of Theorem 1.2.13.

Next, since U is contractible, the map i' above is a homotopy equivalence, and $\mathrm{pr}_1 : f^{-1}(y) \times U \to f^{-1}(y)$ is a homotopy inverse. By Theorem 1.8.10, the object $(j')^*\mathcal{F} \in D^+_{\mathrm{loc}}(f^{-1}(y) \times U, \Bbbk)$ must be isomorphic to $\mathrm{pr}_1^*(i')^*(j')^*\mathcal{F} \cong \mathrm{pr}_1^*(\mathcal{F}|_{f^{-1}(y)})$. Consider the cartesian square

$$\begin{tikzcd} f^{-1}(y) \times U \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\mathrm{pr}_2"'] & f^{-1}(y) \arrow[d, "\mathrm{a}_{f^{-1}(y)}"] \\ U \arrow[r, "\mathrm{a}_U"] & \mathrm{pt} \end{tikzcd}$$

Using Proposition 1.9.2, we find that

$$(f_*\mathcal{F})|_U \cong \mathrm{pr}_{2*}(j')^*\mathcal{F} \cong \mathrm{pr}_{2*}\mathrm{pr}_1^*(\mathcal{F}|_{f^{-1}(y)}) \cong \mathrm{a}_U^* R\Gamma(\mathcal{F}|_{f^{-1}(y)}).$$

We see that $(f_*\mathcal{F})|_U$ has constant cohomology sheaves, so $f_*\mathcal{F}$ has locally constant cohomology sheaves. We have now proved part (1). □

Corollary 1.9.6. *Let $f : X \to Y$ be a covering map of locally contractible spaces. For any $\mathcal{L} \in \mathrm{Loc}(X, \Bbbk)$, we have $\mathsf{H}^i(f_*\mathcal{L}) = 0$ for $i > 0$, and $\mathsf{H}^0(f_*\mathcal{L}) \in \mathrm{Loc}(Y, \Bbbk)$. As a consequence, f_* restricts to a functor $D^{\mathrm{b}}_{\mathrm{loc}}(X, \Bbbk) \to D^{\mathrm{b}}_{\mathrm{loc}}(Y, \Bbbk)$.*

Proof. To prove that $\mathsf{H}^i(f_*\mathcal{L}) = 0$ for $i > 0$, it is enough to show that the stalks $\mathsf{H}^i(f_*\mathcal{L})_y$ vanish for $i > 0$. By Theorem 1.9.5, we have $\mathsf{H}^i(f_*\mathcal{L})_y \cong \mathsf{H}^i(R\Gamma(\mathcal{L}|_{f^{-1}(y)}))$. Since $f^{-1}(y)$ is a discrete space, this vanishes for $i > 0$ by Example 1.1.16. □

Sheaves on aspherical spaces. Recall that an **aspherical space** is a topological space that has a contractible universal cover. Such a space has the property that its higher homotopy groups $\pi_n(X, x_0)$ are all trivial for $n \geq 2$. We conclude this section with the following homological result about aspherical spaces.

Theorem 1.9.7. *Let X be a connected, locally contractible, aspherical space. Then the inclusion functor $\mathrm{Loc}(X, \Bbbk) \hookrightarrow \mathrm{Sh}(X, \Bbbk)$ induces an equivalence of categories $D^{\mathrm{b}}\mathrm{Loc}(X, \Bbbk) \to D^{\mathrm{b}}_{\mathrm{loc}}(X, \Bbbk)$.*

Proof. We will use the criterion from Corollary A.7.19. Let $\mathcal{L}, \mathcal{M} \in \mathrm{Loc}(X, \Bbbk)$, and let $n > 0$. We must show that every morphism $f : \mathcal{L} \to \mathcal{M}[n]$ in $D^{\mathrm{b}}(X, \Bbbk)$ is effaceable, i.e., that one can find a surjective map $g : \mathcal{L}' \to \mathcal{L}$ and an injective map $h : \mathcal{M} \to \mathcal{M}'$ in $\mathrm{Loc}(X, \Bbbk)$ such that $h[n] \circ f \circ g = 0$.

Choose a point $x_0 \in X$, and consider the $\Bbbk[\pi_1(X, x_0)]$-module $\mathrm{Mon}_{x_0}(\mathcal{L})$. This is a quotient of some free $\Bbbk[\pi_1(X, x_0)]$-module V. Let $\mathcal{L}' \in \mathrm{Loc}(X, \Bbbk)$ be the local system corresponding to V under the equivalence of Theorem 1.7.9, and let $g : \mathcal{L}' \to \mathcal{L}$ be the quotient map. Note that V is also free as a $\Bbbk$-module, so the stalks of $\mathcal{L}'$ are free $\Bbbk$-modules.

Next, let $\pi : \tilde{X} \to X$ be the universal covering. By Corollary 1.9.6, the object $\pi_*\pi^*\mathcal{M}$ is a local system on X. Let h be the adjunction map $\mathcal{M} \to \pi_*\pi^*\mathcal{M}$. The description given in Remark 1.7.17 shows that h is an injective map of local systems. We must show the vanishing of $h[n] \circ f \circ g$, which is an element of

$$\mathrm{Hom}(\mathcal{L}', \pi_*\pi^*\mathcal{M}[n]) \cong \mathrm{Hom}(\pi^*\mathcal{L}', \pi^*\mathcal{M}[n]).$$

Both $\pi^*\mathcal{L}'$ and $\pi^*\mathcal{M}$ are local systems on $\tilde{X}$. But since $\tilde{X}$ is simply connected, they are both constant sheaves: there are $\Bbbk$-modules M and N such that $\pi^*\mathcal{L}' \cong \underline{M}_{\tilde{X}}$ and $\pi^*\mathcal{M} \cong \underline{N}_{\tilde{X}}$. Moreover, because the stalks of $\mathcal{L}'$ are free $\Bbbk$-modules, the same holds for $\pi^*\mathcal{L}'$; thus, M is a free $\Bbbk$-module. We have

$$\begin{aligned}\mathrm{Hom}(\pi^*\mathcal{L}', \pi^*\mathcal{M}[n]) &\cong \mathrm{Hom}(\underline{M}_{\tilde{X}}, \underline{N}_{\tilde{X}}[n]) \cong \mathrm{Hom}(\mathrm{a}_{\tilde{X}}^*\underline{M}_{\mathrm{pt}}, \underline{N}_{\tilde{X}}[n]) \\ &\cong \mathrm{Hom}(\underline{M}_{\mathrm{pt}}, \mathrm{a}_{\tilde{X}*}\underline{N}_{\tilde{X}}[n]) \cong \mathrm{Hom}_{D^{\mathrm{b}}(\Bbbk\text{-mod})}(M, R\Gamma(\underline{N}_{\tilde{X}})[n]).\end{aligned}$$

Now, Corollary 1.8.5 tells us that $R\Gamma(\underline{N}_{\tilde{X}}) \cong N$, so the last computation reduces to $\mathrm{Hom}_{D^{\mathrm{b}}(\Bbbk\text{-mod})}(M, N[n]) \cong \mathrm{Ext}^n_{\Bbbk}(M, N)$. Since M is a free $\Bbbk$-module, this Ext-group vanishes. We conclude that $h[n] \circ f \circ g = 0$, as desired. $\square$

Remark 1.9.8. Suppose X satisfies the assumptions of Theorem 1.9.7. Combining that result with Theorem 1.7.9, we obtain an equivalence of categories

$$\mathrm{Mon}_{x_0} : D^{\mathrm{b}}_{\mathrm{loc}}(X, \Bbbk) \xrightarrow{\sim} D^{\mathrm{b}}(\Bbbk[\pi_1(X, x_0)]\text{-mod}).$$

In this setting, a number of results from Section 1.7 can be upgraded to the derived level. For instance, if $\Bbbk$ has finite global dimension, then Proposition 1.7.11 together with the equivalence above implies that for $\mathcal{F}, \mathcal{G} \in D^{\mathrm{b}}_{\mathrm{loc}}(X, \Bbbk)$, we have

$$\mathrm{Mon}_{x_0}(\mathcal{F}) \overset{L}{\otimes} \mathrm{Mon}_{x_0}(\mathcal{G}) \cong \mathrm{Mon}_{x_0}(\mathcal{F} \overset{L}{\otimes} \mathcal{G}),$$
$$\mathrm{Mon}_{x_0}(R\mathscr{H}om(\mathcal{F}, \mathcal{G})) \cong R\mathrm{Hom}_{\Bbbk}(\mathrm{Mon}_{x_0}(\mathcal{F}), \mathrm{Mon}_{x_0}(\mathcal{G})).$$

1.10. Additional notes and exercises

Notes. Sheaf theory originated in the work of Leray in the 1940s. Following contributions by Cartan, Serre, Borel, and others, the foundations largely achieved their present form in a monograph of Godement [**84**], which covers most of the sheaf operations discussed in this chapter, makes use of flabby and c-soft resolutions, and contains statements equivalent to the proper base change theorem and the projection formula, among others. The latter were reformulated in terms of $f_!$ by Verdier [**238**], who also established the existence and basic properties of its right adjoint $f^!$. The concept of a local system was introduced by Steenrod [**232**]. For a comprehensive account of the early decades of sheaf theory, see [**107**].

The exercises below involve a number of examples of objects $\mathcal{F} \in D^{\mathrm{b}}(X, \Bbbk)$ with the following property: the space X can be written as a disjoint union of locally closed subsets $X = X_1 \cup \cdots \cup X_k$ such that each $\mathsf{H}^i(\mathcal{F})|_{X_j}$ is a local system. (These are examples of **constructible complexes**, to be defined in Chapter 2.) It is convenient to record this information in a **table of stalks**:

$$\mathcal{F}: \quad \begin{array}{c|c|c|c|c|c} & X_1 & \cdots & X_j & \cdots & X_k \\ \hline \vdots & \vdots & & \vdots & & \vdots \\ \hline i & \mathsf{H}^i(\mathcal{F})|_{X_1} & \cdots & \mathsf{H}^i(\mathcal{F})|_{X_j} & \cdots & \mathsf{H}^i(\mathcal{F})|_{X_k} \\ \hline \vdots & \vdots & & \vdots & & \vdots \end{array}$$

In the examples we consider, the subsets X_j are all smooth complex algebraic varieties. Typically, we will list them in order of decreasing dimension.

In general, of course, the object $\mathcal{F}$ cannot be recovered from its table of stalks. Nevertheless, in these problems, the phrases "compute $\mathcal{F}$" or "determine $\mathcal{F}$" should be understood to mean "compute its table of stalks."

EXERCISE 1.10.1. Let $j : \mathbb{C}^\times \hookrightarrow \mathbb{C}$ be the inclusion map. Use Lemma 1.2.10 to show that $(j_*\underline{\Bbbk}_{\mathbb{C}^\times})|_0 \cong R\Gamma(\underline{\Bbbk}_{\mathbb{C}^\times})$. Thus, its table of stalks is given by

$$j_*\underline{\Bbbk}_{\mathbb{C}^\times}: \quad \begin{array}{c|c|c} & \mathbb{C}^\times & \{0\} \\ \hline 1 & & \underline{\Bbbk} \\ \hline 0 & \underline{\Bbbk} & \underline{\Bbbk} \end{array}$$

EXERCISE 1.10.2. Let $j : \mathbb{C}^\times \hookrightarrow \mathbb{C}$ be the inclusion map. Let $f : \mathbb{C} \to \mathbb{C}$ be the map given by $f(z) = z^2$, and let $g = f|_{\mathbb{C}^\times} : \mathbb{C}^\times \to \mathbb{C}^\times$. Assume that $\Bbbk$ is a field whose characteristic is not 2.

(a) Show that $g_*\underline{\Bbbk}_{\mathbb{C}^\times}$ is isomorphic to $\underline{\Bbbk}_{\mathbb{C}^\times} \oplus \mathcal{Q}$, where $\mathcal{Q}$ is a certain nontrivial local system that is locally free of rank 1. (If $\Bbbk = \mathbb{C}$, the sheaf $\mathcal{Q}$ is the "square-root sheaf" from Exercise 1.1.8.)

(b) Use the commutative diagram

$$\begin{tikzcd} \mathbb{C}^\times \arrow[r, "j"] \arrow[d, "g"'] & \mathbb{C} \arrow[d, "f"] & \{0\} \arrow[l] \arrow[d, equal] \\ \mathbb{C}^\times \arrow[r, "j"] & \mathbb{C} & \{0\} \arrow[l] \end{tikzcd}$$

to show that $f_*\underline{\Bbbk}_{\mathbb{C}^\times}$ is given by

$$f_*\underline{\Bbbk}_{\mathbb{C}^\times}: \quad \begin{array}{c|c|c} & \mathbb{C}^\times & \{0\} \\ \hline 0 & \underline{\Bbbk}_{\mathbb{C}^\times} \oplus \mathcal{Q} & \underline{\Bbbk} \end{array}$$

Deduce from this that $j_!\mathcal{Q} \cong j_*\mathcal{Q}$. In other words, $j_*\mathcal{Q}$ is given by

$$j_*\mathcal{Q}: \quad \begin{array}{c|c|c} & \mathbb{C}^\times & \{0\} \\ \hline 0 & \mathcal{Q} & 0 \end{array}$$

EXERCISE 1.10.3. This problem generalizes the preceding two problems. Let $j : \mathbb{C}^\times \hookrightarrow \mathbb{C}$ be the inclusion map. Let $\mathcal{L}$ be a local system on $\mathbb{C}^\times$. Identify $\Bbbk[\pi_1(\mathbb{C}^\times, 1)]$ with the ring $R = \Bbbk[T, T^{-1}]$, and suppose $\mathcal{L}$ corresponds to the R-module M via Theorem 1.7.9. Show that there is a natural isomorphism

$$(j_*\mathcal{L})_0 \cong R\mathrm{Hom}_R(\Bbbk, M).$$

Then show that $j_*\mathcal{L}$ is given by

$$j_*\mathcal{L}: \quad \begin{array}{c|c|c} & \mathbb{C}^\times & \{0\} \\ \hline 1 & & M_T \\ \hline 0 & \mathcal{L} & M^T \end{array}$$

where M^T and M_T are the T-invariants and T-coinvariants, respectively:

$$M^T = \{m \in M \mid T \cdot m = m\}, \qquad M_T = M/(m - T \cdot m : m \in M).$$

As a special case, if $\Bbbk$ is a field and $\mathcal{L}$ is a nontrivial irreducible local system, then we obtain the following table of stalks, which shows that $j_*\mathcal{L} \cong j_!\mathcal{L}$:

$$j_*\mathcal{L}: \quad \begin{array}{c|c|c} & \mathbb{C}^\times & \{0\} \\ \hline 0 & \mathcal{L} & 0 \end{array}$$

For a generalization of Exercise 1.10.3 to $(\mathbb{C}^\times)^n$, see Proposition B.2.3.

Remark. Part of Exercise 1.10.3 involves computing $R\mathrm{Hom}_R(\Bbbk, M)$. This can be done using the following projective resolution of $\Bbbk$ as an R-module:

$$0 \to R \xrightarrow{1 \mapsto 1-T} R \xrightarrow{T \mapsto 1} \Bbbk \to 0.$$

This resolution involves a choice: namely, the choice of generator $1 - T$ for the kernel of the map $\epsilon : R \to \Bbbk$. Let us try to rewrite this calculation in a more "intrinsic" way, without making such a choice. Note that the ideal $\ker \epsilon \subset R$ is a free R-module of rank 1, and $\ker \epsilon/(\ker \epsilon)^2$ is a free $\Bbbk$-module of rank 1. We call this module the **Tate module**, and we introduce the new notation

$$\Bbbk(1) = \ker \epsilon/(\ker \epsilon)^2.$$

For more on the Tate module, see Section 2.2. For any $\Bbbk$-module V, we put

$$V(1) = \Bbbk(1) \otimes V \qquad \text{and} \qquad V(-1) = \mathrm{Hom}_\Bbbk(\Bbbk(1), V).$$

Now, $\mathrm{Ext}^1_R(\Bbbk, M)$ is canonically isomorphic to the cokernel of the natural map $\mathrm{Hom}_R(R, M) \to \mathrm{Hom}_R(\ker \epsilon, M)$. One can check that the sequence

$$\mathrm{Hom}_R(R, M) \to \mathrm{Hom}_R(\ker \epsilon, M) \to \mathrm{Hom}_R(\ker \epsilon, M_T) \to 0$$

is exact, and that there is a natural isomorphism $M_T(-1) \xrightarrow{\sim} \mathrm{Hom}_R(\ker \epsilon, M_T)$. Thus, a "better" answer to Exercise 1.10.3 is

$$j_*\mathcal{L}: \quad \begin{array}{c|c|c} & \mathbb{C}^\times & \{0\} \\ \hline 1 & & M_T(-1) \\ \hline 0 & \mathcal{L} & M^T \end{array}$$

As a special case, a solution to Exercise 1.10.1 that avoids the choice of a generator for $\mathbf{H}^1(\mathbb{C}^\times; \Bbbk)$ is

$$j_*\underline{\Bbbk}_{\mathbb{C}^\times}: \quad \begin{array}{c|c|c} & \mathbb{C}^\times & \{0\} \\ \hline 1 & & \Bbbk(-1) \\ \hline 0 & \underline{\Bbbk} & \Bbbk \end{array}$$

Exercise 1.10.4. Let $\mathcal{N}$ be the space of 2×2 complex matrices with trace 0 and determinant 0:

$$\mathcal{N} = \left\{ \begin{bmatrix} x & y \\ z & -x \end{bmatrix} \middle| \, x^2 + yz = 0 \right\}.$$

In other words, $\mathcal{N}$ is the set of nilpotent 2×2 matrices. Let $U = \mathcal{N} \smallsetminus \{0\}$, and let $j : U \hookrightarrow \mathcal{N}$ be the inclusion map.

(a) Consider the map $p : \mathbb{C}^2 \to \mathcal{N}$ given by $p(x,y) = \left[\begin{smallmatrix} xy & x^2 \\ -y^2 & -xy \end{smallmatrix}\right]$. Show that p restricts to a covering map $\mathbb{C}^2 \smallsetminus \{0\} \to U$. Deduce that for any point $u_0 \in U$, we have $\pi_1(U, u_0) \cong \mathbb{Z}/2\mathbb{Z}$.

(b) Compute $j_*\underline{\Bbbk}_U$ when $\Bbbk = \mathbb{Z}$ and when $\Bbbk$ is a field. *Hint*: Use the map p to show that U is homotopy-equivalent to $\mathbb{RP}^3$. The answers are as follows (you may ignore the Tate twists for now):

$j_*\underline{\mathbb{Z}}_U$:

	U	$\{0\}$
3		$\underline{\mathbb{Z}}(-2)$
2		$\underline{\mathbb{Z}/2\mathbb{Z}}(-1)$
1		
0	$\underline{\mathbb{Z}}$	$\underline{\mathbb{Z}}$

When $\Bbbk$ is a field, we have

$j_*\underline{\Bbbk}_U$, $\operatorname{char} \Bbbk = 2$:

	U	$\{0\}$
3		$\underline{\Bbbk}(-2)$
2		$\underline{\Bbbk}(-1)$
1		$\underline{\Bbbk}(-1)$
0	$\underline{\Bbbk}$	$\underline{\Bbbk}$

$j_*\underline{\Bbbk}_U$, $\operatorname{char} \Bbbk \neq 2$:

	U	$\{0\}$
3		$\underline{\Bbbk}(-2)$
2		
1		
0	$\underline{\Bbbk}$	$\underline{\Bbbk}$

EXERCISE 1.10.5. This question involves the complex projective line $\mathbb{P}^1$. We indicate points on $\mathbb{P}^1$ in homogeneous coordinates $[u : v]$. Let

$$\tilde{\mathcal{N}} = \left\{ \left(\begin{bmatrix} x & y \\ z & -x \end{bmatrix}, [u : v] \right) \in \mathcal{N} \times \mathbb{P}^1 \,\middle|\, \begin{bmatrix} x & y \\ z & -x \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = 0 \right\} \subset \mathcal{N} \times \mathbb{P}^1.$$

Let $\mu : \tilde{\mathcal{N}} \to \mathcal{N}$ be the projection map onto the first factor.

(a) Compute $\mu_*\underline{\Bbbk}_{\tilde{\mathcal{N}}}$. *Answer*:

$\mu_*\underline{\Bbbk}_{\tilde{\mathcal{N}}}$:

	U	$\{0\}$
2		$\underline{\Bbbk}(-1)$
1		
0	$\underline{\Bbbk}$	$\underline{\Bbbk}$

(b) Let $i : \{0\} \hookrightarrow \mathcal{N}$ be the inclusion map. Compute $i^!\mu_*\underline{\Bbbk}$. Then compute the canonical map $i^!\mu_*\underline{\Bbbk} \to i^*\mu_*\underline{\Bbbk}$. *Answer*:

$i^!\mu_*\underline{\Bbbk}_{\tilde{\mathcal{N}}}$:

	$\{0\}$
4	$\underline{\Bbbk}(-2)$
3	
2	$\underline{\Bbbk}(-1)$

$i^!\mu_*\underline{\Bbbk}_{\tilde{\mathcal{N}}} \to i^*\mu_*\underline{\Bbbk}_{\tilde{\mathcal{N}}}$:

	$\{0\}$
4	$\underline{\Bbbk}(-2)$
3	
2	$\underline{\Bbbk}(-1) \xrightarrow{\times 2} \underline{\Bbbk}(-1)$
1	
0	$\underline{\Bbbk}$

(c) Part (a) shows that $\mathsf{H}^0(\mu_*\underline{\Bbbk}_{\tilde{\mathcal{N}}})$ has the same stalks as the constant sheaf $\underline{\Bbbk}_{\mathcal{N}}$. Show that, in fact, we have $\mathsf{H}^0(\mu_*\underline{\Bbbk}_{\tilde{\mathcal{N}}}) \cong \underline{\Bbbk}_{\mathcal{N}}$. Deduce that there is a truncation distinguished triangle

$$\underline{\Bbbk}_{\mathcal{N}} \to \mu_*\underline{\Bbbk}_{\tilde{\mathcal{N}}} \to i_*\underline{\Bbbk}_{\mathrm{pt}}(-1)[-2] \to .$$

(d) Show that this truncation distinguished triangle splits if and only if 2 is invertible in $\Bbbk$.

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CHAPTER 2

Constructible sheaves on complex algebraic varieties

Starting from this chapter, we will be mainly interested in sheaves on complex algebraic varieties, equipped with the analytic topology. Some minimal familiarity with concepts from basic algebraic geometry (say at the level of [**210**, **217**]) is helpful, but most of the definitions and results we need will be reviewed in Section 2.1.

In Section 2.3, we introduce the fundamental notion of **constructible** (complexes of) sheaves. From then on and for the rest of the book, almost all complexes of sheaves we consider will be constructible. The goal of this chapter is to establish a number of remarkable properties of constructible sheaves. Here are some of the highlights of this chapter:

- We establish vanishing bounds for the hypercohomology of a constructible sheaf (Section 2.6).
- We will show that in the algebraic setting, all sheaf functors send constructible complexes to constructible complexes (Section 2.7).
- We will show that for constructible complexes, Verdier duality is an involution (Section 2.8), and external tensor product and extension of scalars both commute with all sheaf operations (Section 2.9).

Once these fundamental results on constructible complexes have been established, we apply them to various constructions in Sections 2.10 and 2.11.

From this chapter on, the ring $\Bbbk$ should always be assumed to be noetherian and of finite global dimension.

2.1. Preliminaries from complex algebraic geometry

Let $\mathbb{A}^n$ and $\mathbb{P}^n$ denote complex affine and projective space, respectively, of (complex) dimension n. More generally, if V is a complex vector space, we denote by $\mathbb{P}(V)$ the variety of (complex) lines in V. Of course, $\mathbb{A}^n$ is just another notation for $\mathbb{C}^n$. We will use both notations in this book: generally speaking, we write $\mathbb{A}^n$ in algebro-geometric contexts, and $\mathbb{C}^n$ in nonalgebraic contexts.

Both $\mathbb{A}^n$ and $\mathbb{P}^n$ can be equipped with either the Zariski topology or the analytic (or "strong") topology. In this book, a **variety** will mean a quasiprojective complex algebraic variety, i.e., a subset of some projective space $\mathbb{P}^n$ that is locally closed in the Zariski topology. Thus, any variety comes with both a Zariski topology and an analytic topology. We emphasize that *sheaves will always be considered with respect to the analytic topology.* The analytic topology is much better suited to applying results from Chapter 1, in part thanks to Theorem 2.1.6 below. Nevertheless, from now on, the terms "open" and "closed" should generally be understood to refer to the Zariski topology; when the analytic topology is needed, we will usually explicitly say "analytic open set."

Of course, a variety is **affine** if it is isomorphic to a Zariski-closed subset of some affine space $\mathbb{A}^n$. The **dimension** of a variety should always be understood to be its algebraic or complex dimension. Smooth varieties are always assumed to be **equidimensional**, i.e., to have all connected components of the same dimension. The rest of this section collects additional definitions and theorems from algebraic geometry that we will need, mostly without proofs.

Topology of algebraic varieties. We begin by listing some essential topological facts about varieties.

LEMMA 2.1.1. *Let X be a variety, and let $Y \subset X$ be a locally closed subset in the Zariski topology. Its closure in the analytic topology is equal to its closure in the Zariski topology.*

See [**183**, Section I.10] for a proof. In particular, this lemma tells us that (Zariski-)dense open subvarieties remain dense in the analytic topology.

LEMMA 2.1.2. *Every variety can be covered by affine open subvarieties. In the Zariski topology, any variety is a noetherian topological space.*

For the first assertion, see, for instance, [**210**, Section I.4.2]. For the second part, recall that a **noetherian topological space** is a topological space in which descending chains of closed subsets are eventually constant. The fact that any variety is noetherian follows from the fact that $\mathbb{P}^n$ itself is noetherian; and this in turn follows from the definition of the Zariski topology and the fact that a polynomial ring in finitely many variables is a noetherian ring.

We sometimes use the second part of Lemma 2.1.2 to carry out proofs by **noetherian induction**. In this framework, to prove a statement $P(X)$ for all varieties X, one carries the following two steps:

- Prove $P(X)$ in the case where $X = \varnothing$ (or perhaps $X = \mathrm{pt}$).
- Prove that if $P(Z)$ is true for every proper closed subvariety $Z \subset X$, then $P(X)$ is true.

LEMMA 2.1.3. *Let X be a variety, and let $U_1, U_2 \subset X$ be two locally closed affine subvarieties. Then $U_1 \cap U_2$ is also an affine subvariety.*

PROOF. Consider the following cartesian square, where the right-hand vertical map is the diagonal map:

$$\begin{array}{ccc} U_1 \cap U_2 & \longrightarrow & X \\ \downarrow & & \downarrow \\ U_1 \times U_2 & \longrightarrow & X \times X \end{array}$$

Because the right-hand vertical map is a closed embedding, the left-hand vertical map is as well. In other words, $U_1 \cap U_2$ is identified with a closed subvariety of the affine variety $U_1 \times U_2$, so it is affine. □

THEOREM 2.1.4. *Every variety has a smooth dense open subvariety.*

See [**97**, Theorem I.5.3] or [**210**, Section II.2.2] for a proof.

THEOREM 2.1.5. *A variety is connected in the Zariski topology if and only if it is connected in the analytic topology.*

See [**211**, Section VII.2] for a proof.

THEOREM 2.1.6. *In the analytic topology, every variety is locally compact, locally contractible, and of finite c-soft dimension.*

Since a variety is a locally closed subset of a manifold (namely, $\mathbb{P}^n$), it is clearly locally compact, and it is of finite c-soft dimension by Proposition 1.5.3. It is well known that any variety is homeomorphic to a finite simplicial complex [**101**], and therefore locally contractible (see [**98**, Proposition A.4]).

Proper and finite morphisms. For morphisms of varieties, the definition of **properness** can be found in [**210**, Remark after Theorem 1.11] (see also, for instance, [**97**, Section II.4]). We will not need the details of the definition, thanks to the fact that a proper morphism of varieties is also a proper map of locally compact topological spaces in the sense of Chapter 1. This can be proved using Chow's lemma ([**91**, Théorème 5.6.1]); see [**183**, Theorem I.10.2] for an argument along these lines. We will freely apply results on proper maps from Chapter 1 to proper morphisms. The following important result will let us reduce some questions about arbitrary morphisms to the proper case.

THEOREM 2.1.7 (Nagata's compactification theorem). *Let $f : X \to Y$ be a morphism of varieties. There exists a variety $\tilde{X}$ and morphisms $j : X \to \tilde{X}$ and $\tilde{f} : \tilde{X} \to Y$ such that following conditions hold:*

(1) *We have $f = \tilde{f} \circ j$.*
(2) *The map j is an open embedding.*
(3) *The map $\tilde{f}$ is proper.*

See [**53, 163**] for proofs in a scheme-theoretic setting.

Recall that a morphism of varieties $f : X \to Y$ is **quasi-finite** if $f^{-1}(y)$ is a finite set for each $y \in Y$. The definition of a **finite morphism** can be found in [**210**, Section 1.5.3] or [**97**, Section II.3]. Again, we will not need the details of the definition; instead, we will rely on the characterization found in [**93**, Corollaire 18.12.4]: a morphism is finite if and only if it is proper and quasi-finite. See [**210**, Section 5.4] for a proof of the following result.

LEMMA 2.1.8 (Noether normalization lemma). *Let X be an affine variety of dimension n. Then there exists a finite morphism $f : X \to \mathbb{A}^n$.*

LEMMA 2.1.9. *Let $f : X \to Y$ be a finite morphism. Then ${}^\circ f_* : \mathrm{Sh}(X, \Bbbk) \to \mathrm{Sh}(Y, \Bbbk)$ is an exact functor. Moreover, for any sheaf $\mathcal{F} \in \mathrm{Sh}(X, \Bbbk)$, we have $\operatorname{supp} {}^\circ f_* \mathcal{F} = f(\operatorname{supp} \mathcal{F})$.*

PROOF. Let $\mathcal{F} \in \mathrm{Sh}(X, \Bbbk)$. To prove exactness, we must show that $f_* \mathcal{F} \in \mathrm{Sh}(Y, \Bbbk)$. It is enough to show that the stalks $(f_* \mathcal{F})_y$ are concentrated in degree 0 for all $y \in Y$. Since f is proper, we have

$$(f_* \mathcal{F})_y \cong R\Gamma(\mathcal{F}|_{f^{-1}(y)}) \cong \prod_{x \in f^{-1}(y)} \mathcal{F}_x,$$

as desired. It is clear that $\operatorname{supp} {}^\circ f_* \mathcal{F} \subset f(\operatorname{supp} \mathcal{F})$. For the opposite containment, let $S = \{x \in X \mid \mathcal{F}_x \neq 0\}$. By definition, S is dense in $\operatorname{supp} \mathcal{F}$. The calculation above shows that ${}^\circ f_* \mathcal{F}$ has nonzero stalks at all points of $f(S)$, which is dense in $f(\operatorname{supp} \mathcal{F})$. Therefore, $\operatorname{supp} {}^\circ f_* \mathcal{F} \supset f(\operatorname{supp} \mathcal{F})$. $\square$

Smooth and étale morphisms. We will review an approach to smooth morphisms of quasiprojective varieties following [**18**, Definition 18.1.7]. See [**18**, Theorem 18.1.10] for comments on how to relate this to the usual definition in, say, [**97**, Section III.10].

Suppose first that $Y \subset \mathbb{A}^k$ is an affine variety. Let $\mathrm{pr}_1 : \mathbb{A}^{k+m} \to \mathbb{A}^k$ be the projection onto the first k coordinates, and suppose we have an algebraic map $g = (g_1, \ldots, g_n) : \mathbb{A}^{k+m} \to \mathbb{A}^n$, with $0 \le n \le m$. Let $X \subset \mathbb{A}^{k+m}$ be the affine variety defined by

$$X = \mathrm{pr}_1^{-1}(Y) \cap g^{-1}(0).$$

The map pr_1 restricts to a morphism $f : X \to Y$ of algebraic varieties. This morphism is **smooth** of **relative dimension** $m-n$ at a point $p = (p_1, \ldots, p_{k+m}) \in X$ if the $n \times m$ matrix

$$\left(\frac{\partial g_i}{\partial x_j}\right)_{1\le i\le n,\ k+1\le j\le k+m}\Bigg|_{(x_j)=(p_j)} \tag{2.1.1}$$

has rank n. (Here, $x_1, \ldots, x_{k+m}$ are the coordinates on $\mathbb{A}^{k+m}$.)

More generally, a morphism of varieties $f : X \to Y$ is **smooth** at a point $p \in X$ if p and $f(p)$ have affine neighborhoods on which f can be written in the form described above. In this book, we adopt the convention that smooth morphisms are assumed to be **equidimensional**, meaning that the relative dimension is the same at all points of X. If $f : X \to Y$ is smooth of relative dimension d, then for every point $y \in Y$, the variety $f^{-1}(y)$ is a smooth variety of dimension d.

It is well known that the composition of smooth morphisms is smooth. We also have the following partial converse, which follows from [**93**, Théorème 17.11.1].

Proposition 2.1.10. *Let $f : X \to Y$ be a smooth surjective morphism, and let $g : Y \to Z$ be another morphism of varieties. If $g \circ f$ is smooth, then g is smooth.*

The proof of the following lemma is left as an exercise.

Lemma 2.1.11. *Let $f : X \to Y$ be a smooth surjective morphism with connected fibers. If Y is connected, then X is connected.*

Remark 2.1.12. If $f : X \to Y$ is a smooth, proper morphism of smooth varieties, and if X is nonempty and Y is connected, then f is automatically surjective. Indeed, a smooth map is an open map, and a proper map is closed, so the image of f must be both open and closed. Since it is nonempty, it must be all of Y.

Theorem 2.1.13 (Generic smoothness). *Let $f : X \to Y$ be a morphism of varieties, and assume that X is a smooth variety. There is a nonempty Zariski-open subset $U \subset Y$ such that $f|_{f^{-1}(U)} : f^{-1}(U) \to U$ is a smooth morphism.*

See [**97**, Corollary 10.7] for a proof.

An **étale morphism** is a smooth morphism of relative dimension 0. Since a smooth variety of dimension 0 is a finite set of points, étale morphisms are automatically quasi-finite. The following fact follows easily from the definition above.

Lemma 2.1.14. *An étale morphism of varieties is a local homeomorphism. In particular, a surjective, proper, étale morphism is a covering map.*

(The last assertion uses the fact that a surjective, proper local homeomorphism is a covering map; see [**102**, Lemma 2].) For a proof of the following complement to Proposition 2.1.10, see [**93**, Corollaire 17.3.5].

PROPOSITION 2.1.15. *Let $g : Y \to Z$ be an étale morphism, and let $f : X \to Y$ be another morphism. If $g \circ f$ is smooth, then f is smooth (of the same relative dimension).*

LEMMA 2.1.16. *Let $f : X \to Y$ be a finite morphism. There is a nonempty Zariski-open subset $U \subset Y$ such that $f|_{f^{-1}(U)} : f^{-1}(U) \to U$ is finite and étale.*

PROOF. If f is not surjective, then $U = Y \smallsetminus f(X)$ trivially satisfies the conclusion of the lemma. Assume henceforth that f is surjective. Since f is finite, we must have $\dim X = \dim Y$. Let Z be the singular locus of X. Then $\dim Z < \dim Y$, and $f(Z)$ is a proper closed subset of Y. Let $V = Y \smallsetminus f(Z)$. Then $f^{-1}(V)$ is a smooth open subset of X, and $f|_{f^{-1}(V)} : f^{-1}(V) \to V$ is still finite. Apply Theorem 2.1.13 to this map to obtain a smaller open subset $U \subset V$ such that $f|_{f^{-1}(U)} : f^{-1}(U) \to U$ is smooth (necessarily of relative dimension 0) and finite. ☐

Embedded resolution of singularities. It is a famous theorem of Hironaka [**100**] that any irreducible complex variety admits a resolution of singularities, i.e., a proper birational map from a smooth variety. We will need a version of this result which gives additional control over the fibers of the map, involving the following notion.

DEFINITION 2.1.17. Let X be a smooth variety of dimension n. Let $Z \subset X$ be a closed subvariety of dimension $n - 1$, with irreducible components $Z_1, \ldots, Z_k$. The variety Z is said to be a **divisor with simple normal crossings** if the following conditions hold:

(1) Each component Z_i is smooth.
(2) For each $x \in X$, let $I(x) = \{i \in \{1, \ldots, k\} \mid x \in Z_i\}$. Then x has an affine neighborhood U and regular functions $\{f_i \in \mathbb{C}[U] \mid i \in I(x)\}$ such that

$$Z_i \cap U = \{y \in U \mid f_i(y) = 0\} \qquad \text{for all } i \in I(x),$$

and such that the differentials $\{df_i|_x \mid i \in I(x)\}$ form a linearly independent subset of the cotangent space $T^*_x X$.

Given a subset $I \subset \{1, \ldots, k\}$, we sometimes use the notation

$$Z_I = \bigcap_{i \in I} Z_i. \tag{2.1.2}$$

When Z_I is nonempty, it is a smooth variety of codimension $|I|$.

Note that a smooth subvariety $Z \subset X$ of dimension $n - 1$ is trivially a divisor with simple normal crossings: any point $x \in X$ sits on at most one component of Z, and a smooth irreducible subvariety of codimension 1 is locally defined by a function with nonzero differential. We will study this notion in more detail in Section 2.4.

THEOREM 2.1.18 (Resolution of singularities). *Let X be an irreducible variety, and let $Z \subset X$ be a closed subvariety whose complement is nonempty and smooth. There exists a proper map $f : \tilde{X} \to X$ with the following properties:*

(1) *The variety $\tilde{X}$ is smooth.*
(2) *The map f restricts to an isomorphism $f^{-1}(X \smallsetminus Z) \to X \smallsetminus Z$.*
(3) *The preimage $f^{-1}(Z)$ is a divisor with simple normal crossings.*

Many expository sources give a version in which Z is the singular locus of X, but we will require this more general version. For a discussion, see [**151**, Theorem 4.1.3] and the references therein.

Locally trivial fibrations. See the discussion preceding Theorem 1.9.5 for the definition of a locally trivial fibration. If X and Y are smooth manifolds, a differentiable map $f : X \to Y$ is said to be a **differentiable locally trivial fibration** (with fiber F) if for each point $y \in Y$, there is a neighborhood U and a diffeomorphism $b : f^{-1}(U) \xrightarrow{\sim} F \times U$ making the diagram

$$\begin{tikzcd} f^{-1}(U) \arrow[r, "b", "\sim"'] \arrow[d, "f|_{f^{-1}(U)}"'] & F \times U \arrow[dl, "\mathrm{pr}_2"] \\ U & \end{tikzcd} \tag{2.1.3}$$

commute. The following result tells us about the occurrence of such maps in algebraic geometry.

Theorem 2.1.19 (Ehresmann's fibration theorem). *Let $f : X \to Y$ be a smooth, surjective, proper morphism of smooth varieties. Then f is a differentiable locally trivial fibration.*

For the original proof, see [**71**]; for an exposition, see [**70**, Theorem 8.5.10]. We emphasize that the open set U in (2.1.3) is an analytic open set, and that the map b is only a diffeomorphism; it cannot, in general, be chosen to be a biholomorphism.

We will require a mild generalization of Ehresmann's theorem, involving the following notion.

Definition 2.1.20. Let X and Y be smooth varieties, and let $Z \subset X$ be a divisor with simple normal crossings, and with irreducible components $Z_1, \dots, Z_k$. A map $f : X \to Y$ is said to be **transverse to** Z if for each $I \subset \{1, \dots, k\}$ with $Z_I \neq \varnothing$, the map $f|_{Z_I} : Z_I \to Y$ (see (2.1.2)) is smooth and surjective.

The map f is said to be a **transverse locally trivial fibration** if each point $y \in Y$ admits an analytic neighborhood U and a diffeomorphism $b : f^{-1}(U) \to f^{-1}(y) \times U$ that restricts to a diffeomorphism

$$b|_{f^{-1}(U) \cap Z_I} : f^{-1}(U) \cap Z_I \to (f^{-1}(y) \cap Z_I) \times U$$

for each subset $I \subset \{1, \dots, k\}$, and that makes (2.1.3) commute.

Theorem 2.1.21. *Let $f : X \to Y$ be a smooth morphism of smooth varieties that is transverse to a divisor with simple normal crossings $Z \subset X$. If f is proper, then it is a transverse locally trivial fibration.*

This result can be proved by a minor variation on the proof of Theorem 2.1.19; see [**94**, Section 4] for a discussion of ideas along these lines. Theorem 2.1.21 can also be seen as a special case of a much more general result known as Thom's first isotopy lemma [**170**, **236**] (see [**88**, Section I.1.5] for a discussion). However, the latter result requires the language of Whitney stratifications, which we will not develop in this book (cf. Remark 2.3.21).

Fundamental groups. We conclude this section with some results on fundamental groups of smooth varieties.

LEMMA 2.1.22. *Let X be a smooth, connected variety. For any Zariski open subset $U \subset X$ and any point $x_0 \in U$, the natural map $\pi_1(U, x_0) \to \pi_1(X, x_0)$ is surjective.*

PROOF. We will need the following theorem about fundamental groups of manifolds: If M is a connected real manifold, and $N \subset M$ is a closed, connected submanifold of (real) codimension ≥ 2, then for any $m_0 \in M \smallsetminus N$, the map $\pi_1(M \smallsetminus N, m_0) \to \pi_1(M, m_0)$ is surjective. For a proof, see [**83**, Théorème 2.3].

Let $Z = X \smallsetminus U$. To prove the lemma, we proceed by noetherian induction on Z. If $Z = \varnothing$, the statement is trivial. Otherwise, choose a smooth connected open subvariety $V \subset Z$, and let $Z' = Z \smallsetminus V$. Since X is smooth and connected, it is irreducible, so $\dim Z < \dim X$, and hence $\dim V < \dim X = \dim(U \cup V)$. Regarding $U \cup V$ and V as real manifolds, we see that V is a closed, connected submanifold of (real) codimension ≥ 2. By the result mentioned above, the first map below is surjective:

$$\pi_1(U, x_0) \to \pi_1(U \cup V, x_0) \to \pi_1(X, x_0).$$

The second map is surjective by induction, so we are done. □

PROPOSITION 2.1.23. *Let $f : X \to Y$ be a smooth, surjective morphism with connected fibers, and assume that Y is smooth and connected. For any $x_0 \in X$, the induced map $\pi_1(X, x_0) \to \pi_1(Y, f(x_0))$ is surjective.*

PROOF. Apply Nagata's compactification theorem (Theorem 2.1.7) to factor f as an open embedding followed by a proper map:

$$X \hookrightarrow X' \xrightarrow{\tilde{f}} Y.$$

Our assumptions imply that X is smooth. We may also assume that X' is smooth; if it is not, replace it by a resolution of singularities that is an isomorphism over the open subset $X \subset X'$. Now apply Theorem 2.1.13 to $\tilde{f}$: there is an open subset $V \subset Y$ such that $\tilde{f}|_{\tilde{f}^{-1}(V)} : \tilde{f}^{-1}(V) \to V$ is a smooth morphism. Let $\tilde{U} = \tilde{f}^{-1}(V)$, and let $\tilde{f}' = \tilde{f}|_{\tilde{U}} : \tilde{U} \to V$. Also let $U = f^{-1}(V) = X \cap \tilde{U}$, and let $j : U \hookrightarrow \tilde{U}$ be the open embedding. The following commutative diagram summarizes the spaces we have introduced so far:

$$\begin{tikzcd}
U \arrow[r, hook, "j"] \arrow[d, hook] & \tilde{U} \arrow[r, "\tilde{f}'"] \arrow[d, hook] & V \arrow[d, hook] \\
X \arrow[r, hook] \arrow[rr, bend right, "f"'] & X' \arrow[r, "\tilde{f}"] & Y
\end{tikzcd}$$

Since $\tilde{f}$ is proper and surjective, $\tilde{f}'$ is smooth, proper, and surjective. For any point $v \in V$, the fiber $(\tilde{f}')^{-1}(v)$ contains the dense open connected subset $(\tilde{f}')^{-1}(v) \cap U = f^{-1}(v)$, so it is connected.

Note that X is connected (by Lemma 2.1.11), so we may as well assume that $x_0 \in U$. We obtain the following diagram of fundamental groups:

$$\begin{array}{ccccc} \pi_1(U, x_0) & \xrightarrow{\pi_1(j)} & \pi_1(\tilde{U}, x_0) & \xrightarrow{\pi_1(\tilde{f}')} & \pi_1(V, f(x_0)) \\ \downarrow & & & & \downarrow \\ \pi_1(X, x_0) & & \xrightarrow{\qquad\pi_1(f)\qquad} & & \pi_1(Y, f(x_0)) \end{array}$$

The vertical maps and $\pi_1(j)$ are surjective by Lemma 2.1.22, so to show that $\pi_1(f)$ is surjective, it is enough to show that $\pi_1(\tilde{f}')$ is surjective. By Ehresmann's fibration theorem (Theorem 2.1.19), $\tilde{f}'$ is a locally trivial fibration. Since $\tilde{f}'$ has connected fibers, the long exact sequence of homotopy groups associated to this fibration (see [**98**, Theorem 4.41]) shows that $\pi_1(\tilde{f}) : \pi_1(\tilde{U}, x_0) \to \pi_1(V, f(x_0))$ is surjective. □

2.2. Smooth pullback and smooth base change

In this section, we establish a number of properties of smooth morphisms.

Smooth base change. The following lemma implies that a smooth morphism of relative dimension d is a topological submersion of relative (real) dimension $2d$.

LEMMA 2.2.1. *Let $f : X \to Y$ be a smooth morphism of relative dimension d. For any point $p \in X$, there is an analytic neighborhood U of p, an analytic neighborhood V of $f(p)$, an analytic open subset $M \subset \mathbb{C}^d$ that is homeomorphic to $\mathbb{C}^d$, and a biholomorphism $b : U \xrightarrow{\sim} V \times M$ such that the following diagram commutes:*

$$\begin{array}{ccc} U & \xrightarrow[\sim]{b} & V \times M \\ {\scriptstyle f|_U}\big\downarrow & \swarrow{\scriptstyle \mathrm{pr}_1} & \\ V & & \end{array}$$

Since b is a biholomorphism and not just a homeomorphism, the condition in this lemma is stronger than that in Definition 1.5.10.

PROOF. This lemma is essentially an exercise in using the implicit function theorem. We may assume that $Y \subset \mathbb{C}^k$ and $X \subset \mathbb{C}^{k+m}$, and that f and $g : \mathbb{C}^{k+m} \to \mathbb{C}^n$ are as in the discussion preceding (2.1.1). We have $d = m - n$. By renumbering the coordinates on $\mathbb{C}^{k+n}$ if necessary, we may also assume that the last n columns of the matrix (2.1.1) are linearly independent. In other words, the $n \times n$ matrix

$$\left(\frac{\partial g_i}{\partial x_j}\right)_{1 \le i \le n,\ k+d+1 \le j \le k+d+n}\Bigg|_{(x_j)=(p_j)}$$

is nonsingular. Write p as a triple (q, p', p''), where $q \in \mathbb{C}^k$, $p' \in \mathbb{C}^d$, and $p'' \in \mathbb{C}^n$. The holomorphic implicit function theorem then tells us that there is an (analytic) open subset $W_1 \subset \mathbb{C}^{k+d}$ containing (q, p'), an open subset $W_2 \subset \mathbb{C}^n$ containing p'', and a holomorphic map $h_0 : W_1 \to W_2$ such that the map

$$h : W_1 \to g^{-1}(0) \cap (W_1 \times W_2) \qquad \text{given by} \qquad h(u) = (u, h_0(u))$$

is a bijection. Moreover, h has a holomorphic (in fact, algebraic) inverse b, given by the projection map $\mathbb{C}^{k+d+n} \to \mathbb{C}^{k+d}$ onto the first $k + d$ coordinates.

Now, replace W_1 by a smaller open set of the form $V_0 \times M$, where $V_0 \subset \mathbb{C}^k$ is an open set containing q, and $M \subset \mathbb{C}^d$ is an open set that contains p' and is itself homeomorphic to $\mathbb{C}^d$. Let $U_0 = V_0 \times M \times W_2$. We have the following commutative diagram:

$$\begin{tikzcd} g^{-1}(0) \cap U_0 \arrow[d, "\mathrm{pr}_1"'] & V_0 \times M \arrow[l, "h"', "\sim"] \arrow[dl, "\mathrm{pr}_1"] \\ V_0 & \end{tikzcd}$$

The lemma follows by intersecting each space here with an appropriate preimage of Y. Specifically, let $U = U_0 \cap X$ and $V = V_0 \cap Y$. $\square$

As an immediate consequence, by Theorem 1.9.3, we have the following.

THEOREM 2.2.2 (Smooth base change). *Suppose we have a cartesian square of varieties:*

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

Assume that g is smooth. Then, for $\mathcal{F} \in D^+(X, \Bbbk)$, the base change map $g^* f_* \mathcal{F} \xrightarrow{\sim} f'_*(g')^*\mathcal{F}$ is an isomorphism.

The Tate module and smooth pullback. Our next task is to give a concrete description of the functor $f^!$ for a smooth morphism f. In view of Lemma 2.2.1, Theorem 1.5.11 reduces this problem to that of computing the relative orientation sheaf or_f. We will express the answer in terms the following definition, which relies on the well-known fact that $\mathbf{H}^2_c(\mathbb{A}^1; \Bbbk)$ is a free $\Bbbk$-module of rank 1 (see Exercise 1.8.3).

DEFINITION 2.2.3. The **Tate module** with coefficients in $\Bbbk$ is the free $\Bbbk$-module $\Bbbk(1)$ given by
$$\Bbbk(1) = \mathrm{Hom}(\mathbf{H}^2_c(\mathbb{A}^1; \Bbbk), \Bbbk).$$
We also put $\Bbbk(-1) = \mathbf{H}^2(\mathbb{A}^1; \Bbbk)$, and, more generally, for any $n \in \mathbb{Z}$, we let

$$\Bbbk(n) = \begin{cases} \underbrace{\Bbbk(1) \otimes \cdots \otimes \Bbbk(1)}_{n \text{ copies}} & \text{if } n \geq 0, \\ \underbrace{\Bbbk(-1) \otimes \cdots \otimes \Bbbk(-1)}_{-n \text{ copies}} & \text{if } n \leq 0. \end{cases}$$

For any complex of sheaves $\mathcal{F}$ and any $n \in \mathbb{Z}$, the nth **Tate twist** of $\mathcal{F}$, denoted by $\mathcal{F}(n)$, is the complex of sheaves $\mathcal{F}(n) = \mathcal{F} \otimes^L \underline{\Bbbk(n)}$.

There is a canonical isomorphism $\Bbbk(-1) \otimes \Bbbk(1) \xrightarrow{\sim} \Bbbk$, and this gives rise to isomorphisms

$$\Bbbk(n) \otimes \Bbbk(m) \cong \Bbbk(n+m), \qquad \mathrm{Hom}(\Bbbk(n), \Bbbk(m)) \cong \Bbbk(m-n)$$

for any $n, m \in \mathbb{Z}$. Also, by the universal coefficient theorem (Corollary 1.4.18), there is a natural isomorphism

$$\Bbbk(1) = \Bbbk \otimes_{\mathbb{Z}} \mathbb{Z}(1).$$

REMARK 2.2.4. Of course, one could simply choose an isomorphism $\Bbbk(1) \cong \Bbbk$ and abandon the notion of Tate twists. (Such a choice is equivalent to the choice of a $\Bbbk$-orientation on the manifold $\mathbb{C}$.) However, a number of results in the sequel have (in my opinion) more natural formulations if we avoid making such a choice. By keeping track of Tate twists, one can see how various constructions depend on this choice. Using Tate twists now can also prepare us for Chapter 5, where they are essential and cannot be omitted.

Nevertheless, some readers may object that the manifold $\mathbb{C}$ carries a *canonical* orientation, and hence that there is a canonical isomorphism $\Bbbk(1) \cong \Bbbk$. Well, I cannot stop you from ignoring Tate twists if you insist!

Below are some examples in which Tate twists naturally appear.

EXAMPLE 2.2.5. The Künneth formula (Corollary 1.4.22) implies that

$$\mathbf{H}^i_{\mathrm{c}}(\mathbb{A}^n;\Bbbk) \cong \begin{cases} \underline{\Bbbk}(-n) & \text{if } i = 2n, \\ 0 & \text{otherwise.} \end{cases}$$

EXAMPLE 2.2.6. Consider an open embedding $j : \mathbb{A}^1 \hookrightarrow \mathbb{P}^1$ whose complement is some point, say $i : \mathrm{pt} \hookrightarrow \mathbb{P}^1$. Apply $R\Gamma \cong R\Gamma_{\mathrm{c}}$ to the distinguished triangle $j_!\underline{\Bbbk}_{\mathbb{A}^1} \to \underline{\Bbbk}_{\mathbb{P}^1} \to i_*\underline{\Bbbk}_{\mathrm{pt}} \to$ to obtain the distinguished triangle $R\Gamma_{\mathrm{c}}(\underline{\Bbbk}_{\mathbb{A}^1}) \to R\Gamma(\underline{\Bbbk}_{\mathbb{P}^1}) \to \Bbbk \to$. The long exact sequence in cohomology gives us an isomorphism

$$\mathbf{H}^2(\mathbb{P}^1;\Bbbk) \cong \mathbf{H}^2_{\mathrm{c}}(\mathbb{A}^1;\Bbbk) = \Bbbk(-1).$$

It can be deduced from Corollary 2.2.10 below (alternatively, see Lemma 2.11.8) that this isomorphism is canonical, i.e., independent of the choice of open embedding $j : \mathbb{A}^1 \hookrightarrow \mathbb{P}^1$. More generally, for $0 \le r \le n$, Exercise 2.11.5 yields a canonical isomorphism

$$\mathbf{H}^{2r}(\mathbb{P}^n;\Bbbk) \cong \Bbbk(-r).$$

EXAMPLE 2.2.7. Let $j : \mathbb{A}^1 \smallsetminus \{0\} \hookrightarrow \mathbb{A}^1$ and $i : \{0\} \hookrightarrow \mathbb{A}^1$ be the inclusion maps, and let $\mathcal{F} = j_*\underline{\Bbbk}_{\mathbb{A}^1 \smallsetminus \{0\}}$. By Exercise 1.10.1 (see also Lemma B.2.2), there is a natural isomorphism $i^*j_*\underline{\Bbbk}_{\mathbb{A}^1 \smallsetminus \{0\}} \cong R\Gamma(\underline{\Bbbk}_{\mathbb{A}^1 \smallsetminus \{0\}})$. By Example 1.7.6, we have ${}^\circ j_*\underline{\Bbbk}_{\mathbb{A}^1 \smallsetminus \{0\}} \cong \underline{\Bbbk}_{\mathbb{A}^1}$, so there is a truncation distinguished triangle

$$\underline{\Bbbk}_{\mathbb{A}^1} \to j_*\underline{\Bbbk}_{\mathbb{A}^1 \smallsetminus \{0\}} \to i_*\mathbf{H}^1(\mathbb{A}^1 \smallsetminus \{0\};\Bbbk) \to .$$

Now apply $R\Gamma_{\mathrm{c}}$. By Lemma B.2.6, $R\Gamma_{\mathrm{c}}(j_*\underline{\Bbbk}_{\mathbb{A}^1 \smallsetminus \{0\}}) = 0$. It follows that

$$\mathbf{H}^1(\mathbb{A}^1 \smallsetminus \{0\};\Bbbk) \cong \underline{\Bbbk}(-1).$$

We have thus reconciled Definition 2.2.3 with the remark following Exercise 1.10.3.

The following result relies on some calculations from Section B.1.

PROPOSITION 2.2.8. *Let $f : X \to Y$ be a smooth morphism of relative dimension d. There is a canonical isomorphism $\mathit{or}_f \cong \underline{\Bbbk}_X(d)$.*

PROOF. Choose an open covering $(U_\alpha)_{\alpha \in I}$ of X consisting of open sets obtained from Lemma 2.2.1. For brevity, let us write $f_{U_\alpha} = f|_{U_\alpha} : U_\alpha \to f(U_\alpha)$. Thus, for each $\alpha \in I$, there exists an open subset $M_\alpha \subset \mathbb{C}^d$ that is homeomorphic to $\mathbb{C}^d$, a biholomorphism $\phi_\alpha : U_\alpha \to f(U_\alpha) \times M_\alpha$, and a commutative diagram

$$\begin{tikzcd}
U_\alpha \arrow[r, "\phi_\alpha"] \arrow[d, "f_{U_\alpha}"'] & f(U_\alpha) \times M_\alpha \arrow[d, "\mathrm{pr}_1"] \\
f(U_\alpha) \arrow[r, equal] & f(U_\alpha)
\end{tikzcd}$$

Let us also assume that each U_α is connected, and that the collection $(U_\alpha)_{\alpha\in I}$ contains a basis for the topology of X. By Proposition 1.4.21(3), we have

$$\begin{aligned}
\mathrm{pr}_{1!}\underline{\Bbbk}_{f(U_\alpha)\times M_\alpha} &\cong \mathrm{pr}_{1!}(\underline{\Bbbk}_{f(U_\alpha)} \boxtimes \underline{\Bbbk}_{M_\alpha}) \\
&\qquad\qquad \cong \underline{\Bbbk}_{f(U_\alpha)} \overset{L}{\otimes} \mathrm{a}^*_{f(U_\alpha)} R\Gamma_{\mathrm{c}}(\underline{\Bbbk}_{M_\alpha}) \cong \mathrm{a}^*_{f(U_\alpha)} R\Gamma_{\mathrm{c}}(\underline{\Bbbk}_{M_\alpha}).
\end{aligned}$$

Applying $\mathsf{H}^{2d}(-)$ to this, we can identify

$$\mathsf{H}^{2d}(\mathrm{pr}_{1!}\underline{\Bbbk}_{f(U_\alpha)\times M_\alpha}) \cong \underline{\mathbf{H}^{2d}_{\mathrm{c}}(M_\alpha;\Bbbk)}_{f(U_\alpha)}.$$

In particular, the construction from Remark 1.3.8 gives us an isomorphism

$$\phi_{\alpha\sharp} : \mathsf{H}^{2d}(f_{U_\alpha!}\underline{\Bbbk}_{U_\alpha}) \xrightarrow{\sim} \underline{\mathbf{H}^{2d}_{\mathrm{c}}(M_\alpha;\Bbbk)}_{f(U_\alpha)}.$$

Similarly, from the inclusion map $j_\alpha : U_\alpha \times M_\alpha \to U_\alpha \times \mathbb{C}^d$, we obtain a map

$$j_{\alpha\sharp} : \underline{\mathbf{H}^{2d}_{\mathrm{c}}(M_\alpha;\Bbbk)}_{f(U_\alpha)} \to \underline{\mathbf{H}^{2d}_{\mathrm{c}}(\mathbb{C}^d;\Bbbk)}_{f(U_\alpha)} = \underline{\Bbbk}_{f(U_\alpha)}(-d).$$

A minor variant of Lemma B.1.2 shows that this is an isomorphism.

Take another open set $U_\beta \subset U_\alpha$, and choose a biholomorphism $\phi_\beta : U_\beta \to f(U_\beta) \times M_\beta$ as above. Let $h : U_\beta \hookrightarrow U_\alpha$ and $k : f(U_\beta) \hookrightarrow f(U_\alpha)$ be the inclusion maps. Consider the commutative diagram

(2.2.1)
$$\begin{tikzcd}
U_\beta \arrow[rr, "h"] \arrow[dr, "\phi_\beta"] \arrow[dd, "f_{U_\beta}"'] & & U_\alpha \arrow[dr, "\phi_\alpha"] \arrow[dd, "f_{U_\alpha}"] & \\
& f(U_\beta) \times M_\beta \arrow[rr, "q=\phi_\alpha h \phi_\beta^{-1}"] \arrow[dd, "\mathrm{pr}_1"] & & f(U_\alpha) \times M_\alpha \arrow[dd, "\mathrm{pr}_1"] \\
f(U_\beta) \arrow[rr, "k"] \arrow[dr, equal] & & f(U_\alpha) \arrow[dr, equal] & \\
& f(U_\beta) \arrow[rr, "k"] & & f(U_\alpha)
\end{tikzcd}$$

We claim that the following diagram commutes:

(2.2.2)
$$\begin{tikzcd}
\mathsf{H}^{2d}(f_{U_\beta!}\underline{\Bbbk}_{U_\beta}) \arrow[r, "h_\sharp"] \arrow[d, "\phi_{\beta\sharp}"'] & \mathsf{H}^{2d}(f_{U_\alpha}\underline{\Bbbk}_{U_\alpha})|_{f(U_\beta)} \arrow[d, "\phi_{\alpha\sharp}"] \\
\underline{\mathbf{H}^{2d}_{\mathrm{c}}(M_\beta;\Bbbk)}_{f(U_\beta)} \arrow[r, "q_\sharp"] \arrow[d, "j_{\beta\sharp}"'] & \underline{\mathbf{H}^{2d}_{\mathrm{c}}(M_\alpha;\Bbbk)}_{f(U_\alpha)}|_{f(U_\beta)} \arrow[d, "j_{\alpha\sharp}"] \\
\underline{\Bbbk}_{f(U_\beta)} \arrow[r, equal] & \underline{\Bbbk}_{f(U_\alpha)}|_{f(U_\beta)}(-d)
\end{tikzcd}$$

The upper square commutes by several applications of Exercise 1.3.6 to (2.2.1), and the lower square commutes by Lemma B.1.4.

In the special case where $U_\beta = U_\alpha$, this diagram shows that we have an isomorphism $\mathsf{H}^{2d}(f_{U_\alpha !}\underline{\Bbbk}_{U_\alpha}) \cong \underline{\Bbbk}_{f(U_\alpha)}(-d)$ that is independent of the choice of ϕ_α and M_α, and hence a canonical isomorphism $\mathit{or}_{f,\mathrm{pre}}(U_\alpha) \cong \Bbbk(d)$. Thanks to the commutativity of (2.2.2), we can apply the criterion from Lemma 1.7.5 to obtain an isomorphism

$$(\mathit{or}_f)|_{U_\alpha} \cong \underline{\Bbbk}_{U_\alpha}(d). \tag{2.2.3}$$

Now consider two such isomorphisms $(\mathit{or}_f)|_{U_\alpha} \cong \underline{\Bbbk}_{U_\alpha}(d)$ and $(\mathit{or}_f)|_{U_\gamma} \cong \underline{\Bbbk}_{U_\gamma}(d)$. The commutativity of (2.2.2) implies that the restrictions of these isomorphisms to $U_\alpha \cap U_\gamma$ are equal. By Exercise 1.1.11, we can glue the various isomorphisms (2.2.3) to obtain the desired isomorphism $\mathit{or}_f \cong \underline{\Bbbk}_X(d)$. □

As an immediate consequence, by Theorem 1.5.11, we have the following.

THEOREM 2.2.9. *Let $f : X \to Y$ be a smooth morphism of relative dimension d. For $\mathcal{F} \in D^+(Y, \Bbbk)$, there is a natural isomorphism $f^*\mathcal{F}[2d](d) \xrightarrow{\sim} f^!\mathcal{F}$.*

We also obtain the following description of the dualizing complex (see Definition 1.5.13) of a smooth variety.

COROLLARY 2.2.10 (Poincaré duality). *For a smooth variety X of dimension n, the dualizing complex is given by $\omega_X \cong \underline{\Bbbk}_X[2n](n)$.*

For the relationship between this result and what it is usually called "Poincaré duality" in algebraic topology, see Exercise 2.8.1 and Remark 2.11.2.

An important "meta-consequence" of the results so far in this section is the following.

PRINCIPLE 2.2.11. *Smooth pullback commutes with all sheaf operations.*

Let us spell out what this means. Let $f : X \to Y$ be a smooth morphism of relative dimension d, and suppose we have a cartesian square

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

Then f' is automatically smooth (of relative dimension d). For $\mathcal{F}, \mathcal{G} \in D^{\mathrm{b}}(Y, \Bbbk)$ and $\mathcal{H} \in D^{\mathrm{b}}(Y', \Bbbk)$, we have the following natural isomorphisms:

$$\begin{aligned} f^*(\mathcal{F} \overset{L}{\otimes} \mathcal{G}) &\cong f^*\mathcal{F} \overset{L}{\otimes} f^*\mathcal{G} & f^* R\mathscr{H}om(\mathcal{F}, \mathcal{G}) &\cong R\mathscr{H}om(f^*\mathcal{F}, f^*\mathcal{G}) \\ (g')^* f^*\mathcal{F} &\cong (f')^* g^*\mathcal{F} & (g')^! f^*\mathcal{F} &\cong (f')^* g^!\mathcal{F} \\ f^* g_! \mathcal{H} &\cong g'_!(f')^*\mathcal{H} & f^* g_* \mathcal{H} &\cong g'_*(f')^*\mathcal{H} \\ & & f^*\mathbb{D}(\mathcal{F}) &\cong \mathbb{D}(f^*\mathcal{F})[-2d](-d). \end{aligned}$$

Here, the isomorphisms in the first column are copied verbatim from Chapter 1 (and do not involve the assumption that f is smooth), but those in the second column are obtained by combining results from Chapter 1 with Theorem 2.2.2 and Theorem 2.2.9. The idea expressed in Principle 2.2.11 will be especially important in Sections 3.6–3.7 and in Chapter 6.

Smooth pairs. We will now apply Theorem 2.2.9 to study the restriction with supports functor for certain closed embeddings.

DEFINITION 2.2.12. Let $f : X \to S$ be a smooth morphism of relative dimension d, and let $Z \subset X$ be a closed subvariety. The pair (Z, X) is said to be a **smooth pair** of codimension r (with respect to f) if $f|_Z : Z \to S$ is a smooth morphism of relative dimension $d - r$.

As an important special case, if f is the constant map $\mathrm{a}_X : X \to \mathrm{pt}$, then to say that (Z, X) is a smooth pair just means that Z and X are both smooth varieties.

THEOREM 2.2.13. *Let (Z, X) be a smooth pair of codimension r (with respect to some smooth morphism $f : X \to S$), and let $i : Z \to X$ be the inclusion map. For any $\mathcal{F} \in D^+(X, \Bbbk)$, there is a natural map*

$$i^*\mathcal{F}[-2r](-r) \to i^!\mathcal{F}.$$

Moreover, this map is an isomorphism if one of the following conditions holds:

- *$\mathcal{F}$ lies in $\mathrm{Loc}^{\mathrm{ft}}(X, \Bbbk)$ or, more generally, in $D^{\mathrm{b}}_{\mathrm{locf}}(X, \Bbbk)$, or*
- *$\mathcal{F} \cong f^*\mathcal{G}$ for some $\mathcal{G} \in D^+(S, \Bbbk)$.*

PROOF. Let d be the relative dimension of f. Then $f \circ i$ is smooth of relative dimension $d - r$. Applying Theorem 2.2.9 twice, we have

$$i^!\underline{\Bbbk}_X \cong i^! f^*\underline{\Bbbk}_S \cong i^! f^!\underline{\Bbbk}_S[-2d](-d) \cong i^* f^*\underline{\Bbbk}_S[-2r](-r) \cong \underline{\Bbbk}_Z[-2r](-r).$$

By Proposition 1.5.9, we obtain the desired natural map, along with the claim that it is an isomorphism for $\mathcal{F} \in \mathrm{Loc}^{\mathrm{ft}}(X, \Bbbk)$. A straightforward argument with truncation and induction on the number of nonzero cohomology sheaves shows that it is also an isomorphism for $\mathcal{F} \in D^{\mathrm{b}}_{\mathrm{locf}}(X, \Bbbk)$.

If $\mathcal{F} = f^*\mathcal{G}$, then the map described above is the first map in the sequence

$$i^* f^*\mathcal{G}[-2r](-r) \to i^! f^*\mathcal{G} \xrightarrow{\sim} i^! f^!\mathcal{G}[-2d](-d).$$

By Theorem 2.2.9, the second map above is an isomorphism, as is the composition of the two maps. The first map is therefore also an isomorphism. ☐

Theorem 2.2.13 easily generalizes to the case of locally closed subvarieties: if $f : X \to S$ is smooth, and if $h : Y \hookrightarrow X$ is a locally closed embedding such that $f \circ h$ is also smooth, then h can be factored as $Y \xrightarrow{i} U \xrightarrow{j} X$, where i is a closed embedding, and j is an open embedding. Combining Theorem 2.2.13 with Lemma 1.3.3, we obtain a natural map

$$h^*\mathcal{F}[-2r](-r) \to h^!\mathcal{F} \tag{2.2.4}$$

that is an isomorphism if $\mathcal{F} \in D^{\mathrm{b}}_{\mathrm{locf}}(X, \Bbbk)$ or if $\mathcal{F}$ is pulled back from S.

COROLLARY 2.2.14. *Let X be a smooth variety, and let Y be a locally closed subvariety. Let $h : Y \hookrightarrow X$ be the inclusion map, and let $r = \dim X - \dim Y$. For $\mathcal{L} \in \mathrm{Loc}^{\mathrm{ft}}(X, \Bbbk)$, we have $h^!\mathcal{L} \in D^+(Y, \Bbbk)^{\geq 2r}$.*

PROOF. We proceed by noetherian induction on Y. Let $Y_0 \subset Y$ be a smooth open subset of Y, and let $Y_1 = Y \smallsetminus Y_0$. Let $i : Y_1 \hookrightarrow Y$ and $j : Y_0 \hookrightarrow Y$ be the inclusion maps. Consider the distinguished triangle

$$i_*i^!h^!\mathcal{L} \to h^!\mathcal{L} \to j_*j^*h^!\mathcal{L} \to .$$

It is enough to show that the first and last terms belong to $D^+(Y,\Bbbk)^{\geq 2r}$. By induction, we have

$$i^! h^! \mathcal{L} \cong (h \circ i)^! \mathcal{L} \in D^{\mathrm{b}}(Y_1, \Bbbk)^{\geq 2(\dim X - \dim Y_1)} \subset D^{\mathrm{b}}(Y_1, \Bbbk)^{\geq 2r}.$$

On the other hand, the version of Theorem 2.2.13 discussed in (2.2.4) implies that

$$j^* h^! \mathcal{L} \cong (h \circ j)^! \mathcal{L} \cong \mathcal{L}|_{Y_0}[2(\dim Y_0 - \dim X)](\dim Y_0 - \dim X) \in D^+(Y_0, \Bbbk)^{\geq 2r}.$$

Since ${}^\circ i_*$ is exact and ${}^\circ j_*$ is left exact, we have $i_* i^! h^! \mathcal{L}, j_* j^* h^! \mathcal{L} \in D^+(Y,\Bbbk)^{\geq 2r}$, as desired. □

Exercises.

2.2.1. Suppose we have a cartesian square

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

in which f and f' are smooth, say of relative dimension d.

(a) Show that there is a canonical isomorphism $(g')^* \mathit{or}_f \cong \mathit{or}_{f'}$.
(b) Show that for any $\mathcal{F} \in D^+(Y,\Bbbk)$, the natural map $(g')^* f^! \mathcal{F} \to (f')^! g^* \mathcal{F}$ from Proposition 1.5.7 is an isomorphism.
(c) Show that for $\mathcal{G} \in D^+(Y',\Bbbk)$, the following diagram commutes:

$$\begin{tikzcd} f^* g_* \mathcal{G}[2d](d) \arrow[r, "\text{Thm. 2.2.2}", "\sim"'] \arrow[d, "\text{Thm. 2.2.9}"'] & g'_*(f')^* \mathcal{G}[2d](d) \arrow[d, "\text{Thm. 2.2.9}"] \\ f^! g_* \mathcal{G} & g'_*(f')^! \mathcal{G} \arrow[l, "\text{Prop. 1.5.7}"'] \end{tikzcd}$$

Hint: Use Exercise 1.6.2.

2.3. Stratifications and constructible sheaves

In this section, we introduce the main new concepts of this chapter, and we establish a number of preliminary results about them.

DEFINITION 2.3.1. Let X be a variety. A **stratification** of X is a finite collection $(X_s)_{s \in \mathscr{S}}$ of disjoint smooth, connected, locally closed subvarieties such that $X = \bigcup_{s \in \mathscr{S}} X_s$, and such that for any two $s, t \in \mathscr{S}$, we have that $\overline{X_s} \cap X_t$ is either empty or X_t.

The subvarieties X_s are called the **strata** of the stratification. The set $\mathscr{S}$ carries a natural partial order called the **closure partial order**, given by

$$t \leq s \qquad \text{if} \qquad X_t \subset \overline{X_s}.$$

The notion defined above is sometimes called an **algebraic stratification**, to distinguish it from more general notions in which the strata might be allowed to be analytic spaces, or even just manifolds, rather than algebraic varieties.

In a minor abuse of language, if X is equipped with a stratification $(X_s)_{s \in \mathscr{S}}$, we will say that "$\mathscr{S}$ is a stratification of X."

EXAMPLE 2.3.2. Let X be a smooth connected variety. The **trivial stratification** on X is the stratification consisting of a single stratum, namely X itself.

EXAMPLE 2.3.3. Let G be a connected algebraic group, and let X be a variety equipped with an action of G. Suppose this action has finitely many orbits. Then the G-orbits constitute a stratification of X.

The condition on $\overline{X_s} \cap X_t$ can be rephrased as follows: we require that the closure of each stratum be a union of strata. It will sometimes be useful to weaken this condition a bit:

DEFINITION 2.3.4. Let X be a variety, and let $(X_s)_{s \in \mathscr{S}}$ be a finite collection of disjoint smooth, connected, locally closed subvarieties such that $X = \bigcup_{s \in \mathscr{S}} X_s$. This collection is called a **filtration of X by smooth varieties** if the elements of $\mathscr{S}$ can be ordered as $\mathscr{S} = \{s_1, s_2, \ldots, s_k\}$ in such a way that for each i, the subset

$$X_{s_1} \cup X_{s_2} \cup \cdots \cup X_{s_i}$$

is a closed subset of X.

In [**57**, Section 1.3], this notion was called an **alpha-partition**. In the case where each stratum is an affine space, this is usually called an **affine paving**. Note that any stratification is automatically a filtration by smooth varieties: if $(X_s)_{s \in \mathscr{S}}$ is a stratification, we can simply order $\mathscr{S}$ by any total order that refines the closure partial order.

It is easy to see that any variety X admits a filtration by smooth varieties: for instance, by noetherian induction, one can take a smooth, connected, open subset $U \subset X$, combined with a filtration by smooth varieties of the complement $X \smallsetminus U$.

DEFINITION 2.3.5. Let X be a variety, and let $\mathscr{S}$ and $\mathscr{T}$ be two filtrations of X by smooth varieties. Then $\mathscr{T}$ is said to be a **refinement** of $\mathscr{S}$ if every stratum of $\mathscr{T}$ is contained in a stratum of $\mathscr{S}$.

The following easy lemma is left to the reader.

LEMMA 2.3.6. *Let X be a variety.*

1. *Let $\mathscr{S}$ be a filtration of X by smooth varieties. Then $\mathscr{S}$ admits a refinement that is a stratification.*
2. *Let $\mathscr{S}$ and $\mathscr{T}$ be two stratifications of a variety X. There exists a stratification that is a refinement of both $\mathscr{S}$ and $\mathscr{T}$.*
3. *Let $Y \subset X$ be a locally closed subvariety. There exists a stratification of X such that Y is a union of strata.*

DEFINITION 2.3.7. Let X be a variety, and let $(X_s)_{s \in \mathscr{S}}$ be a stratification. A sheaf $\mathcal{F} \in \mathrm{Sh}(X, \Bbbk)$ is said to be **weakly constructible with respect to $\mathscr{S}$** if for each $s \in \mathscr{S}$, the restriction $\mathcal{F}|_{X_s}$ is a local system. It is **constructible with respect to $\mathscr{S}$** if each $\mathcal{F}|_{X_s}$ is a local system of finite type. More generally, a sheaf $\mathcal{F}$ is simply said to be **weakly constructible** or **constructible** if there exists a stratification with respect to which it is weakly constructible or constructible, respectively. The full subcategory of $\mathrm{Sh}(X, \Bbbk)$ consisting of constructible sheaves is denoted by $\mathrm{Sh}_{\mathrm{c}}(X, \Bbbk)$.

An object $\mathcal{F} \in D^{\mathrm{b}}(X, \Bbbk)$ is said to be **constructible with respect to $\mathscr{S}$** (or simply **constructible**) if each cohomology sheaf $\mathsf{H}^k(\mathcal{F})$ has the same property. The full subcategory of $D^{\mathrm{b}}(X, \Bbbk)$ consisting of objects that are constructible with respect to $\mathscr{S}$ is denoted by $D^{\mathrm{b}}_{\mathscr{S}}(X, \Bbbk)$. The full subcategory of $D^{\mathrm{b}}(X, \Bbbk)$ consisting of constructible complexes is denoted by $D^{\mathrm{b}}_{\mathrm{c}}(X, \Bbbk)$.

Remark 2.3.8. Under the equivalence of Remark 1.1.10, the category of constructible sheaves on a point can be identified with the category $\Bbbk\text{-mod}^{\mathrm{fg}}$ of finitely generated $\Bbbk$-modules. Moreover, by Proposition A.7.20, we may identify

$$D^{\mathrm{b}}_{\mathrm{c}}(\mathrm{pt}, \Bbbk) \cong D^{\mathrm{b}}(\Bbbk\text{-mod}^{\mathrm{fg}}).$$

If $\mathscr{S}$ is the trivial stratification, then $D^{\mathrm{b}}_{\mathscr{S}}(X, \Bbbk)$ is equal to the category

$$D^{\mathrm{b}}_{\mathrm{locf}}(X, \Bbbk) = \{\mathcal{F} \in D^{\mathrm{b}}(X, \Bbbk) \mid \mathsf{H}^k(\mathcal{F}) \in \mathrm{Loc}^{\mathrm{ft}}(X, \Bbbk) \text{ for all } k \in \mathbb{Z}\}$$

(see Section 1.8). More generally, for any stratification $(X_s)_{s \in \mathscr{S}}$, an object $\mathcal{F} \in D^{\mathrm{b}}(X, \Bbbk)$ is constructible with respect to $\mathscr{S}$ if and only if each $\mathcal{F}|_{X_s}$ belongs to $D^{\mathrm{b}}_{\mathrm{locf}}(X_s, \Bbbk)$.

Note that by definition, a constructible complex must belong to the bounded derived category. We will occasionally speak of **weakly constructible complexes** in $D^+(X, \Bbbk)$: such an object is allowed to have infinitely many nonzero cohomology sheaves, as well as stalks that are not of finite type. We will not introduce a separate notation for the category of weakly constructible complexes.

If $\mathcal{F}$ is a constructible sheaf with respect to $\mathscr{S}$, it makes sense to speak of the **grade** of $\mathcal{F}|_{X_s}$ for each stratum X_s, in the sense of Definition 1.7.2.

Definition 2.3.9. Let X be a variety with a stratification $(X_s)_{s \in \mathscr{S}}$, and let $\mathcal{F}$ be a constructible sheaf with respect to $\mathscr{S}$. The **modified dimension of support** of $\mathcal{F}$, denoted by $\mathrm{mdsupp}\,\mathcal{F}$, is defined by

$$\mathrm{mdsupp}\,\mathcal{F} = \sup_{s \in \mathscr{S}} \{\dim X_s - \mathrm{grade}(\mathcal{F}|_{X_s})\}.$$

If $\mathcal{F} = 0$, then by definition, $\mathrm{grade}(\mathcal{F}|_{X_s}) = \infty$ for all s, so

$$\mathrm{mdsupp}\, 0 = -\infty.$$

If $\mathcal{F} \neq 0$, then there is at least one stratum X_s such that $\mathcal{F}|_{X_s} \neq 0$. In this case, recall from Corollary A.10.5 that $\mathrm{grade}(\mathcal{F}|_{X_s}) \leq \mathrm{gldim}\,\Bbbk$. It follows that for nonzero $\mathcal{F}$, we have

$$-\mathrm{gldim}\,\Bbbk \leq \mathrm{mdsupp}\,\mathcal{F} \leq \dim X.$$

In particular, $\mathrm{mdsupp}\,\mathcal{F}$ may be negative. However, if $\Bbbk$ is a field, then every nonzero $\Bbbk$-module has grade 0. Therefore,

$$\mathrm{mdsupp}\,\mathcal{F} = \dim \mathrm{supp}\,\mathcal{F} \qquad \text{if } \Bbbk \text{ is a field.}$$

The proofs of the next few results are left to the reader.

Lemma 2.3.10. *The modified dimension of support of a constructible sheaf $\mathcal{F}$ is well defined, i.e., it is independent of the choice of stratification.*

Lemma 2.3.11. *Let X be a variety, and let $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ be a short exact sequence of constructible sheaves. We have*

$$\dim \mathrm{supp}\,\mathcal{F} = \max\{\dim \mathrm{supp}\,\mathcal{F}', \dim \mathrm{supp}\,\mathcal{F}''\},$$
$$\mathrm{mdsupp}\,\mathcal{F} = \max\{\mathrm{mdsupp}\,\mathcal{F}', \mathrm{mdsupp}\,\mathcal{F}''\}.$$

Corollary 2.3.12. *Let X be a variety, and let $\mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to$ be a distinguished triangle in $D^{\mathrm{b}}_{\mathrm{c}}(X, \Bbbk)$. Suppose there is an integer n such that*

$$\mathrm{mdsupp}\,\mathsf{H}^k(\mathcal{F}') \leq n - k \qquad \text{and} \qquad \mathrm{mdsupp}\,\mathsf{H}^k(\mathcal{F}'') \leq n - k$$

for all k. Then $\mathrm{mdsupp}\,\mathsf{H}^k(\mathcal{F}) \leq n - k$ for all k as well.

LEMMA 2.3.13. *Let X be a variety.*

(1) *For any stratification $(X_s)_{s\in\mathscr{S}}$ of X, the category $D^{\mathrm{b}}_{\mathscr{S}}(X,\Bbbk)$ is a full triangulated subcategory of $D^{\mathrm{b}}(X,\Bbbk)$.*
(2) *The category $D^{\mathrm{b}}_{\mathrm{c}}(X,\Bbbk)$ is a full triangulated subcategory of $D^{\mathrm{b}}(X,\Bbbk)$.*

First results on constructibility. One of the main goals of this chapter is to show that all sheaf functors send constructible complexes to constructible complexes. In this section, we will cover a few easy cases.

PROPOSITION 2.3.14. *Let $f : X \to Y$ be a morphism of varieties. For any $\mathcal{F} \in D^{\mathrm{b}}_{\mathrm{c}}(Y,\Bbbk)$, we have $f^*\mathcal{F} \in D^{\mathrm{b}}_{\mathrm{c}}(X,\Bbbk)$.*

PROOF. Let $(Y_t)_{t\in\mathscr{T}}$ be a stratification of Y with respect to which $\mathcal{F}$ is constructible. Choose a stratification $(X_s)_{s\in\mathscr{S}}$ of X such that each preimage $f^{-1}(Y_t)$ of a stratum of Y is a union of strata. Since $\mathcal{F}|_{Y_t}$ is a bounded complex with locally constant cohomology sheaves of finite type, the same holds for $(f^*\mathcal{F})|_{f^{-1}(Y_t)}$, and hence for each $(f^*\mathcal{F})|_{X_s}$. Thus, $f^*\mathcal{F}$ is constructible with respect to $\mathscr{S}$. □

PROPOSITION 2.3.15. *Let X be a variety, and let $h : Y \hookrightarrow X$ be an inclusion of a locally closed subvariety. For any $\mathcal{F} \in D^{\mathrm{b}}_{\mathrm{c}}(Y,\Bbbk)$, we have $h_!\mathcal{F} \in D^{\mathrm{b}}_{\mathrm{c}}(X,\Bbbk)$. In particular, if Y is closed, then $h_*\mathcal{F} \in D^{\mathrm{b}}_{\mathrm{c}}(X,\Bbbk)$.*

PROOF. Choose a stratification $(X_s)_{s\in\mathscr{S}}$ of X such that Y is a union of strata, and such that $\mathcal{F}$ is constructible with respect to the induced stratification of Y. If $X_s \subset Y$, then $(h_!\mathcal{F})|_{X_s} \cong \mathcal{F}|_{X_s}$, and this lies in $D^{\mathrm{b}}_{\mathrm{locf}}(X_s,\Bbbk)$ by assumption. If $X_s \not\subset Y$, then $X_s \cap Y = \varnothing$, so $(h_!\mathcal{F})|_{X_s} = 0$, and this trivially lies in $D^{\mathrm{b}}_{\mathrm{locf}}(X_s,\Bbbk)$. □

LEMMA 2.3.16. *Let X be a variety, and let $U \subset X$ be an open subset. If $\mathcal{F} \in D^{\mathrm{b}}(X,\Bbbk)$ is such that $\mathcal{F}|_U$ and $\mathcal{F}|_{X\smallsetminus U}$ are constructible, then $\mathcal{F}$ is constructible.*

PROOF. This follows from Proposition 2.3.15 and the distinguished triangle

$$j_!(\mathcal{F}|_U) \to \mathcal{F} \to i_*(\mathcal{F}|_{X\smallsetminus U}) \to,$$

where $j : U \hookrightarrow X$ and $i : X \smallsetminus U \hookrightarrow X$ are the inclusion maps. □

LEMMA 2.3.17. *Let X be a smooth, connected variety. For $\mathcal{F},\mathcal{G} \in D^{\mathrm{b}}_{\mathrm{locf}}(X,\Bbbk)$, both $\mathcal{F} \otimes^L \mathcal{G}$ and $R\mathscr{H}om(\mathcal{F},\mathcal{G})$ lie in $D^{\mathrm{b}}_{\mathrm{locf}}(X,\Bbbk)$.*

PROOF SKETCH. A routine argument with truncation and induction on the number of nonzero cohomology sheaves lets us reduce these claims to the case where $\mathcal{F}$ is a sheaf (and hence a local system of finite type). We assume this from now on. It is enough to show that each point x has an analytic neighborhood U such that the objects $(\mathcal{F}\otimes^L\mathcal{G})|_U \cong (\mathcal{F}|_U)\otimes^L(\mathcal{G}|_U)$ and $R\mathscr{H}om(\mathcal{F},\mathcal{G})|_U \cong R\mathscr{H}om(\mathcal{F}|_U,\mathcal{G}|_U)$ lie in $D^{\mathrm{b}}_{\mathrm{locf}}(U,\Bbbk)$. In fact, let us choose U to be a contractible neighborhood of X. Then $\mathcal{F}|_U$ is a constant sheaf, say $\mathcal{F}|_U = \underline{M}_U$.

We have reduced the problem to showing that for any finitely generated $\Bbbk$-module, the objects $\underline{M}_U \otimes^L (\mathcal{G}|_U)$ and $R\mathscr{H}om(\underline{M}_U,\mathcal{G}|_U)$ lie in $D^{\mathrm{b}}_{\mathrm{locf}}(U,\Bbbk)$. If M is a free $\Bbbk$-module, these claims follow from Proposition 1.4.4. For general M, the claim follows by induction on the projective dimension of M (see the proof of Proposition 1.5.9 for a similar argument). □

PROPOSITION 2.3.18. *Let X be a variety. If $\mathcal{F}$ and $\mathcal{G}$ are objects in $D^{\mathrm{b}}_{\mathrm{c}}(X,\Bbbk)$ (resp. $D^{\mathrm{b}}_{\mathscr{S}}(X,\Bbbk)$ for some stratification $\mathscr{S}$), then $\mathcal{F} \otimes^L \mathcal{G}$ also lies in $D^{\mathrm{b}}_{\mathrm{c}}(X,\Bbbk)$ (resp. $D^{\mathrm{b}}_{\mathscr{S}}(X,\Bbbk)$).*

Proof. If $\mathcal{F}$ and $\mathcal{G}$ are both constructible with respect to $(X_s)_{s\in\mathscr{S}}$, then Lemma 2.3.17 implies that $(\mathcal{F}\otimes^L\mathcal{G})|_{X_s}\cong(\mathcal{F}|_{X_s})\otimes^L(\mathcal{G}|_{X_s})$ lies in $D^{\mathrm{b}}_{\mathrm{locf}}(X_s,\Bbbk)$ for each s. In other words, $\mathcal{F}\otimes^L\mathcal{G}$ is also constructible with respect to $\mathscr{S}$. □

Lemma 2.3.19. *Let $f: X\to Y$ be a finite morphism. If $\mathcal{F}\in D^{\mathrm{b}}_{\mathrm{c}}(X,\Bbbk)$, then $f_*\mathcal{F}\in D^{\mathrm{b}}_{\mathrm{c}}(Y,\Bbbk)$. If $\mathcal{F}$ is a sheaf, then* $\dim\operatorname{supp} f_*\mathcal{F}=\dim\operatorname{supp}\mathcal{F}$ *and* $\operatorname{mdsupp} f_*\mathcal{F}=\operatorname{mdsupp}\mathcal{F}$.

Proof. Although the first assertion is stated for general objects of $D^{\mathrm{b}}_{\mathrm{c}}(X,\Bbbk)$, it follows from the case of a constructible sheaf. We therefore assume throughout that $\mathcal{F}$ is a constructible sheaf.

We proceed by noetherian induction on X. Since f is a closed map, it can be factored as a finite map $X\to f(X)$ followed by a closed embedding $f(X)\hookrightarrow Y$. In view of Proposition 2.3.15, it is enough to prove the present lemma in the case where f is surjective. We henceforth assume that this is the case.

Choose a smooth, connected, open subset $V\subset Y$ such that $f|_{f^{-1}(V)}: f^{-1}(V)\to V$ is étale (see Lemma 2.1.16). Then $f^{-1}(V)$ is smooth. Since $\mathcal{F}$ is constructible, there is an open subset $U\subset f^{-1}(V)$ such that $\mathcal{F}|_U$ is a local system. The set $\overline{U}\smallsetminus U$ has smaller dimension than U (or V), so $f(\overline{U}\smallsetminus U)\cap V$ is a proper closed subset of V. Let $V'=V\smallsetminus f(\overline{U}\smallsetminus U)$, and let $U'=U\cap f^{-1}(V')$. Our construction implies that U' is both open and closed as a subset of $f^{-1}(V')$, and that $f|_{U'}: U'\to V'$ is finite, étale, and surjective, and hence a covering map. Let $\mathcal{L}=\mathcal{F}|_{U'}$. By Corollary 1.9.6, $(f|_{U'})_*\mathcal{L}$ is a local system on V'. Moreover, a stalk calculation like that in the proof of Lemma 2.1.9 shows that it is a local system of finite type, and that

$$\operatorname{grade}(f|_{U'})_*\mathcal{L}=\operatorname{grade}\mathcal{L}. \tag{2.3.1}$$

Let $j: U'\hookrightarrow X$ and $h: V'\hookrightarrow Y$ be the inclusion maps, and let $i: Z\hookrightarrow X$ be the inclusion of the complement of U'. We have a short exact sequence

$$0\to j_!\mathcal{L}\to\mathcal{F}\to i_*i^*\mathcal{F}\to 0. \tag{2.3.2}$$

By Lemma 2.1.9, f_* sends this to a short exact sequence; namely,

$$0\to h_!((f|_{U'})_*\mathcal{L})\to f_*\mathcal{F}\to(f\circ i)_*i^*\mathcal{F}\to 0. \tag{2.3.3}$$

By noetherian induction, push-forward along $f\circ i: Z\to Y$ sends constructible sheaves to constructible sheaves and preserves dim supp and mdsupp. On the other hand, $h_!((f|_{U'})_*\mathcal{L})$ is constructible; its support has the same dimension as $j_!\mathcal{L}$; and (2.3.1) implies that $\operatorname{mdsupp} h_!((f|_{U'})_*\mathcal{L})=\operatorname{mdsupp} j_!\mathcal{L}$. Combining these observations with Lemma 2.3.11 applied to (2.3.2) and (2.3.3), we conclude that $f_*\mathcal{F}$ is constructible, and that $\dim\operatorname{supp}\mathcal{F}=\dim\operatorname{supp} f_*\mathcal{F}$ and $\operatorname{mdsupp}\mathcal{F}=\operatorname{mdsupp} f_*\mathcal{F}$. □

Good stratifications. Stratifications with the following property are particularly desirable.

Definition 2.3.20. Let X be a variety equipped with a stratification $(X_s)_{s\in\mathscr{S}}$. For each $s\in\mathscr{S}$, let $j_s: X_s\hookrightarrow X$ be the inclusion map. This stratification is said to be a **good stratification** if for any $s\in\mathscr{S}$ and any local system of finite type $\mathcal{L}$ on X_s, the object $j_{s*}\mathcal{L}$ belongs to $D^{\mathrm{b}}_{\mathscr{S}}(X,\Bbbk)$.

For examples of good stratifications, see Lemma 2.4.2 and Exercise 6.5.2.

REMARK 2.3.21. Since we will work almost exclusively with constructible complexes throughout the rest of the book, it would seem desirable to have some criteria for checking whether a given stratification is good. This is a difficult problem whose study requires substantial input from analysis and differential geometry. Perhaps the best known sufficient conditions for ensuring that a given stratification is good are the **Whitney regularity conditions** [**243**]. For a comprehensive discussion of this and other regularity conditions on stratifications, see [**206**, Chapter 4]. If one wishes to study constructible sheaves outside the setting of algebraic varieties (say, with respect to analytic or topological stratifications), then a thorough study of regularity conditions is indispensable.

In this book, however, we will completely ignore the problem of determining which stratifications are good (except for one easy case in Lemma 2.4.2). We can get away with this thanks to a special feature of the algebraic situation: if $j : X_s \hookrightarrow X$ is the inclusion of a stratum in an *arbitrary* (algebraic) stratification, then we will see in Theorem 2.7.1 that $j_{s*}\mathcal{L}$ is *always* constructible, possibly with respect to a finer (but still algebraic) stratification.

LEMMA 2.3.22. *Let X be a variety, and let $(X_s)_{s\in\mathscr{S}}$ be a good stratification. Let $Y \subset X$ be a locally closed subvariety that is a union of strata, and let $h : Y \hookrightarrow X$ be the inclusion map. The functors h_*, $h_!$, h^*, and $h^!$ all take constructible complexes with respect to $\mathscr{S}$ to constructible complexes with respect to $\mathscr{S}$.*

PROOF. In this proof, the term "constructible" should always be implicitly understood to mean "with respect to $\mathscr{S}$." For h^*, the result is immediate from the definitions, and for $h_!$, it holds by the same reasoning as in Proposition 2.3.15.

Next, consider h_*. We wish to show that for $\mathcal{F} \in D^{\mathrm{b}}_{\mathscr{S}}(Y, \Bbbk)$, we have $h_*\mathcal{F} \in D^{\mathrm{b}}_{\mathscr{S}}(X, \Bbbk)$. We proceed by induction on the number of strata in Y. Suppose first that Y is a single stratum, say $Y = X_s$, so that $h = j_s$, and $\mathcal{F} \in D^{\mathrm{b}}_{\mathrm{locf}}(X_s, \Bbbk)$. By truncation and induction on the number of nonzero cohomology sheaves, one can reduce to the case where $\mathcal{F}$ is a local system. In this case, $j_{s*}\mathcal{F}$ lies in $D^{\mathrm{b}}_{\mathscr{S}}(X, \Bbbk)$ by the definition of a good stratification.

If Y contains more than one stratum, choose a stratum X_s contained in Y that is minimal for the closure partial order. In other words, X_s is closed in Y. Let $Y' = Y \smallsetminus X_s$, and let $j : Y' \hookrightarrow Y$ and $i : X_s \hookrightarrow Y$ be the inclusion maps. We also let $h' = h \circ j : Y' \hookrightarrow X$. By induction, h'_* preserves constructibility.

Consider the triangle

$$i_*i^!\mathcal{F} \to \mathcal{F} \to j_*j^*\mathcal{F} \to .$$

The last term can be identified with $(h'_*j^*\mathcal{F})|_Y$, so it is constructible. It follows that the first term is as well, as is $i^!\mathcal{F} \cong i^*(i_*i^!\mathcal{F})$. Now apply h_* to obtain

$$j_{s*}i^!\mathcal{F} \to h_*\mathcal{F} \to h'_*j^*\mathcal{F} \to .$$

We already know that j_{s*} and h'_* preserve constructibility, so the first and last terms are constructible, and hence so is $h_*\mathcal{F}$.

Finally, let us consider $h^!$. Factoring h as an open embedding followed by a closed embedding, it of course suffices to treat these cases separately. The case of an open embedding is clear (because it is the same as h^*). Assume, therefore, that $h : Y \hookrightarrow X$ is a closed embedding. Let $j : X \smallsetminus Y \hookrightarrow X$ be the complementary open embedding. We have seen above that j_* preserves constructibility, so we see from the triangle

$$h_*h^!\mathcal{F} \to \mathcal{F} \to j_*j^*\mathcal{F} \to$$

that $h_*h^!\mathcal{F}$ and hence $h^!\mathcal{F} \cong h^*(h_*h^!\mathcal{F})$ are constructible. □

Remark 2.3.23. The finiteness condition in the definition of constructibility did not play any role in the proof of Lemma 2.3.22, so the same argument can be used to prove a similar statement for weakly constructible complexes. Explicitly: suppose $(X_s)_{s\in\mathscr{S}}$ has the property that for any local system $\mathcal{L}$ on X_s (not necessarily of finite type), the object $j_{s*}\mathcal{L}$ is a weakly constructible complex in $D^+(X,\Bbbk)$. Then, for $h : Y \hookrightarrow X$ as in Lemma 2.3.22, the functors h_*, $h_!$, h^*, and $h^!$ all take weakly constructible complexes with respect to $\mathscr{S}$ to weakly constructible complexes with respect to $\mathscr{S}$.

Exercises.

2.3.1. Let $(X_s)_{s\in\mathscr{S}}$ be a stratification of X, and for each $s \in \mathscr{S}$, let $j_s : X_s \hookrightarrow X$ be the inclusion map. Show that a morphism $\phi : \mathcal{F} \to \mathcal{G}$ in $D^+(X,\Bbbk)$ is an isomorphism if and only if $j_s^!\phi : j_s^!\mathcal{F} \to j_s^!\mathcal{G}$ is an isomorphism for all $s \in \mathscr{S}$.

2.3.2. Let $f : X \to Y$ be a smooth morphism, and let $(Y_s)_{s\in\mathscr{S}}$ be a stratification of Y. Show that the set of connected components of preimages of strata of Y constitutes a stratification of X.

2.4. Divisors with simple normal crossings

Let X be a smooth variety of dimension n. Given a divisor with simple normal crossings $Z \subset X$ with irreducible components $Z_1, \ldots, Z_k$, we can partition X into subvarieties labelled by subsets $I \subset \{1, \ldots, k\}$ by setting

$$X_I = \{x \in X \mid x \in Z_i \text{ if and only if } i \in I\} = \{x \in X \mid I(x) = I\}. \tag{2.4.1}$$

The nonempty sets in the collection $(X_I)_{I\subset\{1,\ldots,k\}}$ constitute a stratification of X, called the **normal crossings stratification**. The closure of X_I is the set Z_I defined in (2.1.2). The set $X_\varnothing = X \smallsetminus Z$ is the unique open stratum. More generally, for any subset $I \subset \{1, \ldots, k\}$, the set

$$X_{\subset I} = \bigcup_{J\subset I} X_J$$

is open. In this section, we will prove that a normal crossings stratification is always a good stratification, and we will prove some base change results for smooth maps that are transverse to a divisor with simple normal crossings.

For the following lemma, recall that a **polydisc** in $\mathbb{C}^n$ is an open subset that is the product of n Euclidean balls in $\mathbb{C}$.

Lemma 2.4.1. *Let X be a smooth variety of dimension n, and let $Z \subset X$ be a divisor with simple normal crossings, with irreducible components $Z_1, \ldots, Z_k$. Let $J \subset \{1, \ldots, k\}$, and suppose $J = \{i_1, i_2, \ldots, i_j\}$. For any point $x \in X_J$, there is an analytic open set $V \subset X_{\subset J}$ containing x, a polydisc $D \subset \mathbb{C}^n$, and a biholomorphism*

$$\phi : V \xrightarrow{\sim} D$$

such that for $K \subset J$, we have

$$\phi(V \cap X_K) = \left\{ (x_1, \ldots, x_n) \in D \;\middle|\; \begin{array}{c} x_k = 0 \textit{ if } 1 \le k \le j \textit{ and } i_k \in K, \textit{ and} \\ x_k \ne 0 \textit{ if } 1 \le k \le j \textit{ and } i_k \notin K \end{array} \right\}. \tag{2.4.2}$$

A neighborhood V with these properties is called a **normal crossings coordinate chart**. A point x can belong to at most n components of Z, so the assumption that $x \in X_J$ forces it to be the case that $j \le n$. The description of $\phi(V \cap X_K)$ places no restriction on the coordinates x_k when $j < k \le n$.

Of course, there is a homeomorphism $D \xrightarrow{\sim} \mathbb{C}^n$ that respects the vanishing conditions on coordinates. It is sometimes convenient to pass through such a homeomorphism: then the map ϕ in the lemma is replaced by a homeomorphism (no longer a biholomorphism)

$$\phi : V \xrightarrow{\sim} \mathbb{C}^n$$

such that $\phi(V \cap X_K)$ has a description much like (2.4.2). In the extreme cases where $K = \varnothing$ or $K = J$, we obtain the following picture:

$$\begin{tikzcd}
V \cap X_\varnothing \arrow[r, "\phi|_{V\cap X_\varnothing}", "\sim"'] \arrow[d, hook] & (\mathbb{C}^\times)^j \times \mathbb{C}^{n-j} \arrow[d, hook] \\
V \arrow[r, "\phi", "\sim"'] & \mathbb{C}^n = \mathbb{C}^j \times \mathbb{C}^{n-j} \\
V \cap X_J \arrow[r, "\phi|_{V\cap X_J}", "\sim"'] \arrow[u, hook] & \{0\} \times \mathbb{C}^{n-j} \arrow[u, hook]
\end{tikzcd} \tag{2.4.3}$$

Proof. For simplicity, assume that $J = \{1, \ldots, j\}$. Let U be an affine neighborhood of x such that for $i \in J$, $Z_i \cap U$ is defined by a function $f_i \in \mathbb{C}[U]$ with nonzero differential at x. By definition, the differentials $df_1, \ldots, df_j \in T^*_x X$ are linearly independent. By replacing U by a smaller open set if necessary, we may assume that U does not meet the Z_i with $i \notin J$.

Choose functions $g_{j+1}, g_{j+2}, \ldots, g_n$ that vanish at x and such that together, the differentials

$$\{df_1, \ldots, df_j, dg_{j+1}, \ldots, dg_n\}$$

form a basis for $T^*_x X$. Define $\phi : U \to \mathbb{C}^n$ to be the map given by

$$\phi(u) = (f_1(u), \ldots, f_j(u), g_{j+1}(u), \ldots, g_n(u)).$$

Then $\phi(x) = 0$, and by construction, ϕ has nonsingular Jacobian matrix at x, so by the holomorphic inverse function theorem, there is an analytic neighborhood V of x and an analytic neighborhood V' of $0 \in \mathbb{C}^n$ such that ϕ restricts to a biholomorphism $\phi : V \xrightarrow{\sim} V'$. Moreover, if $K \subset J$, then we clearly have

$$\begin{aligned}
\phi(V \cap X_K) &= \phi(\{x \in V \mid \text{for } k \in J,\ f_k(x) = 0 \text{ if and only if } k \in K\}) \\
&= \left\{ (x_1, \ldots, x_n) \in V' \;\middle|\; \begin{array}{r} x_k = 0 \text{ if } 1 \le k \le j \text{ and } k \in K, \text{ and} \\ x_k \ne 0 \text{ if } 1 \le k \le j \text{ and } k \notin K \end{array} \right\}.
\end{aligned}$$

Replacing V' (and V) by smaller open sets if necessary, we may assume that V' is a product of Euclidean discs in $\mathbb{C}$. $\square$

Lemma 2.4.2. *Let X be a smooth variety, and let Z be a divisor with simple normal crossings. Let $j_I : X_I \hookrightarrow X$ be the inclusion of a stratum in the normal crossings stratification. If $\mathcal{L}$ is a local system on the stratum X_I, then $j_{I*}\mathcal{L}$ is weakly constructible with respect to the normal crossings stratification. If $\mathcal{L}$ is of finite* *type, then* $j_{I*}\mathcal{L}$ *is constructible. Moreover,* $\mathsf{H}^i(j_*\mathcal{L})$ *is supported on a subvariety of codimension at least* $|I| + i$.

In particular, this lemma says that the normal crossings stratification is a good stratification.

PROOF. We first treat the special case $I = \varnothing$. For brevity, let us put $j = j_\varnothing : X_\varnothing \hookrightarrow X$. We will show below that for any subset $J \subset \{1, \ldots, k\}$, the sheaf $\mathsf{H}^i(j_*\mathcal{L})|_{X_J}$ is a local system (of finite type if $\mathcal{L}$ is), and that it vanishes unless $0 \le i \le |J|$. Note that this implies that $\mathsf{H}^i(j_*\mathcal{L})$ is supported on the union of the X_J with $|J| \ge i$. Since $\operatorname{codim} X_J = |J|$, it follows that $\operatorname{codim}\operatorname{supp}\mathsf{H}^i(j_*\mathcal{L}) \ge i$.

For simplicity, assume that $J = \{1, \ldots, j\}$. Given a point $x \in X_J$, it is enough to exhibit an analytic neighborhood V such that each $\mathsf{H}^i(j_*\mathcal{L})|_{V \cap X_J}$ is a local system (of finite type if $\mathcal{L}$ is), vanishing unless $0 \le i \le |J|$.

Let V be a normal crossings coordinate chart around x, as in Lemma 2.4.1. We can then transfer our problem to the right-hand column of the commutative diagram (2.4.3). The desired properties hold there by Corollary B.2.5. This completes the proof when $I = \varnothing$.

For general I, $j_{I*}\mathcal{L}$ is supported on the closed smooth variety Z_I. The result follows by applying the special case above with X and Z replaced by Z_I and $Z_I \smallsetminus X_I$, respectively. □

REMARK 2.4.3. Lemma 2.4.2 tells us that a normal crossings stratification satisfies the hypotheses of Remark 2.3.23, so the weakly constructible analogue of Lemma 2.3.22 holds for normal crossings stratifications.

LEMMA 2.4.4. *Let* $f : X \to Y$ *be a smooth morphism of smooth varieties, and let* $Z \subset X$ *be a divisor with simple normal crossings. Assume that* f *is a transverse locally trivial fibration with respect to* Z. *Let* $h : W \hookrightarrow X$ *be the inclusion map of a locally closed union of strata for the normal crossings stratification* $\mathscr{S}$. *Let* $y \in Y$, *and consider the cartesian diagram*

$$\begin{array}{ccc} f^{-1}(y) \cap W & \longrightarrow & W \\ \Big\downarrow{\scriptstyle h_y} & & \Big\downarrow{\scriptstyle h} \\ f^{-1}(y) & \longrightarrow & X \end{array}$$

If $\mathcal{F} \in D^+(W, \Bbbk)$ *is weakly constructible with respect to* $\mathscr{S}$, *then the base change map* $(h_*\mathcal{F})|_{f^{-1}(y)} \xrightarrow{\sim} h_{y*}(\mathcal{F}|_{f^{-1}(y)})$ *is an isomorphism.*

PROOF. We proceed by induction on the number of strata in W. If W consists of a single stratum X_I, then $\mathcal{F}$ belongs to $D^+_{\mathrm{loc}}(X_I, \Bbbk)$. Let U be a contractible analytic neighborhood of y such that there is a diffeomorphism $b : f^{-1}(U) \to f^{-1}(y) \times U$ as in Definition 2.1.20. We decompose the cartesian square into the following larger diagram:

$$\begin{array}{ccccccc} f^{-1}(y) \cap X_I & \hookrightarrow & (f^{-1}(y) \cap X_I) \times U & \xrightarrow[\sim]{b^{-1}} & f^{-1}(U) \cap X_I & \hookrightarrow & X_I \\ \Big\downarrow & & \Big\downarrow & & \Big\downarrow & & \Big\downarrow \\ f^{-1}(y) & \hookrightarrow & f^{-1}(y) \times U & \xrightarrow[\sim]{b^{-1}} & f^{-1}(U) & \hookrightarrow & X \end{array}$$

Here, the base change morphism for the rightmost square is an isomorphism by Proposition 1.2.16; for the middle square, it is an isomorphism because the horizontal maps are homeomorphisms; and for the leftmost square, it is an isomorphism by Proposition 1.9.4.

Now suppose that W contains more than one stratum. Let X_I be a stratum that is closed in W, and let $W' = W \smallsetminus X_I$. Let $u : W' \hookrightarrow W$ and $i : X_I \hookrightarrow W$ be the inclusion maps, so that we have a distinguished triangle

$$i_* i^! \mathcal{F} \to \mathcal{F} \to u_*(\mathcal{F}|_{W'}) \to . \tag{2.4.4}$$

Next, let $h' = h \circ u$ and $j_I = h \circ i$. We also let $h'_y : f^{-1}(y) \cap W' \hookrightarrow f^{-1}(y)$ and $j_y : f^{-1}(y) \cap X_I \hookrightarrow f^{-1}(y)$ be the inclusion maps. Consider the following commutative diagram:

$$\begin{array}{ccccccc} (j_{I*} i^! \mathcal{F})|_{f^{-1}(y)} & \longrightarrow & (h_* \mathcal{F})|_{f^{-1}(y)} & \longrightarrow & (h'_*(\mathcal{F}|_{W'}))|_{f^{-1}(y)} & \longrightarrow \\ \downarrow & & \downarrow & & \downarrow & \\ j_{y*}((i^! \mathcal{F})|_{f^{-1}(y) \cap X_I}) & \longrightarrow & h_{y*}(\mathcal{F}|_{f^{-1}(y) \cap W}) & \longrightarrow & h'_{y*}(\mathcal{F}|_{f^{-1}(y) \cap W'}) & \longrightarrow \end{array}$$

Here, the top row is obtained by applying $(h_*(-))|_{f^{-1}(y)}$ to (2.4.4), and the bottom row by applying $h_{y*}((-)|_{f^{-1}(y) \cap W})$. The vertical maps are base change morphisms. The left- and right-hand vertical maps are isomorphisms by induction, so the middle one is as well. $\square$

Lemma 2.4.5. *Let $f : X \to Y$ be a smooth morphism of smooth varieties, and let $Z \subset X$ be a divisor with simple normal crossings. Assume that f is a transverse locally trivial fibration with respect to Z. If $\mathcal{F} \in D^+(X, \Bbbk)$ is weakly constructible with respect to the normal crossings stratification, then $f_* \mathcal{F} \in D^+_{\mathrm{loc}}(Y, \Bbbk)$. Moreover, for any point $y \in Y$, the base change map $(f_* \mathcal{F})_y \xrightarrow{\sim} R\Gamma(\mathcal{F}|_{f^{-1}(y)})$ is an isomorphism.*

Later in this chapter, we will see that if $\mathcal{F}$ is constructible, then $f_* \mathcal{F}$ actually belongs to $D^{\mathrm{b}}_{\mathrm{locf}}(Y, \Bbbk)$ (i.e., it is constructible).

Proof. Suppose Z has irreducible components $Z_1, \ldots, Z_k$. We proceed by induction on k. If $k = 0$, i.e., if Z is empty, then $\mathcal{F}$ lies in $D^+_{\mathrm{loc}}(X, \Bbbk)$. In this case, the conclusions hold by Theorem 1.9.5.

Otherwise, let $X' = X \smallsetminus Z_1$, and let $Z' = Z \cap X'$. Let $j : X' \hookrightarrow X$ and $i : Z_1 \hookrightarrow X$ be the inclusion maps. Observe that $Z' \subset X'$ is a divisor with simple normal crossings and $k - 1$ components. Similarly, the variety

$$Z_1 \smallsetminus X_1 = \bigcup_{j \neq 1} Z_1 \cap Z_j$$

is a divisor with simple normal crossings in Z_1 with at most $k - 1$ components. Our assumptions imply that the restrictions

$$f|_{X'} : X' \to Y \qquad \text{and} \qquad f|_{Z_1} : Z_1 \to Y$$

satisfy the assumptions of the lemma. In particular, they are transverse locally trivial fibrations with respect to $Z' \subset X'$ and $Z_1 \smallsetminus X_1 \subset Z_1$, respectively. By induction, the result holds on these spaces.

Form the distinguished triangle

$$i_* i^! \mathcal{F} \to \mathcal{F} \to j_* j^* \mathcal{F} \to, \tag{2.4.5}$$

and then apply f_* to obtain the triangle

$$(f|_{Z_1})_* i^! \mathcal{F} \to f_* \mathcal{F} \to (f|_{X'})_* j^* \mathcal{F} \to .$$

The objects $i^!\mathcal{F}$ and $j^*\mathcal{F}$ are weakly constructible with respect to the normal crossings stratification (for the former, use Remark 2.4.3), so by induction, the first and last terms of this triangle belong to $D^+_{\mathrm{loc}}(Y, \Bbbk)$, so the middle term does as well.

It remains to compute the stalk of $f_*\mathcal{F}$ at a point $y \in Y$. Let $X'_y = X' \cap f^{-1}(y)$ and $Z_{1,y} = Z_1 \cap f^{-1}(y)$, and let $j_y : X'_y \hookrightarrow f^{-1}(y)$ and $i_y : Z_{1,y} \hookrightarrow f^{-1}(y)$ be the inclusion maps. Restricting (2.4.5) to $f^{-1}(y)$ gives us the distinguished triangle

$$i_{y*}((i^!\mathcal{F})|_{Z_{1,y}}) \to \mathcal{F}|_{f^{-1}(y)} \to j_{y*}(\mathcal{F}|_{X'_y}) \to,$$

where, for the last term, we have used the isomorphism $(j_*j^*\mathcal{F})|_{f^{-1}(y)} \cong j_{y*}(\mathcal{F}|_{X'_y})$ from Lemma 2.4.4. Now, by the naturality of the base change map, we get a commuative diagram

$$\begin{array}{ccccccc}
((f|_{Z_1})_* i^!\mathcal{F})_y & \longrightarrow & (f_*\mathcal{F})_y & \longrightarrow & ((f|_{X'})_* j^*\mathcal{F})_y & \longrightarrow \\
\downarrow{\wr} & & \downarrow & & \downarrow{\wr} & \\
R\Gamma((i^!\mathcal{F})|_{Z_{1,y}}) & \longrightarrow & R\Gamma(\mathcal{F}|_{f^{-1}(y)}) & \longrightarrow & R\Gamma(\mathcal{F}|_{X'_y}) & \longrightarrow
\end{array}$$

The first and third vertical arrows are isomorphisms by induction, so the middle one is as well. $\square$

2.5. Base change and the affine line

Let $f : X \to Y$ be a morphism of varieties, and consider the cartesian square

$$\begin{array}{ccc}
X \times \mathbb{A}^1 & \xrightarrow{f' = f \times \mathrm{id}_{\mathbb{A}^1}} & Y \times \mathbb{A}^1 \\
{\scriptstyle \mathrm{pr}_1}\downarrow & & \downarrow{\scriptstyle \mathrm{pr}_1} \\
X & \xrightarrow{f} & Y
\end{array}$$

For $\mathcal{F} \in D^+(Y \times \mathbb{A}^1, \Bbbk)$, we have a base change map $f^*\mathrm{pr}_{1*}\mathcal{F} \to \mathrm{pr}_{1*}(f')^*\mathcal{F}$. This is certainly not an isomorphism in general, but by Theorem 1.9.5, it is an isomorphism when $\mathcal{F} \in D^+_{\mathrm{loc}}(Y \times \mathbb{A}^1, \Bbbk)$. In this section, we will find some other conditions under which it is an isomorphism, and we will carry out some computations related to $\mathrm{pr}_{1*}\mathcal{F}$. The statements in this section are somewhat technical, but they will play a crucial role in the proofs in Section 2.6.

For brevity, we introduce the following terminology: a closed subvariety $Z \subset X \times \mathbb{A}^1$ will be said to be **finite** (resp. **surjective**, **étale**) **over** X if the map $\mathrm{pr}_1|_Z : Z \to X$ is finite (resp. surjective, étale). The following lemma provides a supply of examples of this situation.

LEMMA 2.5.1. *Let $f : \mathbb{A}^n \to \mathbb{A}^1$ be a nonconstant regular map, and let $Z = f^{-1}(0)$. There is a linear automorphism $\phi : \mathbb{A}^n \xrightarrow{\sim} \mathbb{A}^n$ such that $\phi(Z) \subset \mathbb{A}^{n-1} \times \mathbb{A}^1$ is finite and surjective over $\mathbb{A}^{n-1}$.*

PROOF. The map f is given by some polynomial $\sum c_{a_1,\ldots,a_n} x_1^{a_1} \cdots x_n^{a_n}$. Let d be the degree of f, and let $f_{\max}$ be the sum of monomials in f of maximal degree:

$$f_{\max} = \sum_{\substack{a_1,\ldots,a_n \in \mathbb{Z}_{\geq 0} \\ a_1 + \cdots + a_n = d}} c_{a_1,\ldots,a_n} x_1^{a_1} \cdots x_n^{a_n}.$$

Choose a nonzero point $(u_1, \ldots, u_n) \in \mathbb{A}^n$ such that $f_{\max}(u_1, \ldots, u_n) \neq 0$, and then let $\phi : \mathbb{A}^n \to \mathbb{A}^n$ be a linear automorphism of $\mathbb{A}^n$ such that $\phi(u_1, \ldots, u_n) = (0, \ldots, 0, 1)$. Then $\phi(Z)$ is defined by the polynomial $f \circ \phi^{-1}$. For brevity, let us simply replace Z by $\phi(Z)$ and f by $f \circ \phi^{-1}$. That is, we will not explicitly mention ϕ in the rest of the proof.

Our set-up implies that the monomidal x_n^d occurs with nonzero coefficient in $f_{\max}$. We may assume that this coefficient is 1, i.e., that f is monic as a polynomial in x_n. It follows that the ring $\mathbb{C}[x_1, \ldots, x_n]/(f)$ is finitely generated as a module over $\mathbb{C}[x_1, \ldots, x_{n-1}]$. The coordinate ring $\mathbb{C}[Z]$ of Z is a quotient of the (possibly nonreduced) ring $\mathbb{C}[x_1, \ldots, x_n]/(f)$, so it is also a finitely generated $\mathbb{C}[x_1, \ldots, x_{n-1}]$-module. In other words, the projection map $\mathrm{pr}_1|_Z : Z \to \mathbb{A}^{n-1}$ is finite.

Finally, given $(a_1, \ldots, a_{n-1}) \in \mathbb{A}^{n-1}$, the polynomial $f(a_1, \ldots, a_{n-1}, x_n) \in \mathbb{C}[x_n]$ is nonconstant (again because f is monic in x_n) and thus has solutions. In other words, $\mathrm{pr}_1^{-1}(a_1, \ldots, a_{n-1}) \cap Z$ is nonempty, so Z is surjective over $\mathbb{A}^{n-1}$. $\square$

In the next few statements, we will identify $\mathbb{A}^1$ with the open subset $\{[x : y] \mid y \neq 0\} \subset \mathbb{P}^1$, and we let $\infty = [1 : 0] \in \mathbb{P}^1$ denote the remaining point. More generally, for any variety X, we regard $X \times \mathbb{A}^1$ as an open subset of $X \times \mathbb{P}^1$; its complement is $X \times \{\infty\}$.

LEMMA 2.5.2. *Let Z be a closed subvariety of $X \times \mathbb{A}^1$ that is finite over X. Then Z remains closed as a subset of $X \times \mathbb{P}^1$.*

PROOF. We must show that no point of the form $(x, \infty) \in X \times \mathbb{P}^1$ lies in the closure of Z. Choose an analytic neighborhood B_x of x in X whose closure $\overline{B_x}$ is compact. Because $\mathrm{pr}_1|_Z : Z \to X$ is proper, the set $\mathrm{pr}_1^{-1}(\overline{B_x}) \cap Z$ is compact, so its image $\mathrm{pr}_2(\mathrm{pr}_1^{-1}(\overline{B_x}) \cap Z) \subset \mathbb{A}^1$ is compact and hence bounded. Let B_∞ be an analytic neighborhood of $\infty \in \mathbb{P}^1$ that does not meet $\mathrm{pr}_2(\mathrm{pr}^{-1}(\overline{B_x}) \cap Z)$. Then $B_x \times B_\infty$ is a neighborhood of (x, ∞) that does not meet Z, as desired. $\square$

The main result of this section is the following.

PROPOSITION 2.5.3. *Let $f : X \to Y$ be a morphism of varieties, and let $f' = f \times \mathrm{id}_{\mathbb{A}^1} : X \times \mathbb{A}^1 \to Y \times \mathbb{A}^1$, as shown below:*

$$\begin{tikzcd} X \times \mathbb{A}^1 \arrow[r, "f'"] \arrow[d, "\mathrm{pr}_1"'] & Y \times \mathbb{A}^1 \arrow[d, "\mathrm{pr}_1"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

Let $Z \subset Y \times \mathbb{A}^1$ be a closed subvariety that is finite over Y, and let $U = (Y \times \mathbb{A}^1) \smallsetminus Z$. For $\mathcal{F} \in D^+(Y \times \mathbb{A}^1, \Bbbk)$, if $\mathcal{F}|_U$ lies in $D^+_{\mathrm{loc}}(U, \Bbbk)$, then the base change map $f^*\mathrm{pr}_{1*}\mathcal{F} \to \mathrm{pr}_{1*}(f')^*\mathcal{F}$ is an isomorphism.

PROOF. We will begin by considering the special case where $f : X \to Y$ is the inclusion of a single point $\{y\}$. Let $k : Y \times \mathbb{A}^1 \hookrightarrow Y \times \mathbb{P}^1$ be the inclusion map. Our base change problem breaks up into the following two squares:

$$\begin{tikzcd} \{y\} \times \mathbb{A}^1 \arrow[r] \arrow[d, "k_y"'] & Y \times \mathbb{A}^1 \arrow[d, "k"'] \arrow[dd, bend left, "\mathrm{pr}_1"] \\ \{y\} \times \mathbb{P}^1 \arrow[r] \arrow[d] & Y \times \mathbb{P}^1 \arrow[d, "\mathrm{pr}_1"'] \\ \{y\} \arrow[r] & Y \end{tikzcd} \tag{2.5.1}$$

Since $\mathrm{pr}_1 : Y \times \mathbb{P}^1 \to Y$ is proper, the base change map for the lower square is an isomorphism. It remains to deal with the upper square. It is clear that $(k_*\mathcal{F})|_{\{y\}\times\mathbb{P}^1} \to k_{y*}(\mathcal{F}|_{\{y\}\times\mathbb{A}^1})$ is an isomorphism over $\{y\} \times \mathbb{A}^1$, so we just need to show that it is also an isomorphism in a neighborhood of (y, ∞). By Lemma 2.5.2, (y, ∞) admits an analytic neighborhood not meeting Z. We may take this neighborhood to be of the form $B_y \times B_\infty$, where $B_y \subset Y$ and $B_\infty \subset \mathbb{P}^1$ are contractible analytic open sets containing y and ∞, respectively. Let $B_\infty^\circ = B_\infty \cap \mathbb{A}^1$. Our problem concerning the upper square in (2.5.1) reduces to showing that the base change map for the diagram below is an isomorphism:

$$\begin{array}{ccc} \{y\} \times B_\infty^\circ & \longrightarrow & B_y \times B_\infty^\circ \\ \downarrow & & \downarrow \\ \{y\} \times B_\infty & \longrightarrow & B_y \times B_\infty \end{array}$$

Since $B_y \times B_\infty^\circ$ does not meet Z, $\mathcal{F}|_{B_y\times B_\infty^\circ}$ lies in $D^+_{\mathrm{loc}}(B_y \times B_\infty^\circ, \Bbbk)$, and our claim holds by Proposition 1.9.4.

We now return to the case of a general map $f : X \to Y$. Let $Z' = (f')^{-1}(Z)$, and observe that Z' is finite over X. Moreover, if we let $U' = (X \times \mathbb{A}^1) \smallsetminus Z'$, then $(f')^*\mathcal{F}$ has the property that $((f')^*\mathcal{F})|_{U'}$ lies in $D^+_{\mathrm{loc}}(U', \Bbbk)$. Now choose a point $x \in X$, and consider the diagram

$$\begin{tikzcd} \{x\} \times \mathbb{A}^1 \arrow[r, hook] \arrow[d] \arrow[rr, bend left] & X \times \mathbb{A}^1 \arrow[r, "f'"] \arrow[d, "\mathrm{pr}_1"] & Y \times \mathbb{A}^1 \arrow[d, "\mathrm{pr}_1"] \\ \{x\} \arrow[r, hook] \arrow[rr, bend right] & X \arrow[r, "f"] & Y \end{tikzcd}$$

The base change maps for the left-hand square and for the outer square are isomorphisms by the preceding paragraph, so the base change map for the right-hand square is as well. □

The rest of this section is devoted to applying Proposition 2.5.3 to carry out some computations related to $\mathrm{pr}_{1*}\mathcal{F}$, where $\mathcal{F}$ is a constructible sheaf on a variety of the form $X \times \mathbb{A}^1$.

LEMMA 2.5.4. *Let $Z \subset X \times \mathbb{A}^1$ be a closed subvariety that is finite and surjective over X. There exists a nonempty, irreducible, smooth open subset $V \subset X$ with the following properties:*

(1) *Let $Z' = (V \times \mathbb{A}^1) \cap Z$. Then Z' is a divisor with simple normal crossings in $V \times \mathbb{A}^1$, and it is finite, surjective, and étale over V.*
(2) *The map $\mathrm{pr}_1 : V \times \mathbb{A}^1 \to V$ is a transverse locally trivial fibration with respect to Z'.*

PROOF. Let $p = \mathrm{pr}_1|_Z : Z \to X$. Use Lemma 2.1.16 and Theorem 2.1.4 to find a nonempty, irreducible, smooth open subset $V \subset Y$ such that $p|_{p^{-1}(V)} : p^{-1}(V) \to V$ is finite, surjective, and étale. Note that $Z' = p^{-1}(V)$. Since V is smooth and $p|_{p^{-1}(V)}$ is étale, we see that Z' is smooth, and that $\dim Z' = \dim V = \dim(V \times \mathbb{A}^1) - 1$. In other words, Z' is (trivially) a divisor with simple normal crossings in $V \times \mathbb{A}^1$, and $\mathrm{pr}_1 : V \times \mathbb{A}^1 \to V$ is transverse to it.

For the second assertion, we will work in the larger variety $V \times \mathbb{P}^1$. By Lemma 2.5.2, Z' remains closed as a subset of $V \times \mathbb{P}^1$. Let $Z'' = Z' \cup (V \times \{\infty\})$. Then Z'' is a divisor with simple normal crossings in $V \times \mathbb{P}^1$, and $V \times \{\infty\}$ is an irreducible component of Z''. The projection map

$$\mathrm{pr}_1 : V \times \mathbb{P}^1 \to V \tag{2.5.2}$$

is proper, so by Theorem 2.1.21, it is a transverse locally trivial fibration. Since $\mathrm{pr}_1 : V \times \mathbb{A}^1 \to V$ is obtained by deleting a component of Z'' from the domain of (2.5.2), it remains a transverse locally trivial fibration. $\square$

LEMMA 2.5.5. *Let $h : V \hookrightarrow X$ be an open embedding. Let $Z \subset X \times \mathbb{A}^1$ be a closed subvariety that is finite over X, and let $U = (X \times \mathbb{A}^1) \smallsetminus Z$. Consider the cartesian square*

$$\begin{tikzcd}
V \times \mathbb{A}^1 \arrow[r, "h' = h \times \mathrm{id}_{\mathbb{A}^1}"] \arrow[d, "\mathrm{pr}_1"'] & X \times \mathbb{A}^1 \arrow[d, "\mathrm{pr}_1"] \\
V \arrow[r, "h"] & X
\end{tikzcd}$$

For $\mathcal{F} \in D^+(X \times \mathbb{A}^1, \Bbbk)$, if $\mathcal{F}|_U$ lies in $D^+_{\mathrm{loc}}(U, \Bbbk)$, then there is a natural isomorphism $h_! h^* \mathrm{pr}_{1*}\mathcal{F} \xrightarrow{\sim} \mathrm{pr}_{1*} h'_!(h')^*\mathcal{F}$.

PROOF. Let $X' = X \smallsetminus V$, and let $v : X' \hookrightarrow X$ be the inclusion map. We also let $v' = v \times \mathrm{id}_{\mathbb{A}^1} : X' \times \mathbb{A}^1 \to X \times \mathbb{A}^1$, as shown below:

$$\begin{tikzcd}
X' \times \mathbb{A}^1 \arrow[r, "v' = v \times \mathrm{id}_{\mathbb{A}^1}"] \arrow[d, "\mathrm{pr}_1"'] & X \times \mathbb{A}^1 \arrow[d, "\mathrm{pr}_1"] \\
X' \arrow[r, "v"] & X
\end{tikzcd}$$

By Proposition 2.5.3, there is a natural isomorphism $\theta : v^*\mathrm{pr}_{1*}\mathcal{F} \xrightarrow{\sim} \mathrm{pr}_{1*}(v')^*\mathcal{F}$. Apply v_* to get a natural isomorphism $v_*v^*\mathrm{pr}_{1*}\mathcal{F} \to v_*\mathrm{pr}_{1*}(v')^*\mathcal{F} \cong \mathrm{pr}_{1*}v'_*(v')^*\mathcal{F}$. This map is the third vertical map in the following diagram:

$$\begin{tikzcd}
h_!h^*\mathrm{pr}_{1*}\mathcal{F} \arrow[r] \arrow[d, dashed, "\wr"] & \mathrm{pr}_{1*}\mathcal{F} \arrow[r] \arrow[d, equal] & v_*v^*\mathrm{pr}_{1*}\mathcal{F} \arrow[r] \arrow[d, "v_*\theta", "\wr"'] & {} \\
\mathrm{pr}_{1*}h'_!(h')^*\mathcal{F} \arrow[r] & \mathrm{pr}_{1*}\mathcal{F} \arrow[r] & \mathrm{pr}_{1*}v'_*(v')^*\mathcal{F} \arrow[r] & {}
\end{tikzcd} \tag{2.5.3}$$

Then Lemma A.4.3(2) lets us fill in the first vertical arrow with an isomorphism. $\square$

LEMMA 2.5.6. *Let $Z \subset X \times \mathbb{A}^1$ be a closed subvariety that is finite and surjective over X. Let $U = (X \times \mathbb{A}^1) \smallsetminus Z$, and let $\mathcal{F}$ be a constructible sheaf on $X \times \mathbb{A}^1$ such that $\mathcal{F}|_U$ is a local system.*

1. *If $\mathcal{F}|_Z = 0$, then $\mathsf{H}^k(\mathrm{pr}_{1*}\mathcal{F}) = 0$ for all $k \neq 1$.*
2. *The object $\mathrm{pr}_{1*}\mathcal{F}$ is constructible and satisfies*

$$\operatorname{mdsupp} \mathsf{H}^k(\mathrm{pr}_{1*}\mathcal{F}) \leq \operatorname{mdsupp} \mathcal{F} - k.$$

PROOF. (1) Assume that $\mathcal{F}|_Z = 0$. We must show that for any $x \in X$, we have $\mathsf{H}^k((\mathrm{pr}_{1*}\mathcal{F})_x) = 0$ for $k \neq 1$. By Proposition 2.5.3, this is equivalent to showing that $\mathbf{H}^k(\{x\} \times \mathbb{A}^1, \mathcal{F}|_{\{x\}\times\mathbb{A}^1})$ vanishes for $k \neq 1$.

Let $U_x = (\{x\} \times \mathbb{A}^1) \cap U$, and let $j_x : U_x \hookrightarrow \{x\} \times \mathbb{A}^1$ be the inclusion map. Because $\mathrm{pr}_1|_Z$ is finite and surjective, the complement of U_x in $\{x\} \times \mathbb{A}^1$ is finite and nonempty. The assumption that $\mathcal{F}|_Z = 0$ means that $\mathcal{F}$ is the extension-by-zero of a local system $\mathcal{L}$ on U. By Theorem 1.2.13, we have $\mathcal{F}|_{\{x\}\times\mathbb{A}^1} \cong j_{x!}(\mathcal{L}|_{U_x})$. Our claim about $\mathbf{H}^k(\{x\}\times\mathbb{A}^1, \mathcal{F}|_{\{x\}\times\mathbb{A}^1})$ then holds by Proposition B.3.4.

For later use, we note that Proposition B.3.4 and Remark B.3.5 actually tell us slightly more: $\mathsf{H}^1((\mathrm{pr}_{1*}\mathcal{F})_x)$ is a finitely generated module satisfying

$$\operatorname{grade}\mathsf{H}^1((\mathrm{pr}_{1*}\mathcal{F})_x) \geq \operatorname{grade}(\mathcal{F}|_U). \tag{2.5.4}$$

(2) We now drop the assumption that $\mathcal{F}|_Z = 0$. We will prove this statement by noetherian induction on X. Choose an open set $V \subset X$ as in Lemma 2.5.4. Let $X' = X \smallsetminus V$. We retain the notation from the proof of Lemma 2.5.5. By induction, $\mathrm{pr}_{1*}(v')^*\mathcal{F}$ is constructible and satisfies $\operatorname{mdsupp}\mathsf{H}^k(\mathrm{pr}_{1*}(v')^*\mathcal{F}) \leq \operatorname{mdsupp}(v')^*\mathcal{F} - k \leq \operatorname{mdsupp}\mathcal{F} - k$. The same assertions hold for $v_*\mathrm{pr}_{1*}(v')^*\mathcal{F} \cong v_*v^*\mathrm{pr}_{1*}\mathcal{F}$. Examining the top row of (2.5.3), we see that it is enough to study $h_!h^*\mathrm{pr}_{1*}\mathcal{F}$. By Proposition 2.3.15, we can further reduce to showing the claims for $h^*\mathrm{pr}_{1*}\mathcal{F}$, and then, by Proposition 1.2.16, to showing that $\mathrm{pr}_{1*}(\mathcal{F}|_{V\times\mathbb{A}^1})$ is constructible and satisfies $\operatorname{mdsupp}\mathsf{H}^k(\mathrm{pr}_{1*}(\mathcal{F}|_{V\times\mathbb{A}^1})) \leq \operatorname{mdsupp}\mathcal{F} - k$.

Let $U' = (V \times \mathbb{A}^1) \cap U$ and $Z' = (V \times \mathbb{A}^1) \cap Z$, and let $j : U' \hookrightarrow V \times \mathbb{A}^1$ and $i : Z' \hookrightarrow V \times \mathbb{A}^1$ be the inclusion maps. Let $\mathcal{G} = \mathcal{F}|_{V\times\mathbb{A}^1}$, and note that $\operatorname{mdsupp}\mathcal{G} \leq \operatorname{mdsupp}\mathcal{F}$. Consider the distinguished triangle

$$\mathrm{pr}_{1*}j_!j^*\mathcal{G} \to \mathrm{pr}_{1*}\mathcal{G} \to \mathrm{pr}_{1*}i_*i^*\mathcal{G} \to .$$

By Corollary 2.3.12, it is enough to show that the first and last terms are constructible and obey the desired mdsupp bounds. Because $\mathrm{pr}_1 \circ i : Z' \to V$ is finite, the last term has the required properties by Lemma 2.3.19. On the other hand, $j^*\mathcal{G} \cong \mathcal{F}|_{U'}$ is a local system by assumption, so $j_!j^*\mathcal{F}$ is constructible with respect to the normal crossings stratification of $V \times \mathbb{A}^1$. Lemma 2.4.5 then tells us that $\mathrm{pr}_{1*}j_!j^*\mathcal{G}$ at least belongs to $D^+_{\mathrm{loc}}(V, \Bbbk)$. It remains to show that $\mathsf{H}^k(\mathrm{pr}_{1*}j_!j^*\mathcal{G})$ is of finite type, vanishes for all but finitely many values of k, and satisfies

$$\operatorname{mdsupp}\mathsf{H}^k(\mathrm{pr}_{1*}j_!j^*\mathcal{G}) \leq \operatorname{mdsupp} j^*\mathcal{G} - k \leq \operatorname{mdsupp}\mathcal{F} - k.$$

Note that if this sheaf is nonzero, we have $\dim\operatorname{supp}\mathsf{H}^k(\mathrm{pr}_{1*}j_!j^*\mathcal{G}) = \dim V = \dim\operatorname{supp} j^*\mathcal{G} - 1$. Our problem is thus equivalent to showing that for any $x \in V$, the stalk cohomology $\mathsf{H}^k((\mathrm{pr}_{1*}j_!j^*\mathcal{G})_x)$ is finitely generated, vanishes in all but finitely many degrees, and satisfies

$$\operatorname{grade}\mathsf{H}^k((\mathrm{pr}_{1*}j_!j^*\mathcal{G})_x) + 1 \geq \operatorname{grade} j^*\mathcal{G} + k.$$

These claims hold by part (1) of the lemma (along with (2.5.4)) applied to $\mathrm{pr}_1 : V \times \mathbb{A}^1 \to V$ and to the sheaf $j_!j^*\mathcal{G}$. □

2.6. Artin's vanishing theorem

Since a smooth complex variety X of dimension n is also a manifold of dimension $2n$, one expects that $\mathbf{H}^k(X; \Bbbk) = 0$ for $k > 2n$. (Indeed, this holds without the assumption of smoothness; see Theorem 2.7.5.)

But surprisingly, for *affine* varieties, it turns out that $\mathbf{H}^k(X; \Bbbk) = 0$ for $k > n$. More generally, the hypercohomology of any constructible sheaf vanishes in degrees $> n$. In other words, affine varieties have half the expected cohomological dimension. We will prove this result—known as Artin's vanishing theorem—by an argument adapted from [**187**]. Along the way, we will establish estimates on the grades of various $\Bbbk$-modules. These estimates will be needed in Chapter 3.

PROPOSITION 2.6.1. *Let X be an affine variety of dimension n, and let $\mathcal{F} \in \mathrm{Sh}(X, \Bbbk)$ be a constructible sheaf. There is a Zariski-open subset $j : U \to X$ such that $\mathbf{H}^k(X, j_!(\mathcal{F}|_U)) = 0$ for $k \neq n$. Moreover, $\mathbf{H}^n(X, j_!(\mathcal{F}|_U))$ is a finitely generated $\Bbbk$-module that satisfies*

$$\operatorname{grade} \mathbf{H}^n(X, j_!(\mathcal{F}|_U)) \geq n - \operatorname{mdsupp} \mathcal{F}.$$

PROOF. We begin by reducing to the case where $X = \mathbb{A}^n$. If X is some other affine variety, then by the Noether normalization lemma (Lemma 2.1.8), there is a finite morphism $f : X \to \mathbb{A}^n$. By Lemmas 2.1.9 and 2.3.19, $f_*\mathcal{F}$ is a constructible sheaf on $\mathbb{A}^n$ with $\operatorname{mdsupp} f_*\mathcal{F} = \operatorname{mdsupp} \mathcal{F}$. Assume that the result is known for $\mathbb{A}^n$: let $j' : U' \hookrightarrow \mathbb{A}^n$ be an open inclusion map such that $\mathbf{H}^k(\mathbb{A}^n, j'_!((f_*\mathcal{F})|_{U'}))$ vanishes for $k \neq n$ and is finitely generated for $k = n$, with grade bounded below by $n - \operatorname{mdsupp} \mathcal{F}$. Let $U = f^{-1}(U')$, and let $j : U \hookrightarrow X$ be the inclusion map. Using the cartesian square

$$\begin{tikzcd} U \arrow[r, "j"] \arrow[d, "f|_U"'] & X \arrow[d, "f"] \\ U' \arrow[r, "j'"] & \mathbb{A}^n \end{tikzcd}$$

and the fact that f is proper, we see that

$$f_* j_!(\mathcal{F}|_U) \cong j'_!(f|_U)_*(\mathcal{F}|_U) \cong j'_!((f_*\mathcal{F})|_{U'}).$$

We deduce that

$$R\Gamma(j_!(\mathcal{F}|_U)) \cong R\Gamma(f_* j_!(\mathcal{F}|_U)) \cong R\Gamma(j'_!((f_*\mathcal{F})|_{U'})).$$

The desired properties of $\mathbf{H}^k(X, j_!(\mathcal{F}|_U))$ then follow from the case of $\mathbb{A}^n$.

For the remainder of the proof, we assume that $X = \mathbb{A}^n$. Since $\mathcal{F}$ is constructible, there is a Zariski-open subset $U \subset \mathbb{A}^n$ such that $\mathcal{F}|_U$ is a local system. Let $Z = X \smallsetminus U$. By replacing U by a smaller open set if necessary, we may assume that it is a principal open set, i.e., that Z is defined by the vanishing of some nonconstant polynomial. By Lemma 2.5.1, after a suitable change in coordinates, we may assume that the map $\mathrm{pr}_1|_Z : Z \to \mathbb{A}^{n-1}$ is finite and surjective, where $\mathrm{pr}_1 : \mathbb{A}^n \to \mathbb{A}^{n-1}$ is the projection map onto the first $n-1$ coordinates.

Let $j : U \hookrightarrow \mathbb{A}^n$ be the inclusion map, and consider the sheaf $\mathcal{F}' = j_!(\mathcal{F}|_U)$. In particular, $\mathcal{F}'|_Z = 0$. Then Lemma 2.5.6 tells us that $\mathrm{pr}_{1*}\mathcal{F}'$ is constructible and that $\mathsf{H}^k(\mathrm{pr}_{1*}\mathcal{F}')$ vanishes for $k \neq 1$. Furthermore, letting $\mathcal{F}'' = \mathsf{H}^1(\mathrm{pr}_{1*}\mathcal{F}')$, we have that $\mathcal{F}''$ is a constructible sheaf on $\mathbb{A}^{n-1}$ satisfying

$$\operatorname{mdsupp} \mathcal{F}'' \leq \operatorname{mdsupp} \mathcal{F}' - 1 \leq \operatorname{mdsupp} \mathcal{F} - 1. \tag{2.6.1}$$

If $n = 1$, then $\mathrm{pr}_{1*}\mathcal{F}' = R\Gamma(j_!(\mathcal{F}|_U))$, and

$$\operatorname{grade} \mathbf{H}^1(\mathbb{A}^n, j_!(\mathcal{F}|_U)) = -\operatorname{mdsupp} \mathcal{F}'' \geq 1 - \operatorname{mdsupp} \mathcal{F},$$

so we are done. Otherwise, we proceed by induction on n. There is an open set $h : V \hookrightarrow \mathbb{A}^{n-1}$ such that $\mathbf{H}^k(\mathbb{A}^{n-1}, h_!(\mathcal{F}''|_V))$ vanishes for $k \neq n-1$ and such that $\mathbf{H}^{n-1}(\mathbb{A}^{n-1}, h_!(\mathcal{F}''|_V))$ is finitely generated and satisfies

$$\operatorname{grade} \mathbf{H}^{n-1}(\mathbb{A}^{n-1}, h_!(\mathcal{F}''|_V)) \geq (n-1) - \operatorname{mdsupp} \mathcal{F}'' \geq n - \operatorname{mdsupp} \mathcal{F}.$$

Here, the last inequality uses (2.6.1).

Let $U' = \mathrm{pr}_1^{-1}(V) \cap U = (V \times \mathbb{A}^1) \cap U$, and let h', j', h'', and k be the inclusion maps shown below:

$$\begin{tikzcd} U' \arrow[r, "j'"] \arrow[d, "h''"'] \arrow[dr, "k"] & \mathrm{pr}_1^{-1}(V) \arrow[d, "h'"] \\ U \arrow[r, "j"] & \mathbb{A}^n \end{tikzcd}$$

Using the proper base change theorem and Lemma 2.5.5, we have

$$\begin{aligned} \mathrm{pr}_{1*}k_!(\mathcal{F}|_{U'}) \cong \mathrm{pr}_{1*}h'_!j'_!(\mathcal{F}|_{U'}) \cong \mathrm{pr}_{1*}h'_!(h')^*j_!(\mathcal{F}|_U) \cong \mathrm{pr}_{1*}h'_!(h')^*\mathcal{F}' \\ \cong h_!h^*(\mathrm{pr}_{1*}\mathcal{F}') \cong h_!(\mathcal{F}''|_V)[-1]. \end{aligned}$$

Therefore, $R\Gamma(k_!(\mathcal{F}|_{U'})) \cong R\Gamma(\mathrm{pr}_{1*}k_!(\mathcal{F}|_{U'})) \cong R\Gamma(h_!(\mathcal{F}''|_V))[-1]$. We conclude that $\mathbf{H}^k(\mathbb{A}^n, k_!(\mathcal{F}|_{U'})) = 0$ for $k \neq n$ and that $\mathbf{H}^n(\mathbb{A}^n, k_!(\mathcal{F}|_{U'}))$ has the desired properties. $\square$

The following theorem is the main result of this section.

THEOREM 2.6.2 (Artin's vanishing theorem). *Let X be an affine variety, and let $\mathcal{F}$ be a constructible sheaf on X. Then $\mathbf{H}^k(X, \mathcal{F})$ is a finitely generated $\Bbbk$-module, and it vanishes unless $0 \le k \le \dim \operatorname{supp} \mathcal{F}$. Moreover, if $\operatorname{mdsupp} \mathcal{F} = m$, then for all k, we have $\operatorname{grade} \mathbf{H}^k(X, \mathcal{F}) \ge k - m$.*

In particular, we have $\mathbf{H}^k(X; \Bbbk) = 0$ unless $0 \le k \le \dim X$.

PROOF. We proceed by noetherian induction on X. If X is a point, the statement is trivial. Otherwise, choose a Zariski-open subset $j : U \hookrightarrow X$ as follows: if $\dim \operatorname{supp} \mathcal{F} = \dim X$, invoke Proposition 2.6.1, but if $\dim \operatorname{supp} \mathcal{F} < \dim X$, choose U so that $\mathcal{F}|_U = 0$.

Let $i : Z \hookrightarrow X$ be the complementary closed subset. All three terms of the short exact sequence $0 \to j_!j^*\mathcal{F} \to \mathcal{F} \to i_*i^*\mathcal{F} \to 0$ are constructible. Applying $R\Gamma$, we obtain a distinguished triangle

$$R\Gamma(j_!(\mathcal{F}|_U)) \to R\Gamma(\mathcal{F}) \to R\Gamma(i^*\mathcal{F}) \to \tag{2.6.2}$$

in $D^+(\Bbbk\text{-mod})$. Using the long exact sequence in cohomology and Corollary A.10.12, we see that it is enough to show that if M is the first or last term of this triangle, then $\mathsf{H}^k(M)$ is finitely generated, vanishes unless $0 \le k \le \dim \operatorname{supp} \mathcal{F}$, and satisfies $\operatorname{grade} \mathsf{H}^k(M) \ge k - \operatorname{mdsupp} \mathcal{F}$.

For the the first term in (2.6.2), these claims hold by Proposition 2.6.1 if $\dim \operatorname{supp} \mathcal{F} = \dim X$, and trivially in the case where $\dim \operatorname{supp} \mathcal{F} < \dim X$. For the last term, since $\dim \operatorname{supp} i^*\mathcal{F} \le \dim \operatorname{supp} \mathcal{F}$ and $\operatorname{mdsupp} i^*\mathcal{F} \le \operatorname{mdsupp} \mathcal{F}$, the claims hold by induction. $\square$

Exercises.

2.6.1. It is known that every variety of dimension n admits a covering by at most $n+1$ affine open subsets. Use this fact to prove that if $\mathcal{F}$ is a constructible sheaf on a variety of dimension n, then $\mathbf{H}^k(X, \mathcal{F})$ is finitely generated for all k, and vanishes unless $0 \le k \le 2n$. (We will see a different proof of this later on.)

2.7. Sheaf functors and constructibility

We have already seen that the functors f^* and $\otimes^L$ preserve constructibility (in Propositions 2.3.14 and 2.3.18, respectively). In this section, we will show that f_*, $f_!$, and $R\mathscr{H}om$ also preserve constructibility. (The case of $f^!$ will be dealt with in Section 2.8.)

As an application, we obtain bounds on hypercohomology (analogous to Theorem 2.6.2) for arbitrary varieties.

THEOREM 2.7.1. *Let $f : X \to Y$ be a morphism of varieties. For any $\mathcal{F} \in D^{\mathrm{b}}_{\mathrm{c}}(X, \Bbbk)$, the objects $f_*\mathcal{F}$ and $f_!\mathcal{F}$ lie in $D^{\mathrm{b}}_{\mathrm{c}}(Y, \Bbbk)$.*

PROOF. By Nagata's compactification theorem (Theorem 2.1.7), it is enough to treat two special cases: that of an open embedding and that of a proper map. When f is an open embedding, we dealt with $f_!\mathcal{F}$ in Proposition 2.3.15. When f is a proper map, we have $f_!\mathcal{F} = f_*\mathcal{F}$.

We will prove the constructibility of $f_*\mathcal{F}$ in these two special cases simultaneously, by induction on the dimension of X. When $\dim X = 0$, f is a finite morphism, and the result holds Lemma 2.3.19. Otherwise, we proceed as follows.

Step 1. First reductions. Using truncation distinguished triangles, an induction argument on the number of nonzero cohomology sheaves of $\mathcal{F}$ lets us reduce to the case where $\mathcal{F}$ is actually a constructible sheaf. We assume for the rest of the proof that this is the case.

Next, suppose X is not irreducible. Let $i_1 : X_1 \hookrightarrow X$ be the inclusion of an irreducible component, and let $j_1 : U_1 \hookrightarrow X$ be its complement. Then we have a distinguished triangle

$$f_*j_{1!}j_1^*\mathcal{F} \to f_*\mathcal{F} \to (f \circ i_1)_*i_1^*\mathcal{F} \to .$$

The first term can be rewritten as $f'_*j'_{1!}j_1^*\mathcal{F}$, where $j'_1 : U_1 \hookrightarrow \overline{U_1}$ is the inclusion of U_1 into its closure, and $f' = f|_{\overline{U_1}} : \overline{U_1} \to Y$. Both f' and $(f \circ i_1)$ have domains with fewer irreducible components. Induction on this number lets us reduce to the case where X is irreducible. We henceforth assume that X is irreducible.

We can factor f as a dominant morphism to $\overline{f(X)}$, followed by a closed embedding into Y. We already know that the latter preserves constructibility (Proposition 2.3.15), so we may as well replace Y by $\overline{f(X)}$. We henceforth assume that f is dominant, and hence that Y is also irreducible.

Finally, since $\mathcal{F}$ is constructible, there exists a smooth open set $h : U \hookrightarrow X$ such that $\mathcal{F}|_U$ is a local system of finite type, say $\mathcal{L}$. Let $i : Z \hookrightarrow X$ be the complementary closed subset, and consider the distinguished triangles

$$i_*i^!\mathcal{F} \to \mathcal{F} \to h_*\mathcal{L} \to, \tag{2.7.1}$$

$$(f \circ i)_*i^!\mathcal{F} \to f_*\mathcal{F} \to f_*h_*\mathcal{L}. \tag{2.7.2}$$

We will study these triangles in the following steps.

Step 2. If f is an open embedding, then $f_*h_*\mathcal{L}$ is constructible. Suppose $f : X \to Y$ is an open embedding. Since we have also assumed that f is dominant, we have $Y = \overline{X}$. Let $Z' = Y \smallsetminus U$, and invoke Theorem 2.1.18 to obtain a proper map $p : \tilde{Y} \to Y$ such that $p^{-1}(U) \to U$ is an isomorphism and $p^{-1}(Z')$ is a divisor with simple normal crossings. Let $\tilde{h} : p^{-1}(U) \hookrightarrow \tilde{Y}$ be the inclusion map. Identifying $p^{-1}(U)$ with U, we see that $f_*h_*\mathcal{L} \cong p_*\tilde{h}_*\mathcal{L}$. Now, $(f_*h_*\mathcal{L})|_U \cong \mathcal{L}$ is obviously constructible, so by Lemma 2.3.16, it is enough to check that $(f_*h_*\mathcal{L})|_{Z'}$

is constructible. Let $\tilde{Z}' = p^{-1}(Z')$, and let $p' = p|_{\tilde{Z}'} : \tilde{Z}' \to Z'$. By the proper base change theorem, we have

$$(f_* h_* \mathcal{L})|_{Z'} \cong (p_* \tilde{h}_* \mathcal{L})|_{Z'} \cong p'_*((\tilde{h}_* \mathcal{L})|_{\tilde{Z}'}).$$

Lemma 2.4.2 tells us that $\tilde{h}_* \mathcal{L}$ is constructible, and then $(\tilde{h}_* \mathcal{L})|_{\tilde{Z}'}$ is clearly constructible as well. Since $\dim \tilde{Z}' < \dim U = \dim X$, the functor p'_* preserves constructibility by induction. Thus, $f_* h_* \mathcal{L}$ is constructible.

Step 3. Reduction to the study of $f_* h_* \mathcal{L}$ in general. In this step, f may be either an open embedding or a proper map. As a special case, Step 2 tells us that $h_* \mathcal{L}$ itself is constructible. From (2.7.1), we see that $i_* i^! \mathcal{F}$ is constructible, and hence so is $i^! \mathcal{F}$. Next, since $\dim Z < \dim X$, the functor $(f \circ i)_*$ preserves constructibility by induction. In view of (2.7.2), to prove that $f_* \mathcal{F}$ is constructible, it is enough to show that $f_* h_* \mathcal{L}$ is constructible.

Combining Steps 2 and 3, we conclude that $f_* \mathcal{F}$ is constructible when f is an open embedding.

Step 4. The case of a proper map with $\dim Y = 0$. Since Y is irreducible, it is a single point, and $f_* h_* \mathcal{L}$ can be identified with $R\Gamma(\mathcal{L})$. Replacing U by a smaller open set if necessary, we may assume that U is affine. Then Theorem 2.6.2 tells us that $R\Gamma(\mathcal{L})$ is constructible.

Step 5. The case of a proper map with $\dim Y > 0$. Use Theorem 2.1.18 to obtain a proper map $p : \tilde{X} \to X$ such that $p^{-1}(U) \to U$ is an isomorphism and such that the variety $\tilde{Z} = p^{-1}(Z) \subset \tilde{X}$ is a divisor with simple normal crossings. Let $\tilde{h} : p^{-1}(U) \to \tilde{X}$ be the inclusion map, and let $\tilde{f} = f \circ p : \tilde{X} \to Y$. Identifying $p^{-1}(U)$ with U, we see that $f_* h_* \mathcal{L} \cong \tilde{f}_* \tilde{h}_* \mathcal{L}$.

Let $\tilde{Z}_1, \ldots, \tilde{Z}_k$ be the irreducible components of $\tilde{Z}$. By invoking Theorem 2.1.13 several times, we can find a Zariski-open subset $V \subset Y$ such that for every subset $I \subset \{1, \ldots, k\}$, the map

$$\tilde{f}|_{\tilde{f}^{-1}(V) \cap \tilde{Z}_I} : \tilde{f}^{-1}(V) \cap \tilde{Z}_I \to V$$

is smooth and proper. Its image is thus both open and closed. But V is irreducible (because Y is), and hence connected, so the map above is surjective. Let $\tilde{V} = \tilde{f}^{-1}(V)$, and let $\tilde{f}' = \tilde{f}|_{\tilde{V}}$. The preceding observations say that $\tilde{f}' : \tilde{V} \to V$ is transverse to $\tilde{V} \cap \tilde{Z}$, and hence (by Theorem 2.1.21) a transverse locally trivial fibration.

Let us now show that $(f_* h_* \mathcal{L})|_V$ is constructible. By base change, we have

$$(f_* h_* \mathcal{L})|_V \cong (\tilde{f}_* \tilde{h}_* \mathcal{L})|_V \cong \tilde{f}'_*((\tilde{h}_* \mathcal{L})|_{\tilde{V}}).$$

Lemma 2.4.2 tells us that $\tilde{h}_* \mathcal{L}$ is constructible with respect to the normal crossings stratification of $\tilde{X}$. The open subset $\tilde{V}$ inherits this stratification, and $(\tilde{h}_* \mathcal{L})|_{\tilde{V}}$ remains constructible with respect to it. Then Lemma 2.4.5 says that $\tilde{f}'_*((\tilde{h}_* \mathcal{L})|_{\tilde{V}})$ lies in $D^+_{\mathrm{loc}}(V, \Bbbk)$. To show that it is constructible, it is enough to show that its stalk at any point $y \in V$ is constructible. By proper base change again, we have

$$\tilde{f}'_*((\tilde{h}_* \mathcal{L})|_{\tilde{V}})_y \cong (\mathrm{a}_{\tilde{f}^{-1}(y)})_*((\tilde{h}_* \mathcal{L})|_{\tilde{f}^{-1}(y)}).$$

Since $\tilde{f}$ is smooth, we have $\dim \tilde{f}^{-1}(y) = \dim \tilde{X} - \dim V = \dim X - \dim Y < \dim X$. Thus, $(\mathrm{a}_{\tilde{f}^{-1}(y)})_*$ preserves constructibility by induction. We conclude that $(f_* h_* \mathcal{L})|_V$ is constructible.

Now, let W be the closed set $Y \smallsetminus V$, and let $\tilde{W} = \tilde{f}^{-1}(W)$. To finish the proof that $f_*h_*\mathcal{L}$ is constructible, it is enough to show that $(f_*h_*\mathcal{L})|_W$ is constructible. Using proper base change one more time, we have

$$(f_*h_*\mathcal{L})|_W \cong (\tilde{f}_*\tilde{h}_*\mathcal{L})|_W \cong (\tilde{f}|_{\tilde{W}})_*((\tilde{h}_*\mathcal{L})|_{\tilde{W}}).$$

Since $\dim \tilde{W} < \dim \tilde{X} = \dim X$, we know that $(\tilde{f}|_{\tilde{W}})_*$ preserves constructibility by induction.

Combining Steps 3, 4, and 5, we conclude that $f_*\mathcal{F}$ is constructible when f is a proper map. $\square$

LEMMA 2.7.2. *Let $i : Z \hookrightarrow X$ be a closed embedding. If $\mathcal{F} \in D^{\mathrm{b}}_{\mathrm{c}}(X, \Bbbk)$, then $i^!\mathcal{F} \in D^{\mathrm{b}}_{\mathrm{c}}(Z, \Bbbk)$.*

PROOF. Let $j : U \hookrightarrow X$ be the complementary open embedding. In the distinguished triangle $i_*i^!\mathcal{F} \to \mathcal{F} \to j_*j^*\mathcal{F} \to$, the middle term is constructible by assumption, and the last term is constructible by Theorem 2.7.1. It follows that $i_*i^!\mathcal{F}$ is constructible, and hence so is $i^!\mathcal{F} \cong i^*(i_*i^!\mathcal{F})$. $\square$

PROPOSITION 2.7.3. *Let X be a variety. For any $\mathcal{F}, \mathcal{G} \in D^{\mathrm{b}}_{\mathrm{c}}(X, \Bbbk)$, we have $R\mathscr{H}om(\mathcal{F}, \mathcal{G}) \in D^{\mathrm{b}}_{\mathrm{c}}(X, \Bbbk)$.*

PROOF. We proceed by noetherian induction on X. Choose a smooth, connected Zariski-open subset $j : U \hookrightarrow X$ such that $\mathcal{F}|_U$ and $\mathcal{G}|_U$ both lie in $D^{\mathrm{b}}_{\mathrm{locf}}(U, \Bbbk)$, and let $i : Z \hookrightarrow X$ be the complementary closed subset. By Exercise 1.5.2, we have a distinguished triangle

$$i_*R\mathscr{H}om(i^*\mathcal{F}, i^!\mathcal{G}) \to R\mathscr{H}om(\mathcal{F}, \mathcal{G}) \to j_*R\mathscr{H}om(\mathcal{F}|_U, \mathcal{G}|_U) \to . \tag{2.7.3}$$

Now, $i^!\mathcal{G}$ is constructible by Lemma 2.7.2, so $R\mathscr{H}om(i^*\mathcal{F}, i^!\mathcal{G})$ is constructible by induction. On the other hand, $R\mathscr{H}om(\mathcal{F}|_U, \mathcal{G}|_U)$ is constructible by Lemma 2.3.17. The first and last terms of (2.7.3) are then constructible by Theorem 2.7.1, so the middle term is as well. $\square$

Cohomology vanishing results. The remainder of this section is devoted to vanishing bounds on the hypercohomology (both ordinary and with compact support) of a constructible sheaf.

THEOREM 2.7.4. *Let X be a variety of dimension n, and let $\mathcal{F}$ be a constructible sheaf on X. The $\Bbbk$-module $\mathbf{H}^k_{\mathrm{c}}(X, \mathcal{F})$ is finitely generated for all $\Bbbk$, and it vanishes unless $0 \le k \le 2n$.*

For a refinement of this statement that gives bounds on $\operatorname{grade} \mathbf{H}^k_{\mathrm{c}}(X, \mathcal{F})$, see Proposition 3.8.2.

PROOF. The finite generation claim is contained in the fact that $\mathrm{a}_{X!}\mathcal{F}$ is constructible. To prove the cohomology vanishing bound, we proceed by noetherian induction on X. If X is a point, the theorem is trivial. Otherwise, let $j : U \hookrightarrow X$ be a smooth Zariski-open subset of dimension n, and let $i : Z \hookrightarrow X$ be the inclusion of the complementary closed subset. Apply $R\Gamma_{\mathrm{c}}$ to the triangle $j_!j^*\mathcal{F} \to \mathcal{F} \to i_*i^*\mathcal{F} \to$ to obtain

$$R\Gamma_{\mathrm{c}}(\mathcal{F}|_U) \to R\Gamma_{\mathrm{c}}(\mathcal{F}) \to R\Gamma_{\mathrm{c}}(\mathcal{F}|_Z) \to . \tag{2.7.4}$$

Since $\dim Z \le n$, we know by induction that $\mathbf{H}^k_{\mathrm{c}}(Z, \mathcal{F}|_Z) = 0$ unless $0 \le k \le 2n$. From the long exact sequence in cohomology associated to (2.7.4), we see that it is enough to prove that $\mathbf{H}^k_{\mathrm{c}}(U, \mathcal{F}|_U) = 0$ unless $0 \le k \le 2n$. Suppose instead that there is some $m > 2n$ such that $\mathbf{H}^m_{\mathrm{c}}(U, \mathcal{F}|_U)$. In fact, take m to be as large as possible. (There is a largest such m because $R\Gamma_{\mathrm{c}}(\mathcal{F}|_U) \cong \mathrm{a}_{U!}(\mathcal{F}|_U)$ belongs to $D^{\mathrm{b}}_{\mathrm{c}}(\mathrm{pt}, \Bbbk)$.) Let $M = \mathbf{H}^m_{\mathrm{c}}(U, \mathcal{F}|_U)$. Then there is a nonzero truncation map

$$R\Gamma_{\mathrm{c}}(\mathcal{F}|_U) \to \tau^{\ge m} R\Gamma_{\mathrm{c}}(\mathcal{F}|_U) \cong \mathsf{H}^m(R\Gamma_{\mathrm{c}}(\mathcal{F}|_U))[-m] \cong M[-m]$$

in $D^{\mathrm{b}}_{\mathrm{c}}(\mathrm{pt}, \Bbbk)$. Call this map θ. It is a nonzero element of the Hom-group

$$\mathrm{Hom}(\mathrm{a}_{U!}(\mathcal{F}|_U), M[-m]) \cong \mathrm{Hom}(\mathcal{F}|_U, \mathrm{a}_U^! M[-m]).$$

Since U is smooth, $\mathrm{a}_U : U \to \mathrm{pt}$ is a smooth morphism, so by Theorem 2.2.9, the last term can be rewritten as

$$\mathrm{Hom}(\mathcal{F}|_U, \mathrm{a}_U^* M[2n-m](n)) \cong \mathrm{Ext}^{2n-m}_{\mathrm{Sh}(U,\Bbbk)}(\mathcal{F}|_U, \underline{M}_U(n)).$$

But $2n - m < 0$, so this Ext-group clearly vanishes, contradicting the fact that $\theta \neq 0$. □

THEOREM 2.7.5. *Let X be a variety of dimension n, and let $\mathcal{F}$ be a constructible sheaf on X. The $\Bbbk$-module $\mathbf{H}^k(X, \mathcal{F})$ is finitely generated for all $\Bbbk$, and it vanishes unless $0 \le k \le 2n$.*

PROOF. Again, the finite generation claim is contained in the fact that $\mathrm{a}_{X*}\mathcal{F}$ is constructible. It is obvious that $\mathbf{H}^k(X, \mathcal{F}) = 0$ for $k < 0$. To prove that it vanishes for $k > 2n$, we begin by observing a consequence of the theorem that will be useful at various intermediate steps.

Step 1. Assume that the theorem holds for all closed subvarieties of X, and let $\mathcal{G} \in D^{\mathrm{b}}_{\mathrm{c}}(X, \Bbbk)$ be such that $\operatorname{codim} \operatorname{supp} \mathsf{H}^i(\mathcal{G}) \ge i$ *if $i \le n$, and* $\mathsf{H}^i(\mathcal{G}) = 0$ *if $i > n$. Then $\mathbf{H}^k(X, \mathcal{G}) = 0$ if $k > 2n$.* We proceed by induction on the number of nonzero cohomology sheaves of $\mathcal{G}$. If $\mathsf{H}^i(\mathcal{G}) = 0$ for all i, there is nothing to prove. Otherwise, let m be the largest integer such that $\mathsf{H}^m(\mathcal{G}) \neq 0$. The truncation distinguished triangle $\tau^{\le m-1}\mathcal{G} \to \mathcal{G} \to \mathsf{H}^m(\mathcal{G})[-m] \to$ gives rise to

$$R\Gamma(\tau^{\le m-1}\mathcal{G}) \to R\Gamma(\mathcal{G}) \to R\Gamma(\mathsf{H}^m(\mathcal{G})[-m]) \to . \tag{2.7.5}$$

We know that $\mathbf{H}^k(X, \tau^{\le m-1}\mathcal{G}) = 0$ for $k > 2n$ by induction. Since $\mathsf{H}^m(\mathcal{G})$ is supported on a closed subvariety of dimension at most $n-m$, we have by assumption that $\mathbf{H}^k(X, \mathsf{H}^m(\mathcal{G})) = 0$ if $k > 2n - 2m$. In other words, $\mathbf{H}^k(X, \mathsf{H}^m(\mathcal{G})[-m]) = 0$ if $k > 2n - m$. The long exact sequence in cohomology associated to (2.7.5) then gives us the desired vanishing of $\mathbf{H}^k(X, \mathcal{G})$.

We now turn to proving the theorem in various special cases.

Step 2. The case where X is a projective variety. In this case, $R\Gamma(\mathcal{F}) \cong R\Gamma_{\mathrm{c}}(\mathcal{F})$, so the result holds by Theorem 2.7.4.

Step 3. The case where X is smooth and $\mathcal{F}$ is a local system. We can embed X as an open dense subset of some projective variety $\overline{X}$. Using Theorem 2.1.18, we may in fact assume that $\overline{X}$ is also smooth, and that the complement $\overline{X} \smallsetminus X$ is a divisor with simple normal crossings. Let $j : X \hookrightarrow \overline{X}$ be the inclusion map. By Lemma 2.4.2, the object $j_*\mathcal{F} \in D^{\mathrm{b}}_{\mathrm{c}}(\overline{X}, \Bbbk)$ satisfies the hypotheses of Step 1. The theorem is already known for the projective variety $\overline{X}$ and all its closed subvarieties, so Step 1 tells us that $\mathbf{H}^k(X, \mathcal{F}) \cong \mathbf{H}^k(\overline{X}, j_*\mathcal{F}) = 0$ if $k > 2n$.

Step 4. The general case. We proceed by induction on $\dim X$ and then by noetherian induction on varieties of a given dimension. (In other words, we assume that the result is known for proper closed subvarieties of X and for all varieties of smaller dimension.) If X is a point, the claim is trivial. Otherwise, let $j : U \hookrightarrow X$ be an irreducible smooth Zariski-open subset such that $\mathcal{F}|_U$ is a local system, say $\mathcal{F}|_U = \mathcal{L}$, and let $i : Z \hookrightarrow X$ be the inclusion of the complementary open subset. Apply $R\Gamma$ to the triangle $j_!j^*\mathcal{F} \to \mathcal{F} \to i_*i^*\mathcal{F} \to$ to obtain

$$R\Gamma(j_!\mathcal{L}) \to R\Gamma(\mathcal{F}) \to R\Gamma(\mathcal{F}|_Z) \to .$$

Since $\dim Z \le n$, we know by induction that $\mathbf{H}^k(Z, \mathcal{F}|_Z) = 0$ for $k > 2n$. From the long exact sequence in cohomology associated to (2.7.5), we see that it is enough to prove that $\mathbf{H}^k(X, j_!\mathcal{L}) = 0$ for $k > 2n$. Since j factors through the closed embedding $\overline{U} \hookrightarrow X$, we may assume for simplicity that $X = \overline{U}$. In particular, X is also irreducible.

Use Theorem 2.1.18 to find a proper map $p : \tilde{X} \to X$ with $\tilde{X}$ smooth and such that $p^{-1}(U) \to U$ is an isomorphism and $p^{-1}(Z)$ is a divisor with simple normal crossings. Identify $p^{-1}(U)$ with U, and let $\tilde{\jmath} : U \to \tilde{X}$ be the inclusion map. Note that $j_!\mathcal{L} \cong p_*\tilde{\jmath}_!\mathcal{L}$, so $R\Gamma(j_!\mathcal{L}) \cong R\Gamma(\tilde{\jmath}_!\mathcal{L})$. It is therefore enough to prove that $\mathbf{H}^k(\tilde{X}, \tilde{\jmath}_!\mathcal{L}) = 0$ for $k > 2n$.

Let $\tilde{Z} = p^{-1}(Z)$, and let $\tilde{\imath} : \tilde{Z} \to \tilde{X}$ be the inclusion map. Consider the distinguished triangle

$$\tilde{\jmath}_!\mathcal{L} \to \tilde{\jmath}_*\mathcal{L} \to \tilde{\imath}_*\tilde{\imath}^*\tilde{\jmath}_*\mathcal{L} \to . \tag{2.7.6}$$

Since $\tilde{\jmath}_!\mathcal{L}$ is a sheaf, the long exact sequence of cohomology sheaves shows that $\mathsf{H}^i(\tilde{\jmath}_*\mathcal{L}) \cong \mathsf{H}^i(\tilde{\imath}_*\tilde{\imath}^*\tilde{\jmath}_*\mathcal{L})$ for $i \ge 1$. In particular, using Lemma 2.4.2, we see that

$$\operatorname{codim}_{\tilde{X}} \operatorname{supp} \mathsf{H}^i(\tilde{\imath}_*\tilde{\imath}^*\tilde{\jmath}_*\mathcal{L}) \ge i$$

for $i \ge 1$, or, equivalently,

$$\operatorname{codim}_{\tilde{Z}} \operatorname{supp} \mathsf{H}^i(\tilde{\imath}^*\tilde{\jmath}_*\mathcal{L}[1]) \ge i$$

for $i \ge 0$. In fact, this latter version also holds for $i = -1$, i.e., for all i. By induction, the theorem already holds for all closed subvarieties of $\tilde{Z}$, so we can apply Step 1: we have $\mathbf{H}^k(\tilde{Z}, \tilde{\imath}^*\tilde{\jmath}_*\mathcal{L}[1]) = 0$ for $k > 2(n-1)$. (The bound here involves $\dim \tilde{Z} = n - 1$.) In other words,

$$\mathbf{H}^k(\tilde{Z}, \tilde{\imath}^*\tilde{\jmath}_*\mathcal{L}[-1]) = 0 \qquad \text{for } k > 2n. \tag{2.7.7}$$

Now apply $R\Gamma$ to (a rotation of) the triangle (2.7.6) to obtain

$$R\Gamma(\tilde{\imath}^*\tilde{\jmath}_*\mathcal{L}[-1]) \to R\Gamma(\tilde{\jmath}_!\mathcal{L}) \to R\Gamma(\mathcal{L}) \to .$$

We know from Step 3 that $\mathbf{H}^k(U, \mathcal{L}) = 0$ for $k > 2n$. In view of (2.7.7), the long exact cohomology sequence for the triangle above shows the desired vanishing for $\mathbf{H}^k(\tilde{X}, \tilde{\jmath}_!\mathcal{L})$. ☐

Exercises.

2.7.1. Let X be a variety, and let $(X_s)_{s\in\mathscr{S}}$ be a stratification. A map $f : X \to Y$ is said to be a **stratumwise locally trivial fibration** with respect to $\mathscr{S}$ if for each $s \in \mathscr{S}$, the map $f|_{X_s} : X_s \to Y$ is a locally trivial fibration. Show that if $f : X \to Y$ is a stratumwise locally trivial fibration, then for any $\mathcal{F} \in D^{\mathrm{b}}_{\mathscr{S}}(X, \Bbbk)$, the objects $f_!\mathcal{F}$ and $f_*\mathcal{F}$ lie in $D^{\mathrm{b}}_{\mathrm{locf}}(Y, \Bbbk)$.

2.7.2. Let X and Y be smooth varieties, and let $Z \subset X$ be a divisor with simple normal crossings. Show that if $f : X \to Y$ is a transverse locally trivial fibration, then it is also a stratumwise locally trivial fibration with respect to the normal crossings stratification of X.

2.7.3. Let X be a variety of dimension n, and let M be a finitely generated $\Bbbk$-module. Show that there is a natural isomorphism

$$\mathbf{H}^{2n}_{\mathrm{c}}(X;M) \cong M \otimes \mathbf{H}^{2n}_{\mathrm{c}}(X;\Bbbk).$$

2.8. Verdier duality

The dualizing complex $\omega_X = \mathrm{a}_X^! \underline{\Bbbk}_{\mathrm{pt}}$ and the Verdier duality functor $\mathbb{D} = R\mathscr{H}om(-,\omega_X)$ were introduced in Definition 1.5.13. In this section, we will study these notions in the context of constructible sheaves.

Lemma 2.8.1. *The dualizing complex of any variety X lies in $D^{\mathrm{b}}_{\mathrm{c}}(X,\Bbbk)$. As a consequence, the Verdier duality functor restricts to a functor*

$$\mathbb{D} : D^{\mathrm{b}}_{\mathrm{c}}(X,\Bbbk)^{\mathrm{op}} \to D^{\mathrm{b}}_{\mathrm{c}}(X,\Bbbk).$$

Proof. By Proposition 2.7.3, the second part of this lemma follows from the first. To show that ω_X is constructible, we proceed by noetherian induction on X. If X is a point, then $\omega_X = \underline{\Bbbk}_X$, so the result is clear. Otherwise, choose a smooth open subset $j : U \hookrightarrow X$, and let $i : Z \hookrightarrow X$ be the complementary closed embedding. By Lemma 1.5.14, the triangle $i_*i^!\omega_X \to \omega_X \to j_*j^*\omega_X \to$ can be rewritten as

$$i_*\omega_Z \to \omega_X \to j_*\omega_U \to .$$

Here, ω_Z is constructible by induction, and Corollary 2.2.10 tells us that $\omega_U \cong \underline{\Bbbk}_U[2\dim U](\dim U)$. Since the first and last terms of the distinguished triangle are constructible, ω_X is as well. $\square$

Since $\mathbb{D}$ sends any constructible complex $\mathcal{F}$ to a bounded complex, it makes sense for the object $\mathbb{D}(\mathcal{F}) \otimes^L \mathcal{F}$ to appear on the left-hand side of the isomorphism

$$\mathrm{Hom}(\mathbb{D}(\mathcal{F}) \overset{L}{\otimes} \mathcal{F}, \omega_X) \cong \mathrm{Hom}(\mathcal{F}, R\mathscr{H}om(\mathbb{D}(\mathcal{F}),\omega_X)) = \mathrm{Hom}(\mathcal{F}, \mathbb{D}(\mathbb{D}(\mathcal{F}))).$$

Via this isomorphism, the pairing map from Remark 1.5.16 corresponds to a natural map

$$\mathcal{F} \to \mathbb{D}(\mathbb{D}(\mathcal{F})),$$

called the **evaluation map**. The main goal of this section is to show that the evaluation map is an isomorphism.

Lemma 2.8.2. *Let X be a smooth variety of dimension n, and let $x \in X$. For any object $\mathcal{F} \in D^{\mathrm{b}}_{\mathrm{locf}}(X,\Bbbk)$, there is a natural isomorphism*

$$(\mathbb{D}\mathcal{F})_x \xrightarrow{\sim} \mathbb{D}(\mathcal{F}_x)[2n](n).$$

Proof. Using Lemma 1.2.10, we have

$$\mathsf{H}^k((\mathbb{D}\mathcal{F})_x) \cong \varinjlim_{U \ni x} \mathsf{H}^k(R\Gamma(R\mathscr{H}om(\mathcal{F},\omega_X)|_U)) \cong \varinjlim_{U \ni x} \mathsf{H}^k(R\mathrm{Hom}(\mathcal{F}|_U, \underline{\Bbbk}_U[2n](n))).$$

On the other hand, for any analytic open set U containing x, there is a map

$$\begin{aligned}(2.8.1)\quad & \mathsf{H}^k(R\mathrm{Hom}(\mathcal{F}|_U, \underline{\Bbbk}_U[2n](n))) \\ & \qquad \to \mathsf{H}^k(R\mathrm{Hom}(\mathcal{F}_x, \underline{\Bbbk}_{\mathrm{pt}}[2n](n))) \cong \mathsf{H}^k(\mathbb{D}(\mathcal{F}_x)[2n](n)).\end{aligned}$$

Together, these maps give rise to a natural map

$$\varinjlim_{U \ni x} \mathsf{H}^k(R\mathrm{Hom}(\mathcal{F}|_U, \underline{\Bbbk}_U[2n](n))) \to \mathsf{H}^k(\mathbb{D}(\mathcal{F}_x)[2n](n)). \tag{2.8.2}$$

If U is a contractible neighborhood of x, then Theorem 1.8.10 tells us that the map (2.8.1) is an isomorphism. Since x has a basis of contractible neighborhoods, it follows that (2.8.2) is an isomorphism as well. $\square$

REMARK 2.8.3. Here is another statement in the spirit of Lemma 2.8.2. If X is a smooth variety of dimension n, and if $\mathcal{L} \in \mathrm{Loc}^{\mathrm{ft}}(X, \Bbbk)$ is locally free, then

$$\mathbb{D}\mathcal{L} \cong \mathcal{L}^\vee[2n](n).$$

This follows from Propositions 1.7.11 and 1.7.12 and Corollary 2.2.10.

Recall from Lemma 1.5.15 that for any morphism of varieties $f : X \to Y$, there are natural isomorphisms

$$f_*\mathbb{D}\mathcal{F} \cong \mathbb{D}(f_!\mathcal{F}) \qquad \text{and} \qquad f^!\mathbb{D}\mathcal{G} \cong \mathbb{D}(f^*\mathcal{G}). \tag{2.8.3}$$

The next lemma is quite similar; its proof is left as an exercise.

LEMMA 2.8.4. *Let X be a variety, let $U \subset X$ be an open subset, and let $i : Z \hookrightarrow X$ be the inclusion of a closed subset. For $\mathcal{F} \in D^{\mathrm{b}}_{\mathrm{c}}(X, \Bbbk)$, there is a natural isomorphism $(\mathbb{D}\mathcal{F})|_U \cong \mathbb{D}(\mathcal{F}|_U)$. For $\mathcal{G} \in D^{\mathrm{b}}_{\mathrm{c}}(Z, \Bbbk)$, there is a natural isomorphism $i_*\mathbb{D}\mathcal{G} \cong \mathbb{D}(i_*\mathcal{G})$.*

LEMMA 2.8.5. *Let X be a variety, and let $j : U \hookrightarrow X$ be the inclusion of a smooth, irreducible open subset. For any object $\mathcal{F} \in D^{\mathrm{b}}_{\mathrm{locf}}(U, \Bbbk)$, there is a natural isomorphism $\mathbb{D}(j_*\mathcal{F}) \cong j_!(\mathbb{D}\mathcal{F})$.*

PROOF. We first consider various special cases.

Step 1. The case where X is smooth and U is the complement of a divisor Z with simple normal crossings. Let $n = \dim X$, and let $Z_1, \ldots, Z_k$ be the irreducible components of Z. We already know from Lemma 2.8.4 that $(\mathbb{D}(j_*\mathcal{F}))|_U \cong \mathbb{D}\mathcal{F} \cong (j_!(\mathbb{D}\mathcal{F}))|_U$. Therefore, to prove the isomorphism, it is enough to show that $(\mathbb{D}(j_*\mathcal{F}))|_Z = 0$ or to show that the stalk $(\mathbb{D}(j_*\mathcal{F}))_x$ vanishes for all $x \in Z$. Suppose $x \in X_J$, where $J \subset \{1, \ldots, k\}$ is a nonempty subset, and let $j = |J|$. By Lemma 1.2.10, we have

$$\begin{aligned}
\mathsf{H}^k((\mathbb{D}(j_*\mathcal{F}))_x) &= \varinjlim_{V \ni x} \mathbf{H}^k(V, (\mathbb{D}(j_*\mathcal{F}))|_V) \\
&\cong \varinjlim_{V \ni x} \mathsf{H}^k(R\Gamma(R\mathscr{H}om((j_*\mathcal{F})|_V, \omega_V))) \\
&\cong \varinjlim_{V \ni x} \mathrm{Hom}((j_*\mathcal{F})|_V, \mathrm{a}_V^!\underline{\Bbbk}_{\mathrm{pt}}[k]) \\
&\cong \varinjlim_{V \ni x} \mathrm{Hom}(R\Gamma_{\mathrm{c}}((j_*\mathcal{F})|_V), \underline{\Bbbk}_{\mathrm{pt}}[k]).
\end{aligned} \tag{2.8.4}$$

We claim that if V is a sufficiently small normal crossings coordinate chart (see Lemma 2.4.1), then $R\Gamma_{\mathrm{c}}((j_*\mathcal{F})|_V) = 0$, so the limit above vanishes. Using Proposition 1.2.16 and the commutative diagram (2.4.3), our claim can be rephrased as follows: if we let $h : (\mathbb{C}^\times)^j \times \mathbb{C}^{n-j} \hookrightarrow \mathbb{C}^n$ be the inclusion map, then for any $\mathcal{G} \in D^{\mathrm{b}}_{\mathrm{locf}}((\mathbb{C}^\times)^j \times \mathbb{C}^{n-j}, \Bbbk)$, we have

$$R\Gamma_{\mathrm{c}}(h_*\mathcal{G}) = 0.$$

This holds by Lemma B.2.6.

Step 2. The case where U is dense in X. By Theorem 2.1.18, there exists a proper map $p : \tilde{X} \to X$ such that $\tilde{X}$ is smooth, $p|_{p^{-1}(U)} : p^{-1}(U) \to U$ is an isomorphism, and $p^{-1}(X \smallsetminus U)$ is a divisor with simple normal crossings. Identify $p^{-1}(U)$ with U, and let $\tilde{\jmath} : U \hookrightarrow \tilde{X}$ be the inclusion map. Then $j_*\mathcal{F} \cong p_*\tilde{\jmath}_*\mathcal{F}$. Because p is proper, (2.8.3) tells us that $\mathbb{D}$ commutes with p_*. Using this observation and Step 1, we have

$$\mathbb{D}(j_*\mathcal{F}) \cong \mathbb{D}(p_*\tilde{\jmath}_*\mathcal{F}) \cong p_*\mathbb{D}(\tilde{\jmath}_*\mathcal{F}) \cong p_*\tilde{\jmath}_!(\mathbb{D}\mathcal{F}) \cong j_!(\mathbb{D}\mathcal{F}).$$

Step 3. The general case. Factor $j : U \hookrightarrow X$ as $\bar{\jmath} : U \hookrightarrow \overline{U}$ followed by $i : \overline{U} \hookrightarrow X$. Then the result follows from Step 2 (applied to $\bar{\jmath}$) and Lemma 2.8.4 (applied to i). □

THEOREM 2.8.6 (Verdier duality). *Let X be a variety. For any $\mathcal{F} \in D^{\mathrm{b}}_{\mathrm{c}}(X)$, the evaluation map $\mathcal{F} \to \mathbb{D}(\mathbb{D}(\mathcal{F}))$ is an isomorphism.*

PROOF. Let us first treat the special case where X is smooth and irreducible, and $\mathcal{F} \in D^{\mathrm{b}}_{\mathrm{locf}}(X, \Bbbk)$. To prove that the evaluation map is an isomorphism, it is enough to prove that the induced map on stalks $\mathcal{F}_x \to (\mathbb{D}(\mathbb{D}\mathcal{F}))_x$ is an isomorphism. By Lemma 2.3.17 and Corollary 2.2.10, the object

$$\mathbb{D}\mathcal{F} \cong R\mathscr{H}om(\mathcal{F}, \underline{\Bbbk}_X[2\dim X](\dim X))$$

lies in $D^{\mathrm{b}}_{\mathrm{locf}}(X, \Bbbk)$, so applying Lemma 2.8.2 twice, we find that

$$\begin{aligned}(\mathbb{D}(\mathbb{D}\mathcal{F}))_x &\cong R\mathrm{Hom}((\mathbb{D}\mathcal{F})_x, \underline{\Bbbk}[2n](n)) \\ &\cong R\mathrm{Hom}(R\mathrm{Hom}(\mathcal{F}_x, \underline{\Bbbk}[2n](n)), \underline{\Bbbk}[2n](n)) \\ &\cong R\mathrm{Hom}(R\mathrm{Hom}(\mathcal{F}_x, \underline{\Bbbk}), \underline{\Bbbk}).\end{aligned}$$

The map $\mathcal{F}_x \to R\mathrm{Hom}(R\mathrm{Hom}(\mathcal{F}_x, \underline{\Bbbk}), \underline{\Bbbk})$ is an isomorphism by Theorem A.10.2.

For general X, we proceed by noetherian induction. The case where X is a point is covered by the preceding paragraph. Otherwise, choose a smooth irreducible open subset $j : U \hookrightarrow X$ such that $\mathcal{F}|_U$ lies in $D^{\mathrm{b}}_{\mathrm{locf}}(U, \Bbbk)$, and let $i : Z \hookrightarrow X$ be the complementary closed subset. By the preceding paragraph and Lemma 2.8.4, the natural map

$$j^*\mathcal{F} \xrightarrow{\sim} j^*\mathbb{D}\mathbb{D}\mathcal{F} \cong \mathbb{D}\mathbb{D}(j^*\mathcal{F}) \tag{2.8.5}$$

is an isomorphism. On the other hand, if $\mathcal{G} \in D^{\mathrm{b}}_{\mathrm{c}}(Z, \Bbbk)$, then by induction, the map $\mathcal{G} \xrightarrow{\sim} \mathbb{D}(\mathbb{D}\mathcal{G})$ is an isomorphism, so by Lemma 2.8.4, so is

$$i_*\mathcal{G} \xrightarrow{\sim} \mathbb{D}(\mathbb{D}(i_*\mathcal{G})) \cong i_*\mathbb{D}(\mathbb{D}\mathcal{G}). \tag{2.8.6}$$

Now, for $\mathcal{F} \in D^{\mathrm{b}}_{\mathrm{c}}(X, \Bbbk)$, we claim that there is a commutative diagram

$$\begin{tikzcd} j_!j^*\mathcal{F} \arrow[r] \arrow[d, "a"] & \mathcal{F} \arrow[r] \arrow[d, "b"] & i_*i^*\mathcal{F} \arrow[r] \arrow[d, "c", "\wr"'] & {} \\ \mathbb{D}\mathbb{D}j_!j^*\mathcal{F} \arrow[r] \arrow[d, dashed, "d"] & \mathbb{D}\mathbb{D}\mathcal{F} \arrow[r] \arrow[d, equal] & \mathbb{D}\mathbb{D}i_*i^*\mathcal{F} \arrow[r] \arrow[d, dashed, "e"] & {} \\ j_!j^*\mathbb{D}\mathbb{D}\mathcal{F} \arrow[r] & \mathbb{D}\mathbb{D}\mathcal{F} \arrow[r] & i_*i^*\mathbb{D}\mathbb{D}\mathcal{F} \arrow[r] & {} \end{tikzcd} \tag{2.8.7}$$

in which the rows are distinguished triangles. Note first that the map $c : i_*i^*\mathcal{F} \to \mathbb{D}\mathbb{D}i_*i^*\mathcal{F}$ is an isomorphism by (2.8.6). Next, by (2.8.3) and Lemma 2.8.5, we have $\mathbb{D}\mathbb{D}j_!j^*\mathcal{F} \cong \mathbb{D}j_*\mathbb{D}j^*\mathcal{F} \cong j_!(\mathbb{D}\mathbb{D}j^*\mathcal{F})$. By adjunction, it follows that

$$\mathrm{Hom}(\mathbb{D}\mathbb{D}j_!j^*\mathcal{F}, i_*i^*\mathbb{D}\mathbb{D}\mathcal{F}) = \mathrm{Hom}(\mathbb{D}\mathbb{D}j^*\mathcal{F}, j^*i_*i^*\mathbb{D}\mathbb{D}\mathcal{F}) = 0.$$

The same reasoning shows that $\mathrm{Hom}(\mathbb{D}\mathbb{D}j_!j^*\mathcal{F}, i_*i^*\mathbb{D}\mathbb{D}\mathcal{F}[-1]) = 0$. Therefore, by Lemma A.4.10, there exist unique morphisms d and e making the bottom part of the diagram commute. In fact, Theorem 1.3.10 says that the bottom row is canonical with respect to the property that its first term has zero restriction to Z and its last term is supported on Z. This means that d and e are isomorphisms. Finally, (2.8.5) implies that $d \circ a : j_! j^* \mathcal{F} \to j_! j^* \mathbb{D}\mathbb{D}\mathcal{F}$ is an isomorphism, so the map a is as well. Since a and c are isomorphisms, b is an isomorphism, as desired. $\square$

In other words, Theorem 2.8.6 says that there is a natural isomorphism

$$\mathbb{D} \circ \mathbb{D} \cong \mathrm{id}. \tag{2.8.8}$$

We are now ready to complete our endeavor of showing that all six basic sheaf operations preserve constructibility.

Proposition 2.8.7. *Let $f : X \to Y$ be a morphism of varieties. For any $\mathcal{F} \in D^{\mathrm{b}}_{\mathrm{c}}(Y, \Bbbk)$, the object $f^! \mathcal{F}$ lies in $D^{\mathrm{b}}_{\mathrm{c}}(X, \Bbbk)$.*

Proof. By (2.8.3) and (2.8.8), we have $f^! \mathcal{F} \cong f^!(\mathbb{D}\mathbb{D}\mathcal{F}) \cong \mathbb{D}(f^* \mathbb{D}\mathcal{F})$. Since $\mathbb{D}$ and f^* preserve constructibility, we are done. $\square$

Below, we record a few more formulas involving $\mathbb{D}$ that are easy consequences of (2.8.8). Their proofs are left as exercises.

Corollary 2.8.8. *Let $f : X \to Y$ be a morphism of varieties. For $\mathcal{F} \in D^{\mathrm{b}}_{\mathrm{c}}(X, \Bbbk)$ and $\mathcal{G} \in D^{\mathrm{b}}_{\mathrm{c}}(Y, \Bbbk)$, there are natural isomorphisms*

$$f_*(\mathbb{D}\mathcal{F}) \cong \mathbb{D}(f_! \mathcal{F}), \qquad f^!(\mathbb{D}\mathcal{G}) \cong \mathbb{D}(f^* \mathcal{G}),$$
$$f_!(\mathbb{D}\mathcal{F}) \cong \mathbb{D}(f_* \mathcal{F}), \qquad f^*(\mathbb{D}\mathcal{G}) \cong \mathbb{D}(f^! \mathcal{G}).$$

Corollary 2.8.9. *For $\mathcal{F}, \mathcal{G} \in D^{\mathrm{b}}_{\mathrm{c}}(X, \Bbbk)$, there are natural isomorphisms*

$$R\mathscr{H}om(\mathcal{F}, \mathcal{G}) \cong \mathbb{D}(\mathcal{F} \overset{L}{\otimes} \mathbb{D}\mathcal{G}) \cong R\mathscr{H}om(\mathbb{D}\mathcal{G}, \mathbb{D}\mathcal{F}).$$

Exercises.

2.8.1. Let X be a smooth, projective variety of dimension n. Use Corollary 2.2.10 and Verdier duality to show that when $\Bbbk$ is a field, there is a natural isomorphism $\mathbf{H}^k(X; \Bbbk) \cong \mathbf{H}^{2n-k}(X; \Bbbk)^*$ for any k.

2.8.2. Show that if $(X_s)_{s \in \mathscr{S}}$ is a good stratification of X, then $\mathbb{D}$ restricts to a functor $\mathbb{D} : D^{\mathrm{b}}_{\mathscr{S}}(X, \Bbbk)^{\mathrm{op}} \to D^{\mathrm{b}}_{\mathscr{S}}(X, \Bbbk)$.

2.9. More compatibilities of functors

In Chapter 1, we saw that $\boxtimes$ and extension of scalars commute with three of the six basic sheaf operations (namely, f^*, $f_!$, and $\otimes^L$). In this section, we will show that for constructible complexes, they also commute with the remaining three and with Verdier duality.

External tensor product. The first half of this section is devoted to compatibility results involving external tensor product.

Proposition 2.9.1. *Let $f : X \to X'$ and $g : Y \to Y'$ be maps of varieties. For $\mathcal{F} \in D^{\mathrm{b}}_{\mathrm{c}}(X, \Bbbk)$ and $\mathcal{G} \in D^{\mathrm{b}}_{\mathrm{c}}(Y, \Bbbk)$, there is a natural isomorphism*

$$f_* \mathcal{F} \boxtimes g_* \mathcal{G} \cong (f \times g)_*(\mathcal{F} \boxtimes \mathcal{G}).$$

Proof. Using the factorization $f \times g = (\mathrm{id}_{X'} \times g) \circ (f \times \mathrm{id}_Y)$, we see that it is enough to treat the special case where one of f or g is an identity map. Assume henceforth that $Y = Y'$ and $g = \mathrm{id}_Y$. Let $f' = f \times \mathrm{id} : X \times Y \to X' \times Y$. We must show that $f_*\mathcal{F} \boxtimes \mathcal{G} \cong f'_*(\mathcal{F} \boxtimes \mathcal{G})$.

Step 1. Construction of a natural map $f_*\mathcal{F} \boxtimes \mathcal{G} \to f'_*(\mathcal{F} \boxtimes \mathcal{G})$. We define this map to be the composition

$$f_*\mathcal{F} \boxtimes \mathcal{G} \to f'_*(f')^*(f_*\mathcal{F} \boxtimes \mathcal{G}) \cong f'_*(f^*f_*\mathcal{F} \boxtimes \mathcal{G}) \to f'_*(\mathcal{F} \boxtimes \mathcal{G}).$$

Here, the first map is an adjunction map, the second map comes from Proposition 1.4.21(1), and the last is induced by the adjunction map $f^*f_*\mathcal{F} \to \mathcal{F}$.

Step 2. The case where Y is smooth and connected, and $\mathcal{G}$ is a local system. Note that $X' \times Y$ is covered by analytic open subsets of the form $X' \times B$, where B is a contractible analytic open subset of Y. It is enough to show that the map from Step 1 is an isomorphism after restriction to any such open subset. Let $f'_B = f \times \mathrm{id}_B : X \times B \to X' \times B$. By Propositions 1.4.21(1) and 1.2.16, we have

$$\begin{aligned}(f_*\mathcal{F} \boxtimes \mathcal{G})|_{X' \times B} &\cong f_*\mathcal{F} \boxtimes (\mathcal{G}|_B),\\ (f'_*(\mathcal{F} \boxtimes \mathcal{G}))|_{X' \times B} &\cong f'_{B*}((\mathcal{F} \boxtimes \mathcal{G})|_{X \times B}) \cong f'_{B*}(\mathcal{F} \boxtimes (\mathcal{G}|_B)).\end{aligned}$$

Since B is contractible, $\mathcal{G}|_B$ is a constant sheaf, say $\underline{M}_B$. We are thus reduced to showing that for any finitely generated $\Bbbk$-module M, the map

$$f_*\mathcal{F} \boxtimes \underline{M}_B \to f'_{B*}(\mathcal{F} \boxtimes \underline{M}_B) \tag{2.9.1}$$

is an isomorphism. If M is a free $\Bbbk$-module (necessarily of finite rank), this follows from Proposition 1.9.2. For general M, we proceed by induction on its projective dimension. If its projective dimension is 0, i.e., if M is projective, then it is a direct summand of a free module of finite rank, and the claim follows from the free module case. Otherwise, there is a short exact sequence of finitely generated $\Bbbk$-modules $0 \to M' \to P \to M \to 0$ where P is projective and M' has smaller projective dimension than M. This gives rise to a commutative diagram

$$\begin{tikzcd}
f_*\mathcal{F} \boxtimes \underline{M'}_B \arrow[r] \arrow[d] & f_*\mathcal{F} \boxtimes \underline{P}_B \arrow[r] \arrow[d] & f_*\mathcal{F} \boxtimes \underline{M}_B \arrow[r] \arrow[d] & {} \\
f'_{B*}(\mathcal{F} \boxtimes \underline{M'}_B) \arrow[r] & f'_{B*}(\mathcal{F} \boxtimes \underline{P}_B) \arrow[r] & f'_{B*}(\mathcal{F} \boxtimes \underline{M}_B) \arrow[r] & {}
\end{tikzcd}$$

The first and second vertical arrows are isomorphisms by induction, so the third is as well.

Step 3. The case where Y is smooth and connected, and $\mathcal{G} \in D^{\mathrm{b}}_{\mathrm{locf}}(Y, \Bbbk)$. This follows from Step 2 by truncation and induction on the number of nonzero cohomology sheaves of $\mathcal{G}$.

Step 4. The general case. We proceed by noetherian induction on Y. If Y is a point, the result holds by Step 3. Otherwise, let $j : U \hookrightarrow Y$ be the inclusion of a smooth connected closed subset such that $\mathcal{G}|_U \in D^{\mathrm{b}}_{\mathrm{locf}}(Y, \Bbbk)$, and let $i : Z \hookrightarrow Y$ be the complementary closed subset. We also put $j' = \mathrm{id}_{X'} \times j : X' \times U \hookrightarrow X' \times Y$ and $i' = \mathrm{id}_{X'} \times i : X' \times Z \hookrightarrow X' \times Y$. We have a commutative diagram

$$\begin{tikzcd}
j'_!(j')^*(f_*\mathcal{F} \boxtimes \mathcal{G}) \arrow[r] \arrow[d] & f_*\mathcal{F} \boxtimes \mathcal{G} \arrow[r] \arrow[d] & i'_*(i')^*(f_*\mathcal{F} \boxtimes \mathcal{G}) \arrow[r] \arrow[d] & {} \\
j'_!(j')^*f'_*(\mathcal{F} \boxtimes \mathcal{G}) \arrow[r] & f'_*(\mathcal{F} \boxtimes \mathcal{G}) \arrow[r] & i'_*(i')^*(f_*\mathcal{F} \boxtimes \mathcal{G}) \arrow[r] & {}
\end{tikzcd} \tag{2.9.2}$$

Let $f'_U = f \times \mathrm{id}_U : X \times U \to X' \times U$, and let $f'_Z = f \times \mathrm{id}_Z : X \times Z \to X' \times Z$. By Propositions 1.4.21(1) and 1.2.16, the first vertical map in (2.9.2) is given by applying $j'_!$ to the map

$$f_*\mathcal{F} \boxtimes (\mathcal{G}|_U) \to f'_{U*}(\mathcal{F} \boxtimes (\mathcal{G}|_U)),$$

which is an isomorphism by Step 3. Similarly, the third vertical map in (2.9.2) is given by applying i'_* to the map

$$f_*\mathcal{F} \boxtimes (\mathcal{G}|_Z) \to f'_{Z*}(\mathcal{F} \boxtimes (\mathcal{G}|_Z)),$$

which is an isomorphism by induction. We conclude that the middle vertical map in (2.9.2) is an isomorphism, as desired. □

REMARK 2.9.2. The proof of Step 2 above goes through for arbitrary local systems, not just those of finite type. The only modification needed is this: one must show that (2.9.1) is an isomorphism when M is an arbitrary free module, rather than a free module of finite rank. However, one can show using Exercise 1.4.4 that this stronger claim still follows from Proposition 1.9.2.

COROLLARY 2.9.3 (Künneth formula). *For varieties X and Y, there is a natural isomorphism $R\Gamma(\underline{\Bbbk}_X) \otimes^L R\Gamma(\underline{\Bbbk}_Y) \cong R\Gamma(\underline{\Bbbk}_{X\times Y})$.*

PROPOSITION 2.9.4. *Let X and Y be varieties, and let $\mathrm{pr}_1 : X \times Y \to X$ and $\mathrm{pr}_2 : X \times Y \to Y$ be the projection maps. For any $\mathcal{F} \in D^{\mathrm{b}}_{\mathrm{c}}(X, \Bbbk)$ and $\mathcal{G} \in D^{\mathrm{b}}_{\mathrm{c}}(Y, \Bbbk)$, there is a natual isomorphism $(\mathbb{D}\mathcal{F}) \boxtimes \mathcal{G} \xrightarrow{\sim} R\mathscr{H}om(\mathrm{pr}_1^*\mathcal{F}, \mathrm{pr}_2^!\mathcal{G})$.*

PROOF. Recall from Remark 1.4.11 that there is a canonical map $\mathbb{D}\mathcal{F} \otimes^L \mathcal{F} \to \omega_X = \mathrm{a}_X^!\underline{\Bbbk}_{\mathrm{pt}}$. By adjunction, we obtain a map

$$t : \mathrm{a}_{X!}(\mathbb{D}\mathcal{F} \overset{L}{\otimes} \mathcal{F}) \to \underline{\Bbbk}_{\mathrm{pt}}.$$

Now apply $(-) \boxtimes \mathcal{G}$ to this map. Using the fact that

$$\mathrm{a}_{X!}(\mathbb{D}\mathcal{F} \overset{L}{\otimes} \mathcal{F}) \boxtimes \mathcal{G} \cong \mathrm{pr}_{2!}((\mathbb{D}\mathcal{F} \overset{L}{\otimes} \mathcal{F}) \boxtimes \mathcal{G}),$$

we regard $t \boxtimes \mathrm{id}_\mathcal{G}$ as an element of

$$\begin{aligned} \mathrm{Hom}(\mathrm{pr}_{2!}((\mathbb{D}\mathcal{F} \overset{L}{\otimes} \mathcal{F}) \boxtimes \mathcal{G}), \mathcal{G}) &\cong \mathrm{Hom}((\mathbb{D}\mathcal{F} \overset{L}{\otimes} \mathcal{F}) \boxtimes \mathcal{G}, \mathrm{pr}_2^!\mathcal{G}) \\ &\cong \mathrm{Hom}((\mathbb{D}\mathcal{F}) \boxtimes \mathcal{G}, R\mathscr{H}om(\mathrm{pr}_1^*\mathcal{F}, \mathrm{pr}_2^!\mathcal{G})). \end{aligned} \tag{2.9.3}$$

The resulting element in the last Hom-space above is a natural map

$$(\mathbb{D}\mathcal{F}) \boxtimes \mathcal{G} \to R\mathscr{H}om(\mathrm{pr}_1^*\mathcal{F}, \mathrm{pr}_2^!\mathcal{G}). \tag{2.9.4}$$

We wish to prove that this map is an isomorphism.

Step 1. The case where X is smooth and connected, and $\mathcal{F} = \underline{\Bbbk}_X$. In this case, pr_2 is a smooth map, say of relative dimension n, so $\mathrm{pr}_2^! \cong \mathrm{pr}_2^*[2n](n)$ by Theorem 2.2.9. If we regard pr_2 as a map $X \times Y \to \mathrm{pt} \times Y$, then for $\mathcal{H} \in D^{\mathrm{b}}_{\mathrm{c}}(\mathrm{pt}, \Bbbk)$, using Proposition 1.4.21(1), we have

$$\mathrm{pr}_2^!(\mathcal{H} \boxtimes \mathcal{G}) \cong (\mathrm{a}_X^*\mathcal{H} \boxtimes \mathcal{G})[2n](n) \cong \mathrm{a}_X^!\mathcal{H} \boxtimes \mathcal{G}.$$

We also have $\mathrm{pr}_1^*\mathcal{F} \cong \underline{\Bbbk}_{X\times Y}$, and $R\mathscr{H}om(\mathrm{pr}_1^*\mathcal{F}, -)$ is the identity functor, by Proposition 1.4.4(2). To prove that (2.9.4) is an isomorphism, we can back up one step in (2.9.3) and instead prove that the map

$$(\mathbb{D}\mathcal{F} \overset{L}{\otimes} \mathcal{F}) \boxtimes \mathcal{G} \to \mathrm{pr}_2^!\mathcal{G} \tag{2.9.5}$$

is an isomorphism. Now, $\mathbb{D}\mathcal{F} \otimes^L \mathcal{F} \cong \omega_X \otimes^L \underline{\Bbbk}_X \cong \omega_X \cong \mathrm{a}_X^! \underline{\Bbbk}_{\mathrm{pt}}$. Unraveling the definitions, we see that this map is the composition

$$\begin{aligned}(\mathbb{D}\mathcal{F} \overset{L}{\otimes} \mathcal{F}) \boxtimes \mathcal{G} &\cong \mathrm{a}_X^! \underline{\Bbbk}_{\mathrm{pt}} \boxtimes \mathcal{G} \to \mathrm{pr}_2^! \mathrm{pr}_{2!}(\mathrm{a}_X^! \underline{\Bbbk}_{\mathrm{pt}} \boxtimes \mathcal{G}) \\ &\cong \mathrm{pr}_2^!(\mathrm{a}_{X!}\mathrm{a}_X^! \underline{\Bbbk}_{\mathrm{pt}} \boxtimes \mathcal{G}) \cong \mathrm{a}_X^! \mathrm{a}_{X!}\mathrm{a}_X^! \underline{\Bbbk}_{\mathrm{pt}} \boxtimes \mathcal{G} \to \mathrm{a}_X^! \underline{\Bbbk}_{\mathrm{pt}} \boxtimes \mathcal{G} \cong \mathrm{pr}_2^! \mathcal{G}.\end{aligned}$$

In other words, (2.9.5) is essentially given by applying $(-) \boxtimes \mathcal{G}$ to the composition of adjunction maps $\mathrm{a}_X^! \underline{\Bbbk}_{\mathrm{pt}} \to \mathrm{a}_X^! \mathrm{a}_{X!}\mathrm{a}_X^! \underline{\Bbbk}_{\mathrm{pt}} \to \mathrm{a}_X^! \underline{\Bbbk}_{\mathrm{pt}}$. The latter is the identity map, by the unit–counit relations. Therefore, (2.9.5) is an isomorphism.

Step 2. The case where X is smooth and $\mathcal{F}$ is a local system. Suppose first that $\mathcal{F}$ is a constant sheaf, say $\mathcal{F} = \underline{M}_X$, where M is a finitely generated $\Bbbk$-module. If M is a free module, the claim follows from Step 1. Otherwise, an induction argument on the projective dimension of M, similar to that in the proof of Proposition 2.9.1, establishes the result. Finally, if $\mathcal{F}$ is an arbitrary local system of finite type, then it is enough to prove that (2.9.4) becomes an isomorphism after restricting to an open subset of the form $X \times B$, where B is a contractible analytic open subset of Y. Since $\mathcal{F}|_B$ is a constant sheaf, this case follows from the case of a constant sheaf.

Step 3. The case where X is smooth and $\mathcal{F} \in D^{\mathrm{b}}_{\mathrm{locf}}(X, \Bbbk)$. This case follows from Step 2 by truncation and induction on the number of nonzero cohomology sheaves of $\mathcal{F}$.

Step 4. The general case. We proceed by noetherian induction on X. If X is a point, the claim holds by Step 2. Otherwise, let $j : U \hookrightarrow Z$ be the inclusion of a smooth connected open subset such that $\mathcal{F}|_U \in D^{\mathrm{b}}_{\mathrm{locf}}(U, \Bbbk)$, and let $i : Z \hookrightarrow X$ be the complementary open subset. Let $j' = j \times \mathrm{id} : U \times Y \to X \times Y$ and $i' = i \times \mathrm{id} : Z \times Y \to X \times Y$. We will write $\mathrm{pr}_{1,U}$ and $\mathrm{pr}_{2,U}$ for the projections from $U \times Y$, and $\mathrm{pr}_{1,Z}$ and $\mathrm{pr}_{2,Z}$ for the projections from $Z \times Y$.

We claim that there is a commutative diagram of the form

$$\begin{tikzcd} \mathbb{D}(i_*i^*\mathcal{F}) \boxtimes \mathcal{G} \arrow[r, dashed] \arrow[d] & i'_*(i')^! R\mathscr{H}om(\mathrm{pr}_1^*\mathcal{F}, \mathrm{pr}_2^!\mathcal{G}) \arrow[d] \\ \mathbb{D}\mathcal{F} \boxtimes \mathcal{G} \arrow[r] \arrow[d] & R\mathscr{H}om(\mathrm{pr}_1^*\mathcal{F}, \mathrm{pr}_2^!\mathcal{G}) \arrow[d] \\ \mathbb{D}(j_!j^*\mathcal{F}) \boxtimes \mathcal{G} \arrow[r, dashed] \arrow[d] & j'_*(j')^* R\mathscr{H}om(\mathrm{pr}_1^*\mathcal{F}, \mathrm{pr}_2^!\mathcal{G}) \arrow[d] \\ {} & {} \end{tikzcd} \tag{2.9.6}$$

Here, the columns come from the familiar distinguished triangles of Theorem 1.3.10. At the top of the left column, we have

$$\mathbb{D}(i_*i^*\mathcal{F}) \boxtimes \mathcal{G} \cong i_*\mathbb{D}(i^*\mathcal{F}) \boxtimes \mathcal{G} \cong i'_*(\mathbb{D}(i^*\mathcal{F}) \boxtimes \mathcal{G}), \tag{2.9.7}$$

where the second isomorphism comes from the fact that i is proper. It follows that

$$\mathrm{Hom}(\mathbb{D}(i_*i^*\mathcal{F}) \boxtimes \mathcal{G}, j'_*(j')^* R\mathscr{H}om(\mathrm{pr}_1^*\mathcal{F}, \mathrm{pr}_2^!\mathcal{G})[k]) = 0$$

for all k. By Lemma A.4.10, the top and bottom dotted arrows in (2.9.6) can be filled in uniquely. Moreover, using (2.9.7), one can see that the top arrow in (2.9.6) is given by applying i'_* to the map

$$\mathbb{D}(i^*\mathcal{F}) \boxtimes \mathcal{G} \to R\mathscr{H}om(\mathrm{pr}_{1,Z}^* i^*\mathcal{F}, \mathrm{pr}_{2,Z}^!\mathcal{G}) \cong (i')^! R\mathscr{H}om(\mathrm{pr}_1^*\mathcal{F}, \mathrm{pr}_2^!\mathcal{G}).$$

This map is an isomorphism by induction. Similarly, by Proposition 2.9.1, we have

$$\mathbb{D}(j_!j^*\mathcal{F}) \boxtimes \mathcal{G} \cong j_*\mathbb{D}(j^*\mathcal{F}) \boxtimes \mathcal{G} \cong j'_*(\mathbb{D}(j^*\mathcal{F}) \boxtimes \mathcal{G}).$$

Then the bottom arrow in (2.9.6) is given by applying j'_* to

$$\mathbb{D}(j^*\mathcal{F}) \boxtimes \mathcal{G} \to R\mathscr{H}om(\mathrm{pr}_{1,U} j^*\mathcal{F}, \mathrm{pr}^!_{2,U}\mathcal{G}) \cong (j')^* R\mathscr{H}om(\mathrm{pr}_1^*\mathcal{F}, \mathrm{pr}_2^!\mathcal{G}).$$

This map is an isomorphism by Step 3. We conclude that the middle horizontal arrow in (2.9.6) is an isomorphism, as desired. □

COROLLARY 2.9.5. *Let X and Y be varieties. There is a canonical isomorphism $\omega_X \boxtimes \omega_Y \cong \omega_{X\times Y}$.*

PROOF. Apply Proposition 2.9.4 with $\mathcal{F} = \underline{\Bbbk}_X$ and $\mathcal{G} = \omega_Y$. □

COROLLARY 2.9.6. *Let X and Y be varieties. For $\mathcal{F}, \mathcal{F}' \in D^{\mathrm{b}}_{\mathrm{c}}(X, \Bbbk)$ and $\mathcal{G} \in D^{\mathrm{b}}_{\mathrm{c}}(Y, \Bbbk)$, there is a natural isomorphism*

$$R\mathscr{H}om(\mathcal{F}, \mathcal{F}') \boxtimes \mathcal{G} \cong R\mathscr{H}om(\mathrm{pr}_1^*\mathcal{F}, \mathcal{F}' \boxtimes \mathcal{G}).$$

PROOF. The following calculation uses various adjunctions relations, together with Theorem 2.8.6, Corollary 2.8.9, and Proposition 2.9.4:

$$\begin{aligned} R\mathscr{H}om(\mathcal{F}, \mathcal{F}') \boxtimes \mathcal{G} &\cong \mathbb{D}(\mathcal{F} \overset{L}{\otimes} (\mathbb{D}\mathcal{F}')) \boxtimes \mathcal{G} \cong R\mathscr{H}om(\mathrm{pr}_1^*\mathcal{F} \overset{L}{\otimes} \mathrm{pr}_1^*\mathbb{D}\mathcal{F}', \mathrm{pr}_2^!\mathcal{G}) \\ &\cong R\mathscr{H}om(\mathrm{pr}_1^*\mathcal{F}, R\mathscr{H}om(\mathrm{pr}_1^*\mathbb{D}\mathcal{F}', \mathrm{pr}_2^!\mathcal{G})) \\ &\cong R\mathscr{H}om(\mathrm{pr}_1^*\mathcal{F}, \mathbb{D}(\mathbb{D}\mathcal{F}') \boxtimes \mathcal{G}) \cong R\mathscr{H}om(\mathrm{pr}_1^*\mathcal{F}, \mathcal{F}' \boxtimes \mathcal{G}). \end{aligned}$$ □

PROPOSITION 2.9.7. *Let $f : X \to X'$ and $g : Y \to Y'$ be maps of varieties.*

(1) *For $\mathcal{F}, \mathcal{F}' \in D^{\mathrm{b}}_{\mathrm{c}}(X, \Bbbk)$ and $\mathcal{G}, \mathcal{G}' \in D^{\mathrm{b}}_{\mathrm{c}}(Y, \Bbbk)$, there is a natural isomorphism*

$$R\mathscr{H}om(\mathcal{F}, \mathcal{F}') \boxtimes R\mathscr{H}om(\mathcal{G}, \mathcal{G}') \cong R\mathscr{H}om(\mathcal{F} \boxtimes \mathcal{G}, \mathcal{F}' \boxtimes \mathcal{G}').$$

(2) *For $\mathcal{F} \in D^{\mathrm{b}}_{\mathrm{c}}(X, \Bbbk)$ and $\mathcal{G} \in D^{\mathrm{b}}_{\mathrm{c}}(Y, \Bbbk)$, there is a natural isomorphism*

$$(\mathbb{D}\mathcal{F}) \boxtimes (\mathbb{D}\mathcal{G}) \cong \mathbb{D}(\mathcal{F} \boxtimes \mathcal{G}).$$

(3) *For $\mathcal{F} \in D^{\mathrm{b}}_{\mathrm{c}}(X', \Bbbk)$ and $\mathcal{G} \in D^{\mathrm{b}}_{\mathrm{c}}(Y', \Bbbk)$, there is a natural isomorphism*

$$f^!\mathcal{F} \boxtimes g^!\mathcal{G} \cong (f \times g)^!(\mathcal{F} \boxtimes \mathcal{G}).$$

PROOF. (1) Using Corollary 2.9.6 twice, we have

$$\begin{aligned} &R\mathscr{H}om(\mathcal{F}, \mathcal{F}') \boxtimes R\mathscr{H}om(\mathcal{G}, \mathcal{G}') \cong R\mathscr{H}om(\mathrm{pr}_1^*\mathcal{F}, \mathcal{F}' \boxtimes R\mathscr{H}om(\mathcal{G}, \mathcal{G}')) \\ &\quad\cong R\mathscr{H}om(\mathrm{pr}_1^*\mathcal{F}, R\mathscr{H}om(\mathrm{pr}_2^*\mathcal{G}, \mathcal{F}' \boxtimes \mathcal{G}')) \cong R\mathscr{H}om(\mathrm{pr}_1^*\mathcal{F} \overset{L}{\otimes} \mathrm{pr}_2^*\mathcal{G}, \mathcal{F}' \boxtimes \mathcal{G}') \\ &\quad\cong R\mathscr{H}om(\mathcal{F} \boxtimes \mathcal{G}, \mathcal{F}' \boxtimes \mathcal{G}'). \end{aligned}$$

(2) This follows from part (1) together with Corollary 2.9.5.

(3) From Corollary 2.8.8, we have $f^! \cong \mathbb{D} \circ f^* \circ \mathbb{D}$. This result therefore follows from part (2) above, combined with Proposition 1.4.21(1). □

Extension of scalars. The remainder of this section is devoted to establishing compatibility results for extension of scalars with respect to a homomorphism $\varphi : \Bbbk \to \Bbbk'$ of noetherian rings of finite global dimension.

PROPOSITION 2.9.8. *Let $f : X \to Y$ be a morphism of varieties, and let $\varphi : \Bbbk \to \Bbbk'$ be a ring homomorphism. For $\mathcal{F} \in D^{\mathrm{b}}_{\mathrm{c}}(X, \Bbbk)$, there is a natural isomorphism $\Bbbk' \otimes^L f_*\mathcal{F} \to f_*(\Bbbk' \otimes^L \mathcal{F})$.*

Proof. We define the map $\Bbbk' \otimes^L f_*\mathcal{F} \to f_*(\Bbbk' \otimes^L \mathcal{F})$ to be the composition

$$\Bbbk' \overset{L}{\otimes} f_*\mathcal{F} \to f_*f^*(\Bbbk' \overset{L}{\otimes} f_*\mathcal{F}) \cong f_*(\Bbbk' \overset{L}{\otimes} f^*f_*\mathcal{F}) \to f_*(\Bbbk' \overset{L}{\otimes} \mathcal{F}).$$

Here, the first and last arrows are adjunction maps, and the second isomorphism comes from Proposition 1.4.15. To show that this is an isomorphism, it is enough to show that it remains an isomorphism after applying $\mathrm{res}_{\Bbbk}^{\Bbbk'}$. In other words, we may instead study the corresponding map $\underline{\Bbbk}'_X \otimes^L f_*\mathcal{F} \to f_*(\underline{\Bbbk}'_X \otimes^L \mathcal{F})$ in $D^{\mathrm{b}}(X, \Bbbk)$, where we regard $\Bbbk'$ as a $\Bbbk$-module. This, in turn, can be rewritten as $\underline{\Bbbk}'_{\mathrm{pt}} \boxtimes f_*\mathcal{F} \to f_*(\underline{\Bbbk}'_{\mathrm{pt}} \boxtimes \mathcal{F})$.

We are now almost in the setting of Step 2 of the proof of Proposition 2.9.1, with $Y = \mathrm{pt}$ and $\mathcal{G} = \underline{\Bbbk}'$. The one difficulty is that $\Bbbk'$ may not be finitely generated as a $\Bbbk$-module, so $\underline{\Bbbk}'_{\mathrm{pt}}$ may be not a constructible sheaf in $\mathrm{Sh}(\mathrm{pt}, \Bbbk)$. But the proof still goes through, as we saw in Remark 2.9.2. ☐

Corollary 2.9.9. *Let $i : Z \hookrightarrow X$ be a closed embedding, and let $\varphi : \Bbbk \to \Bbbk'$ be a ring homomorphism. For $\mathcal{F} \in D^{\mathrm{b}}_{\mathrm{c}}(X, \Bbbk)$, there is a natural isomorphism $\Bbbk' \otimes^L i^!\mathcal{F} \xrightarrow{\sim} i^!(\Bbbk' \otimes^L \mathcal{F})$.*

Proof. Let $j : U \hookrightarrow X$ be the complementary open embedding. Propositions 1.4.15 and 2.9.8 give us the vertical isomorphism in the third column of the diagram

$$\begin{tikzcd}
\Bbbk' \otimes^L i_*i^!\mathcal{F} \arrow[r] \arrow[d, dashed] & \Bbbk' \otimes^L \mathcal{F} \arrow[r] \arrow[d, equal] & \Bbbk' \otimes^L j_*j^*\mathcal{F} \arrow[r] \arrow[d, "\wr"] & {} \\
i_*i^!(\Bbbk' \otimes^L \mathcal{F}) \arrow[r] & \Bbbk' \otimes^L \mathcal{F} \arrow[r] & j_*j^*(\Bbbk' \otimes^L \mathcal{F}) \arrow[r] & {}
\end{tikzcd}$$

By Theorem 1.3.10, there is a canonical isomorphism $\Bbbk' \otimes^L i_*i^!\mathcal{F} \xrightarrow{\sim} i_*i^!(\Bbbk' \otimes^L \mathcal{F})$. Combining this with another instance of Proposition 2.9.8, we obtain a natural isomorphism $i_*(\Bbbk' \otimes^L i^!\mathcal{F}) \xrightarrow{\sim} i_*i^!(\Bbbk' \otimes^L \mathcal{F})$. Since i_* is fully faithful, this determines a natural isomorphism $\Bbbk' \otimes^L i^!\mathcal{F} \xrightarrow{\sim} i^!(\Bbbk' \otimes^L \mathcal{F})$. ☐

Proposition 2.9.10. *Let X be a variety, and $\varphi : \Bbbk \to \Bbbk'$ be a ring homomorphism. There is a canonical isomorphism $\Bbbk' \otimes \omega_X^{\Bbbk} \cong \omega_X^{\Bbbk'}$.*

Proof. If X is smooth, this is immediate from Corollary 2.2.10 and the fact that $\Bbbk' \otimes^L \underline{\Bbbk}_X \cong \underline{\Bbbk}'_X$. Otherwise, choose a locally closed embedding $h : X \hookrightarrow Y$, where Y is a smooth variety. Then $\omega_X \cong h^!\omega_Y$. Proposition 1.4.15 and Corollary 2.9.9 imply that $h^!$ commutes with $\Bbbk' \otimes^L (-)$, and the result follows. ☐

Proposition 2.9.11. *Let $f : X \to Y$ be a morphism of varieties, and let $\varphi : \Bbbk \to \Bbbk'$ be a ring homomorphism.*

(1) *For $\mathcal{F} \in D^{\mathrm{b}}_{\mathrm{c}}(X, \Bbbk)$, there is a natural isomorphism*
$$\Bbbk' \overset{L}{\otimes} \mathbb{D}\mathcal{F} \xrightarrow{\sim} \mathbb{D}(\Bbbk' \overset{L}{\otimes} \mathcal{F}).$$

(2) *For $\mathcal{F}, \mathcal{G} \in D^{\mathrm{b}}_{\mathrm{c}}(X, \Bbbk)$, there is a natural isomorphism*
$$\Bbbk' \overset{L}{\otimes} R\mathscr{H}om(\mathcal{F}, \mathcal{G}) \xrightarrow{\sim} R\mathscr{H}om(\Bbbk' \overset{L}{\otimes} \mathcal{F}, \Bbbk' \overset{L}{\otimes} \mathcal{G}).$$

(3) *For $\mathcal{F} \in D^{\mathrm{b}}_{\mathrm{c}}(Y, \Bbbk)$, there is a natural isomorphism*
$$\Bbbk' \overset{L}{\otimes} f^!\mathcal{F} \xrightarrow{\sim} f^!(\Bbbk' \overset{L}{\otimes} \mathcal{F}).$$

Proof sketch. Note that part (2) follows from part (1) combined with Proposition 1.4.16 and Corollary 2.8.9. Similarly, part (3) follows from part (1) combined with Proposition 1.4.15 and Corollary 2.8.8. The rest of the proof is devoted to part (1).

Step 1. Construction of a map $\Bbbk' \otimes^L \mathbb{D}\mathcal{F} \to \mathbb{D}(\Bbbk' \otimes^L \mathcal{F})$. Let $\epsilon : \mathbb{D}\mathcal{F} \otimes^L \mathcal{F} \to \omega_X^{\Bbbk}$ be the pairing map from Remark 1.5.16. Let η be the map given by the composition

$$(\Bbbk' \overset{L}{\otimes} \mathbb{D}\mathcal{F}) \overset{L}{\otimes} (\Bbbk' \overset{L}{\otimes} \mathcal{F}) \to \Bbbk' \overset{L}{\otimes} (\mathbb{D}\mathcal{F} \overset{L}{\otimes} \mathcal{F}) \xrightarrow{\Bbbk' \otimes^L \epsilon} \Bbbk' \overset{L}{\otimes} \omega_X^{\Bbbk} \to \omega_X^{\Bbbk'},$$

where the first and last maps come from Propositions 1.4.16 and 2.9.10, respectively. The map we wish to study is the map corresponding to η under the isomorphism

$$\mathrm{Hom}((\Bbbk' \overset{L}{\otimes} \mathbb{D}\mathcal{F}) \overset{L}{\otimes} (\Bbbk' \overset{L}{\otimes} \mathcal{F}), \omega_X^{\Bbbk'}) \cong \mathrm{Hom}(\Bbbk' \overset{L}{\otimes} \mathbb{D}\mathcal{F}, \mathbb{D}(\Bbbk' \overset{L}{\otimes} \mathcal{F})). \tag{2.9.8}$$

Step 2. The case where X is smooth and connected, and $\mathcal{F} = \underline{\Bbbk}_X$. In this case, by unwinding the details of the construction of the pairing map ϵ (which goes back to Remark 1.4.11) and the adjunction (2.9.8), one can check that the map from Step 1 is an isomorphism.

Step 3. Remainder of the proof. The rest of the proof follows the pattern of Proposition 2.9.4. □

Corollary 2.9.12. *Let X be a variety, and $\varphi : \Bbbk \to \Bbbk'$ be a ring homomorphism. For $\mathcal{F}, \mathcal{G} \in D^{\mathrm{b}}_{\mathrm{c}}(X, \Bbbk)$, there is a natural isomorphism*

$$\Bbbk' \overset{L}{\otimes} R\mathrm{Hom}(\mathcal{F}, \mathcal{G}) \xrightarrow{\sim} R\mathrm{Hom}(\Bbbk' \overset{L}{\otimes} \mathcal{F}, \Bbbk' \overset{L}{\otimes} \mathcal{G}).$$

In particular, if $\Bbbk$ has global dimension ≤ 1, there is a natural short exact sequence

$$0 \to \Bbbk' \otimes \mathrm{Hom}(\mathcal{F}, \mathcal{G}) \to \mathrm{Hom}(\Bbbk' \overset{L}{\otimes} \mathcal{F}, \Bbbk' \overset{L}{\otimes} \mathcal{G}) \to \mathrm{Tor}_1^{\Bbbk}(\Bbbk', \mathrm{Hom}(\mathcal{F}, \mathcal{G}[1])) \to 0.$$

Proof. For the first assertion, combine Propositions 2.9.8 and 2.9.11(2) with Proposition 1.4.6. For the second assertion, apply Proposition A.6.18 to $\Bbbk' \otimes^L R\mathrm{Hom}(\mathcal{F}, \mathcal{G})$. □

2.10. Localization with respect to a $\mathbb{G}_{\mathrm{m}}$-action

Let X be a $\mathbb{G}_{\mathrm{m}}$-variety, i.e., a variety equipped with an action of the group $\mathbb{G}_{\mathrm{m}}$, denoted by $\sigma : \mathbb{G}_{\mathrm{m}} \times X \to X$. Then the set $X^{\mathbb{G}_{\mathrm{m}}}$ of $\mathbb{G}_{\mathrm{m}}$-fixed points is a closed subvariety of X. In this section, we will study various functors from $D^{\mathrm{b}}_{\mathrm{c}}(X, \Bbbk)$ to $D^{\mathrm{b}}_{\mathrm{c}}(X^{\mathbb{G}_{\mathrm{m}}}, \Bbbk)$. These functors are particularly well behaved on objects that satisfy the following condition (see Chapter 6 for an explanation of this terminology).

Definition 2.10.1. Let X be a $\mathbb{G}_{\mathrm{m}}$-variety, and let $\sigma : \mathbb{G}_{\mathrm{m}} \times X \to X$ be the action. An object $\mathcal{F} \in D^{\mathrm{b}}_{\mathrm{c}}(X)$ is said to be **weakly $\mathbb{G}_{\mathrm{m}}$-equivariant** if $\sigma^*\mathcal{F} \cong \underline{\Bbbk} \boxtimes \mathcal{F}$.

Attracting and repelling actions. We begin with a result on $\mathbb{G}_{\mathrm{m}}$-varieties satisfying the following condition.

Definition 2.10.2. Let X be a $\mathbb{G}_{\mathrm{m}}$-variety. The $\mathbb{G}_{\mathrm{m}}$-action on X is said to be **attracting** if the action map $\sigma : \mathbb{G}_{\mathrm{m}} \times X \to X$ admits an extension to a monoid action $\tilde{\sigma} : \mathbb{A}^1 \times X \to X$.

Similarly, σ is said to be **repelling** if the action $\sigma' : \mathbb{G}_{\mathrm{m}} \times X \to X$ given by $\sigma'(t, x) = \sigma(t^{-1}, x)$ is attracting.

Of course, if σ is an attracting action, then the extension $\tilde{\sigma} : \mathbb{A}^1 \times X \to X$ is uniquely determined: indeed, we have

$$\tilde{\sigma}(0, x) = \lim_{t \to 0} \sigma(t, x),$$

where the limit is taken in the analytic topology. In particular, for an attracting action, $\lim_{t\to 0} t \cdot x$ must exist for all $x \in X$. Similarly, for a repelling action, $\lim_{t\to\infty} t \cdot x$ must exist for all $x \in X$. For attracting (or repelling) actions, we have the following result of Springer [**229**, Proposition 1].

THEOREM 2.10.3 (Homotopy for constructible sheaves). *Let X be a variety equipped with an attracting $\mathbb{G}_{\mathrm{m}}$-action. Let $Z = X^{\mathbb{G}_{\mathrm{m}}}$ be the set of fixed points under this action. Let $i : Z \hookrightarrow X$ be the inclusion map, and let $p : X \to Z$ be the map given by $p(x) = \lim_{t\to 0} t \cdot x$. For any $\mathcal{F} \in D^{\mathrm{b}}_{\mathrm{c}}(X)$, there are natural maps*

$$p_*\mathcal{F} \to i^*\mathcal{F} \qquad \text{and} \qquad i^!\mathcal{F} \to p_!\mathcal{F}.$$

If $\mathcal{F}$ is weakly $\mathbb{G}_{\mathrm{m}}$-equivariant, these maps are isomorphisms.

The use of the term "homotopy" refers to the fact that the proof (which will be given below) relies on the following algebraic analogue of Lemma 1.8.2.

LEMMA 2.10.4. *For $t \in \mathbb{A}^1$, let $i_t : X \to \mathbb{A}^1 \times X$ be the map given by $i_t(x) = (t, x)$. The functor $\mathrm{pr}_2^* : D^{\mathrm{b}}_{\mathrm{c}}(X, \Bbbk) \to D^{\mathrm{b}}_{\mathrm{c}}(\mathbb{A}^1 \times X, \Bbbk)$ is fully faithful, and the functors*

$$i_t^* \quad \text{and} \quad \mathrm{pr}_{2*} : D^{\mathrm{b}}_{\mathrm{c}}(\mathbb{A}^1 \times X, \Bbbk) \to D^{\mathrm{b}}_{\mathrm{c}}(X, \Bbbk)$$

are both left inverses to pr_2^*.

See Exercise 2.10.1 for a proof of this lemma. (It can also be proved using the notion of "∞-acyclic maps" from Definition 6.1.18.)

PROOF OF THEOREM 2.10.3. We will prove the statement for p_* and i^*. The statement for $i^!$ and $p_!$ can be proved similarly, or deduced by Verdier duality. Let $U = X \smallsetminus Z$, and let $j : U \hookrightarrow X$ be the inclusion map. Applying p_* to the distinguished triangle $j_!j^*\mathcal{F} \to \mathcal{F} \to i_*i^*\mathcal{F} \to$ yields

$$p_*j_!j^*\mathcal{F} \to p_*\mathcal{F} \to i^*\mathcal{F} \to,$$

since $p \circ i = \mathrm{id}_Z$. The second arrow above is the desired natural map $p_*\mathcal{F} \to i^*\mathcal{F}$.

From now on, we assume that $\mathcal{F}$ is weakly $\mathbb{G}_{\mathrm{m}}$-equivariant. This implies that $j^*\mathcal{F}$ and $j_!j_*\mathcal{F}$ are also weakly $\mathbb{G}_{\mathrm{m}}$-equivariant. (See Exercise 2.10.2.) For brevity, let $\mathcal{F}' = j_!j^*\mathcal{F}$. To prove that $p_*\mathcal{F} \to i^*\mathcal{F}$ is an isomorphism, we must show that

$$p_*\mathcal{F}' = 0.$$

Step 1. Definition of $Q \subset \mathbb{A}^1 \times X \times X$. Let $Q \subset \mathbb{A}^1 \times X \times X$ be the graph of the monoid action map $\mathbb{A}^1 \times X \to X$. Define the maps $q_1, q_2 : Q \to \mathbb{A}^1 \times X$ by

$$q_1(t, x, y) = (t, x), \qquad q_2(t, x, y) = (t, y).$$

Both q_1 and q_2 restrict to isomorphisms over the open subset $\mathbb{G}_{\mathrm{m}} \times X \subset \mathbb{A}^1 \times X$. Moreover, q_1 is proper. (In fact, q_1 is an isomorphism of varieties, but this fact will not be used below.) Identify $\mathbb{G}_{\mathrm{m}} \times U$ with $q_1^{-1}(\mathbb{G}_{\mathrm{m}} \times U) \subset Q$, and let $h : \mathbb{G}_{\mathrm{m}} \times U \hookrightarrow Q$ be the inclusion map.

Step 2. Construction of a morphism $\psi : q_{1!}q_2^*\mathcal{F}' \to \underline{\Bbbk}_{\mathbb{A}^1} \boxtimes \mathcal{F}'$. Let $u : \mathbb{G}_{\mathrm{m}} \hookrightarrow \mathbb{A}^1$ be the inclusion map, and consider the cartesian square

$$\begin{tikzcd} \mathbb{G}_{\mathrm{m}} \times U \arrow[r, hook, "h"] \arrow[dd, "\sigma"'] & Q \arrow[d, "q_2"] \\ & \mathbb{A}^1 \times X \arrow[d, "\mathrm{pr}_2"] \\ U \arrow[r, hook, "j"] & X \end{tikzcd}$$

By proper base change and the fact that $\mathcal{F}|_U$ is weakly $\mathbb{G}_{\mathrm{m}}$-equivariant, we have

$$q_2^*(\underline{\Bbbk}_{\mathbb{A}^1} \boxtimes \mathcal{F}') \cong q_2^*\mathrm{pr}_2^* j_!(\mathcal{F}|_U) \cong h_!\sigma^*(\mathcal{F}|_U) \cong h_!(\underline{\Bbbk}_{\mathbb{G}_{\mathrm{m}}} \boxtimes (\mathcal{F}|_U)),$$

and hence

$$q_{1!}q_2^*(\underline{\Bbbk}_{\mathbb{A}^1} \boxtimes \mathcal{F}') \cong q_{1!}h_!(\underline{\Bbbk}_{\mathbb{G}_{\mathrm{m}}} \boxtimes (\mathcal{F}|_U)) \cong u_!\underline{\Bbbk}_{\mathbb{G}_{\mathrm{m}}} \boxtimes j_!(\mathcal{F}|_U) = u_!\underline{\Bbbk}_{\mathbb{G}_{\mathrm{m}}} \boxtimes \mathcal{F}'.$$

The adjunction map $u_!\underline{\Bbbk}_{\mathbb{G}_{\mathrm{m}}} \to \underline{\Bbbk}_{\mathbb{A}^1}$ gives rise to a natural map

$$\psi : q_{1!}q_2^*(\underline{\Bbbk}_{\mathbb{A}^1} \boxtimes \mathcal{F}') \to \underline{\Bbbk}_{\mathbb{A}^1} \boxtimes \mathcal{F}'.$$

By construction, we have

$$\begin{aligned} &\psi|_{\mathbb{G}_{\mathrm{m}} \times X} \text{ is an isomorphism;} \\ &\psi|_{\{0\} \times X} \text{ is the zero morphism.} \end{aligned} \tag{2.10.1}$$

Next, consider the adjunction map $\eta : \underline{\Bbbk}_{\mathbb{A}^1} \boxtimes \mathcal{F}' \to q_{2*}q_2^*(\underline{\Bbbk}_{\mathbb{A}^1} \boxtimes \mathcal{F}')$. Since q_2 is an isomorphism over $\mathbb{G}_{\mathrm{m}} \times X$, we have

$$\eta|_{\mathbb{G}_{\mathrm{m}} \times X} \text{ is an isomorphism.} \tag{2.10.2}$$

Step 3. Construction of an endomorphism of $p_*\mathcal{F}'$. Let $f : \mathbb{A}^1 \times X \to \mathbb{A}^1 \times Z$ be the map given by $f(t, x) = (t, p(x))$. Using the fact that $f \circ q_1 = f \circ q_2$, along with the properness of q_1, we define a map $\phi : \underline{\Bbbk}_{\mathbb{A}^1} \boxtimes p_*\mathcal{F}' \to \underline{\Bbbk}_{\mathbb{A}^1} \boxtimes p_*\mathcal{F}'$ to be the composition

$$\begin{aligned} \underline{\Bbbk}_{\mathbb{A}^1} \boxtimes p_*\mathcal{F}' \cong f_*(\underline{\Bbbk}_{\mathbb{A}^1} \boxtimes \mathcal{F}') &\xrightarrow{f_*\eta} f_*q_{2*}q_2^*(\underline{\Bbbk}_{\mathbb{A}^1} \boxtimes \mathcal{F}') \\ &\cong f_*q_{1!}q_2^*(\underline{\Bbbk}_{\mathbb{A}^1} \boxtimes \mathcal{F}') \xrightarrow{f_*\psi} f_*(\underline{\Bbbk}_{\mathbb{A}^1} \boxtimes \mathcal{F}') \cong \underline{\Bbbk}_{\mathbb{A}^1} \boxtimes p_*\mathcal{F}'. \end{aligned}$$

For $t \in \mathbb{A}^1$, let $i_t : Z \to \mathbb{A}^1 \times Z$ be the map given by $i_t(z) = (t, z)$. It follows from (2.10.1) and (2.10.2) that

$$i_t^*\phi : p_*\mathcal{F}' \to p_*\mathcal{F}' \text{ is } \begin{cases} \text{an isomorphism} & \text{if } t \in \mathbb{G}_{\mathrm{m}}, \\ \text{the zero morphism} & \text{if } t = 0. \end{cases}$$

But by Lemma 2.10.4, $i_t^*\phi$ is independent of t. We conclude that $p_*\mathcal{F}' = 0$. ☐

The argument above can also be carried out in certain situations where the action is *not* attracting (or repelling).

PROPOSITION 2.10.5. *Let X be a $\mathbb{G}_{\mathrm{m}}$-variety, and let $i : Z \hookrightarrow X$ be the inclusion map of a $\mathbb{G}_{\mathrm{m}}$-stable closed subset. Let $Q \subset \mathbb{A}^1 \times X \times X$ be the closure of the graph of the $\mathbb{G}_{\mathrm{m}}$-action, i.e.,*

$$Q = \overline{\{(t, x, \sigma(t, x)) \in \mathbb{G}_{\mathrm{m}} \times X \times X \mid t \in \mathbb{G}_{\mathrm{m}},\ x \in X\}},$$

and let $q_1, q_2 : Q \to \mathbb{A}^1 \times X$ *be given by*

$$q_1(t,x,y) = (t,x), \qquad q_2(t,x,y) = (t,y).$$

Finally, let $g : Z \to B$ *be a* $\mathbb{G}_\mathrm{m}$*-invariant map, i.e., a map satisfying* $g(\sigma(t,x)) = g(x)$ *for all* $t \in \mathbb{G}_\mathrm{m}$ *and* $x \in Z$*. Assume that:*

- *The map* $i : Z \hookrightarrow X$ *admits a* $\mathbb{G}_\mathrm{m}$*-equivariant left inverse* $p : X \to Z$*.*
- *The map* $q_1 : Q \to \mathbb{A}^1 \times X$ *is proper.*

Then, for any weakly $\mathbb{G}_\mathrm{m}$*-equivariant object* $\mathcal{F} \in D^\mathrm{b}_\mathrm{c}(X, \Bbbk)$*, there are natural isomorphisms*

$$g_* p_* \mathcal{F} \xrightarrow{\sim} g_* i^* \mathcal{F}, \qquad g_! i^! \mathcal{F} \xrightarrow{\sim} g_! p_! \mathcal{F}.$$

The proof is essentially identical to that of Theorem 2.10.3. (One minor difference is that for Step 3, one should define $f : \mathbb{A}^1 \times X \to \mathbb{A}^1 \times B$ by $f(t,x) = (t, g(p(x)))$. Then it is still true that $f \circ q_1 = f \circ q_2$.) We omit further details.

Hyperbolic localization. When the $\mathbb{G}_\mathrm{m}$-action on X is neither attracting nor repelling, we may break it up into attracting and repelling subsets as follows.

Definition 2.10.6. Let X be a $\mathbb{G}_\mathrm{m}$-variety. Let $Z = X^{\mathbb{G}_\mathrm{m}}$ be its set of fixed points, and let $Z_1, \ldots, Z_k$ be the connected components of Z. For each $i \in \{1, \ldots, k\}$, let

$$X_i^+ = \{x \in X \mid \lim_{t \to 0} t \cdot x \in Z_i\} \qquad \text{and} \qquad X_i^- = \{x \in X \mid \lim_{t \to \infty} t \cdot x \in Z_i\}.$$

The **attracting** and **repelling varieties** are the varieties X^+ and X^- given by

$$X^+ = X_1^+ \sqcup \cdots \sqcup X_k^+ \qquad \text{and} \qquad X^- = X_1^- \sqcup \cdots \sqcup X_k^-,$$

respectively. Let $i^\pm : X^\pm \to X$ be the obvious maps, and let $h^\pm : Z \hookrightarrow X^\pm$ be the inclusion maps. The **hyperbolic localization** functors with respect to this $\mathbb{G}_\mathrm{m}$-action on X are the two functors $D^\mathrm{b}_\mathrm{c}(X, \Bbbk) \to D^\mathrm{b}_\mathrm{c}(Z, \Bbbk)$ given by

$$\mathcal{F} \mapsto (h^-)^! (i^-)^* \mathcal{F} \qquad \text{and} \qquad \mathcal{F} \mapsto (h^+)^* (i^+)^! \mathcal{F}.$$

We will not discuss the proof of the claim that the sets X^+ and X^- are, in fact, varieties. A general proof can be found in [**69**, Corollary 1.5.3]. Under the assumption that X is a *normal* variety, this claim can also be deduced from a result of Sumihiro [**234**] (cf. [**69**, Remark 1.5.7]). If X is smooth and projective, this fact goes back to Białynicki-Birula [**28**].

Of course, the $\mathbb{G}_\mathrm{m}$-action on X^+ is attracting, and that on X^- is repelling, so Theorem 2.10.3 yields an alternative description of the hyperbolic localization functors for weakly $\mathbb{G}_\mathrm{m}$-equivariant objects. To wit, let $p^\pm : X^\pm \to Z$ be given by

$$p^+(x) = \lim_{t \to 0} t \cdot x \qquad \text{and} \qquad p^-(x) = \lim_{t \to \infty} t \cdot x.$$

Then, for weakly $\mathbb{G}_\mathrm{m}$-equivariant objects, the hyperbolic localization functors are given by

$$\mathcal{F} \mapsto (p^-)_! (i^-)^* \mathcal{F} \qquad \text{and} \qquad \mathcal{F} \mapsto (p^+)_* (i^+)^! \mathcal{F}. \tag{2.10.3}$$

The following generalization of Theorem 2.10.3 is due to Braden [**43**]. It was motivated in part by earlier results of Kirwan [**137**] and Goresky–MacPherson [**89**] on the hypercohomology of sheaves in the presence of a $\mathbb{G}_\mathrm{m}$-action.

THEOREM 2.10.7 (Hyperbolic localization). *Let X be a $\mathbb{G}_{\mathrm{m}}$-variety, and let $Z = X^{\mathbb{G}_{\mathrm{m}}}$ be the set of fixed points. For any $\mathcal{F} \in D^{\mathrm{b}}_{\mathrm{c}}(X, \Bbbk)$, there is a natural map*

$$(h^+)^*(i^+)^!\mathcal{F} \to (h^-)^!(i^-)^*\mathcal{F}.$$

If $\mathcal{F}$ is weakly $\mathbb{G}_{\mathrm{m}}$-equivariant, this map is an isomorphism.

The argument below, adapted from [**43**], requires the assumption that X is a normal variety. See [**69**] for a proof that does not require this assumption.

PROOF SKETCH FOR A NORMAL VARIETY. Most of the argument outlined below is devoted to studying the case of a linear $\mathbb{G}_{\mathrm{m}}$-action on a vector space. We will return to the general setting of the theorem in Step 5 below.

Step 1. Notation for a linear $\mathbb{G}_{\mathrm{m}}$-action. Let V be a vector space with a linear $\mathbb{G}_{\mathrm{m}}$-action $\sigma : \mathbb{G}_{\mathrm{m}} \times V \to V$. Then, by, say, [**230**, Theorem 3.2.3(c)], V has a basis of **weight vectors**, i.e., a basis $\{e_1, \dots, e_N\}$ such that

$$\sigma(t, e_i) = t^{a_i} e_i$$

for some integers $a_1, \dots, a_N$. Let

$$V_\uparrow = \mathrm{span}\{e_i \mid a_i > 0\}, \quad V_0 = V^{\mathbb{G}_{\mathrm{m}}} = \mathrm{span}\{e_i \mid a_i = 0\}, \quad V_\downarrow = \mathrm{span}\{e_i \mid a_i < 0\}.$$

The attracting variety for this action is $V^+ = V_\uparrow \times V_0$, and the repelling variety is $V^- = V_0 \times V_\downarrow$. The limit map $p^- : V^- \to V_0$ can be identified with projection onto the first factor.

Let $\mathbb{G}_{\mathrm{m}}$ act on $V_\uparrow \times V_\downarrow \times \mathbb{A}^1$ by having it act trivially on the last factor. This gives rise to a $\mathbb{G}_{\mathrm{m}}$-action on the projective space $\mathbb{P}(V_\uparrow \times V_\downarrow \times \mathbb{A}^1)$. There is a $\mathbb{G}_{\mathrm{m}}$-equivariant open embedding $V_\uparrow \times V_\downarrow \hookrightarrow \mathbb{P}(V_\uparrow \times V_\downarrow \times \mathbb{A}^1)$ that sends a vector $(v, w) \in V_\uparrow \times V_\downarrow$ to the line spanned by $(v, w, 1) \in V_\uparrow \times V_\downarrow \times \mathbb{A}^1$. Finally, observe that $\mathbb{P}(V_\uparrow)$ can be identified with a closed subvariety of $\mathbb{P}(V_\uparrow \times V_\downarrow \times \mathbb{A}^1)$, and that

$$V_\uparrow \times V_\downarrow \subset \mathbb{P}(V_\uparrow \times V_\downarrow \times \mathbb{A}^1) \smallsetminus \mathbb{P}(V_\uparrow). \tag{2.10.4}$$

Let

$$Y = V_0 \times (\mathbb{P}(V_\uparrow \times V_\downarrow \times \mathbb{A}^1) \smallsetminus \mathbb{P}(V_\uparrow)).$$

The embedding (2.10.4) extends to an open embedding

$$b : V \hookrightarrow Y. \tag{2.10.5}$$

The subset $V_\uparrow \times \{0\} \subset V_\uparrow \times V_\downarrow$ remains closed as a subset of $\mathbb{P}(V_\uparrow \times V_\downarrow \times \mathbb{A}^1) \smallsetminus \mathbb{P}(V_\uparrow)$ under (2.10.4), and hence V^+ remains closed as a subset of Y under (2.10.5). Let $Y' = Y \smallsetminus V^+$, and let

$$k : V^+ \hookrightarrow Y \qquad \text{and} \qquad h : Y' \hookrightarrow Y \tag{2.10.6}$$

be the inclusion maps. On the other hand, V^- is generally not closed in Y; its closure $\overline{V^-}$ can be identified with $V_0 \times \mathbb{P}(V_\downarrow \times \mathbb{A}^1)$. Let

$$i : \overline{V^-} \hookrightarrow Y$$

be the inclusion map. The linear projection map $V \times \mathbb{A}^1 \to V^- \times \mathbb{A}^1$ induces a map

$$p : Y \to \overline{V^-}$$

satisfying $p \circ i = \mathrm{id}_{\overline{V^-}}$. Finally, the map $p^- : V^- \to V_0$ extends (in a unique way) to a map

$$\hat{p}^- : \overline{V^-} \to V_0.$$

This map is given by projection onto the first factor under the identification $\overline{V^-} \cong V_0 \times \mathbb{P}(V_\downarrow \times \mathbb{A}^1)$.

Step 2. If $\mathcal{G} \in D^{\mathrm{b}}_{\mathrm{c}}(Y, \Bbbk)$ is weakly $\mathbb{G}_{\mathrm{m}}$-equivariant, then there is a natural isomorphism $\hat{p}^-_* p_* \mathcal{G} \xrightarrow{\sim} \hat{p}^-_* i^* \mathcal{G}$. We wish to apply Proposition 2.10.5. To do this, we must check that the map $q_1 : Q \to \mathbb{A}^1 \times Y$ defined in the statement of that proposition is proper. This can be proved using the observation that Q remains closed as a subset of $\mathbb{A}^1 \times Y \times (V_0 \times \mathbb{P}(V_\uparrow \times V_\downarrow \times \mathbb{A}^1))$.

For the next step, we introduce the notation

$$i' = i|_{Y' \cap \overline{V^-}} : Y' \cap \overline{V^-} \hookrightarrow Y', \qquad p' = p|_{Y'} : Y' \to Y' \cap \overline{V^-},$$
$$(\hat{p}^-)' = \hat{p}^-|_{Y' \cap \overline{V^-}} : Y' \cap \overline{V^-} \to V_0.$$

Step 3. If $\mathcal{G} \in D^{\mathrm{b}}_{\mathrm{c}}(Y', \Bbbk)$ is weakly $\mathbb{G}_{\mathrm{m}}$-equivariant, then there is a natural isomorphism $(\hat{p}^-)'_* p'_* \mathcal{G} \xrightarrow{\sim} (\hat{p}^-)'_* (i')^* \mathcal{G}$. This is identical to Step 2.

Step 4. Proof of the theorem in the case of a linear action. We continue with the notation from Step 1. Let $U^\pm = V \smallsetminus V^\pm$, and let $j^\pm : U^\pm \hookrightarrow V$ be the inclusion map. For any $\mathcal{F} \in D^{\mathrm{b}}_{\mathrm{c}}(V, \Bbbk)$, we have a natural distinguished triangle

$$(h^+)^*(i^+)^! j^-_! (j^-)^* \mathcal{F} \to (h^+)^*(i^+)^! \mathcal{F} \to (h^+)^*(i^+)^! i^-_* (i^-)^* \mathcal{F} \to .$$

The cartesian square

$$\begin{array}{ccc} V_0 & \xrightarrow{h^-} & V^- \\ {\scriptstyle h^+}\downarrow & & \downarrow{\scriptstyle i^-} \\ V^+ & \xrightarrow{i^+} & V \end{array}$$

shows that $(i^+)^! i^-_* \cong (h^+)_* (h^-)^!$, and hence $(h^+)^*(i^+)^! i^-_* (i^-)^* \mathcal{F} \cong (h^-)^!(i^-)^* \mathcal{F}$. The distinguished triangle above thus becomes

$$(h^+)^*(i^+)^! j^-_! (j^-)^* \mathcal{F} \to (h^+)^*(i^+)^! \mathcal{F} \to (h^-)^!(i^-)^* \mathcal{F} \to .$$

The second arrow above is the desired natural map.

Assume now that $\mathcal{F}$ is weakly $\mathbb{G}_{\mathrm{m}}$-equivariant. Let $\mathcal{F}' = j^-_!(j^-)^* \mathcal{F}$. To finish the proof, we must show that $(h^+)^*(i^+)^! \mathcal{F}' = 0$. As in (2.10.3), this is equivalent to showing that

(2.10.7) $$(p^+)_*(i^+)^! \mathcal{F}' = 0.$$

Let $\mathcal{G} = b_! \mathcal{F}'$, where b is the open embedding from (2.10.5). From (2.10.6), we have a distinguished triangle

(2.10.8) $$k_* k^! \mathcal{G} \to \mathcal{G} \to h_*(\mathcal{G}|_{Y'}) \to .$$

Note that $k^! \mathcal{G} \cong (i^+)^! b^! \mathcal{G} \cong (i^+)^! \mathcal{F}'$. Let $\mathrm{pr}_0 : Y \to V_0$ be the projection map. Note that $\mathrm{pr}_0 \circ k = p^+$, $\mathrm{pr}_0 = \hat{p}^- \circ p$, and $\mathrm{pr}_0 \circ h = (\hat{p}^-)' \circ p'$. Thus, applying pr_{0*} to (2.10.8), we obtain

(2.10.9) $$(p^+)_*(i^+)^! \mathcal{F}' \to \hat{p}^-_* p_* \mathcal{G} \to (\hat{p}^-)'_*(p')_*(\mathcal{G}|_{Y'}) \to .$$

Since $\mathcal{G} \cong (b \circ j^-)_!(\mathcal{F}|_{U^-})$, we see that $\mathcal{G}|_{Y \smallsetminus U^-} = 0$. In particular, we have $i^* \mathcal{G} = 0$ and $(i')^*(\mathcal{G}|_{Y'}) = 0$. By Steps 2 and 3, the second and third terms in (2.10.9) vanish. We conclude that (2.10.7) holds.

Step 5. Proof in the general case. According to Sumihiro's embedding theorem [**234**], because X is normal, it can be covered by finitely many $\mathbb{G}_{\mathrm{m}}$-stable affine open subsets, each of which is isomorphic to a closed subvariety of some affine space on which $\mathbb{G}_{\mathrm{m}}$ acts linearly. As a consequence, the theorem follows from the special case considered in Step 4. ☐

Exercises.

2.10.1. Let X be a variety, and consider the projection map $\mathrm{pr}_2 : \mathbb{A}^1 \times X \to X$. Here is an outline of the proof of Lemma 2.10.4.

(a) Show that for any $\mathcal{F} \in D^{\mathrm{b}}_{\mathrm{c}}(X, \Bbbk)$, the adjunction map $\mathcal{F} \to \mathrm{pr}_{2*}\mathrm{pr}_2^*\mathcal{F}$ is an isomorphism. (*Hint*: First show that for any $x \in X$, the map $\mathcal{F}_x \to (\mathrm{pr}_{2*}\mathrm{pr}_2^*\mathcal{F})_x$ is an isomorphism.)
(b) Show that $\mathrm{pr}_2^* : D^{\mathrm{b}}_{\mathrm{c}}(X, \Bbbk) \to D^{\mathrm{b}}_{\mathrm{c}}(\mathbb{A}^1 \times X, \Bbbk)$ is fully faithful.
(c) Prove the remaining assertions in Lemma 2.10.4.

2.10.2. Let X and Y be $\mathbb{G}_{\mathrm{m}}$-varieties, and let $f : X \to Y$ be a $\mathbb{G}_{\mathrm{m}}$-equivariant algebraic map. Prove that f_*, $f_!$, f^*, and $f^!$ take weakly $\mathbb{G}_{\mathrm{m}}$-equivariant objects to weakly $\mathbb{G}_{\mathrm{m}}$-equivariant objects.

2.11. Homology and fundamental classes

In this section, we discuss sheaf-theoretic incarnations of a number of notions from algebraic topology, in the spirit of Theorem 1.1.18. We will give proofs that rely on the machinery developed in this chapter, and thus only apply to algebraic varieties. However, many of the statements in this section are true in greater generality.

Homology. The first notion defined below can be shown to coincide with the singular homology groups from algebraic topology, by methods similar to the proof of Theorem 1.1.18. See [**189**, Remark 4.4.2] and [**44**, Section V.12] for further comments.

Definition 2.11.1. Let X be a variety, and let M be a finitely generated $\Bbbk$-module. The kth **homology group** of X with coefficients in M is the $\Bbbk$-module

$$\mathbf{H}_k(X; M) = \mathbf{H}^{-k}_{\mathrm{c}}(X, \mathrm{a}_X^! \underline{M}_{\mathrm{pt}}).$$

The kth **Borel–Moore homology group** of X with coefficients in M is the $\Bbbk$-module

$$\mathbf{H}^{\mathrm{BM}}_k(X; M) = \mathbf{H}^{-k}(X, \mathrm{a}_X^! \underline{M}_{\mathrm{pt}}).$$

Of course, the most important case is $M = \Bbbk$, in which case we have

$$\mathbf{H}_k(X; \Bbbk) = \mathbf{H}^{-k}_{\mathrm{c}}(X, \omega_X) \qquad \text{and} \qquad \mathbf{H}^{\mathrm{BM}}_k(X; \Bbbk) = \mathbf{H}^{-k}(X, \omega_X).$$

If $f : X \to Y$ is a morphism of varieties, then we have an adjunction map $f_!\omega_X \cong f_!f^!\omega_Y \to \omega_Y$. Applying $R\Gamma_{\mathrm{c}}$, we obtain a natural map $R\Gamma_{\mathrm{c}}(\omega_X) \to R\Gamma_{\mathrm{c}}(\omega_Y)$, and hence an induced map

$$f_\sharp : \mathbf{H}_k(X; \Bbbk) \to \mathbf{H}_k(Y; \Bbbk).$$

There is not always an induced map in Borel–Moore homology, but in the special case where f is proper, we can rewrite our adjunction map as $f_*\omega_X \to \omega_Y$, and then apply $R\Gamma$ to obtain an induced map

$$f_\sharp : \mathbf{H}^{\mathrm{BM}}_k(X; \Bbbk) \to \mathbf{H}^{\mathrm{BM}}_k(Y; \Bbbk).$$

On the other hand, if $f : X \to Y$ is smooth of relative dimension d, there is an adjunction map $\omega_Y \to f_*f^*\omega_Y \cong f_*f^!\omega_Y[-2d](-d) \cong f_*\omega_X[-2d](-d)$. Applying $R\Gamma$, we get an induced map

$$f^\sharp : \mathbf{H}_k^{\mathrm{BM}}(Y;\Bbbk) \to \mathbf{H}_{k+2d}^{\mathrm{BM}}(X;\Bbbk)(-d).$$

One can also consider induced maps in (ordinary or Borel–Moore) homology with coefficients in M; the details are left to the reader.

Remark 2.11.2. If X is a smooth variety of dimension n, then by Corollary 2.2.10, we have $\omega_X \cong \underline{\Bbbk}_X[2n](n)$, and hence

$$\mathbf{H}_k(X;\Bbbk) \cong \mathbf{H}_{\mathrm{c}}^{2n-k}(X;\Bbbk)(n) \qquad \text{and} \qquad \mathbf{H}_k^{\mathrm{BM}}(X;\Bbbk) \cong \mathbf{H}^{2n-k}(X;\underline{\Bbbk})(n).$$

These isomorphisms are essentially instances of the Poincaré duality theorem from algebraic topology (cf. [**98**, Theorem 3.35]).

Lemma 2.11.3. *Let X be a variety. Let $U \subset X$ be a Zariski open subset, and let $Z = X \smallsetminus U$. Then there is a long exact sequence*

$$\cdots \to \mathbf{H}_k^{\mathrm{BM}}(Z;\Bbbk) \to \mathbf{H}_k^{\mathrm{BM}}(X;\Bbbk) \to \mathbf{H}_k^{\mathrm{BM}}(U;\Bbbk) \to \mathbf{H}_{k-1}^{\mathrm{BM}}(Z;\Bbbk) \to \cdots.$$

Proof. Let $i : Z \hookrightarrow X$ and $j : U \hookrightarrow X$ be the inclusion maps. The distinguished triangle $i_*i^!\omega_X \to \omega_X \to j_*j^*\omega_X \to$ can be rewritten as $i_*\omega_Z \to \omega_X \to j_*\omega_U \to$. Apply $R\Gamma$ and then take the long exact sequence in cohomology. □

Proposition 2.11.4 (Universal coefficient theorems for homology). *Let X be a variety. There are natural short exact sequences*

$$0 \to \mathrm{Ext}^1_{\mathbb{Z}}(\mathbf{H}^{k+1}(X;\mathbb{Z}),\Bbbk) \to \mathbf{H}_k(X;\Bbbk) \to \mathrm{Hom}_{\mathbb{Z}}(\mathbf{H}^k(X;\mathbb{Z}),\Bbbk) \to 0,$$
$$0 \to \mathrm{Ext}^1_{\mathbb{Z}}(\mathbf{H}_{\mathrm{c}}^{k+1}(X;\mathbb{Z}),\Bbbk) \to \mathbf{H}_k^{\mathrm{BM}}(X;\Bbbk) \to \mathrm{Hom}_{\mathbb{Z}}(\mathbf{H}_{\mathrm{c}}^k(X;\mathbb{Z}),\Bbbk) \to 0.$$

Proof. By Proposition 2.9.8, we have $R\Gamma(\underline{\Bbbk}_X) \cong \Bbbk \otimes^L_{\mathbb{Z}} R\Gamma(\underline{\mathbb{Z}}_X)$, and hence

$$\begin{aligned} R\Gamma_{\mathrm{c}}(\omega_X) &\cong R\mathrm{Hom}_{\Bbbk}(R\Gamma(\underline{\Bbbk}_X),\Bbbk) \\ &\cong R\mathrm{Hom}_{\Bbbk}(\Bbbk \overset{L}{\otimes}_Z R\Gamma(\underline{\mathbb{Z}}_X),\Bbbk) \cong R\mathrm{Hom}_{\mathbb{Z}}(R\Gamma(\underline{\mathbb{Z}}_X),\Bbbk). \end{aligned}$$

Similar reasoning shows that

$$R\Gamma(\omega_X) \cong R\mathrm{Hom}_{\mathbb{Z}}(R\Gamma_{\mathrm{c}}(\underline{\mathbb{Z}}_X),\Bbbk).$$

Apply Proposition A.6.19 to these formulas to obtain the result. □

Lemma 2.11.5. *Let X be a variety of dimension n. Then both $\mathbf{H}_k(X;\Bbbk)$ and $\mathbf{H}_k^{\mathrm{BM}}(X;\Bbbk)$ are finitely generated $\Bbbk$-modules, and they vanish unless $0 \le k \le 2n$.*

Proof. The fact that these modules are finitely generated just comes from the fact that $R\Gamma_{\mathrm{c}}(\omega_X)$ and $R\Gamma(\omega_X)$ are constructible. We also know that these modules vanish for all but finitely many k. If $k \le -1$, the vanishing of $\mathbf{H}_k(X;\Bbbk)$ and $\mathbf{H}_k^{\mathrm{BM}}(X;\Bbbk)$ follows from Proposition 2.11.4. (For $k = -1$, use the fact that $\mathbf{H}^0(X;\mathbb{Z}) = \Gamma(\underline{\mathbb{Z}}_X)$ and $\mathbf{H}_{\mathrm{c}}^0(X;\mathbb{Z}) = \Gamma_{\mathrm{c}}(\underline{\mathbb{Z}}_X)$ are free abelian groups, so the Ext^1-terms in Proposition 2.11.4 vanish.)

Next, let m be the largest integer such that $\mathbf{H}_m(X;\Bbbk)$ is nonzero. Then there is a nonzero map $\Bbbk \to \mathbf{H}_m(X;\Bbbk)$ and hence a nonzero map

$$\Bbbk[m] \to \mathsf{H}^{-m}(R\Gamma_{\mathrm{c}}(\omega_X))[m] \cong \tau^{\le -m} R\Gamma_{\mathrm{c}}(\omega_X).$$

By the adjunction property of truncation, this corresponds to a nonzero map $\Bbbk[m] \to R\Gamma_c(\omega_X)$. In other words, $\mathrm{Hom}(\Bbbk[m], R\Gamma_c(\omega_X)) \neq 0$. We have

$$\begin{aligned}\mathrm{Hom}(\Bbbk[m], R\Gamma_c(\omega_X)) &\cong \mathrm{Hom}(\mathbb{D}(R\Gamma_c(\omega_X)), \mathbb{D}(\Bbbk[m])) \\ &\cong \mathrm{Hom}(R\Gamma(\underline{\Bbbk}_X), \Bbbk[-m]) \cong \mathrm{Hom}(\tau^{\geq m} R\Gamma(\underline{\Bbbk}_X), \Bbbk[-m]).\end{aligned}$$

We deduce that $\tau^{\geq m} R\Gamma(\underline{\Bbbk}_X) \neq 0$. Theorem 2.7.5 then implies that $m \leq 2n$, as desired. The proof for $\mathbf{H}^{\mathrm{BM}}_k(X;\Bbbk)$ is very similar, using Theorem 2.7.4 instead. □

Fundamental classes. We now describe some special elements in the top-degree Borel–Moore homology of a variety.

Lemma 2.11.6. *Let X be an irreducible variety of dimension n, and let $j : U \hookrightarrow X$ be an open embedding. Then $j^\sharp : \mathbf{H}^{\mathrm{BM}}_{2n}(X;\Bbbk) \to \mathbf{H}^{\mathrm{BM}}_{2n}(U;\Bbbk)$ is an isomorphism.*

Proof. Let $i : Z \hookrightarrow X$ be the inclusion of the complementary closed subset. Lemma 2.11.3 gives us a long exact sequence

$$\cdots \to \mathbf{H}^{\mathrm{BM}}_{2n}(Z;\Bbbk) \xrightarrow{i_\sharp} \mathbf{H}^{\mathrm{BM}}_{2n}(X;\Bbbk) \xrightarrow{j^\sharp} \mathbf{H}^{\mathrm{BM}}_{2n}(U;\Bbbk) \to \mathbf{H}^{\mathrm{BM}}_{2n-1}(Z;\Bbbk) \to \cdots.$$

Since X is irreducible, we must have $\dim Z < n$, so the the first and last terms above vanish by Lemma 2.11.5. Thus, $j^\sharp$ is an isomorphism. □

Definition 2.11.7. Let X be an irreducible variety of dimension n, and let $j : U \hookrightarrow X$ be the inclusion of a smooth Zariski open subset. The **fundamental class** of X, denoted by $[X] \in \mathbf{H}^{\mathrm{BM}}_{2n}(X;\Bbbk)(-n)$, is the image of $1 \in \Bbbk$ under the following sequence of maps:

$$1 \in \Bbbk \cong \mathbf{H}^{\mathrm{BM}}_0(\mathrm{pt};\Bbbk) \xrightarrow{\mathrm{a}_U^\sharp} \mathbf{H}^{\mathrm{BM}}_{2n}(U;\Bbbk)(-n) \xleftarrow[\sim]{j^\sharp} \mathbf{H}^{\mathrm{BM}}_{2n}(X;\Bbbk)(-n).$$

In a minor abuse of notation, if X is any (not necessarily irreducible) variety, and $Z \subset X$ is an irreducible closed subvariety of dimension m, then we sometimes write

$$[Z] \in \mathbf{H}^{\mathrm{BM}}_{2m}(X;\Bbbk)(-m)$$

to denote the element $i_\sharp([Z])$, where $i_\sharp : \mathbf{H}^{\mathrm{BM}}_{2m}(Z;\Bbbk) \to \mathbf{H}^{\mathrm{BM}}_{2m}(X;\Bbbk)$ is induced by the inclusion map $i : Z \hookrightarrow X$.

Lemma 2.11.8. *Let X be an irreducible variety of dimension n. The fundamental class $[X] \in \mathbf{H}^{\mathrm{BM}}_{2n}(X;\Bbbk)(-n)$ is independent of the choice of smooth open subset $U \subset X$.*

Proof. In this proof, let $[X]_U$ denote the element obtained by applying Definition 2.11.7 to $j : U \hookrightarrow X$. Let $V \subset X$ be another smooth Zariski open subset, and let $u : U \cap V \hookrightarrow U$ be the inclusion map. The commutativity of the diagram

$$\begin{tikzcd} & \mathbf{H}^{\mathrm{BM}}_{2n}(U\cap V;\Bbbk)(-n) & \\ \mathbf{H}^{\mathrm{BM}}_0(\mathrm{pt};\Bbbk) \arrow[ur, "\mathrm{a}_{U\cap V}^\sharp"] \arrow[r, "\mathrm{a}_U^\sharp"'] & \mathbf{H}^{\mathrm{BM}}_{2n}(U;\Bbbk)(-n) \arrow[u, "u^\sharp"] & \mathbf{H}^{\mathrm{BM}}_{2n}(X;\Bbbk)(-n) \arrow[l, "j^\sharp"] \arrow[ul, "(j\circ u)^\sharp"'] \end{tikzcd}$$

shows that $[X]_U = [X]_{U\cap V}$. The same reasoning shows that $[X]_{U\cap V} = [X]_V$, so $[X]_U = [X]_V$, as desired. □

REMARK 2.11.9. For an irreducible variety X of dimension n, the fundamental class $[X]$ is an element of the group $\mathbf{H}^{\mathrm{BM}}_{2n}(X;\Bbbk)(-n) = \mathsf{H}^{-2n}(R\Gamma(\omega_X(-n))) \cong \mathrm{Hom}(\underline{\Bbbk}_X, \omega_X[-2n](-n))$. In other words, the fundamental class can be thought of as a canonical map

$$[X] : \underline{\Bbbk}_X \to \omega_X[-2n](-n).$$

By unwinding the definitions, one can check that in the special case where X is smooth, this canonical map is the isomorphism from Corollary 2.2.10.

More generally, if Z is an irreducible closed subvariety of dimension m in a (not necessarily irreducible) variety X, then the element $[Z] \in \mathbf{H}^{\mathrm{BM}}_{2m}(X;\Bbbk)(-m)$ is given by the composition

$$\underline{\Bbbk}_X \to i_*\underline{\Bbbk}_Z \xrightarrow{[Z]} i_*\omega_Z[-2m](-m) \cong i_*i^!\omega_X[-2m](-m) \to \omega_X[-2m](-m).$$

REMARK 2.11.10. Let X be a smooth variety of dimension n. Then, via Remark 2.11.2, fundamental classes of irreducible subvarieties can be regarded as elements of cohomology, rather than Borel–Moore homology. If $Z \subset X$ is an irreducible closed subvariety of codimension r, then we write $\mathrm{cl}_X(Z) \in \mathbf{H}^{2r}(X;\Bbbk)(r)$ for the element corresponding to $[Z] \in \mathbf{H}^{\mathrm{BM}}_{2n-2r}(X;\Bbbk)(r-n)$. The resulting map

$$\mathrm{cl}_X : \left\{ \begin{array}{c} \text{irreducible closed} \\ \text{subvarieties of codimension } r \end{array} \right\} \to \mathbf{H}^{2r}(X;\Bbbk)(r)$$

is known as the **cycle class map**.

PROPOSITION 2.11.11. *Let X be a variety of dimension n, and let $X_1, \ldots, X_k$ be its n-dimensional irreducible components. Then $\mathbf{H}^{\mathrm{BM}}_{2n}(X;\Bbbk)(-n)$ is a free $\Bbbk$-module with basis $\{[X_1], \ldots, [X_k]\}$.*

PROOF. For each X_i, choose a smooth Zariski open subset $U_i \subset X_i$ that does not meet any other irreducible component. Let $U = U_1 \cup \cdots \cup U_k$, and let $Z = X \smallsetminus U$. Then Z is a closed subvariety of dimension $< n$, so $\mathbf{H}^{\mathrm{BM}}_{2n}(Z;\Bbbk) = \mathbf{H}^{\mathrm{BM}}_{2n-1}(Z;\Bbbk) = 0$. From the long exact sequence in Lemma 2.11.3, we obtain a canonical isomorphism $\mathbf{H}^{\mathrm{BM}}_{2n}(X;\Bbbk) \xrightarrow{\sim} \mathbf{H}^{\mathrm{BM}}_{2n}(U;\Bbbk)$. Since U is the disjoint union of $U_1, \ldots, U_k$, we have a canonical isomorphism

$$\mathbf{H}^{\mathrm{BM}}_{2n}(U;\Bbbk) \cong \mathbf{H}^{\mathrm{BM}}_{2n}(U_1;\Bbbk) \oplus \cdots \oplus \mathbf{H}^{\mathrm{BM}}_{2n}(U_k;\Bbbk).$$

We have reduced the problem to showing that $\mathbf{H}^{\mathrm{BM}}_{2n}(U_i;\Bbbk)$ is a free $\Bbbk$-module of rank 1, spanned by $[U_i]$. Since U_i is smooth and connected, we have $\mathbf{H}^{\mathrm{BM}}_{2n}(U_i;\Bbbk)(-n) \cong \mathrm{Hom}(\underline{\Bbbk}_{U_i}, \omega_{U_i}[-2n](-n))$ as in Remark 2.11.9, and $[X]$ corresponds to an isomorphism $\underline{\Bbbk}_{U_i} \xrightarrow{\sim} \omega_{U_i}[-2n](-n)$. Our claim thus follows from the fact (from Lemma 1.7.3) that $\mathrm{End}(\underline{\Bbbk}_{U_i})$ is a free $\Bbbk$-module of rank 1, generated by the identity map $\underline{\Bbbk}_{U_i} \to \underline{\Bbbk}_{U_i}$. □

PROPOSITION 2.11.12. *Let X be a variety of dimension n. There is a natural isomorphism*

$$\mathbf{H}^{2n}_{\mathrm{c}}(X;\Bbbk) \cong \mathrm{Hom}_{\mathbb{Z}}(\mathbf{H}^{\mathrm{BM}}_{2n}(X;\mathbb{Z}), \Bbbk).$$

In particular, $\mathbf{H}^{2n}_{\mathrm{c}}(X;\Bbbk)$ is a free $\Bbbk$-module whose rank is the number of n-dimensional irreducible components of X.

PROOF. Recall that $R\Gamma_{\mathrm{c}}(\underline{\Bbbk}_X) \cong R\mathrm{Hom}(R\Gamma(\omega_X), \Bbbk)$. Suppose first that $\Bbbk$ is a field. In this case, the functor $\mathrm{Hom}(-, \Bbbk)$ is exact, so applying $\mathsf{H}^{2n}(-)$ yields an isomorphism $\mathbf{H}^{2n}_{\mathrm{c}}(X;\Bbbk) \cong \mathrm{Hom}(\mathbf{H}^{\mathrm{BM}}_{2n}(X;\Bbbk), \Bbbk)$. In view of Proposition 2.11.11, we conclude that $\mathbf{H}^{2n}_{\mathrm{c}}(X;\Bbbk)$ has dimension equal to the number of n-dimensional irreducible components. In particular, this dimension is independent of $\Bbbk$.

Next, a minor variation on the reasoning in Proposition 2.11.4 shows that $R\Gamma_{\mathrm{c}}(\underline{\Bbbk}_X) \cong R\mathrm{Hom}_{\mathbb{Z}}(R\Gamma(\omega^{\mathbb{Z}}_X),\Bbbk)$. Proposition A.6.19 gives us a short exact sequence

$$0 \to \mathrm{Ext}^1_{\mathbb{Z}}(\mathbf{H}^{\mathrm{BM}}_{2n-1}(X;\mathbb{Z}),\Bbbk) \to \mathbf{H}^{2n}_{\mathrm{c}}(X;\Bbbk) \xrightarrow{\psi} \mathrm{Hom}_{\Bbbk}(\mathbf{H}^{\mathrm{BM}}_{2n}(X;\mathbb{Z}),\Bbbk) \to 0. \tag{2.11.1}$$

If $\Bbbk$ is a field, then by the previous paragraph, the second and third terms of (2.11.1) are $\Bbbk$-vector spaces of the same dimension, so the surjective map ψ is an isomorphism. Therefore,

$$\mathrm{Ext}^1_{\mathbb{Z}}(\mathbf{H}^{\mathrm{BM}}_{2n-1}(X;\mathbb{Z}),\Bbbk) = 0 \tag{2.11.2}$$

for every field $\Bbbk$. Any finitely generated abelian group M with p-torsion for some prime number p must have $\mathrm{Ext}^1_{\mathbb{Z}}(M,\mathbb{F}_p) \neq 0$, so (2.11.2) shows that $\mathbf{H}^{\mathrm{BM}}_{2n-1}(X;\mathbb{Z})$ is torsion-free, and hence free.

We now allow $\Bbbk$ to be an arbitrary noetherian ring of finite global dimension. The previous paragraph implies that the first term of (2.11.1) vanishes, so ψ is an isomorphism. □

Definition 2.11.13. Let X be a variety of dimension n, and let $X_1,\dots,X_k$ be its n-dimensional irreducible components. The **dual fundamental classes** of these components are the elements $[X_1]^*,\dots,[X_k]^* \in \mathbf{H}^{2n}_{\mathrm{c}}(X;\Bbbk)(n)$ that form the dual basis to $[X_1],\dots,[X_k] \in \mathbf{H}^{\mathrm{BM}}_{2n}(X;\mathbb{Z})(-n)$ under the isomorphism $\mathbf{H}^{2n}_{\mathrm{c}}(X;\Bbbk)(n) \cong \mathrm{Hom}(\mathbf{H}^{\mathrm{BM}}_{2n}(X;\mathbb{Z})(-n),\Bbbk)$.

Example 2.11.14. Recall from Example 2.2.6 that we have a canonical identification $\mathbf{H}^2_{\mathrm{c}}(\mathbb{P}^1;\Bbbk)(1) \cong \mathbf{H}^2(\mathbb{P}^1;\Bbbk)(1) \cong \Bbbk$. Under this identification, the dual fundamental class $[\mathbb{P}^1]^*$ is identified with the element $1 \in \Bbbk$.

Remark 2.11.15. Let X be a variety of dimension n. It can be shown that for any $\Bbbk$, there are natural isomorphisms

$$\mathbf{H}^{\mathrm{BM}}_{2n}(X;\Bbbk) \cong \Bbbk \otimes \mathbf{H}^{\mathrm{BM}}_{2n}(X;\mathbb{Z}) \qquad \text{and} \qquad \mathbf{H}^{2n}_{\mathrm{c}}(X;\Bbbk) \cong \Bbbk \otimes \mathbf{H}^{2n}_{\mathrm{c}}(X;\mathbb{Z}),$$

and that these isomorphisms send the (dual) fundamental classes to the (dual) fundamental classes. In this sense, the fundamental classes in $\mathbf{H}^{\mathrm{BM}}_{2n}(X;\Bbbk)$ and the dual fundamental classes in $\mathbf{H}^{2n}_{\mathrm{c}}(X;\Bbbk)$ are independent of $\Bbbk$.

Fundamental classes for smooth pairs. We now discuss a variant of the preceding ideas in the context of Definition 2.2.12.

Definition 2.11.16. Let $f : X \to S$ be a smooth morphism of varieties, and let $i : Z \hookrightarrow X$ be a closed embedding such that (Z,X) is a smooth pair of codimension r. The **cycle class** of Z, denoted by $\mathrm{cl}_X(Z)$, is the map given by the composition

$$\underline{\Bbbk}_X \to i_*\underline{\Bbbk}_Z \cong i_*i^!\underline{\Bbbk}_X[2r](r) \to \underline{\Bbbk}_X[2r](r),$$

regarded as an element of $\mathrm{Hom}(\underline{\Bbbk}_X,\underline{\Bbbk}_X[2r](r)) \cong \mathbf{H}^{2r}(X;\Bbbk)(r)$.

In the case where X itself is smooth, this notation is consistent with that introduced in Remark 2.11.10.

LEMMA 2.11.17. *Let $f : X \to S$ be a smooth morphism, and let $g : X' \to X$ be a morphism such that $f \circ g$ is also smooth. Suppose we have a cartesian square*

$$\begin{tikzcd} Z' \arrow[r, "g'"] \arrow[d, "i'"'] & Z \arrow[d, "i"] & \\ X' \arrow[r, "g"] & X \arrow[r, "f"] & S \end{tikzcd}$$

such that i and i' are both closed embeddings, and such that (Z, X) and (Z', X') are both smooth pairs of codimension r. Then $g^\sharp(\mathrm{cl}_X(Z)) = \mathrm{cl}_{X'}(Z')$.

PROOF SKETCH. The element $g^\sharp(\mathrm{cl}_X(Z))$ is given by the top row of the diagram below, while $\mathrm{cl}_{X'}(Z')$ is given by the bottom row:

$$\begin{tikzcd} g^*\underline{\Bbbk}_X \arrow[r] \arrow[dd, equal] \arrow[dr, phantom, "\text{Prop. 1.6.4}"] & g^*i_*i^*\underline{\Bbbk}_X \arrow[r, "\sim"] \arrow[d, "\wr", "\text{b.ch.}"'] & g^*i_*i^!\underline{\Bbbk}_X[2r](r) \arrow[r] \arrow[d, "\wr", "\text{b.ch.}"'] \arrow[dr, phantom, "\text{Prop. 1.6.5}"] & g^*\underline{\Bbbk}_X[2r](r) \arrow[dd, equal] \\ & i'_*(g')^*i^*\underline{\Bbbk}_X \arrow[r, "\sim"] \arrow[d, "\text{comp.}"', "\wr"] \arrow[dr, phantom, "(*)"] & i'_*(g')^*i^!\underline{\Bbbk}_X[2r](r) \arrow[d, "\text{cart.}"] & \\ g^*\underline{\Bbbk}_X \arrow[r] & i'_*(i')^*g^*\underline{\Bbbk}_X \arrow[r, "\sim"] & i'_*(i')^!g^*\underline{\Bbbk}_X[2r](r) \arrow[r] & g^*\underline{\Bbbk}_X[2r](r) \end{tikzcd}$$

We must show that this diagram commutes. The left- and right-hand squares commute by Propositions 1.6.4 and 1.6.5, respectively. The commutativity of the upper middle square is obvious. Finally, the commutativity of the square labelled $(*)$ can be checked by an argument similar to that outlined in Exercise 2.2.1. ☐

DEFINITION 2.11.18. Let X be a variety, and let $p : E \to X$ be a vector bundle of rank r over X. Let $i : X \to E$ be the zero section of this vector bundle. The **Euler class** of $E \to X$, denoted by $e_E \in \mathbf{H}^{2r}(X; \Bbbk)(r)$, is the image of $\mathrm{cl}_E(X) \in \mathbf{H}^{2r}(E; \Bbbk)(r)$ under the map $\mathbf{H}^{2r}(E; \Bbbk)(r) \xrightarrow{i^\sharp} \mathbf{H}^{2r}(X; \Bbbk)(r)$.

By unwinding the definitions, one can write down the Euler class explicitly as the map $e_E : \underline{\Bbbk}_X \to \underline{\Bbbk}_X[2r](r)$ given by

$$\underline{\Bbbk}_X \xleftarrow[\sim]{\text{Thm. 2.2.13}} i^!\underline{\Bbbk}_E[2r](r) \to i^*\underline{\Bbbk}_E[2r](r) \cong \underline{\Bbbk}_X[2r](r). \tag{2.11.3}$$

Traditionally, the Euler class is defined to be an element of $\mathbf{H}^{2r}(X; \Bbbk)$ that depends on the choice of an **orientation** of the bundle, which can be understood (as in, say, [**205**, Definition 2.4.5]) to be a choice of isomorphism $\underline{\Bbbk}_X \xrightarrow{\sim} i^!\underline{\Bbbk}_E[2r]$. In this book, we avoid this choice by using the canonical isomorphism from Theorem 2.2.13.

A straightforward consequence of Lemma 2.11.17 is the following:

LEMMA 2.11.19. *Suppose we have a cartesian square*

$$\begin{tikzcd} X \times_Y E \arrow[r] \arrow[d, "p'"'] & E \arrow[d, "p"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

where $p : E \to Y$ and $p' : X \times_Y E \to X$ are vector bundles. Then $f^\sharp(e_E) = e_{X \times_Y E}$.

We omit the proof. The following examples rely on properties of the Euler class that we will use without proof.

EXAMPLE 2.11.20. Let X be a smooth variety. Let $p : E \to X$ be a vector bundle of rank r, and let $p' : E' \to X$ be the dual vector bundle. By [**205**, Propositions 4.2.3 and 4.2.5], we have

$$e_{E'} = (-1)^r e_E.$$

EXAMPLE 2.11.21. Let X be a smooth, connected, projective variety of dimension n. Then $\mathbf{H}^{2n}(X; \Bbbk)(n)$ is identified with $\mathbf{H}^{2n}_{\mathrm{c}}(X; \Bbbk)(n)$, so it is a free $\Bbbk$-module on the dual fundamental class $[X]^*$. By [**205**, Corollary 2.7.6], the Euler class of the tangent bundle $TX \to X$ is given by

$$e_{TX} = \chi(X)[X]^* \qquad \text{where} \qquad \chi(X) = \sum_i (-1)^i \dim_{\mathbb{Q}} \mathbf{H}^i(X; \mathbb{Q}).$$

Recall that the integer $\chi(X)$ is known as the **Euler characteristic** of X. This fact is the reason for the name "Euler class." By Example 2.11.20, the Euler class of the cotangent bundle $T^*X \to X$ is given by

$$e_{T^*X} = (-1)^n \chi(X)[X]^*.$$

EXAMPLE 2.11.22. Let $\tilde{X} = \mathbb{A}^2 \smallsetminus \{0\}$, and let $\tilde{E} = \tilde{X} \times \mathbb{A}^1$. Choose an integer $n \in \mathbb{Z}$, and let $\mathbb{G}_{\mathrm{m}}$ act on $\tilde{E}$ by

$$z \cdot ((x, y), v) = ((zx, zy), z^n v).$$

This restricts to an action of $\mathbb{G}_{\mathrm{m}}$ on $\tilde{X}$. Form the quotient spaces $X = \tilde{X}/\mathbb{G}_{\mathrm{m}} = \mathbb{P}^1$ and $E_n = \tilde{E}/\mathbb{G}_{\mathrm{m}}$. (See Section 6.2 for a discussion of the existence of these quotients.) The projection $\tilde{E} \to \tilde{X}$ induces a map $p_n : E_n \to \mathbb{P}^1$, making E_n into a line bundle over $\mathbb{P}^1$. Identify $\mathbf{H}^2(\mathbb{P}^1; \Bbbk)(1)$ with $\Bbbk$ as in Example 2.11.14. We claim that the Euler class of E_n is given by

$$e_{E_n} = n \in \mathbf{H}^2(\mathbb{P}^1; \Bbbk)(1) \cong \Bbbk. \tag{2.11.4}$$

Here is an outline of the proof.

Step 1. By Remark 2.11.15, it is enough to prove (2.11.4) when $\Bbbk = \mathbb{Z}$.

Step 2. For any $n, m \in \mathbb{Z}$, there is an isomorphism of line bundles $E_n \otimes E_m \cong E_{n+m}$. By [**205**, Propositions 4.2.3 and 4.2.11], we have $e_{E_{n+m}} = e_{E_n} + e_{E_m}$.

Step 3. The vector bundle $p_2 : E_2 \to \mathbb{P}^1$ is the tangent bundle of $\mathbb{P}^1$, so by Example 2.11.21, its Euler class is $e_{E_2} = \chi(\mathbb{P}^1) = 2$.

Exercises.

2.11.1 (Artin vanishing for homology). Let X be an affine variety of dimension n. Show that $\mathbf{H}_k(X; \Bbbk) = 0$ unless $0 \le k \le n$.

2.11.2. Let X be a variety of dimension n, and let M be a finitely generated $\Bbbk$-module. Show that there is a natural isomorphism

$$\mathbf{H}^{\mathrm{BM}}_{2n}(X; M) \cong M \otimes \mathbf{H}^{\mathrm{BM}}_{2n}(X; \Bbbk).$$

Hint: Imitate the steps leading to the proof of Proposition 2.11.11.

2.11.3. Fill in the details in the argument sketched in Example 2.11.22.

2.11.4. Let E_1 be the open subset of $\mathbb{P}^{n+1}$ given by

$$E_1 = \{[x_0 : x_1 : \cdots : x_{n+1}] \in \mathbb{P}^{n+1} \mid x_0, \ldots, x_n \text{ are not all } 0\}.$$

The map $p : E_1 \to \mathbb{P}^n$ given by $p([x_0 : \cdots : x_{n+1}]) = [x_0 : \cdots : x_n]$ makes E_1 into a line bundle over $\mathbb{P}^n$, called the **hyperplane bundle**. (This coincides with the bundle denoted by E_1 in Example 2.11.22.) Let $i : \mathbb{P}^n \to E_1$ be the zero section.

(a) Given a hyperplane $H \subset \mathbb{A}^{n+1}$, one can consider the closed subvariety $\mathbb{P}(H) \subset \mathbb{P}^n$ of lines contained in H. Show that there is a bijection

$$\left\{ \begin{matrix} \text{sections of } p : E_1 \to \mathbb{P}^n \\ \text{other than the zero section} \end{matrix} \right\} \Big/ \mathbb{G}_{\mathrm{m}} \xleftrightarrow{\sim} \left\{ \begin{matrix} \text{hyperplanes} \\ \text{in } \mathbb{A}^{n+1} \end{matrix} \right\}.$$

(For the left-hand side, $\mathbb{G}_{\mathrm{m}}$ acts on the space of sections by scaling along the fibers of p; we take the quotient by this action.) Specifically, given a section $s : \mathbb{P}^n \to E_1$, the corresponding hyperplane $H \subset \mathbb{A}^{n+1}$ is determined by the condition that the following square be cartesian:

$$\begin{tikzcd} \mathbb{P}(H) \arrow[r, hook] \arrow[d, hook] & \mathbb{P}^n \arrow[d, "i"] \\ \mathbb{P}^n \arrow[r, "s"] & E_1 \end{tikzcd}$$

(b) Show that the Euler class $e_E \in \mathbf{H}^2(\mathbb{P}^n; \Bbbk)(1)$ is equal to $\mathrm{cl}_{\mathbb{P}^n}(\mathbb{P}(H))$ for any hyperplane $H \subset \mathbb{A}^{n+1}$.

2.11.5. Generalize the preceding exercise as follows: let $V \subset \mathbb{A}^{n+1}$ be a linear subspace of codimension r, and show that the element $\mathrm{cl}_{\mathbb{P}^n}(\mathbb{P}(V)) \in \mathbf{H}^{2r}(\mathbb{P}^n; \Bbbk)(r)$ is independent of V. Deduce that there is a canonical isomorphism

$$\Bbbk \xrightarrow{\sim} \mathbf{H}^{2r}(\mathbb{P}^n; \Bbbk)(r)$$

that sends $1 \in \Bbbk$ to $\mathrm{cl}_{\mathbb{P}^n}(\mathbb{P}(V))$.

2.12. Additional notes and exercises

Notes. Artin's vanishing theorem was proved in the étale setting in [**3**], following an earlier result on the singular cohomology of smooth affine complex varieties, due to Andreotti–Frankel [**17**]. The proof given in this book is based on an argument given by Nori [**187**]. The proof that f_* preserves constructibility in the setting considered in this book is due to Verdier [**239**] (again following a precedent in the étale setting [**3**]). The results on Verdier duality and some of the material in this chapter on fundamental classes also come from [**239**], while the notion of Borel–Moore homology was introduced in [**31**]. Most of the compatibility results from Section 2.9 come from [**32**].

Let $\mathrm{Mat}_{m\times n}(\mathbb{C})$ be the space of $m \times n$ matrices with complex coefficients. The following three exercises involve various examples of closed subvarieties $X \subset \mathrm{Mat}_{m\times n}(\mathbb{C})$, together with a smooth variety $\tilde{X}$ and a proper map $p : \tilde{X} \to X$. In each exercise, the variety X contains $0 \in \mathrm{Mat}_{m\times n}(\mathbb{C})$, and the subset $U = X \smallsetminus \{0\}$ is smooth. Let $j : U \hookrightarrow X$ be the inclusion map. For each exercise, do the following:

- Prove that $\tilde{X}$ is smooth and that p induces an isomorphism $p^{-1}(U) \to U$.
- Compute $p_*\underline{\Bbbk}_{\tilde{X}}$, $j_*\underline{\Bbbk}_U$, and $\omega_X = \mathbb{D}(\underline{\Bbbk}_X)$.

Exercise 2.12.1. Answer the questions above for

$$X = \{x \in \mathrm{Mat}_{2\times 2}(\mathbb{C}) \mid \det x = 0\},$$
$$\tilde{X} = \{(x, L) \in X \times \mathbb{P}^1 \mid L \subset \ker x\}.$$

Answers:

$p_*\underline{\Bbbk}_{\tilde{X}}$:

	U	$\{0\}$
2		$\Bbbk(-1)$
1		
0	$\underline{\Bbbk}$	$\Bbbk$

$j_*\underline{\Bbbk}_U$:

	U	$\{0\}$
5		$\Bbbk(-3)$
4		
3		$\Bbbk(-2)$
2		$\Bbbk(-1)$
1		
0	$\underline{\Bbbk}$	$\Bbbk$

ω_X :

	U	$\{0\}$
-3		$\Bbbk(1)$
-4		$\Bbbk(2)$
-5		
-6	$\underline{\Bbbk}(3)$	$\Bbbk(3)$

Hint: Compute $p_*\underline{\Bbbk}_{\tilde{X}}$ using the proper base change theorem. For $j_*\underline{\Bbbk}_U$, one option is to use Theorem 2.10.3. Another is to first compute $\tilde{\jmath}_*\underline{\Bbbk}_{p^{-1}(U)}$, where $\tilde{\jmath} : p^{-1}(U) \hookrightarrow \tilde{X}$ is the inclusion map. Finally, compute $\omega_X \cong \mathbb{D}(\underline{\Bbbk}_X)$ by showing that $\tau^{\leq 0} j_*\underline{\Bbbk}_U \cong \underline{\Bbbk}_X$, and then applying $\mathbb{D}$ to $\tau^{\leq 0} j_*\underline{\Bbbk}_U \to j_*\underline{\Bbbk}_U \to \tau^{\geq 1} j_*\underline{\Bbbk}_U \to$.

EXERCISE 2.12.2. Answer the questions above for

$$X = \left\{ \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \in \mathrm{Mat}_{2\times 3}(\mathbb{C}) \,\middle|\, ae - bd = af - cd = bf - ce = 0 \right\},$$

$$\tilde{X} = \{(x, L) \in X \times \mathbb{P}^2 \mid L \subset \ker x\}.$$

Hints and answers: Let $i : \{0\} \hookrightarrow X$ be the inclusion map. For this problem, it is useful to also compute $i^! p_*\underline{\Bbbk}_{\tilde{X}}$. Show using the proper base change theorem that

$p_*\underline{\Bbbk}_{\tilde{X}}$:

	U	$\{0\}$
4		$\Bbbk(-2)$
3		
2		$\Bbbk(-1)$
1		
0	$\underline{\Bbbk}$	$\Bbbk$

$i^! p_*\underline{\Bbbk}_{\tilde{X}}$:

	$\{0\}$
8	$\Bbbk(-4)$
7	
6	$\Bbbk(-3)$
5	
4	$\Bbbk(-2)$

To compute $j_*\underline{\Bbbk}_U$, it is useful to study the distinguished triangle $i_* i^! p_*\underline{\Bbbk}_{\tilde{X}} \to p_*\underline{\Bbbk}_{\tilde{X}} \to j_*\underline{\Bbbk}_U \to$, or rather $i^! p_*\underline{\Bbbk}_{\tilde{X}} \to i^* p_*\underline{\Bbbk}_{\tilde{X}} \to i^* j_*\underline{\Bbbk}_U \to$. The first map in this triangle looks like

$i^! p_*\underline{\Bbbk}_{\tilde{X}} \to i^* p_*\underline{\Bbbk}_{\tilde{X}}$:

	$\{0\}$
8	$\Bbbk(-4)$
7	
6	$\Bbbk(-3)$
5	
4	$\Bbbk(-2) \to \Bbbk(-2)$
3	
2	$\Bbbk(-1)$
1	
0	$\Bbbk$

Next, show that $\mathrm{pr}_2 : \tilde{X} \to \mathbb{P}^2$ is a vector bundle and that the map shown above is given in degree 4 by multiplication by the Euler class $e_{\tilde{X}} \in \mathbf{H}^4(\mathbb{P}^2; \Bbbk)(2) \cong \Bbbk$. It turns out that $e_{\tilde{X}}$ is a unit in $\Bbbk$, so this map is an isomorphism in H^4. Finally, use the method from Exercise 2.12.1 to compute ω_X:

$j_*\underline{\Bbbk}_U$:

	U	$\{0\}$
7		$\Bbbk(-4)$
6		
5		$\Bbbk(-3)$
4		
3		
2		$\Bbbk(-1)$
1		
0	$\underline{\Bbbk}$	$\Bbbk$

ω_X :

	U	$\{0\}$
-3		$\Bbbk(1)$
-4		
-5		
-6		$\Bbbk(3)$
-7		
-8	$\underline{\Bbbk}(4)$	$\Bbbk(4)$

Note: If you have some background in characteristic classes, show that the vector bundle $\mathrm{pr}_2 : \tilde{X} \to \mathbb{P}^2$ is the direct sum of two copies of the tautological bundle over $\mathbb{P}^2$. Use this observation to compute its Euler class.

EXERCISE 2.12.3. For this question, assume that $\Bbbk$ is a field. Answer the questions above for

$$X = \{x \in \mathrm{Mat}_{3\times 3}(\mathbb{C}) \mid x^2 = 0\},$$
$$\tilde{X} = \{(x, P) \in X \times \mathrm{Gr}(2,3) \mid P \subset \ker x\}.$$

Here $\mathrm{Gr}(2,3)$ is the Grassmannian of 2-dimensional subspaces of $\mathbb{C}^3$.

Hints and answers: The variety $\mathrm{Gr}(2,3)$ is isomorphic to $\mathbb{P}^2$, and parts of this exercise closely resemble Exercise 2.12.2. Let $i : \{0\} \hookrightarrow X$ be the inclusion map. Then we again have

$p_*\underline{\Bbbk}_{\tilde{X}}$:

	U	$\{0\}$
4		$\Bbbk(-2)$
3		
2		$\Bbbk(-1)$
1		
0	$\underline{\Bbbk}$	$\Bbbk$

$i^! p_*\underline{\Bbbk}_{\tilde{X}}$:

	$\{0\}$
8	$\Bbbk(-4)$
7	
6	$\Bbbk(-3)$
5	
4	$\Bbbk(-2)$

To compute $j_*\underline{\Bbbk}_U$, the problem again comes down to the fact that $\mathsf{H}^4(i^! p_*\underline{\Bbbk}_{\tilde{X}}) \to \mathsf{H}^4(i^* p_*\underline{\Bbbk}_{\tilde{X}})$ is given by multiplication by the Euler class $e_{\tilde{X}} \in \mathbf{H}^4(\mathbb{P}^2;\Bbbk)(2) \cong \Bbbk$.

Prove that $\mathrm{pr}_2 : \tilde{X} \to \mathrm{Gr}(2,3)$ is the cotangent bundle of $\mathrm{Gr}(2,3)$. By Example 2.11.21, we have $e_{\tilde{X}} = \chi(\mathrm{Gr}(2,3)) = 3 \in \Bbbk$:

$j_*\underline{\Bbbk}_U$, $\mathrm{char}\,\Bbbk = 3$:

	U	$\{0\}$
7		$\Bbbk(-4)$
6		
5		$\Bbbk(-3)$
4		$\Bbbk(-2)$
3		$\Bbbk(-2)$
2		$\Bbbk(-1)$
1		
0	$\underline{\Bbbk}$	$\Bbbk$

$j_*\underline{\Bbbk}_U$, $\mathrm{char}\,\Bbbk \neq 3$:

	U	$\{0\}$
7		$\Bbbk(-4)$
6		
5		$\Bbbk(-3)$
4		
3		
2		$\Bbbk(-1)$
1		
0	$\underline{\Bbbk}$	$\Bbbk$

Finally, ω_X is given by

ω_X, $\operatorname{char} \Bbbk = 3$:

	U	$\{0\}$
-3		$\Bbbk(1)$
-4		$\Bbbk(2)$
-5		$\Bbbk(2)$
-6		$\Bbbk(3)$
-7		
-8	$\underline{\Bbbk}(4)$	$\Bbbk(4)$

ω_X, $\operatorname{char} \Bbbk \neq 3$:

	U	$\{0\}$
-3		$\Bbbk(1)$
-4		
-5		
-6		$\Bbbk(3)$
-7		
-8	$\underline{\Bbbk}(4)$	$\Bbbk(4)$

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CHAPTER 3

Perverse sheaves

This chapter contains the definition and a number of foundational results on perverse sheaves. The first two sections of the chapter contain preliminary results, including the fact that perverse sheaves form an abelian category.

Perhaps the most important perverse sheaves are those known as "intersection cohomology complexes," introduced in Section 3.3. When $\Bbbk$ is a field, the intersection cohomology construction produces the simple objects in the category of perverse sheaves. For general noetherian $\Bbbk$, every perverse sheaf still admits a finite filtration by the intersection cohomology complexes. This fact is a step on the way to the proof of the fact that perverse sheaves form a noetherian abelian category.

Sections 3.5–3.8 contain various exactness results for pullback and push-forward functors, in the case of affine, smooth, or semismall (see Definition 3.8.1) morphisms. In Section 3.7, we show that a perverse sheaf can be recovered from its pullback along a smooth surjective morphism.

Finally, Section 3.9 discusses two of the deepest results on perverse sheaves in the case where $\Bbbk = \mathbb{Q}$: the decomposition theorem and the hard Lefschetz theorem. The full proof is beyond the scope of this book, but we will discuss some of the ingredients involved.

3.1. The perverse t-structure

This chapter makes heavy use of the theory of t-structures. For a summary of this theory, see Section A.7. The following notion is the main object of study in this chapter.

Definition 3.1.1. Let X be a variety. The **perverse t-structure** on X is the t-structure on $D^{\mathrm{b}}_{\mathrm{c}}(X,\Bbbk)$ given by

$$\begin{aligned} {}^pD^{\mathrm{b}}_{\mathrm{c}}(X,\Bbbk)^{\le 0} &= \{\mathcal{F} \in D^{\mathrm{b}}_{\mathrm{c}}(X,\Bbbk) \mid \text{for all } i, \text{ we have } \dim\operatorname{supp} \mathsf{H}^i(\mathcal{F}) \le -i\},\\ {}^pD^{\mathrm{b}}_{\mathrm{c}}(X,\Bbbk)^{\ge 0} &= \{\mathcal{F} \in D^{\mathrm{b}}_{\mathrm{c}}(X,\Bbbk) \mid \text{for all } i, \text{ we have } \operatorname{mdsupp} \mathsf{H}^i(\mathbb{D}\mathcal{F}) \le -i\}. \end{aligned}$$

The heart of this t-structure is denoted by

$$\mathrm{Perv}(X,\Bbbk) = {}^pD^{\mathrm{b}}_{\mathrm{c}}(X,\Bbbk)^{\le 0} \cap {}^pD^{\mathrm{b}}_{\mathrm{c}}(X,\Bbbk)^{\ge 0},$$

and objects in the heart are called **perverse sheaves**.

Of course, we need to prove that these categories do indeed define a t-structure. This occupies most of the rest of this section. Once this is done, we will have perverse truncation and cohomology functors, denoted by

$${}^p\tau^{\le n}, {}^p\tau^{\ge n} : D^{\mathrm{b}}_{\mathrm{c}}(X,\Bbbk) \to D^{\mathrm{b}}_{\mathrm{c}}(X,\Bbbk), \qquad {}^p\mathsf{H}^n : D^{\mathrm{b}}_{\mathrm{c}}(X,\Bbbk) \to \mathrm{Perv}(X,\Bbbk).$$

We will simultaneously see that if $(X_s)_{s\in\mathscr{S}}$ is a good stratification (see Definition 2.3.20), then there is an induced t-structure on $D^{\mathrm{b}}_{\mathscr{S}}(X,\Bbbk)$ given by

$$\begin{aligned} {}^pD^{\mathrm{b}}_{\mathscr{S}}(X,\Bbbk)^{\leq 0} &= {}^pD^{\mathrm{b}}_{\mathrm{c}}(X,\Bbbk)^{\leq 0} \cap D^{\mathrm{b}}_{\mathscr{S}}(X,\Bbbk), \\ {}^pD^{\mathrm{b}}_{\mathscr{S}}(X,\Bbbk)^{\geq 0} &= {}^pD^{\mathrm{b}}_{\mathrm{c}}(X,\Bbbk)^{\geq 0} \cap D^{\mathrm{b}}_{\mathscr{S}}(X,\Bbbk). \end{aligned} \tag{3.1.1}$$

(If $\mathscr{S}$ is the trivial stratification on a smooth, connected variety, a variant of this notation will be introduced in Lemma 3.1.3 below.) Its heart is denoted by

$$\mathrm{Perv}_{\mathscr{S}}(X,\Bbbk) = {}^pD^{\mathrm{b}}_{\mathscr{S}}(X,\Bbbk)^{\leq 0} \cap {}^pD^{\mathrm{b}}_{\mathscr{S}}(X,\Bbbk)^{\geq 0}.$$

This notation will not be used for stratifications that are not good.

Recall that if $\Bbbk$ is a field, then $\operatorname{mdsupp}\mathcal{G} = \dim\operatorname{supp}\mathcal{G}$ for any constructible sheaf $\mathcal{G}$. Thus, for field coefficients, Definition 3.1.1 simplifies, and we have

$$\mathcal{F} \in {}^pD^{\mathrm{b}}_{\mathrm{c}}(X,\Bbbk)^{\geq 0} \text{ if and only if } \mathbb{D}\mathcal{F} \in {}^pD^{\mathrm{b}}_{\mathrm{c}}(X,\Bbbk)^{\leq 0} \qquad \text{when } \Bbbk \text{ is a field.} \tag{3.1.2}$$

For another characterization of ${}^pD^{\mathrm{b}}_{\mathrm{c}}(X,\Bbbk)^{\geq 0}$, see Exercise 3.1.1.

The proof of the following consequence of Lemma 2.3.11 is left to the reader.

LEMMA 3.1.2. *The categories* ${}^pD^{\mathrm{b}}_{\mathrm{c}}(X,\Bbbk)^{\leq 0}$ *and* ${}^pD^{\mathrm{b}}_{\mathrm{c}}(X,\Bbbk)^{\geq 0}$ *are stable under extensions.*

LEMMA 3.1.3. *Let X be a smooth connected variety. We have*

$$\begin{aligned} {}^pD^{\mathrm{b}}_{\mathrm{locf}}(X,\Bbbk)^{\leq 0} &= D^{\mathrm{b}}_{\mathrm{locf}}(X,\Bbbk)^{\leq -\dim X}, \\ {}^pD^{\mathrm{b}}_{\mathrm{locf}}(X,\Bbbk)^{\geq 0} &= D^{\mathrm{b}}_{\mathrm{locf}}(X,\Bbbk)^{\geq -\dim X}. \end{aligned}$$

As a consequence, if we let $\mathscr{S}$ denote the trivial stratification on X, then

$$\mathrm{Perv}_{\mathscr{S}}(X,\Bbbk) = \mathrm{Loc}^{\mathrm{ft}}(X,\Bbbk)[\dim X].$$

In particular, we have $\mathrm{Perv}(\mathrm{pt},\Bbbk) = \mathrm{Sh}_{\mathrm{c}}(\mathrm{pt},\Bbbk) \cong \Bbbk\text{-mod}^{\mathrm{fg}}$.

Objects of $\mathrm{Loc}^{\mathrm{ft}}(X,\Bbbk)[\dim X]$ are sometimes called **shifted local systems**.

PROOF. It is obvious that $D^{\mathrm{b}}_{\mathrm{locf}}(X,\Bbbk)^{\leq -\dim X} \subset {}^pD^{\mathrm{b}}_{\mathrm{locf}}(X,\Bbbk)^{\leq 0}$. The opposite containment follows from the observation that for $\mathcal{F} \in D^{\mathrm{b}}_{\mathrm{locf}}(X,\Bbbk)$, each cohomology sheaf $\mathsf{H}^i(\mathcal{F})$ is a local system, so if it is nonzero, then $\dim\operatorname{supp}\mathsf{H}^i(\mathcal{F}) = \dim X$.

To prove that ${}^pD^{\mathrm{b}}_{\mathrm{locf}}(X,\Bbbk)^{\geq 0} = D^{\mathrm{b}}_{\mathrm{locf}}(X,\Bbbk)^{\geq -\dim X}$, we will prove that the following four conditions on an object $\mathcal{F} \in D^{\mathrm{b}}_{\mathrm{locf}}(X,\Bbbk)$ are all equivalent:

(1) $\mathcal{F} \in D^{\mathrm{b}}_{\mathrm{locf}}(X,\Bbbk)^{\geq -\dim X}$.
(2) $\mathcal{F}_x \in D^{\mathrm{b}}(\Bbbk\text{-mod})^{\geq -\dim X}$ for all $x \in X$.
(3) $\operatorname{grade}\mathsf{H}^i(\mathbb{D}(\mathcal{F}_x)) \geq i - \dim X$ for all $x \in X$ and all i.
(4) $\mathcal{F} \in {}^pD^{\mathrm{b}}_{\mathrm{locf}}(X,\Bbbk)^{\geq 0}$.

It is clear that condition (1) implies condition (2), because the functor $\mathcal{F} \mapsto \mathcal{F}_x$ is t-exact. The opposite implication follows from the fact that if $\mathsf{H}^i(\mathcal{F}) \neq 0$, then $\mathsf{H}^i(\mathcal{F}_x) \neq 0$ as well.

The equivalence of conditions (2) and (3) follows from Proposition A.10.6.

Finally, let $n = \dim X$. By Lemma 2.8.2, we have

$$(\mathbb{D}\mathcal{F})_x \cong R\mathrm{Hom}(\mathcal{F}_x, \underline{\Bbbk}[2n](n)) \cong \mathbb{D}(\mathcal{F}_x)[2n](n),$$

so condition (3) is equivalent to the condition that $\operatorname{grade}\mathsf{H}^i((\mathbb{D}\mathcal{F})_x) \geq i+n$ for all i. Since every nonzero cohomology sheaf $\mathsf{H}^i(\mathbb{D}\mathcal{F})$ has support of dimension n, this in turn is equivalent to requiring that $\operatorname{mdsupp}\mathsf{H}^i(\mathbb{D}\mathcal{F}) \leq n - (i+n) = -i$. $\square$

The next few lemmas describe how the perverse t-structure interacts with open and closed embeddings.

LEMMA 3.1.4. *Let $j : U \hookrightarrow X$ be an open embedding, and let $i : Z \hookrightarrow X$ be a closed embedding.*

(1) *The functor j^* sends ${}^pD^{\mathrm{b}}_{\mathrm{c}}(X,\Bbbk)^{\le 0}$ to ${}^pD^{\mathrm{b}}_{\mathrm{c}}(U,\Bbbk)^{\le 0}$ and ${}^pD^{\mathrm{b}}_{\mathrm{c}}(X,\Bbbk)^{\ge 0}$ to ${}^pD^{\mathrm{b}}_{\mathrm{c}}(U,\Bbbk)^{\ge 0}$.*
(2) *The functor $j_!$ sends ${}^pD^{\mathrm{b}}_{\mathrm{c}}(U,\Bbbk)^{\le 0}$ to ${}^pD^{\mathrm{b}}_{\mathrm{c}}(X,\Bbbk)^{\le 0}$.*
(3) *The functor j_* sends ${}^pD^{\mathrm{b}}_{\mathrm{c}}(U,\Bbbk)^{\ge 0}$ to ${}^pD^{\mathrm{b}}_{\mathrm{c}}(X,\Bbbk)^{\ge 0}$.*
(4) *The functor i_* sends ${}^pD^{\mathrm{b}}_{\mathrm{c}}(Z,\Bbbk)^{\le 0}$ to ${}^pD^{\mathrm{b}}_{\mathrm{c}}(X,\Bbbk)^{\le 0}$ and ${}^pD^{\mathrm{b}}_{\mathrm{c}}(Z,\Bbbk)^{\ge 0}$ to ${}^pD^{\mathrm{b}}_{\mathrm{c}}(X,\Bbbk)^{\ge 0}$.*
(5) *The functor i^* sends ${}^pD^{\mathrm{b}}_{\mathrm{c}}(X,\Bbbk)^{\le 0}$ to ${}^pD^{\mathrm{b}}_{\mathrm{c}}(Z,\Bbbk)^{\le 0}$.*
(6) *The functor $i^!$ sends ${}^pD^{\mathrm{b}}_{\mathrm{c}}(X,\Bbbk)^{\ge 0}$ to ${}^pD^{\mathrm{b}}_{\mathrm{c}}(Z,\Bbbk)^{\ge 0}$.*

PROOF. Parts (1) and (4) follow from the fact that j^* and i_* are t-exact (for the natural t-structure), commute with $\mathbb{D}$, and do not increase the ordinary or modified dimension of support of a constructible sheaf. Similarly, parts (2) and (5) hold because $j_!$ and i^* are t-exact and do not increase the dimension of the support.

For part (3) suppose $\mathcal{F} \in {}^pD^{\mathrm{b}}_{\mathrm{c}}(U,\Bbbk)^{\ge 0}$. We have

$$\operatorname{mdsupp} \mathsf{H}^i(\mathbb{D}(j_*\mathcal{F})) = \operatorname{mdsupp} \mathsf{H}^i(j_!\mathbb{D}(\mathcal{F})) \le \operatorname{mdsupp} \mathsf{H}^i(\mathbb{D}(\mathcal{F})) \le -i,$$

so $j_*\mathcal{F} \in {}^pD^{\mathrm{b}}_{\mathrm{c}}(X,\Bbbk)^{\ge 0}$. A similar calculation involving $i^!$ and i^* proves part (6). □

LEMMA 3.1.5. *Let $j : U \hookrightarrow X$ be an open embedding, and let $i : Z \hookrightarrow X$ be the complementary closed embedding. Let $\mathcal{F} \in D^{\mathrm{b}}_{\mathrm{c}}(X,\Bbbk)$.*

(1) *We have $\mathcal{F} \in {}^pD^{\mathrm{b}}_{\mathrm{c}}(X,\Bbbk)^{\le 0}$ if and only if $j^*\mathcal{F} \in {}^pD^{\mathrm{b}}_{\mathrm{c}}(U,\Bbbk)^{\le 0}$ and $i^*\mathcal{F} \in {}^pD^{\mathrm{b}}_{\mathrm{c}}(Z,\Bbbk)^{\le 0}$.*
(2) *We have $\mathcal{F} \in {}^pD^{\mathrm{b}}_{\mathrm{c}}(X,\Bbbk)^{\ge 0}$ if and only if $j^*\mathcal{F} \in {}^pD^{\mathrm{b}}_{\mathrm{c}}(U,\Bbbk)^{\ge 0}$ and $i^!\mathcal{F} \in {}^pD^{\mathrm{b}}_{\mathrm{c}}(Z,\Bbbk)^{\ge 0}$.*

PROOF. For part (1), the "only if" direction is immediate from Lemma 3.1.4. For the "if" direction, apply Lemmas 3.1.2 and 3.1.4 to the distinguished triangle $j_!j^*\mathcal{F} \to \mathcal{F} \to i_*i^*\mathcal{F} \to$. The proof of part (2) is similar. □

The following generalization of Lemmas 3.1.3 and 3.1.5 is proved by induction on the number of strata. We omit the details.

LEMMA 3.1.6. *Let X be a variety, and let $(X_s)_{s\in\mathscr{S}}$ be a stratification. For each $s \in \mathscr{S}$, let $j_s : X_s \hookrightarrow X$ be the inclusion map, and let $\mathcal{F} \in D^{\mathrm{b}}_{\mathrm{c}}(X,\Bbbk)$.*

(1) *Suppose $\mathcal{F}$ is constructible with respect to $\mathscr{S}$. We have $\mathcal{F} \in {}^pD^{\mathrm{b}}_{\mathrm{c}}(X,\Bbbk)^{\le 0}$ if and only if $j_s^*\mathcal{F} \in D^{\mathrm{b}}_{\mathrm{locf}}(X_s,\Bbbk)^{\le -\dim X_s}$ for all $s \in \mathscr{S}$.*
(2) *Suppose $\mathbb{D}\mathcal{F}$ is constructible with respect to $\mathscr{S}$. We have $\mathcal{F} \in {}^pD^{\mathrm{b}}_{\mathrm{c}}(X,\Bbbk)^{\ge 0}$ if and only if $j_s^!\mathcal{F} \in D^{\mathrm{b}}_{\mathrm{locf}}(X_s,\Bbbk)^{\ge -\dim X_s}$ for all $s \in \mathscr{S}$.*

We are nearly ready to prove that Definition 3.1.1 defines a t-structure. The next two lemmas contain part of this proof.

LEMMA 3.1.7. *Let $\mathcal{F} \in D^{\mathrm{b}}(X,\Bbbk)$ be an object such that for each i, the support of $\mathsf{H}^i(\mathcal{F})$ is an algebraic variety, and that $\dim\operatorname{supp}\mathsf{H}^i(\mathcal{F}) \le -i$. For all $\mathcal{G} \in {}^pD^{\mathrm{b}}_{\mathrm{c}}(X,\Bbbk)^{\ge 1}$, we have $\operatorname{Hom}(\mathcal{F},\mathcal{G}) = 0$.*

Note that in this lemma, $\mathcal{F}$ is not required to be constructible. (This additional generality will occasionally be needed later on.) Of course, if it is constructible, the assumptions are equivalent to requiring $\mathcal{F} \in {}^pD^{\mathrm{b}}_{\mathrm{c}}(X,\Bbbk)^{\le 0}$.

Proof. The conditions satisfied by $\mathcal{F}$ are preserved by the (natural) truncation functors. An induction argument with truncation shows that we may reduce to the case where $\mathcal{F}$ is concentrated in a single degree, say $\mathcal{F} \cong \mathsf{H}^j(\mathcal{F})[-j]$.

Choose a stratification with respect to which $\mathbb{D}\mathcal{G}$ is constructible, and then refine it so that the support of $\mathcal{F}$ is a union of strata. Let $(X_s)_{s\in\mathscr{S}}$ be this stratification. We proceed by induction on the number of strata. Let $i : X_t \hookrightarrow X$ be the inclusion of a closed stratum. Let $U = X \smallsetminus X_t$, and let $j : U \hookrightarrow X$ be the inclusion map. (In the base case, U is empty.) From Exercise 1.3.7, we get a long exact sequence

$$\cdots \to \mathrm{Hom}(i^*\mathcal{F}, i^!\mathcal{G}) \to \mathrm{Hom}(\mathcal{F},\mathcal{G}) \to \mathrm{Hom}(j^*\mathcal{F}, j^*\mathcal{G}) \to \cdots. \tag{3.1.3}$$

The last term vanishes by induction. If $i^*\mathcal{F} = 0$, then of course the first and second terms also vanish, and we are done.

If $i^*\mathcal{F} \neq 0$, then X_t is contained in the support of $\mathsf{H}^j(\mathcal{F})$, so $\dim X_t \leq \dim \operatorname{supp} \mathsf{H}^j(\mathcal{F}) \leq -j$. On the other hand, Lemma 3.1.6 tells us that $i^!\mathcal{G} \in D^{\mathrm{b}}_{\mathrm{locf}}(X_t,\Bbbk)^{\geq -\dim X_t+1} \subset D^{\mathrm{b}}(X,\Bbbk)^{\geq j+1}$. Since $i^*\mathcal{F}$ is concentrated in degree j, we have $\mathrm{Hom}(i^*\mathcal{F}, i^!\mathcal{G}) = 0$, and then (3.1.3) shows that $\mathrm{Hom}(\mathcal{F},\mathcal{G}) = 0$. $\square$

Lemma 3.1.8. *Let X be a variety. We have*

$$D^{\mathrm{b}}_{\mathrm{c}}(X,\Bbbk)^{\leq -\dim X} \subset {}^pD^{\mathrm{b}}_{\mathrm{c}}(X,\Bbbk)^{\leq 0} \subset D^{\mathrm{b}}_{\mathrm{c}}(X,\Bbbk)^{\leq 0},$$
$$D^{\mathrm{b}}_{\mathrm{c}}(X,\Bbbk)^{\geq 0} \subset {}^pD^{\mathrm{b}}_{\mathrm{c}}(X,\Bbbk)^{\geq 0} \subset D^{\mathrm{b}}_{\mathrm{c}}(X,\Bbbk)^{\geq -\dim X}.$$

Proof. If $\mathcal{F} \in D^{\mathrm{b}}_{\mathrm{c}}(X,\Bbbk)^{\leq -\dim X}$, then of course $\dim \operatorname{supp} \mathsf{H}^i(\mathcal{F}) \leq \dim X$ for $i \leq -\dim X$, and $\mathsf{H}^i(\mathcal{F}) = 0$ for $i > -\dim X$, so $\mathcal{F} \in {}^pD^{\mathrm{b}}_{\mathrm{c}}(X,\Bbbk)^{\leq 0}$. Next, if $\mathcal{G} \in {}^pD^{\mathrm{b}}_{\mathrm{c}}(X,\Bbbk)^{\geq 0}$, then by Lemma 3.1.7, $\mathrm{Hom}(\mathcal{F},\mathcal{G}) = 0$ for all $\mathcal{F} \in {}^pD^{\mathrm{b}}_{\mathrm{c}}(X,\Bbbk)^{\leq -1}$, and in particular for all $\mathcal{F} \in D^{\mathrm{b}}_{\mathrm{c}}(X,\Bbbk)^{\leq -\dim X-1}$. By Lemma A.7.3, we have $\mathcal{G} \in D^{\mathrm{b}}_{\mathrm{c}}(X,\Bbbk)^{\geq -\dim X}$. The remaining assertions in the lemma are proved similarly. $\square$

Theorem 3.1.9. *Let X be a variety.*

1. *The pair $({}^pD^{\mathrm{b}}_{\mathrm{c}}(X,\Bbbk)^{\leq 0}, {}^pD^{\mathrm{b}}_{\mathrm{c}}(X,\Bbbk)^{\geq 0})$ is a bounded t-structure on $D^{\mathrm{b}}_{\mathrm{c}}(X,\Bbbk)$.*
2. *Let $(X_s)_{s\in\mathscr{S}}$ be a good stratification of X. Then the pair of categories $({}^pD^{\mathrm{b}}_{\mathscr{S}}(X,\Bbbk)^{\leq 0}, {}^pD^{\mathrm{b}}_{\mathscr{S}}(X,\Bbbk)^{\geq 0})$ is a bounded t-structure on $D^{\mathrm{b}}_{\mathscr{S}}(X,\Bbbk)$.*

Proof. We must check the three conditions from Definition A.7.1. The first condition is immediate from the definitions, and the second condition follows from Lemma 3.1.7. The boundedness condition follows from Lemma 3.1.8.

It remains to check the last condition of Definition A.7.1. We proceed by noetherian induction on X. Let $\mathcal{F} \in D^{\mathrm{b}}_{\mathrm{c}}(X,\Bbbk)$. For part (1) of the theorem, choose a stratification $(X_s)_{s\in\mathscr{S}}$ with respect to which $\mathcal{F}$ and $\mathbb{D}\mathcal{F}$ are constructible. (For part (2), just use the given stratification.) Choose an open stratum $j_u : X_u \hookrightarrow X$, and let $i : Z \hookrightarrow X$ be its closed complement. Let $m = \dim X_u$. Using the natural t-structure on $D^{\mathrm{b}}_{\mathrm{locf}}(X_u,\Bbbk)$, form the distinguished triangle

$$j_{u!}\tau^{\leq -m-1} j_u^*\mathcal{F} \to j_{u!}j_u^*\mathcal{F} \to j_{u!}\tau^{\geq -m} j_u^*\mathcal{F} \to.$$

Compose the first map above with the natural map $j_{u!}j_u^*\mathcal{F} \to \mathcal{F}$, and then complete that to a distinguished triangle

$$j_{u!}\tau^{\leq -m-1} j_u^*\mathcal{F} \to \mathcal{F} \to \mathcal{G} \to, \tag{3.1.4}$$

where $\mathcal{G}$ is some new object of $D^{\mathrm{b}}_{\mathrm{c}}(X,\Bbbk)$. Note that applying j_u^* to this triangle yields $\tau^{\leq -m-1} j_u^*\mathcal{F} \to j_u^*\mathcal{F} \to j_u^*\mathcal{G} \to$, where the first map is the canonical truncation map.

We deduce that

$$j_u^*\mathcal{G} \cong \tau^{\geq -m} j_u^*\mathcal{F}. \tag{3.1.5}$$

Since $({}^pD^{\mathrm{b}}_{\mathrm{c}}(Z,\Bbbk)^{\leq 0}, {}^pD^{\mathrm{b}}_{\mathrm{c}}(Z,\Bbbk)^{\geq 0})$ is known by induction to be a t-structure, it has truncation functors ${}^p\tau^{\leq -1}$ and ${}^p\tau^{\geq 0}$. Apply these functors to $i^!\mathcal{G} \in D^{\mathrm{b}}_{\mathrm{c}}(Z,\Bbbk)$, and then apply i_* to obtain a distinguished triangle

$$i_* {}^p\tau^{\leq -1} i^!\mathcal{G} \to i_*i^!\mathcal{G} \to i_* {}^p\tau^{\geq 0} i^!\mathcal{G} \to .$$

Again, we compose the first arrow with the adjunction map $i_*i^!\mathcal{G} \to \mathcal{G}$ and then complete to a distinguished triangle

$$i_* {}^p\tau^{\leq -1} i^!\mathcal{G} \to \mathcal{G} \to \mathcal{F}'' \to . \tag{3.1.6}$$

Applying $i^!$ yields ${}^p\tau^{\leq -1} i^!\mathcal{G} \to i^!\mathcal{G} \to i^!\mathcal{F}'' \to$, so

$$i^!\mathcal{F}'' \cong {}^p\tau^{\geq 0} i^!\mathcal{G}. \tag{3.1.7}$$

Finally, we combine (3.1.4) and (3.1.6) into an octahedral diagram containing a new object $\mathcal{F}'$:

$$\begin{tikzcd}
j_{u!}\tau^{\leq -m-1}j_u^*\mathcal{F} \arrow[dd] & & i_* {}^p\tau^{\leq -1} i^!\mathcal{G} \arrow[ll, squiggly] \arrow[dl] \\
& \mathcal{G} \arrow[ul, squiggly] \arrow[dr] & \\
\mathcal{F} \arrow[ur] \arrow[rr] & & \mathcal{F}'' \arrow[uu, squiggly]
\end{tikzcd}
\qquad
\begin{tikzcd}
j_{u!}\tau^{\leq -m-1}j_u^*\mathcal{F} \arrow[dd] \arrow[dr] & & i_* {}^p\tau^{\leq -1} i^!\mathcal{G} \arrow[ll, squiggly] \\
& \mathcal{F}' \arrow[ur] \arrow[dl] & \\
\mathcal{F} \arrow[rr] & & \mathcal{F}'' \arrow[uu, squiggly] \arrow[ul, squiggly]
\end{tikzcd}$$

We claim that the distinguished triangle $\mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to$ is the one we seek. To prove this, we must show that $\mathcal{F}' \in {}^pD^{\mathrm{b}}_{\mathrm{c}}(X,\Bbbk)^{\leq -1}$ and $\mathcal{F}'' \in {}^pD^{\mathrm{b}}_{\mathrm{c}}(X,\Bbbk)^{\geq 0}$.

For $\mathcal{F}'$, consider the distinguished triangle

$$j_{u!}\tau^{\leq -m-1}j_u^*\mathcal{F} \to \mathcal{F}' \to i_* {}^p\tau^{\leq -1} i^!\mathcal{G} \to .$$

The first and last terms belong to ${}^pD^{\mathrm{b}}_{\mathrm{c}}(X,\Bbbk)^{\leq -1}$ by Lemmas 3.1.4 and 3.1.6. By Lemma 3.1.2, $\mathcal{F}'$ also belongs to ${}^pD^{\mathrm{b}}_{\mathrm{c}}(X,\Bbbk)^{\leq -1}$.

For $\mathcal{F}''$, we will show that $j_s^!\mathcal{F}'' \in D^{\mathrm{b}}_{\mathrm{locf}}(X_s,\Bbbk)^{\geq -\dim X_s}$ for each stratum X_s. For $s = u$, it follows from (3.1.6) that $j_u^!\mathcal{F}'' \cong j_u^!\mathcal{G}$, and since $j_u^! = j_u^*$, our claim holds by (3.1.5). For $s \neq u$, factor j_s as $j_s' : X_s \hookrightarrow Z$ followed by $i : Z \hookrightarrow X$. Then $j_s^!\mathcal{F}'' \cong (j_s')^! i^!\mathcal{F}''$, and our claim holds by applying $(j_s')^!$ to (3.1.7) and induction. We conclude that $\mathcal{F}'' \in {}^pD^{\mathrm{b}}_{\mathrm{c}}(X,\Bbbk)^{\geq 0}$, as desired. □

Proposition 3.1.10. *Let $i : Z \hookrightarrow X$ be the inclusion of a closed subvariety. The functor i_* induces an equivalence of categories*

$$\mathrm{Perv}(Z,\Bbbk) \xrightarrow{\sim} \{\mathcal{F} \in \mathrm{Perv}(X,\Bbbk) \mid \operatorname{supp}\mathcal{F} \subset Z\}.$$

Moreover, the right-hand side is a Serre subcategory of $\mathrm{Perv}(X,\Bbbk)$.

Proof. The first assertion is an immediate consequence of the t-exactness of i_* (Lemma 3.1.4(4)) and Corollary 1.3.11. For the second assertion, let $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ be a short exact sequence of perverse sheaves. It is clear that if the supports of $\mathcal{F}'$ and $\mathcal{F}''$ are contained in Z, then $\operatorname{supp}\mathcal{F} \subset Z$ as well. Conversely, suppose $\operatorname{supp}\mathcal{F} \subset Z$. Let $U = X \smallsetminus Z$. By Lemma 3.1.4(1), the sequence $0 \to \mathcal{F}'|_U \to \mathcal{F}|_U \to \mathcal{F}''|_U \to 0$ is a short exact sequence in $\mathrm{Perv}(U,\Bbbk)$. But the middle term of this sequence is 0, so we must have $\mathcal{F}'|_U = \mathcal{F}''|_U = 0$ as well. In other words, $\mathcal{F}'$ and $\mathcal{F}''$ must have supports contained in Z. □

LEMMA 3.1.11. *If $\Bbbk$ is a field, the Verdier duality functor $\mathbb{D} : D^{\mathrm{b}}_{\mathrm{c}}(X, \Bbbk)^{\mathrm{op}} \to D^{\mathrm{b}}_{\mathrm{c}}(X, \Bbbk)$ is t-exact for the perverse t-structure.*

PROOF. This is immediate from (3.1.2). □

PROPOSITION 3.1.12. *Let $f : X \to Y$ be a finite morphism. The functor $f_* : D^{\mathrm{b}}_{\mathrm{c}}(X, \Bbbk) \to D^{\mathrm{b}}_{\mathrm{c}}(Y, \Bbbk)$ is t-exact for the perverse t-structure.*

PROOF. By Lemmas 2.1.9 and 2.3.19, f_* is exact (for the natural t-structure) and preserves both $\dim \operatorname{supp}$ and mdsupp. Since f is proper, f_* commutes with $\mathbb{D}$. The result then follows from the definition of the perverse t-structure. □

Exercises.

3.1.1. For $x \in X$, let $i_x : \{x\} \hookrightarrow X$ be the inclusion map. The ith **cosupport** of a constructible complex $\mathcal{F} \in D^{\mathrm{b}}_{\mathrm{c}}(X, \Bbbk)$, denoted by $\operatorname{cosupp}^i \mathcal{F}$, is defined to be the following closed subset of X:

$$\operatorname{cosupp}^i \mathcal{F} = \overline{\{x \in X \mid \mathsf{H}^i(i_x^! \mathcal{F}) \neq 0\}} \subset X.$$

Show that for the perverse t-structure, we have

$${}^pD^{\mathrm{b}}_{\mathrm{c}}(X, \Bbbk)^{\geq 0} = \{\mathcal{F} \in D^{\mathrm{b}}_{\mathrm{c}}(X, \Bbbk) \mid \text{for all } i, \text{ we have } \dim \operatorname{cosupp}^i \mathcal{F} \leq i\}.$$

3.1.2. Let X be a variety equipped with a good stratification $(X_s)_{s \in \mathscr{S}}$. For each $s \in \mathscr{S}$, let $j_s : X_s \hookrightarrow X$ be the inclusion map. Show that ${}^pD^{\mathrm{b}}_{\mathscr{S}}(X, \Bbbk)^{\leq 0}$ is generated under extensions by objects of the form

$$j_{s!}\mathcal{L}[n] \qquad \text{where } \mathcal{L} \in \mathrm{Loc}^{\mathrm{ft}}(X_s, \Bbbk) \text{ and } n \geq \dim X_s.$$

Similarly, ${}^pD^{\mathrm{b}}_{\mathscr{S}}(X, \Bbbk)^{\geq 0}$ is generated under extensions by objects of the form

$$j_{s*}\mathcal{L}[n] \qquad \text{where } \mathcal{L} \in \mathrm{Loc}^{\mathrm{ft}}(X_s, \Bbbk) \text{ and } n \leq \dim X_s.$$

3.1.3. Let $h : Y \hookrightarrow X$ be a locally closed embedding. Show that $h_!$ is right t-exact for the perverse t-structure and that h_* is left t-exact.

3.1.4. Let $f : X \to Y$ be a finite morphism, and let $\mathcal{F} \in D^{\mathrm{b}}_{\mathrm{c}}(X, \Bbbk)$. Show that $\mathcal{F}$ is perverse if and only if $f_*\mathcal{F}$ is perverse.

3.1.5. Let $(X_s)_{s \in \mathscr{S}}$ be a good stratification of X, and let $q : \mathscr{S} \to \mathbb{Z}$ be a function, called a **perversity function** (cf. [**24**, Section 2.1] as well as [**85**, **86**]).

(a) Show that the pair of categories

$$\begin{aligned} {}^qD^{\mathrm{b}}_{\mathscr{S}}(X, \Bbbk)^{\leq 0} &= \{\mathcal{F} \in D^{\mathrm{b}}_{\mathscr{S}}(X, \Bbbk) \mid \text{for all } s \in \mathscr{S},\ j_s^*\mathcal{F} \in D^{\mathrm{b}}_{\mathrm{locf}}(X_s, \Bbbk)^{\leq q(s)}\}, \\ {}^qD^{\mathrm{b}}_{\mathscr{S}}(X, \Bbbk)^{\geq 0} &= \{\mathcal{F} \in D^{\mathrm{b}}_{\mathscr{S}}(X, \Bbbk) \mid \text{for all } s \in \mathscr{S},\ j_s^!\mathcal{F} \in D^{\mathrm{b}}_{\mathrm{locf}}(X_s, \Bbbk)^{\geq q(s)}\} \end{aligned}$$

constitutes a t-structure on $D^{\mathrm{b}}_{\mathscr{S}}(X, \Bbbk)$. (Here $j_s : X_s \hookrightarrow X$ is the inclusion map of a stratum.) Denote its heart by ${}^q\mathrm{Perv}_{\mathscr{S}}(X, \Bbbk)$. *Hint*: Use the recollement formalism from Exercises 1.3.4 and A.7.4.

(b) Show that if $\Bbbk$ is a field, then $\mathbb{D}$ swaps ${}^q\mathrm{Perv}_{\mathscr{S}}(X, \Bbbk)$ with ${}^{\bar{q}}\mathrm{Perv}_{\mathscr{S}}(X, \Bbbk)$, where $\bar{q}$ is the perversity function given by $\bar{q}(s) = -2\dim X_s - q(s)$.

(c) Show that the usual category of perverse sheaves $\mathrm{Perv}_{\mathscr{S}}(X, \Bbbk)$ arises from this construction using the perversity function $q(s) = -\dim X_s$.

Some sources use the term "perverse sheaf" for objects of ${}^q\mathrm{Perv}_{\mathscr{S}}(X,\Bbbk)$ for arbitrary q. However, most of the theorems on perverse sheaves in this chapter are false in this generality, and the categories ${}^q\mathrm{Perv}_{\mathscr{S}}(X,\Bbbk)$ seem not to have applications in representation theory except when $q(s) = -\dim X_s$. In this book, the term "perverse sheaf" will always refer to the notion from Definition 3.1.1 or (3.1.1), and not to the notion defined in this exercise.

3.1.6. Let $h : Y \hookrightarrow X$ be a locally closed embedding. Let $\mathcal{F}$ be a perverse sheaf on Y, and let $\mathcal{G}$ be a perverse sheaf on X that is supported on $\overline{Y} \smallsetminus Y$. Show that

$$\mathrm{Hom}({}^p\mathsf{H}^0(h_!\mathcal{F}),\mathcal{G}) = \mathrm{Ext}^1({}^p\mathsf{H}^0(h_!\mathcal{F}),\mathcal{G}) = 0$$

and that

$$\mathrm{Hom}(\mathcal{G},{}^p\mathsf{H}^0(h_*\mathcal{F})) = \mathrm{Ext}^1(\mathcal{G},{}^p\mathsf{H}^0(h_*\mathcal{F})) = 0.$$

3.1.7. Let $(U_\alpha)_{\alpha\in I}$ be a Zariski open covering of X, and let $\mathcal{F} \in D^{\mathrm{b}}_{\mathrm{c}}(X,\Bbbk)$. Show that $\mathcal{F} \in {}^pD^{\mathrm{b}}_{\mathrm{c}}(X,\Bbbk)^{\leq 0}$ (resp. $\mathcal{F} \in {}^pD^{\mathrm{b}}_{\mathrm{c}}(X,\Bbbk)^{\geq 0}$) if and only if $\mathcal{F}|_{U_\alpha} \in {}^pD^{\mathrm{b}}_{\mathrm{c}}(U_\alpha,\Bbbk)^{\leq 0}$ (resp. $\mathcal{F}|_{U_\alpha} \in {}^pD^{\mathrm{b}}_{\mathrm{c}}(U_\alpha,\Bbbk)^{\geq 0}$) for all $\alpha \in I$. As a consequence, $\mathcal{F}$ is perverse if and only if $\mathcal{F}|_{U_\alpha}$ is perverse for all $\alpha \in I$.

3.2. Tensor product and sheaf Hom for perverse sheaves

This section collects a number of lemmas on the interaction of the perverse t-structure with $\otimes^L$, $\boxtimes$, extension of scalars, and $R\mathscr{H}om$.

Lemma 3.2.1. *For any two objects $\mathcal{F} \in {}^pD^{\mathrm{b}}_{\mathrm{c}}(X,\Bbbk)^{\leq 0}$ and $\mathcal{G} \in {}^pD^{\mathrm{b}}_{\mathrm{c}}(X,\Bbbk)^{\geq 0}$, we have $R\mathscr{H}om(\mathcal{F},\mathcal{G}) \in D^{\mathrm{b}}_{\mathrm{c}}(X,\Bbbk)^{\geq 0}$.*

Proof. We proceed by noetherian induction on X. Choose a stratification of X with respect to which both $\mathcal{F}$ and $\mathbb{D}\mathcal{G}$ are constructible, and let $j : U \hookrightarrow X$ be the inclusion of an open stratum of this stratification. Let $i : Z \hookrightarrow X$ be the inclusion of the complementary closed subset. By Lemma 3.1.6, we have

$$j^*\mathcal{F} \in D^{\mathrm{b}}_{\mathrm{locf}}(U,\Bbbk)^{\leq -\dim U} \qquad \text{and} \qquad j^*\mathcal{G} \in D^{\mathrm{b}}_{\mathrm{locf}}(U,\Bbbk)^{\geq -\dim U}.$$

It follows that

$$R\mathscr{H}om(j^*\mathcal{F}, j^*\mathcal{G}) \in D^{\mathrm{b}}_{\mathrm{c}}(U,\Bbbk)^{\geq 0}. \tag{3.2.1}$$

By Exercise 1.5.2, we have a distinguished triangle

$$i_*R\mathscr{H}om(i^*\mathcal{F}, i^!\mathcal{G}) \to R\mathscr{H}om(\mathcal{F},\mathcal{G}) \to j_*R\mathscr{H}om(j^*\mathcal{F}, j^*\mathcal{G}) \to . \tag{3.2.2}$$

Since j_* is a right derived functor, it follows from (3.2.1) that the last term lies in $D^{\mathrm{b}}_{\mathrm{c}}(X,\Bbbk)^{\geq 0}$. For the first term, we have $i^*\mathcal{F} \in {}^pD^{\mathrm{b}}_{\mathrm{c}}(Z,\Bbbk)^{\leq 0}$ and $i^!\mathcal{G} \in {}^pD^{\mathrm{b}}_{\mathrm{c}}(Z,\Bbbk)^{\geq 0}$ by Lemma 3.1.4, so $R\mathscr{H}om(i^*\mathcal{F}, i^!\mathcal{G})$ lies in $D^{\mathrm{b}}_{\mathrm{c}}(Z,\Bbbk)^{\geq 0}$ by induction, and hence $i_*R\mathscr{H}om(i^*\mathcal{F}, i^!\mathcal{G})$ lies in $D^{\mathrm{b}}_{\mathrm{c}}(X,\Bbbk)^{\geq 0}$. We conclude that the middle term of (3.2.2) lies in $D^{\mathrm{b}}_{\mathrm{c}}(X,\Bbbk)^{\geq 0}$ as well, as desired. □

Lemma 3.2.2. *Let X be a variety, and let $\mathcal{L} \in \mathrm{Loc}^{\mathrm{ft}}(X,\Bbbk)$. The functor*

$$(-) \overset{L}{\otimes} \mathcal{L}, \quad \textit{resp.} \quad R\mathscr{H}om(\mathcal{L}, -) : D^{\mathrm{b}}_{\mathrm{c}}(X,\Bbbk) \to D^{\mathrm{b}}_{\mathrm{c}}(X,\Bbbk)$$

is right t-exact, resp. left t-exact, for the perverse t-structure. If $\mathcal{L}$ is locally free, then both functors are t-exact.

In particular, with field coefficients, both functors above are always t-exact.

Proof. We will prove the statement for $(-)\otimes^L \mathcal{L}$. The proof for $R\mathscr{H}om(\mathcal{L},-)$ is similar and left to the reader. Let $\mathcal{F} \in D^{\mathrm{b}}_{\mathrm{c}}(X,\Bbbk)$. Choose a stratification $(X_s)_{s\in\mathscr{S}}$ with respect to which both $\mathcal{F}$ and $\mathbb{D}\mathcal{F}$ are constructible. For each $s \in \mathscr{S}$, let $j_s : X_s \hookrightarrow X$ be the inclusion map.

Suppose that $\mathcal{F} \in {}^pD^{\mathrm{b}}_{\mathrm{c}}(X,\Bbbk)^{\le 0}$. Using Lemma 3.1.6, Proposition 1.4.5, and the fact that $(-)\otimes^L j_s^*\mathcal{L}$ is right t-exact for the natural t-structure, we see that

$$j_s^*(\mathcal{F} \overset{L}{\otimes} \mathcal{L}) \cong j_s^*\mathcal{F} \overset{L}{\otimes} j_s^*\mathcal{L} \in D^{\mathrm{b}}_{\mathrm{locf}}(X_s,\Bbbk)^{\le -\dim X_s},$$

so $\mathcal{F}\otimes^L \mathcal{L} \in {}^pD^{\mathrm{b}}_{\mathrm{c}}(X,\Bbbk)^{\le 0}$, as desired.

Suppose now that $\mathcal{F} \in {}^pD^{\mathrm{b}}_{\mathrm{c}}(X,\Bbbk)^{\ge 0}$ and that $\mathcal{L}$ is locally free. In particular, $\mathcal{L}$ and $j_s^*\mathcal{L}$ are flat sheaves, so $(-)\otimes^L \mathcal{L}$ is t-exact for the natural t-structure. Using this observation along with Lemma 3.1.6 and Proposition 1.5.9, we see that

$$j_s^!(\mathcal{F} \overset{L}{\otimes} \mathcal{L}) \cong j_s^!\mathcal{F} \overset{L}{\otimes} j_s^*\mathcal{L} \in D^{\mathrm{b}}_{\mathrm{locf}}(X_s,\Bbbk)^{\ge -\dim X_s},$$

so $\mathcal{F}\otimes^L \mathcal{L} \in {}^pD^{\mathrm{b}}_{\mathrm{c}}(X,\Bbbk)^{\ge 0}$, as desired. □

Remark 3.2.3. If $\mathcal{F} \in {}^pD^{\mathrm{b}}_{\mathrm{c}}(X,\Bbbk)^{\le 0}$, and if $\mathcal{L}$ is a local system that is not necessarily of finite type, then of course, $\mathcal{F}\otimes^L \mathcal{L}$ need not be constructible. Nevertheless, a minor modification of the argument in Lemma 3.2.2 shows that we still have $\dim \operatorname{supp} \mathsf{H}^i(\mathcal{F}\otimes^L \mathcal{L}) \le -i$ for all i. Thus, by Lemma 3.1.7, if $\mathcal{F} \in {}^pD^{\mathrm{b}}_{\mathrm{c}}(X,\Bbbk)^{\le 0}$ and $\mathcal{G} \in {}^pD^{\mathrm{b}}_{\mathrm{c}}(X,\Bbbk)^{\ge 1}$, then

$$\operatorname{Hom}(\mathcal{F} \overset{L}{\otimes} \mathcal{L}, \mathcal{G}) = 0$$

for any local system $\mathcal{L}$ on X, not necessarily of finite type.

Lemma 3.2.4. *Let $\varphi : \Bbbk \to \Bbbk'$ be a ring homomorphism. The functor $\Bbbk' \otimes^L_{\Bbbk} (-) : D^{\mathrm{b}}_{\mathrm{c}}(X,\Bbbk) \to D^{\mathrm{b}}_{\mathrm{c}}(X,\Bbbk')$ is right t-exact for the perverse t-structure. Moreover, if $\Bbbk'$ is flat as a $\Bbbk$-module, then this functor is t-exact.*

The proof of this statement is very similar to that of Lemma 3.2.2 and will be omitted.

Lemma 3.2.5. *If $\mathcal{F} \in {}^pD^{\mathrm{b}}_{\mathrm{c}}(X,\Bbbk)^{\le 0}$ and $\mathcal{G} \in {}^pD^{\mathrm{b}}_{\mathrm{c}}(Y,\Bbbk)^{\le 0}$, then $\mathcal{F} \boxtimes \mathcal{G} \in {}^pD^{\mathrm{b}}_{\mathrm{c}}(X\times Y,\Bbbk)^{\le 0}$. If $\Bbbk$ is a field, then $\boxtimes$ is t-exact for the perverse t-structure.*

For more information on the field case, see Lemma 3.3.14.

Proof. Choose a stratification $(X_s)_{s\in\mathscr{S}}$ of X with respect to which $\mathcal{F}$ is constructible, and a stratification $(Y_t)_{t\in\mathscr{T}}$ of Y with respect to which $\mathcal{G}$ is constructible. Then $(X_s \times Y_t)_{(s,t)\in\mathscr{S}\times\mathscr{T}}$ is a stratification of $X\times Y$, and $\mathcal{F}\boxtimes\mathcal{G}$ is constructible with respect to this stratification. (This follows from Proposition 2.3.18, using the fact that $\mathcal{F}\boxtimes\mathcal{G} \cong \mathrm{pr}_1^*\mathcal{F}\otimes^L \mathrm{pr}_2^*\mathcal{G}$.) By Lemma 3.1.6, we have

$$\mathcal{F}|_{X_s} \in D^{\mathrm{b}}_{\mathrm{c}}(X_s,\Bbbk)^{\le -\dim X_s} \qquad \text{and} \qquad \mathcal{G}|_{Y_t} \in D^{\mathrm{b}}_{\mathrm{c}}(Y_t,\Bbbk)^{\le -\dim Y_t}.$$

Because $\boxtimes$ is right t-exact for the natural t-structure, we have

$$(\mathcal{F}\boxtimes\mathcal{G})|_{X_s\times Y_t} \cong (\mathcal{F}|_{X_s})\boxtimes(\mathcal{G}|_{Y_t}) \le D^{\mathrm{b}}_{\mathrm{c}}(X_s\times Y_t,\Bbbk)^{\le -\dim(X_s\times Y_t)}$$

for all $(s,t)\in\mathscr{S}\times\mathscr{T}$. Thus, $\mathcal{F}\boxtimes\mathcal{G} \in {}^pD^{\mathrm{b}}_{\mathrm{c}}(X\times Y,\Bbbk)^{\le 0}$, as desired.

If $\Bbbk$ is a field and $\mathcal{F}$ and $\mathcal{G}$ are perverse sheaves, then (3.1.2) and the preceding paragraph tell us that

$$\mathbb{D}(\mathcal{F}\boxtimes\mathcal{G}) \cong (\mathbb{D}\mathcal{F})\boxtimes(\mathbb{D}\mathcal{G}) \in {}^pD^{\mathrm{b}}_{\mathrm{c}}(X\times Y,\Bbbk)^{\le 0},$$

and hence that $\mathcal{F}\boxtimes\mathcal{G} \in \operatorname{Perv}(X\times Y,\Bbbk)$. □

LEMMA 3.2.6. *Suppose $\mathcal{F} \in {}^p\!D^{\mathrm{b}}_{\mathrm{c}}(X,\Bbbk)^{\le 0}$ and $\mathcal{F} \notin {}^p\!D^{\mathrm{b}}_{\mathrm{c}}(X,\Bbbk)^{\le -1}$. Then there exists a ring homomorphism $\varphi : \Bbbk \to \Bbbk'$ with $\Bbbk'$ a field such that ${}^p\mathsf{H}^0(\Bbbk' \otimes^L_{\Bbbk} \mathcal{F}) \neq 0$.*

PROOF. First consider the special case where X is smooth and connected, and $\mathcal{F} \in D^{\mathrm{b}}_{\mathrm{locf}}(X,\Bbbk)$. Then $\mathcal{F} \in D^{\mathrm{b}}_{\mathrm{locf}}(X,\Bbbk)^{\le -\dim X}$ but $\mathcal{F} \notin D^{\mathrm{b}}_{\mathrm{locf}}(X,\Bbbk)^{\le -\dim X - 1}$. By Proposition A.6.11, for any ring homomorphism $\varphi : \Bbbk \to \Bbbk'$, we have

$$\mathsf{H}^{-\dim X}(\Bbbk' \overset{L}{\otimes}_{\Bbbk} \mathcal{F}) \cong \Bbbk' \otimes_{\Bbbk} \mathsf{H}^{-\dim X}(\mathcal{F}).$$

Let M be a stalk of $\mathsf{H}^{-\dim X}(\mathcal{F})$. Then the stalks of $\Bbbk' \otimes_{\Bbbk} \mathsf{H}^{-\dim X}(\mathcal{F})$ are isomorphic to $\Bbbk' \otimes_{\Bbbk} M$. Choose a maximal ideal $\mathfrak{p} \subset \Bbbk$ such that $M_{\mathfrak{p}} \neq 0$, and let $\Bbbk' = \Bbbk_{\mathfrak{p}}/\mathfrak{p}\Bbbk_{\mathfrak{p}}$. Then $\Bbbk' \otimes_{\Bbbk} M \cong \Bbbk' \otimes_{\Bbbk_{\mathfrak{p}}} M_{\mathfrak{p}} \cong M_{\mathfrak{p}}/\mathfrak{p}M_{\mathfrak{p}}$, and this is nonzero by Nakayama's lemma. We conclude that $\mathsf{H}^{-\dim X}(\Bbbk' \otimes^L_{\Bbbk} \mathcal{F}) \neq 0$, and hence ${}^p\mathsf{H}^0(\Bbbk' \otimes^L_{\Bbbk} \mathcal{F}) \neq 0$.

We now return to the general setting. Given $\mathcal{F} \in {}^p\!D^{\mathrm{b}}_{\mathrm{c}}(X,\Bbbk)^{\le 0}$ with $\mathcal{F} \notin {}^p\!D^{\mathrm{b}}_{\mathrm{c}}(X,\Bbbk)^{\le -1}$, choose a stratification $(X_s)_{s\in\mathscr{S}}$ with respect to which $\mathcal{F}$ is constructible, and let $j_s : X_s \hookrightarrow X$ be the inclusion map. By Lemma 3.1.6(1), we have $j_s^*\mathcal{F} \in D^{\mathrm{b}}_{\mathrm{locf}}(X_s,\Bbbk)^{\le -\dim X_s}$ for all s. Moreover, there must exist some s such that $j_s^*\mathcal{F} \notin D^{\mathrm{b}}_{\mathrm{locf}}(X_s,\Bbbk)^{\le -\dim X_s - 1}$. By the previous paragraph, we can find a field $\Bbbk'$ such that $\mathsf{H}^{-\dim X_s}(\Bbbk' \otimes^L_{\Bbbk} (j_s^*\mathcal{F})) \cong \mathsf{H}^{-\dim X_s}(j_s^*(\Bbbk' \otimes^L_{\Bbbk} \mathcal{F}))$ is nonzero, and hence ${}^p\mathsf{H}^0(\Bbbk' \otimes^L_{\Bbbk} \mathcal{F}) \neq 0$. ☐

LEMMA 3.2.7. *For any prime ideal $\mathfrak{p} \subset \Bbbk$, if $\mathcal{F} \in {}^p\!D^{\mathrm{b}}_{\mathrm{c}}(X,\Bbbk)^{\ge 0}$, then $\Bbbk_{\mathfrak{p}}/\mathfrak{p}\Bbbk_{\mathfrak{p}} \otimes^L_{\Bbbk} \mathcal{F} \in {}^p\!D^{\mathrm{b}}_{\mathrm{c}}(X,\Bbbk)^{\ge -\mathrm{ht}(\mathfrak{p})}$. Moreover, if $\mathcal{F} \in {}^p\!D^{\mathrm{b}}_{\mathrm{c}}(X,\Bbbk)^{\ge 0}$ and $\mathcal{F} \notin {}^p\!D^{\mathrm{b}}_{\mathrm{c}}(X,\Bbbk)^{\ge 1}$, there exists a prime ideal $\mathfrak{p} \subset \Bbbk$ such that ${}^p\mathsf{H}^{-\mathrm{ht}(\mathfrak{p})}(\Bbbk_{\mathfrak{p}}/\mathfrak{p}\Bbbk_{\mathfrak{p}} \otimes^L_{\Bbbk} \mathcal{F}) \neq 0$.*

PROOF. For brevity, let $\Bbbk' = \Bbbk_{\mathfrak{p}}/\mathfrak{p}\Bbbk_{\mathfrak{p}}$. First consider the special case where X is a point, so that $D^{\mathrm{b}}_{\mathrm{c}}(X,\Bbbk) \cong D^{\mathrm{b}}(\Bbbk\text{-mod}^{\mathrm{fg}})$ (see Remark 2.3.8). In this case, the perverse t-structure coincides with the natural t-structure. The functor $\Bbbk' \otimes^L_{\Bbbk} (-)$ can be factored as

$$\Bbbk_{\mathfrak{p}}/\mathfrak{p}\Bbbk_{\mathfrak{p}} \overset{L}{\otimes}_{\Bbbk_{\mathfrak{p}}} (\Bbbk_{\mathfrak{p}} \overset{L}{\otimes}_{\Bbbk} (-)) : D^{\mathrm{b}}(\Bbbk\text{-mod}^{\mathrm{fg}}) \to D^{\mathrm{b}}(\Bbbk'\text{-mod}^{\mathrm{fg}}).$$

The ring $\Bbbk_{\mathfrak{p}}$ is flat as a $\Bbbk$-module (see, for instance, [**72**, Proposition 2.5]), so $\Bbbk_{\mathfrak{p}} \otimes^L_{\Bbbk} (-)$ is t-exact. Next, by Lemma A.10.8, $\Bbbk_{\mathfrak{p}}$ has global dimension $\mathrm{ht}(\mathfrak{p})$, and this implies that $\Bbbk' \otimes^L_{\Bbbk_{\mathfrak{p}}} (-)$ has cohomological dimension $\le \mathrm{ht}(\mathfrak{p})$. By Proposition A.6.11 (cf. Proposition A.6.16), if $M \in D^{\mathrm{b}}(\Bbbk\text{-mod}^{\mathrm{fg}})^{\ge 0}$, then $\Bbbk' \otimes^L_{\Bbbk} M \in D^{\mathrm{b}}(\Bbbk\text{-mod}^{\mathrm{fg}})^{\ge -\mathrm{ht}(\mathfrak{p})}$.

Next, we must show that if $M \notin D^{\mathrm{b}}(\Bbbk\text{-mod}^{\mathrm{fg}})^{\ge 1}$, then $\mathfrak{p}$ can be chosen such that $\Bbbk' \otimes^L_{\Bbbk} M \notin D^{\mathrm{b}}(\Bbbk\text{-mod}^{\mathrm{fg}})^{\ge -\mathrm{ht}(\mathfrak{p})+1}$. By Proposition A.10.6, grade $\mathsf{H}^i(\mathbb{D}(M)) \ge i$ for all i. If this inequality were strict for all i, we would have $M \in D^{\mathrm{b}}(\Bbbk\text{-mod}^{\mathrm{fg}})^{\ge 1}$, so in fact there is some n such that grade $\mathsf{H}^n(\mathbb{D}(M)) = n$. Note that if $i > n$, then grade $\mathsf{H}^i(\mathbb{D}(M)) \ge i > n$. By Lemma A.10.10, there exists a prime ideal $\mathfrak{p} \subset \Bbbk$ with $\mathrm{ht}(\mathfrak{p}) = n$ and such that $\mathsf{H}^n(\mathbb{D}(M))_{\mathfrak{p}} \neq 0$. That lemma also tells us that for $i > n$, we have $\mathsf{H}^i(\mathbb{D}(M))_{\mathfrak{p}} = 0$. In other words, $\Bbbk_{\mathfrak{p}} \otimes^L_{\Bbbk} \mathbb{D}(M)$ lies in $D^{\mathrm{b}}(\Bbbk_{\mathfrak{p}}\text{-mod}^{\mathrm{fg}})^{\le n}$, and $\mathsf{H}^n(\Bbbk_{\mathfrak{p}} \otimes^L_{\Bbbk} \mathbb{D}(M)) \neq 0$. It follows that $\Bbbk' \otimes^L_{\Bbbk} \mathbb{D}(M) \cong \Bbbk' \otimes^L_{\Bbbk_{\mathfrak{p}}} (\Bbbk_{\mathfrak{p}} \otimes^L_{\Bbbk} \mathbb{D}(M))$ lies in $D^{\mathrm{b}}(\Bbbk'\text{-mod}^{\mathrm{fg}})^{\le n}$. Moreover, the module

$$\mathsf{H}^n(\Bbbk' \overset{L}{\otimes}_{\Bbbk} \mathbb{D}(M)) \cong \Bbbk_{\mathfrak{p}}/\mathfrak{p}\Bbbk_{\mathfrak{p}} \otimes_{\Bbbk_{\mathfrak{p}}} \mathsf{H}^n(\mathbb{D}(M))_{\mathfrak{p}} \cong \mathsf{H}^n(\mathbb{D}(M))_{\mathfrak{p}}/\mathfrak{p}\mathsf{H}^n(\mathbb{D}(M))_{\mathfrak{p}}$$

is nonzero by Nakayama's lemma. By Proposition 2.9.11(1), we conclude that

$$\mathbb{D}(\Bbbk' \overset{L}{\otimes}_{\Bbbk} \mathbb{D}(M)) \cong \Bbbk' \overset{L}{\otimes}_{\Bbbk} M$$

lies in $D^{\mathrm{b}}(\Bbbk'\text{-mod}^{\mathrm{fg}})^{\geq -n}$ but not in $D^{\mathrm{b}}(\Bbbk'\text{-mod}^{\mathrm{fg}})^{\geq -n+1}$, as desired.

Next, consider the case where X is smooth and connected, and $\mathcal{F} \in D^{\mathrm{b}}_{\mathrm{locf}}(X, \Bbbk)$. Choose a point $x \in X$. The functor $D^{\mathrm{b}}_{\mathrm{locf}}(X, \Bbbk) \to D^{\mathrm{b}}(\Bbbk\text{-mod}^{\mathrm{fg}})$ given by $\mathcal{F} \mapsto \mathcal{F}_x[-\dim X]$ is t-exact, commutes with $\Bbbk' \otimes^L_{\Bbbk} (-)$, and kills no nonzero perverse sheaf. The assertions of the lemma in this case can thus be reduced to the case of a point, which we have already treated above.

Finally, consider the general case. Choose a stratification $(X_s)_{s \in \mathscr{S}}$ with respect to which $\mathbb{D}\mathcal{F}$ is constructible, and let $j_s : X_s \hookrightarrow X$ be the inclusion map. By Lemma 3.1.6(2), if $\mathcal{F} \in {}^p\!D^{\mathrm{b}}_{\mathrm{c}}(X, \Bbbk)^{\geq 0}$, we have $j_s^! \mathcal{F} \in D^{\mathrm{b}}_{\mathrm{locf}}(X_s, \Bbbk)^{\geq -\dim X_s}$ for all s, so by the preceding paragraph,

$$j_s^!(\Bbbk' \overset{L}{\otimes}_{\Bbbk} \mathcal{F}) \cong \Bbbk' \overset{L}{\otimes}_{\Bbbk} (j_s^! \mathcal{F}) \in D^{\mathrm{b}}_{\mathrm{locf}}(X_s, \Bbbk)^{\geq -\dim X_s - \mathrm{ht}(\mathfrak{p})}.$$

We conclude that $\Bbbk' \otimes^L_{\Bbbk} \mathcal{F} \in {}^p\!D^{\mathrm{b}}_{\mathrm{c}}(X, \Bbbk)^{\geq -\mathrm{ht}(\mathfrak{p})}$. Moreover, if $\mathcal{F} \notin {}^p\!D^{\mathrm{b}}_{\mathrm{c}}(X, \Bbbk)^{\geq 1}$, then there must exist some $s \in \mathscr{S}$ such that $j_s^! \mathcal{F} \notin D^{\mathrm{b}}_{\mathrm{locf}}(X_s, \Bbbk)^{\geq -\dim X_s + 1}$. By the previous paragraph, we can find a prime ideal $\mathfrak{p} \subset \Bbbk$ such that ${}^p\mathsf{H}^{-\mathrm{ht}(\mathfrak{p})}(\Bbbk' \otimes^L_{\Bbbk} (j_s^! \mathcal{F})) \cong {}^p\mathsf{H}^{-\mathrm{ht}(\mathfrak{p})}(j_s^!(\Bbbk' \otimes^L_{\Bbbk} \mathcal{F}))$ is nonzero, and hence ${}^p\mathsf{H}^{-\mathrm{ht}(\mathfrak{p})}(\Bbbk' \otimes^L_{\Bbbk} \mathcal{F}) \neq 0$. □

Exercises.

3.2.1. Here is a variant of Lemma 3.2.2. Let X be a variety, and let $M \in D^{\mathrm{b}}(\Bbbk\text{-mod}^{\mathrm{fg}})$ be an object that admits a (bounded) projective resolution with nonzero terms only in degrees $\geq k$. Show that if $\mathcal{F} \in {}^p\!D^{\mathrm{b}}_{\mathrm{c}}(X, \Bbbk)^{\geq 0}$, then $\mathcal{F} \otimes^L \mathrm{a}_X^* M$ lies in ${}^p\!D^{\mathrm{b}}_{\mathrm{c}}(X, \Bbbk)^{\geq k}$.

Then deduce that for any object $A \in D^{\mathrm{b}}(\mathbb{Z}\text{-mod}^{\mathrm{fg}})^{\geq 0}$ and any $\mathcal{F} \in {}^p\!D^{\mathrm{b}}_{\mathrm{c}}(X, \Bbbk)^{\geq 0}$, we have that $\mathcal{F} \otimes^L \mathrm{a}_X^*(\Bbbk \otimes^L_{\mathbb{Z}} A)$ lies in ${}^p\!D^{\mathrm{b}}_{\mathrm{c}}(X, \Bbbk)^{\geq -1}$. If $\mathsf{H}^0(A)$ is a free $\mathbb{Z}$-module, then in fact $(-) \otimes^L \mathrm{a}_X^*(\Bbbk \otimes^L_{\mathbb{Z}} A)$ is left t-exact.

3.3. Intersection cohomology complexes

In this section, we study a way to extend a perverse sheaf along a locally closed embedding embedding $h : Y \hookrightarrow X$ that "interpolates" between $h_!$ and h_*. We emphasize that the following functor is *not* defined for arbitrary objects in $D^{\mathrm{b}}_{\mathrm{c}}(Y, \Bbbk)$.

DEFINITION 3.3.1. Let $h : Y \hookrightarrow X$ be a locally closed embedding. The **intermediate-extension functor** is the functor

$$h_{!*} : \mathrm{Perv}(Y, \Bbbk) \to \mathrm{Perv}(X, \Bbbk)$$

given by

$$h_{!*}(\mathcal{F}) = \mathrm{im}({}^p\mathsf{H}^0(h_! \mathcal{F}) \to {}^p\mathsf{H}^0(h_* \mathcal{F})).$$

LEMMA 3.3.2. *Let $h : Y \hookrightarrow X$ be a locally closed embedding.*

1. *For $\mathcal{F} \in \mathrm{Perv}(Y, \Bbbk)$, there is a natural isomorphism $h^* h_{!*} \mathcal{F} \cong \mathcal{F}$.*
2. *For $\mathcal{F} \in \mathrm{Perv}(Y, \Bbbk)$, the object $h_{!*} \mathcal{F}$ has no nonzero subobjects or quotients supported on $\overline{Y} \smallsetminus Y$.*

PROOF. Since $h_! \mathcal{F}$ and $h_* \mathcal{F}$ are both supported on $\overline{Y}$, we may as well assume without loss of generality that $X = \overline{Y}$. Then $h : Y \hookrightarrow X$ is an open embedding, so by Lemma 3.1.4, h^* is t-exact for the perverse t-structure. Therefore, we have

$$h^* h_{!*}\mathcal{F} = h^* \operatorname{im}({}^p\mathsf{H}^0(h_!\mathcal{F}) \to {}^p\mathsf{H}^0(h_*\mathcal{F})) \cong \operatorname{im}(h^*{}^p\mathsf{H}^0(h_!\mathcal{F}) \to h^*{}^p\mathsf{H}^0(h_*\mathcal{F}))$$
$$\cong \operatorname{im}({}^p\mathsf{H}^0(h^*h_!\mathcal{F}) \to {}^p\mathsf{H}^0(h^*h_*\mathcal{F})) \cong \operatorname{im}(\mathcal{F} \xrightarrow{\mathrm{id}} \mathcal{F}) = \mathcal{F}.$$

Let $Z = X \smallsetminus Y$, and let $i : Z \hookrightarrow X$ be the inclusion map. If $h_{!*}\mathcal{F}$ had a nonzero subobject $\mathcal{G}$ supported on Z, we could regard it via $h_{!*}\mathcal{F} \hookrightarrow {}^p\mathsf{H}^0(h_*\mathcal{F})$ as a subobject of ${}^p\mathsf{H}^0(h_*\mathcal{F})$. But the inclusion map $\mathcal{G} \hookrightarrow {}^p\mathsf{H}^0(h_*\mathcal{F})$ contradicts Exercise 3.1.6. The proof that $h_{!*}\mathcal{F}$ has no quotient supported on Z is similar. $\square$

LEMMA 3.3.3. *Let $h : Y \hookrightarrow X$ be a locally closed embedding. Then the functor $h_{!*} : \mathrm{Perv}(Y, \Bbbk) \to \mathrm{Perv}(X, \Bbbk)$ is fully faithful. For $\mathcal{F} \in \mathrm{Perv}(Y, \Bbbk)$, the object $h_{!*}\mathcal{F}$ is the unique perverse sheaf on X (up to isomorphism) with the following properties:*

- *It is supported on $\overline{Y}$.*
- *Its restriction to Y is isomorphic to $\mathcal{F}$.*
- *It has no nonzero subobjects or quotients supported on $\overline{Y} \smallsetminus Y$.*

PROOF. As in the previous lemma, this statement deals only with objects supported on $\overline{Y}$, so we may as well assume that $X = \overline{Y}$. Again, $h : Y \hookrightarrow X$ is now an open embedding. Let $i : Z \hookrightarrow X$ be the inclusion of the complementary closed subset.

Let $\mathrm{Perv}^\circ(X, \Bbbk) \subset \mathrm{Perv}(X, \Bbbk)$ be the full subcategory consisting of perverse sheaves with no nonzero subobjects or quotients supported on Z. Given an object $\mathcal{F} \in \mathrm{Perv}(X, \Bbbk)$, consider the distinguished triangle

$$h_!h^*\mathcal{F} \to \mathcal{F} \to i_*i^*\mathcal{F} \to . \tag{3.3.1}$$

By Lemma 3.1.4, all three terms are in ${}^pD^{\mathrm{b}}_{\mathrm{c}}(X, \Bbbk)^{\leq 0}$, so in the long exact cohomology sequence, the map $\mathcal{F} = {}^p\mathsf{H}^0(\mathcal{F}) \to {}^p\mathsf{H}^0(i_*i^*\mathcal{F})$ is surjective. The perverse sheaf ${}^p\mathsf{H}^0(i_*i^*\mathcal{F})$ is supported on Z, so if $\mathcal{F} \in \mathrm{Perv}^\circ(X, \Bbbk)$, then we must have ${}^p\mathsf{H}^0(i_*i^*\mathcal{F}) = 0$, or, equivalently, $i_*i^*\mathcal{F} \in {}^pD^{\mathrm{b}}_{\mathrm{c}}(X, \Bbbk)^{\leq -1}$. Since i_* is t-exact and fully faithful, we conclude that

$$i^*\mathcal{F} \in {}^pD^{\mathrm{b}}_{\mathrm{c}}(Z, \Bbbk)^{\leq -1} \qquad \text{if } \mathcal{F} \in \mathrm{Perv}^\circ(X, \Bbbk). \tag{3.3.2}$$

Similar reasoning using $i_*i^!\mathcal{F} \to \mathcal{F} \to h_*h^*\mathcal{F} \to$ shows that

$$i^!\mathcal{F} \in {}^pD^{\mathrm{b}}_{\mathrm{c}}(Z, \Bbbk)^{\geq 1} \qquad \text{if } \mathcal{F} \in \mathrm{Perv}^\circ(X, \Bbbk). \tag{3.3.3}$$

Now consider the long exact sequence from Exercise 1.3.7:

$$\cdots \to \mathrm{Hom}(h^*\mathcal{F}, h^*\mathcal{G}[-1]) \to \mathrm{Hom}(i^*\mathcal{F}, i^!\mathcal{G}) \to \mathrm{Hom}(\mathcal{F}, \mathcal{G})$$
$$\to \mathrm{Hom}(h^*\mathcal{F}, h^*\mathcal{G}) \to \mathrm{Hom}(i^*\mathcal{F}, i^!\mathcal{G}[1]) \to \cdots .$$

The axioms for a t-structure, along with (3.3.2) and (3.3.3) applied to $\mathcal{G}$, imply that the first, second, and fifth terms here vanish, so we get an isomorphism

$$\mathrm{Hom}(\mathcal{F}, \mathcal{G}) \to \mathrm{Hom}(h^*\mathcal{F}, h^*\mathcal{G}).$$

In other words, the functor

$$h^* : \mathrm{Perv}^\circ(\overline{Y}, \Bbbk) \to \mathrm{Perv}(Y, \Bbbk)$$

is fully faithful. It is also essentially surjective: by Lemma 3.3.2, for any $\mathcal{F} \in \mathrm{Perv}(Y, \Bbbk)$, we have $\mathcal{F} \cong h^*h_{!*}\mathcal{F}$. We conclude that h^* is an equivalence of categories and that $h_{!*}$ is its inverse. In particular, $h_{!*}$ is fully faithful.

We rephrase this as follows: suppose $\mathcal{H}$ is a perverse sheaf on X such that $h^*\mathcal{H} \cong \mathcal{F}$ and such that $\mathcal{H}$ has no nonzero subobject or quotient supported on Z. The $\mathcal{H} \in \mathrm{Perv}^\circ(X, \Bbbk)$, and then the previous paragraph tells us that $\mathcal{H} \cong h_{!*}\mathcal{F}$. ☐

The conditions (3.3.2) and (3.3.3) that came up in this proof can be extracted to give another characterization of the intermediate extension functor. The details of the proof of the following lemma are left to the reader.

LEMMA 3.3.4. *Let $\mathcal{F}$ be a perverse sheaf on X. Let $h : Y \hookrightarrow X$ be the inclusion map of a locally closed subvariety, and let $\mathcal{F}'$ be a perverse sheaf on Y. Let $Z = \overline{Y} \smallsetminus Y$, and let $i : Z \hookrightarrow X$ be the inclusion map. The following conditions are equivalent:*

(1) $\mathcal{F} \cong h_{!*}\mathcal{F}'$.
(2) *The support of $\mathcal{F}$ is $\overline{Y}$. Moreover, $\mathcal{F}|_Y \cong \mathcal{F}'$, and we have*

$$i^!\mathcal{F} \in {}^pD^{\mathrm{b}}_{\mathrm{c}}(Z, \Bbbk)^{\geq 1} \qquad \text{and} \qquad i^*\mathcal{F} \in {}^pD^{\mathrm{b}}_{\mathrm{c}}(Z, \Bbbk)^{\leq -1}.$$

LEMMA 3.3.5. *Let $h : Y \hookrightarrow X$ be a locally closed embedding. The functor $h_{!*}$ takes injective maps to injective maps, and surjective maps to surjective maps.*

REMARK 3.3.6. The functor $h_{!*}$ is *not* exact in general, and its image is not a Serre subcategory of $\mathrm{Perv}(X, \Bbbk)$. See Exercise 3.10.4(d) for an example.

PROOF. We saw in the proof of Lemma 3.3.3 that $h_{!*}$ is fully faithful. Let $\phi : \mathcal{F} \to \mathcal{G}$ be an injective map in $\mathrm{Perv}(Y, \Bbbk)$, and let $\mathcal{K}$ be the kernel of $h_{!*}\mathcal{F} \to h_{!*}\mathcal{G}$, so that we have an exact sequence

$$0 \to \mathcal{K} \to h_{!*}\mathcal{F} \xrightarrow{h_{!*}\phi} h_{!*}\mathcal{G}.$$

Note that $\mathcal{K}$ must be supported on $\overline{Y}$, so by Lemma 3.1.4, $h^*\mathcal{K}$ is a perverse sheaf. If $\mathcal{K}$ is nonzero, then Lemma 3.3.2 implies that $\mathcal{K}$ cannot be supported on $\overline{Y} \smallsetminus Y$, so $h^*\mathcal{K}$ must be nonzero. Applying h^* to the sequence above yields the exact sequence

$$0 \to h^*\mathcal{K} \to \mathcal{F} \xrightarrow{\phi} \mathcal{G},$$

but this contradicts the fact that ϕ is injective. We conclude that $\mathcal{K} = 0$. A similar argument shows that $h_{!*}$ takes surjective maps to surjective maps. ☐

LEMMA 3.3.7. *Let $\mathcal{F}$ be a perverse sheaf on X, and let $i : Z \hookrightarrow X$ be the inclusion of a closed subvariety.*

(1) *The natural map ${}^p\mathsf{H}^0(i_*i^!\mathcal{F}) \to \mathcal{F}$ is injective. Moreover, if $\phi : \mathcal{G} \to \mathcal{F}$ is a map from a perverse sheaf $\mathcal{G}$ supported on Z, then there is a unique map ϕ' making the following diagram commute:*

$$\begin{tikzcd}
\mathcal{G} \arrow[dr, dashed, "\phi'"'] \arrow[drr, "\phi"] & & \\
& {}^p\mathsf{H}^0(i_*i^!\mathcal{F}) \arrow[r] & \mathcal{F}
\end{tikzcd}$$

(2) *The natural map $\mathcal{F} \to {}^p\mathsf{H}^0(i_*i^*\mathcal{F})$ is surjective. Moreover, if $\phi : \mathcal{F} \to \mathcal{G}$ is a map to a perverse sheaf $\mathcal{G}$ supported on Z, then there is a unique map*

ϕ' making the following diagram commute:

$$\begin{tikzcd} \mathcal{F} \arrow[r] \arrow[drr, two heads, "\phi"'] & {}^p\mathsf{H}^0(i_*i^*\mathcal{F}) \arrow[dr, dashed, two heads, "\phi'"] & \\ & & \mathcal{G} \end{tikzcd}$$

In more concise terms, this lemma says that ${}^p\mathsf{H}^0(i_*i^!\mathcal{F})$ is the unique maximal subobject of $\mathcal{F}$ that is supported on Z, and that ${}^p\mathsf{H}^0(i_*i^*\mathcal{F})$ is the unique maximal quotient supported on Z.

Proof. We will prove part (1). The proof of part (2) is similar. Let $j : U \hookrightarrow X$ be the inclusion of the open subset complementary to Z. By Lemma 3.1.4, all three terms in the distinguished triangle $i_*i^!\mathcal{F} \to \mathcal{F} \to j_*j^*\mathcal{F} \to$ lie in ${}^pD^{\mathrm{b}}_{\mathrm{c}}(X, \Bbbk)^{\geq 0}$, so the long exact cohomology sequence shows that ${}^p\mathsf{H}^0(i_*i^!\mathcal{F}) \to \mathcal{F}$ is injective. Now let $\mathcal{G}$ be a perverse sheaf supported on Z. Consider the exact sequence

$$\cdots \to \mathrm{Hom}(\mathcal{G}, j_*j^*\mathcal{F}[-1]) \to \mathrm{Hom}(\mathcal{G}, i_*i^!\mathcal{F}) \\ \to \mathrm{Hom}(\mathcal{G}, \mathcal{F}) \to \mathrm{Hom}(\mathcal{G}, j_*j^*\mathcal{F}) \to \cdots .$$

Using adjunction and the fact that $j^*\mathcal{G} = 0$, we see that the first and last terms vanish, so the middle two terms are isomorphic. That is, every map $\phi : \mathcal{G} \to \mathcal{F}$ factors uniquely through $i_*i^!\mathcal{F} \to \mathcal{F}$. Since $\mathcal{G}$ is perverse and $i_*i^!\mathcal{F} \in {}^pD^{\mathrm{b}}_{\mathrm{c}}(X, \Bbbk)^{\geq 0}$, every map $\mathcal{G} \to i_*i^!\mathcal{F}$ factors uniquely through ${}^p\mathsf{H}^0(i_*i^!\mathcal{F}) = {}^p\tau^{\leq 0}(i_*i^!\mathcal{F}) \to i_*i^!\mathcal{F}$, as desired. $\square$

Lemma 3.3.8. *Let X be an irreducible variety. Let $j : U \hookrightarrow X$ be the inclusion map of an open subset, and let $i : Z \hookrightarrow X$ be the complementary closed subset. Let $\mathcal{F}$ be a perverse sheaf on X.*

(1) *If $\mathcal{F}$ has no quotient supported on Z, then there is a natural short exact sequence $0 \to {}^p\mathsf{H}^0(i_*i^!\mathcal{F}) \to \mathcal{F} \to j_{!*}(\mathcal{F}|_U) \to 0$.*
(2) *If $\mathcal{F}$ has no subobject supported on Z, then there is a natural short exact sequence $0 \to j_{!*}(\mathcal{F}|_U) \to \mathcal{F} \to {}^p\mathsf{H}^0(i_*i^*\mathcal{F}) \to 0$.*

Proof. Assume that $\mathcal{F}$ has no quotient supported on Z. Let $\mathcal{K}$ be the cokernel of the injective map ${}^p\mathsf{H}^0(i_*i^!\mathcal{F}) \to \mathcal{F}$ from Lemma 3.3.7. The universal property stated in that lemma implies that $\mathcal{K}$ has no subobject supported on Z. As a quotient of $\mathcal{F}$, $\mathcal{K}$ has no quotient supported on Z either. Then Lemma 3.3.3 implies that $\mathcal{K} \cong j_{!*}(\mathcal{F}|_U)$. The proof of the second assertion is similar. $\square$

The remainder of this section is devoted to a discussion of the following notion, which is essentially just an alternative notation for a special case of the intermediate extension functor.

Definition 3.3.9. Let X be a variety. Let $Y \subset X$ be a smooth, connected, locally closed subvariety, and let $\mathcal{L}$ be a local system of finite type on Y. The **intersection cohomology complex** associated to the pair $(Y, \mathcal{L})$ is the perverse sheaf given by

$$\mathrm{IC}(Y, \mathcal{L}) = h_{!*}(\mathcal{L}[\dim Y]),$$

where $h : Y \hookrightarrow X$ is the inclusion map.

REMARK 3.3.10. Let X be an irreducible variety, and let X_{sm} be its smooth locus. An important special case of Definition 3.3.9 is the object $\mathrm{IC}(X_{\mathrm{sm}}, \underline{\Bbbk}_{X_{\mathrm{sm}}})$. In practice, instead of explicitly naming the smooth locus, this object is usually just denoted by

$$\mathrm{IC}(X; \Bbbk).$$

It is called the **intersection cohomology complex** of X with coefficients in $\Bbbk$. (The minor conflict with the notation in Definition 3.3.9 should not cause any confusion.)

The hypercohomology of $\mathrm{IC}(X;\Bbbk)[-\dim X]$, denoted by

$$\mathbf{IH}^k(X;\Bbbk) = \mathbf{H}^{k-\dim X}(X, \mathrm{IC}(X;\Bbbk)),$$

is called the **intersection cohomology** of X with coefficients in $\Bbbk$. (Note that Lemma 3.1.8 implies that $\mathbf{IH}^\bullet(X;\Bbbk)$ lives in nonnegative degrees.) This notion actually predates the theory of perverse sheaves. It was introduced by Goresky–MacPherson in [**85**], and revisited from the perverse sheaf perspective in [**86**].

Combining Lemma 3.3.4 with Lemma 3.1.6, we obtain the following description of intersection cohomology complexes.

LEMMA 3.3.11. *Let $\mathcal{F}$ be a perverse sheaf on X. Let $(X_s)_{s\in\mathscr{S}}$ be a stratification with respect to which both $\mathcal{F}$ and $\mathbb{D}\mathcal{F}$ are constructible. Let $u \in \mathscr{S}$, and let $\mathcal{L}$ be a local system of finite type on X_u. The following conditions are equivalent:*

(1) *$\mathcal{F} \cong \mathrm{IC}(X_u, \mathcal{L})$.*
(2) *The support of $\mathcal{F}$ is $\overline{X_u}$. Moreover, $\mathcal{F}|_{X_u} \cong \mathcal{L}[\dim X_u]$, and for each stratum $X_t \subset \overline{X_u} \smallsetminus X_u$, we have*

$$j_t^*\mathcal{F} \in D^{\mathrm{b}}_{\mathrm{locf}}(X_t, \Bbbk)^{\le -\dim X_t - 1} \quad \text{and} \quad j_t^!\mathcal{F} \in D^{\mathrm{b}}_{\mathrm{locf}}(X_t, \Bbbk)^{\ge -\dim X_t + 1},$$

where $j_t : X_t \hookrightarrow X$ denotes the inclusion map.

LEMMA 3.3.12. *Let X be a smooth, connected variety of dimension n, and let $\mathcal{L}$ be a local system of finite type on X. For any open subset $U \subset X$, we have $\mathrm{IC}(U, \mathcal{L}|_U) \cong \mathcal{L}[n]$.*

PROOF. Choose a stratification $(X_s)_{s\in\mathscr{S}}$ that contains U as a stratum. Let $j_t : X_t \hookrightarrow X$ be the inclusion map of a stratum other than U, so that $\dim X_t < n$. We certainly have

$$j_t^*\mathcal{L}[n] \in D^{\mathrm{b}}_{\mathrm{locf}}(X_t, \Bbbk)^{\le -n} \subset D^{\mathrm{b}}_{\mathrm{locf}}(X_t, \Bbbk)^{\le -\dim X_t - 1}.$$

On the other hand, by Theorem 2.2.13, we have

$$j_t^!\mathcal{L}[n] \cong j_t^*\mathcal{L}[n + 2(\dim X_t - n)](\dim X_t - n) \in D^{\mathrm{b}}_{\mathrm{locf}}(X_t, \Bbbk)^{\ge -\dim X_t + 1}.$$

By Lemma 3.3.11, we are done. □

LEMMA 3.3.13. *Let $\Bbbk$ be a field. Let $Y \subset X$ be a smooth, connected, locally closed subvariety, and let $\mathcal{L}$ be a local system on Y. Then there is a natural isomorphism*

$$\mathbb{D}(\mathrm{IC}(Y, \mathcal{L})) \cong \mathrm{IC}(Y, \mathcal{L}^\vee)(n).$$

PROOF. It is clear that $\mathbb{D}(\mathrm{IC}(Y, \mathcal{L}))$ is supported on $\overline{Y}$ and that

$$\mathbb{D}(\mathrm{IC}(Y, \mathcal{L}))|_Y \cong \mathbb{D}(\mathrm{IC}(Y, \mathcal{L})|_Y) \cong \mathbb{D}(\mathcal{L}[n]) \cong \mathcal{L}^\vee[n](n).$$

(For the last step, see Remark 2.8.3.) Next, suppose that $\mathbb{D}(\mathrm{IC}(Y, \mathcal{L}))$ had a nonzero subobject $\mathcal{F}$ supported on $\overline{Y} \smallsetminus Y$. Since $\mathbb{D}$ is an exact functor on $\mathrm{Perv}(X, \Bbbk)$ (see

Lemma 3.1.11), applying it to $\mathcal{F} \hookrightarrow \mathbb{D}(\mathrm{IC}(Y, \mathcal{L}))$ yields a surjective map $\mathrm{IC}(Y, \mathcal{L}) \twoheadrightarrow \mathbb{D}\mathcal{F}$. In other words, $\mathbb{D}\mathcal{F}$ is a quotient of $\mathrm{IC}(Y, \mathcal{L})$ supported on $\overline{Y} \smallsetminus Y$. This is impossible, so $\mathbb{D}(\mathrm{IC}(Y, \mathcal{L}))$ has no nonzero subobject supported on $\overline{Y} \smallsetminus Y$. Similar reasoning shows that $\mathbb{D}(\mathrm{IC}(Y, \mathcal{L}))$ has no nonzero quotient on $\overline{Y} \smallsetminus Y$ either. We have shown that $\mathbb{D}(\mathrm{IC}(Y, \mathcal{L}))$ satisfies the properties from Lemma 3.3.3 that uniquely characterize $\mathrm{IC}(Y, \mathcal{L}^\vee(n))$, as desired. □

LEMMA 3.3.14. *Let $Y_1 \subset X_1$ and $Y_2 \subset X_2$ be smooth, connected, locally closed subvarieties, and let $\Bbbk$ be a field. For $\mathcal{L}_1 \in \mathrm{Loc}^{\mathrm{ft}}(Y_1, \Bbbk)$ and $\mathcal{L}_2 \in \mathrm{Loc}^{\mathrm{ft}}(Y_2, \Bbbk)$, we have*

$$\mathrm{IC}(Y_1, \mathcal{L}_1) \boxtimes \mathrm{IC}(Y_2, \mathcal{L}_2) \cong \mathrm{IC}(Y_1 \times Y_2, \mathcal{L}_1 \boxtimes \mathcal{L}_2)$$

in $\mathrm{Perv}(X_1 \times X_2, \Bbbk)$. In particular, if $\Bbbk$ is algebraically closed, the external tensor product of simple perverse sheaves is simple.

The last assertion in this lemma depends on a description of simple perverse sheaves that will be obtained later in Theorem 3.4.5.

PROOF. An argument similar to the proof of Lemma 3.2.5 shows that the perverse sheaf $\mathrm{IC}(Y_1, \mathcal{L}_1) \boxtimes \mathrm{IC}(Y_2, \mathcal{L}_2)$ satisfies the conditions from Lemma 3.3.11 that characterize $\mathrm{IC}(Y_1 \times Y_2, \mathcal{L}_1 \boxtimes \mathcal{L}_2)$. The details are left as an exercise.

Now assume that $\Bbbk$ is algebraically closed and that $\mathcal{L}_1$ and $\mathcal{L}_2$ are irreducible local systems. Choose base points $y_1 \in Y_1$ and $y_2 \in Y_2$, and let $V_1 = \mathrm{Mon}_{y_1}(\mathcal{L}_1)$ and $V_2 = \mathrm{Mon}_{y_2}(\mathcal{L}_2)$ be the corresponding representations. The last assertion in the lemma follows from the fact that $V_1 \otimes V_2$ is an irreducible representation of $\pi_1(Y_1 \times Y_2, (y_1, y_2)) \cong \pi_1(Y_1, y_1) \times \pi_1(Y_2, y_2)$; see, for instance, [**141**, Proposition 2.3.23]. □

Exercises.

3.3.1. Let $k : W \hookrightarrow Y$ and $h : Y \hookrightarrow X$ be locally closed embeddings. Prove that for $\mathcal{F} \in \mathrm{Perv}(W, \Bbbk)$, there is a natural isomorphism $h_{!*}k_{!*}\mathcal{F} \cong (h \circ k)_{!*}\mathcal{F}$.

3.3.2. Let $\mathcal{F} \in \mathrm{Perv}(X, \Bbbk)$, and let $(X_s)_{s \in \mathscr{S}}$ be a stratification with respect to which both $\mathcal{F}$ and $\mathbb{D}\mathcal{F}$ are constructible. Let $h : U \hookrightarrow X$ be the inclusion of an open subset such that $U \cap X_s$ is nonempty for all $s \in \mathscr{S}$. Show that $\mathcal{F} \cong h_{!*}(\mathcal{F}|_U)$.

3.3.3. Let X be an irreducible variety. Show that for any smooth open subset $U \subset X$, we have $\mathrm{IC}(X; \Bbbk) \cong \mathrm{IC}(U, \underline{\Bbbk}_U)$.

3.3.4. This exercise is a continuation of Exercise 3.1.6. Let $h : Y \hookrightarrow X$ be a locally closed embedding. Let $\mathcal{F} \in \mathrm{Perv}(Y, \Bbbk)$ and $\tilde{\mathcal{F}} \in \mathrm{Perv}(X, \Bbbk)$. Show that $\tilde{\mathcal{F}} \cong {}^p\mathsf{H}^0(h_!\mathcal{F})$ if and only if the following conditions both hold:

(a) There is a surjective map $\tilde{\mathcal{F}} \to h_{!*}\mathcal{F}$ whose kernel is supported on $\overline{Y} \smallsetminus Y$.
(b) For any perverse sheaf $\mathcal{G}$ supported on $\overline{Y} \smallsetminus Y$, we have

$$\mathrm{Hom}(\tilde{\mathcal{F}}, \mathcal{G}) = \mathrm{Ext}^1(\tilde{\mathcal{F}}, \mathcal{G}) = 0.$$

Similarly, $\tilde{\mathcal{F}} \cong {}^p\mathsf{H}^0(h_*\mathcal{F})$ if and only if the following conditions both hold:

(a) There is an injective map $h_{!*}\mathcal{F} \to \tilde{\mathcal{F}}$ whose cokernel is supported on $\overline{Y} \smallsetminus Y$.
(b) For any perverse sheaf $\mathcal{G}$ supported on $\overline{Y} \smallsetminus Y$, we have

$$\mathrm{Hom}(\mathcal{G}, \tilde{\mathcal{F}}) = \mathrm{Ext}^1(\mathcal{G}, \tilde{\mathcal{F}}) = 0.$$

3.3.5. Let X be an irreducible variety. Let $\Bbbk$ be a principal ideal domain, and let $\varphi : \Bbbk \to \Bbbk'$ be a ring homomorphism. Show that $\Bbbk' \otimes^L_{\Bbbk} \mathrm{IC}(X;\Bbbk)$ is a perverse sheaf and that there is a surjective map

$$\Bbbk' \overset{L}{\otimes}_{\Bbbk} \mathrm{IC}(X;\Bbbk) \to \mathrm{IC}(X;\Bbbk').$$

Then show that if $\Bbbk'$ is flat as a $\Bbbk$-module, then this map is an isomorphism. In particular, we always have

$$\mathbb{Q} \overset{L}{\otimes}_{\mathbb{Z}} \mathrm{IC}(X;\mathbb{Z}) \cong \mathrm{IC}(X;\mathbb{Q}).$$

3.3.6. Let X be an irreducible variety, and let $(X_s)_{s\in\mathscr{S}}$ be a stratification with respect to which $\mathrm{IC}(X;\mathbb{Z})$ is constructible. Let p be a prime number. Show that the following conditions are equivalent:

(a) We have $\operatorname{rank} \mathsf{H}^i(\mathrm{IC}(X;\mathbb{Q})|_{X_s}) = \operatorname{rank} \mathsf{H}^i(\mathrm{IC}(X;\mathbb{F}_p)|_{X_s})$ for all $i \in \mathbb{Z}$ and all $s \in \mathscr{S}$.
(b) The stalks of $\mathrm{IC}(X;\mathbb{Z})$ and of $\mathbb{D}\mathrm{IC}(X;\mathbb{Z})$ have no p-torsion.
(c) The stalks and costalks of $\mathrm{IC}(X;\mathbb{Z})$ have no p-torsion.

3.4. The noetherian property for perverse sheaves

In this section, we will show that every perverse sheaf admits a finite filtration whose subquotients are intersection cohomology complexes. As a consequence, we will deduce that $\mathrm{Perv}(X,\Bbbk)$ is a noetherian category. When $\Bbbk$ is a field, it is also artinian.

PROPOSITION 3.4.1. *Let X be a smooth, connected variety of dimension n. The category $\mathrm{Loc}^{\mathrm{ft}}(X,\Bbbk)[n]$ is a Serre subcategory of $\mathrm{Perv}(X,\Bbbk)$.*

PROOF. We must show that $\mathrm{Loc}^{\mathrm{ft}}(X,\Bbbk)[n]$ is closed under taking subobjects, quotients, and extensions. The fact that it is closed under extensions follows from Lemma 1.8.6. We will show that it is closed under subobjects; the argument for quotients is similar.

Let $\mathcal{L}$ be a local system of finite type, and let $\mathcal{F}$ be a sub perverse sheaf of $\mathcal{L}[n]$. Choose a stratification $(X_s)_{s\in\mathscr{S}}$ with respect to which $\mathcal{F}$ is constructible, and let U be the unique open stratum of this stratification. Then $\mathcal{F}|_U$ is a shifted local system: say $\mathcal{F}|_U = \mathcal{L}'_U[n]$ for some local system $\mathcal{L}'_U$ on U. This is a sub local system of $\mathcal{L}|_U$.

Let $x_0 \in U$, and let M be the $\Bbbk[\pi_1(X,x_0)]$-module corresponding to $\mathcal{L}$ under Theorem 1.7.9. By Proposition 1.7.10, $\mathcal{L}|_U$ is obtained by regarding M as a $\Bbbk[\pi_1(U,x_0)]$-module via the natural map $\pi_1(U,x_0) \to \pi_1(X,x_0)$. By Lemma 2.1.22, this map is surjective. It follows that every $\Bbbk[\pi_1(U,x_0)]$-submodule of M is also a $\Bbbk[\pi_1(X,x_0)]$-submodule. In terms of local systems, this means that every sub local system of $\mathcal{L}|_U$ is the restriction to U of a sub local system of $\mathcal{L}$. In particular, there exists a sub local system $\mathcal{L}' \subset \mathcal{L}$ such that $\mathcal{L}'|_U = \mathcal{L}'_U$. Let $\mathcal{L}'' = \mathcal{L}/\mathcal{L}'$ be the quotient local system on X.

We will show that $\mathcal{F} \cong \mathcal{L}'[n]$. Since $\mathcal{F}$ is a sub perverse sheaf of $\mathcal{L}[n]$, which is an intersection cohomology complex by Lemma 3.3.12, it (like $\mathcal{L}[n]$) has no subobject supported on $X \smallsetminus U$. By Lemma 3.3.12 and Lemma 3.3.8, there is a short exact sequence

$$0 \to \mathcal{L}'[n] \to \mathcal{F} \to {}^p\mathsf{H}^0(i_*i^*\mathcal{F}) \to 0,$$

where $i : X \smallsetminus U \hookrightarrow X$ is the inclusion map. This shows that ${}^p\mathsf{H}^0(i_*i^*\mathcal{F})$ can be regarded as a subobject of the quotient perverse sheaf $\mathcal{G} = (\mathcal{L}[n])/(\mathcal{L}'[n])$. Since $\mathcal{G}$ is the cone of $\mathcal{L}'[n] \to \mathcal{L}[n]$, it must be isomorphic to $\mathcal{L}''[n]$, so by Lemma 3.3.12, $\mathcal{G}$ has no nonzero subobject supported on $X \smallsetminus U$. In other words, ${}^p\mathsf{H}^0(i_*i^*\mathcal{F}) = 0$, and $\mathcal{F} \cong \mathcal{L}'[n]$, as desired. ☐

THEOREM 3.4.2. *Every perverse sheaf admits a finite filtration whose subquotients are intersection cohomology complexes.*

PROOF. We proceed by noetherian induction. Let $\mathcal{F}$ be a perverse sheaf on X. Let U be an open stratum of a stratification with respect to which $\mathcal{F}$ is constructible, and let $j : U \hookrightarrow X$ be the inclusion map. Then $\mathcal{F}|_U$ is a perverse sheaf, and by Lemma 3.1.3, it is of the form $\mathcal{L}[\dim U]$ for some local system $\mathcal{L}$. Let $i : Z \hookrightarrow X$ be the inclusion of the complementary closed subset. Let $\mathcal{F}_1$ be the cokernel of ${}^p\mathsf{H}^0(i_*i^!\mathcal{F}) \to \mathcal{F}$, so that we have a short exact sequence

$$0 \to {}^p\mathsf{H}^0(i_*i^!\mathcal{F}) \to \mathcal{F} \to \mathcal{F}_1 \to 0. \tag{3.4.1}$$

Lemma 3.3.7 implies that $\mathcal{F}_1$ has no subobject supported on Z. On the other hand, the first term above is supported on Z, so applying j^*, we get an isomorphism

$$\mathcal{L}[n] \cong j^*\mathcal{F} \xrightarrow{\sim} j^*\mathcal{F}_1.$$

Now apply Lemma 3.3.8 to $\mathcal{F}_1$ to obtain a short exact sequence

$$0 \to \mathrm{IC}(U, \mathcal{L}) \to \mathcal{F}_1 \to {}^p\mathsf{H}^0(i_*i^*\mathcal{F}_1) \to 0. \tag{3.4.2}$$

By induction, both ${}^p\mathsf{H}^0(i_*i^!\mathcal{F})$ and ${}^p\mathsf{H}^0(i_*i^*\mathcal{F}_1)$ have finite filtrations by intersection cohomology complexes. Then (3.4.2) shows that $\mathcal{F}_1$ also has such a filtration, and finally (3.4.1) shows that $\mathcal{F}$ does as well. ☐

LEMMA 3.4.3. *Let $Y \subset X$ be a smooth, connected, locally closed subvariety. Let $0 \to \mathcal{L}' \to \mathcal{L} \to \mathcal{L}'' \to 0$ be a short exact sequence of local systems on Y. Then* $\mathrm{IC}(Y, \mathcal{L})$ *admits a three-step filtration*

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \mathcal{F}_2 \subset \mathcal{F}_3 = \mathrm{IC}(Y, \mathcal{L})$$

such that

$$\mathcal{F}_1 \cong \mathrm{IC}(Y, \mathcal{L}') \qquad \textit{and} \qquad \mathcal{F}_3/\mathcal{F}_2 \cong \mathrm{IC}(Y, \mathcal{L}''),$$

and such that $\mathcal{F}_2/\mathcal{F}_1$ is supported on $\overline{Y} \smallsetminus Y$.

PROOF. Let $h : Y \hookrightarrow X$ be the inclusion map. By Lemma 3.3.5, there is a natural injective map $\mathrm{IC}(Y, \mathcal{L}') \hookrightarrow \mathrm{IC}(Y, \mathcal{L})$ and a natural surjective map $IC(Y, \mathcal{L}) \twoheadrightarrow \mathrm{IC}(Y, \mathcal{L}'')$. Let $\mathcal{F}_1$ be the image of the former, and let $\mathcal{F}_2$ be the kernel of the latter. The sequence

$$\mathrm{IC}(Y, \mathcal{L}') \xrightarrow{\phi} \mathrm{IC}(Y, \mathcal{L}) \xrightarrow{\psi} \mathrm{IC}(Y, \mathcal{L}'') \tag{3.4.3}$$

need not be exact, but the composition of the two maps must be zero, so $\mathcal{F}_1 \subset \mathcal{F}_2$. It remains to show that $\mathcal{F}_2/\mathcal{F}_1$ is supported on $\overline{Y} \smallsetminus Y$. Its support is certainly contained in $\overline{Y}$, so we must show that $(\mathcal{F}_2/\mathcal{F}_1)|_Y = 0$ or, equivalently, that $(\ker\psi/\operatorname{im}\phi)|_Y = 0$. This follows from the fact that the restriction to Y of (3.4.3) is a short exact sequence. ☐

THEOREM 3.4.4. *The category* $\mathrm{Perv}(X, \Bbbk)$ *is noetherian.*

PROOF. *Step 1. If U is a smooth, connected variety, then* $\mathrm{Loc}^{\mathrm{ft}}(U, \Bbbk)$ *is noetherian.* If $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots$ is a chain of subsheaves of a local system $\mathcal{L}$, then for any point $x \in U$, the the sequence of stalks $(\mathcal{F}_1)_x \subset (\mathcal{F}_2)_x \subset \cdots$ is eventually constant, because it is an ascending chain of submodules of $\mathcal{L}_x$, a finitely generated module over a noetherian ring. Let N be such that $(\mathcal{F}_N)_x = (\mathcal{F}_{N+i})_x$ for all $i \geq 0$. Then $\mathcal{F}_{N+i}/\mathcal{F}_N$ is a local system on a connected variety whose stalk at some point x is 0, so $\mathcal{F}_{N+i}/\mathcal{F}_N = 0$. In other words, $\mathcal{F}_N = \mathcal{F}_{N+i}$ for all $i \geq 0$.

Step 2. Reduction to the case of an intersection cohomology complex. We prove the theorem by noetherian induction on X. If X is a single point, then the result holds by Step 1. Otherwise, assume the result is known for all proper closed subvarieties of X. A standard argument shows that given a short exact sequence $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ in $\mathrm{Perv}(X, \Bbbk)$, if $\mathcal{F}'$ and $\mathcal{F}''$ are noetherian, then $\mathcal{F}$ is as well. By induction on the length of the filtration from Theorem 3.4.2, we can reduce the problem to showing that an intersection cohomology complex is noetherian. For an intersection cohomology complex supported on a proper closed subvariety, the claim holds by induction and Proposition 3.1.10. It remains to prove the result in the following case.

Step 3. If X is irreducible and $U \subset X$ is a smooth open subset, then for any local system $\mathcal{L}$ of finite type on U, $\mathrm{IC}(U, \mathcal{L})$ *is noetherian.* Let $n = \dim X$, and let $Z = X \smallsetminus U$. Suppose we have a chain of subobjects $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots$ of $\mathrm{IC}(U, \mathcal{L})$. Then $\mathcal{F}_1|_U \subset \mathcal{F}_2|_U \subset \cdots$ is a chain of sub perverse sheaves of $\mathcal{L}[n]$. By Proposition 3.4.1, this is actually a chain of shifted local systems, and then by Step 1, this chain is eventually constant. By discarding finitely many terms from the beginning of the sequence, we may assume that $\mathcal{F}_1|_U = \mathcal{F}_2|_U = \cdots$. Let $\mathcal{L}' = (\mathcal{F}_1|_U)[-n]$. This is a sub local system of $\mathcal{L}$. By Lemma 3.4.3, there is a perverse sheaf $\mathcal{G}$ with

$$\mathrm{IC}(U, \mathcal{L}') \subset \mathcal{G} \subset \mathrm{IC}(U, \mathcal{L}) \tag{3.4.4}$$

where $\mathrm{IC}(U, \mathcal{L})/\mathcal{G} \cong \mathrm{IC}(U, \mathcal{L}/\mathcal{L}')$, and where $\mathcal{G}/\mathrm{IC}(U, \mathcal{L}')$ is supported on Z.

Note that $\mathcal{F}_i$ has no subobject supported on Z (because it is a subobject of $\mathrm{IC}(U, \mathcal{L})$), so Lemma 3.3.8 implies that $\mathrm{IC}(U, \mathcal{L}') \subset \mathcal{F}_i$, and that the quotient $\mathcal{F}_i/\mathrm{IC}(U, \mathcal{L}')$ is supported on Z. It follows that $\mathcal{F}_i/(\mathcal{G} \cap \mathcal{F}_i)$ is also supported on Z. But $\mathcal{F}_i/(\mathcal{G} \cap \mathcal{F}_i) \cong (\mathcal{F}_i + \mathcal{G})/\mathcal{G}$ is a subobject of $\mathrm{IC}(U, \mathcal{L})/\mathcal{G} \cong \mathrm{IC}(U, \mathcal{L}/\mathcal{L}')$, which has no nonzero subobject supported on Z. We conclude that $\mathcal{F}_i/(\mathcal{G} \cap \mathcal{F}_i) = 0$, and hence that $\mathcal{F}_i \subset \mathcal{G}$. In other words, the entire chain is sandwiched between the first two terms of (3.4.4):

$$\mathrm{IC}(U, \mathcal{L}') \subset \mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots \subset \mathcal{G}.$$

To show that the chain is eventually constant, we may instead consider

$$\mathcal{F}_1/\mathrm{IC}(U, \mathcal{L}') \subset \mathcal{F}_2/\mathrm{IC}(U, \mathcal{L}') \subset \cdots \subset \mathcal{G}/\mathrm{IC}(U, \mathcal{L}').$$

This is a chain of subobjects of a perverse sheaf supported on Z, so it is eventually constant by induction. $\square$

THEOREM 3.4.5. *Assume that $\Bbbk$ is a field.*

1. *If $Y \subset X$ is a smooth, connected, locally closed subvariety, and $\mathcal{L}$ is an irreducible local system on Y, then* $\mathrm{IC}(Y, \mathcal{L})$ *is a simple object in* $\mathrm{Perv}(X, \Bbbk)$.
2. *Every perverse sheaf admits a finite filtration whose subquotients are simple intersection cohomlogy complexes.*
3. *The category* $\mathrm{Perv}(X, \Bbbk)$ *is artinian, and every simple object therein is a simple intersection cohomology complex.*

PROOF. (1) To show that $\mathrm{IC}(Y, \mathcal{L})$ is simple, we must show that it has no subobjects other than 0 and itself. Let $\mathcal{F} \subset \mathrm{IC}(Y, \mathcal{L})$ be a nonzero subobject. By Proposition 3.1.10, we may assume for simplicity that $X = \overline{Y}$, so that Y is an open subset of X. Then $\mathcal{F}$ cannot be supported on $X \smallsetminus Y$; in other words, $\mathcal{F}|_Y \neq 0$. By Proposition 3.4.1, $(\mathcal{F}|_Y)[-\dim Y]$ is a sub local system of $\mathcal{L}$. Since $\mathcal{L}$ is irreducible and $\mathcal{F}|_Y$ is nonzero, we conclude that $\mathcal{F}|_Y \cong \mathcal{L}[n]$. Since $\mathcal{F}$ has no nonzero subobject supported on $X \smallsetminus Y$, Lemma 3.3.8 then gives us an injective map $\mathrm{IC}(Y, \mathcal{L}) \hookrightarrow \mathcal{F}$. We now have two injective maps

$$\mathrm{IC}(Y, \mathcal{L}) \hookrightarrow \mathcal{F} \hookrightarrow \mathrm{IC}(Y, \mathcal{L}).$$

Let $\phi : \mathrm{IC}(Y, \mathcal{L}) \to \mathrm{IC}(Y, \mathcal{L})$ be the composition of these two maps. By construction, $\phi|_Y$ is the identity map of $\mathcal{L}[n]$. Since the IC functor is fully faithful (Lemma 3.3.3), ϕ must be the identity map of $\mathrm{IC}(Y, \mathcal{L})$. We conclude that $\mathcal{F} = \mathrm{IC}(Y, \mathcal{L})$.

(2) We proceed by noetherian induction. Assume that the statement is known for all proper closed subvarieties of X.

By Theorem 3.4.2, it is enough to show that every intersection cohomology complex $\mathrm{IC}(Y, \mathcal{L})$ has a finite filtration of the desired form. Choose a point $y_0 \in Y$. The local system $\mathcal{L}$ corresponds via Theorem 1.7.9 to a finite-dimensional representation of $\pi_1(Y, y_0)$ over the field $\Bbbk$. Of course, any such representation has a finite filtration whose subquotients are irreducible representations. Correspondingly, $\mathcal{L}$ has a finite filtration by sub local systems such that the quotients are irreducible local systems. By applying Lemma 3.4.3 repeatedly, we obtain a filtration of $\mathrm{IC}(Y, \mathcal{L})$ whose subquotients are of one of the following two forms:

- $\mathrm{IC}(Y, \mathcal{L}')$ for some irreducible subquotient $\mathcal{L}'$ of $\mathcal{L}$, or
- perverse sheaves supported on $\overline{Y} \smallsetminus Y$.

The former are already simple, and the latter have filtrations of the desired form by induction.

(3) This is an immediate consequence of part (2). □

Exercises. For the following exercises, assume that $\Bbbk$ is a principal ideal domain. An object $\mathcal{F} \in \mathrm{Perv}(X, \Bbbk)$ is said to be **torsion** if there is a nonzero element $a \in \Bbbk$ such that $a \cdot \mathrm{id}_{\mathcal{F}} = 0$. On the other hand, $\mathcal{F}$ is **torsion-free** if for every nonzero $a \in \Bbbk$, the map $a \cdot \mathrm{id}_{\mathcal{F}} : \mathcal{F} \to \mathcal{F}$ is injective.

3.4.1. Show that every perverse sheaf $\mathcal{F} \in \mathrm{Perv}(X, \Bbbk)$ has a unique maximal torsion sub perverse sheaf $\mathcal{F}_{\mathrm{tor}} \subset \mathcal{F}$ and that the quotient $\mathcal{F}/\mathcal{F}_{\mathrm{tor}}$ is torsion-free. Then show that the set of torsion perverse sheaves and the set of torsion-free perverse sheaves are each stable under extensions. (This exercise isn't specific to perverse sheaves; it applies to any noetherian $\Bbbk$-linear abelian category.)

3.4.2. Let $Y \subset X$ be a smooth, connected, locally closed subvariety, and let $\mathcal{L} \in \mathrm{Loc}^{\mathrm{ft}}(Y, \Bbbk)$. Show that $\mathrm{IC}(Y, \mathcal{L})$ is torsion, resp. torsion-free, if and only if the stalks of $\mathcal{L}$ are torsion, resp. torsion-free, $\Bbbk$-modules.

3.4.3. Let $Y \subset X$ be a smooth, connected, locally closed subvariety, and let $\mathcal{L}$ be a locally free (and hence torsion-free) local system of finite type on Y. Show that there is a unique object $\mathrm{IC}^+(Y, \mathcal{L}) \in \mathrm{Perv}(X, \Bbbk)$ characterized by the following properties:

- It is supported on $\overline{Y}$, and its restriction to Y is $\mathcal{L}[\dim Y]$.
- It has no nonzero torsion-free quotient supported on $\overline{Y} \smallsetminus Y$.

- It has no nonzero subobject supported on $\overline{Y} \smallsetminus Y$.
- It admits no nonsplit extensions by torsion perverse sheaves supported on $\overline{Y} \smallsetminus Y$.

Then show that $\mathrm{IC}^+(Y, \mathcal{L}) \cong \mathbb{D}(\mathrm{IC}(Y, \mathcal{L}^\vee)(\dim Y))$.

3.4.4. Show that $\mathcal{F} \in \mathrm{Perv}(X, \Bbbk)$ is torsion-free if and only if $\mathbb{D}\mathcal{F} \in \mathrm{Perv}(X, \Bbbk)$.

3.5. Affine open subsets and affine morphisms

In this section, we will revisit Artin's vanishing theorem (Theorem 2.6.2) from the perspective of perverse sheaves. The main results of this section (Theorems 3.5.3 and 3.5.7) give a new characterization of the perverse t-structure in terms of cohomology vanishing. As a consequence, we obtain an exactness result (Theorem 3.5.8) for push-forwards along affine morphisms.

Artin vanishing and ${}^pD^{\mathrm{b}}_{\mathrm{c}}(X, \Bbbk)^{\le 0}$. We will need some technical statements in the spirit of Sections 2.5 and 2.6, starting with the following "nonvanishing" counterpart to Proposition 2.6.1.

PROPOSITION 3.5.1. *Let X be a variety, and let $\mathcal{F}$ be a nonzero constructible sheaf on X. If* $\dim \operatorname{supp} \mathcal{F} = n$*, then there is an affine open subset $U \subset X$ such that $\mathbf{H}^n(U, \mathcal{F}|_U) \neq 0$, and, moreover,* $\operatorname{grade} \mathbf{H}^n(U, \mathcal{F}|_U) = \operatorname{grade} \mathcal{F}_x$ *for some $x \in U$.*

PROOF. There certainly exists an affine open subset $X' \subset X$ such that $\dim(X' \cap \operatorname{supp} \mathcal{F}) = n$. We may as well replace X by X'; in other words, we assume that X itself is affine. Let $m = \dim X$. Of course, we have $m \ge n$.

If $n = 0$, then $\operatorname{supp} \mathcal{F}$ is a finite set of points. In this case, we simply take $U = X$: we have

$$\mathbf{H}^0(X, \mathcal{F}) = \Gamma(\mathcal{F}) \cong \bigoplus_{x \in \operatorname{supp} \mathcal{F}} \mathcal{F}_x.$$

This is clearly nonzero; moreover, Lemma A.10.11 implies that

$$\operatorname{grade} \Gamma(\mathcal{F}) = \min\{\operatorname{grade} \mathcal{F}_x \mid x \in \operatorname{supp} \mathcal{F}\},$$

so there is some $x \in X$ such that $\operatorname{grade} \mathbf{H}^0(X, \mathcal{F})) = \operatorname{grade} \mathcal{F}_x$.

We assume from now on that $n > 0$, and we proceed by induction on m.

Step 1. Reduction to the case of affine space. By the Noether normalization lemma, there is a finite morphism $f : X \to \mathbb{A}^m$. By Lemmas 2.1.9 and 2.3.19, $f_*\mathcal{F}$ is a constructible sheaf whose support is $f(\operatorname{supp} \mathcal{F})$. Since f is a finite morphism, we have $\dim f(\operatorname{supp} \mathcal{F}) = n$. Assume the result is known for $\mathbb{A}^m$, and let $U' \subset \mathbb{A}^m$ be an affine open subset such that $\mathbf{H}^n(U', (f_*\mathcal{F})|_{U'}) \neq 0$, and suppose $\operatorname{grade} \mathbf{H}^n(U', (f_*\mathcal{F})|_{U'}) = \operatorname{grade}(f_*\mathcal{F})_y$, where $y \in U'$. We claim that $U = f^{-1}(U')$ has the properties we want. Since f is an affine morphism, U is indeed an affine open set. Proposition 1.2.16 implies that $\mathbf{H}^n(U, \mathcal{F}|_U) \cong \mathbf{H}^n(U', (f_*\mathcal{F})|_{U'})$; in particular, this module is nonzero, and its grade is equal to that of

$$(f_*\mathcal{F})_y \cong \bigoplus_{x \in f^{-1}(y)} \mathcal{F}_x.$$

By the same reasoning as above, Lemma A.10.11 implies that there is some $x \in U$ such that $\operatorname{grade} \mathbf{H}^n(U, \mathcal{F}|_U) = \operatorname{grade} \mathcal{F}_x$.

Step 2. Proof for $\mathbb{A}^m$. Assume $X = \mathbb{A}^m$. If $m = 0$, the result is trivial, so we assume from now on that $m \ge 1$. Let $U_1 \subset \mathbb{A}^m$ be a Zariski-open subset such that $\mathcal{F}|_{U_1}$ is a local system (possibly 0), and let $Z_1 = X \smallsetminus U_1$. Arguing as in the proof of Proposition 2.6.1, we may replace U_1 by a smaller open set so that Z_1 is defined by the vanishing of a nonconstant polynomial $f \in \mathbb{C}[x_1, \ldots, x_m]$ that is monic in x_m. Let $p : \mathbb{A}^m \to \mathbb{A}^{m-1}$ be the projection onto the first $m-1$ coordinates. The map $p|_{Z_1} : Z_1 \to \mathbb{A}^{m-1}$ is finite and surjective.

Next, choose a scalar $a \in \mathbb{C}$ such that $f(x_1, \ldots, x_{m-1}, a)$ is not identically 0, and let $g = (x_m - a)f$. Let $Z_2 = Z_1 \cup (\mathbb{A}^{m-1} \times \{a\})$, and let $U_2 = U_1 \smallsetminus (\mathbb{A}^{m-1} \times \{a\})$. Then Z_2 is the vanishing locus of g, and U_2 is its complement. The map $p|_{Z_2} : Z_2 \to \mathbb{A}^{m-1}$ is still finite and surjective, but it now has degree ≥ 2. Note that both U_1 and U_2, being principal open sets, are affine varieties.

If $m > n$, then the local system $\mathcal{F}|_{U_1}$ must be zero, and $\operatorname{supp} \mathcal{F}$ is contained in Z_1. In this case, one can repeat the reasoning of Step 1 using the finite map $p|_{Z_1}$ to deduce the result from the case of $\mathbb{A}^{m-1}$.

Suppose from now on that $m = n$. Apply Lemma 2.5.4 to $Z_2 \subset \mathbb{A}^m$ to obtain an open subset $h : V \hookrightarrow \mathbb{A}^{m-1}$. By replacing V by a smaller open set if necessary, we may assume that it is affine. Let $Z' = Z_2 \cap p^{-1}(V)$, and let $p_V = p|_{p^{-1}(V)} : p^{-1}(V) \to V$. Note that $V \times \{a\}$ is an irreducible component of Z'. Since Z' is smooth, it must actually be a connected component of Z'. In other words, $V \times \{a\}$ and $Z_1 \cap p^{-1}(V)$ are disjoint, and for any point $y \in V$, the set $p^{-1}(y) \cap Z'$ contains at least two points.

Let $U_3 = p^{-1}(V) \cap U_2$, and let $q = p_V|_{U_3} : U_3 \to V$. Since U_3 is the intersection of two affine open sets, it is affine as well. Lemma 2.5.4 implies that q is a locally trivial fibration, so $q_*(\mathcal{F}|_{U_3})$ lies in $D^{\mathrm{b}}_{\mathrm{locf}}(V, \Bbbk)$, and

$$(q_*(\mathcal{F}|_{U_3}))_y \cong R\Gamma(\mathcal{F}|_{q^{-1}(y)}) \tag{3.5.1}$$

for all $y \in V$. Let $\mathcal{F}' = \mathsf{H}^1(q_*(\mathcal{F}|_{U_3}))$; this is a local system on V. Now, the set $q^{-1}(y) = p^{-1}(y) \cap U_2$ is the complement of at least two points in the affine space $p^{-1}(y) \cong \mathbb{A}^1$. Lemma B.3.6 then tells us that there is a surjective map from $\mathbf{H}^1(q^{-1}(y), \mathcal{F}|_{q^{-1}(y)})$ to a stalk of $\mathcal{F}|_{U_1}$. By Lemma A.10.11, we have

$$\operatorname{grade} \mathcal{F}'_y = \operatorname{grade} \mathbf{H}^1(q^{-1}(y), \mathcal{F}|_{q^{-1}(y)}) \leq \operatorname{grade}(\mathcal{F}|_{U_1}). \tag{3.5.2}$$

We deduce that $\mathcal{F}'$ is nonzero and that

$$\operatorname{mdsupp} \mathcal{F}' = (m-1) - \operatorname{grade} \mathcal{F}' \geq (m - \operatorname{grade}(\mathcal{F}|_{U_1})) - 1 = \operatorname{mdsupp} \mathcal{F}|_{U_1} - 1.$$

By induction, apply the proposition to the sheaf $\mathcal{F}'$ on V. Thus, there exists an affine open subset $V_1 \subset V$ such that $\mathbf{H}^{m-1}(V_1, \mathcal{F}'|_{V_1})$ is nonzero and has grade equal to that of (any stalk of) $\mathcal{F}'$. By (3.5.2), we have

$$\operatorname{grade} \mathbf{H}^{m-1}(V_1, \mathcal{F}'|_{V_1}) \leq \operatorname{grade} \mathcal{F}|_{U_1}. \tag{3.5.3}$$

Let us return to (3.5.1). Lemma B.3.6 also implies that $\mathsf{H}^k(q_*(\mathcal{F}|_{U_3}))$ vanishes for $k \geq 2$. In other words, it can be nonzero only for $k = 0, 1$, so the truncation distinguished triangle reduces to

$$\mathsf{H}^0(q_*(\mathcal{F}|_{U_3})) \to q_*(\mathcal{F}|_{U_3}) \to \mathcal{F}'[-1] \to . \tag{3.5.4}$$

Let $W = q^{-1}(V_1) \cap U_3$, and let $r = q|_W : W \to V_1$, so that we have a cartesian square

$$\begin{tikzcd} W \arrow[r] \arrow[d, "r"'] & U_3 \arrow[d, "q"] \\ V_1 \arrow[r] & V \end{tikzcd}$$

By Proposition 1.2.16, we have $q_*(\mathcal{F}|_{U_3})|_{V_1} \cong r_*(\mathcal{F}|_W)$. Restrict the distinguished triangle (3.5.4) to V_1 and then apply $R\Gamma$ to obtain the triangle

$$R\Gamma(\mathsf{H}^0(r_*(\mathcal{F}|_W))) \to R\Gamma(\mathcal{F}|_W) \to R\Gamma(\mathcal{F}'|_{V_1})[-1] \to .$$

Now take the long exact sequence in cohomology. Since $\mathsf{H}^0(r_*(\mathcal{F}|_W))$ is a constructible sheaf on the affine variety V_1 of dimension $m-1$, Theorem 2.6.2 tells us that $\mathbf{H}^k(V_1, \mathsf{H}^0(r_*(\mathcal{F}|_W))) = 0$ for $k \geq m$. It follows that the map

$$\mathbf{H}^m(W, \mathcal{F}|_W) \to \mathbf{H}^m(V_1, \mathcal{F}'|_{V_1}[-1]) = \mathbf{H}^{m-1}(V_1, \mathcal{F}'|_{V_1})$$

is an isomorphism. In particular, by (3.5.3), $\mathbf{H}^m(W, \mathcal{F}|_W)$ is nonzero and satisfies

$$\operatorname{grade} \mathbf{H}^m(W, \mathcal{F}|_W) \leq \operatorname{grade} \mathcal{F}|_{U_1} = \operatorname{grade} \mathcal{F}|_W = m - \operatorname{mdsupp} \mathcal{F}|_W.$$

But Theorem 2.6.2 gives us the opposite inequality, so these grades are equal. □

LEMMA 3.5.2. *Let $\mathcal{F} \in {}^p D^{\mathrm{b}}_{\mathrm{c}}(X, \Bbbk)^{\leq 0}$. For every affine open subset $U \subset X$, we have $R\Gamma(\mathcal{F}|_U) \in D^{\mathrm{b}}(\Bbbk\text{-mod}^{\mathrm{fg}})^{\leq 0}$. Moreover, if $\mathcal{F} \notin {}^p D^{\mathrm{b}}_{\mathrm{c}}(X, \Bbbk)^{\leq -1}$, then there exists an affine open subset $U \subset X$ such that $\mathbf{H}^0(U, \mathcal{F}|_U) \neq 0$.*

PROOF. From the definition of ${}^p D^{\mathrm{b}}_{\mathrm{c}}(X, \Bbbk)^{\leq 0}$, it is clear that if we form a truncation distinguished triangle

$$\tau^{\leq n}\mathcal{F} \to \mathcal{F} \to \tau^{\geq n+1}\mathcal{F} \to \tag{3.5.5}$$

for the natural t-structure, then the first and last terms are also in ${}^p D^{\mathrm{b}}_{\mathrm{c}}(X, \Bbbk)^{\leq 0}$. Therefore, it makes sense to proceed by induction on the number of nonzero cohomology sheaves $\mathsf{H}^i(\mathcal{F})$.

If $\mathcal{F}$ is concentrated in a single degree, then $\mathcal{F} \cong \mathcal{F}'[n]$ for some some sheaf $\mathcal{F}'$ satisfying $\dim \operatorname{supp} \mathcal{F}' \leq n$. Then Theorem 2.6.2 tells us that for any affine open set $U \subset X$, $\mathsf{H}^k(R\Gamma(\mathcal{F}'[n]|_U))$ vanishes for $k > -n + \dim \operatorname{supp} \mathcal{F}'$. In other words, $R\Gamma(\mathcal{F}) \in D^{\mathrm{b}}(\Bbbk\text{-mod}^{\mathrm{fg}})^{\leq 0}$. Next, if $\mathcal{F} \notin {}^p D^{\mathrm{b}}_{\mathrm{c}}(X, \Bbbk)^{\leq -1}$, then we must have $\dim \operatorname{supp} \mathcal{F}' = n$. In this case, Proposition 3.5.1 gives us an affine open subset $U \subset X$ such that $\mathbf{H}^0(U, \mathcal{F}|_U) \cong \mathsf{H}^n(\mathcal{F}'|_U) \neq 0$.

If $\mathcal{F}$ has cohomology sheaves in more than one degree, choose an n so that both the first and last terms of (3.5.5) are nonzero, and thus have fewer nonzero cohomology sheaves than $\mathcal{F}$. For any affine open set $U \subset X$, we can form the triangle

$$R\Gamma(\tau^{\leq n}\mathcal{F}|_U) \to R\Gamma(\mathcal{F}|_U) \to R\Gamma(\tau^{\geq n+1}\mathcal{F}|_U) \to .$$

By induction, the first and last terms belong to $D^{\mathrm{b}}(\Bbbk\text{-mod}^{\mathrm{fg}})^{\leq 0}$, so the middle term does as well. In particular, the long exact cohomology sequence terminates in degree 0:

$$\begin{aligned} \cdots \to \mathbf{H}^{-1}(U, \tau^{\geq n+1}\mathcal{F}|_U) &\to \mathbf{H}^0(U, \tau^{\leq n}\mathcal{F}|_U) \\ &\to \mathbf{H}^0(U, \mathcal{F}|_U) \to \mathbf{H}^0(U, \tau^{\geq n+1}\mathcal{F}|_U) \to 0. \end{aligned} \tag{3.5.6}$$

If $\mathcal{F} \notin {}^p D^{\mathrm{b}}_{\mathrm{c}}(X, \Bbbk)^{\leq -1}$, then the same must hold for at least one of $\tau^{\leq n}\mathcal{F}$ and $\tau^{\geq n+1}\mathcal{F}$. If $\tau^{\geq n+1}\mathcal{F} \notin {}^p D^{\mathrm{b}}_{\mathrm{c}}(X, \Bbbk)^{\leq -1}$, then by induction, there is an affine open subset U such that the last term in (3.5.6) is nonzero, and then $\mathbf{H}^0(U, \mathcal{F}|_U) \neq 0$ as well. On the other hand, suppose that $\tau^{\geq n+1}\mathcal{F} \in {}^p D^{\mathrm{b}}_{\mathrm{c}}(X, \Bbbk)^{\leq -1}$ and that $\tau^{\leq n}\mathcal{F} \notin {}^p D^{\mathrm{b}}_{\mathrm{c}}(X, \Bbbk)^{\leq -1}$. It follows from the definition that $\dim \operatorname{supp} \tau^{\geq n+1}\mathcal{F} \leq -n-2$. Let $X' = X \smallsetminus \operatorname{supp} \tau^{\geq n+1}\mathcal{F}$. Now, $\tau^{\leq n}\mathcal{F}$ has some nonzero cohomology sheaf $\mathsf{H}^k(\tau^{\leq n}\mathcal{F})$ whose support has dimension $-k \geq -n$. It follows that $\dim \operatorname{supp} \mathsf{H}^k(\tau^{\leq n}\mathcal{F})|_{X'} = -k$, so $\tau^{\leq n}\mathcal{F}|_{X'} \notin {}^p D^{\mathrm{b}}_{\mathrm{c}}(X', \Bbbk)^{\leq -1}$. Again by induction, there is an affine open subset $U \subset X'$ such that $\mathbf{H}^0(U, \tau^{\leq n}\mathcal{F}|_U) \neq 0$. Because we have chosen $U \subset X'$, the first and last terms in (3.5.6) vanish, and we deduce that $\mathbf{H}^0(U, \mathcal{F}|_U) \neq 0$. $\square$

THEOREM 3.5.3. *Let X be a variety, and let $\mathcal{F} \in D^{\mathrm{b}}_{\mathrm{c}}(X, \Bbbk)$. The following conditions are equivalent:*

(1) $\mathcal{F} \in {}^pD^{\mathrm{b}}_{\mathrm{c}}(X, \Bbbk)^{\leq 0}$.
(2) *For any affine open subset $U \subset X$, we have $R\Gamma(\mathcal{F}|_U) \in D^{\mathrm{b}}(\Bbbk\text{-mod}^{\mathrm{fg}})^{\leq 0}$.*

PROOF. Lemma 3.5.2 tells us that condition (1) implies condition (2). For the opposite implication, suppose $\mathcal{F} \notin {}^pD^{\mathrm{b}}_{\mathrm{c}}(X, \Bbbk)^{\leq 0}$. Let n be the smallest integer such that ${}^pD^{\mathrm{b}}_{\mathrm{c}}(X, \Bbbk)^{\leq n}$. (Such an integer exists because the perverse t-structure is bounded.) We have $n > 0$, and $\mathcal{F} \notin {}^pD^{\mathrm{b}}_{\mathrm{c}}(X, \Bbbk)^{\leq n-1}$. Then Lemma 3.5.2 tells us that there exists an affine open subset $U \subset X$ such that $\mathsf{H}^n(R\Gamma(\mathcal{F}|_U)) \neq 0$; in particular, $R\Gamma(\mathcal{F}|_U) \notin D^{\mathrm{b}}(\Bbbk\text{-mod}^{\mathrm{fg}})^{\leq 0}$. $\square$

Cohomology with compact support and ${}^pD^{\mathrm{b}}_{\mathrm{c}}(X, \Bbbk)^{\geq 0}$. Our next goal is to prove a counterpart of Theorem 3.5.3 for the positive part of the perverse t-structure. In the case where $\Bbbk$ is a field, this counterpart (Theorem 3.5.7 below) follows immediately from Theorem 3.5.3 by Verdier duality. But for general coefficients, additional work along the lines of Proposition 3.5.1 is required.

PROPOSITION 3.5.4. *Let X be a variety, and let $i : Z \hookrightarrow X$ be the inclusion of a smooth, connected, closed subvariety. Let $\mathcal{F} \in D^{\mathrm{b}}_{\mathrm{locf}}(Z, \Bbbk)$. Assume that* $\operatorname{mdsupp} \mathsf{H}^i(\mathcal{F}) \leq -i$ *for all i, with equality for at least one i. Then there exists an affine open subset $U \subset X$ and an integer k such that $\mathbf{H}^k(U, i_*\mathcal{F}|_U) \neq 0$ and* $\operatorname{grade} \mathbf{H}^k(U, i_*\mathcal{F}|_U) = k$.

PROOF. *Step 1. Reduction to the case where X is affine.* Let $X' \subset X$ be an affine open subset such that $X' \cap Z$ is nonempty. Then it is enough to prove the result after replacing X, Z, and $\mathcal{F}$ by X', $X' \cap Z$, and $\mathcal{F}|_{X' \cap Z}$, respectively.

Step 2. Reduction to the case where X is an open subset of $\mathbb{A}^m$. Assume that X is affine, and let $m = \dim X$. By the Noether normalization lemma, there is a finite morphism $f : X \to \mathbb{A}^m$. Choose a stratification of $\mathbb{A}^m$ with respect to which $f_*i_*\mathcal{F}$ is constructible, and then choose a stratum Z' of maximal dimension such that $(f_*i_*\mathcal{F})|_{Z'}$ is nonzero. Let $X' \subset \mathbb{A}^m$ be the union of all strata whose closures contain Z'. Then X' is an open subset of $\mathbb{A}^m$, Z' is a closed subset of X', and $(f_*i_*\mathcal{F})|_{X'}$ is supported on Z'. In other words, if we let $i' : Z' \hookrightarrow X'$ be the inclusion map and $\mathcal{F}' = (f_*i_*\mathcal{F})|_{Z'}$, then $(f_*i_*\mathcal{F})|_{X'} \cong i'_*\mathcal{F}'$. By construction, we have $\mathcal{F}' \in D^{\mathrm{b}}_{\mathrm{locf}}(Z', \Bbbk)$.

Let $\tilde{Z}' = f^{-1}(X') \cap Z$. A stalk calculation like that in the proof of Lemma 2.1.9 shows that $f(\tilde{Z}')$ must be contained in Z'. That is, f restricts to a (finite) map $f' : \tilde{Z}' \to Z'$, and we have $\mathcal{F}' \cong f'_*(\mathcal{F}|_{\tilde{Z}'})$. We actually have $f(\tilde{Z}') = Z'$, since $\mathcal{F}'$ can have nonzero stalks only at points of $f(\tilde{Z}')$. It follows that

$$\dim Z' = \dim \tilde{Z}' = \dim Z.$$

Next, Lemmas 2.1.9 and 2.3.19 imply that

$$\operatorname{grade} \mathsf{H}^i(\mathcal{F}') = \operatorname{grade} f'_*\mathsf{H}^i(\mathcal{F}|_{\tilde{Z}'}) = \operatorname{grade} \mathsf{H}^i(\mathcal{F}|_{\tilde{Z}'}) = \operatorname{grade} \mathsf{H}^i(\mathcal{F}),$$

and hence $\operatorname{mdsupp} \mathsf{H}^i(\mathcal{F}') = \operatorname{mdsupp} \mathsf{H}^i(\mathcal{F})$. In particular, $\operatorname{mdsupp} \mathsf{H}^i(\mathcal{F}') \leq -i$, with equality for at least one i. Thus, together, X', Z', and $\mathcal{F}'$ satisfy the assumptions of the proposition. Suppose the result is known in this case: there is an affine open subset $U' \subset X'$ such that $\mathbf{H}^k(U', (i'_*\mathcal{F}')|_{U'})$ has the desired properties. Since f is affine, the set $U = f^{-1}(U')$ is affine. We have

$$R\Gamma((i'_*\mathcal{F}')|_{U'}) \cong R\Gamma((f_*i_*\mathcal{F})|_{U'}) \cong R\Gamma((f|_U)_*(i_*\mathcal{F}|_U)) \cong R\Gamma((i_*\mathcal{F})|_U),$$

and so $\mathbf{H}^k(U, (i_*\mathcal{F})|_U)$ has the desired properties as well.

Step 3. Reduction to the case where $\dim X = \dim Z$. Assume that X is an affine open subset of $\mathbb{A}^m$ and that $\dim X > \dim Z$. Then $\overline{Z}$ is a proper closed subset of $\mathbb{A}^m$, so as in the proof of Proposition 2.6.1, one can find (perhaps after a suitable change of coordinates) a nonconstant polynomial $f \in \mathbb{C}[x_1, \ldots, x_m]$ that is monic in x_m and that vanishes on $\overline{Z}$. Let $p : \mathbb{A}^m \to \mathbb{A}^{m-1}$ be the projection onto the first $m-1$ coordinates. The map $p|_{\overline{Z}} : \overline{Z} \to \mathbb{A}^{m-1}$ is finite. Let $W = \overline{Z} \smallsetminus Z$. Both $p(\overline{Z})$ and $p(W)$ are closed subsets of $\mathbb{A}^m$. Moreover, $p(\overline{Z}) \smallsetminus p(W)$ is nonempty because $\dim p(W) = \dim W$ is strictly smaller than $\dim p(\overline{Z}) = \dim \overline{Z}$.

Let Z' be an open subset of $p(\overline{Z}) \smallsetminus p(W)$. Let $\tilde{Z}' = p^{-1}(Z') \cap \overline{Z}$ and consider the finite map $p' : \tilde{Z}' \to Z'$. Our setup implies that $\tilde{Z}' \subset Z$, so it makes sense to consider the object $\mathcal{F}' = p'_*(\mathcal{F}|_{\tilde{Z}'})$. By replacing Z' by a smaller open subset, we may assume that $\mathcal{F}'$ belongs to $D^{\mathrm{b}}_{\mathrm{locf}}(Z', \Bbbk)$. The reasoning of Step 2 again shows that $\operatorname{grade} \mathsf{H}^i(\mathcal{F}') = \operatorname{grade} \mathsf{H}^i(\mathcal{F})$ for all i. Finally, let X' be an open subset of $\mathbb{A}^{m-1}$ such that $X' \cap p(\overline{Z}) = Z'$, and let $i' : Z' \hookrightarrow X'$ be the inclusion map. Then X', Z', and $\mathcal{F}'$ satisfy the assumptions of the proposition. Moreover, $\dim Z' = \dim Z$, but $\dim X' < \dim X$.

To finish this step, we proceed by induction on $\dim X - \dim Z$. Suppose the result is known for X', Z', and $\mathcal{F}'$, and let $U' \subset X'$ be a suitable affine open subset. Then $p^{-1}(U')$ is affine, and by Lemma 2.1.3, so is $U = p^{-1}(U') \cap X$. A calculation like that at the end of Step 2 shows that $R\Gamma((i'_*\mathcal{F}')|_{U'}) \cong R\Gamma((i_*\mathcal{F})|_U)$, and hence that the result holds for X, Z, and $\mathcal{F}$.

Step 4. Induction argument for open subsets of affine space. Assume now that X is an open subset of $\mathbb{A}^m$ and that $\dim Z = \dim X$. Since X is irreducible, we must in fact have $Z = X$. We proceed by induction on m. If $m = 0$, the result holds trivially. Assume now that $m > 0$. We will show that if the result holds for open subsets of $\mathbb{A}^{m-1}$, then it holds for X.

By replacing X by a smaller open set if necessary, we may assume that it is the principal open subset of $\mathbb{A}^m$ associated to a nonconstant polynomial $f \in \mathbb{C}[x_1, \ldots, x_m]$ that is monic in x_m. Let $U_1 = X$, and let $Z_1 = \mathbb{A}^m \smallsetminus X$. Then repeat the construction of Step 2 of the proof of Proposition 3.5.1: this gives a map $p : \mathbb{A}^m \to \mathbb{A}^{m-1}$, a scalar $a \in \mathbb{C}$, and new subsets $Z_2, U_2, U_3 \subset \mathbb{A}^m$ and $V \subset \mathbb{A}^{m-1}$, all of which are affine varieties. We also have the affine morphism $q : U_3 \to V$, defined by restricting p. Let $\mathcal{F}' = q_*(\mathcal{F}|_{U_3})$. As in the proof of Proposition 3.5.1, we have that $\mathcal{F}' \in D^{\mathrm{b}}_{\mathrm{locf}}(V, \Bbbk)$ and that

$$\mathcal{F}'_y \cong R\Gamma(\mathcal{F}|_{q^{-1}(y)}) \tag{3.5.7}$$

for all $y \in V$. The same calculation that led to (3.5.2) shows that for all i, we have

$$\operatorname{grade} \mathbf{H}^1(q^{-1}(y), \mathsf{H}^i(\mathcal{F})|_{q^{-1}(y)}) \le \operatorname{grade} \mathsf{H}^i(\mathcal{F}). \tag{3.5.8}$$

We will show below that $\operatorname{mdsupp} \mathsf{H}^i(\mathcal{F}') \le -i$, or, equivalently, that

$$\operatorname{grade} \mathsf{H}^i(\mathcal{F}')_y \ge m - 1 + i \tag{3.5.9}$$

for all i, with equality for at least one i. In other words, we will show that $\mathcal{F}'$ satisfies the assumptions of the proposition. Reasoning similar to that at the end of Step 3 then shows that if the result holds for open subsets of $\mathbb{A}^{m-1}$, it holds for open subsets of $\mathbb{A}^m$ as well.

Let $U_y = q^{-1}(y)$. This is a nonempty open subset of $p^{-1}(y) \cong \mathbb{A}^1$. We claim that there is a short exact sequence

$$0 \to \mathbf{H}^1(U_y, \mathsf{H}^{i-1}(\mathcal{F})|_{U_y}) \to \mathbf{H}^i(U_y, \mathcal{F}|_{U_y}) \to \mathbf{H}^0(U_y, \mathsf{H}^i(\mathcal{F})|_{U_y}) \to 0. \tag{3.5.10}$$

To see this, form the distinguished triangle $\tau^{\leq i-1}\mathcal{F} \to \mathcal{F} \to \tau^{\geq i}\mathcal{F} \to$, and then apply $R\Gamma((-)|_{U_y})$. Part of the long exact sequence in cohomology is

$$\begin{aligned} \cdots \to \mathbf{H}^{i-1}(U_y, \tau^{\geq i}\mathcal{F}|_{U_y}) \to \mathbf{H}^i(U_y, \tau^{\leq i-1}\mathcal{F}|_{U_y}) \to \mathbf{H}^i(U_y, \mathcal{F}|_{U_y}) \\ \to \mathbf{H}^i(U_y, \tau^{\geq i}\mathcal{F}|_{U_y}) \to \mathbf{H}^{i+1}(U_y, \tau^{\leq i-1}\mathcal{F}|_{U_y}) \to \cdots. \end{aligned} \tag{3.5.11}$$

By Proposition A.6.11, the first term in (3.5.11) vanishes, and the fourth term is identified with $\Gamma(\mathsf{H}^i(\mathcal{F})|_{U_y}) \cong \mathbf{H}^0(U_y, \mathsf{H}^i(\mathcal{F})|_{U_y})$. On the other hand, because U_y is an affine variety of dimension 1, the functor $\Gamma((-)|_{U_y})$ has cohomological dimension ≤ 1 (by Theorem 2.6.2), so Proposition A.6.11 (see also Remark A.7.15) tells us that the last term of (3.5.11) vanishes, and identifies the second term with $\mathbf{H}^1(U_y, \mathsf{H}^{i-1}(\mathcal{F})|_{U_y})$. We have thus produced the short exact sequence (3.5.10).

Now, U_y is an affine variety of dimension 1, and each $\mathsf{H}^i(\mathcal{F})|_{U_y}$ is a local system satisfying $\operatorname{mdsupp} \mathsf{H}^i(\mathcal{F})|_{U_y} \leq -m+1-i$. Theorem 2.6.2 tells us that the first and third terms of (3.5.10) both have grade $\geq m-1+i$, so by Lemma A.10.11, the middle term does as well. We have proved the inequality (3.5.9).

Finally, let j be an integer such that $\operatorname{mdsupp} \mathsf{H}^j(\mathcal{F}) = -j$, or in other words, $\operatorname{grade} \mathsf{H}^j(\mathcal{F})|_{U_y} = m+j$. Consider (3.5.10) with $i = j+1$. Then (3.5.8) implies that the first term of (3.5.10) has grade equal to $m+j = m-1+i$, and hence so does the middle term. In other words, (3.5.9) is an equality for $i = j+1$. ☐

LEMMA 3.5.5. *Suppose $\mathcal{F} \in D^{\mathrm{b}}_{\mathrm{c}}(X, \Bbbk)$ satisfies $\operatorname{mdsupp} \mathsf{H}^i(\mathcal{F}) \leq -i$ for all i. For any affine open subset $U \subset X$ and any $k \in \mathbb{Z}$, we have $\operatorname{grade} \mathbf{H}^k(U, \mathcal{F}|_U) \geq k$.*

PROOF. The proof is by induction on the number of nonzero cohomology sheaves of $\mathcal{F}$. If $\mathcal{F}$ is concentrated in a single degree, then $\mathcal{F} \cong \mathcal{F}'[n]$ for some some sheaf $\mathcal{F}'$ satisfying $\operatorname{mdsupp} \mathcal{F}' \leq n$. Then Theorem 2.6.2 tells us that for any affine open subset $U \subset X$ and any integer k, we have

$$\operatorname{grade} \mathbf{H}^k(U, \mathcal{F}|_U) = \operatorname{grade} \mathbf{H}^{k+n}(U, \mathcal{F}'|_U) \geq k+n-n = k.$$

If $\mathcal{F}$ has cohomology in more than one degree, we can find a truncation distinguished triangle $\tau^{\leq n}\mathcal{F} \to \mathcal{F} \to \tau^{\geq n+1}\mathcal{F} \to$ with respect to the natural t-structure such that the first and last terms are both nonzero. Apply $R\Gamma((-)|_U)$:

$$R\Gamma(\tau^{\leq n}\mathcal{F}|_U) \to R\Gamma(\mathcal{F}|_U) \to R\Gamma(\tau^{\geq n+1}\mathcal{F}|_U) \to .$$

By induction, both $\mathsf{H}^k(R\Gamma(\tau^{\leq n}\mathcal{F}|_U))$ and $\mathsf{H}^k(R\Gamma(\tau^{\geq n+1}\mathcal{F}|_U))$ have grade $\geq k$, so by Corollary A.10.12, $\mathsf{H}^k(R\Gamma(\mathcal{F}|_U))$ does as well. ☐

LEMMA 3.5.6. *Suppose $\mathcal{F} \in D^{\mathrm{b}}_{\mathrm{c}}(X, \Bbbk)$ satisfies $\operatorname{mdsupp} \mathsf{H}^k(\mathcal{F}) \leq -k$ for all k, with equality for at least one k. Then there exists an affine open subset $U \subset X$ and an integer k such that $\operatorname{grade} \mathbf{H}^k(U, \mathcal{F}|_U) = k$.*

PROOF. Choose a stratification $(X_s)_{s \in \mathscr{S}}$ of X with respect to which $\mathcal{F}$ is constructible. Let m be an integer such that $\operatorname{mdsupp} \mathsf{H}^m(\mathcal{F}) = -m$. For each X_s, we have $\operatorname{mdsupp} \mathsf{H}^m(\mathcal{F}|_{X_s}) \leq -m$; moreover, by definition, there exists a stratum X_t such that $\operatorname{mdsupp} \mathsf{H}^m(\mathcal{F}|_{X_t}) = -m$. We may as well replace X by the open subset consisting of strata whose closure contains X_t; in other words, we assume without loss of generality that X_t is a closed stratum in X.

Let $i : X_t \hookrightarrow X$ be the inclusion map. Note that $i^*\mathcal{F}$ belongs to $D^{\mathrm{b}}_{\mathrm{locf}}(X_t, \Bbbk)$ and satisfies $\operatorname{mdsupp} \mathsf{H}^r(i^*\mathcal{F}) \le -r$ for all r, with equality for $r = m$. Let $j : V \hookrightarrow X$ be the inclusion of the open complement of X_t. By Proposition 3.5.4, there exists an affine open subset $U \subset X$ and an integer k such that $\operatorname{grade} \mathbf{H}^k(U, (i_*i^*\mathcal{F})|_U) = k$. The distinguished triangle $j_!j^*\mathcal{F} \to \mathcal{F} \to i_*i^*\mathcal{F} \to$ gives rise to a long exact sequence

$$\cdots \to \mathbf{H}^k(U, \mathcal{F}|_U) \xrightarrow{p} \mathbf{H}^k(U, (i_*i^*\mathcal{F})|_U) \xrightarrow{q} \mathbf{H}^{k+1}(U, (j_!j^*\mathcal{F})|_U) \to \cdots .$$

Now, the object $j_!j^*\mathcal{F}$ satisfies $\operatorname{mdsupp} \mathsf{H}^r(j_!j^*\mathcal{F}) \le -r$ for all r, so according to Lemma 3.5.5, the last term above has grade $\ge k + 1$, and hence so does the image of the map q. Since the middle term has grade k, it follows from Lemma A.10.11 that the image of p has grade exactly k as well. The first term must therefore have grade $\le k$. But Lemma 3.5.5 again tells us that the first term has grade $\ge k$, so in fact its grade is exactly k. $\square$

THEOREM 3.5.7. *Let X be a variety, and let $\mathcal{F} \in D^{\mathrm{b}}_{\mathrm{c}}(X, \Bbbk)$. The following conditions are equivalent:*

(1) $\mathcal{F} \in {}^pD^{\mathrm{b}}_{\mathrm{c}}(X, \Bbbk)^{\ge 0}$.
(2) *For any affine open subset $U \subset X$, we have $R\Gamma_{\mathrm{c}}(\mathcal{F}|_U) \in D^{\mathrm{b}}(\Bbbk\text{-mod}^{\mathrm{fg}})^{\ge 0}$.*

PROOF. (1) $\Longrightarrow$ (2) Suppose $\mathcal{F} \in {}^pD^{\mathrm{b}}_{\mathrm{c}}(X, \Bbbk)^{\ge 0}$, and let $\mathcal{G} = \mathbb{D}\mathcal{F}$. Then $\mathcal{G}$ satisfies the assumptions of Lemma 3.5.5, so for any affine open subset $U \subset X$, we have $\operatorname{grade} \mathsf{H}^k(R\Gamma(\mathcal{G}|_U)) \ge k$ for all k. By Proposition A.10.6, this means that

$$R\Gamma(\mathcal{G}|_U) \in \mathbb{D}(D^{\mathrm{b}}(\Bbbk\text{-mod}^{\mathrm{fg}})^{\ge 0}), \qquad \text{or} \qquad \mathbb{D}R\Gamma(\mathcal{G}|_U) \in D^{\mathrm{b}}(\Bbbk\text{-mod}^{\mathrm{fg}})^{\ge 0}.$$

Since $\mathbb{D}R\Gamma(\mathcal{G}|_U) \cong R\Gamma_{\mathrm{c}}(\mathbb{D}\mathbb{D}\mathcal{F}|_U) \cong R\Gamma_{\mathrm{c}}(\mathcal{F}|_U)$, we are done.

(2) $\Longrightarrow$ (1) Suppose $\mathcal{F} \notin {}^pD^{\mathrm{b}}_{\mathrm{c}}(X, \Bbbk)^{\ge 0}$. Let n be the largest integer such that $\mathcal{F} \in {}^pD^{\mathrm{b}}_{\mathrm{c}}(X, \Bbbk)^{\ge n}$. Of course, we have $n < 0$. Then $\mathcal{F}[n]$ lies in ${}^pD^{\mathrm{b}}_{\mathrm{c}}(X, \Bbbk)^{\ge 0}$, but $\mathcal{F}[n+1]$ does not. Let $\mathcal{G} = \mathbb{D}(\mathcal{F}[n])$. Then $\mathcal{G}$ satisfies $\operatorname{mdsupp} \mathsf{H}^k(\mathcal{G}) \le -k$ for all k, with equality for at least one k. By Lemma 3.5.6, there exists an affine open subset $U \subset X$ and an integer r such that $\operatorname{grade} \mathsf{H}^r(R\Gamma(\mathcal{G}|_U)) = r$. By Proposition‘A.10.6, this means that $R\Gamma(\mathcal{G}|_U) \notin \mathbb{D}(D^{\mathrm{b}}(\Bbbk\text{-mod}^{\mathrm{fg}})^{\ge 1})$ and hence, since $-n \ge 1$, that

$$R\Gamma(\mathcal{G}|_U) \notin \mathbb{D}(D^{\mathrm{b}}(\Bbbk\text{-mod}^{\mathrm{fg}})^{\ge -n}) \qquad \text{or} \qquad (\mathbb{D}R\Gamma(\mathcal{G}|_U))[-n] \notin D^{\mathrm{b}}(\Bbbk\text{-mod}^{\mathrm{fg}})^{\ge 0}.$$

Since $(\mathbb{D}R\Gamma(\mathcal{G}|_U))[-n] \cong R\Gamma_{\mathrm{c}}(\mathbb{D}\mathbb{D}\mathcal{F}|_U) \cong R\Gamma_{\mathrm{c}}(\mathcal{F}|_U)$, we are done. $\square$

Exactness results. We conclude this section with some useful consequences of Theorems 3.5.3 and 3.5.7.

THEOREM 3.5.8. *Let $f : X \to Y$ be an affine morphism. With respect to the perverse t-structure, the functor $f_* : D^{\mathrm{b}}_{\mathrm{c}}(X, \Bbbk) \to D^{\mathrm{b}}_{\mathrm{c}}(Y, \Bbbk)$ is right t-exact and the functor $f_! : D^{\mathrm{b}}_{\mathrm{c}}(X, \Bbbk) \to D^{\mathrm{b}}_{\mathrm{c}}(Y, \Bbbk)$ is left t-exact.*

PROOF. For the first assertion, suppose $\mathcal{F} \in D^{\mathrm{b}}_{\mathrm{c}}(X, \Bbbk)^{\le 0}$. We wish to prove that $f_*\mathcal{F} \in D^{\mathrm{b}}_{\mathrm{c}}(Y, \Bbbk)^{\le 0}$. By Theorem 3.5.3, it is enough to show that for any affine open subset $U \subset Y$, we have

$$R\Gamma((f_*\mathcal{F})|_U) \in D^{\mathrm{b}}(\Bbbk\text{-mod}^{\mathrm{fg}})^{\le 0}. \tag{3.5.12}$$

Let $\tilde{U} = f^{-1}(U)$, and let $\tilde{f} = f|_{\tilde{U}} : \tilde{U} \to U$. We have

$$R\Gamma((f_*\mathcal{F})|_U) \cong R\Gamma(\tilde{f}_*(\mathcal{F}|_{\tilde{U}})) \cong R\Gamma(\mathcal{F}|_{\tilde{U}}).$$

Since f is an affine morphism, $\tilde{U}$ is an affine open subset of X, and then another application of Theorem 3.5.3 tells us that $R\Gamma(\mathcal{F}|_{\tilde{U}}) \in D^{\mathrm{b}}(\Bbbk\text{-mod}^{\mathrm{fg}})^{\leq 0}$. Therefore, (3.5.12) holds.

The proof of the left t-exactness of $f_!$ is similar, using Theorem 3.5.7. □

Corollary 3.5.9. *Let $h : Y \hookrightarrow X$ be a locally closed embedding that is also an affine morphism. The functors $h_!$ and h_* are t-exact for the perverse t-structure.*

In particular, this result applies to any locally closed embedding $h : Y \hookrightarrow X$ in which Y is an affine variety (cf. Lemma 2.1.3).

Proof. By Lemma 3.1.4, $h_!$ is right t-exact and h_* is left t-exact. The opposite exactness claims hold by Theorem 3.5.8. □

Exercises.

3.5.1. Let $X \subset \mathbb{A}^n$ be an affine variety of dimension m. Recall that X is said to be a **complete intersection** if it is defined by exactly $n - m$ polynomials $f_1, \ldots, f_{n-m} \in \mathbb{C}[x_1, \ldots, x_n]$. Show that if X is a complete intersection, then $\underline{\Bbbk}_X[m]$ is a perverse sheaf.

3.5.2 (Lefschetz hyperplane theorem). Let $X \subset \mathbb{P}^n$ be a projective variety of dimension m. Let $H \subset \mathbb{A}^{n+1}$ be a linear hyperplane, and assume that $X \smallsetminus \mathbb{P}(H)$ is smooth. Show that the natural map

$$\mathbf{H}^k(X; \Bbbk) \to \mathbf{H}^k(X \cap \mathbb{P}(H); \Bbbk)$$

is an isomorphism for $0 \leq k \leq m-2$ and that it is injective for $k = m-1$.

3.6. Smooth pullback

Perverse sheaves behave exceptionally well with respect to pullback along a smooth morphism. This section and the next one are devoted to describing this behavior. Perhaps the most important result in this section is the full faithfulness result in Theorem 3.6.6.

Functors for smooth morphisms. The starting point is the following exactness result.

Proposition 3.6.1. *Let $f : X \to Y$ be a smooth morphism of relative dimension d. The functor $f^*[d] \cong f^![-d](-d) : D^{\mathrm{b}}_{\mathrm{c}}(Y, \Bbbk) \to D^{\mathrm{b}}_{\mathrm{c}}(X, \Bbbk)$ is t-exact for the perverse t-structure.*

Proof. For any smooth, connected, locally closed subvariety $Z \subset Y$, we have that $f^{-1}(Z)$ is either empty or satisfies $\dim f^{-1}(Z) = \dim Z + d$. It easy to deduce from this that if $\mathcal{F}$ is a constructible sheaf on Y, then

$$\dim \operatorname{supp} f^*\mathcal{F} \leq \dim \operatorname{supp} \mathcal{F} + d \quad \text{and} \quad \operatorname{mdsupp} f^*\mathcal{F} \leq \operatorname{mdsupp} \mathcal{F} + d. \tag{3.6.1}$$

Now let $\mathcal{F} \in {}^{p}D^{\mathrm{b}}_{\mathrm{c}}(Y, \Bbbk)^{\leq 0}$, so that $\dim \operatorname{supp} \mathsf{H}^i(\mathcal{F}) \leq -i$ for all i. It follows immediately from (3.6.1) that $\dim \operatorname{supp} \mathsf{H}^i(f^*\mathcal{F}[d]) = \dim \operatorname{supp} f^*\mathsf{H}^{i+d}(\mathcal{F}) \leq -(i + d) + d = -i$, so $f^*\mathcal{F}[d] \in {}^{p}D^{\mathrm{b}}_{\mathrm{c}}(X, \Bbbk)^{\leq 0}$. In other words, $f^*[d]$ is right t-exact.

A similar calculation using mdsupp shows that if $\mathcal{G} \in \mathbb{D}({}^{p}D^{\mathrm{b}}_{\mathrm{c}}(Y, \Bbbk)^{\geq 0})$, then $f^*\mathcal{G}[d]$, and hence $f^*\mathcal{G}[d](d)$, both lie in $\mathbb{D}({}^{p}D^{\mathrm{b}}_{\mathrm{c}}(X, \Bbbk)^{\geq 0})$. In other words, for $\mathcal{F} \in$

${}^p D^{\mathrm{b}}_{\mathrm{c}}(Y,\Bbbk)^{\geq 0}$, we have $\mathbb{D}(f^*(\mathbb{D}\mathcal{F})[d](d)) \in {}^p D^{\mathrm{b}}_{\mathrm{c}}(X,\Bbbk)^{\geq 0}$. Since $\mathbb{D}(f^*(\mathbb{D}\mathcal{F})[d](d)) \cong (\mathbb{D}f^*\mathbb{D}\mathcal{F})[-d](-d) \cong f^!\mathcal{F}[-d](-d) \cong f^*\mathcal{F}[d]$, we conclude that $f^*[d]$ is left t-exact, and hence t-exact. ☐

For a smooth morphism $f : X \to Y$ of relative dimension d, we introduce the notation

$$f^\dagger = f^*[d] : D^{\mathrm{b}}_{\mathrm{c}}(Y,\Bbbk) \to D^{\mathrm{b}}_{\mathrm{c}}(X,\Bbbk), \qquad f_\dagger = f_*[-d] : D^{\mathrm{b}}_{\mathrm{c}}(X,\Bbbk) \to D^{\mathrm{b}}_{\mathrm{c}}(Y,\Bbbk).$$

Of course, $f_\dagger$ is right adjoint to $f^\dagger$. Proposition 3.6.1 tells us that $f^\dagger$ restricts to an exact functor

$$f^\dagger : \mathrm{Perv}(Y,\Bbbk) \to \mathrm{Perv}(X,\Bbbk).$$

We also introduce the notation

$${}^p f_\dagger = {}^p\mathsf{H}^0 \circ f_\dagger : \mathrm{Perv}(X,\Bbbk) \to \mathrm{Perv}(Y,\Bbbk).$$

Lemma 3.6.2. *Let $f : X \to Y$ be a smooth morphism of relative dimension d.*

(1) *The functor $f_\dagger$ is left t-exact for the perverse t-structure. Equivalently, the functor ${}^p f_\dagger$ is left exact.*
(2) *For $\mathcal{F} \in \mathrm{Perv}(Y,\Bbbk)$ and $\mathcal{G} \in \mathrm{Perv}(X,\Bbbk)$, there is a natural isomorphism*

$$\mathrm{Hom}(f^\dagger\mathcal{F},\mathcal{G}) \cong \mathrm{Hom}(\mathcal{F}, {}^p f_\dagger\mathcal{G}).$$

Proof. (1) Let $\mathcal{G} \in {}^p D^{\mathrm{b}}_{\mathrm{c}}(X,\Bbbk)^{\geq 0}$. We must show that $f_\dagger\mathcal{G} \in {}^p D^{\mathrm{b}}_{\mathrm{c}}(Y,\Bbbk)^{\geq 0}$. This is equivalent to the claim that $\mathrm{Hom}(\mathcal{F}, f_\dagger\mathcal{G}) = 0$ for all $\mathcal{F} \in {}^p D^{\mathrm{b}}_{\mathrm{c}}(Y,\Bbbk)^{\leq -1}$. By adjunction, this is in turn equivalent to the claim that $\mathrm{Hom}(f^\dagger\mathcal{F},\mathcal{G}) = 0$. This latter Hom-group vanishes because, by Proposition 3.6.1, we have $f^\dagger\mathcal{F} \in {}^p D^{\mathrm{b}}_{\mathrm{c}}(X,\Bbbk)^{\leq -1}$.

(2) By part (1), we know that for $\mathcal{G} \in \mathrm{Perv}(X,\Bbbk)$, we have ${}^p\tau^{\leq 0} f_\dagger\mathcal{G} \cong {}^p\mathsf{H}^0(f_\dagger\mathcal{G})$. Using the adjunction properties of truncation, we have

$$\mathrm{Hom}(f^\dagger\mathcal{F},\mathcal{G}) \cong \mathrm{Hom}(\mathcal{F}, f_\dagger\mathcal{G}) \cong \mathrm{Hom}(\mathcal{F}, {}^p\tau^{\leq 0} f_\dagger\mathcal{G}) \cong \mathrm{Hom}(\mathcal{F}, {}^p\mathsf{H}^0(f_\dagger\mathcal{G})),$$

as desired. ☐

Lemma 3.6.3. *Let $f : X \to Y$ be a smooth morphism, and let $h : V \hookrightarrow Y$ be the inclusion of a locally closed subvariety. Form the cartesian square*

$$\begin{tikzcd} f^{-1}(V) \arrow[r, "h'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ V \arrow[r, "h"] & Y \end{tikzcd}$$

For any $\mathcal{F} \in \mathrm{Perv}(V,\Bbbk)$, there is a natural isomorphism

$$f^\dagger h_{!*}\mathcal{F} \cong h'_{!*}(f')^\dagger\mathcal{F}.$$

Proof. Consider the commutative diagram from Lemma 1.2.12:

$$\begin{tikzcd} f^\dagger h_!\mathcal{F} \arrow[r, "\sim"] \arrow[d] & h'_!(f')^\dagger\mathcal{F} \arrow[d] \\ f^\dagger h_*\mathcal{F} \arrow[r, "\sim"] & h'_*(f')^\dagger\mathcal{F} \end{tikzcd}$$

Here, both horizontal arrows are isomorphisms because f is smooth. Now apply ${}^p\mathsf{H}^0$. Since $f^\dagger$ is t-exact, it commutes with ${}^p\mathsf{H}^0$, and we obtain the diagram

$$\begin{tikzcd} f^\dagger {}^p\mathsf{H}^0(h_!\mathcal{F}) \arrow[r, "\sim"] \arrow[d] & {}^p\mathsf{H}^0(h'_!(f')^\dagger\mathcal{F}) \arrow[d] \\ f^\dagger {}^p\mathsf{H}^0(h_*\mathcal{F}) \arrow[r, "\sim"] & {}^p\mathsf{H}^0(h'_*(f')^\dagger\mathcal{F}) \end{tikzcd}$$

in the abelian category $\mathrm{Perv}(X, \Bbbk)$. The image of the left-hand vertical arrow is $f^\dagger h_{!*}(\mathcal{F})$, while the image of the right-hand vertical arrow is $h'_{!*}(f')^\dagger\mathcal{F}$. $\square$

Smooth surjective morphisms. Let $f : X \to Y$ be a smooth surjective morphism, and consider the cartesian square

$$\begin{tikzcd} X \times_Y X \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"'] \arrow[dr, "f'"] & X \arrow[d, "f"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

where $f' = f \circ \mathrm{pr}_1 = f \circ \mathrm{pr}_2$. Note that pr_1 and pr_2 are also smooth surjective morphisms. For $\mathcal{F} \in D^{\mathrm{b}}(Y, \Bbbk)$ and $i = 1, 2$, let ρ_i denote the composition

$$f_*f^*\mathcal{F} \to f_*\mathrm{pr}_{i*}\mathrm{pr}_i^*f^*\mathcal{F} \xrightarrow{\sim} f'_*(f')^*\mathcal{F},$$

where the first map is the adjunction map. We also let $\eta : \mathcal{F} \to f_*f^*\mathcal{F}$ be the adjunction map. According to Proposition 1.6.1, the composition $\rho_i \circ \eta$ is equal to the adjunction map $\mathcal{F} \to f'_*(f')^*\mathcal{F}$; in particular, $\rho_1 \circ \eta = \rho_2 \circ \eta$. Now let $\rho = \rho_1 - \rho_2$, and consider the sequence

$$\mathcal{F} \xrightarrow{\eta} f_*f^*\mathcal{F} \xrightarrow{\rho} f'_*(f')^*\mathcal{F}. \tag{3.6.2}$$

The observations above say that $\rho \circ \eta = 0$.

Lemma 3.6.4. *Let $f : X \to Y$ be a smooth surjective morphism, and let f' and ρ be as above. For any sheaf $\mathcal{F} \in \mathrm{Sh}(Y, \Bbbk)$, the sequence*

$$0 \to \mathcal{F} \xrightarrow{\eta} {}^\circ f_*f^*\mathcal{F} \xrightarrow{\rho} {}^\circ f'_*(f')^*\mathcal{F} \tag{3.6.3}$$

is exact. If f has connected fibers, then $\eta : \mathcal{F} \to {}^\circ f_*f^*\mathcal{F}$ is an isomorphism.

Proof. It follows from Lemma 2.2.1 that for every point $y \in Y$, there exists an analytic neighborhood V of y and an analytic map $e : V \to f^{-1}(V)$ such that $f|_{f^{-1}(V)} \circ e = \mathrm{id}_V$. We call such a map a *local right inverse* to f over V. Every point $y \in Y$ has a basis of analytic neighborhoods over which f has a local right inverse.

Let $V \subset Y$ be an analytic open set over which f has a local right inverse e, and let $f_V = f|_{f^{-1}(V)} : f^{-1}(V) \to V$ and $f'_V = f'|_{(f')^{-1}(V)} : (f')^{-1}(V) \to V$. We have a cartesian square

$$\begin{tikzcd} (f')^{-1}(V) = f^{-1}(V) \times_V f^{-1}(V) \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"'] \arrow[dr, "f'_V"] & f^{-1}(V) \arrow[d, "f_V"] \\ f^{-1}(V) \arrow[r, "f_V"] & V \end{tikzcd}$$

The procedure used to define (3.6.3) can be repeated to give a sequence

$$0 \to \mathcal{F}|_V \xrightarrow{\eta} {}^\circ f_{V*}f_V^*(\mathcal{F}|_V) \xrightarrow{\rho} {}^\circ f'_{V*}(f'_V)^*(\mathcal{F}|_V);$$

indeed, this sequence is just the restriction to V of (3.6.3). Now apply Γ: since $\Gamma \circ {}^\circ f_{V*} \cong \Gamma$ and $\Gamma \circ {}^\circ f'_{V*} \cong \Gamma$, our sequence becomes

$$0 \to \Gamma(\mathcal{F}|_V) \xrightarrow{p} \Gamma(f_V^*(\mathcal{F}|_V)) \xrightarrow{q} \Gamma((f'_V)^*(\mathcal{F}|_V)). \tag{3.6.4}$$

We will show that this sequence is exact. This claim implies the lemma: since (3.6.4) is the sequence of sections over V of (3.6.3), and since it is exact for all sufficiently small V, it follows that (3.6.3) is exact on all stalks, so it is exact.

Combining the adjunction maps for f_V and for e, we obtain a sequence of maps

$$\mathcal{F}|_V \to {}^\circ f_{V*} f_V^*(\mathcal{F}|_V) \to {}^\circ f_{V*} {}^\circ e_* e^* f_V^*(\mathcal{F}|_V) \cong {}^\circ(f_V \circ e)_*(f_V \circ e)^*(\mathcal{F}|_V). \tag{3.6.5}$$

Since $f_V \circ e = \mathrm{id}_V$, the functors ${}^\circ(f_V \circ e)_*$ and $(f_V \circ e)^*$ are just identity functors, and the composition of the maps in (3.6.5) is the identity map. In particular, the map $\mathcal{F}|_V \to {}^\circ f_{V*} f_V^*(\mathcal{F}|_V)$ is injective. Now apply Γ:

$$\Gamma(\mathcal{F}|_V) \xrightarrow{p} \Gamma(f_V^*(\mathcal{F}|_V)) \to \Gamma(e^* f_V^*(\mathcal{F}|_V)) \cong \Gamma(\mathcal{F}|_V). \tag{3.6.6}$$

Since Γ is left exact, the map p here (and in (3.6.4)) is injective.

It remains to prove that (3.6.4) is exact at $\Gamma(f_V^*\mathcal{F})$. We at least have $\operatorname{im} p \subset \ker q$ (since $\rho \circ \eta = 0$), so (3.6.6) can be replaced by the diagram

$$\Gamma(\mathcal{F}|_V) \xrightarrow{p} \ker q \xrightarrow{r} \Gamma(e^* f_V^*(\mathcal{F}|_V)) \cong \Gamma(\mathcal{F}|_V). \tag{3.6.7}$$

The composition here is still the identity, so the map $r : \ker q \to \Gamma(e^* f_V^*(\mathcal{F}|_V))$ is surjective. To show that $\operatorname{im} p = \ker q$, it is enough to show that r is injective.

Suppose instead that there is a nonzero section $s \in \ker q \subset \Gamma(f_V^*(\mathcal{F}|_V))$ such that $r(s) = 0$. The latter condition means that

$$s|_{e(V)} = 0. \tag{3.6.8}$$

On the other hand, since s is nonzero, there exists some $y \in V$ such that $s|_{f_V^{-1}(y)}$ is nonzero. Let $M = \mathcal{F}_y$, and let $K = f_V^{-1}(y)$. Then the sheaf $(f_V^*(\mathcal{F}|_V))|_K$ is isomorphic to the constant sheaf $\underline{M}_K$, and the section $s' = s|_K$ can be thought of as a locally constant function $s' : K \to M$. Let $K_1, \ldots, K_n$ be the connected components of K; thus, s' is constant on each K_i. Consider the cartesian square

$$\begin{array}{ccc} K \times K = \coprod_{i,j} K_i \times K_j & \xrightarrow{\mathrm{pr}_2} & K = \coprod_i K_i \\ {\scriptstyle \mathrm{pr}_1}\downarrow & & \downarrow{\scriptstyle \mathrm{a}_K} \\ K = \coprod_i K_i & \xrightarrow{\mathrm{a}_K} & \{y\} \end{array}$$

Recall that the condition $s \in \ker q$ means that its images under $\rho_1, \rho_2 : f_* f^* \mathcal{F} \to f'_*(f')^*\mathcal{F}$ coincide. After restricting to K, this means that our function s' has the property that the locally constant functions $s' \circ \mathrm{pr}_1, s' \circ \mathrm{pr}_2 : K \times K \to M$ coincide. This, in turn, implies that $s'(K_i) = s'(K_j)$ for all i, j. In other words, s' is a nonzero constant function, so it has nonzero germ at every point of K. In particular, at the point $e(y) \in K \cap e(V)$, the germ $s'_{e(y)} = s_{e(y)}$ is nonzero. But this contradicts (3.6.8). We conclude that both maps in (3.6.7) are isomorphisms and that (3.6.4) is exact.

Finally, suppose that f has connected fibers. In this case, an argument like that of the preceding paragraph can be applied directly to (3.6.6) rather than (3.6.7), with the conclusion that both maps in (3.6.6) are isomorphisms. As a consequence, the map p in (3.6.4) is an isomorphism, as is the map η in (3.6.3). $\square$

LEMMA 3.6.5. *Let $f : X \to Y$ be a smooth surjective morphism. For $\mathcal{F} \in {}^pD^{\mathrm{b}}_{\mathrm{c}}(Y, \Bbbk)^{\leq 0}$ and $\mathcal{G} \in {}^pD^{\mathrm{b}}_{\mathrm{c}}(Y, \Bbbk)^{\geq 0}$, there is an exact sequence*

$$0 \to \mathrm{Hom}(\mathcal{F}, \mathcal{G}) \xrightarrow{f^*} \mathrm{Hom}(f^*\mathcal{F}, f^*\mathcal{G}) \xrightarrow{\mathrm{pr}_1^* - \mathrm{pr}_2^*} \mathrm{Hom}((f')^*\mathcal{F}, (f')^*\mathcal{G}).$$

Moreover, if f has connected fibers, then the map induced by f^* is an isomorphism.

PROOF. Apply the construction of (3.6.2) to $R\mathscr{H}om(\mathcal{F}, \mathcal{G})$ to get a sequence

$$R\mathscr{H}om(\mathcal{F}, \mathcal{G}) \xrightarrow{\eta} f_*f^*R\mathscr{H}om(\mathcal{F}, \mathcal{G}) \xrightarrow{\rho} f'_*(f')^*R\mathscr{H}om(\mathcal{F}, \mathcal{G}). \tag{3.6.9}$$

Lemma 3.2.1 implies that all three terms above lie in $D^{\mathrm{b}}_{\mathrm{c}}(Y, \Bbbk)^{\geq 0}$, so Proposition A.6.11 tells us that on the second term, we have

$$\mathsf{H}^0(f_*f^*R\mathscr{H}om(\mathcal{F}, \mathcal{G})) \cong {}^\circ f_*\mathsf{H}^0(f^*R\mathscr{H}om(\mathcal{F}, \mathcal{G})),$$

and likewise for the third term. Thus, applying H^0 to (3.6.9) yields

$$\begin{aligned} 0 \to \mathsf{H}^0 R\mathscr{H}om(\mathcal{F}, \mathcal{G}) \to {}^\circ f_*\mathsf{H}^0(f^*R\mathscr{H}om(\mathcal{F}, \mathcal{G})) \\ \to {}^\circ f'_*\mathsf{H}^0((f')^*R\mathscr{H}om(\mathcal{F}, \mathcal{G})), \end{aligned} \tag{3.6.10}$$

which is exact by Lemma 3.6.4. Using Principle 2.2.11, we rewrite this sequence as

$$\begin{aligned} 0 \to \mathsf{H}^0 R\mathscr{H}om(\mathcal{F}, \mathcal{G}) \to {}^\circ f_*\mathsf{H}^0(R\mathscr{H}om(f^*\mathcal{F}, f^*\mathcal{G})) \\ \to {}^\circ f'_*\mathsf{H}^0(R\mathscr{H}om((f')^*\mathcal{F}, (f')^*\mathcal{G})). \end{aligned} \tag{3.6.11}$$

Next, apply the left exact functor Γ to obtain another exact sequence. Note that $\Gamma \circ {}^\circ f_* \cong \Gamma$, and likewise for ${}^\circ f'_*$. In addition, Proposition A.6.11 tells us that on each term in (3.6.11), we have $\Gamma \circ \mathsf{H}^0 \cong \mathsf{H}^0 \circ R\Gamma$. We thus obtain the exact sequence

$$\begin{aligned} 0 \to \mathsf{H}^0 R\Gamma R\mathscr{H}om(\mathcal{F}, \mathcal{G}) \to \mathsf{H}^0 R\Gamma R\mathscr{H}om(f^*\mathcal{F}, f^*\mathcal{G}) \\ \to \mathsf{H}^0 R\Gamma R\mathscr{H}om((f')^*\mathcal{F}, (f')^*\mathcal{G}). \end{aligned}$$

The lemma follows by Proposition 1.4.6 and Proposition A.6.13(2). ☐

An immediate consequence of Lemma 3.6.5 is the following.

THEOREM 3.6.6. *Let $f : X \to Y$ be a smooth surjective morphism. The functor*

$$f^\dagger : \mathrm{Perv}(Y, \Bbbk) \to \mathrm{Perv}(X, \Bbbk)$$

is faithful. If f has connected fibers, then $f^\dagger$ is fully faithful.

For additional information on the image of $f^\dagger$, see Theorem 3.7.6.

REMARK 3.6.7. For shifted local systems, Theorem 3.6.6 can be understood from another perspective. Let $f : X \to Y$ be a smooth surjective morphism with connected fibers, and assume that Y is smooth and connected. Then X is also smooth and connected (see Lemma 2.1.11). Choose a point $x_0 \in X$, and let $y_0 = f(x_0)$. Via Proposition 1.7.10, the functor

$$f^* : \mathrm{Loc}^{\mathrm{ft}}(Y, \Bbbk) \to \mathrm{Loc}^{\mathrm{ft}}(X, \Bbbk)$$

corresponds to the functor

$$\mathrm{Res}^{\Bbbk[\pi_1(Y, y_0)]}_{\Bbbk[\pi_1(X, x_0)]} : \Bbbk[\pi_1(Y, y_0)]\text{-mod}^{\mathrm{fg}} \to \Bbbk[\pi_1(X, x_0)]\text{-mod}^{\mathrm{fg}} \tag{3.6.12}$$

induced by the homomorphism

$$\pi_1(f) : \pi_1(X, x_0) \to \pi_1(Y, y_0). \tag{3.6.13}$$

According to Proposition 2.1.23, our assumptions imply that (3.6.13) is surjective. It follows immediately that (3.6.12) is fully faithful.

The surjectivity of (3.6.13) also implies that the functor (3.6.12) takes irreducible $\Bbbk[\pi_1(Y, y_0)]$-modules to irreducible $\Bbbk[\pi_1(X, x_0)]$-modules. In other words, we obtain the following result.

LEMMA 3.6.8. *Let Y be a smooth, connected variety, and let $f : X \to Y$ be a smooth surjective map with connected fibers. Assume that $\Bbbk$ is a field. If $\mathcal{L} \in \mathrm{Loc}^{\mathrm{ft}}(Y, \Bbbk)$ is an irreducible local system, then $f^*\mathcal{L} \in \mathrm{Loc}^{\mathrm{ft}}(X, \Bbbk)$ is irreducible as well.*

COROLLARY 3.6.9. *Let $f : X \to Y$ be a smooth surjective morphism with connected fibers. If $\Bbbk$ is a field, then $f^\dagger$ sends any simple perverse sheaf to a simple perverse sheaf.*

PROOF. Recall from Theorem 3.4.5 that every simple perverse sheaf is the intermediate extension of an irreducible local system on some smooth, connected, locally closed subvariety. The claim then follows from Lemmas 3.6.3 and 3.6.8. $\square$

Exercises.

3.6.1. Let $f : X \to Y$ be a smooth but not necessarily surjective morphism whose nonempty fibers are connected. Let $\Bbbk$ be a field. Show that $f^\dagger$ sends any simple perverse sheaf to either 0 or a simple perverse sheaf.

3.7. Smooth descent

Suppose $f : X \to Y$ is a smooth surjective morphism. Given a perverse sheaf $\mathcal{F}$ on X, how can we tell whether $\mathcal{F}$ comes from a perverse sheaf on Y via $f^\dagger$? When this is the case, how can we recover the original perverse sheaf on Y? The answer to these questions involves the notion of *descent data*.

In the discussion below, we will require a number of maps to and from double and triple fiber products of X over Y, summarized in the following diagram:

$$\begin{tikzcd}
X \times_Y X \times_Y X \arrow[rrr, "\mathrm{pr}_{23}"] \arrow[dr, "\mathrm{pr}_{13}"] \arrow[dd, "\mathrm{pr}_{12}"'] & & & X \times_Y X \arrow[dr, "\mathrm{pr}_2"] \arrow[dd, "\mathrm{pr}_1" near start] & \\
& X \times_Y X \arrow[rrr, "\mathrm{pr}_2" near start, crossing over] & & & X \arrow[dd, "f"] \\
X \times_Y X \arrow[rrr, "\mathrm{pr}_2" near start] \arrow[dr, "\mathrm{pr}_1"'] & & & X \arrow[dr, "f"] & \\
& X \arrow[rrr, "f"] \arrow[from=uu, "\mathrm{pr}_1" near start, crossing over] & & & Y
\end{tikzcd} \tag{3.7.1}$$

Every face of this cube is cartesian, and every map in this cube is surjective and smooth of relative dimension d.

We also denote the various projection maps from $X \times_Y X \times_Y X$ to X by

$$\mathrm{pr}'_1, \mathrm{pr}'_2, \mathrm{pr}'_3 : X \times_Y X \times_Y X \to X.$$

Note that $\mathrm{pr}'_1 = \mathrm{pr}_1 \circ \mathrm{pr}_{12} = \mathrm{pr}_1 \circ \mathrm{pr}_{13}$, and similarly for the other maps. These maps are smooth of relative dimension $2d$.

Finally, there are obvious maps from these fiber products to Y, denoted by

$$f' : X \times_Y X \to Y \qquad \text{and} \qquad f'' : X \times_Y X \times_Y X \to Y.$$

DEFINITION 3.7.1. Let $f : X \to Y$ be a smooth surjective morphism of varieties of relative dimension d, and let $\mathcal{F} \in \mathrm{Perv}(X, \Bbbk)$. A **descent datum** for $\mathcal{F}$ with respect to f is an isomorphism

$$\phi : \mathrm{pr}_1^\dagger \mathcal{F} \xrightarrow{\sim} \mathrm{pr}_2^\dagger \mathcal{F} \qquad \text{in } \mathrm{Perv}(X \times_Y X, \Bbbk) \tag{3.7.2}$$

such that the following diagram in $\mathrm{Perv}(X \times_Y X \times_Y X, \Bbbk)$ commutes:

$$\begin{tikzcd} (\mathrm{pr}_1')^\dagger \mathcal{F} \arrow[rr, "\mathrm{pr}_{13}^\dagger \phi"] \arrow[dr, "\mathrm{pr}_{12}^\dagger \phi"'] & & (\mathrm{pr}_3')^\dagger \mathcal{F} \\ & (\mathrm{pr}_2')^\dagger \mathcal{F} \arrow[ur, "\mathrm{pr}_{23}^\dagger \phi"'] & \end{tikzcd} \tag{3.7.3}$$

Next, let $\mathcal{G}$ be another perverse sheaf on X, and let ϕ' be a descent datum for $\mathcal{G}$. A **morphism of descent data** $q : (\mathcal{F}, \phi) \to (\mathcal{G}, \phi')$ is a morphism of perverse sheaves $q : \mathcal{F} \to \mathcal{G}$ such that the diagram

$$\begin{tikzcd} \mathrm{pr}_1^\dagger \mathcal{F} \arrow[r, "\mathrm{pr}_1^\dagger q"] \arrow[d, "\phi"'] & \mathrm{pr}_1^\dagger \mathcal{G} \arrow[d, "\phi'"] \\ \mathrm{pr}_2^\dagger \mathcal{F} \arrow[r, "\mathrm{pr}_2^\dagger q"] & \mathrm{pr}_2^\dagger \mathcal{G} \end{tikzcd} \tag{3.7.4}$$

commutes. The **category of descent data** with respect to the map f, denoted by $\mathrm{Desc}(f, \Bbbk)$, is the category whose objects are pairs $(\mathcal{F}, \phi)$ as above, and whose morphisms are morphisms of descent data.

There is an obvious forgetful functor

$$\mathrm{Desc}(f, \Bbbk) \to \mathrm{Perv}(X, \Bbbk).$$

This functor is clearly faithful. From the definitions, one sees that for any two objects $(\mathcal{F}, \phi), (\mathcal{G}, \phi') \in \mathrm{Desc}(f, \Bbbk)$, there is an exact sequence

$$\begin{aligned} 0 \to \mathrm{Hom}_{\mathrm{Desc}(f,\Bbbk)}((\mathcal{F}, \phi), (\mathcal{G}, \phi')) \to \mathrm{Hom}(\mathcal{F}, \mathcal{G}) \\ \xrightarrow{q \mapsto \phi' \circ \mathrm{pr}_1^\dagger q - \mathrm{pr}_2^\dagger q \circ \phi} \mathrm{Hom}(\mathrm{pr}_1^\dagger \mathcal{F}, \mathrm{pr}_2^\dagger \mathcal{G}). \end{aligned} \tag{3.7.5}$$

LEMMA 3.7.2. *Let $f : X \to Y$ be a smooth surjective morphism of relative dimension d. For any $\mathcal{G} \in \mathrm{Perv}(Y, \Bbbk)$, the perverse sheaf $f^\dagger \mathcal{G}$ admits a canonical descent datum. Moreover, $f^\dagger$ gives rise to a fully faithful functor*

$$f^\dagger : \mathrm{Perv}(Y, \Bbbk) \to \mathrm{Desc}(f, \Bbbk).$$

PROOF. Recall that $f' = f \circ \mathrm{pr}_1 = f \circ \mathrm{pr}_2$. Proposition 1.2.8(1) gives us natural isomorphisms

$$\mathrm{pr}_1^\dagger f^\dagger \mathcal{G} \xrightarrow{\sim} (f')^\dagger \mathcal{G} \xrightarrow{\sim} \mathrm{pr}_2^\dagger f^\dagger \mathcal{G}.$$

We define $\phi : \mathrm{pr}_1^\dagger f^\dagger \mathcal{G} \to \mathrm{pr}_2^\dagger f^\dagger \mathcal{G}$ to be the composition of these maps. To check that this is indeed a descent datum, we consider the following diagram:

$$
\begin{tikzcd}[column sep=small]
(\mathrm{pr}_1')^\dagger f^\dagger \mathcal{G} \arrow[rr, "\sim"] \arrow[d, "\wr"] \arrow[ddrrrr, "\sim"] \arrow[rrrrrrrr, bend left, "\mathrm{pr}_{13}^\dagger \phi"] \arrow[dddrrrr, bend right=60, "\mathrm{pr}_{12}^\dagger \phi"'] & & \mathrm{pr}_{13}^\dagger \mathrm{pr}_1^\dagger f^\dagger \mathcal{G} \arrow[rr, "\sim"] & & \mathrm{pr}_{13}^\dagger (f')^\dagger \mathcal{G} \arrow[rr, "\sim"] \arrow[d, "\wr"] & & \mathrm{pr}_{13}^\dagger \mathrm{pr}_2^\dagger f^\dagger \mathcal{G} \arrow[rr, "\sim"] & & (\mathrm{pr}_3')^\dagger f^\dagger \mathcal{G} \\
\mathrm{pr}_{12}^\dagger \mathrm{pr}_1^\dagger f^\dagger \mathcal{G} \arrow[drr, "\sim"] & & & & (f'')^\dagger \mathcal{G} \arrow[urrrr, "\sim"] \arrow[drr, "\sim"] & & & & \mathrm{pr}_{23}^\dagger \mathrm{pr}_2^\dagger f^\dagger \mathcal{G} \arrow[u, "\wr"] \\
& & \mathrm{pr}_{12}^\dagger (f')^\dagger \mathcal{G} \arrow[urr, "\sim"] \arrow[d, "\wr"] & & & & \mathrm{pr}_{23}^\dagger (f')^\dagger \mathcal{G} \arrow[urr, "\sim"] & & \\
& & \mathrm{pr}_{12}^\dagger \mathrm{pr}_2^\dagger f^\dagger \mathcal{G} \arrow[rr, "\sim"] & & (\mathrm{pr}_2')^\dagger f^\dagger \mathcal{G} \arrow[rr, "\sim"] \arrow[uu, "\wr"] \arrow[uuurrrr, bend right=60, "\mathrm{pr}_{23}^\dagger \phi"'] & & \mathrm{pr}_{23}^\dagger \mathrm{pr}_1^\dagger f^\dagger \mathcal{G} \arrow[u, "\wr"] & &
\end{tikzcd}
$$

This diagram is the union of numerous small squares, each of which has $(f'')^\dagger \mathcal{G}$ as a vertex. Each such square commutes by Proposition 1.6.2, so the entire diagram commutes.

To show that $f^\dagger$ determines a functor taking values in $\mathrm{Desc}(f, \Bbbk)$, we must show that it takes any morphism in $\mathrm{Perv}(Y, \Bbbk)$ to a morphism of descent data. That is, we must check that the diagram (3.7.4) commutes. But this is obvious, since ϕ is defined as a composition of two natural maps.

It is immediate from Theorem 3.6.6 that $f^\dagger : \mathrm{Perv}(Y, \Bbbk) \to \mathrm{Desc}(f, \Bbbk)$ is faithful. To show that it is fully faithful, we will study the following diagram:

$$
\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}(\mathcal{F}, \mathcal{G}) \arrow[r, "f^*"] \arrow[d, "f^\dagger"'] & \mathrm{Hom}(f^*\mathcal{F}, f^*\mathcal{G}) \arrow[r] \arrow[d, "\wr"] & \mathrm{Hom}((f')^*\mathcal{F}, (f')^*\mathcal{G}) \arrow[d, "\wr"] \\
0 \arrow[r] & \mathrm{Hom}_{\mathrm{Desc}(f,\Bbbk)}(f^\dagger\mathcal{F}, f^\dagger\mathcal{G}) \arrow[r] & \mathrm{Hom}(f^\dagger\mathcal{F}, f^\dagger\mathcal{G}) \arrow[r] & \mathrm{Hom}(\mathrm{pr}_1^\dagger f^\dagger\mathcal{F}, \mathrm{pr}_2^\dagger f^\dagger\mathcal{G})
\end{tikzcd}
$$

Here, the top row is from Lemma 3.6.5, and the bottom row is an instance of (3.7.5). All Hom-groups except the one on the lower left are to be computed in the appropriate derived category of sheaves. The commutativity of the left-hand square is obvious, and a straightforward calculation unwinding the definitions shows that the right-hand square also commutes. The five lemma then implies that the leftmost vertical arrow is an isomorphism, so our functor is fully faithful. $\square$

Let $\mathcal{F} \in \mathrm{Perv}(Y, \Bbbk)$, and consider the sequence of maps defined in (3.6.2). Recall that $f_* f^* \cong f_\dagger f^\dagger$. In view of Lemma 3.6.2, if we apply ${}^p\mathsf{H}^0$ to that sequence, we obtain the sequence of maps in the following lemma, which is a perverse analogue of Lemma 3.6.4.

LEMMA 3.7.3. *Let $f : X \to Y$ be a smooth surjective morphism. For any perverse sheaf $\mathcal{F} \in \mathrm{Perv}(Y, \Bbbk)$, the sequence*

$$0 \to \mathcal{F} \xrightarrow{\eta} {}^p f_\dagger f^\dagger \mathcal{F} \xrightarrow{\rho} {}^p f'_\dagger (f')^\dagger \mathcal{F} \tag{3.7.6}$$

is exact. If f has connected fibers, then $\eta : \mathcal{F} \to {}^p f_\dagger f^\dagger \mathcal{F}$ is an isomorphism.

PROOF. For any $\mathcal{G} \in \mathrm{Perv}(Y, \Bbbk)$, we can consider the sequence

$$0 \to \mathrm{Hom}(\mathcal{G}, \mathcal{F}) \to \mathrm{Hom}(\mathcal{G}, {}^p f_\dagger f^\dagger \mathcal{F}) \to \mathrm{Hom}(\mathcal{G}, {}^p f'_\dagger (f')^\dagger \mathcal{F}). \tag{3.7.7}$$

Note that if (3.7.6) fails to be exact at either $\mathcal{F}$ or ${}^p f_\dagger f^\dagger \mathcal{F}$, then one can choose a $\mathcal{G}$ such that (3.7.7) also fails to be exact (for instance, take $\mathcal{G}$ to be the kernel of one of the maps in (3.7.6)). Thus, it is enough to prove that (3.7.7) is exact for all $\mathcal{G}$.

By adjunction, (3.7.7) can be rewritten as

$$0 \to \mathrm{Hom}(\mathcal{G}, \mathcal{F}) \to \mathrm{Hom}(f^\dagger \mathcal{G}, f^\dagger \mathcal{F}) \to \mathrm{Hom}((f')^\dagger \mathcal{G}, (f')^\dagger \mathcal{F}),$$

and this in turn can be identified with the exact sequence of Lemma 3.6.5. □

THEOREM 3.7.4. *Let $f : X \to Y$ be a smooth surjective morphism. The functor*

$$f^\dagger : \mathrm{Perv}(Y, \Bbbk) \to \mathrm{Desc}(f, \Bbbk)$$

is an equivalence of categories.

PROOF. We have already seen that this functor is fully faithful; we just need to prove that it is essentially surjective. To do this, we will construct a functor in the opposite direction. Let $(\mathcal{F}, \phi) \in \mathrm{Desc}(f, \Bbbk)$. For $i = 1, 2$, let $\eta_i : \mathcal{F} \to {}^p\mathrm{pr}_{i\dagger}\mathrm{pr}_i^\dagger \mathcal{F}$ denote the adjunction map. Let $\tilde{\rho}_1 : {}^p f_\dagger \mathcal{F} \to {}^p f'_\dagger \mathrm{pr}_2^\dagger \mathcal{F}$ be the composition of the following maps:

$${}^p f_\dagger \mathcal{F} \xrightarrow{{}^p f_\dagger \eta_1} {}^p f_\dagger {}^p\mathrm{pr}_{1\dagger}\mathrm{pr}_1^\dagger \mathcal{F} \xrightarrow{\sim} {}^p f'_\dagger \mathrm{pr}_1^\dagger \mathcal{F} \xrightarrow{{}^p f'_\dagger \phi} {}^p f'_\dagger \mathrm{pr}_2^\dagger \mathcal{F}.$$

Similarly, let $\tilde{\rho}_2 : {}^p f_\dagger \mathcal{F} \to {}^p f'_\dagger \mathrm{pr}_2^\dagger \mathcal{F}$ be the composition of the following maps:

$${}^p f_\dagger \mathcal{F} \xrightarrow{{}^p f_\dagger \eta_2} {}^p f_\dagger {}^p\mathrm{pr}_{2\dagger}\mathrm{pr}_2^\dagger \mathcal{F} \xrightarrow{\sim} {}^p f'_\dagger \mathrm{pr}_2^\dagger \mathcal{F}.$$

Finally, we define a functor

$$Q : \mathrm{Desc}(f, \Bbbk) \to \mathrm{Perv}(Y, \Bbbk) \qquad \text{by} \qquad Q(\mathcal{F}, \phi) = \ker(\tilde{\rho}_1 - \tilde{\rho}_2).$$

We claim that $f^\dagger Q(\mathcal{F}, \phi)$ is naturally isomorphic to $(\mathcal{F}, \phi)$. Before proving this, we need another description of $\mathcal{F}$. Let us apply Lemma 3.7.3 to the smooth map $\mathrm{pr}_1 : X \times_Y X \to X$ and the cartesian square

$$\begin{tikzcd} X \times_Y X \times_Y X \arrow[r, "\mathrm{pr}_{13}"] \arrow[d, "\mathrm{pr}_{12}"'] \arrow[dr, "\mathrm{pr}'_1"] & X \times_Y X \arrow[d, "\mathrm{pr}_1"] \\ X \times_Y X \arrow[r, "\mathrm{pr}_1"] & X \end{tikzcd}$$

Explicitly, let $\rho'_1, \rho'_2 : {}^p\mathrm{pr}_{1\dagger}\mathrm{pr}_1^\dagger \mathcal{F} \to {}^p\mathrm{pr}'_{1\dagger}(\mathrm{pr}'_1)^\dagger \mathcal{F}$ be the maps given by the compositions

$${}^p\mathrm{pr}_{1\dagger}\mathrm{pr}_1^\dagger \mathcal{F} \to {}^p\mathrm{pr}_{1\dagger}{}^p\mathrm{pr}_{12\dagger}\mathrm{pr}_{12}^\dagger \mathrm{pr}_1^\dagger \mathcal{F} \to {}^p\mathrm{pr}'_{1\dagger}(\mathrm{pr}'_1)^\dagger \mathcal{F},$$

$${}^p\mathrm{pr}_{1\dagger}\mathrm{pr}_1^\dagger \mathcal{F} \to {}^p\mathrm{pr}_{1\dagger}{}^p\mathrm{pr}_{13\dagger}\mathrm{pr}_{13}^\dagger \mathrm{pr}_1^\dagger \mathcal{F} \to {}^p\mathrm{pr}'_{1\dagger}(\mathrm{pr}'_1)^\dagger \mathcal{F},$$

respectively. Then Lemma 3.7.3 gives us a natural isomorphism

$$\mathcal{F} \xrightarrow{\sim} \ker(\rho'_1 - \rho'_2).$$

To relate $\mathcal{F}$ to $f^\dagger Q(\mathcal{F}, \phi)$, we must study the relationships between the $\tilde{\rho}_i$'s and the ρ'_i's. These relationships are shown in the first two large diagrams in Figure 3.7.1. In that figure, the left superscript "p"s have been omitted to reduce clutter. The small regions of each diagram commute by naturality, by (3.7.3), or by some result from Section 1.6.

The outer parts of those two diagrams have the same objects and the same morphisms, except that ρ'_1 and $\tilde{\rho}_1$ in the first diagram are replaced by ρ'_2 and $\tilde{\rho}_2$

$$
\begin{tikzcd}
\mathrm{pr}_{1\dagger}\mathrm{pr}_1^\dagger\mathcal{F} \arrow[r] \arrow[d, "\phi"', "\wr"] \arrow[rrr, bend left, "\rho_1'"] & \mathrm{pr}_{1\dagger}\mathrm{pr}_{12\dagger}\mathrm{pr}_{12}^\dagger\mathrm{pr}_1^\dagger\mathcal{F} \arrow[rr, "\mathrm{comp.}", "\sim"'] \arrow[d, "\phi"', "\wr"] & & \mathrm{pr}'_{1\dagger}(\mathrm{pr}'_1)^\dagger\mathcal{F} \arrow[r, "\mathrm{pr}_{13}^\dagger\phi", "\sim"'] \arrow[d, "\mathrm{pr}_{12}^\dagger\phi"', "\wr"] & \mathrm{pr}'_{1\dagger}(\mathrm{pr}'_3)^\dagger\mathcal{F} \arrow[dd, "\wr"] \\
\mathrm{pr}_{1\dagger}\mathrm{pr}_2^\dagger\mathcal{F} \arrow[r, "\mathrm{adj.}"] \arrow[dd, "\mathrm{b.ch.}"', "\wr"] \arrow[dr, "\mathrm{adj.}"'] & \mathrm{pr}_{1\dagger}\mathrm{pr}_{12\dagger}\mathrm{pr}_{12}^\dagger\mathrm{pr}_2^\dagger\mathcal{F} \arrow[r, "\mathrm{comp.}", "\sim"'] \arrow[d, phantom, "\text{Prop. 1.6.4}"] & \mathrm{pr}_{1\dagger}\mathrm{pr}_{12\dagger}(\mathrm{pr}'_2)^\dagger\mathcal{F} \arrow[r, "\mathrm{comp.}", "\sim"'] \arrow[d, "\wr"'] & \mathrm{pr}'_{1\dagger}(\mathrm{pr}'_2)^\dagger\mathcal{F} \arrow[d] & \\
 & \mathrm{pr}_{1\dagger}\mathrm{pr}_2^\dagger\mathrm{pr}_{1\dagger}\mathrm{pr}_1^\dagger\mathcal{F} \arrow[r, "\sim", "\mathrm{b.ch.}"'] \arrow[d, "\mathrm{b.ch.}"', "\wr"] & \mathrm{pr}_{1\dagger}\mathrm{pr}_{12\dagger}\mathrm{pr}_{23}^\dagger\mathrm{pr}_1^\dagger\mathcal{F} \arrow[r, "\mathrm{comp.}", "\sim"'] \arrow[d, phantom, "\text{Prop. 1.6.3}"] & \mathrm{pr}'_{1\dagger}\mathrm{pr}_{23}^\dagger\mathrm{pr}_1^\dagger\mathcal{F} \arrow[r, "\mathrm{pr}_{23}^\dagger\phi", "\sim"'] \arrow[d, "\mathrm{b.ch.}"', "\wr"] & \mathrm{pr}'_{1\dagger}\mathrm{pr}_{23}^\dagger\mathrm{pr}_2^\dagger\mathcal{F} \arrow[d, "\wr"', "\mathrm{b.ch.}"] \\
f^\dagger f_\dagger\mathcal{F} \arrow[r, "\mathrm{adj.}"] \arrow[rrrr, bend right, "f^\dagger\tilde{\rho}_1"'] & f^\dagger f_\dagger\mathrm{pr}_{1\dagger}\mathrm{pr}_1^\dagger\mathcal{F} \arrow[rr, "\mathrm{comp.}", "\sim"'] & & f^\dagger f'_\dagger\mathrm{pr}_1^\dagger\mathcal{F} \arrow[r, "\phi", "\sim"'] & f^\dagger f'_\dagger\mathrm{pr}_2^\dagger\mathcal{F}
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathrm{pr}_{1\dagger}\mathrm{pr}_1^\dagger\mathcal{F} \arrow[r] \arrow[d, "\phi"', "\wr"] \arrow[rrr, bend left, "\rho_2'"] & \mathrm{pr}_{1\dagger}\mathrm{pr}_{13\dagger}\mathrm{pr}_{13}^\dagger\mathrm{pr}_1^\dagger\mathcal{F} \arrow[rr, "\mathrm{comp.}", "\sim"'] \arrow[d, "\phi"', "\wr"] & & \mathrm{pr}'_{1\dagger}(\mathrm{pr}'_1)^\dagger\mathcal{F} \arrow[r, "\mathrm{pr}_{13}^\dagger\phi", "\sim"'] & \mathrm{pr}'_{1\dagger}(\mathrm{pr}'_3)^\dagger\mathcal{F} \arrow[dd, "\wr"] \\
\mathrm{pr}_{1\dagger}\mathrm{pr}_2^\dagger\mathcal{F} \arrow[r, "\mathrm{adj.}"] \arrow[dd, "\mathrm{b.ch.}"', "\wr"] \arrow[dr, "\mathrm{adj.}"'] & \mathrm{pr}_{1\dagger}\mathrm{pr}_{13\dagger}\mathrm{pr}_{13}^\dagger\mathrm{pr}_2^\dagger\mathcal{F} \arrow[r, "\mathrm{comp.}", "\sim"'] \arrow[d, phantom, "\text{Prop. 1.6.4}"] & \mathrm{pr}_{1\dagger}\mathrm{pr}_{13\dagger}(\mathrm{pr}'_3)^\dagger\mathcal{F} \arrow[urr, "\mathrm{comp.}", "\sim"'] \arrow[d, "\wr"'] & & \\
 & \mathrm{pr}_{1\dagger}\mathrm{pr}_2^\dagger\mathrm{pr}_{2\dagger}\mathrm{pr}_2^\dagger\mathcal{F} \arrow[r, "\sim", "\mathrm{b.ch.}"'] \arrow[d, "\mathrm{b.ch.}"', "\wr"] & \mathrm{pr}_{1\dagger}\mathrm{pr}_{13\dagger}\mathrm{pr}_{23}^\dagger\mathrm{pr}_2^\dagger\mathcal{F} \arrow[rr, "\mathrm{comp.}", "\sim"'] \arrow[d, phantom, "\text{Prop. 1.6.3}"] & & \mathrm{pr}'_{1\dagger}\mathrm{pr}_{23}^\dagger\mathrm{pr}_2^\dagger\mathcal{F} \arrow[d, "\mathrm{b.ch.}"', "\wr"] \\
f^\dagger f_\dagger\mathcal{F} \arrow[r, "\mathrm{adj.}"] \arrow[rrrr, bend right, "f^\dagger\tilde{\rho}_2"'] & f^\dagger f_\dagger\mathrm{pr}_{2\dagger}\mathrm{pr}_2^\dagger\mathcal{F} \arrow[rrr, "\mathrm{comp.}", "\sim"'] & & & f^\dagger f'_\dagger\mathrm{pr}_2^\dagger\mathcal{F}
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathrm{pr}_1^\dagger\mathcal{F} \arrow[rrrr, "\phi", "\sim"'] \arrow[d, hook, "\mathrm{adj.}"'] \arrow[dr, "\mathrm{adj.}"] \arrow[ddddd, bend right=60, "\mathrm{pr}_1^\dagger\theta"'] & & & & \mathrm{pr}_2^\dagger\mathcal{F} \arrow[dl, "\mathrm{adj.}"'] \arrow[d, hook, "\mathrm{adj.}"] \arrow[ddddd, bend left=60, "\mathrm{pr}_2^\dagger\theta"] \\
\mathrm{pr}_1^\dagger\mathrm{pr}_{1\dagger}\mathrm{pr}_1^\dagger\mathcal{F} \arrow[d, "\mathrm{b.ch.}"', "\wr"] \arrow[dd, bend right, "\phi"'] \arrow[r, phantom, "\text{Prop. 1.6.4}"] & \mathrm{pr}_{12\dagger}\mathrm{pr}_{12}^\dagger\mathrm{pr}_1^\dagger\mathcal{F} \arrow[rr, "\mathrm{pr}_{12}^\dagger\phi", "\sim"'] \arrow[d, "\wr"', "\mathrm{comp.}"] & & \mathrm{pr}_{12\dagger}\mathrm{pr}_{12}^\dagger\mathrm{pr}_2^\dagger\mathcal{F} \arrow[d, "\mathrm{comp.}"', "\wr"] \arrow[r, phantom, "\text{Prop. 1.6.4}"] & \mathrm{pr}_2^\dagger\mathrm{pr}_{1\dagger}\mathrm{pr}_1^\dagger\mathcal{F} \arrow[d, "\wr"', "\mathrm{b.ch.}"] \arrow[dd, bend left, "\phi"] \\
\mathrm{pr}_{12\dagger}\mathrm{pr}_{13}^\dagger\mathrm{pr}_1^\dagger\mathcal{F} \arrow[r, "\mathrm{comp.}", "\sim"'] \arrow[dr, "\sim"'] & \mathrm{pr}_{12\dagger}(\mathrm{pr}'_1)^\dagger\mathcal{F} \arrow[dr, "\mathrm{pr}_{13}^\dagger\phi", "\sim"'] & & \mathrm{pr}_{12\dagger}(\mathrm{pr}'_2)^\dagger\mathcal{F} \arrow[r, "\mathrm{comp.}", "\sim"'] \arrow[dl, "\sim"', "\mathrm{pr}_{23}^\dagger\phi"] & \mathrm{pr}_{12\dagger}\mathrm{pr}_{23}^\dagger\mathrm{pr}_1^\dagger\mathcal{F} \arrow[dl, "\sim"'] \\
\mathrm{pr}_1^\dagger\mathrm{pr}_{1\dagger}\mathrm{pr}_2^\dagger\mathcal{F} \arrow[r, "\mathrm{b.ch.}", "\sim"'] \arrow[d, "\mathrm{b.ch.}"', "\wr"] & \mathrm{pr}_{12\dagger}\mathrm{pr}_{13}^\dagger\mathrm{pr}_2^\dagger\mathcal{F} \arrow[r, "\mathrm{comp.}", "\sim"'] \arrow[d, phantom, "\text{Prop 1.6.3}"] & \mathrm{pr}_{12\dagger}(\mathrm{pr}'_3)^\dagger\mathcal{F} \arrow[r, "\mathrm{comp.}", "\sim"'] \arrow[d, "\mathrm{b.ch.}"', "\wr"] & \mathrm{pr}_{12\dagger}\mathrm{pr}_{23}^\dagger\mathrm{pr}_2^\dagger\mathcal{F} \arrow[r, "\mathrm{b.ch.}", "\sim"'] \arrow[d, phantom, "\text{Prop 1.6.3}"] & \mathrm{pr}_2^\dagger\mathrm{pr}_{1\dagger}\mathrm{pr}_2^\dagger\mathcal{F} \arrow[d, "\wr"', "\mathrm{b.ch.}"] \\
\mathrm{pr}_1^\dagger f^\dagger f_\dagger\mathcal{F} \arrow[rr] & & (f')^\dagger f_\dagger\mathcal{F} \arrow[rr] & & \mathrm{pr}_2^\dagger f^\dagger f_\dagger\mathcal{F} \\
\mathrm{pr}_1^\dagger f^\dagger Q(\mathcal{F}) \arrow[rr] \arrow[u, hook] \arrow[rrrr, bend right, "\phi'"'] & & (f')^\dagger Q(\mathcal{F}) \arrow[rr] \arrow[u, hook] & & \mathrm{pr}_2^\dagger f^\dagger Q(\mathcal{F}) \arrow[u, hook]
\end{tikzcd}
$$

FIGURE 3.7.1. Diagrams for the proof of Theorem 3.7.4

in the second. Combining these two diagrams, we deduce that there is a unique isomorphism $\theta : \mathcal{F} \xrightarrow{\sim} f^\dagger Q(\mathcal{F}, \phi)$ that makes the following diagram commute:

$$
\begin{tikzcd}
\mathcal{F} \cong \ker(\rho_1' - \rho_2') \arrow[r] \arrow[dd, dashed, "\theta"', "\wr"] & {}^p\mathrm{pr}_{1\dagger}\mathrm{pr}_1^\dagger\mathcal{F} \arrow[d, "\wr"] \\
 & {}^p\mathrm{pr}_{1\dagger}\mathrm{pr}_2^\dagger\mathcal{F} \arrow[d, "\wr"] \\
f^\dagger Q(\mathcal{F}, \phi) \cong \ker(f^\dagger\tilde{\rho}_1 - f^\dagger\tilde{\rho}_2) \arrow[r] & f^{\dagger p}f_\dagger\mathcal{F}
\end{tikzcd}
$$

The map θ is an isomorphism of perverse sheaves. To complete the proof, we must show that θ is in fact an isomorphism of descent data. That is, we must show that the isomorphism θ constructed above intertwines ϕ with the new descent datum $\phi' : \mathrm{pr}_1^\dagger f^\dagger Q(\mathcal{F}, \phi) \to \mathrm{pr}_2^\dagger f^\dagger Q(\mathcal{F}, \phi)$ produced by Lemma 3.7.2. This claim is proved by the third diagram in Figure 3.7.1.

Having exhibited an isomorphism of descent data $(\mathcal{F}, \phi) \cong f^\dagger Q(\mathcal{F}, \phi)$, we conclude that $f^\dagger$ is essentially surjective, and hence an equivalence of categories. $\square$

REMARK 3.7.5. Let $f : X \to Y$ be a smooth, surjective morphism with connected fibers. Theorem 3.6.6 and Lemma 3.7.3 have the following consequences:

(1) Any perverse sheaf on X admits at most one descent datum.
(2) The functor $Q : \mathrm{Desc}(f, \Bbbk) \xrightarrow{\sim} \mathrm{Perv}(Y, \Bbbk)$ inverse to $f^\dagger$ is given by
$$Q(\mathcal{F}, \phi) = {}^p f_\dagger \mathcal{F}.$$

As an application, we prove the following complement to Theorem 3.6.6.

THEOREM 3.7.6. *Let $f : X \to Y$ be a smooth, surjective morphism with connected fibers. The image of the fully faithful functor $f^\dagger : \mathrm{Perv}(Y, \Bbbk) \to \mathrm{Perv}(X, \Bbbk)$ is closed under subobjects and quotients.*

PROOF. We must show that if $\mathcal{F} \in \mathrm{Perv}(X, \Bbbk)$ admits a descent datum with respect to f, then so do all of its subobjects and quotients. We proceed by noetherian induction on Y: assume that the theorem is known when Y is replaced by any proper closed subvariety. Explicitly, let $Y' \subset Y$ be a closed subset, and let $Z = f^{-1}(Y')$. If $\mathcal{F}$ admits a descent datum and is supported on Z, then all its subobjects and quotients admit descent data by induction.

Let us fix some additional notation related to open and closed subsets of X and Y. Let $V = Y \smallsetminus Y'$, let $U = f^{-1}(V)$, and let $n = \dim U$. Assume that V is smooth and connected. We also let i, j, h, k, f_U, and f_Z be the maps indicated in the following diagram:

$$\begin{tikzcd} U \arrow[r, "j"] \arrow[d, "f_U"'] & X \arrow[d, "f"'] & Z \arrow[l, "i"'] \arrow[d, "f_Z"] \\ V \arrow[r, "h"] & Y & Y' \arrow[l, "k"'] \end{tikzcd} \tag{3.7.8}$$

Later in the proof, we will make specific choices of V, Y', etc.

Step 1. Let $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ be a short exact sequence of perverse sheaves on X, and assume that $\mathcal{F}$ admits a descent datum. Then $\mathcal{F}'$ admits a descent datum if and only if $\mathcal{F}''$ does. Consider the diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{pr}_1^\dagger \mathcal{F}' \arrow[r] \arrow[d, dashed, "\phi_{\mathcal{F}'}"'] & \mathrm{pr}_1^\dagger \mathcal{F} \arrow[r] \arrow[d, "\phi_{\mathcal{F}}"'] & \mathrm{pr}_1^\dagger \mathcal{F}'' \arrow[r] \arrow[d, dashed, "\phi_{\mathcal{F}''}"] & 0 \\ 0 \arrow[r] & \mathrm{pr}_2^\dagger \mathcal{F}' \arrow[r] & \mathrm{pr}_2^\dagger \mathcal{F} \arrow[r] & \mathrm{pr}_2^\dagger \mathcal{F}'' \arrow[r] & 0 \end{tikzcd} \tag{3.7.9}$$

To show that $\mathcal{F}'$ admits a descent datum, we claim that it is enough to show that an isomorphism $\phi_{\mathcal{F}'}$ making the left-hand square commute exists. Indeed, it is easy to see that if such a map exists, it is unique, and that it satisfies (3.7.3) because $\phi_{\mathcal{F}}$ does. Similarly, $\mathcal{F}''$ admits a descent datum if and only if there is a map $\phi_{\mathcal{F}''}$

making the right-hand square commute. Finally, a diagram chase shows that $\phi_{\mathcal{F}'}$ exists if and only if $\phi_{\mathcal{F}''}$ exists.

Step 2. Let $\mathcal{G}$ be a subobject or quotient of $\mathcal{F}$, and suppose we have a short exact sequence $0 \to \mathcal{G}' \to \mathcal{G} \to \mathcal{G}'' \to 0$. If $\mathcal{F}$, $\mathcal{G}'$, and $\mathcal{G}''$ admit descent data, then $\mathcal{G}$ does as well. Consider the case where $\mathcal{G}$ is a subobject of $\mathcal{F}$. Then $\mathcal{G}'$ is also a subobject of $\mathcal{F}$, and if we set $\mathcal{F}'' = \mathcal{F}/\mathcal{G}'$, then $\mathcal{G}''$ is a subobject of $\mathcal{F}''$. Since $\mathcal{G}'$ admits a descent datum, Step 1 tells us that $\mathcal{F}''$ does as well. The descent data we have can be arranged into a diagram as follows:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{pr}_1^\dagger\mathcal{G}' \arrow[rr] \arrow[dr, "\phi_{\mathcal{G}'}"] \arrow[dd, equal] & & \mathrm{pr}_1^\dagger\mathcal{G} \arrow[rr] \arrow[dr, dashed] \arrow[dd, hook] & & \mathrm{pr}_1^\dagger\mathcal{G}'' \arrow[rr] \arrow[dr, "\phi_{\mathcal{G}''}"] \arrow[dd, hook] & & 0 & \\
& 0 \arrow[r] & \mathrm{pr}_2^\dagger\mathcal{G}' \arrow[rr] \arrow[dd, equal] & & \mathrm{pr}_2^\dagger\mathcal{G} \arrow[rr] \arrow[dd, hook] & & \mathrm{pr}_2^\dagger\mathcal{G}'' \arrow[rr] \arrow[dd, hook] & & 0 \\
0 \arrow[r] & \mathrm{pr}_1^\dagger\mathcal{G}' \arrow[rr] \arrow[dr, "\phi_{\mathcal{G}'}"] & & \mathrm{pr}_1^\dagger\mathcal{F} \arrow[rr] \arrow[dr, "\phi_{\mathcal{F}}"] & & \mathrm{pr}_1^\dagger\mathcal{F}'' \arrow[rr] \arrow[dr, "\phi_{\mathcal{F}''}"] & & 0 & \\
& 0 \arrow[r] & \mathrm{pr}_2^\dagger\mathcal{G}' \arrow[rr] & & \mathrm{pr}_2^\dagger\mathcal{F} \arrow[rr] & & \mathrm{pr}_2^\dagger\mathcal{F}'' \arrow[rr] & & 0
\end{tikzcd}$$

An easy diagram chase shows that the dotted arrow can be filled in (in a unique way) to make the diagram commute, so $\mathcal{G}$ admits a descent datum. The proof in the case where $\mathcal{G}$ is a quotient of $\mathcal{F}$ is similar.

Step 3. If $\mathcal{G} \in \mathrm{Perv}(U, \Bbbk)$ admits a descent datum with respect to f_U, then $j_{!*}\mathcal{G}$ admits a descent datum with respect to f. If $\mathcal{G} \cong f_U^\dagger \bar{\mathcal{G}}$, then by Lemma 3.6.3, we have $j_{!*}\mathcal{G} \cong f^\dagger h_{!*}\bar{\mathcal{G}}$.

Step 4. If $\mathcal{F} \in \mathrm{Perv}(X, \Bbbk)$ admits a descent datum, then so do ${}^p\mathsf{H}^0(i_*i^!\mathcal{F})$ and ${}^p\mathsf{H}^0(i_*i^*\mathcal{F})$. Suppose $\mathcal{F} \cong f^\dagger\bar{\mathcal{F}}$. Then

$$i_*i^!\mathcal{F} \cong i_*i^! f^\dagger \bar{\mathcal{F}} \cong i_* f_Z^\dagger k^! \bar{\mathcal{F}} \cong f^\dagger k_* k^! \bar{\mathcal{F}}.$$

We conclude that ${}^p\mathsf{H}^0(i_*i^!\mathcal{F}) \cong f^{\dagger\,p}\mathsf{H}^0(k_*k^!\bar{\mathcal{F}})$. The proof for ${}^p\mathsf{H}^0(i_*i^*\mathcal{F})$ is siimlar.

Step 5. Let $0 \to \mathcal{L}' \to \mathcal{L} \to \mathcal{L}'' \to 0$ be a short exact sequence of local systems on U. If $\mathcal{L}$ admits a descent datum with respect to f_U, then $\mathcal{L}'$ and $\mathcal{L}''$ do as well. Choose a point $u_0 \in U$, and let $v_0 = f(u_0)$. Recall from Remark 3.6.7 that $f_U^* : \mathrm{Loc}^{\mathrm{ft}}(V, \Bbbk) \to \mathrm{Loc}^{\mathrm{ft}}(U, \Bbbk)$ can be identified with the functor

$$\Bbbk[\pi_1(V, v_0)]\text{-mod}^{\mathrm{fg}} \to \Bbbk[\pi_1(U, u_0)]\text{-mod}^{\mathrm{fg}} \tag{3.7.10}$$

coming from the natural map $\pi_1(U, u_0) \to \pi_1(V, v_0)$. The latter map is surjective (see Proposition 2.1.23), so the image of (3.7.10) consists precisely of $\Bbbk[\pi_1(U, u_0)]$-modules that factor through the quotient map to $\Bbbk[\pi_1(V, v_0)]$. The category of such modules is closed under subobjects and quotients.

Step 6. Suppose $\mathcal{F} \in \mathrm{Perv}(X, \Bbbk)$ has no subobject or quotient supported on Z, and that $\mathcal{F}|_U$ is a shifted local system. If $\mathcal{F}$ admits a descent datum, then every subobject or quotient of $\mathcal{F}$ does as well. Our assumptions on $\mathcal{F}$ are equivalent to requiring that $\mathcal{F} \cong j_{!*}(\mathcal{L}[n])$ for some local system $\mathcal{L}$ on U. By Step 1, it is enough to prove that every subobject admits a descent datum. Let $\mathcal{F}' \subset \mathcal{F}$ be a sub perverse sheaf, and let $\mathcal{L}' = \mathcal{F}'|_U[-n]$. This is a sub local system of $\mathcal{L}$. Let $\mathcal{L}'' = \mathcal{L}/\mathcal{L}'$. By Steps 3 and 5, $j_{!*}(\mathcal{L}''[n])$ admits a descent datum. By Lemma 3.3.5, the surjective map $\mathcal{L} \to \mathcal{L}''$ gives rise to a surjective map $\mathcal{F} \to j_{!*}(\mathcal{L}''[n])$. Let $\mathcal{K} \subset \mathcal{F}$ be its kernel. By Step 1, $\mathcal{K}$ admits a descent datum. By Step 4, ${}^p\mathsf{H}^0(i_*i^*\mathcal{K})$ also admits a descent datum. Since the latter object is supported on Z, by noetherian induction, every subobject and quotient of ${}^p\mathsf{H}^0(i_*i^*\mathcal{K})$ admits a descent datum.

It is easy to see that the composition $\mathcal{F}' \hookrightarrow \mathcal{F} \twoheadrightarrow j_{!*}(\mathcal{L}''[n])$ is zero (because its restriction to U is zero). In other words, $\mathcal{F}'$ is contained in $\mathcal{K}$. Let $\mathcal{K}' = \mathcal{K}/\mathcal{F}'$, so that we have a short exact sequence

$$0 \to \mathcal{F}' \to \mathcal{K} \to \mathcal{K}' \to 0.$$

By construction, we have $\mathcal{F}'|_U \cong \mathcal{K}|_U \cong \mathcal{L}'$. It follows that $\mathcal{K}'$ is supported on Z. By Lemma 3.3.7, $\mathcal{K}'$ is a quotient of ${}^p\mathsf{H}^0(i_*i^*\mathcal{K})$, so by the previous paragraph, it admits a descent datum. Finally, by Step 1, $\mathcal{F}'$ does as well.

Step 7. Suppose $\mathcal{F} \in \mathrm{Perv}(X, \Bbbk)$ *has no quotient supported on* Z, *and that* $\mathcal{F}|_U$ *is a shifted local system. If* $\mathcal{F}$ *admits a descent datum, then every subobject or quotient of* $\mathcal{F}$ *does as well.* Again, it is enough to consider the case of a subobject $\mathcal{F}'$. By Lemma 3.3.8, we have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & {}^p\mathsf{H}^0(i_*i^!\mathcal{F}') \arrow[r] \arrow[d] & \mathcal{F}' \arrow[r] \arrow[d] & j_{!*}(\mathcal{F}'|_U) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & {}^p\mathsf{H}^0(i_*i^!\mathcal{F}) \arrow[r] & \mathcal{F} \arrow[r] & j_{!*}(\mathcal{F}|_U) \arrow[r] & 0
\end{tikzcd}$$

Note that all three vertical maps are injective (for the rightmost vertical map, see Lemma 3.3.5). By Step 4, ${}^p\mathsf{H}^0(i_*i^!\mathcal{F})$ admits a descent datum, and then by induction, so does its subobject ${}^p\mathsf{H}^0(i_*i^!\mathcal{F}')$. By Steps 3 and 5, $j_{!*}(\mathcal{F}'|_U)$ admits a descent datum. Finally, by Step 2, $\mathcal{F}'$ admits a descent datum.

Step 8. Conclusion of the proof. Suppose $\mathcal{F} \in \mathrm{Perv}(X, \Bbbk)$ admits a descent datum so that $\mathcal{F} \cong f^\dagger \bar{\mathcal{F}}$ for some $\bar{\mathcal{F}} \in \mathrm{Perv}(Y, \Bbbk)$. Choose a smooth, connected open subset $V \subset Y$ such that $\bar{\mathcal{F}}|_V$ is a shifted local system, and then define the other spaces and maps in (3.7.8) accordingly. Let $\mathcal{K}$ be the kernel of the natural map $\mathcal{F} \to {}^p\mathsf{H}^0(i_*i^*\mathcal{F})$. Lemma 3.3.7 implies that the latter map is surjective and that $\mathcal{K}$ has no nonzero quotient supported on Z.

Let $\mathcal{F}' \subset \mathcal{F}$ be a subobject. Let $\mathcal{G}' = \mathcal{F}' \cap \mathcal{K}$, and let $\mathcal{G}'' = \mathcal{F}'/\mathcal{G}'$. We have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{G}' \arrow[r] \arrow[d] & \mathcal{F}' \arrow[r] \arrow[d] & \mathcal{G}'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{K} \arrow[r] & \mathcal{F} \arrow[r] & {}^p\mathsf{H}^0(i_*i^*\mathcal{F}) \arrow[r] & 0
\end{tikzcd}$$

By construction, all three vertical maps are injective. By Steps 1 and 4, $\mathcal{K}$ and ${}^p\mathsf{H}^0(i_*i^*\mathcal{F})$ admit descent data. By Step 7, $\mathcal{G}'$ admits a descent datum, and by noetherian induction, so does $\mathcal{G}''$. Finally, Step 2 tells us that $\mathcal{F}'$ admits a descent datum. Thus, all subobjects of $\mathcal{F}$ admit descent data, and by Step 1, so do all quotients. □

Exercises.

3.7.1. If we work at the level of derived categories rather than perverse sheaves, then the main results of this section are false in general.

(a) Give an example of a smooth surjective morphism $f : X \to Y$ with connected fibers such that $f^* : D^{\mathrm{b}}_{\mathrm{c}}(Y, \Bbbk) \to D^{\mathrm{b}}_{\mathrm{c}}(X, \Bbbk)$ is not fully faithful.

(b) Copy the definition of descent datum for objects in the derived category. Show that for any smooth surjective morphism $f : X \to Y$ and any $\mathcal{F} \in D^{\mathrm{b}}_{\mathrm{c}}(Y, \Bbbk)$, the object $f^*\mathcal{F}$ admits a canonical descent datum. Nevertheless, the resulting functor from $D^{\mathrm{b}}_{\mathrm{c}}(Y, \Bbbk)$ to descent data is *not* an equivalence of categories.

Both parts of this problem involve the same difficulty: roughly, the cohomology of the fibers of f can lead to "extra" morphisms in $D^{\mathrm{b}}_{\mathrm{c}}(X, \Bbbk)$ or in the category of descent data.

3.7.2 (Acyclic descent). There is a way around the difficulties of the previous exercise if we restrict to an appropriate class of morphisms. Let $n \geq 0$. A smooth morphism of varieties $f : X \to Y$ is said to be **n-acyclic** if for every smooth morphism $Y' \to Y$, the base change $f' : X \times_Y Y' \to Y'$ has the property that for every perverse sheaf $\mathcal{F} \in \mathrm{Perv}(Y', \Bbbk)$, the natural map

$$\mathcal{F} \to {}^p\tau^{\leq n} f'_*(f')^*\mathcal{F}$$

is an isomorphism.

(a) Show that an n-acyclic morphism is surjective and has connected fibers.
(b) Show that a composition of n-acyclic morphisms is n-acyclic.
(c) Let $k \in \mathbb{Z}$. Show that if $f : X \to Y$ is n-acyclic, then

$$f^\dagger : {}^pD^{\mathrm{b}}_{\mathrm{c}}(Y, \Bbbk)^{[k,k+n]} \to {}^pD^{\mathrm{b}}_{\mathrm{c}}(X, \Bbbk)^{[k,k+n]}$$

is fully faithful.
(d) Let $\mathrm{Desc}(f, \Bbbk)^{[k,k+n]}$ be the category of pairs $(\mathcal{F}, \phi)$ as in Definition 3.7.1, but with $\mathcal{F}$ allowed to be any object of ${}^pD^{\mathrm{b}}_{\mathrm{c}}(X, \Bbbk)^{[k,k+n]}$. Show that $f^\dagger$ induces an equivalence of categories

$$f^\dagger : {}^pD^{\mathrm{b}}_{\mathrm{c}}(Y, \Bbbk)^{[k,k+n]} \to \mathrm{Desc}(f, \Bbbk)^{[k,k+n]}.$$

Hint: The inverse functor should be ${}^p\tau^{\leq k+n} f_\dagger$ (cf. Remark 3.7.5).
(e) Show that if $f : X \to Y$ is ∞-acyclic, then for all $\mathcal{F} \in D^{\mathrm{b}}_{\mathrm{c}}(Y, \Bbbk)$, the adjunction maps $\mathcal{F} \to f_*f^*\mathcal{F}$ and $f_!f^!\mathcal{F} \to \mathcal{F}$ are both isomorphisms.

3.8. Semismall maps

This section contains t-exactness results for push-forward along maps satisfying certain "combinatorial" conditions on the dimensions of fibers. Compared to the results of the preceding three sections, this section is much more elementary. The conditions described in Definitions 3.8.1 and 3.8.8 hold in some important examples in Chapters 8 and 9.

DEFINITION 3.8.1. Let X be an irreducible variety. A morphism of varieties $f : X \to Y$ is said to be **semismall** if Y admits a stratification $(Y_t)_{t \in \mathscr{T}}$ such that for each stratum Y_t and each point $y \in Y_t \cap f(X)$, we have

$$\dim f^{-1}(y) \leq \tfrac{1}{2}(\dim X - \dim Y_t).$$

Note that there is no condition on points $y \in Y$ that are not in the image of f. The reader should be aware that many sources (e.g., [**40**]) require semismall maps to obey the additional condition that $\dim X = \dim Y$.

Before studying semismall maps in detail, we require a grade estimate for constructible sheaves. The following result thematically belongs to Section 2.7, but it is more convenient to prove it using the technology of the present chapter.

PROPOSITION 3.8.2. *Let X be a variety of dimension n, and let $\mathcal{F}$ be a constructible sheaf on X. For all k, we have* $\operatorname{grade} \mathbf{H}^k_{\mathrm{c}}(X, \mathcal{F}) \geq k - n - \operatorname{mdsupp} \mathcal{F}$.

PROOF. *Step 1. The case where X is smooth and connected, and $\mathcal{F}$ is a local system.* Let $m = \operatorname{grade} \mathcal{F}$, so $\operatorname{mdsupp} \mathcal{F} = n - m$. Then $\mathcal{F}[n-m]$ belongs to $\mathbb{D}({}^p D^{\mathrm{b}}_{\mathrm{locf}}(X, \Bbbk)^{\geq 0})$, or equivalently, $(\mathbb{D}\mathcal{F})[m-n] \in {}^p D^{\mathrm{b}}_{\mathrm{locf}}(X, \Bbbk)^{\geq 0}$. The morphism $\mathrm{a}_X : X \to \mathrm{pt}$ is smooth of relative dimension n, so by Lemma 3.6.2, the functor $\mathrm{a}_{X\dagger} = \mathrm{a}_{X*}[-n]$ is left t-exact for the perverse t-structure. In particular, we have $R\Gamma(\mathbb{D}\mathcal{F})[m-2n] \in D^{\mathrm{b}}(\Bbbk\text{-mod}^{\mathrm{fg}})^{\geq 0}$ or, equivalently,

$$\mathbb{D}(R\Gamma(\mathbb{D}\mathcal{F}))[2n-m] \cong R\Gamma_{\mathrm{c}}(\mathcal{F})[2n-m] \in \mathbb{D}(D^{\mathrm{b}}(\Bbbk\text{-mod}^{\mathrm{fg}})^{\geq 0}).$$

Therefore, by Proposition A.10.6, we have

$$\begin{aligned} \operatorname{grade} \mathbf{H}^k_{\mathrm{c}}(X, \mathcal{F}) &= \operatorname{grade} \mathbf{H}^{k-2n+m}_{\mathrm{c}}(X, \mathcal{F}[2n-m]) \\ &\geq k - 2n + m = k - n - \operatorname{mdsupp} \mathcal{F}. \end{aligned}$$

Step 2. The general case. We proceed by noetherian induction. Let $j : U \hookrightarrow X$ be the inclusion of a smooth, connected open subset of dimension n with the property that $\mathcal{F}|_U$ is a local system. Let $i : Z \hookrightarrow X$ be the inclusion of the complementary closed subset. Apply $R\Gamma_{\mathrm{c}}$ to the distinguished triangle $j_!(\mathcal{F}|_U) \to \mathcal{F} \to i_*(\mathcal{F}|_Z) \to$ to obtain the distinguished triangle

$$R\Gamma_{\mathrm{c}}(\mathcal{F}|_U) \to R\Gamma_{\mathrm{c}}(\mathcal{F}) \to R\Gamma_{\mathrm{c}}(\mathcal{F}|_Z) \to .$$

By Corollary A.10.12, the result follows from the following two observations:

$$\operatorname{grade} \mathbf{H}^k_{\mathrm{c}}(U, \mathcal{F}|_U) \geq k - n - \operatorname{mdsupp}(\mathcal{F}|_U) \geq k - n - \operatorname{mdsupp} \mathcal{F},$$
$$\operatorname{grade} \mathbf{H}^k_{\mathrm{c}}(Z, \mathcal{F}|_Z) \geq k - \dim Z - \operatorname{mdsupp}(\mathcal{F}|_Z) \geq k - n - \operatorname{mdsupp} \mathcal{F}.$$

Here, the first inequality holds by Step 1, and the second by induction. □

LEMMA 3.8.3. *Let $f : X \to Y$ be a semismall morphism, and assume that X is smooth and connected.*

1. *If $\mathcal{F} \in {}^p D^{\mathrm{b}}_{\mathrm{locf}}(X, \Bbbk)^{\leq 0}$, then $f_!\mathcal{F} \in {}^p D^{\mathrm{b}}_{\mathrm{c}}(Y, \Bbbk)^{\leq 0}$.*
2. *If $\mathcal{F} \in \mathbb{D}({}^p D^{\mathrm{b}}_{\mathrm{locf}}(X, \Bbbk)^{\geq 0})$, then $f_!\mathcal{F} \in \mathbb{D}({}^p D^{\mathrm{b}}_{\mathrm{c}}(Y, \Bbbk)^{\geq 0})$.*
3. *If $\mathcal{F} \in {}^p D^{\mathrm{b}}_{\mathrm{locf}}(X, \Bbbk)^{\geq 0}$, then $f_*\mathcal{F} \in {}^p D^{\mathrm{b}}_{\mathrm{c}}(Y, \Bbbk)^{\geq 0}$.*

PROOF. Part (3) follows from part (2) by Verdier duality. We will prove parts (1) and (2) simultaneously. By truncation and induction on the number of nonzero cohomology sheaves of $\mathcal{F}$, we can reduce to the case where $\mathcal{F}$ is concentrated in a single degree, say $\mathcal{F} = \mathcal{L}[n]$ for some local system $\mathcal{L}$. Let $m = \operatorname{grade} \mathcal{L}$. In case (1), we must have

$$\dim \operatorname{supp} \mathcal{L} = \dim X \leq n. \tag{3.8.1}$$

In case (2), we instead have

$$\operatorname{mdsupp} \mathcal{L} = \dim X - m \leq n. \tag{3.8.2}$$

Let $y \in Y_t$. By proper base change, we have

$$\mathsf{H}^i(f_!\mathcal{F})_y \cong \mathbf{H}^i_{\mathrm{c}}(f^{-1}(y), \mathcal{L}[n]|_{f^{-1}(y)}) \cong \mathbf{H}^{i+n}_{\mathrm{c}}(f^{-1}(y), \mathcal{L}|_{f^{-1}(y)}).$$

Using the bound on $\dim f^{-1}(y)$, Theorem 2.7.4 tells us that this vanishes unless $i + n \leq \dim X - \dim Y_t$. In other words, the support of $\mathsf{H}^i(f_!\mathcal{L}[n])$ is contained in the union of strata satisfying $\dim Y_t \leq -i + \dim X - n$. We deduce that

$$\dim \operatorname{supp} \mathsf{H}^i(f_!\mathcal{F}) \leq -i + \dim X - n.$$

In case (1), by (3.8.1), we have $\dim \operatorname{supp} \mathsf{H}^i(f_!\mathcal{F}) \le -i$, so $f_!\mathcal{F} \in {}^pD^{\mathrm{b}}_{\mathrm{c}}(Y,\Bbbk)^{\le 0}$, as desired.

Now suppose we are in case (2). By Proposition 3.8.2, for $y \in Y_t$, we have

$$\begin{aligned}\operatorname{grade} \mathsf{H}^i(f_!\mathcal{F})_y &= \operatorname{grade} \mathbf{H}^{i+n}_{\mathrm{c}}(f^{-1}(y), \mathcal{L}|_{f^{-1}(y)}) \\ &\ge i + n - \dim f^{-1}(y) - \operatorname{mdsupp}(\mathcal{L}|_{f^{-1}(y)}) \\ &= i + n - \dim f^{-1}(y) - (\dim f^{-1}(y) - m) \\ &\ge i + n - \dim X + \dim Y_t + m.\end{aligned}$$

It follows that

$$\begin{aligned}\operatorname{mdsupp} \mathsf{H}^i(f_!\mathcal{F})|_{Y_t} \le \dim Y_t - (i + n - \dim X + \dim Y_t + m) \\ = -i + \dim X - m - n.\end{aligned}$$

Since this bound holds for all strata Y_t, it is also a bound on $\operatorname{mdsupp} \mathsf{H}^i(f_!\mathcal{F})$. By (3.8.2), we conclude that $\operatorname{mdsupp} \mathsf{H}^i(f_!\mathcal{F}) \le -i$, so $f_!\mathcal{F} \in \mathbb{D}({}^pD^{\mathrm{b}}_{\mathrm{c}}(Y,\Bbbk)^{\ge 0})$. □

As an immediate consequence, we have the following result.

THEOREM 3.8.4. *Let $f : X \to Y$ be a proper, semismall morphism. Assume that X is smooth and connected. For any local system $\mathcal{L}$ of finite type on X, we have $f_*\mathcal{L}[\dim X] \in \operatorname{Perv}(Y,\Bbbk)$.*

The following variant of Definition 3.8.1 involves a strict inequality.

DEFINITION 3.8.5. Let X and Y be varieties, and let $W \subset Y$ be an open, dense subvariety. Assume that X is irreducible. A morphism of varieties $f : X \to Y$ is said to be **small** with respect to W if the following conditions hold:

(1) For each $y \in W$, $f^{-1}(y)$ is a finite set.
(2) There exists a stratification $(Y_t)_{t\in\mathscr{T}}$ of Y such that W is a union of strata and such that for each stratum $Y_t \subset Y \smallsetminus W$ and each point $y \in Y_t \cap f(X)$, we have
$$\dim f^{-1}(y) < \tfrac{1}{2}(\dim X - \dim Y_t).$$

LEMMA 3.8.6. *Let X be a smooth, connected variety, and let $f : X \to Y$ be a small morphism with respect to $W \subset Y$. Let $i : \overline{W} \smallsetminus W \hookrightarrow Y$ be the inclusion map.*

(1) *If $\mathcal{F} \in {}^pD^{\mathrm{b}}_{\mathrm{locf}}(X,\Bbbk)^{\le 0}$, then $i^*(f_!\mathcal{F}) \in {}^pD^{\mathrm{b}}_{\mathrm{c}}(\overline{W} \smallsetminus W,\Bbbk)^{\le -1}$.*
(2) *If $\mathcal{F} \in \mathbb{D}({}^pD^{\mathrm{b}}_{\mathrm{locf}}(X,\Bbbk)^{\ge 0})$, then $i^*(f_!\mathcal{F}) \in \mathbb{D}({}^pD^{\mathrm{b}}_{\mathrm{c}}(\overline{W} \smallsetminus W,\Bbbk)^{\ge 1})$.*
(3) *If $\mathcal{F} \in {}^pD^{\mathrm{b}}_{\mathrm{locf}}(X,\Bbbk)^{\ge 0}$, then $i^!(f_*\mathcal{F}) \in {}^pD^{\mathrm{b}}_{\mathrm{c}}(\overline{W} \smallsetminus W,\Bbbk)^{\ge 1}$.*

PROOF SKETCH. The proof is almost identical to that of Lemma 3.8.3. Using the strict inequality from Definition 3.8.5, one can show that in case (1), we have $\dim \operatorname{supp} \mathsf{H}^k(i^* f_!\mathcal{F}) < -k$, while in case (2), we have $\operatorname{mdsupp} \mathsf{H}^k(i^* f_!\mathcal{F}) < -k$. Part (3) follows by Verdier duality. □

PROPOSITION 3.8.7. *Let X be a smooth, connected variety, and let $f : X \to Y$ be a proper, small morphism with respect to $W \subset Y$. Let $f' = f|_{f^{-1}(W)} : f^{-1}(W) \to W$, and let $h : W \hookrightarrow Y$ be the inclusion map. Then, for any local system $\mathcal{L}$ of finite type on X, we have*

$$f_*\mathcal{L}[\dim X] \cong h_{!*}(f'_*\mathcal{L}|_{f^{-1}(W)}[\dim X]).$$

PROOF. Our assumptions imply that $f' : f^{-1}(W) \to W$ is a finite morphism, so the object

$$(f_*\mathcal{L}[\dim X])|_W \cong f'_*\mathcal{L}|_{f^{-1}(W)}[\dim W]$$

is perverse by Proposition 3.1.12. In view of parts (1) and (3) of Lemma 3.8.6, Lemma 3.1.5 tells us that $f_*\mathcal{L}[\dim X]$ is a perverse sheaf, and Lemma 3.3.4 tells us that $f_*\mathcal{L}[\dim X] \cong h_{!*}(f'_*\mathcal{L}|_{f^{-1}(W)}[\dim X])$, as desired. □

We conclude with a variant of Definitions 3.8.1 and 3.8.5 that is suited to maps $f : X \to Y$ where X is not necessarily smooth, or where one would like to consider perverse sheaves on X other than shifted local systems.

DEFINITION 3.8.8. Let X be a variety equipped with a good stratification $(X_s)_{s\in\mathscr{S}}$. A morphism of varieties $f : X \to Y$ is said to be **stratified semismall** if Y admits a stratification $(Y_t)_{t\in\mathscr{T}}$ such that for any $s \in \mathscr{S}$, any $t \in \mathscr{T}$, and any point $y \in Y_t \cap f(X_s)$,

$$\dim(f^{-1}(y) \cap X_s) \le \tfrac{1}{2}(\dim X_s - \dim Y_t).$$

Similarly, it is said to be **stratified small** with respect to a dense open subset $W \subset Y$ if the following conditions hold:

1. For each $y \in W$, $f^{-1}(y)$ is a finite set.
2. There exists a stratification $(Y_t)_{t\in\mathscr{T}}$ of Y such that W is a union of strata, and such that for any $s \in \mathscr{S}$, any $Y_t \subset Y \smallsetminus W$, and any point $y \in Y_t \cap f(X_s)$, we have

$$\dim(f^{-1}(y) \cap X_s) < \tfrac{1}{2}(\dim X_s - \dim Y_t).$$

In other words, $f : X \to Y$ is stratified semismall (resp. stratified small) if for every stratum X_s, the map $f|_{X_s} : X_s \to Y$ is semismall (resp. small).

THEOREM 3.8.9. *Let X be a variety with a good stratification $(X_s)_{s\in\mathscr{S}}$, and let $f : X \to Y$ be a proper, stratified semismall morphism. Then $f_* : D^{\mathrm{b}}_{\mathscr{S}}(X, \Bbbk) \to D^{\mathrm{b}}_{\mathrm{c}}(Y, \Bbbk)$ is t-exact for the perverse t-structure.*

PROOF. Let $j_s : X_s \hookrightarrow X$ be the inclusion map of a stratum. To show that f_* is right t-exact, using Exercise 3.1.2, it is enough to prove the following claim: for any local system $\mathcal{L}$ of finite type on a stratum X_s, and any $n \ge \dim X_s$, we have $f_*j_{s!}\mathcal{L}[n] \in {}^pD^{\mathrm{b}}_{\mathrm{c}}(Y, \Bbbk)^{\le 0}$. Definition 3.8.8 implies that $f \circ j_s : X_s \to Y$ is semismall. Since f is proper, we have $f_! = f_*$. We clearly have $\mathcal{L}[n] \in {}^pD^{\mathrm{b}}_{\mathrm{locf}}(X_s, \Bbbk)^{\le 0}$, so

$$f_*j_{s!}\mathcal{L}[n] \cong (f \circ j_s)_!\mathcal{L}[n] \in {}^pD^{\mathrm{b}}_{\mathrm{c}}(Y, \Bbbk)^{\le 0}$$

by Lemma 3.8.3. Similar reasoning (using $f_*j_{s*}\mathcal{L}[n]$ with $n \le \dim X_s$) shows that f_* is also left t-exact, so it is t-exact. □

The proof of the following proposition is very similar and is left as an exercise.

PROPOSITION 3.8.10. *Let X be a variety with a good stratification $(X_s)_{s\in\mathscr{S}}$, and let $f : X \to Y$ be a proper, stratified small morphism with respect to $W \subset Y$. Let $f' = f|_{f^{-1}(W)} : f^{-1}(W) \to W$, and let $h : W \hookrightarrow Y$ be the inclusion map. Then, for any perverse sheaf $\mathcal{F} \in \mathrm{Perv}_{\mathscr{S}}(X, \Bbbk)$, there is a natural isomorphism*

$$f_*\mathcal{F} \cong h_{!*}f'_*(\mathcal{F}|_{f^{-1}(W)}).$$

Exercises.

3.8.1. Let $f : X \to Y$ be a proper morphism of varieties, where X is irreducible. Show that f is semismall if and only if $\dim X \times_Y X \le \dim X$. *Hint*: It may be helpful to consider the fiber dimension theorem, as in [**97**, Exercise II.3.22(c)] or [**210**, Theorem 1.25].

3.9. The decomposition theorem and the hard Lefschetz theorem

In this section, we discuss two major and closely related theorems that hold when $\Bbbk$ is a field of characteristic 0. In the statements and discussion below, we assume that $\Bbbk = \mathbb{Q}$. See Exercise 3.9.1 for an extension to number fields, and see [**68**, Sections 1.7 and 2.5] for a discussion of how to extend these results to arbitrary fields of characteristic 0.

Definition 3.9.1. Let X be a variety. An object $\mathcal{F} \in D^{\mathrm{b}}_{\mathrm{c}}(X, \mathbb{Q})$ is said to be a **semisimple complex** if it is isomorphic to a (finite) direct sum of shifts of simple perverse sheaves. The additive category of semisimple complexes is denoted by $\mathrm{Semis}(X, \mathbb{Q})$.

Let us now state the two theorems of this section. The first result, which plays a crucial role in many applications of perverse sheaves, is the following.

Theorem 3.9.2 (Decomposition theorem). *Let $f : X \to Y$ be a proper morphism of varieties. For any $\mathcal{F} \in \mathrm{Semis}(X, \mathbb{Q})$, we have $f_*\mathcal{F} \in \mathrm{Semis}(Y, \mathbb{Q})$.*

As an important special case, if X is smooth, then the constant sheaf $\underline{\mathbb{Q}}_X$ is a semisimple complex, so $f_*\underline{\mathbb{Q}}_X$ is a semisimple complex. If, in addition, f is semismall, then $f_*\underline{\mathbb{Q}}_X[\dim X]$ is a semisimple perverse sheaf.

The second result deals with projective morphisms. Recall that a morphism of varieties $f : X \to Y$ is **projective** if it factors as

$$X \xrightarrow{\lambda} Y \times \mathbb{P}(V) \xrightarrow{p} Y \tag{3.9.1}$$

where λ is a closed embedding, p is the projection map, and V is some finite-dimensional complex vector space. Over $\mathbb{P}(V)$, one can consider the **hyperplane bundle** $E_1^V \to \mathbb{P}(V)$, defined as in Exercise 2.11.3.

Theorem 3.9.3 (Relative hard Lefschetz theorem). *Let $f : X \to Y$ be a projective morphism of varieties. Choose a factorization $f = p \circ \lambda$ as in* (3.9.1), *and let $e_{Y \times E_1^V} \in \mathbf{H}^2(Y \times \mathbb{P}(V); \mathbb{Q})(1)$ be the Euler class of the hyperplane bundle $Y \times E_1^V \to Y \times \mathbb{P}(V)$. Let $\mathcal{F} \in \mathrm{Perv}(X, \mathbb{Q})$ be a semisimple perverse sheaf, and let*

$$\eta = \eta_{\mathcal{F}} = f_*(\lambda^\sharp(e_{Y \times E_1^V}) \overset{L}{\otimes} \mathrm{id}_{\mathcal{F}}) : f_*\mathcal{F} \to f_*\mathcal{F}[2](1).$$

Then, for any $k \ge 0$, the map

$${}^p\mathsf{H}^{-k}(\eta^k) : {}^p\mathsf{H}^{-k}(f_*\mathcal{F}) \to {}^p\mathsf{H}^k(f_*\mathcal{F})(k)$$

is an isomorphism.

Complete proofs of these theorems are beyond the scope of this book. However, we will discuss some aspects of the proof of Theorem 3.9.2 in Section 5.7, and we will explain how to deduce Theorem 3.9.3 from Theorem 3.9.2 later in this section.

Here are some comments on the history of these results. Both theorems were first proved by Beilinson–Bernstein–Deligne–Gabber [**24**] by transferring the problem to the setting of varieties over a finite field. This transfer does not go through for arbitrary semisimple complexes; instead, in [**24**], these results are stated only when $\mathcal{F}$ is "of geometric origin."

The second proof of these theorems was obtained by Saito [**199**, **201**] as part of his theory of mixed Hodge modules. Again, this framework does not allow arbitrary semisimple complexes; one must require the simple summands of $\mathcal{F}$ to come from "polarizable variations of Hodge structures." For further discussion of these two approaches, see Section 5.7.

A third proof, focused on the case where X is irreducible and $\mathcal{F} = \mathrm{IC}(X;\mathbb{Q})$, is due to de Cataldo–Migliorini [**55**, **56**]. This proof is much more elementary than those of [**24**, **199**], as it relies only on classical Hodge theory and algebraic geometry, and not on any "mixed" technology.

Finally, the most general version of these statements—with $\mathcal{F}$ an arbitrary semisimple complex—was explicitly conjectured by Kashiwara [**123**]. This conjecture has now been proved: there is an arithmetic approach due to Drinfeld [**68**] (based on [**29**, **80**]), as well as an analytic approach, as part of the theory of "twistor $\mathscr{D}$-modules" of Sabbah [**198**] and Mochizuki [**180**, **181**]. See also [**54**].

Proof of the relative Hard Lefschetz theorem. The remainder of this section is devoted to showing (following [**24**]) how to deduce Theorem 3.9.3 from (part of) Theorem 3.9.2. More precisely, if $f : X \to Y$ is a proper map and $\mathcal{F} \in \mathrm{Semis}(X, \mathbb{Q})$, then the conclusion of the decomposition theorem can be broken up into two parts:

$$\text{Each perverse cohomology sheaf } {}^p\mathsf{H}^i(f_*\mathcal{F}) \text{ is semisimple.} \tag{3.9.2}$$

$$\text{We have } f_*\mathcal{F} \cong \bigoplus_{i\in\mathbb{Z}} {}^p\mathsf{H}^i(f_*\mathcal{F})[-i]. \tag{3.9.3}$$

The argument below will only use (3.9.2). (See Exercise 3.9.2.)

Before beginning the proof, we need some additional notation and two lemmas. Given a projective morphism $f : X \to Y$, fix a factorization as in (3.9.1). Let V^* be the dual vector space to V, and consider the projective space $\mathbb{P}(V^*)$. Let $d = \dim V - 1 = \dim \mathbb{P}(V)$. Define two subvarieties of $\mathbb{P}(V) \times \mathbb{P}(V^*)$ as follows:

$$F_V = \{(L, L') \in \mathbb{P}(V) \times \mathbb{P}(V^*) \mid L' \text{ annihilates } L\},$$
$$S_V = (\mathbb{P}(V) \times \mathbb{P}(V^*)) \smallsetminus F_V.$$

Note that F_V is closed in $\mathbb{P}(V) \times \mathbb{P}(V^*)$ and that S_V is open. Moreover, F_V is a smooth variety of dimension $2d-1$, so that $(F_V, \mathbb{P}(V) \times \mathbb{P}(V^*))$ is a smooth pair of codimension 1.

LEMMA 3.9.4. *The projection maps* $\mathrm{pr}_1 : S_V \to \mathbb{P}(V)$ *and* $\mathrm{pr}_2 : S_V \to \mathbb{P}(V^*)$ *are both affine morphisms.*

PROOF. We will prove the lemma for pr_1; the proof for pr_2 is similar. Choose a hyperplane $H \subset V$ and a (nonzero) vector $v_0 \in V \smallsetminus H$. Let $U_H = \mathbb{P}(V) \smallsetminus \mathbb{P}(H)$. This is an affine open subset of $\mathbb{P}(V)$; indeed, there is an isomorphism

$$H \xrightarrow{\sim} U_H \qquad \text{given by} \qquad v \mapsto \mathrm{span}\{v + v_0\}.$$

Next, let $H^\vee \subset V^*$ be the hyperplane that annihilates v_0, and let $\lambda_0 \in V^*$ be the element such that $\lambda_0|_H = 0$ and $\lambda_0(v_0) = 1$. Define a map

$$g : H \times H^\vee \to S_V \qquad \text{by} \qquad (v, \lambda) \mapsto (\mathrm{span}\{v + v_0\}, \mathrm{span}\{\lambda + (1 - \lambda(v))\lambda_0\}).$$

It is left to the reader to check that g is an open embedding and that it induces an isomorphism $H \times H^\vee \xrightarrow{\sim} \mathrm{pr}_1^{-1}(U_H)$. In particular, $\mathrm{pr}_1^{-1}(U_H)$ is affine. Since open sets of the form U_H cover $\mathbb{P}(V)$, this shows that pr_1 is affine. □

LEMMA 3.9.5. *Let $q_1 = \mathrm{id}_Y \times \mathrm{pr}_1 : Y \times S_V \to Y \times \mathbb{P}(V)$. For any $\mathcal{F} \in D^{\mathrm{b}}_{\mathrm{c}}(Y \times \mathbb{P}(V), \mathbb{Q})$, the adjunction maps $\mathcal{F} \to q_{1*}q_1^*\mathcal{F}$ and $q_{1!}q_1^!\mathcal{F} \to \mathcal{F}$ are isomorphisms.*

We will give a direct proof of this lemma. It can also be deduced from the claim that q_1 is an "∞-acyclic map"; see Exercise 3.7.2(e) and Definition 6.1.18.

PROOF. Retain the notation from the proof of Lemma 3.9.4. It is enough to show that the adjunction maps become isomorphisms over the open set $Y \times U_H$. Since $\mathrm{pr}_1^{-1}(U_H) \cong H \times H^\vee$, the map $q^{-1}(Y \times U_H) \to Y \times U_H$ can be identified with the projection map $Y \times H \times H^\vee \to Y \times H$. It follows that $(q_{1*}q_1^*\mathcal{F})|_{Y \times U_H} \cong \mathcal{F}|_{Y \times U_H} \boxtimes R\Gamma(\underline{\mathbb{Q}}_{H^\vee}) \cong \mathcal{F}|_{Y \times U_H}$, and similarly for $q_{1!}q_1^!\mathcal{F}$. □

We are now ready for the main proof of this section.

PROOF OF THEOREM 3.9.3. We can identify $\lambda_*(\lambda^\sharp(e_{Y \times E_1^V}) \otimes^L \mathrm{id}_{\mathcal{F}}) : \lambda_*\mathcal{F} \to \lambda_*\mathcal{F}[2](1)$ with $e_{Y \times E_1^V} \otimes^L \mathrm{id}_{\lambda_*\mathcal{F}}$ by the projection formula. Thus, we may as well replace X by $Y \times \mathbb{P}(V)$ and $\mathcal{F}$ by $\lambda_*\mathcal{F}$. For the rest of the proof, we assume that $X = Y \times \mathbb{P}(V)$ and that $f : Y \times \mathbb{P}(V) \to Y$ is the projection map. Much of the proof deals with the following diagram:

$$\begin{tikzcd}
Y \times F_V \arrow[r, "i"] & Y \times \mathbb{P}(V) \times \mathbb{P}(V^*) \arrow[dl, "p'"'] \arrow[dr, "u'"] & Y \times S_V \arrow[l, "j"'] \arrow[d, "q_1 = u' \circ j"] \arrow[dll, "q_2 = p' \circ j"] \\
Y \times \mathbb{P}(V^*) \arrow[dr, "u"'] & & X = Y \times \mathbb{P}(V) \arrow[dl, "f = p"] \\
& Y &
\end{tikzcd} \tag{3.9.4}$$

Here, i and j are inclusion maps, and u, u', p, and p' are projection maps. The latter four are all proper and smooth of relative dimension d. Moreover, $u' \circ i : Y \times F_V \to Y \times \mathbb{P}(V)$ is smooth of relative dimension $d - 1$, and $(Y \times F_V, Y \times \mathbb{P}(V) \times \mathbb{P}(V^*))$ is a smooth pair of codimension 1. Finally, let $\mathcal{F}' = (u')^\dagger\mathcal{F}$, so that by smooth base change, we have a natural isomorphism

$$u^\dagger p_*\mathcal{F} \cong p'_*\mathcal{F}'. \tag{3.9.5}$$

We now proceed as follows.

Step 1. Study of the Euler class on $Y \times \mathbb{P}(V) \times \mathbb{P}(V^*)$. For any object $\mathcal{G} \in D^{\mathrm{b}}_{\mathrm{c}}(Y \times \mathbb{P}(V) \times \mathbb{P}(V^*), \mathbb{Q})$, we can consider the map

$$\eta_{\mathcal{G}} = p'_*(e_{Y \times E_1^V \times \mathbb{P}(V^*)} \overset{L}{\otimes} \mathrm{id}_{\mathcal{G}}) : p'_*\mathcal{G} \to p'_*\mathcal{G}[2](1)$$

in $D^{\mathrm{b}}_{\mathrm{c}}(Y \times \mathbb{P}(V^*), \mathbb{Q})$. On the other hand, one can also consider the cycle class

$$c = \mathrm{cl}_{Y \times \mathbb{P}(V) \times \mathbb{P}(V^*)}(Y \times F_V) \in \mathbf{H}^2(Y \times \mathbb{P}(V) \times \mathbb{P}(V^*); \mathbb{Q})(1).$$

For any $\mathcal{G} \in D^{\mathrm{b}}_{\mathrm{c}}(Y \times \mathbb{P}(V) \times \mathbb{P}(V^*), \mathbb{Q})$, we set

$$\theta_{\mathcal{G}} = p'_*(c \overset{L}{\otimes} \mathrm{id}_{\mathcal{G}}) : p'_*\mathcal{G} \to p'_*\mathcal{G}[2](1).$$

We claim that $\eta_{\mathcal{G}}$ and $\theta_{\mathcal{G}}$ induce equal maps in perverse cohomology; that is,

$$({\rm 3.9.6}) \qquad {}^p\mathsf{H}^i(\eta_{\mathcal{G}}) = {}^p\mathsf{H}^i(\theta_{\mathcal{G}}) : {}^p\mathsf{H}^i(p'_*\mathcal{G}) \to {}^p\mathsf{H}^{i+2}(p'_*\mathcal{G})(1).$$

Let $V^\circ = V^* \smallsetminus \{0\}$, and let

$$\tilde{F}_V = \{(L, \lambda) \in \mathbb{P}(V) \times V^\circ \mid \lambda|_L = 0\}.$$

We have obvious quotient maps $V^\circ \to \mathbb{P}(V^*)$ and $\tilde{F}_V \to F_V$. Let $\mu : V \times V^\circ \to V \times \mathbb{A}^1 \times \mathbb{P}(V^*)$ be the map given by $\mu(v, \lambda) = (v, \lambda(v), \mathrm{span}\{\lambda\})$. This induces a map $\bar{\mu} : \mathbb{P}(V) \times V^\circ \to E^V_1 \times \mathbb{P}(V^*)$. Using these maps, we construct the following three cartesian squares:

$$({\rm 3.9.7}) \qquad \begin{tikzcd}
Y \times F_V \arrow[d] & Y \times \tilde{F}_V \arrow[l] \arrow[r] \arrow[d] & Y \times \mathbb{P}(V) \times \mathbb{P}(V^*) \arrow[d] \\
Y \times \mathbb{P}(V) \times \mathbb{P}(V^*) \arrow[d, "p'"'] & Y \times \mathbb{P}(V) \times V^\circ \arrow[l, "v'"'] \arrow[r, "\mathrm{id}_Y \times \bar{\mu}"] \arrow[d, "p''"] & Y \times E^V_1 \times \mathbb{P}(V^*) \\
Y \times \mathbb{P}(V^*) & Y \times V^\circ \arrow[l, "v"']
\end{tikzcd}$$

Let $\tilde{c} = \mathrm{cl}_{Y \times \mathbb{P}(V) \times V^\circ}(Y \times \tilde{F}_V)$. By Lemma 2.11.17 applied to the upper left-hand cartesian square above, we have $(v')^\sharp(c) = \tilde{c}$. By smooth base change for the lower left-hand square, we have

$$v^\dagger \theta_{\mathcal{G}} = p''_*(\tilde{c} \overset{L}{\otimes} \mathrm{id}_{(v')^\dagger \mathcal{G}}).$$

On the other hand, from the right-hand square in (3.9.7), a minor variation on Exercise 2.11.3 shows that $e_{Y \times E^V_1 \times V^\circ} = \tilde{c}$. So by smooth base change again, we have

$$v^\dagger \eta_{\mathcal{G}} = p''_*(\tilde{c} \overset{L}{\otimes} \mathrm{id}_{(v')^\dagger \mathcal{G}}).$$

Thus, $v^\dagger \eta_{\mathcal{G}} = v^\dagger \theta_{\mathcal{G}}$, and hence ${}^p\mathsf{H}^i(v^\dagger \eta_{\mathcal{G}}) = {}^p\mathsf{H}^i(v^\dagger \theta_{\mathcal{G}})$. Since $v^\dagger$ is t-exact and fully faithful on perverse sheaves, we deduce that (3.9.6) holds.

Step 2. Study of $\theta_{\mathcal{F}'}$ and $\theta_{i_*i^*\mathcal{F}'}$. Consider the diagram

$$({\rm 3.9.8}) \qquad \begin{tikzcd}
\mathcal{F}' \arrow[rr, "\mathrm{adj.}"] \arrow[dd, "c \otimes^L \mathrm{id}_{\mathcal{F}'}"'] & & i_*i^*\mathcal{F}' \arrow[dl, "\mathrm{Thm.\ 2.2.13}"', "\sim"] \arrow[dd, "c \otimes^L \mathrm{id}_{i_*i^*\mathcal{F}'}"] \\
& i_*i^!\mathcal{F}'[2](1) \arrow[dl, "\mathrm{adj.}"'] & \\
\mathcal{F}'[2](1) \arrow[rr, "\mathrm{adj.}"] & & i_*i^*\mathcal{F}'[2](1)
\end{tikzcd}$$

Here, the arrows labelled "adj." are adjunction maps. The outer square obviously commutes, and the upper-left triangle commutes by the definition of c (see Definition 2.11.16). Therefore, the lower-right triangle at least commutes after composing with the top horizontal arrow. But the map

$$\mathrm{Hom}(i_*i^*\mathcal{F}', i_*i^*\mathcal{F}'[2](1)) \xrightarrow{\sim} \mathrm{Hom}(\mathcal{F}', i_*i^*\mathcal{F}'[2](1))$$

induced by $\mathcal{F}' \to i_*i^*\mathcal{F}'$ is an (adjunction) isomorphism. We conclude that the lower-right triangle of (3.9.8) commutes. Define α and β to be the maps indicated in the diagram below, obtained by applying p'_* to (3.9.8):

$$\begin{tikzcd} p'_*\mathcal{F}' \arrow[r, "\alpha"] \arrow[d, "\theta_{\mathcal{F}'}"'] & p'_*i_*i^*\mathcal{F}' \arrow[d, "\theta_{i_*i^*\mathcal{F}'}"] \arrow[dl, "\beta"'] \\ p'_*\mathcal{F}'[2](1) \arrow[r, "\alpha[2](1)"] & p'_*i_*i^*\mathcal{F}'[2](1) \end{tikzcd} \tag{3.9.9}$$

Step 3. The map ${}^p\mathsf{H}^{-1}(\alpha) : {}^p\mathsf{H}^{-1}(p'_*\mathcal{F}') \to {}^p\mathsf{H}^{-1}(p'_*i_*i^*\mathcal{F}')$ *is injective, and it identifies* ${}^p\mathsf{H}^{-1}(p'_*\mathcal{F}')$ *with the largest subobject of* ${}^p\mathsf{H}^{-1}(p'_*i_*i^*\mathcal{F}')$ *that lies in the essential image of* $u^\dagger$. Apply $p'_! = p'_*$ to the distinguished triangle $j_!j^*\mathcal{F}' \to \mathcal{F}' \to i_*i^*\mathcal{F}' \to$ to obtain

$$q_{2!}(j^*\mathcal{F}') \to p'_*\mathcal{F}' \xrightarrow{\alpha} p'_*i_*i^*\mathcal{F} \to .$$

Since q_2 is affine (Lemma 3.9.4) and $j^*\mathcal{F}'$ is perverse, Theorem 3.5.8 tells us that

$$q_{2!}(j^*\mathcal{F}') \in {}^pD^{\mathrm{b}}_{\mathrm{c}}(Y \times \mathbb{P}(V^*), \mathbb{Q})^{\geq 0}. \tag{3.9.10}$$

The long exact sequence in perverse cohomology then shows that

$$ {}^p\mathsf{H}^{-k}(\alpha) : {}^p\mathsf{H}^{-k}(p'_*\mathcal{F}') \to {}^p\mathsf{H}^{-k}(p'_*i_*i^*\mathcal{F}') \quad \text{is} \quad \begin{cases} \text{injective} & \text{for } k = 1, \\ \text{an isomorphism} & \text{for } k > 1. \end{cases} \tag{3.9.11}$$

It is immediate from (3.9.5) that ${}^p\mathsf{H}^{-1}(p'_*\mathcal{F}')$ lies in the image of $u^\dagger$. It remains to show that the cokernel of ${}^p\mathsf{H}^{-1}(\alpha)$ has no nonzero subobject in the image of $u^\dagger$. We will instead show that ${}^p\mathsf{H}^0(q_{2!}j^*\mathcal{F}')$ has no such subobject. Let $\mathcal{H} \in \mathrm{Perv}(Y, \mathbb{Q})$. Using (3.9.10), we have

$$\begin{aligned} \mathrm{Hom}(u^\dagger\mathcal{H}, {}^p\mathsf{H}^0(q_{2!}j^*\mathcal{F}')) &\cong \mathrm{Hom}(u^\dagger\mathcal{H}, {}^p\tau^{\leq 0}(q_{2!}j^*\mathcal{F}')) \cong \mathrm{Hom}(u^\dagger\mathcal{H}, q_{2!}j^*\mathcal{F}') \\ &\cong \mathrm{Hom}(\mathcal{H}, u_\dagger p'_*j_!j^*\mathcal{F}') \cong \mathrm{Hom}(\mathcal{H}, p_\dagger u'_*j_!j^*(u')^*\mathcal{F}) \\ &\cong \mathrm{Hom}(p^\dagger\mathcal{H}, q_{1!}q_1^!\mathcal{F}[-2d](-d)) \cong \mathrm{Hom}(p^\dagger\mathcal{H}, \mathcal{F}[-2d](-d)), \end{aligned}$$

where the last isomorphism holds by Lemma 3.9.5. Since $p^\dagger\mathcal{H}$ and $\mathcal{F}$ are both perverse, we have $\mathrm{Hom}(p^\dagger\mathcal{H}, \mathcal{F}[-2d](-d)) = 0$, so we are done.

Step 4. The map ${}^p\mathsf{H}^{-1}(\beta) : {}^p\mathsf{H}^{-1}(p'_*i_*i^*\mathcal{F}') \to {}^p\mathsf{H}^1(p'_*\mathcal{F}')(1)$ *is surjective, and it identifies* ${}^p\mathsf{H}^1(p'_*\mathcal{F}')(1)$ *with the largest quotient of* ${}^p\mathsf{H}^{-1}(p'_*i_*i^*\mathcal{F}')$ *that lies in the essential image of* $u^\dagger$. This is very similar to Step 3, using the distinguished triangle $i_*i^!\mathcal{F}' \to \mathcal{F}' \to j_*j^*\mathcal{F}' \to$. Along the way, one shows that

$$ {}^p\mathsf{H}^{k}(\beta) : {}^p\mathsf{H}^{k}(p'_*i_*i^*\mathcal{F}') \to {}^p\mathsf{H}^{k+2}(p'_*\mathcal{F}')(1) \quad \text{is} \quad \begin{cases} \text{surjective} & \text{for } k = -1, \\ \text{an isomorphism} & \text{for } k > -1. \end{cases} \tag{3.9.12}$$

Step 5. Conclusion of the proof. By Lemma 2.11.19 and smooth base change for the square in (3.9.4), the map $u^\dagger\eta_{\mathcal{F}}$ is identified with $\eta_{\mathcal{F}'} : p'_*\mathcal{F}' \to p'_*\mathcal{F}'[2](1)$. Since $u^\dagger$ is fully faithful on perverse sheaves, the theorem is equivalent to the claim that

$$ {}^p\mathsf{H}^{-k}(\eta^k_{\mathcal{F}'}) : {}^p\mathsf{H}^{-k}(p'_*\mathcal{F}') \to {}^p\mathsf{H}^{k}(p'_*\mathcal{F}')(k)$$

is an isomorphism. By (3.9.6), this is in turn equivalent to the claim that

$$ {}^p\mathsf{H}^{-k}(\theta^k_{\mathcal{F}'}) : {}^p\mathsf{H}^{-k}(p'_*\mathcal{F}') \to {}^p\mathsf{H}^{k}(p'_*\mathcal{F}')(k) \tag{3.9.13}$$

is an isomorphism. Next, observe that since $u' \circ i$ is smooth, the object $i^*\mathcal{F}'[-1] \cong (u' \circ i)^\dagger \mathcal{F}$ is a semisimple perverse sheaf, and hence so is the object

$$i_* i^* \mathcal{F}'[-1] \in \mathrm{Perv}(Y \times \mathbb{P}(V) \times \mathbb{P}(V^*), \mathbb{Q}). \tag{3.9.14}$$

We will prove that (3.9.13) is an isomorphism by induction on k. The case $k = 0$ is trivial.

Now suppose that $k = 1$. Apply the part of the decomposition theorem stated in (3.9.2) to (3.9.14) to conclude that ${}^p\mathsf{H}^{-1}(p'_* i_* i^* \mathcal{F})$ is a semisimple perverse sheaf. In particular, the largest subobject of ${}^p\mathsf{H}^{-1}(p'_* i_* i^* \mathcal{F})$ that lies in the image of $u^\dagger$ is in fact a direct summand, and it is identified with the largest quotient that lies in the image of $u^\dagger$. Since $\theta_{\mathcal{F}'} = \beta \circ \alpha$ (see (3.9.9)), we see from Steps 3 and 4 that ${}^p\mathsf{H}^{-1}(\theta_{\mathcal{F}'})$ is an isomorphism.

Finally, suppose $k > 1$. We expand $\theta^k_{\mathcal{F}'}$ using (3.9.9) as follows (for brevity, we omit Tate twists):

$$\theta^k_{\mathcal{F}'} = \beta[2k-2] \circ \alpha[2k-2] \cdots \circ \beta[2] \circ \alpha[2] \circ \beta \circ \alpha = \beta[2k-2] \circ \theta^{k-1}_{i_* i^* \mathcal{F}'} \circ \alpha.$$

Therefore,

$$\begin{aligned} {}^p\mathsf{H}^{-k}(\theta^k_{\mathcal{F}'}) &= {}^p\mathsf{H}^{k-2}(\beta) \circ {}^p\mathsf{H}^{-k}(\theta^{k-1}_{i_* i^* \mathcal{F}'}) \circ {}^p\mathsf{H}^{-k}(\alpha) \\ &= {}^p\mathsf{H}^{k-2}(\beta) \circ {}^p\mathsf{H}^{-k+1}(\eta^{k-1}_{i_* i^* \mathcal{F}'[-1]}) \circ {}^p\mathsf{H}^{-k}(\alpha), \end{aligned}$$

where the second equality comes from (3.9.6). The maps ${}^p\mathsf{H}^{-k}(\alpha)$ and ${}^p\mathsf{H}^{k-2}(\beta)$ are isomorphisms by (3.9.11) and (3.9.12), respectively, while ${}^p\mathsf{H}^{k-2}(\eta^{k-1}_{i_* i^* \mathcal{F}'[-1]})$ is an isomorphism by induction. □

Exercises.

3.9.1. Let $\Bbbk'$ be a finite extension of $\mathbb{Q}$. In this exercise, you may use the following claim without proof: *For a smooth, connected variety X, a local system $\mathcal{L} \in \mathrm{Loc}^{\mathrm{ft}}(X, \mathbb{Q})$ is semisimple if and only if $\Bbbk' \otimes^L_{\mathbb{Q}} \mathcal{L} \in \mathrm{Loc}^{\mathrm{ft}}(X, \Bbbk')$ is semisimple.* Show that Theorems 3.9.2 and 3.9.3 imply the analogous statements for semisimple complexes with coefficients in $\Bbbk'$.

Note: The claim above is a consequence of the following more general fact:

> Let G be a group, let $\Bbbk \subset \Bbbk'$ be a separable algebraic field extension, and let M be a finite-dimensional $\Bbbk[G]$-module. Then M is semisimple if and only if $\Bbbk' \otimes_\Bbbk M$ is a semisimple $\Bbbk'[G]$-module.

Here is a sketch of the proof of this fact. Let R be the image of the ring homomorphism $\Bbbk[G] \to \mathrm{End}_\Bbbk(M)$. It is enough to show that M is a semisimple R-module if and only if $\Bbbk' \otimes_\Bbbk M$ is a semisimple $\Bbbk' \otimes_\Bbbk R$-module. Since R is an artinian ring, M is semisimple if and only if its Jacobson radical $J(R)$ acts by 0 on M. Similarly, $\Bbbk' \otimes_\Bbbk M$ is semisimple if and only if $J(\Bbbk' \otimes_\Bbbk R)$ acts by 0. Finally, according to [**144**, Theorem 5.17], $J(\Bbbk' \otimes_\Bbbk R)$ can be identified with $\Bbbk' \otimes_\Bbbk J(R)$.

3.9.2. Let $f : X \to Y$ be a projective morphism. Assuming (3.9.2) and the hard Lefschetz theorem, prove (3.9.3).

3.10. Additional notes and exercises

Notes. The notion of intersection (co)homology predates the theory of perverse sheaves by several years, having been introduced by Goresky–MacPherson in [**85**, **86**]. The perverse t-structure was first defined in the groundbreaking monograph [**24**] of Beilinson–Bernstein–Deligne–Gabber. That monograph is also where most of the main results of this chapter first appeared, including those of Sections 3.5–3.7 and Section 3.9. (However, the proofs presented in this chapter are sometimes quite different from those in [**24**]: the latter are generally adapted to the étale setting, and sometimes restricted to the case where $\Bbbk = \overline{\mathbb{Q}}_\ell$.) The notion of a semismall map was introduced by Borho–MacPherson [**40**]. The notion of a stratified semismall map was known implicitly for quite some time (at least going back to [**159**]), but seems not to have been written down explicitly until [**178**]. For a comprehensive history of the development of perverse sheaves, see [**138**].

Exercise 3.10.1. (**Deligne's construction**) Let X be a variety, and let $\Bbbk$ be a field. Let $U \subset X$ be a smooth, connected open subset, and let $\mathcal{L}$ be a local system of finite type on U. Choose a stratification with respect to which $\mathrm{IC}(U, \mathcal{L})$ is constructible, and label the strata as $X_1, X_2, \ldots, X_k$ in such a way that $X_1 = U$, and $X_1 \cup X_2 \cup \cdots \cup X_i$ is open for all i. Let $U_i = X_1 \cup \cdots \cup X_i$, and let $j_i : U_i \hookrightarrow U_{i+1}$ be the inclusion map, so that we have a sequence of open embeddings

$$U = U_1 \xrightarrow{j_1} U_2 \xrightarrow{j_2} \cdots \xrightarrow{j_{k-1}} U_k = X.$$

Show that

$$\mathrm{IC}(U, \mathcal{L}) \cong \tau^{\le -\dim X_k - 1} j_{k-1,*} \cdots \tau^{\le -\dim X_3 - 1} j_{2*} \tau^{\le -\dim X_2 - 1}(j_{1*}\mathcal{L}[\dim U]).$$

Exercise 3.10.2. Show that if X is an irreducible variety, then

$$\tau^{\le -\dim X} \mathrm{IC}(X; \Bbbk) \cong \underline{\Bbbk}_X[\dim X].$$

Deduce that there is a natural map $\mathbf{H}^\bullet(X; \Bbbk) \to \mathbf{IH}^\bullet(X; \Bbbk)$.

Exercise 3.10.3. Let X be an irreducible variety, and let $\Bbbk$ be a field. Show that the following three conditions are equivalent:

(a) We have $\omega_X \cong \underline{\Bbbk}_X[2 \dim X](\dim X)$.
(b) We have $\mathrm{IC}(X; \Bbbk) \cong \underline{\Bbbk}_X[\dim X]$.
(c) For all $x \in X$, we have

$$\dim \mathsf{H}^i(\mathrm{IC}(X; \Bbbk)_x) = \begin{cases} 1 & \text{if } i = -\dim X, \\ 0 & \text{otherwise.} \end{cases}$$

Then show that if these conditions hold for one field, then they hold for all other fields of the same characteristic.

If the conditions above hold for $\Bbbk = \mathbb{Q}$, the variety X is said to be **rationally smooth** or a **rational homology manifold**. If they hold for $\Bbbk$ of characteristic p, then X is said to be p-**smooth**.

Exercise 3.10.4. Let $U = \mathbb{A}^1 \smallsetminus \{0\}$, and let $Z = \{0\}$. Let $j : U \hookrightarrow \mathbb{A}^1$ and $i : Z \hookrightarrow \mathbb{A}^1$ be the inclusion maps. Let $\mathscr{S}$ denote the stratification $\mathbb{A}^1 = U \cup Z$. Assume throughout that $\Bbbk$ is a field.

(a) Verify that $j_!\underline{\Bbbk}[1]$ and $j_*\underline{\Bbbk}[1]$ are perverse sheaves, and determine composition series for each of them. *Answer*: There are short exact sequences

$$0 \to \mathrm{IC}(Z, \Bbbk) \to j_!\underline{\Bbbk}[1] \to \mathrm{IC}(U, \Bbbk) \to 0,$$
$$0 \to \mathrm{IC}(U, \Bbbk) \to j_*\underline{\Bbbk}[1] \to \mathrm{IC}(Z, \Bbbk)(-1) \to 0.$$

Alternatively, we can record the Loewy layers of these objects as follows:

$$j_!\underline{\Bbbk}[1] : \begin{array}{r|c|} \cline{2-2} \text{head:} & \mathrm{IC}(U,\Bbbk) \\ \cline{2-2} \text{socle:} & \mathrm{IC}(Z,\Bbbk) \\ \cline{2-2} \end{array} \qquad j_*\underline{\Bbbk}[1] : \begin{array}{r|c|} \cline{2-2} \text{head:} & \mathrm{IC}(Z,\Bbbk)(-1) \\ \cline{2-2} \text{socle:} & \mathrm{IC}(U,\Bbbk) \\ \cline{2-2} \end{array}$$

(b) Show that for any integer $n \geq 1$, there is a unique (up to isomorphism) indecomposable rank-n local system $\mathscr{J}_n$ on U that admits a nonzero map $\mathscr{J}_n \to \underline{\Bbbk}_U$. Its composition factors are $\underline{\Bbbk}_U(n-1), \underline{\Bbbk}_U(n-2), \ldots, \underline{\Bbbk}_U(1), \underline{\Bbbk}_U$. (For more on this local system, see Section 4.4.)

(c) Compute the stalks of $\mathrm{IC}(U, \mathscr{J}_n)$. Then determine a Loewy series for this perverse sheaf. *Answer*:

$$\mathrm{IC}(U,\mathscr{J}_n) : \begin{array}{c|c|c} & U & Z \\ \hline -1 & \mathscr{J}_n & \Bbbk(n-1) \end{array} \qquad \text{Loewy series:} \quad \begin{array}{r|c|} \cline{2-2} \text{head:} & \mathrm{IC}(U,\Bbbk) \\ \cline{2-2} & \mathrm{IC}(Z,\underline{\Bbbk}) \\ \cline{2-2} & \mathrm{IC}(U,\Bbbk)(1) \\ \cline{2-2} & \mathrm{IC}(Z,\underline{\Bbbk})(1) \\ \cline{2-2} & \vdots \\ \cline{2-2} & \mathrm{IC}(Z,\underline{\Bbbk})(n-2) \\ \cline{2-2} \text{socle:} & \mathrm{IC}(U,\underline{\Bbbk})(n-1) \\ \cline{2-2} \end{array}$$

(d) Deduce that $j_{!*} : \mathrm{Perv}(U,\Bbbk) \to \mathrm{Perv}(\mathbb{A}^1,\Bbbk)$ is neither left exact nor right exact, and that its image is not a Serre subcategory of $\mathrm{Perv}(\mathbb{A}^1,\Bbbk)$.

Exercise 3.10.5. Let Y be a smooth, connected, locally closed subvariety of X, and let $h : Y \hookrightarrow X$ be the inclusion map. A local system $\mathcal{L}$ on Y is said to be **clean** if the natural map

$$h_!\mathcal{L} \to h_*\mathcal{L}$$

is an isomorphism. Of course, if Y is closed in X, then every local system is clean.

(a) Show that if $\mathcal{L}$ is clean, then $h_!\mathcal{L}[n] \cong h_*\mathcal{L}[n]$ is perverse, and hence

$$\mathrm{IC}(Y,\mathcal{L}) \cong h_!\mathcal{L}[n] \cong h_*\mathcal{L}[n].$$

(b) Show that $\underline{\Bbbk}_Y$ is clean if and only if Y is closed in X.

(c) Let $\Bbbk$ be a field, and let $U = \mathbb{A}^1 \smallsetminus \{0\} \subset \mathbb{A}^1$. Show that any nonconstant rank-1 $\Bbbk$-local system on U is clean.

Exercise 3.10.6. This question deals with various closed subvarieties X of the space $\mathrm{Mat}_{m\times n}(\mathbb{C})$ of $m \times n$ matrices with complex coefficients. In each case, let $U = X \smallsetminus \{0\}$, and let $Z = \{0\}$. Let $\mathscr{S}$ denote the stratification $X = U \cup Z$. Let $\Bbbk$ be a field. Determine all the simple perverse sheaves in $\mathrm{Perv}_{\mathscr{S}}(X,\Bbbk)$. *Note*: In all but part (a), U is simply connected. In part (a), we have $\pi_1(U,u_0) \cong \mathbb{Z}/2\mathbb{Z}$.

(a) $X = \{x \in \mathrm{Mat}_{2\times 2}(\mathbb{C}) \mid \det x = \operatorname{tr} x = 0\}$.
(b) $X = \{x \in \mathrm{Mat}_{2\times 2}(\mathbb{C}) \mid \det x = 0\}$.
(c) $X = \{x \in \mathrm{Mat}_{2\times 3}(\mathbb{C}) \mid \operatorname{rank} x \leq 1\}$.
(d) $X = \{x \in \mathrm{Mat}_{3\times 3}(\mathbb{C}) \mid x \text{ is nilpotent and } \operatorname{rank} x \leq 1\}$.

Answers: In all cases, there is a skyscraper simple perverse sheaf:

$$\mathrm{IC}(Z,\underline{\Bbbk}) = i_*\underline{\Bbbk}_Z : \quad \begin{array}{c|c|c} & U & Z \\ \hline 0 & & \underline{\Bbbk} \end{array}$$

(a) We have

$\mathrm{IC}(U, \underline{\Bbbk})$, $\operatorname{char} \Bbbk = 2$:

	U	Z
-1		$\underline{\Bbbk}(-1)$
-2	$\underline{\Bbbk}$	$\underline{\Bbbk}$

$\mathrm{IC}(U, \underline{\Bbbk})$, $\operatorname{char} \Bbbk \neq 2$:

	U	Z
-2	$\underline{\Bbbk}$	$\underline{\Bbbk}$

If $\operatorname{char} \Bbbk \neq 2$, there is a nontrivial irreducible local system $\mathcal{L}$ on U, corresponding to the nontrivial irreducible representation of $\mathbb{Z}/2\mathbb{Z}$. We have

$\mathrm{IC}(U, \mathcal{L})$:

	U	Z
-2	$\mathcal{L}$	0

(b) We have

$\mathrm{IC}(U, \underline{\Bbbk})$:

	U	Z
-1		$\underline{\Bbbk}(-1)$
-2		
-3	$\underline{\Bbbk}$	$\underline{\Bbbk}$

(c) We have

$\mathrm{IC}(U, \underline{\Bbbk})$:

	U	Z
-2		$\underline{\Bbbk}(-1)$
-3		
-4	$\underline{\Bbbk}$	$\underline{\Bbbk}$

(d) We have

$\mathrm{IC}(U, \underline{\Bbbk})$, $\operatorname{char} \Bbbk = 3$:

	U	Z
-1		$\underline{\Bbbk}(-2)$
-2		$\underline{\Bbbk}(-1)$
-3		
-4	$\underline{\Bbbk}$	$\underline{\Bbbk}$

$\mathrm{IC}(U, \underline{\Bbbk})$, $\operatorname{char} \Bbbk \neq 3$:

	U	Z
-2		$\underline{\Bbbk}(-1)$
-3		
-4	$\underline{\Bbbk}$	$\underline{\Bbbk}$

EXERCISE 3.10.7. For which of the varieties X in Exercise 3.10.6 is $\underline{\Bbbk}_X[\dim X]$ a perverse sheaf? When it is perverse, what are its composition factors?

EXERCISE 3.10.8. Let $\mu : \tilde{\mathcal{N}} \to \mathcal{N}$ be the map from Exercise 1.10.5, and let $\Bbbk$ be a field. (Here, $\mathcal{N}$ is the same as the variety considered in Exercise 3.10.6(a).)

(a) Show that μ is a semismall map. Deduce that $\mu_* \underline{\Bbbk}_{\tilde{\mathcal{N}}}[2]$ is a perverse sheaf.
(b) Determine a Loewy series for $\mu_* \underline{\Bbbk}_{\tilde{\mathcal{N}}}[2]$. Show that this perverse sheaf is semisimple if and only if $\operatorname{char} \Bbbk \neq 2$.

This exercise shows that the decomposition theorem (Theorem 3.9.2) may fail for coefficients not in a field of characteristic 0.

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CHAPTER 4

Nearby and vanishing cycles

Let $f : X \to \mathbb{A}^1$ be a morphism of varieties. In this chapter, we will associate to such a map a new sheaf functor Ψ_f, called the **nearby cycles functor**. (We will also study a companion functor called the **vanishing cycles functor**.) Although the setting is algebraic, the definition of Ψ_f involves the (nonalgebraic) exponential map $\exp : \mathbb{C} \to \mathbb{C}^\times$, so the results of Chapter 2 cannot be applied to it directly. We will prove that Ψ_f takes constructible complexes to constructible complexes, that it is t-exact for the perverse t-structure, and that it commutes with Verdier duality and with extension of scalars. As an application of this theory, we study in Section 4.3 the problem of building a perverse sheaf from data on a hypersurface and on its complement.

In the last two sections of the chapter, we restrict to field coefficients. Section 4.4 gives an alternative description of the "unipotent part" of Ψ_f. In Section 4.5, we show that the derived category of the abelian category of perverse sheaves $D^{\mathrm{b}}\mathrm{Perv}(X, \Bbbk)$ is in fact equivalent to $D^{\mathrm{b}}_{\mathrm{c}}(X, \Bbbk)$. This result anticipates some phenomena in Chapter 5.

4.1. Definitions and preliminaries

In this section, we temporarily leave the world of algebraic varieties. All topological spaces in this section are assumed to be locally compact, locally contractible, and of finite c-soft dimension. The coefficient ring $\Bbbk$ will still be assumed to be noetherian and of finite global dimension.

Definitions. Let $f : X \to \mathbb{C}$ be a continuous map. Throughout this chapter, we will use the notation

$$X_{\mathbf{s}} = f^{-1}(0) \qquad \text{and} \qquad X_{\boldsymbol{\eta}} = f^{-1}(\mathbb{C}^\times).$$

The space $X_{\mathbf{s}}$ is called the **special fiber** of f. Of course, $X_{\mathbf{s}}$ is closed, $X_{\boldsymbol{\eta}}$ is open, and they are complementary to one another. We denote the inclusion maps by

$$i_{\mathbf{s}} : X_{\mathbf{s}} \hookrightarrow X \qquad \text{and} \qquad j_{\boldsymbol{\eta}} : X_{\boldsymbol{\eta}} \hookrightarrow X.$$

We also define $f_{\boldsymbol{\eta}} : X_{\boldsymbol{\eta}} \to \mathbb{C}^\times$ to be the restriction of f, as in the following cartesian square:

$$\begin{array}{ccc} X_{\boldsymbol{\eta}} & \xrightarrow{j_{\boldsymbol{\eta}}} & X \\ {\scriptstyle f_{\boldsymbol{\eta}}}\downarrow & & \downarrow{\scriptstyle f} \\ \mathbb{C}^\times & \xrightarrow{u} & \mathbb{C} \end{array} \tag{4.1.1}$$

Let $\exp : \mathbb{C} \to \mathbb{C}^\times$ be the analytic map given by $\exp(t) = e^t$, and let $T : \mathbb{C} \to \mathbb{C}$ be the deck transformation given by $T(z) = z + 2\pi i$. This is a generator of fundamental group $\pi_1(\mathbb{C}^\times, 1)$. Consider the ring $\Bbbk[\pi_1(\mathbb{C}^\times, 1)] = \Bbbk[T, T^{-1}]$, and let $\epsilon : \Bbbk[T, T^{-1}] \to \Bbbk$ be the map given by $\epsilon(T) = 1$. Note that $\ker \epsilon$ is a free $\Bbbk[T, T^{-1}]$-module, and recall from the remark following Exercise 1.10.3 (see also Example 2.2.7) that there is a canonical identification $\Bbbk(1) = \ker \epsilon / (\ker \epsilon)^2$. Lift this to an isomorphism

$$\Bbbk[T, T^{-1}](1) \cong \ker \epsilon. \tag{4.1.2}$$

This isomorphism involves a choice. Fix this choice as follows: consider the element $1 - T \in \ker \epsilon$, and let $[1 - T]$ denote its image in $\ker \epsilon / (\ker \epsilon)^2 = \Bbbk(1)$. Let us take (4.1.2) to be the isomorphism that sends $1 \otimes [1 - T] \in \Bbbk[T, T^{-1}] \otimes \Bbbk(1)$ to $1 - T \in \ker \epsilon$. We thus have a short exact sequence

$$0 \to \Bbbk[T, T^{-1}](1) \to \Bbbk[T, T^{-1}] \xrightarrow{\epsilon} \Bbbk \to 0. \tag{4.1.3}$$

By Proposition 1.7.16 (see also Remark 1.7.15), the sheaf $\exp_! \underline{\Bbbk}_{\mathbb{C}}$ (a local system of infinite rank) corresponds to $\Bbbk[T, T^{-1}]$ as a module over itself. The short exact sequence above then corresponds to the short exact sequence

$$0 \to \exp_! \underline{\Bbbk}_{\mathbb{C}}(1) \xrightarrow{d} \exp_! \underline{\Bbbk}_{\mathbb{C}} \xrightarrow{\epsilon} \underline{\Bbbk}_{\mathbb{C}^\times} \to 0 \tag{4.1.4}$$

of local systems on $\mathbb{C}^\times$. Next, let $u : \mathbb{C}^\times \hookrightarrow \mathbb{C}$ and $s : \{0\} \hookrightarrow \mathbb{C}$ be the inclusion maps, so that we have a canonical short exact sequence

$$0 \to u_! \underline{\Bbbk}_{\mathbb{C}^\times} \to \underline{\Bbbk}_{\mathbb{C}} \to s_* \underline{\Bbbk}_{\mathrm{pt}} \to 0. \tag{4.1.5}$$

Let $t : u_! \exp_! \underline{\Bbbk}_{\mathbb{C}} \to \underline{\Bbbk}_{\mathbb{C}}$ be the composition of $u_! \epsilon : u_! \exp_! \underline{\Bbbk}_{\mathbb{C}} \to u_! \underline{\Bbbk}_{\mathbb{C}^\times}$ with the first map in (4.1.5). Thus, there is a commutative diagram of sheaves

(4.1.6)
$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] & \underline{\Bbbk}_{\mathbb{C}} \arrow[r, equal] & \underline{\Bbbk}_{\mathbb{C}} \arrow[r] & 0 \\
0 \arrow[r] & u_! \exp_! \underline{\Bbbk}_{\mathbb{C}}(1) \arrow[r, "u_! d"] \arrow[u] & u_! \exp_! \underline{\Bbbk}_{\mathbb{C}} \arrow[r, "u_! \epsilon"] \arrow[u, "t"] & u_! \underline{\Bbbk}_{\mathbb{C}^\times} \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

in which the rows are exact. Let us now regard this diagram as a short exact sequence of chain complexes, with the top row in degree 0 and the bottom row in degree -1. Let $\mathcal{K}$ be the middle chain complex in this diagram; in other words, $\mathcal{K} = \mathrm{chcone}(t)$. From (4.1.5), we see that the third chain complex in (4.1.6) is quasi-isomorphic to $s_* \underline{\Bbbk}_{\mathrm{pt}}$. The diagram (4.1.6) gives us a distinguished triangle

$$u_! \exp_! \underline{\Bbbk}_{\mathbb{C}}[1](1) \to \mathcal{K} \to s_* \underline{\Bbbk}_{\mathrm{pt}} \to \tag{4.1.7}$$

in $D^{\mathrm{b}}(\mathbb{C}, \Bbbk)$. On the other hand, from the definition of $\mathcal{K}$, there is an obvious distinguished triangle

$$\underline{\Bbbk}_{\mathbb{C}} \to \mathcal{K} \to u_! \exp_! \underline{\Bbbk}_{\mathbb{C}}[1] \to . \tag{4.1.8}$$

The various short exact sequences and distinguished triangles above can be collected into the following octahedral diagram:
(4.1.9)

$$\begin{tikzcd}
s_* \underline{\Bbbk}_{\mathrm{pt}} \arrow[dd, squiggly] & & \underline{\Bbbk}_{\mathbb{C}} \arrow[ll] \arrow[dl] \\
& \mathcal{K} \arrow[ul] \arrow[dr] & \\
u_! \exp_! \underline{\Bbbk}_{\mathbb{C}}[1](1) \arrow[ur] \arrow[rr, "u_! d"] & & u_! \exp_! \underline{\Bbbk}_{\mathbb{C}}[1] \arrow[uu, squiggly]
\end{tikzcd}
\qquad
\begin{tikzcd}
s_* \underline{\Bbbk}_{\mathrm{pt}} \arrow[dd, squiggly] \arrow[dr, squiggly] & & \underline{\Bbbk}_{\mathbb{C}} \arrow[ll] \\
& u_! \underline{\Bbbk}_{\mathbb{C}^\times} \arrow[ur] \arrow[dl] & \\
u_! \exp_! \underline{\Bbbk}_{\mathbb{C}}[1](1) \arrow[rr, "u_! d"] & & u_! \exp_! \underline{\Bbbk}_{\mathbb{C}}[1] \arrow[uu, squiggly] \arrow[ul, squiggly, "u_! \epsilon"']
\end{tikzcd}$$

DEFINITION 4.1.1. Let $f : X \to \mathbb{C}$ be a continuous map. The **nearby cycles functor** associated to f is the functor $\Psi_f : D^+(X_{\boldsymbol{\eta}}, \Bbbk) \to D^+(X_{\mathbf{s}}, \Bbbk)$ given by

$$\Psi_f(\mathcal{F}) = i_{\mathbf{s}}^* j_{\boldsymbol{\eta}*} R\mathscr{H}om(f_{\boldsymbol{\eta}}^* \exp_! \underline{\Bbbk}_{\mathbb{C}}[1], \mathcal{F}).$$

The **vanishing cycles functor** associated to f is the functor $\Phi_f : D^+(X, \Bbbk) \to D^+(X_{\mathbf{s}}, \Bbbk)$ given by

$$\Phi_f(\mathcal{F}) = i_{\mathbf{s}}^* R\mathscr{H}om(f^*\mathcal{K}, \mathcal{F}).$$

As in Definition 1.7.21, the deck transformation $T : z \mapsto z + 2\pi i$ gives rise to an automorphism

$$\mathrm{mon}_!(T) : \exp_! \underline{\Bbbk}_{\mathbb{C}} \to \exp_! \underline{\Bbbk}_{\mathbb{C}}. \tag{4.1.10}$$

Using Proposition 1.7.20, one can see that the diagram

$$\begin{tikzcd} \underline{\Bbbk}_{\mathbb{C}^\times} \arrow[r, equal, "\mathrm{id}"] & \underline{\Bbbk}_{\mathbb{C}^\times} \\ \exp_! \underline{\Bbbk}_{\mathbb{C}} \arrow[u, "\epsilon"] \arrow[r, "\mathrm{mon}_!(T)"] & \exp_! \underline{\Bbbk}_{\mathbb{C}} \arrow[u, "\epsilon"'] \end{tikzcd}$$

commutes. Now apply $u_!$ and compose the top part of the diagram with the first map from (4.1.5) to obtain the commutative diagram

$$\begin{tikzcd} \underline{\Bbbk}_{\mathbb{C}} \arrow[r, equal, "\mathrm{id}"] & \underline{\Bbbk}_{\mathbb{C}} \\ u_! \exp_! \underline{\Bbbk}_{\mathbb{C}} \arrow[u, "t"] \arrow[r, "u_! \mathrm{mon}_!(T)"] & u_! \exp_! \underline{\Bbbk}_{\mathbb{C}} \arrow[u, "t"'] \end{tikzcd} \tag{4.1.11}$$

This diagram can be regarded as a chain map $\mathcal{K} \to \mathcal{K}$.

DEFINITION 4.1.2. Let $f : X \to \mathbb{C}$ be a continuous map. For $\mathcal{F} \in D^+(X_{\boldsymbol{\eta}}, \Bbbk)$, the **monodromy automorphism** of $\Psi_f(\mathcal{F})$, denoted by

$$\mathrm{mon}_\Psi(T) : \Psi_f(\mathcal{F}) \to \Psi_f(\mathcal{F}),$$

is the natural automorphism induced by (4.1.10). Similarly, for $\mathcal{G} \in D^+(X, \Bbbk)$, the **monodromy automorphism** of $\Phi_f(\mathcal{G})$ is the natural automorphism

$$\mathrm{mon}_\Phi(T) : \Phi_f(\mathcal{F}) \to \Phi_f(\mathcal{F})$$

induced by (4.1.11).

More generally, for any $h \in \pi_1(\mathbb{C}^\times, 1)$, or even any $h \in \Bbbk[\pi_1(\mathbb{C}^\times, 1)] = \Bbbk[T, T^{-1}]$, one can consider the maps $\mathrm{mon}_\Psi(h)$ and $\mathrm{mon}_\Phi(h)$.

Basic properties. Given a continuous map $f : X \to \mathbb{C}$, let $\tilde{X}_{\boldsymbol{\eta}} = X_{\boldsymbol{\eta}} \times_{\mathbb{C}} \mathbb{C}^\times$, and let $\exp_X : \tilde{X}_{\boldsymbol{\eta}} \to X_{\boldsymbol{\eta}}$ be the natural map, so that we have a cartesian square

$$\begin{tikzcd} \tilde{X}_{\boldsymbol{\eta}} \arrow[r, "\exp_X"] \arrow[d] & X_{\boldsymbol{\eta}} \arrow[d, "f_{\boldsymbol{\eta}}"] \\ \mathbb{C} \arrow[r, "\exp"] & \mathbb{C}^\times \end{tikzcd} \tag{4.1.12}$$

The following lemma gives some alternative descriptions of Ψ_f.

LEMMA 4.1.3. *Let $f : X \to \mathbb{C}$ be a continuous map.*

(1) *For $\mathcal{F} \in D^+(X_{\boldsymbol{\eta}}, \Bbbk)$, there is a natural isomorphism*

$$\Psi_f(\mathcal{F}) \cong i_{\mathbf{s}}^* j_{\boldsymbol{\eta}*} \exp_{X*} \exp_X^* \mathcal{F}[-1].$$

(2) *For $\mathcal{F} \in D^+(X, \Bbbk)$, there is a natural isomorphism*

$$\Psi_f(\mathcal{F}|_{X_{\boldsymbol{\eta}}}) \cong i_{\mathbf{s}}^* R\mathscr{H}om(f^* u_! \exp_! \underline{\Bbbk}_{\mathbb{C}}[1], \mathcal{F}).$$

Proof. (1) The cartesian square (4.1.12) shows that $f_{\boldsymbol{\eta}}^* \exp_! \underline{\Bbbk}_{\mathbb{C}} \cong \exp_{X!} \underline{\Bbbk}_{\tilde{X}_{\boldsymbol{\eta}}}$. Next, since $\exp_X$ is a covering map, we have $\exp_X^! \cong \exp_X^*$. The result then follows from the following calculation:

$$\begin{aligned} R\mathscr{H}om(f_{\boldsymbol{\eta}}^* \exp_! \underline{\Bbbk}_{\mathbb{C}}[1], \mathcal{F}) &\cong R\mathscr{H}om(\exp_{X!} \underline{\Bbbk}_{\tilde{X}_{\boldsymbol{\eta}}}, \mathcal{F}) \\ &\cong \exp_{X*} R\mathscr{H}om(\underline{\Bbbk}_{\tilde{X}_{\boldsymbol{\eta}}}, \exp_X^* \mathcal{F}) \cong \exp_{X*} \exp_X^* \mathcal{F}. \end{aligned}$$

(2) Since $j_{\boldsymbol{\eta}}$ is an open embedding, we have $j_{\boldsymbol{\eta}}^! \cong j_{\boldsymbol{\eta}}^*$. By (4.1.1), we have

$$\begin{aligned} j_{\boldsymbol{\eta}*} R\mathscr{H}om(f_{\boldsymbol{\eta}}^* \exp_! \underline{\Bbbk}_{\mathbb{C}}[1], j_{\boldsymbol{\eta}}^* \mathcal{F}) &\cong R\mathscr{H}om(j_{\boldsymbol{\eta}!} f_{\boldsymbol{\eta}}^* \exp_! \underline{\Bbbk}_{\mathbb{C}}[1], \mathcal{F}) \\ &\cong R\mathscr{H}om(f^* u_! \exp_! \underline{\Bbbk}_{\mathbb{C}}[1], \mathcal{F}). \end{aligned}$$ ☐

Example 4.1.4. Let us work out the nearby cycles functor for the map $\mathrm{id} : \mathbb{C} \to \mathbb{C}$ using the formula from Lemma 4.1.3(1). Let $\mathcal{L}$ be a local system of finite type on $\mathbb{C}^\times$. Let $R = \Bbbk[T, T^{-1}] = \Bbbk[\pi_1(\mathbb{C}^\times, 1)]$, and let M be the R-module corresponding to $\mathcal{L}$ via Theorem 1.7.9. By Proposition 1.7.16 and Corollary 1.9.6, we have that $\exp_* \exp^* \mathcal{L}$ is the local system corresponding to the R-module $\mathrm{Hom}_\Bbbk(R, M)$.

Let us spell this out a bit more. There are actually two obvious R-actions on $\mathrm{Hom}_\Bbbk(R, M)$: given a function $s : R \to M$, there is the "induction action" given by

$$(T \cdot s)(r) = s(rT)$$

and the "internal action" (technically an action of the opposite ring R^{op}, although this can be ignored since R is commutative) given by

$$\mathrm{Int_r}(T)(s)(r) = T^{-1} s(Tr).$$

These two actions commute. Under Theorem 1.7.9, $\exp_* \exp^* \mathcal{L}$ corresponds to $\mathrm{Hom}_\Bbbk(R, M)$ with the induction action. By Proposition 1.7.20, the monodromy map $\mathrm{mon}_*(T)$ on $\exp_* \exp^* \mathcal{L}$ then corresponds to $\mathrm{Int_r}(T)$.

Since R is a free $\Bbbk$-module, we may replace $\mathrm{Hom}_\Bbbk(R, M)$ by $R\mathrm{Hom}_\Bbbk(R, M)$. Next, Proposition B.2.3 tells us that

$$\begin{aligned} \Psi_{\mathrm{id}}(\mathcal{L}) &\cong i_{\mathbf{s}}^* j_{\boldsymbol{\eta}*} \exp_* \exp^* \mathcal{L}[-1] \cong R\mathrm{Hom}_R(\Bbbk, R\mathrm{Hom}_\Bbbk(R, M))[-1] \\ &\cong R\mathrm{Hom}_\Bbbk(R \otimes_R \Bbbk, M)[-1] \cong R\mathrm{Hom}_\Bbbk(\Bbbk, M)[-1] \cong M[-1]. \end{aligned}$$

To compute $\mathrm{mon}_\Psi(T)$, we follow the map induced by $\mathrm{Int_r}(T)$ through this calculation. The result is that $\mathrm{mon}_\Psi(T) : \Psi_{\mathrm{id}}(\mathcal{L}[1]) \to \Psi_{\mathrm{id}}(\mathcal{L}[1])$ corresponds to the given action of T on the R-module M.

For more examples, see Section B.5.

Thanks to Lemma 4.1.3(2), any map between $u_! \exp_! \underline{\Bbbk}_{\mathbb{C}}[1]$ and $\mathcal{K}$ induces a natural transformation between Ψ_f and Φ_f. In particular, by applying the functor $i_{\mathbf{s}}^* R\mathscr{H}om(f^*(-), \mathcal{F})$ to (4.1.9), we obtain the following result.

Lemma 4.1.5. *Let $f : X \to \mathbb{C}$ be a continuous map. For $\mathcal{F} \in D^+(X, \Bbbk)$, there are natural transformations*

$$\mathrm{can} : \Psi_f(\mathcal{F}|_{X_{\boldsymbol{\eta}}}) \to \Phi_f(\mathcal{F}) \qquad \textit{and} \qquad \mathrm{var} : \Phi_f(\mathcal{F}) \to \Psi_f(\mathcal{F}|_{X_{\boldsymbol{\eta}}})(-1)$$

that fit into a natural octahedral diagram

$$
\begin{tikzcd}
i_{\mathbf{s}}^*\mathcal{F} \arrow[dd, rightsquigarrow] & & i_{\mathbf{s}}^!\mathcal{F} \arrow[ll] \arrow[dl] \\
& \Phi_f(\mathcal{F}) \arrow[ul] \arrow[dr, "\mathrm{var}"] & \\
\Psi_f(\mathcal{F}|_{X_{\boldsymbol{\eta}}}) \arrow[ur, "\mathrm{can}"] \arrow[rr, "\mathrm{var}\circ\mathrm{can}"] & & \Psi_f(\mathcal{F}|_{X_{\boldsymbol{\eta}}})(-1) \arrow[uu, rightsquigarrow]
\end{tikzcd}
\qquad
\begin{tikzcd}
i_{\mathbf{s}}^*\mathcal{F} \arrow[dd, rightsquigarrow] \arrow[dr] & & i_{\mathbf{s}}^!\mathcal{F} \arrow[ll] \\
& i_{\mathbf{s}}^* j_{\boldsymbol{\eta}*}(\mathcal{F}|_{X_{\boldsymbol{\eta}}}) \arrow[ur, rightsquigarrow] \arrow[dl, rightsquigarrow] & \\
\Psi_f(\mathcal{F}|_{X_{\boldsymbol{\eta}}}) \arrow[rr, "\mathrm{var}\circ\mathrm{can}"] & & \Psi_f(\mathcal{F}|_{X_{\boldsymbol{\eta}}})(-1) \arrow[uu, rightsquigarrow] \arrow[ul]
\end{tikzcd}
$$

Moreover, this diagram is compatible with the monodromy automorphisms of Definition 4.1.2*.*

The maps denoted by can and var above are called the **canonical** and **variation maps**, respectively. The canonical map is induced by the second map in (4.1.8), while the variation map is induced by the first map in (4.1.7).

The last assertion means that the maps in the diagram intertwine the monodromy automorphisms of Definition 4.1.2 with each other or with the identity maps of $i_{\mathbf{s}}^*\mathcal{F}$, $i_{\mathbf{s}}^!\mathcal{F}$, or $i_{\mathbf{s}}^* j_{\boldsymbol{\eta}*}(\mathcal{F}|_{X_{\boldsymbol{\eta}}})$, as appropriate.

In a minor abuse of notation, for $\mathcal{G} \in D^+(X_{\boldsymbol{\eta}}, \Bbbk)$, we write

$$\mathrm{var} \circ \mathrm{can} : \Psi_f(\mathcal{G}) \to \Psi_f(\mathcal{G})(-1)$$

for the map induced by the first map in (4.1.4), even though var and can themselves do not make sense for $\mathcal{G}$. Moreover, the bottom distinguished triangle in the lower cap in Lemma 4.1.5 makes sense even if we start with an object $\mathcal{G} \in D^+(X_{\boldsymbol{\eta}}, \Bbbk)$, rather than an object on X:

$$\Psi_f(\mathcal{G}) \xrightarrow{\mathrm{var}\circ\mathrm{can}} \Psi_f(\mathcal{G})(-1) \to i_{\mathbf{s}}^* j_{\boldsymbol{\eta}*}\mathcal{G} \to . \tag{4.1.13}$$

LEMMA 4.1.6. *Let $f : X \to \mathbb{C}$ be a continuous map. Choose an isomorphism $b : \Bbbk \to \Bbbk(1)$. There exists a scalar $a \in \Bbbk^\times$ (depending on b) such that for any $\mathcal{F} \in D^+(X, \Bbbk)$ and any $\mathcal{G} \in D^+(X_{\boldsymbol{\eta}}, \Bbbk)$, we have*

$$\mathrm{can} \circ \mathrm{var} = a\, \mathrm{mon}_\Phi(1 - T) : \Phi_f(\mathcal{F}) \to \Phi_f(\mathcal{F}),$$
$$\mathrm{var} \circ \mathrm{can} = a\, \mathrm{mon}_\Psi(1 - T) : \Psi_f(\mathcal{G}) \to \Psi_f(\mathcal{G}).$$

In this statement, we use b to suppress Tate twists.

PROOF. By the discussion preceding (4.1.4), the map $d : \exp_! \underline{\Bbbk}_{\mathbb{C}}(1) \to \exp_! \underline{\Bbbk}_{\mathbb{C}}$ in that short exact sequence can be identified with $a\, \mathrm{mon}_!(\mathrm{id} - T) : \exp_! \underline{\Bbbk}_{\mathbb{C}} \to \underline{\Bbbk}_! \underline{\Bbbk}_C$ for some scalar $a \in \Bbbk^\times$. The result follows. □

Compatibilities with sheaf functors. We have the following relationship between nearby cycles and pullback and push-forward.

LEMMA 4.1.7. *Let $g : X \to Y$ and $f : Y \to \mathbb{C}$ be continuous maps. Let $g_{\mathbf{s}} : X_{\mathbf{s}} \to Y_{\mathbf{s}}$ and $g_{\boldsymbol{\eta}} : X_{\boldsymbol{\eta}} \to Y_{\boldsymbol{\eta}}$ be the maps obtained by restricting g. (Here, $X_{\mathbf{s}}$ is the special fiber of $f \circ g$, and $Y_{\mathbf{s}}$ is the special fiber of f.)*

(1) *If g is proper, then for any $\mathcal{F} \in D^+(X_{\boldsymbol{\eta}}, \Bbbk)$, there is a natural isomorphism*

$$\Psi_f(g_{\boldsymbol{\eta}*}\mathcal{F}) \xrightarrow{\sim} g_{\mathbf{s}*}\Psi_{f\circ g}(\mathcal{F}).$$

(2) *If g is a topological submersion (e.g., a smooth morphism of varieties), then for any $\mathcal{G} \in D^+(Y_{\boldsymbol{\eta}}, \Bbbk)$, there is a natural isomorphism*

$$g_{\mathbf{s}}^*\Psi_f(\mathcal{G}) \xrightarrow{\sim} \Psi_{f\circ g}(g_{\boldsymbol{\eta}}^*\mathcal{G}).$$

PROOF. Consider the cartesian squares

$$\begin{tikzcd} \tilde{X}_{\boldsymbol{\eta}} \arrow[r, "\exp_X"] \arrow[d, "\tilde{g}_{\boldsymbol{\eta}}"'] & X_{\boldsymbol{\eta}} \arrow[r, "j^X_{\boldsymbol{\eta}}"] \arrow[d, "g_{\boldsymbol{\eta}}"'] & X \arrow[d, "g"'] & X_{\mathbf{s}} \arrow[l, "i^X_{\mathbf{s}}"] \arrow[d, "g_{\mathbf{s}}"] \\ \tilde{Y}_{\boldsymbol{\eta}} \arrow[r, "\exp_Y"] & Y_{\boldsymbol{\eta}} \arrow[r, "j^Y_{\boldsymbol{\eta}}"] & Y & Y_{\mathbf{s}} \arrow[l, "i^Y_{\mathbf{s}}"'] \end{tikzcd}$$

For part (1), all vertical maps are proper, and we have a sequence of natural isomorphisms

$$\begin{aligned}(i^Y_{\mathbf{s}})^* j^Y_{\boldsymbol{\eta}*} \exp_{Y*} \exp_Y^* g_{\boldsymbol{\eta}*}\mathcal{F}[-1] &\xrightarrow{\sim} (i^Y_{\mathbf{s}})^* j^Y_{\boldsymbol{\eta}*} \exp_{Y*} \tilde{g}_{\boldsymbol{\eta}*} \exp_X^* \mathcal{F}[-1] \\ \xrightarrow{\sim} (i^Y_{\mathbf{s}})^* j^Y_{\boldsymbol{\eta}*} g_{\boldsymbol{\eta}*} \exp_{X*} \exp_X^* \mathcal{F}[-1] &\xrightarrow{\sim} (i^Y_{\mathbf{s}})^* g_* j^X_{\boldsymbol{\eta}*} \exp_{X*} \exp_X^* \mathcal{F}[-1] \\ &\xrightarrow{\sim} g_{\mathbf{s}*} (i^X_{\mathbf{s}})^* j^X_{\boldsymbol{\eta}*} \exp_{X*} \exp_X^* \mathcal{F}[-1].\end{aligned}$$

The proof of part (2) is similar, using base change for topological submersions. Further details are left to the reader. □

The remaining statements in this section describe natural maps (not necessarily isomorphisms) relating nearby cycles to tensor products, Verdier duality, and extension of scalars. In Section 4.2, we will see some circumstances under which the latter two are isomorphisms.

LEMMA 4.1.8. *Let $h : X \to \mathbb{C}$ be a continuous map. For $\mathcal{F}, \mathcal{G} \in D^+(X_{\boldsymbol{\eta}}, \Bbbk)$, there is a natural map*

$$\Psi_h(\mathcal{F}) \overset{L}{\otimes} \Psi_h(\mathcal{G})[1] \to \Psi_h(\mathcal{F} \overset{L}{\otimes} \mathcal{G}) \tag{4.1.14}$$

such that the following diagram commutes:

$$\begin{tikzcd} \Psi_h(\mathcal{F}) \otimes^L \Psi_h(\mathcal{G})[1] \arrow[r] \arrow[d, "\mathrm{mon}_\Psi(T) \otimes^L \mathrm{mon}_\Psi(T)[1]"'] & \Psi_h(\mathcal{F} \otimes^L \mathcal{G}) \arrow[d, "\mathrm{mon}_\Psi(T)"] \\ \Psi_h(\mathcal{F}) \otimes^L \Psi_f(\mathcal{G})[1] \arrow[r] & \Psi_h(\mathcal{F} \otimes^L \mathcal{G}). \end{tikzcd}$$

Moreover, if h is the composition of two maps $X \xrightarrow{g} Y \xrightarrow{f} \mathbb{C}$ with g proper, then the following diagram commutes:

$$\begin{tikzcd} \Psi_f(g_{\boldsymbol{\eta}*}\mathcal{F}) \otimes^L \Psi_f(g_{\boldsymbol{\eta}*}\mathcal{G})[1] \arrow[r, "(4.1.14)"] \arrow[d, "\wr", "\text{Lem. 4.1.7}"'] & \Psi_f(g_{\boldsymbol{\eta}*}\mathcal{F} \otimes^L g_{\boldsymbol{\eta}*}\mathcal{G}) \arrow[r, "\text{Rmk. 1.4.12}"] & \Psi_f(g_{\boldsymbol{\eta}*}(\mathcal{F} \otimes^L \mathcal{G})) \arrow[d, "\text{Lem. 4.1.7}"', "\wr"] \\ g_{\mathbf{s}*}\Psi_h(\mathcal{F}) \otimes^L g_{\mathbf{s}*}\Psi_h(\mathcal{G})[1] \arrow[r, "\text{Rmk. 1.4.12}"] & g_{\mathbf{s}*}(\Psi_h(\mathcal{F}) \otimes^L \Psi_h(\mathcal{G}))[1] \arrow[r, "(4.1.14)"] & g_{\mathbf{s}*}\Psi_h(\mathcal{F} \otimes^L \mathcal{G}) \end{tikzcd}$$

PROOF SKETCH. By Proposition 1.4.5, we have $\exp_X^*(\mathcal{F} \otimes^L \mathcal{G}) \cong \exp_X^* \mathcal{F} \otimes^L \exp_X^* \mathcal{G}$, and then Remark 1.4.12 gives us a natural map

$$j_{\boldsymbol{\eta}*} \exp_{X*} \exp_X^* \mathcal{F} \overset{L}{\otimes} j_{\boldsymbol{\eta}*} \exp_{X*} \exp_X^* \mathcal{G} \to j_{\boldsymbol{\eta}*} \exp_{X*} \exp_X^* (\mathcal{F} \overset{L}{\otimes} \mathcal{G}).$$

Applying $i_{\mathbf{s}}^*[-1]$ and then using Proposition 1.4.5 again, we obtain a natural map

$$\Psi_h(\mathcal{F}) \overset{L}{\otimes} \Psi_h(\mathcal{G})[1] \to \Psi_h(\mathcal{F} \overset{L}{\otimes} \mathcal{G}).$$

The proof that this map is compatible with monodromy is a lengthy but routine application of Propositions 1.6.1 and 1.6.7, combined with the description of $\mathrm{mon}_\Psi(T)$ given in Exercise 4.1.1 below.

The proof of compatibility with Lemma 4.1.7 is similar, using Propositions 1.6.3 and 1.6.4. We omit further details. □

Corollary 4.1.9. *Let $h : X \to \mathbb{C}$ be a continuous map. Let $\mathcal{F} \in D^{\mathrm{b}}(X_{\boldsymbol{\eta}}, \Bbbk)$. Assume that $\mathbb{D}\mathcal{F} \in D^{\mathrm{b}}(X_{\boldsymbol{\eta}}, \Bbbk)$, and that $\Psi_h(\mathcal{F})$ and $\Psi_h(\mathbb{D}\mathcal{F})$ both lie in $D^{\mathrm{b}}(X_{\mathbf{s}}, \Bbbk)$. Then there is a natural map*

$$\Psi_h(\mathbb{D}\mathcal{F}) \to \mathbb{D}(\Psi_h(\mathcal{F}))(1). \tag{4.1.15}$$

Moreover, if h is the composition of two maps $X \xrightarrow{g} Y \xrightarrow{f} \mathbb{C}$ with g proper, then the following diagram commutes:

$$\begin{tikzcd}
\Psi_f(\mathbb{D}g_{\boldsymbol{\eta}*}\mathcal{F}) \arrow[r, "(4.1.15)"] \arrow[d, "\wr"'] & \mathbb{D}(\Psi_f(g_{\boldsymbol{\eta}*}\mathcal{F}))(1) \arrow[d, "\wr"] \\
g_{\mathbf{s}*}\Psi_h(\mathbb{D}\mathcal{F}) \arrow[r, "(4.1.15)"] & g_{\mathbf{s}*}\mathbb{D}(\Psi_h(\mathcal{F}))(1)
\end{tikzcd}$$

Proof sketch. Part of Lemma 4.1.5 gives us a natural map $\Psi_h(\omega_{X_{\boldsymbol{\eta}}}) \to i_{\mathbf{s}}^! \omega_X[1](1) \cong \omega_{X_{\mathbf{s}}}[1](1)$. Combining this with Lemma 4.1.8 and Remark 1.5.16, we obtain a sequence of natural maps

$$\Psi_h(\mathbb{D}\mathcal{F}) \overset{L}{\otimes} \Psi_h(\mathcal{F}) \to \Psi_h(\mathbb{D}\mathcal{F} \overset{L}{\otimes} \mathcal{F})[-1] \to \Psi_h(\omega_{X_{\boldsymbol{\eta}}})[-1] \to \omega_{X_{\mathbf{s}}}(1). \tag{4.1.16}$$

The composition is an element of the left-hand side of the isomorphism

$$\begin{aligned}\mathrm{Hom}(\Psi_h(\mathbb{D}\mathcal{F}) \overset{L}{\otimes} \Psi_h(\mathcal{F}), \omega_{X_{\mathbf{s}}}(1)) &\cong \mathrm{Hom}(\Psi_h(\mathbb{D}\mathcal{F}), R\mathscr{H}om(\Psi_h(\mathcal{F}), \omega_{X_{\mathbf{s}}})(1)) \\ &= \mathrm{Hom}(\Psi_h(\mathbb{D}\mathcal{F}), \mathbb{D}(\Psi_h(\mathcal{F}))(1)).\end{aligned}$$

The desired natural map is the corresponding element of the right-hand side.

When h factors as $f \circ g$, the isomorphism $\omega_{X_{\boldsymbol{\eta}}} \cong g_{\boldsymbol{\eta}}^! \omega_{Y_{\boldsymbol{\eta}}}$ gives rise, by adjunction, to a natural map $g_{\boldsymbol{\eta}*}\omega_{X_{\boldsymbol{\eta}}} \to \omega_{Y_{\boldsymbol{\eta}}}$. We likewise have a natural map $g_{\mathbf{s}*}\omega_{X_{\mathbf{s}}} \to \omega_{Y_{\mathbf{s}}}$. Then Lemma 4.1.8 lets us construct the following commutative diagram:

$$\begin{tikzcd}[column sep=small]
\Psi_f(\mathbb{D}g_{\boldsymbol{\eta}*}\mathcal{F}) \overset{L}{\otimes} \Psi_f(g_{\boldsymbol{\eta}*}\mathcal{F}) \arrow[r] \arrow[d, "\wr"'] \arrow[rrr, bend left, "(4.1.16)"] & \Psi_f((\mathbb{D}g_{\boldsymbol{\eta}*}\mathcal{F}) \overset{L}{\otimes} g_{\boldsymbol{\eta}*}\mathcal{F})[-1] \arrow[r] \arrow[d] & \Psi_f(\omega_{Y_{\boldsymbol{\eta}}})[-1] \arrow[r] & \omega_{Y_{\mathbf{s}}}(1) \\
g_{\mathbf{s}*}\Psi_h(\mathbb{D}\mathcal{F}) \overset{L}{\otimes} g_{\mathbf{s}*}\Psi_h(\mathcal{F}) \arrow[d] & \Psi_f(g_{\boldsymbol{\eta}*}(\mathbb{D}\mathcal{F} \overset{L}{\otimes} \mathcal{F}))[-1] \arrow[r] \arrow[d, "\wr"] & \Psi_f(g_{\boldsymbol{\eta}*}\omega_{X_{\boldsymbol{\eta}}})[-1] \arrow[u] \arrow[d, "\wr"] & \\
g_{\mathbf{s}*}(\Psi_h(\mathbb{D}\mathcal{F}) \overset{L}{\otimes} \Psi_h(\mathcal{F})) \arrow[r] \arrow[rrr, bend right, "(4.1.16)"'] & g_{\mathbf{s}*}\Psi_h(\mathbb{D}\mathcal{F} \overset{L}{\otimes} \mathcal{F})[-1] \arrow[r] & g_{\mathbf{s}*}\Psi_h(\omega_{X_{\boldsymbol{\eta}}})[-1] \arrow[r] & g_{\mathbf{s}*}\omega_{X_{\mathbf{s}}}(1) \arrow[uu]
\end{tikzcd}$$

The last assertion in the corollary can be deduced from this. □

The proof of the next statement is very similar and will be omitted.

Corollary 4.1.10. *Let $h : X \to \mathbb{C}$ be a continuous map, and let $\varphi : \Bbbk \to \Bbbk'$ be a ring homomorphism. For $\mathcal{F} \in D^+(X_{\boldsymbol{\eta}}, \Bbbk)$, there is a natural map*

$$\Bbbk' \overset{L}{\otimes}_{\Bbbk} \Psi_h(\mathcal{F}) \to \Psi_h(\Bbbk' \overset{L}{\otimes}_{\Bbbk} \mathcal{F}). \tag{4.1.17}$$

Moreover, if h is the composition of two maps $X \xrightarrow{g} Y \xrightarrow{f} \mathbb{C}$ with g proper, then the following diagram commutes:

$$\begin{array}{ccc}
\Bbbk' \otimes^L_{\Bbbk} \Psi_f(g_{\boldsymbol{\eta}*}\mathcal{F}) & \xrightarrow{(4.1.17)} & \Psi_f(\Bbbk' \otimes^L_{\Bbbk} g_{\boldsymbol{\eta}*}\mathcal{F}) \\
\wr\downarrow & & \downarrow\wr \\
g_{\mathbf{s}*}(\Bbbk' \otimes^L_{\Bbbk} \Psi_h(\mathcal{F})) & \xrightarrow{(4.1.17)} & g_{\mathbf{s}*}\Psi_h(\Bbbk' \otimes^L_{\Bbbk} \mathcal{F})
\end{array}$$

Exercises.

4.1.1. Given a continuous function $f : X \to \mathbb{C}$, let $T_X : \tilde{X}_{\boldsymbol{\eta}} \to \tilde{X}_{\boldsymbol{\eta}}$ be the deck transformation of $\exp_X : \tilde{X}_{\boldsymbol{\eta}} \to X_{\boldsymbol{\eta}}$ induced by $T : \mathbb{C} \to \mathbb{C}$ (see (4.1.12) for notation). Let $\mathcal{F} \in D^+(X_{\boldsymbol{\eta}}, \Bbbk)$. Show that under the isomorphism of Lemma 4.1.3(1), the monodromy automorphism $\mathrm{mon}_\Psi(T)$ is identified with the composition

$$\begin{aligned}
i_{\mathbf{s}}^* j_{\boldsymbol{\eta}*} \exp_{X*} \exp_X^* \mathcal{F}[-1] &\to i_{\mathbf{s}}^* j_{\boldsymbol{\eta}*} \exp_{X*} T_{X*} T_X^* \exp_X^* \mathcal{F}[-1] \\
&\cong i_{\mathbf{s}}^* j_{\boldsymbol{\eta}*} (\exp_X \circ T_X)_* (\exp_X \circ T_X)^* \mathcal{F}[-1] = i_{\mathbf{s}}^* j_{\boldsymbol{\eta}*} \exp_{X*} \exp_X^* \mathcal{F}[-1].
\end{aligned}$$

4.1.2. Show that the natural map in Lemma 4.1.8 is compatible with restriction to an open subset. That is, if $U \subset X$ is an open subset, show that there is a commutative diagram

$$\begin{array}{ccc}
\Psi_h(\mathcal{F})|_{U_{\mathbf{s}}} \otimes^L \Psi_h(\mathcal{G})|_{U_{\mathbf{s}}}[1] & \longrightarrow & \Psi_h(\mathcal{F} \otimes^L \mathcal{G})|_{U_{\mathbf{s}}} \\
\wr\downarrow & & \downarrow\wr \\
\Psi_{h|_U}(\mathcal{F}|_{U_{\boldsymbol{\eta}}}) \otimes^L \Psi_{h|_U}(\mathcal{G}|_{U_{\boldsymbol{\eta}}})[1] & \longrightarrow & \Psi_{h|_U}(\mathcal{F}|_{U_{\boldsymbol{\eta}}} \otimes^L \mathcal{G}|_{U_{\boldsymbol{\eta}}})
\end{array}$$

Then show that Corollaries 4.1.9 and 4.1.10 are also compatible with restriction to an open subset.

4.1.3. Let $f : X \to \mathbb{C}$ and $g : X \to \mathbb{C}$ be two continuous functions. For $\mathcal{F} \in D^+(X_{\boldsymbol{\eta}}, \Bbbk)$ and $\mathcal{G} \in D^+(Y_{\boldsymbol{\eta}}, \Bbbk)$, define their **relative external tensor product** over $\mathbb{C}^\times$ to be the object

$$\mathcal{F} \boxtimes_{\mathbb{C}^\times} \mathcal{G} = \mathrm{pr}_1^* \mathcal{F} \overset{L}{\otimes} \mathrm{pr}_2^* \mathcal{G}[1] \in D^+(X_{\boldsymbol{\eta}} \times_{\mathbb{C}^\times} Y_{\boldsymbol{\eta}}, \Bbbk),$$

where $\mathrm{pr}_1 : X_{\boldsymbol{\eta}} \times_{\mathbb{C}^\times} Y_{\boldsymbol{\eta}} \to X_{\boldsymbol{\eta}}$ and $\mathrm{pr}_2 : X_{\boldsymbol{\eta}} \times_{\mathbb{C}^\times} Y_{\boldsymbol{\eta}} \to Y_{\boldsymbol{\eta}}$ are the projection maps. Use Lemma 4.1.8 to define a natural transformation

$$\Psi_f(\mathcal{F}) \boxtimes \Psi_g(\mathcal{G}) \to \Psi_{f \times_{\mathbb{C}} g}(\mathcal{F} \boxtimes_{\mathbb{C}^\times} \mathcal{G}).$$

4.2. Properties of algebraic nearby cycles

We now return to the setting of algebraic varieties. The goal of this section is to establish the fundamental properties of the nearby cycles functor with respect to a regular function $f : X \to \mathbb{A}^1$: it preserves constructibility; it commutes with Verdier duality and extension of scalars; and it is t-exact for the perverse t-structure. Some of the proofs in this section rely on calculations carried out in Section B.5.

We begin with the following observations.

LEMMA 4.2.1. *Let $f : X \to \mathbb{A}^1$ be a regular function. The map $j_{\boldsymbol{\eta}} : X_{\boldsymbol{\eta}} \hookrightarrow X$ is an affine morphism. As a consequence, for any $\mathcal{F} \in \mathrm{Perv}(X_{\boldsymbol{\eta}}, \Bbbk)$, the objects $j_{\boldsymbol{\eta}!}\mathcal{F}$ and $j_{\boldsymbol{\eta}*}\mathcal{F}$ are perverse.*

PROOF. The map $j_{\boldsymbol{\eta}}$ arises by base change from the affine open embedding $u : \mathbb{A}^1 \smallsetminus \{0\} \hookrightarrow \mathbb{A}$, so it is affine. The second assertion holds by Corollary 3.5.9. $\square$

LEMMA 4.2.2. *Let X be an irreducible variety of dimension n, and let $f : X \to \mathbb{A}^1$ be a nonconstant regular function. Then every irreducible component of $X_{\mathbf{s}}$ has dimension $n-1$.*

PROOF. It is enough to show that the irreducible components of the intersection of $X_{\mathbf{s}}$ with any affine open subset have dimension $n-1$. In other words, we may assume that X itself is an affine variety, say $X \subset \mathbb{A}^N$. Then f extends to a regular function $\tilde{f} : \mathbb{A}^N \to \mathbb{A}^1$. By, say, [**210**, Theorem 1.20], every component of $\tilde{f}^{-1}(0)$ has dimension $N-1$; then, by [**210**, Theorems 1.19 and 1.24], every component of $X_{\mathbf{s}} = X \cap \tilde{f}^{-1}(0)$ has dimension $n-1$. $\square$

The proofs of the next three statements are nearly identical, and we will present the argument only once below. Note, however, that the statements of Propositions 4.2.4 and 4.2.5 depend on the conclusion of Theorem 4.2.3. To avoid circular reasoning, the reader should read the argument once for Theorem 4.2.3, and then again for the remaining two propositions.

THEOREM 4.2.3. *Let $f : X \to \mathbb{A}^1$ be a regular function. For any $\mathcal{F} \in D^{\mathrm{b}}_{\mathrm{c}}(X_{\boldsymbol{\eta}}, \Bbbk)$, the object $\Psi_f(\mathcal{F})$ lies in $D^{\mathrm{b}}_{\mathrm{c}}(X_{\mathbf{s}}, \Bbbk)$.*

PROPOSITION 4.2.4. *Let $f : X \to \mathbb{A}^1$ be a regular function. For any $\mathcal{F} \in D^{\mathrm{b}}_{\mathrm{c}}(X_{\boldsymbol{\eta}}, \Bbbk)$, there is a natural isomorphism $\Psi_f(\mathbb{D}\mathcal{F}) \xrightarrow{\sim} \mathbb{D}(\Psi_f(\mathcal{F}))(1)$.*

PROPOSITION 4.2.5. *Let $f : X \to \mathbb{A}^1$ be a regular function, and let $\varphi : \Bbbk \to \Bbbk'$ be a ring homomorphism. For any $\mathcal{F} \in D^{\mathrm{b}}_{\mathrm{c}}(X_{\boldsymbol{\eta}}, \Bbbk)$, there is a natural isomorphism $\Bbbk' \otimes^L_{\Bbbk} \Psi_f(\mathcal{F}) \xrightarrow{\sim} \Psi_f(\Bbbk' \otimes^L_{\Bbbk} \mathcal{F})$.*

PROOF. *Step 1. Proof in the case of a divisor with simple normal crossings.* For this step, we make the following assumptions:

- Assume that X is smooth and that there is a divisor with simple normal crossings $Z \subset X$ such that $X_{\mathbf{s}} \subset Z$. Let $U = X \smallsetminus Z$, and let $h : U \hookrightarrow X_{\boldsymbol{\eta}}$ be the inclusion map.
- Assume that $\mathcal{F} = h_!\mathcal{G}$, where $\mathcal{G} \in D^{\mathrm{b}}_{\mathrm{locf}}(U, \Bbbk)$.

Let $Z_1, \ldots, Z_k$ be the irreducible components of Z, and consider the normal crossings stratification $\{X_I\}_{I \subset \{1,\ldots,k\}}$. By Lemma 4.2.2, $X_{\mathbf{s}}$ is a union of components of Z, and hence a union of strata of the normal crossings stratification.

Choose a point $x \in X_{\mathbf{s}}$. Let D be a normal crossings coordinate chart around x, and identify D (biholomorphically) with a polydisc $D_1 \times \cdots \times D_n \subset \mathbb{C}^n$. Then $f|_D : D \to \mathbb{C}^\times$ is holomorphic. Let $U' = U \cap D$, and let $h' = h|_{U'} : U' \hookrightarrow (D \cap X_{\boldsymbol{\eta}})$ be the inclusion map. We also let $\mathcal{G}' = \mathcal{G}|_{U'}$. By Lemma 4.1.7(2), we have

$$\Psi_f(h_!\mathcal{G})|_{D \cap X_{\mathbf{s}}} \cong \Psi_{f|_D}(h'_!\mathcal{G}').$$

We will now apply the results of Section B.5 to study $\Psi_g(h'_!\mathcal{G}')$. By construction, the map $f|_D : D \to \mathbb{C}$ satisfies assumption (B.5.1). Since $f|_D : D \to \mathbb{C}$ is holomorphic, Lemma B.5.9 (see also Remark B.5.10) tells us that it also satisfies assumption (B.5.6), so by Proposition B.5.12,

$$\Psi_f(h_!\mathcal{G})|_{D \cap X_I \cap X_{\mathbf{s}}} \in D^{\mathrm{b}}_{\mathrm{locf}}(D \cap X_I \cap X_{\mathbf{s}}, \Bbbk) \qquad \text{for all } I \subset \{1, \ldots, k\}. \tag{4.2.1}$$

Next, by Propositions B.5.15 and B.5.16 (along with Exercise 4.1.2), the natural maps

$$
\begin{aligned}
\Psi_f(\mathbb{D}(h_!\mathcal{G}))|_{D\cap X_{\mathbf{s}}} &\xrightarrow{\sim} \mathbb{D}\Psi_f(h_!\mathcal{G})(1)|_{D\cap X_{\mathbf{s}}},\\
\mathbb{k}' \overset{L}{\otimes}_{\mathbb{k}} \Psi_f(h_!\mathcal{G})|_{D\cap X_{\mathbf{s}}} &\xrightarrow{\sim} \Psi_f(\mathbb{k}' \overset{L}{\otimes}_{\mathbb{k}} h_!\mathcal{G})|_{D\cap X_{\mathbf{s}}}
\end{aligned}
\tag{4.2.2}
$$

are isomorphisms.

We have shown that every point $x \in X_{\mathbf{s}}$ admits an analytic neighborhood D such that (4.2.1) and (4.2.2) hold. The former implies that $\Psi_f(h_!\mathcal{G})$ is constructible (with respect to the normal crossings stratification), and the latter implies that Propositions 4.2.4 and 4.2.5 hold in our special case.

Step 2. Proof in the general case. We proceed by noetherian induction on $X_{\boldsymbol{\eta}}$. Let $\mathcal{F} \in D^{\mathrm{b}}_{\mathrm{c}}(X_{\boldsymbol{\eta}}, \mathbb{k})$, and let U be a smooth, connected open subset of $X_{\boldsymbol{\eta}}$ such that $\mathcal{F}|_U \in D^{\mathrm{b}}_{\mathrm{locf}}(U, \mathbb{k})$. Let $Z = X_{\boldsymbol{\eta}} \smallsetminus U$, and let $\overline{U}$ and $\overline{Z}$ be their closures in X. Let $j : U \hookrightarrow X_{\boldsymbol{\eta}}$ and $i : Z \hookrightarrow X_{\boldsymbol{\eta}}$ be the inclusion maps, and apply Ψ_f to the distinguished triangle $j_!(\mathcal{F}|_U) \to \mathcal{F} \to i_*i^*\mathcal{F} \to$. Applying Lemma 4.1.7(1) to the closed embedding $\bar{\imath} : \overline{Z} \cup X_{\mathbf{s}} \hookrightarrow X$, this triangle can be rewritten as

$$\Psi_f(j_!(\mathcal{F}|_U)) \to \Psi_f(\mathcal{F}) \to \Psi_{f\circ\bar{\imath}}(i^*\mathcal{F}) \to .$$

By induction, the desired conclusions hold for $\Psi_{f\circ\bar{\imath}}(i^*\mathcal{F})$. That is, $\Psi_{f\circ\bar{\imath}}(i^*\mathcal{F})$ is constructible, and the third vertical arrow in each of the following diagrams is an isomorphism:

$$
\begin{tikzcd}
{} & \Psi_f(\mathbb{D}(j_!(\mathcal{F}|_U))) \arrow[l] \arrow[d] & \Psi_f(\mathbb{D}\mathcal{F}) \arrow[l] \arrow[d] & \Psi_{f\circ\bar{\imath}}(\mathbb{D}(i^*\mathcal{F})) \arrow[l] \arrow[d, "\wr"] \\
{} & \mathbb{D}\Psi_f(j_!(\mathcal{F}|_U))(1) \arrow[l] & \mathbb{D}\Psi_f(\mathcal{F})(1) \arrow[l] & \mathbb{D}\Psi_{f\circ\bar{\imath}}(i^*\mathcal{F})(1) \arrow[l]
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathbb{k}' \otimes^L_{\mathbb{k}} \Psi_f(j_!(\mathcal{F}|_U)) \arrow[r] \arrow[d] & \mathbb{k}' \otimes^L_{\mathbb{k}} \Psi_f(\mathcal{F}) \arrow[r] \arrow[d] & \mathbb{k}' \otimes^L_{\mathbb{k}} \Psi_{f\circ\bar{\imath}}(i^*\mathcal{F}) \arrow[r] \arrow[d, "\wr"] & {} \\
\Psi_f(\mathbb{k}' \otimes^L_{\mathbb{k}} j_!(\mathcal{F}|_U)) \arrow[r] & \Psi_f(\mathbb{k}' \otimes^L_{\mathbb{k}} \mathcal{F}) \arrow[r] & \Psi_{f\circ\bar{\imath}}(\mathbb{k}' \otimes^L_{\mathbb{k}} i^*\mathcal{F}) \arrow[r] & {}
\end{tikzcd}
$$

To finish the proof, we must show that the conclusions of Theorem 4.2.3, Proposition 4.2.4, and Proposition 4.2.5 hold for $\Psi_f(j_!(\mathcal{F}|_U))$. By Theorem 2.1.18, there exists a smooth variety $\tilde{X}$ and a proper map $\tilde{X} \to \overline{U}$ that is an isomorphism over U, and such that the preimage of $\overline{Z} \cap X_{\mathbf{s}}$ is a divisor with simple normal crossings. Let $p : \tilde{X} \to X$ be the composition of this map with the inclusion map $\overline{U} \hookrightarrow X$. Identify U with $p^{-1}(U)$, and let $h : U \hookrightarrow \tilde{X}_{\boldsymbol{\eta}}$ be the inclusion map. Then $j_!(\mathcal{F}|_U) \cong p_{\boldsymbol{\eta}*}h_!(\mathcal{F}|_U)$, and by Lemma 4.1.7(1), we have

$$\Psi_f(j_!(\mathcal{F}|_U)) \cong p_{\mathbf{s}*}\Psi_{f\circ p}(h_!(\mathcal{F}|_U)).$$

The desired conclusions hold for $\Psi_{f\circ p}(h_!(\mathcal{F}|_U))$ by Step 1. By Theorem 2.7.1, we conclude that $\Psi_f(j_!(\mathcal{F}|_U))$ is constructible. Since the natural maps of Corollaries 4.1.9 and 4.1.10 are compatible with $p_{\mathbf{s}*}$, the fact that Propositions 4.2.4 and 4.2.5 hold for $\Psi_{f\circ p}(h_!(\mathcal{F}|_U))$ implies that they hold for $\Psi_f(j_!(\mathcal{F}|_U))$. $\square$

In view of Theorem 4.2.3, the distinguished triangles in Lemma 4.1.5 immediately imply the following statement.

COROLLARY 4.2.6. *Let $f : X \to \mathbb{A}^1$ be a regular function. For any $\mathcal{F} \in D^{\mathrm{b}}_{\mathrm{c}}(X, \Bbbk)$, the object $\Phi_f(\mathcal{F})$ lies in $D^{\mathrm{b}}_{\mathrm{c}}(X_{\mathbf{s}}, \Bbbk)$.*

The following statement is similar in spirit to Propositions 4.2.4 and 4.2.5. (See Exercise 4.1.3 for the definition of the relative external tensor product $\mathcal{F} \boxtimes_{\mathbb{A}^1 \smallsetminus \{0\}} \mathcal{G}$.)

PROPOSITION 4.2.7 (Künneth formula for nearby cycles). *Let $f : X \to \mathbb{A}^1$ and $g : X \to \mathbb{A}^1$ be regular functions. For $\mathcal{F} \in D^{\mathrm{b}}_{\mathrm{c}}(X_{\boldsymbol{\eta}}, \Bbbk)$ and $\mathcal{G} \in D^{\mathrm{b}}_{\mathrm{c}}(X_{\boldsymbol{\eta}}, \Bbbk)$, there is a natural isomorphism $\Psi_f(\mathcal{F}) \boxtimes \Psi_g(\mathcal{G}) \xrightarrow{\sim} \Psi_{f \times_{\mathbb{A}^1} g}(\mathcal{F} \boxtimes_{\mathbb{A}^1 \smallsetminus \{0\}} \mathcal{G})$.*

For a proof of this proposition, see [**206**, Theorems 1.0.4 and 1.3.1]. A key technical point in the proof is the existence of a certain family of compact neighborhoods of each point of $X \times_{\mathbb{A}^1} Y$ related to Milnor fibrations; see [**206**, Assumption 1.1.1 and Example 1.1.3]. Unfortunately, this proof is beyond the scope of the present book. (I do not know whether Proposition 4.2.7 has an "algebraic" proof similar to those of Propositions 4.2.4 and 4.2.5.)

THEOREM 4.2.8. *Let $f : X \to \mathbb{A}^1$ be a regular function. The functors Ψ_f and Φ_f are t-exact for the perverse t-structure.*

PROOF. We begin with the observation that for $\mathcal{G} \in \mathrm{Perv}(X, \Bbbk)$, we have

$$ {}^p\mathsf{H}^i(i_{\mathbf{s}}^*\mathcal{G}) = 0 \qquad \text{unless} \qquad i = -1, 0. \tag{4.2.3}$$

To see this, note first that by Lemma 4.2.1, the object $j_{\boldsymbol{\eta}!}(\mathcal{G}|_{X_{\boldsymbol{\eta}}})$ is perverse. Then (4.2.3) follows from the long exact sequence in perverse cohomology associated to the distinguished triangle $j_{\boldsymbol{\eta}!}(\mathcal{G}|_{X_{\boldsymbol{\eta}}}) \to \mathcal{G} \to i_{\mathbf{s}*}i_{\mathbf{s}}^*\mathcal{G} \to$. Similar reasoning using $i_{\mathbf{s}*}i_{\mathbf{s}}^!\mathcal{G} \to \mathcal{G} \to j_{\boldsymbol{\eta}*}(\mathcal{G}|_{X_{\boldsymbol{\eta}}}) \to$ shows that

$$ {}^p\mathsf{H}^i(i_{\mathbf{s}}^!\mathcal{G}) = 0 \qquad \text{unless} \qquad i = 0, 1. \tag{4.2.4}$$

We now proceed as follows.

Step 1. t-exactness of Ψ_f for field coefficients. Assume that $\Bbbk$ is a field, and let $\mathcal{F} \in \mathrm{Perv}(X_{\boldsymbol{\eta}}, \Bbbk)$. Since $\Psi_f(\mathcal{F})$ is constructible, we know that $\mathrm{End}(\Psi_f(\mathcal{F}))$ is a finite-dimensional $\Bbbk$-vector space. Consider the ring homomorphism

$$\mathrm{mon}_\Psi : \Bbbk[T, T^{-1}] \to \mathrm{End}(\Psi_f(\mathcal{F})).$$

This map has some nontrivial kernel $\mathfrak{a} \subset \Bbbk[T, T^{-1}]$. Since $\Bbbk$ is a field, the ring $\Bbbk[T, T^{-1}]$ is a principal ideal domain, and $\mathfrak{a}$ is the principal ideal generated by some Laurent polynomial $h(T)$. Consider the short exact sequence of $\Bbbk[T, T^{-1}]$-modules

$$0 \to \Bbbk[T, T^{-1}] \xrightarrow{h(T)} \Bbbk[T, T^{-1}] \to \Bbbk[T, T^{-1}]/(h(T)) \to 0.$$

Let $\mathcal{L}_h$ be the local system on $\mathbb{C}^\times$ corresponding to $\Bbbk[T, T^{-1}]/(h(T))$. We thus have a short exact sequence of local systems

$$0 \to \exp_! \underline{\Bbbk}_{\mathbb{C}} \xrightarrow{\mathrm{mon}_!(h(T))} \exp_! \underline{\Bbbk}_{\mathbb{C}} \to \mathcal{L}_h \to 0. \tag{4.2.5}$$

Let $\mathcal{H} = R\mathscr{H}om(f_{\boldsymbol{\eta}}^*\mathcal{L}_h, \mathcal{F})$. By Lemma 3.2.2, $\mathcal{H}$ is again a perverse sheaf. Apply the functor $i_{\mathbf{s}}^* j_{\boldsymbol{\eta}*} R\mathscr{H}om(f_{\boldsymbol{\eta}}^*(-)[1], \mathcal{F})$ to (4.2.5) to obtain a distinguished triangle

$$\Psi_f(\mathcal{F}) \xrightarrow{\mathrm{mon}_\Psi(h))} \Psi_f(\mathcal{F}) \to i_{\mathbf{s}}^* j_{\boldsymbol{\eta}*}\mathcal{H} \to . \tag{4.2.6}$$

Note that $j_{\boldsymbol{\eta}*}\mathcal{H}$ is perverse, so by (4.2.3), we have

$$ {}^p\mathsf{H}^i(i_{\mathbf{s}}^* j_{\boldsymbol{\eta}*}\mathcal{H}) = 0 \qquad \text{unless} \qquad i = -1, 0. \tag{4.2.7}$$

On the other hand, by the definition of h, $\mathrm{mon}_\Psi(h)$ is the zero morphism, so the triangle (4.2.6) splits, and

$$i_{\mathbf{s}}^* j_{\boldsymbol{\eta}*}\mathcal{H} \cong \Psi_f(\mathcal{F}) \oplus \Psi_f(\mathcal{F})[1].$$

If ${}^p\mathsf{H}^i(\Psi_f(\mathcal{F})) \neq 0$ for some $i \neq 0$, we would get a contradiction with (4.2.7). We conclude that $\Psi_f(\mathcal{F})$ is perverse.

Step 2. Right t-exactness of Ψ_f in general. Let $\mathcal{F} \in \mathrm{Perv}(X_{\boldsymbol{\eta}}, \Bbbk)$, and let n by the largest integer such that ${}^p\mathsf{H}^n(\Psi_f(\mathcal{F})) \neq 0$. By Lemmas 3.2.4 and 3.2.6, we can find a ring homomorphism $\varphi : \Bbbk \to \Bbbk'$ with $\Bbbk'$ a field such that

$$\Bbbk' \overset{L}{\otimes} \Psi_f(\mathcal{F}) \in {}^pD^{\mathrm{b}}_{\mathrm{c}}(X_{\mathbf{s}}, \Bbbk')^{\leq n} \qquad \text{and} \qquad {}^p\mathsf{H}^n(\Bbbk' \overset{L}{\otimes} \Psi_f(\mathcal{F})) \neq 0.$$

By Proposition 4.2.5, these conditions also hold for $\Psi_f(\Bbbk' \otimes^L_{\Bbbk} \mathcal{F})$. By Lemma 3.2.4 again, $\Bbbk' \otimes^L_{\Bbbk} \mathcal{F}$ lies in ${}^pD^{\mathrm{b}}_{\mathrm{c}}(X_{\boldsymbol{\eta}}, \Bbbk')^{\leq 0}$. Since $\Bbbk'$ is a field, Step 1 tells us that $\Psi_f(\Bbbk' \otimes^L_{\Bbbk} \mathcal{F}) \in {}^pD^{\mathrm{b}}_{\mathrm{c}}(X_{\mathbf{s}}, \Bbbk')^{\leq 0}$. We conclude that $n \leq 0$, and hence that $\Psi_f(\mathcal{F}) \in {}^pD^{\mathrm{b}}_{\mathrm{c}}(X_{\mathbf{s}}, \Bbbk)^{\leq 0}$.

Step 3. Left t-exactness of Ψ_f in general. Let $\mathcal{F} \in \mathrm{Perv}(X_{\boldsymbol{\eta}}, \Bbbk)$, and let n be the smallest integer such that ${}^p\mathsf{H}^n(\Psi_f(\mathcal{F})) \neq 0$. By Lemma 3.2.7, there is a prime ideal $\mathfrak{p} \subset \Bbbk$ such that if we set $\Bbbk' = \Bbbk_{\mathfrak{p}}/\mathfrak{p}\Bbbk_{\mathfrak{p}}$, then

$$\Bbbk' \overset{L}{\otimes} \Psi_f(\mathcal{F}) \in {}^pD^{\mathrm{b}}_{\mathrm{c}}(X_{\mathbf{s}}, \Bbbk)^{\geq n-\mathrm{ht}(\mathfrak{p})} \qquad \text{and} \qquad {}^p\mathsf{H}^{n-\mathrm{ht}(\mathfrak{p})}(\Bbbk' \overset{L}{\otimes} \Psi_f(\mathcal{F})) \neq 0.$$

By Proposition 4.2.5, these conditions also hold for $\Psi_f(\Bbbk' \otimes^L_{\Bbbk} \mathcal{F})$. Similar reasoning to that in Step 2, using Lemma 3.2.7 instead of Lemma 3.2.4, shows that $n \geq 0$, and hence that $\Psi_f(\mathcal{F}) \in {}^pD^{\mathrm{b}}_{\mathrm{c}}(X_{\mathbf{s}}, X)^{\geq 0}$.

Step 4. t-exactness of Φ_f. By Lemma 4.1.5, for any $\mathcal{G} \in \mathrm{Perv}(X, \Bbbk)$, we have distinguished triangles

$$\Psi_f(\mathcal{G}|_{X_{\boldsymbol{\eta}}}) \xrightarrow{\mathrm{can}} \Phi_f(\mathcal{G}) \to i_{\mathbf{s}}^*\mathcal{G} \to,$$
$$i_{\mathbf{s}}^!\mathcal{G} \to \Phi_f(\mathcal{G}) \xrightarrow{\mathrm{var}} \Psi_f(\mathcal{G}|_{X_{\boldsymbol{\eta}}})(-1) \to .$$

By Steps 2 and 3, $\Psi_f(\mathcal{G}|_{X_{\boldsymbol{\eta}}})$ is perverse. Then, by (4.2.3) and (4.2.4), the first triangle above implies that $\Phi_f(\mathcal{G}) \in {}^pD^{\mathrm{b}}_{\mathrm{c}}(X_{\mathbf{s}}, \Bbbk)^{\leq 0}$, and the second implies that $\Phi_f(\mathcal{G}) \in {}^pD^{\mathrm{b}}_{\mathrm{c}}(X_{\mathbf{s}}, \Bbbk)^{\geq 0}$. Thus, Φ_f is t-exact. $\square$

4.3. Extension across a hypersurface

Let $f : X \to \mathbb{A}^1$ be a regular function. In this section, we consider the following question: given a perverse sheaf $\mathcal{F} \in \mathrm{Perv}(X_{\boldsymbol{\eta}}, \Bbbk)$, what are all the possible ways to extend it to a perverse sheaf on all of X? The answer will be given in Theorem 4.3.9.

The maximal extension functor. We start by describing a new functorial way to extend perverse sheaves from $X_{\boldsymbol{\eta}}$ to X. In general, this construction is different from the functors $j_{\boldsymbol{\eta}!}$, $j_{\boldsymbol{\eta}*}$, and $j_{\boldsymbol{\eta}!*}$ that we already know.

PROPOSITION 4.3.1. *Let $f : X \to \mathbb{A}^1$ be a regular function. There is an exact functor*

$$\Xi_f : \mathrm{Perv}(X_{\boldsymbol{\eta}}, \Bbbk) \to \mathrm{Perv}(X, \Bbbk)$$

such that for any $\mathcal{F} \in \mathrm{Perv}(X_{\boldsymbol{\eta}}, \Bbbk)$, there are natural short exact sequences

$$0 \to i_{\mathbf{s}*}\Psi_f(\mathcal{F}) \xrightarrow{\mathrm{can}_\Xi} \Xi_f(\mathcal{F}) \to j_{\boldsymbol{\eta}*}\mathcal{F} \to 0,$$
$$0 \to j_{\boldsymbol{\eta}!}\mathcal{F} \to \Xi_f(\mathcal{F}) \xrightarrow{\mathrm{var}_\Xi} i_{\mathbf{s}*}\Psi_f(\mathcal{F})(-1) \to 0,$$

where $\mathrm{var}_\Xi \circ \mathrm{can}_\Xi = i_{\mathbf{s}*}(\mathrm{var} \circ \mathrm{can})$.

DEFINITION 4.3.2. Let $f : X \to \mathbb{A}^1$ be a regular function. The exact functor $\Xi_f : \mathrm{Perv}(X_{\boldsymbol{\eta}}, \Bbbk) \to \mathrm{Perv}(X, \Bbbk)$ is called the **maximal extension functor**.

PROOF. *Step 1. Definition of* Ξ_f*.* Recall the map $d : \exp_! \underline{\Bbbk}_{\mathbb{C}}(1) \to \exp_! \underline{\Bbbk}_{\mathbb{C}}$ from (4.1.4). For $\mathcal{F} \in \mathrm{Perv}(X_{\boldsymbol{\eta}}, \Bbbk)$, this gives rise to a natural map

$$j_{\boldsymbol{\eta}!} R\mathscr{H}\!om(f_{\boldsymbol{\eta}}^* \exp_! \underline{\Bbbk}_{\mathbb{C}}[1], \mathcal{F}) \to j_{\boldsymbol{\eta}*} R\mathscr{H}\!om(f_{\boldsymbol{\eta}}^* \exp_! \underline{\Bbbk}_{\mathbb{C}}[1](1), \mathcal{F}).$$

Complete this map to a distinguished triangle, and denote the third term by $\Xi_f(\mathcal{F})$. Then $j_{\boldsymbol{\eta}}^* \Xi_f(\mathcal{F})$ is isomorphic to the third term of the distinguished triangle

$$R\mathscr{H}\!om(f_{\boldsymbol{\eta}}^* \exp_! \underline{\Bbbk}_{\mathbb{C}}[1], \mathcal{F}) \to R\mathscr{H}\!om(f_{\boldsymbol{\eta}}^* \exp_! \underline{\Bbbk}_{\mathbb{C}}[1](1), \mathcal{F}) \to R\mathscr{H}\!om(f_{\boldsymbol{\eta}}^* \underline{\Bbbk}_{\mathbb{C}^\times}, \mathcal{F}) \to \tag{4.3.1}$$

induced by (4.1.4). In other words, we have

$$j_{\boldsymbol{\eta}}^* \Xi_f(\mathcal{F}) \cong \mathcal{F}. \tag{4.3.2}$$

To make the assignment $\mathcal{F} \mapsto \Xi(\mathcal{F})$ into a functor, we must explain what to do with a morphism $\phi : \mathcal{F} \to \mathcal{G}$ in $\mathrm{Perv}(X_{\boldsymbol{\eta}}, \Bbbk)$. We claim that there is a unique morphism $\Xi_f(\phi)$ such that the diagram

$$\begin{tikzcd}
\Xi_f(\mathcal{F}) \arrow[r] \arrow[d, "\Xi_f(\phi)"'] & j_{\boldsymbol{\eta}!} R\mathscr{H}\!om(f_{\boldsymbol{\eta}}^* \exp_! \underline{\Bbbk}_{\mathbb{C}}, \mathcal{F}) \arrow[r] \arrow[d] & j_{\boldsymbol{\eta}*} R\mathscr{H}\!om(f_{\boldsymbol{\eta}}^* \exp_! \underline{\Bbbk}_{\mathbb{C}}(1), \mathcal{F}) \arrow[r] \arrow[d] & {} \\
\Xi_f(\mathcal{G}) \arrow[r] & j_{\boldsymbol{\eta}!} R\mathscr{H}\!om(f_{\boldsymbol{\eta}}^* \exp_! \underline{\Bbbk}_{\mathbb{C}}, \mathcal{G}) \arrow[r] & j_{\boldsymbol{\eta}*} R\mathscr{H}\!om(f_{\boldsymbol{\eta}}^* \exp_! \underline{\Bbbk}_{\mathbb{C}}(1), \mathcal{G}) \arrow[r] & {}
\end{tikzcd} \tag{4.3.3}$$

commutes. Indeed, this claim follows from Lemma A.4.10 and the observation that

$$\begin{aligned}
\mathrm{Hom}(\Xi_f(\mathcal{F}), & j_{\boldsymbol{\eta}*} R\mathscr{H}\!om(f_{\boldsymbol{\eta}}^* \exp_! \underline{\Bbbk}_{\mathbb{C}}(1), \mathcal{G})[-1]) \\
&\cong \mathrm{Hom}(j_{\boldsymbol{\eta}}^* \Xi_f(\mathcal{F}), R\mathscr{H}\!om(f_{\boldsymbol{\eta}}^* \exp_! \underline{\Bbbk}_{\mathbb{C}}(1), \mathcal{G})[-1]) \\
&\cong \mathrm{Hom}(\mathcal{F} \overset{L}{\otimes} f_{\boldsymbol{\eta}}^* \exp_! \underline{\Bbbk}_{\mathbb{C}}(1), \mathcal{G}[-1]) = 0,
\end{aligned}$$

where the last term vanishes by Remark 3.2.3. We have now defined Ξ_f as an additive functor $\mathrm{Perv}(X_{\boldsymbol{\eta}}, \Bbbk) \to D^+(X, \Bbbk)$. We will see in Step 4 that it actually takes values in $\mathrm{Perv}(X, \Bbbk)$.

Step 2. There is a natural octahedral diagram

$$\begin{tikzcd}[column sep=small]
i_{\mathbf{s}*} i_{\mathbf{s}}^* j_{\boldsymbol{\eta}*} R\mathscr{H}\!om(f_{\boldsymbol{\eta}}^* \exp_! \underline{\Bbbk}_{\mathbb{C}}[1], \mathcal{F}) \arrow[dd, rightsquigarrow] & & j_{\boldsymbol{\eta}*}\mathcal{F} \arrow[ll, rightsquigarrow] \arrow[dl, rightsquigarrow] \\
& j_{\boldsymbol{\eta}*} R\mathscr{H}\!om(f_{\boldsymbol{\eta}}^* \exp_! \underline{\Bbbk}_{\mathbb{C}}[1], \mathcal{F}) \arrow[ul] \arrow[dr] & \\
j_{\boldsymbol{\eta}!} R\mathscr{H}\!om(f_{\boldsymbol{\eta}}^* \exp_! \underline{\Bbbk}_{\mathbb{C}}[1], \mathcal{F}) \arrow[ur] \arrow[rr] & & j_{\boldsymbol{\eta}*} R\mathscr{H}\!om(f_{\boldsymbol{\eta}}^* \exp_! \underline{\Bbbk}_{\mathbb{C}}[1](1), \mathcal{F}) \arrow[uu]
\end{tikzcd}$$

$$\begin{tikzcd}[column sep=small]
i_{\mathbf{s}*} i_{\mathbf{s}}^* j_{\boldsymbol{\eta}*} R\mathscr{H}\!om(f_{\boldsymbol{\eta}}^* \exp_! \underline{\Bbbk}_{\mathbb{C}}[1], \mathcal{F}) \arrow[dd, rightsquigarrow] \arrow[dr] & & j_{\boldsymbol{\eta}*}\mathcal{F} \arrow[ll, rightsquigarrow] \\
& \Xi_f(\mathcal{F}) \arrow[ur] \arrow[dl, rightsquigarrow] & \\
j_{\boldsymbol{\eta}!} R\mathscr{H}\!om(f_{\boldsymbol{\eta}}^* \exp_! \underline{\Bbbk}_{\mathbb{C}}[1], \mathcal{F}) \arrow[rr] & & j_{\boldsymbol{\eta}*} R\mathscr{H}\!om(f_{\boldsymbol{\eta}}^* \exp_! \underline{\Bbbk}_{\mathbb{C}}[1](1), \mathcal{F}) \arrow[uu] \arrow[ul]
\end{tikzcd}$$

In the upper cap, the left-hand distinguished triangle comes from the open and closed embeddings $j_{\boldsymbol{\eta}}$ and $i_{\mathbf{s}}$, and the right-hand distinguished triangle comes from (4.3.1). Both of these distinguished triangles are clearly natural. In the lower cap, the bottom distinguished triangle is the definition of $\Xi_f(\mathcal{F})$; we saw its naturality in (4.3.3). Finally, observe that in the top distinguished triangle in the lower cap, the first term is supported on $X_{\mathbf{s}}$, and the last term involves $j_{\boldsymbol{\eta}*}$. Theorem 1.3.10 tells us that this triangle is natural.

Step 3. There exist natural morphisms $\mathrm{can}_\Xi : i_{\mathbf{s}*}\Psi_f(\mathcal{F}) \to \Xi_f(\mathcal{F})$ *and* $\mathrm{var}_\Xi : \Xi_f(\mathcal{F}) \to i_{\mathbf{s}*}\Psi_f(\mathcal{F})(-1)$ *such that* $\mathrm{var}_\Xi \circ \mathrm{can}_\Xi = i_{\mathbf{s}*}(\mathrm{var} \circ \mathrm{can})$. The object in the upper-left corner of the diagrams in Step 2 is $i_{\mathbf{s}*}\Psi_f(\mathcal{F})$, and we define $\mathrm{can}_\Xi : i_{\mathbf{s}*}\Psi_f(\mathcal{F}) \to \Xi_f(\mathcal{F})$ to be the map from the lower cap. Now apply $i_{\mathbf{s}}^*$ to that octahedron. We obtain

$$\begin{tikzcd}
\Psi_f(\mathcal{F}) \arrow[dd, rightsquigarrow] \arrow[dr, equal] & & i_{\mathbf{s}}^* j_{\boldsymbol{\eta}*}\mathcal{F} \arrow[ll, rightsquigarrow] \arrow[dl, rightsquigarrow] \\
& \Psi_f(\mathcal{F}) \arrow[dr, "\mathrm{var}\circ\mathrm{can}"] & \\
0 \arrow[ur] \arrow[rr] & & \Psi_f(\mathcal{F})(-1) \arrow[uu]
\end{tikzcd}
\qquad
\begin{tikzcd}
\Psi_f(\mathcal{F}) \arrow[dd, rightsquigarrow] \arrow[dr, "i_{\mathbf{s}}^*\mathrm{can}_\Xi"] & & i_{\mathbf{s}}^* j_{\boldsymbol{\eta}*}\mathcal{F} \arrow[ll, rightsquigarrow] \\
& i_{\mathbf{s}}^*\Xi_f(\mathcal{F}) \arrow[ur, rightsquigarrow] \arrow[dl, rightsquigarrow] & \\
0 \arrow[rr] & & \Psi_f(\mathcal{F})(-1) \arrow[uu] \arrow[ul, "\sim"]
\end{tikzcd}
\tag{4.3.4}$$

The lower cap gives us a natural isomorphism $i_{\mathbf{s}}^*\Xi_f(\mathcal{F}) \cong \Psi_f(\mathcal{F})(-1)$. Define var_Ξ to be the composition

$$\Xi_f(\mathcal{F}) \to i_{\mathbf{s}*}i_{\mathbf{s}}^*\Xi_f(\mathcal{F}) \cong i_{\mathbf{s}*}\Psi_f(\mathcal{F})(-1).$$

The fact that $\mathrm{var}_\Xi \circ \mathrm{can}_\Xi = i_{\mathbf{s}*}(\mathrm{var} \circ \mathrm{can})$ follows from the commutativity of the square in (4.3.4) involving $\Psi_f(\mathcal{F})$, $\Psi_f(\mathcal{F})(-1)$, and $i_{\mathbf{s}}^*\Xi_f(\mathcal{F})$.

Step 4. Conclusion of the proof. We saw in Step 3 that $i_{\mathbf{s}}^*\Xi_f(\mathcal{F}) \cong \Psi_f(\mathcal{F})(-1)$. Similarly, applying $i_{\mathbf{s}}^!$ to the lower cap of the octahedron from Step 2 yields (using Corollary 1.3.12) the following natural isomorphisms:

$$i_{\mathbf{s}}^!\Xi_f(\mathcal{F}) \cong i_{\mathbf{s}}^! j_{\boldsymbol{\eta}!} R\mathscr{H}om(f_{\boldsymbol{\eta}}^* \exp_! \underline{\Bbbk}_{\mathbb{C}}, \mathcal{F}) \cong i_{\mathbf{s}}^* j_{\boldsymbol{\eta}*} R\mathscr{H}om(f_{\boldsymbol{\eta}}^* \exp_! \underline{\Bbbk}_{\mathbb{C}}, \mathcal{F})[-1] \cong \Psi_f(\mathcal{F}).$$

Combine these observations with (4.3.2), and then invoke Theorem 1.3.10 to obtain natural distinguished triangles

$$i_{\mathbf{s}*}\Psi_f(\mathcal{F}) \xrightarrow{\mathrm{can}_\Xi} \Xi_f(\mathcal{F}) \to j_{\boldsymbol{\eta}*}\mathcal{F} \to,$$
$$j_{\boldsymbol{\eta}!}\mathcal{F} \to \Xi_f(\mathcal{F}) \xrightarrow{\mathrm{var}_\Xi} i_{\mathbf{s}*}\Psi_f(\mathcal{F})(-1) \to$$

in $D^+(X, \Bbbk)$. These distinguished triangles show that $\Xi_f(\mathcal{F})$ is constructible and that it takes values in $\mathrm{Perv}(X, \Bbbk)$.

Finally, recall that the functors $\mathcal{F} \mapsto i_{\mathbf{s}*}\Psi_f(\mathcal{F})$ and $\mathcal{F} \mapsto j_{\boldsymbol{\eta}*}\mathcal{F}$ are both exact (by Theorem 4.2.8 and by Lemma 4.2.1, respectively). A diagram chase shows that Ξ_f is also exact. $\square$

REMARK 4.3.3. Consider the composition of the two natural maps

$$j_{\boldsymbol{\eta}!}\mathcal{F} \to \Xi_f(\mathcal{F}) \to j_{\boldsymbol{\eta}*}\mathcal{F} \tag{4.3.5}$$

from Proposition 4.3.1. The last step of the preceding proof shows that the restriction of (4.3.5) to $X_{\boldsymbol{\eta}}$ is identified (via (4.3.2)) with the identity morphism of $\mathcal{F}$. In other words, (4.3.5) is equal to the canonical map $j_{\boldsymbol{\eta}!}\mathcal{F} \to j_{\boldsymbol{\eta}*}\mathcal{F}$ from (1.2.1).

Beilinson's vanishing cycles functor. We will now use Ξ_f to construct an exact functor $\Phi'_f : \mathrm{Perv}(X, \Bbbk) \to \mathrm{Perv}(X_{\mathbf{s}}, \Bbbk)$ that has similar formal properties to the vanishing cycles functor Φ_f of Section 4.1. Given a perverse sheaf $\mathcal{F} \in \mathrm{Perv}(X, \Bbbk)$, consider the adjunction maps

$$\epsilon : j_{\boldsymbol{\eta}!}(\mathcal{F}|_{X_{\boldsymbol{\eta}}}) \to \mathcal{F}, \qquad \eta : \mathcal{F} \to j_{\boldsymbol{\eta}*}(\mathcal{F}|_{X_{\boldsymbol{\eta}}}).$$

Their composition $\eta \circ \epsilon$ is the canonical map $j_{\boldsymbol{\eta}!}(\mathcal{F}|_{X_{\boldsymbol{\eta}}}) \to j_{\boldsymbol{\eta}*}(\mathcal{F}|_{X_{\boldsymbol{\eta}}})$. Next, let

$$\epsilon' : j_{\boldsymbol{\eta}!}(\mathcal{F}|_{X_{\boldsymbol{\eta}}}) \to \Xi_f(\mathcal{F}|_{X_{\boldsymbol{\eta}}}), \qquad \eta' : \Xi_f(\mathcal{F}|_{X_{\boldsymbol{\eta}}}) \to j_{\boldsymbol{\eta}*}(\mathcal{F}|_{X_{\boldsymbol{\eta}}})$$

be the maps from the short exact sequences in Proposition 4.3.1.

Definition 4.3.4. Let $f : X \to \mathbb{A}^1$ be a regular function. **Beilinson's vanishing cycles functor** associated to f, denoted by

$$\Phi'_f : \mathrm{Perv}(X, \Bbbk) \to \mathrm{Perv}(X_{\mathbf{s}}, \Bbbk),$$

is defined by $\Phi_f(\mathcal{F}) = i_{\mathbf{s}}^* \mathsf{H}^0(V(\mathcal{F}))$, where $V(\mathcal{F}) \in \mathit{Ch}^{\mathrm{b}}\mathrm{Perv}(X, \Bbbk)$ is the chain complex

$$\cdots \to 0 \to j_{\boldsymbol{\eta}!}(\mathcal{F}|_{X_{\boldsymbol{\eta}}}) \xrightarrow{\left[\begin{smallmatrix}\epsilon \\ \epsilon'\end{smallmatrix}\right]} \mathcal{F} \oplus \Xi_f(\mathcal{F}|_{X_{\boldsymbol{\eta}}}) \xrightarrow{[\,-\eta\ \eta'\,]} j_{\boldsymbol{\eta}*}(\mathcal{F}|_{X_{\boldsymbol{\eta}}}) \to 0 \to \cdots$$

which is concentrated in degrees -1, 0, and 1.

According to Remark 4.3.3, we have $\eta' \circ \epsilon' = \eta \circ \epsilon$, so the diagram above really is a chain complex. However, there is a well-definedness issue: we have to check that $i_{\mathbf{s}}^* \mathsf{H}^0(V(\mathcal{F}))$ is a perverse sheaf. We will do this in the course of proving Proposition 4.3.5 below.

For the relationship between Φ'_f and Φ_f, see Exercise 4.3.2.

Proposition 4.3.5. *Let $f : X \to \mathbb{A}^1$ be a regular function. The functor Φ'_f : $\mathrm{Perv}(X, \Bbbk) \to \mathrm{Perv}(X_{\mathbf{s}}, \Bbbk)$ is exact.*

Proof. The functors $j_{\boldsymbol{\eta}!}(\mathcal{F}|_{X_{\boldsymbol{\eta}}})$, $\Xi_f(\mathcal{F}|_{X_{\boldsymbol{\eta}}})$, and $j_{\boldsymbol{\eta}*}(\mathcal{F}|_{X_{\boldsymbol{\eta}}})$ are all exact, so the assignment $\mathcal{F} \mapsto V(\mathcal{F})$ is an exact functor $\mathrm{Perv}(X, \Bbbk) \to \mathit{Ch}^{\mathrm{b}}\mathrm{Perv}(X, \Bbbk)$. Thus, if $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ is a short exact sequence in $\mathrm{Perv}(X, \Bbbk)$, then

$$0 \to V(\mathcal{F}) \to V(\mathcal{G}) \to V(\mathcal{H}) \to 0 \tag{4.3.6}$$

is a short exact sequence of chain complexes. It gives rise to a long exact sequence in cohomology.

We claim that $\mathsf{H}^i(V(\mathcal{F})) = 0$ for $i \neq 0$. If $i < -1$ or $i > 1$, this is obvious. Since ϵ' is injective, so is $\left[\begin{smallmatrix}\epsilon \\ \epsilon'\end{smallmatrix}\right]$, so $\mathsf{H}^{-1}(V(\mathcal{F})) = 0$. Similarly, the surjectivity of η' implies that $\mathsf{H}^1(V(\mathcal{F})) = 0$. The same remarks apply to the other terms in (4.3.6), so in fact our long exact sequence in cohomology reduces to a short exact sequence

$$0 \to \mathsf{H}^0(V(\mathcal{F})) \to \mathsf{H}^0(V(\mathcal{G})) \to \mathsf{H}^0(V(\mathcal{H})) \to 0.$$

Finally, we must show that $i_{\mathbf{s}}^*$ takes each of these terms to a perverse sheaf. This would follow if we knew that $\mathsf{H}^0(V(\mathcal{F}))|_{X_{\boldsymbol{\eta}}} = 0$ (and likewise for the other terms). Since restriction to $X_{\boldsymbol{\eta}}$ is an exact functor, we have $\mathsf{H}^0(V(\mathcal{F}))|_{X_{\boldsymbol{\eta}}} \cong \mathsf{H}^0(V(\mathcal{F})|_{X_{\boldsymbol{\eta}}})$. But $V(\mathcal{F})|_{X_{\boldsymbol{\eta}}}$ can be identified with the chain complex

$$\cdots \to 0 \to \mathcal{F}|_{X_{\boldsymbol{\eta}}} \xrightarrow{\left[\begin{smallmatrix}\mathrm{id} \\ \mathrm{id}\end{smallmatrix}\right]} \mathcal{F}|_{X_{\boldsymbol{\eta}}} \oplus \mathcal{F}|_{X_{\boldsymbol{\eta}}} \xrightarrow{[\,-\mathrm{id}\ \mathrm{id}\,]} \mathcal{F}|_{X_{\boldsymbol{\eta}}} \to 0 \to \cdots,$$

which is clearly acyclic. ∎

In the course of the preceding proof, we saw that there is a natural isomorphism $i_{\mathbf{s}*}\Phi'_f(\mathcal{F}) \cong \mathsf{H}^0(V(\mathcal{F}))$.

Definition 4.3.6. Let $f : X \to \mathbb{A}^1$ be a regular function, and let $\mathcal{F} \in \mathrm{Perv}(X_{\boldsymbol{\eta}}, \Bbbk)$. The **canonical** and **variation maps**, denoted by

$$\Psi_f(\mathcal{F}) \xrightarrow{\mathrm{can}'} \Phi'_f(\mathcal{F}) \xrightarrow{\mathrm{var}'} \Psi_f(\mathcal{F})(-1),$$

are the maps obtained by applying $i_{\mathbf{s}}^* \mathsf{H}^0(-)$ to the following maps of chain complexes in $\mathit{Ch}^{\mathrm{b}}\mathrm{Perv}(X, \Bbbk)$:

$$\begin{tikzcd} & j_{\boldsymbol{\eta}*}(\mathcal{F}|_{X_{\boldsymbol{\eta}}}) & \\ i_{\mathbf{s}*}\Psi_f(\mathcal{F}|_{X_{\boldsymbol{\eta}}}) \arrow[r, "\left[\begin{smallmatrix} 0 \\ \mathrm{can}_\Xi \end{smallmatrix}\right]"] & \mathcal{F} \oplus \Xi_f(\mathcal{F}|_{X_{\boldsymbol{\eta}}}) \arrow[u, "{[\,-\eta\ \eta'\,]}"] \arrow[r, "{[\,0\ \mathrm{var}_\Xi\,]}"] & i_{\mathbf{s}*}\Psi_f(\mathcal{F}|_{X_{\boldsymbol{\eta}}})(-1) \\ & j_{\boldsymbol{\eta}!}(\mathcal{F}|_{X_{\boldsymbol{\eta}}}) \arrow[u, "\left[\begin{smallmatrix} \epsilon \\ \epsilon' \end{smallmatrix}\right]"] & \end{tikzcd}$$

It is immediate from the definition that

$$\mathrm{var}' \circ \mathrm{can}' = i_{\mathbf{s}}^*(\mathrm{var}_\Xi \circ \mathrm{can}_\Xi) = \mathrm{var} \circ \mathrm{can}.$$

Lemma 4.3.7. *Let $f : X \to \mathbb{A}^1$ be a regular map, and let $\Upsilon_f : \mathrm{Perv}(X, \Bbbk) \to \mathrm{Perv}(X, \Bbbk)$ be the functor given by*

$$\Upsilon_f(\mathcal{F}) = \ker(\mathcal{F} \oplus \Xi_f(\mathcal{F}|_{X_{\boldsymbol{\eta}}}) \xrightarrow{[\,-\eta\ \eta'\,]} j_{\boldsymbol{\eta}*}(\mathcal{F}|_{X_{\boldsymbol{\eta}}})).$$

Then Υ_f is an exact functor. Moreover, for $\mathcal{F} \in \mathrm{Perv}(X, \Bbbk)$, there are natural commutative diagrams

$$\begin{tikzcd} 0 \arrow[r] & i_{\mathbf{s}*}\Psi_f(\mathcal{F}|_{X_{\boldsymbol{\eta}}}) \arrow[r] \arrow[d, equal] & \Upsilon_f(\mathcal{F}) \arrow[r] \arrow[d] & \mathcal{F} \arrow[r] \arrow[d, "\eta"] & 0 \\ 0 \arrow[r] & i_{\mathbf{s}*}\Psi_f(\mathcal{F}|_{X_{\boldsymbol{\eta}}}) \arrow[r, "\mathrm{can}_\Xi"] & \Xi_f(\mathcal{F}|_{X_{\boldsymbol{\eta}}}) \arrow[r, "\eta'"] & j_{\boldsymbol{\eta}*}(\mathcal{F}|_{X_{\boldsymbol{\eta}}}) \arrow[r] & 0 \end{tikzcd} \tag{4.3.7}$$

and

$$\begin{tikzcd} 0 \arrow[r] & j_{\boldsymbol{\eta}!}(\mathcal{F}|_{X_{\boldsymbol{\eta}}}) \arrow[r, "\epsilon'"] \arrow[d, "0"'] & \Xi_f(\mathcal{F}|_{X_{\boldsymbol{\eta}}}) \arrow[r, "\mathrm{var}_\Xi"] \arrow[d] & i_{\mathbf{s}*}\Psi_f(\mathcal{F}|_{X_{\boldsymbol{\eta}}})(-1) \arrow[r] \arrow[d, "\mathrm{can}'"] & 0 \\ 0 \arrow[r] & j_{\boldsymbol{\eta}!}(\mathcal{F}|_{X_{\boldsymbol{\eta}}})(-1) \arrow[r] & \Upsilon_f(\mathcal{F})(-1) \arrow[r] & i_{\mathbf{s}*}\Phi'_f(\mathcal{F})(-1) \arrow[r] & 0 \end{tikzcd} \tag{4.3.8}$$

Proof. Since $j_{\boldsymbol{\eta}*}((-)|_{X_{\boldsymbol{\eta}}})$ and Ξ_f are exact functors, an easy diagram chase shows that Υ_f is also exact. Using the fact that η' is surjective, another diagram chase shows that the induced map $\Upsilon_f(\mathcal{F}) \to \mathcal{F}$ is surjective and that its kernel is naturally identified with $\ker \eta' \cong i_{\mathbf{s}*}\Psi_f(\mathcal{F}|_{X_{\boldsymbol{\eta}}})$. This establishes (4.3.7).

The bottom short exact sequence in (4.3.8) comes from the definition of $\Phi'_f(\mathcal{F})$, and the middle vertical arrow is induced by $\mathrm{can}_\Xi \circ \mathrm{var}_\Xi$. The commutativity of the diagram follows easily. $\square$

Vanishing cycles data. The main result of this section describes perverse sheaves on X in terms of the following notion.

DEFINITION 4.3.8. Let $f : X \to \mathbb{A}^1$ be a regular function. A **vanishing cycles datum** for f is a quadruple $(\mathcal{F}, \mathcal{G}, \alpha, \beta)$, where $\mathcal{F} \in \mathrm{Perv}(X_{\boldsymbol{\eta}}, \Bbbk)$, $\mathcal{G} \in \mathrm{Perv}(X_{\mathbf{s}}, \Bbbk)$, and α and β are maps

$$\Psi_f(\mathcal{F}) \xrightarrow{\alpha} \mathcal{G} \xrightarrow{\beta} \Psi_f(\mathcal{F})(-1) \qquad \text{such that} \qquad \beta \circ \alpha = \mathrm{var} \circ \mathrm{can}. \tag{4.3.9}$$

A **morphism** of vanishing cycles data $\phi : (\mathcal{F}, \mathcal{G}, \alpha, \beta) \to (\mathcal{F}', \mathcal{G}', \alpha', \beta')$ is a pair of morphisms $\phi_{\boldsymbol{\eta}} : \mathcal{F} \to \mathcal{F}'$, $\phi_{\mathbf{s}} : \mathcal{G} \to \mathcal{G}'$ such that the diagram

$$\begin{tikzcd}
\Psi_f(\mathcal{F}) \arrow[r, "\alpha"] \arrow[d, "\Psi_f(\phi_{\boldsymbol{\eta}})"'] & \mathcal{G} \arrow[r, "\beta"] \arrow[d, "\phi_{\mathbf{s}}"'] & \Psi_f(\mathcal{F})(-1) \arrow[d, "\Psi_f(\phi_{\boldsymbol{\eta}})(-1)"] \\
\Psi_f(\mathcal{F}') \arrow[r, "\alpha'"] & \mathcal{G}' \arrow[r, "\beta'"] & \Psi_f(\mathcal{F}')(-1)
\end{tikzcd}$$

commutes. The category of vanishing cycles data for f with coefficients in $\Bbbk$ is denoted by $\mathrm{VC}(X, f, \Bbbk)$.

THEOREM 4.3.9. *Let $f : X \to \mathbb{A}^1$ be a regular function. There is an equivalence of categories* $\mathrm{Perv}(X, \Bbbk) \cong \mathrm{VC}(X, f, \Bbbk)$.

PROOF SKETCH. Define a functor $F : \mathrm{Perv}(X, \Bbbk) \to \mathrm{VC}(X, f, \Bbbk)$ by

$$F(\mathcal{F}) = (\mathcal{F}|_{X_{\boldsymbol{\eta}}}, \Phi'_f(\mathcal{F}), \mathrm{can}', \mathrm{var}').$$

Next, given a vanishing cycles datum $M = (\mathcal{F}, \mathcal{G}, \alpha, \beta)$, let $E(M)$ be the chain complex

$$\cdots \to 0 \to i_{\mathbf{s}*}\Psi_f(\mathcal{F}) \xrightarrow{\left[\begin{smallmatrix} i_{\mathbf{s}*}\alpha \\ -\mathrm{can}_\Xi \end{smallmatrix}\right]} i_{\mathbf{s}*}\mathcal{G} \oplus \Xi_f(\mathcal{F}|_{X_{\boldsymbol{\eta}}}) \xrightarrow{[\, i_{\mathbf{s}*}\beta \ \ \mathrm{var}_\Xi \,]} i_{\mathbf{s}*}\Psi_f(\mathcal{F}) \to 0 \to \cdots .$$

(Since $i_{\mathbf{s}*}(\beta \circ \alpha) = i_{\mathbf{s}*}(\mathrm{var} \circ \mathrm{can}) = \mathrm{var}_\Xi \circ \mathrm{can}_\Xi$, this is indeed a chain complex.) Define a functor $G : \mathrm{VC}(X, f, \Bbbk) \to \mathrm{Perv}(X, \Bbbk)$ by $G(M) = \mathsf{H}^0(E(M))$.

We claim that the functors F and G are inverse to one another. We will outline a sketch of the proof that $G(F(\mathcal{F})) \cong \mathcal{F}$. Given $\mathcal{F} \in \mathrm{Perv}(X, \Bbbk)$, let $A = A(\mathcal{F})$ denote the following chain complex:

$$\begin{tikzcd}
j_{\boldsymbol{\eta}*}(\mathcal{F}|_{X_{\boldsymbol{\eta}}}) \oplus \Xi_f(\mathcal{F}|_{X_{\boldsymbol{\eta}}}) \\
j_{\boldsymbol{\eta}!}(\mathcal{F}|_{X_{\boldsymbol{\eta}}}) \oplus \mathcal{F} \oplus \Xi_f(\mathcal{F}|_{X_{\boldsymbol{\eta}}}) \oplus \Xi_f(\mathcal{F}|_{X_{\boldsymbol{\eta}}}) \oplus j_{\boldsymbol{\eta}*}(\mathcal{F}|_{X_{\boldsymbol{\eta}}}) \arrow[u, "\left[\begin{smallmatrix} 0 & \eta & \eta' & 0 & \mathrm{id} \\ \epsilon' & 0 & \mathrm{id} & \mathrm{id} & 0 \end{smallmatrix}\right]"] \\
j_{\boldsymbol{\eta}!}(\mathcal{F}|_{X_{\boldsymbol{\eta}}}) \oplus \Xi_f(\mathcal{F}|_{X_{\boldsymbol{\eta}}}) \arrow[u, "\left[\begin{smallmatrix} \mathrm{id} & 0 \\ \epsilon & 0 \\ -\epsilon' & \mathrm{id} \\ 0 & -\mathrm{id} \\ 0 & -\eta' \end{smallmatrix}\right]"]
\end{tikzcd}$$

This chain complex admits a filtration $A_1 \subset A_2 \subset A$ as shown below, where for brevity, we simply write $j_{\boldsymbol{\eta}*}$ for $j_{\boldsymbol{\eta}*}(\mathcal{F}|_{X_{\boldsymbol{\eta}}})$, and likewise for $j_{\boldsymbol{\eta}!}$ and Ξ_f:

$$\begin{array}{ccccc}
j_{\boldsymbol{\eta}*} & & j_{\boldsymbol{\eta}*} \oplus \Xi_f & & j_{\boldsymbol{\eta}*} \oplus \Xi_f \\
\uparrow & & \uparrow & & \uparrow \\
j_{\boldsymbol{\eta}*} & \subset & \mathcal{F} \oplus \Xi_f \oplus \Xi_f \oplus j_{\boldsymbol{\eta}*} & \subset & j_{\boldsymbol{\eta}!} \oplus \mathcal{F} \oplus \Xi_f \oplus \Xi_f \oplus j_{\boldsymbol{\eta}*} \\
 & & \uparrow & & \uparrow \\
 & & \Xi_f & & j_{\boldsymbol{\eta}!} \oplus \Xi_f
\end{array}$$

It is easy to see directly that A_1 and A/A_2 are acyclic, so A is naturally quasi-isomorphic to A_2/A_1. The obvious map

$$A_2/A_1 = \left(\begin{tikzcd} \Xi_f \\ \mathcal{F} \oplus \Xi_f \oplus \Xi_f \arrow[u, "{[0\ \mathrm{id}\ \mathrm{id}]}"] \\ \Xi_f \arrow[u, "{\left[\begin{smallmatrix} 0 \\ \mathrm{id} \\ -\mathrm{id} \end{smallmatrix}\right]}"] \end{tikzcd} \right) \xrightarrow{[\mathrm{id}\ 0\ 0]} \mathcal{F}$$

is easily seen to be a quasi-isomorphism as well.

On the other hand, there is also a pair of chain maps

$$\begin{tikzcd} j_{\boldsymbol{\eta}*} \oplus \Xi_f & & \Xi_f & & i_{\mathbf{s}*}\Psi_f(-1) \\ j_{\boldsymbol{\eta}!} \oplus \mathcal{F} \oplus \Xi_f \oplus \Xi_f \oplus j_{\boldsymbol{\eta}*} \arrow[u] & \supset & j_{\boldsymbol{\eta}!} \oplus \Upsilon_f(\mathcal{F}) \oplus \Xi_f \arrow[u] \arrow[rr, two heads] & & i_{\mathbf{s}*}\Phi'_f(\mathcal{F}) \oplus \Xi_f \arrow[u] \\ j_{\boldsymbol{\eta}!} \oplus \Xi_f \arrow[u] & & j_{\boldsymbol{\eta}!} \oplus i_{\mathbf{s}*}\Psi_f \arrow[u] & & i_{\mathbf{s}*}\Psi_f \arrow[u] \end{tikzcd}$$

Let A' denote the middle chain complex above. Observe that the rightmost chain complex is $G(F(\mathcal{F}))$. Combining the observations above, we have a sequence of natural quasi-isomorphisms

$$G(F(\mathcal{F})) \twoheadleftarrow A' \hookrightarrow A \hookleftarrow A_2 \twoheadrightarrow A_2/A_1 \twoheadrightarrow \mathcal{F},$$

and hence a natural isomorphism $G(F(\mathcal{F})) \cong \mathcal{F}$.

Finally, given a vanishing cycles datum $M = (\mathcal{F}, \mathcal{G}, \alpha, \beta)$, one can construct a natural isomorphism $F(G(M)) \cong M$ by an explicit calculation very similar to the one above. We omit further details. ☐

As a corollary of Theorem 4.3.9, we obtain a solution to the problem raised at the beginning of the section: for a fixed perverse sheaf $\mathcal{F} \in \mathrm{Perv}(X_{\boldsymbol{\eta}}, \Bbbk)$, the category of extensions of $\mathcal{F}$ to a perverse sheaf on X is equivalent to the category of triples $(\mathcal{G}, \alpha, \beta)$, where α and β are as in (4.3.9).

Exercises. Let $f : X \to \mathbb{A}^1$ be a regular function.

4.3.1. Show that for $\mathcal{F} \in \mathrm{Perv}(X_{\boldsymbol{\eta}}, \Bbbk)$, there is a natural "monodromy" automorphism $\mathrm{mon}_{\Xi}(T) : \Xi_f(\mathcal{F}) \to \Xi_f(\mathcal{F})$ such that the short exact sequences of Proposition 4.3.1 are compatible with monodromy. (That is, the maps in these short exact sequences intertwine $\mathrm{mon}_{\Psi}(T)$, $\mathrm{mon}_{\Xi}(T)$, and the identity maps of $j_{\boldsymbol{\eta}!}\mathcal{F}$ and $j_{\boldsymbol{\eta}*}\mathcal{F}$, as appropriate.)

4.3.2. Show that for any $\mathcal{F} \in \mathrm{Perv}(X, \Bbbk)$, there is a natural octahedral diagram

$$\begin{tikzcd} i_{\mathbf{s}}^*\mathcal{F} \arrow[dd, squiggly] & & i_{\mathbf{s}}^!\mathcal{F} \arrow[ll] \arrow[dl] \\ & \Phi'_f(\mathcal{F}) \arrow[ul] \arrow[dr, "\mathrm{var}'"] & \\ \Psi_f(\mathcal{F}|_{X_{\eta}}) \arrow[ur, "\mathrm{can}'"] \arrow[rr, "\mathrm{varocan}"] & & \Psi_f(\mathcal{F}|_{X_{\eta}})(-1) \arrow[uu, squiggly] \end{tikzcd} \qquad \begin{tikzcd} i_{\mathbf{s}}^*\mathcal{F} \arrow[dd, squiggly] \arrow[dr] & & i_{\mathbf{s}}^!\mathcal{F} \arrow[ll] \\ & i_{\mathbf{s}}^* j_{\boldsymbol{\eta}*}(\mathcal{F}|_{X_{\eta}}) \arrow[ur, squiggly] \arrow[dl, squiggly] & \\ \Psi_f(\mathcal{F}|_{X_{\eta}}) \arrow[rr, "\mathrm{varocan}"] & & \Psi_f(\mathcal{F}|_{X_{\eta}})(-1) \arrow[uu, squiggly] \arrow[ul] \end{tikzcd}$$

Then, by comparing with Lemma 4.1.5, deduce that $\Phi_f(\mathcal{F})$ and $\Phi'_f(\mathcal{F})$ are isomorphic. (I do not know whether this isomorphism can be made natural.)

4.3.3. Let $\mathcal{F} \in \mathrm{Perv}(X_{\boldsymbol{\eta}}, \Bbbk)$. Determine the vanishing cycles data corresponding to $j_{\boldsymbol{\eta}!}\mathcal{F}$, $j_{\boldsymbol{\eta}*}\mathcal{F}$, $j_{\boldsymbol{\eta}!*}\mathcal{F}$, and $\Xi_f(\mathcal{F})$.

4.4. Unipotent nearby cycles

In this section, we assume that $\Bbbk$ is a field. Let $f : X \to \mathbb{A}^1$ be a regular function. By Example A.8.6, the category $D^{\mathrm{b}}_{\mathrm{c}}(X_{\mathbf{s}}, \Bbbk)$ is a Krull–Schmidt category, so for any $\mathcal{F} \in D^{\mathrm{b}}_{\mathrm{c}}(X_{\boldsymbol{\eta}}, \Bbbk)$, we can apply Proposition A.8.9 to form the "generalized eigenspace decomposition" of $\Psi_f(\mathcal{F})$ with respect to the automorphism $\mathrm{mon}_\Psi(T)$. We have a decomposition

$$\Psi_f(\mathcal{F}) = \bigoplus_{\substack{\alpha(x) \in \Bbbk[x] \\ \text{irreducible monic}}} \Psi^\alpha_f(\mathcal{F}) \tag{4.4.1}$$

such that $\mathrm{mon}_\Psi(T)$ is the direct sum of automorphisms $\mathrm{mon}^\alpha_\Psi(T) : \Psi^\alpha_f(\mathcal{F}) \to \Psi^\alpha_f(\mathcal{F})$. These have the property that for any irreducible monic polynomial $\beta(x) \in \Bbbk[x]$, we have

$$\beta(\mathrm{mon}^\alpha_\Psi(T)) \text{ is } \begin{cases} \text{nilpotent} & \text{if } \alpha = \beta, \\ \text{invertible} & \text{if } \alpha \neq \beta. \end{cases}$$

Moreover, Lemma A.8.11 tells us that the decomposition (4.4.1) is natural, so that for each irreducible monic polynomial $\alpha(x) \in \Bbbk[x]$, we obtain a functor

$$\Psi^\alpha_f : D^{\mathrm{b}}_{\mathrm{c}}(X_{\boldsymbol{\eta}}, \Bbbk) \to D^{\mathrm{b}}_{\mathrm{c}}(X_{\mathbf{s}}, \Bbbk),$$

along with a natural automorphism $\mathrm{mon}^\alpha_\Psi(T) : \Psi^\alpha_f(\mathcal{F}) \to \Psi^\alpha_f(\mathcal{F})$. In the special case where $\alpha(x) = x - 1$, we denote this functor by

$$\Psi^{\mathrm{un}}_f : D^{\mathrm{b}}_{\mathrm{c}}(X_{\boldsymbol{\eta}}, \Bbbk) \to D^{\mathrm{b}}_{\mathrm{c}}(X_{\mathbf{s}}, \Bbbk).$$

This functor is called the **unipotent nearby cycles functor**. Since it is a direct summand of Ψ_f, it follows immediately from Theorem 4.2.8 that it is t-exact. One can similarly define the **unipotent vanishing cycles functor** $\Phi^{\mathrm{un}}_f : D^{\mathrm{b}}_{\mathrm{c}}(X_{\boldsymbol{\eta}}, \Bbbk) \to D^{\mathrm{b}}_{\mathrm{c}}(X_{\mathbf{s}}, \Bbbk)$.

The goal of this section is to give an alternative construction of Ψ^{un}_f due to Beilinson [**22**] that stays in the realm of algebraic maps and constructible complexes, avoiding the use of $\exp : \mathbb{C} \to \mathbb{C}^\times$.

Jordan block local systems. Let $\mathfrak{m}$ be the maximal ideal of $\Bbbk[T, T^{-1}] = \Bbbk[\pi_1(\mathbb{A}^1 \smallsetminus \{0\}, 1)]$ generated by $T - 1$. For $n \geq 0$, let

$$J_n = \Bbbk[T, T^{-1}]/\mathfrak{m}^n.$$

This $\Bbbk[T, T^{-1}]$-module has dimension n as a vector space over $\Bbbk$. The local system on $\mathbb{A}^1 \smallsetminus \{0\}$ corresponding to J_n via Theorem 1.7.9 is denoted by

$$\mathcal{J}_n \in \mathrm{Loc}^{\mathrm{ft}}(\mathbb{A}^1 \smallsetminus \{0\}, \Bbbk).$$

Recall from the remark following Exercise 1.10.3 (see also Example 2.2.7) that there is a canonical isomorphism $\mathfrak{m}/\mathfrak{m}^2 \cong \Bbbk(1) \cong J_1(1)$. More generally, we have $\mathfrak{m}^n/\mathfrak{m}^{n+1} \cong \Bbbk(n)$. In particular, there is a natural short exact sequence

$$0 \to \underline{\Bbbk}_{\mathbb{A}^1 \smallsetminus \{0\}}(n-1) \to \mathcal{J}_n \to \mathcal{J}_{n-1} \to 0. \tag{4.4.2}$$

For some purposes, it is convenient to replace $\Bbbk[T, T^{-1}]$ by the larger ring $\Bbbk[[1-T]]$ of formal power series in $1-T$. Let $\hat{\mathfrak{m}} \subset \Bbbk[[1-T]]$ be the unique maximal ideal. We can then identify

$$J_n = \Bbbk[[1-T]]/\hat{\mathfrak{m}}^n.$$

Now choose a **uniformizer**, i.e., a generator ϖ of the principal ideal $\hat{\mathfrak{m}}$. Such a generator is a power series

$$\varpi = \sum_{i \geq 1} a_i (1-T)^i$$

with $a_1 \neq 0$. Then the class of ϖ gives a basis element for $\hat{\mathfrak{m}}/\hat{\mathfrak{m}}^2 \cong \Bbbk(1)$. Denote this basis element by $[\varpi]$. For $k \geq 0$, there is an isomorphism of vector spaces

$$\sigma_\varpi : \Bbbk[[1-T]](k) = \Bbbk[[1-T]] \otimes \underbrace{\Bbbk(1) \otimes \cdots \otimes \Bbbk(1)}_{k \text{ factors}} \xrightarrow{\sim} \hat{\mathfrak{m}}^k$$

given by $\sigma_\varpi(h \otimes [\varpi] \otimes \cdots \otimes [\varpi]) = h\varpi^k$, and this induces an isomorphism

$$\sigma_\varpi : J_n(k) \xrightarrow{\sim} \hat{\mathfrak{m}}^k/\hat{\mathfrak{m}}^{k+n}.$$

A very similar formula makes sense for $k \leq 0$ as well, if we understand $\hat{\mathfrak{m}}^k$ to be a fractional ideal in the field of formal Laurent series $\Bbbk((1-T))$. Next, define a map

$$N_\varpi : J_n \to J_n(-1)$$

to be the composition $J_n = \Bbbk[[1-T]]/\hat{\mathfrak{m}}^n \hookrightarrow \hat{\mathfrak{m}}^{-1}/\hat{\mathfrak{m}}^n \twoheadrightarrow \hat{\mathfrak{m}}^{-1}/\hat{\mathfrak{m}}^{n-1} \xrightarrow[\sim]{\sigma_\varpi^{-1}} J_n(-1)$.

Lemma 4.4.1. *There is a nondegenerate T-invariant pairing $J_n \otimes J_n(1-n) \to \Bbbk$.*

Proof. There is an automorphism of the field $\Bbbk((1-T))$ given by $T \mapsto T^{-1}$ or, equivalently, by

$$1 - T \mapsto -(1-T) - (1-T)^2 - (1-T)^3 - \cdots .$$

Denote this automorphism by $q \mapsto \bar{q}$. Define a pairing $\langle -, - \rangle : \Bbbk((1-T)) \otimes \Bbbk((1-T)) \to \Bbbk$ by

$$\langle f, g \rangle = \text{constant term of } f\bar{g}.$$

Here, "constant term" means the coefficient a_0 in the expression $f\bar{g} = \sum a_i(1-T)^i$. This pairing is $\Bbbk$-bilinear and T-invariant (i.e., $\langle T \cdot f, T \cdot g \rangle = \langle f, g \rangle$), and it restricts to a pairing

$$\langle -, - \rangle : \hat{\mathfrak{m}}^0 \otimes \hat{\mathfrak{m}}^{1-n} \to \Bbbk.$$

Assume from now on that $f \in \hat{\mathfrak{m}}^0 = \Bbbk[[1-T]]$ and $g \in \hat{\mathfrak{m}}^{1-n}$. If, in addition, we have either $f \in \hat{\mathfrak{m}}^n$ or $g \in \hat{\mathfrak{m}}$, then $f\bar{g} \in \hat{\mathfrak{m}}$, so $\langle f, g \rangle = 0$. It follows that $\langle -, - \rangle$ induces a pairing

$$\langle -, - \rangle : \hat{\mathfrak{m}}^0/\hat{\mathfrak{m}}^n \otimes \hat{\mathfrak{m}}^{1-n}/\hat{\mathfrak{m}} \to \Bbbk.$$

We regard this as a pairing $J_n \otimes J_n(1-n) \to \Bbbk$ via σ_ϖ. It is left to the reader to check that it is nondegenerate. $\square$

Lemma 4.4.2. *Let $f : X \to \mathbb{A}^1$ be a regular function. For $\mathcal{F} \in D^{\mathrm{b}}_{\mathrm{c}}(X_{\boldsymbol{\eta}}, \Bbbk)$, there is a natural isomorphism $R\mathscr{H}om(f^*_{\boldsymbol{\eta}}\mathcal{J}_n, \mathcal{F}) \cong f^*_{\boldsymbol{\eta}}\mathcal{J}_n(1-n) \otimes^L \mathcal{F}$.*

PROOF. By Remark 1.4.12, there is a natural map

$$f_{\boldsymbol{\eta}}^* R\mathscr{H}om(\mathcal{J}_n, \underline{\Bbbk}_{\mathbb{A}^1 \smallsetminus \{0\}}) \to R\mathscr{H}om(f_{\boldsymbol{\eta}}^* \mathcal{J}_n, \underline{\Bbbk}_{X_{\boldsymbol{\eta}}}). \tag{4.4.3}$$

This map is easily seen to be an isomorphism when $n = 1$ (i.e., for $\mathcal{J}_1 = \underline{\Bbbk}_{\mathbb{A}^1 \smallsetminus \{0\}}$). Then, using the short exact sequence (4.4.2), we see by induction on n that it is an isomorphism for all n. Similar reasoning shows that the natural map (see Remark 1.4.11)

$$R\mathscr{H}om(f_{\boldsymbol{\eta}}^* \mathcal{J}_n, \underline{\Bbbk}_{X_{\boldsymbol{\eta}}}) \overset{L}{\otimes} R\mathscr{H}om(\underline{\Bbbk}_{X_{\boldsymbol{\eta}}}, \mathcal{F}) \to R\mathscr{H}om(f_{\boldsymbol{\eta}}^* \mathcal{J}_n, \mathcal{F}) \tag{4.4.4}$$

is an isomorphism for all n.

Lemma 4.4.1 gives us an isomorphism $R\mathscr{H}om(\mathcal{J}_n, \underline{\Bbbk}_{\mathbb{A}^1 \smallsetminus \{0\}}) \cong \mathcal{J}_n(1-n)$. Combining this observation with (4.4.3) and (4.4.4), we obtain a natural isomorphism $f_{\boldsymbol{\eta}}^* \mathcal{J}_n(1-n) \otimes^L \mathcal{F} \xrightarrow{\sim} R\mathscr{H}om(f_{\boldsymbol{\eta}}^* \mathcal{J}_n, \mathcal{F})$, as desired. □

PROPOSITION 4.4.3. *Let $f : X \to \mathbb{A}^1$ be a regular function, and let $\mathcal{F} \in$* Perv$(X_{\boldsymbol{\eta}}, \Bbbk)$*. For n sufficiently large, there are natural isomorphisms*

$$\begin{aligned} \Psi_f^{\mathrm{un}}(\mathcal{F}) &\cong i_{\mathbf{s}}^* \ker(j_{\boldsymbol{\eta}!}(f_{\boldsymbol{\eta}}^* \mathcal{J}_n(1-n) \overset{L}{\otimes} \mathcal{F}) \to j_{\boldsymbol{\eta}*}(f_{\boldsymbol{\eta}}^* \mathcal{J}_n(1-n) \overset{L}{\otimes} \mathcal{F})) \\ &\cong i_{\mathbf{s}}^* \operatorname{cok}(j_{\boldsymbol{\eta}!}(f_{\boldsymbol{\eta}}^* \mathcal{J}_n(1) \overset{L}{\otimes} \mathcal{F}) \to j_{\boldsymbol{\eta}*}(f_{\boldsymbol{\eta}}^* \mathcal{J}_n(1) \overset{L}{\otimes} \mathcal{F})). \end{aligned}$$

PROOF. Generalizing (4.1.3), we have a short exact sequence of $\Bbbk[T, T^{-1}]$-modules $0 \to \Bbbk[T, T^{-1}](n) \to \Bbbk[T, T^{-1}] \to J_n \to 0$, and a corresponding sequence of local systems

$$0 \to \exp_! \underline{\Bbbk}_{\mathbb{C}}(n) \xrightarrow{d^n} \exp_! \underline{\Bbbk}_{\mathbb{C}} \to \mathcal{J}_n \to 0.$$

Apply $i_{\mathbf{s}}^* j_{\boldsymbol{\eta}*} R\mathscr{H}om(f_{\boldsymbol{\eta}}^*(-)[1], \mathcal{F})$ to obtain the distinguished triangle

$$i_{\mathbf{s}}^* j_{\boldsymbol{\eta}*} R\mathscr{H}om(f_{\boldsymbol{\eta}}^* \mathcal{J}_n[1], \mathcal{F}) \to \Psi_f(\mathcal{F}) \xrightarrow{(\mathrm{var} \circ \mathrm{can})^n} \Psi_f(\mathcal{F})(-n) \to .$$

After rotating and invoking Lemma 4.4.2, this can be rewritten as

$$\Psi_f(\mathcal{F}) \xrightarrow{(\mathrm{var} \circ \mathrm{can})^n} \Psi_f(\mathcal{F})(-n) \to i_{\mathbf{s}}^* j_{\boldsymbol{\eta}*}(f_{\boldsymbol{\eta}}^* \mathcal{J}_n(1-n) \overset{L}{\otimes} \mathcal{F}) \to .$$

The last term above is equipped with an automorphism induced by the action of T on J_n, and of course the first and second terms are equipped with the automorphism $\mathrm{mon}_\Psi(T)$. The maps in the distinguished triangle are compatible with these automorphisms, so when we form the "eigenspace decomposition" as in (4.4.1), we get a collection of distinguished triangles

$$\Psi_f^\alpha(\mathcal{F}) \xrightarrow{(\mathrm{var} \circ \mathrm{can})_\alpha^n} \Psi_f^\alpha(n) \to (i_{\mathbf{s}}^* j_{\boldsymbol{\eta}*}(f_{\boldsymbol{\eta}}^* \mathcal{J}_n(1-n) \overset{L}{\otimes} \mathcal{F}))_\alpha \to .$$

But the action of T on $\mathcal{J}_n$ is unipotent, so the last term above vanishes if $\alpha(x) \neq x - 1$. (Alternatively, by Lemma 4.1.6, $(\mathrm{var} \circ \mathrm{can})_\alpha^n$ can be identified with $\mathrm{mon}_\Psi^\alpha(1-T)^n$, and this is an automorphism if $\alpha(x) \neq x - 1$.) When $\alpha(x) = x - 1$, we obtain a natural distinguished triangle

$$\Psi_f^{\mathrm{un}}(\mathcal{F}) \xrightarrow{(\mathrm{var} \circ \mathrm{can})^n} \Psi_f^{\mathrm{un}}(\mathcal{F})(-n) \to i_{\mathbf{s}}^* j_{\boldsymbol{\eta}*}(f_{\boldsymbol{\eta}}^* \mathcal{J}_n(1-n) \overset{L}{\otimes} \mathcal{F}) \to .$$

If n is large enough, then the endomorphism $\mathrm{mon}_\Psi(1-T)^n$ of $\Psi_f^{\mathrm{un}}(\mathcal{F})$ is 0, so the triangle above splits. From the long exact sequence in perverse cohomology, we obtain

$$\Psi_f^{\mathrm{un}}(\mathcal{F}) \cong {}^p\mathsf{H}^{-1}(i_{\mathbf{s}}^* j_{\boldsymbol{\eta}*}(f_{\boldsymbol{\eta}}^* \mathcal{J}_n(1-n) \overset{L}{\otimes} \mathcal{F})) \cong {}^p\mathsf{H}^0(i_{\mathbf{s}}^* j_{\boldsymbol{\eta}*}(f_{\boldsymbol{\eta}}^* \mathcal{J}_n(1) \overset{L}{\otimes} \mathcal{F})).$$

On the other hand, we have a distinguished triangle

$$j_{\boldsymbol{\eta}!}(f_{\boldsymbol{\eta}}^*\mathcal{J}_n(1-n)\overset{L}{\otimes}\mathcal{F}) \xrightarrow{\theta} j_{\boldsymbol{\eta}*}(f_{\boldsymbol{\eta}}^*\mathcal{J}_n(1-n)\overset{L}{\otimes}\mathcal{F}) \to i_{\mathbf{s}*}i_{\mathbf{s}}^*j_{\boldsymbol{\eta}*}(f_{\boldsymbol{\eta}}^*\mathcal{J}_n(1-n)\overset{L}{\otimes}\mathcal{F}) \to .$$

The first and second terms are perverse by Lemmas 3.2.2 and 4.2.1, and so the long exact sequence in perverse cohomology shows that

$$
\begin{aligned}
{}^p\mathsf{H}^{-1}(i_{\mathbf{s}}^*j_{\boldsymbol{\eta}*}(f_{\boldsymbol{\eta}}^*\mathcal{J}_n(1-n)\overset{L}{\otimes}\mathcal{F})) &\cong i_{\mathbf{s}}^*(\ker\theta),\\
{}^p\mathsf{H}^{0}(i_{\mathbf{s}}^*j_{\boldsymbol{\eta}*}(f_{\boldsymbol{\eta}}^*\mathcal{J}_n(1)\overset{L}{\otimes}\mathcal{F})) &\cong i_{\mathbf{s}}^*(\operatorname{cok}\theta(n)),
\end{aligned}
$$

as desired. □

DEFINITION 4.4.4. Let $f : X \to \mathbb{A}^1$ be a regular function, and let $\varpi \in \hat{\mathfrak{m}}$ be a uniformizer. For $\mathcal{F} \in \operatorname{Perv}(X_{\boldsymbol{\eta}}, \Bbbk)$, the **nilpotent monodromy map** is the map

$$N_\varpi : \Psi_f^{\mathrm{un}}(\mathcal{F}) \to \Psi_f^{\mathrm{un}}(\mathcal{F})(-1)$$

induced by the map

$$f_{\boldsymbol{\eta}}^*N_{\bar{\varpi}}\overset{L}{\otimes}\mathrm{id}_{\mathcal{F}} : f_{\boldsymbol{\eta}}^*\mathcal{J}_n\overset{L}{\otimes}\mathcal{F} \to f_{\boldsymbol{\eta}}^*\mathcal{J}_n(-1)\overset{L}{\otimes}\mathcal{F}$$

via either of the isomorphisms of Proposition 4.4.3.

The following description of N_ϖ is an easy consequence of the construction. (In this statement, we suppress Tate twists in the same way as in Lemma 4.1.6.)

PROPOSITION 4.4.5. *Let $f : X \to \mathbb{A}^1$ be a regular function. Choose an isomorphism $b : \Bbbk \to \Bbbk(1)$. There exists a scalar $a \in \Bbbk^\times$ (depending on b) such that*

$$N_\varpi = a\operatorname{mon}_\Psi(\varpi) : \Psi_f^{\mathrm{un}}(\mathcal{F}) \to \Psi_f^{\mathrm{un}}(\mathcal{F}).$$

In particular, we have $N_{1-T} = \mathrm{var}\circ\mathrm{can} : \Psi_f^{\mathrm{un}}(\mathcal{F}) \to \Psi_f^{\mathrm{un}}(\mathcal{F})$.

REMARK 4.4.6. If $\Bbbk$ has characteristic 0, there is a special choice of ϖ that has particularly nice properties: namely,

$$\varpi = \log T = -(1-T) - \tfrac{1}{2}(1-T)^2 - \tfrac{1}{3}(1-T)^3 - \cdots .$$

In this case, the map $N_\varpi : \Psi_f^{\mathrm{un}}(\mathcal{F}) \to \Psi_f^{\mathrm{un}}(\mathcal{F})(-1)$ is often denoted simply by N, and it is called the **logarithm of the monodromy action**.

4.5. Beilinson's theorem

We continue to assume that $\Bbbk$ is a field. In this section, we will prove that the derived category of $\operatorname{Perv}(X, \Bbbk)$ is equivalent to $D_{\mathrm{c}}^{\mathrm{b}}(X, \Bbbk)$.

Effaceability for local systems. We will first prove a derived equivalence statement for local systems. The next few lemmas will use the following notion. For the relationship with the notion of "effaceability" from Section A.7, see the proof of Theorem 4.5.5.

DEFINITION 4.5.1. Let $\mathscr{T}$ be a triangulated category equipped with a bounded t-structure, and let $\mathscr{C}$ be its heart. Let $\mathscr{A}$ be an abelian category, and let $F : \mathscr{T} \to \mathscr{A}$ be a cohomological functor. We say that F is **right effaceable** if the following conditions hold:

1. For $X \in \mathscr{C}$ and $k < 0$, $F(X[k]) = 0$.
2. For $X \in \mathscr{C}$, there exists an injective map $X \to Y$ in $\mathscr{C}$ such that for all $k > 0$, the induced map $F(X[k]) \to F(Y[k])$ is the zero map.

LEMMA 4.5.2. *Let X be a smooth, connected variety, and let $\Bbbk$ be a field. The following three conditions on X are equivalent:*

(1) *The functor $\mathbf{H}^0(X,-) : D^{\mathrm{b}}_{\mathrm{locf}}(X,\Bbbk) \to \Bbbk\text{-mod}^{\mathrm{fg}}$ is right effaceable.*
(2) *For any $\mathcal{L} \in \mathrm{Loc}^{\mathrm{ft}}(X,\Bbbk)$, the functor $\mathrm{Hom}(\mathcal{L},-) : D^{\mathrm{b}}_{\mathrm{locf}}(X,\Bbbk) \to \Bbbk\text{-mod}^{\mathrm{fg}}$ is right effaceable.*
(3) *For any $\mathcal{F} \in D^{\mathrm{b}}_{\mathrm{locf}}(X,\Bbbk)^{\geq 0}$, there exists a local system $\mathcal{I} \in \mathrm{Loc}^{\mathrm{ft}}(X,\Bbbk)$ and a map $i : \mathcal{F} \to \mathcal{I}$ such that $\mathsf{H}^0(i) : \mathsf{H}^0(\mathcal{F}) \to \mathcal{I}$ is injective and such that the induced map $\mathbf{H}^k(X,\mathcal{F}) \to \mathbf{H}^k(X,\mathcal{I})$ is zero for all $k > 0$.*

PROOF. Because $\Bbbk$ is a field, Proposition 1.7.12 tells us that for any two local systems $\mathcal{L},\mathcal{L}' \in \mathrm{Loc}^{\mathrm{ft}}(X,\Bbbk)$, we have $R\mathscr{H}om(\mathcal{L},\mathcal{L}') \cong \mathscr{H}om(\mathcal{L},\mathcal{L}')$, and $\mathcal{L} \otimes^L \mathcal{L}' \cong \mathcal{L} \otimes \mathcal{L}'$. This observation will be used throughout the proof: we will write $\mathscr{H}om$ and $\otimes$, but we will often treat them as derived functors. Note, in particular, that

$$\begin{aligned}(4.5.1)\quad \mathbf{H}^n(X,\mathscr{H}om(\mathcal{L},\mathcal{L}')) &= \mathsf{H}^n(R\Gamma(R\mathscr{H}om(\mathcal{L},\mathcal{L}'))) \\ &\cong \mathsf{H}^n(R\mathrm{Hom}(\mathcal{L},\mathcal{L}')) \cong \mathrm{Hom}(\mathcal{L},\mathcal{L}'[n]).\end{aligned}$$

(1) $\Longrightarrow$ (2) Given $\mathcal{L}' \in \mathrm{Loc}^{\mathrm{ft}}(X,\Bbbk)$, consider the local system $\mathscr{H}om(\mathcal{L},\mathcal{L}')$. By assumption, there exists an injective map $j : \mathscr{H}om(\mathcal{L},\mathcal{L}') \to \mathcal{J}$ such that the induced map

$$(4.5.2)\qquad \mathrm{Hom}(\mathcal{L},\mathcal{L}'[n]) \cong \mathbf{H}^n(X,\mathscr{H}om(\mathcal{L},\mathcal{L}')) \to \mathbf{H}^n(X,\mathcal{J})$$

vanishes for $n > 0$. Consider the natural map $\epsilon : \mathcal{L} \otimes \mathscr{H}om(\mathcal{L},\mathcal{L}') \to \mathcal{L}'$. Let $\mathcal{I}$ be the cokernel of the map

$$\mathcal{L} \otimes \mathscr{H}om(\mathcal{L},\mathcal{L}') \xrightarrow{(\epsilon,-\mathrm{id}\otimes j)} \mathcal{L}' \oplus (\mathcal{L} \otimes \mathcal{J}),$$

and let $i : \mathcal{L}' \to \mathcal{I}$ be the induced map. We then have a commutative diagram

$$\begin{tikzcd}
\mathcal{L} \otimes \mathscr{H}om(\mathcal{L},\mathcal{L}') \arrow[r, "\epsilon"] \arrow[d, "\mathrm{id}_{\mathcal{L}}\otimes j"'] & \mathcal{L}' \arrow[d, "i"] \\
\mathcal{L} \otimes \mathcal{J} \arrow[r] & \mathcal{I}
\end{tikzcd}$$

Since j is injective and $\mathcal{L} \otimes (-)$ is an exact functor, the map $\mathrm{id}_{\mathcal{L}} \otimes j$ is injective. A diagram chase then shows that i is injective. Now apply $\mathscr{H}om(\mathcal{L},-)$ to this diagram. Recall also that for any $\mathcal{F} \in \mathrm{Loc}^{\mathrm{ft}}(X,\Bbbk)$, there is a natural map $\eta : \mathcal{F} \to \mathscr{H}om(\mathcal{L},\mathcal{L} \otimes \mathcal{F})$. We obtain the following commutative diagram:

$$\begin{tikzcd}
\mathscr{H}om(\mathcal{L},\mathcal{L}') \arrow[r, "\eta"'] \arrow[rr, bend left, "\mathrm{id}"] \arrow[d, "j"'] & \mathscr{H}om(\mathcal{L},\mathcal{L} \otimes \mathscr{H}om(\mathcal{L},\mathcal{L}')) \arrow[r, "\epsilon"'] \arrow[d, "\mathrm{id}_{\mathcal{L}}\otimes j"'] & \mathscr{H}om(\mathcal{L},\mathcal{L}') \arrow[d, "i"] \\
\mathcal{J} \arrow[r, "\eta"] & \mathscr{H}om(\mathcal{L},\mathcal{L} \otimes \mathcal{J}) \arrow[r] & \mathscr{H}om(\mathcal{L},\mathcal{I})
\end{tikzcd}$$

In other words, the map $\mathscr{H}om(\mathcal{L},\mathcal{L}') \to \mathscr{H}om(\mathcal{L},\mathcal{I})$ induced by i factors through $\mathcal{J}$. Finally, apply $\mathbf{H}^n(X,-)$. Using (4.5.1), we see that the map

$$\mathrm{Hom}(\mathcal{L},\mathcal{L}'[n]) \to \mathrm{Hom}(\mathcal{L},\mathcal{I}[n])$$

induced by i factors through (4.5.2), so it vanishes for $n > 0$, as desired.

(2) $\Longrightarrow$ (1) This is immediate from the fact that

$$\mathbf{H}^0(X,-) = \mathsf{H}^0 \circ R\Gamma(-) \cong \mathsf{H}^0 \circ R\mathrm{Hom}(\underline{\Bbbk}_X,-) \cong \mathrm{Hom}(\underline{\Bbbk}_X,-).$$

(1) $\Longrightarrow$ (3) Assume $\mathcal{F}$ is nonzero (otherwise there is nothing to prove), and let n be the largest integer such that $\mathsf{H}^n(\mathcal{F}) \neq 0$. We proceed by induction on n. If $n = 0$, then $\mathcal{F}$ is a local system, and we are in the setting of part (1). Otherwise, form the truncation distinguished triangle

$$\tau^{\leq n-1}\mathcal{F} \to \mathcal{F} \to \mathsf{H}^n(\mathcal{F})[-n] \to .$$

By induction, there is a map $j_1 : \tau^{\leq n-1}\mathcal{F} \to \mathcal{J}_1$ satisfying the conclusions of the statement. Next, since we already know that conditions (1) and (2) are equivalent, we can apply the latter to $\mathrm{Hom}(\mathsf{H}^n(\mathcal{F}), -)$: there is an injective map $j_2 : \mathcal{J}_1 \to \mathcal{J}_2$ such that

$$\mathrm{Hom}(\mathsf{H}^n(\mathcal{F})[-k], \mathcal{J}_1) \to \mathrm{Hom}(\mathsf{H}^n(\mathcal{F})[-k], \mathcal{J}_2)$$

is zero for $k > 0$. In particular, the composition

$$\mathsf{H}^n(\mathcal{F})[-n-1] \to \tau^{\leq n-1}\mathcal{F} \xrightarrow{j_1} \mathcal{J}_1 \xrightarrow{j_2} \mathcal{J}_2$$

is zero, so $j_2 \circ j_1$ factors through $\tau^{\leq n-1}\mathcal{F} \to \mathcal{F}$. We thereby obtain a new map $j_2' : \mathcal{F} \to \mathcal{J}_2$. Finally, apply condition (1) to $\mathcal{J}_2$: find an injective map $j_3 : \mathcal{J}_2 \to \mathcal{I}$ such that the induced maps $\mathbf{H}^k(X, \mathcal{J}_2) \to \mathbf{H}^k(X, \mathcal{I})$ vanish for $k > 0$. Let $i = j_3 \circ j_2' : \mathcal{F} \to \mathcal{I}$. Its induced map in hypercohomology is the composition

$$\mathbf{H}^k(X, \mathcal{F}) \to \mathbf{H}^k(X, \mathcal{J}_2) \to \mathbf{H}^k(X, \mathcal{I}),$$

which vanishes. It remains to check that the map $\mathsf{H}^0(i) : \mathsf{H}^0(\mathcal{F}) \to \mathcal{I}$ is injective. This holds because it is the composition of the following injective maps:

$$\mathsf{H}^0(\mathcal{F}) = \mathsf{H}^0(\tau^{\leq n-1}\mathcal{F}) \xrightarrow{\mathsf{H}^0(j_1)} \mathcal{J}_1 \xrightarrow{j_2} \mathcal{J}_2 \xrightarrow{j_3} \mathcal{I}.$$

(3) $\Longrightarrow$ (1) The definition of right effaceability for $\mathbf{H}^0(X, -)$ is just the special case of condition (3) where $\mathcal{F} \in \mathrm{Loc}^{\mathrm{ft}}(X, \Bbbk)$. $\square$

LEMMA 4.5.3. *Let $f : X \to Y$ be a smooth morphism of smooth, connected varieties. Assume that f is a locally trivial fibration and that its fibers are affine varieties of dimension ≤ 1. If $\mathbf{H}^0(Y, -) : D^{\mathrm{b}}_{\mathrm{locf}}(Y, \Bbbk) \to \Bbbk\text{-mod}^{\mathrm{fg}}$ is right effaceable, then $\mathbf{H}^0(X, -) : D^{\mathrm{b}}_{\mathrm{locf}}(X, \Bbbk) \to \Bbbk\text{-mod}^{\mathrm{fg}}$ is also right effaceable.*

PROOF. Let $\mathcal{L} \in \mathrm{Loc}^{\mathrm{ft}}(X, \Bbbk)$. By Theorems 1.9.5 and 2.7.1, the object $f_*\mathcal{L}$ lies in $D^{\mathrm{b}}_{\mathrm{locf}}(Y, \Bbbk)^{\geq 0}$, and for $y \in Y$, we have $\mathsf{H}^i(f_*\mathcal{L})_y \cong \mathbf{H}^i(f^{-1}(y), \mathcal{L}|_{f^{-1}(y)})$. Because $f^{-1}(y)$ is an affine variety of dimension ≤ 1, Theorem 2.6.2 implies that

$$\mathsf{H}^i(f_*\mathcal{L}) = 0 \qquad \text{for } i \geq 2. \tag{4.5.3}$$

Let us apply condition (3) from Lemma 4.5.2 to Y: there exists a local system $\mathcal{J} \in \mathrm{Loc}^{\mathrm{ft}}(Y, \Bbbk)$ and a map $j : f_*\mathcal{L} \to \mathcal{J}$ such that $\mathsf{H}^0(j) : {}^\circ f_*\mathcal{L} \to \mathcal{J}$ is injective, and such that the map $\mathbf{H}^k(Y, f_*\mathcal{L}) \to \mathbf{H}^k(Y, \mathcal{J})$ is zero for $k > 0$. Let $\epsilon : f^*f_*\mathcal{L} \to \mathcal{L}$ be the adjunction map, and consider the map

$$q = (\epsilon, f^*(j)) : f^*f_*\mathcal{L} \to \mathcal{L} \oplus f^*\mathcal{J}.$$

Complete this to a distinguished triangle

$$f^*f_*\mathcal{L} \xrightarrow{q} \mathcal{L} \oplus f^*\mathcal{J} \xrightarrow{h} \mathcal{J}' \to .$$

Since the first two terms belong to $D^{\mathrm{b}}_{\mathrm{locf}}(X, \Bbbk)$, the last one does as well. Using (4.5.3) and the long exact sequence in cohomology, we see that

$$\mathsf{H}^i(\mathcal{J}') = 0 \qquad \text{unless } i = -1, 0. \tag{4.5.4}$$

The long exact sequence also shows that $\mathsf{H}^0(h)$ factors as

$$\mathcal{L} \oplus f^*\mathcal{J} \twoheadrightarrow \mathrm{cok}(\mathsf{H}^0(q)) \hookrightarrow \mathsf{H}^0(\mathcal{J}').$$

Write h in components, as the sum of $h_1 : \mathcal{L} \to \mathcal{J}'$ and $h_2 : f^*\mathcal{J} \to \mathcal{J}'$. Then both $\mathsf{H}^0(h_1)$ and $\mathsf{H}^0(h_2)$ also factor through $\mathrm{cok}(\mathsf{H}^0(q))$, say as $h_1' : \mathcal{L} \to \mathrm{cok}(\mathsf{H}^0(q))$ and $h_2' : f^*\mathcal{J} \to \mathrm{cok}(\mathsf{H}^0(q))$, respectively. Let $u : \mathrm{cok}(\mathsf{H}^0(q)) \to \mathsf{H}^0(\mathcal{J}')$ be the inclusion map. We then have the following commutative diagram:

$$\begin{tikzcd}
\mathsf{H}^0(f^*f_*\mathcal{L}) \cong f^*({}^\circ f_*\mathcal{L}) \arrow[r, "\mathsf{H}^0(\epsilon)"] \arrow[d, "\mathsf{H}^0(f^*(j))"'] & \mathcal{L} \arrow[d, "h_1'"] \arrow[dr, "\mathsf{H}^0(h_1)=u\circ h_1'"] & \\
f^*\mathcal{J} \arrow[r, "-h_2'"] & \mathrm{cok}(\mathsf{H}^0(q)) \arrow[r, hook, "u"] & \mathsf{H}^0(\mathcal{J}')
\end{tikzcd}$$

Note that $\mathsf{H}^0(f^*(j))$ is injective, because $\mathsf{H}^0(j)$ is, and f^* is exact. A diagram chase then shows that h_1' is also injective. Therefore, $\mathsf{H}^0(h_1) : \mathcal{L} \to \mathsf{H}^0(\mathcal{J}')$ is injective.

Let us introduce some new notation: we set $\mathcal{I} = \mathsf{H}^0(\mathcal{J}')$ and $i = \mathsf{H}^0(h_1) : \mathcal{L} \to \mathcal{I}$. Thanks to (4.5.4), we have a truncation map $\mathcal{J}' \to \tau^{\geq 0}\mathcal{J}' \cong \mathsf{H}^0(\mathcal{J}') = \mathcal{I}$. We now consider the following related commutative diagram:

$$\begin{tikzcd}
f^*f_*\mathcal{L} \arrow[r, "\epsilon"] \arrow[d, "f^*(j)"'] & \mathcal{L} \arrow[d, "h_1"] \arrow[dr, "i=\mathsf{H}^0(h_1)"] & \\
f^*\mathcal{J} \arrow[r, "-h_2"] & \mathcal{J}' \arrow[r] & \tau^{\geq 0}\mathcal{J}' = \mathcal{I}
\end{tikzcd}$$

Apply f_* to this diagram. We also draw in some new arrows, coming from the adjunction map $\eta : \mathrm{id} \to f_*f^*$:

$$\begin{tikzcd}
f_*\mathcal{L} \arrow[r, "\eta f_*"'] \arrow[d, "j"'] \arrow[rr, bend left, "\mathrm{id}"] & f_*f^*f_*\mathcal{L} \arrow[r, "f_*\epsilon"'] \arrow[d, "f_*f^*(j)"'] & f_*\mathcal{L} \arrow[d, "f_*(h_1)"'] \arrow[dr, "f_*(i)"] & \\
\mathcal{J} \arrow[r, "\eta"] & f_*f^*\mathcal{J} \arrow[r, "f_*(-h_2)"] & f_*\mathcal{J}' \arrow[r] & f_*\mathcal{I}
\end{tikzcd}$$

In other words, the map $f_*(i) : f_*\mathcal{L} \to f_*\mathcal{I}$ factors through $j : f_*\mathcal{L} \to \mathcal{J}$. It follows that for $k > 0$, the map

$$\mathbf{H}^k(Y, f_*\mathcal{L}) \to \mathbf{H}^k(Y, f_*\mathcal{I})$$

induced by $f_*(i)$ is zero. But this can be identified with the map

$$\mathbf{H}^k(X, \mathcal{L}) \to \mathbf{H}^k(Y, \mathcal{I})$$

induced by i. Thus, $\mathbf{H}^0(X, -)$ is right effaceable. $\square$

PROPOSITION 4.5.4. *For any smooth, connected variety X, there exists an affine open subset $U \subset X$ with the property that $\mathbf{H}^0(U, -) : D^{\mathrm{b}}_{\mathrm{locf}}(U, \Bbbk) \to \Bbbk\text{-mod}^{\mathrm{fg}}$ is right effaceable.*

PROOF. If $\dim X = 0$, i.e., if X is a point, then of course $\mathbf{H}^n(X, \mathcal{F}) = 0$ for any $\mathcal{F} \in \mathrm{Loc}^{\mathrm{ft}}(X, \Bbbk)$ and any $n > 0$, so there is nothing to prove. Assume for the rest of the proof that $\dim X > 0$. We will first prove the result in the case where X is an open subset of $\mathbb{A}^n$ by induction on n, and then we will reduce the general case to this case.

Step 1. The case where X is an open subset of $\mathbb{A}^1$. In this case, X is the complement of a finite number of points in $\mathbb{A}^1$, and it is already an affine variety. Let $\mathcal{L} \in \mathrm{Loc}^{\mathrm{ft}}(X, \Bbbk)$. By Theorem 2.6.2, we already know that $\mathbf{H}^n(X, \mathcal{L}) = 0$ unless

$n = 0, 1$. If $\mathbf{H}^1(X, \mathcal{L}) = 0$ as well, there is nothing to prove. Otherwise, choose a basis $e_1, \ldots, e_k$ for $\mathbf{H}^1(X, \mathcal{L})$. Via the isomorphism $\mathbf{H}^1(X, \mathcal{L}) \cong \mathrm{Hom}(\underline{\Bbbk}_X, \mathcal{L}[1])$, each e_i can be regarded as a morphism $\underline{\Bbbk}_X \to \mathcal{L}[1]$. Let $e = (e_1, \ldots, e_k) : \underline{\Bbbk}_X^{\oplus k} \to \mathcal{L}[1]$. Complete this morphism to a distinguished triangle, and rotate it to get

$$\mathcal{L} \xrightarrow{h} \mathcal{F} \to \underline{\Bbbk}_X^{\oplus k} \xrightarrow{e} .$$

Using Lemma 1.8.6, this triangle shows us that $\mathcal{F}$ is a local system of finite type and that h is injective. By construction, we have $h[1] \circ e = 0$, and hence $h[1] \circ e_i = 0$ for all i. In other words, the map

$$\mathbf{H}^1(X, \mathcal{L}) \to \mathbf{H}^1(X, \mathcal{F}) \qquad \text{or} \qquad \mathrm{Hom}(\underline{\Bbbk}_X, \mathcal{L}[1]) \to \mathrm{Hom}(\underline{\Bbbk}_X, \mathcal{F}[1])$$

induced by h is the zero map. Thus, $\mathbf{H}^0(X, -)$ is right effaceable.

Step 2. The case where X is an open subset of $\mathbb{A}^n$. Assume the result is known for open subsets of $\mathbb{A}^{n-1}$. By replacing X by a smaller open subset of $\mathbb{A}^n$ if necessary, we may assume that X is a principal affine open set; in other words, its complement $Z = \mathbb{A}^n \smallsetminus X$ is defined by the vanishing of some nonconstant polynomial $f \in \mathbb{C}[x_1, \ldots, x_n]$. Let $p : \mathbb{A}^n \to \mathbb{A}^{n-1}$ be the projection onto the first $n-1$ coordinates. By Lemma 2.5.1, after a suitable linear change of coordinates (depending on f), we may assume that the map $p|_Z : Z \to \mathbb{A}^{n-1}$ is finite and surjective.

By Lemma 2.5.4, there is an open subset $V \subset \mathbb{A}^{n-1}$ such that $Z \cap (V \times \mathbb{A}^1)$ is a divisor with simple normal crosssings in $V \times \mathbb{A}^1 \subset \mathbb{A}^n$. Moreover, the map $p|_{V\times\mathbb{A}^1} : V \times \mathbb{A}^1 \to V$ is a transverse locally trivial fibration with respect to $Z \cap (V \times \mathbb{A}^1)$. In particular, if we restrict p to the complement of this divisor, i.e., to $X \cap (V \times \mathbb{A}^1)$, the resulting map

$$p|_{X\cap(V\times\mathbb{A}^1)} : X \cap (V \times \mathbb{A}^1) \to V$$

is a smooth morphism and a locally trivial fibration. Its fibers are open subsets of $\mathbb{A}^1$, so they are affine varieties of dimension 1.

By induction, after replacing V by a smaller open set if necessary, we may assume that it is affine and that $\mathbf{H}^0(V, -) : D^{\mathrm{b}}_{\mathrm{locf}}(V, \Bbbk) \to \Bbbk\text{-mod}^{\mathrm{fg}}$ is right effaceable. Let $U = X \cap (V \times \mathbb{A}^1)$. Then U is affine, and by Lemma 4.5.3, the functor $\mathbf{H}^0(U, -) : D^{\mathrm{b}}_{\mathrm{locf}}(U, \Bbbk) \to \Bbbk\text{-mod}^{\mathrm{fg}}$ is right effaceable.

Step 3. The general case. Assume the result is known for open subsets of any affine space. By replacing X by an affine open subset, we may assume that X itself is affine, say of dimension n. By the Noether normalization lemma (Lemma 2.1.8), there is a finite morphism $f : X \to \mathbb{A}^n$. By Lemma 2.1.16, there is an open subset $V \subset \mathbb{A}^n$ such that $f|_{f^{-1}(V)} : f^{-1}(V) \to V$ is finite and étale. In particular, it is smooth and proper. By Ehresmann's fibration theorem (see Theorem 2.1.19 and Remark 2.1.12), $f|_{f^{-1}(V)} : f^{-1}(V) \to V$ is a locally trivial fibration. Its fibers are 0-dimensional (affine) varieties.

By replacing V by a smaller open set if necessary, we may assume that it is affine and that $\mathbf{H}^0(V, -) : D^{\mathrm{b}}_{\mathrm{locf}}(V, \Bbbk) \to \Bbbk\text{-mod}^{\mathrm{fg}}$ is right effaceable. Let $U = f^{-1}(V)$. Since f is an affine morphism, U is an affine open set. Finally, by Lemma 4.5.3, the functor $\mathbf{H}^0(U, -) : D^{\mathrm{b}}_{\mathrm{locf}}(U, \Bbbk) \to \Bbbk\text{-mod}^{\mathrm{fg}}$ is right effaceable. $\square$

THEOREM 4.5.5. *For any variety X, there exists a smooth, connected, affine open subset $U \subset X$ for which the natural functor*

$$D^{\mathrm{b}}\mathrm{Loc}^{\mathrm{ft}}(U, \Bbbk) \to D^{\mathrm{b}}_{\mathrm{locf}}(U, \Bbbk)$$

is an equivalence of categories.

PROOF. Choose a smooth, connected, open subset $V \subset X$, and then apply Proposition 4.5.4 to V to obtain a smooth, connected, affine open subset U on which $\mathbf{H}^0(U, -) : D^{\mathrm{b}}_{\mathrm{locf}}(U, \Bbbk) \to \Bbbk\text{-mod}^{\mathrm{fg}}$ is right effaceable. By Lemma 4.5.2, for any $\mathcal{L} \in \mathrm{Loc}^{\mathrm{ft}}(U, \Bbbk)$, the functor $\mathrm{Hom}(\mathcal{L}, -) : D^{\mathrm{b}}_{\mathrm{locf}}(U, \Bbbk) \to \Bbbk\text{-mod}^{\mathrm{fg}}$ is right effaceable. Let us spell out what this means: given a local system $\mathcal{L}'$ and an integer $k > 0$, there is an injective map $\mathcal{L}' \to \mathcal{I}$ to another local system such that the induced map

$$\mathrm{Hom}(\mathcal{L}, \mathcal{L}'[k]) \to \mathrm{Hom}(\mathcal{L}, \mathcal{I}[k])$$

is zero. In other words, every morphism $\mathcal{L} \to \mathcal{L}'[k]$ is effaceable in the sense of Definition A.7.17. The theorem then follows by Corollary A.7.19. □

Ext-groups on open and closed subvarieties. If $i : Z \hookrightarrow X$ and $j : U \hookrightarrow X$ are open and closed embeddings, then i_* and j^* are t-exact for the perverse t-structure, and so they give rise to functors on the derived category of perverse sheaves. The next few statements are devoted to studying these functors.

LEMMA 4.5.6. *Let $j : U \hookrightarrow X$ be the inclusion of an affine open subset. For $\mathcal{F} \in \mathrm{Perv}(X, \Bbbk)$ and $\mathcal{G} \in \mathrm{Perv}(U, \Bbbk)$, and for any $k \geq 0$, there are natural isomorphisms*

$$\mathrm{Ext}^k_{\mathrm{Perv}(U,\Bbbk)}(j^*\mathcal{F}, \mathcal{G}) \cong \mathrm{Ext}^k_{\mathrm{Perv}(X,\Bbbk)}(\mathcal{F}, j_*\mathcal{G}),$$
$$\mathrm{Ext}^k_{\mathrm{Perv}(X,\Bbbk)}(j_!\mathcal{G}, \mathcal{F}) \cong \mathrm{Ext}^k_{\mathrm{Perv}(U,\Bbbk)}(\mathcal{G}, j^*\mathcal{F}).$$

PROOF. Note that (for now) these statements are different from Theorems 1.2.4 and 1.3.6: they involve $D^{\mathrm{b}}\mathrm{Perv}(U, \Bbbk)$ and $D^{\mathrm{b}}\mathrm{Perv}(X, \Bbbk)$, rather than $D^{\mathrm{b}}_{\mathrm{c}}(U, \Bbbk)$ and $D^{\mathrm{b}}_{\mathrm{c}}(X, \Bbbk)$. Nevertheless, those theorems at least give us adjunction maps $\eta : \mathcal{F} \to j_*j^*\mathcal{F}$ and $\epsilon : j^*j_*\mathcal{G} \to \mathcal{G}$. Because j is an affine morphism, the functors $j_!$ and j_* are exact on perverse sheaves. In particular, $j_*j^*\mathcal{F}$ and $j^*j_*\mathcal{G}$ are perverse sheaves, and η and ϵ are morphisms of perverse sheaves. We will now follow the usual recipe for constructing the Hom-isomorphism for a pair of adjoint functors from the unit and counit. Because all our functors are exact, the recipe makes sense for Ext-groups as well: there are natural maps

$$\mathrm{Ext}^k(j^*\mathcal{F}, \mathcal{G}) \xrightarrow{j_*} \mathrm{Ext}^k(j_*j^*\mathcal{F}, j_*\mathcal{G}) \xrightarrow{\eta} \mathrm{Ext}^k(\mathcal{F}, j_*\mathcal{G}),$$
$$\mathrm{Ext}^k(\mathcal{F}, j_*\mathcal{G}) \xrightarrow{j^*} \mathrm{Ext}^k(j^*\mathcal{F}, j^*j_*\mathcal{G}) \xrightarrow{\epsilon} \mathrm{Ext}^k(j^*\mathcal{F}, \mathcal{G}),$$

and the unit–counit equations imply that these maps are both isomorphisms, inverse to one another. The proof of the second isomorphism in the lemma is similar. □

PROPOSITION 4.5.7. *Let $i : Z \hookrightarrow X$ be the inclusion of a closed subset, where $Z = f^{-1}(0)$ for some regular function $f : X \to \mathbb{A}^1$. Then the functor*

$$i_* : D^{\mathrm{b}}\mathrm{Perv}(Z, \Bbbk) \to D^{\mathrm{b}}\mathrm{Perv}(X, \Bbbk)$$

is fully faithful. Its image consists of chain complexes $\mathcal{F} \in D^{\mathrm{b}}\mathrm{Perv}(X, \Bbbk)$ such that for all k, $\mathsf{H}^k(\mathcal{F})$ is supported on Z.

Note that despite the similarity in notation, the content of this lemma is different from that of Corollary 1.3.11.

PROOF. Let $\mathscr{T} \subset D^{\mathrm{b}}\mathrm{Perv}(X, \Bbbk)$ be the full triangulated subcategory consisting of objects $\mathcal{F}$ such that $\mathsf{H}^k(\mathcal{F})$ is supported on Z for all k. It is obvious that i_* takes values in $\mathscr{T}$. We wish to prove that the induced functor

$$i_* : D^{\mathrm{b}}\mathrm{Perv}(Z, \Bbbk) \to \mathscr{T}$$

is an equivalence of categories.

Step 1. The functor $i_* : D^{\mathrm{b}}\mathrm{Perv}(Z, \Bbbk) \to \mathscr{T}$ *is essentially surjective.* We will use the exact functor Υ_f from Lemma 4.3.7. Let $\mathcal{F}$ be a chain complex in $Ch^{\mathrm{b}}\mathrm{Perv}(X, \Bbbk)$. The natural maps $\Upsilon_f \to \mathrm{id}$ and $\Upsilon_f \to i_*\Phi'_f$ that appeared in Lemma 4.3.7 give rise to chain maps

$$\mathcal{F} \leftarrow \Upsilon_f(\mathcal{F}) \to i_*\Phi'_f(\mathcal{F}). \tag{4.5.5}$$

Now apply H^k to this diagram. Because Υ_f and $i_*\Phi'_f$ are both exact functors, the resulting diagram can be written as

$$\mathsf{H}^k(\mathcal{F}) \leftarrow \Upsilon_f(\mathsf{H}^k(\mathcal{F})) \to i_*\Phi'_f(\mathsf{H}^k(\mathcal{F})). \tag{4.5.6}$$

Now suppose that each $\mathsf{H}^k(\mathcal{F})$ is supported on Z. In this case, the short exact sequences in Lemma 4.3.7 show that both maps in (4.5.6) are isomorphisms, so both maps in (4.5.5) are quasi-isomorphisms. In other words, in $D^{\mathrm{b}}\mathrm{Perv}(X, \Bbbk)$, any object $\mathcal{F} \in \mathscr{T}$ is isomorphic to $i_*\Phi'_f(\mathcal{F})$, as desired.

Step 2. For any $\mathcal{F}, \mathcal{G} \in \mathrm{Perv}(Z, \Bbbk)$*, the map*

$$i_* : \mathrm{Ext}^k_{\mathrm{Perv}(Z,\Bbbk)}(\mathcal{F}, \mathcal{G}) \to \mathrm{Ext}^k_{\mathrm{Perv}(X,\Bbbk)}(i_*\mathcal{F}, i_*\mathcal{G}) \tag{4.5.7}$$

is surjective. Let $\phi : i_*\mathcal{F} \to i_*\mathcal{G}[k]$ be a morphism in $D^{\mathrm{b}}\mathrm{Perv}(X, \Bbbk)$. Then the cone of ϕ lies in $\mathscr{T}$, and hence in the image of i_*, by Step 1. Since $\mathrm{Perv}(Z, \Bbbk)$ is a Serre subcategory of $\mathrm{Perv}(X, \Bbbk)$, Lemma A.7.21 tells us that ϕ is in the image of (4.5.7), as desired.

Step 3. Conclusion of the proof. Using Proposition A.7.18, an easy induction argument on k shows that (4.5.7) is an isomorphism for all k. Since $\mathrm{Perv}(Z, \Bbbk)$ generates $D^{\mathrm{b}}\mathrm{Perv}(Z, \Bbbk)$ (Exercise A.7.2), Proposition A.4.16 tells us that i_* is fully faithful, as desired. □

Lemma 4.5.8. *Let* $\mathcal{F}, \mathcal{G} \in \mathrm{Perv}(X, \Bbbk)$*, and let* $\phi \in \mathrm{Ext}^k_{\mathrm{Perv}(X,\Bbbk)}(\mathcal{F}, \mathcal{G})$*. There exists an affine open subset* $U \subset X$ *such that* $\mathcal{F}|_U$ *and* $\mathcal{G}|_U$ *are both shifted local systems, and such that the restriction of*

$$\mathrm{real} : \mathrm{Ext}^k_{\mathrm{Perv}(U,\Bbbk)}(\mathcal{F}|_U, \mathcal{G}|_U) \to \mathrm{Hom}_{D^{\mathrm{b}}_{\mathrm{c}}(U,\Bbbk)}(\mathcal{F}|_U, \mathcal{G}|_U[k])$$

to the span of $\phi|_U \in \mathrm{Ext}^k_{\mathrm{Perv}(U,\Bbbk)}(\mathcal{F}|_U, \mathcal{G}|_U)$ *is injective.*

Note that $\phi|_U$ may be zero: this lemma just tells us that if $\phi|_U \neq 0$, then $\mathrm{real}(\phi|_U) \neq 0$ as well.

Proof. Let $V \subset X$ be a smooth, connected, open subset such that both $\mathcal{F}|_V$ and $\mathcal{G}|_V$ are shifted local systems. There is an obvious functor

$$\iota_V : D^{\mathrm{b}}(\mathrm{Loc}^{\mathrm{ft}}(V, \Bbbk)[\dim V]) \to D^{\mathrm{b}}\mathrm{Perv}(V, \Bbbk). \tag{4.5.8}$$

We claim that V can be chosen so that $\phi|_V$ is in the image of this functor. Indeed, let $\mathcal{K}$ be the cone of $\phi|_V$. This object is some bounded chain complex of perverse sheaves. Since it has only finitely many nonzero terms, there exists some Zariski open subset $V' \subset V$ such that $\mathcal{K}|_{V'}$ is a chain complex of shifted local systems. By replacing V by V', we may assume that $\mathcal{K}$ is itself a chain complex of shifted local systems; in other words, $\mathcal{K}$ is in the image of (4.5.8). Since $\mathrm{Loc}^{\mathrm{ft}}(V, \Bbbk)[\dim V]$ is a Serre subcategory of $\mathrm{Perv}(V, \Bbbk)$ (Proposition 3.4.1), we can apply Lemma A.7.21 to conclude that $\phi|_V$ is in the image of (4.5.8).

Now apply Theorem 4.5.5 to V to find an affine open subset $U \subset V$ such that $D^{\mathrm{b}}\mathrm{Loc}^{\mathrm{ft}}(U, \Bbbk) \to D^{\mathrm{b}}_{\mathrm{c}}(U, \Bbbk)$ is an equivalence of categories. Of course, the same statement holds if we replace local systems by shifted local systems. We thus have a sequence of natural functors

$$D^{\mathrm{b}}(\mathrm{Loc}^{\mathrm{ft}}(U, \Bbbk)[\dim U]) \xrightarrow{\iota_U} D^{\mathrm{b}}\mathrm{Perv}(U, \Bbbk) \xrightarrow{\mathrm{real}} D^{\mathrm{b}}_{\mathrm{c}}(U, \Bbbk)$$

such that $\mathrm{real} \circ \iota_U$ is an equivalence of categories. The reasoning of the previous paragraph shows that $\phi|_U$ (a morphism in $D^{\mathrm{b}}\mathrm{Perv}(U, \Bbbk)$) lies in the image of ι_U, since $\mathcal{K}|_U$ is again a chain complex of shifted local systems. Suppose $\phi|_U = \iota_U(\psi)$. If $\phi|_U \neq 0$, then $\psi \neq 0$, so $\mathrm{real}(\phi|_U) = \mathrm{real}(\iota_U(\psi)) \neq 0$ because $\mathrm{real} \circ \iota_U$ is fully faithful. □

Beilinson's theorem. The main result of this section is the following.

THEOREM 4.5.9 (Beilinson's theorem). *Let X be a variety, and let $\Bbbk$ be a field. The realization functor*

$$\mathrm{real} : D^{\mathrm{b}}\mathrm{Perv}(X, \Bbbk) \to D^{\mathrm{b}}_{\mathrm{c}}(X, \Bbbk)$$

is an equivalence of categories.

PROOF. By Propositions A.4.16 and A.4.17, it is enough to show that for any two perverse sheaves $\mathcal{F}, \mathcal{G} \in \mathrm{Perv}(X, \Bbbk)$ and any $k \geq 0$, the map

$$\mathrm{real} : \mathrm{Ext}^k_{\mathrm{Perv}(X,\Bbbk)}(\mathcal{F}, \mathcal{G}) \to \mathrm{Hom}_{D^{\mathrm{b}}_{\mathrm{c}}(X,\Bbbk)}(\mathcal{F}, \mathcal{G}[k]) \tag{4.5.9}$$

induced by real is an isomorphism. The first few steps of the proof treat the case where X is an affine variety. In this case, we will proceed by noetherian induction.

Step 1. Assume that X is an affine variety and that $\mathrm{supp}\,\mathcal{F} \cup \mathrm{supp}\,\mathcal{G}$ *is a proper closed subvariety.* Choose a nonzero function $f : X \to \mathbb{A}^1$ in the defining ideal of $\mathrm{supp}\,\mathcal{F} \cup \mathrm{supp}\,\mathcal{G}$. Let $Z = f^{-1}(0)$, and let $i : Z \hookrightarrow X$ be the inclusion map. Since $\mathcal{F}$ and $\mathcal{G}$ are supported on Z, we have $\mathcal{F} \cong i_*\mathcal{F}'$ and $\mathcal{G} \cong i_*\mathcal{G}'$ for some $\mathcal{F}', \mathcal{G}' \in \mathrm{Perv}(Z, \Bbbk)$. Now consider the following diagram:

$$\begin{tikzcd}
\mathrm{Ext}^k_{\mathrm{Perv}(Z,\Bbbk)}(\mathcal{F}', \mathcal{G}') \arrow[r, "\mathrm{real}"] \arrow[d, "i_*"'] & \mathrm{Hom}_{D^{\mathrm{b}}_{\mathrm{c}}(Z,\Bbbk)}(\mathcal{F}', \mathcal{G}'[k]) \arrow[d, "i_*"] \\
\mathrm{Ext}^k_{\mathrm{Perv}(X,\Bbbk)}(i_*\mathcal{F}', i_*\mathcal{G}') \arrow[r, "\mathrm{real}"] & \mathrm{Hom}_{D^{\mathrm{b}}_{\mathrm{c}}(X,\Bbbk)}(i_*\mathcal{F}', i_*\mathcal{G}')
\end{tikzcd}$$

The top arrow is an isomorphism by induction. The left-hand vertical arrow is an isomorphism by Proposition 4.5.7, and the right-hand vertical arrow is an isomorphism by Corollary 1.3.11. It can be shown from the construction of the realization functor that the square commutes, and hence that the bottom arrow is an isomorphism.

Step 2. Assume that X is an affine variety and that $\mathrm{supp}\,\mathcal{G}$ *is a proper closed subvariety.* We proceed by induction on the number of composition factors of $\mathcal{F}$. First assume that $\mathcal{F}$ is simple. If $\mathrm{supp}\,\mathcal{F} \cup \mathrm{supp}\,\mathcal{G}$ is a proper closed subvariety, then we are done by Step 1. Otherwise, the support of $\mathcal{F}$ contains an open subset of X that does not meet $\mathrm{supp}\,\mathcal{G}$. Choose a connected affine open subset $U \subset X$ such that $\mathcal{G}|_U = 0$ and such that $\mathcal{F}|_U$ is a shifted local system. Let $j : U \hookrightarrow X$ be the inclusion map. Then $j_!(\mathcal{F}|_U)$ is a perverse sheaf. The adjunction map $j_!(\mathcal{F}|_U) \to \mathcal{F}$ is nonzero, and since $\mathcal{F}$ is simple, it is surjective. Let $\mathcal{K}$ be its kernel.

By construction, $\mathcal{K}$ is supported on the complement of U, so $\operatorname{supp}\mathcal{K} \cup \operatorname{supp}\mathcal{G}$ is a proper closed subset. Consider the commutative diagram

$$\begin{tikzcd}
\operatorname{Ext}^{k-1}(j_!(\mathcal{F}|_U), \mathcal{G}) \arrow[r] \arrow[d] & \operatorname{Ext}^{k-1}(\mathcal{K}, \mathcal{G}) \arrow[r] \arrow[d] & \operatorname{Ext}^{k}(\mathcal{F}, \mathcal{G}) \arrow[r] \arrow[d] & \operatorname{Ext}^{k}(j_!(\mathcal{F}|_U), \mathcal{G}) \arrow[d] \\
\operatorname{Hom}(j_!(\mathcal{F}|_U), \mathcal{G}[k-1]) \arrow[r] & \operatorname{Hom}(\mathcal{K}, \mathcal{G}[k-1]) \arrow[r] & \operatorname{Hom}(\mathcal{F}, \mathcal{G}[k]) \arrow[r] & \operatorname{Hom}(j_!(\mathcal{F}|_U), \mathcal{G}[k])
\end{tikzcd}$$

All terms in the first and fourth columns vanish, by Lemma 4.5.6 and the usual adjunction property of $j_!$. The second column is an isomorphism by Step 1, so the third column is an isomorphism as well.

If $\mathcal{F}$ is not simple, then there is some short exact sequence $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ in which $\mathcal{F}'$ and $\mathcal{F}''$ both have fewer composition factors. The result follows by applying the five lemma to the commutative diagram

$$\begin{tikzcd}
\operatorname{Ext}^{k-1}(\mathcal{F}', \mathcal{G}) \arrow[r] \arrow[d] & \operatorname{Ext}^{k}(\mathcal{F}'', \mathcal{G}) \arrow[r] \arrow[d] & \operatorname{Ext}^{k}(\mathcal{F}, \mathcal{G}) \arrow[r] \arrow[d] & \operatorname{Ext}^{k}(\mathcal{F}', \mathcal{G}) \arrow[r] \arrow[d] & \operatorname{Ext}^{k+1}(\mathcal{F}'', \mathcal{G}) \arrow[d] \\
\operatorname{Hom}(\mathcal{F}', \mathcal{G}[k-1]) \arrow[r] & \operatorname{Hom}(\mathcal{F}'', \mathcal{G}[k]) \arrow[r] & \operatorname{Hom}(\mathcal{F}, \mathcal{G}[k]) \arrow[r] & \operatorname{Hom}(\mathcal{F}', \mathcal{G}[k]) \arrow[r] & \operatorname{Hom}(\mathcal{F}'', \mathcal{G}[k+1])
\end{tikzcd}$$

Step 3. Conclusion of the proof for an affine variety. Assume for now that $\mathcal{G}$ is a simple perverse sheaf. If the support of $\mathcal{G}$ is a proper closed subvariety, then by Step 2, there is nothing to prove. Otherwise, choose an open subset $V \subset X$ such that both $\mathcal{F}|_V$ and $\mathcal{G}|_V$ are shifted local systems. Apply Theorem 4.5.5 to V to find an affine open subset $U \subset V$ on which $D^{\mathrm{b}}\mathrm{Loc}^{\mathrm{ft}}(U, \Bbbk) \to D^{\mathrm{b}}_{\mathrm{locf}}(U, \Bbbk)$ is an equivalence of categories. Let $j : U \hookrightarrow X$ be the inclusion map. Since j is an affine morphism, $j_*(\mathcal{G}|_U)$ is a perverse sheaf. Since $\mathcal{G}$ is simple, the adjunction map $\mathcal{G} \to j_*(\mathcal{G}|_U)$ is injective (because it is nonzero). Let $\mathcal{K}$ be its cokernel, and consider the diagram

(4.5.10)

$$\begin{tikzcd}
\operatorname{Ext}^{k-1}(\mathcal{F}, j_*(\mathcal{G}|_U)) \arrow[r] \arrow[d] & \operatorname{Ext}^{k-1}(\mathcal{F}, \mathcal{K}) \arrow[r] \arrow[d] & \operatorname{Ext}^{k}(\mathcal{F}, \mathcal{G}) \arrow[r] \arrow[d] & \operatorname{Ext}^{k}(\mathcal{F}, j_*(\mathcal{G}|_U)) \arrow[r] \arrow[d] & \operatorname{Ext}^{k}(\mathcal{F}, \mathcal{K}) \arrow[d] \\
\operatorname{Hom}(\mathcal{F}, j_*(\mathcal{G}|_U)[k-1]) \arrow[r] & \operatorname{Hom}(\mathcal{F}, \mathcal{K}[k-1]) \arrow[r] & \operatorname{Hom}(\mathcal{F}, \mathcal{G}[k]) \arrow[r] & \operatorname{Hom}(\mathcal{F}, j_*(\mathcal{G}|_U)[k]) \arrow[r] & \operatorname{Hom}(\mathcal{F}, \mathcal{K}[k])
\end{tikzcd}$$

Since $\mathcal{K}$ is supported on the complement of U, the second and fifth vertical maps are isomorphisms by Step 2.

To study the first and fourth vertical maps, consider the commutative diagram

$$\begin{tikzcd}
\operatorname{Ext}^{k}_{\mathrm{Perv}(U,\Bbbk)}(\mathcal{F}|_U, \mathcal{G}|_U) \arrow[r, "\sim"] \arrow[d] & \operatorname{Ext}^{k}_{\mathrm{Perv}(X,\Bbbk)}(\mathcal{F}, j_*(\mathcal{G}|_U)) \arrow[d] \\
\operatorname{Hom}(\mathcal{F}|_U, \mathcal{G}|_U[k]) \arrow[r, "\sim"] & \operatorname{Hom}(\mathcal{F}, j_*(\mathcal{G}|_U)[k])
\end{tikzcd} \tag{4.5.11}$$

Here, the top horizontal arrow is an isomorphism by Lemma 4.5.6 (and, of course, the bottom horizontal arrow is an isomorphism by Theorem 1.2.4). The left-hand vertical map fits into the sequence

$$\operatorname{Ext}^{k}_{\mathrm{Loc}(U,\Bbbk)}(\mathcal{F}|_U[-\dim U], \mathcal{G}|_U[-\dim U]) \\ \to \operatorname{Ext}^{k}_{\mathrm{Perv}(U,\Bbbk)}(\mathcal{F}|_U, \mathcal{G}|_U) \to \operatorname{Hom}(\mathcal{F}|_U, \mathcal{G}|_U[k]).$$

Because we used Theorem 4.5.5 to choose U, the composition of these maps is an isomorphism. It follows that both vertical maps in (4.5.11) are surjective, so the first and fourth vertical maps in (4.5.10) are surjective. By the four lemma, we conclude that (4.5.9) is surjective.

To show that it is injective, consider an element $\phi \in \mathrm{Ext}^k(\mathcal{F},\mathcal{G})$. Let us replace U with an affine open set from Lemma 4.5.8. Let $\phi' \in \mathrm{Ext}^k(\mathcal{F}, j_*(\mathcal{G}|_U))$ be the composition of ϕ with $\mathcal{G} \to j_*(\mathcal{G}|_U)$. In (4.5.11), ϕ' corresponds to $\phi|_U \in \mathrm{Ext}^k_{\mathrm{Perv}(U,\Bbbk)}(\mathcal{F}|_U, \mathcal{G}|_U)$. Lemma 4.5.8 implies that if $\phi' \neq 0$, then its image in $\mathrm{Hom}(\mathcal{F}, j_*(\mathcal{G}|_U)[k])$ is nonzero. A diagram chase in (4.5.10) then shows that if $\phi \neq 0$, its image in $\mathrm{Hom}(\mathcal{F},\mathcal{G}[k])$ is nonzero. We conclude that (4.5.9) is an isomorphism when $\mathcal{G}$ is simple.

Finally, for general $\mathcal{G}$, an induction argument on the number of composition factors (similar to that at the end of Step 2) shows that (4.5.9) is an isomorphism.

Step 4. Conclusion of the proof in general. Using Proposition A.7.18 and an induction argument on k, to prove that (4.5.9) is an isomorphism, it is enough to show that every morphism $\phi \in \mathrm{Hom}(\mathcal{F},\mathcal{G}[k])$ is effaceable in the sense of Definition A.7.17. Let $(U_1,\ldots,U_n)$ be a finite collection of affine open subsets that cover X. Because the theorem is already proved for each U_i, Lemma A.5.17 tells us that there is a surjective map $p_i : \mathcal{P}_i \to \mathcal{F}|_{U_i}$ such that $\phi|_{U_i} \circ p_i = 0$.

For each i, let $j_i : U_i \hookrightarrow X$ be the inclusion map, and let $\bar{p}_i : j_{i!}\mathcal{P}_i \to \mathcal{F}$ be the map corresponding to p_i by adjunction. The diagram

$$\begin{tikzcd}
j_{i!}\mathcal{P}_i \arrow[r, "j_{i!}(p_i)"'] \arrow[rr, bend left, "\bar{p}_i"] \arrow[dr, "j_{i!}(\phi|_{U_1}\circ p_i)=0"'] & j_{i!}(\mathcal{F}|_{U_i}) \arrow[r] \arrow[d, "j_{i!}(\phi|_{U_i})"] & \mathcal{F} \arrow[d, "\phi"] \\
& j_{i!}(\mathcal{G}|_{U_i})[k] \arrow[r] & \mathcal{G}[k]
\end{tikzcd}$$

shows that $\phi \circ \bar{p}_i = 0$. Summing over i, we obtain a map

$$\bar{p} : \bigoplus_{i=1}^{n} j_{i!}\mathcal{P}_i \to \mathcal{F} \qquad \text{such that} \qquad \phi \circ \bar{p} = 0.$$

Finally, the surjectivity of p_i implies that $\bar{p}|_{U_i}$ is surjective. That is, $(\operatorname{cok}\bar{p})|_{U_i}$ vanishes. Since this holds for all the U_i, $\operatorname{cok}\bar{p}$ must be the zero perverse sheaf. In other words, $\bar{p}$ is surjective, and hence ϕ is effaceable, as desired. ☐

Remark 4.5.10. It is natural to ask if the analogue of Theorem 4.5.9 is true when $\Bbbk$ is not a field. I don't know the answer to this question in general, but in the special case where X is a point, the answer is given by Proposition A.7.20. Indeed, using Lemma 3.1.3, that proposition can be rephrased as saying that for any noetherian ring $\Bbbk$ of finite global dimension, there is an equivalence of categories

$$D^{\mathrm{b}}\mathrm{Perv}(\mathrm{pt},\Bbbk) \xrightarrow{\sim} D^{\mathrm{b}}_{\mathrm{c}}(\mathrm{pt},\Bbbk).$$

4.6. Additional notes and exercises

Notes. The starting point for the notion of nearby cycles is the **Milnor fibration** associated to an analytic map $f : X \to \mathbb{C}$ and a point $x \in X_{\mathbf{s}}$, introduced by Milnor in [**176**], and generalized by Lê [**152**]. The singular cohomology $\mathbf{H}^\bullet(F_x;\mathbb{Z})$ of the Milnor fiber F_x is known as **nearby cycle cohomology**. The functor Ψ_f was introduced by Deligne [**5**]. In this setting, the nearby cycle cohomology $\mathbf{H}^\bullet(F_x;\mathbb{Z})$ at x is identified with the cohomology of the stalk of $\Psi_f(\underline{\mathbb{Z}}_{X_\eta})$ at x.

The t-exactness of Ψ_f and its commutativity with Verdier duality were both asserted in [**24**], but without complete proofs. These questions have subsequently had contributions from Goresky–MacPherson [**87**], Brylinski [**49**], Kashiwara–Schapira [**125**], Illusie [**109**], Massey [**169**], and others.

The content of Sections 4.3–4.5 is based on two short papers of Beilinson [**22**, **23**], along with related work of Verdier [**240**] and MacPherson–Vilonen [**166**] (see also the exegesis in [**191**]).

EXERCISE 4.6.1. Let $U = \mathbb{A}^1 \smallsetminus \{0\}$, and let $Z = \{0\}$. Let $j : U \hookrightarrow \mathbb{A}^1$ and $i : Z \hookrightarrow \mathbb{A}^1$ be the inclusion maps. Let $\mathscr{S}$ denote the stratification $\mathbb{A}^1 = U \cup Z$. The goal of this problem is to show that when $\Bbbk$ is a field, $\mathrm{Perv}_{\mathscr{S}}(\mathbb{A}^1, \Bbbk)$ can be described in the language of "representations of a quiver with relations." Let $\mathscr{C}$ be the category whose objects are diagrams

$$V_1 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} V_2$$

where V_1 and V_2 are finite-dimensional vector spaces, and α and β are linear maps such that $\mathrm{id}_{V_1} - \beta \circ \alpha$ and $\mathrm{id}_{V_2} - \alpha \circ \beta$ are invertible. Morphisms in $\mathscr{C}$ are defined in the obvious way. Use Theorem 4.3.9 to show that there is an equivalence of categories $\mathrm{Perv}_{\mathscr{S}}(\mathbb{A}^1, \Bbbk) \xrightarrow{\sim} \mathscr{C}$.

EXERCISE 4.6.2. This exercise is a continuation of Exercise 4.6.1. Here are a number of conditions that a perverse sheaf $\mathcal{F} \in \mathrm{Perv}_{\mathscr{S}}(\mathbb{A}^1, \Bbbk)$ may satisfy:

(a) $i^*\mathcal{F} = 0$.
(b) $i^!\mathcal{F} = 0$.
(c) $\mathcal{F}$ is supported on Z.
(d) $\mathcal{F}$ has no subobject supported on Z.
(e) $\mathcal{F}$ has no quotient supported on Z.

For each of these, find the corresponding condition on objects of $\mathscr{C}$.

EXERCISE 4.6.3. Throughout this problem, let $\Bbbk$ be a field.

(a) Give an example of a variety X with a good stratification $\mathscr{S}$ such that $D^{\mathrm{b}}\mathrm{Perv}_{\mathscr{S}}(X, \Bbbk)$ and $D^{\mathrm{b}}_{\mathscr{S}}(X, \Bbbk)$ are not equivalent. If this question is too easy, give three examples.
(b) Let $\mathscr{S}$ be the stratification of $\mathbb{A}^1$ from Exercise 4.6.1. Show that real : $D^{\mathrm{b}}\mathrm{Perv}_{\mathscr{S}}(\mathbb{A}^1, \Bbbk) \to D^{\mathrm{b}}_{\mathscr{S}}(\mathbb{A}^1, \Bbbk)$ is an equivalence of categories. *Hint*: Show first that for $\mathcal{F}, \mathcal{G} \in \mathrm{Perv}_{\mathscr{S}}(\mathbb{A}^1, \Bbbk)$, we have

$$\mathrm{Hom}_{D^{\mathrm{b}}_{\mathscr{S}}(\mathbb{A}^1, \Bbbk)}(\mathcal{F}, \mathcal{G}[k]) = 0 \qquad \text{for all } k \geq 2.$$

EXERCISE 4.6.4. Let X be a variety. Let $i : Z \hookrightarrow X$ be the inclusion of a closed subvariety, and let $j : U \hookrightarrow X$ be the inclusion of the complementary open subset. Let $\mathcal{F} \in \mathrm{Perv}(U, \Bbbk)$. A perverse sheaf $\mathcal{G} \in \mathrm{Perv}(X, \Bbbk)$ is called a **tilting extension** of $\mathcal{F}$ if the following conditions hold:

$$\mathcal{G}|_U \cong \mathcal{F}, \qquad i^*\mathcal{G} \in \mathrm{Perv}(Z, \Bbbk), \qquad i^!\mathcal{G} \in \mathrm{Perv}(Z, \Bbbk).$$

Show that if $\Bbbk$ is a field and $j : U \hookrightarrow X$ is an affine morphism, then every perverse sheaf in $\mathrm{Perv}(U, \Bbbk)$ admits a tilting extension. *Hint*: First show that ${}^p\mathsf{H}^i(i^*j_*\mathcal{F}) = 0$ unless $i = 0$ or $i = -1$, and then use Theorem 4.5.9 (applied to Z) to show that there exists a distinguished triangle $\mathcal{K}_1 \to i^*j_*\mathcal{F} \to \mathcal{K}_2[1] \to$ with $\mathcal{K}_1, \mathcal{K}_2 \in \mathrm{Perv}(Z, \Bbbk)$.

Finally, construct a tilting extension of X by considering the upper cap diagram

$$\begin{tikzcd}
j_!\mathcal{F} \arrow[dd] & & i_*\mathcal{K}_1 \arrow[ll, squiggly] \arrow[dl] \\
& i_*i^*j_*\mathcal{F} \arrow[ul, squiggly] \arrow[dr, squiggly] & \\
j_*\mathcal{F} \arrow[ur] \arrow[rr, squiggly] & & i_*\mathcal{K}_2 \arrow[uu]
\end{tikzcd}$$

Of course, $\mathcal{K}_1$ and $\mathcal{K}_2$ are not unique; there are many tilting extensions of $\mathcal{F}$.

EXERCISE 4.6.5. Apply the construction of Exercise 4.6.4 to $X = \mathbb{P}^1$, $Z = \{[0 : 1]\}$, and $U = \mathbb{P}^1 \smallsetminus Z \cong \mathbb{A}^1$. Show that the perverse sheaf $\underline{\Bbbk}_U[1] \in \mathrm{Perv}(U, \Bbbk)$ admits a unique *indecomposable* tilting extension $\mathcal{T} \in \mathrm{Perv}(\mathbb{P}^1, \Bbbk)$, and that all other tilting extensions are of the form $\mathcal{T} \oplus i_*\underline{\Bbbk}_Z^{\oplus n}$ for some $n \geq 0$.

Then show that $\mathcal{T}$ is uniserial and has length 3. That is, it has a unique composition series $0 = \mathcal{T}_0 \subset \mathcal{T}_1 \subset \mathcal{T}_2 \subset \mathcal{T}_3 = \mathcal{T}$. Moreover, we have

$$\mathcal{T}_1 \cong \mathcal{T}_3/\mathcal{T}_2 \cong \mathrm{IC}(Z, \underline{\Bbbk}) \qquad \text{and} \qquad \mathcal{T}_2/\mathcal{T}_1 \cong \mathrm{IC}(U, \underline{\Bbbk}).$$

EXERCISE 4.6.6. Let $U, Z \subset \mathbb{P}^1$ be as in Exercise 4.6.5, and let $j : U \hookrightarrow \mathbb{P}^1$ and $i : Z \hookrightarrow \mathbb{P}^1$ be the inclusion maps. Let $\mathscr{S}$ denote the stratification $\mathbb{P}^1 = U \cup Z$, and let $\Bbbk$ be a field. Classify the indecomposable perverse sheaves in $\mathrm{Perv}_{\mathscr{S}}(\mathbb{P}^1, \Bbbk)$. Which among them are projective objects of $\mathrm{Perv}_{\mathscr{S}}(\mathbb{P}^1, \Bbbk)$? Which are injective?

Answer: There are five indecomposable perverse sheaves, up to isomorphism:

$$\mathrm{IC}(U, \underline{\Bbbk}) \cong \underline{\Bbbk}_{\mathbb{P}^1}[1], \quad \mathrm{IC}(Z, \underline{\Bbbk}) \cong i_*\underline{\Bbbk}_Z, \quad j_!\underline{\Bbbk}_U[1], \quad j_*\underline{\Bbbk}_U[1], \quad \mathcal{T}.$$

Here, $\mathcal{T}$ is the indecomposable tilting extension of $\underline{\Bbbk}_U[1]$ from Exercise 4.6.5. Among these, $j_!\underline{\Bbbk}_U$ is projective, $j_*\underline{\Bbbk}_U$ is injective, and $\mathcal{T}$ is both projective and injective.

EXERCISE 4.6.7. Continue with the setting of Exercise 4.6.6. Prove that real : $D^{\mathrm{b}}\mathrm{Perv}_{\mathscr{S}}(\mathbb{P}^1, \Bbbk) \to D^{\mathrm{b}}_{\mathscr{S}}(\mathbb{P}^1, \Bbbk)$ is an equivalence of categories.

Hint: Deduce from Exercise 4.6.6 that $\mathrm{Perv}_{\mathscr{S}}(\mathbb{P}^1, \Bbbk)$ has enough projectives and enough injectives. Using the classification of indecomposable objects, show that if $\mathcal{F}$ is a projective perverse sheaf and $\mathcal{G}$ is an injective perverse sheaf, then $\mathrm{Hom}_{D^{\mathrm{b}}_{\mathscr{S}}(\mathbb{P}^1, \Bbbk)}(\mathcal{F}, \mathcal{G}[n]) = 0$ for all $n \geq 1$.

EXERCISE 4.6.8. Let $X = \mathbb{A}^2$, and let $f : X \to \mathbb{A}^1$ be the map $f(x_1, x_2) = x_1x_2$. In this problem, you will compute $\Psi_f^{\mathrm{un}}(\underline{\Bbbk}_{X_{\boldsymbol{\eta}}})$ using Proposition 4.4.3. For $I \subset \{1, 2\}$, let

$$X_I = \{(x_1, x_2) \in \mathbb{A}^2 \mid x_i = 0 \text{ if and only if } i \in I\}.$$

Note that $X_{\boldsymbol{\eta}} = X_{\varnothing}$. Each closure $\overline{X_I}$ is an affine space of dimension $2 - |I|$, so we have $\mathrm{IC}(X_I, \underline{\Bbbk}) \cong \underline{\Bbbk}_{\overline{X_I}}[2 - |I|]$. For brevity, we put

$$\mathrm{IC}_I = \mathrm{IC}(X_I, \underline{\Bbbk}).$$

(a) Compute the perverse sheaf $j_{\boldsymbol{\eta}*}\underline{\Bbbk}[2]$, and find a Loewy series for it. *Answer*:

$j_{\boldsymbol{\eta}*}\underline{\Bbbk}[2]$:

	$X_\varnothing$	X_1	X_2	$X_{1,2}$
0				$\Bbbk(-2)$
-1		$\underline{\Bbbk}(-1)$	$\underline{\Bbbk}(-1)$	$\underline{\Bbbk}(-1) \oplus \underline{\Bbbk}(-1)$
-2	$\underline{\Bbbk}$	$\underline{\Bbbk}$	$\underline{\Bbbk}$	$\underline{\Bbbk}$

Loewy series:

head:	$\mathrm{IC}_{1,2}(-2)$
	$\mathrm{IC}_1(-1) \oplus \mathrm{IC}_2(-1)$
socle:	$\mathrm{IC}_\varnothing$

(b) Compute $j_{\boldsymbol{\eta}*}f_{\boldsymbol{\eta}}^*\mathcal{J}_n[2]$ and $\mathrm{IC}(X_\varnothing, f_{\boldsymbol{\eta}}^*\mathcal{J}_n)$. *Hints and answers*:

$j_{\boldsymbol{\eta}*}f_{\boldsymbol{\eta}}^*\mathcal{J}_n[2]$:

	$X_\varnothing$	X_1	X_2	$X_{1,2}$
0				$\Bbbk(-2)$
-1		$\underline{\Bbbk}(-1)$	$\underline{\Bbbk}(-1)$	$\underline{\Bbbk}(n-2) \oplus \underline{\Bbbk}(-1)$
-2	$f_{\boldsymbol{\eta}}^*\mathcal{J}_n$	$\underline{\Bbbk}(n-1)$	$\underline{\Bbbk}(n-1)$	$\underline{\Bbbk}(n-1)$

To compute $\mathrm{IC}(X_\varnothing, f_{\boldsymbol{\eta}}^*\mathcal{J}_n)$, for $i = 1, 2$, let $j_i : X_i \hookrightarrow \mathbb{A}^2$ be the inclusion map, and consider the perverse sheaf $j_{i*}\underline{\Bbbk}[1](-1)$. By Exercise 3.10.4(a), there is a surjective map $\tau_i : j_{i*}\underline{\Bbbk}[1](-1) \to \mathrm{IC}_{1,2}(-2)$. Let $\mathcal{G}$ be the kernel of $\tau = (\tau_1, -\tau_2) : j_{1*}\underline{\Bbbk}[1](-1) \oplus j_{2*}\underline{\Bbbk}[1](-1) \to \mathrm{IC}_{1,2}(-2)$. It looks like:

$\mathcal{G}$:

	$X_\varnothing$	X_1	X_2	$X_{1,2}$
0				$\Bbbk(-2)$
-1		$\underline{\Bbbk}(-1)$	$\underline{\Bbbk}(-1)$	$\underline{\Bbbk}(-1) \oplus \underline{\Bbbk}(-1)$

Use adjunction to produce a natural map $\phi : j_{\boldsymbol{\eta}*}f_{\boldsymbol{\eta}}^*\mathcal{J}_n[2] \to j_{1*}\underline{\Bbbk}[1](-1) \oplus j_{2*}\underline{\Bbbk}[1](-1)$. Then show that $\tau \circ \phi = 0$, so ϕ induces a map $\psi : j_{\boldsymbol{\eta}*}f_{\boldsymbol{\eta}}^*\mathcal{J}_n \to \mathcal{G}$. Show that ψ is surjective, and that if $n \geq 2$, its kernel is

$\mathcal{G}'$:

	$X_\varnothing$	X_1	X_2	$X_{1,2}$
0				$\Bbbk(-1)$
-1				$\underline{\Bbbk}(n-2)$
-2	$f_{\boldsymbol{\eta}}^*\mathcal{J}_n$	$\underline{\Bbbk}(n-1)$	$\underline{\Bbbk}(n-1)$	$\underline{\Bbbk}(n-1)$

(What happens if $n = 1$?) Finally, show that there is a surjective map $\mathcal{G}' \to \mathrm{IC}_{1,2}(-1)$, and that its kernel is

$\mathrm{IC}(X_\varnothing, f_{\boldsymbol{\eta}}^*\mathcal{J}_n)$:

	$X_\varnothing$	X_1	X_2	$X_{1,2}$
0				
-1				$\underline{\Bbbk}(n-2)$
-2	$f_{\boldsymbol{\eta}}^*\mathcal{J}_n$	$\underline{\Bbbk}(n-1)$	$\underline{\Bbbk}(n-1)$	$\underline{\Bbbk}(n-1)$

(c) Compute the stalks and Loewy series for $\Psi_f^{\mathrm{un}}(\underline{\Bbbk}_{X_\varnothing})$. *Hint*: The calculations in the preceding part give rise to short exact sequences

$$0 \to \mathcal{G}' \to j_{\boldsymbol{\eta}*}f_{\boldsymbol{\eta}}^*\mathcal{J}_n[2] \to \mathcal{G} \to 0,$$
$$0 \to \mathrm{IC}(X_\varnothing, f_{\boldsymbol{\eta}}^*\mathcal{J}_n) \to \mathcal{G}' \to \mathrm{IC}_{1,2}(-1) \to 0.$$

Proposition 4.4.3 implies that there is a short exact sequence

$$0 \to \mathrm{IC}(X_\varnothing, f_{\boldsymbol{\eta}}^*\mathcal{J}_n) \to j_{\boldsymbol{\eta}*}f_{\boldsymbol{\eta}}^*\mathcal{J}_n[2] \to i_{\mathbf{s}*}\Psi_f^{\mathrm{un}}(\underline{\Bbbk}_{X_\varnothing})(-1) \to 0.$$

Using these calculations, deduce that

$$\Psi_f^{\mathrm{un}}(\underline{\Bbbk}_{X_\varnothing}) : \begin{array}{c|c|c|c} & X_1 & X_2 & X_{1,2} \\ \hline 0 & & & \underline{\Bbbk}(-1) \\ \hline -1 & \underline{\Bbbk} & \underline{\Bbbk} & \underline{\Bbbk} \end{array} \qquad \begin{array}{r|c|} \cline{2-2} \text{head:} & \mathrm{IC}_{1,2}(-1) \\ \cline{2-2} \text{Loewy series:} & \mathrm{IC}_1 \oplus \mathrm{IC}_2 \\ \cline{2-2} \text{socle:} & \mathrm{IC}_{1,2} \\ \cline{2-2} \end{array}$$

As a bonus, show that the monodromy map N_ϖ is a scalar multiple of

$$\Psi_f^{\mathrm{un}}(\underline{\Bbbk}_{X_\varnothing}) \twoheadrightarrow \mathrm{IC}_{1,2}(-2) \hookrightarrow \Psi_f^{\mathrm{un}}(\underline{\Bbbk}_{X_\varnothing})(-1).$$

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CHAPTER 5

Mixed sheaves

The focus of the preceding chapters has been on complex algebraic varieties equipped with the analytic topology. This chapter gives an overview (with almost no proofs) of other settings in which one can develop a theory of perverse sheaves. The first half of the chapter (Sections 5.1–5.4) deals with sheaves in the étale topology on algebraic varieties over a finite or algebraically closed field. The second half (Sections 5.5 and 5.6) is about algebraic $\mathscr{D}$-modules on complex varieties.

Both of these settings lead to an enriched ("mixed") version of perverse sheaves. Mixed sheaves—which always require coefficients in a field of characteristic 0—come equipped with a canonical finite filtration (called the **weight filtration**), and all morphisms are strictly compatible with the filtration. This imposes strong constraints on Hom- and Ext-groups. In Section 5.7, we discuss some ways in which these constraints can be exploited, especially in the context of hyperbolic localization (cf. Section 2.10) and push-forward along a proper map (cf. Section 3.9).

5.1. Étale and ℓ-adic sheaves

This section gives a brief overview of the étale topology and constructible étale complexes. There are no proofs in this section. For further reading, see [**75**, **136**].

Étale sheaves. Let $\mathbb{F}$ be either a finite field or an algebraically closed field. In the former case, fix an algebraic closure $\bar{\mathbb{F}}$ of $\mathbb{F}$. By a **variety** over $\mathbb{F}$, we mean a reduced, separated scheme of finite type over $\mathbb{F}$. For a variety X over $\mathbb{F}$, we write

$$\mathrm{a}_X : X \to \operatorname{Spec} \mathbb{F}$$

for the structure map. For any algebraic field extension $\mathbb{F}'$ of $\mathbb{F}$, recall that an $\mathbb{F}'$-point of X is a morphism of schemes $x : \operatorname{Spec} \mathbb{F}' \to X$. In particular, a **geometric point** is an $\bar{\mathbb{F}}$-point. The set of $\mathbb{F}'$-points of X is denoted by $X(\mathbb{F}')$.

Definition 5.1.1. Let X be a variety. An **étale open set** of X is an étale morphism of varieties $j : U \to X$. An **étale covering** of X is a collection of étale open sets $(j_\alpha : U_\alpha \to X)_{\alpha \in I}$ such that the union of the images $\bigcup_{\alpha\in I} j_\alpha(U_\alpha)$ is equal to X.

Given two étale open sets $j_1 : U_1 \to X$ and $j_2 : U_2 \to X$, a **morphism of étale open sets** is simply a morphism of varieties $h : U_1 \to U_2$ such that the triangle

$$\begin{tikzcd} U_1 \arrow[rr, "h"] \arrow[dr, "j_1"'] & & U_2 \arrow[dl, "j_2"] \\ & X & \end{tikzcd}$$

commutes. The category of étale open sets of X is denoted by $\mathrm{Op}_{\text{ét}}(X)$.

By Proposition 2.1.15, any morphism of étale open sets is again étale. Because the composition of two étale maps is étale, if $j : U \to X$ is an étale open set of X, then any étale open set of U gives rise to an étale open set of X.

Definition 5.1.2. Let X be a variety over $\mathbb{F}$. An **étale presheaf** of $\Bbbk$-modules on X is a functor $\mathcal{F} : \mathrm{Op}_{\text{ét}}(X)^{\mathrm{op}} \to \Bbbk\text{-mod}$. A **morphism** of étale presheaves $\phi : \mathcal{F} \to \mathcal{G}$ is a natural transformation of functors. The category of étale presheaves of $\Bbbk$-modules on X is denoted by $\mathrm{Presh}_{\text{ét}}(X, \Bbbk)$.

To say this in more concrete terms, an étale presheaf $\mathcal{F}$ is a rule that assigns:

- to each étale open set $U \xrightarrow{j} X$ a $\Bbbk$-module $\mathcal{F}(U \xrightarrow{j} X)$,
- to each morphism $h : (U \to X) \to (V \to X)$ of étale open sets a homomorphism
$$\mathrm{res}_{V \xleftarrow{h} U} : \mathcal{F}(V \to X) \to \mathcal{F}(U \to X),$$
called a **restriction map**.

It does not make sense to "intersect" to étale open sets. Instead, we form fiber products: if $U \to X$ and $V \to X$ are two étale open sets, then so is $U \times_X V$. Moreover, the maps pr_1 and pr_2 in the square

$$\begin{array}{ccc} U \times_X V & \xrightarrow{\mathrm{pr}_2} & V \\ {\scriptstyle \mathrm{pr}_1}\downarrow & & \downarrow \\ U & \longrightarrow & X \end{array}$$

are morphisms of étale open sets.

Definition 5.1.3. Let X be a variety over $\mathbb{F}$. An étale presheaf of $\Bbbk$-modules $\mathcal{F}$ on X is called an **étale sheaf** if it satisfies the following two axioms:

(1) (Gluing) Given an étale open covering $(U_\alpha \to U)_{\alpha \in I}$ of an étale open set $U \to X$ and a collection of sections $(s_\alpha \in \mathcal{F}(U_\alpha \to X))_{\alpha \in I}$ satisfying
$$\mathrm{res}_{U_\alpha \xleftarrow{\mathrm{pr}_1} U_\alpha \times_X U_\beta}(s_\alpha) = \mathrm{res}_{U_\beta \xleftarrow{\mathrm{pr}_2} U_\alpha \times_X U_\beta}(s_\beta)$$
for all $\alpha, \beta \in I$, there exists a section $s \in \mathcal{F}(U \to X)$ with the property that $\mathrm{res}_{U \leftarrow U_\alpha}(s) = s_\alpha$ for all $\alpha \in I$.

(2) (Local identity) Given an étale open covering $(U_\alpha \to U)_{\alpha \in I}$ of an étale open set $U \to X$ and two sections $s, t \in \mathcal{F}(U \to X)$, if $\mathrm{res}_{U \leftarrow U_\alpha}(s) = \mathrm{res}_{U \leftarrow U_\alpha}(t)$ for all $\alpha \in I$, then $s = t$.

The category of étale sheaves of $\Bbbk$-modules on X is denoted by $\mathrm{Sh}_{\text{ét}}(X, \Bbbk)$.

As in Chapter 1, there is a functor $\mathrm{Presh}_{\text{ét}}(X, \Bbbk) \to \mathrm{Sh}_{\text{ét}}(X, \Bbbk)$, called **sheafification**, that is left adjoint to the inclusion functor $\mathrm{Sh}_{\text{ét}}(X, \Bbbk) \hookrightarrow \mathrm{Presh}_{\text{ét}}(X, \Bbbk)$. See [**75**, Remark I.1.11 and Definition I.2.4].

Definition 5.1.4. Let $\bar{x}$ be a geometric point of X. An **étale neighborhood** of $\bar{x}$ is a pair $(j : U \to X, \bar{y})$ where $j : U \to X$ is an étale open set of X, and $\bar{y}$ is a geometric point of U such that $j \circ \bar{y} = \bar{x}$. A **morphism** of étale neighborhoods $h : (U_1 \to X, \bar{y}_1) \to (U_2 \to X, \bar{y}_2)$ is a morphism of étale open sets $h : U_1 \to U_2$ such that $h \circ \bar{y}_1 = \bar{y}_2$.

DEFINITION 5.1.5. Let $\mathcal{F}$ be an étale presheaf of $\Bbbk$-modules on X, and let $\bar{x}$ be a geometric point of X. The **stalk** of $\mathcal{F}$ at $\bar{x}$, denoted by $\mathcal{F}_{\bar{x}}$, is the $\Bbbk$-module

$$\mathcal{F}_{\bar{x}} = \varinjlim_{\substack{(U \to X, \bar{y}) \text{ an étale} \\ \text{neighborhood of } \bar{x}}} \mathcal{F}(U \to X).$$

As with Definition 1.1.3, we can make this explicit as follows. An element of $\mathcal{F}_{\bar{x}}$ is an equivalence class of pairs $((U \to X, \bar{y}), s)$, where $(U \to X, \bar{y})$ is an étale neighborhood of $\bar{x}$, and $s \in \mathcal{F}(U \to X)$, where the equivalence relation is given by $((U \to X, \bar{y}), s) \sim ((U' \to X, \bar{y}'), s')$ if there exists an étale neighborhood $(V \to X, \bar{z})$ and morphisms of étale neighborhoods $h : (V \to X, \bar{z}) \to (U \to X, \bar{y})$ and $h' : (V \to X, \bar{z}) \to (U' \to X, \bar{y}')$ such that $\mathrm{res}_h(s) = \mathrm{res}_{h'}(s')$ in $\mathcal{F}(V \to X)$.

We have the following counterparts of basic results from Section 1.1. For a discussion of these facts, see [**75**, Remarks I.1.12 and I.2.12].

- The category $\mathrm{Sh}_{\text{ét}}(X, \Bbbk)$ is an abelian category.
- For any geometric point $\bar{x}$ of X, the functor $\mathrm{Sh}_{\text{ét}}(X, \Bbbk) \to \Bbbk\text{-mod}$ given by $\mathcal{F} \mapsto \mathcal{F}_{\bar{x}}$ is exact.
- A morphism $\phi : \mathcal{F} \to \mathcal{G}$ of étale sheaves is injective, resp. surjective, resp. an isomorphism, if and only if for every geometric point $\bar{x}$ of X, the induced map on stalks $\phi_{\bar{x}} : \mathcal{F}_{\bar{x}} \to \mathcal{G}_{\bar{x}}$ is injective, resp. surjective, resp. an isomorphism.

Below are a few examples of étale sheaves. These examples are developed further in the exercises.

EXAMPLE 5.1.6. Let M be a $\Bbbk$-module. The **constant étale sheaf** on X with value M, denoted by $\underline{M}_X$, is the étale sheaf given by

$$\underline{M}_X(U \to X) = \{\text{locally constant functions } U \to X\}.$$

Example 5.1.6 leads to the following notion.

DEFINITION 5.1.7. An étale sheaf $\mathcal{F}$ is said to be **locally constant** if there is an étale open covering $(U_\alpha \to X)_{\alpha \in I}$ such that $\mathcal{F}|_{U_\alpha}$ is a constant sheaf for all $\alpha \in I$. It is said to be of **finite type** if its stalks are finitely generated $\Bbbk$-modules.

EXAMPLE 5.1.8. Assume that the characteristic of $\mathbb{F}$ is not 2. Let $U = X = \mathbb{A}^1 \smallsetminus \{0\}$, and let $j : U \to X$ be the map $j(z) = z^2$. Then $U \times_X U$ can be identified with the disjoint union of two copies of U. Explicitly, let $U_1 = \{(z, z) \in U \times_X U\}$ be the "diagonal" copy of U, and let $U_2 = \{(z, -z) \in U \times_X U\}$ be the "antidiagonal" copy. Consider the diagram

$$\begin{tikzcd} U_1 \sqcup U_2 \arrow[r, "\sim"] & U \times_X U \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"] & U \arrow[d, "j"] \\ & U \arrow[r, "j"] & X \end{tikzcd}$$

and consider the constant sheaf $\underline{\Bbbk}_U$ on U. Define an isomorphism

$$\theta : \mathrm{pr}_1^* \underline{\Bbbk}_U \xrightarrow{\sim} \mathrm{pr}_2^* \underline{\Bbbk}_U \qquad \text{by} \qquad \begin{cases} \theta|_{U_1} = \mathrm{id}_{\underline{\Bbbk}_{U_1}}, \\ \theta|_{U_2} = -\mathrm{id}_{\underline{\Bbbk}_{U_2}}. \end{cases}$$

It can be checked that θ satisfies a condition like that in Exercise 1.1.12 or Definition 3.7.1. By carrying out a "gluing" or "descent" procedure, the pair $(\underline{\Bbbk}_U, \theta)$ determines an étale sheaf $\mathcal{Q}$ on X, called the **square-root sheaf**.

EXAMPLE 5.1.9. Let $\mathbb{F}$ be a finite field, and let n be a positive integer that is relatively prime to $\operatorname{char}\mathbb{F}$. If $j : U \to \operatorname{Spec}\mathbb{F}$ is an étale map with U connected, then U must be $\operatorname{Spec}\mathbb{F}'$ for some finite field extension $\mathbb{F}'$ of $\mathbb{F}$. (See, for instance, [**75**, Remark 1.2].) We define a sheaf $\underline{\mathbb{Z}/n\mathbb{Z}}(1)$ on $\operatorname{Spec}\mathbb{F}$, called the **Tate module**, by the following rule:

$$\underline{\mathbb{Z}/n\mathbb{Z}}(1)(\operatorname{Spec}\mathbb{F}' \to \operatorname{Spec}\mathbb{F}) = \{n\text{th roots of unity in } \mathbb{F}'\}.$$

Note that the (multiplicative) group of nth roots of unity is naturally a $\mathbb{Z}/n\mathbb{Z}$-module. The Tate module is a sheaf in $\mathrm{Sh}_{\text{ét}}(\operatorname{Spec}\mathbb{F}, \mathbb{Z}/n\mathbb{Z})$.

More generally, if X is a variety over $\mathbb{F}$, then we define the **Tate sheaf** $\underline{\mathbb{Z}/n\mathbb{Z}}_X(1)$ by

$$\underline{\mathbb{Z}/n\mathbb{Z}}_X(1)(U \to X) = \{f \in \mathbb{F}[U] \mid f^n = 1\}.$$

It can be shown that $\underline{\mathbb{Z}/n\mathbb{Z}}_X(1)$ is a locally constant sheaf, all of whose stalks are isomorphic to $\mathbb{Z}/n\mathbb{Z}$. But if $\mathbb{F}$ itself does not contain the nth roots of unity, then $\underline{\mathbb{Z}/n\mathbb{Z}}_X(1)$ is *not* isomorphic to the constant sheaf $\underline{\mathbb{Z}/n\mathbb{Z}}_X$.

This example shows that even though $\operatorname{Spec}\mathbb{F}$ consists of a single point as a topological space, its étale topology may be rich enough to allow nonconstant sheaves. For more on the Tate sheaf, see Examples 5.1.17, 5.2.2, and 5.2.10.

Sheaf functors. The category of étale sheaves has enough injectives and enough flat sheaves. One can form its derived categories

$$D^{\mathrm{b}}_{\text{ét}}(X, \Bbbk) = D^{\mathrm{b}}\mathrm{Sh}_{\text{ét}}(X, \Bbbk) \qquad \text{and} \qquad D^{+}_{\text{ét}}(X, \Bbbk) = D^{+}\mathrm{Sh}_{\text{ét}}(X, \Bbbk).$$

If $f : X \to Y$ is a morphism of varieties, then the usual definitions of f^*, ${}^\circ f_*$, $\otimes$, and $\mathscr{H}om$ can be repeated. They give rise to derived functors f^*, f_*, $\otimes^L$, and $R\mathscr{H}om$, and these functors satisfy the usual adjunction properties.

The definition of $f_!$ in the étale setting is a bit trickier. If we just copy Definition 1.2.1(3) and take its right-derived functor, the result does not behave as expected. (See [**75**, Section I.8].) Instead, if f is an open embedding, we define $f_!$ to be the extension-by-zero functor, i.e., we take the formula in Lemma 1.3.1 as the definition. If f is proper, we define $f_! = f_*$. Finally, for a general morphism of varieties, we use Nagata's compactification theorem (Theorem 2.1.7) and then define $f_!$ in terms of the two special cases considered above. (We will discuss $f^!$ later on.)

Note that if $\mathbb{F}$ is a finite field, *the functor* a_{X*} *and the functor* $R\Gamma$ *cannot be identified.* Indeed, the former takes values values in the derived category of $\mathrm{Sh}_{\text{ét}}(\operatorname{Spec}\mathbb{F}, \Bbbk)$, which (as we saw above) is *not* equivalent to $\Bbbk$-mod. Thus, statements from Chapters 1–3 involving either a_{X*} or $R\Gamma$ can be generalized in two different ways. Motivated by Proposition 1.4.6, we introduce the following new functor.

DEFINITION 5.1.10. Let X be a variety over a field $\mathbb{F}$. The (derived) **geometric Hom functor** is the functor

$$R\underline{\mathrm{Hom}} : D^{-}_{\text{ét}}(X, \Bbbk)^{\mathrm{op}} \times D^{+}_{\text{ét}}(X, \Bbbk) \to D^{+}_{\text{ét}}(\operatorname{Spec}\mathbb{F}, \Bbbk)$$

given by

$$R\underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{G}) = \mathrm{a}_{X*}R\mathscr{H}om(\mathcal{F}, \mathcal{G}).$$

For $\mathcal{F} \in D^{-}_{\text{ét}}(X, \Bbbk)$ and $\mathcal{G} \in D^{+}_{\text{ét}}(X, \Bbbk)$, we also put

$$\underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{G}) = \mathsf{H}^0(R\underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{G})) \qquad \text{and} \qquad \underline{\mathbf{H}}^k(X, \mathcal{G}) = \mathsf{H}^k(R\underline{\mathrm{Hom}}(\underline{\Bbbk}_X, \mathcal{G})).$$

For more on $\underline{\mathrm{Hom}}$, see Remark 5.3.5.

Once we have defined all these functors, we would like to prove that they behave like their analogues in the topological setting. In more detail:

DESIDERATUM 5.1.11. The derived sheaf functors f^*, f_*, $f_!$, $\otimes^L$ and $R\mathscr{H}om$ should have the following properties:

(1) The various natural isomorphisms from Chapters 1 and 2 should hold, such as the proper base change theorem, the projection formula, and the smooth base change theorem.
(2) All of these functors should take constructible complexes to constructible complexes.
(3) The functor $f_! : D^{\mathrm{b}}_{\mathrm{c}}(X, \Bbbk) \to D^{\mathrm{b}}_{\mathrm{c}}(Y, \Bbbk)$ should have a right adjoint $f^! : D^{\mathrm{b}}_{\mathrm{c}}(Y, \Bbbk) \to D^{\mathrm{b}}_{\mathrm{c}}(X, \Bbbk)$.
(4) The functor $\mathbb{D} = R\mathscr{H}om(-, \mathrm{a}_X^! \underline{\Bbbk})$ should satisfy $\mathbb{D} \circ \mathbb{D} \cong \mathrm{id}$.

Unfortunately, we now run into serious problems: the proper and smooth base change theorems only hold for "torsion sheaves," and the results on constructibility only work for sheaves of finite sets!

ℓ-adic sheaves. All the statements from Desideratum 5.1.11 can be proved if we require $\Bbbk$ to be a finite ring in which $\mathrm{char}\,\mathbb{F}$ is invertible. But unfortunately, étale sheaves of $\mathbb{Z}$- or $\mathbb{Q}$-modules are not reasonable objects to work with, because the theorems of Chapters 1 and 2 do not hold.

A work-around is possible if one is willing to restrict to rings that can be "approximated" by finite rings. For example, the ring $\mathbb{Z}_\ell$ of ℓ-adic integers is the inverse limit of the finite rings $\mathbb{Z}/\ell^n\mathbb{Z}$. Instead of working with étale sheaves of $\mathbb{Z}_\ell$-modules, one could instead try to work formally with inverse limits of sheaves of $\mathbb{Z}/\ell^i\mathbb{Z}$-modules, since the latter do behave well. For the remainder of this section, the coefficient ring $\Bbbk$ will be required to come from the following list.

DEFINITION 5.1.12. Let $\mathbb{F}$ be a finite field or an algebraically closed field, and let $p = \mathrm{char}\,\mathbb{F}$. A commutative ring $\Bbbk$ is said to be a **permitted ring of coefficients** for varieties over $\mathbb{F}$ if it is one of the following forms:

(1) $\Bbbk$ is a finite self-injective ring, and if $p > 0$, then p is invertible in $\Bbbk$.
(2) $\Bbbk$ is the ring of integers in a finite extension of $\mathbb{Q}_\ell$, where ℓ is a prime number, and $\ell \neq p$.
(3) $\Bbbk$ is a finite field extension of the ℓ-adic numbers $\mathbb{Q}_\ell$, where ℓ is a prime number, and $\ell \neq p$.
(4) $\Bbbk$ is an algebraic closure $\overline{\mathbb{Q}}_\ell$ of $\mathbb{Q}_\ell$, where ℓ is a prime number, and $\ell \neq p$.

Recall that a **self-injective** ring is a ring that is an injective module over itself. Every finite cyclic ring $\mathbb{Z}/n\mathbb{Z}$ is self-injective. Similarly, if $\mathfrak{o}$ is the ring of integers in a finite extension of $\mathbb{Q}_\ell$, and $\mathfrak{m}$ is its maximal ideal, then every quotient $\mathfrak{o}/\mathfrak{m}^i$ is a finite self-injective ring. In other words, $\mathfrak{o}$ is an inverse limit of finite self-injective rings. For a discussion of the role of this condition in the theory, see Remark 5.1.20.

We define constructible sheaves first with coefficients in a finite self-injective ring, and later for other permitted coefficient rings.

DEFINITION 5.1.13. Let X be a variety over $\mathbb{F}$, and let $\Bbbk$ be a finite self-injective ring. If $\mathrm{char}\,\mathbb{F} > 0$, assume that $\mathrm{char}\,\mathbb{F}$ is invertible in $\Bbbk$. A sheaf $\mathcal{F} \in \mathrm{Sh}_{\mathrm{\acute{e}t}}(X, \Bbbk)$ is said to be **constructible** if there exists a stratification $(X_s)_{s \in \mathscr{S}}$ such that for each $s \in \mathscr{S}$, the restriction $\mathcal{F}|_{X_s}$ is a locally constant sheaf of finite type.

An object $\mathcal{F} \in D^{\mathrm{b}}_{\text{ét}}(X, \Bbbk)$ is said to be **constructible** if each of its cohomology sheaves is constructible. The full subcategory of $D^{\mathrm{b}}_{\text{ét}}(X, \Bbbk)$ consisting of constructible complexes is denoted by $D^{\mathrm{b}}_{\mathrm{c}}(X, \Bbbk)$.

Furthermore, an object $\mathcal{F} \in D^{\mathrm{b}}_{\mathrm{c}}(X, \Bbbk)$ is said to be **Tor-finite** if it is a quasi-isomorphic to a bounded complex of flat sheaves. The full subcategory of $D^{\mathrm{b}}_{\mathrm{c}}(X, \Bbbk)$ consisting of Tor-finite constructible complexes is denoted by $D^{\mathrm{b}}_{\mathrm{ctf}}(X, \Bbbk)$.

From Chapter 2 on, we have always worked with a coefficient ring $\Bbbk$ that was assumed to be of finite global dimension. But we are now forced to consider constructible sheaves with coefficients in such rings as $\mathbb{Z}/\ell^n\mathbb{Z}$, which has infinite global dimension for all $n > 1$. The purpose of the notion of "Tor-finite complexes" is to get around this problem.

Definition 5.1.14. Let X be a variety over $\mathbb{F}$, and let $\Bbbk$ be a permitted ring of coefficients. The **constructible derived category** of $\Bbbk$-sheaves on X, denoted by $D^{\mathrm{b}}_{\mathrm{c}}(X, \Bbbk)$, is defined as follows:

(1) If $\Bbbk$ is a finite self-injective ring, see Definition 5.1.13.

(2) If $\Bbbk$ is the ring of integers in a finite extension of $\mathbb{Q}_\ell$, let $\mathfrak{m} \subset \Bbbk$ be the maximal ideal. An object $\mathcal{F} \in D^{\mathrm{b}}_{\mathrm{c}}(X, \Bbbk)$ is a sequence of pairs $(\mathcal{F}_i, \phi_i)_{i \geq 1}$, where each $\mathcal{F}_i$ is a Tor-finite constructible complex in $D^{\mathrm{b}}_{\mathrm{ctf}}(X, \Bbbk/\mathfrak{m}^i)$ and where each ϕ_i is an isomorphism
$$\phi_i : \mathcal{F}_i \xrightarrow{\sim} (\Bbbk/\mathfrak{m}^i) \overset{L}{\otimes}_{\Bbbk/\mathfrak{m}^{i+1}} \mathcal{F}_{i+1}$$
in $D^{\mathrm{b}}_{\mathrm{ctf}}(X, \Bbbk/\mathfrak{m}^i)$. A morphism $f : \mathcal{F} \to \mathcal{G}$ in $D^{\mathrm{b}}_{\mathrm{c}}(X, \Bbbk)$ is a sequence of morphisms $(f_i : \mathcal{F}_i \to \mathcal{G}_i)_{i \geq 1}$ such that the following diagram commutes:
$$\begin{tikzcd}
\mathcal{F}_i \arrow[rr, "f_i"] \arrow[d, "\wr"'] & & \mathcal{G}_i \arrow[d, "\wr"] \\
(\Bbbk/\mathfrak{m}^i) \otimes_{\Bbbk/\mathfrak{m}^{i+1}} \mathcal{F}_{i+1} \arrow[rr, "(\Bbbk/\mathfrak{m}^i)\otimes_{\Bbbk/\mathfrak{m}^{i+1}} f_{i+1}"] & & (\Bbbk/\mathfrak{m}^i) \otimes_{\Bbbk/\mathfrak{m}^{i+1}} \mathcal{G}_{i+1}
\end{tikzcd}$$

(3) If $\Bbbk$ is a finite extension of $\mathbb{Q}_\ell$, let $\mathfrak{o} \subset \Bbbk$ be its ring of integers, so that $D^{\mathrm{b}}_{\mathrm{c}}(X, \mathfrak{o})$ has been defined above. Define $D^{\mathrm{b}}_{\mathrm{c}}(X, \Bbbk)$ to be the category with the same objects as $D^{\mathrm{b}}_{\mathrm{c}}(X, \mathfrak{o})$, but with Hom-groups given by
$$\mathrm{Hom}_{D^{\mathrm{b}}_{\mathrm{c}}(X,\Bbbk)}(\mathcal{F}, \mathcal{G}) = \Bbbk \otimes_{\mathfrak{o}} \mathrm{Hom}_{D^{\mathrm{b}}_{\mathrm{c}}(X,\mathfrak{o})}(\mathcal{F}, \mathcal{G}).$$

(4) Suppose $\Bbbk = \overline{\mathbb{Q}}_\ell$. An object $\mathcal{F} \in D^{\mathrm{b}}_{\mathrm{c}}(X, \overline{\mathbb{Q}}_\ell)$ is a pair $(E, \mathcal{F}_E)$, where $E \subset \overline{\mathbb{Q}}_\ell$ is a finite extension of $\mathbb{Q}_\ell$, and where $\mathcal{F}_E \in D^{\mathrm{b}}_{\mathrm{c}}(X, E)$. The space of morphisms from $\mathcal{F} = (E, \mathcal{F}_E)$ to $\mathcal{G} = (E', \mathcal{G}_{E'})$ is given by
$$\mathrm{Hom}_{D^{\mathrm{b}}_{\mathrm{c}}(X,\overline{\mathbb{Q}}_\ell)}(\mathcal{F}, \mathcal{G}) = \varinjlim_{\substack{E'' \subset \overline{\mathbb{Q}}_\ell \text{ a finite extension} \\ \text{of } \mathbb{Q}_\ell \text{ containing both } E \text{ and } E'}} \mathrm{Hom}_{D^{\mathrm{b}}_{\mathrm{c}}(X,E'')}(E'' \otimes_E \mathcal{F}_E, E'' \otimes_{E'} \mathcal{G}_{E'}).$$

In all cases, an object of $D^{\mathrm{b}}_{\mathrm{c}}(X, \Bbbk)$ is called a **constructible $\Bbbk$-complex**.

In cases (2)–(4), the objects of $D^{\mathrm{b}}_{\mathrm{c}}(X, \Bbbk)$ are also known informally as **constructible ℓ-adic complexes**.

Theorem 5.1.15. *Let X be a variety over $\mathbb{F}$, and let $\Bbbk$ be a permitted ring of coefficients. The category $D^{\mathrm{b}}_{\mathrm{c}}(X, \Bbbk)$ has the structure of a triangulated category, equipped with a* ***natural t-structure****.*

If $\Bbbk$ is a finite self-injective ring, there is nothing to prove here, but for ℓ-adic coefficients, even the fact that $D^{\mathrm{b}}_{\mathrm{c}}(X,\Bbbk)$ is a triangulated category is not entirely trivial. For a proof of this theorem, see [**136**, Sections II.5 and II.6]. The "natural t-structure" mentioned in this theorem plays the same conceptual role as that in Example A.7.2, but in the ℓ-adic case there are technical difficulties to be overcome before one can write down the definition, as $D^{\mathrm{b}}_{\mathrm{c}}(X,\Bbbk)$ does not come with a ready-made notion of "cohomology sheaf." Part of the content of Theorem 5.1.15 is explaining how to define this notion.

Definition 5.1.16. Let X be a variety over $\mathbb{F}$, and let $\Bbbk$ be a permitted ring of coefficients. A **constructible $\Bbbk$-sheaf** is an object in the heart of the natural t-structure on $D^{\mathrm{b}}_{\mathrm{c}}(X,\Bbbk)$. The category of constructible $\Bbbk$-sheaves is denoted by $\mathrm{Sh}_{\mathrm{c}}(X,\Bbbk)$.

It is worth reiterating what is going on here: Definitions 5.1.3 and 5.1.13 make sense for arbitrary coefficients, but they give undesirable results except when $\Bbbk$ is a finite ring. There is such a thing as a "constructible étale sheaf of $\mathbb{Z}_\ell$-modules," and *it is not the same concept* as a "constructible $\mathbb{Z}_\ell$-sheaf."

Built into the definitions are a number of change-of-scalars functors:

(1) If $\varphi : \Bbbk \to \Bbbk'$ is a homomorphism of finite self-injective rings (with the assumption that if $\operatorname{char}\mathbb{F} > 0$, then it is invertible in both), then the definition from Chapter 1 can be used directly to give a functor
$$\Bbbk' \overset{L}{\otimes} (-) : D^{\mathrm{b}}_{\mathrm{ctf}}(X,\Bbbk) \to D^{\mathrm{b}}_{\mathrm{ctf}}(X,\Bbbk').$$

(2) If $\Bbbk$ is the ring of integers in a finite extension of $\mathbb{Q}_\ell$, let $\mathfrak{m} \subset \Bbbk$ be the maximal ideal. Then the functor
$$(\Bbbk/\mathfrak{m}^k) \overset{L}{\otimes}_{\Bbbk} (-) : D^{\mathrm{b}}_{\mathrm{c}}(X,\Bbbk) \to D^{\mathrm{b}}_{\mathrm{ctf}}(X,\Bbbk/\mathfrak{m}^i)$$
sends an object $\mathcal{F} = (\mathcal{F}_i, \phi_i)_{i\ge 1}$ to $\mathcal{F}_k$. This functor is right t-exact.

(3) If $\Bbbk$ is a finite extension of $\mathbb{Q}_\ell$, and $\mathfrak{o} \subset \Bbbk$ is its ring of integers, there is an obvious functor
$$\Bbbk \otimes_{\mathfrak{o}} (-) : D^{\mathrm{b}}_{\mathrm{c}}(X,\mathfrak{o}) \to D^{\mathrm{b}}_{\mathrm{c}}(X,\Bbbk).$$
This functor is t-exact.

(4) If $\Bbbk = \overline{\mathbb{Q}}_\ell$, and $E \subset \overline{\mathbb{Q}}_\ell$ is a finite extension of $\mathbb{Q}_\ell$, there is an obvious functor
$$\overline{\mathbb{Q}}_\ell \otimes_E (-) : D^{\mathrm{b}}_{\mathrm{c}}(X,E) \to D^{\mathrm{b}}_{\mathrm{c}}(X,\overline{\mathbb{Q}}_\ell).$$
This functor is t-exact.

Example 5.1.17. Let X be a variety over a finite field $\mathbb{F}$, and suppose $\Bbbk = \mathbb{Z}_\ell$, where ℓ is a prime different from $\operatorname{char}\mathbb{F}$. Let $\underline{\mathbb{Z}_\ell}_X(1)$ denote the constructible $\mathbb{Z}_\ell$-sheaf on X given by
$$\underline{\mathbb{Z}_\ell}_X(1) = \varprojlim_i (\underline{\mathbb{Z}/\ell^i\mathbb{Z}}_X)(1).$$
Here, the right-hand side involves the sheaves from Example 5.1.9. Let us write this out explicitly in the notation of Definition 5.1.14: we define $\underline{\mathbb{Z}_\ell}_X(1)$ to be the sequence $(\underline{\mathbb{Z}/\ell^i\mathbb{Z}}_X(1), \phi_i)_{i\ge 1}$, where $\phi_i : \underline{\mathbb{Z}/\ell^i\mathbb{Z}}_X(1) \xrightarrow{\sim} \mathbb{Z}/\ell^i\mathbb{Z} \otimes_{\mathbb{Z}/\ell^{i+1}\mathbb{Z}} \underline{\mathbb{Z}/\ell^{i+1}\mathbb{Z}}_X(1)$ is the inverse of the sheafification of the presheaf map
$$\mathbb{Z}/\ell^i\mathbb{Z} \otimes_{\mathrm{pre},\mathbb{Z}/\ell^{i+1}\mathbb{Z}} \underline{\mathbb{Z}/\ell^{i+1}\mathbb{Z}}_X(1) \to \underline{\mathbb{Z}/\ell^i\mathbb{Z}}_X(1) \qquad \text{given by} \qquad 1 \otimes f \mapsto f^\ell,$$

where f is a section of $\underline{\mathbb{Z}/\ell^{i+1}\mathbb{Z}}_X(1)$, i.e., an element $f \in \mathbb{F}[U]$ for some étale open set $U \to X$ satisfying $f^{\ell^{i+1}} = 1$. This sheaf is called the **Tate sheaf** with coefficients in $\mathbb{Z}_\ell$. More generally, if $\Bbbk$ is the ring of integers in a finite extension of $\mathbb{Q}_\ell$, or if $\Bbbk$ is a finite extension or the algebraic closure of $\mathbb{Q}_\ell$, we define the Tate module with coefficients in $\Bbbk$ by change of scalars from $\mathbb{Z}_\ell$:

$$\underline{\Bbbk}_X(1) = \Bbbk \otimes_{\mathbb{Z}_\ell} \underline{\mathbb{Z}_\ell}_X(1).$$

Finally, for any constructible $\Bbbk$-complex $\mathcal{F}$, we define its **Tate twist** by

$$\mathcal{F}(1) = \mathcal{F} \overset{L}{\otimes} \underline{\Bbbk}_X(1).$$

For further discussion of the Tate sheaf, see Examples 5.2.2 and 5.2.10.

THEOREM 5.1.18. *Let $f : X \to Y$ be a morphism of varieties over $\mathbb{F}$, and let $\Bbbk$ be a permitted ring of coefficients.*

(1) *The functors f^*, f_*, $f_!$, $\otimes^L$ and $R\mathscr{H}om$ take constructible complexes to constructible complexes. If $\Bbbk$ is a finite self-injective ring, these functors also take Tor-finite constructible complexes to Tor-finite constructible complexes.*
(2) *The functor $f_! : D^{\mathrm{b}}_{\mathrm{c}}(X, \Bbbk) \to D^{\mathrm{b}}_{\mathrm{c}}(Y, \Bbbk)$ has a right adjoint $f^! : D^{\mathrm{b}}_{\mathrm{c}}(Y, \Bbbk) \to D^{\mathrm{b}}_{\mathrm{c}}(X, \Bbbk)$. If $\Bbbk$ is a finite ring, this restricts to a functor $f^! : D^{\mathrm{b}}_{\mathrm{ctf}}(Y, \Bbbk) \to D^{\mathrm{b}}_{\mathrm{ctf}}(X, \Bbbk)$.*
(3) *The various natural isomorphisms in Chapters* 1 *and* 2 *hold.*

For a discussion and references, see [**136**, Appendix D]. The pattern of the proof is that one must treat the case where $\Bbbk$ is finite first, including especially the statements about Tor-finite constructible complexes. The results for ℓ-adic coefficients follow from this. For finite coefficients, the results for f^*, $\otimes^L$, and the extension-by-zero functor for an open embedding are easy. For f_*, and $R\mathscr{H}om$, see [**136**, Corollary D.6], and for $f^!$, see [**136**, Sections II.7–II.9].

EXAMPLE 5.1.19. In many familiar examples, calculations in the setting of Theorem 5.1.18 look just like their classical analogues from Chapter 2. For instance, if $X = \mathbb{A}^n$, we have $\mathrm{a}_{X!}\underline{\Bbbk}_X \cong \underline{\Bbbk}_{\mathrm{pt}}[-2n](-n)$, and hence

$$\underline{\mathbf{H}}^i_{\mathrm{c}}(\mathbb{A}^n; \Bbbk) \cong \begin{cases} \underline{\Bbbk}(-n) & \text{if } i = 2n, \\ 0 & \text{otherwise.} \end{cases}$$

(Compare with Example 2.2.5.) Such calculations can be seen as instances of far-reaching comparison theorems (beyond the scope of this book) relating the étale and analytic settings.

Note that when $\mathbb{F}$ is a finite field, the Tate twist above cannot be omitted, because the Tate sheaf is not isomorphic to the constant sheaf. Similarly, the Tate twists in the étale analogues of Theorem 2.2.9 and related statements cannot be omitted.

REMARK 5.1.20. One striking difference between Theorem 5.1.18 and most of the rest of the book is that here, $\Bbbk$ is not required to have finite global dimension! This observation is relevant only to the case of finite rings: of course, fields have global dimension 0, and the ring of integers in a finite extension of $\mathbb{Q}_\ell$ has global dimension 1 (because it is a principal ideal domain).

One key reason for the omnipresent assumption of finite global dimension is to make Verdier duality work; ultimately, this goes back to Theorem A.10.2. It turns out that Theorem A.10.2 also holds for self-injective rings of infinite global dimension. This is why finite rings were assumed to be self-injective in Definition 5.1.12.

Perverse sheaves. We adapt Definition 3.1.1 to the étale setting as follows:

DEFINITION 5.1.21. Let X be a variety over $\mathbb{F}$, and let $\Bbbk$ be a permitted ring of coefficients. The **perverse t-structure** on $D^{\mathrm{b}}_{\mathrm{c}}(X,\Bbbk)$ is the t-structure given by

$$ {}^{p}D^{\mathrm{b}}_{\mathrm{c}}(X,\Bbbk)^{\leq 0} = \{\mathcal{F} \in D^{\mathrm{b}}_{\mathrm{c}}(X,\Bbbk) \mid \text{for all } i, \text{ we have } \dim\operatorname{supp} \mathsf{H}^i(\mathcal{F}) \leq -i\}. $$

The heart of this t-structure is denoted by $\mathrm{Perv}(X,\Bbbk)$, and objects in the heart are called **perverse sheaves**.

We have not discussed the notions of "grade" or "modified dimension of support" in the étale setting, so the description of ${}^{p}D^{\mathrm{b}}_{\mathrm{c}}(X,\Bbbk)^{\geq 0}$ given in Definition 3.1.1 is not available. Instead, by Lemma A.7.3, we just have

$$ {}^{p}D^{\mathrm{b}}_{\mathrm{c}}(X,\Bbbk)^{\geq 0} = \{\mathcal{F} \in D^{\mathrm{b}}_{\mathrm{c}}(X,\Bbbk) \mid \text{for all } \mathcal{G} \in {}^{p}D^{\mathrm{b}}_{\mathrm{c}}(X,\Bbbk)^{\leq 0},\ \mathrm{Hom}(\mathcal{G}[1],\mathcal{F}) = 0\}. $$

As usual, if $\Bbbk$ is a field, we have

$$ {}^{p}D^{\mathrm{b}}_{\mathrm{c}}(X,\Bbbk)^{\geq 0} = \{\mathcal{F} \in D^{\mathrm{b}}_{\mathrm{c}}(X,\Bbbk) \mid \text{for all } i, \text{ we have } \dim\operatorname{supp} \mathsf{H}^i(\mathbb{D}\mathcal{F}) \leq -i\}. $$

The main results from Chapter 3 remain true in the étale setting.

Exercises.

5.1.1. Show that the formula in Example 5.1.6 really does define an étale sheaf.

5.1.2. Consider the square-root sheaf $\mathcal{Q}$ from Example 5.1.8.
(a) Describe its sections over an étale open set $h : V \to X$ explicitly.
(b) Show that the square root sheaf is locally constant and that its stalks are isomorphic to $\Bbbk$.
(c) The same terminology was used in Exercise 1.1.8. What do these two "square-root sheaves" have to do with each other?

5.1.3. This exercise is about the Tate sheaf from Example 5.1.9.
(a) Show that for any morphism $f : X \to Y$ of varieties over $\mathbb{F}$, we have $f^*\underline{\mathbb{Z}/n\mathbb{Z}}_Y(1) \cong \underline{\mathbb{Z}/n\mathbb{Z}}_X(1)$. In particular, $\underline{\mathbb{Z}/n\mathbb{Z}}_X(1) \cong \mathrm{a}_X^*\underline{\mathbb{Z}/n\mathbb{Z}}_{\operatorname{Spec} F}(1)$.
(b) Show that the Tate sheaf is locally constant and that its stalks are isomorphic to $\mathbb{Z}/n\mathbb{Z}$.

5.2. Local systems and the étale fundamental group

This section gives an overview of the étale counterparts to the content of Section 1.7. As in the previous section, most proofs are omitted.

DEFINITION 5.2.1. Let X be a variety over $\mathbb{F}$, and let $\Bbbk$ be a permitted ring of coefficients. A constructible $\Bbbk$-sheaf $\mathcal{F}$ is said to be a **lisse sheaf**, or a **local system of finite type**, under the following conditions:
(1) If $\Bbbk$ is a finite ring, then $\mathcal{F}$ is lisse if it is locally constant of finite type, in the sense of Definition 5.1.7.
(2) If $\Bbbk$ is the ring of integers in a finite extension of $\mathbb{Q}_\ell$, let $\mathfrak{m} \subset \Bbbk$ be the maximal ideal. Then $\mathcal{F}$ is lisse if each $\mathsf{H}^0(\Bbbk/\mathfrak{m}^i \otimes^L_{\Bbbk} \mathcal{F}) \in \mathrm{Sh}_{\mathrm{c}}(X,\Bbbk/\mathfrak{m}^i)$ is locally constant of finite type.

(3) If $\Bbbk$ is a finite extension of $\mathbb{Q}_\ell$, with ring of integers $\mathfrak{o}$, then $\mathcal{F}$ is lisse if it is isomorphic to $\Bbbk \otimes^L_{\mathfrak{o}} \mathcal{F}'$ for some lisse $\mathfrak{o}$-sheaf $\mathcal{F}' \in \mathrm{Sh_c}(X, \mathfrak{o})$.
(4) If $\Bbbk = \overline{\mathbb{Q}}_\ell$, then $\mathcal{F}$ is lisse if there is some finite extension $E \subset \overline{\mathbb{Q}}_\ell$ of $\mathbb{Q}_\ell$ and some lisse sheaf $\mathcal{F}' \in \mathrm{Sh_c}(X, E)$ such that $\mathcal{F} \cong \overline{\mathbb{Q}}_\ell \otimes^L_E \mathcal{F}'$.

The category of lisse $\Bbbk$-sheaves on X is denoted by $\mathrm{Loc}^{\mathrm{ft}}(X, \Bbbk)$.

In some sources, lisse sheaves are called "locally constant," but there is a caveat: except in the finite ring case, it need not be true that every point has an étale neighborhood over which the sheaf is constant.

Example 5.2.2. The Tate sheaf $\underline{\Bbbk}_X(1)$ from Example 5.1.17 is lisse.

The following statement is obvious when $\Bbbk$ is a finite self-injective ring (and of course its analogue in Chapter 2 was a definition, not a theorem). For a proof in the ℓ-adic case, see [**75**, Proposition I.12.10].

Proposition 5.2.3. *Let X be a variety over $\mathbb{F}$, and let $\Bbbk$ be a permitted ring of coefficients. For any constructible $\Bbbk$-sheaf $\mathcal{F}$, there exists a stratification $(X_s)_{s \in \mathscr{S}}$ such that each $\mathcal{F}|_{X_s}$ is a lisse sheaf.*

Below, we will state an étale version of Theorem 1.7.9. The next few definitions serve as preparation for this statement.

Definition 5.2.4. A morphism of varieties $f : X \to Y$ is said to be a **covering map** if it is étale, proper, and surjective. If $f : X \to Y$ is a covering map of connected varieties, the **Galois group** of f, also known as the group of **deck transformations**, is the group

$$\mathrm{Gal}(X/Y) = \{\text{isomorphisms } h : X \to X \mid f \circ h = f\},$$

but where the group structure is *opposite* to that given by composition of maps $X \to X$.

Remark 5.2.5. Note that the group structure in Definition 5.2.4 is opposite to the convention used in Section 1.7. This reflects the predominant conventions in the literature: Section 1.7 is consistent with common references for algebraic topology ([**98**, **225**]), while Definition 5.2.4 is consistent with the étale sheaf theory literature [**4**, **75**, **136**].

In the étale setting, one reason to prefer the convention of Definition 5.2.4 is that when X and Y are affine varieties, one can identify

$$\mathrm{Gal}(X/Y) \cong \{\mathbb{F}[Y]\text{-algebra isomorphisms } \mathbb{F}[X] \to \mathbb{F}[X]\}, \tag{5.2.1}$$

where the group structure is given by composition of algebra maps $\mathbb{F}[X] \to \mathbb{F}[X]$.

Definition 5.2.6. Let X be a connected variety, and let $\bar{x}$ be a geometric point of X. The **étale fundamental group** is the group

$$\pi_1^{\text{ét}}(X, \bar{x}) = \varprojlim_{(f:U \to X, \bar{u})} \mathrm{Gal}(U/X),$$

where the limit is over pairs $(f : U \to X, \bar{u})$ such that f is a Galois covering map satisfying $f \circ \bar{u} = \bar{x}$.

For basic properties of the étale fundamental group, see [**75**, Appendix A.I]. As a consequence of (5.2.1), if $\mathbb{F}$ is a perfect field, the étale fundamental group of $\operatorname{Spec} \mathbb{F}$ is identified with its usual absolute Galois group (see, for instance, (5.2.2) below). The following fact comes from [**1**, Théorème IX.6.1 and Corollaire IX.6.4].

PROPOSITION 5.2.7. *Let X_0 be a connected variety over a finite field $\mathbb{F}$, and let $X = \operatorname{Spec} \bar{\mathbb{F}} \times_{\operatorname{Spec} \mathbb{F}} X_0$. For any geometric point $\bar{x}$ of X, there is a natural short exact sequence*

$$1 \to \pi_1^{\text{ét}}(X, \bar{x}) \to \pi_1^{\text{ét}}(X_0, \bar{x}) \to \operatorname{Gal}(\bar{\mathbb{F}}/\mathbb{F}) \to 1.$$

Moreover, if $\bar{x}$ factors through an $\mathbb{F}$-point of X_0, then this sequence splits.

By definition, $\pi_1^{\text{ét}}(X, \bar{x})$ has the structure of a **profinite group**, i.e., an inverse limit of finite groups. As such, it carries a natural topology, making it into a topological group.

Each of the permitted rings $\Bbbk$ from Definition 5.1.12 also has a natural topology: finite self-injective rings have the discrete topology, while each of the ℓ-adic cases has the ℓ-adic topology. It makes sense to speak of topological modules over these topological rings.

DEFINITION 5.2.8. Let X be a variety over $\mathbb{F}$, and let $\Bbbk$ be a permitted ring of coefficients. A **continuous** $\Bbbk[\pi_1^{\text{ét}}(X, \bar{x})]$-module is a $\Bbbk[\pi_1^{\text{ét}}(X, \bar{x})]$-module M with the following properties:

1. It is finitely generated over $\Bbbk$.
2. It is equipped with a topology that makes it into a topological $\Bbbk$-module.
3. The action map $\pi_1(X, \bar{x}) \times M \to M$ is continuous.

The category of continuous $\Bbbk[\pi_1^{\text{ét}}(X, \bar{x})]$-modules and continuous homomorphisms between them is denoted by $\Bbbk[\pi_1^{\text{ét}}(X, \bar{x})]\text{-mod}^{\text{fc}}$.

THEOREM 5.2.9. *Let X be a connected variety, and let $\bar{x}$ be a geometric point of X. Let $\Bbbk$ be a permitted ring of coefficients. There is an equivalence of categories*

$$\operatorname{Mon}_{\bar{x}} : \operatorname{Loc}^{\text{ft}}(X, \Bbbk) \xrightarrow{\sim} \Bbbk[\pi_1^{\text{ét}}(X, \bar{x})]\text{-mod}^{\text{fc}}.$$

For a proof of Theorem 5.2.9, see [**75**, Propositions A.I.7 and A.I.8].

An important special case of Theorem 5.2.9 is that in which $X = \operatorname{Spec} \mathbb{F}_q$. Then $\operatorname{Spec} \bar{\mathbb{F}}_q$ is a geometric point of $\operatorname{Spec} \mathbb{F}_q$, and we have

$$\pi_1^{\text{ét}}(\operatorname{Spec} \mathbb{F}_q, \operatorname{Spec} \bar{\mathbb{F}}_q) = \operatorname{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q). \tag{5.2.2}$$

It is well known that this group is isomorphic to $\hat{\mathbb{Z}}$, the profinite completion of the integers. (See, for instance, [**77**, Section 1.5].) The **arithmetic Frobenius element** $\mathsf{Fr}_q^{\text{arith}} \in \operatorname{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q)$ is the field automorphism $\mathsf{Fr}_q^{\text{arith}} : \bar{\mathbb{F}}_q \to \bar{\mathbb{F}}_q$ defined by $\mathsf{Fr}_q(x) = x^q$. Its inverse is called the **geometric Frobenius element** and is denoted simply by

$$\mathsf{Fr}_q = (\mathsf{Fr}_q^{\text{arith}})^{-1}.$$

Both $\mathsf{Fr}_q^{\text{arith}}$ and Fr_q are **topological generators** of $\operatorname{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q)$: this means that the (infinite cyclic) subgroup generated by either of these elements is dense in $\operatorname{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q)$. As a consequence, any continuous $\Bbbk[\operatorname{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q)]$-module is determined by the action of Fr_q.

EXAMPLE 5.2.10. Let $X = \operatorname{Spec} \mathbb{F}_q$, and choose a map $\bar{x} : \operatorname{Spec} \bar{\mathbb{F}}_q \to \operatorname{Spec} \mathbb{F}_q$. Consider the Tate module $\underline{\mathbb{Z}/m\mathbb{Z}}(1)$ from Example 5.1.9, where $\operatorname{char} \mathbb{F}_q \nmid m$. From the definition, we see that its stalk at $\bar{x}$ is

$$\underline{\mathbb{Z}/m\mathbb{Z}}(1)_{\bar{x}} = \{f \in \bar{\mathbb{F}}_q \mid f^m = 1\},$$

regarded as an abelian group (or a $\mathbb{Z}/m\mathbb{Z}$-module) under multiplication. The Galois group action on this stalk is induced by the action on $\bar{\mathbb{F}}_q$. In particular, $\mathsf{Fr}_q^{\mathrm{arith}}$ acts by $f \mapsto f^q$, or, converting to additive language,

$$\mathsf{Fr}_q^{\mathrm{arith}} : \underline{\mathbb{Z}/m\mathbb{Z}}(1)_{\bar{x}} \to \underline{\mathbb{Z}/m\mathbb{Z}}(1)_{\bar{x}} \qquad \text{is given by multiplication by } q.$$

Now let ℓ be prime different from $\operatorname{char} \mathbb{F}_q$, and apply the preceding discussion to $m = \ell^i$. Taking the inverse limit over i, we obtain the corresponding statement for the ℓ-adic Tate module:

$$\begin{aligned} \mathsf{Fr}_q^{\mathrm{arith}} : \underline{\mathbb{Z}_\ell}(1)_{\bar{x}} \to \underline{\mathbb{Z}_\ell}(1)_{\bar{x}} &\qquad \text{is given by multiplication by } q, \text{ and} \\ \mathsf{Fr}_q : \underline{\mathbb{Z}_\ell}(1)_{\bar{x}} \to \underline{\mathbb{Z}_\ell}(1)_{\bar{x}} &\qquad \text{is given by multiplication by } q^{-1}. \end{aligned}$$

Of course, the same descriptions hold for the Tate sheaf with coefficients in a finite extension of $\mathbb{Q}_\ell$, or in $\overline{\mathbb{Q}}_\ell$.

The main results from Section 1.7 hold for lisse sheaves and étale fundamental groups. However, results involving deck transformations must be modified to take into account the difference in conventions discussed in Remark 5.2.5. For instance, let $f : (X, \bar{x}) \to (Y, \bar{y})$ be a covering map of connected varieties. As in (1.7.5), there is an isomorphism

$$\theta : \mathrm{Gal}(X/Y) \to \mathrm{N}_{\pi_1^{\text{ét}}(Y,\bar{y})} \pi_1^{\text{ét}}(X, \bar{x}) / \pi_1^{\text{ét}}(X, \bar{x}). \tag{5.2.3}$$

In place of Lemma 1.7.18, we have the following result.

LEMMA 5.2.11. *Let X and Y be connected varieties over $\mathbb{F}$, and let $f : (X, \bar{x}) \to (Y, \bar{y})$ be a covering map. Let $h : X \to X$ be a deck transformation. For $\mathcal{L} \in \mathrm{Loc}^{\mathrm{ft}}(X, \Bbbk)$, there is a natural isomorphism*

$$\mathrm{Mon}_{\bar{x}}(h^* \mathcal{L}) \cong \mathrm{Mon}_{\bar{x}}(\mathcal{L})^{\theta(h^{-1})}.$$

Moreover, for $\mathcal{F} \in \mathrm{Loc}(Y, \Bbbk)$, we have a commutative diagram

$$\begin{tikzcd} \mathrm{Mon}_{\bar{x}}(h^* f^* \mathcal{F}) \arrow[r, "\sim"] \arrow[d, "\wr"'] & (\mathrm{Res}^{\Bbbk[\pi_1^{\text{ét}}(Y,\bar{y})]}_{\Bbbk[\pi_1^{\text{ét}}(X,\bar{x})]} \mathrm{Mon}_{\bar{y}}(\mathcal{F}))^{\theta(h^{-1})} \arrow[d, "\wr"] \\ \mathrm{Mon}_{\bar{x}}(f^* \mathcal{F}) \arrow[r, "\sim"] & \mathrm{Res}^{\Bbbk[\pi_1^{\text{ét}}(Y,\bar{y})]}_{\Bbbk[\pi_1^{\text{ét}}(X,\bar{x})]} \mathrm{Mon}_{\bar{y}}(\mathcal{F}) \end{tikzcd}$$

As in Lemma 1.7.18, one can make the right-hand part of this diagram explicit by choosing a representative $\tilde{h} \in \mathrm{N}_{\pi_1^{\text{ét}}(Y,\bar{y})} \pi_1^{\text{ét}}(X, \bar{x})$ of $\theta(h)$. Then the module in the upper right corner is $(\mathrm{Res}^{\Bbbk[\pi_1^{\text{ét}}(Y,\bar{y})]}_{\Bbbk[\pi_1^{\text{ét}}(X,\bar{x})]} \mathrm{Mon}_{\bar{y}}(\mathcal{F}))^{\tilde{h}^{-1}}$, and the right-hand vertical arrow is multiplication by $\tilde{h}$.

EXAMPLE 5.2.12. Let $\mathbb{F}$ be a field of characteristic p, and let $X = \mathbb{A}^1 = \operatorname{Spec} \mathbb{F}[x]$. The ring map $\mathbb{F}[x] \to \mathbb{F}[x]$ given by $x \mapsto x^p - x$ determines a morphism of varieties $f : \mathbb{A}^1 \to \mathbb{A}^1$, called an **Artin–Schreier covering map**. This map is a Galois covering map, with Galois group $\mathbb{Z}/p\mathbb{Z}$. In particular, there is a surjective homomorphism $\pi_!^{\text{ét}}(\mathbb{A}^1, \bar{x}) \twoheadrightarrow \mathbb{Z}/p\mathbb{Z}$.

Now let $\Bbbk$ be a permitted ring of coefficients. For any group homomorphism $\psi : \mathbb{Z}/p\mathbb{Z} \to \Bbbk^\times$, we obtain, via Theorem 5.2.9, a lisse sheaf $\mathcal{L}_\psi \in \mathrm{Loc}^{\mathrm{ft}}(\mathbb{A}^1, \Bbbk)$ with the property that $f^* \mathcal{L}_\psi \cong \underline{\Bbbk}_{\mathbb{A}^1}$. The sheaf $\mathcal{L}_\psi$ is called the **Artin–Schreier local system** associated to ψ.

The following fact can be seen as analogous to Theorem 1.9.7.

PROPOSITION 5.2.13. *Let $\mathbb{F}$ be a finite or algebraically closed field, and let $\Bbbk$ be a permitted ring of coefficients. There is an equivalence of categories*

$$D^{\mathrm{b}}\mathrm{Loc}^{\mathrm{ft}}(\operatorname{Spec}\mathbb{F},\Bbbk) \xrightarrow{\sim} D^{\mathrm{b}}_{\mathrm{c}}(\operatorname{Spec}\mathbb{F},\Bbbk).$$

When $\Bbbk$ is a finite self-injective ring, this follows easily from the definition. For ℓ-adic coefficients, this is proved at the beginning of [**136**, Section II.6].

Given a continuous $\Bbbk[\mathrm{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q)]$-module M, let

$$M^{\mathsf{Fr}} = \{m \in M \mid \mathsf{Fr}\cdot m = m\}$$

be the module of **Frobenius invariants** in M, and let

$$M_{\mathsf{Fr}} = M/(\{m - \mathsf{Fr}\cdot m \mid m \in M\})$$

be the module **Frobenius coinvariants**.

LEMMA 5.2.14. *Let M be a continuous $\Bbbk[\mathrm{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q)]$-module, where $\Bbbk$ is a permitted ring of coefficients. We have natural isomorphisms*

$$\begin{aligned}
&\mathrm{Hom}(\Bbbk, M) \cong M^{\mathsf{Fr}},\\
&\mathrm{Ext}^1(\Bbbk, M) \cong M_{\mathsf{Fr}},\\
&\mathrm{Ext}^k(\Bbbk, M) = 0 \qquad \textit{for } k \geq 2.
\end{aligned}$$

PROOF SKETCH. If $\Bbbk$ is a finite self-injective ring, then M has the discrete topology, and these claims come from the study of the Galois cohomology of $\hat{\mathbb{Z}}$; see, for instance, [**209**, Section XIII.1]. The ℓ-adic cases follow from this case. □

LEMMA 5.2.15. *Let $\Bbbk$ be a permitted ring of coefficients for varieties over the finite field $\mathbb{F}_q$. For any $\mathcal{F} \in D^{\mathrm{b}}_{\mathrm{c}}(\operatorname{Spec}\mathbb{F}_q,\Bbbk)$, there is a natural short exact sequence*

$$0 \to \mathsf{H}^{-1}(\mathcal{F})_{\mathsf{Fr}} \to \mathrm{Hom}(\Bbbk,\mathcal{F}) \to \mathsf{H}^0(\mathcal{F})^{\mathsf{Fr}} \to 0.$$

PROOF. Lemma 5.2.14 implies that

$$\mathrm{Hom}(\underline{\Bbbk}_{\operatorname{Spec}\mathbb{F}_q},\mathcal{G}) = 0 \qquad \text{if } \mathcal{G} \in D^{\mathrm{b}}_{\mathrm{c}}(\operatorname{Spec}\mathbb{F}_q,\Bbbk)^{\leq -2}. \tag{5.2.4}$$

Apply $\mathrm{Hom}(\Bbbk,-)$ to $\tau^{\leq -1}\mathcal{F} \to \mathcal{F} \to \tau^{\geq 0}\mathcal{F} \to$ to get the long exact sequence

$$\begin{aligned}
\cdots \to \mathrm{Hom}(\Bbbk,(\tau^{\geq 0}\mathcal{F})[-1]) \to \mathrm{Hom}(\Bbbk,\tau^{\leq -1}\mathcal{F})&\\
\to \mathrm{Hom}(\Bbbk,\mathcal{F}) \to \mathrm{Hom}(\Bbbk,\tau^{\geq 0}\mathcal{F}) \to \mathrm{Hom}(\Bbbk,(\tau^{\leq -1}\mathcal{F})[1]) \to \cdots .&
\end{aligned} \tag{5.2.5}$$

The first term obviously vanishes, and the last term vanishes by (5.2.4). By the adjunction relationship for truncation, combined with Lemma 5.2.14, the fourth term is given by

$$\mathrm{Hom}(\Bbbk,\tau^{\geq 0}\mathcal{F}) \cong \mathrm{Hom}(\Bbbk,\tau^{\leq 0}\tau^{\geq 0}\mathcal{F}) \cong \mathsf{H}^0(\mathcal{F})^{\mathsf{Fr}}.$$

Finally, applying $\mathrm{Hom}(\Bbbk,-)$ to the distinguished triangle $\tau^{\leq -2}\mathcal{F} \to \tau^{\leq -1}\mathcal{F} \to \mathsf{H}^{-1}(\mathcal{F})[1] \to$ and using (5.2.4) again, we can identify the second term of (5.2.5) with $\mathrm{Hom}(\Bbbk,\mathsf{H}^{-1}(\mathcal{F})[1]) \cong \mathrm{Ext}^1(\Bbbk,\mathsf{H}^{-1}(\mathcal{F})) \cong \mathsf{H}^{-1}(\mathcal{F})_{\mathsf{Fr}}$. □

5.3. Passage to the algebraic closure

In this section, we study the relationship between sheaves on a variety over a finite field and those over its algebraic closure, using the following functor.

Definition 5.3.1. Let X_0 be a variety over a finite field $\mathbb{F}_q$, and let $X = \operatorname{Spec}\bar{\mathbb{F}}_q \times_{\operatorname{Spec}\mathbb{F}_q} X_0$. There is a natural map of schemes $\gamma_{X_0} : X \to X_0$. The **extension of ground field** functor is the pullback along γ_{X_0}. It is denoted by

$$\mathrm{egf} = \gamma_{X_0}^* : D^{\mathrm{b}}_{\mathrm{c}}(X_0, \Bbbk) \to D^{\mathrm{b}}_{\mathrm{c}}(X, \Bbbk).$$

The map $\operatorname{Spec}\bar{\mathbb{F}}_q \to \operatorname{Spec}\mathbb{F}_q$ is not a smooth map (because $\bar{\mathbb{F}}_q$ is not finitely generated as an $\mathbb{F}_q$-algebra), but it is an inverse limit of smooth (in fact, étale) maps; namely, the finite field extensions $\operatorname{Spec}\mathbb{F}_{q^n} \to \operatorname{Spec}\mathbb{F}_q$. The map $X \to X_0$ is thus likewise an inverse limit of étale maps. One goal of this section to explain a kind of descent theory for egf, modeled on Section 3.7.

First properties. According to the comments after [**75**, Theorem 7.3], the smooth base change theorem still holds if "smooth morphism" is replaced by "inverse limit of smooth morphisms." In particular, Principle 2.2.11 applies to $\gamma_{X_0} : X \to X_0$, and we obtain the following fact.

Proposition 5.3.2. *The extension of ground field functor commutes with all sheaf functors.*

Proposition 5.3.3. *The extension of ground field functor is t-exact for both the natural and perverse t-structures.*

Proof sketch. For the natural t-structure, this follows from the way the t-structure is defined in the proof of Theorem 5.1.15. For the perverse t-structure, this follows from the fact that pullback along an étale morphism (cf. Theorem 2.2.9) is t-exact. □

Proposition 5.3.4. *Let X_0 be a variety over $\mathbb{F}_q$, and let $\Bbbk$ be a permitted ring of coefficients. For $\mathcal{F}, \mathcal{G} \in D^{\mathrm{b}}_{\mathrm{c}}(X_0, \Bbbk)$, there is a natural short exact sequence*

$$0 \to \underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{G}[-1])_{\mathsf{Fr}} \to \mathrm{Hom}(\mathcal{F}, \mathcal{G}) \to \underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{G})^{\mathsf{Fr}} \to 0.$$

Proof. We have

$$\begin{aligned}\mathrm{Hom}(\mathcal{F}, \mathcal{G}) &\cong \mathrm{Hom}(\underline{\Bbbk} \overset{L}{\otimes} \mathcal{F}, \mathcal{G}) \cong \mathrm{Hom}(\mathrm{a}_{X_0}^* \underline{\Bbbk}, R\mathscr{H}om(\mathcal{F}, \mathcal{G})) \\ &\cong \mathrm{Hom}(\underline{\Bbbk}, \mathrm{a}_{X_0*} R\mathscr{H}om(\mathcal{F}, \mathcal{G})) \cong \mathrm{Hom}(\underline{\Bbbk}, R\underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{G})).\end{aligned}$$

The result then follows from Lemma 5.2.15. □

Remark 5.3.5. As a special case of Proposition 5.3.2, we have

$$\mathrm{egf}\, R\underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{G}) \cong R\mathrm{Hom}(\mathrm{egf}(\mathcal{F}), \mathrm{egf}(\mathcal{G})).$$

Now apply H^0. This commutes with egf by Proposition 5.3.3, so we obtain a natural isomorphism

$$\mathrm{egf}(\underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{G})) \cong \mathrm{Hom}(\mathrm{egf}(\mathcal{F}), \mathrm{egf}(\mathcal{G})).$$

In other words, the underlying $\Bbbk$-module of the $\mathrm{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q)$-module $\underline{\mathrm{Hom}}(\mathcal{F},\mathcal{G})$ is identified with $\mathrm{Hom}(\mathrm{egf}(\mathcal{F}),\mathrm{egf}(\mathcal{G}))$. Combining this observation with Proposition 5.3.4, we obtain the following diagram:

$$\begin{tikzcd}
\mathrm{Hom}(\mathcal{F},\mathcal{G}) \arrow[rr, "\mathrm{egf}"] \arrow[dr, two heads] & & \mathrm{Hom}(\mathrm{egf}(\mathcal{F}),\mathrm{egf}(\mathcal{G})) \\
& \underline{\mathrm{Hom}}(\mathcal{F},\mathcal{G})^{\mathsf{Fr}} \arrow[ur, hook] &
\end{tikzcd}$$

This diagram describes the image of $\mathrm{egf} : \mathrm{Hom}(\mathcal{F},\mathcal{G}) \to \mathrm{Hom}(\mathrm{egf}(\mathcal{F}),\mathrm{egf}(\mathcal{G}))$.

Frobenius morphisms. In Section 5.2, we discussed Frobenius elements in a Galois group. On the other hand, there are various notions of "Frobenius morphisms" of schemes, to be discussed below. The relationship between them will be clarified following Definition 5.3.7. For now, we point out a typographic distinction: Frobenius elements in Galois groups are denoted with sans serif type ("Fr_q"), while Frobenius morphisms of schemes are denoted with roman type ("Fr_q").

Let X be an affine variety over a field $\mathbb{F}$ of characteristic p, say $X = \operatorname{Spec} A$ for some $\mathbb{F}$-algebra A. Let q be a power of p. Then the map $A \to A$ given by $f \mapsto f^q$ is a ring homomorphism, so it gives rise to a map

$$\mathrm{Fr}_q^{\mathrm{abs}} = \mathrm{Fr}_{X,q}^{\mathrm{abs}} : X \to X,$$

called the **absolute q-Frobenius morphism**. This is a morphism of schemes, but it is usually *not* a morphism of $\mathbb{F}$-varieties, since the ring map $A \to A$ need not be $\mathbb{F}$-linear. (It is $\mathbb{F}$-linear if and only if all elements of $\mathbb{F}$ are fixed by $f \mapsto f^q$, i.e., if $\mathbb{F}$ is a finite field of order $\le q$.) At the level of topological spaces, $\mathrm{Fr}_q^{\mathrm{abs}}$ is the identity map. This construction easily generalizes to nonaffine varieties (or indeed, to any scheme over $\mathbb{F}$).

Next, let X_0 be a variety over the finite field $\mathbb{F}_q$, and let

$$X = \operatorname{Spec} \bar{\mathbb{F}}_q \times_{\operatorname{Spec} \mathbb{F}_q} X_0.$$

The **arithmetic q-Frobenius morphism** of X is the map

$$\mathrm{Fr}_q^{\mathrm{arith}} : X \to X \qquad \text{given by} \qquad \mathrm{Fr}_q^{\mathrm{arith}} = \mathrm{Fr}^{\mathrm{abs}}_{\operatorname{Spec} \bar{\mathbb{F}}_q, q} \times \mathrm{id}_{X_0},$$

and the **relative q-Frobenius morphism** is the map

$$\mathrm{Fr}_q : X \to X \qquad \text{given by} \qquad \mathrm{Fr}_q = \mathrm{id}_{\operatorname{Spec} \bar{\mathbb{F}}_q} \times \mathrm{Fr}^{\mathrm{abs}}_{X_0,q}.$$

(This map will be the most important one for our purposes, so we use unadorned notation for it.) In concrete terms, if X_0 is an affine variety, say given by $X_0 = \operatorname{Spec} \mathbb{F}_q[x_1,\ldots,x_k]/(f_1,\ldots,f_j)$, then we can identify X with the affine variety $\operatorname{Spec} \bar{\mathbb{F}}_q[x_1,\ldots,x_k]/(f_1,\ldots,f_j)$. The arithmetic and relative Frobenius morphisms correspond to the ring homomorphisms given by

$$\begin{aligned}
\mathrm{Fr}^{\mathrm{arith}}_{q\sharp}\left(\sum a_i x_1^{n_{i1}} \cdots x_k^{n_{ik}}\right) &= \sum a_i^q x_1^{n_{i1}} \cdots x_k^{n_{ik}}, \\
\mathrm{Fr}_{q\sharp}\left(\sum a_i x_1^{n_{i1}} \cdots x_k^{n_{ik}}\right) &= \sum a_i x_1^{q n_{i1}} \cdots x_k^{q n_{ik}},
\end{aligned}$$

respectively. In particular, we have

$$\mathrm{Fr}_q \circ \mathrm{Fr}_q^{\mathrm{arith}} = \mathrm{Fr}_q^{\mathrm{arith}} \circ \mathrm{Fr}_q = \mathrm{Fr}^{\mathrm{abs}}_{X,q}. \tag{5.3.1}$$

Note that the relative q-Frobenius morphism (unlike the absolute or arithmetic ones) *is* a morphism of $\bar{\mathbb{F}}_q$-varieties.

Lemma 5.3.6. *Let X be a variety over a field $\mathbb{F}$ of characteristic p, and let q be a power of p. The functor* $(\mathrm{Fr}_q^{\mathrm{abs}})^* : \mathrm{Sh}_{\text{ét}}(X, \Bbbk) \to \mathrm{Sh}_{\text{ét}}(X, \Bbbk)$ *is canonically isomorphic to the identity functor.*

In practice, when working with étale or ℓ-adic sheaves, this lemma lets us treat $\mathrm{Fr}_q^{\mathrm{abs}}$ as though it were the identity map.

Proof sketch. We noted above that $\mathrm{Fr}_q^{\mathrm{abs}}$ is the identity map at the level of topological spaces. The key point is that it is also isomorphic to the identity map at the level of the étale topology: for any étale open set $j : U \to X$, the diagram

$$\begin{tikzcd} U \arrow[r, "\mathrm{Fr}_{U,q}^{\mathrm{abs}}"] \arrow[d, "j"'] & U \arrow[d, "j"] \\ X \arrow[r, "\mathrm{Fr}_{X,q}^{\mathrm{abs}}"] & X \end{tikzcd}$$

is cartesian. (See [**4**, Exposé XV.1, Proposition 2(c)].) It follows from this that $(\mathrm{Fr}_q^{\mathrm{abs}})^*\mathcal{F}(U \to X) = \mathcal{F}(U \to X)$. □

Weil structures. The following notion can be thought of as a weak kind of descent datum for the extension of ground field functor.

Definition 5.3.7. Let X_0 be a variety over $\mathbb{F}_q$, and let $X = \mathrm{Spec}\,\bar{\mathbb{F}}_q \times_{\mathrm{Spec}\,\mathbb{F}_q} X_0$. For $\mathcal{F} \in D^{\mathrm{b}}_{\mathrm{c}}(X, \Bbbk)$, a **Weil structure** on $\mathcal{F}$ is an isomorphism $\theta : \mathrm{Fr}_q^*\mathcal{F} \xrightarrow{\sim} \mathcal{F}$. A **morphism of Weil structures** $\phi : (\mathcal{F}, \theta_{\mathcal{F}}) \to (\mathcal{G}, \theta_{\mathcal{G}})$ is a morphism $\phi : \mathcal{F} \to \mathcal{G}$ in $D^{\mathrm{b}}_{\mathrm{c}}(X, \Bbbk)$ such that the following diagram commutes:

$$\begin{tikzcd} \mathrm{Fr}_q^*\mathcal{F} \arrow[r, "\mathrm{Fr}_q^*\phi"] \arrow[d, "\theta_{\mathcal{F}}"'] & \mathrm{Fr}_q^*\mathcal{G} \arrow[d, "\theta_{\mathcal{G}}"] \\ \mathcal{F} \arrow[r, "\phi"] & \mathcal{G} \end{tikzcd}$$

If $\mathcal{F} \in \mathrm{Perv}(X, \Bbbk)$, the pair $(\mathcal{F}, \theta)$ is called a **Weil perverse sheaf**. The category of Weil perverse sheaves on X is denoted by $\mathrm{Perv}_{\mathrm{W}}(X, \Bbbk)$.

It is easy to check that $\mathrm{Perv}_{\mathrm{W}}(X, \Bbbk)$ is an abelian category. (In contrast, the category of all Weil structures is *not* a triangulated category.) Note that when $X_0 = \mathrm{Spec}\,\mathbb{F}_q$, the map $\mathrm{Fr}_q : X \to X$ is the identity map, so a Weil structure on $\mathrm{Spec}\,\bar{\mathbb{F}}_q$ just consists of an object $\mathcal{F} \in D^{\mathrm{b}}_{\mathrm{c}}(\mathrm{Spec}\,\bar{\mathbb{F}}_q, \Bbbk)$ together with an automorphism $\mathcal{F} \to \mathcal{F}$. In this special case, we usually denote this automorphism by

$$\mathsf{Fr}_q : \mathcal{F} \to \mathcal{F}$$

instead of θ (see the discussion following Lemma 5.3.8 below).

If $(\mathcal{F}, \theta_{\mathcal{F}})$ is a Weil structure on X, then for any Fr_{q^n}-fixed geometric point x, the stalk $\mathcal{F}_x$ inherits an $\mathbb{F}_{q^n}$-Weil structure, given by

$$\mathsf{Fr}_{q^n} = (\theta_{\mathcal{F}})_x^n : \mathcal{F}_x \to \mathcal{F}_x.$$

If $(\mathcal{G}, \theta_{\mathcal{G}})$ is another Weil structure, then the $\Bbbk$-module $\mathrm{Hom}(\mathcal{F}, \mathcal{G})$ also inherits a Weil structure, given by

$$\mathsf{Fr}_q(\phi) = \theta_{\mathcal{G}} \circ (\mathrm{Fr}_q^*\phi) \circ \theta_{\mathcal{F}}^{-1} \qquad \text{for } \phi : \mathcal{F} \to \mathcal{G}.$$

We denote this Weil structure by $\underline{\mathrm{Hom}}(\mathcal{F},\mathcal{G})$, and call it the **geometric Hom space**. It is immediate from the definition that

$$\{\text{morphisms of Weil structures } (\mathcal{F},\theta_{\mathcal{F}}) \to (\mathcal{G},\theta_{\mathcal{G}})\} = \underline{\mathrm{Hom}}(\mathcal{F},\mathcal{G})^{\mathsf{Fr}}. \tag{5.3.2}$$

LEMMA 5.3.8. *Let X_0 be a variety over $\mathbb{F}_q$, and let $X = \operatorname{Spec}\bar{\mathbb{F}}_q \times_{\operatorname{Spec}\mathbb{F}_q} X_0$. For any $\mathcal{F} \in D^{\mathrm{b}}_{\mathrm{c}}(X_0,\Bbbk)$, the object $\mathrm{egf}(\mathcal{F}) \in D^{\mathrm{b}}_{\mathrm{c}}(X,\Bbbk)$ is equipped with a canonical Weil structure. In particular,* egf *induces an exact functor*

$$\mathrm{egf} : \mathrm{Perv}(X_0,\Bbbk) \to \mathrm{Perv}_{\mathrm{W}}(X,\Bbbk).$$

PROOF. From the commutative triangle

$$\begin{tikzcd} X \arrow[rr, "\mathrm{Fr}_q^{\mathrm{arith}}"] \arrow[dr] & & X \arrow[dl] \\ & X_0 & \end{tikzcd}$$

we obtain a natural isomorphism $\mathrm{egf}(\mathcal{F}) \cong (\mathrm{Fr}_q^{\mathrm{arith}})^* \,\mathrm{egf}(\mathcal{F})$. By Lemma 5.3.6, we have $(\mathrm{Fr}_q^{\mathrm{abs}})^* \,\mathrm{egf}(\mathcal{F}) \cong (\mathrm{Fr}_q^{\mathrm{arith}})^* \,\mathrm{egf}(\mathcal{F})$. Since $\mathrm{Fr}_q^{\mathrm{arith}}$ is an isomorphism of schemes, the functor $(\mathrm{Fr}_q^{\mathrm{arith}})^*$ is an equivalence of categories; its inverse is given by $((\mathrm{Fr}_q^{\mathrm{arith}})^{-1})^*$. We thus obtain a natural isomorphism

$$((\mathrm{Fr}_q^{\mathrm{arith}})^{-1})^*(\mathrm{Fr}_q^{\mathrm{abs}})^* \,\mathrm{egf}(\mathcal{F}) \cong \mathrm{egf}(\mathcal{F}).$$

By (5.3.1), the left-hand side is naturally isomorphic to $\mathrm{Fr}_q^* \,\mathrm{egf}(\mathcal{F})$. □

Let us work out what Lemma 5.3.8 says when $X_0 = \operatorname{Spec}\mathbb{F}_q$. We have $\mathrm{Perv}(\operatorname{Spec}\mathbb{F}_q,\Bbbk) = \Bbbk[\mathrm{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q)]\text{-mod}^{\mathrm{fc}}$, while $\mathrm{Perv}_{\mathrm{W}}(\operatorname{Spec}\bar{\mathbb{F}}_q,\Bbbk)$ is identified with $\Bbbk[\langle \mathsf{Fr}_q\rangle]\text{-mod}^{\mathrm{fg}}$, where $\langle \mathsf{Fr}_q\rangle \subset \mathrm{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q)$ is the infinite cyclic group generated by Fr_q. The functor $\mathrm{egf} : \mathrm{Perv}(\operatorname{Spec}\mathbb{F}_q,\Bbbk) \to \mathrm{Perv}_{\mathrm{W}}(\operatorname{Spec}\bar{\mathbb{F}}_q,\Bbbk)$ is identified with the obvious functor

$$\Bbbk[\mathrm{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q)]\text{-mod}^{\mathrm{fc}} \to \Bbbk[\langle \mathsf{Fr}_q\rangle]\text{-mod}^{\mathrm{fg}}.$$

We can generalize this situation as follows. Suppose X_0 is smooth and connected, and let $\bar{x}$ be an Fr_q-fixed geometric point of X. Using Proposition 5.2.7, we have

$$\mathrm{Loc}^{\mathrm{ft}}(X_0,\Bbbk) \cong \Bbbk[\mathrm{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q) \ltimes \pi_1^{\text{ét}}(X,\bar{x})]\text{-mod}^{\mathrm{fc}}. \tag{5.3.3}$$

On the other hand, the category of Weil local systems on X can be described as

$$\mathrm{Loc}^{\mathrm{ft}}_{\mathrm{W}}(X,\Bbbk) \cong \left\{ \begin{matrix} \Bbbk[\langle \mathsf{Fr}_q\rangle \ltimes \pi_1^{\text{ét}}(X,\bar{x})]\text{-modules on which} \\ \text{the action of } \pi_1^{\text{ét}}(X,\bar{x}) \text{ is continuous} \end{matrix} \right\}. \tag{5.3.4}$$

As the functor egf resembles pullback along a smooth morphism, the following proposition can be seen as analogous to Theorems 3.6.6, 3.7.4, and 3.7.6.

PROPOSITION 5.3.9. *Let $\Bbbk$ be a permitted ring of coefficients. The functor*

$$\mathrm{egf} : \mathrm{Perv}(X_0,\Bbbk) \to \mathrm{Perv}_{\mathrm{W}}(X,\Bbbk) \tag{5.3.5}$$

is fully faithful, and its image is a Serre subcategory of $\mathrm{Perv}_{\mathrm{W}}(X,\Bbbk)$. If $\Bbbk$ is a finite self-injective ring or the ring of integers in a finite extension of $\mathbb{Q}_\ell$, this functor is an equivalence of categories.

Proof sketch. *Step 1. Full faithfulness and stability under extensions.* Let $\mathcal{F}, \mathcal{G} \in \mathrm{Perv}(X_0, \Bbbk)$. Then $\underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{G}[-1]) = 0$ (by Remark 5.3.5 and the fact that $\mathrm{Hom}(\mathrm{egf}(\mathcal{F}), \mathrm{egf}(\mathcal{G})[-1]) = 0$), so Proposition 5.3.4 gives us a natural isomorphism

$$\mathrm{Hom}(\mathcal{F}, \mathcal{G}) \xrightarrow{\sim} \underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{G})^{\mathsf{Fr}}.$$

In view of (5.3.2), we see that (5.3.5) is fully faithful. Another instance of Proposition 5.3.4 gives us a short exact sequence

$$0 \to \underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{G})_{\mathsf{Fr}} \to \mathrm{Ext}^1_{\mathrm{Perv}(X_0,\Bbbk)}(\mathcal{F}, \mathcal{G}) \to \underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{G}[1])^{\mathsf{Fr}} \to 0.$$

This sequence has a natural map to the sequence in Exercise 5.3.1 below. The five lemma then gives us an isomorphism

$$\mathrm{Ext}^1_{\mathrm{Perv}(X_0,\Bbbk)}(\mathcal{F}, \mathcal{G}) \xrightarrow{\sim} \mathrm{Ext}^1_{\mathrm{Perv_W}(X,\Bbbk)}(\mathcal{F}, \mathcal{G}),$$

and this implies that the image of (5.3.5) is stable under extensions.

Step 2. Stability under subobjects and quotients. The proof of this claim is very similar in structure to the proof of Theorem 3.7.6: it is based on noetherian induction and reduction to the case of local systems. We omit most of this argument. The problem ultimately comes down to showing that when X_0 is smooth and connected, the image of

$$\mathrm{egf} : \mathrm{Loc}^{\mathrm{ft}}(X_0, \Bbbk) \to \mathrm{Loc}^{\mathrm{ft}}_{\mathrm{W}}(X, \Bbbk) \tag{5.3.6}$$

is closed under subobjects and quotients. This is immediate from a comparison of (5.3.3) and (5.3.4), since $\langle \mathsf{Fr}_q \rangle$ is a dense subgroup of $\mathrm{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q)$.

Step 3. Essential surjectivity for certain coefficients. A noetherian induction argument lets us again reduce the problem to the setting of (5.3.6). If $\Bbbk$ is a finite self-injective ring, then any module on the right-hand side of (5.3.4) is a finite set, so the action of Fr_q automatically extends to a continuous action of $\mathrm{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q)$. Thus, (5.3.6) is essentially surjective.

If $\Bbbk$ is the ring of integers in a finite extension of $\mathbb{Q}_\ell$, then any module on the right-hand side of (5.3.4) is an inverse limit of finite modules, so the reasoning above applies in this case as well. □

Remark 5.3.10. When $\Bbbk$ is an ℓ-adic field, the functor (5.3.5) is *not* essentially surjective. For example, let $X_0 = \mathrm{Spec}\, \mathbb{F}_q$ and $X = \mathrm{Spec}\, \bar{\mathbb{F}}_q$, and equip $\underline{\mathbb{Q}_\ell} \in D^{\mathrm{b}}_{\mathrm{c}}(X, \mathbb{Q}_\ell)$ with a Weil structure by making Fr_q act by multiplication by ℓ. Then the closure of the image of $\langle \mathsf{Fr}_q \rangle$ inside $\mathbb{Q}_\ell^\times$ is not compact. Since $\mathrm{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q)$ is compact, this action of Fr_q does not extend to a continuous action of $\mathrm{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q)$.

The sheaf–function correspondence. We conclude this section by showing how various constructions with sheaves on a variety X_0 over $\mathbb{F}_q$ can be turned into more elementary operations involving just functions on the set $X_0(\mathbb{F}_{q^n})$, which we implicitly identify with the set of Fr_{q^n}-fixed geometric points of X_0.

Definition 5.3.11. Let X_0 be a variety over $\mathbb{F}_q$, and let $\mathcal{F} \in D^{\mathrm{b}}_{\mathrm{c}}(X_0, \Bbbk)$, where $\Bbbk$ is a permitted field of coefficients. For any $n \geq 1$, the nth **characteristic function** of $\mathcal{F}$ is the function

$$\chi_{\mathcal{F}} : X_0(\mathbb{F}_{q^n}) \to \Bbbk \qquad \text{given by} \qquad \chi_{\mathcal{F}}(x) = \sum_{i \in \mathbb{Z}} (-1)^i \operatorname{tr}(\mathsf{Fr}_{q^n}, \mathsf{H}^i(\mathcal{F}_x)).$$

(The same formula can be used to define a characteristic function for any Weil complex on $X = \operatorname{Spec} \bar{\mathbb{F}}_q \times_{\operatorname{Spec} \mathbb{F}_q} X_0$, but we will not need this extra generality.) The following lemma is a straightforward exercise.

LEMMA 5.3.12. *If $\mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to$ is a distinguished triangle in $D^{\mathrm{b}}_{\mathrm{c}}(X_0, \Bbbk)$, then $\chi_{\mathcal{F}} = \chi_{\mathcal{F}'} + \chi_{\mathcal{F}''}$. As a consequence, the characteristic function construction defines a map*

$$\chi : \mathrm{K}_0(D^{\mathrm{b}}_{\mathrm{c}}(X_0, \Bbbk)) \to \{\textit{functions } X_0(\mathbb{F}_{q^n}) \to \Bbbk\}.$$

THEOREM 5.3.13 (The sheaf–function correspondence). *Let $f : X_0 \to Y_0$ be a morphism of $\mathbb{F}_q$-varieties, and let $\Bbbk$ be a permitted field of coefficients.*

(1) *For any two objects $\mathcal{F}, \mathcal{F}' \in D^{\mathrm{b}}_{\mathrm{c}}(X_0, \Bbbk)$, we have $\chi_{\mathcal{F} \otimes^L \mathcal{F}'} = \chi_{\mathcal{F}} \chi_{\mathcal{F}'}$.*
(2) *For any object $\mathcal{G} \in D^{\mathrm{b}}_{\mathrm{c}}(Y_0, \Bbbk)$, we have $\chi_{f^*\mathcal{G}} = \chi_{\mathcal{G}} \circ f$.*
(3) *For any object $\mathcal{F} \in D^{\mathrm{b}}_{\mathrm{c}}(X_0, \Bbbk)$, we have $\chi_{f_!\mathcal{F}} = \int_f \chi_{\mathcal{F}}$.*

The integral in the last part of this theorem is defined as follows:

$$\int_f \chi_{\mathcal{F}} : Y_0(\mathbb{F}_{q^n}) \to \Bbbk \qquad \text{is given by} \qquad \left(\int_f \chi_{\mathcal{F}}\right)(y) = \sum_{\substack{x \in X_0(\mathbb{F}_{q^n}) \\ x \in f^{-1}(y)}} \chi_{\mathcal{F}}(x).$$

PROOF SKETCH. Part (3) is perhaps the deepest part of this result. By proper base change, it can be reduced to the special case where $f = \mathrm{a}_{X_0} : X_0 \to \operatorname{Spec} \mathbb{F}_q$. In that special case, the statement above is equivalent to the **Grothendieck–Lefschetz trace formula**, which states that for any $\mathcal{F} \in D^{\mathrm{b}}_{\mathrm{c}}(X_0, \Bbbk)$, we have

$$\sum_{i \in \mathbb{Z}} (-1)^i \operatorname{tr}(\mathsf{Fr}_{q^n}, \mathbf{H}^i_{\mathrm{c}}(X, \mathcal{F})) = \sum_{x \in X(\mathbb{F}_{q^n})} \chi_{\mathcal{F}}(x).$$

For a proof of this formula, see [**75**, Corollary II.3.15].

Parts (1) and (2) are much easier: they can be shown by elementary computations from the definitions. □

Exercises.

5.3.1. Let $\mathcal{F}, \mathcal{G} \in \mathrm{Perv}_{\mathrm{W}}(X, \Bbbk)$. Show that there is a natural exact sequence

$$0 \to \underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{G})_{\mathsf{Fr}} \to \mathrm{Ext}^1_{\mathrm{Perv}_{\mathrm{W}}(X,\Bbbk)}(\mathcal{F}, \mathcal{G}) \to \underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{G}[1])^{\mathsf{Fr}}.$$

5.4. Mixed ℓ-adic sheaves

Fix a prime number ℓ that is different from $p = \operatorname{char} \mathbb{F}_q$. In this section, we will study constructible $\overline{\mathbb{Q}}_\ell$-sheaves on $\mathbb{F}_q$-varieties that obey an additional constraint on the eigenvalues of the Fr_{q^n}-action on their stalks at Fr_{q^n}-fixed points.

Mixed complexes. For the following definition, recall that $\overline{\mathbb{Q}}_\ell$ is isomorphic as a field to $\mathbb{C}$, as they are both algebraically closed fields of characteristic 0 and of the same transcendence degree over $\mathbb{Q}$.

DEFINITION 5.4.1. An element $\lambda \in \overline{\mathbb{Q}}_\ell$ is said to have **complex absolute value** z if under every isomorphism $\iota : \overline{\mathbb{Q}}_\ell \to \mathbb{C}$, we have $|\iota(\lambda)| = z$.

Of course, it is possible for $|\iota(\lambda)|$ to depend on the choice of ι; in that case, λ does not have a well-defined complex absolute value. On the other hand, rational numbers and roots of unity do have well-defined complex absolute values.

DEFINITION 5.4.2. Let X_0 be a variety over $\mathbb{F}_q$. A sheaf $\mathcal{F} \in \mathrm{Sh}_c(X_0, \overline{\mathbb{Q}}_\ell)$ is said to be **pointwise pure** of weight w if for every $n \geq 1$ and every $\mathbb{F}_{q^n}$-point x of X_0, the eigenvalues of the Frobenius action on $\mathcal{F}_x$ have complex absolute value $q^{wn/2}$.

A sheaf $\mathcal{F} \in \mathrm{Sh}_c(X_0, \overline{\mathbb{Q}}_\ell)$ is said to be **mixed** if there is a finite filtration $0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_k = \mathcal{F}$ such that each subquotient $\mathcal{F}_i/\mathcal{F}_{i-1}$ is pointwise pure (not necessarily of the same weight). The **pointwise weights** of $\mathcal{F}$ are the integers that occur as weights of the subquotients $\mathcal{F}_i/\mathcal{F}_{i-1}$.

A constructible $\overline{\mathbb{Q}}_\ell$-complex $\mathcal{F} \in D^{\mathrm{b}}_{\mathrm{c}}(X_0, \overline{\mathbb{Q}}_\ell)$ is said to be a **mixed complex** if each $\mathsf{H}^i(\mathcal{F})$ is a mixed sheaf. The full triangulated subcategory of $D^{\mathrm{b}}_{\mathrm{c}}(X_0, \overline{\mathbb{Q}}_\ell)$ consisting of mixed complexes is denoted by $D^{\mathrm{b}}_{\mathrm{m}}(X_0, \overline{\mathbb{Q}}_\ell)$.

Both the preceding definition and the following fundamental fact are due to Deligne [**60**]. For an exposition, see [**136**, Theorem II.12.2].

THEOREM 5.4.3. *Let $f : X_0 \to Y_0$ be a morphism of varieties over $\mathbb{F}_q$. The functors f^*, f_*, $f_!$, $f^!$, $\otimes^L$, $R\mathscr{H}om$, $\mathbb{D}$, $\tau^{\leq n}$ and $\tau^{\geq n}$ take mixed complexes to mixed complexes.*

DEFINITION 5.4.4. Let $w \in \mathbb{Z}$, and let $\mathcal{F} \in D^{\mathrm{b}}_{\mathrm{m}}(X_0, \overline{\mathbb{Q}}_\ell)$. The complex $\mathcal{F}$ is said to **have weights** $\leq w$ if each cohomology sheaf $\mathsf{H}^i(\mathcal{F})$ has pointwise weights $\leq w+i$. It is said to **have weights** $\geq w$ if its Verdier dual $\mathbb{D}\mathcal{F}$ has weights $\leq -w$. The full subcategory of $D^{\mathrm{b}}_{\mathrm{m}}(X_0, \overline{\mathbb{Q}}_\ell)$ consisting of objects with weights $\leq w$, resp. $\geq w$, is denoted by

$$D^{\mathrm{b}}_{\leq w}(X_0, \overline{\mathbb{Q}}_\ell), \qquad \text{resp.} \qquad D^{\mathrm{b}}_{\geq w}(X_0, \overline{\mathbb{Q}}_\ell).$$

Finally, $\mathcal{F}$ is said to be **pure of weight** w if it has weights $\leq w$ and $\geq w$.

It is immediate from the definition that

$$\begin{aligned} D^{\mathrm{b}}_{\leq w}(X_0, \overline{\mathbb{Q}}_\ell)[1] &= D^{\mathrm{b}}_{\leq w+1}(X_0, \overline{\mathbb{Q}}_\ell), \\ D^{\mathrm{b}}_{\geq w}(X_0, \overline{\mathbb{Q}}_\ell)[1] &= D^{\mathrm{b}}_{\geq w+1}(X_0, \overline{\mathbb{Q}}_\ell). \end{aligned} \tag{5.4.1}$$

EXAMPLE 5.4.5. Consider the Tate sheaf $\underline{\overline{\mathbb{Q}}_\ell}_{X_0}(1)$. We saw in Example 5.2.10 that the Frobenius action on any stalk is given by multiplication by q^{-1}, so $\underline{\overline{\mathbb{Q}}_\ell}_{X_0}(1)$ is pointwise pure of weight -2, and hence mixed with weights ≤ -2. More generally, for any integer n, the sheaf $\underline{\overline{\mathbb{Q}}_\ell}_{X_0}(n)$ is mixed with weights $\leq -2n$.

If X_0 is smooth of dimension d, then $\underline{\overline{\mathbb{Q}}_\ell}_{X_0}(n)$ is in fact pure of weight $-2n$, since its Verdier dual $\mathbb{D}(\underline{\overline{\mathbb{Q}}_\ell}_{X_0}(n)) \cong \underline{\overline{\mathbb{Q}}_\ell}[2d](d-n)$ has weights $\leq 2n$.

REMARK 5.4.6. Let X_0 be a smooth variety of dimension d. One can generalize Example 5.4.5 as follows: by an analogue of Remark 2.8.3, if $\mathcal{L}$ is a local system on X_0 with pointwise weights $\leq w$, then $\mathbb{D}(\mathcal{L})[-2d]$ is a local system with pointwise weights $\geq -2d - w$.

More generally, suppose X_0 is smooth and that $\mathcal{F} \in D^{\mathrm{b}}_{\mathrm{m}}(X_0, \overline{\mathbb{Q}}_\ell)$ has locally constant cohomology sheaves. Then $\mathcal{F}$ has weights $\geq w$ if and only if each cohomology sheaf $\mathsf{H}^i(\mathcal{F})$ has pointwise weights $\geq w + i$. (But in general, this is false if we drop the assumptions on X_0 and $\mathcal{F}$.)

The following major theorem is Deligne's reformulation of the Weil conjectures. The original proof was published in [**60**]. Some years later, another proof, which makes crucial use of the Fourier–Deligne transform, was given by Laumon [**149**]. For a textbook account of the latter, see [**136**].

THEOREM 5.4.7. *Let $f : X_0 \to Y_0$ be a morphism of varieties over $\mathbb{F}_q$.*

(1) *The functors f^* and $f_!$ take mixed complexes with weights $\leq w$ to mixed complexes with weights $\leq w$.*
(2) *The functors f_* and $f^!$ take mixed complexes with weights $\geq w$ to mixed complexes with weights $\geq w$.*
(3) *If $\mathcal{F} \in D^{\mathrm{b}}_{\leq w}(X_0, \overline{\mathbb{Q}}_\ell)$ and $\mathcal{G} \in D^{\mathrm{b}}_{\leq v}(X_0, \overline{\mathbb{Q}}_\ell)$, then*

$$\mathcal{F} \overset{L}{\otimes} \mathcal{G} \in D^{\mathrm{b}}_{\leq v+w}(X_0, \overline{\mathbb{Q}}_\ell).$$

(4) *If $\mathcal{F} \in D^{\mathrm{b}}_{\leq w}(X_0, \overline{\mathbb{Q}}_\ell)$ and $\mathcal{G} \in D^{\mathrm{b}}_{\geq v}(X_0, \overline{\mathbb{Q}}_\ell)$, then*

$$R\mathscr{H}om(\mathcal{F}, \mathcal{G}) \in D^{\mathrm{b}}_{\geq v-w}(X_0, \overline{\mathbb{Q}}_\ell).$$

COROLLARY 5.4.8. *Let $\mathcal{F} \in D^{\mathrm{b}}_{\mathrm{m}}(X_0, \overline{\mathbb{Q}}_\ell)$ and $\mathcal{G} \in D^{\mathrm{b}}_{\mathrm{m}}(Y_0, \overline{\mathbb{Q}}_\ell)$.*

(1) *If $\mathcal{F} \in D^{\mathrm{b}}_{\leq w}(X_0, \overline{\mathbb{Q}}_\ell)$ and $\mathcal{G} \in D^{\mathrm{b}}_{\leq v}(Y_0, \overline{\mathbb{Q}}_\ell)$, then*

$$\mathcal{F} \boxtimes \mathcal{G} \in D^{\mathrm{b}}_{\leq v+w}(X_0 \times Y_0, \overline{\mathbb{Q}}_\ell).$$

(2) *If $\mathcal{F} \in D^{\mathrm{b}}_{\geq w}(X_0, \overline{\mathbb{Q}}_\ell)$ and $\mathcal{G} \in D^{\mathrm{b}}_{\geq v}(Y_0, \overline{\mathbb{Q}}_\ell)$, then*

$$\mathcal{F} \boxtimes \mathcal{G} \in D^{\mathrm{b}}_{\geq v+w}(X_0 \times Y_0, \overline{\mathbb{Q}}_\ell).$$

PROOF. The first assertion follows immediately from Theorem 5.4.7. The second assertion then follows because $\boxtimes$ commutes with $\mathbb{D}$. □

COROLLARY 5.4.9. *Let $f : X_0 \to Y_0$ be a morphism of varieties over $\mathbb{F}_q$.*

(1) *If f is proper, then f_* takes pure complexes of weight w to pure complexes of weight w.*
(2) *If f is smooth, then f^* takes pure complexes of weight w to pure complexes of weight w.*

Mixed perverse sheaves. Because truncation functors send mixed complexes to mixed complexes, the natural t-structure on $D^{\mathrm{b}}_{\mathrm{c}}(X_0, \overline{\mathbb{Q}}_\ell)$ induces a t-structure (again called "natural") on $D^{\mathrm{b}}_{\mathrm{m}}(X_0, \overline{\mathbb{Q}}_\ell)$. Its heart is the category of mixed sheaves. For the same reason, the following definition makes sense.

DEFINITION 5.4.10. Let X_0 be a variety over $\mathbb{F}_q$, and let

$$\begin{aligned} {}^p\!D^{\mathrm{b}}_{\mathrm{m}}(X_0, \overline{\mathbb{Q}}_\ell)^{\leq 0} &= {}^p\!D^{\mathrm{b}}_{\mathrm{c}}(X_0, \overline{\mathbb{Q}}_\ell)^{\leq 0} \cap D^{\mathrm{b}}_{\mathrm{m}}(X_0, \overline{\mathbb{Q}}_\ell), \\ {}^p\!D^{\mathrm{b}}_{\mathrm{m}}(X_0, \overline{\mathbb{Q}}_\ell)^{\geq 0} &= {}^p\!D^{\mathrm{b}}_{\mathrm{c}}(X_0, \overline{\mathbb{Q}}_\ell)^{\geq 0} \cap D^{\mathrm{b}}_{\mathrm{m}}(X_0, \overline{\mathbb{Q}}_\ell). \end{aligned}$$

The pair $({}^p\!D^{\mathrm{b}}_{\mathrm{m}}(X_0, \overline{\mathbb{Q}}_\ell)^{\leq 0}, {}^p\!D^{\mathrm{b}}_{\mathrm{m}}(X_0, \overline{\mathbb{Q}}_\ell)^{\geq 0})$ is called the **perverse t-structure** on $D^{\mathrm{b}}_{\mathrm{m}}(X_0, \overline{\mathbb{Q}}_\ell)$. Its heart, denoted by

$$\mathrm{Perv}_{\mathrm{m}}(X_0, \overline{\mathbb{Q}}_\ell) = {}^p\!D^{\mathrm{b}}_{\mathrm{m}}(X_0, \overline{\mathbb{Q}}_\ell)^{\leq 0} \cap {}^p\!D^{\mathrm{b}}_{\mathrm{m}}(X_0, \overline{\mathbb{Q}}_\ell)^{\geq 0},$$

is the category of **mixed perverse sheaves** on X_0.

PROPOSITION 5.4.11. *Let X_0 be a variety over $\mathbb{F}_q$, and let $\mathcal{F} \in \mathrm{Perv}_{\mathrm{m}}(X_0, \overline{\mathbb{Q}}_\ell)$. If $\mathcal{F}$ has weights $\leq w$, resp. $\geq w$, then the same holds for every subquotient of $\mathcal{F}$.*

PROOF SKETCH. For a proof that subquotients of $\mathcal{F}$ are at least mixed, see [**24**, 5.1.7(ii)]. To show that they obey the same weight inequalities as $\mathcal{F}$, we must make use of the following fact:

(5.4.2) A mixed perverse sheaf $\mathcal{F}$ has weights $\geq w$ if and only if for every affine étale open set $j : U_0 \to X_0$, the cohomology group $\underline{\mathbf{H}}^0(U_0, j^*\mathcal{F})$ has weights $\geq w$.

This fact can be seen as a weight-theoretic analogue of Theorem 3.5.3. For a proof, see [**24**, Théorème 5.2.1].

It is enough to prove the proposition in the special case where $w = 0$. Assume that $\mathcal{F}$ has weights ≥ 0. Let $\mathcal{F}' \subset \mathcal{F}$ be a subobject of $\mathcal{F}$, and let $\mathcal{F}'' = \mathcal{F}/\mathcal{F}'$. Let $j : U_0 \to X_0$ be an affine étale open set. By Theorem 3.5.3, $\underline{\mathbf{H}}^k(U_0, j^*\mathcal{F}') = 0$ for $k \geq 1$, so the long exact sequence in cohomology becomes

$$\cdots \to \underline{\mathbf{H}}^{-1}(U_0, j^*\mathcal{F}'') \to \underline{\mathbf{H}}^0(U_0, j^*\mathcal{F}') \to \underline{\mathbf{H}}^0(U_0, j^*\mathcal{F}) \to \underline{\mathbf{H}}^0(U_0, j^*\mathcal{F}'') \to 0.$$

Our assumptions imply that $\underline{\mathbf{H}}^0(U_0, j^*\mathcal{F})$ has weights ≥ 0. Recall that this condition is a constraint on the eigenvalues of the Frobenius action on the underlying $\overline{\mathbb{Q}}_\ell$-vector space of $\underline{\mathbf{H}}^0(U_0, j^*\mathcal{F})$. Since $\underline{\mathbf{H}}^0(U_0, j^*\mathcal{F}'')$ is a quotient of $\underline{\mathbf{H}}^0(U_0, j^*\mathcal{F})$, the eigenvalues of its Frobenius action must satisfy the same constraint. In other words, $\underline{\mathbf{H}}^0(U_0, j^*\mathcal{F}'')$ has weights ≥ 0. Since this holds for all U_0, we conclude that $\mathcal{F}''$ has weights ≥ 0.

By Theorem 5.4.7, $\mathrm{a}_{U_0*}j^*\mathcal{F}''$ also has weights ≥ 0, so the cohomology group $\underline{\mathbf{H}}^{-1}(U_0, j^*\mathcal{F}'') = \mathsf{H}^{-1}(\mathrm{a}_{U_0*}j^*\mathcal{F}'')$ has weights ≥ -1. The long exact sequence above shows that $\underline{\mathbf{H}}^0(U_0, j^*\mathcal{F}')$ has weights ≥ -1. Since this holds for all U_0, we conclude that $\mathcal{F}'$ has weights ≥ -1.

We have shown that any quotient of $\mathcal{F}$ has weights ≥ 0 and any subobject has weights ≥ -1. To improve the latter estimate, suppose that $\mathcal{F}'$ does *not* have weights ≥ 0. Then there exists an affine étale open set $j : U_0 \to X_0$ such that -1 occurs as weight of $\underline{\mathbf{H}}^0(U_0, j^*\mathcal{F}')$. It follows that -2 occurs as a weight of $\underline{\mathbf{H}}^0(U_0, j^*\mathcal{F}') \otimes \underline{\mathbf{H}}^0(U_0, j^*\mathcal{F}')$. It is an easy exercise to check that

$$\underline{\mathbf{H}}^0(U_0, j^*\mathcal{F}') \otimes \underline{\mathbf{H}}^0(U_0, j^*\mathcal{F}') \cong \underline{\mathbf{H}}^0(U_0 \times U_0, (j \times j)^*(\mathcal{F}' \boxtimes \mathcal{F}')).$$

By Lemma 3.2.5, $\mathcal{F}' \boxtimes \mathcal{F}'$ is a perverse sheaf. So this calculation shows that $\mathcal{F}' \boxtimes \mathcal{F}'$ does *not* have weights ≥ -1.

On the other hand, $\mathcal{F}' \boxtimes \mathcal{F}'$ is a subobject of $\mathcal{F} \boxtimes \mathcal{F}$, which has weights ≥ 0 by Corollary 5.4.8. The second and third paragraphs of the proof apply to this object, and tell us that any subobject of $\mathcal{F} \boxtimes \mathcal{F}$ has weights ≥ -1. We have a contradiction, so $\mathcal{F}'$ must have weights ≥ 0. ☐

THEOREM 5.4.12 (Purity). *Let X_0 be a variety over $\mathbb{F}_q$.*

(1) *Let $h : Y_0 \hookrightarrow X_0$ be a locally closed embedding. If $\mathcal{F} \in \mathrm{Perv}_{\mathrm{m}}(Y_0, \overline{\mathbb{Q}}_\ell)$ is pure of weight w, then $h_{!*}\mathcal{F} \in \mathrm{Perv}_{\mathrm{m}}(X_0, \overline{\mathbb{Q}}_\ell)$ is also pure of weight w.*
(2) *Every simple mixed perverse sheaf is pure.*

PROOF SKETCH. (1) One can first reduce to the case where h is an affine morphism, so that $h_!\mathcal{F}$ and $h_*\mathcal{F}$ are perverse (by Corollary 3.5.9). By Theorem 5.4.7, the former has weights $\leq w$, and the latter has weights $\geq w$. Since $h_{!*}\mathcal{F}$ is a quotient of $h_!\mathcal{F}$ and a subobject of $h_*\mathcal{F}$, Proposition 5.4.11 implies that it is pure.

(2) Note that a mixed irreducible local system on a smooth variety is necessarily pointwise pure. The same applies to its Verdier dual, so in fact a mixed irreducible

local system on a smooth variety is pure. The result therefore follows from part (1) and Theorem 3.4.5. □

EXAMPLE 5.4.13. Let X_0 be an irreducible variety, and let $h : X_{\mathrm{sm}} \hookrightarrow X_0$ be the inclusion of its smooth locus. By (5.4.1) and Example 5.4.5, the object $\underline{\Bbbk}_{X_{\mathrm{sm}}}[\dim X_0]$ is pure of weight $\dim X_0$. Thus, the intersection cohomology complex (cf. Remark 3.3.10)

$$\mathrm{IC}(X_0; \overline{\mathbb{Q}}_\ell) = h_{!*}\underline{\overline{\mathbb{Q}}_\ell}_{X_{\mathrm{sm}}}[\dim X_0]$$

is also pure of weight $\dim X_0$. This object is not self-dual; its Verdier dual is $\mathrm{IC}(X_0; \overline{\mathbb{Q}}_\ell)(\dim X_0)$, which is pure of weight $-\dim X_0$. But if $\dim X_0$ is even, we may consider the object

$$\mathrm{IC}(X_0; \overline{\mathbb{Q}}_\ell)(\tfrac{1}{2}\dim X_0),$$

which is pure of weight 0 and self-dual under Verdier duality.

LEMMA 5.4.14. *Let $\mathcal{F} \in D^{\mathrm{b}}_{\leq w}(X_0, \overline{\mathbb{Q}}_\ell)$ and $\mathcal{G} \in D^{\mathrm{b}}_{\mathrm{m}}(X_0, \overline{\mathbb{Q}}_\ell)$.*

(1) *If $\mathcal{G}$ has weights $\geq w + 1$, then for any morphism $\phi : \mathcal{F} \to \mathcal{G}$, the map $\mathrm{egf}(\phi) : \mathrm{egf}(\mathcal{F}) \to \mathrm{egf}(\mathcal{G})$ is the zero map.*
(2) *If $\mathcal{F} \in {}^p D^{\mathrm{b}}_{\mathrm{m}}(X_0, \overline{\mathbb{Q}}_\ell)^{\leq n}$, $\mathcal{G} \in {}^p D^{\mathrm{b}}_{\mathrm{m}}(X_0, \overline{\mathbb{Q}}_\ell)^{\geq n}$, and $\mathcal{G}$ has weights $\geq w + 1$, then $\mathrm{Hom}(\mathcal{F}, \mathcal{G}) = 0$.*
(3) *If $\mathcal{G}$ has weights $\geq w + 2$, then $\mathrm{Hom}(\mathcal{F}, \mathcal{G}) = 0$.*

PROOF. Suppose first that $\mathcal{G}$ has weights $\geq w + 1$. By Theorem 5.4.7, the object $R\underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{G}) = \mathrm{a}_{X_0 *} R\mathscr{H}om(\mathcal{F}, \mathcal{G})$ has weights $\geq (w + 1) - w = 1$. By Remark 5.4.6, this implies that $\underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{G}) = \mathsf{H}^0(R\underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{G}))$ also has weights $\geq$ 1. In particular, $q^0 = 1$ is not an eigenvalue of the Frobenius action on this space, so $\underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{G})^{\mathrm{Fr}} = 0$. By Remark 5.3.5, we conclude that egf sends every morphism $\mathcal{F} \to \mathcal{G}$ to 0.

If, in addition, we have $\mathcal{F} \in {}^p D^{\mathrm{b}}_{\mathrm{m}}(X_0, \overline{\mathbb{Q}}_\ell)^{\leq n}$ and $\mathcal{G} \in {}^p D^{\mathrm{b}}_{\mathrm{m}}(X_0, \overline{\mathbb{Q}}_\ell)^{\geq n}$, then $\underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{G}[-1]) = 0$, since its underlying vector space $\mathrm{Hom}(\mathrm{egf}(\mathcal{F}), \mathrm{egf}(\mathcal{G})[-1])$ vanishes by the axioms for a t-structure. By Proposition 5.3.4, $\mathrm{Hom}(\mathcal{F}, \mathcal{G}) = 0$.

Finally, suppose that $\mathcal{G}$ has weights $\geq w + 2$. Then $R\underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{G})$ has weights $\geq$ 2, so in addition to the conclusions in the first paragraph above, we find that $\underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{G}[-1]) \cong \mathsf{H}^{-1}(R\underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{G}))$ has weights ≥ 1, so $\underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{G}[-1])_{\mathrm{Fr}} = 0$. Once again, Proposition 5.3.4 tells us that $\mathrm{Hom}(\mathcal{F}, \mathcal{G}) = 0$. □

By Corollary A.7.12, we have the following special case of Lemma 5.4.14.

COROLLARY 5.4.15. *Let $\mathcal{F}$ and $\mathcal{G}$ be simple mixed perverse sheaves on X_0. Assume that they are pure of weights w and v, respectively. If $w < v$, then $\mathrm{Ext}^1(\mathcal{F}, \mathcal{G}) = 0$.*

This corollary is the key ingredient in the proof of the following theorem.

THEOREM 5.4.16 (Weight filtration). *Every mixed perverse sheaf $\mathcal{F}$ is equipped with a canonical filtration*

$$\cdots \subset W_{i-1}\mathcal{F} \subset W_i\mathcal{F} \subset W_{i+1}\mathcal{F} \subset\!\!\to \cdots$$

such that $W_i\mathcal{F}/W_{i-1}\mathcal{F}$ is pure of weight i. Moreover, every morphism $\phi : \mathcal{F} \to \mathcal{G}$ of mixed perverse sheaves is strictly compatible with this filtration.

PROOF. *Step 1. Let $\mathcal{F}$ be a perverse sheaf, and let $\mathcal{P}$ be a composition factor. If the weight of $\mathcal{P}$ is minimal among the weights of composition factors of $\mathcal{F}$, then $\mathcal{P}$ is isomorphic to a subobject of $\mathcal{F}$.* Let w be the weight of $\mathcal{P}$. We proceed by induction on the length of $\mathcal{F}$. If $\mathcal{F}$ has length 1, then $\mathcal{F} = \mathcal{P}$, and there is nothing to prove. If $\mathcal{F}$ has length 2, then $\mathcal{P}$ is either a subobject or a quotient of $\mathcal{F}$. In the latter case, let $\mathcal{K}$ be the kernel of $\mathcal{F} \twoheadrightarrow \mathcal{P}$, so that we have a short exact sequence

$$0 \to \mathcal{K} \to \mathcal{F} \to \mathcal{P} \to 0. \tag{5.4.3}$$

Here, $\mathcal{K}$ is a simple perverse sheaf whose weight (by assumption) is $\geq w$. If $\mathcal{K}$ has weight $> w$, then Corollary 5.4.15 implies that (5.4.3) splits, so $\mathcal{P}$ is a subobject of $\mathcal{F}$. If $\mathcal{K}$ has weight exactly w, then (5.4.3) may or may not split. If it splits, we are done. If it does not split, then $\mathcal{F}$ is indecomposable and pure of weight w. Exercise 5.4.2 below implies that $\mathcal{K}$ and $\mathcal{P}$ are isomorphic, so we are done.

Finally, if $\mathcal{F}$ has length ≥ 3, choose some simple subobject $\mathcal{K} \subset \mathcal{F}$. If $\mathcal{K}$ is not isomorphic to $\mathcal{P}$, then $\mathcal{P}$ is a composition factor of $\mathcal{F}/\mathcal{K}$, and hence, by induction, a subobject. The preimage of that subobject under the quotient map $\mathcal{F} \to \mathcal{F}/\mathcal{K}$ is a length-2 subobject $\mathcal{F}' \subset \mathcal{F}$ that fits into a short exact sequence $0 \to \mathcal{K} \to \mathcal{F}' \to \mathcal{P} \to 0$. The reasoning above shows that $\mathcal{P}$ is a subobject of $\mathcal{F}'$, and hence of $\mathcal{F}$.

Step 2. Existence and uniqueness of the weight filtration. If $\mathcal{F}', \mathcal{F}'' \subset \mathcal{F}$ are two sub perverse sheaves with weights $\leq i$, then $\mathcal{F}' + \mathcal{F}''$ is as well, since it is a quotient of $\mathcal{F}' \oplus \mathcal{F}''$. Since $\mathrm{Perv}_{\mathrm{m}}(X_0, \overline{\mathbb{Q}}_\ell)$ is a noetherian category, it follows that there is a unique maximal subobject $W_i\mathcal{F} \subset \mathcal{F}$ that has weights $\leq i$.

It remains to show that $W_i\mathcal{F}/W_{i-1}\mathcal{F}$ is pure. This quotient at least has weights $\leq i$. Let $\mathcal{P}$ be a composition factor of $W_i\mathcal{F}/W_{i-1}\mathcal{F}$ of minimal weight. By Step 1, $\mathcal{P}$ is in fact a subobject. If it has weight $\leq i-1$, then its preimage in $W_i\mathcal{F}$ would be a subobject that strictly contains $W_{i-1}\mathcal{F}$ and has weights $\leq i-1$, contradicting the maximality of $W_{i-1}\mathcal{F}$. So $\mathcal{P}$ must have weight i, and hence $W_i\mathcal{F}/W_{i-1}\mathcal{F}$ is pure of weight i.

Step 3. Strict compatibility of morphisms. Let $\phi : \mathcal{F} \to \mathcal{G}$ be a morphism of mixed perverse sheaves. We must show that

$$\phi(W_i\mathcal{F}) = \phi(\mathcal{F}) \cap W_i\mathcal{G}. \tag{5.4.4}$$

It follows from the construction in Step 2 that both sides of (5.4.4) are equal to the unique maximal subobject of $\phi(\mathcal{F})$ with weights $\leq i$. $\square$

PROPOSITION 5.4.17. *Let $\mathcal{F} \in D^{\mathrm{b}}_{\mathrm{m}}(X_0, \overline{\mathbb{Q}}_\ell)$.*

1. *The object $\mathcal{F}$ has weights $\leq w$ if and only if each perverse cohomology sheaf ${}^p\mathsf{H}^i(\mathcal{F})$ has weights $\leq w + i$.*
2. *The object $\mathcal{F}$ has weights $\geq w$ if and only if each perverse cohomology sheaf ${}^p\mathsf{H}^i(\mathcal{F})$ has weights $\geq w + i$.*

PROOF. Since $\mathbb{D}$ is t-exact for field coefficients, the second part of this proposition follows from the first by applying $\mathbb{D}$. Let us prove the first part. We proceed by induction on the number of nonzero perverse cohomology sheaves of $\mathcal{F}$. If $\mathcal{F} = 0$, there is nothing to prove. Otherwise, let n be the largest integer such that ${}^p\mathsf{H}^n(\mathcal{F}) \neq 0$, and form the truncation distinguished triangle

$${}^p\tau^{\leq n-1}\mathcal{F} \to \mathcal{F} \to {}^p\mathsf{H}^n(\mathcal{F})[-n] \to .$$

Assume first that $\mathcal{F}$ has weights $\leq w$. If ${}^p\mathsf{H}^n(\mathcal{F})$ did not have weights $\leq w+n$, then by Theorem 5.4.16, it would have some quotient ${}^p\mathsf{H}^n(\mathcal{F}) \twoheadrightarrow \mathcal{G}$ where $\mathcal{G}$ is pure of some weight $> w+n$. The adjunction properties of truncation show that the composition $\mathcal{F} \to {}^p\mathsf{H}^n(\mathcal{F})[-n] \to \mathcal{G}[-n]$ is nonzero. But since $\mathcal{G}[-n]$ has weights $>$ w, this contradicts Lemma 5.4.14(2). We conclude that ${}^p\mathsf{H}^n(\mathcal{F})$ has weights $\leq w+n$. The distinguished triangle above then shows that ${}^p\tau^{\leq n-1}\mathcal{F}$ has weights $\leq w$, so by induction, we see that ${}^p\mathsf{H}^i(\mathcal{F})$ has weights $\leq w+i$ for all i.

For the opposite implication, if each ${}^p\mathsf{H}^i(\mathcal{F})$ has weights $\leq w+i$, then by induction, ${}^p\tau^{\leq n-1}\mathcal{F}$ has weights $\leq w$, and then the distinguished triangle above shows us that $\mathcal{F}$ does as well. $\square$

Pure complexes and semisimplicity. We conclude the étale portion of the chapter with results on the behavior of egf on pure objects.

PROPOSITION 5.4.18. *If $\mathcal{F} \in \mathrm{Perv_m}(X_0, \overline{\mathbb{Q}}_\ell)$ is a pure perverse sheaf, then $\mathrm{egf}(\mathcal{F}) \in \mathrm{Perv}(X, \overline{\mathbb{Q}}_\ell)$ is semisimple.*

PROOF. Let $\mathcal{F}' \subset \mathrm{egf}(\mathcal{F})$ be the unique maximal semisimple subobject (i.e., the socle) of $\mathrm{egf}(\mathcal{F})$. The $\mathrm{Fr}_q^*\mathcal{F}'$ is the unique maximal semisimple subobject of $\mathrm{Fr}_q^*\,\mathrm{egf}(\mathcal{F})$, and the Weil structure $\theta : \mathrm{Fr}_q^*\,\mathrm{egf}(\mathcal{F}) \xrightarrow{\sim} \mathrm{egf}(\mathcal{F})$ on $\mathrm{egf}(\mathcal{F})$ restricts to a Weil structure on $\mathcal{F}'$. That is, $(\mathcal{F}', \theta)$ is a sub Weil perverse sheaf of $(\mathrm{egf}(\mathcal{F}), \theta)$.

Recall from Proposition 5.3.9 that the image of egf is a Serre subcategory of $\mathrm{Perv_W}(X, \overline{\mathbb{Q}}_\ell)$. Therefore, $(\mathcal{F}', \theta)$ must arise by applying egf to some sub perverse sheaf $\tilde{\mathcal{F}}' \subset \mathcal{F}$ in $\mathrm{Perv_m}(X_0, \overline{\mathbb{Q}}_\ell)$. Let $\mathcal{G} = \mathcal{F}/\tilde{\mathcal{F}}'$, and let δ be the third morphism in the distinguished triangle

$$\tilde{\mathcal{F}}' \to \mathcal{F} \to \mathcal{G} \xrightarrow{\delta} \tilde{\mathcal{F}}'[1].$$

By Proposition 5.4.11, $\tilde{\mathcal{F}}'$ and $\mathcal{G}$ are both pure of the same weight as $\mathcal{F}$, say w, and then $\tilde{\mathcal{F}}'[1]$ is pure of weight $w+1$. By Lemma 5.4.14(1), $\mathrm{egf}(\delta) = 0$, so the distinguished triangle

$$\mathcal{F}' \to \mathrm{egf}(\mathcal{F}) \to \mathrm{egf}(\mathcal{G}) \xrightarrow{0}$$

in $D^{\mathrm{b}}_{\mathrm{c}}(X, \overline{\mathbb{Q}}_\ell)$ splits. Thus, $\mathrm{egf}(\mathcal{F}) \cong \mathcal{F}' \oplus \mathrm{egf}(\mathcal{G})$. But if $\mathrm{egf}(\mathcal{G}) \neq 0$, this contradicts the assumption that $\mathcal{F}'$ is the maximal semisimple subobject of $\mathrm{egf}(\mathcal{F})$. We conclude that $\mathrm{egf}(\mathcal{G}) = 0$, and hence that $\mathrm{egf}(\mathcal{F}) \cong \mathcal{F}'$. $\square$

THEOREM 5.4.19 (Semisimplicity). *If $\mathcal{F} \in D^{\mathrm{b}}_{\mathrm{m}}(X_0, \overline{\mathbb{Q}}_\ell)$ is pure, then $\mathrm{egf}(\mathcal{F}) \in D^{\mathrm{b}}_{\mathrm{c}}(X, \overline{\mathbb{Q}}_\ell)$ is a semisimple complex.*

PROOF. In view of Proposition 5.4.18, it is enough to show that

$$\mathrm{egf}(\mathcal{F}) \cong \bigoplus_{i \in \mathbb{Z}} {}^p\mathsf{H}^i(\mathrm{egf}(\mathcal{F}))[-i].$$

To prove this, we proceed by induction on the number of nonzero perverse cohomology sheaves of $\mathcal{F}$. If $\mathcal{F}$ is concentrated in a single perverse degree, there is nothing to prove. Otherwise, let n be the largest integer such that ${}^p\mathsf{H}^n(\mathcal{F}) \neq 0$, and consider the truncation distinguished triangle

$${}^p\tau^{\leq n-1}\mathcal{F} \to \mathcal{F} \to {}^p\mathsf{H}^n(\mathcal{F})[-n] \xrightarrow{\delta} ({}^p\tau^{\leq n-1}\mathcal{F})[1].$$

If $\mathcal{F}$ is pure of weight w, then Proposition 5.4.17 implies that $({}^p\tau^{\leq n-1}\mathcal{F})[1]$ is pure of weight $w+1$. By Lemma 5.4.14(1), $\mathrm{egf}(\delta) = 0$, so the distinguished triangle

$$\mathrm{egf}({}^p\tau^{\leq n-1}\mathcal{F}) \to \mathrm{egf}(\mathcal{F}) \to {}^p\mathsf{H}^n(\mathrm{egf}(\mathcal{F}))[-n] \xrightarrow{0}$$

splits. The result follows. □

Exercises.

5.4.1. Let $\mathcal{F} \in D^{\mathrm{b}}_{\mathrm{m}}(\operatorname{Spec} \mathbb{F}_q, \overline{\mathbb{Q}}_\ell)$ be a pure object of weight 0. Suppose there is a Laurent polynomial $g(v) \in \mathbb{Z}[v, v^{-1}]$ such that for all $n \geq 1$, we have

$$\chi_{\mathcal{F},\mathbb{F}_{q^n}} = g(q^{n/2}).$$

Show that all eigenvalues of Fr_{q^n} on $\mathsf{H}^i(\mathcal{F})$ are equal to $q^{in/2}$, and that

$$g(v) = \sum_{i\in\mathbb{Z}} (-1)^i \dim \mathsf{H}^i(\mathcal{F}) v^i.$$

5.4.2. In this exercise, you will classify indecomposable pure perverse sheaves.

(a) Show that for any $n \geq 1$, there is a unique (up to isomorphism) indecomposable continuous $\overline{\mathbb{Q}}_\ell[\mathrm{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q)]$-module S_n of dimension n on which Fr_q acts with eigenvalue 1. With a suitable choice of basis for S_n, Fr_q acts by the matrix

$$\begin{bmatrix} 1 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 1 \end{bmatrix}.$$

(b) Let $\mathcal{F} \in \mathrm{Perv}_{\mathrm{m}}(X_0, \overline{\mathbb{Q}}_\ell)$ be an indecomposable pure perverse sheaf. Show that $\mathcal{F}$ is of the form $S_n \boxtimes \mathrm{IC}(Y_0, \mathcal{L})$, where $\mathcal{L}$ is an irreducible pure local system on a smooth, connected, locally closed subvariety $Y_0 \subset X_0$.

5.4.3. Let $\mathcal{F} \in D^{\mathrm{b}}_{\mathrm{m}}(X_0, \overline{\mathbb{Q}}_\ell)$ be a pure complex, and let $\mathcal{G} \in \mathrm{Perv}_{\mathrm{m}}(X_0, \overline{\mathbb{Q}}_\ell)$ be a subquotient of some perverse cohomology sheaf ${}^p\mathsf{H}^k(\mathcal{F})$. Show that for all $i \in \mathbb{Z}$, $\underline{\mathbf{H}}^i(X_0, \mathcal{G})$ is a subquotient of $\underline{\mathbf{H}}^i(X_0, \mathcal{F})$.

5.5. $\mathscr{D}$-modules and the Riemann–Hilbert correspondence

We now leave the étale setting and return to the world of complex algebraic varieties. This section contains a cursory overview of $\mathscr{D}$-modules and the Riemann–Hilbert correspondence.

There are no proofs in this section. Proofs (and additional references) for all statements in this section can be found in [**106**, Part I]. Some other possible references for the foundations of the theory of $\mathscr{D}$-modules include [**37**, **124**].

Unlike in earlier chapters, we will consider sheaves on complex varieties in both the Zariski topology and the analytic topology. When we say "Let X be a complex algebraic variety," we retain the convention that X should be thought of as a topological space with the analytic topology. We denote by

$$X^{\mathrm{Zar}}$$

the topological space obtained by equipping the same set with the Zariski topology.

Much of this section involves the notion of a sheaf of rings or of a sheaf of modules over a sheaf of rings (see Exercises 1.1.10 and 1.4.3). If $\mathcal{R}$ is a sheaf of $\mathbb{C}$-algebras and $\mathcal{M}_1$ and $\mathcal{M}_2$ are $\mathcal{R}$-modules, we may consider both

$$\mathrm{Hom}_{\mathcal{R}}(\mathcal{M}_1, \mathcal{M}_2) \qquad \text{and} \qquad \mathrm{Hom}_{\mathbb{C}}(\mathcal{M}_1, \mathcal{M}_2).$$

The former is the space of morphisms in the category of $\mathcal{R}$-modules, and the latter is the space of morphisms in the category of sheaves of $\mathbb{C}$-vector spaces. Similarly, we may consider both

$$\mathscr{H}om_{\mathcal{R}}(\mathcal{M}_1, \mathcal{M}_2) \qquad \text{and} \qquad \mathscr{H}om_{\mathbb{C}}(\mathcal{M}_1, \mathcal{M}_2).$$

Differential operators. For any algebraic variety X, let $\mathcal{O}_X$ be its **structure sheaf**. Recall that this is the sheaf in $\mathrm{Sh}(X^{\mathrm{Zar}}, \mathbb{C})$ that assigns to each open set $U \subset X^{\mathrm{Zar}}$ the ring of regular functions on U. We assume some familiarity with this notion. Since $\mathcal{O}_X$ is a sheaf of rings, it can act on itself by multiplication. There are natural (injective) maps

$$\Gamma(\mathcal{O}_X) \to \mathrm{End}_{\mathbb{C}}(\mathcal{O}_X) \qquad \text{and} \qquad \mathcal{O}_X \to \mathscr{H}om_{\mathbb{C}}(\mathcal{O}_X, \mathcal{O}_X).$$

Definition 5.5.1. Let X be a smooth algebraic variety. A **derivation** of $\mathcal{O}_X$ is a morphism $\phi : \mathcal{O}_X \to \mathcal{O}_X$ in $\mathrm{End}_{\mathbb{C}}(\mathcal{O}_X)$ such that for any open set $U \subset X^{\mathrm{Zar}}$ and any two sections $s, t \in \mathcal{O}_X(U)$, we have

$$\phi_U(st) = s\phi_U(t) + \phi_U(s)t.$$

The space of derivations of $\mathcal{O}_X$ is denoted by $\Theta(X) \subset \mathrm{End}_{\mathbb{C}}(\mathcal{O}_X)$. The **sheaf of derivations** of $\mathcal{O}_X$ is the subsheaf $\Theta_X \subset \mathscr{H}om_{\mathbb{C}}(\mathcal{O}_X, \mathcal{O}_X)$ given by

$$\Theta_X(U) = \Theta(U).$$

The ring of **differential operators** on X is the subring $\mathscr{D}(X) \subset \mathrm{End}_{\mathbb{C}}(\mathcal{O}_X)$ generated by $\Gamma(\mathcal{O}_X)$ and $\Theta(X)$. The **sheaf of differential operators** on X, denoted by $\mathscr{D}_X$, is the subsheaf of rings of $\mathscr{H}om_{\mathbb{C}}(\mathcal{O}_X, \mathcal{O}_X)$ given by

$$\mathscr{D}_X(U) = \mathscr{D}(U).$$

Example 5.5.2. Consider the variety $\mathbb{A}^1 = \mathrm{Spec}\,\mathbb{C}[x]$. The differentiation operator $\partial_x : \mathbb{C}[x] \to \mathbb{C}[x]$ given by

$$\partial_x(x^n) = nx^{n-1}$$

is a derivation, and it can be shown that every derivation is a $\mathbb{C}[x]$-multiple of this. For any polynomial p, note that $\partial_x(xp) - x\partial_x(p) = p$. This observation leads to an explicit description of the ring $\mathscr{D}(\mathbb{A}^1)$, as follows: let $\mathbb{C}\langle x, \partial_x\rangle$ denote the free (noncommutative) associative $\mathbb{C}$-algebra on the symbols x and ∂_x. Then

$$\mathscr{D}(\mathbb{A}^1) \cong \mathbb{C}\langle x, \partial_x\rangle/(\partial_x x - x\partial_x - 1).$$

The right-hand side is known as the (first) **Weyl algebra**.

Similar considerations show that for $\mathbb{A}^1 \smallsetminus \{0\} = \mathrm{Spec}\,\mathbb{C}[x, x^{-1}]$, we have

$$\mathscr{D}(\mathbb{A}^1 \smallsetminus \{0\}) \cong \mathbb{C}\langle x, x^{-1}, \partial_x\rangle/(xx^{-1} - 1, x^{-1}x - 1, \partial_x x - x\partial_x - 1).$$

Definition 5.5.3. The **order filtration** on $\mathscr{D}(X)$ is the increasing filtration $F_\bullet\mathscr{D}(X)$ defined as follows:

(1) For $p < 0$, we set $F_p\mathscr{D}(X) = 0$, and we set $F_0\mathscr{D}(X) = \Gamma(\mathcal{O}_X)$.
(2) For $p > 0$, we set $F_p\mathscr{D}(X) = F_{p-1}\mathscr{D}(X) + \Theta(X)F_{p-1}\mathscr{D}(X)$.

Elements of $F_p\mathscr{D}(X)$ are called **differential operators of order** $\leq p$.

Similarly, the **order filtration** on the sheaf $\mathscr{D}_X$ is the filtration $F_\bullet\mathscr{D}_X$ given by $(F_p\mathscr{D}_X)(U) = F_p\mathscr{D}(U)$.

It is easy to see that this filtration is compatible with the ring structure, i.e., that $F_p\mathscr{D}(X)F_q\mathscr{D}(X) \subset F_{p+q}\mathscr{D}(X)$. As a consequence, the associated graded space for the order filtration has the structure of a ring. The associated graded sheaf for the order filtration on $\mathscr{D}_X$ is a sheaf of rings, described by the following theorem.

THEOREM 5.5.4. *The associated graded sheaf* $\mathrm{gr}^F \mathscr{D}_X$ *for the order filtration on* $\mathscr{D}_X$ *is a sheaf of commutative* $\mathcal{O}_X$*-algebras. Its relative spectrum is given by*

$$\underline{\mathrm{Spec}}(\mathrm{gr}^F \mathscr{D}_X) \cong T^*X,$$

where T^*X *is the cotangent bundle of* X*.*

$\mathscr{D}$-modules and integrable connections. We denote by

$$\mathscr{D}_X\text{-mod}$$

the category of (left) $\mathscr{D}_X$-modules. We will also consider its bounded derived category $D^{\mathrm{b}}(\mathscr{D}_X\text{-mod})$. If $f : X \to Y$ is a morphism of smooth varieties, there is a pullback functor

$$f^* : D^{\mathrm{b}}(\mathscr{D}_Y\text{-mod}) \to D^{\mathrm{b}}(\mathscr{D}_X\text{-mod}). \tag{5.5.1}$$

Although this is philosophically related to the pullback functor of Section 1.2, its definition is rather more complicated. (If we naively apply the construction of Definition 1.2.1(1) to a $\mathscr{D}_Y$-module, the resulting object does not have the structure of a $\mathscr{D}_X$-module.) At the level of abelian categories, pullback of $\mathscr{D}$-modules is not exact; the functor (5.5.1) is a left derived functor. There are also $\mathscr{D}$-module versions of the other sheaf functors from Chapter 1.

Since $\mathcal{O}_X$ is a subsheaf of $\mathscr{D}_X$, any $\mathscr{D}_X$-module $\mathcal{M}$ can be regarded as an $\mathcal{O}_X$-module. We assume some familiarity with the notions of **coherent** and **quasicoherent** $\mathcal{O}_X$-modules from algebraic geometry. We are mainly interested in the following class of $\mathscr{D}$-modules.

DEFINITION 5.5.5. A $\mathscr{D}_X$-module $\mathcal{M}$ is said to be **coherent** if it is quasicoherent as an $\mathcal{O}_X$-module and locally finitely generated as a $\mathscr{D}_X$-module, i.e., if for every open set $U \subset X^{\mathrm{Zar}}$, $\mathcal{M}(U)$ is a finitely generated $\mathscr{D}_X(U)$-module. The category of coherent $\mathscr{D}_X$-modules is denoted by $\mathscr{D}_X\text{-mod}^{\mathrm{coh}}$. Let $D^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X\text{-mod})$ be the full triangulated subcategory of $D^{\mathrm{b}}(\mathscr{D}_X\text{-mod})$ given by

$$D^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X\text{-mod}) = \{\mathcal{M} \mid \text{for all } i,\ \mathsf{H}^i(\mathcal{M}) \text{ is a coherent } \mathscr{D}_X\text{-module}\}.$$

(Note that in many sources, the definition of "coherent $\mathscr{D}_X$-module" is a different condition, and the description given above is instead a theorem.) The structure sheaf $\mathcal{O}_X$ always has a natural structure of a coherent $\mathscr{D}_X$-module. This is sometimes known as the trivial $\mathscr{D}_X$-module. We will see more examples below.

Certain coherent $\mathscr{D}_X$-modules can be described using the following notion from differential geometry.

DEFINITION 5.5.6. Let $\mathcal{M}$ be a coherent $\mathcal{O}_X$-module. An **integrable connection** on $\mathcal{M}$ is a morphism

$$\nabla : \Theta_X \to \mathscr{H}om_{\mathbb{C}}(\mathcal{M}, \mathcal{M})$$

such that for any open set $U \subset X^{\mathrm{Zar}}$ and any $f \in \mathcal{O}_X(U)$, $\theta, \theta' \in \Theta_X(U)$, and $s \in \mathcal{M}(U)$, the following three conditions hold:

$$\begin{aligned}
\nabla_U(f\theta)(s) &= f\nabla_U(\theta)(s),\\
\nabla_U(\theta)(fs) &= \theta(f)(s) + f\nabla_U(\theta)(s),\\
\nabla_U([\theta_1,\theta_2])(s) &= \nabla_U(\theta_1)(\nabla_U(\theta_2)(s)) - \nabla_U(\theta_2)(\nabla_U(\theta_1)(s)).
\end{aligned}$$

THEOREM 5.5.7. *Let $\mathcal{M}$ be a coherent $\mathcal{O}_X$-module.*

(1) *Equipping $\mathcal{M}$ with an integrable connection is equivalent to giving it the structure of a $\mathscr{D}_X$-module.*
(2) *If $\mathcal{M}$ admits an integrable connection, then it is locally free as an $\mathcal{O}_X$-module.*

As a consequence of this theorem, we will use the term "integrable connection" as a synonym for a "$\mathscr{D}_X$-module that is coherent over $\mathcal{O}_X$."

EXAMPLE 5.5.8. Below are some examples of $\mathscr{D}$-modules on $\mathbb{A}^1$ or $\mathbb{A}^1 \smallsetminus \{0\}$. Each example is of the form $\mathcal{M} = \mathscr{D}/\mathcal{I}$, where $\mathcal{I}$ is some left ideal in $\mathscr{D}$. Let ν be the image of the generator $1 \in \mathscr{D}$. In each example, we will write down a $\mathbb{C}$-basis for $\mathcal{M}$ in terms of ν, and we will indicate how the generators $x, \partial_x \in \mathscr{D}$ act on the basis elements.

(1) On $\mathbb{A}^1$, let $\mathcal{M} = \mathscr{D}/\mathscr{D}\cdot x$. The following set is a $\mathbb{C}$-basis for this module:
$$\nu, \partial_x\nu, \partial_x^2\nu, \ldots.$$
We have
$$x \cdot \partial_x^k\nu = (-k)\partial_x^{k-1}\nu, \qquad \partial_x \cdot \partial_x^k\nu = \partial_x^{k+1}\nu.$$
This module is finitely generated over $\mathscr{D}$, but not over $\mathbb{C}[x]$. Thus, it is a coherent $\mathscr{D}$-module but not an integrable connection.
(2) On $\mathbb{A}^1 \smallsetminus \{0\}$, let $\lambda \in \mathbb{C}$, and let $\mathcal{M}_\lambda = \mathscr{D}/\mathscr{D}\cdot(x\partial_x - \lambda)$. The following set is a $\mathbb{C}$-basis for this module:
$$\ldots, x^{-2}\nu, x^{-1}\nu, \nu, x\nu, x^2\nu, \ldots.$$
We have
$$x \cdot x^k\nu = x^{k+1}\nu, \qquad \partial_x \cdot x^k\nu = (k+\lambda)x^{k-1}\nu.$$
This module is finitely generated over $\mathbb{C}[x, x^{-1}]$, so it is an integrable connection. Note that if $\lambda - \mu \in \mathbb{Z}$, then $\mathcal{M}_\lambda \cong \mathcal{M}_\mu$. If $\lambda \in \mathbb{Z}$, $\mathcal{M}_\lambda$ is isomorphic to the trivial $\mathscr{D}$-module $\mathcal{O}_{\mathbb{A}^1\smallsetminus\{0\}}$. This example is related to Exercise 1.7.3: see Example 5.5.23.
(3) On $\mathbb{A}^1 \smallsetminus \{0\}$, let $\lambda \in \mathbb{C}$, and let $\mathcal{M} = \mathscr{D}/\mathscr{D}\cdot(x^2\partial_x - 1)$. The following set is a $\mathbb{C}$-basis for this module:
$$\ldots, x^{-2}\nu, x^{-1}\nu, \nu, x\nu, x^2\nu, \ldots.$$
We have
$$x \cdot x^k\nu = x^{k+1}\nu, \qquad \partial_x \cdot x^k\nu = kx^{k-1}\nu + x^{k-2}\nu.$$
This module is again an integrable connection.

REMARK 5.5.9 (Regular integrable connections). An important subclass of the category of integrable connections is the category of **regular integrable connections**. For a discussion of this notion, see [**106**, Section 5.3]. The general definition is, unfortunately, beyond the scope of this book. However, on the variety $X = \mathbb{A}^1 \smallsetminus \{0\}$, we have the following characterization:

Let $\mathcal{M}$ be a $\mathscr{D}(X)$-module that is finitely generated as a module over $\mathcal{O}(X) = \mathbb{C}[x, x^{-1}]$. This integrable connection is regular if and only if both of the following conditions hold:

- *(Regularity at 0) There is a subspace $\mathcal{M}^+ \subset \mathcal{M}$ that is preserved by x and by $x\partial_x$, and that generates $\mathcal{M}$ as a $\mathbb{C}[x, x^{-1}]$-module.*
- *(Regularity at ∞) There is a subspace $\mathcal{M}^- \subset \mathcal{M}$ that is preserved by x^{-1} and by $x\partial_x$, and that generates $\mathcal{M}$ as a $\mathbb{C}[x, x^{-1}]$-module.*

It can be checked that Example 5.5.8(2) is a regular integrable connection. On the other hand, Example 5.5.8(3) is regular at ∞, but not at 0.

PROPOSITION 5.5.10. *Let $h : Y \hookrightarrow X$ be the inclusion map of a smooth, locally closed subvariety, and let $\mathcal{M}$ be an integrable connection on Y. There is, up to isomorphism, a unique $\mathscr{D}_X$-module $\mathscr{L}(Y, \mathcal{M})$ that is characterized by the following properties:*

1. *The support of $\mathscr{L}(Y, \mathcal{M})$ is $\overline{Y}$.*
2. *The $\mathscr{D}_X$-module $\mathscr{L}(Y, \mathcal{M})$ has no nonzero submodule or quotient supported on $\overline{Y} \smallsetminus Y$.*
3. *We have $h^*\mathscr{L}(Y, \mathcal{M})[\dim X - \dim Y] \cong \mathcal{M}$.*

Moreover, if $\mathcal{M}$ is a simple $\mathscr{D}_Y$-module, then $\mathscr{L}(Y, \mathcal{M})$ is a simple $\mathscr{D}_X$-module.

Recall that the functor h^* appearing above is the $\mathscr{D}$-module pullback functor from (5.5.1). One might observe that Proposition 5.5.10 is reminiscent of Lemma 3.3.3. This presages some aspects of the Riemann–Hilbert correspondence.

DEFINITION 5.5.11. The $\mathscr{D}_X$-module $\mathscr{L}(Y, \mathcal{M})$ given by Proposition 5.5.10 is called the **minimal extension** of $\mathcal{M}$.

Holonomic $\mathscr{D}$-modules. We now introduce a condition on $\mathscr{D}$-modules that leads to an analogue of Theorem 3.4.5.

DEFINITION 5.5.12. Let $\mathcal{M}$ be a coherent $\mathscr{D}_X$-module. A **good filtration** on $\mathcal{M}$ is an increasing filtration $F_\bullet\mathcal{M}$ by $\mathcal{O}_X$-submodules with the following properties:

1. For all $p, q \in \mathbb{Z}$, we have $(F_p\mathscr{D}_X)(F_q\mathcal{M}) \subset F_{p+q}\mathcal{M}$.
2. For $p \ll 0$, we have $F_p\mathcal{M} = 0$, and $\mathcal{M} = \bigcup_{p\in\mathbb{Z}} F_p\mathcal{M}$.
3. Each $F_p\mathcal{M}$ is a coherent $\mathcal{O}_X$-module.
4. There exists a $q_0 \in \mathbb{Z}$ such that for all $p \geq 0$ and all $q \geq q_0$, we have $(F_p\mathscr{D}_X)(F_q\mathcal{M}) = F_{p+q}\mathcal{M}$.

PROPOSITION 5.5.13. *Every coherent $\mathscr{D}_X$-module admits a good filtration.*

Given a coherent $\mathscr{D}_X$-module $\mathcal{M}$ with a good filtration $F_\bullet\mathcal{M}$, the associated graded sheaf $\operatorname{gr}^F \mathcal{M}$ is naturally a sheaf of modules over $\operatorname{gr}^F \mathscr{D}_X$. Via the relative Spec construction, $\operatorname{gr}^F \mathcal{M}$ can be regarded as a coherent $\mathcal{O}_{T^*X}$-module on the cotangent bundle T^*X.

DEFINITION 5.5.14. Let $\mathcal{M}$ be a coherent $\mathscr{D}_X$-module, and let $F_\bullet\mathcal{M}$ be a good filtration on it. The **characteristic variety** of $(\mathcal{M}, F_\bullet)$, denoted by $\mathrm{CV}(\mathcal{M})$, is the support of the coherent $\mathcal{O}_{T^*X}$-module corresponding to $\operatorname{gr}^F \mathcal{M}$.

THEOREM 5.5.15. *Let $\mathcal{M}$ be a coherent $\mathscr{D}_X$-module.*

(1) *The characteristic variety of $\mathcal{M}$ is independent of the choice of good filtration on $\mathcal{M}$.*
(2) *If $\mathcal{M} \neq 0$, then* $\dim \mathrm{CV}(\mathcal{M}) \geq \dim X$.

DEFINITION 5.5.16. A coherent $\mathscr{D}_X$-module is said to be **holonomic** if it is either the zero module or if $\dim \mathrm{CV}(\mathcal{M}) = \dim X$. The category of holonomic $\mathscr{D}_X$-modules is denoted by $\mathscr{D}_X\text{-mod}^{\mathrm{hol}}$. Let $D^{\mathrm{b}}_{\mathrm{hol}}(\mathscr{D}_X\text{-mod})$ be the full triangulated subcategory of $D^{\mathrm{b}}(\mathscr{D}_X\text{-mod})$ given by

$$D^{\mathrm{b}}_{\mathrm{hol}}(\mathscr{D}_X\text{-mod}) = \{\mathcal{M} \mid \text{for all } i,\ \mathsf{H}^i(\mathcal{M}) \text{ is a holonomic } \mathscr{D}_X\text{-module}\}.$$

This is a rather opaque definition, especially for the reader who has not seen characteristic varieties before. Fortunately, the theorem below offers another perspective. It turns out that every integrable connection is holonomic. The class of holonomic $\mathscr{D}$-modules can be thought of, in some sense, as the class of $\mathscr{D}$-modules "generated" by integrable connections on smooth locally closed subvarieties.

THEOREM 5.5.17. *For $\mathcal{M} \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X\text{-mod})$, the following conditions are equivalent:*

(1) *We have $\mathcal{M} \in D^{\mathrm{b}}_{\mathrm{hol}}(\mathscr{D}_X\text{-mod})$.*
(2) *There exists a stratification $(X_s)_{s\in\mathscr{S}}$ of X such that for each stratum $i_s : X_s \hookrightarrow X$ and each $k \in \mathbb{Z}$, the $\mathscr{D}_{X_s}$-module $\mathsf{H}^k(i_s^*\mathcal{M})$ is an integrable connection.*

PROPOSITION 5.5.18. *Let $Y \subset X$ be a smooth, locally closed subvariety, and let $\mathcal{M}$ be an integrable connection on Y. Then the minimal extension $\mathscr{L}(Y, \mathcal{M})$ is a holonomic $\mathscr{D}_X$-module.*

THEOREM 5.5.19. (1) *Every holonomic $\mathscr{D}_X$-module has finite length.*
(2) *Every simple holonomic $\mathscr{D}$-module is of the form $\mathscr{L}(Y, \mathcal{M})$, where $\mathcal{M}$ is a simple integrable connection on a smooth, locally closed subvariety $Y \subset X$.*

By imposing the condition from Remark 5.5.9 on the integrable connections appearing in the second part of this theorem, we obtain the following notion.

DEFINITION 5.5.20. Let X be a smooth variety. A holonomic $\mathscr{D}_X$-module is said to be **regular holonomic** if all of its composition factors are of the form $\mathscr{L}(Y, \mathcal{M})$, where $\mathcal{M}$ is a simple regular integrable connection on a smooth, locally closed subvariety $Y \subset X$. The category of regular holonomic $\mathscr{D}_X$-modules is denoted by $\mathscr{D}_X\text{-mod}^{\mathrm{rh}}$. Let $D^{\mathrm{b}}_{\mathrm{rh}}(\mathscr{D}_X\text{-mod})$ be the full triangulated subcategory of $D^{\mathrm{b}}(\mathscr{D}_X\text{-mod})$ given by

$$D^{\mathrm{b}}_{\mathrm{rh}}(\mathscr{D}_X\text{-mod}) = \{\mathcal{M} \mid \text{for all } i,\ \mathsf{H}^i(\mathcal{M}) \text{ is a regular holonomic } \mathscr{D}_X\text{-module}\}.$$

The Riemann–Hilbert correspondence. So far, we have been considering algebraic $\mathscr{D}$-modules, which are sheaves in the Zariski topology on X. Let us now switch back to the analytic topology. In $\mathrm{Sh}(X, \mathbb{C})$, we may consider the **holomorphic structure sheaf** $\mathcal{O}_X^{\mathrm{an}}$, given by

$$\mathcal{O}_X^{\mathrm{an}}(U) = \{\text{holomorphic functions } U \to \mathbb{C}\}.$$

Imitating Definition 5.5.1, we may define a sheaf of analytic derivations Θ_X^{an}, and then the sheaf of analytic differential operators $\mathscr{D}_X^{\mathrm{an}} \subset \mathscr{H}om_{\mathbb{C}}(\mathcal{O}_X^{\mathrm{an}}, \mathcal{O}_X^{\mathrm{an}})$.

Finally, let Ω_X^{an} be the sheaf of top-degree holomorphic differential forms on X. This sheaf naturally has the structure of a *right* $\mathscr{D}_X^{\mathrm{an}}$-module.

Consider the obvious continuous map $\iota : X \to X^{\mathrm{Zar}}$ that is the identity map on the underlying sets. Given a sheaf $\mathcal{F} \in \mathrm{Sh}(X^{\mathrm{Zar}}, \mathbb{C})$, we denote its pullback to X in the sense of Definition 1.2.1(1) by $\iota^{-1}\mathcal{F}$ rather than by $\iota^*\mathcal{F}$, to avoid confusion with the $\mathscr{D}$-module pullback functor (5.5.1).

DEFINITION 5.5.21. Let X be a smooth complex variety. Given a $\mathscr{D}_X$-module $\mathcal{M} \in \mathscr{D}_X$-mod, its **analytification** is defined to be the $\mathscr{D}_X^{\mathrm{an}}$-module given by

$$\mathcal{M}^{\mathrm{an}} = \mathscr{D}_X^{\mathrm{an}} \otimes_{\iota^{-1}\mathscr{D}_X} \iota^{-1}\mathcal{M}.$$

It can be shown that the assignment $\mathcal{M} \mapsto \mathcal{M}^{\mathrm{an}}$ is an exact functor, so it gives rise to a functor of derived categories

$$(\mathcal{M} \mapsto \mathcal{M}^{\mathrm{an}}) : D^{\mathrm{b}}(\mathscr{D}_X\text{-mod}) \to D^{\mathrm{b}}(\mathscr{D}_X^{\mathrm{an}}\text{-mod}).$$

DEFINITION 5.5.22. Let X be a smooth complex variety. The **de Rham functor** for X is the functor $\mathrm{dR} : D^{\mathrm{b}}(\mathscr{D}_X\text{-mod}) \to D^{\mathrm{b}}(X, \mathbb{C})$ given by

$$\mathrm{dR}(\mathcal{M}) = \Omega_X^{\mathrm{an}} \overset{L}{\otimes}_{\mathscr{D}_X^{\mathrm{an}}} \mathcal{M}^{\mathrm{an}}.$$

The **solutions functor** is the functor $\mathrm{Sol} : D^{\mathrm{b}}(\mathscr{D}_X\text{-mod})^{\mathrm{op}} \to D^{\mathrm{b}}(X, \mathbb{C})$ given by

$$\mathrm{Sol}(\mathcal{M}) = R\mathscr{H}om_{\mathscr{D}_X^{\mathrm{an}}}(\mathcal{M}^{\mathrm{an}}, \mathcal{O}_X^{\mathrm{an}}).$$

EXAMPLE 5.5.23. Let $\lambda \in \mathbb{C}$, and let $\mathcal{M}_\lambda$ be the $\mathscr{D}$-module on $\mathbb{A}^1 \smallsetminus \{0\}$ defined in Example 5.5.8(2). Then $\mathrm{Sol}(\mathcal{M}_\lambda)$ can be identified with the local system $\mathcal{Q}_\lambda$ defined in Exercise 1.7.3.

For holonomic $\mathscr{D}_X$-modules, we have the following result.

THEOREM 5.5.24 (Kashiwara's constructibility theorem). *Let X be a smooth complex variety. If $\mathcal{M} \in D^{\mathrm{b}}_{\mathrm{hol}}(\mathscr{D}_X\text{-mod})$, then $\mathrm{dR}(\mathcal{M})$ and $\mathrm{Sol}(\mathcal{M})$ lie in $D^{\mathrm{b}}_{\mathrm{c}}(X, \mathbb{C})$.*

In the setting of constructible sheaves, the de Rham and solutions functors are related by Verdier duality.

PROPOSITION 5.5.25. *For $\mathcal{M} \in D^{\mathrm{b}}_{\mathrm{hol}}(\mathscr{D}_X\text{-mod})$, there is a natural isomorphism $\mathrm{Sol}(\mathcal{M})[\dim X] \cong \mathbb{D}(\mathrm{dR}(\mathcal{M}))$.*

After further restricting to regular holonomic $\mathscr{D}_X$-modules, we arrive at the main statement of this section, due to Kashiwara [**121**] and Mebkhout [**175**].

THEOREM 5.5.26 (Riemann–Hilbert correspondence). *Let X be a smooth complex variety. The de Rham functor gives an equivalence of categories*

$$\mathrm{dR} : D^{\mathrm{b}}_{\mathrm{rh}}(\mathscr{D}_X\text{-mod}) \xrightarrow{\sim} D^{\mathrm{b}}_{\mathrm{c}}(X, \mathbb{C}).$$

Moreover, this functor restricts to an equivalence of abelian categories

$$\mathrm{dR} : \mathscr{D}_X\text{-mod}^{\mathrm{rh}} \xrightarrow{\sim} \mathrm{Perv}(X, \mathbb{C}),$$

under which regular integrable connections correspond to shifted local systems.

Of course, by Proposition 5.5.25, these equivalences can be rephrased in terms of the solutions functor.

REMARK 5.5.27. The foundations of the theory of $\mathscr{D}$-modules require working on a smooth variety, so this is included as an assumption in Theorem 5.5.26. However, one can adapt this statement to singular varieties as follows.

Let X be a (possibly singular) algebraic variety. It is always possible to embed X as a closed subvariety of some smooth variety X'. Let $D^{\mathrm{b}}_{\mathrm{rh},X}(\mathscr{D}_{X'}\text{-mod})$ be the full subcategory of $D^{\mathrm{b}}_{\mathrm{rh}}(\mathscr{D}_{X'}\text{-mod})$ consisting of objects supported on $X \subset X'$. Then the Riemann–Hilbert equivalence for X' induces an equivalence

$$D^{\mathrm{b}}_{\mathrm{rh},X}(\mathscr{D}_{X'}\text{-mod}) \xrightarrow{\sim} D^{\mathrm{b}}_{\mathrm{c}}(X,\mathbb{C}).$$

Exercises.

5.5.1. Let X be either $\mathbb{A}^1$ or $\mathbb{A}^1 \smallsetminus \{0\}$, and let $\phi \in \mathscr{D}(X)$ be a differential operator. Show that there is an isomorphism

$$\mathrm{Hom}_{\mathscr{D}_X^{\mathrm{an}}}((\mathscr{D}_X/\mathscr{D}_X\phi)^{\mathrm{an}}, \mathscr{O}_X^{\mathrm{an}}) \cong \left\{ \begin{array}{c} \text{solutions to the differential} \\ \text{equation } \phi f = 0 \end{array} \right\}.$$

Deduce that for $\phi = x\partial_x - \lambda$ (cf. Example 5.5.8(2)), we have

$$\mathrm{Sol}(\mathscr{D}_X/\mathscr{D}_X\phi) \cong \mathcal{Q}_\lambda,$$

where $\mathcal{Q}_\lambda$ is the local system defined in Exercise 1.7.3.

5.6. Mixed Hodge modules

Both sides of the Riemann–Hilbert equivalence $\mathscr{D}_X\text{-mod}^{\mathrm{rh}} \cong \mathrm{Perv}(X,\mathbb{C})$ have features not available on the other: for instance, "good filtrations" make sense only for $\mathscr{D}$-modules; on the other hand, perverse sheaves make sense with coefficients in, say, $\mathbb{Q}$ or $\mathbb{R}$, whereas $\mathscr{D}$-modules are necessarily $\mathbb{C}$-linear. The theory of mixed Hodge modules, due to Saito, exploits a delicate interplay between these differing aspects to produce a new category with remarkable properties. This section gives a brief survey of the main ingredients and main results in this theory, with no proofs. Throughout this section, we assume that

$$\Bbbk \text{ is a subfield of } \mathbb{R}.$$

Mixed Hodge structures. We begin with an antecedent of the theory of mixed Hodge modules, due to Deligne [**58**].

DEFINITION 5.6.1. A **pure Hodge structure** of weight $n \in \mathbb{Z}$ is a triple (M, F, K), where M is a finite-dimensional $\mathbb{C}$-vector space, K is a $\Bbbk$-vector space equipped with an isomorphism

$$\mathbb{C} \otimes_{\Bbbk} K \cong M, \tag{5.6.1}$$

and $F = F^\bullet M$ is a decreasing filtration (called the **Hodge filtration**) on M satisfying

$$F^pM \oplus \overline{F^qM} = M \qquad \text{if } p+q = n+1.$$

Here, $\overline{}$ is the complex conjugation on M induced by that on $\mathbb{C}$ via (5.6.1).

As a consequence of the definition, in a pure Hodge structure (M, F, K), there is a canonical decomposition

$$M = \bigoplus_{p,q \in \mathbb{Z}} F^pM \cap \overline{F^qM},$$

called the **Hodge decomposition**. It is possible to rephrase the definition above in terms of this decomposition, instead of using the filtration F.

EXAMPLE 5.6.2. The **trivial Hodge structure** is the pure Hodge structure of weight 0 consisting of the triple $(\mathbb{C}, F, \Bbbk)$ with

$$F^0\mathbb{C} = \mathbb{C}, \qquad F^1\mathbb{C} = 0.$$

The trivial Hodge structure is often denoted simply by $\Bbbk$.

EXAMPLE 5.6.3. The **Tate module**, denoted by $\Bbbk(1)$, is the pure Hodge structre of weight -2 consisting of the triple $(\mathbb{C}, F, (2\pi\sqrt{-1})\Bbbk)$ with

$$F^{-1}\mathbb{C} = \mathbb{C}, \qquad F^0\mathbb{C} = 0.$$

More generally, for any $n \in \mathbb{Z}$, we define $\Bbbk(n)$ to be the pure Hodge structure of weight $-2n$ given by $(\mathbb{C}, F, (2\pi\sqrt{-1})^n\Bbbk)$, where

$$F^{-n}\mathbb{C} = \mathbb{C}, \qquad F^{-n+1}\mathbb{C} = 0.$$

DEFINITION 5.6.4. Let (M, F, K) be a pure Hodge structure of weight n. A **polarization** of this pure Hodge structure is a nondegenerate $\Bbbk$-bilinear form $Q : K \otimes_{\Bbbk} K \to \Bbbk$ satisfying the following three conditions:

(1) We have $Q(u, v) = (-1)^n Q(v, u)$.
(2) If $u \in F^pM$ and $v \in F^{n+1-p}M$, then $Q(u, v) = 0$.
(3) The Hermitian form $(u, v) \mapsto Q(C(u), \overline{v})$ on M is positive-definite. Here, $C : M \to M$ is the $\mathbb{C}$-linear map given by

$$C(u) = (\sqrt{-1})^{p-q}u \qquad \text{if } u \in F^pM \cap \overline{F^qM}.$$

We say that (M, F, K) is **polarizable** if it admits a polarization.

DEFINITION 5.6.5. A **mixed Hodge structure** is a quadruple (M, F, K, W), where M is a finite-dimensional $\mathbb{C}$-vector space, K is a $\Bbbk$-vector space equipped with an isomorphism

$$\mathbb{C} \otimes_{\Bbbk} K \cong M,$$

$F = F^\bullet M$ is a decreasing filtration on M (called the **Hodge filtration**), and $W = W_\bullet K$ is an increasing filtration on K (called the **weight filtration**) such that for each k, the triple $(\mathrm{gr}^W_k M, \mathrm{gr}^W_k F, \mathrm{gr}^W_k K)$ is a pure polarizable Hodge structure of weight k. A **morphism** of mixed Hodge structures $(M, F, K, W) \to (M', F', K', W')$ is a $\Bbbk$-linear map $K \to K'$ that is compatible with both the Hodge and weight filtrations. The category of mixed Hodge structures (with coefficients in $\Bbbk$) is denoted by $\mathrm{MHS}(\Bbbk)$.

(In some sources, these objects are called "polarizable mixed Hodge structures.") Any pure polarizable Hodge structure can be regarded as a mixed Hodge structure, by equipping it with a weight filtration that is concentrated in one step.

THEOREM 5.6.6. *The category* $\mathrm{MHS}(\Bbbk)$ *is abelian.*

For a proof, see [**188**, Section 3.1]. A key step in the proof of this theorem involves showing that all morphisms of mixed Hodge structures are strictly compatible with both the Hodge and weight filtrations. Suppose $M, N \in \mathrm{MHS}(\Bbbk)$ are objects such that the weight filtration of M is concentrated in degrees $a, a+1, \ldots, b$, and the weight filtration of N is concentrated in degrees $c, c+1, \dots, d$. It is immediate from Definition 5.6.5 that

$$\mathrm{Hom}(M, N) = 0 \tag{5.6.2}$$

if $b < c$. Because morphisms in $\mathrm{MHS}(\Bbbk)$ are strictly compatible with the weight filtration, (5.6.2) also holds if $d < a$. In particular, we have

$$\mathrm{Hom}(\Bbbk(m), \Bbbk(n)) = 0 \qquad \text{if } m \neq n.$$

Along the way to the proof of Theorem 5.6.6, one obtains the following statement.

PROPOSITION 5.6.7. *Let $M, N \in \mathrm{MHS}(\Bbbk)$. If M has weights $\leq n$ and N has weights $\geq n$, then $\mathrm{Ext}^1_{\mathrm{MHS}(\Bbbk)}(M, N) = 0$. In particular, the category of pure polarizable Hodge structures of weight n is a semisimple abelian category.*

For a proof, see [**188**, Corollary 2.12]. We also have the following facts about Ext-groups in $\mathrm{MHS}(\Bbbk)$. See [**188**, Example 3.34 and Proposition 3.35].

PROPOSITION 5.6.8. (1) *We have $\mathrm{Ext}^k_{\mathrm{MHS}(\Bbbk)}(M, N) = 0$ for all $k \geq 2$ and all $M, N \in \mathrm{MHS}(\Bbbk)$.*
(2) *We have*

$$\mathrm{Ext}^1_{\mathrm{MHS}(\Bbbk)}(\Bbbk(m), \Bbbk(n)) \cong \begin{cases} \mathbb{C}/(2\pi\sqrt{-1})^{n-m}\Bbbk & \text{if } n > m, \\ 0 & \text{if } n \leq m. \end{cases}$$

One striking aspect of Proposition 5.6.8 is that Ext-groups in $\mathrm{MHS}(\Bbbk)$ may easily be infinite-dimensional as $\Bbbk$-vector spaces.

DEFINITION 5.6.9. Let $M \in \mathrm{MHS}(\Bbbk)$ be a mixed Hodge structure. The kth **Hodge cohomology** of M is the $\Bbbk$-vector space $\mathsf{H}^k_{\mathrm{Hdg}}(M)$ given by

$$\mathsf{H}^k_{\mathrm{Hdg}}(M) = \mathrm{Ext}^k_{\mathrm{MHS}(\Bbbk)}(\Bbbk, M).$$

LEMMA 5.6.10. *For $M \in D^{\mathrm{b}}\mathrm{MHS}(\Bbbk)$, there is a natural short exact sequence*

$$0 \to \mathsf{H}^1_{\mathrm{Hdg}}(\mathsf{H}^{-1}(M)) \to \mathrm{Hom}_{D^{\mathrm{b}}\mathrm{MHS}(\Bbbk)}(\Bbbk, M) \to \mathsf{H}^0_{\mathrm{Hdg}}(\mathsf{H}^0(M)) \to 0.$$

PROOF. It follows from Proposition 5.6.8(1) that

$$\mathrm{Hom}(\Bbbk, N) = 0 \qquad \text{if } N \in D^{\mathrm{b}}\mathrm{MHS}(\Bbbk)^{\leq -2}. \tag{5.6.3}$$

Apply $\mathrm{Hom}(\Bbbk, -)$ to the distinguished triangle $\tau^{\leq -1}M \to M \to \tau^{\geq 0}M \to$ to get the sequence

$$\begin{aligned} \cdots \to \mathrm{Hom}(\Bbbk, (\tau^{\geq 0}M)[-1]) &\to \mathrm{Hom}(\Bbbk, \tau^{\leq -1}M) \\ &\to \mathrm{Hom}(\Bbbk, M) \to \mathrm{Hom}(\Bbbk, \tau^{\geq 0}M) \to \mathrm{Hom}(\Bbbk, (\tau^{\leq -1}M)[1]) \to \cdots. \end{aligned} \tag{5.6.4}$$

The first term obviously vanishes, and the last term vanishes by (5.6.3). By the adjunction relationship for truncation, the fourth term is given by

$$\mathrm{Hom}(\Bbbk, \tau^{\geq 0}M) \cong \mathrm{Hom}(\Bbbk, \tau^{\leq 0}\tau^{\geq 0}M) \cong \mathsf{H}^0_{\mathrm{Hdg}}(\mathsf{H}^0(M)).$$

Finally, applying $\mathrm{Hom}(\Bbbk, -)$ to the distinguished triangle $\tau^{\leq -2}M \to \tau^{\leq -1}M \to \mathsf{H}^{-1}(M)[1] \to$ and using (5.6.3) again, we can identify the second term of (5.6.4) with $\mathrm{Hom}(\Bbbk, \mathsf{H}^{-1}(M)[1]) \cong \mathsf{H}^1_{\mathrm{Hdg}}(\mathsf{H}^{-1}(M))$. ∎

Mixed Hodge modules. We will now give a brief survey of some of the highlights of Saito's theory of mixed Hodge modules, developed in [**199**–**201**]. Let X be a smooth complex variety, and let $\Bbbk$ be a subfield of $\mathbb{R}$. Consider a quadruple $(\mathcal{M}, F, \mathcal{K}, W)$, where:

- $\mathcal{M}$ is a regular holonomic $\mathscr{D}_X$-module;
- $F = F_\bullet\mathcal{M}$ is a good filtration on $\mathcal{M}$, called the **Hodge filtration**;
- $\mathcal{K} \in \mathrm{Perv}(X, \Bbbk)$ is a perverse sheaf equipped with an isomorphism
$$\mathbb{C} \otimes_{\Bbbk} \mathcal{K} \cong \mathrm{dR}(\mathcal{M});$$
- $W = W_\bullet\mathcal{K}$ is an increasing filtration on $\mathcal{K}$, called the **weight filtration**.

Quadruples of this form that satisfy certain additional conditions are called **mixed Hodge modules**. These additional conditions are rather complicated and depend on a delicate induction on the dimension of X, as well as on several related auxiliary notions, including **pure (polarizable) Hodge modules**, and on pure and mixed **variations of Hodge structure**. These topics are well beyond the scope of this book. For a guide to the development of this notion, see [**188**, Chapter 14].

Here are some key features of mixed Hodge modules:

(1) Mixed Hodge modules form a finite-length abelian category, denoted by $\mathrm{MHM}(X, \Bbbk)$. There is an exact, faithful functor
$$\mathrm{rat} : \mathrm{MHM}(X, \Bbbk) \to \mathrm{Perv}(X, \Bbbk)$$
given by $\mathrm{rat}(\mathcal{M}, F, \mathcal{K}, W) = \mathcal{K}$. It gives rise to a functor of derived categories $\mathrm{rat} : D^{\mathrm{b}}\mathrm{MHM}(X, \Bbbk) \to D^{\mathrm{b}}\mathrm{Perv}(X, \Bbbk)$. By composing this with the equivalence $D^{\mathrm{b}}\mathrm{Perv}(X, \Bbbk) \cong D^{\mathrm{b}}_{\mathrm{c}}(X, \Bbbk)$ from Beilinson's theorem (Theorem 4.5.9), we regard rat as a functor
$$\mathrm{rat} : D^{\mathrm{b}}\mathrm{MHM}(X, \Bbbk) \to D^{\mathrm{b}}_{\mathrm{c}}(X, \Bbbk).$$

(2) For $X = \mathrm{pt}$, the category $\mathrm{MHM}(X, \Bbbk)$ can be identified with the category $\mathrm{MHS}(\Bbbk)$ of mixed Hodge structures.

(3) Let $\mathcal{F} = (\mathcal{M}, F, \mathcal{K}, W)$ be a mixed Hodge module. The weight filtration W on $\mathcal{K}$ determines a canonical filtration $W_\bullet\mathcal{F}$ in $\mathrm{MHM}(X, \Bbbk)$, again called the **weight filtration**. Moreover, every morphism in $\mathrm{MHM}(X)$ is strictly compatible with this filtration.

(4) Let Z be a (not necessarily smooth) complex variety, and choose a closed embedding $i : Z \hookrightarrow X$. Let $\mathrm{MHM}_Z(X, \Bbbk)$ be the full subcategory of $\mathrm{MHM}(X, \Bbbk)$ consisting of objects supported on Z. This abelian category is independent (up to equivalence) of the choice of embedding $i : Z \hookrightarrow X$.

The last point above makes it possible to speak of $\mathrm{MHM}(X, \Bbbk)$ even when X is not smooth.

A mixed Hodge module $\mathcal{F} \in \mathrm{MHM}(X, \Bbbk)$ is said to **have weights** $\leq w$ if $\mathcal{F} = W_w\mathcal{F}$. It is said to **have weights** $\geq w$ if $W_{w-1}\mathcal{F} = 0$. It is **pure** of weight w if it has weights $\leq w$ and $\geq w$.

EXAMPLE 5.6.11. It is a nontrivial theorem to show that on a smooth variety X, the shifted constant sheaf $\underline{\Bbbk}_X[\dim X]$ comes from a mixed Hodge module. In fact, the results of [**199**] imply that there is a canonical way to upgrade it to a mixed Hodge module that is pure of weight $\dim X$. More generally, if X is irreducible, then the intersection cohomology complex (cf. Remark 3.3.10)
$$\mathrm{IC}(X; \Bbbk)$$

can be upgraded in a canonical way to a mixed Hodge module of weight $\dim X$. As in Example 5.4.13, this mixed Hodge module is not self-dual, but if $\dim X$ is even, then $\mathrm{IC}(X;\Bbbk)(\frac{1}{2}\dim X)$ is self-dual.

THEOREM 5.6.12. *Let $f : X \to Y$ be a morphism of complex varieties. There are functors*

$$\begin{aligned} f_*, f_! &: D^{\mathrm{b}}\mathrm{MHM}(X,\Bbbk) \to D^{\mathrm{b}}\mathrm{MHM}(Y,\Bbbk), \\ f^*, f^! &: D^{\mathrm{b}}\mathrm{MHM}(Y,\Bbbk) \to D^{\mathrm{b}}\mathrm{MHM}(X,\Bbbk), \\ \overset{L}{\otimes} &: D^{\mathrm{b}}\mathrm{MHM}(X,\Bbbk) \times D^{\mathrm{b}}\mathrm{MHM}(X,\Bbbk) \to D^{\mathrm{b}}\mathrm{MHM}(X,\Bbbk), \\ R\mathscr{H}om &: D^{\mathrm{b}}\mathrm{MHM}(X,\Bbbk)^{\mathrm{op}} \times D^{\mathrm{b}}\mathrm{MHM}(X,\Bbbk) \to D^{\mathrm{b}}\mathrm{MHM}(X,\Bbbk) \end{aligned}$$

that commute with rat. *They satisfy analogues of the various natural isomorphisms from Chapters* 1 *and* 2.

DEFINITION 5.6.13. Let X be a complex variety. The (derived) **geometric Hom functor** is the functor

$$R\underline{\mathrm{Hom}} : D^{\mathrm{b}}\mathrm{MHM}(X,\Bbbk)^{\mathrm{op}} \times D^{\mathrm{b}}\mathrm{MHM}(X,\Bbbk) \to D^{\mathrm{b}}\mathrm{MHS}(\Bbbk)$$

given by

$$R\underline{\mathrm{Hom}}(\mathcal{F},\mathcal{G}) = \mathrm{a}_{X*}R\mathscr{H}om(\mathcal{F},\mathcal{G}).$$

For $\mathcal{F},\mathcal{G} \in D^{\mathrm{b}}\mathrm{MHM}(X,\Bbbk)$, we also put

$$\underline{\mathrm{Hom}}(\mathcal{F},\mathcal{G}) = \mathsf{H}^0(R\underline{\mathrm{Hom}}(\mathcal{F},\mathcal{G})) \qquad \text{and} \qquad \underline{\mathbf{H}}^k(X,\mathcal{G}) = \mathsf{H}^k(R\underline{\mathrm{Hom}}(\underline{\Bbbk}_X,\mathcal{G})).$$

EXAMPLE 5.6.14. Let $X = \mathbb{C}^n$. In the mixed Hodge module setting (and much like in Examples 2.2.5 and 5.1.19), we have $\mathrm{a}_{X!}\underline{\Bbbk}_X \cong \underline{\Bbbk}_{\mathrm{pt}}[-2n](-n)$, and hence

$$\underline{\mathbf{H}}^i_{\mathrm{c}}(\mathbb{C}^n;\Bbbk) \cong \begin{cases} \underline{\Bbbk}(-n) & \text{if } i = 2n, \\ 0 & \text{otherwise.} \end{cases}$$

As in Example 5.1.19, the Tate twist above cannot be omitted. Similarly, the Tate twists in the mixed Hodge module analogues of Theorem 2.2.9 and related statements cannot be omitted.

REMARK 5.6.15. By Theorem 5.6.12, for $\mathcal{F},\mathcal{G} \in D^{\mathrm{b}}\mathrm{MHM}(X,\Bbbk)$, there is a natural isomorphism

$$\mathrm{rat}\, R\underline{\mathrm{Hom}}(\mathcal{F},\mathcal{G}) \cong R\mathrm{Hom}(\mathrm{rat}(\mathcal{F}),\mathrm{rat}(\mathcal{G})).$$

Since rat is t-exact, applying H^0 yields

$$\mathrm{rat}\,\underline{\mathrm{Hom}}(\mathcal{F},\mathcal{G}) \cong \mathrm{Hom}(\mathrm{rat}(\mathcal{F}),\mathrm{rat}(\mathcal{G})).$$

PROPOSITION 5.6.16. *Let X be a complex variety. For $\mathcal{F},\mathcal{G} \in D^{\mathrm{b}}\mathrm{MHM}(X,\Bbbk)$, there is a natural short exact sequence*

$$0 \to \mathsf{H}^1_{\mathrm{Hdg}}(\underline{\mathrm{Hom}}(\mathcal{F},\mathcal{G}[-1])) \to \mathrm{Hom}(\mathcal{F},\mathcal{G}) \to \mathsf{H}^0_{\mathrm{Hdg}}(\underline{\mathrm{Hom}}(\mathcal{F},\mathcal{G})) \to 0.$$

PROOF. By adjunction, we have

$$\begin{aligned} \mathrm{Hom}(\mathcal{F},\mathcal{G}) &\cong \mathrm{Hom}(\underline{\Bbbk}_X, R\mathscr{H}om(\mathcal{F},\mathcal{G})) \\ &\cong \mathrm{Hom}(\underline{\Bbbk}_{\mathrm{pt}}, \mathrm{a}_{X*}R\mathscr{H}om(\mathcal{F},\mathcal{G})) \cong \mathrm{Hom}(\underline{\Bbbk}_{\mathrm{pt}}, R\underline{\mathrm{Hom}}(\mathcal{F},\mathcal{G})). \end{aligned}$$

The last Hom-group is computed in the derived category of mixed Hodge structures. The proposition now follows from Lemma 5.6.10. ☐

Definition 5.6.17. Let $\mathcal{F} \in D^{\mathrm{b}}\mathrm{MHM}(X)$. The complex $\mathcal{F}$ is said to **have weights** $\leq w$ if each cohomology object $\mathsf{H}^i(\mathcal{F})$ has weights $\leq w+i$. It is said to **have weights** $\geq w$ if each cohomology object $\mathsf{H}^i(\mathcal{F})$ has weights $\geq w+i$. The full subcategory of $D^{\mathrm{b}}\mathrm{MHM}(X)$ consisting of objects with weights $\leq w$, resp. $\geq w$, is denoted by

$$D^{\mathrm{b}}_{\leq w}\mathrm{MHM}(X), \qquad \text{resp.} \qquad D^{\mathrm{b}}_{\geq w}\mathrm{MHM}(X).$$

Finally, $\mathcal{F}$ is **pure** of weight w if it has weights $\leq w$ and $\geq w$.

It is immediate from the definition that

$$D^{\mathrm{b}}_{\leq w}\mathrm{MHM}(X,\Bbbk)[1] = D^{\mathrm{b}}_{\leq w+1}\mathrm{MHM}(X,\Bbbk),$$
$$D^{\mathrm{b}}_{\geq w}\mathrm{MHM}(X,\Bbbk)[1] = D^{\mathrm{b}}_{\geq w+1}\mathrm{MHM}(X,\Bbbk).$$

The following major theorem is the main result of [**201**].

Theorem 5.6.18. *Let $f : X \to Y$ be a morphism of complex varieties.*

(1) *The functors f^* and $f_!$ take complexes with weights $\leq w$ to complexes with weights $\leq w$.*
(2) *The functors f_* and $f^!$ take complexes with weights $\geq w$ to complexes with weights $\geq w$.*
(3) *If $\mathcal{F} \in D^{\mathrm{b}}_{\leq w}\mathrm{MHM}(X,\Bbbk)$ and $\mathcal{G} \in D^{\mathrm{b}}_{\leq v}\mathrm{MHM}(X,\Bbbk)$, then*

$$\mathcal{F} \overset{L}{\otimes} \mathcal{G} \in D^{\mathrm{b}}_{\leq v+w}\mathrm{MHM}(X,\Bbbk).$$

(4) *If $\mathcal{F} \in D^{\mathrm{b}}_{\leq w}\mathrm{MHM}(X,\Bbbk)$ and $\mathcal{G} \in D^{\mathrm{b}}_{\geq v}\mathrm{MHM}(X,\Bbbk)$, then*

$$R\mathscr{H}om(\mathcal{F},\mathcal{G}) \in D^{\mathrm{b}}_{\geq v-w}\mathrm{MHM}(X,\Bbbk).$$

Corollary 5.6.19. *Let $\mathcal{F} \in D^{\mathrm{b}}\mathrm{MHM}(X,\Bbbk)$ and $\mathcal{G} \in D^{\mathrm{b}}\mathrm{MHM}(Y,\Bbbk)$.*

(1) *If $\mathcal{F} \in D^{\mathrm{b}}_{\leq w}\mathrm{MHM}(X,\Bbbk)$ and $\mathcal{G} \in D^{\mathrm{b}}_{\leq v}\mathrm{MHM}(Y,\Bbbk)$, then*

$$\mathcal{F} \boxtimes \mathcal{G} \in D^{\mathrm{b}}_{\leq v+w}\mathrm{MHM}(X \times Y,\Bbbk).$$

(2) *If $\mathcal{F} \in D^{\mathrm{b}}_{\geq w}\mathrm{MHM}(X,\Bbbk)$ and $\mathcal{G} \in D^{\mathrm{b}}_{\geq v}\mathrm{MHM}(Y,\Bbbk)$, then*

$$\mathcal{F} \boxtimes \mathcal{G} \in D^{\mathrm{b}}_{\geq v+w}\mathrm{MHM}(X \times Y,\Bbbk).$$

Proof. Identical to Corollary 5.4.8. □

Corollary 5.6.20. *Let $f : X \to Y$ be a morphism of complex varieties.*

(1) *If f is proper, then f_* takes pure complexes of weight w to pure complexes of weight w.*
(2) *If f is smooth, then f^* takes pure complexes of weight w to pure complexes of weight w.*

Compared to the ℓ-adic setting, the behavior of weights of mixed Hodge modules is a little bit "better." For instance, the following vanishing statement is stronger than its ℓ-adic analogue (Lemma 5.4.14).

Lemma 5.6.21. *Let $\mathcal{F},\mathcal{G} \in D^{\mathrm{b}}\mathrm{MHM}(X,\Bbbk)$. If $\mathcal{F}$ has weights $\leq w$ and $\mathcal{G}$ has weights $\geq w+1$, then* $\mathrm{Hom}(\mathcal{F},\mathcal{G}) = 0$.

PROOF. Theorem 5.6.18 implies that $R\underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{G})$ has weights ≥ 1. Therefore, $\underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{G})$ has weights ≥ 1 and $\underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{G}[-1])$ has weights ≥ 0. Considering the weight filtration in $\mathrm{MHS}(\Bbbk)$, we see that

$$\mathsf{H}^0_{\mathrm{Hdg}}(\underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{G})) = \mathrm{Hom}_{\mathrm{MHS}(\Bbbk)}(\Bbbk, \underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{G})) = 0,$$

while Proposition 5.6.7 implies that

$$\mathsf{H}^1_{\mathrm{Hdg}}(\underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{G}[-1])) = \mathrm{Ext}^1_{\mathrm{MHS}(\Bbbk)}(\Bbbk, \underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{G}[-1])) = 0.$$

By Proposition 5.6.16, we are done. □

As a corollary, we obtain the following generalization of Proposition 5.6.7.

COROLLARY 5.6.22. *Let $\mathcal{F}, \mathcal{G} \in \mathrm{MHM}(X, \Bbbk)$. If $\mathcal{F}$ has weights $\leq w$ and $\mathcal{G}$ has weights $\geq w$, then $\mathrm{Ext}^1(\mathcal{F}, \mathcal{G}) = 0$. In particular, the category of pure Hodge structures of a given weight is semisimple.*

Again, this statement is stronger than its ℓ-adic analogue. Indeed, the category of pure ℓ-adic perverse sheaves is *not* semisimple; see Exercise 5.4.2.

We conclude this section by briefly mentioning the mixed Hodge module counterparts of Proposition 5.4.18 and Theorem 5.4.19.

PROPOSITION 5.6.23. *If $\mathcal{F} \in \mathrm{MHM}(X, \Bbbk)$ is a pure Hodge module, then the perverse sheaf $\mathrm{rat}(\mathcal{F}) \in \mathrm{Perv}(X, \Bbbk)$ is semisimple.*

This statement, which is part of the results of [**199**], ultimately comes from the fact that the underlying local system of a polarizable variation of Hodge structure is semisimple. The latter statement is due to Deligne [**59**], Griffiths [**90**], and Schmid [**204**]. See [**61**, Section 1.12] for an overview.

THEOREM 5.6.24 (Semisimplicity). *If $\mathcal{F} \in D^{\mathrm{b}}\mathrm{MHM}(X, \Bbbk)$ is pure, then the object $\mathrm{rat}(\mathcal{F}) \in D^{\mathrm{b}}_{\mathrm{c}}(X, \Bbbk)$ is a semisimple complex.*

The proof is similar to that of Theorem 5.4.19, substituting Lemma 5.6.21 for Lemma 5.4.14.

5.7. Further topics around purity

We conclude this chapter with a discussion of various topics related to the notion of pure complexes. Most statements in this section have both ℓ-adic and mixed Hodge module versions; we will treat these cases simultaneously. In the mixed Hodge module case, we will state the results only for $\Bbbk = \mathbb{Q}$, but they hold for any subfield of $\mathbb{R}$.

Pointwise purity. In Definition 5.4.2, we defined the notion of pointwise purity for a constructible sheaf $\mathcal{F} \in \mathrm{Sh}_{\mathrm{c}}(X_0, \overline{\mathbb{Q}}_\ell)$. The following generalization is sometimes useful.

DEFINITION 5.7.1.

(1) Let X_0 be a variety over a finite field $\mathbb{F}_q$. An object $\mathcal{F} \in D^{\mathrm{b}}_{\mathrm{m}}(X_0, \overline{\mathbb{Q}}_\ell)$ is said to be **pointwise pure** of weight w if for every $n \geq 1$ and every $\mathbb{F}_{q^n}$-point x of X_0, the stalk $\mathcal{F}_x \in D^{\mathrm{b}}_{\mathrm{m}}(\mathrm{Spec}\,\mathbb{F}_{q^n}, \overline{\mathbb{Q}}_\ell)$ is pure of weight w.
(2) Let X be a complex variety. An object $\mathcal{F} \in D^{\mathrm{b}}\mathrm{MHM}(X, \mathbb{Q})$ is said to be **pointwise pure** of weight w if for every point $x \in X$, the stalk $\mathcal{F}_x \in D^{\mathrm{b}}\mathrm{MHM}(\mathrm{pt}, \mathbb{Q})$ is pure of weight w.

(In the ℓ-adic case, since x is not a geometric point, the word "stalk" here is not quite the same as in Definition 5.1.5. Instead, the point x is a morphism $x : \operatorname{Spec} \mathbb{F}_{q^n} \to X_0$, so it gives rise to a functor $x^* : D^{\mathrm{b}}_{\mathrm{m}}(X_0, \overline{\mathbb{Q}}_\ell) \to D^{\mathrm{b}}_{\mathrm{m}}(\operatorname{Spec} \mathbb{F}_{q^n}, \overline{\mathbb{Q}}_\ell)$. We write $\mathcal{F}_x$ as a shorthand for $x^*\mathcal{F}$.)

In general, purity and pointwise purity are independent concepts; neither one implies the other. However, in applications in geometric representation theory, it often turns out that the simple perverse sheaves of interest are pointwise pure. (See Sections 7.4, 10.5, and 10.6, and Exercise 8.6.2.) Furthermore, this pointwise purity is typically a key fact that is needed for extracting representation-theoretic information from these perverse sheaves (say, via the sheaf–function correspondence).

Below, we give a criterion for pointwise purity in the equivariant setting. The statement and proof of Proposition 5.7.3 require some familiarity with notions from Chapter 6.

Definition 5.7.2. Let G be an algebraic group, and let X be an irreducible G-variety. Let $x \in X$, and let $C = G \cdot x$ be its orbit. A **transverse slice** to C at x is a locally closed subvariety $S \subset X$ with the following properties:

(1) S contains x.
(2) The action map $\sigma : G \times S \to X$ is a smooth morphism.
(3) We have $\dim X = \dim C + \dim S$.

We say that S admits an **attracting cocharacter** if there is a homomorphism $\lambda : \mathbb{G}_{\mathrm{m}} \to G$ such that S is preserved by $\lambda(\mathbb{G}_{\mathrm{m}})$ and such that the resulting $\mathbb{G}_{\mathrm{m}}$-action on S is attracting (cf. Definition 2.10.2), with x as its unique fixed point.

If there is a transverse slice to C at one point, then of course the G-action gives transverse slices to all other points. Thus, the existence of a transverse slice is a property of the orbit, not of an individual point. Similarly, the existence of a transverse slice with an attracting cocharacter is a property of the orbit. The following pointwise purity result is due to Springer [**229**] (see also [**168**, Proposition 2.3.3]).

Proposition 5.7.3. *Let G be an algebraic group, and let X be an irreducible G-variety. Assume that every orbit admits a transverse slice with an attracting cocharacter. If $\mathcal{F} \in \mathrm{Perv}_{\mathrm{m}}(X, \overline{\mathbb{Q}}_\ell)$ or $\mathcal{F} \in \mathrm{MHM}(X, \mathbb{Q})$ is pure and G-equivariant, then it is pointwise pure.*

Proof sketch. Let $x \in X$. Let $C = G \cdot x$ be its orbit, and let S be a transverse slice to C at x. Let G act on $G \times S$ by $g \cdot (h, s) = (gh, s)$. The action map $\sigma : G \times S \to X$ is G-equivariant and smooth of relative dimension $\dim G - \dim C$, so $\sigma^*\mathcal{F}[\dim G - \dim C]$ is G-equivariant, perverse, and pure.

Next, let $e : S \to G \times S$ be the map given by $e(s) = (1, s)$. By Proposition 6.2.11, $e^*[-\dim G]$ is an equivalence of categories between G-equivariant perverse sheaves on $G \times S$ and all perverse sheaves on S. In particular, $e^*[-\dim G]$ takes pure perverse sheaves to pure perverse sheaves.

Let $f : S \hookrightarrow X$ be the inclusion map. Since $f^* \cong e^* \circ \sigma^*$, the preceding paragraphs imply that $f^*\mathcal{F}[-\dim C]$ is perverse and pure. The map f is $\mathbb{G}_{\mathrm{m}}$-equivariant with respect to the action of an attracting cocharacter $\lambda : \mathbb{G}_{\mathrm{m}} \to G$, so $f^*\mathcal{F}[-\dim C]$ is $\mathbb{G}_{\mathrm{m}}$-equivariant.

Let $i : \{x\} \to S$ be the inclusion map, and let $p : S \to \{x\}$ be the constant map. By Theorem 2.10.3, we have an isomorphism

$$p_* f^*\mathcal{F}[-\dim C] \xrightarrow{\sim} i^* f^*\mathcal{F}[-\dim C].$$

Suppose $f^*\mathcal{F}[-\dim C]$ is pure of weight w. Then the first term above has weights $\geq w$ and the second term has weights $\leq w$, so both terms must in fact be pure. In particular, $i^*f^*\mathcal{F} \cong \mathcal{F}_x$ is pure, so $\mathcal{F}$ is pointwise pure. □

Objects of Tate type. The "nicest" pure objects on a point are those of the following form.

DEFINITION 5.7.4. (1) An object $\mathcal{F} \in D^{\mathrm{b}}_{\mathrm{m}}(\operatorname{Spec}\mathbb{F}_q, \overline{\mathbb{Q}}_\ell)$ is of **Tate type** with weight w if

$$\mathsf{H}^i(\mathcal{F}) \cong \begin{cases} 0 & \text{if } i \not\equiv w \pmod 2, \\ \text{a direct sum of copies of } \overline{\mathbb{Q}}_\ell(-\frac{w+i}{2}) & \text{if } i \equiv w \pmod 2. \end{cases}$$

(2) An object $M \in D^{\mathrm{b}}\mathrm{MHS}(\mathrm{pt}, \mathbb{Q})$ is of **Tate type** with weight w if

$$\mathsf{H}^i(M) \cong \begin{cases} 0 & \text{if } i \not\equiv w \pmod 2, \\ \text{a direct sum of copies of } \mathbb{Q}(-\frac{w+i}{2}) & \text{if } i \equiv w \pmod 2. \end{cases}$$

It is sometimes useful to know whether the stalks or the hypercohomology of some mixed object are of Tate type. Requiring stalks of Tate type is a very strong form of pointwise purity: instead of constraining the absolute values of the eigenvalues of Fr_{q^n}, one requires that Fr_{q^n} act on $\mathsf{H}^i(\mathcal{F}_x)$ by the scalar $q^{n(w+i)/2}$. For examples of objects with stalks of Tate type, see Theorem 7.4.8 and Exercise 8.6.2.

The proposition below describes one situation in which the cohomology of a variety is of Tate type. (For the notion of an affine paving, see the discussion following Definition 2.3.4.)

PROPOSITION 5.7.5. (1) *Let X_0 be a projective variety over $\mathbb{F}_q$. If X_0 admits an affine paving, then $\underline{\mathbf{H}}^\bullet(X_0; \overline{\mathbb{Q}}_\ell)$ is of Tate type with weight* 0.

(2) *Let X be a complex projective variety. If X admits an affine paving, then $\underline{\mathbf{H}}^\bullet(X; \mathbb{Q})$ is of Tate type with weight* 0.

PROOF. We will prove this in the ℓ-adic case. See the end of the proof for brief comments on the mixed Hodge module case.

We proceed by induction on the number of strata in the affine paving. If X_0 consists of a single stratum X_s, then, since it is projective, the stratum X_s must be a 0-dimensional affine space $\mathbb{A}^0 \cong \operatorname{Spec}\mathbb{F}_q$, and we trivially have $\mathrm{a}_{X_0*}\underline{\overline{\mathbb{Q}}_\ell}_{X_0} \cong \overline{\mathbb{Q}}_\ell$.

Otherwise, let $X_u \subset X_0$ be an open stratum, say of dimension d. Let $Z_0 = X_0 \smallsetminus X_u$. The remaining strata give an affine paving of Z. Since Z_0 is a closed subvariety of X_0, it is projective, so by induction, $\mathrm{a}_{Z_0*}\underline{\overline{\mathbb{Q}}_\ell}_{Z_0}$ is of Tate type. Let $j : X_u \to X_0$ and $i : Z_0 \to X_0$ be the inclusion maps, and consider the distinguished triangle $j_!\underline{\overline{\mathbb{Q}}_\ell}_{X_u} \to \underline{\overline{\mathbb{Q}}_\ell}_{X_0} \to i_*\underline{\overline{\mathbb{Q}}_\ell}_{Z_0} \to$. Apply $\mathrm{a}_{X_0*} \cong \mathrm{a}_{X_0!}$ and then take the long exact sequence in cohomology:

$$\cdots \to \underline{\mathbf{H}}^i_{\mathrm{c}}(\mathbb{A}^d; \overline{\mathbb{Q}}_\ell) \to \underline{\mathbf{H}}^i(X_0; \overline{\mathbb{Q}}_\ell) \to \underline{\mathbf{H}}^i(Z_0; \overline{\mathbb{Q}}_\ell) \to \underline{\mathbf{H}}^{i+1}_{\mathrm{c}}(\mathbb{A}^d; \overline{\mathbb{Q}}_\ell) \to \cdots.$$

Using Example 5.1.19 and induction, we see that

$$\underline{\mathbf{H}}^i(X_0; \overline{\mathbb{Q}}_\ell) \cong \underline{\mathbf{H}}^i(Z_0; \overline{\mathbb{Q}}_\ell) \qquad \text{for } i \neq 2d.$$

On the other hand, for $i = 2d$, we have a short exact sequence

$$0 \to \underline{\mathbf{H}}^{2d}_{\mathrm{c}}(X_u; \overline{\mathbb{Q}}_\ell) \to \underline{\mathbf{H}}^i(X_0; \overline{\mathbb{Q}}_\ell) \to \underline{\mathbf{H}}^{2d}(Z_0; \overline{\mathbb{Q}}_\ell) \to 0. \tag{5.7.1}$$

Since the first and last terms of this sequence are both isomorphic to direct sums of copies of $\overline{\mathbb{Q}}_\ell(-d)$, to finish the proof, we must show that (5.7.1) splits.

To do this, let us consider the variety $Y_0 = \overline{X_u}$. Let $W_0 = Y_0 \smallsetminus X_u$. Then we have an open embedding $h : X_u \hookrightarrow Y_0$ and a closed embedding $W_0 \hookrightarrow Y_0$. The same reasoning as above leads to a long exact sequence

$$\cdots \to \underline{\mathbf{H}}^i_{\mathrm{c}}(X_u; \overline{\mathbb{Q}}_\ell) \to \underline{\mathbf{H}}^i(Y_0; \overline{\mathbb{Q}}_\ell) \to \underline{\mathbf{H}}^i(W_0; \overline{\mathbb{Q}}_\ell) \to \underline{\mathbf{H}}^{i+1}_{\mathrm{c}}(X_u; \overline{\mathbb{Q}}_\ell) \to \cdots.$$

But this time, we have $\dim W_0 < d$, so $\underline{\mathbf{H}}^{2d}(W_0; \overline{\mathbb{Q}}_\ell) = 0$, and the natural map

$$\underline{\mathbf{H}}^{2d}_{\mathrm{c}}(X_u; \overline{\mathbb{Q}}_\ell) \xrightarrow{\sim} \underline{\mathbf{H}}^{2d}(Y_0; \overline{\mathbb{Q}}_\ell) \tag{5.7.2}$$

is an isomorphism.

Now let $y : Y_0 \hookrightarrow X_0$ be the inclusion map, and consider the commutative diagram

$$\begin{tikzcd} j_! \underline{\overline{\mathbb{Q}}_\ell}_{X_u} \arrow[r, "\sim"] \arrow[d] & y_* h_! \underline{\overline{\mathbb{Q}}_\ell}_{X_u} \arrow[d] \\ \underline{\overline{\mathbb{Q}}_\ell}_{X_0} \arrow[r] & y_* \underline{\overline{\mathbb{Q}}_\ell}_{Y_0} \end{tikzcd}$$

Apply $\underline{\mathbf{H}}^{2d}(X_0, -)$ to this diagram to obtain

$$\begin{tikzcd} \underline{\mathbf{H}}^{2d}_{\mathrm{c}}(X_u; \overline{\mathbb{Q}}_\ell) \arrow[r, "\sim"] \arrow[d] & \underline{\mathbf{H}}^{2d}_{\mathrm{c}}(X_u; \overline{\mathbb{Q}}_\ell) \arrow[d, "\wr"] \\ \underline{\mathbf{H}}^{2d}(X_0; \overline{\mathbb{Q}}_\ell) \arrow[r] & \underline{\mathbf{H}}^{2d}(Y_0; \overline{\mathbb{Q}}_\ell) \end{tikzcd}$$

Here, the left-hand vertical arrow is the first map in (5.7.1), and the right-hand vertical arrow is the isomorphism from (5.7.2). This diagram provides a splitting of (5.7.1), as desired.

In the mixed Hodge module case, the proof begins the same way, but as soon as we get to (5.7.1), we are done: the sequence splits automatically by Proposition 5.6.7. $\square$

REMARK 5.7.6. It is sometimes convenient to work with objects that behave as though they were of Tate type, but that have cohomology in degrees $\not\equiv w \pmod 2$. To make this precise, we must introduce the **square root of the Tate sheaf**.

In the ℓ-adic case, we literally choose a square root $q^{1/2}$ of q in the field $\overline{\mathbb{Q}}_\ell$. There exists a continuous $\overline{\mathbb{Q}}_\ell[\mathrm{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q)]$-module of dimension 1 on which $\mathsf{Fr}_q^{\mathrm{arith}}$ acts by multiplication by $q^{1/2}$ (cf. Example 5.2.10). Let $\overline{\mathbb{Q}}_\ell(\frac{1}{2})$ denote the corresponding lisse sheaf on $\operatorname{Spec} \mathbb{F}_q$ via Theorem 5.2.9. This is a pure sheaf of weight -1. An object $M \in D^{\mathrm{b}}_{\mathrm{m}}(\operatorname{Spec} \mathbb{F}, \overline{\mathbb{Q}}_\ell)$ is of (possibly half-integral) Tate type if

$$\mathsf{H}^i(M) \cong \text{a direct sum of copies of } \overline{\mathbb{Q}}_\ell(-\tfrac{w+i}{2}) \qquad \text{for all } i.$$

In the mixed Hodge module case, introducing a square root of the Tate twist requires enlarging the category we are working in. For any complex variety X, let

$$D^{\mathrm{b}}\mathrm{MHM}(X, \mathbb{Q})' = D^{\mathrm{b}}\mathrm{MHM}(X, \mathbb{Q}) \oplus D^{\mathrm{b}}\mathrm{MHM}(X, \mathbb{Q}).$$

For $\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2) \in D^{\mathrm{b}}\mathrm{MHM}(X, \mathbb{Q})'$, we define the half-integer Tate twist by

$$\mathcal{F}(\tfrac{1}{2}) = (\mathcal{F}_2(1), \mathcal{F}_1).$$

It is left to the reader to determine how to define sheaf functors on the category $D^{\mathrm{b}}\mathrm{MHM}(X,\mathbb{Q})'$. An object $M \in D^{\mathrm{b}}\mathrm{MHM}(\mathrm{pt},\mathbb{Q})'$ is said to be of (possibly half-integral) Tate type if

$$\mathsf{H}^i(M) \cong \text{a direct sum of copies of } \mathbb{Q}(-\tfrac{w+i}{2}) \qquad \text{for all } i.$$

In both the ℓ-adic and mixed Hodge module cases, introducing a square root of the Tate sheaf makes it possible to work with self-dual intersection cohomology complexes $\mathrm{IC}(X)(\frac{1}{2}\dim X)$ for any variety X, rather than just those of even dimension (see Examples 5.4.13 and 5.6.11).

Applications of purity. Any ordinary (i.e., nonmixed) constructible complex that is obtained by applying egf or rat to a pure complex must be semisimple, by Theorems 5.4.19 and 5.6.24. We conclude this chapter with two applications of this observation. The first application involves the decomposition theorem (Section 3.9) and requires the following definition.

Definition 5.7.7. A simple perverse $\overline{\mathbb{Q}}_\ell$-sheaf $\mathcal{F}$ on a variety over $\bar{\mathbb{F}}_p$ is said to be **of geometric origin** if it can be obtained from the constant sheaf $\underline{\overline{\mathbb{Q}}_\ell}_{\mathrm{pt}}$ on a point by a finite sequence of the following operations:

- Given a simple perverse sheaf $\mathcal{F}$, take a simple composition factor of ${}^p\mathsf{H}^i(f_!\mathcal{F})$, ${}^p\mathsf{H}^i(f_*\mathcal{F})$, ${}^p\mathsf{H}^i(f^*\mathcal{F})$, or ${}^p\mathsf{H}^i(f^!\mathcal{F})$ (where f is an appropriate morphism of varieties).
- Given two simple perverse sheaves $\mathcal{F}$ and $\mathcal{G}$, take a simple composition factor of ${}^p\mathsf{H}^i(\mathcal{F} \otimes^L \mathcal{G})$ or ${}^p\mathsf{H}^i(R\mathscr{H}om(\mathcal{F},\mathcal{G}))$.

More generally, a semisimple complex is said to be **of geometric origin** if it is a direct sum of shifts of simple perverse sheaves of geometric origin.

Of course, this definition also makes sense in the setting of perverse $\mathbb{Q}$-sheaves on a complex algebraic variety.

Theorem 5.7.8 (Decomposition theorem). (1) *Let $f : X \to Y$ be a proper morphism of varieties over $\bar{\mathbb{F}}_p$, and let $\mathcal{F} \in \mathrm{Perv}(X,\overline{\mathbb{Q}}_\ell)$ be a simple perverse sheaf. If $\mathcal{F}$ is of geometric origin, then $f_*\mathcal{F}$ is a semisimple complex.*

(2) *Let $f : X \to Y$ be a proper morphism of complex varieties, and let $\mathcal{F} \in \mathrm{Perv}(X,\mathbb{Q})$ be a simple perverse sheaf. If there exists a simple mixed Hodge module $\tilde{\mathcal{F}} \in \mathrm{MHM}(X,\mathbb{Q})$ such that $\mathcal{F}$ is a direct summand of* $\mathrm{rat}\,\tilde{\mathcal{F}}$*, then $f_*\mathcal{F}$ is a semisimple complex.*

Proof sketch. (1) The idea is to "lift" the problem to the mixed setting, i.e., to show that for some prime power $q = p^n$, there is a map of $\mathbb{F}_q$-varieties $f_0 : X_0 \to Y_0$, and a pure perverse sheaf $\mathcal{F}_0 \in \mathrm{Perv}_{\mathrm{m}}(X_0,\overline{\mathbb{Q}}_\ell)$ such that:

- The map $f : X \to Y$ is obtained by extension of the ground field from $f_0 : X_0 \to Y_0$.
- The perverse sheaf $\mathcal{F}$ is a direct summand of $\mathrm{egf}(\mathcal{F}_0)$.

Assuming these statements hold, the object $f_{0*}\mathcal{F}_0$ is pure by Corollary 5.4.9, and then $\mathrm{egf}(f_{0*}\mathcal{F})$ is a semisimple complex by Theorem 5.4.19. Since $f_*\mathcal{F}$ is a direct summand of $\mathrm{egf}(f_{0*}\mathcal{F})$, we are done.

The proof of the existence of $\mathcal{F}_0$ is where the "geometric origin" hypothesis is used: the constant sheaf $\underline{\overline{\mathbb{Q}}_\ell}$ on $\mathrm{Spec}\,\bar{\mathbb{F}}_p$ can certainly be lifted to $\mathrm{Spec}\,\mathbb{F}_q$, and then the property of being able to be lifted to the mixed setting is stable under each of the sheaf operations in Definition 5.7.7.

(2) The object $f_*\tilde{\mathcal{F}}$ is pure by Corollary 5.6.20, and then $\mathrm{rat}(f_*\tilde{\mathcal{F}})$ is a semisimple complex by Theorem 5.6.24. Since $f_*\mathcal{F}$ is a direct summand of $\mathrm{rat}(f_*\tilde{\mathcal{F}})$, we are done. □

Most simple perverse $\overline{\mathbb{Q}}_\ell$- or $\mathbb{Q}$-sheaves that come up in practice (including all those that appear in Chapters 7–10) are of geometric origin. On a complex variety, any simple perverse $\mathbb{Q}$-sheaf of geometric origin automatically satisfies the assumption in Theorem 5.7.8(2).

The second application involves the hyperbolic localization functor from Section 2.10. See Definition 2.10.6 for the definitions of the attracting and repelling varieties X^+ and X^- associated to a $\mathbb{G}_\mathrm{m}$-variety X, and the definitions of the maps

$$i^\pm : X^\pm \to X \qquad \text{and} \qquad p^\pm : X^\pm \to X^{\mathbb{G}_\mathrm{m}}.$$

THEOREM 5.7.9. (1) *Let X be a $\mathbb{G}_\mathrm{m}$-variety over $\bar{\mathbb{F}}_p$. Then the hyperbolic localization of the intersection cohomology complex*

$$p^+_*(i^+)^!\mathrm{IC}(X;\overline{\mathbb{Q}}_\ell) \cong p^-_!(i^-)^*\mathrm{IC}(X;\overline{\mathbb{Q}}_\ell) \in D^\mathrm{b}_\mathrm{c}(X^{\mathbb{G}_\mathrm{m}},\overline{\mathbb{Q}}_\ell)$$

is a semisimple complex.

(2) *Let X be a complex $\mathbb{G}_\mathrm{m}$-variety. Then the hyperbolic localization of the intersection cohomology complex*

$$p^+_*(i^+)^!\mathrm{IC}(X;\mathbb{Q}) \cong p^-_!(i^-)^*\mathrm{IC}(X;\mathbb{Q}) \in D^\mathrm{b}_\mathrm{c}(X^{\mathbb{G}_\mathrm{m}},\mathbb{Q})$$

is a semisimple complex.

PROOF SKETCH. In the ℓ-adic case, the first step is to write down $\mathbb{F}_q$-versions of the various varieties and maps. Denote them by $i_0^\pm : X_0^\pm \to X_0$ and $p_0^\pm : X_0^\pm \to X_0^{\mathbb{G}_\mathrm{m}}$. Then $\mathrm{IC}(X_0;\overline{\mathbb{Q}}_\ell)$ is a simple mixed perverse sheaf, pure of weight $\dim X_0$. By Theorem 5.4.7, $p^+_{0*}(i_0^+)^!\mathrm{IC}(X_0;\overline{\mathbb{Q}}_\ell)$ has weights $\geq \dim X_0$, while $p^-_{0!}(i_0^-)^*\mathrm{IC}(X_0;\overline{\mathbb{Q}}_\ell)$ has weights $\leq \dim X_0$. Since $\mathrm{IC}(X_0;\overline{\mathbb{Q}}_\ell)$ is weakly $\mathbb{G}_\mathrm{m}$-equivariant (by reasoning similar to Exercise 2.10.2), these two objects are isomorphic, and hence pure of weight $\dim X_0$. The hyperbolic localization of $\mathrm{IC}(X;\overline{\mathbb{Q}}_\ell)$ is obtained by applying egf to a pure object, so it is a semisimple complex.

The proof in the mixed Hodge module case is very similar. □

Exercises.

5.7.1. Let X be an irreducible complex projective variety with an affine paving. Show that the natural map $\mathbf{H}^\bullet(X;\mathbb{Q}) \to \mathbf{IH}^\bullet(X;\mathbb{Q})$ (see Exercise 3.10.2) is injective.

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CHAPTER 6

Equivariant derived categories

Suppose X is a variety equipped with an action of an algebraic group G. A "G-equivariant sheaf" on X is, roughly speaking, a sheaf with a compatible action of G. It is not too difficult to make this precise; see Section 6.2 for the definition of $\mathrm{Sh}_G(X, \Bbbk)$.

However, it is much trickier to find the correct notion at the level of derived categories, partly because injective objects in $\mathrm{Sh}_G(X, \Bbbk)$ typically do not remain injective in $\mathrm{Sh}(X, \Bbbk)$. Another perspective is that the correct notion of equivariance at the derived level ought to "see" the topology (via singular cohomology) of G, but this is invisible in the abelian category $\mathrm{Sh}_G(X, \Bbbk)$. See Section 6.4 for a detailed discussion of the difficulties.

The solution to this problem is due to Bernstein–Lunts, and the heart of this chapter (especially Sections 6.4–6.7) is based on their monograph [**27**]. We will see how to define a new triangulated category $D^{\mathrm{b}}_G(X, \Bbbk)$ that has desirable properties, but is typically *not* equivalent to the derived category of $\mathrm{Sh}_G(X, \Bbbk)$: specifically, Hom-groups in $D^{\mathrm{b}}_G(X, \Bbbk)$ tend to be larger than in $D^{\mathrm{b}}\mathrm{Sh}_G(X, \Bbbk)$ or in $D^{\mathrm{b}}(X, \Bbbk)$.

In Section 6.8, we describe an alternative language for some of the main results of the chapter, in Section 6.9, we apply this machinery to develop a kind of Fourier transform for $\mathbb{G}_{\mathrm{m}}$-equivariant sheaves on vector bundles, due to Laumon [**150**].

6.1. Preliminaries on algebraic groups, actions, and quotients

In this book, an **algebraic group** is defined to be an affine variety G equipped with a morphism of varieties $G \times G \to G$ that gives it the structure of a group. (That is, we will never consider group structures on varieties that are not affine.) The term **closed subgroup** should always be understood to mean a subgroup that is closed in the Zariski topology. Thus, a closed subgroup of an algebraic group is again an algebraic group.

It is well known that every algebraic group is smooth (see, for instance, [**36**, Proposition 1.2(a)]) and **linear**, i.e., isomorphic to a closed subgroup of some GL_n (see [**36**, Proposition 1.10] or [**230**, Proposition 2.3.7]). The connected component of G containing e is denoted by G°. This is a normal subgroup of finite index; the connected components of G (which coincide with its irreducible components) are precisely the cosets of G°.

LEMMA 6.1.1. *Let G be an algebraic group, and let d be the largest integer such that $\mathbf{H}_d(G; \mathbb{Z})$ is nonzero. Then $d \leq \dim G$, and $\mathbf{H}_d(G; \mathbb{Z})$ is a free abelian group of rank equal to the number of connected components of G.*

Proof sketch. Since G is an affine variety, we have $d \leq \dim G$ by Exercise 2.11.1. Now regard G as a Lie group. Then it has a maximal compact subgroup $K \subset G$, and the inclusion $K \hookrightarrow G$ is a homotopy equivalence (see [**99**, Theorems 14.1.3 and 14.3.11]). We therefore have $\mathbf{H}_i(G;\mathbb{Z}) \cong \mathbf{H}_i(K;\mathbb{Z})$ for all i. Since K is a compact, orientable manifold (see [**99**, Proposition 10.3.53]), we conclude that d is the (real) dimension of K and that $\mathbf{H}_d(K;\mathbb{Z})$ is free (see [**98**, Theorem 3.26]). □

A G**-variety** is a variety X equipped with an algebraic G-action

$$\sigma : G \times X \to X.$$

(Throughout this chapter, "σ" will be the default notation for the action map of a group G on a G-variety.) If X is a G-variety, then for any point $x \in X$, the orbit $G \cdot x \subset X$ is a smooth, locally closed subvariety, satisfying

$$\dim G \cdot x = \dim G - \dim G^x,$$

where $G^x \subset G$ is the stabilizer of x (which is a closed subgroup). The closure $\overline{G \cdot x}$ is the union of $G \cdot x$ and other orbits of lower dimension. In particular, G-orbits of minimal dimension are closed.

Since G is a smooth variety, the map $\mathrm{pr}_2 : G \times X \to X$ is a smooth morphism of relative dimension $\dim G$. The action map $\sigma : G \times X \to X$ is also smooth of relative dimension $\dim G$: to see this, write σ as $\mathrm{pr}_2 \circ \sigma'$, where $\sigma' : G \times X \to G \times X$ is the isomorphism $\sigma'(g,x) = (g, g \cdot x)$.

Quotients and principal bundles. We now discuss what it means to take a "quotient of a G-variety by the G-action."

Definition 6.1.2. Let G be an algebraic group, and let X be a G-variety. A **geometric quotient** of X by G is a map $\pi : X \to Y$ with the following properties:

(1) We have $\pi \circ \sigma = \pi \circ \mathrm{pr}_2$.
(2) The map π is surjective, and the image of $(\sigma, \mathrm{pr}_2) : G \times X \to X \times X$ is identified with $X \times_Y X$.
(3) A subset $U \subset Y$ is a Zariski open subset if and only if $\pi^{-1}(U)$ is a Zariski open subset of X.
(4) For any Zariski open subset $U \subset Y$, the ring of regular functions $\mathbb{C}[U]$ is identified with the ring $\mathbb{C}[\pi^{-1}(U)]^G$ of G-invariant functions on $\pi^{-1}(U)$.

Let us unpack the parts of this definition. The first condition simply says that π is constant on G-orbits. It follows immediately that the image of (σ, pr_2) is contained in the subset $X \times_Y X$. Requiring these to be equal means that if $\pi(x_1) = \pi(x_2)$, then there exists a $g \in G$ such that $g \cdot x_2 = x_1$. In other words, conditions (1) and (2) together say that π induces a bijection

$$\{G\text{-orbits in } X\} \overset{\sim}{\longleftrightarrow} Y.$$

Condition (3) says that the Zariski topology on Y is the quotient topology.

The first three conditions imply that every G-invariant algebraic function $f : X \to \mathbb{C}$ factors in a unique way through a continuous function $\bar{f} : Y \to \mathbb{C}$—but a priori, $\bar{f}$ is not necessarily algebraic. Condition (4) imposes the requirement that $\bar{f}$ always be algebraic. Geometric quotients satisfy the following universal property.

Lemma 6.1.3. *Let G be an algebraic group, and let X be a G-variety. Suppose X has a geometric quotient $\pi : X \to Y$. Let Z be another variety, and suppose*

$f : X \to Z$ is a morphism of varieties satisfying $f(g \cdot x) = f(x)$ for all $g \in G$ and $x \in X$. Then there exists a unique map $\bar{f} : Y \to Z$ such that $\bar{f} \circ \pi = f$.

For a proof, see [**184**, Proposition 0.1]. As usual, this universal property means that if a geometric quotient exists, it is unique up to canonical isomorphism.

Definition 6.1.4. An action of an algebraic group G on a variety X is said to be **free** if the map $(\sigma, \mathrm{pr}_2) : G \times X \to X \times X$ is a closed embedding.

Remark 6.1.5. Recall that in abstract group theory, a group action on a set X is called "free" if the stabilizer of every element in X is trivial. It is easy to see that Definition 6.1.4 implies this condition, but it is strictly stronger. See [**184**, Example 0.4] for an example an algebraic group action that has trivial stabilizers, but is not free in the sense of Definition 6.1.4.

Definition 6.1.6. Let X be a G-variety. A morphism of varieties $\pi : X \to Y$ is said to be a **principal G-bundle** over Y if the G-action on X is free and if π is a geometric quotient.

We sometimes simply say "X is a principal G-variety" to mean that X is a free G-variety that admits a geometric quotient. In this case, we may denote the quotient map by

$$\pi : X \to X/G.$$

In many sources (e.g., [**45**, **184**]), the definition of "principal bundle" is stated somewhat differently, as expressed in the following statement.

Proposition 6.1.7. *Let X be a G-variety. A morphism of varieties $\pi : X \to Y$ is a principal G-bundle if and only if the following conditions hold:*

(1) *We have $\pi \circ \sigma = \pi \circ \mathrm{pr}_2$.*
(2) *The map π is surjective, and the diagram*

$$\begin{tikzcd} G \times X \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\sigma"'] & X \arrow[d, "\pi"] \\ X \arrow[r, "\pi"] & Y \end{tikzcd}$$

is cartesian. In other words, $(\sigma, \mathrm{pr}_2) : G \times X \to X \times X$ induces an isomorphism $G \times X \xrightarrow{\sim} X \times_Y X$.
(3) *The map π is a flat morphism of varieties.*

For a proof of the "if" direction, see [**45**, Proposition 2.3.3 and Remark 2.3.4]. For a proof of the "only if" direction, see [**184**, Proposition 0.9].

Example 6.1.8. Perhaps the most important example is this: if H is a closed subgroup of G, then the left and right translation actions of H on G each make G into a principal H-variety. The quotients of G by these actions are denoted by

$$H\backslash G \qquad \text{and} \qquad G/H,$$

respectively. For details, see [**36**, Theorem 6.8] or [**230**, Theorem 5.5.5].

Example 6.1.9. Let $X = \mathbb{A}^n \smallsetminus \{0\}$. Let $G = \mathbb{G}_{\mathrm{m}}$ act on X by $z \cdot x = z^k x$ for some integer k. If $k \neq 0$, then the obvious map $\pi : X \to \mathbb{P}^{n-1}$ is a geometric quotient. If $k = \pm 1$, then π is a principal $\mathbb{G}_{\mathrm{m}}$-bundle.

Example 6.1.10. Here is a generalization of Example 6.1.9. Fix two positive integers $0 < k \le m$, and let $W_{m,k}$ denote the variety of injective linear maps $\mathbb{C}^k \to \mathbb{C}^m$. (To see that this set is a variety, observe that it is a Zariski open subset of the set $\mathbb{A}^{mk}$ of all $m \times k$ matrices.) It is known as the **(noncompact) complex Stiefel manifold**.

There is an obvious action of GL_k on $W_{m,k}$, and it is straightforward to check that this action is free. In fact, $W_{m,k}$ is a principal GL_k-variety: the geometric quotient is the Grassmannian $\mathrm{Gr}(k, m)$ of k-dimensional linear subspaces of $\mathbb{C}^m$.

Proposition 6.1.11. *Let $\pi : X \to Y$ be a principal G-bundle. Then π is smooth (of relative dimension* $\dim G$*), affine, and a holomorphic locally trivial fibration.*

Proof sketch. For a proof that π is smooth and affine, see [**45**, Proposition 2.3.3]. Next, Lemma 2.2.1 implies that any point $y \in Y$ has an analytic neighborhood V over which π has an analytic right inverse $s : V \to \pi^{-1}(V)$. Define a map $G \times V \to \pi^{-1}(V)$ by $(g, v) \mapsto g \cdot s(v)$. One can check that this map is a homeomorphism (in fact, a biholomorphism). The commutative diagram below shows that π is a locally trivial fibration:

$$\begin{tikzcd} \pi^{-1}(V) \arrow[r, "\sim"] \arrow[d, "\pi|_{\pi^{-1}(V)}"'] & G \times V \arrow[dl, "\mathrm{pr}_2"] \\ V & \end{tikzcd} \tag{6.1.1}$$

□

Remark 6.1.12. A principal G-bundle $\pi : X \to Y$ is said to be **Zariski locally trivial** if for any point $y \in Y$, one can construct a diagram like (6.1.1) in which V is a Zariski open set containing y, and b is a G-equivariant isomorphism of varieties. (It is a **trivial bundle** if one can take $V = Y$.) Not every principal bundle is Zariski-locally trivial. See Exercise 6.1.1 for a counterexample, and see [**208**, Section 4] for some situations where Zariski-local triviality is guaranteed.

However, every principal bundle is at least **étale-locally trivial**. This means that every point $y \in Y$ admits an étale neighborhood V (see Definition 5.1.4) such that there is a commutative diagram

$$\begin{tikzcd} X \times_Y V \arrow[r, "b", "\sim"'] \arrow[d, "\pi|_{\pi^{-1}(V)}"'] & G \times V \arrow[dl, "\mathrm{pr}_2"] \\ V & \end{tikzcd} \tag{6.1.2}$$

where b is a G-equivariant isomorphism of varieties. The proof is very similar to that of Proposition 6.1.11, using the fact that every point $y \in Y$ admits an étale neighborhood over which π has a right inverse; see [**93**, Corollaire 17.16.3(ii)].

In fact, a stronger claim is true: every principal bundle is **locally isotrivial**. This means that in the diagram (6.1.2), the étale neighborhood V can be chosen to be finite over its image in Y. For a proof, see [**190**, Lemme XIV.1.4].

Proposition 6.1.13. *Let P be a principal G-variety, and let X be another G-variety. Then $P \times X$ is a principal G-variety.*

Proof sketch. It is easy to see that the G-action on $P \times X$ is again free; the hard part is showing the existence of the geometric quotient $(P \times X)/G$. This is proved in [**208**, Proposition 4] under two additional conditions:

(1) The map $P \to P/G$ is locally isotrivial (see Remark 6.1.12).
(2) Every finite subset of X is contained in an affine open subset.

Because G is affine, the first condition holds by [**190**, Lemme XIV.1.4], and because X is quasiprojective, the second condition holds by, for instance, [**154**, Proposition 3.3.36]. □

We emphasize that the preceding proposition depends crucially on our two running assumptions: that G is affine and that X is quasiprojective. If one allows more general group schemes or non-quasiprojective varieties, there are counterexamples to Proposition 6.1.13.

In view of Example 6.1.8, Proposition 6.1.13 implies that the quotient in the following definition always exists.

Definition 6.1.14. Let G be an algebraic group. Let $H \subset G$ be a closed subgroup, and let X be an H-variety. The **induction space** for this action is the geometric quotient of $G \times X$ by the H-action given by

$$h \cdot (g, x) = (gh^{-1}, h \cdot x). \tag{6.1.3}$$

It is denoted by $G \times^H X$.

Lemma 6.1.15. *Let X be a principal G-variety, and let $H \subset G$ be a closed subgroup. Then X is also a principal H-variety. If H is a normal subgroup of G, then X/H is a principal G/H-variety, and there is a canonical isomorphism $(X/H)/(G/H) \cong X/G$.*

Proof sketch. It is easy to see that the H-action on X is free. Let G act on $H\backslash G$ by multiplication on the right. By Proposition 6.1.13, $X \times H\backslash G$ is a principal G-variety. Let $Y = (X \times H\backslash G)/G$, and let $\pi : X \to Y$ be the composition

$$X \xrightarrow{x \mapsto (x, He)} X \times H\backslash G \to Y.$$

Then one can check that $\pi : X \to Y$ is a geometric quotient for the H-action on X. In the case where H is normal, the left multiplication action of G/H on $H\backslash G$ induces an action on Y. One shows that this action is again free and that X/G is a geometric quotient for it. □

Proposition 6.1.16. *Let X and Y be principal G-varieties, and let $f : X \to Y$ be a G-equivariant morphism of varieties. Then the diagram*

$$\begin{tikzcd} X \arrow[r, "\pi_X"] \arrow[d, "f"'] & X/G \arrow[d, "\bar{f}"] \\ Y \arrow[r, "\pi_Y"] & Y/G \end{tikzcd}$$

is cartesian. Moreover, $\bar{f}$ is smooth if and only if f is smooth.

Proof sketch. The existence of $\bar{f}$ comes from Lemma 6.1.3, and if f is smooth, Proposition 2.1.10 implies that $\bar{f}$ is smooth. Next, the fiber product $Y \times_{Y/G} (X/G)$ is also a principal G-bundle over X/G, and the natural map $i : X \to Y \times_{Y/G} (X/G)$ is G-equivariant. It remains to show that i is an isomorphism. In the case where $X \to X/G$ is a trivial bundle, this is easy. Otherwise, since $X \to X/G$ is at least étale-locally trivial (see Remark 6.1.12), i becomes an isomorphism after base change along a surjective étale map $U \to X/G$. By [**92**, Proposition 2.7.1(viii)], i is an isomoprhism. □

Resolutions and acyclic maps. One theme of this chapter is that principal G-varieties are the "best behaved" G-varieties. For G-varieties that are not principal, we will often make use of the following notion.

DEFINITION 6.1.17. Let G be an algebraic group, and let X be a G-variety. A **G-resolution** (or simply a **resolution**) of X consists of a principal G-variety P together with a smooth G-equivariant map $p : P \to X$.

Given two G-resolutions $p : P \to X$ and $q : Q \to X$, a **smooth morphism of resolutions** $\nu : (P \xrightarrow{p} X) \to (Q \xrightarrow{q} X)$ is a smooth G-equivariant map $\nu : P \to Q$ such that the following diagram commutes:

$$\begin{tikzcd} P \arrow[rr, "\nu"] \arrow[dr, "p"'] & & Q \arrow[dl, "q"] \\ & X & \end{tikzcd}$$

We are especially interested in resolutions that enjoy the following property, previously introduced in Exercise 3.7.2.

DEFINITION 6.1.18. Let $f : X \to Y$ be a smooth morphism of varieties, and let n be a nonnegative integer. The map f is said to be **n-acyclic** if for every smooth morphism $Y' \to Y$, the base change $f' : X \times_Y Y' \to Y'$ has the property that for every perverse sheaf $\mathcal{F} \in \mathrm{Perv}(Y', \Bbbk)$, the natural map

$$\mathcal{F} \to {}^p\tau^{\leq n} f'_\dagger (f')^\dagger \mathcal{F}$$

is an isomorphism. The map f is **acyclic** if it is 0-acyclic, and it is **∞-acyclic** if it is n-acyclic for all $n \geq 0$.

(This definition is slightly different from the one used in [**27**]. See Remark 6.4.11 for a comparison.) The most important fact from Exercise 3.7.2 that we will need is the following.

LEMMA 6.1.19. *Let $f : X \to Y$ be an n-acyclic morphism of varieties. For any $k \in \mathbb{Z}$, the functor $f^\dagger : {}^pD^{\mathrm{b}}_{\mathrm{c}}(Y, \Bbbk)^{[k,k+n]} \to {}^pD^{\mathrm{b}}_{\mathrm{c}}(X, \Bbbk)^{[k,k+n]}$ is fully faithful.*

LEMMA 6.1.20. *Let X be a smooth, connected variety such that $\mathbf{H}^k(X; \mathbb{Z}) = 0$ for $1 \leq k \leq n$, and such that $\mathbf{H}^{n+1}(X; \mathbb{Z})$ is a free abelian group. Then $\mathrm{a}_X : X \to \mathrm{pt}$ is n-acyclic.*

PROOF. The base change of $\mathrm{a}_X : X \to \mathrm{pt}$ along any map $\mathrm{a}_Y : Y \to \mathrm{pt}$ is just the projection map $\mathrm{pr}_2 : X \times Y \to Y$. For $\mathcal{F} \in \mathrm{Perv}(Y, \Bbbk)$, we have

$$\begin{aligned} \mathrm{pr}_{2\dagger}\mathrm{pr}_2^\dagger \mathcal{F} \cong \mathrm{pr}_{2*}\mathrm{pr}_2^* \mathcal{F} &\cong (\mathrm{a}_X \times \mathrm{id}_Y)_*(\underline{\Bbbk}_X \boxtimes \mathcal{F}) \\ &\cong \mathrm{a}_Y^* R\Gamma(\underline{\Bbbk}_X) \overset{L}{\otimes} \mathcal{F} \cong \mathrm{a}_Y^*(\Bbbk \overset{L}{\otimes}_{\mathbb{Z}} R\Gamma(\underline{\mathbb{Z}}_X)) \overset{L}{\otimes} \mathcal{F}. \end{aligned}$$

Since X is connected, $\mathbf{H}^0(X; \mathbb{Z}) \cong \mathbb{Z}$. Our assumptions on $\mathbf{H}^k(X; \mathbb{Z})$ imply that $\tau^{\leq n} R\Gamma(\underline{\mathbb{Z}}_X) \cong \mathbb{Z}$, so truncation gives us a distinguished triangle

$$\mathbb{Z} \to R\Gamma(\underline{\mathbb{Z}}_X) \to \tau^{\geq n+1} R\Gamma(\underline{\mathbb{Z}}_X) \to .$$

Apply $\mathrm{a}_Y^*(\Bbbk \otimes^L_{\mathbb{Z}} (-)) \otimes^L \mathcal{F}$ to this distinguished triangle. The calculations above let us rewrite this as

$$\mathcal{F} \to \mathrm{pr}_{2\dagger}\mathrm{pr}_2^\dagger \mathcal{F} \to \mathrm{a}_Y^*(\Bbbk \overset{L}{\otimes}_{\mathbb{Z}} \tau^{\geq n+1} R\Gamma(\underline{\mathbb{Z}}_X)) \otimes \mathcal{F} \to .$$

Since $\mathsf{H}^{n+1}(R\Gamma(\underline{\mathbb{Z}}_X))$ is a free abelian group, we can apply Exercise 3.2.1 to conclude that the third term above lies in ${}^pD^{\mathrm{b}}_{\mathrm{c}}(Y,\Bbbk)^{\geq n+1}$. It follows that the whole triangle is canonically isomorphic to a truncation distinguished triangle, and in particular that $\mathcal{F} \cong {}^p\tau^{\leq n}\mathrm{pr}_{2\dagger}\mathrm{pr}_2^{\dagger}\mathcal{F}$. $\square$

EXAMPLE 6.1.21. Let $X = \mathbb{A}^n \smallsetminus \{0\}$, regarded as a $\mathbb{G}_{\mathrm{m}}$-variety with the usual scaling action. We have seen in Example 6.1.9 that X is a principal $\mathbb{G}_{\mathrm{m}}$-variety. As a topological space, X is homotopy equivalent to the $(2n-1)$-sphere, so $\mathbf{H}^i(X;\mathbb{Z})$ is a free abelian group for $i = 0$ and $i = 2n-1$, and it vanishes otherwise. By Lemma 6.1.20, $\mathrm{a}_X : X \to \mathrm{pt}$ is a $(2n-2)$-acyclic $\mathbb{G}_{\mathrm{m}}$-resolution.

EXAMPLE 6.1.22. The reasoning of Example 6.1.21 can be generalized to the Stiefel manifold $W_{m,k}$ from Example 6.1.10. We claim that

$$\mathrm{a}_{W_{m,k}} : W_{m,k} \to \mathrm{pt}$$

is a $(2m-2k)$-acyclic GL_k-resolution of a point. Indeed, as explained in [**113**, Chapter 2], $W_{m,k}$ is homotopy-equivalent to the *compact* complex Stiefel manifold $W'_{m,k}$, defined to be the set of orthonormal k-frames in $\mathbb{C}^n$. The singular cohomology of $W'_{m,k}$ (and hence also of $W_{m,k}$) is described in [**33**, Proposition 9.1]. In particular, $\mathbf{H}^i(W_{m,k};\mathbb{Z})$ vanishes for $1 \leq i \leq 2m-2k$, and it is a free abelian group for $i = 2m-2k+1$.

The following result will be an important technical tool in Section 6.4.

PROPOSITION 6.1.23. *Let X be a G-variety. For any $n \geq 0$, there exists an n-acyclic G-resolution $p : P \to X$.*

PROOF. Choose an embedding of G as a closed subgroup of GL_k. By Example 6.1.22, there exists an n-acyclic GL_k-resolution $p_0 : P_0 \to \mathrm{pt}$. By Lemma 6.1.15, P_0 is also a principal G-variety, so $p_0 : P_0 \to \mathrm{pt}$ is an n-acyclic G-resolution as well.

The property of being n-acyclic is stable under base change, so the map $p = \mathrm{pr}_2 : P_0 \times X \to X$ is n-acyclic. By Proposition 6.1.13, $P_0 \times X$ is still a principal G-variety, so $p : P_0 \times X \to X$ is an n-acyclic G-resolution of X. $\square$

We conclude this section with a lemma on the stability of acyclicity under group quotients.

LEMMA 6.1.24. *Let $f : X \to Y$ be a G-equivariant n-acyclic map of principal G-varieties. Then the induced map $\bar{f} : X/G \to Y/G$ is again n-acyclic.*

PROOF. Given a smooth morphism $Y' \to Y/G$, let $\tilde{Y}' = Y \times_{Y/G} Y'$, and consider the diagram

$$\begin{tikzcd}
X \times_Y \tilde{Y}' \arrow[rr] \arrow[dr] \arrow[dd, "f'"'] & & X/G \times_{Y/G} Y' \arrow[dr] \arrow[dd, "\bar{f}'" near start] & \\
& X \arrow[rr, crossing over] & & X/G \arrow[dd, "\bar{f}"] \\
\tilde{Y}' \arrow[rr, "\tilde{\pi}" near end] \arrow[dr] & & Y' \arrow[dr] & \\
& Y \arrow[rr, "\pi"] \arrow[from=uu, crossing over, "f" near start] & & Y/G
\end{tikzcd}$$

Given $\mathcal{F} \in \mathrm{Perv}(Y',\Bbbk)$, we must show that $\mathcal{F} \to {}^p\tau^{\leq n}\bar{f}'_{\dagger}(\bar{f}')^{\dagger}\mathcal{F}$ is an isomorphism. Because $\tilde{\pi}^{\dagger}$ is t-exact and faithful on perverse sheaves, it is enough to check this after applying $\tilde{\pi}^\dagger$. By base change, our claim follows from the fact that for $\mathcal{G} \in \mathrm{Perv}(\tilde{Y}', \Bbbk)$, the map $\mathcal{G} \to {}^p\tau^{\le n} f'_\dagger (f')^\dagger \mathcal{G}$ is an isomorphism. $\square$

Exercises.

6.1.1. Let $X = \mathbb{A}^1 \smallsetminus \{0\}$, and let $G = \{\pm 1\}$ act on X by multiplication. Let $\pi : X \to X$ be the map $\pi(z) = z^2$. Show that π is a principal G-bundle, but that it is not Zariski-locally trivial.

6.1.2. Let G be an algebraic group. Let $H \subset G$ be a closed subgroup, and let X be an H-variety. Consider the map $p : G \times^H X \to G/H$ given by $(g, x) \mapsto gH$. Show that this map is a locally trivial fibration with fibers isomorphic to X. If $G \to G/H$ is Zariski-locally trivial, then so is $G \times^H X \to G/H$.

6.1.3. Let X be a G-variety, and let $H \subset G$ be a closed subgroup. Show that there is a G-equivariant isomorphism $G \times^H X \cong G/H \times X$.

6.1.4. Let $f : X \to Y$ be a smooth morphism of varieties. Assume that f is a locally trivial fibration whose fibers $f^{-1}(y)$ are connected and have the property that $\mathbf{H}^k(f^{-1}(y); \mathbb{Z}) = 0$ for $1 \le k \le n$, and that $\mathbf{H}^{n+1}(f^{-1}(y); \mathbb{Z})$ is a free abelian group. Show that f is n-acyclic.

6.2. Equivariant sheaves and perverse sheaves

Prologue. Let G be a topological group, and let X be a topological space equipped with a continuous G-action $\sigma : G \times X \to X$. In this prologue, we seek to motivate the definition of "equivariant sheaf" (Definition 6.2.1). The reader who is happy with this definition can skip ahead to the next subsection.

We would like to capture the notion of "a sheaf equipped with an action of G that is compatible with the G-action on X."

Naïve definition, first attempt. Define a "G-equivariant sheaf" on X to be a sheaf $\mathcal{F}$ together with a rule θ that assigns to each group element $g \in G$ and each open set $U \subset X$ an isomorphism

$$\theta_{g,U} : \mathcal{F}(U) \to \mathcal{F}(g \cdot U).$$

Moreover, θ should satisfy the following "associativity condition":

$$\theta_{g,h\cdot U} \circ \theta_{h,U} = \theta_{gh,U} : \mathcal{F}(U) \to \mathcal{F}(gh \cdot U). \tag{6.2.1}$$

This notation suggests a slightly more efficient way to encode the same data: as U varies, the $\theta_{g,U}$ should together form a morphism of sheaves θ_g from $\mathcal{F}$ to another sheaf. Specifically, let $\sigma_g : X \to X$ be the map given by $\sigma_g(x) = g \cdot x$. Since σ_g is a homeomorphism, we have $(\sigma_g^* \mathcal{F})(U) = \mathcal{F}(\sigma_g(U)) = \mathcal{F}(g \cdot U)$.

Naïve definition, second attempt. Define a "G-equivariant sheaf" on X to be a sheaf $\mathcal{F}$ together with a rule θ that assigns to each group element $g \in G$ an isomorphism of sheaves

$$\theta_g : \mathcal{F} \to \sigma_g^* \mathcal{F}.$$

Moreover, θ should satisfy the following "associativity condition":

$$(\sigma_h^* \theta_g) \circ \theta_h = \theta_{gh} : \mathcal{F} \to \sigma_{gh}^* \mathcal{F}. \tag{6.2.2}$$

Note that this definition implicitly makes use of the isomorphism $\sigma_h^* \sigma_g^* \cong \sigma_{gh}^*$.

The main problem with the naïve definitions above is that they do not take into account the topology of G. To do this, let us try to interpret θ_g as the restriction to $\{g\} \times X \subset G \times X$ of some isomorphism of sheaves $\theta : \mathrm{pr}_2^*\mathcal{F} \to \sigma^*\mathcal{F}$. To write down the "associativity condition" in this language, we will need the following maps:

$$\begin{aligned} m &: G \times G \times X \to G \times X & m(g,h,x) &= (gh, x), \\ b &: G \times G \times X \to G \times X & b(g,h,x) &= (g, h \cdot x), \\ \mathrm{pr}_{23} &: G \times G \times X \to G \times X & \mathrm{pr}_{23}(g,h,x) &= (h, x). \end{aligned} \tag{6.2.3}$$

We are now ready to state the actual definition of "equivariant sheaf."

Definition 6.2.1. Let G be a topological group, and let X be a topological space equipped with a continuous G-action $\sigma : G \times X \to X$. Let m, b, and pr_{23} be as in (6.2.3). A **G-equivariant sheaf** on X is a pair $(\mathcal{F}, \theta)$, where $\mathcal{F} \in \mathrm{Sh}(X, \Bbbk)$ and θ is an isomorphism

$$\theta : \mathrm{pr}_2^*\mathcal{F} \xrightarrow{\sim} \sigma^*\mathcal{F}$$

in $\mathrm{Sh}(G \times X, \Bbbk)$ that satisfies

$$b^*\theta \circ \mathrm{pr}_{23}^*\theta = m^*\theta. \tag{6.2.4}$$

A **morphism** of G-equivariant sheaves $\phi : (\mathcal{F}, \theta_\mathcal{F}) \to (\mathcal{G}, \theta_\mathcal{G})$ is a morphism of sheaves $\phi : \mathcal{F} \to \mathcal{G}$ such that the following diagram commutes:

$$\begin{tikzcd} \mathrm{pr}_2^*\mathcal{F} \arrow[r, "\theta_\mathcal{F}"] \arrow[d, "\mathrm{pr}_2^*\phi"'] & \sigma^*\mathcal{F} \arrow[d, "\sigma^*\phi"] \\ \mathrm{pr}_2^*\mathcal{G} \arrow[r, "\theta_\mathcal{G}"] & \sigma^*\mathcal{G} \end{tikzcd}$$

The category of G-equivariant sheaves on X is denoted by $\mathrm{Sh}_G(X, \Bbbk)$.

(In some sources, θ is required to satisfy an additional axiom, corresponding to the idea that in a group action, and the identity element $e \in G$ should act as the identity operator. It turns out that this axiom is actually redundant; see Exercise 6.2.2.)

Note that if we identify $\{g\} \times \{h\} \times X$ with X, then we have

$$\sigma_h^*\theta_g = (b^*\theta)|_{\{g\}\times\{g\}\times X}, \quad \theta_h = (\mathrm{pr}_{23}^*\theta)|_{\{g\}\times\{h\}\times X}, \quad \theta_{gh} = (m^*\theta)|_{\{g\}\times\{h\}\times X}.$$

In view of these equalities, if we restrict (6.2.4) to $\{g\} \times \{h\} \times X$, we recover (6.2.2). Nevertheless, it is important to note that the two "naïve" definitions above are *not* equivalent to Definition 6.2.1, except in the case of a discrete group (see Exercise 6.2.1).

The proof of the following result is left as an exercise.

Proposition 6.2.2. *Let G be a topological group, and let X be a topological space equipped with a continuous G-action. Then* $\mathrm{Sh}_G(X, \Bbbk)$ *is an abelian category.*

Equivariant perverse sheaves. We now return to the algebraic setting of Section 6.1. The following definition is modeled on Definition 6.2.1.

Definition 6.2.3. Let G be an algebraic group, and let X be a G-variety. Let m, b, and pr_{23} be as in (6.2.3). A **G-equivariant perverse sheaf** on X (with coefficients in $\Bbbk$) is a pair $(\mathcal{F}, \theta)$, where $\mathcal{F} \in \mathrm{Perv}(X, \Bbbk)$, and θ is an isomorphism

$$\theta : \mathrm{pr}_2^*\mathcal{F} \xrightarrow{\sim} \sigma^*\mathcal{F}$$

in $D^{\mathrm{b}}_{\mathrm{c}}(G \times X, \Bbbk)$ that satisfies

$$b^*\theta \circ \mathrm{pr}_{23}^*\theta = m^*\theta. \tag{6.2.5}$$

A **morphism** of G-equivariant perverse sheaves $\phi : (\mathcal{F}, \theta_{\mathcal{F}}) \to (\mathcal{G}, \theta_{\mathcal{G}})$ is a morphism of perverse sheaves $\phi : \mathcal{F} \to \mathcal{G}$ such that the following diagram commutes:

$$\begin{tikzcd} \mathrm{pr}_2^*\mathcal{F} \arrow[r, "\theta_{\mathcal{F}}"] \arrow[d, "\mathrm{pr}_2^*\phi"'] & \sigma^*\mathcal{F} \arrow[d, "\sigma^*\phi"] \\ \mathrm{pr}_2^*\mathcal{G} \arrow[r, "\theta_{\mathcal{G}}"] & \sigma^*\mathcal{G} \end{tikzcd}$$

The category of G-equivariant perverse sheaves on X with coefficients in $\Bbbk$ is denoted by $\mathrm{Perv}_G(X, \Bbbk)$.

REMARK 6.2.4. As noted in Section 6.1, the maps $\sigma, \mathrm{pr}_2 : G \times X \to X$ are both smooth of relative dimension $\dim G$. The same holds for the maps in (6.2.3). Thus, the definition above could as well have been formulated using $\sigma^\dagger = \sigma^*[\dim G]$ and $\mathrm{pr}_2^\dagger = \mathrm{pr}_2^![\dim G]$ instead (along with $m^\dagger$, $b^\dagger$, and $\mathrm{pr}_{23}^\dagger$). This approach would have the aesthetic advantage that all objects and morphisms appearing in the definition would live in the abelian category of perverse sheaves. We will occasionally use this alternative formulation of the definition.

REMARK 6.2.5. If G is the trivial group $\{e\}$, then $\mathrm{pr}_2 = \sigma$, and the definition forces θ to be the identity map. Any perverse sheaf $\mathcal{F}$ can be regarded as an equivariant perverse sheaf by taking $\theta = \mathrm{id}$, and every morphism of perverse sheaves is also a morphism of equivariant perverse sheaves. In other words, there is a canonical equivalence (in fact, an isomorphism) of categories

$$\mathrm{Perv}_{\{e\}}(X, \Bbbk) \cong \mathrm{Perv}(X, \Bbbk).$$

Let G be an algebraic group, and let $H \subset G$ be a closed subgroup. If X is a G-variety, consider the diagram

$$H \times X \xrightarrow{i \times \mathrm{id}_X} G \times X \underset{\mathrm{pr}_2}{\overset{\sigma}{\rightrightarrows}} X$$

where $i : H \hookrightarrow G$ is the inclusion map. If $(\mathcal{F}, \theta)$ is a G-equivariant perverse sheaf, then it is straightforward to check (mainly using Proposition 1.2.8(1)) that the morphism $(i \times \mathrm{id}_X)^*\theta$ in $D^{\mathrm{b}}_{\mathrm{c}}(H \times X, \Bbbk)$ satisfies the axioms of Definition 6.2.3, but with G replaced by H. That is, $(\mathcal{F}, (i \times \mathrm{id}_X)^*\theta)$ is an H-equivariant perverse sheaf. This construction defines a functor

$$\mathrm{For}_H^G : \mathrm{Perv}_G(X, \Bbbk) \to \mathrm{Perv}_H(X, \Bbbk), \tag{6.2.6}$$

called the **forgetful functor**. When H is the trivial group, this becomes

$$\mathrm{For} \text{ or } \mathrm{For}^G : \mathrm{Perv}_G(X, \Bbbk) \to \mathrm{Perv}(X, \Bbbk).$$

Next, let $H \lhd G$ be a normal subgroup, and let X be a G/H-variety. Consider the diagram

$$G \times X \xrightarrow{q \times \mathrm{id}_X} G/H \times X \underset{\mathrm{pr}_2}{\overset{\sigma}{\rightrightarrows}} X$$

where $q : G \to G/H$ is the quotient map. Similar reasoning to that above shows that if $(\mathcal{F}, \theta)$ is a G/H-equivariant perverse sheaf, then $(\mathcal{F}, (q \times \mathrm{id}_X)^*\theta)$ is a G-equivariant perverse sheaf. This construction defines the **inflation functor**, denoted by

$$\mathrm{Infl}_{G/H}^G : \mathrm{Perv}_{G/H}(X, \Bbbk) \to \mathrm{Perv}_G(X, \Bbbk). \tag{6.2.7}$$

Smooth pullback. We will discuss equivariant sheaf functors in general later, but for now, we consider one easy case. If $f : X \to Y$ is a smooth G-equivariant morphism of varieties, and if $(\mathcal{F}, \theta) \in \mathrm{Perv}_G(Y, \Bbbk)$, then we claim that the perverse sheaf $f^\dagger\mathcal{F}$ naturally has a G-equivariant structure. Indeed, the diagrams

$$\begin{array}{ccc} G \times X & \xrightarrow{\mathrm{id}\times f} & G \times Y \\ {\scriptstyle \mathrm{pr}_2}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{pr}_2} \\ X & \xrightarrow{\quad f \quad} & Y \end{array} \qquad \text{and} \qquad \begin{array}{ccc} G \times X & \xrightarrow{\mathrm{id}\times f} & G \times Y \\ {\scriptstyle \sigma}\big\downarrow & & \big\downarrow{\scriptstyle \sigma} \\ X & \xrightarrow{\quad f \quad} & Y \end{array} \tag{6.2.8}$$

are both cartesian, and $(\mathrm{id} \times f)^\dagger\theta$ can be regarded as a map $\mathrm{pr}_2^*(f^\dagger\mathcal{F}) \to \sigma^*(f^\dagger\mathcal{F})$. This map satisfies (6.2.5), so $f^\dagger$ defines a functor $\mathrm{Perv}_G(Y, \Bbbk) \to \mathrm{Perv}_G(X, \Bbbk)$.

PROPOSITION 6.2.6. *Let X and Y be G-varieties, and let $f : X \to Y$ be a smooth surjective G-equivariant morphism. The functor*

$$f^\dagger : \mathrm{Perv}_G(Y, \Bbbk) \to \mathrm{Perv}_G(X, \Bbbk)$$

is faithful. If f has connected fibers, then $f^\dagger$ is fully faithful.

PROOF. The faithfulness of $f^\dagger$ is immediate from the faithfulness of the non-equivariant version (Theorem 3.6.6). Now assume that f has connected fibers. Let $\mathcal{F}, \mathcal{G} \in \mathrm{Perv}_G(Y, \Bbbk)$, and let $\psi : f^\dagger\mathcal{F} \to f^\dagger\mathcal{G}$ be a morphism in $\mathrm{Perv}_G(X, \Bbbk)$. Because the nonequivariant version of $f^\dagger$ is fully faithful (Theorem 3.6.6 again), we at least have a morphism $\phi : \mathrm{For}(\mathcal{F}) \to \mathrm{For}(\mathcal{G})$ in $\mathrm{Perv}(Y, \Bbbk)$ such that $f^\dagger\phi = \mathrm{For}(\psi)$. To finish the proof, we must show that ϕ is an equivariant morphism, i.e., that

$$\sigma^*\phi \circ \theta_{\mathcal{F}} = \theta_{\mathcal{G}} \circ \mathrm{pr}_2^*\phi. \tag{6.2.9}$$

Apply $(\mathrm{id} \times f)^\dagger$ to both sides. Using the commutativity of the squares in (6.2.8), the fact that $f^\dagger\phi = \psi$, and the definition of the equivariant structure on $f^\dagger\mathcal{F}$ and $f^\dagger\mathcal{G}$, the two sides become

$$\sigma^*\psi \circ \theta_{f^\dagger\mathcal{F}} \qquad \text{and} \qquad \theta_{f^\dagger\mathcal{G}} \circ \mathrm{pr}_2^*\psi.$$

These are equal because ψ is an equivariant morphism. Since (the nonequivariant version of) $(\mathrm{id} \times f)^\dagger$ is faithful, we conclude that (6.2.9) holds. ☐

PROPOSITION 6.2.7. *Let X and Y be G-varieties, and let $f : X \to Y$ be a smooth, surjective, G-equivariant morphism with connected fibers. The image of the fully faithful functor $f^\dagger : \mathrm{Perv}_G(Y, \Bbbk) \to \mathrm{Perv}_G(X, \Bbbk)$ is closed under subobjects and quotients.*

PROOF. We will prove the statement for subobjects; the proof for quotients is similar. Let $\mathcal{F} \in \mathrm{Perv}_G(X, \Bbbk)$ be in the image of this functor, say $\mathcal{F} \cong f^\dagger\bar{\mathcal{F}}$. Let $\mathcal{F}' \subset \mathcal{F}$ be a sub G-equivariant perverse sheaf. By Theorem 3.7.6, we at least know that $\mathrm{For}(\mathcal{F}')$ lies in the image of (the nonequivariant version of) $f^\dagger$, say $\mathcal{F}' \cong f^\dagger\bar{\mathcal{F}}'$. We must show how to equip $\bar{\mathcal{F}}'$ with a G-equivariant structure.

In this proof, it will be convenient to use the "† version" of Definition 6.2.3, as in Remark 6.2.4. On $G \times X$, we have a commutative diagram

$$\begin{array}{ccc} \mathrm{pr}_2^\dagger\mathcal{F}' & \xrightarrow{\theta_{\mathcal{F}'}} & \sigma^\dagger\mathcal{F}' \\ \big\downarrow & & \big\downarrow \\ \mathrm{pr}_2^\dagger\mathcal{F} & \xrightarrow{\theta_{\mathcal{F}}} & \sigma^\dagger\mathcal{F} \end{array}$$

Using the commutative diagrams in (6.2.8), we can rewrite this as

$$\begin{tikzcd}
(\mathrm{id}\times f)^\dagger \mathrm{pr}_2^\dagger \bar{\mathcal{F}}' \arrow[r, "\theta_{\mathcal{F}'}"] \arrow[d, hook] & (\mathrm{id}\times f)^\dagger \sigma^\dagger \bar{\mathcal{F}}' \arrow[d, hook] \\
(\mathrm{id}\times f)^\dagger \mathrm{pr}_2^\dagger \bar{\mathcal{F}} \arrow[r, "(\mathrm{id}\times f)^\dagger\theta_{\bar{\mathcal{F}}}", "\sim"'] & (\mathrm{id}\times f)^\dagger \sigma^\dagger \bar{\mathcal{F}}
\end{tikzcd}$$

Since $\mathrm{id} \times f$ is smooth and has connected fibers, $(\mathrm{id} \times f)^\dagger$ is fully faithful, by Theorem 3.6.6. The map $\theta_{\mathcal{F}'}$ across the top of the diagram above can therefore be written as $(\mathrm{id} \times f)^\dagger \theta_{\bar{\mathcal{F}}'}$ for some map $\theta_{\bar{\mathcal{F}}'} : \mathrm{pr}_2^\dagger \bar{\mathcal{F}}' \xrightarrow{\sim} \sigma^\dagger \bar{\mathcal{F}}'$. We now have a commutative diagram

$$\begin{tikzcd}
\mathrm{pr}_2^\dagger \bar{\mathcal{F}}' \arrow[r, "\theta_{\bar{\mathcal{F}}'}"] \arrow[d, hook] & \sigma^\dagger \bar{\mathcal{F}}' \arrow[d, hook] \\
\mathrm{pr}_2^\dagger \bar{\mathcal{F}} \arrow[r, "\theta_{\bar{\mathcal{F}}}"] & \sigma^\dagger \bar{\mathcal{F}}
\end{tikzcd}$$

Because the vertical maps are injective, the fact that $\theta_{\bar{\mathcal{F}}'}$ satisfies (6.2.5) follows from the corresponding fact for $\theta_{\bar{\mathcal{F}}}$. ☐

PROPOSITION 6.2.8. *Let G be an algebraic group, and let $H \lhd G$ be a connected normal subgroup. For any G/H-variety X, the inflation functor*

$$\mathrm{Infl}_{G/H}^G : \mathrm{Perv}_{G/H}(X, \Bbbk) \to \mathrm{Perv}_G(X, \Bbbk)$$

is an equivalence of categories.

For a similar statement when G is connected, see Proposition 6.2.16.

PROOF. This proposition is again an application of Theorem 3.6.6. Consider the following diagram, where $q : G \to G/H$ is the quotient map:

$$\begin{tikzcd}
G \times X \arrow[r, "q\times\mathrm{id}_X"] \arrow[rr, bend left, "\sigma'=\sigma\circ(q\times\mathrm{id}_X)"] \arrow[rr, bend right, "\mathrm{pr}_2'=\mathrm{pr}_2\circ(q\times\mathrm{id}_X)"'] & G/H \times X \arrow[r, shift left, "\sigma"] \arrow[r, shift right, "\mathrm{pr}_2"'] & X
\end{tikzcd}$$

We will use the version of Definition 6.2.3 from Remark 6.2.4. If $(\mathcal{F}, \theta' : (\mathrm{pr}_2')^\dagger \mathcal{F} \to (\sigma')^\dagger \mathcal{F})$ is a G-equivariant perverse sheaf, then by Theorem 3.6.6, there is a unique morphism $\theta : \mathrm{pr}_2^\dagger \mathcal{F} \to \sigma^\dagger \mathcal{F}$ such that $\theta' = (q \times \mathrm{id})^\dagger \theta$. Moreover, it is straightforward to check that θ satisfies (6.2.5), so $(\mathcal{F}, \theta)$ is a G/H-equivariant perverse sheaf. We have shown that $(\mathcal{F}, \theta') \cong \mathrm{Infl}_{G/H}^G(\mathcal{F}, \theta)$, so $\mathrm{Infl}_{G/H}^G$ is essentially surjective.

The proof that it is fully faithful is very similar to the proof of Proposition 6.2.6, and will be omitted. ☐

REMARK 6.2.9. As a special case, if G is a connected group that acts trivially on X, Proposition 6.2.8 says that $\mathrm{Infl}_1^G : \mathrm{Perv}(X, \Bbbk) \xrightarrow{\sim} \mathrm{Perv}_G(X, \Bbbk)$ is an equivalence of categories. In other words, in this situation, every perverse sheaf admits a unique equivariant structure.

Quotients and induction spaces. The next few results relate equivariant perverse sheaves to geometric quotients.

PROPOSITION 6.2.10. *Let X be a principal G-variety, and let $\pi : X \to X/G$ be the quotient map. Then $\pi^\dagger$ induces an equivalence of categories*

$$\pi^\dagger : \mathrm{Perv}(X/G, \Bbbk) \xrightarrow{\sim} \mathrm{Perv}_G(X, \Bbbk).$$

PROOF. Consider the cartesian diagram from Proposition 6.1.7. According to Theorem 3.7.4, $\mathrm{Perv}(X/G, \Bbbk)$ is equivalent via $\pi^\dagger$ to the category $\mathrm{Desc}(\pi, \Bbbk)$ of descent data for the smooth morphism $\pi : X \to X/G$. To understand what a descent datum for π is, we will study the following diagram of cartesian squares, which can be identified with (3.7.1):

$$\begin{tikzcd}
G \times G \times X \arrow[rrr, "b"] \arrow[dr, "m"] \arrow[dd, "\mathrm{pr}_{23}"'] & & & G \times X \arrow[dr, "\sigma"] \arrow[dd, "\mathrm{pr}_2"] & \\
& G \times X \arrow[rrr, "\sigma"] \arrow[dd, "\mathrm{pr}_2"] & & & X \arrow[dd, "\pi"] \\
G \times X \arrow[rrr, "\sigma"] \arrow[dr, "\mathrm{pr}_2"'] & & & X \arrow[dr, "\pi"] & \\
& X \arrow[rrr, "\pi"] & & & X/G
\end{tikzcd}$$

According to Definition 3.7.1, an object of $\mathrm{Desc}(\pi, \Bbbk)$ is a pair $(\mathcal{F}, \phi)$, where $\mathcal{F} \in \mathrm{Perv}(X, \Bbbk)$ and ϕ is an isomorphism

$$\phi : \mathrm{pr}_2^\dagger \mathcal{F} \to \sigma^\dagger \mathcal{F}$$

such that the diagram

$$\begin{tikzcd}
\mathrm{pr}_{23}^\dagger \mathrm{pr}_2^\dagger \mathcal{F} \cong m^\dagger \mathrm{pr}_2^\dagger \mathcal{F} \arrow[rr, "m^\dagger \phi"] \arrow[dr, "\mathrm{pr}_{23}^\dagger \phi"'] & & m^\dagger \sigma^\dagger \mathcal{F} \cong b^\dagger \sigma^\dagger \mathcal{F} \\
& \mathrm{pr}_{23}^\dagger \sigma^\dagger \mathcal{F} \cong b^\dagger \mathrm{pr}_2^\dagger \mathcal{F} \arrow[ur, "b^\dagger \phi"'] &
\end{tikzcd}$$

commutes. In other words, a descent datum for π is the same thing as a G-equivariant perverse sheaf on X. □

PROPOSITION 6.2.11. *Let G be an algebraic group, and let $H \subset G$ be a closed subgroup. Let X be an H-variety, and let $i : X \to G \times^H X$ be the map $i(x) = (e, x)$. Then $i^*[-\dim G/H]$ induces an equivalence of categories*

$$i^*[-\dim G/H] : \mathrm{Perv}_G(G \times^H X, \Bbbk) \xrightarrow{\sim} \mathrm{Perv}_H(X, \Bbbk).$$

This will follow later on from the derived version (Theorem 6.5.10). For a proof in the abelian category setting, see Exercise 6.2.4.

PROPOSITION 6.2.12. *Let G be an algebraic group, and let G° be its identity component. There is an equivalence of categories* $\mathrm{Perv}_G(\mathrm{pt}, \Bbbk) \cong \Bbbk[G/G^\circ]\text{-mod}^{\mathrm{fg}}$.

PROOF. Assume first that G is a finite group and is thus identified with G/G°. In particular, G is a discrete topological group, so by Exercise 6.2.1, $\mathrm{Sh}_G(\mathrm{pt}, \Bbbk)$ can be described by the first "naïve" definition at the beginning of the section. That description gives us an equivalence $\mathrm{Sh}_G(\mathrm{pt}, \Bbbk) \cong \Bbbk[G]\text{-mod}$. Using Lemma 3.1.3, we deduce that $\mathrm{Perv}_G(\mathrm{pt}, \Bbbk) \cong \Bbbk[G/G^\circ]\text{-mod}^{\mathrm{fg}}$.

For general G, we have $\mathrm{Perv}_{G/G^\circ}(\mathrm{pt}, \Bbbk) \cong \mathrm{Perv}_G(\mathrm{pt}, \Bbbk)$ by Proposition 6.2.8, so the result follows from the preceding paragraph. □

PROPOSITION 6.2.13. *Let G be an algebraic group, and let X be a homogeneous space for G. Then every G-equivariant perverse sheaf on X is a shifted local system. Moreover, there is an equivalence of categories*

$$\mathrm{Loc}^{\mathrm{ft}}_G(X, \Bbbk) \cong \Bbbk[G^x/(G^x)^\circ]\text{-mod}^{\mathrm{fg}}$$

where $G^x \subset G$ is the stabilizer of some point $x \in X$ and where $(G^x)^\circ \subset G^x$ is its identity component.

Due to the resemblance between this statement and Theorem 1.7.9, the group $(G^x)/(G^x)^\circ$ is sometimes called the **equivariant fundamental group** of X.

PROOF. Let $\mathcal{F} \in \mathrm{Perv}_G(X, \Bbbk)$. Any cohomology sheaf $\mathsf{H}^i(\mathcal{F})$ is constructible, so there exists an open subset $U \subset X$ such that $\mathsf{H}^i(\mathcal{F})|_U$ is locally constant. Because $\mathsf{H}^i(\mathcal{F})$ is an equivariant sheaf, for any $g \in G$, we have $\mathsf{H}^i(\mathcal{F})|_{g \cdot U} \cong \sigma^*_{g^{-1}}(\mathsf{H}^i(\mathcal{F})|_U)$. In particular, $\mathsf{H}^i(\mathcal{F})|_{g \cdot U}$ is also locally constant. Since X is a homogeneous space, it is covered by open sets of the form $g \cdot U$ for $g \in G$, so $\mathsf{H}^i(\mathcal{F})$ is a local system. A perverse sheaf all of whose cohomology sheaves are local systems must itself be a shifted local system.

Now, choose a point $x \in X$. By assumption, we have $X \cong G/G^x \cong G \times^{G^x} \mathrm{pt}$. By Propositions 6.2.11 and 6.2.12, we have $\mathrm{Perv}_G(X, \Bbbk) \cong \mathrm{Perv}_{G^x}(\mathrm{pt}, \Bbbk) \cong \Bbbk[G^x/(G^x)^\circ]\text{-mod}^{\mathrm{fg}}$. □

REMARK 6.2.14. Suppose G is a connected group. Let X be a homogeneous space for G, and let $x \in X$. Identify X with G/G^x, and consider the space $\tilde{X} = G/(G^x)^\circ$. The obvious map $\tilde{X} \to X$ is a covering map. The group $G^x/(G^x)^\circ$ acts on $\tilde{X}$ by multiplication on the right, and this action identifies $G^x/(G^x)^\circ$ with the group of deck transformations $\mathrm{Gal}(\tilde{X}/X)$. Indeed, this is a regular covering, so there is a short exact sequence

$$1 \to \pi_1(\tilde{X}, \tilde{x}) \to \pi_1(X, x) \to (G^x)/(G^x)^\circ \to 1.$$

Since G is connected, Proposition 6.2.15 below tells us that the functor $\mathrm{For}^G : \mathrm{Loc}^{\mathrm{ft}}_G(X, \Bbbk) \to \mathrm{Loc}^{\mathrm{ft}}(X, \Bbbk)$ is fully faithful. Proposition 6.2.13 says that a local system on X admits a G-equivariant structure if and only if the corresponding $\Bbbk[\pi_1(X, x)]$-module descends to a module for $\Bbbk[(G^x)/(G^x)^\circ]$.

Connected groups. We finish this section with a study of equivariant perverse sheaves for a connected group. This case is especially nice, mainly as a consequence of Theorem 3.6.6.

PROPOSITION 6.2.15. *Let G be a connected algebraic group. For any G-variety X, the functor*

$$\mathrm{For} : \mathrm{Perv}_G(X, \Bbbk) \to \mathrm{Perv}(X, \Bbbk)$$

is fully faithful, and its image is closed under subobjects and quotients.

PROOF. Let G act on $G \times X$ by $g \cdot (h, x) = (gh, x)$. This action is free, and the map $\mathrm{pr}_2 : G \times X \to X$ is a geometric quotient, so by Proposition 6.2.10, the functor

$$\mathrm{pr}_2^\dagger : \mathrm{Perv}(X, \Bbbk) \to \mathrm{Perv}_G(G \times X, \Bbbk)$$

is an equivalence of categories. Now consider the commutative diagram

$$\begin{tikzcd}
\mathrm{Perv}(X,\Bbbk) \arrow[rr, "\mathrm{pr}_2^\dagger"] \arrow[dr, "\mathrm{pr}_2^\dagger"', "\sim"] & & \mathrm{Perv}(G\times X,\Bbbk) \\
& \mathrm{Perv}_G(G\times X,\Bbbk) \arrow[ur, "\mathrm{For}"'] &
\end{tikzcd}$$

By Theorems 3.6.6 and 3.7.6, the top arrow (i.e., the nonequivariant version of $\mathrm{pr}_2^\dagger$) is fully faithful, and its image is closed under subobjects and quotients. Therefore, the same properties hold for $\mathrm{For} : \mathrm{Perv}_G(G \times X, \Bbbk) \to \mathrm{Perv}(G \times X, \Bbbk)$. (We have just proved a special case of the proposition.)

We now turn to X. The action map $\sigma : G \times X \to X$ is G-equivariant, so by Propositions 6.2.6 and 6.2.7, the functor

$$\sigma^\dagger : \mathrm{Perv}_G(X,\Bbbk) \to \mathrm{Perv}_G(G\times X,\Bbbk)$$

is fully faithful, and its image is closed under subobjects and quotients. The map $\theta : \mathrm{pr}_2^\dagger \mathcal{F} \xrightarrow{\sim} \sigma^\dagger\mathcal{F}$ with which every equivariant perverse sheaf is equipped can be regarded as a natural transformation that makes the following diagram commute:

$$\begin{tikzcd}
\mathrm{Perv}_G(X,\Bbbk) \arrow[r, "\sigma^\dagger"] \arrow[d, "\mathrm{For}"'] & \mathrm{Perv}_G(G\times X,\Bbbk) \arrow[d, "\mathrm{For}"] \\
\mathrm{Perv}(X,\Bbbk) \arrow[r, "\mathrm{pr}_2^\dagger"] & \mathrm{Perv}(G\times X,\Bbbk)
\end{tikzcd}$$

We have already seen that the top, bottom, and right-hand arrows are fully faithful, and that they have images that are closed under subobjects and quotients. Therefore, the same properties hold for $\mathrm{For} : \mathrm{Perv}_G(X, \Bbbk) \to \mathrm{Perv}(X, \Bbbk)$. □

The preceding proposition says that if G is connected, then for perverse sheaves, G-equivariance is a condition, rather than additional data.

Proposition 6.2.16. *Let G be a connected algebraic group, and let $H \lhd G$ be a normal subgroup. For any G/H-variety X, the inflation functor*

$$\mathrm{Infl}_{G/H}^G : \mathrm{Perv}_{G/H}(X,\Bbbk) \to \mathrm{Perv}_G(X,\Bbbk)$$

is fully faithful.

For a similar statement when H is connected, see Proposition 6.2.8.

Proof. By Proposition 6.2.15, both forgetful functors in the commutative diagram below are fully faithful. The result follows:

$$\begin{tikzcd}
\mathrm{Perv}_{G/H}(X,\Bbbk) \arrow[rr, "\mathrm{For}^{G/H}"] \arrow[dr, "\mathrm{Infl}_{G/H}^G"'] & & \mathrm{Perv}(X,\Bbbk) \\
& \mathrm{Perv}_G(X,\Bbbk) \arrow[ur, "\mathrm{For}^G"'] &
\end{tikzcd}$$

□

Proposition 6.2.17. *Let G be a connected algebraic group, and let X be a G-variety. A perverse sheaf $\mathcal{F} \in \mathrm{Perv}(X, \Bbbk)$ lies in $\mathrm{Perv}_G(X, \Bbbk)$ if and only if $\mathrm{pr}_2^\dagger\mathcal{F}$ is isomorphic to $\sigma^\dagger\mathcal{F}$.*

Proof. The "only if" direction of this statement is true by definition; we must prove the "if" direction. For this proof, we temporarily introduce the following terminology: let us say that $\mathcal{F} \in \mathrm{Perv}(X, \Bbbk)$ is **weakly G-equivariant** if $\mathrm{pr}_2^* \mathcal{F} \cong \sigma^* \mathcal{F}$. Let $\mathrm{Perv}_G^{\mathrm{weak}}(X, \Bbbk) \subset \mathrm{Perv}(X, \Bbbk)$ be the full subcategory consisting of weakly G-equivariant perverse sheaves. We clearly have $\mathrm{Perv}_G(X, \Bbbk) \subset \mathrm{Perv}_G^{\mathrm{weak}}(X, \Bbbk)$; we must show that these two categories are in fact equal.

Step 1. If $f : X \to Y$ is a smooth G-equivariant map with connected fibers, then $f^\dagger$ induces a fully faithful functor $\mathrm{Perv}_G^{\mathrm{weak}}(Y, \Bbbk) \to \mathrm{Perv}_G^{\mathrm{weak}}(X, \Bbbk)$. This is immediate from Theorem 3.6.6 and the commutative diagrams in (6.2.8).

Step 2. Let G act on $G \times X$ by $g \cdot (h, x) = (gh, x)$. We have $\mathrm{Perv}_G(G \times X, \Bbbk) = \mathrm{Perv}_G^{\mathrm{weak}}(G \times X, \Bbbk)$. Moreover, the functor $\mathrm{pr}_2^\dagger : \mathrm{Perv}(X, \Bbbk) \to \mathrm{Perv}_G^{\mathrm{weak}}(G \times X, \Bbbk)$ is an equivalence of categories. Let $\mathcal{G} \in \mathrm{Perv}_G^{\mathrm{weak}}(G \times X, \Bbbk)$. This means that $\mathrm{pr}_{23}^* \mathcal{G} \cong m^* \mathcal{G}$, where m and pr_{23} are as in (6.2.3). Next, define maps

$$
\begin{aligned}
&s : X \to G \times X && s(x) = (e, x), \\
&u : G \times X \to G \times G \times X && u(g, x) = (g, e, x).
\end{aligned}
$$

Observe that $m \circ u = \mathrm{id}_{G \times X}$ and $\mathrm{pr}_{23} \circ u = s \circ \mathrm{pr}_2$. We therefore have

$$\mathcal{G} \cong u^* m^* \mathcal{G} \cong u^* \mathrm{pr}_{23}^* \mathcal{G} \cong \mathrm{pr}_2^* s^* \mathcal{G} = \mathrm{pr}_2^\dagger s^* \mathcal{G}[-\dim G].$$

Since $\mathrm{pr}_2^\dagger$ is t-exact and faithful on perverse sheaves, the fact that $\mathrm{pr}_2^\dagger(s^* \mathcal{G}[-\dim G])$ is perverse implies that $s^* \mathcal{G}[-\dim G]$ is perverse. With respect to the trivial G-action on X, $s^* \mathcal{G}[-\dim G]$ is of course G-equivariant (cf. Remark 6.2.9). Moreover, $\mathrm{pr}_2 : G \times X \to X$ is G-equivariant, so $\mathrm{pr}_2^\dagger(s^* \mathcal{G}[-\dim G]) \cong \mathcal{G}$ is a G-equivariant perverse sheaf on $G \times X$. We conclude that $\mathrm{Perv}_G(G \times X, \Bbbk) = \mathrm{Perv}_G^{\mathrm{weak}}(G \times X, \Bbbk)$. By Proposition 6.2.10, $\mathrm{pr}_2^\dagger : \mathrm{Perv}(X, \Bbbk) \to \mathrm{Perv}_G^{\mathrm{weak}}(G \times X, \Bbbk)$ is an equivalence of categories.

Step 3. The functor $s^*[-\dim G]$ gives an equivalence of categories $\mathrm{Perv}_G^{\mathrm{weak}}(G \times X, \Bbbk) \to \mathrm{Perv}(X, \Bbbk)$, inverse to the equivalence from Step 2. Since $\mathrm{pr}_2^\dagger$ is an equivalence of categories, this claim is immediate from the observation that $s^*[-\dim G] \circ \mathrm{pr}_2^\dagger \cong s^* \mathrm{pr}_2^* \cong \mathrm{id}_X^*$.

Step 4. If $\mathcal{F} \in \mathrm{Perv}_G^{\mathrm{weak}}(X, \Bbbk)$, then there exists an isomorphism $\theta : \mathrm{pr}_2^\dagger \mathcal{F} \xrightarrow{\sim} \sigma^\dagger \mathcal{F}$ such that $s^* \theta[-\dim G] = \mathrm{id}_{\mathcal{F}}$. To make sense of this claim, note that $\mathrm{pr}_2 \circ s = \sigma \circ s = \mathrm{id}_X$. By assumption, there exists some isomorphism $\beta : \mathrm{pr}_2^\dagger \mathcal{F} \xrightarrow{\sim} \sigma^\dagger \mathcal{F}$. Apply $s^*[-\dim G]$ to obtain an automorphism $\beta_1 = s^* \beta[-\dim G] : \mathcal{F} \xrightarrow{\sim} \mathcal{F}$, and then set $\theta = \sigma^\dagger \beta_1^{-1} \circ \beta$. This map has the desired property.

Step 5. Conclusion of the proof. We will show that the map θ from Step 4 satisfies (6.2.5). Let $G \times G$ act on $G \times G \times X$ by $(g_1, g_2) \cdot (h_1, h_2, x) = (g_1 h_1, g_2 h_2, x)$. Then the map $\mathrm{pr}_2 \circ \mathrm{pr}_{23} : G \times G \times X \to X$ is $G \times G$-equivariant with respect to the trivial action on X, so for any perverse sheaf $\mathcal{F}$ on X, we see that $\mathrm{pr}_{23}^\dagger \mathrm{pr}_2^\dagger \mathcal{F}$ is (weakly) $G \times G$-equivariant.

Now let $\mathcal{F} \in \mathrm{Perv}_G^{\mathrm{weak}}(X, \Bbbk)$, and let θ be as in Step 4. To finish the proof, we must show that the diagram

(6.2.10)
$$
\begin{tikzcd}
\mathrm{pr}_{23}^\dagger \mathrm{pr}_2^\dagger \mathcal{F} \cong m^\dagger \mathrm{pr}_2^\dagger \mathcal{F} \arrow[rr, "m^\dagger \theta", "\sim"'] \arrow[dr, "\sim", "\mathrm{pr}_{23}^\dagger \theta"'] & & m^\dagger \sigma^\dagger \mathcal{F} \cong b^\dagger \sigma^\dagger \mathcal{F} \\
& \mathrm{pr}_{23}^\dagger \sigma^\dagger \mathcal{F} \cong b^\dagger \mathrm{pr}_2^\dagger \mathcal{F} \arrow[ur, "\sim", "b^\dagger \theta"'] &
\end{tikzcd}
$$

commutes. Since θ is an isomorphism, all three arrows in this diagram are isomorphisms as well, so all three objects are (weakly) $G \times G$-equivariant. Let $s' : X \to G \times G \times X$ be the map given by $s'(x) = (e, e, x)$. By Step 3 (with G replaced by $G \times G$), the commutativity of (6.2.10) can be checked after applying $(s')^*[-2 \dim G]$. Since $m \circ s' = b \circ s' = \mathrm{pr}_{23} \circ s' = s$, the commutativity of (6.2.10) would follow if we knew that

$$s^*\theta \circ s^*\theta = s^*\theta.$$

But this is obvious from Step 4. □

Exercises.

6.2.1. Let G be a discrete topological group, and let X be a topological space with a continuous G-action. Show that Definition 6.2.1 is equivalent to the two "naïve" definitions proposed at the beginning of this section.

6.2.2. Let X be a G-variety, and let $(\mathcal{F}, \theta)$ be an equivariant sheaf or perverse sheaf. Let $s : X \to G \times X$ be the map $s(x) = (e, x)$, where $e \in G$ is the identity element. Note that $\sigma \circ s = \mathrm{pr}_2 \circ s = \mathrm{id}_X$, so the map $s^*\theta : s^*\mathrm{pr}_2^*\mathcal{F} \to s^*\sigma^*\mathcal{F}$ can be regarded as an automorphism of $\mathcal{F}$. Show that $s^*\theta = \mathrm{id}_{\mathcal{F}}$.

6.2.3. Let X be a G-variety. Let $H \lhd G$ be a normal subgroup such that X is a principal H-variety, and let $\pi : X \to X/H$ be the quotient map. Prove that $\pi^\dagger$ induces an equivalence of categories

$$\pi^\dagger : \mathrm{Perv}_{G/H}(X/H, \Bbbk) \xrightarrow{\sim} \mathrm{Perv}_G(X, \Bbbk).$$

6.2.4. Prove Proposition 6.2.11. *Hint*: Let $G \times H$ act on $G \times X$ by $(g, h) \cdot (g', x) = (gg'h^{-1}, h \cdot x)$, and then compare both sides to $\mathrm{Perv}_{G \times H}(G \times X, \Bbbk)$ using Exercise 6.2.3.

6.2.5. Let $\varphi : G \to H$ be a homomorphism of algebraic groups, and let X be an H-variety. Explain how to define a functor $\mathrm{Res}_\varphi : \mathrm{Perv}_H(X, \Bbbk) \to \mathrm{Perv}_G(X, \Bbbk)$ that generalizes both the forgetful and inflation functors.

6.3. Twisted equivariance

This section is devoted to a variant of Definition 6.2.3. Neither this variant nor the results of this section will be used elsewhere in this chapter. (They will, however, be needed in Chapter 8.)

Let G be a connected algebraic group. In this section, we will call a homomorphism $\chi : \pi_1(G, 1) \to \Bbbk^\times$ a **character** of $\pi_1(G, 1)$. Any character χ of $\pi_1(G, 1)$ gives rise to a rank-1 locally free local system on G, denoted by $\mathcal{A}_\chi$. Characters themselves form a group, under pointwise multiplication in $\Bbbk^\times$. To make the notation a bit less cluttered, we will write $\mathcal{A}_\chi^{-1}$ instead of $\mathcal{A}_{\chi^{-1}}$ for the local system associated to the inverse of a character χ. It follows from Proposition 1.7.11 that

$$\mathcal{A}_{\chi_1} \otimes \mathcal{A}_{\chi_2} \cong \mathcal{A}_{\chi_1\chi_2}.$$

If χ is the trivial character, then $\mathcal{A}_\chi \cong \underline{\Bbbk}_G$. More generally, for any character χ, we have $\mathcal{A}_\chi \otimes \mathcal{A}_\chi^{-1} \cong \underline{\Bbbk}_G$.

LEMMA 6.3.1. *Let $m_G : G \times G \to G$ be the multiplication map, and let $i_G : G \to G$ be the inverse map. For any character $\chi : \pi_1(G, 1) \to \Bbbk^\times$, there are canonical isomorphisms*

$$m_G^*\mathcal{A}_\chi \cong \mathcal{A}_\chi \boxtimes \mathcal{A}_\chi \qquad \textit{and} \qquad i_G^*\mathcal{A}_\chi \cong \mathcal{A}_{\chi^{-1}}.$$

Proof. We will prove the statement for m_G. The proof for i_G is very similar (and somewhat easier). Identify $\pi_1(G \times G, 1)$ with $\pi_1(G,1) \times \pi_1(G,1)$. It is well known that the homomorphism

$$\pi_1(m_G) : \pi_1(G,1) \times \pi_1(G,1) \to \pi_1(G,1) \tag{6.3.1}$$

induced by m_G is none other than the multiplication map for the group structure on $\pi_1(G,1)$. (The multiplication map is a group homomorphism because $\pi_1(G,1)$ is necessarily abelian; see, for instance, [**98**, Exercise 3.C.5].)

Let $\Bbbk_\chi$ be the module $\Bbbk$, regarded as a $\pi_1(G,1)$-module via χ. Observe that the $\pi_1(G,1) \times \pi_1(G,1)$-module $\Bbbk_\chi \otimes \Bbbk_\chi$ is isomorphic to $\Bbbk_\chi$, regarded as a $\pi_1(G,1) \times \pi_1(G,1)$-module via (6.3.1). Propositions 1.7.10 and 1.7.11 convert this observation into the desired isomorphism of local systems. $\square$

Definition 6.3.2. Let G be a connected group, and let $\chi : \pi_1(G,1) \to \Bbbk^\times$ be a character. Let X be a G-variety, and let m, b, and pr_{23} be as in (6.2.3). A (G,χ)**-equivariant perverse sheaf** on X is a pair $(\mathcal{F},\theta)$, where $\mathcal{F} \in \mathrm{Perv}(X,\Bbbk)$ and θ is an isomorphism

$$\theta : \mathcal{A}_\chi \boxtimes \mathcal{F} \xrightarrow{\sim} \sigma^*\mathcal{F}$$

such that the following diagram commutes:

$$\begin{tikzcd}[column sep=small]
& m_G^*\mathcal{A}_\chi \boxtimes \mathcal{F} \arrow[dr, "m^*\theta"] & \\
\mathcal{A}_\chi \boxtimes \mathcal{A}_\chi \boxtimes \mathcal{F} \arrow[ur, "\text{Lemma 6.3.1}"] \arrow[dr, "\mathrm{id}_{\mathcal{A}_\chi} \boxtimes \theta"'] & & m^*\sigma^*\mathcal{F} \cong b^*\sigma^*\mathcal{F} \\
& \mathcal{A}_\chi \boxtimes \sigma^*\mathcal{F} \cong b^*(\mathcal{A}_\chi \boxtimes \mathcal{F}) \arrow[ur, "b^*\theta"'] &
\end{tikzcd}$$

A **morphism** of (G,χ)-equivariant perverse sheaves $\phi : (\mathcal{F},\theta_\mathcal{F}) \to (\mathcal{G},\theta_\mathcal{G})$ is a morphism of perverse sheaves $\phi : \mathcal{F} \to \mathcal{G}$ such that the following diagram commutes:

$$\begin{tikzcd}
\mathcal{A}_\chi \boxtimes \mathcal{F} \arrow[r, "\theta_\mathcal{F}"] \arrow[d, "\mathrm{id}_{\mathcal{A}_\chi} \boxtimes \phi"'] & \sigma^*\mathcal{F} \arrow[d, "\sigma^*\phi"] \\
\mathcal{A}_\chi \boxtimes \mathcal{G} \arrow[r, "\theta_\mathcal{G}"] & \sigma^*\mathcal{G}
\end{tikzcd} \tag{6.3.2}$$

The category of (G,χ)-equivariant perverse sheaves on X with coefficients in $\Bbbk$ is denoted by $\mathrm{Perv}_{G,\chi}(X,\Bbbk)$.

It is easy to see that if χ is the trivial character (so that $\mathcal{A}_\chi \cong \underline{\Bbbk}_G$), then this definition reduces to Definition 6.2.3. There is an obvious forgetful functor

$$\mathrm{For} = \mathrm{For}^{G,\chi} : \mathrm{Perv}_{G,\chi}(X,\Bbbk) \to \mathrm{Perv}(X,\Bbbk).$$

Lemma 6.3.3. *Let G be a connected algebraic group, and let X be a variety. Let G act on $G \times X$ by $g \cdot (h,x) = (gh,x)$.*

1. *Let $\chi_1,\chi_2 : \pi_1(G,1) \to \Bbbk^\times$ be two characters. There is an equivalence of categories $\mathrm{Perv}_{G,\chi_1}(G \times X,\Bbbk) \to \mathrm{Perv}_{G,\chi_2}(G \times X,\Bbbk)$ given on the underlying perverse sheaves by*

$$\mathcal{F} \mapsto (\mathcal{A}_{\chi_2\chi_1^{-1}} \boxtimes \underline{\Bbbk}_X) \overset{L}{\otimes} \mathcal{F}. \tag{6.3.3}$$

2. *Let $\chi : \pi_1(G,1) \to \Bbbk^\times$ be a character. There is an equivalence of categories $\mathrm{Perv}(X,\Bbbk) \to \mathrm{Perv}_{G,\chi}(G \times X,\Bbbk)$ given by $\mathcal{F} \mapsto \mathcal{A}_\chi \boxtimes \mathcal{F}$.*

Proof. (1) Let $(\mathcal{F}, \theta)$ be a (G, χ_1)-equivariant perverse sheaf on $G \times X$. Let $\mathcal{F}' = (\mathcal{A}_{\chi_2\chi_1^{-1}} \boxtimes \underline{\Bbbk}_X) \otimes^L \mathcal{F}$. The action map $\sigma : G \times G \times X \to G \times X$ is just $m_G \times \mathrm{id}_X$. Consider the following sequence of isomorphisms:

$$\begin{aligned}\sigma^*\mathcal{F}' &\cong \sigma^*(\mathcal{A}_{\chi_2\chi_1^{-1}} \boxtimes \underline{\Bbbk}_X) \overset{L}{\otimes} \sigma^*\mathcal{F} \cong (m_G^*\mathcal{A}_{\chi_2\chi_1^{-1}} \boxtimes \underline{\Bbbk}_X) \overset{L}{\otimes} \sigma^*\mathcal{F} \\ &\xrightarrow[\sim]{\text{Lemma 6.3.1}} (\mathcal{A}_{\chi_2\chi_1^{-1}} \boxtimes \mathcal{A}_{\chi_2\chi_1^{-1}} \boxtimes \underline{\Bbbk}_X) \overset{L}{\otimes} \sigma^*\mathcal{F} \\ &\xrightarrow[\sim]{\mathrm{id}\otimes\theta^{-1}} (\mathcal{A}_{\chi_2\chi_1^{-1}} \boxtimes \mathcal{A}_{\chi_2\chi_1^{-1}} \boxtimes \underline{\Bbbk}_X) \overset{L}{\otimes} (\mathcal{A}_{\chi_1} \boxtimes \mathcal{F}) \\ &\cong \mathcal{A}_{\chi_2} \boxtimes ((\mathcal{A}_{\chi_2\chi_1^{-1}} \boxtimes \underline{\Bbbk}_X) \overset{L}{\otimes} \mathcal{F}) \cong \mathcal{A}_{\chi_2} \boxtimes \mathcal{F}'.\end{aligned}$$

Define $\theta' : \mathcal{A}_{\chi_2} \boxtimes \mathcal{F}' \to \sigma^*\mathcal{F}'$ to be the inverse of the composition of the isomorphisms above. It is left to the reader to check that $(\mathcal{F}', \theta')$ satisfies the condition in (6.3.2). This shows that the formula (6.3.3) gives a well-defined functor from $\mathrm{Perv}_{G,\chi_1}(G \times X, \Bbbk) \to \mathrm{Perv}_{G,\chi_2}(G \times X, \Bbbk)$. It has an obvious inverse (given by exchanging the roles of χ_1 and χ_2 in (6.3.3)), so it is an equivalence of categories.

(2) In the special case where χ is trivial, this follows from Proposition 6.2.10. By composing this equivalence with one from part (1), we obtain the desired result for any χ. □

The proof of the following proposition is left as an exercise.

Proposition 6.3.4. *Let G be a connected algebraic group, and let $\chi, \chi' : \pi_1(G, 1) \to \Bbbk^\times$ be characters. Let $f : X \to Y$ be a G-equivariant map of G-varieties.*

1. *For $\mathcal{F} \in \mathrm{Perv}_{G,\chi}(X, \Bbbk)$, the objects ${}^p\mathsf{H}^i(f_*\mathrm{For}(\mathcal{F}))$ and ${}^p\mathsf{H}^i(f_!\mathrm{For}(\mathcal{F}))$ can naturally be regarded as (G, χ)-equivariant perverse sheaves.*
2. *For $\mathcal{F} \in \mathrm{Perv}_{G,\chi}(Y, \Bbbk)$, the objects ${}^p\mathsf{H}^i(f^*\mathrm{For}(\mathcal{F}))$ and ${}^p\mathsf{H}^i(f^!\mathrm{For}(\mathcal{F}))$ can naturally be regarded as (G, χ)-equivariant perverse sheaves.*
3. *For $\mathcal{F} \in \mathrm{Perv}_{G,\chi}(X, \Bbbk)$ and $\mathcal{G} \in \mathrm{Perv}_{G,\chi'}(X, \Bbbk)$, the object ${}^p\mathsf{H}^i(\mathrm{For}(\mathcal{F}) \otimes^L \mathrm{For}(\mathcal{G}))$ can naturally be regarded as a $(G, \chi\chi')$-equivariant perverse sheaf. Similarly, the object ${}^p\mathsf{H}^i R\mathscr{H}om(\mathrm{For}(\mathcal{F}), \mathrm{For}(\mathcal{G}))$ can naturally be regarded as a $(G, \chi^{-1}\chi')$-equivariant perverse sheaf.*

Remark 6.3.5. One can obtain twisted versions of many other sheaf-theoretic operations by combining various ingredients from Proposition 6.3.4. For instance, if $h : X \hookrightarrow Y$ is a G-equivariant locally closed embedding and if $\mathcal{F} \in \mathrm{Perv}(X, \Bbbk)$ admits a (G, χ)-equivariant structure, then $h_{!*}\mathcal{F}$ can naturally be regarded as a (G, χ)-equivariant perverse sheaf.

The next three propositions are twisted counterparts of Propositions 6.2.6, 6.2.7, 6.2.10, 6.2.15, and 6.2.17 (see also Exercise 6.2.3). The proofs are nearly identical to those in the untwisted case: the only change is that all occurrences of $\mathrm{pr}_2^\dagger$ must be replaced by $\mathcal{A}_\chi \boxtimes (-)$. We omit further details.

Proposition 6.3.6. *Let G be a connected group, and let $\chi : \pi_1(G, 1) \to \Bbbk^\times$ be a character. Let $f : X \to Y$ be a smooth surjective G-equivariant morphism of G-varieties. The functor*

$$f^\dagger : \mathrm{Perv}_{G,\chi}(Y, \Bbbk) \to \mathrm{Perv}_{G,\chi}(X, \Bbbk)$$

is faithful. If f has connected fibers, it is fully faithful, and its image is closed under subobjects and quotients.

Proposition 6.3.7. *Let G be a connected group, and let $\chi : \pi_1(G,1) \to \Bbbk^\times$ be a character. For any G-variety X, the forgetful functor*

$$\mathrm{For}^{G,\chi} : \mathrm{Perv}_{G,\chi}(X,\Bbbk) \to \mathrm{Perv}(X,\Bbbk)$$

is fully faithful, and its image is closed under subobjects and quotients. A perverse sheaf $\mathcal{F} \in \mathrm{Perv}(X,\Bbbk)$ lies in the image of this functor if and only if $\sigma^\dagger\mathcal{F}$ is isomorphic to $\mathcal{A}_\chi \boxtimes \mathcal{F}$ in $\mathrm{Perv}(G \times X, \Bbbk)$.

Proposition 6.3.8. *Let G and H be connected algebraic groups. Let X be a $G \times H$-variety that is also a principal G-variety, and let $\pi : X \to X/G$ be the quotient map. For any character $\chi : \pi_1(H,1) \to \Bbbk^\times$, the functor $\pi^\dagger$ induces an equivalence of categories*

$$\pi^\dagger : \mathrm{Perv}_{H,\chi}(X/G,\Bbbk) \xrightarrow{\sim} \mathrm{Perv}_{G\times H,1\otimes\chi}(X,\Bbbk).$$

We also have a twisted variant of Proposition 6.2.11.

Proposition 6.3.9. *Let G be a connected algebraic group, and let $H \subset G$ be a connected closed subgroup. Let X be an H-variety, and let $i : X \to G \times^H X$ be the map $i(x) = (e,x)$. For any character $\chi : \pi_1(G,1) \to \Bbbk^\times$, the functor $i^*[-\dim G/H]$ induces an equivalence of categories*

$$i^*[-\dim G/H] : \mathrm{Perv}_{G,\chi}(G \times^H X,\Bbbk) \xrightarrow{\sim} \mathrm{Perv}_{H,\chi\circ\iota}(X,\Bbbk).$$

Proof. Let $G \times H$ act on $G \times X$ by $(g,h)\cdot(g',x) = (gg'h^{-1}, h\cdot x)$. Using Lemma 6.3.1 and the fact that that $\mathcal{A}_\chi|_H \cong \mathcal{A}_{\chi\iota}$, one can check that $\mathcal{A}_\chi \boxtimes \underline{\Bbbk}_X$ is $(G\times H, \chi\otimes(\chi\iota)^{-1})$-equivariant. Let $\mathrm{pr}_2 : G\times X \to X$ be the projection map, and let $\pi : G \times X \to G\times^H X$ be the quotient map. Consider the following composition of functors:

$$\begin{aligned}(6.3.4)\quad & \mathrm{Perv}_{H,\chi\iota}(X,\Bbbk) \xrightarrow[\text{Prop. 6.3.8}]{\mathrm{pr}_2^\dagger} \mathrm{Perv}_{G\times H,1\otimes\chi\iota}(G\times X,\Bbbk) \\ & \xrightarrow[\text{Prop. 6.3.4}]{(\mathcal{A}_\chi\boxtimes\underline{\Bbbk}_X)\otimes^L(-)} \mathrm{Perv}_{G\times H,\chi\otimes 1}(G\times X,\Bbbk) \xrightarrow[\text{Prop. 6.3.8}]{(\pi^\dagger)^{-1}} \mathrm{Perv}_{G,\chi}(G\times^H X,\Bbbk).\end{aligned}$$

Every functor here is an equivalence. The composition of the first two functors can be written concisely as $\mathcal{A}_\chi[\dim G]\boxtimes(-)$. Let $\mathrm{in}_2 : X \to G\times X$ be the map given by $\mathrm{in}_2(x) = (e,x)$. For any $\mathcal{F} \in D^{\mathrm{b}}_{\mathrm{c}}(X,\Bbbk)$, we have $\mathrm{in}_2^*(\mathcal{A}_\chi \boxtimes \mathcal{F}) \cong (\mathcal{A}_\chi)_e \otimes^L \mathcal{F} \cong \mathcal{F}$. From this, one can deduce that the inverse of the composition of all three functors in (6.3.4) is

$$\mathrm{in}_2^*[-\dim G]\pi^\dagger \cong i^*[-\dim G/H] : \mathrm{Perv}_{G,\chi}(G\times^H X,\Bbbk) \to \mathrm{Perv}_{H,\chi\iota}(X,\Bbbk),$$

as desired. □

The next two statements establish a kind of "rigidity" for characters of twisted equivariant perverse sheaves with field coefficents.

Lemma 6.3.10. *Let G be a connected algebraic group, and let $\Bbbk$ be a field. If $\chi : \pi_1(G,1) \to \Bbbk^\times$ is a nontrivial character, then $\mathrm{Perv}_{G,\chi}(\mathrm{pt},\Bbbk)$ consists of just the zero object.*

Proof. Since $\Bbbk$ is a field, any perverse sheaf on a point is a direct sum of copies of $\underline{\Bbbk}_{\mathrm{pt}}$. If $\mathrm{Perv}_{G,\chi}(\mathrm{pt},\Bbbk)$ contains a nonzero object, it must contain $\underline{\Bbbk}_{\mathrm{pt}}$. But $\sigma^*\underline{\Bbbk}_{\mathrm{pt}} \cong \underline{\Bbbk}_G$ and $\mathcal{A}_\chi \boxtimes \underline{\Bbbk}_{\mathrm{pt}} \cong \mathcal{A}_\chi$ are not isomorphic to one another. □

Proposition 6.3.11. *Let G be a connected algebraic group, let X be a G-variety, and let $\Bbbk$ be a field.*

(1) *Let $\mathcal{F} \in \mathrm{Perv}_{G,\chi_1}(X,\Bbbk)$ and $\mathcal{G} \in \mathrm{Perv}_{G,\chi_2}(X,\Bbbk)$. If $\chi_1 \neq \chi_2$, then $\mathrm{Hom}_{D^{\mathrm{b}}_{\mathrm{c}}(X,\Bbbk)}(\mathrm{For}(\mathcal{F}), \mathrm{For}(\mathcal{G})[n]) = 0$ for all $n \in \mathbb{Z}$.*
(2) *If $\mathcal{F} \in \mathrm{Perv}(X,\Bbbk)$ is nonzero, there is at most one character $\chi : \pi_1(G,1) \to \Bbbk^\times$ such that $\mathcal{F}$ admits the structure of a (G,χ)-equivariant perverse sheaf.*

Proof. (1) By Proposition 6.3.4, the $\Bbbk$-vector space

$$\begin{aligned}\mathrm{Hom}(\mathrm{For}(\mathcal{F}), \mathrm{For}(\mathcal{G})[n]) &\cong \mathsf{H}^n R\mathrm{Hom}(\mathrm{For}(\mathcal{F}), \mathrm{For}(\mathcal{G})) \\ &\cong \mathsf{H}^n(\mathrm{a}_{X*} R\mathscr{H}om(\mathrm{For}(\mathcal{F}), \mathrm{For}(\mathcal{G})))\end{aligned}$$

naturally has the structure of a $(G, \chi_1^{-1}\chi_2)$-equivariant perverse sheaf on a point. But $\chi_1^{-1}\chi_2$ is a nontrivial character, so by Lemma 6.3.10, this vector space is zero.

(2) If $\mathcal{F}$ were both (G,χ_1)- and (G,χ_2)-equivariant, the identity map $\mathrm{id} : \mathcal{F} \to \mathcal{F}$ would contradict part (1). □

Twisted equivariance from central extensions. One situation in which twisted equivariance can emerge from the study of ordinary equivariance is the following. Let G be a connected group, and let $F \lhd G$ be a finite normal subgroup. Let X be a G/F-variety. A G-equivariant perverse sheaf on X may or may not admit a G/F-equivariant structure, i.e., it may or may not be in the image of the inflation functor $\mathrm{Infl}^G_{G/F}$ (see Proposition 6.2.16). In some cases, a G-equivariant perverse sheaf that does not admit a G/F-equivariant structure may nevertheless admit a *twisted* G/F-equivariant structure.

The quotient map $\varphi : G \to G/F$ is a covering map. By considering the multiplication action of F on G, one can see that it is in fact a regular covering whose Galois group is identified with F, and hence that there is a short exact sequence

$$1 \to \pi_1(G,1) \xrightarrow{\pi_1(\varphi)} \pi_1(G/F,1) \to F \to 1. \tag{6.3.5}$$

Any character $\chi : F \to \Bbbk^\times$ gives rise to a character of $\pi_1(G/F,1)$ by composing with the second map above. In a minor abuse of language, we may speak of $(G/F,\chi)$-equivariant perverse sheaves where χ is a character of F rather than of $\pi_1(G/F,1)$. Lemma 6.3.12 below highlights the importance of characters of this kind.

For the remainder of this section, we work with field coefficients, so that Proposition 6.3.11 is available.

Lemma 6.3.12. *Let G be a connected algebraic group, and let $F \lhd G$ be a finite normal subgroup. Let X be a G/F-variety, and let $\mathcal{F} \in \mathrm{Perv}_G(X,\Bbbk)$, where $\Bbbk$ is a field. If $\mathrm{For}^G(\mathcal{F}) \in \mathrm{Perv}(X,\Bbbk)$ admits a $(G/F,\chi)$-equivariant structure for some $\chi : \pi_1(G/F,1) \to \Bbbk^\times$, then χ factors through the map $\pi_1(G/F,1) \to F$.*

Proof. According to Exercise 6.3.1 below, one can define an inflation functor $\mathrm{Infl}^G_{G/F} : \mathrm{Perv}_{G/F,\chi}(X,\Bbbk) \to \mathrm{Perv}_{G,\chi\circ\pi_1(\varphi)}(X,\Bbbk)$, where $\pi_1(\varphi)$ is the first map in (6.3.5). Now consider the diagram

$$\begin{tikzcd} \mathrm{Perv}_G(X,\Bbbk) \arrow[dr, "\mathrm{For}"'] & \mathrm{Perv}_{G,\chi\circ\pi_1(\varphi)}(X,\Bbbk) \arrow[d, "\mathrm{For}"'] & \mathrm{Perv}_{G/F,\chi}(X,\Bbbk) \arrow[l, "\mathrm{Infl}^G_{G/F}"'] \arrow[dl, "\mathrm{For}"] \\ & \mathrm{Perv}(X,\Bbbk) & \end{tikzcd}$$

The assumptions of the lemma imply that $\mathrm{For}^G(\mathcal{F}) \in \mathrm{Perv}(X, \Bbbk)$ is actually in the image of all three forgetful functors. But then Proposition 6.3.11(2) tells us that $\chi \circ \pi_1(\varphi)$ must be trivial. In other words, χ factors through $\pi_1(G/F, 1) \to F$. $\square$

DEFINITION 6.3.13. Let G be a connected algebraic group, and let $F \lhd G$ be a finite normal subgroup. Let X be a G/F-variety, and let $\chi : F \to \Bbbk^\times$ be a character, where $\Bbbk$ is a field. A G-equivariant perverse sheaf $\mathcal{F} \in \mathrm{Perv}_G(X, \Bbbk)$ is said to have **F-character** χ if $\mathrm{For}^G(\mathcal{F}) \in \mathrm{Perv}(X, \Bbbk)$ admits a $(G/F, \chi)$-equivariant structure.

PROPOSITION 6.3.14. *Let G be a connected algebraic group, and let $F \lhd G$ be a finite normal subgroup. Let X be a homogeneous space for G/F, say $X = (G/F)/(H/F) = G/H$. Let $\Bbbk$ be a field, let $E \in \Bbbk[H/H^\circ]\text{-mod}^{\mathrm{fg}}$, and let $\mathcal{L}_E \in \mathrm{Loc}^{\mathrm{ft}}_G(X, \Bbbk)$ be the corresponding local system. Let $\chi : F \to \Bbbk^\times$ be a character. The following conditions are equivalent:*

1. *The local system $\mathcal{L}_E$ has F-character χ.*
2. *Via the natural map $F \to H/H^\circ$, F acts on E by the character χ.*

In the case where χ is trivial, this proposition just says that $\mathcal{L}_E$ is G/F-equivariant if and only if E descends to a representation of $(H/F)/(H/F)^\circ$.

PROOF. As usual, we let $\varphi : G \to G/F$ be the quotient map. We also let $q : G \to X$ and $q' : G/F \to X$ be the quotient maps, and we let $\sigma : G \times X \to X$ and $\sigma' : G/F \times X \to X$ be the action maps. We will write 1 for the base point of X (as well as for the identity elements of G and G/F). Since G is connected, we saw in Remark 6.2.14 that H/H° is naturally a quotient of $\pi_1(X, 1)$. It is left as an exercise to show that the following diagram commutes:

$$\begin{tikzcd} \pi_1(G/F,1) \arrow[r, "\pi_1(q')"] \arrow[d, "u"'] & \pi_1(X,1) \arrow[d, "v"] \\ F \arrow[r] & H/H^\circ \end{tikzcd} \tag{6.3.6}$$

Consider the following commutative diagram of spaces, and its accompanying diagram of fundamental groups:

$$\begin{tikzcd} G \times G \arrow[r, "m_G"] \arrow[d, "\varphi \times q"'] & G \arrow[d, "q"] \\ G/F \times X \arrow[r, "\sigma'"] & X \end{tikzcd} \qquad \begin{tikzcd} \pi_1(G,1) \times \pi_1(G,1) \arrow[r, "\pi_1(m_G)"] \arrow[d, "\pi_1(\varphi)\times\pi_1(q)"'] & \pi_1(G,1) \arrow[d, "\pi_1(q)"] \\ \pi_1(G/F,1) \times \pi_1(X,1) \arrow[r, "\pi_1(\sigma')"] & \pi_1(X,1) \end{tikzcd}$$

As in the proof of Lemma 6.3.1, $\pi_1(m_G)$ is just the multiplication map on $\pi_1(G, 1)$. Similar considerations show that

$$\pi_1(\sigma') : \pi_1(G/F, 1) \times \pi_1(X, 1) \to \pi_1(X, 1) \tag{6.3.7}$$

is given by $(\gamma, \xi) \mapsto \pi_1(q')(\gamma) \cdot \xi$. (For this to make sense as a group homomorphism, we need to know that the image of the map $\pi_1(q') : \pi_1(G/F, 1) \to \pi_1(X, 1)$ is contained in the center of $\pi_1(X, 1)$. This can be proved by a variant of the usual proof that $\pi_1(G/F, 1)$ is abelian; see [**98**, Exercise 3.C.5].)

Combining (6.3.6) and (6.3.7), we obtain a commutative diagram

$$\begin{tikzcd} \pi_1(G/F,1)\times\pi_1(X,1) \arrow[r, "\pi_1(\sigma')"] \arrow[d, "u\times v"'] & \pi_1(X,1) \arrow[d, "v"] \\ F\times H/H^\circ \arrow[r, "\varsigma"] & H/H^\circ \end{tikzcd} \tag{6.3.8}$$

where ς is given by multiplication.

Consider the H/H°-representation E. We denote its pullback along ς by ς^*E, and likewise for the other maps in this diagram. Observe that F acts on E via χ if and only if $\varsigma^*E \cong \chi\otimes E$. On the other hand, using the characterization at the end of Proposition 6.3.7, we see that $\mathcal{L}_E$ is $(G/F,\chi)$-equivariant if and only if $\pi_1(\sigma')^*v^*E$ is isomorphic to $\chi\otimes v^*E$. To finish the proof, we must show that $\pi_1(\sigma')^*v^*E \cong \chi\otimes v^*E$ if and only if $\varsigma^*E\cong\chi\otimes E$. This claim follows from the commutativity of (6.3.8). □

Exercises.

6.3.1. Let G be a connected algebraic group, and let $H \lhd G$ be a normal subgroup. The quotient map $\varphi : G \to G/H$ induces a map

$$\pi_1(\varphi) : \pi_1(G,1) \to \pi_1(G/H,1).$$

Let $\chi : \pi_1(G/H,1) \to \Bbbk^\times$ be a character, and let X be a G/H-variety. Explain how to define an inflation functor

$$\mathrm{Infl}_{G/H}^G : \mathrm{Perv}_{G/H,\chi}(X,\Bbbk) \to \mathrm{Perv}_{G,\chi\circ\pi_1(\varphi)}(X,\Bbbk),$$

and show that it is fully faithful.

6.3.2. Give an example showing that when $\Bbbk$ is not a field, the conclusion of Lemma 6.3.10 can be false.

6.4. Equivariant derived categories

The derived category problem. In Section 6.2, we have defined the abelian category of equivariant perverse sheaves, but for many purposes, we need to be able to work in a suitable derived or triangulated category.

As a first guess, one might try simply copying Definitions 6.2.1 or 6.2.3, but allow $\mathcal{F}$ to be a general object of $D^{\mathrm{b}}(X,\Bbbk)$ or $D^{\mathrm{b}}_{\mathrm{c}}(X,\Bbbk)$. This turns out not to work, because the category defined in this way is not, in general, triangulated! (See Exercise 6.4.2.)

A second guess might be to work in the derived category of the abelian category of equivariant sheaves or perverse sheaves: either $D^{\mathrm{b}}\mathrm{Sh}_G(X,\Bbbk)$ or $D^{\mathrm{b}}\mathrm{Perv}_G(X,\Bbbk)$. These are both at least triangulated categories, but they lack some desirable properties. To make this explicit, let us spell out what the "desirable properties" are.

Desideratum 6.4.1. For each G-variety X, there should exist a triangulated category $D^{\mathrm{b}}_G(X,\Bbbk)$ with the following properties:

1. (1) It has a bounded t-structure whose heart is identified with $\mathrm{Perv}_G(X,\Bbbk)$.
2. (2) There is a t-exact "forgetful functor" $\mathrm{For}^G : D^{\mathrm{b}}_G(X,\Bbbk) \to D^{\mathrm{b}}_{\mathrm{c}}(X,\Bbbk)$ that restricts to the forgetful functor $\mathrm{For}^G : \mathrm{Perv}_G(X,\Bbbk) \to \mathrm{Perv}(X,\Bbbk)$.

In addition, the usual sheaf operations $\otimes^L$, $R\mathscr{H}om$, f^*, f_*, $f_!$, $f^!$ should exist. (The latter four are functors between $D^{\mathrm{b}}_G(X)$ and $D^{\mathrm{b}}_G(Y)$, where Y is another G-variety, and $f : X \to Y$ is a G-equivariant map.) All six operations should satisfy the adjunction properties and other natural relations from Chapters 1–3, and they should commute with For.

The following example shows that the derived category $D^{\mathrm{b}}\mathrm{Perv}_G(X, \Bbbk)$ may fail to satisfy Desideratum 6.4.1.

EXAMPLE 6.4.2. Consider the variety $\mathbb{P}^1$, equipped with the action of GL_2 that is induced by the obvious action on $\mathbb{C}^2$. This action is transitive; indeed, $\mathbb{P}^1$ can be identified with GL_2/B, where B is the stabilizer of the first coordinate axis in $\mathbb{C}^2$. (Thus, B is the group of upper triangular matrices.) Since B and GL_2 are both connected, by Proposition 6.2.13, we have

$$\mathrm{Perv}_{\mathrm{GL}_2}(\mathbb{P}^1, \Bbbk) \cong \Bbbk\text{-mod}^{\mathrm{fg}} \qquad \text{and} \qquad \mathrm{Perv}_{\mathrm{GL}_2}(\mathrm{pt}, \Bbbk) \cong \Bbbk\text{-mod}^{\mathrm{fg}}.$$

Now consider the map $\mathrm{a}_{\mathbb{P}^1} : \mathbb{P}^1 \to \mathrm{pt}$. We will show that there do not exist functors

$$(\mathrm{a}_{\mathbb{P}^1})_* : D^{\mathrm{b}}\mathrm{Perv}_{\mathrm{GL}_2}(\mathbb{P}^1, \Bbbk) \to D^{\mathrm{b}}\mathrm{Perv}_{\mathrm{GL}_2}(\mathrm{pt}, \Bbbk),$$
$$\mathrm{a}^*_{\mathbb{P}^1} : D^{\mathrm{b}}\mathrm{Perv}_{\mathrm{GL}_2}(\mathrm{pt}, \Bbbk) \to D^{\mathrm{b}}\mathrm{Perv}_{\mathrm{GL}_2}(\mathbb{P}^1, \Bbbk)$$

satisfying Desideratum 6.4.1. Suppose there did exist such functors. Assume for simplicity that $\Bbbk$ is a field, so that $\Bbbk\text{-mod}^{\mathrm{fg}}$ is a semisimple category, and the higher Ext-groups vanish in both $\mathrm{Perv}_{\mathrm{GL}_2}(\mathbb{P}^1, \Bbbk)$ and $\mathrm{Perv}_{\mathrm{GL}_2}(\mathrm{pt}, \Bbbk)$.

Because these functors commute with $\mathrm{For}^{\mathrm{GL}_2}$, $\mathrm{a}^*_{\mathbb{P}^1}[1]$ is t-exact, and it is easy to see from the descriptions above that

$$\mathrm{a}^*_{\mathbb{P}^1}[1] : \mathrm{Perv}_{\mathrm{GL}_2}(\mathrm{pt}, \Bbbk) \to \mathrm{Perv}_{\mathrm{GL}_2}(\mathbb{P}^1, \Bbbk)$$

is an equivalence of categories. Then, for any $\mathcal{F}, \mathcal{G} \in \mathrm{Perv}_{\mathrm{GL}_2}(\mathrm{pt}, \Bbbk)$, the map

$$\mathrm{Ext}^n_{\mathrm{Perv}_{\mathrm{GL}_2}(\mathrm{pt},\Bbbk)}(\mathcal{F}, \mathcal{G}) \to \mathrm{Ext}^n_{\mathrm{Perv}_{\mathrm{GL}_2}(\mathbb{P}^1,\Bbbk)}(\mathrm{a}^*_{\mathbb{P}^1}\mathcal{F}[1], \mathrm{a}^*_{\mathbb{P}^1}\mathcal{G}[1])$$

is an isomorphism for all n (both sides vanish if $n \neq 0$). Therefore, by Propositions A.4.16 and A.4.17, the functor

$$\mathrm{a}^*_{\mathbb{P}^1} : D^{\mathrm{b}}\mathrm{Perv}_{\mathrm{GL}_2}(\mathrm{pt}, \Bbbk) \xrightarrow{\sim} D^{\mathrm{b}}\mathrm{Perv}_{\mathrm{GL}_2}(\mathbb{P}^1, \Bbbk)$$

is an equivalence of categories.

Finally, the desideratum tells us that $(\mathrm{a}_{\mathbb{P}^1})_*$ should be the right adjoint to $\mathrm{a}^*_{\mathbb{P}^1}$, so it must just be the inverse equivalence. In particular, we must have

$$(\mathrm{a}_{\mathbb{P}^1})_*\underline{\Bbbk}_{\mathbb{P}_1} \cong \underline{\Bbbk}_{\mathrm{pt}}.$$

But this disagrees with the usual (nonequivariant) setting, where we know that $\mathsf{H}^k((\mathrm{a}_{\mathbb{P}^1})_*\underline{\Bbbk}_{\mathbb{P}_1}) = \mathbf{H}^k(\mathbb{P}^1; \Bbbk)$ is nonzero for $k = 0$ and $k = 2$. In other words, we have found that $\mathrm{a}^*_{\mathbb{P}^1}$ does not have a right adjoint that commutes with $\mathrm{For}^{\mathrm{GL}_2}$.

EXAMPLE 6.4.3. Let X be a principal G-variety. Then Proposition 6.2.10 suggests that we might define $D^{\mathrm{b}}_G(X, \Bbbk)$ to be $D^{\mathrm{b}}_{\mathrm{c}}(X/G, \Bbbk)$. Indeed, according to that proposition, the heart of the perverse t-structure on $D^{\mathrm{b}}_{\mathrm{c}}(X/G, \Bbbk)$ is equivalent to $\mathrm{Perv}_G(X, \Bbbk)$. We can define For^G to be $\pi^\dagger : D^{\mathrm{b}}_{\mathrm{c}}(X/G, \Bbbk) \to D^{\mathrm{b}}_{\mathrm{c}}(X, \Bbbk)$, where $\pi : X \to X/G$ is the quotient map.

With this definition, For^G commutes with $\otimes^L$ and $R\mathscr{H}om$. Moreover, if Y is another principal G-variety, and $f : X \to Y$ is a G-equivariant morphism, then we can define the functor $f_* : D^{\mathrm{b}}_G(X, \Bbbk) \to D^{\mathrm{b}}_G(Y, \Bbbk)$ to simply be $\bar{f}_* : D^{\mathrm{b}}_{\mathrm{c}}(X/G, \Bbbk) \to$

$D^{\mathrm{b}}_{\mathrm{c}}(Y/G, \Bbbk)$, where $\bar{f} : X/G \to Y/G$ is the map from Proposition 6.1.16. Define the equivariant versions of f^*, $f_!$, and $f^!$ similarly. The functors f_*, f^*, $f_!$, and $f^!$ all commute with For^G, by the proper and smooth base change theorems applied to the cartesian square in Proposition 6.1.16.

Definition of the equivariant derived category. According to Desideratum 6.4.1, we expect theorems about sheaf functors from earlier in the book, such as Lemma 6.1.19, to remain true in (the as-yet undefined) equivariant derived categories. If $p : P \to X$ is an n-acyclic G-resolution (see Definitions 6.1.17 and 6.1.18), then by combining the anticipated equivariant version of Lemma 6.1.19 with Example 6.4.3, we should expect to obtain a fully faithful functor ${}^pD^{\mathrm{b}}_G(X, \Bbbk)^{[k,k+n]} \to {}^pD^{\mathrm{b}}_{\mathrm{c}}(P/G, \Bbbk)^{[k,k+n]}$. Thus, as n grows, we should be able to "see" more and more of $D^{\mathrm{b}}_G(X, \Bbbk)$ in terms of full subcategories of ordinary constructible derived categories.

This observation is the motivating principle behind the following definition, which is the main concept of this chapter.

Definition 6.4.4. Let X be a G-variety. A **G-equivariant complex** on X is a rule $\mathcal{F}$ consisting of the following data:

- for each G-resolution $p : P \to X$, an object $\mathcal{F}(p) \in D^{\mathrm{b}}_{\mathrm{c}}(P/G, \Bbbk)$;
- for any smooth morphism of resolutions $\nu : (P \xrightarrow{p} X) \to (Q \xrightarrow{q} X)$, an isomorphism $\alpha_\nu : \bar{\nu}^*\mathcal{F}(q) \xrightarrow{\sim} \mathcal{F}(p)$ (where $\bar{\nu} : P/G \to Q/G$ is the map induced by ν).

These data are subject to the following conditions:

(1) For the identity map $\mathrm{id} : (P \xrightarrow{p} X) \to (P \xrightarrow{p} X)$, we have $\alpha_{\mathrm{id}} = \mathrm{id}_{\mathcal{F}(p)}$.

(2) For any sequence $(P \xrightarrow{p} X) \xrightarrow{\nu} (Q \xrightarrow{q} X) \xrightarrow{\xi} (R \xrightarrow{r} X)$ of smooth morphisms of resolutions, we have

$$\alpha_\nu \circ \bar{\nu}^*\alpha_\xi = \alpha_{\xi\circ\nu} : \bar{\nu}^*\bar{\xi}^*\mathcal{F}(r) \xrightarrow{\sim} \mathcal{F}(p). \tag{6.4.1}$$

A **morphism** of G-equivariant complexes $\phi : \mathcal{F} \to \mathcal{G}$ consists of a morphism $\phi(p) : \mathcal{F}(p) \to \mathcal{G}(p)$ for each G-resolution $p : P \to X$, subject to the condition that for any smooth morphism of resolutions $\nu : (P \xrightarrow{p} X) \to (Q \xrightarrow{q} X)$, the following diagram commutes:

$$\begin{tikzcd} \bar{\nu}^*\mathcal{F}(q) \arrow[r, "\alpha_\nu"] \arrow[d, "\bar{\nu}^*\phi(q)"'] & \mathcal{F}(p) \arrow[d, "\phi(p)"] \\ \bar{\nu}^*\mathcal{G}(q) \arrow[r, "\alpha_\nu"] & \mathcal{G}(p) \end{tikzcd}$$

The category of G-equivariant complexes is called the **G-equivariant derived category** of X, and is denoted by $D^{\mathrm{b}}_G(X, \Bbbk)$.

Example 6.4.5. Let X be a G-variety, and let M be a finitely generated $\Bbbk$-module. The **equivariant constant sheaf** on X with value M is the object $\underline{M}_X \in D^{\mathrm{b}}_G(X, \Bbbk)$ defined as follows: for any G-resolution $p : P \to X$, we set

$$\underline{M}_X(p) = \underline{M}_P,$$

and for any smooth morphism of resolutions $\nu : (P \xrightarrow{p} X) \to (Q \xrightarrow{q} X)$, we define $\alpha_\nu : \bar{\nu}^*\underline{M}_Q \xrightarrow{\sim} \underline{M}_P$ to be the isomorphism from (1.2.6).

If we let G act on $G \times X$ by $g \cdot (h, x) = (hg^{-1}, g \cdot x)$, then the map $\mathrm{pr}_2 : G \times X \to X$ is a G-resolution of X, called the **trivial G-resolution**. Moreover, the quotient map $G \times X \to (G \times X)/G$ can be identified with the action map $\sigma : G \times X \to X$. Define the **forgetful functor**

$$\text{For} : D^{\mathrm{b}}_G(X, \Bbbk) \to D^{\mathrm{b}}_{\mathrm{c}}(X, \Bbbk) \tag{6.4.2}$$

by $\mathcal{F} \mapsto \mathcal{F}(\mathrm{pr}_2)$. The following lemma provides a link to Definition 6.2.3. (See Theorem 6.4.10 and Exercise 6.5.8 for more context.)

Lemma 6.4.6. *Let X be a G-variety. For any $\mathcal{F} \in D^{\mathrm{b}}_G(X, \Bbbk)$, there is a natural isomorphism $\theta : \mathrm{pr}_2^* \mathrm{For}(\mathcal{F}) \xrightarrow{\sim} \sigma^* \mathrm{For}(\mathcal{F})$ that satisfies* (6.2.5).

Proof. Make the space

$$G^n \times X = \underbrace{G \times \cdots \times G}_{n \text{ copies}} \times X$$

into a G-variety by the rule $g \cdot (h_1, \ldots, h_n, x) = (h_1 g^{-1}, \ldots, h_n g^{-1}, g \cdot x)$. Then the projection map $\mathrm{pr}_{n+1} : G^n \times X \to X$ is a G-resolution of X. (In the case where $n = 1$, this is the trivial resolution.) Next, let $\pi_n : G^n \times X \to G^{n-1} \times X$ be the map given by $\pi_n(h_1, \ldots, h_n, x) = (h_1 h_2^{-1}, h_2 h_3^{-1}, \ldots, h_{n-1} h_n^{-1}, h_n \cdot x)$. One can check that π_n is the quotient by the G-action on $G^n \times X$. Note that $\pi_1 = \sigma : G \times X \to X$.

In each of the diagrams below, the left-hand triangle is a smooth morphism of resolutions, the right-hand square is cartesian, and the rightmost column is the quotient by G of the middle column:

$$\begin{tikzcd} & G \times G \times X \arrow[r, "\pi_2"] \arrow[dl, "\mathrm{pr}_3"'] \arrow[d, "\mathrm{pr}_{23}"] & G \times X \arrow[d, "\mathrm{pr}_2"] \\ X & G \times X \arrow[l, "\mathrm{pr}_2"] \arrow[r, "\sigma"] & X \end{tikzcd} \qquad \begin{tikzcd} & G \times G \times X \arrow[r, "\pi_2"] \arrow[dl, "\mathrm{pr}_3"'] \arrow[d, "\mathrm{pr}_{13}"] & G \times X \arrow[d, "\sigma"] \\ X & G \times X \arrow[l, "\mathrm{pr}_2"] \arrow[r, "\sigma"] & X \end{tikzcd}$$

Thus, for any $\mathcal{F} \in D^{\mathrm{b}}_G(X, \Bbbk)$, we have maps $\alpha_{\mathrm{pr}_{23}} : \mathrm{pr}_2^* \mathrm{For}(\mathcal{F}) \xrightarrow{\sim} \mathcal{F}(\mathrm{pr}_3)$ and $\alpha_{\mathrm{pr}_{13}} : \sigma^* \mathrm{For}(\mathcal{F}) \xrightarrow{\sim} \mathcal{F}(\mathrm{pr}_3)$. We set

$$\theta = \alpha_{\mathrm{pr}_{13}}^{-1} \circ \alpha_{\mathrm{pr}_{23}} : \mathrm{pr}_2^* \mathrm{For}(\mathcal{F}) \to \sigma^* \mathrm{For}(\mathcal{F}).$$

To prove that θ satisfies (6.2.5), we consider the diagrams

$$\begin{tikzcd} G^3 \times X \arrow[r, "\pi_3"] \arrow[d, "\mathrm{pr}_{124}"'] & G^2 \times X \arrow[d, "b"] \\ G^2 \times X \arrow[r, "\pi_2"] & G \times X \end{tikzcd} \quad \begin{tikzcd} G^3 \times X \arrow[r, "\pi_3"] \arrow[d, "\mathrm{pr}_{134}"'] & G^2 \times X \arrow[d, "m"] \\ G^2 \times X \arrow[r, "\pi_2"] & G \times X \end{tikzcd} \quad \begin{tikzcd} G^3 \times X \arrow[r, "\pi_3"] \arrow[d, "\mathrm{pr}_{234}"'] & G^2 \times X \arrow[d, "\mathrm{pr}_{23}"] \\ G^2 \times X \arrow[r, "\pi_2"] & G \times X \end{tikzcd}$$

Again, the left-hand column of each of these squares is a smooth morphism of resolutions, and the right-hand column (with notation following (6.2.3)) is its quotient by G. Equation (6.2.5) is equivalent to the claim that

$$b^* \alpha_{\mathrm{pr}_{13}}^{-1} \circ b^* \alpha_{\mathrm{pr}_{23}} \circ \mathrm{pr}_{23}^* \alpha_{\mathrm{pr}_{13}}^{-1} \circ \mathrm{pr}_{23}^* \alpha_{\mathrm{pr}_{23}} = m^* \alpha_{\mathrm{pr}_{13}}^{-1} \circ m^* \alpha_{\mathrm{pr}_{23}}.$$

It is left as an exercise to deduce this claim from (6.4.1). $\square$

Triangulated structure. It is not immediately obvious from Definition 6.4.4 that $D^{\mathrm{b}}_G(X, \Bbbk)$ is a triangulated category! The rest of this section outlines the main steps in showing that it is triangulated. We omit most details of proofs, however, and refer the reader to [**27**] instead (see also Remark 6.4.11).

To get started, for $\mathcal{F} \in D^{\mathrm{b}}_G(X,\Bbbk)$, define $\mathcal{F}[1]$ to be the object given by $(\mathcal{F}[1])(p) = \mathcal{F}(p)[1]$ for any resolution $p : P \to X$. A diagram

$$\mathcal{F} \to \mathcal{G} \to \mathcal{H} \to \mathcal{F}[1]$$

in $D^{\mathrm{b}}_G(X,\Bbbk)$ is said to be a **distinguished triangle** if for every resolution $p : P \to X$, the corresponding diagram

$$\mathcal{F}(p) \to \mathcal{G}(p) \to \mathcal{H}(p) \to \mathcal{F}(p)[1]$$

is a distinguished triangle in $D^{\mathrm{b}}_{\mathrm{c}}(P/G,\Bbbk)$. Next, given two integers $a \le b$, let

$${}^pD^{\mathrm{b}}_G(X,\Bbbk)^{[a,b]} = \{\mathcal{F} \in D^{\mathrm{b}}_G(X,\Bbbk) \mid \mathrm{For}(\mathcal{F}) \in {}^pD^{\mathrm{b}}_{\mathrm{c}}(X,\Bbbk)^{[a,b]}\}.$$

LEMMA 6.4.7. *Let X be a G-variety. Let $p : P \to X$ be an n-acyclic resolution of relative dimension d, and let $d' = d - \dim G$. For any $k \in \mathbb{Z}$, the functor*

$${}^pD^{\mathrm{b}}_G(X,\Bbbk)^{[k,k+n]} \to {}^pD^{\mathrm{b}}_{\mathrm{c}}(P/G,\Bbbk)^{[k+d',k+d'+n]} \qquad \text{given by} \qquad \mathcal{F} \mapsto \mathcal{F}(p)$$

is fully faithful, and its image is the full subcategory

$$\left\{ \mathcal{G} \in {}^pD^{\mathrm{b}}_{\mathrm{c}}(P/G,\Bbbk)^{[k+d',k+d'+n]} \;\middle|\; \begin{array}{c} \text{there is an object } \mathcal{G}' \in {}^pD^{\mathrm{b}}_{\mathrm{c}}(X,\Bbbk)^{[k,k+n]} \\ \text{such that } p^*\mathcal{G}' \cong \pi_P^*\mathcal{G} \end{array} \right\},$$

where $\pi_P : P \to P/G$ is the quotient map.

PROOF SKETCH. Let $\mathcal{F}, \mathcal{G} \in {}^pD^{\mathrm{b}}_G(X,\Bbbk)^{[k,k+n]}$. Given a resolution $q : Q \to X$, consider the following maps:

$$\begin{array}{ccc} Q \times_X P & \xrightarrow{u} & P \\ {\scriptstyle v}\downarrow & & \downarrow{\scriptstyle p} \\ Q & \xrightarrow[q]{} & X \end{array} \qquad \begin{array}{ccc} (Q \times_X P)/G & \xrightarrow{\bar{u}} & P/G \\ {\scriptstyle \bar{v}}\downarrow & & \\ Q/G & & \end{array}$$

Let $r = pu = qv : Q \times_X P \to X$. The map v is n-acyclic, so by Lemma 6.1.24, $\bar{v}$ is as well, and then by Lemma 6.1.19, the map $\bar{v}^*$ below is an isomorphism:

$$\mathrm{Hom}(\mathcal{F}(p),\mathcal{G}(p)) \xrightarrow{\bar{u}^*} \mathrm{Hom}(\mathcal{F}(r),\mathcal{G}(r)) \xleftarrow[\sim]{\bar{v}^*} \mathrm{Hom}(\mathcal{F}(q),\mathcal{G}(q)).$$

Any morphism $\phi : \mathcal{F}(p) \to \mathcal{G}(p)$ determines a unique morphism $\theta : \mathcal{F} \to \mathcal{G}$ by the rule $\bar{v}^*\theta(q) = \bar{u}^*\phi$. This shows that $\mathcal{F} \mapsto \mathcal{F}(p)$ is fully faithful.

Given $\mathcal{G} \in {}^pD^{\mathrm{b}}_{\mathrm{c}}(P/G,\Bbbk)^{[k+d',k+d'+n]}$, one can try to define a candidate object $\mathcal{F} \in D^{\mathrm{b}}_G(X,\Bbbk)$ giving rise to it by the rule

$$\mathcal{F}(q) = {}^p\tau^{\le k+e-\dim G+n}(\bar{v}_*\bar{u}^*\mathcal{G}),$$

where e is the relative dimension of q. If there is some $\mathcal{G}' \in D^{\mathrm{b}}_{\mathrm{c}}(X,\Bbbk)$ with $p^*\mathcal{G}' \cong \pi_P^*\mathcal{G}$, then one can use $\mathcal{G}'$ to show that $\mathcal{F}$ is indeed a well-defined object of $D^{\mathrm{b}}_G(X,\Bbbk)$ and that $\mathcal{F}(p) \cong \mathcal{G}$. Thus, the image of the functor is as claimed in the lemma. $\square$

Lemma 6.4.7 says that part of $D^{\mathrm{b}}_G(X,\Bbbk)$ can be recovered from a given resolution. The same argument shows that all of $D^{\mathrm{b}}_G(X,\Bbbk)$ can be recovered from a suitable collection of resolutions. To make this precise, given a set $\mathscr{P}$ of G-resolutions of X, define

$$D^{\mathrm{b}}_G(X,\Bbbk,\mathscr{P})$$

by copying Definition 6.4.4, but only considering resolutions from $\mathscr{P}$, rather than all G-resolutions. For any such $\mathscr{P}$, there is an obvious functor $D^{\mathrm{b}}_G(X,\Bbbk) \to D^{\mathrm{b}}_G(X,\Bbbk,\mathscr{P})$.

Lemma 6.4.8. *Let X be a G-variety, and let $\mathscr{P}$ be a set of G-resolutions of X. Assume that the following conditions hold:*

(1) *If $p : P \to X$ and $q : Q \to X$ lie in $\mathscr{P}$, then $P \times_X Q \to X$ also lies in $\mathscr{P}$.*
(2) *For every $n \geq 0$, $\mathscr{P}$ contains an n-acyclic resolution.*

Then the functor $D^{\mathrm{b}}_G(X, \Bbbk) \to D^{\mathrm{b}}_G(X, \Bbbk, \mathscr{P})$ is fully faithful. If $\mathscr{P}$ also satisfies

(3) *The trivial resolution $\mathrm{pr}_2 : G \times X \to X$ lies in $\mathscr{P}$,*

then $D^{\mathrm{b}}_G(X, \Bbbk) \to D^{\mathrm{b}}_G(X, \Bbbk, \mathscr{P})$ is an equivalence of categories.

We omit the proof, as it is very close to that of Lemma 6.4.7.

Corollary 6.4.9. *Let X be a principal G-variety, and let $\pi : X \to X/G$ be the quotient map. There is an equivalence of categories $Q : D^{\mathrm{b}}_G(X, \Bbbk) \xrightarrow{\sim} D^{\mathrm{b}}_{\mathrm{c}}(X/G, \Bbbk)$ with the property that for $\mathcal{F} \in D^{\mathrm{b}}_G(X, \Bbbk)$, there is a natural isomorphism $\pi^* Q(\mathcal{F}) \cong \mathrm{For}(\mathcal{F})$.*

Proof sketch. This follows from Lemma 6.4.7 using the observation that for a principal G-variety X, the identity map $X \to X$ is an ∞-acyclic resolution. □

It is a common abuse of notation to denote the inverse functor of Q simply by

$$\pi^* : D^{\mathrm{b}}_{\mathrm{c}}(X/G, \Bbbk) \xrightarrow{\sim} D^{\mathrm{b}}_G(X, \Bbbk). \tag{6.4.3}$$

Explicitly, this functor can be described as follows: given $\mathcal{F} \in D^{\mathrm{b}}_{\mathrm{c}}(X/G, \Bbbk)$, and given a G-resolution $p : P \to X$, we have

$$Q^{-1}(\mathcal{F})(p) = \pi^*(\mathcal{F})(p) = \bar{p}^* \mathcal{F},$$

where $\bar{p} : P/G \to X/G$ is the map induced by p.

Theorem 6.4.10. *Let X be a G-variety. Then $D^{\mathrm{b}}_G(X, \Bbbk)$ has the structure of a triangulated category. In addition:*

(1) *The pair $({}^p D^{\mathrm{b}}_G(X, \Bbbk)^{\leq 0}, {}^p D^{\mathrm{b}}_G(X, \Bbbk)^{\geq 0})$ is a bounded t-structure whose heart is equivalent to $\mathrm{Perv}_G(X, \Bbbk)$.*
(2) *The functor $\mathrm{For} : D^{\mathrm{b}}_G(X, \Bbbk) \to D^{\mathrm{b}}_{\mathrm{c}}(X, \Bbbk)$ is a t-exact triangulated functor.*

Proof sketch. In view of Lemma 6.4.7, the axioms for a triangulated category hold in $D^{\mathrm{b}}_G(X, \Bbbk)$ because they hold in $D^{\mathrm{b}}_{\mathrm{c}}(P/G, \Bbbk)$ for every resolution $P \to X$. See [**27**, Proposition 2.5.2] for further details. According to [**27**, Section 5.1], there is a unique bounded t-structure on $D^{\mathrm{b}}_G(X, \Bbbk)$ whose heart $\mathscr{C}$ is given by

$$\mathscr{C} = \{\mathcal{F} \in D^{\mathrm{b}}_G(X, \Bbbk) \mid \mathrm{For}(\mathcal{F}) \in \mathrm{Perv}(X, \Bbbk)\}. \tag{6.4.4}$$

However, [**27**] does not contain a proof that $\mathscr{C}$ is equivalent to $\mathrm{Perv}_G(X, \Bbbk)$ as defined in Definition 6.2.3. Let us briefly outline the proof of this equivalence. Lemma 6.4.6 gives us an obvious functor $S : \mathscr{C} \to \mathrm{Perv}_G(X, \Bbbk)$.

Next, we will define a functor $T : \mathrm{Perv}_G(X, \Bbbk) \to \mathscr{C}$. Given $\mathcal{F} \in \mathrm{Perv}_G(X, \Bbbk)$, we must describe the value of $T(\mathcal{F})$ on any G-resolution $p : P \to X$. Suppose p is of relative dimension d, and let $\pi_P : P \to P/G$ be the quotient map. We define $T(\mathcal{F})(p) \in D^{\mathrm{b}}_{\mathrm{c}}(P/G, \Bbbk)$ to be the object such that $T(\mathcal{F})(p)[d - \dim G]$ is the unique perverse sheaf satisfying

$$\pi_P^\dagger T(\mathcal{F})(p)[d - \dim G] \cong p^\dagger \mathcal{F}$$

in $\mathrm{Perv}_G(P, \Bbbk)$. (The uniqueness of this object comes from Proposition 6.2.10.) It is left to the reader to check that S and T are inverse to each other. □

REMARK 6.4.11. The development of the equivariant derived category outlined in this section essentially follows [**27**]; in particular, Definition 6.4.4 and Theorem 6.4.10 above are very close to [**27**, Propositions 2.4.3 and 2.5.2], respectively. However, the reader who wishes to compare technical details should be aware of the following differences in notation and conventions between this book and [**27**]:

- In [**27**], neither resolutions nor morphisms between them are required to be smooth. However, imposing smoothness does not change the resulting category, as explained in [**27**, Section 3.1].
- In [**27**], the notation "$D^{\mathrm{b}}_G(X, \Bbbk)$" is used for a category whose objects are not necessarily constructible, and a separate notation is used for the constructible case (see [**27**, Section 2.8]). In this book, we will never encounter nonconstructible equivariant complexes, and so we include constructibility as part of the definition of $D^{\mathrm{b}}_G(X, \Bbbk)$.
- In [**27**], the term "n-acyclic map" is defined using the natural t-structure, rather than the perverse t-structure, and there is a version of Lemma 6.4.7 involving the natural t-structure (see [**27**, Proposition 2.2.1]). See Exercise 6.4.1 for a comparison of the two notions of acyclic maps.
- In [**27**], the notation "$\mathrm{Perv}_G(X, \Bbbk)$" is defined as in (6.4.4); Definition 6.2.3 is not used. (For the natural t-structure, however, see [**27**, Proposition 2.5.3].)

REMARK 6.4.12. One can copy Definition 6.4.4 in the "mixed" settings considered in Chapter 5. Specifically:

- Let G be an algebraic group over a finite field $\mathbb{F}_q$, and let X be a G-variety. For any permitted ring of coefficients $\Bbbk$ (see Definition 5.1.12), there is a **G-equivariant mixed derived category** denoted by $D^{\mathrm{b}}_{\mathrm{m},G}(X, \Bbbk)$, together with a forgetful functor $\mathrm{For} : D^{\mathrm{b}}_{\mathrm{m},G}(X, \Bbbk) \to D^{\mathrm{b}}_{\mathrm{m}}(X, \Bbbk)$.
- Let G be a complex algebraic group, and let X be a G-variety. For any subfield $\Bbbk$ of $\mathbb{R}$, there is a **G-equivariant mixed derived category** denoted by $D^{\mathrm{b}}_G\mathrm{MHM}(X, \Bbbk)$, together with a forgetful functor $\mathrm{For} : D^{\mathrm{b}}_G\mathrm{MHM}(X, \Bbbk) \to D^{\mathrm{b}}\mathrm{MHM}(X, \Bbbk)$.

In each of these settings, one can define an **equivariant geometric Hom functor** (see Definitions 5.1.10 and 5.6.13) using mixed analogues of Lemma 6.4.7. Specifically, if $\mathcal{F}, \mathcal{G} \in {}^pD^{\mathrm{b}}_{\mathrm{m},G}(X, \Bbbk)^{[k,k+n]}$ or $\mathcal{F}, \mathcal{G} \in D^{\mathrm{b}}_G\mathrm{MHM}(X, \Bbbk)^{[k,k+n]}$, then we define

$$\underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{G}) = \underline{\mathrm{Hom}}(\mathcal{F}(p), \mathcal{G}(p))$$

where $p : P \to X$ is some n-acyclic G-resolution. A routine argument shows that this is independent of the choice of p up to canonical isomorphism.

Exercises.

6.4.1. Let $f : X \to Y$ be a smooth morphism of relative dimension d. Let us say that f is **n-acyclic for the natural t-structure** if for any smooth morphism $Y' \to Y$, the base change $f' : X \times_Y Y' \to Y'$ has the property that for every sheaf $\mathcal{F} \in \mathrm{Sh_c}(Y', \Bbbk)$, the natural map

$$\mathcal{F} \to \tau^{\leq n} f'_*(f')^*\mathcal{F}$$

is an isomorphism.

(a) Show that if f is $(n + d)$-acyclic in the sense of Definition 6.1.18, then it is n-acyclic for the natural t-structure.

(b) Show that if f is $(n+d)$-acyclic for the natural t-structure, then it is n-acyclic in the sense of Definition 6.1.18.

6.4.2. Let X be a G-variety, and let $\mathscr{C}$ be the category whose objects are pairs $(\mathcal{F}, \theta)$, where $\mathcal{F} \in D^{\mathrm{b}}_{\mathrm{c}}(X, \Bbbk)$, and θ is an isomorphism $\mathrm{pr}_2^* \mathcal{F} \to \sigma^* \mathcal{F}$ satisfying the conditions from Definition 6.2.3. Morphisms in $\mathscr{C}$ are also defined as in Definition 6.2.3. We have an obvious functor $\mathrm{For}^G : \mathscr{C} \to D^{\mathrm{b}}_{\mathrm{c}}(X, \Bbbk)$. Let us say that a diagram $\mathcal{F} \to \mathcal{G} \to \mathcal{H} \to \mathcal{F}[1]$ in $\mathscr{C}$ is a **distinguished triangle** if applying For^G gives a distinguished triangle in $D^{\mathrm{b}}_{\mathrm{c}}(X, \Bbbk)$. Give an example showing that $\mathscr{C}$ need not be a triangulated category. (*Hint*: Let $X = \mathrm{pt}$, let $G = \mathbb{Z}/2\mathbb{Z}$, and let $\Bbbk$ be a field of characteristic 2. Exhibit a distinguished triangle that satisfies some but not all of the conditions of Lemma A.4.5.)

6.4.3. Let us say that an object $\mathcal{F} \in \mathrm{Sh}_G(X, \Bbbk)$ is **constructible** if $\mathrm{For}(\mathcal{F}) \in \mathrm{Sh}(X, \Bbbk)$ is a constructible sheaf. Show that $D^{\mathrm{b}}_G(X, \Bbbk)$ also has a "natural t-structure": that is, a bounded t-structure whose heart is equivalent to the category of constructible objects in $\mathrm{Sh}_G(X, \Bbbk)$. Moreover, the functor $\mathrm{For} : D^{\mathrm{b}}_G(X, \Bbbk) \to D^{\mathrm{b}}_{\mathrm{c}}(X, \Bbbk)$ is t-exact with respect to the natural t-structures on both categories.

6.5. Equivariant sheaf functors

The six operations. Definition 6.4.4 is built around data and conditions involving pullback along smooth morphisms. We can therefore rely on Principle 2.2.11 to define sheaf functors in the equivariant setting.

Let X be a G-variety. For $\mathcal{F}, \mathcal{G} \in D^{\mathrm{b}}_G(X, \Bbbk)$, we define $\mathcal{F} \otimes^L \mathcal{G}$ and $R\mathscr{H}om(\mathcal{F}, \mathcal{G})$ as follows: for any G-resolution $p : P \to X$, we set

$$(\mathcal{F} \overset{L}{\otimes} \mathcal{G})(p) = \mathcal{F}(p) \overset{L}{\otimes} \mathcal{G}(p) \qquad \text{and} \qquad R\mathscr{H}om(\mathcal{F}, \mathcal{G})(p) = R\mathscr{H}om(\mathcal{F}(p), \mathcal{G}(p)).$$

Similarly, we define $\mathbb{D}\mathcal{F}$ by

$$(\mathbb{D}\mathcal{F})(p) = \mathbb{D}(\mathcal{F}(p))[2 \dim G - 2d](\dim G - d),$$

where d is the relative dimension of p.

Next, let Y be another G-variety, and let $f : X \to Y$ be a G-equivariant map. Let $p : P \to Y$ be a G-resolution, and then form the following diagram with cartesian squares:

$$\begin{tikzcd} X \arrow[d, "f"'] & P' = P \times_Y X \arrow[l, "p'"'] \arrow[r, "\pi_{P'}"] \arrow[d, "f_P"] & P'/G \arrow[d, "\bar{f}_P"] \\ Y & P \arrow[l, "p"'] \arrow[r, "\pi_P"] & P/G \end{tikzcd} \tag{6.5.1}$$

Note that p' is a G-resolution of X. Moreover, if p is n-acyclic, then p' is as well.

For $\mathcal{F} \in D^{\mathrm{b}}_G(X, \Bbbk)$, we define $f_*\mathcal{F}$ and $f_!\mathcal{F}$ by

$$(f_*\mathcal{F})(p) = \bar{f}_{P*}(\mathcal{F}(p')) \qquad \text{and} \qquad (f_!\mathcal{F})(p) = \bar{f}_{P!}(\mathcal{F}(p')).$$

Finally, given $\mathcal{G} \in D^{\mathrm{b}}_G(Y, \Bbbk)$, we wish to define $f^*\mathcal{G}$ and $f^!\mathcal{G}$. let $\mathscr{P}$ be the set of G-resolutions of X that arise by base change from G-resolutions of Y, i.e., resolutions like p' in (6.5.1). Then $\mathscr{P}$ satisfies all three conditions in Lemma 6.4.8. It is therefore enough to define $f^*\mathcal{G}$ and $f^!\mathcal{G}$ on resolutions belonging to $\mathscr{P}$. We set

$$(f^*\mathcal{G})(p') = \bar{f}_P^*(\mathcal{G}(p)) \qquad \text{and} \qquad (f^!\mathcal{G})(p') = \bar{f}_P^!(\mathcal{G}(p)).$$

Once the pullback functor is defined, one can copy Definition 1.4.20 to define a functor

$$\boxtimes : D^{\mathrm{b}}_G(X,\Bbbk) \to D^{\mathrm{b}}_G(Y,\Bbbk) \to D^{\mathrm{b}}_G(X\times Y,\Bbbk), \tag{6.5.2}$$

where X and Y are both G-varieties.

There is also another kind of external tensor product, involving two groups G and H. Let X be a G-variety, and let Y be an H-variety. If $p : P \to X$ is an n-acyclic G-resolution of X, and $q : Q \to Y$ is an n-acyclic H-resolution of Y, then $p \times q : P \times Q \to X \times Y$ is an n-acyclic $(G \times H)$-resolution of $X \times Y$. The set of $(G \times H)$-resolutions of $X \times Y$ obtained in this way satisfies all three conditions in Lemma 6.4.8. Given $\mathcal{F} \in D^{\mathrm{b}}_G(X,\Bbbk)$ and $\mathcal{G} \in D^{\mathrm{b}}_H(Y,\Bbbk)$, we define an object $\mathcal{F}\boxtimes\mathcal{G} \in D^{\mathrm{b}}_{G\times H}(X\times Y,\Bbbk)$ by

$$(\mathcal{F}\boxtimes\mathcal{G})(p\times q) = \mathcal{F}(p)\boxtimes\mathcal{G}(q).$$

This construction defines a functor

$$\boxtimes : D^{\mathrm{b}}_G(X,\Bbbk) \times D^{\mathrm{b}}_H(Y,\Bbbk) \to D^{\mathrm{b}}_{G\times H}(X\times Y,\Bbbk). \tag{6.5.3}$$

Even though we have used the same notation for (6.5.2) and (6.5.3), there is usually no ambiguity in context about which version is meant.

The functors defined above enjoy the following key properties.

Principle 6.5.1. (1) *All equivariant sheaf functors commute with the forgetful functor.*
(2) *All the usual natural isomorphisms of (nonequivariant) sheaf functors (e.g., adjunction relationships, proper and smooth base change, projection formula, Verdier duality, etc.) hold for equivariant sheaf functors.*

Let us briefly comment on why these assertions are true. Part (1) of Principle 6.5.1 is a special case of the more general claim that for any G-resolution $p : P \to Y$, the equivariant sheaf functors commute with the functor $\mathcal{F} \mapsto \mathcal{F}(p)$ (or perhaps with $\mathcal{F} \mapsto \mathcal{F}(p')$ as in (6.5.1)). This latter claim is true by definition. The idea for part (2) is that the construction of these natural maps can be repeated in $D^{\mathrm{b}}_G(X,\Bbbk)$ (thanks to Principle 2.2.11). If a morphism in $D^{\mathrm{b}}_G(X,\Bbbk)$ is an isomorphism on all resolutions $p : P \to X$, then it is an isomorphism.

Forgetful and inflation functors. In addition to the six operations from Chapter 1, the equivariant setting allows for various new operations, based on the idea of "change of equivariance." We will introduce two (easy) change-of-equivariance functors now; two others will be introduced in Section 6.6.

Definition 6.5.2. Let $H \subset G$ be a closed subgroup, and let X be a G-variety. Let $\mathscr{P}$ be the set of H-resolutions of X consisting of G-resolutions regarded as H-resolutions. For any resolution $p \in \mathscr{P}$, the quotient map $\pi_P : P \to P/G$ factors through P/H as

$$P \xrightarrow{\pi'_P} P/H \xrightarrow{\pi''_P} P/G.$$

We define the **forgetful functor**

$$\mathrm{For}^G_H : D^{\mathrm{b}}_G(X,\Bbbk) \to D^{\mathrm{b}}_H(X,\Bbbk)$$

as follows: for $\mathcal{F} \in D^{\mathrm{b}}_G(X,\Bbbk)$ and $p \in \mathscr{P}$, we set $\mathrm{For}^G_H(\mathcal{F})(p) = (\pi''_P)^*(\mathcal{F}(p))$.

The set $\mathscr{P}$ in the definition above only satisfies the first two assumptions of Lemma 6.4.8, so there is a well-definedness issue to address: we must check that For_H^G actually takes values in $D_H^{\mathrm{b}}(X,\Bbbk)$, and not just in the larger category $D_H^{\mathrm{b}}(X,\Bbbk,\mathscr{P})$. Since $D_G^{\mathrm{b}}(X,\Bbbk)$ and $D_H^{\mathrm{b}}(X,\Bbbk)$ are generated as triangulated categories by $\mathrm{Perv}_G(X,\Bbbk)$ and $\mathrm{Perv}_H(X,\Bbbk)$, respectively, it is enough to show that For_H^G sends $\mathrm{Perv}_G(X,\Bbbk)$ to $\mathrm{Perv}_H(X,\Bbbk)$. This claim follows from the observation that under the embedding $\mathrm{Perv}_G(X,\Bbbk) \hookrightarrow D_G^{\mathrm{b}}(X,\Bbbk)$ described in the proof of Theorem 6.4.10, Definition 6.5.2 agrees with the definition given in (6.2.6).

In the case where H is the trivial group, Lemma 6.4.6 implies that the functor For_1^G from Definition 6.5.2 is naturally isomorphic to the functor defined in (6.4.2).

The next two statements are immediate consequences of the definition together with Principle 2.2.11. We omit their proofs.

LEMMA 6.5.3. *Let $K \subset H \subset G$ be closed subgroups, and let X be a G-variety. For $\mathcal{F} \in D_G^{\mathrm{b}}(X,\Bbbk)$, there is a natural isomorphism $\mathrm{For}_K^G(\mathcal{F}) \cong \mathrm{For}_K^H \mathrm{For}_H^G(\mathcal{F})$.*

PROPOSITION 6.5.4. *Let $H \subset G$ be a subgroup.*

1. *Let X be a G-variety. Then $\otimes^L$ and $R\mathscr{H}om$ commute with For_H^G.*
2. *Let $f : X \to Y$ be a G-equivariant map of G-varieties. Then f_*, $f_!$, f^*, and $f^!$ commute with For_H^G.*

DEFINITION 6.5.5. Let $H \lhd G$ be a normal subgroup, and let X be a G/H-variety (and hence also a G-variety). For any G-resolution $p : P \to X$, let $\bar{p}$ be the unique map such that the diagram

$$\begin{tikzcd} P \arrow[rr] \arrow[dr, "p"'] & & P/H \arrow[dl, "\bar{p}"] \\ & X & \end{tikzcd}$$

commutes. We define the **inflation functor**

$$\mathrm{Infl}_{G/H}^G : D_{G/H}^{\mathrm{b}}(X,\Bbbk) \to D_G^{\mathrm{b}}(X,\Bbbk)$$

as follows: for $\mathcal{F} \in D_{G/H}^{\mathrm{b}}(X,\Bbbk)$, we set $\mathrm{Infl}_{G/H}^G(\mathcal{F})(p) = \mathcal{F}(\bar{p})$.

Let us check that this definition makes sense. The map $\bar{p}$ exists by Lemma 6.1.3, and it is smooth by Proposition 2.1.10. Lemma 6.1.15 then implies that $\bar{p}$ is a G/H-resolution of X, and also tells us that $(P/H)/(G/H) \cong P/G$. It is left as an exercise to check that this functor agrees with (6.2.7) for perverse sheaves.

In the special case where H is the trivial group, the functor Infl_G^G is just the identity functor. The following two statements are immediate from the definitions.

LEMMA 6.5.6. *Let $K \subset H \subset G$ be normal subgroups, and let X be a G/H-variety. For $\mathcal{F} \in D_{G/H}^{\mathrm{b}}(X,\Bbbk)$, there is a natural isomorphism $\mathrm{Infl}_{G/H}^G(\mathcal{F}) \cong \mathrm{Infl}_{G/K}^G \mathrm{Infl}_{G/H}^{G/K}(\mathcal{F})$.*

PROPOSITION 6.5.7. *Let $H \lhd G$ be a normal subgroup.*

1. *Let X be a G/H-variety. Then $\otimes^L$ and $R\mathscr{H}om$ commute with $\mathrm{Infl}_{G/H}^G$.*
2. *Let $f : X \to Y$ be a G/H-equivariant map of G/H-varieties. Then f_*, $f_!$, f^*, and $f^!$ commute with $\mathrm{Infl}_{G/H}^G$.*

Lemma 6.5.8. *Let G be an algebraic group. Let $H, K \subset G$ be subgroups, and assume that K is a normal subgroup of G. Let X be a G/K-variety. For any $\mathcal{F} \in D^{\mathrm{b}}_{G/K}(X, \Bbbk)$, there is a natural isomorphism*

$$\mathrm{For}^G_H \mathrm{Infl}^G_{G/K}(\mathcal{F}) \cong \mathrm{Infl}^H_{H/H\cap K} \mathrm{For}^{G/K}_{H/H\cap K}(\mathcal{F}).$$

Proof. Let $p : P \to X$ be a G-resolution of X, regarded as an H-resolution. It is enough to show that both sides agree on H-resolutions of this form. Consider the following diagram:

$$\begin{tikzcd}
P \arrow[r] \arrow[dr] \arrow[ddr, bend right, "p"'] & P/(H \cap K) \arrow[r] \arrow[d, "\nu"] \arrow[dd, bend right=60, "p_1"'] & P/H \arrow[d, "\bar{\nu}"'] \arrow[dr, "q"] & \\
& P/K \arrow[r] \arrow[d, "p_2"] & P/HK \arrow[r, "r"'] & P/G = \frac{P/K}{G/K} \\
& X & &
\end{tikzcd}$$

Note that $\mathcal{F}(p_2)$ is an object in $D^{\mathrm{b}}_{\mathrm{c}}(P/G, \Bbbk)$. By definition, $\mathrm{For}^G_H \mathrm{Infl}^G_{G/K}(\mathcal{F})(p) = q^*\mathcal{F}(p_2)$. On the other hand, observe that $\nu : P/H \cap K \to P/K$ is a morphism of principal bundles for the group $H/H \cap K \cong HK/K$. We therefore have

$$\begin{aligned}
\mathrm{Infl}^H_{H/H\cap K} \mathrm{For}^{G/K}_{H/H\cap K}(\mathcal{F})(p) &= \mathrm{For}^{G/K}_{H/H\cap K}(\mathcal{F})(p_1) \\
&\cong \bar{\nu}^* \mathrm{For}^{G/K}_{H/H\cap K}(\mathcal{F})(p_2) = \bar{\nu}^* r^* \mathcal{F}(p_2) \cong q^*\mathcal{F}(p_2),
\end{aligned}$$

as desired. ☐

Quotient and induction equivalences. We conclude this section with two useful theorems in which a change of equivariance combined with a suitable change of topological space turns out to be an equivalence of categories.

Theorem 6.5.9 (Quotient equivalence). *Let X be a G-variety, and let $H \lhd G$ be a normal subgroup such that X is a principal H-variety. Then the functor*

$$\pi^* \circ \mathrm{Infl}^G_{G/H} : D^{\mathrm{b}}_{G/H}(X/H, \Bbbk) \xrightarrow{\sim} D^{\mathrm{b}}_G(X, \Bbbk)$$

is an equivalence of categories.

In a minor abuse of notation, one often just writes $\pi^* : D^{\mathrm{b}}_{G/H}(X/H, \Bbbk) \xrightarrow{\sim} D^{\mathrm{b}}_G(X, \Bbbk)$ for this equivalence. The inverse functor $(\pi^* \mathrm{Infl}^G_{G/H})^{-1} : D^{\mathrm{b}}_G(X, \Bbbk) \to D^{\mathrm{b}}_{G/H}(X/H, \Bbbk)$ is sometimes called the **equivariant descent** functor.

Proof. Let $p : P \to X$ be a G-resolution, and consider the diagram

$$\begin{tikzcd}
P \arrow[r, "\pi'_P"'] \arrow[rr, bend left, "\pi_P"] \arrow[d, "p"'] & P/H \arrow[r, "\pi''_P"'] \arrow[d, "\bar{p}"'] & P/G = \frac{P/H}{P/H} \\
X \arrow[r, "\pi"] & X/H &
\end{tikzcd} \tag{6.5.4}$$

Let $\mathscr{P}$ be the set of G/H-resolutions of X/H arising from this construction. We claim that $\mathscr{P}$ satisfies the assumptions of Lemma 6.4.8. It is easy to see that $\mathscr{P}$ is closed under fiber products over X/H, and Lemma 6.1.24 implies that $\mathscr{P}$ contains

n-acyclic resolutions for every $n \geq 0$. Finally, let G act on $G/H \times X$ by $g \cdot (g'H, x) = (g'g^{-1}H, g \cdot x)$. One can check that this makes $G/H \times X$ into a principal G-variety; furthermore, we have $(G/H \times X)/G \cong X/H$ and $(G/H \times X)/H \cong G/H \times X/H$. In particular, we see that $\mathscr{P}$ contains the trivial G/H-resolution of X/H.

We will now define a functor $Q : D^{\mathrm{b}}_G(X, \Bbbk) \to D^{\mathrm{b}}_{G/H}(X/H, \Bbbk)$. By the discussion above, it is enough to define it on resolutions in $\mathscr{P}$. For $\mathcal{F} \in D^{\mathrm{b}}_G(X, \Bbbk)$ and $\bar{p}$ as in (6.5.4), we set $Q(\mathcal{F})(\bar{p}) = \mathcal{F}(p)$. The diagram

$$\begin{tikzcd}
D^{\mathrm{b}}_G(X,\Bbbk) \arrow[rr, "Q"] \arrow[dr, "\mathcal{F} \mapsto \mathcal{F}(p)"'] & & D^{\mathrm{b}}_{G/H}(X/H,\Bbbk) \arrow[dl, "\mathcal{F} \mapsto \mathcal{F}(\bar{p})"] \\
& D^{\mathrm{b}}_{\mathrm{c}}(P/G,\Bbbk) &
\end{tikzcd}$$

commutes by construction, and hence so does

$$\begin{tikzcd}
\mathrm{Hom}_{D^{\mathrm{b}}_G(X,\Bbbk)}(\mathcal{F},\mathcal{G}) \arrow[rr, "Q"] \arrow[dr] & & \mathrm{Hom}_{D^{\mathrm{b}}_{G/H}(X/H,\Bbbk)}(Q(\mathcal{F}),Q(\mathcal{G})) \arrow[dl] \\
& \mathrm{Hom}_{D^{\mathrm{b}}_{\mathrm{c}}(P/G,\Bbbk)}(\mathcal{F}(p),\mathcal{G}(p)) &
\end{tikzcd} \tag{6.5.5}$$

Now choose k and n such that $\mathcal{F}$ and $\mathcal{G}$ lie in ${}^pD^{\mathrm{b}}_G(X,\Bbbk)^{[k,k+n]}$, and $Q(\mathcal{F})$ and $Q(\mathcal{G})$ lie in ${}^pD^{\mathrm{b}}_{G/H}(X/H,\Bbbk)^{[k,k+n]}$. Then choose $p : P \to X$ to be n-acyclic. By Lemma 6.4.7, the two diagonal arrows in (6.5.5) are isomorphisms, and hence so is the top arrow. We conclude that Q is fully faithful.

Finally, let $r : R \to X/H$ be a G-resolution of X/H, and set $P = R \times_{X/H} X$. Consider the following diagram:

$$\begin{tikzcd}
& P = R \times_{X/H} X \arrow[dl, "p"'] \arrow[dd] \arrow[dr] & & \\
X \arrow[dr] \arrow[dd, "\pi"'] & & P/H \arrow[r] \arrow[dd, "\nu"] \arrow[ddll, bend left, "\bar{p}"'] & P/G \arrow[dd, "\bar{\nu}"] \\
& R \arrow[dl, "r"'] \arrow[dr] & & \\
X/H & & R/H \arrow[r] \arrow[ll, bend left, "\bar{r}"] & R/G
\end{tikzcd}$$

Unwinding the definitions, we see that for $\mathcal{G} \in D^{\mathrm{b}}_{G/H}(X/H,\Bbbk)$, we have

$$Q(\pi^*\mathrm{Infl}^G_{G/H}\mathcal{G})(\bar{p}) = (\pi^*\mathrm{Infl}^G_{G/H}\mathcal{G})(p) = \bar{\nu}^*\mathcal{G}(\bar{r}).$$

The map $\alpha_\nu : \bar{\nu}^*\mathcal{G}(\bar{r}) \xrightarrow{\sim} \mathcal{G}(\bar{p})$ thus gives a natural isomorphism $Q(\pi^*\mathrm{Infl}^G_{G/H}\mathcal{G}) \xrightarrow{\sim} \mathcal{G}$. We conclude that Q is essentially surjective, and hence an equivalence of categories, and that $\pi^* \circ \mathrm{Infl}^G_{G/H}$ is its inverse. $\square$

THEOREM 6.5.10 (Induction equivalence). *Let G be an algebraic group, and let $H \subset G$ be a closed subgroup. Let X be an H-variety, and let $i : X \to G \times^H X$ be the map $i(x) = (e, x)$. There is a natural isomorphism of functors*

$$i^*[-\dim G/H] \circ \mathrm{For}^G_H \cong i^![\dim G/H](\dim G/H) \circ \mathrm{For}^G_H :$$
$$D^{\mathrm{b}}_G(G \times^H X, \Bbbk) \xrightarrow{\sim} D^{\mathrm{b}}_H(X, \Bbbk).$$

Moreover, these functors are t-exact equivalences of categories.

PROOF. Let $G \times H$ act on $G \times X$ by $(g,h) \cdot (g', x) = (gg'h^{-1}, h \cdot x)$. Then $H \times X \subset G \times X$ is an $(H \times H)$-stable subvariety. Let $\tilde{\imath} : H \times X \hookrightarrow G \times X$ be the inclusion map. Consider the commutative diagram

$$\begin{tikzcd} D^{\mathrm{b}}_H(X, \Bbbk) \arrow[rr, "\mathrm{pr}_2^\dagger \mathrm{Infl}_{1\times H}^{H\times H}", "\sim"'] \arrow[dr, "\sim", "\mathrm{pr}_2^\dagger \mathrm{Infl}_{1\times H}^{G\times H}"'] & & D^{\mathrm{b}}_{H\times H}(H \times X, \Bbbk) \\ & D^{\mathrm{b}}_{G\times H}(G \times X, \Bbbk) \arrow[ur, "\tilde{\imath}^* \mathrm{For}_{H\times H}^{G\times H}[-\dim G/H]"'] & \end{tikzcd}$$

By Theorem 6.5.9, the top and lower left arrows are both t-exact equivalences of categories, so the lower right arrow is as well. Moreover, the diagram still commutes if one replaces the lower right arrow by $\tilde{\imath}^![\dim G/H](\dim G/H)$. We deduce that there is a natural isomorphism of t-exact equivalences

$$\tilde{\imath}^! \mathrm{For}_{H\times H}^{G\times H}[\dim G/H](\dim G/H) \cong \tilde{\imath}^* \mathrm{For}_{H\times H}^{G\times H}[-\dim G/H] : \\ D^{\mathrm{b}}_{G\times H}(G \times X, \Bbbk) \xrightarrow{\sim} D^{\mathrm{b}}_{H\times H}(H \times X, \Bbbk).$$

Now let $\pi : G \times X \to G \times^H X$ be the quotient map, and consider the diagram

$$\begin{tikzcd} H \times X \arrow[r, "\tilde{\imath}"] \arrow[d, "\sigma"'] & G \times X \arrow[d, "\pi"] \\ X \arrow[r, "i"] & G \times^H X \end{tikzcd}$$

This gives rise to a commutative diagram

$$\begin{tikzcd} D^{\mathrm{b}}_{H\times H}(H \times X, \Bbbk) & D^{\mathrm{b}}_{G\times H}(G \times X, \Bbbk) \arrow[l, "\tilde{\imath}^* \mathrm{For}_{H\times H}^{G\times H}[-\dim G/H]"', "\sim"] \\ D^{\mathrm{b}}_H(X, \Bbbk) \arrow[u, "\sigma^\dagger \mathrm{Infl}_{H\times 1}^{H\times H}", "\wr"'] & D^{\mathrm{b}}_G(G \times^H X, \Bbbk) \arrow[l, "i^* \mathrm{For}_H^G[-\dim G/H]"'] \arrow[u, "\pi^\dagger \mathrm{Infl}_{G\times 1}^{G\times H}"', "\wr"] \end{tikzcd}$$

Both vertical arrows are t-exact equivalences by Theorem 6.5.9, and the top horizontal arrow is an equivalence by the discussion above. Therefore, $i^* \mathrm{For}_H^G[-\dim G/H]$ is a t-exact equivalence of categories. The same reasoning as above shows that it is isomorphic to $i^! \mathrm{For}_H^G[\dim G/H](\dim G/H)$. $\square$

Exercises.

6.5.1. Prove the following equivariant version of Lemma 6.1.19: If $f : X \to Y$ is an n-acyclic G-equivariant morphism of G-varieties, then for any $k \in \mathbb{Z}$, the functor $f^\dagger : {}^pD^{\mathrm{b}}_G(Y, \Bbbk)^{[k,k+n]} \to {}^pD^{\mathrm{b}}_G(X, \Bbbk)^{[k,k+n]}$ is fully faithful.

6.5.2. Let X be a G-variety, and let $(X_s)_{s\in\mathscr{S}}$ be a stratification of X whose strata are stable under G. For each $s \in \mathscr{S}$, let $j_s : X_s \hookrightarrow X$ be the inclusion map. This stratification is said to be a **G-equivariantly good stratification** if for any $s \in \mathscr{S}$ and any $\mathcal{L} \in \mathrm{Loc}^{\mathrm{ft}}_G(X_s, \Bbbk)$, the object $j_{s*}\mathcal{L} \in D^{\mathrm{b}}_G(X, \Bbbk)$ is constructible with respect to $(X_s)_{s\in\mathscr{S}}$.

(a) Assume that G acts on X with finitely many orbits, each of which is connected. Show that the stratification by G-orbits is a G-equivariantly good stratification.

(b) Now assume in addition that G is connected, and that for each G-orbit $Y \subset X$, every local system of finite type on Y admits a G-equivariant structure. (For instance, this holds if every G-orbit is simply connected.) Show that the stratification of X by G-orbits is a good stratification.

6.5.3. Let G be an algebraic group, and let M be a $\Bbbk[G/G^\circ]$-module that is finitely generated over $\Bbbk$, regarded as an object of $D^{\mathrm{b}}_G(\mathrm{pt}, \Bbbk)$ via Proposition 6.2.12. Now let X be a G-variety, and let $\underline{M}_X = \mathrm{a}_X^* M \in D^{\mathrm{b}}_G(X, \Bbbk)$. Give a concrete description of $\underline{M}_X$ in the spirit of Example 6.4.5.

6.5.4. Let X be a G-variety, and let Y be an H-variety. Show that for any $\mathcal{F} \in D^{\mathrm{b}}_G(X, \Bbbk)$ and any $\mathcal{G} \in D^{\mathrm{b}}_H(Y, \Bbbk)$, there is a natural isomorphism $\mathcal{F} \boxtimes \mathcal{G} \cong \mathrm{pr}_1^* \mathrm{Infl}_G^{G\times H}(\mathcal{F}) \otimes^L \mathrm{pr}_2^* \mathrm{Infl}_H^{G\times H}(\mathcal{G})$ in $D^{\mathrm{b}}_{G\times H}(X \times Y, \Bbbk)$.

6.5.5. Let X be a G-variety. Let $p : P \to X$ be a resolution, and let $\pi_P : P \to P/G$ be the quotient map. Show that for any $\mathcal{F} \in D^{\mathrm{b}}_G(X, \Bbbk)$, there is a natural isomorphism $\beta_p : p^*\mathcal{F} \xrightarrow{\sim} \pi_P^* \mathcal{F}(p)$ in $D^{\mathrm{b}}_G(P, \Bbbk)$. Moreover, if $\nu : (P \xrightarrow{p} X) \to (Q \xrightarrow{q} X)$ is a smooth morphism of resolutions, then $\pi_P^* \alpha_\nu \circ \nu^* \beta_q = \beta_p$.

6.5.6. Let P and X be $(G \times H)$-varieties, and assume that P is also a principal H-variety. Let $p : P \to X$ be a smooth $(G \times H)$-equivariant map (and hence also an H-resolution of X).

(a) Show that the functor $D^{\mathrm{b}}_H(X, \Bbbk) \to D^{\mathrm{b}}_{\mathrm{c}}(P/H, \Bbbk)$ given by $\mathcal{F} \mapsto \mathcal{F}(p)$ can be upgraded to a functor $D^{\mathrm{b}}_{G\times H}(X, \Bbbk) \to D^{\mathrm{b}}_G(P/H, \Bbbk)$.
(b) Upgrade Exercise 6.5.5 as follows: for $\mathcal{F} \in D^{\mathrm{b}}_{G\times H}(X, \Bbbk)$, there is a natural isomorphism $\beta_p : p^*\mathcal{F} \xrightarrow{\sim} \pi_P^* \mathrm{Infl}_G^{G\times H} \mathcal{F}(p)$ in $D^{\mathrm{b}}_{G\times H}(P, \Bbbk)$ such that if $\nu : p \to q$ is a $(G \times H)$-equivariant smooth morphism of H-resolutions, then $\pi_P^* \alpha_\nu \circ \nu^* \beta_q = \beta_p$.

6.5.7. Let X be a $(G\times H)$-variety, and let H act on $G\times X$ by $h\cdot(g, x) = (g, h\cdot x)$. Then $\mathrm{pr}_2 : G \times X \to X$ and $\sigma : G \times X \to X$ are both H-equivariant. Upgrade Lemma 6.4.6 as follows: for any $\mathcal{F} \in D^{\mathrm{b}}_{G\times H}(X, \Bbbk)$, there is a natural isomorphism $\theta : \mathrm{pr}_2^* \mathrm{For}_H^{G\times H}(\mathcal{F}) \to \sigma^* \mathrm{For}_H^{G\times H}(\mathcal{F})$ in $D^{\mathrm{b}}_H(G \times X, \Bbbk)$ that satisfies (6.2.5).

6.5.8. Let X be a G-variety. Show that for any element $z \in \mathrm{Z}(G)$ (where $\mathrm{Z}(G)$ is the center of G), there is a natural isomorphism $\theta_z : \mathcal{F} \xrightarrow{\sim} \sigma_z^* \mathcal{F}$ in $D^{\mathrm{b}}_G(X, \Bbbk)$ satisfying (6.2.2).

6.5.9. Let $\varphi : H \to G$ be a homomorphism of algebraic groups, and let X be a G-variety. We can regard it as an H-variety via φ. Define a functor

$$\mathrm{Res}_\varphi : D^{\mathrm{b}}_G(X, \Bbbk) \to D^{\mathrm{b}}_H(X, \Bbbk) \qquad \text{by} \qquad \mathrm{Res}_\varphi = \mathrm{Infl}^H_{\varphi(H)} \circ \mathrm{For}^G_{\varphi(H)}.$$

Now let $\psi : K \to H$ be another group homomorphism. Show that for $\mathcal{F} \in D^{\mathrm{b}}_G(X, \Bbbk)$, there is a natural isomorphism

$$\mathrm{Res}_\psi(\mathrm{Res}_\varphi(\mathcal{F})) \cong \mathrm{Res}_{\varphi\circ\psi}(\mathcal{F}).$$

6.6. Averaging, invariants, and applications

In this section, we introduce change-of-equivariance functors that go in the opposite direction to the forgetful and inflation functors. As an application, we obtain some structural results on equivariant derived categories for finite groups and for unipotent groups.

Averaging and invariants functors. We begin by constructing adjoints to the forgetful functor.

THEOREM 6.6.1. *Let X be a G-variety, and let $H \subset G$ be a closed subgroup. The functor* $\mathrm{For}^G_H : D^{\mathrm{b}}_G(X,\Bbbk) \to D^{\mathrm{b}}_H(X,\Bbbk)$ *has left and right adjoints, denoted by*

$$\mathrm{Av}^G_{H!} : D^{\mathrm{b}}_H(X,\Bbbk) \to D^{\mathrm{b}}_G(X,\Bbbk) \qquad \text{and} \qquad \mathrm{Av}^G_{H*} : D^{\mathrm{b}}_H(X,\Bbbk) \to D^{\mathrm{b}}_G(X,\Bbbk),$$

respectively.

DEFINITION 6.6.2. The functors $\mathrm{Av}^G_{H!}, \mathrm{Av}^G_{H*} : D^{\mathrm{b}}_H(X,\Bbbk) \to D^{\mathrm{b}}_G(X,\Bbbk)$ in Theorem 6.6.1 are called **averaging functors**.

PROOF. Let $\bar{\sigma} : G \times^H X \to X$ be the map induced by the action map $\sigma : G \times X \to X$. Let $i : X \to G \times^H X$ be as in Theorem 6.5.10. Then $\bar{\sigma} \circ i = \mathrm{id}_X$, and (in the nonequivariant setting) $i^* \circ \bar{\sigma}^*$ is isomorphic to the identity functor. It follows that the composition

$$D^{\mathrm{b}}_G(X,\Bbbk) \xrightarrow{\bar{\sigma}^*} D^{\mathrm{b}}_G(G \times^H X,\Bbbk) \xrightarrow{i^* \circ \mathrm{For}^G_H} D^{\mathrm{b}}_H(X,\Bbbk)$$

is isomorphic to For^G_H. The second arrow is an equivalence of categories, and σ^* has a right adjoint, so we conclude that For^G_H has a right adjoint given by

$$\mathrm{Av}^G_{H*} = \bar{\sigma}_* \circ (i^* \circ \mathrm{For}^G_H)^{-1}.$$

Similar reasoning using $\sigma^!$ and $i^!$ shows that For^G_H has a left adjoint given by

$$\mathrm{Av}^G_{H!} = \bar{\sigma}_! \circ (i^! \circ \mathrm{For}^G_H)^{-1},$$

as desired. $\square$

In some situations, it is useful to consider the commutative diagram

$$\begin{tikzcd} G \times^H X \arrow[rr, "(g,x)\mapsto(gH,g\cdot x)"] \arrow[dr, "\bar{\sigma}"'] & & G/H \times X \arrow[dl, "\mathrm{pr}_2"] \\ & X & \end{tikzcd} \tag{6.6.1}$$

The map along the top is a G-equivariant isomorphism of varieties. As a consequence, questions about $\bar{\sigma}$ can be turned into questions about $\mathrm{pr}_2 : G/H \times X \to X$.

COROLLARY 6.6.3. *Let X be a G-variety, and let $H \subset G$ be a closed subgroup of finite index. Then $\mathrm{Av}^G_{H!} \cong \mathrm{Av}^G_{H*}$. Moreover, the functor*

$$\mathrm{Av}^G_{H!} \cong \mathrm{Av}^G_{H*} : D^{\mathrm{b}}_H(X,\Bbbk) \to D^{\mathrm{b}}_G(X,\Bbbk)$$

is t-exact.

PROOF. If G/H is a finite set, then Theorem 6.5.10 tells us that $i^*\mathrm{For}^G_H \cong i^!\mathrm{For}^G_H$ and that this functor is t-exact. The diagram (6.6.1) shows us that $\bar{\sigma}$ is a finite (and hence proper) map, so $\bar{\sigma}_! \cong \bar{\sigma}_*$, and this functor is t-exact by Proposition 3.1.12. The result follows. $\square$

LEMMA 6.6.4. *Let $K \subset H \subset G$ be algebraic groups, and assume that K is a normal subgroup of G. Let X be a G/K-variety. For $\mathcal{F} \in D^{\mathrm{b}}_{H/K}(X,\Bbbk)$, there are natural isomorphisms*

$$\mathrm{Infl}^G_{G/K}\mathrm{Av}^{G/K}_{H/K!}(\mathcal{F}) \cong \mathrm{Av}^G_{H!}\mathrm{Infl}^H_{H/K}(\mathcal{F}),$$

$$\mathrm{Infl}^G_{G/K}\mathrm{Av}^{G/K}_{H/K*}(\mathcal{F}) \cong \mathrm{Av}^G_{H*}\mathrm{Infl}^H_{H/K}(\mathcal{F}).$$

Proof. Note that $G \times^H X \cong (G/K) \times^{H/K} X$. Let $i : X \to G \times^H X$ and $\bar{\sigma} : G \times^H X \to X$ be as in the proof of Theorem 6.6.1. By Lemma 6.5.8 and Proposition 6.5.7, we have $\mathrm{Infl}^H_{H/K} \circ i^! \circ \mathrm{For}^{G/K}_{H/K} \cong i^! \circ \mathrm{For}^G_H \circ \mathrm{Infl}^G_{G/K}$, and hence

$$\mathrm{Infl}^G_{G/K} \circ (i^! \circ \mathrm{For}^{G/K}_{H/K})^{-1} \cong (i^! \circ \mathrm{For}^G_H)^{-1} \circ \mathrm{Infl}^H_{H/K}.$$

Now compose both sides with $\bar{\sigma}_!$. By Proposition 6.5.7 again, $\mathrm{Infl}^G_{G/K} \circ \mathrm{Av}^{G/K}_{H/K!} \cong \mathrm{Av}^G_{H!} \circ \mathrm{Infl}^H_{H/K}$. The proof of the second isomorphism is similar. $\square$

Example 6.6.5. Suppose G acts trivially on X. Let us compute $\mathrm{Av}^G_{1*}\underline{\Bbbk}_X$ and $\mathrm{Av}^G_{1!}\underline{\Bbbk}_X$. The induction space is just the product $G \times X$, and the action map $\bar{\sigma} : G \times X \to X$ is just the projection onto X. With $i : X \to G \times X$ as in the proof of Theorem 6.6.1, we have

$$i^*\mathrm{For}^G \underline{\Bbbk}_{G\times X} \cong \underline{\Bbbk}_X \qquad \text{and} \qquad i^!\mathrm{For}^G \underline{\Bbbk}_{G\times X}[2\dim G](\dim G) \cong \underline{\Bbbk}_X.$$

(The latter isomorphism follows from Theorem 6.5.10, or else from Theorem 2.2.13.) From the definition of the averaging functors, we have

$$\mathrm{Av}^G_{1*}\underline{\Bbbk}_X \cong \mathrm{a}_{G*}\underline{\Bbbk}_G \boxtimes \underline{\Bbbk}_X \cong \mathrm{a}_X^*\mathrm{a}_{G*}\underline{\Bbbk}_G,$$
$$\mathrm{Av}^G_{1!}\underline{\Bbbk}_X \cong \mathrm{a}_{G!}\underline{\Bbbk}_G \boxtimes \underline{\Bbbk}_X[2\dim G](\dim G) \cong \mathrm{a}_X^*\mathrm{a}_{G!}\underline{\Bbbk}_G[2\dim G](\dim G).$$

Our next task is to study adjoints to the inflation functor. Unfortunately, the following example shows that these adjoints can fail to exist.

Example 6.6.6. Consider the functor $\mathrm{Infl}^{\mathbb{G}_\mathrm{m}}_1 : D^\mathrm{b}_\mathrm{c}(\mathrm{pt}, \Bbbk) \to D^\mathrm{b}_{\mathbb{G}_\mathrm{m}}(\mathrm{pt}, \Bbbk)$. If it had a right adjoint $\mathrm{Inv}_{\mathbb{G}_\mathrm{m}*}$, then we would have

$$\mathrm{Hom}_{D^\mathrm{b}_{\mathbb{G}_\mathrm{m}}(\mathrm{pt},\Bbbk)}(\underline{\Bbbk}_\mathrm{pt}, \underline{\Bbbk}_\mathrm{pt}[k]) \cong \mathrm{Hom}_{D^\mathrm{b}_\mathrm{c}(\mathrm{pt},\Bbbk)}(\underline{\Bbbk}_\mathrm{pt}, \mathrm{Inv}_{\mathbb{G}_\mathrm{m}*}(\underline{\Bbbk}_\mathrm{pt})[k]) \cong \mathsf{H}^k(\mathrm{Inv}_{\mathbb{G}_\mathrm{m}*}(\underline{\Bbbk}_\mathrm{pt})).$$

But we will see in Proposition 6.7.6 that the left-hand side is nonzero for all even nonnegative k, while $\mathsf{H}^k(\mathrm{Inv}_{\mathbb{G}_\mathrm{m}*}(\underline{\Bbbk}_\mathrm{pt}))$ can of course be nonzero only for finitely many k. Thus, $\mathrm{Inv}_{\mathbb{G}_\mathrm{m}*} : D^\mathrm{b}_{\mathbb{G}_\mathrm{m}}(\mathrm{pt}, \Bbbk) \to D^\mathrm{b}_\mathrm{c}(\mathrm{pt}, \Bbbk)$ cannot exist.

Nevertheless, the following proposition asserts that some sort of "approximations" to (possibly nonexistent) adjoint functors do exist.

Proposition 6.6.7. *Let $H \lhd G$ be a normal subgroup, and let X be a G/H-variety.*

(1) *For any $n \in \mathbb{Z}$, there exists a functor $\mathrm{Inv}^{\le n}_{H*} : D^\mathrm{b}_G(X, \Bbbk) \to {}^pD^\mathrm{b}_{G/H}(X, \Bbbk)^{\le n}$ such that for any $\mathcal{F} \in {}^pD^\mathrm{b}_{G/H}(X, \Bbbk)^{\le n}$ and any $\mathcal{G} \in D^\mathrm{b}_G(X, \Bbbk)$, there is a natural isomorphism*

$$\mathrm{Hom}(\mathrm{Infl}^G_{G/H}(\mathcal{F}), \mathcal{G}) \cong \mathrm{Hom}(\mathcal{F}, \mathrm{Inv}^{\le n}_{H*}(\mathcal{G})).$$

(2) *For any $n \in \mathbb{Z}$, there exists a functor $\mathrm{Inv}^{\ge n}_{H!} : D^\mathrm{b}_G(X, \Bbbk) \to {}^pD^\mathrm{b}_{G/H}(X, \Bbbk)^{\ge n}$ such that for any $\mathcal{F} \in D^\mathrm{b}_G(X, \Bbbk)$ and any $\mathcal{G} \in {}^pD^\mathrm{b}_{G/H}(X, \Bbbk)^{\ge n}$, there is a natural isomorphism*

$$\mathrm{Hom}(\mathcal{F}, \mathrm{Infl}^G_{G/H}(\mathcal{G})) \cong \mathrm{Hom}(\mathrm{Inv}^{\ge n}_{H!}(\mathcal{F}), \mathcal{G}).$$

Definition 6.6.8. Let $H \lhd G$ be a normal subgroup, and let X be a G/H-variety. For any integer n, the functor $\mathrm{Inv}^{\le n}_{H*} : D^\mathrm{b}_G(X, \Bbbk) \to {}^pD^\mathrm{b}_{G/H}(X, \Bbbk)^{\le n}$ is called a **truncated functor of H-invariants**, and the functor $\mathrm{Inv}^{\ge n}_{H!} : D^\mathrm{b}_G(X, \Bbbk) \to {}^pD^\mathrm{b}_{G/H}(X, \Bbbk)^{\ge n}$ is called a **truncated functor of H-coinvariants**.

PROOF. We will prove the first part; the proof of the second part is similar. Given $\mathcal{F} \in {}^pD^{\mathrm{b}}_{G/H}(X,\Bbbk)^{\le n}$ and $\mathcal{G} \in D^{\mathrm{b}}_G(X,\Bbbk)$, choose an integer $a \le n$ such that $\mathcal{F} \in {}^pD^{\mathrm{b}}_{G/H}(X,\Bbbk)^{\ge a}$ and $\mathcal{G} \in {}^pD^{\mathrm{b}}_G(X,\Bbbk)^{\ge a}$, and then choose an $(n-a)$-acyclic G-resolution $p : P \to X$. This gives rise to a commutative triangle

$$\begin{tikzcd} P \arrow[rr, "\pi"] \arrow[dr, "p"'] & & P/H \arrow[dl, "\bar{p}"] \\ & X & \end{tikzcd}$$

where $\bar{p}$ is smooth by Proposition 2.1.10. By Theorem 6.5.9, we have an equivalence of categories $\pi^\dagger \mathrm{Infl}^G_{G/H} : D^{\mathrm{b}}_{G/H}(P/H,\Bbbk) \to D^{\mathrm{b}}_G(P,\Bbbk)$. Using this together with Exercise 6.5.1 and the adjunction properties of truncation, we have

$$\begin{aligned}
\mathrm{Hom}(\mathrm{Infl}^G_{G/H}(\mathcal{F}),\mathcal{G}) &\cong \mathrm{Hom}(\mathrm{Infl}^G_{G/H}(\mathcal{F}), {}^p\tau^{\le n}(\mathcal{G})) \\
&\cong \mathrm{Hom}(p^\dagger \mathrm{Infl}^G_{G/H}(\mathcal{F}), p^\dagger {}^p\tau^{\le n}(\mathcal{G})) \\
&\cong \mathrm{Hom}(\pi^\dagger \mathrm{Infl}^G_{G/H}(\bar{p}^\dagger\mathcal{F}), p^\dagger {}^p\tau^{\le n}(\mathcal{G})) \\
&\cong \mathrm{Hom}(\bar{p}^\dagger\mathcal{F}, (\pi^\dagger \mathrm{Infl}^G_{G/H})^{-1} p^\dagger {}^p\tau^{\le n}(\mathcal{G})) \\
&\cong \mathrm{Hom}(\mathcal{F}, {}^p\tau^{\le n}\bar{p}_\dagger(\pi^\dagger \mathrm{Infl}^G_{G/H})^{-1} p^\dagger {}^p\tau^{\le n}(\mathcal{G})).
\end{aligned}$$

We therefore set $\mathrm{Inv}^{\le n}_{H*}(\mathcal{G}) = {}^p\tau^{\le n}\bar{p}_\dagger(\pi^\dagger \mathrm{Infl}^G_{G/H})^{-1} p^\dagger {}^p\tau^{\le n}(\mathcal{G})$. A routine argument with Yoneda's lemma shows that this is independent of the choice of p up to canonical isomorphism. □

For $\mathcal{F} \in {}^pD^{\mathrm{b}}_{G/H}(X,\Bbbk)^{\le n}$ and $\mathcal{G} \in D^{\mathrm{b}}_G(X,\Bbbk)$, there is a natural isomorphism

$$\mathrm{Hom}(\mathcal{F}, \mathrm{Inv}^{\le n+1}_{H*}(\mathcal{G})) \cong \mathrm{Hom}(\mathcal{F}, {}^p\tau^{\le n}\mathrm{Inv}^{\le n+1}_{H*}(\mathcal{G})),$$

so Yoneda's lemma implies that $\mathrm{Inv}^{\le n}_{H*}(\mathcal{G}) \cong {}^p\tau^{\le n}\mathrm{Inv}^{\le n+1}_{H*}(\mathcal{G})$. By the adjunction properties of truncation, we get a sequence of natural maps

$$\cdots \to \mathrm{Inv}^{\le -1}_{H*}(\mathcal{G}) \to \mathrm{Inv}^{\le 0}_{H*}(\mathcal{G}) \to \mathrm{Inv}^{\le 1}_{H*}(\mathcal{G}) \to \cdots. \tag{6.6.2}$$

If this sequence stabilizes, i.e., if there is an integer N such that the map $\mathrm{Inv}^{\le n}_{H*}(\mathcal{G}) \to \mathrm{Inv}^{\le n+1}_{H*}(\mathcal{G})$ is an isomorphism for all $n \ge N$, then we denote that object simply by

$$\mathrm{Inv}_{H*}(\mathcal{G}) = \mathrm{Inv}^{\le n}_{H*}(\mathcal{G}) \qquad \text{for any } n \ge N.$$

By construction, this object (if it exists) satisfies

$$\mathrm{Inv}^{\le n}_{H*}(\mathcal{G}) \cong {}^p\tau^{\le n}\mathrm{Inv}_{H*}(\mathcal{G}) \tag{6.6.3}$$

for all n. In a minor abuse of language, we sometimes say "$\mathrm{Inv}_{H*}(\mathcal{G})$ **is bounded**" to mean that (6.6.2) stabilizes.

Similarly, for $\mathcal{F} \in D^{\mathrm{b}}_G(X,\Bbbk)$, there is a sequence of natural maps

$$\cdots \to \mathrm{Inv}^{\ge -1}_{H!}(\mathcal{F}) \to \mathrm{Inv}^{\ge 0}_{H!}(\mathcal{F}) \to \mathrm{Inv}^{\ge 1}_{H!}(\mathcal{F}) \to \cdots. \tag{6.6.4}$$

If there is an integer N such that $\mathrm{Inv}^{\ge n-1}_{H!}(\mathcal{F}) \to \mathrm{Inv}^{\ge n}_{H!}(\mathcal{F})$ is an isomorphism for all $n \le N$, we set

$$\mathrm{Inv}_{H!}(\mathcal{F}) = \mathrm{Inv}^{\ge n}_{H!}(\mathcal{F}) \qquad \text{for any } n \le N,$$

and we say that "$\mathrm{Inv}_{H!}(\mathcal{F})$ **is bounded**." This object (if it exists) satisfies

$$\mathrm{Inv}^{\ge n}_{H!}(\mathcal{F}) \cong {}^p\tau^{\ge n}\mathrm{Inv}_{H!}(\mathcal{F}) \tag{6.6.5}$$

for all n.

Remark 6.6.9. When (6.6.2) or (6.6.4) fail to stabilize, the discussion above suggests that there ought to be objects

$$\mathrm{Inv}_{H*}(\mathcal{G}) \in D^+_G(X,\Bbbk), \qquad \mathrm{Inv}_{H!}(\mathcal{F}) \in D^-_G(X,\Bbbk)$$

that satisfy (6.6.3) and (6.6.5), respectively, in general. But there is a technical obstacle to making this precise: we have not defined $D^+_G(X,\Bbbk)$ or $D^-_G(X,\Bbbk)$!

This is a nontrivial problem, mainly because the approach described in Section 6.4 depends crucially on Lemma 6.4.7, which in general has no unbounded analogue. (In the case where X admits an ∞-acyclic G-resolution, however, one can follow that approach at least to define $D^+_G(X,\Bbbk)$; see [**27**, Section 2.9].)

It is possible to define $D^+_G(X,\Bbbk)$ and $D^-_G(X,\Bbbk)$ in general using the notion of "cartesian sheaves on a simplicial space," explained in [**27**, Appendix B]. Moreover, in this framework, there do indeed exist triangulated functors

$$\mathrm{Inv}_{H*} : D^{\mathrm{b}}_{G/H}(X,\Bbbk) \to D^+_G(X,\Bbbk), \qquad \mathrm{Inv}_{H!} : D^{\mathrm{b}}_{G/H}(X,\Bbbk) \to D^-_G(X,\Bbbk)$$

that are adjoint to $\mathrm{Infl}^G_{G/H}$ in an appropriate sense. However, we will not develop the theory of $D^+_G(X,\Bbbk)$ or $D^-_G(X,\Bbbk)$ in this book.

Lemma 6.6.10. *Let $K \subset H \subset G$ be algebraic groups, and assume that K is a normal subgroup of G. Let X be a G/H-variety. For $\mathcal{F} \in D^{\mathrm{b}}_G(X,\Bbbk)$, there are natural isomorphisms*

$$\mathrm{For}^{G/K}_{H/K}(\mathrm{Inv}^{\le n}_{K*}\mathcal{F}) \cong \mathrm{Inv}^{\le n}_{K*}(\mathrm{For}^G_H(\mathcal{F})),$$
$$\mathrm{For}^{G/K}_{H/K}(\mathrm{Inv}^{\ge n}_{K!}\mathcal{F}) \cong \mathrm{Inv}^{\ge n}_{K!}(\mathrm{For}^G_H(\mathcal{F})).$$

In particular, $\mathrm{Inv}_{K*}(\mathcal{F})$ *is bounded if and only if* $\mathrm{Inv}_{K*}(\mathrm{For}^G_H(\mathcal{F}))$ *is bounded, and likewise for* $\mathrm{Inv}_{K!}(\mathcal{F})$.

Proof. Let $\mathcal{G} \in {}^pD^{\mathrm{b}}_{H/K}(X,\Bbbk)^{\le n}$. We then have natural isomorphisms

$$\mathrm{Hom}(\mathcal{G}, \mathrm{For}^{G/K}_{H/K}(\mathrm{Inv}^{\le n}_{K*}\mathcal{F})) \cong \mathrm{Hom}(\mathrm{Infl}^G_{G/K}\mathrm{Av}^{G/K}_{H/K!}(\mathcal{G}), \mathcal{F}),$$
$$\mathrm{Hom}(\mathcal{G}, \mathrm{Inv}^{\le n}_{K*}(\mathrm{For}^G_H(\mathcal{F}))) \cong \mathrm{Hom}(\mathrm{Av}^G_{H!}\mathrm{Infl}^H_{H/K}(\mathcal{G}), \mathcal{F}).$$

The right-hand sides are isomorphic to each other by Lemma 6.6.4. The result for $\mathrm{Inv}^{\le n}_{K*}$ follows by Yoneda's lemma. The argument for $\mathrm{Inv}^{\ge n}_{K!}$ is similar. ☐

Proposition 6.6.11. *Let G be a connected group, let $H \lhd G$ be a normal subgroup, and let d be the largest integer such that $\mathbf{H}_d(H;\mathbb{Z})$ is nonzero. Let X be a G/H-variety, and let $\mathcal{F} \in D^{\mathrm{b}}_G(X,\Bbbk)$.*

(1) *Assume that* $\mathrm{Inv}_{H*}\mathcal{F}$ *is bounded. If $\mathcal{F}$ lies in ${}^pD^{\mathrm{b}}_G(X,\Bbbk)^{\le n}$, then* $\mathrm{Inv}_{H*}\mathcal{F}$ *lies in ${}^pD^{\mathrm{b}}_{G/H}(X,\Bbbk)^{\le n-d}$.*
(2) *Assume that* $\mathrm{Inv}_{H!}\mathcal{F}$ *is bounded. If $\mathcal{F}$ lies in ${}^pD^{\mathrm{b}}_G(X,\Bbbk)^{\ge n}$, then* $\mathrm{Inv}_{H!}\mathcal{F}$ *lies in ${}^pD^{\mathrm{b}}_{G/H}(X,\Bbbk)^{\ge n+d}$.*

The proof will make use of formulas from Exercises 6.8.2 and 6.8.6. However, the proofs of those formulas rely only on facts we have already discussed, so there is no logical problem in proving this proposition now.

Proof. We will prove the first assertion. At the end of the proof, we will briefly indicate how to adapt the argument to prove the second assertion. Assume throughout that $\mathrm{Inv}_{H*}(\mathcal{F})$ is bounded and nonzero. Let N be the largest integer such that ${}^p\mathsf{H}^N(\mathrm{Inv}_{H*}(\mathcal{F})) \neq 0$.

Step 1. For any object $A \in D^{\mathrm{b}}_G(\mathrm{pt}, \mathbb{Z})^{\geq k}$, the object

$$\mathrm{Inv}_{H^*}(R\mathscr{H}om(\Bbbk \overset{L}{\otimes}_{\mathbb{Z}} \mathrm{a}_X^* A, \mathcal{F}))$$

is bounded and lies in $D^{\mathrm{b}}_{G/H}(X, \Bbbk)^{\leq N-k+1}$. By truncation and induction on the number of nonzero cohomology objects of A, this claim reduces to the following assertion: *Given an abelian group $A \in \mathbb{Z}\text{-mod}^{\mathrm{fg}}$, the object $\mathrm{Inv}_{H*}(R\mathscr{H}om(\Bbbk \otimes^L \underline{A}_X, \mathcal{F}))$ is bounded and lies in $D^{\mathrm{b}}_{G/H}(X, \Bbbk)^{\leq N+1}$.* To prove this, recall that A admits a resolution $0 \to P_1 \to P_0 \to A \to 0$ where P_1 and P_0 are finite-rank free abelian groups. Of course, $R\mathscr{H}om(\Bbbk \otimes^L \underline{P_0}_X, \mathcal{F})$ is a direct sum of copies of $\mathcal{F}$, so $\mathrm{Inv}_{H*}(R\mathscr{H}om(\Bbbk \otimes^L \underline{P_0}_X, \mathcal{F}))$ is bounded and lies in $D^{\mathrm{b}}_{G/H}(X, \Bbbk)^{\leq N}$, and likewise for P_1. The claim then follows from the distinguished triangle

$$\mathrm{Inv}_{H*}(R\mathscr{H}om(\Bbbk \overset{L}{\otimes} \underline{A}_X, \mathcal{F})) \to \mathrm{Inv}_{H*}(R\mathscr{H}om(\Bbbk \overset{L}{\otimes} \underline{P_0}_X, \mathcal{F})) \\ \to \mathrm{Inv}_{H*}(R\mathscr{H}om(\Bbbk \overset{L}{\otimes} \underline{P_1}_X, \mathcal{F})) \to .$$

Step 2. If $A \in D^{\mathrm{b}}_G(\mathrm{pt}, \mathbb{Z})^{\geq k}$ has the property that $\mathsf{H}^k(A)$ is a free abelian group, then $\mathrm{Inv}_{H^*}(R\mathscr{H}om(\Bbbk \otimes^L_{\mathbb{Z}} \mathrm{a}_X^* A, \mathcal{F}))$ lies in $D^{\mathrm{b}}_{G/H}(X, \Bbbk)^{\leq N-k}$, and

$${}^p\mathsf{H}^{N-k}(\mathrm{Inv}_{H^*}(R\mathscr{H}om(\Bbbk \overset{L}{\otimes}_{\mathbb{Z}} \mathrm{a}_X^* A, \mathcal{F}))) \neq 0.$$

The distinguished triangle $\mathsf{H}^k(A)[-k] \to A \to \tau^{\geq k+1}(A) \to$ gives rise to a distingushed triangle

$$\mathrm{Inv}_{H^*}(R\mathscr{H}om(\Bbbk \overset{L}{\otimes}_{\mathbb{Z}} \mathrm{a}_X^*(\tau^{\geq k+1}A), \mathcal{F})) \to \mathrm{Inv}_{H^*}(R\mathscr{H}om(\Bbbk \overset{L}{\otimes}_{\mathbb{Z}} \mathrm{a}_X^* A, \mathcal{F})) \\ \to \mathrm{Inv}_{H^*}(R\mathscr{H}om(\Bbbk \overset{L}{\otimes}_{\mathbb{Z}} \underline{\mathsf{H}^k(A)}_X, \mathcal{F}))[k] \to . \tag{6.6.6}$$

The first term lies in $D^{\mathrm{b}}_{G/H}(X, \Bbbk)^{\leq N-k}$ by Step 1. On the other hand, $R\mathscr{H}om(\Bbbk \otimes^L_{\mathbb{Z}} \underline{\mathsf{H}^k(A)}_X, \mathcal{F})$ is a direct sum of copies of $\mathcal{F}$. By our assumptions on $\mathrm{Inv}_{H*}(\mathcal{F})$, the third term of (6.6.6) lies in $D^{\mathrm{b}}_{G/H}(X, \Bbbk)^{\leq N-k}$, and its perverse cohomology in degree $N-k$ is nonzero. The claim then follows from the long exact sequence in perverse cohomology of (6.6.6).

Step 3. Proof in the case $G = H$. Assume $G = H$. By Exercises 6.8.2 and 6.8.6, we have

$$\mathrm{Inv}_{H*} R\mathscr{H}om(\mathrm{Av}^H_{1!}\underline{\Bbbk}_X, \mathcal{F}) \cong \mathrm{Inv}_{H*}\mathrm{Av}^H_{1*} R\mathscr{H}om(\underline{\Bbbk}_X, \mathrm{For}(\mathcal{F})) \cong \mathrm{For}(\mathcal{F}).$$

On the other hand, by Example 6.6.5, we have

$$\mathrm{Inv}_{H*} R\mathscr{H}om(\mathrm{Av}^H_{1!}\underline{\Bbbk}_X, \mathcal{F}) \cong \mathrm{Inv}_{H*} R\mathscr{H}om(\mathrm{a}_X^* \mathrm{a}_{H!}\underline{\Bbbk}_H[2\dim H](\dim H), \mathcal{F}) \\ \cong \mathrm{Inv}_{H*} R\mathscr{H}om(\Bbbk \overset{L}{\otimes}_{\mathbb{Z}} \mathrm{a}_X^* \mathrm{a}_{H!}\underline{\mathbb{Z}}_H[2\dim H](\dim H), \mathcal{F}).$$

Recall that we defined d to be the largest integer such that the homology group $\mathbf{H}_d(H; \mathbb{Z}) \cong \mathbf{H}_{\mathrm{c}}^{2\dim H - d}(H; \mathbb{Z})(\dim H)$ is nonzero, so that $\mathrm{a}_{H!}\underline{\mathbb{Z}}_H[2\dim H](\dim H)$ lies in $D^{\mathrm{b}}_H(\mathrm{pt}, \Bbbk)^{\geq -d}$. Moreover, by Lemma 6.1.1,

$$\mathsf{H}^{-d}(\mathrm{a}_{H!}\underline{\mathbb{Z}}_H[2\dim H](\dim H)) \cong \mathbf{H}_{\mathrm{c}}^{2\dim H - d}(H; \mathbb{Z})(\dim H) \cong \mathbf{H}_d(H; \mathbb{Z})$$

is a free abelian group. By Step 2, we conclude that $\mathrm{For}(\mathcal{F}) \in D^{\mathrm{b}}_{\mathrm{c}}(X, \Bbbk)^{\leq N+d}$ and that ${}^p\mathsf{H}^{N+d}(\mathrm{For}(\mathcal{F})) \neq 0$. Since $\mathcal{F} \in D^{\mathrm{b}}_G(X, \Bbbk)^{\leq n}$, we conclude that $N + d \leq n$, and hence that $\mathrm{Inv}_{H*}(\mathcal{F}) \in D^{\mathrm{b}}_{\mathrm{c}}(X, \Bbbk)^{\leq n-d}$.

Step 4. Proof in the general case. Observe that the object $\mathrm{Inv}_{H*}(\mathcal{F})$ lies in ${}^pD^{\mathrm{b}}_{G/H}(X, \Bbbk)^{\leq n-d}$ if and only if $\mathrm{For}^{G/H}\mathrm{Inv}_{H*}(\mathcal{F})$ lies in ${}^pD^{\mathrm{b}}_{\mathrm{c}}(X, \Bbbk)^{\leq n-d}$. By Lemma 6.6.10, we have $\mathrm{For}^{G/H}\mathrm{Inv}_{H*}(\mathcal{F}) \cong \mathrm{Inv}_{H*}\mathrm{For}^G_H(\mathcal{F})$, and the latter object lies in ${}^pD^{\mathrm{b}}_{\mathrm{c}}(X, \Bbbk)^{\leq n-d}$ by Step 3.

The argument for $\mathrm{Inv}_{H!}\mathcal{F}$ is similar, with Steps 1 and 2 replaced by claims about $\mathrm{Inv}_{H!}((\Bbbk \otimes^L_{\mathbb{Z}} \mathrm{a}_X^* A) \otimes^L \mathcal{F})$. $\square$

Applications of averaging functors. We conclude this section with some results on equivariant derived categories for finite groups and for unipotent groups.

THEOREM 6.6.12. *Let G be a finite group, and let X be a G-variety. If $\Bbbk$ is a field, then the realization functor*

$$\mathrm{real} : D^{\mathrm{b}}\mathrm{Perv}_G(X, \Bbbk) \to D^{\mathrm{b}}_G(X, \Bbbk)$$

is an equivalence of categories.

PROOF. Let $\mathcal{F} \in \mathrm{Perv}_G(X, \Bbbk)$. By Corollary 6.6.3, $\mathrm{Av}^G_{1!}\mathrm{For}^G(\mathcal{F})$ is a perverse sheaf. We claim that the adjunction map $\epsilon : \mathrm{Av}^G_{1!}\mathrm{For}^G(\mathcal{F}) \to \mathcal{F}$ is surjective. Since $\mathrm{For}^G(\mathrm{cok}\,\epsilon) = 0$ if and only if $\mathrm{cok}\,\epsilon = 0$, it is enough to show the surjectivity of

$$\mathrm{For}^G\mathrm{Av}^G_{1!}\mathrm{For}^G(\mathcal{F}) \xrightarrow{\mathrm{For}^G(\epsilon)} \mathrm{For}^G(\mathcal{F}).$$

The surjectivity of $\mathrm{For}^G(\epsilon)$ follows from the unit–counit equations.

Now let $\mathcal{F}, \mathcal{G} \in \mathrm{Perv}_G(X, \Bbbk)$, and let $\phi : \mathcal{F} \to \mathcal{G}[n]$ be a morphism in $D^{\mathrm{b}}_G(X, \Bbbk)$. By Theorem 4.5.9 (and Lemma A.5.17), the morphism $\mathrm{For}(\phi)$ in $D^{\mathrm{b}}_{\mathrm{c}}(X, \Bbbk)$ is effaceable: there exists a surjective morphism $p : \mathcal{P} \to \mathrm{For}(\mathcal{F})$ such that $\mathrm{For}(\phi) \circ p = 0$. Now consider the commutative diagram

$$\begin{tikzcd}
\mathrm{Av}^G_{1!}(\mathcal{P}) \arrow[r, "\mathrm{Av}^G_{1!}(p)"] & \mathrm{Av}^G_{1!}\mathrm{For}^G(\mathcal{F}) \arrow[r, "\epsilon_{\mathcal{F}}"] \arrow[d, "\mathrm{Av}^G_{1!}\mathrm{For}^G(\phi)"'] & \mathcal{F} \arrow[d, "\phi"] \\
& \mathrm{Av}^G_{1!}\mathrm{For}^G(\mathcal{G}[n]) \arrow[r, "\epsilon_{\mathcal{G}[n]}"] & \mathcal{G}[n]
\end{tikzcd}$$

where the arrows labelled ϵ are adjunction maps. By the preceding paragraph, $\epsilon_{\mathcal{F}}$ is surjective. Since $\mathrm{Av}^G_{1!}$ is a t-exact functor, $\mathrm{Av}^G_{1!}(p)$ is also surjective, and hence so is the composition $\epsilon \circ \mathrm{Av}^G_{1!}(p)$. The diagram shows that $\phi \circ (\epsilon \circ \mathrm{Av}^G_{1!}(p)) = 0$, so ϕ is effaceable. The result follows by Corollary A.7.19. $\square$

PROPOSITION 6.6.13. *Let G be a finite group. There are equivalences of categories*

$$D^{\mathrm{b}}(\Bbbk[G]\text{-}\mathrm{mod}^{\mathrm{fg}}) \cong D^{\mathrm{b}}\mathrm{Perv}_G(\mathrm{pt}, \Bbbk) \xrightarrow{\sim} D^{\mathrm{b}}_G(\mathrm{pt}, \Bbbk).$$

PROOF. The first equivalence comes from Proposition 6.2.12. For the second, copy the proof of Theorem 6.6.12, but instead of invoking Theorem 4.5.9, use Remark 4.5.10. $\square$

REMARK 6.6.14. Here is another perspective on why Proposition 6.6.13 is true. Suppose P is a contractible topological space with a free action of a finite group

G, and suppose that the quotient map $P \to P/G$ is a covering map. Then a minor variant of Theorem 1.9.7 says that there is an equivalence of categories

$$D^{\mathrm{b}}\mathrm{Loc}^{\mathrm{ft}}(P/G, \Bbbk) \to D^{\mathrm{b}}_{\mathrm{locf}}(P/G, \Bbbk). \tag{6.6.7}$$

Imagine now that P is a smooth algebraic variety and a principal G-variety. Then $\mathrm{a}_P : P \to \mathrm{pt}$ is an ∞-acyclic resolution, and Lemma 6.4.7 implies that there is an equivalence of categories

$$D^{\mathrm{b}}_G(\mathrm{pt}, \Bbbk) \xrightarrow{\sim} D^{\mathrm{b}}_{\mathrm{locf}}(P/G, \Bbbk)$$

that restricts to an equivalence of categories $\mathrm{Perv}_G(\mathrm{pt}, \Bbbk) \xrightarrow{\sim} \mathrm{Loc}^{\mathrm{ft}}(P/G, \Bbbk)$. Proposition 6.6.13 then amounts to a restatement of (6.6.7).

(In reality, although there is a close analogy between Theorem 1.9.7 and Proposition 6.6.13, the situation described above cannot literally occur: every nontrivial finite group has infinite cohomological dimension, but the existence of an algebraic ∞-acyclic resolution $P \to \mathrm{pt}$ implies that the cohomological dimension of G is bounded by $2 \dim P/G$.)

THEOREM 6.6.15. *Let G be an algebraic group, and let $H \subset G$ be a closed subgroup such that G/H is contractible. For any G-variety X, the functor $\mathrm{For}^G_H : D^{\mathrm{b}}_G(X, \Bbbk) \to D^{\mathrm{b}}_H(X, \Bbbk)$ is fully faithful.*

PROOF. Let G act on $G/H \times X$ by $g \cdot (g'H, x) = (gg'H, g \cdot x)$. Consider the projection map $\mathrm{pr}_2 : G/H \times X \to X$. For $\mathcal{F} \in D^{\mathrm{b}}_G(X, \Bbbk)$, we have

$$\mathrm{pr}_{2*}\mathrm{pr}_2^*\mathcal{F} \cong (\mathrm{a}_{G/H})_*\underline{\Bbbk}_{G/H} \boxtimes \mathcal{F}.$$

Since G/H is contractible, we have $R\Gamma(\underline{\Bbbk}_{G/H}) \cong \Bbbk$, and hence $\mathrm{pr}_{2*}\mathrm{pr}_2^*\mathcal{F} \cong \mathcal{F}$. It follows by adjunction that pr_2^* is fully faithful.

Let $\bar{\sigma} : G \times^H X \to X$ and $i : X \to G \times^H X$ be as in the proof of Theorem 6.6.1. We saw in the proof of Theorem 6.6.1 that

$$\mathrm{For}^G_H \cong (i^* \circ \mathrm{For}^G_H) \circ \bar{\sigma}^*,$$

where $i^* \circ \mathrm{For}^G_H$ is an equivalence of categories. By the previous paragraph, diagram (6.6.1) shows us that $\bar{\sigma}^*$ is fully faithful, so For^G_H is as well. ☐

THEOREM 6.6.16. *Let G be an algebraic group, and let $U \lhd G$ be a unipotent normal subgroup. For any G/U-variety X, the functor $\mathrm{Infl}^G_{G/U} : D^{\mathrm{b}}_{G/U}(X, \Bbbk) \to D^{\mathrm{b}}_G(X, \Bbbk)$ is an equivalence of categories. If $G = H \ltimes U$, so that $G/U \cong H$, then $\mathrm{For}^G_H : D^{\mathrm{b}}_G(X, \Bbbk) \to D^{\mathrm{b}}_H(X, \Bbbk)$ is the inverse equivalence.*

PROOF SKETCH. Recall that any complex unipotent group is isomorphic as a variety to some affine space. In particular, U is contractible. For the first assertion, it is enough to show that for any two integers $a \le b$, the functor $\mathrm{Infl}^G_{G/U} : {}^pD^{\mathrm{b}}_{G/U}(X, \Bbbk)^{[a,b]} \to {}^pD^{\mathrm{b}}_G(X, \Bbbk)^{[a,b]}$ is an equivalence of categories. Choose an n-acyclic G-resolution $p : P \to X$ with $n \ge b - a$, say of relative dimension d, and consider the diagram

$$\begin{tikzcd} P \arrow[r, "\pi'"] \arrow[dr, "p"'] & P/U \arrow[r] \arrow[d, "\bar{p}"] & P/G = \frac{P/U}{G/U} \\ & X & \end{tikzcd}$$

The map $\pi' : P \to P/U$ is a smooth, locally trivial fibration with contractible fibers, so by Exercise 6.1.4, it is ∞-acyclic. A short calculation then shows that $\bar{p}$ is n-acyclic. By the definition of the inflation functor, the diagram

$$\begin{tikzcd} {}^pD^{\mathrm{b}}_{G/U}(X,\Bbbk)^{[a,b]} \arrow[d, "\mathrm{Infl}^G_{G/U}"'] \arrow[dr, "\mathcal{F}\mapsto\mathcal{F}(\bar{p})"] & \\ {}^pD^{\mathrm{b}}_{G}(X,\Bbbk)^{[a,b]} \arrow[r, "\mathcal{F}\mapsto\mathcal{F}(p)"] & {}^pD^{\mathrm{b}}_{\mathrm{c}}(P/G,\Bbbk)^{[a+d-\dim G, b+d-\dim G]} \end{tikzcd}$$

commutes. By Lemma 6.4.7, both diagonal arrows are fully faithful. Moreover, using the ∞-acyclicity of π', one can show that both diagonal arrows have the same image. It follows that $\mathrm{Infl}^G_{G/U}$ is an equivalence of categories.

Finally, if $G = H \ltimes U$, then Lemma 6.5.8 implies that $\mathrm{For}^G_H \mathrm{Infl}^G_{G/U}(\mathcal{F}) \cong \mathcal{F}$ for all $\mathcal{F} \in D^{\mathrm{b}}_{G/U}(X,\Bbbk)$. $\square$

6.7. Equivariant cohomology

The notion of equivariant cohomology, introduced by Borel [**34**] in the late 1950s, predates most of the content of this chapter by decades. Nevertheless, Borel's definition of $\mathbf{H}^\bullet_G(X;\Bbbk)$ (essentially, it is the singular cohomology of P/G, where $P \to X$ is an ∞-acyclic resolution) fits neatly into the framework of Section 6.4. As in [**27**, Section 13], we place it in a sheaf-theoretic context via the following analogue of Definition 1.1.17.

Definition 6.7.1. Let X be a G-variety, and let $\mathcal{F} \in D^{\mathrm{b}}_G(X,\Bbbk)$. The kth **equivariant hypercohomology** of $\mathcal{F}$, denoted by $\mathbf{H}^k_G(X,\mathcal{F})$, is the $\Bbbk$-module given by

$$\mathbf{H}^k_G(X,\mathcal{F}) = \mathrm{Hom}_{D^{\mathrm{b}}_G(\mathrm{pt},\Bbbk)}(\underline{\Bbbk}_{\mathrm{pt}}, \mathrm{a}_{X*}\mathcal{F}[k]).$$

Similarly, the kth **equivariant hypercohomology with compact support** of $\mathcal{F}$, denoted by $\mathbf{H}^k_{\mathrm{c},G}(\mathcal{F})$, is the $\Bbbk$-module given by

$$\mathbf{H}^k_{\mathrm{c},G}(X,\mathcal{F}) = \mathrm{Hom}_{D^{\mathrm{b}}_G(\mathrm{pt},\Bbbk)}(\underline{\Bbbk}_{\mathrm{pt}}, \mathrm{a}_{X!}\mathcal{F}[k]).$$

Now let M be a finitely generated $\Bbbk$-module. The kth **equivariant cohomology** (resp. **equivariant cohomology with compact support**) of X with coefficients in M is the kth equivariant hypercohomology (resp. equivariant cohomology with compact support) of the constant sheaf $\underline{M}_X$:

$$\mathbf{H}^k_G(X;M) = \mathbf{H}^k_G(X,\underline{M}_X) \qquad \text{and} \qquad \mathbf{H}^k_{\mathrm{c},G}(X;M) = \mathbf{H}^k_{\mathrm{c},G}(X,\underline{M}_X).$$

By adjunction, one obtains the following alternative descriptions of equivariant hypercohomology:

$$\begin{aligned} \mathbf{H}^k_G(X,\mathcal{F}) &\cong \mathrm{Hom}_{D^{\mathrm{b}}_G(X,\Bbbk)}(\underline{\Bbbk}_X, \mathcal{F}[k]) \cong \mathsf{H}^k(\mathrm{Inv}^{\le k}_{G*}\mathrm{a}_{X*}\mathcal{F}), \\ \mathbf{H}^k_{\mathrm{c},G}(X,\mathcal{F}) &\cong \mathsf{H}^k(\mathrm{Inv}^{\le k}_{G*}\mathrm{a}_{X!}\mathcal{F}). \end{aligned}$$

More concretely, let $p : P \to \mathrm{pt}$ be a G-resolution of a point, and let $p' = \mathrm{pr}_2 : P \times X \to X$ be the corresponding G-resolution of X (cf. (6.5.1)). If p is n-acyclic with n sufficiently large, then by unwinding the definitions, one can show that

$$\mathbf{H}^k_G(X,\mathcal{F}) \cong \mathbf{H}^k((P \times X)/G, \mathcal{F}(p')). \tag{6.7.1}$$

The condition that n be "sufficiently large" can be made explicit as follows: there should exist an integer a such that both $\underline{\Bbbk}_X$ and $\mathcal{F}[k]$ lie in ${}^pD^{\mathrm{b}}_G(X,\Bbbk)^{[a,a+n]}$.

There is a ring structure on the total equivariant cohomology

$$\mathbf{H}^\bullet_G(X;\Bbbk) = \bigoplus_{k\geq 0}\mathbf{H}^k_G(X;\Bbbk) \cong \bigoplus_{k\geq 0}\mathrm{Hom}_{D^{\mathrm{b}}_G(X,\Bbbk)}(\underline{\Bbbk}_X,\underline{\Bbbk}_X[k])$$

given by composition of morphisms in $D^{\mathrm{b}}_G(X,\Bbbk)$ (cf. Remark 1.2.5). Any G-equivariant map $f : X \to Y$ induces a graded ring homomorphism

$$f^\sharp : \mathbf{H}^\bullet_G(Y;\Bbbk) \to \mathbf{H}^\bullet_G(X;\Bbbk).$$

In particular, via $\mathrm{a}_X^\sharp$, the equivariant cohomology ring $\mathbf{H}^\bullet_G(X;\Bbbk)$ is canonically an algebra over $\mathbf{H}^\bullet_G(\mathrm{pt};\Bbbk)$. More generally, if $\mathcal{F} \in D^{\mathrm{b}}_G(X,\Bbbk)$, then

$$\mathbf{H}^\bullet_G(X,\mathcal{F}) = \bigoplus_{k\geq 0}\mathbf{H}^k_G(X,\mathcal{F}) \cong \bigoplus_{k\geq 0}\mathrm{Hom}_{D^{\mathrm{b}}_G(X,\Bbbk)}(\underline{\Bbbk}_X,\mathcal{F}[k])$$

naturally has the structure of a right $\mathbf{H}^\bullet_G(X;\Bbbk)$-module, and hence of a right $\mathbf{H}^\bullet_G(\mathrm{pt};\Bbbk)$-module as well.

We begin by proving some elementary lemmas about equivariant hypercohomology on a point.

LEMMA 6.7.2. *Let G be a connected algebraic group. For any $\mathcal{F} \in D^{\mathrm{b}}_G(\mathrm{pt},\Bbbk)$, we have $\mathbf{H}^\bullet_G(\mathrm{pt},\mathcal{F}) = 0$ if and only if $\mathcal{F} = 0$.*

PROOF. If $\mathcal{F} \neq 0$, let k be the smallest integer such that $\mathsf{H}^k(\mathcal{F}) \neq 0$. This is an object of $\mathrm{Loc}^{\mathrm{ft}}_G(\mathrm{pt},\Bbbk) \cong \Bbbk\text{-mod}^{\mathrm{fg}}$ (see Proposition 6.2.13), so there is some nonzero map $\underline{\Bbbk}_{\mathrm{pt}} \to \mathsf{H}^k(\mathcal{F})$. Our assumptions imply that $\mathsf{H}^k(\mathcal{F})[-k] \cong \tau^{\leq k}(\mathcal{F})$, so by the adjunction properties of truncation, there is a nonzero map $\underline{\Bbbk}_{\mathrm{pt}}[-k] \to \mathcal{F}$. In other words, the group $\mathrm{Hom}(\underline{\Bbbk}_{\mathrm{pt}}[-k],\mathcal{F}) \cong \mathbf{H}^k_G(\mathrm{pt},\mathcal{F})$ is nonzero. □

LEMMA 6.7.3. *Let G be a connected algebraic group. For $\mathcal{F} \in D^{\mathrm{b}}_G(\mathrm{pt},\Bbbk)$, the following conditions are equivalent:*

1. *$\mathbf{H}^\bullet_G(\mathrm{pt},\mathcal{F})$ is a free graded $\mathbf{H}^\bullet_G(\mathrm{pt};\Bbbk)$-module of rank r.*
2. *There exist integers $n_1,\ldots,n_r$ such that $\mathcal{F} \cong \underline{\Bbbk}_{\mathrm{pt}}[n_1] \oplus \cdots \oplus \underline{\Bbbk}_{\mathrm{pt}}[n_r]$.*

PROOF. It is obvious that condition (2) implies condition (1). For the opposite implication, suppose $\mathbf{H}^\bullet_G(\mathrm{pt},\mathcal{F})$ is free over $\mathbf{H}^\bullet_G(\mathrm{pt};\Bbbk)$. Choose a homogeneous basis $\gamma_1,\ldots,\gamma_r$, and suppose γ_i is an element of $\mathbf{H}^{-n_i}_G(\mathrm{pt},\mathcal{F}) \cong \mathrm{Hom}(\underline{\Bbbk}_{\mathrm{pt}}[n_i],\mathcal{F})$. Together, the maps $(\gamma_1,\ldots,\gamma_n)$ give us a map

$$\gamma : \underline{\Bbbk}_{\mathrm{pt}}[n_1] \oplus \cdots \oplus \underline{\Bbbk}_{\mathrm{pt}}[n_r] \to \mathcal{F}.$$

By construction, the map γ induces an isomorphism on equivariant hypercohomology, so its cone must have zero equivariant hypercohomology. By Lemma 6.7.2, its cone is the zero object, and thus γ is an isomorphism. □

Tensor product formulas. Let $\varphi : H \to G$ be a homomorphism of algebraic groups. Then the functor $\mathrm{Res}_\varphi : D^{\mathrm{b}}_G(\mathrm{pt},\Bbbk) \to D^{\mathrm{b}}_H(\mathrm{pt},\Bbbk)$ (see Exercise 6.5.9) induces a graded ring homomorphism

$$\varphi^\sharp : \mathbf{H}^\bullet_G(\mathrm{pt};\Bbbk) \to \mathbf{H}^\bullet_H(\mathrm{pt};\Bbbk),$$

making $\mathbf{H}^\bullet_H(\mathrm{pt};\Bbbk)$ into an algebra over $\mathbf{H}^\bullet_G(\mathrm{pt};\Bbbk)$. More generally, if X is a G-variety, then for any $\mathcal{F} \in D^{\mathrm{b}}_G(X,\Bbbk)$, the functor Res_φ gives rise to a map $\mathbf{H}^\bullet_G(X,\mathcal{F}) \to \mathbf{H}^\bullet_H(X,\mathrm{Res}_\varphi(\mathcal{F}))$, and hence to a natural map of $\mathbf{H}^\bullet_H(\mathrm{pt};\Bbbk)$-modules

$$\mathbf{H}^\bullet_G(X,\mathcal{F}) \otimes_{\mathbf{H}^\bullet_G(\mathrm{pt};\Bbbk)} \mathbf{H}^\bullet_H(\mathrm{pt};\Bbbk) \to \mathbf{H}^\bullet_H(X,\mathrm{Res}_\varphi(\mathcal{F})). \tag{6.7.2}$$

LEMMA 6.7.4. *Let $\varphi : H \to G$ be a homomorphism of algebraic groups, with G connected. Let X be a G-variety, and let $\mathcal{F} \in D^{\mathrm{b}}_G(X,\Bbbk)$ be such that $\mathbf{H}^\bullet_G(X,\mathcal{F})$ is a free graded $\mathbf{H}^\bullet_G(\mathrm{pt};\Bbbk)$-module. Then the natural map* (6.7.2) *is an isomorphism.*

PROOF. Assume first that $X = \mathrm{pt}$. By Lemma 6.7.3, we may reduce to the special case where $\mathcal{F} = \underline{\Bbbk}_{\mathrm{pt}}$. In this case, the lemma is obvious.

For general X, we have $\mathbf{H}^\bullet_G(X,\mathcal{F}) \cong \mathbf{H}^\bullet_G(\mathrm{pt},\mathrm{a}_{X*}\mathcal{F})$ and $\mathbf{H}^\bullet_H(X,\mathrm{Res}_\varphi(\mathcal{F})) \cong \mathbf{H}^\bullet_H(\mathrm{pt},\mathrm{Res}_\varphi(\mathrm{a}_{X*}\mathcal{F}))$, so the lemma follows from the special case above. ☐

PROPOSITION 6.7.5. *Let G and H be algebraic groups. Let X be a G-variety, let Y be an H-variety, and let $\mathcal{G} \in D^{\mathrm{b}}_G(X,\Bbbk)$ and $\mathcal{H} \in D^{\mathrm{b}}_H(Y,\Bbbk)$. Assume that at least one of $\mathbf{H}^\bullet_G(X,\mathcal{G})$ or $\mathbf{H}^\bullet_H(Y,\mathcal{H})$ is flat over $\Bbbk$. Then there is a natural isomorphism of $\Bbbk$-modules*

$$\mathbf{H}^\bullet_{G\times H}(X\times Y,\mathcal{G}\boxtimes\mathcal{H}) \cong \mathbf{H}^\bullet_G(X,\mathcal{G}) \otimes_\Bbbk \mathbf{H}^\bullet_H(Y,\mathcal{H}). \tag{6.7.3}$$

In particular, if at least one of $\mathbf{H}^\bullet_G(X;\Bbbk)$ or $\mathbf{H}^\bullet_H(Y;\Bbbk)$ is flat over $\Bbbk$, then there is a natural isomorphism of rings

$$\mathbf{H}^\bullet_{G\times H}(X\times Y;\Bbbk) \cong \mathbf{H}^\bullet_G(X;\Bbbk) \otimes_\Bbbk \mathbf{H}^\bullet_H(Y;\Bbbk). \tag{6.7.4}$$

When (6.7.3) and (6.7.4) both hold, one can check that (6.7.3) is in fact an isomorphism of $\mathbf{H}^\bullet_{G\times H}(X\times Y;\Bbbk)$-modules.

PROOF. The idea of the proof is to apply the Künneth formula along with Lemma A.6.17 on a suitably chosen resolution. Since

$$\mathbf{H}^\bullet_{G\times H}(X\times Y,\mathcal{G}\boxtimes\mathcal{H}) \cong \mathbf{H}^\bullet_{G\times H}(\mathrm{pt},\mathrm{a}_{X*}\mathcal{G}\boxtimes\mathrm{a}_{Y*}\mathcal{H}),$$

it is enough to prove the proposition in the special case where X and Y are both single points. We assume henceforth that this is the case, so that $\mathcal{G} \in D^{\mathrm{b}}_G(\mathrm{pt},\Bbbk)$ and $\mathcal{H} \in D^{\mathrm{b}}_H(\mathrm{pt},\Bbbk)$. After applying a suitable shift if necessary, we further assume that there is an integer $b \geq 0$ such that

$$\mathcal{G} \in D^{\mathrm{b}}_G(\mathrm{pt},\Bbbk)^{[0,b]} \qquad \text{and} \qquad \mathcal{H} \in D^{\mathrm{b}}_H(\mathrm{pt},\Bbbk)^{[0,b]}.$$

Finally, we assume that $\mathbf{H}^\bullet_G(\mathrm{pt},\mathcal{G})$ is flat over $\Bbbk$.

Choose an integer $n \geq b$, and let $d = \mathrm{gldim}\,\Bbbk$. Let $p : P \to \mathrm{pt}$ be an $(n+d)$-acyclic G-resolution, and let $q : Q \to \mathrm{pt}$ be an $(n+d)$-acyclic H-resolution. By (6.7.1), we have

$$\mathbf{H}^k_G(\mathrm{pt},\mathcal{G}) \cong \mathbf{H}^k(P/G,\mathcal{G}(p)) = \mathsf{H}^k(R\Gamma(\mathcal{G}(p))) \qquad \text{if } k \leq n+d, \tag{6.7.5}$$

and likewise for $\mathbf{H}^k_H(\mathrm{pt},\mathcal{H})$ and $\mathbf{H}^k_{G\times H}(\mathrm{pt},\mathcal{G}\boxtimes\mathcal{H})$. The cohomology objects of $\tau^{\leq n+d}R\Gamma(\mathcal{G}(p))$ are flat over $\Bbbk$, since they agree with certain equivariant hypercohomology groups of $\mathcal{G}$. Consider the distinguished triangle

$$\tau^{\leq n+d}R\Gamma(\mathcal{G}(p)) \overset{L}{\otimes} R\Gamma(\mathcal{H}(q)) \to R\Gamma(\mathcal{G}(p)) \overset{L}{\otimes} R\Gamma(\mathcal{H}(q))$$
$$\to \tau^{\geq n+d+1}R\Gamma(\mathcal{G}(p)) \overset{L}{\otimes} R\Gamma(\mathcal{H}(q)) \to .$$

Since $R\Gamma(\mathcal{H}(q)) \in D^{\mathrm{b}}(\Bbbk\text{-mod}^{\mathrm{fg}})^{\geq 0}$, Proposition A.6.16 tells us that the last term lies in $D^{\mathrm{b}}(\Bbbk\text{-mod}^{\mathrm{fg}})^{\geq n+1}$. The long exact cohomology sequence thus shows that

$$\mathsf{H}^k(\tau^{\leq n+d} R\Gamma(\mathcal{G}(p)) \overset{L}{\otimes} R\Gamma(\mathcal{H}(q))) \cong \mathsf{H}^k(R\Gamma(\mathcal{G}(p)) \overset{L}{\otimes} R\Gamma(\mathcal{H}(q))) \qquad \text{for } k \leq n.$$

Now, the analogue of (6.7.5) for $\mathcal{G} \boxtimes \mathcal{H}$ (along with the definition of $\boxtimes$ in (6.5.3)) implies that $\mathbf{H}^k_{G\times H}(\mathrm{pt}, \mathcal{G} \boxtimes \mathcal{H}) \cong \mathsf{H}^k(R\Gamma(\mathcal{G}(p)) \otimes^L R\Gamma(\mathcal{H}(q)))$ for $k \leq n+d$, so

$$\mathbf{H}^k_{G\times H}(\mathrm{pt}, \mathcal{G} \boxtimes \mathcal{H}) \cong \mathsf{H}^k(\tau^{\leq n+d} R\Gamma(\mathcal{G}(p)) \overset{L}{\otimes} R\Gamma(\mathcal{H}(q))) \qquad \text{for } k \leq n.$$

Since $\mathsf{H}^i(\tau^{\leq n+d}(R\Gamma(\mathcal{G}(p)))$ is flat over $\Bbbk$ for all i, Lemma A.6.17 tells us that

$$\begin{aligned} &\mathbf{H}^k_{G\times H}(\mathrm{pt}, \mathcal{G} \boxtimes \mathcal{H}) \\ &\qquad \cong \bigoplus_{i=0}^{k} \mathsf{H}^i(R\Gamma(\mathcal{G}(p))) \otimes \mathsf{H}^{k-i}(R\Gamma(\mathcal{H}(q))) \cong \bigoplus_{i=0}^{k} \mathbf{H}^i_G(X, \mathcal{G}) \otimes_{\Bbbk} \mathbf{H}^{k-i}_H(Y, \mathcal{H}) \end{aligned}$$

for $k \leq n$. By choosing n large enough early in the proof, we see that this isomorphism holds for all k. In other words, we have proved (6.7.3). It is left to the reader to check that (6.7.4) is in fact an isomorphism of rings. $\square$

Equivariant cohomology of a point. The next two results describe the equivariant cohomology of a point in the case of a torus action.

PROPOSITION 6.7.6. *There is a canonical isomorphism of graded rings*

$$\mathbf{H}^\bullet_{\mathbb{G}_\mathrm{m}}(\mathrm{pt}; \Bbbk) \cong \mathrm{Sym}(\Bbbk(-1)),$$

where the symmetric algebra $\mathrm{Sym}(\Bbbk(-1))$ *is made into a graded ring by setting* $\deg \Bbbk(-1) = 2$. *Moreover, if* $\varphi : \mathbb{G}_\mathrm{m} \to \mathbb{G}_\mathrm{m}$ *is the map* $\varphi(t) = t^d$, *then*

$$\varphi^\sharp : \mathbf{H}^\bullet_{\mathbb{G}_\mathrm{m}}(\mathrm{pt}; \Bbbk) \to \mathbf{H}^\bullet_{\mathbb{G}_\mathrm{m}}(\mathrm{pt}; \Bbbk)$$

is the unique $\Bbbk$*-algebra homomorphism such that* $\varphi^\sharp : \mathbf{H}^2_{\mathbb{G}_\mathrm{m}}(\mathrm{pt}; \Bbbk) \to \mathbf{H}^2_{\mathbb{G}_\mathrm{m}}(\mathrm{pt}; \Bbbk)$ *is multiplication by* d.

PROOF. Let $P_{2n} = \mathbb{A}^{n+1} \smallsetminus \{0\}$, equipped with the usual scaling action of $\mathbb{G}_\mathrm{m}$. By Example 6.1.21, $P_{2n} \to \mathrm{pt}$ is a $2n$-acyclic $\mathbb{G}_\mathrm{m}$-resolution. Recall that $P_{2n}/\mathbb{G}_\mathrm{m} \cong \mathbb{P}^n$. By (6.7.1), we have

$$\mathbf{H}^k_{\mathbb{G}_\mathrm{m}}(\mathrm{pt}; \Bbbk) \cong \mathbf{H}^k(\mathbb{P}^n; \Bbbk) \qquad \text{for } 0 \leq k \leq 2n. \tag{6.7.6}$$

It is well known (see, for instance, [**98**, Theorem 3.12]) that if we choose a basis element $x \in \mathbf{H}^2(\mathbb{P}^n; \Bbbk)$, then there is a ring isomorphism $\mathbf{H}^\bullet(\mathbb{P}^n; \Bbbk) \cong \Bbbk[x]/(x^{n+1})$. Since (6.7.6) holds for all n, we conclude that $\mathbf{H}^\bullet_{\mathbb{G}_\mathrm{m}}(\mathrm{pt}; \Bbbk)$ is isomorphic to a polynomial ring $\Bbbk[x]$. To rephrase this in a way that does not involve a choice, $\mathbf{H}^\bullet_{\mathbb{G}_\mathrm{m}}(\mathrm{pt}; \Bbbk)$ is the symmetric algebra on the free $\Bbbk$-module $\mathbf{H}^2_{\mathbb{G}_\mathrm{m}}(\mathrm{pt}; \Bbbk) \cong \mathbf{H}^2(\mathbb{P}^1; \Bbbk)$. By Example 2.2.6, $\mathbf{H}^2(\mathbb{P}^1; \Bbbk)$ is canonically identified with $\Bbbk(-1)$.

Now consider the homomorphism $\varphi : \mathbb{G}_\mathrm{m} \to \mathbb{G}_\mathrm{m}$ given by $\varphi(t) = t^d$. To compute $\varphi^\sharp$ on $\mathbf{H}^2_{\mathbb{G}_\mathrm{m}}(\mathrm{pt}; \Bbbk)$, consider the map $f : P_2 \to P_2$ given by $f(x, y) = (x^d, y^d)$. Then f is φ-equivariant, and of course $f \circ \mathrm{a}_{P_2} = \mathrm{a}_{P_2}$ is still a 2-acyclic resolution. Unwinding the construction of Res_φ, we see that computing $\varphi^\sharp$ comes down to computing

$$f^\sharp : \mathbf{H}^2(\mathbb{P}^1; \Bbbk) \to \mathbf{H}^2(\mathbb{P}^1; \Bbbk).$$

This is an exercise in algebraic topology (see, for instance, [**98**, Exercise 3.2.6]). $\square$

THEOREM 6.7.7. *Let T be an algebraic torus, and let $\mathbf{X}(T)$ be its character lattice. There is a canonical isomorphism of graded rings*

$$\mathbf{H}^\bullet_T(\mathrm{pt};\Bbbk) \cong \mathrm{Sym}(\Bbbk(-1) \otimes_{\mathbb{Z}} \mathbf{X}(T)),$$

where the symmetric algebra $\mathrm{Sym}(\Bbbk(-1) \otimes_{\mathbb{Z}} \mathbf{X}(T))$ *is made into a graded ring by setting* $\deg \Bbbk(-1) \otimes_{\mathbb{Z}} \mathbf{X}(T) = 2$. *Moreover, if $\varphi : T \to T'$ is a homomorphism of tori, let $\mathbf{X}(\varphi) : \mathbf{X}(T') \to \mathbf{X}(T)$ be the induced map of character lattices. Then the following diagram commutes:*

$$\begin{tikzcd}
\mathbf{H}^\bullet_{T'}(\mathrm{pt};\Bbbk) \arrow[r, "\varphi^\sharp"] \arrow[d, "\wr"'] & \mathbf{H}^\bullet_T(\mathrm{pt};\Bbbk) \arrow[d, "\wr"] \\
\mathrm{Sym}(\Bbbk(-1) \otimes_{\mathbb{Z}} \mathbf{X}(T')) \arrow[r, "\text{induced by } \mathbf{X}(\varphi)"] & \mathrm{Sym}(\Bbbk(-1) \otimes_{\mathbb{Z}} \mathbf{X}(T))
\end{tikzcd}$$

PROOF. Let $m = \dim T$, and choose a $\mathbb{Z}$-basis $\alpha_1, \ldots, \alpha_m$ for $\mathbf{X}(T)$. These characters determine an isomorphism of groups $\alpha = (\alpha_1, \ldots, \alpha_m) : T \to \mathbb{G}_{\mathrm{m}}^m$. Proposition 6.7.5 and Proposition 6.7.6 together imply that $\mathbf{H}^\bullet_{\mathbb{G}_{\mathrm{m}}^m}(\mathrm{pt};\Bbbk)$ is the symmetric algebra on $\Bbbk^m(-1)$. In particular, we have a canonical isomorphism

$$\mathbf{H}^\bullet_T(\mathrm{pt};\Bbbk) \cong \mathrm{Sym}(\mathbf{H}^2_T(\mathrm{pt};\Bbbk)),$$

as well as a noncanonical (because it depends on the choice of α) isomorphism

$$\mathbf{H}^2_T(\mathrm{pt};\Bbbk)(1) \cong \Bbbk^m. \tag{6.7.7}$$

Let $\nu : \mathbb{G}_{\mathrm{m}} \to T$ be a cocharacter of T, and consider the induced map

$$\nu^\sharp : \mathbf{H}^2_T(\mathrm{pt};\Bbbk)(1) \to \mathbf{H}^2_{\mathbb{G}_{\mathrm{m}}}(\mathrm{pt};\Bbbk)(1).$$

Via Proposition 6.7.6 and (6.7.7), we can think of this as a map $\Bbbk^m \to \Bbbk$. In fact, by the second part of Proposition 6.7.6,

$$\nu^\sharp : \Bbbk^m \to \Bbbk \qquad \text{is given by} \qquad (a_1, \ldots, a_m) \mapsto \sum_{i=1}^m a_i \langle \alpha_i, \nu \rangle.$$

Let $\mathbf{Y}(T)$ be the cocharacter lattice of T. The formula above implies that the map

$$\mathbf{H}^2_T(\mathrm{pt};\Bbbk)(1) \times (\Bbbk \otimes_{\mathbb{Z}} \mathbf{Y}(T)) \to \Bbbk \qquad \text{given by} \qquad (v, \nu) \mapsto \nu^\sharp(v) \tag{6.7.8}$$

is a perfect pairing. We therefore have a canonical isomorphism

$$\mathbf{H}^2_T(\mathrm{pt};\Bbbk)(1) \cong \Bbbk \otimes_{\mathbb{Z}} \mathbf{X}(T).$$

It remains to establish the commutative diagram. Since all four objects are commutative $\Bbbk$-algebras generated by their degree-2 components, it is enough to show that the diagram commutes in degree 2. Choose a cocharacter $\nu : \mathbb{G}_{\mathrm{m}} \to T$, and consider the diagram

$$\begin{tikzcd}
\mathbf{H}^2_{T'}(\mathrm{pt};\Bbbk)(1) \arrow[r, "\varphi^\sharp"'] \arrow[rr, bend left, "(\varphi\circ\nu)^\sharp"] \arrow[d, equal] & \mathbf{H}^2_T(\mathrm{pt};\Bbbk)(1) \arrow[r, "\nu^\sharp"'] \arrow[d, equal] & \mathbf{H}^2_{\mathbb{G}_{\mathrm{m}}}(\mathrm{pt};\Bbbk)(1) \arrow[d, equal] \\
\Bbbk \otimes_{\mathbb{Z}} \mathbf{X}(T') \arrow[r, dashed] \arrow[rr, bend right, "\langle -,\varphi\circ\nu\rangle"'] & \Bbbk \otimes_{\mathbb{Z}} \mathbf{X}(T) \arrow[r, "\langle -,\nu\rangle"] & \Bbbk
\end{tikzcd}$$

We wish to compute the dotted arrow. The fact that $\nu^\sharp$ can be identified with the pairing map $\langle -, \nu\rangle$ follows from (6.7.8). For the same reason, $(\varphi \circ \nu)^\sharp$ is identified with $\langle -, \varphi\circ\nu\rangle$, so for $\lambda \in \mathbf{X}(T')$, we have

$$\langle \varphi^\sharp(\lambda), \nu\rangle = \langle \lambda, \varphi \circ \nu\rangle.$$

Since this holds for all $\nu \in \mathbf{Y}(T)$, we have $\varphi^\sharp(\lambda) = \lambda \circ \varphi = \mathbf{X}(\varphi)(\lambda)$. □

We conclude this section by stating without proof a description of the equivariant cohomology of a point for a connected reductive group. Theorem 6.7.9 below involves a restriction on the coefficient ring $\Bbbk$, coming from the following result of Borel (see [**35**, Théorème B]):

THEOREM 6.7.8. *Let G be a connected reductive group. For any prime number p, there is p-torsion in $\mathbf{H}^\bullet(G;\mathbb{Z})$ if and only if there is p-torsion in $\mathbf{H}^\bullet_G(\mathrm{pt};\mathbb{Z})$.*

A prime number satisfying the equivalent conditions of Theorem 6.7.8 is called a **torsion prime** for G. By [**35**, Corollaire 3.2 and Lemme 3.3] (see also [**63**, Proposition 6]), a prime number p is a torsion prime for G if and only if one of the following conditions holds:

- The fundamental group $\pi_1(G, 1)$ has p-torsion.
- The number p is a torsion prime for some simply connected, quasisimple group whose root system is an irreducible component in that of G.

An explicit list of torsion primes for simply connected, quasisimple groups can be found in [**63**, Proposition 8] or [**231**, Section 4.4]. There are very few of these: only 2, 3, and 5 occur.

THEOREM 6.7.9. *Let G be a reductive group, and let $T \subset G$ be a maximal torus. Assume that all torsion primes of G are invertible in $\Bbbk$.*

(1) *The natural map $\mathbf{H}^\bullet_G(\mathrm{pt};\Bbbk) \to \mathbf{H}^\bullet_T(\mathrm{pt};\Bbbk)$ is injective.*
(2) *Let $W = \mathrm{N}_G(T)/T$ be the Weyl group. There is a natural isomorphism*

$$\mathbf{H}^\bullet_G(\mathrm{pt};\Bbbk) \cong \Bbbk \otimes_{\mathbb{Z}} \mathrm{Sym}(\mathbf{X}(T)(-1))^W.$$

(3) *The ring $\mathbf{H}^\bullet_G(\mathrm{pt};\Bbbk)$ is isomorphic to a graded polynomial ring with generators in degrees $2d_1, \dots, 2d_r$, where $d_1, \dots, d_r$ are the degrees of W.*
(4) *Let Φ be the root system of G with respect to T. If either $2\mathbf{X}(T) \cap \Phi = \varnothing$ or 2 is invertible in $\Bbbk$, then there is a natural isomorphism*

$$\mathbf{H}^\bullet_G(\mathrm{pt};\Bbbk) \cong \mathrm{Sym}(\Bbbk \otimes_{\mathbb{Z}} \mathbf{X}(T)(-1))^W.$$

In parts (2) and (4), the superscript "W" indicates the ring of W-invariants in the appropriate symmetric algebra. (The formulas differ in whether we tensor with $\Bbbk$ before or after taking W-invariants.) For part (3), the **degrees** of W are traditionally defined to be the degrees of generators of $\mathrm{Sym}(\mathbb{R} \otimes_{\mathbb{Z}} \mathbf{X}(T))^W$; see, for instance, [**108**, Section 3.7]. Thus, part (3) says that the degrees of the generators of $\mathbf{H}^\bullet_G(\mathrm{pt};\Bbbk)$ are independent of $\Bbbk$ (as long as the torsion primes are invertible).

For a proof of parts (1) and (2), see [**33**, Proposition 29.2(a)]. Parts (3) and (4) then follow from [**63**, Théorème 3 and Corollaire 2], respectively.

REMARK 6.7.10. Suppose G, X, and $\Bbbk$ are as in one of the mixed settings considered in Remark 6.4.12. Let $\mathcal{F}$ be an object of the appropriated equivariant mixed derived category, either $D^{\mathrm{b}}_{\mathrm{m},G}(X,\Bbbk)$ or $D^{\mathrm{b}}_G\mathrm{MHM}(X,\Bbbk)$. Imitating Definition 6.7.1, one can define the **geometric equivariant hypercohomology** $\underline{\mathbf{H}}^k_G(X,\mathcal{F})$ by

$$\underline{\mathbf{H}}^k_G(X,\mathcal{F}) = \underline{\mathrm{Hom}}(\underline{\Bbbk}_{\mathrm{pt}}, \mathrm{a}_{X*}\mathcal{F}[k]),$$

and likewise for geometric equivariant hypercohomology with compact support.

In particular, one can consider the geometric equivariant cohomology of a point. Analogues of Theorems 6.7.7 and 6.7.9 hold in the mixed setting. Moreover, if T is a torus, one can check that

$$\underline{\mathbf{H}}^\bullet_T(\mathrm{pt};\Bbbk) \text{ is pure and of Tate type.}$$

(This comes down to the fact that $\underline{\mathbf{H}}^\bullet(\mathbb{P}^n;\Bbbk)$, which appears in the proof of Proposition 6.7.6, is of Tate type.) Similarly, if G is a connected reductive group whose torsion primes are invertible in $\Bbbk$, then Theorem 6.7.9(1) implies that

$$\underline{\mathbf{H}}^\bullet_G(\mathrm{pt};\Bbbk) \text{ is pure and of Tate type.}$$

Exercises.

6.7.1. Let X be a G-variety. Show that for $\mathcal{F},\mathcal{G} \in D^{\mathrm{b}}_G(X,\Bbbk)$, there is a natural isomorphism $\mathrm{Hom}(\mathcal{F},\mathcal{G}[k]) \cong \mathbf{H}^k_G(X, R\mathscr{H}\!om(\mathcal{F},\mathcal{G}))$.

6.7.2. Let G be an algebraic group, and let $G^\circ \subset G$ be its identity component. Show that if $|G/G^\circ|$ is invertible in $\Bbbk$, then there is a natural isomorphism $\mathbf{H}^\bullet_G(\mathrm{pt};\Bbbk) \xrightarrow{\sim} \mathbf{H}^\bullet_{G^\circ}(\mathrm{pt};\Bbbk)$.

6.7.3. Let T be an algebraic torus, and let T act linearly on $\mathbb{A}^1$ by a character $\lambda \in \mathbf{X}(T)$. Let $i : \{0\} \hookrightarrow \mathbb{A}^1$ be the inclusion map. Regard λ as an element of $\mathbf{H}^2_T(\mathrm{pt};\Bbbk)(1) \cong \mathrm{Hom}(\underline{\Bbbk}_{\mathrm{pt}}, \underline{\Bbbk}_{\mathrm{pt}}[2](1))$, and show that $i_*\lambda$ is equal to the composition

$$i_*\underline{\Bbbk}_{\mathrm{pt}} \cong i_*i^!\underline{\Bbbk}_{\mathbb{A}^1}[2](1) \to \underline{\Bbbk}_{\mathbb{A}^1}[2](1) \to i_*i^*\underline{\Bbbk}_{\mathbb{A}^1}[2](1) \cong i_*\underline{\Bbbk}_{\mathrm{pt}}[2](1).$$

Hint: Consider first the special case where $T = \mathbb{G}_{\mathrm{m}}$ and where λ is an integer $n \in \mathbb{Z}$. Let $P = \mathbb{A}^2 \smallsetminus \{0\}$, and let $E_n = (P \times \mathbb{A}^1)/\mathbb{G}_{\mathrm{m}}$. We have a commutative diagram

$$\begin{tikzcd}
\mathrm{pt} \arrow[d, "i"] & P \arrow[l] \arrow[r] \arrow[d, "\tilde{\imath}"] & P/\mathbb{G}_{\mathrm{m}} = \mathbb{P}^1 \arrow[d, "\bar{\imath}"] \\
\mathbb{A}^1 & P \times \mathbb{A}^1 \arrow[l] \arrow[r] & (P \times \mathbb{A}^1)/\mathbb{G}_{\mathrm{m}} = E_n
\end{tikzcd}$$

Show first that the problem is equivalent to that of computing the map

$$\bar{\imath}_*\underline{\Bbbk}_{\mathbb{P}^1} \cong \bar{\imath}_*\bar{\imath}^!\underline{\Bbbk}_{E_n}[2](1) \to \underline{\Bbbk}_{E_n}[2](1) \to \bar{\imath}_*\bar{\imath}^*\underline{\Bbbk}_{E_n}[2](1) \cong \bar{\imath}_*\underline{\Bbbk}_{\mathbb{P}^1}[2](1).$$

Now, E_n has the structure of a vector bundle over $\mathbb{P}^1$, and the map above is given by the Euler class $e_{E_n} \in \mathbf{H}^2(\mathbb{P}^1;\Bbbk)(1)$. To finish the special case, use Example 2.11.22. Finally, for a general torus action, use Theorem 6.7.7.

6.8. The language of stacks

The forgetful and inflation functors behave in some ways like pullback along a smooth morphism, while the averaging and (co)invariants functors behave like push-forwards. These parallels can be emphasized by borrowing some notation and terminology from the theory of algebraic stacks. Moreover, as we will see in Section 6.9, the use of "stacky" language can sometimes make arguments with equivariant derived categories significantly more concise.

In this section, we briefly explain how to translate the content of the preceding sections into stacky language. There are no theorems or proofs in this section.

Background on algebraic stacks. Let $f : X \to Y$ be a smooth morphism of varieties. The main idea in the theory of smooth descent (see Section 3.7) is that perverse sheaves on Y can be described in terms of data involving a pair of smooth morphisms $X \times_Y X \rightrightarrows X$.

Roughly speaking, the theory of **algebraic stacks** is designed to make it possible to carry out "smooth descent" starting from a pair of smooth maps $R \rightrightarrows X$ satisfying suitable axioms (captured by the notion of a **smooth equivalence relation**), even when R does not arise as a fiber product. Any smooth equivalence relation $R \rightrightarrows X$ determines an algebraic stack $\mathcal{X}$.

Any algebraic variety (or more generally, any scheme) can be regarded as a stack. In particular, the stack determined by the smooth equivalence relation $X \times_Y X \rightrightarrows X$ associated to a smooth morphism of varieties $f : X \to Y$ is identified with Y. Many properties of morphisms of varieties also make sense for morphisms of stacks, including: smooth, proper, surjective, open embedding, and closed embedding. It is possible to form fiber products of algebraic stacks. A morphism of stacks $f : \mathcal{X} \to \mathcal{Y}$ is said to be **representable** if for any map $Y' \to \mathcal{Y}$ with Y' a scheme, the fiber product stack $\mathcal{X} \times_{\mathcal{Y}} Y'$ is a scheme.

Now let G be an algebraic group, and let X be a G-variety. The pair of maps $\mathrm{pr}_2, \sigma : G \times X \rightrightarrows X$ is a smooth equivalence relation. The corresponding stack, called a **quotient stack**, is denoted by

$$X/G.$$

If X is a principal G-variety, this notation is consistent with that introduced after Definition 6.1.6: the geometric quotient is canonically isomorphic to the quotient stack. (However, when the G-action is not free, it can happen that the geometric quotient exists as a variety, but it is *not* isomorphic to the quotient stack.) The quotient stack comes equipped with a morphism

$$\pi : X \to X/G \tag{6.8.1}$$

that satisfies a universal property similar to that in Lemma 6.1.3. This morphism is representable, surjective, and smooth of relative dimension equal to $\dim G$. The quotient stack X/G is a smooth stack if and only if X is a smooth variety. In this case, we have

$$\dim X/G + \dim G = \dim X.$$

In particular, the stack X/G may have negative dimension. If $f : X \to Y$ is a G-equivariant morphism of G-varieties, there is an induced morphism of stacks

$$\bar{f} : X/G \to Y/G. \tag{6.8.2}$$

Moreover, Proposition 6.1.16 holds: the diagram

$$\begin{array}{ccc} X & \xrightarrow{\pi_X} & X/G \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \bar{f}} \\ Y & \xrightarrow{\pi_Y} & Y/G \end{array}$$

is cartesian, and f is smooth if and only if $\bar{f}$ is smooth.

As a generalization of (6.8.1), if H is a closed subgroup of G, then there is a morphism of stacks

$$p : X/H \to X/G. \tag{6.8.3}$$

This morphism is again representable, surjective, and smooth of relative dimension $\dim G - \dim H$.

Now let $K \lhd G$ be a normal subgroup, and suppose the G-variety X is also a principal K-variety. Then X/K is a G/K-variety, and one can form the quotient stack $(X/K)/(G/K)$. In analogy with Lemma 6.1.15, there is a canonical isomorphism of stacks

$$X/G \xrightarrow{\sim} (X/K)/(G/K). \tag{6.8.4}$$

See Exercise 6.8.1 for another statement in this spirit.

Finally, with $K \lhd G$ as above, let Y be a G/K-variety. Then there is a morphism

$$q : Y/G \to Y/(G/K). \tag{6.8.5}$$

This morphism is surjective and smooth, but of *negative* relative dimension (namely, $-\dim K$). In particular, this morphism is *not* representable in general.

The various cases described above can be combined as follows: let $\varphi : G \to G'$ be a homomorphism of algebraic groups. Let X be a G-variety, let Y be a G'-variety, and let $\tilde{h} : X \to Y$ be a φ-equivariant morphism of varieties. (That is, for $g \in G$ and $x \in X$, we have $\tilde{h}(g \cdot x) = \varphi(g) \cdot \tilde{h}(x)$.) Then there is an induced morphism of stacks

$$h : X/G \to Y/G'. \tag{6.8.6}$$

If $\tilde{h}$ is smooth of relative dimension d, then h is smooth of relative dimension $d + \dim G' - \dim G$. It is automatically representable if φ is injective. If φ is not injective, h may or may not be representable.

Constructible derived categories. Given a quotient stack X/G, we define its constructible derived category by taking Corollary 6.4.9 as a definition, and set $D^{\mathrm{b}}_{\mathrm{c}}(X/G, \Bbbk) = D^{\mathrm{b}}_{G}(X, \Bbbk)$. (See Remark 6.8.3 for comments on an "intrinsic" approach to constructible sheaves on stacks.) It is nevertheless convenient to distinguish between the two sides. Denote the identity functor by

$$Q : D^{\mathrm{b}}_{G}(X, \Bbbk) \xrightarrow{=} D^{\mathrm{b}}_{\mathrm{c}}(X/G, \Bbbk). \tag{6.8.7}$$

As in (6.4.3), we expect the inverse Q^{-1} to behave like "pullback along a smooth morphism of relative dimension $\dim G$." We equip $D^{\mathrm{b}}_{\mathrm{c}}(X/G, \Bbbk)$ with additional structure as follows:

- In $D^{\mathrm{b}}_{\mathrm{c}}(X/G, \Bbbk)$, the functors $\otimes^L$ and $R\mathscr{H}om$ are defined in such a way that they commute with Q.
- The category $D^{\mathrm{b}}_{\mathrm{c}}(X/G, \Bbbk)$ is equipped with both natural and perverse t-structures. The functor Q is t-exact for the natural t-structure, while $Q[-\dim G]$ is t-exact for the perverse t-structure.
- The Verdier duality functor on $D^{\mathrm{b}}_{\mathrm{c}}(X/G, \Bbbk)$ is defined by the rule that for $\mathcal{F} \in D^{\mathrm{b}}_{G}(X, \Bbbk)$, we have $\mathbb{D}(Q(\mathcal{F})) \cong Q(\mathbb{D}(\mathcal{F}))(-\dim G)[-2\dim G]$.

Now let $h : X/G \to Y/G'$ be a morphism of quotient stacks as in (6.8.6). Then h can be written as a composition of morphisms of the three types (6.8.2), (6.8.3), or (6.8.5). The functors h^*, $h^!$, h_*, and $h_!$ are defined by (compositions of) the functors indicated in Table 6.8.1.

TABLE 6.8.1. Equivariant derived categories and stacks

Equivariant derived categories	*Stacky language*
$f : X \to Y$ an equivariant map of G-varieties, inducing $\bar{f} : X/G \to Y/G$	
$f_*, f_! : D^{\mathrm{b}}_G(X, \Bbbk) \to D^{\mathrm{b}}_G(Y, \Bbbk)$	$\bar{f}_*, \bar{f}_! : D^{\mathrm{b}}_{\mathrm{c}}(X/G, \Bbbk) \to D^{\mathrm{b}}_{\mathrm{c}}(Y/G, \Bbbk)$
$f^*, f^! : D^{\mathrm{b}}_G(Y, \Bbbk) \to D^{\mathrm{b}}_G(X, \Bbbk)$	$\bar{f}^*, \bar{f}^! : D^{\mathrm{b}}_{\mathrm{c}}(Y/G, \Bbbk) \to D^{\mathrm{b}}_{\mathrm{c}}(X/G, \Bbbk)$
$H \subset G$ a closed subgroup, X a G-variety, $p : X/H \to X/G$	
$\mathrm{For}^G_H : D^{\mathrm{b}}_G(X, \Bbbk) \to D^{\mathrm{b}}_H(X, \Bbbk)$	$p^* : D^{\mathrm{b}}_{\mathrm{c}}(X/G, \Bbbk) \to D^{\mathrm{b}}_{\mathrm{c}}(X/H, \Bbbk)$
$\mathrm{Av}^G_{H*} : D^{\mathrm{b}}_H(X, \Bbbk) \to D^{\mathrm{b}}_G(X, \Bbbk)$	$p_* : D^{\mathrm{b}}_{\mathrm{c}}(X/H, \Bbbk) \to D^{\mathrm{b}}_{\mathrm{c}}(X/G, \Bbbk)$
$\mathrm{Av}^G_{H!} : D^{\mathrm{b}}_H(X, \Bbbk) \to D^{\mathrm{b}}_G(X, \Bbbk)$	$p_![2\dim G/H](\dim G/H)$ $: D^{\mathrm{b}}_{\mathrm{c}}(X/H, \Bbbk) \to D^{\mathrm{b}}_{\mathrm{c}}(X/G, \Bbbk)$
$K \lhd G$ a normal subgroup, Y a G/K-variety, $q : Y/G \to Y/(G/K)$	
$\mathrm{Infl}^G_{G/K} : D^{\mathrm{b}}_{G/K}(Y, \Bbbk) \to D^{\mathrm{b}}_G(Y, \Bbbk)$	$q^* : D^{\mathrm{b}}_{\mathrm{c}}(Y/(G/K), \Bbbk) \to D^{\mathrm{b}}_{\mathrm{c}}(Y/G, \Bbbk)$
$\mathrm{Inv}_{K*} : D^{\mathrm{b}}_G(Y, \Bbbk) \to D^{+}_{G/K}(Y, \Bbbk)$	$q_* : D^{\mathrm{b}}_{\mathrm{c}}(Y/G, \Bbbk) \to D^{+}_{\mathrm{c}}(Y/(G/K), \Bbbk)$
$\mathrm{Inv}_{K!} : D^{\mathrm{b}}_G(Y, \Bbbk) \to D^{-}_{G/K}(Y, \Bbbk)$	$q_![-2\dim K](-\dim K)$ $: D^{\mathrm{b}}_{\mathrm{c}}(Y/G, \Bbbk) \to D^{-}_{\mathrm{c}}(Y/(G/K), \Bbbk)$

One important caveat is that when h is not representable, the functors h_* and $h_!$ need not send bounded complexes to bounded complexes: for this reason, some entries in Table 6.8.1 involve $D^+_{\mathrm{c}}(X/G, \Bbbk)$ or $D^-_{\mathrm{c}}(X/G, \Bbbk)$. See Remark 6.6.9 for a nonstacky perspective on this.

The definitions sketched above raise a number of well-definedness questions. We will not address these, as our main goal is simply to explain how to use stacky notation as a shorthand for various constructions from Sections 6.4–6.6. The intent is that it should be possible to treat equivariant operations, such as inflation or averaging, together with ordinary sheaf operations in a uniform way such that the following desideratum holds.

PRINCIPLE 6.8.1. *All natural isomorphisms among sheaf functors from Chapters 1–3 hold for morphisms of stacks. This includes adjunction properties, proper and smooth base change, the projection formula, and Verdier duality.*

This principle isn't quite a theorem (although it could be in the settings discussed in Remark 6.8.3); instead, it is a framework for understanding various results from this chapter as generalizations of results from Chapters 1–3, and for predicting new results based on those earlier chapters. Here are some examples of Principle 6.8.1:

- The quotient equivalence (Theorem 6.5.9) says that pullback along (6.8.4) is an equivalence of categories. Similarly, the induction equivalence (Theorem 6.5.10) is related to the isomorphism of stacks from Exercise 6.8.1.
- Lemma 6.5.8 is a generalization of Proposition 1.2.8(1).
- Propositions 6.5.4 and 6.5.7 generalize Principle 2.2.11.

- Lemmas 6.6.4 and 6.6.10 generalize Theorem 2.2.2 to the cartesian square

$$\begin{array}{ccc} X/H & \longrightarrow & X/(H/K) \\ \big\downarrow & & \big\downarrow \\ X/G & \longrightarrow & X/(G/K) \end{array}$$

In a similar vein, Proposition 6.6.11 can be rephrased as follows:

PROPOSITION 6.8.2. *Let $H \lhd G$ be a normal subgroup, and let d be the largest integer such that $\mathbf{H}_d(H;\mathbb{Z})$ is nonzero. Let X be a G/H-variety, let $q : X/G \to X/(G/H)$ be the natural map, and let $\mathcal{F} \in D^{\mathrm{b}}_{\mathrm{c}}(X/G, \Bbbk)$.*

(1) *Assume that $q_*\mathcal{F}$ is bounded. If $\mathcal{F}$ lies in ${}^pD^{\mathrm{b}}_{\mathrm{c}}(X/G,\Bbbk)^{\le n}$, then $q_*\mathcal{F}$ lies in ${}^pD^{\mathrm{b}}_{\mathrm{c}}(X/(G/H),\Bbbk)^{\le n+\dim H-d}$.*
(2) *Assume that $q_!\mathcal{F}$ is bounded. If $\mathcal{F}$ lies in ${}^pD^{\mathrm{b}}_{\mathrm{c}}(X/G,\Bbbk)^{\ge n}$, then $q_!\mathcal{F}$ lies in ${}^pD^{\mathrm{b}}_{\mathrm{c}}(X/(G/H),\Bbbk)^{\ge n-\dim H+d}$.*

The exercises at the end of this section give a large number of additional statements in a similar spirit, comprising generalizations of Proposition 1.2.8, Theorem 1.2.13, Proposition 1.4.7, Theorem 1.4.9, Corollary 2.8.8, and others.

REMARK 6.8.3. In place of the "piecemeal" approach to constructible sheaves and functors on quotient stacks described above, it would be better to start from scratch and then redevelop the full theory of the "six operations" for arbitrary algebraic stacks (and not just quotient stacks). In the étale topology, such a theory is available [**146–148**] (of course, this setting requires $\Bbbk$ to obey Definition 5.1.12). There is also a theory of $\mathscr{D}$-modules on stacks [**25**, Chapter 7], which can serve as a substitute for constructible sheaves (with $\Bbbk = \mathbb{C}$) via the Riemann–Hilbert correspondence (Theorem 5.5.26).

For our purposes, it would be desirable to have a theory of constructible sheaves in the analytic topology on complex algebraic stacks, with $\Bbbk$ allowed to be an arbitrary noetherian ring of finite global dimension. Unfortunately, I am not aware of a reference that does this (see [**235**, Section 3] for some initial steps, however).

Exercises.

6.8.1. Let $H \subset G$ be a closed subgroup, and let X be an H-variety. Deduce from (6.8.4) that there is an isomorphism of stacks $X/H \cong (G \times^H X)/G$. (*Hint*: Regard $G \times X$ as a $(G \times H)$-variety as in the proof of Theorem 6.5.10.)

6.8.2. Let G be an algebraic group. Let $H, K \subset G$ be subgroups, and assume that K is a normal subgroup of G. Let X be a G/K-variety, and let $\mathcal{F} \in D^{\mathrm{b}}_H(X,\Bbbk)$.

(a) If $\mathrm{Inv}_{H\cap K*}(\mathcal{F})$ is bounded, so is $\mathrm{Inv}_{K*}\mathrm{Av}^G_{H*}(\mathcal{F})$, and $\mathrm{Inv}_{K*}\mathrm{Av}^G_{H*}(\mathcal{F}) = \mathrm{Av}^{G/K}_{H/H\cap K*}\mathrm{Inv}_{H\cap K*}(\mathcal{F})$.
(b) If $\mathrm{Inv}_{H\cap K!}(\mathcal{F})$ is bounded, so is $\mathrm{Inv}_{K!}\mathrm{Av}^G_{H!}(\mathcal{F})$, and $\mathrm{Inv}_{K!}\mathrm{Av}^G_{H!}(\mathcal{F}) = \mathrm{Av}^{G/K}_{H/H\cap K!}\mathrm{Inv}_{H\cap K!}(\mathcal{F})$.

6.8.3. Let X be a G-variety, and let $H \subset G$ be a closed subgroup. For $\mathcal{F} \in D^{\mathrm{b}}_H(X,\Bbbk)$, show that there is a natural isomorphism $\mathrm{Av}^G_{H*}(\mathbb{D}\mathcal{F}) \cong \mathbb{D}(\mathrm{Av}^G_{H!}\mathcal{F})$.

6.8.4. Let $H \subset G$ be a closed subgroup, and let $f : X \to Y$ be a G-equivariant map of G-varieties. For $\mathcal{F} \in D^{\mathrm{b}}_H(X,\Bbbk)$ and $\mathcal{G} \in D^{\mathrm{b}}_H(Y,\Bbbk)$, show that there are natural isomorphisms

$$f_! \mathrm{Av}^G_{H!}(\mathcal{F}) \cong \mathrm{Av}^G_{H!}(f_!\mathcal{F}), \qquad f_* \mathrm{Av}^G_{H*}(\mathcal{F}) \cong \mathrm{Av}^G_{H*}(f_*\mathcal{F}),$$
$$f^* \mathrm{Av}^G_{H!}(\mathcal{G}) \cong \mathrm{Av}^G_{H!}(f^*\mathcal{G}), \qquad f^! \mathrm{Av}^G_{H*}(\mathcal{G}) \cong \mathrm{Av}^G_{H*}(f^!\mathcal{G}).$$

6.8.5. Let $H \lhd G$ be a normal subgroup, and let $f : X \to Y$ be a G/H-equivariant map of G/H-varieties. Let $\mathcal{F} \in D^{\mathrm{b}}_G(X, \Bbbk)$ and $\mathcal{G} \in D^{\mathrm{b}}_G(Y, \Bbbk)$.

(a) If $\mathrm{Inv}_{H*}\mathcal{F}$ is bounded, so is $\mathrm{Inv}_{H*} f_*\mathcal{F}$, and $\mathrm{Inv}_{H*} f_*\mathcal{F} \cong f_* \mathrm{Inv}_{H*}\mathcal{F}$.
(b) If $\mathrm{Inv}_{H!}\mathcal{F}$ is bounded, so is $\mathrm{Inv}_{H!} f_!\mathcal{F}$, and $\mathrm{Inv}_{H!} f_!\mathcal{F} \cong f_! \mathrm{Inv}_{H!}\mathcal{F}$.
(c) If $\mathrm{Inv}_{H*}\mathcal{G}$ is bounded, so is $\mathrm{Inv}_{H*} f^!\mathcal{G}$, and $\mathrm{Inv}_{H*} f^!\mathcal{G} \cong f^! \mathrm{Inv}_{H*}\mathcal{G}$.
(d) If $\mathrm{Inv}_{H!}\mathcal{G}$ is bounded, so is $\mathrm{Inv}_{H!} f^*\mathcal{G}$, and $\mathrm{Inv}_{H!} f^*\mathcal{G} \cong f^* \mathrm{Inv}_{H!}\mathcal{G}$.

6.8.6. Let $H \subset G$ be a closed subgroup, and let X be a G-variety. For $\mathcal{F} \in D^{\mathrm{b}}_H(X, \Bbbk)$ and $\mathcal{G} \in D^{\mathrm{b}}_G(X, \Bbbk)$, show that there are natural isomorphisms

$$\mathrm{Av}^G_{H*} R\mathscr{H}om(\mathrm{For}^G_H(\mathcal{G}), \mathcal{F}) \cong R\mathscr{H}om(\mathcal{G}, \mathrm{Av}^G_{H*}(\mathcal{F})),$$
$$\mathrm{Av}^G_{H*} R\mathscr{H}om(\mathcal{F}, \mathrm{For}^G_H(\mathcal{G})) \cong R\mathscr{H}om(\mathrm{Av}^G_{H!}(\mathcal{F}), \mathcal{G}),$$
$$\mathrm{Av}^G_{H!}(\mathcal{F} \overset{L}{\otimes} \mathrm{For}^G_H(\mathcal{G})) \cong \mathrm{Av}^G_{H!}(\mathcal{F}) \overset{L}{\otimes} \mathcal{G}.$$

6.8.7. Let $H \lhd G$ be a normal subgroup, and let X be a G/H-variety. Let $\mathcal{F} \in D^{\mathrm{b}}_G(X, \Bbbk)$ and $\mathcal{G} \in D^{\mathrm{b}}_{G/H}(X, \Bbbk)$.

(a) If $\mathrm{Inv}_{H*}(\mathcal{F})$ is bounded, so is $\mathrm{Inv}_{H*} R\mathscr{H}om(\mathrm{Infl}^G_{G/H}(\mathcal{G}), \mathcal{F})$, and
$$\mathrm{Inv}_{H*} R\mathscr{H}om(\mathrm{Infl}^G_{G/H}(\mathcal{G}), \mathcal{F}) \cong R\mathscr{H}om(\mathcal{G}, \mathrm{Inv}_{H*}(\mathcal{F})).$$
(b) If $\mathrm{Inv}_{H!}(\mathcal{F})$ is bounded, so are the objects $\mathrm{Inv}_{H*} R\mathscr{H}om(\mathcal{F}, \mathrm{Infl}^G_{G/H}(\mathcal{G}))$ and $\mathrm{Inv}_{H!}(\mathcal{F} \otimes^L \mathrm{Infl}^G_{G/H}(\mathcal{G}))$. We have
$$\mathrm{Inv}_{H*} R\mathscr{H}om(\mathcal{F}, \mathrm{Infl}^G_{G/H}(\mathcal{G})) \cong R\mathscr{H}om(\mathrm{Inv}_{H!}(\mathcal{F}), \mathcal{G}),$$
$$\mathrm{Inv}_{H!}(\mathcal{F} \overset{L}{\otimes} \mathrm{Infl}^G_{G/H}(\mathcal{G})) \cong \mathrm{Inv}_{H!}(\mathcal{F}) \overset{L}{\otimes} \mathcal{G}.$$

6.8.8. Let $H \lhd G$ be a normal subgroup, and let $f : X \to Y$ be a G-equivariant map of G-varieties. Assume that X is a principal H-variety and that H acts trivially on Y. Show that for any $\mathcal{F} \in D^{\mathrm{b}}_G(X, \Bbbk)$, both $\mathrm{Inv}_{H*} f_*\mathcal{F}$ and $\mathrm{Inv}_{H!} f_!\mathcal{F}$ are bounded. *Hint*: Consider the commutative diagram

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\pi"'] & & Y \\ & X/H \arrow[ur, "\bar{f}"'] & \end{tikzcd}$$

6.8.9. What do you call a garden party hosted by an algebraic stack?

6.9. Fourier–Laumon transform

Let S be a variety. Let $p : V \to S$ be a vector bundle over S of rank r, and let $p' : V' \to S$ be the dual vector bundle. Let $\mathbb{G}_{\mathrm{m}}$ act on both V and V' by scaling along the fibers of p and p', respectively. In this section, we will construct (based on [**150**]) an equivalence of categories

$$\mathrm{Four} : D^{\mathrm{b}}_{\mathbb{G}_{\mathrm{m}}}(V, \Bbbk) \to D^{\mathrm{b}}_{\mathbb{G}_{\mathrm{m}}}(V', \Bbbk),$$

called the **Fourier–Laumon transform**.

We will carry out the construction and the proofs using stack-theoretic language, mainly because this will make it substantially easier to keep track of the numerous changes of equivariance that are involved. But we emphasize that everything in this section can be rewritten in the language of Sections 6.4–6.6.

Definition of the Fourier–Laumon transform. Let $u : \mathbb{A}^1 \smallsetminus \{0\} \hookrightarrow \mathbb{A}^1$ be the inclusion map, and let $\mathbb{G}_\mathrm{m}$ act on $\mathbb{A}^1$ by scaling. Then $(\mathbb{A}^1 \smallsetminus \{0\})/\mathbb{G}_\mathrm{m} \cong \mathrm{pt}$, and u induces an open embedding of stacks

$$\bar{u} : \mathrm{pt} \hookrightarrow \mathbb{A}^1/\mathbb{G}_\mathrm{m}.$$

Define the **Fourier–Laumon kernel** to be the object

$$\Psi = \bar{u}_* \underline{\Bbbk}_{\mathrm{pt}} \in D^\mathrm{b}_\mathrm{c}(\mathbb{A}^1/\mathbb{G}_\mathrm{m}, \Bbbk).$$

The scaling actions of $\mathbb{G}_\mathrm{m}$ on V and V' give rise to an action of $\mathbb{G}_\mathrm{m}^2$ on $V \times_S V'$. The pairing map $\tilde{m} : V \times_S V' \to \mathbb{A}^1$ is μ-equivariant, where $\mu : \mathbb{G}_\mathrm{m}^2 \to \mathbb{G}_\mathrm{m}$ is the homomorphism $\mu(z_1, z_2) = z_1 z_2$. There is hence an induced map of stacks

$$m : (V \times_S V')/\mathbb{G}_\mathrm{m}^2 \to \mathbb{A}^1/\mathbb{G}_\mathrm{m}.$$

The stack $(V \times_S V')/\mathbb{G}_\mathrm{m}^2$ can be identified with $(V/\mathbb{G}_\mathrm{m}) \times_S (V'/\mathbb{G}_\mathrm{m})$. Consider the following diagram of stacks:

$$\begin{tikzcd}[column sep=small] & V/\mathbb{G}_\mathrm{m} \times_S V'/\mathbb{G}_\mathrm{m} \arrow[dl, "\mathrm{pr}_1"'] \arrow[dr, "\mathrm{pr}_2"] \arrow[rr, "m"] & & \mathbb{A}^1/\mathbb{G}_\mathrm{m} \\ V/\mathbb{G}_\mathrm{m} & & V'/\mathbb{G}_\mathrm{m} & \end{tikzcd} \tag{6.9.1}$$

The functor $\mathrm{Four} : D^\mathrm{b}_\mathrm{c}(V/\mathbb{G}_\mathrm{m}, \Bbbk) \to D^\mathrm{b}_\mathrm{c}(V'/\mathbb{G}_\mathrm{m}, \Bbbk)$ is defined by

$$\mathrm{Four}(\mathcal{F}) = \mathrm{pr}_{2!}(\mathrm{pr}_1^* \mathcal{F} \overset{L}{\otimes} m^* \Psi)[r-1]. \tag{6.9.2}$$

Note that $\mathrm{pr}_{2!}$ does not necessarily take values in $D^\mathrm{b}_\mathrm{c}(V'/\mathbb{G}_\mathrm{m}, \Bbbk)$. The claim that Four takes bounded complexes to bounded complexes will be proved in Lemma 6.9.8.

Throughout this section, the (stacky) zero sections of the vector bundles $p : V \to S$ and $p' : V \to S$ will be denoted by

$$i : S/\mathbb{G}_\mathrm{m} \hookrightarrow V/\mathbb{G}_\mathrm{m} \qquad \text{and} \qquad i' : S/\mathbb{G}_\mathrm{m} \hookrightarrow V'/\mathbb{G}_\mathrm{m}.$$

REMARK 6.9.1. Let us write out the nonstacky version of (6.9.2) explicitly, using the diagram of varieties

$$\begin{tikzcd} & V \times_S V' \arrow[dl, "\tilde{\mathrm{pr}}_1"'] \arrow[dr, "\tilde{\mathrm{pr}}_2"] \arrow[rr, "\tilde{m}"] & & \mathbb{A}^1 \\ V & & V' & \end{tikzcd} \tag{6.9.3}$$

Define three subgroups of $\mathbb{G}_\mathrm{m}^2$ as follows:

$$K_1 = \{(1, z) \mid z \in \mathbb{G}_\mathrm{m}\}, \quad K_2 = \{(z, 1) \mid z \in \mathbb{G}_\mathrm{m}\}, \quad K_\mu = \{(z, z^{-1}) \mid z \in \mathbb{G}_\mathrm{m}\}.$$

The Fourier–Laumon transform is given by

$$\mathrm{Four}(\mathcal{F}) = \mathrm{Inv}_{K_2!} \tilde{\mathrm{pr}}_{2!}(\tilde{\mathrm{pr}}_1^* \mathrm{Infl}_{\mathbb{G}_\mathrm{m}^2/K_1}^{\mathbb{G}_\mathrm{m}^2}(\mathcal{F}) \overset{L}{\otimes} \tilde{m}^* \mathrm{Infl}_{\mathbb{G}_\mathrm{m}^2/K_\mu}^{\mathbb{G}_\mathrm{m}^2} \mathcal{J})[r],$$

where $\mathcal{J} = u_* \underline{\Bbbk}_{\mathbb{A}^1 \smallsetminus \{0\}}[1](1) \in D^\mathrm{b}_{\mathbb{G}_\mathrm{m}}(\mathbb{A}^1, \Bbbk)$.

REMARK 6.9.2. Starting from the nonstacky diagram (6.9.3), one might also consider the functor given by simplified formula

$$\mathcal{F} \mapsto \tilde{\mathrm{pr}}_{2!}(\tilde{\mathrm{pr}}_1^* \mathcal{F} \otimes \tilde{m}^* \mathcal{J})[r], \tag{6.9.4}$$

ignoring $\mathbb{G}_{\mathrm{m}}$- or $\mathbb{G}_{\mathrm{m}}^2$-equivariance. The reader should beware that this simplified formula does *not* give an equivalence of categories. However, away from the world of constructible sheaves on complex varieties, there are several other kinds of sheaf-theoretic Fourier transforms that are given by a formula resembling (6.9.4):

- The Fourier–Deligne transform [**149**] for varieties over a field of positive characteristic: in this case, $\mathcal{J}$ is an Artin–Schreier local system.
- The Fourier–Sato transform [**125**, Section 3.7] for so-called "conic" sheaves on real vector bundles: in this case, $\mathcal{J}$ is the constant sheaf on the (non-algebraic) subset $\{z \in \mathbb{C} \mid \Re(z) \leq 0\} \subset \mathbb{C}$.
- The Fourier–Laplace transform for $\mathscr{D}$-modules [**106**, Section 3.2.2] is defined by taking $\mathcal{J}$ to be an exponential $\mathscr{D}$-module.

In each of these cases, the key point is that $\mathcal{J}$ satisfies a version of the "multiplicativity" property given in Lemma 6.9.4 below. In the setting of nonequivariant constructible complexes, there is no such multiplicative object, so working in the $\mathbb{G}_{\mathrm{m}}$-equivariant setting is essential. For this reason, the Fourier–Laumon transform is also known as the **homogeneous Fourier transform**.

REMARK 6.9.3. Suppose S is a G-variety, where G is some algebraic group, and that $p : V \to S$ and $p' : V' \to S$ are G-equivariant vector bundles. Then one can study the functor

$$\mathrm{Four} : D^{\mathrm{b}}_{\mathbb{G}_{\mathrm{m}} \times G}(V, \Bbbk) \to D^{\mathrm{b}}_{\mathbb{G}_{\mathrm{m}} \times G}(V', \Bbbk)$$

given by the formula (6.9.2) with respect to the diagram

$$\begin{tikzcd}[column sep=small]
 & V/(\mathbb{G}_{\mathrm{m}} \times G) \times_{S/G} V'/(\mathbb{G}_{\mathrm{m}} \times G) \arrow[dl, "\mathrm{pr}_1"'] \arrow[dr, "\mathrm{pr}_2"] \arrow[rr, "m"] & & \mathbb{A}^1/(\mathbb{G}_{\mathrm{m}} \times G) \\
V/(\mathbb{G}_{\mathrm{m}} \times G) & & V'/(\mathbb{G}_{\mathrm{m}} \times G) &
\end{tikzcd}$$

(Here G acts trivially on $\mathbb{A}^1$.) All results in this section remain valid in this G-equivariant setting; indeed, all the proofs go through essentially verbatim. The G-equivariant Fourier–Laumon transform will be needed in Chapters 8 and 10.

Properties of the Fourier–Laumon kernel. The next few lemmas rely on computations carried out in Section B.6.

LEMMA 6.9.4. *Let $\alpha : \mathbb{A}^2/\mathbb{G}_{\mathrm{m}} \to \mathbb{A}^1/\mathbb{G}_{\mathrm{m}}$ be the morphism of stacks induced by the linear map $\mathbb{A}^2 \to \mathbb{A}^1$ given by $(x, y) \mapsto x + y$. Let $q : \mathbb{A}^2/\mathbb{G}_{\mathrm{m}} \to \mathbb{A}^2/\mathbb{G}_{\mathrm{m}}^2$ be the map induced by the diagonal embedding $\mathbb{G}_{\mathrm{m}} \to \mathbb{G}_{\mathrm{m}}^2$ given by $z \mapsto (z, z)$. Then there is an isomorphism $q_! \alpha^* \Psi[1] \cong \Psi \boxtimes \Psi$.*

PROOF. This is just a restatement of Proposition B.6.3 in stacky language. □

LEMMA 6.9.5. *Let $b : V'/\mathbb{G}_{\mathrm{m}} \to V'/\mathbb{G}_{\mathrm{m}}^2 = S/\mathbb{G}_{\mathrm{m}} \times_S V'/\mathbb{G}_{\mathrm{m}}$ be the map induced by the homomorphism $\mathbb{G}_{\mathrm{m}} \to \mathbb{G}_{\mathrm{m}}^2$ given by $z \mapsto (z^{-1}, z)$. Then $(i \times_S \mathrm{id}_{V'/\mathbb{G}_{\mathrm{m}}})^* m^* \Psi \cong b_! \underline{\Bbbk}_{V'/\mathbb{G}_{\mathrm{m}}}[1]$.*

PROOF. The restriction of the pairing map $V \times_S V' \to \mathbb{A}^1$ to $S \times_S V'$ is the constant map with value 0. Thus, $m \circ (i \times_S \mathrm{id}_{V'/\mathbb{G}_{\mathrm{m}}})$ factors through the inclusion $s : \{0\}/\mathbb{G}_{\mathrm{m}} \hookrightarrow \mathbb{A}^1/\mathbb{G}_{\mathrm{m}}$, as shown in the bottom half of the diagram below:

$$\begin{tikzcd}
V'/\mathbb{G}_{\mathrm{m}} \arrow[rr] \arrow[d, "b"'] & & \mathrm{pt} \arrow[d, "v"] & & \\
S/\mathbb{G}_{\mathrm{m}} \times_S V'/\mathbb{G}_{\mathrm{m}} \arrow[rr, "z"] \arrow[dr, "i \times_S \mathrm{id}_{V'/\mathbb{G}_{\mathrm{m}}}"'] & & \{0\}/\mathbb{G}_{\mathrm{m}} \arrow[rr, "s"] & & \mathbb{A}/\mathbb{G}_{\mathrm{m}} \\
& V/\mathbb{G}_{\mathrm{m}} \times_S V'/\mathbb{G}_{\mathrm{m}} \arrow[urrr, "m"'] & & &
\end{tikzcd}$$

The upper square in this diagram is cartesian. According to Lemma B.6.2, we have $s^*\Psi \cong v_!\underline{\Bbbk}_{\mathrm{pt}}[1]$, so $(i \times_S \mathrm{id}_{V'/\mathbb{G}_{\mathrm{m}}})^* m^*\Psi \cong z^*s^*\Psi \cong z^*v_!\underline{\Bbbk}_{\mathrm{pt}}[1] \cong b_!\underline{\Bbbk}_{V'/\mathbb{G}_{\mathrm{m}}}[1]$. ☐

LEMMA 6.9.6. *There is a natural isomorphism* $\mathrm{pr}_{1!}m^*\Psi \cong i_*\underline{\Bbbk}_{S/\mathbb{G}_{\mathrm{m}}}[1-2r](-r)$.

PROOF. Let $V^\circ \subset V$ be the complement of the zero section of $p : V \to S$, and let $j : V^\circ/\mathbb{G}_{\mathrm{m}} \to V/\mathbb{G}_{\mathrm{m}}$ be the inclusion map. Consider the distinguished triangle

$$j_!j^*(\mathrm{pr}_{1!}m^*\Psi) \to \mathrm{pr}_{1!}m^*\Psi \to i_*i^*(\mathrm{pr}_{1!}m^*\Psi) \to .$$

The lemma follows from the claims in the two steps below.

Step 1. We have $j^*(\mathrm{pr}_{1!}m^*\Psi) = 0$. It is enough to show that any "stalk" of $j^*(\mathrm{pr}_{1!}m^*\Psi)$ (i.e., any pullback to a point of V°) vanishes. Let $v \in V^\circ$, and let $t = p(v) \in S$. Consider the following diagram, in which the square is cartesian:

$$\begin{tikzcd}
\{v\} \times (p')^{-1}(t)/\mathbb{G}_{\mathrm{m}} \arrow[r, "g'"] \arrow[d] & V/\mathbb{G}_{\mathrm{m}} \times_S V'/\mathbb{G}_{\mathrm{m}} \arrow[r, "m"] \arrow[d, "\mathrm{pr}_1"] & \mathbb{A}^1/\mathbb{G}_{\mathrm{m}} \\
\{v\} \arrow[r, "g"] & V/\mathbb{G}_{\mathrm{m}} &
\end{tikzcd}$$

Here $(p')^{-1}(t)$ is an affine space of dimension r, and the map $m \circ g' : \{v\} \times (p')^{-1}(t)/\mathbb{G}_{\mathrm{m}} \to \mathbb{A}^1/\mathbb{G}_{\mathrm{m}}$ is induced by a nonzero linear map $(p')^{-1}(t) \to \mathbb{A}^1$ (namely, pairing with v). In particular, $m \circ g'$ is ∞-acyclic, so by Exercise 3.7.2(e), the adjunction map $(m \circ g')_!(m \circ g)^!\Psi \to \Psi$ is an isomorphism. Using this observation, we have

$$\begin{aligned}
(j^*(\mathrm{pr}_{1!}m^*\Psi))_v &= g^*\mathrm{pr}_{1!}m^*\Psi \cong \mathrm{a}_{(p')^{-1}(t)/\mathbb{G}_{\mathrm{m}},!}((g')^*m^*\Psi) \\
&\cong \mathrm{a}_{\mathbb{A}^1/\mathbb{G}_{\mathrm{m}},!}((m \circ g')_!(m \circ g')^!\Psi)[2-2r](1-r) \cong \mathrm{a}_{\mathbb{A}^1/\mathbb{G}_{\mathrm{m}},!}(\Psi)[2-2r](1-r).
\end{aligned}$$

Lemma B.6.1 implies that $\mathrm{a}_{\mathbb{A}^1/\mathbb{G}_{\mathrm{m}},!}(\Psi) = 0$, so we are done.

Step 2. We have $i^*(\mathrm{pr}_{1!}m^*\Psi) \cong \underline{\Bbbk}_{S/\mathbb{G}_{\mathrm{m}}}[1-2r](-r)$. Consider the following diagram, where b is as in Lemma 6.9.5, and $\bar{p}'$ is induced by $p' : V' \to S$:

$$\begin{tikzcd}
V'/\mathbb{G}_{\mathrm{m}} \arrow[r, "b"] \arrow[dr, "\bar{p}'"'] & S/\mathbb{G}_{\mathrm{m}} \times_S V'/\mathbb{G}_{\mathrm{m}} \arrow[r, "\tilde{\imath} = i \times_S \mathrm{id}_{V'/\mathbb{G}_{\mathrm{m}}}"] \arrow[d, "\mathrm{pr}_1"'] & V/\mathbb{G}_{\mathrm{m}} \times_S V'/\mathbb{G}_{\mathrm{m}} \arrow[r, "m"] \arrow[d, "\mathrm{pr}_1"] & \mathbb{A}^1 \\
& S/\mathbb{G}_{\mathrm{m}} \arrow[r, "i"] & V/\mathbb{G}_{\mathrm{m}} &
\end{tikzcd} \tag{6.9.5}$$

Using Lemma 6.9.5, we have

$$i^*\mathrm{pr}_{1!}m^*\Psi \cong \mathrm{pr}_{1!}\tilde{\imath}^*m^*\Psi \cong \mathrm{pr}_{1!}b_!\underline{\Bbbk}_{V'/\mathbb{G}_{\mathrm{m}}}[1] \cong \bar{p}'_!\underline{\Bbbk}_{V'/\mathbb{G}_{\mathrm{m}}}[1].$$

By Theorem 2.10.3, we have $\bar{p}'_!\underline{\Bbbk}_{V'/\mathbb{G}_{\mathrm{m}}}[1] \cong (i')^!\underline{\Bbbk}_{V'/\mathbb{G}_{\mathrm{m}}}[1]$. By Theorem 2.2.13, $(i')^!\underline{\Bbbk}_{V'/\mathbb{G}_{\mathrm{m}}}[1] \cong \underline{\Bbbk}_{S/\mathbb{G}_{\mathrm{m}}}[1-2r](-r)$. ☐

Exchanging the roles of V and V' in Lemma 6.9.6, we obtain the following.

COROLLARY 6.9.7. *We have* $\mathrm{Four}(\underline{\Bbbk}_V) \cong i'_* \underline{\Bbbk}_{S/\mathbb{G}_\mathrm{m}}[-r](-r)$.

LEMMA 6.9.8. *For* $\mathcal{F} \in D^\mathrm{b}_\mathrm{c}(V/\mathbb{G}_\mathrm{m}, \Bbbk)$*, the object* $\mathrm{Four}(\mathcal{F}) \in D^-(V'/\mathbb{G}_\mathrm{m}, \Bbbk)$ *is bounded.*

PROOF. Let $j : V^\circ/\mathbb{G}_\mathrm{m} \to V/\mathbb{G}_\mathrm{m}$ be as in the proof of Lemma 6.9.6. For any $\mathcal{F} \in D^\mathrm{b}_\mathrm{c}(V/\mathbb{G}_\mathrm{m}, \Bbbk)$, we have a distinguished triangle

$$\mathrm{Four}(j_! j^* \mathcal{F}) \to \mathrm{Four}(\mathcal{F}) \to \mathrm{Four}(i_* i^* \mathcal{F}) \to . \tag{6.9.6}$$

It is enough to show that the first and third terms are bounded.

For the first term, consider the diagram

$$\begin{tikzcd}
 & V^\circ/\mathbb{G}_\mathrm{m} \times_S V'/\mathbb{G}_\mathrm{m} \arrow[rr, "h = j \times_S \mathrm{id}"] \arrow[dl, "\mathrm{pr}_1^\circ"'] & & V/\mathbb{G}_\mathrm{m} \times_S V'/\mathbb{G}_\mathrm{m} \arrow[dl, "\mathrm{pr}_1"] \arrow[dr, "\mathrm{pr}_2"] & \\
V^\circ/\mathbb{G}_\mathrm{m} \arrow[rr, "j"] & & V/\mathbb{G}_\mathrm{m} & & V'/\mathbb{G}_\mathrm{m}
\end{tikzcd}$$

Note that the left-hand square is cartesian. Let $\mathcal{F}' = j^* \mathcal{F}$. We have

$$\begin{aligned}
\mathrm{Four}(j_! \mathcal{F}') &= \mathrm{pr}_{2!}(\mathrm{pr}_1^* j_! \mathcal{F}' \overset{L}{\otimes} m^* \Psi)[r-1] \\
&\cong \mathrm{pr}_{2!}(h_! \mathrm{pr}_1^{\circ *} \mathcal{F}' \overset{L}{\otimes} m^* \Psi)[r-1] \cong (\mathrm{pr}_2 \circ h)_! (\mathrm{pr}_1^{\circ *} \mathcal{F}' \overset{L}{\otimes} h^* m^* \Psi)[r-1].
\end{aligned}$$

We claim that $\mathrm{pr}_2 \circ h$ is a representable morphism of stacks. Indeed, V° is a principal $\mathbb{G}_\mathrm{m}$-variety, so the stack $V^\circ/\mathbb{G}_\mathrm{m}$ is a variety, and the map $V^\circ/\mathbb{G}_\mathrm{m} \to S$ is (trivially) representable. Since $\mathrm{pr}_2 \circ h$ is the base change of $V^\circ/\mathbb{G}_\mathrm{m} \to S$ along $V'/\mathbb{G}_\mathrm{m} \to S$, it is representable. We conclude that the first term in (6.9.6) is bounded.

We will study the third term in (6.9.6) using the diagram (6.9.5) from the proof of Lemma 6.9.6. Let $\mathcal{F}'' = i^* \mathcal{F}$. Using Lemma 6.9.5, we have

$$\begin{aligned}
\mathrm{Four}(i_* \mathcal{F}'')[1-r] &= \mathrm{pr}_{2!}((\mathrm{pr}_1^* i_* \mathcal{F}'') \overset{L}{\otimes} m^* \Psi) \cong \mathrm{pr}_{2!}(\tilde{\imath}_* \mathrm{pr}_1^* \mathcal{F}'' \overset{L}{\otimes} m^* \Psi) \\
&\cong \mathrm{pr}_{2!} \tilde{\imath}_! (\mathrm{pr}_1^* \mathcal{F}'' \overset{L}{\otimes} \tilde{\imath}^* m^* \Psi) \cong \mathrm{pr}_{2!} \tilde{\imath}_! (\mathrm{pr}_1^* \mathcal{F}'' \overset{L}{\otimes} b_! \underline{\Bbbk}_{V'/\mathbb{G}_\mathrm{m}}[1]) \\
&\cong \mathrm{pr}_{2!} \tilde{\imath}_! b_! (b^* \mathrm{pr}_1^* \mathcal{F}'' \overset{L}{\otimes} \underline{\Bbbk}_{V'/\mathbb{G}_\mathrm{m}}[1]) \cong (\bar{p}')^* \mathcal{F}''[1],
\end{aligned}$$

where the last step uses the fact that $\mathrm{pr}_2 \circ \tilde{\imath} \circ b = \mathrm{id} : V'/\mathbb{G}_\mathrm{m} \to V'/\mathbb{G}_\mathrm{m}$. This calculation (which is a special case of Corollary 6.9.14 below) shows that the third term in (6.9.6) is bounded. □

Main results. We are now ready to prove the main properties of the Fourier–Laumon transform.

THEOREM 6.9.9. *For any* $\mathcal{F} \in D^\mathrm{b}_{\mathbb{G}_\mathrm{m} \times G}(V, \Bbbk)$*, there is a natural isomorphism*

$$\mathrm{Four}_{V'}(\mathrm{Four}_V(\mathcal{F})) \cong \mathcal{F}(-r).$$

In particular, Four_V *is an equivalence of categories.*

PROOF. The proof is a straightforward (but lengthy) calculation with the commutative diagram shown in Figure 6.9.1. Let us begin by describing all the maps in this diagram.

The bottom part of this figure is built from two copies of (6.9.1). All maps labelled "pr" (with various super- or subscripts) are projection maps onto some factor of a fiber product, as are t, v_1, v_2, e_1, and e_2. The maps q and s are induced by the diagonal embedding $\mathbb{G}_{\mathrm{m}} \hookrightarrow \mathbb{G}_{\mathrm{m}}^2$. We have

$$v_1 \circ t = \mathrm{pr}_1^{13} \circ \mathrm{pr}_{13} \circ s \qquad \text{and} \qquad v_2 \circ t = \mathrm{pr}_1^{23} \circ \mathrm{pr}_{23} \circ s. \tag{6.9.7}$$

As in (6.9.1), m is the pairing map. The maps M and $\tilde{M}$ are both induced by the map $V \times_S V \times_S V' \to \mathbb{A}^2$ given by $(x, y, \lambda) \mapsto (\langle x, \lambda\rangle, \langle y, \lambda\rangle)$, so that

$$e_1 \circ M = m \circ \mathrm{pr}_{13} \qquad \text{and} \qquad e_2 \circ M = m \circ \mathrm{pr}_{23}. \tag{6.9.8}$$

The maps α and α_V are both addition maps (given by $(x, y) \mapsto x + y$), and $\beta = \alpha_V \times_S \mathrm{id}_{V'/\mathbb{G}_{\mathrm{m}}}$. Next, $\bar{p}$ is the quotient of $p : V \to S$ by $\mathbb{G}_{\mathrm{m}}$ (where $\mathbb{G}_{\mathrm{m}}$ acts trivially on S), and i is the zero section. Finally, δ is induced by the antidiagonal map $V \to V \times_S V$ given by $x \mapsto (x, -x)$. We have

$$v_1 \circ \delta = \mathrm{id}_{V/\mathbb{G}_{\mathrm{m}}} \qquad \text{and} \qquad v_2 \circ \delta = \sigma_{-1}, \tag{6.9.9}$$

where $\sigma_{-1} : V' \to V'$ is the negation map $x \mapsto -x$.

We are now ready to start the calculation. By definition,

$$\begin{aligned}
\mathrm{Four}_{V'}(\mathrm{Four}_V(\mathcal{F}))[2-2r] &= \mathrm{pr}_{1!}^{23}(\mathrm{pr}_2^{23*}\mathrm{pr}_{2!}^{13}(\mathrm{pr}_1^{13*}(\mathcal{F}) \overset{L}{\otimes} m^*\Psi) \overset{L}{\otimes} m^*\Psi) \\
&\cong \mathrm{pr}_{1!}^{23}(\mathrm{pr}_{23!}\mathrm{pr}_{13}^*(\mathrm{pr}_1^{13*}(\mathcal{F}) \overset{L}{\otimes} m^*\Psi) \overset{L}{\otimes} m^*\Psi) \\
&\cong \mathrm{pr}_{1!}^{23}\mathrm{pr}_{23!}(\mathrm{pr}_{13}^*\mathrm{pr}_1^{13*}(\mathcal{F}) \overset{L}{\otimes} \mathrm{pr}_{13}^*m^*\Psi \overset{L}{\otimes} \mathrm{pr}_{23}^*m^*\Psi) \\
&\cong \mathrm{pr}_{1!}^{23}\mathrm{pr}_{23!}(\mathrm{pr}_{13}^*\mathrm{pr}_1^{13*}(\mathcal{F}) \overset{L}{\otimes} M^*(\Psi \boxtimes \Psi)),
\end{aligned}$$

where the last step uses (6.9.8). By Lemma 6.9.4 and (6.9.7), we rewrite this as

$$\begin{aligned}
\mathrm{Four}_{V'}(\mathrm{Four}_V(\mathcal{F}))[1-2r] &\cong \mathrm{pr}_{1!}^{23}\mathrm{pr}_{23!}(\mathrm{pr}_{13}^*\mathrm{pr}_1^{13*}(\mathcal{F}) \overset{L}{\otimes} M^*q_!\alpha^*\Psi) \\
&\cong \mathrm{pr}_{1!}^{23}\mathrm{pr}_{23!}(\mathrm{pr}_{13}^*\mathrm{pr}_1^{13*}(\mathcal{F}) \overset{L}{\otimes} s_!\tilde{M}^*\alpha^*\Psi) \\
&\cong \mathrm{pr}_{1!}^{23}\mathrm{pr}_{23!}s_!(s^*\mathrm{pr}_{13}^*\mathrm{pr}_1^{13*}(\mathcal{F}) \overset{L}{\otimes} \beta^*m^*\Psi) \\
&\cong v_{2!}t_!(t^*v_1^*(\mathcal{F}) \overset{L}{\otimes} \beta^*m^*\Psi) \\
&\cong v_{2!}(v_1^*(\mathcal{F}) \overset{L}{\otimes} t_!\beta^*m^*\Psi) \\
&\cong v_{2!}(v_1^*(\mathcal{F}) \overset{L}{\otimes} \alpha_V^*\mathrm{pr}_{1!}m^*\Psi).
\end{aligned}$$

By Lemma 6.9.6, we obtain

$$\begin{aligned}
\mathrm{Four}_{V'}(\mathrm{Four}_V(\mathcal{F})) &\cong v_{2!}(v_1^*(\mathcal{F}) \overset{L}{\otimes} \alpha_V^*i_!\underline{\Bbbk}_{S/\mathbb{G}_{\mathrm{m}}})(-r) \\
&\cong v_{2!}(v_1^*(\mathcal{F}) \overset{L}{\otimes} \delta_!\underline{\Bbbk}_{V/\mathbb{G}_{\mathrm{m}}})(-r) \\
&\cong v_{2!}\delta_!(\delta^*v_1^*(\mathcal{F}) \overset{L}{\otimes} \underline{\Bbbk}_{V/\mathbb{G}_{\mathrm{m}}})(-r).
\end{aligned}$$

By (6.9.9), this simplifies to $\mathrm{Four}_{V'}(\mathrm{Four}_V(\mathcal{F})) \cong (\sigma_{-1})_*\mathcal{F}(-r)$. By Exercise 6.5.8, we are done. $\square$

LEMMA 6.9.10. *For any $\mathcal{F} \in D_{\mathrm{c}}^{\mathrm{b}}(V/\mathbb{G}_{\mathrm{m}}, \Bbbk)$, there is a natural isomorphism $\mathrm{Four}_V(\mathcal{F}) \cong \mathrm{pr}_{2*}R\mathscr{H}om(m^*\Psi, \mathrm{pr}_1^!\mathcal{F})[1-r](-r)$.*

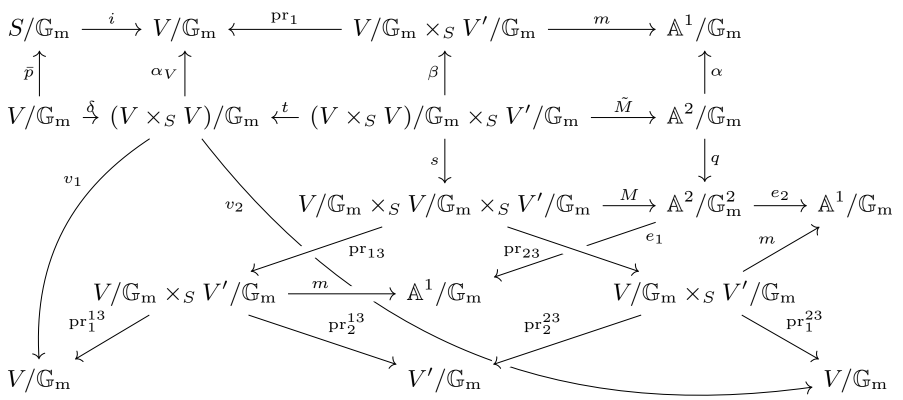

FIGURE 6.9.1. Diagram for the proof of Theorem 6.9.9

PROOF. The functor $\mathrm{Four}_{V'}(\mathcal{G}) = \mathrm{pr}_{1!}(\mathrm{pr}_2^*\mathcal{G} \otimes^L m^*\Psi)[r-1]$ is a composition of several functors that have right adjoints, so $\mathrm{Four}_{V'}$ has a right adjoint as well: it is the functor given by $\mathcal{F} \mapsto \mathrm{pr}_{2*} R\mathscr{H}om(m^*\Psi, \mathrm{pr}_1^!\mathcal{F})[1-r]$. On the other hand, Theorem 6.9.9 implies that $\mathcal{F} \mapsto \mathrm{Four}_V(\mathcal{F})(r)$ is also right adjoint to $\mathrm{Four}_{V'}$. $\square$

Comparing Lemma 6.9.10 with the definition of the Fourier–Laumon transform, we immediately obtain the following consequence.

PROPOSITION 6.9.11. *For $\mathcal{F} \in D^{\mathrm{b}}_{\mathrm{c}}(V/\mathbb{G}_m, \Bbbk)$, there is a natural isomorphism $\mathbb{D}\mathrm{Four}_V(\mathcal{F}) \cong \mathrm{Four}_V(\mathbb{D}\mathcal{F})(r)$.*

THEOREM 6.9.12. *The functor $\mathrm{Four}_V : D^{\mathrm{b}}_{\mathrm{c}}(V/\mathbb{G}_m, \Bbbk) \to D^{\mathrm{b}}_{\mathrm{c}}(V'/\mathbb{G}_m, \Bbbk)$ is t-exact for the perverse t-structure.*

PROOF. *Step 1. For any $\mathcal{G} \in {}^pD^{\mathrm{b}}_{\mathrm{c}}(V/\mathbb{G}_m \times_S V'/\mathbb{G}_m, \Bbbk)^{\geq 0}$, we have $\mathcal{G} \otimes^L m^*\Psi \in {}^pD^{\mathrm{b}}(V/\mathbb{G}_m \times_S V'/\mathbb{G}_m, \Bbbk)^{\geq 0}$.* Recall the open embedding $\bar{u} : (\mathbb{A}^1 \smallsetminus \{0\})/\mathbb{G}_m \hookrightarrow \mathbb{A}^1/\mathbb{G}_m$, and let $s : \{0\}/\mathbb{G}_m \hookrightarrow \mathbb{A}^1/\mathbb{G}_m$ be the complementary closed embedding. Let W and Y be their base changes along m, as shown here:

$$\begin{tikzcd}
W \arrow[r, "v"] \arrow[d] & V/\mathbb{G}_m \times_S V'/\mathbb{G}_m \arrow[d, "m"] & Y \arrow[l, "y"'] \arrow[d] \\
(\mathbb{A}^1 \smallsetminus \{0\})/\mathbb{G}_m \arrow[r, "\bar{u}"] & \mathbb{A}^1/\mathbb{G}_m & \{0\}/\mathbb{G}_m \arrow[l, "s"']
\end{tikzcd}$$

For any $\mathcal{G} \in {}^pD^{\mathrm{b}}_{\mathrm{c}}(V/\mathbb{G}_m \times_S V'/\mathbb{G}_m, \Bbbk)^{\geq 0}$, we have a distinguished triangle

$$v_!v^*\mathcal{G} \to \mathcal{G} \to y_*y^*\mathcal{G} \to .$$

Since $\bar{u}$ is an affine morphism, v is as well, so by Theorem 3.5.8, $v_!v^*\mathcal{G}$ lies in ${}^pD^{\mathrm{b}}_{\mathrm{c}}(V/\mathbb{G}_m \times_S V'/\mathbb{G}_m, \Bbbk)^{\geq 0}$. We deduce that

$$y_*y^*\mathcal{G} \in {}^pD^{\mathrm{b}}_{\mathrm{c}}(V/\mathbb{G}_m \times_S V'/\mathbb{G}_m, \Bbbk)^{\geq -1}. \tag{6.9.10}$$

Next, by Theorem 2.2.13, we have $s^!\underline{\Bbbk}_{\mathbb{A}^1/\mathbb{G}_m} \cong \underline{\Bbbk}_{\mathrm{pt}/\mathbb{G}_m}[-2](-1)$, so there is a distinguished triangle

$$s_*\underline{\Bbbk}_{\mathrm{pt}/\mathbb{G}_m}[-2](-1) \to \underline{\Bbbk}_{\mathbb{A}^1/\mathbb{G}_m} \to \Psi \to . \tag{6.9.11}$$

By proper base change, we have $m^* s_* \underline{\Bbbk}_{\mathrm{pt}/\mathbb{G}_{\mathrm{m}}} \cong y_* \underline{\Bbbk}_Y$, and then the projection formula implies that $\mathcal{G} \otimes^L y_* \underline{\Bbbk}_Y \cong y_* y^* \mathcal{G}$. Thus, applying $\mathcal{G} \otimes^L m^*(-)$ to (6.9.11), we obtain a distinguished triangle

$$\mathcal{G} \to \mathcal{G} \overset{L}{\otimes} m^*\Psi \to y_* y^* \mathcal{G}[-1](-1) \to .$$

If $\mathcal{G} \in {}^p D^{\mathrm{b}}_{\mathrm{c}}(V/\mathbb{G}_{\mathrm{m}} \times_S V'/\mathbb{G}_{\mathrm{m}}, \Bbbk)^{\geq 0}$, then in view of (6.9.10), we deduce that $\mathcal{G} \otimes^L m^*\Psi$ lies in ${}^p D^{\mathrm{b}}_{\mathrm{c}}(V/\mathbb{G}_{\mathrm{m}} \times_S V'/\mathbb{G}_{\mathrm{m}}, \Bbbk)^{\geq 0}$ as well.

Step 2. For any $\mathcal{G} \in {}^p D^{\mathrm{b}}_{\mathrm{c}}(V/\mathbb{G}_{\mathrm{m}} \times_S V'/\mathbb{G}_{\mathrm{m}}, \Bbbk)^{\geq 0}$, if $\mathrm{pr}_{2!}\mathcal{G}$ is bounded, then it lies in ${}^p D^{\mathrm{b}}_{\mathrm{c}}(V'/\mathbb{G}_{\mathrm{m}}, \Bbbk)^{\geq 0}$. Factor $\mathrm{pr}_2 : V/\mathbb{G}_{\mathrm{m}} \times_S V'/\mathbb{G}_{\mathrm{m}} \to V'/\mathbb{G}_{\mathrm{m}}$ as

$$V/\mathbb{G}_{\mathrm{m}} \times_S V'/\mathbb{G}_{\mathrm{m}} \xrightarrow{b} S/\mathbb{G}_{\mathrm{m}} \times_S V'/\mathbb{G}_{\mathrm{m}}, \xrightarrow{c} V'/\mathbb{G}_{\mathrm{m}}.$$

The map b is an affine morphism (it arises by base change from $V/\mathbb{G}_{\mathrm{m}} \to S/\mathbb{G}_{\mathrm{m}}$), so by Theorem 3.5.8, $b_!\mathcal{G}$ lies in ${}^p D^{\mathrm{b}}_{\mathrm{c}}(S/\mathbb{G}_{\mathrm{m}} \times_S V'/\mathbb{G}_{\mathrm{m}}, \Bbbk)^{\geq 0}$. Next, we can identify $S/\mathbb{G}_{\mathrm{m}} \times_S V'/\mathbb{G}_{\mathrm{m}}$ with $V'/\mathbb{G}_{\mathrm{m}}^2$, where the first copy of $\mathbb{G}_{\mathrm{m}}$ acts trivially, and the second by scaling. The object $c_!(b_!\mathcal{G}) \cong \mathrm{pr}_{2!}\mathcal{G}$ is bounded by assumption, so by Proposition 6.8.2 applied to $c : V'/\mathbb{G}_{\mathrm{m}}^2 \to V'/\mathbb{G}_{\mathrm{m}}$, we have $\mathrm{pr}_{2!}\mathcal{G} \in {}^p D^{\mathrm{b}}_{\mathrm{c}}(V'/\mathbb{G}_{\mathrm{m}}, \Bbbk)^{\geq 0}$.

Step 3. Conclusion of the proof. Since pr_1 is smooth of relative dimension $r-1$, Proposition 3.6.1 tells us that $\mathrm{pr}_1^*[r-1]$ is t-exact. Combining this observation with Steps 1 and 2, we see that Four_V is at least left t-exact.

The argument for right t-exactness is very similar, but uses the formula from Lemma 6.9.10 in place of (6.9.2). We omit the details. $\square$

PROPOSITION 6.9.13. *Let $f : V_1 \to V_2$ be a morphism of vector bundles over S, of ranks r_1 and r_2, respectively. Let $f' : V_2' \to V_1'$ be the dual morphism, and let $d = r_2 - r_1$. For $\mathcal{F} \in D^{\mathrm{b}}_{\mathbb{G}_{\mathrm{m}}}(V_1, \Bbbk)$ and $\mathcal{G} \in D^{\mathrm{b}}_{\mathbb{G}_{\mathrm{m}}}(V_2, \Bbbk)$, there are natural isomorphisms*

$$\mathrm{Four}_{V_2}(f_!\mathcal{F}) \cong f'^*\mathrm{Four}_{V_1}(\mathcal{F})[d], \qquad \mathrm{Four}_{V_2}(f_*\mathcal{F}) \cong f'^!\mathrm{Four}_{V_1}(\mathcal{F})[-d](-d),$$
$$\mathrm{Four}_{V_1}(f^!\mathcal{G}) \cong f'_*\mathrm{Four}_{V_2}(\mathcal{G})[-d], \qquad \mathrm{Four}_{V_1}(f^*\mathcal{G}) \cong f'_!\mathrm{Four}_{V_2}(\mathcal{G})[d](d).$$

PROOF. Let $\bar{f} : V_1/\mathbb{G}_{\mathrm{m}} \to V_2/\mathbb{G}_{\mathrm{m}}$ and $\bar{f}' : V_2'/\mathbb{G}_{\mathrm{m}} \to V_1'/\mathbb{G}_{\mathrm{m}}$ be the maps induced by f and f', respectively. In the diagram below, m_1 and m_2 are pairing maps, and s_1 and s_2 arise by base change from $\bar{f}'$ and $\bar{f}$:

$$\begin{tikzcd}
 & & V_1/\mathbb{G}_{\mathrm{m}} \times_S V_2'/\mathbb{G}_{\mathrm{m}} \arrow[dl, "s_1"'] \arrow[dr, "s_2"] & & \\
 & V_1/\mathbb{G}_{\mathrm{m}} \times_S V_1'/\mathbb{G}_{\mathrm{m}} \arrow[dl, "\mathrm{pr}_1^1"'] \arrow[ddrr, "\mathrm{pr}_2^1"'] \arrow[rrr, bend left, "m_1"] & & V_2/\mathbb{G}_{\mathrm{m}} \times V_2'/\mathbb{G}_{\mathrm{m}} \arrow[r, "m_2"'] \arrow[dr, "\mathrm{pr}_2^2"] \arrow[ddll, "\mathrm{pr}_1^2"] & \mathbb{A}^1/\mathbb{G}_{\mathrm{m}} \\
V_1/\mathbb{G}_{\mathrm{m}} \arrow[dr, "\bar{f}"] & & & & V_2'/\mathbb{G}_{\mathrm{m}} \arrow[dl, "\bar{f}'"'] \\
 & V_2/\mathbb{G}_{\mathrm{m}} & & V_1'/\mathbb{G}_{\mathrm{m}} &
\end{tikzcd}$$

In other words, the square formed by $\bar{f}$ and s_2 is cartesian, as is the square formed by $\bar{f}'$ and s_1. For $\mathcal{F} \in D^{\mathrm{b}}_{\mathrm{c}}(V_1/\mathbb{G}_{\mathrm{m}}, \Bbbk)$, we have

$$\begin{aligned}
\mathrm{Four}_{V_2}(\bar{f}_!\mathcal{F})[1-r_2] &= \mathrm{pr}^2_{2!}(\mathrm{pr}_1^{2*}\bar{f}_!\mathcal{F} \overset{L}{\otimes} m_2^*\Psi) \cong \mathrm{pr}^2_{2!}(s_{2!}s_1^*\mathrm{pr}_1^{1*}\mathcal{F} \overset{L}{\otimes} m_2^*\Psi) \\
&\cong \mathrm{pr}^2_{2!}s_{2!}(s_1^*\mathrm{pr}_1^{1*}\mathcal{F} \overset{L}{\otimes} s_2^*m_2^*\Psi) \cong \mathrm{pr}^2_{2!}s_{2!}(s_1^*\mathrm{pr}_1^{1*}\mathcal{F} \overset{L}{\otimes} s_1^*m_1^*\Psi) \\
&\cong \mathrm{pr}^2_{2!}s_{2!}s_1^*(\mathrm{pr}_1^{1*}\mathcal{F} \overset{L}{\otimes} m_1^*\Psi) \cong f'^*\mathrm{pr}^1_{2!}(\mathrm{pr}_1^{1*}\mathcal{F} \overset{L}{\otimes} m_1^*\Psi) = f'^*\mathrm{Four}_{V_1}(\mathcal{F})[1-r_1].
\end{aligned}$$

We have established the first isomorphism in the proposition. The remaining three can be deduced using Proposition 6.9.11 and Theorem 6.9.9. $\square$

COROLLARY 6.9.14. *Let $f : V_1 \hookrightarrow V$ be the inclusion of a subbundle of rank r_1, and let $f^\perp : V_1^\perp \hookrightarrow V'$ be the inclusion of the annihilator of V_1. Then* $\mathrm{Four}(f_*\underline{\Bbbk}_{V_1}[\dim V_1]) \cong f_*^\perp \underline{\Bbbk}_{V_1^\perp}[\dim V_1^\perp](-r_1)$.

PROOF. The dual of f is the quotient map $f' : V' \to V'/V_1^\perp$. By Proposition 6.9.13 and Corollary 6.9.7, we find that

$$\mathrm{Four}_V(f_*\underline{\Bbbk}_{V_1}[\dim V_1])[r_1 - r] \cong f'^*\mathrm{Four}_{V_1}(\underline{\Bbbk}_{V_1}[\dim V_1]) \cong f'^* i'_{1*}\underline{\Bbbk}_S[\dim S](-r_1),$$

where $i'_1 : S \hookrightarrow V'/V_1^\perp$ is the zero section. The last expression is identified with $f_*^\perp \underline{\Bbbk}_{V_1^\perp}[\dim V_1^\perp + r_1 - r](-r_1)$. $\square$

Lastly, the following statement is similar in spirit to Proposition 6.9.13. Its proof is left as an exercise.

PROPOSITION 6.9.15. *Let $p : V \to S$ be a vector bundle, and let $g : T \to S$ be a morphism of varieties. Consider the cartesian squares*

$$\begin{array}{ccc} V \times_S T & \xrightarrow{g_V} & V \\ \downarrow & & \downarrow{\scriptstyle p} \\ T & \xrightarrow{g} & S \end{array} \qquad \textit{and} \qquad \begin{array}{ccc} V' \times_S T & \xrightarrow{g_{V'}} & V \\ \downarrow & & \downarrow{\scriptstyle p'} \\ T & \xrightarrow{g} & S \end{array}$$

Then, for $\mathcal{F} \in D^{\mathrm{b}}_{\mathbb{G}_{\mathrm{m}}}(V \times_S T, \Bbbk)$ and $\mathcal{G} \in D^{\mathrm{b}}_{\mathbb{G}_{\mathrm{m}}}(V, \Bbbk)$, there are natural isomorphisms

$$\mathrm{Four}_V(g_{V!}\mathcal{F}) \cong g_{V'!}\mathrm{Four}_{V\times_S T}(\mathcal{F}), \qquad \mathrm{Four}_{V\times_S T}(g_V^*\mathcal{G}) \cong g_{V'}^*\mathrm{Four}_V(\mathcal{G}).$$

6.10. Additional exercises

EXERCISE 6.10.1. Let G be a (possibly disconnected) group, and let $H \subset G$ be a closed subgroup such that G/H is connected. Show that for any G-variety X, the functor $\mathrm{For}_H^G : \mathrm{Perv}_G(X, \Bbbk) \to \mathrm{Perv}_H(X, \Bbbk)$ is fully faithful.

EXERCISE 6.10.2. Suppose $\mathcal{F} \in D^{\mathrm{b}}_{\mathbb{G}_{\mathrm{m}}}(\mathrm{pt}, \Bbbk)$ is an object that satisfies

$$\mathbf{H}^i_{\mathbb{G}_{\mathrm{m}}}(\mathrm{pt}, \mathcal{F}) \cong \begin{cases} \Bbbk & \text{if } i = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Prove that

$$\mathrm{Hom}(\mathcal{F}, \underline{\Bbbk}[i]) \cong \begin{cases} \Bbbk(1) & \text{if } i = -1, \\ 0 & \text{otherwise,} \end{cases} \qquad \mathrm{Hom}(\mathcal{F}, \mathcal{F}[i]) \cong \begin{cases} \Bbbk & \text{if } i = 0, \\ \Bbbk(1) & \text{if } i = -1, \\ 0 & \text{otherwise.} \end{cases}$$

Hint: Choose a generator of $\mathbf{H}^0_{\mathbb{G}_{\mathrm{m}}}(\mathrm{pt}, \mathcal{F}) \cong \mathrm{Hom}(\underline{\Bbbk}_{\mathrm{pt}}, \mathcal{F})$ to get a map $\phi : \underline{\Bbbk}_{\mathrm{pt}} \to \mathcal{F}$, and then form a distinguished triangle

$$\mathcal{G} \to \underline{\Bbbk}_{\mathrm{pt}} \to \mathcal{F} \to . \tag{6.10.1}$$

Show that $\mathbf{H}^\bullet_{\mathbb{G}_{\mathrm{m}}}(\mathrm{pt}, \mathcal{G})$ is a free $\mathbf{H}^\bullet_{\mathbb{G}_{\mathrm{m}}}(\mathrm{pt}; \Bbbk)$-module, and then that $\mathcal{G} \cong \underline{\Bbbk}_{\mathrm{pt}}[-2](-1)$ (cf. Lemma 6.7.3). Then use (6.10.1) to compute the required Hom-groups.

The next three problems deal with perverse sheaves on $\mathbb{A}^1$. Let $j : U \hookrightarrow \mathbb{A}^1$, $i : Z \hookrightarrow \mathbb{A}^1$, and $\mathscr{S}$ be as in Exercise 3.10.4.

EXERCISE 6.10.3. Let $\mathbb{G}_{\mathrm{m}}$ act on $\mathbb{A}^1$ by $t \cdot x = tx$.

(a) Show that

$$\mathbf{H}^i_{\mathbb{G}_{\mathrm{m}}}(\mathrm{pt}, i^* j_* \underline{\Bbbk}) \cong \begin{cases} \Bbbk & \text{if } i = 0, \\ 0 & \text{otherwise,} \end{cases} \qquad \mathbf{H}^i_{\mathbb{G}_{\mathrm{m}}}(\mathrm{pt}, i^! j_! \underline{\Bbbk}) \cong \begin{cases} \Bbbk(-1) & \text{if } i = 1, \\ 0 & \text{otherwise.} \end{cases}$$

(b) Show that both $j_!\underline{\Bbbk}[1]$ and $j_*\underline{\Bbbk}[1]$ are both injective and projective as objects of $\mathrm{Perv}_{\mathbb{G}_{\mathrm{m}}}(\mathbb{A}^1, \Bbbk)$. Deduce that $\mathrm{Perv}_{\mathbb{G}_{\mathrm{m}}}(\mathbb{A}^1, \Bbbk)$ has enough injectives and enough projectives.

EXERCISE 6.10.4. In Exercise 4.6.1, we saw that $\mathrm{Perv}_{\mathscr{S}}(\mathbb{A}^1, \Bbbk)$ is equivalent to a certain category $\mathscr{C}$ of "representations of a quiver with relations." According to Proposition 6.2.15, we can think of $\mathrm{Perv}_{\mathbb{G}_{\mathrm{m}}}(\mathbb{A}^1, \Bbbk)$ as a full subcategory of $\mathrm{Perv}_{\mathscr{S}}(\mathbb{A}^1, \Bbbk)$. Show that under the equivalence of Exercise 4.6.1, the full subcategory of $\mathscr{C}$ corresponding to $\mathrm{Perv}_{\mathbb{G}_{\mathrm{m}}}(\mathbb{A}^1, \Bbbk)$ consists of objects

$$V_1 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} V_2$$

with $\beta \circ \alpha = 0$ and $\alpha \circ \beta = 0$.

EXERCISE 6.10.5. Prove that there is an equivalence of categories

$$\mathrm{real} : D^{\mathrm{b}}\mathrm{Perv}_{\mathbb{G}_{\mathrm{m}}}(\mathbb{A}^1, \Bbbk) \to D^{\mathrm{b}}_{\mathbb{G}_{\mathrm{m}}}(\mathbb{A}^1, \Bbbk).$$

EXERCISE 6.10.6. Let n be a positive integer, and let $\mathbb{G}_{\mathrm{m}}$ act on $\mathbb{P}^1$ by $t \cdot [u : v] = [t^n u : v]$. Let $\Bbbk$ be a field in which n is invertible. Compute the equivariant cohomology groups $\mathbf{H}^\bullet_{\mathbb{G}_{\mathrm{m}}}(\mathbb{P}^1; \Bbbk)$. *Answer*:

$$\mathbf{H}^i_{\mathbb{G}_{\mathrm{m}}}(\mathbb{P}^1; \Bbbk) \cong \begin{cases} \Bbbk & \text{if } i = 0, \\ \Bbbk(-i/2) \oplus \Bbbk(-i/2) & \text{if } i > 0 \text{ and } i \text{ is even,} \\ 0 & \text{otherwise.} \end{cases}$$

Hint: Let U_1 and U_2 be the open subsets obtained by deleting the points $[0 : 1]$ and $[1 : 0]$, respectively. Use the Mayer–Vietoris sequence for U_1 and U_2.

EXERCISE 6.10.7. Let $X = \mathbb{A}^2 \smallsetminus \{0\}$. Let $T = \mathbb{G}_{\mathrm{m}} \times \mathbb{G}_{\mathrm{m}}$, and let T act on X by $(t_1, t_2) \cdot (x_1, x_2) = (t_1 t_2 x, t_1^{-1} t_2 x)$. To distinguish the two copies of $\mathbb{G}_{\mathrm{m}}$, let us denote the first factor of T by $\mathbb{G}'_{\mathrm{m}}$, and the second factor by $\mathbb{G}''_{\mathrm{m}}$. Then we have

$$\mathbf{H}^\bullet_T(\mathrm{pt}; \Bbbk) \cong \mathbf{H}^\bullet_{\mathbb{G}'_{\mathrm{m}}}(\mathrm{pt}; \Bbbk) \otimes_{\Bbbk} \mathbf{H}^\bullet_{\mathbb{G}''_{\mathrm{m}}}(\Bbbk). \tag{6.10.2}$$

Assume now that $\Bbbk$ is a field of characteristic $\neq 2$. Let $S = \mathrm{Sym}(\Bbbk(-2))$, regarded as a graded ring with $\deg \Bbbk(-2) = 4$. Identify S with the subring of both $\mathbf{H}^\bullet_{\mathbb{G}'_{\mathrm{m}}}(\mathrm{pt}; \Bbbk)$ and $\mathbf{H}^\bullet_{\mathbb{G}''_{\mathrm{m}}}(\mathrm{pt}; \Bbbk)$ generated by elements of degree 4. Prove that there is an isomorphism of $\mathbf{H}^\bullet_T(\mathrm{pt}; \Bbbk)$-modules

$$\mathbf{H}^\bullet_T(X; \Bbbk) \cong \mathbf{H}^\bullet_{\mathbb{G}'_{\mathrm{m}}}(\mathrm{pt}; \Bbbk) \otimes_S \mathbf{H}^\bullet_{\mathbb{G}''_{\mathrm{m}}}(\mathrm{pt}; \Bbbk).$$

Here, $\mathbf{H}^\bullet_T(X; \Bbbk)$ acts on the right-hand side via (6.10.2): $\mathbf{H}^\bullet_{\mathbb{G}'_{\mathrm{m}}}(\mathrm{pt}; \Bbbk)$ acts by multiplication on the left, and $\mathbf{H}^\bullet_{\mathbb{G}''_{\mathrm{m}}}(\mathrm{pt}; \Bbbk)$ acts by multiplication on the right.

Note that $X/\mathbb{G}''_{\mathrm{m}} \cong \mathbb{P}^1$, and the induced $\mathbb{G}'_{\mathrm{m}}$-action on $\mathbb{P}^1$ is by $t \cdot [u : v] = [tu : t^{-1}v] = [t^2 u : v]$. This problem thus gives a new perspective on the $n = 2$ case of Exercise 6.10.6:

$$\mathbf{H}^\bullet_{\mathbb{G}'_{\mathrm{m}}}(\mathbb{P}^1; \Bbbk) \cong \mathbf{H}^\bullet_{\mathbb{G}'_{\mathrm{m}}}(\mathrm{pt}; \Bbbk) \otimes_S \mathbf{H}^\bullet_{\mathbb{G}''_{\mathrm{m}}}(\mathrm{pt}; \Bbbk).$$

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CHAPTER 7

Kazhdan–Lusztig theory

In this chapter, we study sheaves on the flag variety G/B of a reductive group G. Section 7.1 gives a brief overview of the facts we will need on the geometry of G/B. We also review key facts about the Hecke algebra, including the definition of Kazhdan–Lusztig polynomials. The main goal of Sections 7.2 and 7.3 is to show that Kazhdan–Lusztig polynomials encode the tables of stalks of simple perverse $\mathbb{Q}$-sheaves on G/B. This fact, which goes back to [**130**], is a key ingredient in the original proof of the Kazhdan–Lusztig conjectures [**129**] on representations of complex semisimple Lie algebras (see Remark 7.3.10).

One can also consider sheaves with coefficients in a field $\Bbbk$ of characteristic $p > 0$. Sections 7.5 and 7.6 discuss two approaches to the relationship between the Hecke algebra and $\Bbbk$-sheaves on G/B. The results in these sections are closely tied to recent developments in modular representation theory; see Remark 7.5.12.

7.1. Flag varieties and Hecke algebras

The Bruhat decomposition. Let G be a connected reductive complex algebraic group. Choose a Borel subgroup $B \subset G$ and a maximal torus $T \subset B$. Let $\mathrm{N}_G(T)$ be the normalizer of T in G, and let $W = \mathrm{N}_G(T)/T$ be the Weyl group. Let S be the set of simple reflections in W. Recall that there is a **length function**

$$\ell : W \to \mathbb{Z}$$

defined as follows: $\ell(w)$ is the smallest integer k such that there is an expression $w = s_1 s_2 \cdots s_k$ with $s_i \in S$. (Such an expression is called a **reduced expression**.)

For each $w \in W$, choose a representative $\dot{w} \in \mathrm{N}_G(T) \subset G$. The **Bruhat decomposition** says that the double cosets of B in G are in bijection with W:

$$G = \bigsqcup_{w \in W} B\dot{w}B.$$

The **flag variety** of G is the homogeneous space

$$\mathcal{B} = G/B.$$

For each $w \in W$, let $\mathcal{B}_w = B\dot{w}B/B \subset \mathcal{B}$. (Of course, this variety is independent of the choice of representative $\dot{w}$.) Denote the inclusion map by

$$j_w : \mathcal{B}_w \hookrightarrow \mathcal{B}.$$

Let B act on G/B by multiplication on the left. Under this action, $\mathcal{B}_w$ is precisely the B-orbit of the point $\dot{w}B \in G/B$. The Bruhat decomposition implies that

$$\mathcal{B} = \bigsqcup_{w \in W} \mathcal{B}_w.$$

A B-orbit $\mathcal{B}_w \subset \mathcal{B}$ is called a **Schubert cell**, and the closure $\overline{\mathcal{B}_w}$ of a Schubert cell is called a **Schubert variety**. Each Schubert cell is isomorphic as a variety to an affine space; in fact,

$$\mathcal{B}_w \cong \mathbb{A}^{\ell(w)}.$$

Since the Schubert cells are the orbits of a group action and are simply connected, they form a good stratification (see Exercise 6.5.2), called the **Bruhat stratification**. The **Bruhat order** on W coincides with the closure order on strata in this stratification.

We will primarily work with the B-equivariant derived category

$$D^{\mathrm{b}}_B(\mathcal{B}, \Bbbk),$$

but we will occasionally work in the ordinary (nonequivariant) derived category as well. The full subcategory of $D^{\mathrm{b}}_{\mathrm{c}}(\mathcal{B}, \Bbbk)$ consisting of complexes that are constructible with respect to the Bruhat stratification is denoted by

$$D^{\mathrm{b}}_{(B)}(\mathcal{B}, \Bbbk).$$

We will use a slightly abbreviated notation for intersection cohomology complexes on $\mathcal{B}$: instead of $\mathrm{IC}(\mathcal{B}_w, \Bbbk)$, we will write

$$\mathrm{IC}_w \qquad \text{or} \qquad \mathrm{IC}_w(\Bbbk).$$

For a simple reflection $s \in S$, let $P_s = B \cup B\dot{s}B$. This is a parabolic subgroup of G. We have

$$\overline{\mathcal{B}_s} \cong P_s/B \cong \mathbb{P}^1.$$

In particular, $\overline{\mathcal{B}_s}$ is smooth, so by Lemma 3.3.12, we have

$$\mathrm{IC}_s(\Bbbk) \cong i_{s*}\underline{\Bbbk}_{\overline{\mathcal{B}_s}}[1], \tag{7.1.1}$$

where $i_s : \overline{\mathcal{B}_s} \hookrightarrow \mathcal{B}$ is the inclusion map.

Lemma 7.1.1. *Let $i_w : \{\dot{w}B\} \hookrightarrow \mathcal{B}_w$ be the inclusion map. The following four functors are all equivalences, and all isomorphic to one another:*

$$\begin{tikzcd}[column sep=huge] D^{\mathrm{b}}_B(\mathcal{B}_w, \Bbbk) \arrow[r, bend left, "i_w^![2\ell(w)](\ell(w))\mathrm{For}^B_T"] \arrow[r, shift left, "i_w^*\mathrm{For}^B_T"] \arrow[r, shift right, "\mathrm{a}_{\mathcal{B}_w *}\mathrm{For}^B_T"'] \arrow[r, bend right, "\mathrm{a}_{\mathcal{B}_w !}[2\ell(w)](\ell(w))\mathrm{For}^B_T"'] & D^{\mathrm{b}}_T(\mathrm{pt}; \Bbbk) \end{tikzcd}$$

As a consequence, there is a natural isomorphism $\mathbf{H}^\bullet_B(\mathcal{B}_w; \Bbbk) \cong \mathbf{H}^\bullet_T(\mathrm{pt}; \Bbbk)$. In particular, $\mathbf{H}^k_B(\mathcal{B}_w, \Bbbk)$ vanishes for k odd.

Proof sketch. The orbit $\mathcal{B}_w$ can be identified with $B \times^{B\cap\dot{w}B\dot{w}^{-1}} \{\dot{w}B\}$, so Theorem 6.5.10 tells us that the functors below are equivalences of categories:

$$i_w^*\mathrm{For}^B_{B\cap\dot{w}B\dot{w}^{-1}} \cong i_w^!\mathrm{For}^B_{B\cap\dot{w}B\dot{w}^{-1}}[2\ell(w)](\ell(w)) : D^{\mathrm{b}}_B(\mathcal{B}_w, \Bbbk) \to D^{\mathrm{b}}_{B\cap\dot{w}B\dot{w}^{-1}}(\mathrm{pt}, \Bbbk).$$

Since $B \cap \dot{w}B\dot{w}^{-1}$ is the semidirect product of T with a unipotent group, by Theorem 6.6.16, $\mathrm{For}^{B\cap\dot{w}B\dot{w}^{-1}}_T : D^{\mathrm{b}}_{B\cap\dot{w}B\dot{w}^{-1}}(\mathrm{pt}, \Bbbk) \xrightarrow{\sim} D^{\mathrm{b}}_T(\mathrm{pt}, \Bbbk)$ is also an equivalence of categories. It follows that $i_w^*\mathrm{For}^B_T \cong i_w^!\mathrm{For}^B_T[2\ell(w)](\ell(w)) : D^{\mathrm{b}}_B(\mathcal{B}_w, \Bbbk) \to D^{\mathrm{b}}_T(\mathrm{pt}, \Bbbk)$ is an equivalence.

Next, as part of the basic theory of the Bruhat decomposition (see [**230**, Section 8.3], for instance), one can show that the point $\dot{w}B \in \mathcal{B}_w$ is the unique fixed point for the action of T on $\mathcal{B}_w$. One can show that if we make $\mathbb{G}_{\mathrm{m}}$ act on $\mathcal{B}_w$ via a "generic" dominant cocharacter $\mathbb{G}_{\mathrm{m}} \to T$ (i.e., a dominant cocharacter whose image is not contained in the kernel of any root), then this $\mathbb{G}_{\mathrm{m}}$-action is attracting and $\dot{w}B$ is its unique fixed point. By Theorem 2.10.3, for any object $\mathcal{F} \in D^{\mathrm{b}}_T(\mathcal{B}_w, \Bbbk)$, there are natural isomorphisms $\mathrm{a}_{\mathcal{B}_w *}\mathcal{F} \xrightarrow{\sim} i_w^*\mathcal{F}$ and $i_w^!\mathcal{F} \xrightarrow{\sim} \mathrm{a}_{\mathcal{B}_w !}\mathcal{F}$ in $D^{\mathrm{b}}_T(\mathrm{pt}, \Bbbk)$.

The last assertion holds by Theorem 6.7.7. □

The **Bott–Samelson variety** associated to a sequence of simple reflections $s_1, \ldots, s_k$ is the variety given by

$$\mathcal{B}(s_1, \ldots, s_k) = P_{s_1} \times^B P_{s_2} \times^B \cdots \times^B P_{s_{k-1}} \times^B P_{s_k}/B. \tag{7.1.2}$$

Since $\mathcal{B}(s_1, \ldots, s_k)$ is the quotient of a smooth variety by a free group action, it is smooth. The **Bott–Samelson map** is the map

$$m : \mathcal{B}(s_1, \ldots, s_k) \to \mathcal{B} \tag{7.1.3}$$

given by $m(p_1, p_2, \ldots, p_k B) = p_1 p_2 \cdots p_k B$. The following key fact is due to Demazure [**64**] and Hansen [**95**].

Proposition 7.1.2. *Let $w \in W$, and let $w = s_1 \cdots s_k$ be a reduced expression. The image of $m : \mathcal{B}(s_1, \ldots, s_k) \to \mathcal{B}$ is $\overline{\mathcal{B}_w}$, and the map $m : \mathcal{B}(s_1, \ldots, s_k) \to \overline{\mathcal{B}_w}$ is a resolution of singularities. In particular, this map induces an isomorphism $m^{-1}(\mathcal{B}_w) \xrightarrow{\sim} \mathcal{B}_w$.*

The Hecke algebra. Let q be an indeterminate, and consider the ring $\mathbb{Z}[q^{\pm\frac{1}{2}}]$ of Laurent polynomials in $q^{\frac{1}{2}}$. The **Hecke algebra** of W (with respect to S), denoted by $\mathcal{H}$ or $\mathcal{H}_W$, is a $\mathbb{Z}[q^{\pm\frac{1}{2}}]$-algebra defined as follows. As a $\mathbb{Z}[q^{\pm\frac{1}{2}}]$-module, $\mathcal{H}$ is free with basis

$$\{T_w\}_{w \in W}.$$

The multiplication in $\mathcal{H}_W$ is determined by the following rules:

$$T_w T_v = T_{wv} \qquad \text{if } \ell(wv) = \ell(w) + \ell(v), \tag{7.1.4}$$

$$T_s^2 = (q-1)T_s + qT_1 \qquad \text{if } s \in S. \tag{7.1.5}$$

For a proof that these rules determine an algebra structure on $\mathcal{H}$, see [**108**, Chapter 8]. Note that (7.1.4) implies that T_1 is the multiplicative identity in $\mathcal{H}$. We therefore sometimes simply denote it by 1. The relation (7.1.4) implies that $\mathcal{H}$ is generated as an algebra by $\{T_s\}_{s \in S}$, and that

$$T_w = T_{s_1} \cdots T_{s_k} \qquad \text{if } w = s_1 \cdots s_k \text{ is a reduced expression.}$$

The formulas above also imply that for any $w \in W$ and $s \in S$, we have

$$T_w T_s = \begin{cases} T_{ws} & \text{if } ws > w, \\ (q-1)T_w + qT_{ws} & \text{if } ws < w. \end{cases} \tag{7.1.6}$$

Note that (7.1.5) implies that T_s is invertible: indeed, we have

$$T_s^{-1} = q^{-1}T_s + q^{-1} - 1.$$

More generally, all the T_w are invertible. We define a map

$$\overline{} : \mathcal{H} \to \mathcal{H} \qquad \text{by} \qquad \overline{q^{1/2}} = q^{-1/2}, \quad \overline{T_w} = (T_{w^{-1}})^{-1}. \tag{7.1.7}$$

This map is called the **bar involution**.

Theorem 7.1.3. *There is a unique basis $\{\underline{H}_w\}_{w \in W}$ for $\mathcal{H}$ with the following properties:*

(1) *For all* $w \in W$*, we have* $\overline{\underline{H}_w} = \underline{H}_w$.
(2) *If we write* $\underline{H}_w$ *as*

$$\underline{H}_w = q^{-\ell(w)/2} \sum_{x \in W} P_{x,w} T_x,$$

then the coefficients $P_{x,w} \in \mathbb{Z}[q^{\pm\frac{1}{2}}]$ *satisfy*
   (a) $P_{x,w} = 0$ *unless* $x \leq w$.
   (b) $P_{w,w} = 1$.
   (c) *If* $x < w$*, then* $P_{x,w}$ *lies in* $\mathbb{Z}[q]$ *and has degree* $\leq \frac{1}{2}(\ell(w) - \ell(x) - 1)$.

This result was proved in [**129**] (where the elements $\underline{H}_w$ were denoted by C'_w). For a textbook-level exposition of this proof, see [**108**, Sections 7.9–7.11]. Another proof can be found in [**220**].

Definition 7.1.4. The set of elements $\{\underline{H}_w\}_{w \in W}$ from Theorem 7.1.3 is called the **canonical basis** or the **Kazhdan–Lusztig basis**, and the polynomials $P_{x,w} \in \mathbb{Z}[q]$ are called **Kazhdan–Lusztig polynomials**.

Example 7.1.5. The canonical basis elements $\underline{H}_w$ in the case where $w = e$ or where w is a simple reflection $s \in S$ are given by

$$\underline{H}_e = 1 \qquad \text{and} \qquad \underline{H}_s = q^{-1/2}(T_s + 1).$$

Exercises.

7.1.1. This exercise deals with the flag variety for GL_n. Let $B \subset \mathrm{GL}_n$ be the subgroup of upper-triangular matrices, and let $T \subset B$ be the subgroup of diagonal matrices. The Weyl group W is the symmetric group on n letters.

Let $\mathbf{e}_1, \ldots, \mathbf{e}_n$ be the standard basis for $\mathbb{C}^n$, where $\mathbf{e}_1 = (1, 0, \ldots, 0)$, $\mathbf{e}_2 = (0, 1, 0, \ldots, 0)$, etc. Recall that a **flag** F in $\mathbb{C}^n$ is a sequence of subspaces $0 = F_0 \subset F_1 \subset \cdots \subset F_n = \mathbb{C}^n$, where $\dim F_i = i$. The **standard flag** F^{std} is the flag whose ith step is given by $F^{\mathrm{std}}_i = \langle \mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_i \rangle$. The group GL_n acts transitively on the set of flags, and B is the stabilizer of F^{std}. The assignment $gB \mapsto g \cdot F^{\mathrm{std}}$ thus defines a bijection

$$\mathcal{B} = \mathrm{GL}_n/B \xrightarrow{\sim} \{\text{flags in } \mathbb{C}^n\}.$$

(a) For $w \in W$, what is the flag corresponding to $\dot{w}B$?
(b) Given $w \in W$ and $i, j \in \{1, \ldots, n\}$, let

$$r_w(i,j) = |\{k \in \{1, \ldots, i\} \mid w(k) \leq j\}|.$$

(If we think of w as a permutation matrix, $r_w(i,j)$ is the rank of the submatrix in the upper left corner of w with i columns and j rows.) Show that

$$\mathcal{B}_w = \{F \in \mathcal{B} \mid \dim F_i \cap F^{\mathrm{std}}_j = r_w(i,j) \text{ for all } i, j\},$$
$$\overline{\mathcal{B}_w} = \{F \in \mathcal{B} \mid \dim F_i \cap F^{\mathrm{std}}_j \geq r_w(i,j) \text{ for all } i, j\}.$$

(c) For $i \in \{1, \ldots, n-1\}$, let $s_i \in W$ be the permutation of $\{1, \ldots, n\}$ that exchanges i and $i+1$. (The elements $s_1, \ldots, s_{n-1}$ are the simple reflections.) Show that

$$\overline{\mathcal{B}_{s_i}} = \{F \in \mathcal{B} \mid F_j = F^{\mathrm{std}}_j \text{ for all } j \neq i\}.$$

Deduce that $\overline{\mathcal{B}_{s_i}} \cong \mathbb{P}^1$.

7.2. Convolution

In this section, we will define a functor

$$\star : D^{\mathrm{b}}_B(\mathcal{B},\Bbbk) \times D^{\mathrm{b}}_B(\mathcal{B},\Bbbk) \to D^{\mathrm{b}}_B(\mathcal{B},\Bbbk)$$

that makes $D^{\mathrm{b}}_B(\mathcal{B},\Bbbk)$ into a monoidal category. The definition of $\star$ involves the following **convolution diagram**:

$$G/B \times G/B \xleftarrow{p} G \times G/B \xrightarrow{q} G \times^B G/B \xrightarrow{m} G/B. \tag{7.2.1}$$

Here, p and q are the obvious quotient maps, and m is the "multiplication map," given by $m(g, hB) = ghB$. Let $B \times B$ act on $G/B \times G/B$ by $(b_1, b_2) \cdot (g_1B, g_2B) = (b_1g_1B, b_2g_2B)$ and on $G \times G/B$ by $(b_1, b_2) \cdot (g_1, g_2B) = (b_1g_1b_2^{-1}, b_2g_2B)$. Then p is $B \times B$-equivariant, and q is the quotient by the second copy of B. The map m is B-equivariant.

Given $\mathcal{F}, \mathcal{G} \in D^{\mathrm{b}}_B(G/B,\Bbbk)$, consider the object $\mathcal{F} \boxtimes \mathcal{G} \in D^{\mathrm{b}}_{B\times B}(G/B \times G/B,\Bbbk)$. The **twisted external tensor product** of $\mathcal{F}$ and $\mathcal{G}$ is the object

$$\mathcal{F} \tilde{\boxtimes} \mathcal{G} = (q^*\mathrm{Infl}_B^{B\times B})^{-1}(p^*(\mathcal{F} \boxtimes \mathcal{G})) \in D^{\mathrm{b}}_B(G \times^B G/B,\Bbbk)$$

where $(q^*\mathrm{Infl}_B^{B\times B})^{-1}$ is the inverse of the quotient equivalence (Theorem 6.5.9). Finally, the convolution product of $\mathcal{F}$ and $\mathcal{G}$ is defined by

$$\mathcal{F} \star \mathcal{G} = m_*(\mathcal{F} \tilde{\boxtimes} \mathcal{G}).$$

Remark 7.2.1. One can drop the equivariance requirement for $\mathcal{F}$, and carry out essentially the same construction to produce a functor

$$\star : D^{\mathrm{b}}_{\mathrm{c}}(\mathcal{B},\Bbbk) \times D^{\mathrm{b}}_B(\mathcal{B},\Bbbk) \to D^{\mathrm{b}}_{\mathrm{c}}(\mathcal{B},\Bbbk).$$

However, the B-equivariance of $\mathcal{G}$ cannot be dropped: it is needed for the equivariant descent functor $(q^*\mathrm{Infl}_B^{B\times B})^{-1}$ to make sense.

Remark 7.2.2. It is sometimes convenient to work with variants of (7.2.1). Let U and V be locally closed B-stable subvarieties of $\mathcal{B}$. Let $\tilde{U}$ be the preimage of U under $G \to G/B$. Then one can consider the diagram

$$U \times V \xleftarrow{p} \tilde{U} \times V \xrightarrow{q} \tilde{U} \times^B V \xrightarrow{m} G/B. \tag{7.2.2}$$

We sometimes use the notation

$$U \tilde{\times} V = \tilde{U} \times^B V.$$

Here are some observations regarding (7.2.2):

- If U and V are *closed* subsets, then for $\mathcal{F}$ supported on U and $\mathcal{G}$ supported on V, one can use (7.2.2) instead of (7.2.1) to compute $\mathcal{F} \star \mathcal{G}$.
- The collection $\{\mathcal{B}_v \tilde{\times} \mathcal{B}_w\}_{v,w\in W}$ forms a stratification of $G \times^B G/B$.

The following lemma makes use of the ideas in Remark 7.2.2.

Lemma 7.2.3. *For any $\mathcal{F} \in D^{\mathrm{b}}_B(\mathcal{B},\Bbbk)$, there are natural isomorphisms*

$$\mathrm{IC}_e \star \mathcal{F} \cong \mathcal{F} \star \mathrm{IC}_e \cong \mathcal{F}.$$

Proof. Since IC_e is supported on the point $\mathcal{B}_e = B/B$, $\mathrm{IC}_e \star \mathcal{F}$ can be computed using the following variant of (7.2.1):

$$\mathcal{B}_e \times G/B \xleftarrow{p} B \times G/B \xrightarrow{q} B \times^B G/B \xrightarrow{m} G/B.$$

Here, both $\mathcal{B}_e \times G/B$ and $B \times^B G/B$ can be identified with G/B itself. With this identification, the maps p and q are equal, and m is the identity map, so

$$\mathrm{IC}_e \star \mathcal{F} \cong m_*(q^*)^{-1}p^*(\mathcal{F}) \cong \mathcal{F}.$$

Similar reasoning using the diagram

$$G/B \times \mathcal{B}_e \xleftarrow{p} G \times \mathcal{B}_e \xrightarrow{q} G \times^B \mathcal{B}_e \xrightarrow{m} G/B$$

shows that $\mathcal{F} \star \mathrm{IC}_e \cong \mathcal{F}$. $\square$

Lemma 7.2.4. *For $\mathcal{F}, \mathcal{G}, \mathcal{H} \in D^{\mathrm{b}}_B(\mathcal{B}, \Bbbk)$, there is a natural isomorphism $(\mathcal{F} \star \mathcal{G}) \star \mathcal{H} \cong \mathcal{F} \star (\mathcal{G} \star \mathcal{H})$.*

Proof. Consider the commutative diagram

$$\begin{tikzcd}
G/B \times G/B \times G/B & & & \\
G \times G/B \times G/B \arrow[u] \arrow[d] & G \times G \times G/B \arrow[l] \arrow[ul, "\tilde{p}"'] \arrow[d] \arrow[dr, "\tilde{q}"] & & \\
G \times^B G/B \times G/B \arrow[d] & G \times^B G \times G/B \arrow[l] \arrow[r] \arrow[d] & G \times^B G \times^B G/B \arrow[d] \arrow[dr, "\tilde{m}"] & \\
G/B \times G/B & G \times G/B \arrow[l] \arrow[r] & G \times^B G/B \arrow[r] & G/B
\end{tikzcd}$$

Now, $(\mathcal{F} \star \mathcal{G}) \star \mathcal{H}$ is computed by going down the left side and across the bottom of this diagram. Using the proper and smooth base change theorems and the composition identities several times, we see that there is a natural isomorphism

$$(\mathcal{F} \star \mathcal{G}) \star \mathcal{H} \cong \tilde{m}_*(\tilde{q}^*)^{-1}\tilde{p}^*(\mathcal{F} \boxtimes \mathcal{G} \boxtimes \mathcal{H}).$$

A similar diagram shows that the right-hand side of this is also naturally isomorphic to $\mathcal{F} \star (\mathcal{G} \star \mathcal{H})$. $\square$

The preceding proof showed that it is possible to convolve three objects "in a single step." More generally, given $\mathcal{F}_1, \ldots, \mathcal{F}_k \in D^{\mathrm{b}}_B(\mathcal{B}, \Bbbk)$, we can form their convolution product using the diagram

$$\underbrace{G/B \times \cdots \times G/B}_{k \text{ factors}} \leftarrow G \times \cdots \times G \times G/B \to G \times^B \cdots \times^B G/B \to G/B. \tag{7.2.3}$$

Theorem 7.2.5. *The category $D^{\mathrm{b}}_B(\mathcal{B}, \Bbbk)$ is a monoidal category with respect to the convolution product.*

Proof sketch. Lemmas 7.2.3 and 7.2.4 tell us that IC_e is the unit object and that there is an associativity constraint. It remains to check the pentagon and triangle diagrams (see Definition A.2.1). Using the $k = 3$ and $k = 4$ cases of (7.2.3), these axioms can be deduced from the corresponding facts about $\boxtimes$. $\square$

For characteristic 0 coefficients, we have the following additional property.

Proposition 7.2.6. *The category $\mathrm{Semis}_B(\mathcal{B}, \mathbb{Q}) \subset D^{\mathrm{b}}_B(\mathcal{B}, \mathbb{Q})$ is closed under the convolution product.*

Proof. This follows from the observation that each step of the construction of $\star$ sends semisimple complexes to semisimple complexes. Indeed, for $\boxtimes$, this follows from Lemma 3.3.14; for p^* and q^*, it follows from Corollary 3.6.9; and for m_*, this is the decomposition theorem (Theorem 3.9.2). $\square$

For another perspective on this result, see Proposition 7.5.9. The remainder of this section is devoted to some elementary facts about the convolution product.

LEMMA 7.2.7. *For $\mathcal{F}, \mathcal{G} \in D^{\mathrm{b}}_B(\mathcal{B}, \Bbbk)$, there is a natural isomorphism $(\mathbb{D}\mathcal{F}) \star (\mathbb{D}\mathcal{G}) \cong \mathbb{D}(\mathcal{F} \star \mathcal{G})$.*

PROOF. Recall from Proposition 2.9.7 that $(\mathbb{D}\mathcal{F}) \boxtimes (\mathbb{D}\mathcal{G}) \cong \mathbb{D}(\mathcal{F} \boxtimes \mathcal{G})$. Since the maps p and q in (7.2.1) are both smooth of relative dimension $\dim B$, there are natural isomorphisms $\mathbb{D} \circ p^* \cong p^*[2\dim B](\dim B) \circ \mathbb{D}$ and $\mathbb{D} \circ q^* \cong q^*[2\dim B](\dim B) \circ \mathbb{D}$. It follows that

$$(\mathbb{D}\mathcal{F}) \mathbin{\tilde{\boxtimes}} (\mathbb{D}\mathcal{G}) \cong (q^*)^{-1} p^* \mathbb{D}(\mathcal{F} \boxtimes \mathcal{G}) \cong \mathbb{D}(\mathcal{F} \mathbin{\tilde{\boxtimes}} \mathcal{G}).$$

Finally, since m is proper, we have

$$(\mathbb{D}\mathcal{F}) \star (\mathbb{D}\mathcal{G}) = m_*((\mathbb{D}\mathcal{F}) \mathbin{\tilde{\boxtimes}} (\mathbb{D}\mathcal{G})) \cong \mathbb{D}(m_*(\mathcal{F} \mathbin{\tilde{\boxtimes}} \mathcal{G})) \cong \mathbb{D}(\mathcal{F} \star \mathcal{G}),$$

as desired. □

LEMMA 7.2.8. *For any $s \in S$, there is a cartesian square*

$$\begin{tikzcd} G \times^B P_s/B \arrow[r, "m"] \arrow[d, "\mathrm{pr}_1"'] & G/B \arrow[d, "\pi_s"] \\ G/B \arrow[r, "\pi_s"] & G/P_s \end{tikzcd}$$

Moreover, for any $\mathcal{F} \in D^{\mathrm{b}}_B(\mathcal{B}, \Bbbk)$ or $D^{\mathrm{b}}_{\mathrm{c}}(\mathcal{B}, \Bbbk)$, there is a natural isomorphism

$$\mathcal{F} \star \mathrm{IC}_s \cong \pi_s^* \pi_{s*} \mathcal{F}[1].$$

PROOF. Let $f : G \times P_s/B \to G \times_{G/P_s} G/B$ be the map given by $f(g, pB) = (g, gpB)$. It is easy to see that f is an isomorphism of varieties. It is equivariant with respect to appropriate actions of B, so it induces an isomorphism of varieties $\bar{f} : G \times^B P_s/B \to G/B \times_{G/P_s} G/B$. Composing $\bar{f}$ with the first and second projection maps $G/B \times_{G/P_s} G/B \to G/B$ yields pr_1 and m, respectively. This completes the proof that the diagram above is cartesian.

Now consider the diagram

$$\begin{tikzcd} G/B \times P_s/B \arrow[d, "\mathrm{pr}_1'"'] & G \times P_s/B \arrow[l, "p"'] \arrow[r, "q"] \arrow[d, "\mathrm{pr}_1''"'] & G \times^B P_s/B \arrow[r] \arrow[d, "\mathrm{pr}_1"'] & G/B \\ G/B & G \arrow[l] \arrow[r] & G/B & \end{tikzcd}$$

Recall that P_s/B can be identified with the Schubert variety $\overline{\mathcal{B}_s}$, and that $\mathrm{IC}_s \cong \underline{\Bbbk}_{P_s/B}[1]$ (see (7.1.1)). Given $\mathcal{F} \in D^{\mathrm{b}}_B(\mathcal{B}, \Bbbk)$ or $D^{\mathrm{b}}_{\mathrm{c}}(\mathcal{B}, \Bbbk)$, we see that $\mathcal{F} \boxtimes \underline{\Bbbk}_{P_s/B} \cong (\mathrm{pr}_1')^* \mathcal{F}$. The commutativity of the two squares shows that $\mathcal{F} \mathbin{\tilde{\boxtimes}} \underline{\Bbbk}_{P_s/B} \cong \mathrm{pr}_1^* \mathcal{F}$, so

$$\mathcal{F} \star \underline{\Bbbk}_{P_s/B} = m_*(\mathcal{F} \mathbin{\tilde{\boxtimes}} \underline{\Bbbk}_{P_s/B}) \cong m_* \mathrm{pr}_1^* \mathcal{F} \cong \pi_s^* \pi_{s*} \mathcal{F},$$

where the last step uses the fact that m and π_s are proper. □

The Bott–Samelson map (7.1.3) can be seen as part of the multiplication map from (7.2.3), so it can be used to compute convolution products as follows.

LEMMA 7.2.9. *Let $s_1, \ldots, s_k$ be a sequence of simple reflections, and let $m : \mathcal{B}(s_1, \ldots, s_k) \to \mathcal{B}$ be the Bott–Samelson morphism.*

(1) *We have* $\mathrm{IC}_{s_1} \star \cdots \star \mathrm{IC}_{s_k} \cong m_* \underline{\Bbbk}_{\mathcal{B}(s_1, \ldots, s_k)}[k]$.

(2) *If $s_1 \cdots s_k$ is a reduced expression for $w \in W$, then* $\mathrm{IC}_{s_1} \star \cdots \star \mathrm{IC}_{s_k}$ *is supported on* $\overline{\mathcal{B}_w}$, *and*

$$(\mathrm{IC}_{s_1} \star \cdots \star \mathrm{IC}_{s_k})|_{\mathcal{B}_w} \cong \underline{\Bbbk}_{\mathcal{B}_w}[\ell(w)].$$

As a consequence, IC_w *is a subquotient of* ${}^p\mathsf{H}^0(\mathrm{IC}_{s_1} \star \cdots \star \mathrm{IC}_{s_k})$.

PROOF SKETCH. Since $\mathrm{IC}_{s_i} \cong \underline{\Bbbk}_{\overline{\mathcal{B}_{s_i}}}[1]$ for each i (see (7.1.1)), it is easy to see that $\mathrm{IC}_{s_1} \tilde{\boxtimes} \cdots \tilde{\boxtimes} \mathrm{IC}_{s_k} \cong \underline{\Bbbk}_{\mathcal{B}(s_1,\ldots,s_k)}[k]$. Part (1) of the lemma follows. It is left as an exercise to deduce part (2) from Proposition 7.1.2. □

LEMMA 7.2.10. *Let $s_1, \ldots, s_k$ be a sequence of simple reflections.*

(1) *The object* $\mathrm{IC}_{s_1}(\mathbb{Q}) \star \cdots \star \mathrm{IC}_{s_k}(\mathbb{Q})$ *is a semisimple complex.*

(2) *If $s_1 \cdots s_k$ is a reduced expression for $w \in W$, then the multiplicity of* $\mathrm{IC}_w(\mathbb{Q})[n]$ *as a direct summand of* $\mathrm{IC}_{s_1}(\mathbb{Q}) \star \cdots \star \mathrm{IC}_{s_k}(\mathbb{Q})$ *is*

$$\begin{cases} 1 & \text{if } n = 0, \\ 0 & \text{otherwise.} \end{cases}$$

PROOF. Part (1) is immediate from Proposition 7.2.6. Now assume that $w = s_1 \cdots s_k$ is a reduced expression. Since $\mathrm{IC}_{s_1} \star \cdots \star \mathrm{IC}_{s_k}$ is supported on $\overline{\mathcal{B}_w}$, its indecomposable summands are of the form $\mathrm{IC}_v[n]$ with $v \le w$ in the Bruhat order. The restriction of any such summand to $\mathcal{B}_w$ vanishes unless $v = w$, in which case it is isomorphic to $\underline{\mathbb{Q}}_{\mathcal{B}_w}[\ell(w) + n]$. In other words, for part (2), we must determine the multiplicity of $\underline{\mathbb{Q}}_{\mathcal{B}_w}[\ell(w) + n]$ as a summand of $(\mathrm{IC}_{s_1} \star \cdots \star \mathrm{IC}_{s_k})|_{\mathcal{B}_w}$. This can be read from Lemma 7.2.9(2). □

Exercises.

7.2.1. Let $\mathrm{pr}_1 : G \times^B G/B \to G/B$ be the map given by $\mathrm{pr}_1(g, hB) = gB$. Show that $(\mathrm{pr}_1, m) : G \times^B G/B \to G/B \times G/B$ is an isomorphism of varieties.

7.2.2. This exercise deals with GL_n. We use the notation from Exercise 7.1.1. Let $s_{i_1}, \ldots, s_{i_k}$ be a sequence of simple reflections. Show that the Bott–Samelson variety $\mathcal{B}(s_{i_1}, \ldots, s_{i_k})$ can be identified with the set

$$\left\{ (F^{(1)}, \ldots, F^{(k)}) \in \mathcal{B} \times \cdots \times \mathcal{B} \,\middle|\, \begin{matrix} \text{for all } j,\ F^{(j)} \text{ agrees with } F^{(j-1)} \\ \text{except possibly at the } i_j\text{th term} \end{matrix} \right\},$$

where we adopt the convention that $F^{(0)}$ denotes the standard flag.

7.2.3. For GL_3, use the preceding exercise to describe the fibers of the map $m : \mathcal{B}(s_1, s_2, s_1) \to \mathcal{B}$. Then compute $\mathrm{IC}_{s_1} \star \mathrm{IC}_{s_2} \star \mathrm{IC}_{s_1}$.

7.2.4. Let s be a simple reflection, and let $\Delta_s = j_{s!}\underline{\Bbbk}_{\mathcal{B}_s}[1]$ and $\nabla_s = j_{s*}\underline{\Bbbk}_{\mathcal{B}_s}[1]$. Show that Δ_s and ∇_s are perverse sheaves, and that there are short exact sequences

$$0 \to \mathrm{IC}_e \to \Delta_s \to \mathrm{IC}_s \to 0,$$
$$0 \to \mathrm{IC}_s \to \nabla_s \to \mathrm{IC}_e(-1) \to 0.$$

7.2.5. Show directly from the definition of convolution that $\mathrm{IC}_s \star \nabla_s \cong \mathrm{IC}_s[1]$ and that $\mathrm{IC}_s \star \Delta_s \cong \mathrm{IC}_s[-1](-1)$. Also compute $\Delta_s \star \mathrm{IC}_s$ and $\nabla_s \star \mathrm{IC}_s$ using Lemma 7.2.8.

7.2.6. Show that $\Delta_s \star \nabla_s \cong \nabla_s \star \Delta_s \cong \mathrm{IC}_e(-1)$. *Hint*: Convolve the short exact sequences in Exercise 7.2.4 with Δ_s and ∇_s.

7.2.7. For any $w \in W$, let $\Delta_w = j_{w!}\underline{\Bbbk}_{\mathcal{B}_w}[\ell(w)]$ and $\nabla_w = j_{w*}\underline{\Bbbk}_{\mathcal{B}_w}[\ell(w)]$. Let $w = s_1 \cdots s_k$ be a reduced expression. Show that

$$\Delta_w \cong \Delta_{s_1} \star \cdots \star \Delta_{s_k} \qquad \text{and} \qquad \nabla_w \cong \nabla_{s_1} \star \cdots \star \nabla_{s_k}.$$

Then show that Δ_w and ∇_w are perverse sheaves.

7.3. The categorification theorem

In this section, we will prove the main results of the chapter for the case where $\Bbbk = \mathbb{Q}$. However, roughly the first half of this section consists of lemmas that are valid for arbitrary $\Bbbk$ and which will be used again in Section 7.5.

Parity considerations. We will study objects in $\mathcal{B}$ obeying the following vanishing conditions. (The terminology comes from [**120**].)

Definition 7.3.1. An object $\mathcal{F} \in D^{\mathrm{b}}_B(\mathcal{B},\Bbbk)$ or $D^{\mathrm{b}}_{(B)}(\mathcal{B},\Bbbk)$ is said to be **$*$-even** if for all $w \in W$, $\mathsf{H}^i(j_w^*\mathcal{F})$ is a locally free local system when i is even and $\mathsf{H}^i(j_w^*\mathcal{F}) = 0$ when i is odd. It is said to be **$*$-odd** if $\mathcal{F}[1]$ is $*$-even.

Similarly, $\mathcal{F}$ is **!-even** if for all $w \in W$, $\mathsf{H}^i(j_w^!\mathcal{F})$ is a locally free local system when i is even and $\mathsf{H}^i(j_w^!\mathcal{F}) = 0$ when i is odd. It is **!-odd** if $\mathcal{F}[1]$ is !-even.

Since each stratum $\mathcal{B}_w$ is simply connected, a locally free local system is just a direct sum of copies of the constant sheaf $\underline{\Bbbk}_{\mathcal{B}_w}$. If $\Bbbk$ is a field or a local ring, then any direct summand of a $*$-even object is $*$-even (and likewise for $*$-odd), because any direct summand of a free $\Bbbk$-module is free.

The proof of the following fact is left as an exercise.

Lemma 7.3.2. *An object $\mathcal{F} \in D^{\mathrm{b}}_B(\mathcal{B},\Bbbk)$ or $D^{\mathrm{b}}_{(B)}(\mathcal{B},\Bbbk)$ is $*$-even if and only if $\mathbb{D}\mathcal{F}$ is !-even. Similarly, $\mathcal{F}$ is $*$-odd if and only if $\mathbb{D}\mathcal{F}$ is !-odd.*

Lemma 7.3.3. *The category of $*$-even objects (in $D^{\mathrm{b}}_B(\mathcal{B},\Bbbk)$ or $D^{\mathrm{b}}_{(B)}(\mathcal{B},\Bbbk)$) is generated under extensions by objects of the form $j_{w!}\underline{\Bbbk}_{\mathcal{B}_w}[2n]$.*

Similarly, the category of !-even objects is generated under extensions by objects of the form $j_{w*}\underline{\Bbbk}_{\mathcal{B}_w}[2n]$.

Of course, a similar statement holds for $*$- and !-odd objects, using odd shifts.

Proof. We will prove the statement for $*$-even objects. The proof for !-even objects is similar.

Given a distinguished triangle $\mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to$, it is easy to see that if $\mathcal{F}'$ and $\mathcal{F}''$ are $*$-even, then $\mathcal{F}$ is as well. It is clear that $j_{w!}\underline{\Bbbk}_{\mathcal{B}_w}[2n]$ is $*$-even. Let $\mathscr{C}$ be the category generated under extensions by objects of this form. Then every object in $\mathscr{C}$ is $*$-even.

We must now show that every $*$-even object $\mathcal{F}$ belongs to $\mathscr{C}$. We proceed by induction on the support of $\mathcal{F}$. If its support is empty, i.e., if $\mathcal{F} = 0$, then there is nothing to prove. Otherwise, choose a stratum $\mathcal{B}_w$ that is open in the support of $\mathcal{F}$. Let $Z = \operatorname{supp}\mathcal{F} \smallsetminus \mathcal{B}_w$, and let $i : Z \hookrightarrow \mathcal{B}$ be the inclusion map. In the triangle

$$j_{w!}(\mathcal{F}|_{\mathcal{B}_w}) \to \mathcal{F} \to i_*i^*\mathcal{F} \to,$$

note that all three terms are still $*$-even. By induction, $i_*i^*\mathcal{F}$ belongs to $\mathscr{C}$, so it is enough to prove that $j_{w!}(\mathcal{F}|_{\mathcal{B}_w}) \in \mathscr{C}$. To do this, consider a truncation distinguished triangle of the form

$$j_{w!}(\tau^{\leq n-1}(\mathcal{F}|_{\mathcal{B}_w})) \to j_{w!}(\mathcal{F}|_{\mathcal{B}_w}) \to j_{w!}(\tau^{\geq n}(\mathcal{F}|_{\mathcal{B}_w})) \to .$$

Again, all three terms are $*$-even. By induction on the number of nonzero cohomology sheaves on $\mathcal{F}|_{\mathcal{B}_w}$, we may reduce to the case where $\mathcal{F}|_{\mathcal{B}_w}$ is concentrated in a single (even) degree, say $2n$. Then $\mathcal{F}|_{\mathcal{B}_w}[2n]$ is a locally free local system on $\mathcal{B}_w$, so it is a direct sum of copies of $\underline{\Bbbk}_{\mathcal{B}_w}$. In this case, $j_{w!}(\mathcal{F}|_{\mathcal{B}_w})$ is a direct sum of copies of $j_{w!}\underline{\Bbbk}_{\mathcal{B}_w}[-2n]$, so it lies in $\mathscr{C}$, as desired. $\square$

LEMMA 7.3.4. *Let $\mathcal{F}$ and $\mathcal{G}$ be objects of $D^{\mathrm{b}}_B(\mathcal{B},\Bbbk)$ or $D^{\mathrm{b}}_{(B)}(\mathcal{B},\Bbbk)$. If $\mathcal{F}$ is $*$-even and $\mathcal{G}$ is $!$-odd, then* $\mathrm{Hom}(\mathcal{F},\mathcal{G}) = 0$. *Similarly, if $\mathcal{F}$ is $*$-odd and $\mathcal{G}$ is $!$-even, then* $\mathrm{Hom}(\mathcal{F},\mathcal{G}) = 0$.

PROOF. In view of Lemma 7.3.3, it is enough to prove this in the special case where

$$\mathcal{F} = j_{w!}\underline{\Bbbk}_{\mathcal{B}_w}[2n] \qquad \text{and} \qquad \mathcal{G} = j_{v*}\underline{\Bbbk}_{\mathcal{B}_v}[2m+1]$$

for some $w, v \in W$ and some $n, m \in \mathbb{Z}$. In this case, we have

$$\begin{aligned}\mathrm{Hom}(\mathcal{F},\mathcal{G}) &= \mathrm{Hom}(j_{w!}\underline{\Bbbk}_{\mathcal{B}_w}[2n], j_{v*}\underline{\Bbbk}_{\mathcal{B}_v}[2m+1]) \\ &\cong \mathrm{Hom}(j_v^* j_{w!}\underline{\Bbbk}_{\mathcal{B}_w}, \underline{\Bbbk}_{\mathcal{B}_v}[2m-2n+1]).\end{aligned}$$

If $v \neq w$, this vanishes because $j_v^* j_{w!}\underline{\Bbbk}_{\mathcal{B}_w} = 0$. On the other hand, if $v = w$, then $j_v^* j_{w!}\underline{\Bbbk}_{\mathcal{B}_w} \cong \underline{\Bbbk}_{\mathcal{B}_w} \cong \mathrm{a}_{\mathcal{B}_w}^*\underline{\Bbbk}_{\mathrm{pt}}$. We thus have

$$\mathrm{Hom}(\mathcal{F},\mathcal{G}) \cong \mathrm{Hom}(\underline{\Bbbk}_{\mathrm{pt}}, \mathrm{a}_{\mathcal{B}_w*}\underline{\Bbbk}_{\mathcal{B}_w}[2m-2n+1]).$$

Since $\mathcal{B}_w \cong \mathbb{A}^{\ell(w)}$, we know that $\mathrm{a}_{\mathcal{B}_w*}\underline{\Bbbk}_{\mathcal{B}_w} \cong \underline{\Bbbk}_{\mathrm{pt}}$. If we are in $D^{\mathrm{b}}_{(B)}(\mathcal{B},\Bbbk)$, we conclude that $\mathrm{Hom}(\mathcal{F},\mathcal{G}) \cong \mathbf{H}^{2m-2n+1}(\mathrm{pt};\Bbbk) = 0$.

Finally, if we are working in $D^{\mathrm{b}}_B(\mathcal{B},\Bbbk)$, then we instead have $\mathrm{Hom}(\mathcal{F},\mathcal{G}) \cong \mathbf{H}_B^{2m-2n-1}(\mathcal{B}_w;\Bbbk)$. This vanishes by Lemma 7.1.1. $\square$

LEMMA 7.3.5. *Let $v, w \in W$, and let $s \in S$. Then $\mathsf{H}^i((j_{w!}\underline{\Bbbk}_{\mathcal{B}_w} \star \mathrm{IC}_s)|_{\mathcal{B}_v})$ is a (locally) free constant sheaf on $\mathcal{B}_v$ whose rank is given by*

$$\operatorname{rank} \mathsf{H}^i((j_{w!}\underline{\Bbbk}_{\mathcal{B}_w} \star \mathrm{IC}_s)|_{\mathcal{B}_v}) \cong \begin{cases} 1 & \text{if } ws > w,\ i = -1, \text{ and } v \in \{w, ws\}, \\ 1 & \text{if } ws < w,\ i = 1, \text{ and } v \in \{w, ws\}, \\ 0 & \text{otherwise.} \end{cases}$$

PROOF. Throughout the proof, we will assume that $ws > w$, but we will compute both $j_{w!}\underline{\Bbbk}_{\mathcal{B}_w} \star \mathrm{IC}_s$ and $j_{ws!}\underline{\Bbbk}_{\mathcal{B}_{ws}} \star \mathrm{IC}_s$. Choose a reduced expression $w = s_1 \cdots s_k$; then we must have $s_k \neq s$. Let $U_{s_i} \subset B$ be the root subgroup corresponding to the simple reflection s_i. The basic theory of the Bruhat decomposition gives rise to isomorphisms of varieties

$$\begin{aligned}\mathcal{B}_w &\cong U_{s_1}\dot{s}_1 \times \cdots \times U_{s_k}\dot{s}_k \cong \mathbb{A}^k, \\ \mathcal{B}_{ws} &\cong U_{s_1}\dot{s}_1 \times \cdots \times U_{s_k}\dot{s}_k \times U_s\dot{s} \cong \mathbb{A}^{k+1}.\end{aligned}$$

Now let $Y = B\dot{w}P_s/P_s = \pi_s(\mathcal{B}_w) = \pi_s(\mathcal{B}_{ws}) \subset G/P_s$. Using the fact that $P_s = B \cup U_s\dot{s}B$ (see, for instance, [**230**, Theorem 8.4.3(i)]), one can see that π_s restricts to an isomorphism

$$\pi_s|_{\mathcal{B}_w} : \mathcal{B}_w \xrightarrow{\sim} Y.$$

On the other hand, the map

$$\pi_s|_{\mathcal{B}_{ws}} : \mathcal{B}_{ws} \to Y \tag{7.3.1}$$

can be identified with the map $\mathbb{A}^{k+1} \to \mathbb{A}^k$ that projects onto the first k coordinates.

Let $i : Y \hookrightarrow G/P_s$ be the inclusion map. Then

$$\pi_{s*} j_{w!} \underline{\Bbbk}_{\mathscr{B}_w} \cong i_! (\pi_s|_{\mathscr{B}_w})_! \underline{\Bbbk}_{\mathscr{B}_w} \cong i_! \underline{\Bbbk}_Y.$$

Next, let $h : \pi_s^{-1}(Y) \hookrightarrow \mathscr{B}$ be the inclusion map, and consider the cartesian square

$$\begin{tikzcd}
\pi_s^{-1}(Y) = \mathscr{B}_w \cup \mathscr{B}_{ws} \arrow[r, "h"] \arrow[d, "\pi_s|_{\pi_s^{-1}(Y)}"'] & \mathscr{B} \arrow[d, "\pi_s"] \\
Y \arrow[r, "i"] & G/P_s
\end{tikzcd}$$

We then have

$$\pi_s^* \pi_{s*} j_{w!} \underline{\Bbbk}_{\mathscr{B}_w} \cong \pi_s^* i_! \underline{\Bbbk}_Y \cong h_! \underline{\Bbbk}_{\pi_s^{-1}(Y)}.$$

In particular, we have

$$\mathsf{H}^i((\pi_s^* \pi_{s*} j_{w!} \underline{\Bbbk}_{\mathscr{B}_w})|_{\mathscr{B}_v}) \cong \begin{cases} \Bbbk & \text{if } i = 0 \text{ and } v \in \{w, ws\}, \\ 0 & \text{otherwise.} \end{cases}$$

Using Lemma 7.2.8, we obtain the desired description of $j_{w!}\underline{\Bbbk}_{\mathscr{B}_w} \star \mathrm{IC}_s$.

Let us now turn to $\mathscr{B}_{ws}$. The description of (7.3.1) given above lets us identify $\mathscr{B}_{ws}$ with $Y \times \mathbb{A}^1$, so

$$(\pi_s|_{\mathscr{B}_{ws}})_! \underline{\Bbbk}_{\mathscr{B}_{ws}} \cong \underline{\Bbbk}_Y \boxtimes R\Gamma_{\mathrm{c}}(\underline{\Bbbk}_{\mathbb{A}^1}) \cong \underline{\Bbbk}_Y[-2](-1).$$

It follows that $\pi_{s*} j_{ws!} \underline{\Bbbk}_{\mathscr{B}_{ws}} \cong i_! \underline{\Bbbk}_Y[-2](-1)$, and then that

$$\pi_s^* \pi_{s*} j_{ws!} \underline{\Bbbk}_{\mathscr{B}_{ws}} \cong h_! \underline{\Bbbk}_{\pi_s^{-1}(Y)}[-2](-1).$$

The desired description of $j_{ws!}\underline{\Bbbk}_{\mathscr{B}_{ws}} \star \mathrm{IC}_s$ then follows as above. □

PROPOSITION 7.3.6. *Let $\mathcal{F}$ be an object in $D^{\mathrm{b}}_B(\mathscr{B}, \Bbbk)$ or $D^{\mathrm{b}}_{(B)}(\mathscr{B}, \Bbbk)$, and let $s \in S$. If $\mathcal{F}$ is $*$-even, then $\mathcal{F} \star \mathrm{IC}_s$ is $*$-odd, and vice versa.*

PROOF. For $\mathcal{F} = j_{w!}\underline{\Bbbk}_{\mathscr{B}_w}[2n]$, this was established in Lemma 7.3.5. It follows for arbitrary $*$-even objects $\mathcal{F}$ by Lemma 7.3.3. □

COROLLARY 7.3.7. *Let $w \in W$. If $\ell(w)$ is even, then $\mathrm{IC}_w(\mathbb{Q})$ is both $*$-even and $!$-even. If $\ell(w)$ is odd, then $\mathrm{IC}_w(\mathbb{Q})$ is both $*$-odd and $!$-odd.*

PROOF. Choose a reduced expression $w = s_1 \cdots s_k$. Assume for now that k is even. Since each $\mathrm{IC}_{s_i}(\mathbb{Q})$ is $*$-odd, Proposition 7.3.6 implies that $\mathrm{IC}_{s_1}(\mathbb{Q}) \star \cdots \star \mathrm{IC}_{s_k}(\mathbb{Q})$ is $*$-even. Since we are working with field coefficients, its direct summand $\mathrm{IC}_w(\mathbb{Q})$ (see Lemma 7.2.10) is also $*$-even. Finally, $\mathrm{IC}_w(\mathbb{Q})$ is isomorphic to its own Verdier dual (by Lemma 3.3.13), so by Lemma 7.3.2, it is also !-even.

If k is odd, the same reasoning shows that $\mathrm{IC}_w(\mathbb{Q})$ is both $*$- and !-odd. □

The categorification theorem. Since $\mathrm{Semis}_B(\mathscr{B}, \mathbb{Q})$ is a monoidal category (Proposition 7.2.6), its split Grothendieck group $\mathrm{K}_\oplus(\mathrm{Semis}_B(\mathscr{B}, \mathbb{Q}))$ is naturally a ring. The following theorem identifies this ring.

THEOREM 7.3.8. *The map $\mathrm{ch} : \mathrm{K}_\oplus(\mathrm{Semis}_B(\mathscr{B}, \mathbb{Q})) \to \mathcal{H}$ given by*

$$\mathrm{ch}([\mathcal{F}]) = \sum_{\substack{w \in W \\ i \in \mathbb{Z}}} (\mathrm{rank}\, \mathsf{H}^i(\mathcal{F}|_{\mathscr{B}_w})) q^{i/2} T_w$$

is an isomorphism of rings. It satisfies

$$\mathrm{ch}([\mathcal{F}[1]]) = q^{-1/2}\mathrm{ch}([\mathcal{F}]), \quad \mathrm{ch}([\mathbb{D}(\mathcal{F})]) = \overline{\mathrm{ch}([\mathcal{F}])}, \quad \mathrm{ch}([\mathrm{IC}_w(\mathbb{Q})]) = \underline{H}_w.$$

PROOF. It is clear from the definition that $\mathrm{ch}([\mathcal{F} \oplus \mathcal{G}]) = \mathrm{ch}([\mathcal{F}]) + \mathrm{ch}([\mathcal{G}])$, so ch is at least a well-defined homomorphism of abelian groups. Moreover, the definition of ch makes sense on the larger split Grothendieck group $\mathrm{K}_\oplus(D^{\mathrm{b}}_B(\mathcal{B}, \mathbb{Q}))$. Some steps of the proof involve evaluating ch on objects not in $\mathrm{Semis}_B(\mathcal{B}, \mathbb{Q})$.

Step 1. We have $\mathrm{ch}([\mathcal{F}[1]]) = q^{-1/2}\mathrm{ch}([\mathcal{F}])$. *Moreover,* ch *is an isomorphism of abelian groups.* The first assertion is immediate from the definition of ch. To say this another way, make $\mathrm{K}_\oplus(\mathrm{Semis}_B(\mathcal{B}, \mathbb{Q}))$ into a $\mathbb{Z}[q^{\pm\frac{1}{2}}]$-module by setting $q^{-1/2}[\mathcal{F}] = [\mathcal{F}[1]]$. Then ch is a homomorphism of $\mathbb{Z}[q^{\pm\frac{1}{2}}]$-modules.

We will now show that ch is an isomorphism of $\mathbb{Z}[q^{\pm\frac{1}{2}}]$-modules. By Proposition A.9.2, the set $\{[\mathrm{IC}_w(\mathbb{Q})]\}_{w\in W}$ is a $\mathbb{Z}[q^{\pm\frac{1}{2}}]$-basis for $\mathrm{K}_\oplus(\mathrm{Semis}_B(\mathcal{B}, Q))$. We must show that ch sends it to a basis for $\mathcal{H}$. Since $\mathrm{IC}_w(\mathbb{Q})$ is supported on $\overline{\mathcal{B}_w}$, we have

$$\mathrm{ch}([\mathrm{IC}_w(\mathbb{Q})]) \in q^{-\ell(w)/2}T_w + \sum_{x<w} \mathbb{Z}[q^{\pm\frac{1}{2}}]T_x.$$

Since $\{T_w\}_{w\in W}$ is a $\mathbb{Z}[q^{\pm\frac{1}{2}}]$-basis for $\mathcal{H}$, it follows that $\{\mathrm{ch}[\mathrm{IC}_w(\mathbb{Q})]\}$ is as well.

Step 2. Let $\mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to$ *be a distinguished triangle in which all three terms are $*$-even (or all three terms are $*$-odd). Then* $\mathrm{ch}([\mathcal{F}]) = \mathrm{ch}([\mathcal{F}']) + \mathrm{ch}([\mathcal{F}''])$. Because all three terms are $*$-even, for any $w \in W$, the long exact sequence in cohomology associated to $\mathcal{F}'|_{\mathcal{B}_w} \to \mathcal{F}|_{\mathcal{B}_w} \to \mathcal{F}''|_{\mathcal{B}_w} \to$ breaks up into a bunch of short exact sequences

$$0 \to \mathsf{H}^i(\mathcal{F}'|_{\mathcal{B}_w}) \to \mathsf{H}^i(\mathcal{F}|_{\mathcal{B}_w}) \to \mathsf{H}^i(\mathcal{F}''|_{\mathcal{B}_w}) \to 0.$$

Therefore, the rank of the middle term is the sum of the ranks of the first and last terms. The claim follows.

Step 3. Let $\mathcal{F} \in D^{\mathrm{b}}_B(\mathcal{B}, \mathbb{Q})$ *be a $*$-even object. Then we have* $\mathrm{ch}([\mathcal{F} \star \mathrm{IC}_s]) = \mathrm{ch}([\mathcal{F}])\mathrm{ch}([\mathrm{IC}_s])$. Suppose we have a distinguished triangle

$$\mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to$$

in which all three terms are $*$-even. By Proposition 7.3.6, $\mathcal{F}' \star \mathrm{IC}_s \to \mathcal{F} \star \mathrm{IC}_s \to \mathcal{F}'' \star \mathrm{IC}_s \to$ is a distinguished triangle all of whose terms are $*$-odd. Applying Step 2 twice, we see that

$$\begin{aligned}\mathrm{ch}([\mathcal{F} \star \mathrm{IC}_s]) &= \mathrm{ch}([\mathcal{F}' \star \mathrm{IC}_s]) + \mathrm{ch}([\mathcal{F}'' \star \mathrm{IC}_s]),\\ \mathrm{ch}([\mathcal{F}]) &= \mathrm{ch}([\mathcal{F}']) + \mathrm{ch}([\mathcal{F}'']).\end{aligned}$$

Thus, if the claim is already known to hold for $\mathcal{F}'$ and $\mathcal{F}''$, then it holds for $\mathcal{F}$ as well. Next, Lemma 7.3.3 lets us reduce the problem to proving it in the special case where $\mathcal{F} = j_{w!}\underline{\mathbb{Q}}_{\mathcal{B}_w}[2n]$ for some $w \in W$ and some $n \in \mathbb{Z}$. Using Lemma 7.3.5, we compute that

$$\mathrm{ch}([j_{w!}\underline{\mathbb{Q}}_{\mathcal{B}_w}[2n] \star \mathrm{IC}_s]) = \begin{cases} q^{-n-\frac{1}{2}}(T_w + T_{ws}) & \text{if } ws > w,\\ q^{-n+\frac{1}{2}}(T_w + T_{ws}) & \text{if } ws < w.\end{cases}$$

On the other hand, we see directly from the definition that

$$\mathrm{ch}([j_{w!}\underline{\mathbb{Q}}_{\mathcal{B}_w}[2n]]) = q^{-n}T_w \qquad \text{and} \qquad \mathrm{ch}([\mathrm{IC}_s]) = q^{-1/2}(T_s + 1). \tag{7.3.2}$$

Using (7.1.6), one can check that $\mathrm{ch}([j_{w!}\underline{\mathbb{Q}}_{\mathcal{B}_w}[2n]])\mathrm{ch}([\mathrm{IC}_s])$ is also given by the formulas above.

Step 4. For $\mathcal{F}, \mathcal{G} \in \mathrm{Semis}_B(\mathcal{B}, \mathbb{Q})$, *we have* $\mathrm{ch}([\mathcal{F} \star \mathcal{G}]) = \mathrm{ch}([\mathcal{F}])\mathrm{ch}([\mathcal{G}])$. *Therefore,* ch *is an isomorphism of rings.* By Corollary 7.3.7, each indecomposable summand of $\mathcal{F}$ is either $*$-even or $*$-odd. In other words, $\mathcal{F}$ is a direct sum of a $*$-even object and a $*$-odd object. It is enough to treat these cases separately. Assume without loss of generality that $\mathcal{F}$ is $*$-even. It is also enough to prove the claim for $\mathcal{G} = \mathrm{IC}_w$. We proceed by induction on w with respect to the Bruhat order. If $w = e$, we have $\mathrm{ch}([\mathrm{IC}_e]) = T_e = 1$, and the claim holds by Lemma 7.2.3. If w is a simple reflection $s \in S$, the claim holds by Step 3.

Finally, if $\ell(w) > 1$, choose a reduced expression $w = s_1 \cdots s_k$, and then choose a decomposition

$$\mathrm{IC}_{s_1} \star \cdots \star \mathrm{IC}_{s_k} = \mathcal{G}' \oplus \mathrm{IC}_w,$$

where $\mathcal{G}'$ is some equivariant semisimple complex supported on $\overline{\mathcal{B}_w} \smallsetminus \mathcal{B}_w$. In other words, $\mathcal{G}'$ is a direct sum of objects of the form $\mathrm{IC}_v[k]$ with $v < w$. By induction, we have

$$\mathrm{ch}([\mathcal{F} \star \mathcal{G}']) = \mathrm{ch}([\mathcal{F}])\mathrm{ch}([\mathcal{G}']).$$

On the other hand, the repeated application of Step 3 shows that

$$\mathrm{ch}([\mathcal{F} \star \mathrm{IC}_{s_1} \star \cdots \star \mathrm{IC}_{s_k}]) = \mathrm{ch}([\mathcal{F}])\mathrm{ch}([\mathrm{IC}_{s_1}]) \cdots \mathrm{ch}([\mathrm{IC}_{s_k}]).$$

Finally, we also have

$$\mathrm{ch}([\mathrm{IC}_{s_1} \star \cdots \star \mathrm{IC}_{s_k}]) = \mathrm{ch}([\mathcal{G}']) + \mathrm{ch}([\mathrm{IC}_w]),$$
$$\mathrm{ch}([\mathcal{F} \star \mathrm{IC}_{s_1} \star \cdots \star \mathrm{IC}_{s_k}]) = \mathrm{ch}([\mathcal{F} \star \mathcal{G}']) + \mathrm{ch}([\mathcal{F} \star \mathrm{IC}_w]).$$

Combining these equations, we deduce that $\mathrm{ch}([\mathcal{F} \star \mathrm{IC}_w]) = \mathrm{ch}([\mathcal{F}])\mathrm{ch}([\mathrm{IC}_w])$, as desired.

Step 5. We have $\mathrm{ch}([\mathbb{D}(\mathcal{F})]) = \overline{\mathrm{ch}([\mathcal{F}])}$. Lemma 7.2.7 implies that $\mathbb{D}$ induces a ring homomorphism $d : \mathrm{K}_\oplus(\mathrm{Semis}_B(\mathcal{B}, \mathbb{Q})) \to \mathrm{K}_\oplus(\mathrm{Semis}(\mathcal{B}, \mathbb{Q}))$. This map is not $\mathbb{Z}[q^{\pm\frac{1}{2}}]$-linear; rather, since $\mathbb{D}(\mathcal{F}[1]) \cong (\mathbb{D}\mathcal{F})[-1]$, we have

$$d(q^{1/2}[\mathcal{F}]) = q^{-1/2}d([\mathcal{F}]). \tag{7.3.3}$$

Since $\mathbb{D}(\mathrm{IC}_s(\mathbb{Q})) \cong \mathrm{IC}_s(\mathbb{Q})$, we also have

$$d([\mathrm{IC}_s]) = [\mathrm{IC}_s]. \tag{7.3.4}$$

By Lemma 7.2.10, $\mathrm{K}_\oplus(\mathrm{Semis}_B(\mathcal{B}, \mathbb{Q}))$ is generated as a $\mathbb{Z}[q^{\pm\frac{1}{2}}]$-algebra by the elements $\{[\mathrm{IC}_s]\}_{s \in S}$. The ring map d is thus the unique ring map satisfying (7.3.3) and (7.3.4).

We saw in (7.3.2) that $\mathrm{ch}([\mathrm{IC}_s]) = \underline{H}_s$ (see Example 7.1.5), so d corresponds to the unique automorphism of $\mathcal{H}$ given by $q^{1/2} \mapsto q^{-1/2}$ and $\underline{H}_s \mapsto \underline{H}_s$. Since the bar involution has these properties, we are done.

Step 6. We have $\mathrm{ch}([\mathrm{IC}_w]) = \underline{H}_w$. Since $\mathbb{D}(\mathrm{IC}_w) \cong \mathrm{IC}_w$, Step 5 tells us that $\mathrm{ch}([\mathrm{IC}_w])$ is preserved by the bar involution. Next, for any $x, w \in W$, let $P'_{x,w} \in \mathbb{Z}[q^{\pm\frac{1}{2}}]$ be the element given by

$$P'_{x,w} = \sum_{i \in \mathbb{Z}} (\mathrm{rank}\, \mathsf{H}^i(j_x^* \mathrm{IC}_w)) q^{(i+\ell(w))/2},$$

so that

$$\mathrm{ch}([\mathrm{IC}_w(\mathbb{Q})]) = q^{-\ell(w)/2} \sum_{w \in W} P'_{x,w} T_x.$$

We clearly have $P'_{x,w} = 0$ if $x \not\leq w$, and $P'_{w,w} = 1$. If $x < w$, then $\mathsf{H}^i(j_x^* \mathrm{IC}_w(\mathbb{Q})) = 0$ unless $-\ell(w) \leq i \leq -\ell(x) - 1$: the first holds by Lemma 3.1.8, and the second by Lemma 3.3.11. Corollary 7.3.7 tells us that $\mathsf{H}^i(j_x^* \mathrm{IC}_w(\mathbb{Q})) = 0$ unless $i \equiv \ell(w) \pmod 2$. Together, these observations imply that $P'_{x,w}$ lies in $\mathbb{Z}[q]$ and that it has degree $\leq \frac{1}{2}(\ell(w)-\ell(x)-1)$. We have just shown that $\{\mathrm{ch}([\mathrm{IC}_w])\}_{w\in W}$ satisfies the conditions that uniquely characterize the canonical basis in Theorem 7.1.3, so we must have $\mathrm{ch}([\mathrm{IC}_w]) = \underline{H}_w$. □

Extracting the formula from Step 6 of the preceding proof, we obtain the following formula.

COROLLARY 7.3.9. *For $x, w \in W$, we have*

$$P_{x,w} = q^{\ell(w)/2} \sum_{i\in\mathbb{Z}} (\operatorname{rank} \mathsf{H}^i(\mathrm{IC}_w(\mathbb{Q})|_{\mathcal{B}_x}))q^{i/2}.$$

REMARK 7.3.10. The formula in Corollary 7.3.9 is an ingredient in one of the most important early applications of the theory of perverse sheaves: namely, the proof of the **Kazhdan–Lusztig conjecture**. Here is a very brief overview of this topic. Let $\mathfrak{g}$ be a complex semisimple Lie algebra, and let $\mathfrak{h}$ be a Cartan subalgebra. For $\lambda \in \mathfrak{h}^*$, let $M(\lambda)$ denote the Verma module of highest weight λ, and let $L(\lambda)$ denote its unique irreducible quotient. A major open problem in the 1970s was that of computing the character of $L(\lambda)$, especially in the important special case where $L(\lambda)$ belongs to the "principal block." This means that λ is of the form $-w\rho - \rho$, where $w \in W$ and where ρ is the half-sum of the positive roots. In 1979, Kazhdan and Lusztig conjectured [**129**] the following solution to this problem:

$$\mathrm{ch}L(-w\rho-\rho) = \sum_{x\in W} (-1)^{\ell(w)-\ell(x)} P_{x,w}|_{q=1} \mathrm{ch}M(-x\rho-\rho). \tag{7.3.5}$$

Around the same time, they proved Corollary 7.3.9 in [**130**].

The formula (7.3.5) was proved independently by Beĭlinson–Bernstein [**26**] and by Brylinski–Kashiwara [**50**]. The key point is that there is a surjective map

$$\mathrm{U}(\mathfrak{g}) \to \Gamma(\mathscr{D}_{\mathcal{B}}),$$

where $\mathrm{U}(\mathfrak{g})$ is the universal enveloping algebra of $\mathfrak{g}$ and where $\mathscr{D}_{\mathcal{B}}$ is the sheaf of differential operators on $\mathcal{B}$. Using this, it is possible to construct a functor from certain $\mathfrak{g}$-modules to $\mathscr{D}$-modules on $\mathcal{B}$ that fits into the following diagram:

$$\left\{\begin{matrix}\mathfrak{g}\text{-modules with}\\ \text{trivial central character}\end{matrix}\right\} \xrightarrow{[\mathbf{26},\mathbf{50}]} \mathscr{D}_{\mathcal{B}}\text{-mod}^{\mathrm{rh}} \xrightarrow[\text{correspondence}]{\text{Riemann–Hilbert}} \mathrm{Perv}(\mathcal{B}, \mathbb{C}).$$

Following simple and Verma modules through these functors, one shows that computing $\mathrm{ch}L(-w\rho-\rho)$ in terms of Verma modules is equivalent to computing the stalks of $\mathrm{IC}_w(\mathbb{Q})$: compare (7.3.5) with the following theorem.

THEOREM 7.3.11. *The map* $\mathrm{ch} : \mathrm{K}_0(D^{\mathrm{b}}_B(\mathcal{B}, \mathbb{Q})) \to \mathbb{Z}[W]$ *given by*

$$\mathrm{ch}([\mathcal{F}]) = \sum_{\substack{w\in W\\ i\in\mathbb{Z}}} (-1)^i (\operatorname{rank} \mathsf{H}^i(\mathcal{F}|_{\mathcal{B}_w}))w$$

is an isomorphism of rings. It satisfies

$$\mathrm{ch}([j_{w!}\underline{\mathbb{Q}}_{\mathcal{B}_w}]) = w \qquad \textit{and} \qquad \mathrm{ch}([\mathrm{IC}_w(\mathbb{Q})]) = \underline{H}_w|_{q^{1/2}=-1}.$$

As a consequence, in $\mathrm{K}_0(D^{\mathrm{b}}_B(\mathcal{B}, \mathbb{Q}))$, *we have*

$$[\mathrm{IC}_w] = \sum_{x\in W} (-1)^{\ell(w)-\ell(x)} P_{x,w}|_{q=1} [j_{w!}\underline{\mathbb{Q}}_{\mathcal{B}_w}[\ell(w)]].$$

Proof. In this proof, in order to distinguish the map ch defined here from the one defined in Theorem 7.3.8, we denote the latter by ch_q. Consider the diagram

$$\begin{array}{ccc} \mathrm{K}_\oplus(\mathrm{Semis}_B(\mathcal{B},\mathbb{Q})) & \xrightarrow[\sim]{\mathrm{ch}_q} & \mathcal{H} \\ \Big\downarrow & & \Big\downarrow{\scriptstyle q^{1/2}\mapsto -1} \\ \mathrm{K}_0(D^{\mathrm{b}}_B(\mathcal{B},\mathbb{Q})) & \xrightarrow{\mathrm{ch}} & \mathbb{Z}[W] \end{array} \tag{7.3.6}$$

It is clear from the definitions that the diagram commutes. It follows immediately that ch is surjective. Recall that the classes of simple perverse sheaves form a basis for $\mathrm{K}_0(D^{\mathrm{b}}_B(\mathcal{B},\mathbb{Q}))$ (see Propositon A.9.5). Thus, both groups in the bottom row of the diagram are free abelian groups of rank $|W|$. A surjective map between them is necessarily injective as well. Thus, ch is at least an isomorphism of abelian groups.

The formula for $\mathrm{ch}([j_{w!}\underline{\mathbb{Q}}_{\mathcal{B}_w}])$ is immediate from the definition, and the formula for $\mathrm{ch}([\mathrm{IC}_w(\mathbb{Q})])$ follows from (7.3.6) and Theorem 7.3.8. To show that ch is a ring homomorphism, it is enough to check that $\mathrm{ch}([\mathrm{IC}_w \star \mathrm{IC}_v]) = \mathrm{ch}([\mathrm{IC}_w])\mathrm{ch}([\mathrm{IC}_v])$. This, too, follows from (7.3.6) and Theorem 7.3.8. The last equation in the theorem is obtained by evaluating the formula from Theorem 7.1.3 at $q^{1/2} = -1$ to get an equation in $\mathbb{Z}[W]$, and then applying ch^{-1} to get an equation in $\mathrm{K}_0(D^{\mathrm{b}}_B(\mathcal{B},\mathbb{Q}))$. $\square$

Exercises.

7.3.1. Let $\mathcal{F} \in D^{\mathrm{b}}_B(\mathcal{B},\Bbbk)$ be a $*$-even object. Let $x \in W$, and assume that $xs > x$. Show that

$$\begin{aligned} \operatorname{rank} \mathsf{H}^i((\mathcal{F}\star \mathrm{IC}_s)|_{\mathcal{B}_x}) &= \operatorname{rank} \mathsf{H}^i((\mathcal{F}\star \mathrm{IC}_s)|_{\mathcal{B}_{xs}}) \\ &= \operatorname{rank} \mathsf{H}^{i+1}(\mathcal{F}|_{\mathcal{B}_x}) + \operatorname{rank} \mathsf{H}^{i-1}(\mathcal{F}|_{\mathcal{B}_{xs}}). \end{aligned}$$

7.3.2. Show that if $ws > w$, then $\mathrm{IC}_w \star \mathrm{IC}_s$ is perverse. In the special case where $\Bbbk = \mathbb{Q}$, show that

$$\mathrm{IC}_w(\mathbb{Q}) \star \mathrm{IC}_s(\mathbb{Q}) \cong \mathrm{IC}_{ws}(\mathbb{Q}) \oplus \bigoplus_{\substack{z<ws \\ \ell(z)=\ell(ws) \pmod 2}} \mathrm{IC}_z(\mathbb{Q})^{\oplus m_{z,ws}},$$

where $m_{z,ws} = \operatorname{rank} \mathsf{H}^{-\ell(z)}((\mathrm{IC}_w(\mathbb{Q}) \star \mathrm{IC}_s(\mathbb{Q}))|_{\mathcal{B}_z})$.

7.3.3. Using the preceding exercise, show that Corollary 7.3.9 is equivalent to the claim that in the Hecke algebra, if $ws > w$, then

$$\underline{H}_{ws} = \underline{H}_w \underline{H}_s - \sum_{z<ws} a_{z,ws} \underline{H}_z,$$

where $a_{z,ws} \in \mathbb{Z}$ is the coefficient of $q^{(\ell(ws)-\ell(z))/2} T_z$ in $\underline{H}_w \underline{H}_s$.

7.4. Mixed sheaves on the flag variety

In this section, we change settings and let G be a split reductive group over the finite field $\mathbb{F}_p$. We likewise replace B, T, etc., by groups over $\mathbb{F}_p$, and we let $\mathcal{B}$ be the flag variety over $\mathbb{F}_p$. For any prime power p^n, we may consider the finite group $G(\mathbb{F}_{p^n})$, as well as the set of $\mathbb{F}_{p^n}$-points of the flag variety $\mathcal{B}(\mathbb{F}_{p^n})$, which can be identified with $G(\mathbb{F}_{p^n})/B(\mathbb{F}_{p^n})$.

The goal of this section is to understand the Hecke algebra from the perspective of the sheaf–function correspondence (Theorem 5.3.13). Let $\mathcal{L}(G, B, \mathbb{F}_{p^n})$ be the space of $B(\mathbb{F}_{p^n})$-biinvariant integer-valued functions on $G(\mathbb{F}_{p^n})$. That is,

$$\mathcal{L}(G, B, \mathbb{F}_{p^n}) = \left\{ f : G(\mathbb{F}_{p^n}) \to \mathbb{Z} \,\middle|\, \begin{array}{c}\text{for } b_1, b_2 \in B(\mathbb{F}_{p^n}) \text{ and } g \in G(\mathbb{F}_{p^n}),\\ \text{we have } f(b_1 g b_2) = f(g)\end{array} \right\}. \tag{7.4.1}$$

This abelian group can be made into a ring using the **convolution product**

$$(f_1 \star f_2)(g) = \frac{1}{|B(\mathbb{F}_{p^n})|} \sum_{\substack{h_1, h_2 \in G(\mathbb{F}_{p^n})\\ h_1 h_2 = g}} f_1(h_1) f_2(h_2). \tag{7.4.2}$$

(It is an exercise to check that $f_1 \star f_2$ is still $\mathbb{Z}$-valued and $B(\mathbb{F}_{p^n})$-biinvariant.) The ring $\mathcal{L}(G, B, \mathbb{F}_{p^n})$ is free over $\mathbb{Z}$ with basis $\{S_w \mid w \in W\}$, where the S_w are the characteristic functions of the $B(\mathbb{F}_{p^n})$-double cosets:

$$S_w(g) = \begin{cases} 1 & \text{if } g \in B(\mathbb{F}_{p^n}) \dot{w} B(\mathbb{F}_{p^n}),\\ 0 & \text{otherwise.} \end{cases}$$

The element S_e is the unit element for the ring structure on $\mathcal{L}(G, B, \mathbb{F}_{p^n})$. The following theorem of Iwahori [**110**] describes this ring.

THEOREM 7.4.1 (Iwahori). *There is a ring isomorphism* $\theta : \mathcal{L}(G, B, \mathbb{F}_{p^n}) \xrightarrow{\sim} \mathcal{H}|_{q=p^n}$ *given by* $\theta(S_w) = T_w|_{q=p^n}$.

Here is an alternative description of $\mathcal{L}(G, B, \mathbb{F}_{p^n})$. A $B(\mathbb{F}_{p^n})$-biinvariant function on $G(\mathbb{F}_{p^n})$ is the same thing as a $B(\mathbb{F}_{p^n})$-invariant function on $\mathcal{B}(\mathbb{F}_{p^n})$, so we may identify

$$\mathcal{L}(G, B, \mathbb{F}_{p^n}) \cong \left\{ f : \mathcal{B}(\mathbb{F}_{p^n}) \to \mathbb{Z} \,\middle|\, \begin{array}{c}\text{for } b \in B(\mathbb{F}_{p^n}) \text{ and } gB(\mathbb{F}_{p^n}) \in \mathcal{B}(\mathbb{F}_{p^n}),\\ \text{we have } f(bgB(\mathbb{F}_{p^n})) = f(gB(\mathbb{F}_{p^n}))\end{array} \right\}. \tag{7.4.3}$$

To describe the convolution product in this language, we need an $\mathbb{F}_{p^n}$-version of the convolution diagram:

$$\mathcal{B}(\mathbb{F}_{p^n}) \times \mathcal{B}(\mathbb{F}_{p^n}) \xleftarrow{p} G(\mathbb{F}_{p^n}) \times \mathcal{B}(\mathbb{F}_{p^n}) \xrightarrow{q} G(\mathbb{F}_{p^n}) \times^{B(\mathbb{F}_{p^n})} \mathcal{B}(\mathbb{F}_{p^n}) \xrightarrow{m} \mathcal{B}(\mathbb{F}_{p^n}). \tag{7.4.4}$$

Given two $B(\mathbb{F}_{p^n})$-invariant functions $f_1, f_2 : \mathcal{B}(\mathbb{F}_{p^n}) \to \mathbb{Z}$, define a function

$$f_1 \times f_2 : \mathcal{B}(\mathbb{F}_{p^n}) \times \mathcal{B}(\mathbb{F}_{p^n}) \to \mathbb{Z}$$

by $(f_1 \times f_2)(g_1 B(\mathbb{F}_{p^n}), g_2 B(\mathbb{F}_{p^n})) = f_1(g_1 B(\mathbb{F}_{p^n})) f_2(g_2 B(\mathbb{F}_{p^n}))$. Next, observe that there is a unique $B(\mathbb{F}_{p^n})$-invariant function

$$f_1 \tilde{\times} f_2 : G(\mathbb{F}_{p^n}) \times^{B(\mathbb{F}_{p^n})} \mathcal{B}(\mathbb{F}_{p^n}) \to \mathbb{Z}$$

such that $(f_1 \tilde{\times} f_2) \circ q = (f_1 \times f_2) \circ p$. The convolution product is given by

$$f_1 \star f_2 = \int_m (f_1 \tilde{\times} f_2), \tag{7.4.5}$$

where the integral is defined by $(\int_m f)(x) = \sum_{y \in m^{-1}(x)} f(y)$. The proof of the following lemma is left to the reader.

LEMMA 7.4.2. *Under the natural bijection between* (7.4.1) *and* (7.4.3)*, the convolution product defined in* (7.4.2) *coincides with the one defined in* (7.4.5).

Recall from Definition 5.3.11 that for any object $\mathcal{F} \in D^{\mathrm{b}}_{\mathrm{m}}(\mathcal{B}, \overline{\mathbb{Q}}_\ell)$ and any prime power p^n, one can define the characteristic function

$$\chi_{\mathcal{F}} : \mathcal{B}(\mathbb{F}_{p^n}) \to \overline{\mathbb{Q}}_\ell.$$

If $\mathcal{F}$ is instead an object of the equivariant derived category $D^{\mathrm{b}}_{\mathrm{m},B}(\mathcal{B}, \overline{\mathbb{Q}}_\ell)$, the same definition makes sense, and the resulting function is then automatically $B(\mathbb{F}_{p^n})$-invariant. By Lemma 5.3.12, we get a map

$$\chi : \mathrm{K}_0(D^{\mathrm{b}}_{\mathrm{m},B}(\mathcal{B}, \overline{\mathbb{Q}}_\ell)) \to \overline{\mathbb{Q}}_\ell \otimes_{\mathbb{Z}} \mathcal{L}(G, B, \mathbb{F}_{p^n}). \tag{7.4.6}$$

PROPOSITION 7.4.3. *The characteristic function map* $\chi : \mathrm{K}_0(D^{\mathrm{b}}_{\mathrm{m},B}(\mathcal{B}, \overline{\mathbb{Q}}_\ell)) \to \overline{\mathbb{Q}}_\ell \otimes_{\mathbb{Z}} \mathcal{L}(G, B, \mathbb{F}_{p^n})$ *is a ring homomorphism.*

PROOF. This result amounts to comparing the convolution product of sheaves from Section 7.2 with the convolution product of functions in (7.4.5), and then applying the sheaf–function correspondence (Theorem 5.3.13). We omit the details. □

The rest of this section is devoted to computing the characteristic functions of intersection cohomology complexes. We will have to pay attention to weights. Recall from Example 5.4.13 that

$$\mathrm{IC}_w = \mathrm{IC}(\mathcal{B}_w, \underline{\overline{\mathbb{Q}}_\ell})$$

is pure of weight $\ell(w) = \dim \mathcal{B}_w$.

LEMMA 7.4.4. *Let* $\mathcal{F}, \mathcal{G} \in D^{\mathrm{b}}_{\mathrm{m},B}(\mathcal{B}, \overline{\mathbb{Q}}_\ell)$. *If* $\mathcal{F}$ *is pure of weight* n *and* $\mathcal{G}$ *is pure of weight* m, *then* $\mathcal{F} \star \mathcal{G}$ *is pure of weight* $n + m$.

PROOF. In the convolution diagram (7.2.1), the maps p and q are both smooth of relative dimension $\dim B$, and the map m is proper. The claim follows immediately from the formula for convolution together with Corollaries 5.4.8 and 5.4.9. □

LEMMA 7.4.5. *Let* $w \in W$, *and let* $w = s_1 \dots s_k$ *be a reduced expression. Then*

$$\underline{\mathbf{H}}^\bullet(\mathcal{B}, \mathrm{IC}_{s_1} \star \cdots \star \mathrm{IC}_{s_k})$$

is pure of weight k *and of Tate type.*

PROOF. Let $\mathcal{B}(s_1, \dots, s_k)$ be as in (7.1.2), but defined over $\mathbb{F}_q$. As in Remark 7.2.2, this variety has a stratification whose strata are of the form

$$\mathcal{B}_{w_1} \tilde{\times} \cdots \tilde{\times} \mathcal{B}_{w_k} \qquad \text{where } w_i \in \{e, s_i\} \text{ for each } i.$$

As in the proof of Lemma 7.3.5, one can show that this variety is isomorphic to

$$X_{w_1} \times \cdots \times X_{w_k} \qquad \text{where} \qquad X_{w_i} = \begin{cases} \{\dot{s}_i\} & \text{if } w_i = e, \\ U_{s_i}\dot{s}_i & \text{if } w_i = s_i. \end{cases}$$

In particular, each stratum $\mathcal{B}_{w_1} \tilde{\times} \cdots \tilde{\times} \mathcal{B}_{w_k}$ is an affine space. By Proposition 5.7.5, $\mathrm{a}_{\mathcal{B}(s_1,\dots,s_k)*}\underline{\overline{\mathbb{Q}}_\ell}$ is pure of weight 0 and of Tate type. Let $m : \mathcal{B}(s_1, \dots, s_k) \to \mathcal{B}$ be as in (7.1.3). Since $\mathrm{a}_{\mathcal{B}*}(\mathrm{IC}_{s_1} \star \cdots \star \mathrm{IC}_{s_k}) \cong \mathrm{a}_{\mathcal{B}*}m_*\underline{\overline{\mathbb{Q}}_\ell}[k] \cong \mathrm{a}_{\mathcal{B}(s_1,\dots,s_k)*}\underline{\overline{\mathbb{Q}}_\ell}[k]$, we are done. □

PROPOSITION 7.4.6. *For every* $w \in W$, $\underline{\mathbf{H}}^\bullet(\mathcal{B}, \mathrm{IC}_w)$ *is pure of weight* $\ell(w)$ *and of Tate type.*

PROOF. Choose a reduced expression $w = s_1 \cdots s_k$. By Lemma 7.2.9(2), IC_w is a composition factor of ${}^p\mathsf{H}^0(\mathrm{IC}_{s_1} \star \cdots \star \mathrm{IC}_{s_k})$. Then, by Exercise 5.4.3, $\underline{\mathbf{H}}^i(\mathcal{B}, \mathrm{IC}_w)$ is a subquotient of $\underline{\mathbf{H}}^i(\mathcal{B}, \mathrm{IC}_{s_1} \star \cdots \star \mathrm{IC}_{s_k})$. Since the latter is a direct sum of copies of $\overline{\mathbb{Q}}_\ell(-\frac{k+i}{2})$, the former is as well. □

LEMMA 7.4.7. *Suppose $\mathcal{F} \in D^{\mathrm{b}}_{\mathrm{m},B}(\mathcal{B}, \overline{\mathbb{Q}}_\ell)$ is $*$-even (or $*$-odd), and that $\mathrm{a}_{\mathcal{B}*}\mathcal{F}$ is pure of weight k and of Tate type. Then, for all $w \in W$, $\mathrm{a}_{\mathcal{B}_w!}(\mathcal{F}|_{\mathcal{B}_w})$ is pure of weight k and of Tate type.*

PROOF. We first claim that for any $*$-even object $\mathcal{G} \in D^{\mathrm{b}}_{\mathrm{m},B}(\mathcal{B}, \overline{\mathbb{Q}}_\ell)$, $\underline{\mathbf{H}}^k(\mathcal{B}, \mathcal{G}) = 0$ for k odd. The proof of this claim is left to the reader.

We now prove the lemma by induction on the support of $\mathcal{F}$ with respect to the Bruhat order. If $\mathcal{F}$ is supported on $\mathcal{B}_e$, then of course $\mathrm{a}_{\mathcal{B}*}\mathcal{F} \cong \mathrm{a}_{\mathcal{B}_e!}(\mathcal{F}|_{\mathcal{B}_e})$, and there is nothing to prove.

Otherwise, let w be a maximal element such that $\mathcal{F}|_{\mathcal{B}_w} \neq 0$. Let $Z = \operatorname{supp} \mathcal{F} \smallsetminus \mathcal{B}_w$, and suppose the lemma is already known for objects supported on Z. Let $i : Z \hookrightarrow \mathcal{B}$ be the inclusion map, and consider the distinguished triangle

$$j_{w!}j_w^*\mathcal{F} \to \mathcal{F} \to i_*i^*\mathcal{F} \to .$$

Apply $\mathrm{a}_{\mathcal{B}*} = \mathrm{a}_{\mathcal{B}!}$ and then form the long exact sequence in cohomology. All three terms in the distinguished triangle are $*$-even, so by the claim at the beginning of the proof, the terms of the long exact sequence vanish in odd degrees, and in even degrees, we get a short exact sequence

$$0 \to \underline{\mathbf{H}}^i_c(\mathcal{B}_w, \mathcal{F}|_{\mathcal{B}_w}) \to \underline{\mathbf{H}}^i(\mathcal{B}, \mathcal{F}) \to \underline{\mathbf{H}}^i(\mathcal{B}, i_*i^*\mathcal{F}) \to 0.$$

Since the middle term is a direct sum of copies of $\overline{\mathbb{Q}}_\ell(-\frac{k+i}{2})$, the first term is as well. □

THEOREM 7.4.8. *Let $w \in W$.*

(1) *The object IC_w is pointwise pure of weight $\ell(w)$, and its stalks are of Tate type.*
(2) *The characteristic function $\chi_{\mathrm{IC}_w} \in \overline{\mathbb{Q}}_\ell \otimes_{\mathbb{Z}} \mathcal{L}(G, B, \mathbb{F}_{p^n})$ is given by*

$$\chi_{\mathrm{IC}_w} = (-1)^{\ell(w)} \sum_{x \in W} P_{x,w}|_{q=p^n} S_x.$$

PROOF. For part (1), since IC_w is B-equivariant, it is enough to describe its stalk at one point of each Schubert cell $\mathcal{B}_x$, say at $\dot{x}B$. By Lemma 7.1.1, the problem is equivalent to showing that $\mathrm{a}_{\mathcal{B}_x!}(\mathrm{IC}_w|_{\mathcal{B}_x})[2\ell(x)](\ell(x))$ is pure and of Tate type for all $x \in W$. In view of Corollary 7.3.7 and Proposition 7.4.6, this follows from Lemma 7.4.7.

Let us now compute the characteristic function χ_{IC_w} as a linear combination of the basis elements S_x. From the definition, the coefficient of S_x is

$$\chi_{\mathrm{IC}_w}(\dot{x}B) = \sum_{i \in \mathbb{Z}} (-1)^i \operatorname{tr}(\mathsf{Fr}_{p^n}, \mathsf{H}^i((\mathrm{IC}_w)_{\dot{x}B})).$$

Since $\mathsf{H}^i((\mathrm{IC}_w)_{\dot{x}B})$ is a direct sum of copies of $\overline{\mathbb{Q}}_\ell(-\frac{\ell(w)+i}{2})$, Example 5.2.10 tells us that the trace of Fr_{p^n} is $p^{n(\ell(w)+i)/2} \dim \mathsf{H}^i((\mathrm{IC}_w)_{\dot{x}B})$. We deduce that

$$\chi_{\mathrm{IC}_w}(\dot{x}B) = (-1)^{\ell(w)} \sum_{x \in W} p^{n(\ell(w)+i)/2} \operatorname{rank} \mathsf{H}^i(\mathrm{IC}_w|_{\mathcal{B}_x}).$$

Comparing this with Corollary 7.3.9, we obtain part (2) of the theorem. □

Exercises.

7.4.1. Choose a square root of the Tate module $\overline{\mathbb{Q}}_\ell(\frac{1}{2})$, as in Remark 5.7.6. Let $D^{\mathrm{b}}_{\mathrm{m},B}(\mathscr{B}, \overline{\mathbb{Q}}_\ell)'$ be the full triangulated subcategory of $D^{\mathrm{b}}_{\mathrm{m},B}(\mathscr{B}, \overline{\mathbb{Q}}_\ell)$ generated by objects of the form $\mathrm{IC}_w(\frac{m}{2})$ with $w \in W$ and $m \in \mathbb{Z}$. Show that $D^{\mathrm{b}}_{\mathrm{m},B}(\mathscr{B}, \overline{\mathbb{Q}}_\ell)'$ is closed under convolution. As a consequence, the Grothendieck group $\mathrm{K}_0(D^{\mathrm{b}}_{\mathrm{m},B}(\mathscr{B}, \overline{\mathbb{Q}}_\ell)')$ has the structure of a ring.

7.4.2. For all $w \in W$, prove that $j_{w!}\underline{\overline{\mathbb{Q}}}_{\ell\mathscr{B}_w}$ and $j_{w*}\underline{\overline{\mathbb{Q}}}_{\ell\mathscr{B}_w}$ lie in $D^{\mathrm{b}}_{\mathrm{m},B}(\mathscr{B}, \overline{\mathbb{Q}}_\ell)'$. *Hint*: Use Exercise 7.2.7 and induction on $\ell(w)$.

7.4.3. Prove that there is a unique isomorphism of rings

$$\mathrm{ch} : \mathrm{K}_0(D^{\mathrm{b}}_{\mathrm{m},B}(\mathscr{B}, \overline{\mathbb{Q}}_\ell)') \xrightarrow{\sim} \mathcal{H}$$

such that $\mathrm{ch}([j_{w!}\underline{\overline{\mathbb{Q}}}_{\ell\mathscr{B}_w}]) = T_w$ for all $w \in W$. Moreover, this map satisfies

$$\mathrm{ch}([\mathcal{F}[1](\tfrac{1}{2})]) = q^{-1/2}\mathrm{ch}([\mathcal{F}]), \quad \mathrm{ch}([\mathbb{D}(\mathcal{F})]) = \overline{\mathrm{ch}([\mathcal{F}])}, \quad \mathrm{ch}([\mathrm{IC}_w]) = \underline{H}_w.$$

7.5. Parity sheaves

In this section, we will study objects that satisfy the following self-dual version of the conditions in Definition 7.3.1.

DEFINITION 7.5.1. An object $\mathcal{F} \in D^{\mathrm{b}}_B(\mathscr{B}, \Bbbk)$ is said to be **even** if it is both $*$-even and !-even. It is said to be **odd** if it is both $*$-odd and !-odd.

A **parity sheaf** is an object $\mathcal{F} \in D^{\mathrm{b}}_B(\mathscr{B}, \Bbbk)$ that admits a decomposition $\mathcal{F} \cong \mathcal{F}_0 \oplus \mathcal{F}_1$, where $\mathcal{F}_0$ is even and $\mathcal{F}_1$ is odd. The full additive subcategory of $D^{\mathrm{b}}_B(\mathscr{B}, \Bbbk)$ consisting of parity sheaves is denoted by $\mathrm{Parity}_B(\mathscr{B}, \Bbbk)$.

Definitions 7.3.1 and 7.5.1 were both introduced by Juteau–Mautner–Williamson [**120**], not just for $\mathscr{B}$, but for quite general stratified varieties. We refer the reader to [**120**, Section 4] for an overview of applications of parity sheaves on spaces other than $\mathscr{B}$. (The reader should beware that the definition of "parity sheaf" given in [**120**] differs slightly from that given above: it includes the additional condition that the object be indecomposable.) For the case of $\mathscr{B}$, a number of basic facts about parity sheaves appeared earlier (with different terminology) in [**221**].

Classification of parity sheaves. The goal for the first part of this section is to classify the indecomposable parity sheaves on $\mathscr{B}$.

LEMMA 7.5.2. *Let $\mathcal{F}, \mathcal{G} \in D^{\mathrm{b}}_B(\mathscr{B}, \Bbbk)$. Suppose that $\mathcal{F}$ is $*$-even and that $\mathcal{G}$ is !-even (or that $\mathcal{F}$ is $*$-odd and that $\mathcal{G}$ is !-odd). If $U \subset \mathscr{B}$ is a B-stable open subset, then the natural map*

$$\mathrm{Hom}(\mathcal{F}, \mathcal{G}) \to \mathrm{Hom}(\mathcal{F}|_U, \mathcal{G}|_U)$$

is surjective.

PROOF. Suppose that $\mathcal{F}$ is $*$-even and that $\mathcal{G}$ is !-even. Let $Z = \mathscr{B} \smallsetminus U$, and let $i : Z \hookrightarrow \mathscr{B}$ and $j : U \hookrightarrow \mathscr{B}$ be the inclusion maps. Then $i_*i^!\mathcal{G}$ and $j_*j^*\mathcal{G}$ are both !-even. Apply $\mathrm{Hom}(\mathcal{F}, -)$ to the distinguished triangle $i_*i^!\mathcal{G} \to \mathcal{G} \to j_*j^*\mathcal{G} \to$ to get a long exact sequence

$$\cdots \to \mathrm{Hom}(\mathcal{F}, \mathcal{G}) \to \mathrm{Hom}(\mathcal{F}, j_*j^*\mathcal{G}) \to \mathrm{Hom}(\mathcal{F}, i_*i^!\mathcal{G}[1]) \to \cdots.$$

By Lemma 7.3.4, the last term vanishes. By adjunction, the middle term can be identified with $\mathrm{Hom}(\mathcal{F}|_U, \mathcal{G}|_U)$, and the result follows. $\square$

LEMMA 7.5.3. *Let $w \in W$, and let $\mathcal{F} \in D^{\mathrm{b}}_B(\mathcal{B}_w, \Bbbk)$. Assume that $\mathsf{H}^i(\mathcal{F}) = 0$ if i is odd and that it is a locally free local system of finite type if i is even. Then $\mathcal{F}$ is isomorphic to a direct sum of objects of the form $\underline{\Bbbk}_{\mathcal{B}_w}[2n]$.*

PROOF. We proceed by induction on the number of nonvanishing cohomology sheaves of $\mathcal{F}$. If $\mathcal{F} = 0$, there is nothing to prove. Otherwise, let m be the largest integer such that $\mathsf{H}^m(\mathcal{F}) \neq 0$. Then m is even, and since $\mathcal{B}_w$ is simply connected,

$$\mathsf{H}^m(\mathcal{F}) \text{ is a direct sum of copies of } \underline{\Bbbk}_{\mathcal{B}_w}. \tag{7.5.1}$$

Consider the truncation distinguished triangle

$$\tau^{\leq m-1}\mathcal{F} \to \mathcal{F} \to \mathsf{H}^m(\mathcal{F})[-m] \to . \tag{7.5.2}$$

The object $\tau^{\leq m-1}\mathcal{F}$ has fewer nonzero cohomology sheaves, so by induction, it is a direct sum of objects of the form $\underline{\Bbbk}_{\mathcal{B}_w}[2n]$.

We claim that the third morphism in (7.5.2) vanishes. In view of (7.5.1) and the preceding observation, our claim follows from the assertion that the group

$$\mathrm{Hom}(\underline{\Bbbk}_{\mathcal{B}_w}[-m], \underline{\Bbbk}_{\mathcal{B}_w}[2n+1]) \cong \mathbf{H}^{m+2n+1}_B(\mathcal{B}_w; \Bbbk)$$

vanishes. Since m is even, this vanishing holds by Lemma 7.1.1. We conclude that (7.5.2) splits and that $\mathcal{F}$ is a direct sum as claimed. $\square$

LEMMA 7.5.4. *Let $\mathcal{F}, \mathcal{G} \in D^{\mathrm{b}}_B(\mathcal{B}, \Bbbk)$. If $\mathcal{F}$ is $*$-even and $\mathcal{G}$ is $!$-even, then $\mathrm{Hom}^\bullet(\mathcal{F}, \mathcal{G})$ is a free $\mathbf{H}^\bullet_B(\mathrm{pt}; \Bbbk)$-module, and it is concentrated in even degrees.*

PROOF. Suppose first that $\mathcal{F} = j_{w!}\underline{\Bbbk}_{\mathcal{B}_w}[2n]$ for some $w \in W$ and some integer n. Then of course $\mathrm{Hom}^\bullet(\mathcal{F}, \mathcal{G}) \cong \mathrm{Hom}^\bullet(\underline{\Bbbk}_{\mathcal{B}_w}[2n], j_w^!\mathcal{G})$. By Lemma 7.5.3, $j_w^!\mathcal{G}$ is a direct sum of objects of the form $\underline{\Bbbk}_{\mathcal{B}_w}[2m]$, so by Lemma 7.1.1, $\mathrm{Hom}(\mathcal{F}, \mathcal{G})$ is a direct sum of copies of $\mathbf{H}^\bullet_B(\mathrm{pt}; \Bbbk)$ with even shifts. The lemma is proved in this case.

For general $\mathcal{F}$, by Lemma 7.3.3, we may suppose that there is a distinguished triangle $j_{w!}\underline{\Bbbk}_{\mathcal{B}_w}[2n] \to \mathcal{F} \to \mathcal{F}' \to$, where $\mathcal{F}'$ is also $*$-even, and for which the lemma is already known to hold. Consider the long exact sequence

$$\cdots \to \mathrm{Hom}(\mathcal{F}', \mathcal{G}[k]) \to \mathrm{Hom}(\mathcal{F}, \mathcal{G}[k]) \to \mathrm{Hom}(j_{w!}\underline{\Bbbk}_{\mathcal{B}_w}[2n], \mathcal{G}[k]) \to \cdots .$$

All three terms vanish when k is odd, by Lemma 7.3.4, so our long exact sequence breaks up into a collection of short exact sequences in even degrees. Thus, we have a short exact sequence of $\mathbf{H}^\bullet_B(\mathrm{pt}; \Bbbk)$-modules

$$0 \to \mathrm{Hom}^\bullet(\mathcal{F}', \mathcal{G}) \to \mathrm{Hom}^\bullet(\mathcal{F}, \mathcal{G}) \to \mathrm{Hom}^\bullet(j_{w!}\underline{\Bbbk}_{\mathcal{B}_w}[2n], \mathcal{G}) \to 0.$$

By induction, the first and last terms are free $\mathbf{H}^\bullet_B(\mathrm{pt}; \Bbbk)$-modules concentrated in even degrees, so the middle term is as well. $\square$

LEMMA 7.5.5. *Let $\mathcal{F}$ be an object of $\mathrm{Parity}_B(\mathcal{B}, \Bbbk)$, and let $s \in S$. Then $\mathcal{F} \star \mathrm{IC}_s$ is a parity sheaf.*

PROOF. Assume without loss of generality that $\mathcal{F}$ is even. By Proposition 7.3.6, $\mathcal{F} \star \mathrm{IC}_s$ is at least $*$-odd. By Lemma 7.3.2, $\mathbb{D}\mathcal{F}$ is again a parity sheaf, so $(\mathbb{D}\mathcal{F}) \star \mathrm{IC}_s \cong \mathbb{D}(\mathcal{F} \star \mathrm{IC}_s)(-1)$ is also $*$-odd. Thus, $\mathcal{F} \star \mathrm{IC}_s$ is $!$-odd, and hence a parity sheaf, as desired. $\square$

THEOREM 7.5.6. *Assume that* $\Bbbk$ *is a field or a complete noetherian local ring of finite global dimension.*

(1) *For each* $w \in W$*, there is, up to isomorphism, a* unique *indecomposable object* $\mathcal{E}_w(\Bbbk) \in \mathrm{Parity}_B(\mathcal{B}, \Bbbk)$ *that is supported on* $\overline{\mathcal{B}_w}$ *and satisfies*

$$\mathcal{E}_w(\Bbbk)|_{\mathcal{B}_w} \cong \underline{\Bbbk}_{\mathcal{B}_w}[\ell(w)].$$

(2) *Every indecomposable object in* $\mathrm{Parity}_B(\mathcal{B}, \Bbbk)$ *is isomorphic to* $\mathcal{E}_w(\Bbbk)[n]$ *for some* $w \in W$ *and some* $n \in \mathbb{Z}$*.*

PROOF. Our assumptions on $\Bbbk$ imply that $D^{\mathrm{b}}_B(\mathcal{B}, \Bbbk)$ is a Krull–Schmidt category (see Example A.8.6) and that $\mathrm{Parity}_B(\mathcal{B}, \Bbbk)$ is closed under direct summands (see the remarks following Definition 7.3.1). Therefore, $\mathrm{Parity}_B(\mathcal{B}, \Bbbk)$ is also Krull–Schmidt. We will establish the properties of the $\mathcal{E}_w$ in the following steps.

Step 1. Existence. Given $w \in W$, choose a reduced expression $w = s_1 \cdots s_k$. The skyscraper sheaf IC_e is clearly a parity sheaf, so by Lemma 7.5.5, the object $\mathrm{IC}_{s_1} \star \cdots \star \mathrm{IC}_{s_k}$ is as well. Of course, any direct summand of a parity sheaf is still a parity sheaf. By Lemma 7.2.9(2), every such summand is supported on $\overline{\mathcal{B}_w}$, and there exists a summand $\mathcal{E}_w$ such that $\mathcal{E}_w|_{\mathcal{B}_w} \cong \underline{\Bbbk}_{\mathcal{B}_w}[\ell(w)]$.

Step 2. Uniqueness. Suppose $\mathcal{E}$ and $\mathcal{E}'$ are two indecomposable parity sheaves that are both supported on $\overline{\mathcal{B}_w}$ and that satisfy $\mathcal{E}|_{\mathcal{B}_w} \cong \mathcal{E}'|_{\mathcal{B}_w} \cong \underline{\Bbbk}_{\mathcal{B}_w}[\ell(w)]$. Let $Z = \overline{\mathcal{B}_w} \smallsetminus \mathcal{B}_w$, and let $U = \mathcal{B} \smallsetminus Z$. Thus, U is a B-stable open subset of $\mathcal{B}$, and $\mathcal{B}_w$ is closed as a subset of U. Our assumptions imply that $\mathcal{E}|_U \cong \mathcal{E}'|_U$. By Lemma 7.5.2, there exists a map $\phi : \mathcal{E} \to \mathcal{E}'$ such that $\phi|_U : \mathcal{E}|_U \to \mathcal{E}'|_U$ is an isomorphism. There is also a map $\psi : \mathcal{E}' \to \mathcal{E}$ such that $\psi|_U$ is the inverse of $\phi|_U$.

Using Lemma 7.5.2 again, we have that the natural map

$$\mathrm{End}(\mathcal{E}) \to \mathrm{End}(\mathcal{E}|_U) \tag{7.5.3}$$

is surjective. Consider the element $\psi \circ \phi \in \mathrm{End}(\mathcal{E})$. Since its image in $\mathrm{End}(\mathcal{E}|_U)$ is the identity map, we have $\psi \circ \phi = \mathrm{id}_{\mathcal{E}} + m$ for some element m in the kernel of (7.5.3). Since $\mathcal{E}$ is indecomposable, $\mathrm{End}(\mathcal{E})$ is a local ring (see Corollary A.8.8), so m is contained in its unique maximal ideal. By general properties of local rings, $\psi \circ \phi$ is a unit in $\mathrm{End}(\mathcal{E})$; i.e., it is an automorphism of $\mathcal{E}$. Similarly, $\phi \circ \psi$ is an automorphism of $\mathcal{E}'$. From this, it can be deduced that ϕ and ψ are themselves isomorphisms. Thus, $\mathcal{E} \cong \mathcal{E}'$.

We have completed the proof of part (1) of the theorem. The remaining steps are concerned with the proof of part (2).

Step 3. Support. We will now show that the support of any indecomposable parity sheaf $\mathcal{E}$ is the closure of a single Schubert cell. If not, there are (at least) two different Schubert cells, say $\mathcal{B}_w$ and $\mathcal{B}_v$, that are open in its support. Here, w and v must be incomparable in the Bruhat order. Let

$$U = \bigcup_{\substack{y \in W \\ y \geq w \text{ or } y \geq v}} \mathcal{B}_y \qquad \text{and} \qquad Z = \mathcal{B} \smallsetminus U,$$

and let $i : Z \hookrightarrow \mathcal{B}$ and $j : U \hookrightarrow \mathcal{B}$ be the inclusion maps. Thus, U is a B-stable open subset, and $\mathcal{E}|_U$ is supported on the disconnected closed subset $\mathcal{B}_w \cup \mathcal{B}_v$. This implies that

$$\mathrm{End}(\mathcal{E}|_U) \cong \mathrm{End}(\mathcal{E}|_{\mathcal{B}_w}) \oplus \mathrm{End}(\mathcal{E}|_{\mathcal{B}_v}). \tag{7.5.4}$$

Once again, $\mathrm{End}(\mathcal{E})$ is a local ring. By Lemma 7.5.2, the ring $\mathrm{End}(\mathcal{E}|_U)$ is a quotient of $\mathrm{End}(\mathcal{E})$, so it too is a local ring. But this contradicts (7.5.4), whose right-hand side is manifestly not a local ring.

Step 4. Conclusion of the proof. Let $\mathcal{E}$ be an indecomposable parity sheaf, and suppose its support is $\overline{\mathcal{B}_w}$. Assume for simplicity that $\mathcal{E}$ is even. Let U be as in Step 2, so that $\mathcal{B}_w$ is a closed subset of U. As in Step 3, the ring $\mathrm{End}(\mathcal{E}|_U)$ must be a local ring (because it is a quotient of $\mathrm{End}(\mathcal{E})$), so $\mathcal{E}|_U$ is an indecomposable object. Since $\mathcal{E}|_U$ is supported on $\mathcal{B}_w$, we actually have that $\mathcal{E}|_{\mathcal{B}_w}$ is indecomposable. This object also satisfies the hypothesis of Lemma 7.5.3, which tells us that $\mathcal{E}|_{\mathcal{B}_w} \cong \underline{\Bbbk}_{\mathcal{B}_w}[n]$ for some n, as desired. ☐

EXAMPLE 7.5.7. We have $\mathcal{E}_e(\Bbbk) \cong \mathrm{IC}_e(\Bbbk)$. For any $s \in S$, (7.1.1) implies that $\mathcal{E}_s(\Bbbk) \cong \mathrm{IC}_s(\Bbbk)$.

For any $w \in W$, Corollary 7.3.7 says that $\mathrm{IC}_w(\mathbb{Q}) \cong \mathcal{E}_w(\mathbb{Q})$. More generally, this holds if $\mathbb{Q}$ is replaced by any field of characteristic 0. For general $\Bbbk$, however, $\mathrm{IC}_w(\Bbbk)$ and $\mathcal{E}_w(\Bbbk)$ may not be isomorphic.

For future reference, let us extract the conclusion of Step 1 of the proof of Theorem 7.5.6.

COROLLARY 7.5.8. *Assume that $\Bbbk$ is a field or a complete noetherian local ring of finite global dimension. Every indecomposable parity sheaf on $\mathcal{B}$ is a shift of a direct summand of some object of the form $\mathrm{IC}_{s_1} \star \cdots \star \mathrm{IC}_{s_k}$, where $s_1, \ldots, s_k \in S$.*

PROPOSITION 7.5.9. *Let $\Bbbk$ be a field or a complete noetherian local ring of finite global dimension. The category $\mathrm{Parity}_B(\mathcal{B}, \Bbbk) \subset D^{\mathrm{b}}_B(\mathcal{B}, \Bbbk)$ is closed under the convolution product.*

For $\Bbbk = \mathbb{Q}$, the argument below is an alternative proof of Proposition 7.2.6.

PROOF. Thanks to the classification of indecomposable parity sheaves in Theorem 7.5.6, it is enough to show that for any two elements $w, v \in W$, we have $\mathcal{E}_w \star \mathcal{E}_v \in \mathrm{Parity}_B(\mathcal{B}, \Bbbk)$. By Corollary 7.5.8, $\mathcal{E}_v$ is a direct summand of some object of the form $\mathrm{IC}_{s_1} \star \cdots \star \mathrm{IC}_{s_k}$, where $s_1, \ldots, s_k \in S$. Therefore, $\mathcal{E}_w \star \mathcal{E}_v$ is a direct summand of $\mathcal{E}_w \star \mathrm{IC}_{s_1} \star \cdots \star \mathrm{IC}_{s_k}$, and this is a parity sheaf by Lemma 7.5.5. ☐

The categorification theorem. Here is the main result of this section.

THEOREM 7.5.10. *Assume that $\Bbbk$ is a field or a complete noetherian local ring of finite global dimension. The map $\mathrm{ch} : \mathrm{K}_\oplus(\mathrm{Parity}_B(\mathcal{B}, \Bbbk)) \to \mathcal{H}$ given by*

$$\mathrm{ch}([\mathcal{F}]) = \sum_{\substack{w \in W \\ i \in \mathbb{Z}}} (\mathrm{rank}\, \mathsf{H}^i(\mathcal{F}|_{\mathcal{B}_w})) q^{i/2} T_w$$

is an isomorphism of rings. It satisfies

$$\mathrm{ch}([\mathcal{F}[1]]) = q^{-1/2}\mathrm{ch}([\mathcal{F}]) \qquad \text{and} \qquad \mathrm{ch}([\mathbb{D}(\mathcal{F})]) = \overline{\mathrm{ch}([\mathcal{F}])}.$$

The set $\{\mathrm{ch}([\mathcal{E}_w(\Bbbk)])\}_{w \in W}$ forms a $\mathbb{Z}[q^{\pm\frac{1}{2}}]$-basis for $\mathcal{H}$.

PROOF SKETCH. The proof is essentially identical to Steps 1–5 of the proof of Theorem 7.3.8. The only modifications are as follows:

- Every mention of $\mathrm{Semis}_B(\mathcal{B}, \mathbb{Q})$ should be replaced by $\mathrm{Parity}_B(\mathcal{B}, \Bbbk)$.

- Every mention of $\mathrm{IC}_w(\mathbb{Q})$ should be replaced by $\mathcal{E}_w(\Bbbk)$. However, in the special cases where $w = e$ or $w \in S$, one may instead use $\mathrm{IC}_w(\Bbbk)$, thanks to Example 7.5.7. □

The basis from Theorem 7.5.10 depends only on the (residue) characteristic of $\Bbbk$: see Exercise 7.5.2 below. Here is some terminology related to this basis.

Definition 7.5.11. Let p be 0 or a prime number, and let $\Bbbk$ be either a field of characteristic p or a complete noetherian local ring of finite global dimension and residue characteristic p. Let

$$ {}^p\underline{H}_w = \mathrm{ch}([\mathcal{E}_w(\Bbbk)]) \in \mathcal{H}. $$

The basis $\{{}^p\underline{H}_w\}_{w \in W}$ for $\mathcal{H}$ is called the **p-canonical basis**. Write it in terms of the standard basis as

$$ {}^p\underline{H}_w = q^{-\ell(w)/2} \sum_{x \in W} {}^pP_{x,w} T_w. $$

The coefficients ${}^pP_{x,w} \in \mathbb{Z}[q^{\pm 1}]$ are called **p-Kazhdan–Lusztig polynomials**.

Remark 7.5.12. The p-canonical basis has played an important role in recent developments related to the **Lusztig conjecture** for semisimple algebraic groups in characteristic p [**156**]. This conjecture proposed a character formula for certain irreducible representations in terms of affine Kazhdan–Lusztig polynomials, modeled on the Kazhdan–Lusztig conjecture for complex semisimple Lie algebras (see Remark 7.3.10). In the 1990s, work of Kazhdan–Lusztig [**131**], Kashiwara–Tanisaki [**127**], and Andersen–Jantzen–Soergel [**16**] established the Lusztig conjecture when p is "sufficiently large." Unfortunately, the method of proof in those papers gave no way to decide whether or not a given explicit prime number p was large enough. (Some years later, an explicit, but very large, bound was obtained by Fiebig [**74**].)

Around 2000, Soergel proposed a framework [**221**] for understanding a portion of the Lusztig conjecture in terms of ordinary (rather than affine) Kazhdan–Lusztig polynomials. Let $\Bbbk$ be an algebraically closed field of characteristic p, and let $\check{G}_\Bbbk$ be the reductive algebraic group over $\Bbbk$ that is Langlands dual to the (complex) reductive group G. Assume that p is larger than the Coxeter number h for $\check{G}_\Bbbk$. In modern language, the main result of [**221**] can be stated as follows: "If the Lusztig character formula holds for $\check{G}_\Bbbk$ in characteristic p, then the p-canonical basis for $\mathcal{H}$ coincides with the canonical basis." To check the latter condition in a specific example, one must compute the indecomposable parity sheaves on $\mathcal{B}$.

It is not difficult to find examples for small p where the p-canonical basis differs from the canonical basis (see Exercise 7.7.7), but it was widely expected that they would agree when $p > h$. However, in 2013, Williamson announced [**245**] the discovery of many examples of p-canonical basis elements for GL_n with $p > n$ that differ from the corresponding canonical basis elements. See [**244**] for a sheaf-theoretic account of these examples, and see [**12, 193, 194**] for further results on the p-canonical basis in representation theory.

Exercises.

7.5.1. Let $\varphi : \Bbbk \to \Bbbk'$ be a ring homomorphism, where each of $\Bbbk$ and $\Bbbk'$ is a field or a complete noetherian local ring of finite global dimension.

(a) Show that if $\mathcal{F}$ is $*$-even, resp. !-even, then $\Bbbk' \otimes^L_{\Bbbk} \mathcal{F}$ has the same property. As a consequence, $\Bbbk' \otimes^L_{\Bbbk} (-)$ sends parity sheaves to parity sheaves.

(b) Show that if that $\mathcal{F}$ is $*$-even and $\mathcal{G}$ is !-even, then

$$\mathrm{Hom}^\bullet(\Bbbk' \overset{L}{\otimes}_{\Bbbk} \mathcal{F}, \Bbbk' \overset{L}{\otimes}_{\Bbbk} \mathcal{G}) \cong \Bbbk' \otimes_{\Bbbk} \mathrm{Hom}^\bullet(\mathcal{F}, \mathcal{G}).$$

7.5.2. Let $\varphi : \Bbbk \to \Bbbk'$ be a ring homomorphism satisfying one of the following conditions:

(a) both $\Bbbk$ and $\Bbbk'$ are fields (of the same characteristic) or

(b) $\Bbbk$ is a complete local ring of finite global dimension, and $\Bbbk'$ is its residue field.

Show that for all $w \in W$, we have $\Bbbk' \otimes^L_{\Bbbk} \mathcal{E}_w(\Bbbk) \cong \mathcal{E}_w(\Bbbk')$. As a consequence, the p-canonical basis really is independent of the choice of coefficient ring $\Bbbk$ in Definition 7.5.11.

7.5.3. Show that for any $w \in W$ and any prime $p > 0$, we have

$${}^p\underline{H}_w = \underline{H}_w + \sum_{y<w} a_{y,w}\underline{H}_y$$

where $a_{y,w} \in \mathbb{Z}[q^{\pm\frac{1}{2}}]$ satisfies $\overline{a_{y,w}} = a_{y,w}$. Show, moreover, that the coefficients of $a_{y,w}$ are nonnegative integers.

7.5.4. Let p be a prime number. Show that the following conditions on $w \in W$ are equivalent (cf. Exercise 3.3.6):

(a) The stalks and costalks of $\mathrm{IC}_w(\mathbb{Z})$ have no p-torsion.

(b) We have $\mathcal{E}_w(\mathbb{F}_p) \cong \mathrm{IC}_w(\mathbb{F}_p)$.

(c) We have ${}^p\underline{H}_w = \underline{H}_w$.

7.6. Soergel bimodules

In this section, we study one more categorification of the Hecke algebra, based on the equivariant cohomology of parity sheaves, studied in the guise of certain bimodules over the ring

$$R = \mathbf{H}^\bullet_B(\mathrm{pt}; \Bbbk) \cong \mathbf{H}^\bullet_T(\mathrm{pt}; \Bbbk) \cong \mathrm{Sym}(\Bbbk \otimes_{\mathbb{Z}} X(T)).$$

(We will suppress Tate twists throughout this section.) These bimodules (known as **Soergel bimodules**) were introduced in the case $\Bbbk = \mathbb{C}$ in [**219**] (following earlier work in [**218**]). The content of this section is based on the setting considered in [**221**]. See Remark 7.6.13 for additional context.

We assume throughout this section that $\Bbbk$ is a field whose characteristic obeys the following condition (depending on the group G under consideration):

(7.6.1) The characteristic of $\Bbbk$ is not 2. If the root system of G contains a factor of type G_2, then the characteristic of $\Bbbk$ is also not 3.

This condition has the following consequence.

LEMMA 7.6.1. *Let G be a connected reductive group, and let $\Bbbk$ be a field satisfying* (7.6.1). *For any two distinct roots $\alpha, \beta \in \mathbf{X}(T)$, their images in $\Bbbk \otimes_{\mathbb{Z}} \mathbf{X}(T)$ remain distinct. In particular, the image of any root α in $\Bbbk \otimes_{\mathbb{Z}} \mathbf{X}(T)$ is nonzero.*

See Exercise 7.6.1 for a sketch of the proof.

Preliminaries on Soergel bimodules. There is a natural action of W on R, induced by its action on $\mathbf{X}(T)$. If $s \in S$ is a simple reflection, let

$$R^s = \{r \in R \mid s \cdot r = r\}$$

be the subring of s-invariants in R.

Let (R, R)-gmod denote the category of graded (R, R)-bimodules. Given two graded (R, R)-bimodules M and N, one can form the new (R, R)-bimodule

$$M \otimes_R N,$$

where the tensor product identifies the right R-action on M with the left R-action on N. For any graded (R, R)-bimodule $M = \bigoplus_{n \in \mathbb{Z}} M^n$, let $M[1]$ be the graded (R, R)-bimodule given by $M[1]^n = M^{n+1}$. Here are some examples of (R, R)-bimodules:

(1) Of course, R itself is an (R, R)-bimodule in the obvious way.
(2) For $s \in S$, let B_s denote the (R, R)-bimodule $(R \otimes_{R^s} R)[1]$.
(3) Let $w \in W$. Let R_w be the (R, R)-bimodule whose underlying left R-module is just R itself, and on which the right R-action is defined as follows: for $r \in R$ and $m \in R_w$, $m \cdot r = (wr) \cdot m$.

Definition 7.6.2. The category of **Soergel bimodules**, denoted by SBim, is the smallest strictly full additive subcategory of (R, R)-gmod with the following properties:

(1) It is Karoubian and closed under direct sums, $[1]$, and $(-) \otimes_R (-)$.
(2) It contains R and B_s for all $s \in S$.

Below are some preliminary lemmas about (R, R)-bimodules.

Lemma 7.6.3. *Let $w \in W$, and assume that $w \neq e$. For any $n \in \mathbb{Z}$, we have $\mathrm{Hom}_{R \otimes R}(R, R_w[n]) = 0$.*

Proof. Suppose we had a nonzero (R, R)-bimodule homomorphism $f : R \to R_w[n]$. Choose a root α such that $w\alpha \neq \alpha$. The left and right R-actions show that $f(\alpha) = \alpha f(1)$ and $f(\alpha) = (w\alpha) f(1)$, so $(w\alpha - \alpha) f(1) = 0$. Since R is a domain, this implies that $w\alpha - \alpha = 0$ in $\Bbbk \otimes_{\mathbb{Z}} \mathbf{X}(T)$, contradicting Lemma 7.6.1. $\square$

In the following lemma, regard R_s as a ring with the same ring structure as R.

Lemma 7.6.4. *Let $s \in S$, and let α be the corresponding simple root.*

(1) *The left R-algebra homomorphisms*
$$\psi_1 : R \to R/\alpha \qquad \textit{and} \qquad \psi_2 : R_s \to R/\alpha,$$
both given by $r \mapsto r + \alpha R$, are in fact (R, R)-bimodule homomorphisms.
(2) *There are unique (R, R)-bimodule homomorphisms*
$$\phi_1 : R \otimes_{R^s} R \to R \qquad \textit{and} \qquad \phi_2 : R \otimes_{R^s} R \to R_s$$
that are also ring homomorphisms.
(3) *These maps fit into a short exact sequence of (R, R)-bimodules*
$$0 \to R \otimes_{R^s} R \xrightarrow{\phi = \left[\begin{smallmatrix} \phi_1 \\ -\phi_2 \end{smallmatrix}\right]} R \oplus R_s \xrightarrow{\psi = [\,\psi_1 \ \psi_2\,]} R/\alpha \to 0.$$

Proof sketch. (1) The only nontrivial content is the assertion that ψ_2 is a right R-module homomorphism. For any $r \in R$, recall that $r - sr \in \alpha R$. We study the right R-action for $m \in R_s$ and $r \in R$ as follows:

$$\psi_2(m \cdot r) = \psi_2((sr)m) = (sr)m + \alpha R = (sr)m + (r - rs)m + \alpha R = (m + \alpha R)r.$$

(2) The desired maps are given by $\phi_1(r_1 \otimes r_2) = r_1 r_2$ and $\phi_2(r_1 \otimes r_2) = r_1(sr_2)$.

(3) It is clear that ψ is surjective. Moreover, we have

$$\psi(\phi(r_1 \otimes r_2)) = r_1 r_2 - r_1(sr_2) + \alpha R = 0,$$

since $r_2 - sr_2 \in \alpha R$. Thus, ϕ at least defines a map $R \otimes_{R^s} R \to \ker \psi$. Next, note that for $(r_1, r_2) \in \ker \psi$, we have $r_1 + r_2 \in \alpha R$, so the element $\frac{r_1 + r_2}{\alpha} \in R$ makes sense. Define a map $\phi' : \ker \psi \to R \otimes_{R^s} R$ by

$$\phi'(r_1, r_2) = \tfrac{1}{2}(\tfrac{r_1 + r_2}{\alpha} \otimes \alpha + (r_1 - r_2) \otimes 1).$$

It is left to the reader to check that ϕ' is an (R, R)-bimodule homomorphism and that it is inverse to ϕ'. □

Lemma 7.6.5. (1) *Every Soergel bimodule is finitely generated as a left (or right) R-module.*
(2) *For $M, N \in \mathrm{SBim}$, $\mathrm{Hom}(M, N)$ is a finite-dimensional $\Bbbk$-vector space.*
(3) *The category* SBim *is Krull–Schmidt.*

Proof. (1) Lemma 7.6.4(3) shows that B_s is finitely generated as a left (or right) R-module, and the property of being finitely generated as a left (or right) R-module is preserved when taking direct summands or direct sums, and when applying [1] or $(-) \otimes_R (-)$. The desired claim then holds by the definition of SBim.

(1) Each graded component of a finitely generated (R, R)-bimodule is finite-dimensional, so this follows from part (1).

(3) The category SBim is Karoubian by definition. For every $M \in \mathrm{SBim}$, the ring $\mathrm{End}(M)$ is a finite-dimensional $\Bbbk$-algebra, and hence a semiperfect ring, by [**144**, Example 23.3] (see also Example A.8.6). By Proposition A.8.5, SBim is Krull–Schmidt. □

Equivariant cohomology. In addition to the usual ("left") flag variety $\mathscr{B} = G/B$, we will also consider the "right" flag variety $\mathscr{B}' = B\backslash G$. Let

$$\pi : G \to G/B \qquad \text{and} \qquad \pi' : G \to B\backslash G$$

be the quotient maps. Let $B \times B$ act on G by $(b_1, b_2) \cdot g = b_1 g b_2^{-1}$. By Theorem 6.5.9, the functors π^* and π'^* give rise to equivalences of categories

$$D^{\mathrm{b}}_B(\mathscr{B}, \Bbbk) \xrightarrow{\sim} D^{\mathrm{b}}_{B \times B}(G, \Bbbk) \xleftarrow{\sim} D^{\mathrm{b}}_B(\mathscr{B}', \Bbbk). \tag{7.6.2}$$

For any $\tilde{\mathcal{F}} \in D^{\mathrm{b}}_{B \times B}(G, \Bbbk)$, the equivariant cohomology $\mathbf{H}^\bullet_{B \times B}(G, \tilde{\mathcal{F}})$ has the structure of a $\mathbf{H}^\bullet_{B \times B}(\mathrm{pt}; \Bbbk)$-module. By Proposition 6.7.5, we have

$$\mathbf{H}^\bullet_{B \times B}(\mathrm{pt}; \Bbbk) \cong \mathbf{H}^\bullet_B(\mathrm{pt}; \Bbbk) \otimes \mathbf{H}^\bullet_B(\mathrm{pt}; \Bbbk) = R \otimes R.$$

We define a functor

$$\mathbb{H} : D^{\mathrm{b}}_B(G/B, \Bbbk) \to (R, R)\text{-gmod} \qquad \text{by} \qquad \mathbb{H}(\mathcal{F}) = \mathbf{H}^\bullet_{B \times B}(G, \pi^* \mathcal{F}).$$

The main goal of this section is to study this functor on parity sheaves.

Let $\mathcal{F} \in D^{\mathrm{b}}_B(\mathscr{B}, \Bbbk)$, and let $\mathcal{F}'$ be the corresponding object in $D^{\mathrm{b}}_B(\mathscr{B}', \Bbbk)$ (so that $\pi^* \mathcal{F} \cong \pi'^* \mathcal{F}'$). The equivalences (7.6.2) induce isomorphisms of $\Bbbk$-modules

$$\mathbf{H}^\bullet_B(\mathscr{B}, \mathcal{F}) \cong \mathbf{H}^\bullet_{B \times B}(G, \pi^* \mathcal{F}) \cong \mathbf{H}^\bullet_B(\mathscr{B}', \mathcal{F}'). \tag{7.6.3}$$

As noted above, the middle term is naturally an $R \otimes R$-module, but the left- and right-hand terms are only modules over one copy of R. We have

$$\mathbf{H}^\bullet_B(\mathcal{B}, \mathcal{F}) \cong \mathbb{H}(\mathcal{F}), \text{ regarded as a module over } R \otimes \Bbbk \subset R \otimes R,$$
$$\mathbf{H}^\bullet_B(\mathcal{B}', \mathcal{F}') \cong \mathbb{H}(\mathcal{F}), \text{ regarded as a module over } \Bbbk \otimes R \subset R \otimes R.$$

LEMMA 7.6.6. *For any $w \in W$, we have $\mathbb{H}(j_{w*}\underline{\Bbbk}_{\mathcal{B}_w}) \cong R_w$.*

PROOF. From the definitions, we have $\mathbb{H}(j_{w*}\underline{\Bbbk}_{\mathcal{B}_w}) \cong \mathbf{H}^\bullet_{B\times B}(B\dot{w}B; \Bbbk)$. By Theorem 6.6.15, we may instead work with $T\times T$-equivariant cohomology, where $T\times T$ acts on $B\dot{w}B$ by $(t_1, t_2)\cdot g = t_1 g t_2^{-1}$. Let U be the unipotent radical of B, and let $\bar{U}$ be the unipotent radical of the opposite Borel subgroup. Let $U_w = \dot{w}^{-1}\bar{U}\dot{w} \cap U$. By [**230**, Lemma 8.3.6(ii)], the multiplication map

$$f : U_w \times \dot{w}T \times U \to B\dot{w}B \qquad \text{given by} \qquad f(u_1, \dot{w}t, u_2) = u_1\dot{w}tu_2$$

is an isomorphism of varieties. Let $T \times T$ act on $U_w \times \dot{w}T \times U$ by

$$(t_1, t_2) \cdot (u_1, \dot{w}t, u_2) = (t_1 u t_1^{-1}, t_1\dot{w}tt_2^{-1}, t_2u_2t_2^{-1}).$$

Then f is $T\times T$-equivariant. The middle part of this formula defines a $T\times T$-action on $\dot{w}T$. With respect to this action, the projection map

$$g : U_w \times \dot{w}T \times U \to \dot{w}T \qquad \text{given by} \qquad g(u_1, \dot{w}t, u_2) = \dot{w}$$

is also $T\times T$-equivariant. Since U_w and U are contractible (they are affine spaces), we have

$$g_*\underline{\Bbbk}_{U_w\times\dot{w}T\times U} \cong \underline{\Bbbk}_{\dot{w}T}.$$

The reasoning gives us a sequence of natural isomorphisms

$$\begin{aligned}\mathbb{H}(j_{w*}\underline{\Bbbk}_{\mathcal{B}_w}) \cong \mathbf{H}^\bullet_{B\times B}(B\dot{w}B;\Bbbk) &\cong \mathbf{H}^\bullet_{T\times T}(U_w \times \dot{w}T \times U; \Bbbk)\\ &\cong \mathbf{H}^\bullet_{T\times T}(\dot{w}T, g_*\underline{\Bbbk}_{U_w\times\dot{w}T\times U}) \cong \mathbf{H}^\bullet_{T\times T}(\dot{w}T;\Bbbk).\end{aligned}$$

Now, let $T^w = \{(t_1, t_2) \in T \times T \mid t_1 = \dot{w}t_2\dot{w}^{-1}\}$. The action of $T \times T$ on $\dot{w}T$ gives rise to a $T \times T$-equivariant isomorphism of varieties

$$(T \times T)/T^w \xrightarrow{\sim} \dot{w}T.$$

By Theorem 6.5.10, we deduce that there are natural isomorphisms

$$\mathbf{H}^\bullet_{T\times T}(\dot{w}T;\Bbbk) \cong \mathbf{H}^\bullet_{T\times T}((T\times T)/T^w;\Bbbk) \cong \mathbf{H}^\bullet_{T^w}(\mathrm{pt};\Bbbk).$$

The structure of this space as a $\mathbf{H}^\bullet_{T\times T}(\mathrm{pt};\Bbbk)$-module comes from the ring homomorphism

$$R \otimes R = \mathbf{H}^\bullet_{T\times T}(\mathrm{pt};\Bbbk) \to \mathbf{H}^\bullet_{T^w}(\mathrm{pt};\Bbbk)$$

induced by the inclusion $\iota : T^w \hookrightarrow T \times T$. Consider the projection maps $\mathrm{pr}_1, \mathrm{pr}_2 : T \times T \to T$. Identify T^w with T via the isomorphism $\mathrm{pr}_1 \circ \iota : T^w \xrightarrow{\sim} T$. Then $\mathbf{H}^\bullet_{T^w}(\mathrm{pt};\Bbbk)$ is identified with R, and the left copy of R in $R\otimes R$ acts on it in the obvious way. The map $\mathrm{pr}_2 \circ \iota : T^w \to T$ is then identified with the map $T \to T$ given by $t \mapsto \dot{w}^{-1}t\dot{w}$. The induced map on characters $\mathbf{X}(\mathrm{pr}_2 \circ \iota) : \mathbf{X}(T) \to \mathbf{X}(T)$ is the action of w. By Theorem 6.7.7, the right copy of R in $R\otimes R$ acts on $\mathbf{H}^\bullet_{T^w}(\mathrm{pt};\Bbbk) \cong R$ by the formula $m \cdot r = (wr) \cdot m$, as desired. ☐

LEMMA 7.6.7. *For any $s \in S$, there is a canonical isomorphism $\mathbb{H}(\mathcal{E}_s) \cong \mathsf{B}_s$.*

PROOF. This statement is equivalent to showing that $\mathbf{H}^\bullet_B(\overline{\mathscr{B}_s}, \Bbbk) \cong R \otimes_{R^s} R$. Recall that $\overline{\mathscr{B}_s}$ is isomorphic to $\mathbb{P}^1$. There are exactly two T-fixed points on $\overline{\mathscr{B}_s}$: namely, $1B$ and $\dot{s}B$. Let $U_1 = \overline{\mathscr{B}_s} \smallsetminus \{\dot{s}B\}$, and let $U_s = \mathscr{B}_s = \overline{\mathscr{B}_s} \smallsetminus \{1B\}$. By Lemma 7.6.6, we have

$$\mathbf{H}^\bullet_B(U_s; \Bbbk) = \mathbf{H}^\bullet_B(\mathscr{B}_s; \Bbbk) \cong R_s. \tag{7.6.4}$$

Next, let $i_1 : \{1B\} \hookrightarrow \overline{\mathscr{B}_s}$ be the inclusion map, and recall that $i_1^! \underline{\Bbbk}_{\overline{\mathscr{B}_s}} \cong \underline{\Bbbk}_{\mathrm{pt}}[-2]$. The long exact hypercohomology sequence associated to $i_* \underline{\Bbbk}_{\mathscr{B}_1}[-2] \to \underline{\Bbbk}_{\overline{\mathscr{B}_s}} \to j_{s*} \underline{\Bbbk}_{\mathscr{B}_s} \to$, together with Lemma 7.1.1 or Lemma 7.6.6, shows that

$$\mathbf{H}^k_B(\overline{\mathscr{B}_s}; \Bbbk) = 0 \qquad \text{if } k \text{ is odd.}$$

As in the proof of Lemma 7.1.1, $1B$ is the unique fixed point in U_1 for the action of $\mathbb{G}_\mathrm{m}$ through a suitable "generic" cocharacter $\mathbb{G}_\mathrm{m} \to T$, so Theorem 2.10.3 implies that $\mathrm{a}_{U_1 *} \underline{\Bbbk}_{U_1} \cong i_1^* \underline{\Bbbk}_{U_1} \cong \underline{\Bbbk}_{\mathrm{pt}}$. We deduce that

$$\mathbf{H}^\bullet_B(U_1; \Bbbk) \cong \mathbf{H}^\bullet_B(\mathscr{B}_1; \Bbbk) \cong R. \tag{7.6.5}$$

Finally, let $U = U_1 \cap U_s$. This is isomorphic to $\mathbb{G}_\mathrm{m}$. It can be deduced from the Bruhat decomposition that there is a T-equivariant isomorphism

$$T/\ker\alpha \xrightarrow{\sim} U,$$

where α is the simple root corresponding to s. Now, $\ker\alpha$ may be a disconnected group. Characters of the finite abelian group $(\ker\alpha)/(\ker\alpha)^\circ$ are given by the torsion subgroup of $\mathbf{X}(T)/(\alpha)$. The order of this group must be invertible in $\Bbbk$, as otherwise the image of α in $\Bbbk \otimes_\mathbb{Z} \mathbf{X}(T)$ would be 0, contradicting Lemma 7.6.1. As a consequence, using Exercise 6.7.2 and Theorem 6.7.7, we find that

$$\mathbf{H}^\bullet_T(U; \Bbbk) \cong \mathbf{H}^\bullet_{\ker\alpha}(\mathrm{pt}; \Bbbk) \cong \mathbf{H}^\bullet_{(\ker\alpha)^\circ}(\mathrm{pt}; \Bbbk) \cong R/\alpha. \tag{7.6.6}$$

We will now use the Mayer–Vietoris sequence associated to the covering $\overline{\mathscr{B}_s} = U_1 \cup U_s$ to compute the cohomology of $\overline{\mathscr{B}_s}$. The discussion above shows that all the cohomology groups we are interested in vanish in odd degrees, so the long exact sequence is actually a short exact sequence

$$0 \to \mathbf{H}^\bullet_B(\overline{\mathscr{B}_s}; \Bbbk) \to \mathbf{H}^\bullet_B(U_1; \Bbbk) \oplus \mathbf{H}^\bullet_B(U_s; \Bbbk) \to \mathbf{H}^\bullet_B(U; \Bbbk) \to 0. \tag{7.6.7}$$

By Remark 1.3.14, the second map in this short exact sequence is the sum of the natural maps $\mathbf{H}^\bullet_B(U_1; \Bbbk) \to \mathbf{H}^\bullet_B(U; \Bbbk)$ and $\mathbf{H}^\bullet_B(U_s; \Bbbk) \to \mathbf{H}^\bullet_B(U; \Bbbk)$. Both of these maps are ring homomorphisms. Moreover, every map in (7.6.7) is an (R, R)-bimodule homomorphism. From these observations combined with (7.6.4), (7.6.5), and (7.6.6), we see that (7.6.7) must be identified with the short exact sequence from Lemma 7.6.4(3). □

LEMMA 7.6.8. *For any parity sheaf $\mathcal{F} \in \mathrm{Parity}_B(\mathscr{B}, \Bbbk)$, the module $\mathbb{H}(\mathcal{F})$ is free as a left R-module, and also free as a right R-module.*

PROOF. Assume without loss of generality that $\mathcal{F}$ is even. To study the left R-module structure, we may replace $\mathbb{H}(\mathcal{F})$ by $\mathbf{H}^\bullet_B(\mathscr{B}, \mathcal{F}) \cong \mathrm{Hom}^\bullet(\underline{\Bbbk}_\mathscr{B}, \mathcal{F})$. The claim then holds by Lemma 7.5.4.

All the results of Section 7.5 have obvious analogues for $\mathscr{B}'$. If we let $\mathcal{F}' \in \mathrm{Parity}_B(\mathscr{B}', \Bbbk)$ be the object corresponding to $\mathcal{F}$ as in (7.6.3), then the same reasoning as above yields the claim about the right R-module structure. □

Proposition 7.6.9. *Let $\mathcal{F}, \mathcal{G} \in \mathrm{Parity}_B(\mathcal{B}, \Bbbk)$. Then there is a natural isomorphism*
$$\mathbb{H}(\mathcal{F} \star \mathcal{G}) \cong \mathbb{H}(\mathcal{F}) \otimes_R \mathbb{H}(\mathcal{G}).$$

Proof. Let us revisit the convolution diagram (7.2.1). We will need to distinguish three different copies of B acting in various ways, so we will denote them by B_1, B_2, B_3. Let $B_1 \times B_2 \times B_3$ act on $G \times G/B$ by
$$(b_1, b_2, b_3) \cdot (g_1, g_2 b) = (b_1 g_1 b_2^{-1}, b_3 g_2 b).$$
We also let B_Δ be the diagonal copy of B inside $B_2 \times B_3$:
$$B_\Delta = \{(b_2, b_3) \in B_2 \times B_3 \mid b_2 = b_3\}.$$
We now consider the diagram
$$\begin{tikzcd} G/B_2 \times G/B & G \times G/B \arrow[l, "p"'] \arrow[r, "q"] \arrow[d, "p'"'] & G \times^{B_\Delta} G/B \arrow[r, "m"] & G/B \\ & B_1\backslash G \times G/B & & \end{tikzcd}$$

Suppose $\mathcal{F} \in D^{\mathrm{b}}_{B_1}(G/B_2, \Bbbk)$ and $\mathcal{G} \in D^{\mathrm{b}}_{B_3}(G/B, \Bbbk)$. Let $\mathcal{F}' \in D^{\mathrm{b}}_{B_2}(B_1\backslash G, \Bbbk)$ be the object corresponding to $\mathcal{F}$ as in (7.6.3). Since the composition $B_1 \times B_\Delta \to B_1 \times B_2 \times B_3 \to B_1 \times B_3$ is an isomorphism, we can rephrase the definition of $\mathcal{F} \tilde{\boxtimes} \mathcal{G}$ as follows: it is the object characterized by an isomorphism
$$\mathrm{For}^{B_1 \times B_2 \times B_3}_{B_1 \times B_\Delta} p^* \mathrm{Infl}^{B_1 \times B_2 \times B_3}_{B_1 \times B_3}(\mathcal{F} \boxtimes \mathcal{G}) \cong q^* \mathrm{Infl}^{B_1 \times B_\Delta}_{B_1}(\mathcal{F} \tilde{\boxtimes} \mathcal{G}).$$
Let $\mathcal{H} = p^* \mathrm{Infl}^{B_1 \times B_2 \times B_3}_{B_1 \times B_3}(\mathcal{F} \boxtimes \mathcal{G})$. From the definition of convolution, we have
$$\begin{aligned} \mathbb{H}(\mathcal{F} \star \mathcal{G}) &\cong \mathbf{H}^\bullet_{B_1}(G/B, \mathcal{F} \star \mathcal{G}) \cong \mathbf{H}^\bullet_{B_1}(G \times^{B_\Delta} G/B, \mathcal{F} \tilde{\boxtimes} \mathcal{G}) \\ &\cong \mathbf{H}^\bullet_{B_1 \times B_\Delta}(G \times G/B, \mathrm{For}^{B_1 \times B_2 \times B_3}_{B_1 \times B_\Delta}(\mathcal{H})). \end{aligned} \tag{7.6.8}$$
Next, consider the equivalences of categories
$$(p')^* \mathrm{Infl}^{B_1 \times B_2 \times B_3}_{B_2 \times B_3} : D^{\mathrm{b}}_{B_2 \times B_3}(B_1\backslash G \times G/B, \Bbbk) \to D^{\mathrm{b}}_{B_1 \times B_2 \times B_3}(G \times G/B, \Bbbk),$$
$$(p')^* \mathrm{Infl}^{B_1 \times B_\Delta}_{B_\Delta} : D^{\mathrm{b}}_{B_\Delta}(B_1\backslash G \times G/B, \Bbbk) \to D^{\mathrm{b}}_{B_1 \times B_\Delta}(G \times G/B, \Bbbk).$$
We have $\mathcal{H} \cong (p')^* \mathrm{Infl}^{B_1 \times B_2 \times B_3}_{B_2 \times B_3}(\mathcal{F}' \boxtimes \mathcal{G})$. Continuing the calculation from (7.6.8), we have
$$\mathbb{H}(\mathcal{F} \star \mathcal{G}) \cong \mathbf{H}^\bullet_{B_\Delta}(B_1\backslash G \times G/B, \mathrm{For}^{B_2 \times B_3}_{B_\Delta}(\mathcal{F}' \boxtimes \mathcal{G})).$$
Thanks to Lemma 7.6.8, Proposition 6.7.5 tells us that
$$\mathbf{H}^\bullet_{B_2 \times B_3}(B_1\backslash G \times G/B, \mathcal{F}' \boxtimes \mathcal{G}) \cong \mathbf{H}^\bullet_{B_2}(B_1\backslash G, \mathcal{F}') \otimes_\Bbbk \mathbf{H}^\bullet_{B_3}(G/B, \mathcal{G}).$$
In particular, this cohomology is free as a module over $\mathbf{H}^\bullet_{B_2 \times B_3}(\mathrm{pt}; \Bbbk)$. Then Lemma 6.7.4 tells us how to apply $\mathrm{For}^{B_2 \times B_3}_{B_\Delta}$. Writing R_Δ, R_2, and R_3 for the various copies of $\mathbf{H}^\bullet_B(\mathrm{pt}; \Bbbk)$, we have
$$\begin{aligned} \mathbb{H}(\mathcal{F} \star \mathcal{G}) &\cong \mathbf{H}^\bullet_{B_\Delta}(\mathrm{pt}; \Bbbk) \underset{\mathbf{H}^\bullet_{B_2 \times B_3}(\mathrm{pt}; \Bbbk)}{\otimes} (\mathbf{H}^\bullet_{B_2}(B_1\backslash G, \mathcal{F}') \otimes_\Bbbk \mathbf{H}^\bullet_{B3}(G/B, \mathcal{G})) \\ &\cong R_\Delta \underset{R_2 \otimes R_3}{\otimes} (\mathbb{H}(\mathcal{F}) \otimes_\Bbbk \mathbb{H}(\mathcal{G})) \cong \mathbb{H}(\mathcal{F}) \otimes_R \mathbb{H}(\mathcal{G}), \end{aligned}$$
as desired. $\square$

PROPOSITION 7.6.10. *For $\mathcal{F}, \mathcal{G} \in D^{\mathrm{b}}_B(\mathscr{B}, \Bbbk)$ and $s \in S$, there is a natural commutative diagram*

$$\begin{tikzcd}
\mathrm{Hom}_{D^{\mathrm{b}}_B(\mathscr{B},\Bbbk)}(\mathcal{F} \star \mathcal{E}_s, \mathcal{G}) \arrow[r] \arrow[d, "\wr"'] & \mathrm{Hom}_{R\otimes R}(\mathbb{H}(\mathcal{F}) \otimes_R \mathsf{B}_s, \mathbb{H}(\mathcal{G})) \arrow[d, "\wr"] \\
\mathrm{Hom}_{D^{\mathrm{b}}_B(\mathscr{B},\Bbbk)}(\mathcal{F}, \mathcal{G} \star \mathcal{E}_s) \arrow[r] & \mathrm{Hom}_{R\otimes R}(\mathbb{H}(\mathcal{F}), \mathbb{H}(\mathcal{G}) \otimes_R \mathsf{B}_s)
\end{tikzcd}$$

in which the vertical arrows are isomorphisms.

PROOF. Let us first establish the isomorphism along the left side of this diagram. We must show that the functor $(-)\star\mathcal{E}_s$ is its own adjoint (on both sides). This follows from Lemma 7.2.8: since $\pi_s : G/B \to G/P_s$ is proper and smooth of relative dimension 1, the right adjoint to $(-) \star \mathcal{E}_s \cong \pi_s^* \circ \pi_{s*}[1]$ is $\pi_s^! \circ \pi_{s*}[-1] \cong \pi_s^* \circ \pi_{s*}[1]$.

This adjunction gives us a unit natural transformation $\eta : \mathrm{id} \to (-) \star \mathcal{E}_s \star \mathcal{E}_s$ and a counit natural transformation $\epsilon : (-) \star \mathcal{E}_s \star \mathcal{E}_s \to \mathrm{id}$. By applying these to the skyscraper sheaf $\mathcal{E}_e$, we may think of them as ordinary morphisms

$$\eta : \mathcal{E}_e \to \mathcal{E}_s \star \mathcal{E}_s, \qquad \epsilon : \mathcal{E}_s \star \mathcal{E}_s \to \mathcal{E}_e$$

in $D^{\mathrm{b}}_B(\mathscr{B}, \Bbbk)$. The unit–counit equations say that

$$(\mathcal{E}_s \star \epsilon) \circ (\eta \star \mathcal{E}_s) = (\epsilon \star \mathcal{E}_s) \circ (\mathcal{E}_s \star \eta) = \mathrm{id}_{\mathcal{E}_s}.$$

Apply $\mathbb{H}$ to the equations above. In view of Lemma 7.6.7 and Proposition 7.6.9, we get maps

$$\eta : R \to \mathsf{B}_s \otimes_R \mathsf{B}_s, \qquad \epsilon : \mathsf{B}_s \otimes_R \mathsf{B}_s \to R$$

that again satisfy the unit–counit equations:

$$(\mathsf{B}_s \otimes_R \epsilon) \circ (\eta \otimes_R \mathsf{B}_s) = (\epsilon \otimes_R \mathsf{B}_s) \circ (\mathsf{B}_s \otimes_R \eta) = \mathrm{id}_{\mathsf{B}_s}.$$

Therefore, the functor $(-) \otimes_R \mathsf{B}_s$ on the category of (R, R)-bimodules is its own adjoint.

Now consider the diagram

$$\begin{tikzcd}
\mathrm{Hom}_{D^{\mathrm{b}}_B(\mathscr{B},\Bbbk)}(\mathcal{F} \star \mathcal{E}_s, \mathcal{G}) \arrow[r] \arrow[d, "(-)\star\mathcal{E}_s"'] & \mathrm{Hom}_{R\otimes R}(\mathbb{H}(\mathcal{F}) \otimes_R \mathsf{B}_s, \mathbb{H}(\mathcal{G})) \arrow[d, "(-)\otimes_R \mathsf{B}_s"] \\
\mathrm{Hom}_{D^{\mathrm{b}}_B(\mathscr{B},\Bbbk)}(\mathcal{F} \star \mathcal{E}_s \star \mathcal{E}_s, \mathcal{G} \star \mathcal{E}_s) \arrow[r] \arrow[d, "\eta"'] & \mathrm{Hom}_{R\otimes R}(\mathbb{H}(\mathcal{F}) \otimes_R \mathsf{B}_s \otimes_R \mathsf{B}_s, \mathbb{H}(\mathcal{G}) \otimes_R \mathsf{B}_s) \arrow[d, "\eta"] \\
\mathrm{Hom}_{D^{\mathrm{b}}_B(\mathscr{B},\Bbbk)}(\mathcal{F}, \mathcal{G} \star \mathcal{E}_s) \arrow[r] & \mathrm{Hom}_{R\otimes R}(\mathbb{H}(\mathcal{F}), \mathbb{H}(\mathcal{G}) \otimes_R \mathsf{B}_s)
\end{tikzcd}$$

The upper square commutes by the naturality of the isomorphism in Proposition 7.6.9, and the lower square commutes because $\mathbb{H}$ is a functor. The composition of the two arrows along the left is an adjunction isomorphism, and likewise for the two arrows along the right. $\square$

The categorification theorem. We are now ready for the main results of this section.

THEOREM 7.6.11. *The functor $\mathbb{H}$ induces an equivalence of monoidal categories*

$$\mathbb{H} : \mathrm{Parity}_B(\mathscr{B}, \Bbbk) \to \mathrm{SBim}.$$

PROOF. Let $s_1, \ldots, s_k$ be a sequence of simple reflections. From the definition, any shift of a direct summand of $\mathsf{B}_{s_1} \otimes_R \cdots \otimes_R \mathsf{B}_{s_k}$ belongs to SBim. Combining Corollary 7.5.8, Lemma 7.6.7, and Proposition 7.6.9, we deduce that $\mathbb{H}$ sends parity sheaves to Soergel bimodules. Proposition 7.6.9 also tells us that $\mathbb{H} : \mathrm{Parity}_B(\mathcal{B}, \Bbbk) \to \mathrm{SBim}$ is a monoidal functor. It remains to show that it is an equivalence of categories.

Recall that $\mathbb{H}(\mathcal{E}_e) \cong R$. It is easy to see that the map

$$\mathrm{Hom}^\bullet_{\mathrm{Parity}_B(\mathcal{B},\Bbbk)}(\mathcal{E}_e, \mathcal{E}_e) \to \mathrm{Hom}^\bullet_{\mathrm{SBim}}(\mathbb{H}(\mathcal{E}_e), \mathbb{H}(\mathcal{E}_e)) \tag{7.6.9}$$

is an isomorphism, as both sides are identified with R.

Let $U = \mathcal{B} \smallsetminus \mathcal{B}_1$, and let $i : \mathcal{B}_1 \hookrightarrow \mathcal{B}$ and $j : U \hookrightarrow \mathcal{B}$ be the inclusion maps. Let $\mathscr{C} \subset D^{\mathrm{b}}_B(\mathcal{B}, \Bbbk)$ be the category of !-even objects $\mathcal{H}$ that satisfy $i^!\mathcal{H} = 0$. A minor variation on Lemma 7.3.3 shows that $\mathscr{C}$ is generated under extensions by objects of the form

$$j_{w*}\underline{\Bbbk}_{\mathcal{B}_w}[2n] \qquad \text{with } w \neq e.$$

Recall from Lemma 7.6.6 that $\mathbb{H}(j_{w*}\underline{\Bbbk}_{\mathcal{B}_w}[2n]) \cong R_w[2n]$. In particular, this (R, R)-bimodule is concentrated in even degrees. It follows by induction that for any object $\mathcal{H} \in \mathscr{C}$, the (R, R)-bimodule $\mathbb{H}(\mathcal{H})$ is concentrated in even degrees, and that it admits a filtration whose subquotients are of the form $R_w[2n]$ with $w \neq e$. By Lemma 7.6.3, we conclude that

$$\mathrm{Hom}_{R\otimes R}(R, \mathbb{H}(\mathcal{H})) = 0 \qquad \text{for all } \mathcal{H} \in \mathscr{C}. \tag{7.6.10}$$

Now let $\mathcal{G} \in D^{\mathrm{b}}_B(\mathcal{B}, \Bbbk)$ be an even object. Consider the distinguished triangle

$$i_*i^!\mathcal{G} \to \mathcal{G} \to j_*(\mathcal{G}|_U) \to . \tag{7.6.11}$$

Using the fact that $\mathcal{B}_1$ is a closed orbit, one may check that $i_*i^!\mathcal{G}$ is again even. Since it is supported on $\mathcal{B}_1$, $i_*i^!\mathcal{G}$ must be a direct sum of objects of the form $\mathcal{E}_e[2n]$. On the other hand, $j_*(\mathcal{G}|_U)$ is !-even and belongs to $\mathscr{C}$. The functor $\mathbb{H}$ takes the first and third terms of (7.6.11) to (R, R)-bimodules concentrated in even degrees, so the same holds for the middle term as well. We conclude that the long exact sequence in hypercohomology is in fact a short exact sequence

$$0 \to \mathbb{H}(i_*i^!\mathcal{G}) \to \mathbb{H}(\mathcal{G}) \to \mathbb{H}(j_*(\mathcal{G}|_U)) \to 0 \tag{7.6.12}$$

of (R, R)-bimodules.

Consider the following commutative diagram:

$$\begin{array}{ccc}
\mathrm{Hom}(\mathcal{E}_e, j_*(\mathcal{G}|_U)[-1]) & & 0 \\
\downarrow & & \downarrow \\
\mathrm{Hom}(\mathcal{E}_e, i_*i^!\mathcal{G}) & \xrightarrow{\sim} & \mathrm{Hom}(\mathbb{H}(\mathcal{E}_e), \mathbb{H}(i_*i^!\mathcal{G})) \\
\downarrow & & \downarrow \\
\mathrm{Hom}(\mathcal{E}_e, \mathcal{G}) & \longrightarrow & \mathrm{Hom}(\mathbb{H}(\mathcal{E}_e), \mathbb{H}(\mathcal{G})) \\
\downarrow & & \downarrow \\
\mathrm{Hom}(\mathcal{E}_e, j_*(\mathcal{G}|_U)) & \longrightarrow & \mathrm{Hom}(\mathbb{H}(\mathcal{E}_e), \mathbb{H}(j_*(\mathcal{G}_U)))
\end{array}$$

Here, the left-hand column is exact, and the right-hand column is exact because the sequence (7.6.12) is exact. In the left-hand column, the first and fourth terms vanish by adjunction. The bottom entry in the right-hand column vanishes by (7.6.10). By (7.6.9), the horizontal map involving $i_*i^!\mathcal{G}$ is an isomorphism. We conclude that

$$\mathrm{Hom}(\mathcal{E}_e, \mathcal{G}) \to \mathrm{Hom}(\mathbb{H}(\mathcal{E}_e), \mathbb{H}(\mathcal{G})) \tag{7.6.13}$$

is an isomorphism for any even object $\mathcal{G}$. Similar reasoning applies to odd objects, so (7.6.13) is an isomorphism for any parity sheaf $\mathcal{G}$. (If $\mathcal{G}$ is odd, both sides of (7.6.13) vanish.)

We are now ready to prove that

$$\mathrm{Hom}(\mathcal{F}, \mathcal{G}) \to \mathrm{Hom}(\mathbb{H}(\mathcal{F}), \mathbb{H}(\mathcal{G}))$$

is an isomorphism for all $\mathcal{F}, \mathcal{G} \in \mathrm{Parity}_B(\mathscr{B}, \Bbbk)$. By Corollary 7.5.8, it is enough to consider the special case where $\mathcal{F} = \mathcal{E}_{s_1} \star \cdots \star \mathcal{E}_{s_k}$ for some simple reflections $s_1, \ldots, s_k \in S$. We proceed by induction on k. The case $k = 0$ is covered by (7.6.13). For $k > 0$, Proposition 7.6.10 gives us a commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_{D^{\mathrm{b}}_B(\mathscr{B},\Bbbk)}(\mathcal{E}_{s_1} \star \cdots \star \mathcal{E}_{s_k}, \mathcal{G}) \arrow[r] \arrow[d, "\wr"'] & \mathrm{Hom}_{R\otimes R}(\mathbb{H}(\mathcal{E}_{s_1} \star \cdots \star \mathcal{E}_{s_{k-1}}) \otimes_R \mathsf{B}_{s_k}, \mathbb{H}(\mathcal{G})) \arrow[d, "\wr"] \\
\mathrm{Hom}_{D^{\mathrm{b}}_B(\mathscr{B},\Bbbk)}(\mathcal{E}_{s_1} \star \cdots \star \mathcal{E}_{s_{k-1}}, \mathcal{G} \star \mathcal{E}_{s_k}) \arrow[r] & \mathrm{Hom}_{R\otimes R}(\mathbb{H}(\mathcal{E}_{s_1} \star \cdots \star \mathcal{E}_{s_{k-1}}), \mathbb{H}(\mathcal{G}) \otimes_R \mathsf{B}_{s_k})
\end{tikzcd}$$

The bottom horizontal map is an isomorphism by induction, so the top horizontal map is as well.

We have now shown that $\mathbb{H} : \mathrm{Parity}_B(\mathscr{B}, \Bbbk) \to \mathrm{SBim}$ is fully faithful. To show that it is essentially surjective, we must check that its image satisfies the conditions in Definition 7.6.2. All of these conditions except Karoubianness are either obvious or follow from Lemma 7.6.7 and Proposition 7.6.9. We must show that if $\mathcal{F} \in \mathrm{Parity}_B(\mathscr{B}, \Bbbk)$, and if M is a direct summand of $\mathbb{H}(\mathcal{F})$ (as an (R, R)-bimodule), then M is also in the image of $\mathbb{H}$. Since SBim is a Krull–Schmidt category (see Lemma 7.6.5), it is enough to show that $\mathbb{H}$ sends indecomposable parity sheaves to indecomposable (R, R)-bimodules. Using the criterion from Corollary A.8.8, this follows from the fact that $\mathbb{H}$ is fully faithful. $\square$

For $w \in W$, we now let

$$\mathsf{B}_w = \mathbb{H}(\mathcal{E}_w).$$

Theorem 7.6.11 implies that this is an indecomposable Soergel bimodule. Moreover, every indecomposable Soergel bimodule is isomorphic to some $B_w[n]$.

Theorem 7.6.12. *Assume that* $\Bbbk$ *is a field satisfying* (7.6.1). *There is a unique isomorphism of rings* $\mathrm{ch} : \mathrm{K}_\oplus(\mathrm{SBim}) \to \mathcal{H}$ *that satisfies*

$$\mathrm{ch}([R[1]]) = q^{-1/2}, \qquad \textit{and} \qquad \mathrm{ch}(\mathsf{B}_s) = \underline{H}_s \quad \textit{for } s \in S.$$

Moreover, for all $w \in W$, *we have* $\mathrm{ch}([B_w]) = {}^p\underline{H}_w$.

Proof. By Theorem 7.6.11, the functor $\mathbb{H}$ induces an isomorphism of split Grothendieck groups $\mathrm{K}_\oplus(\mathrm{Parity}_B(\mathscr{B}, \Bbbk)) \xrightarrow{\sim} \mathrm{K}_\oplus(\mathrm{SBim})$. Moreover, this is in fact a ring isomorphism. The theorem above follows from these observations and Theorem 7.5.10. $\square$

Remark 7.6.13. Since this is a book on sheaf theory, this section has taken a sheaf-theoretic approach to the main results about Soergel bimodules. But one of the main reasons to study Soergel bimodules is to have a *non-sheaf-theoretic* categorification of the Hecke algebra. Much of the literature on Soergel bimodules begins with a much more general set-up: rather than starting with the Weyl group of a reductive algebraic group, one can begin with an arbitrary Coxeter group, along with a representation V satisfying some assumptions (this will play the role of $\Bbbk \otimes_{\mathbb{Z}} \mathbf{X}(T)$). One can define R to be $\mathrm{Sym}(V)$, and then repeat Definition 7.6.2. For a version of Theorem 7.6.12 in this generality, see [**222**].

Another approach to non-sheaf-theoretic categorification of the Hecke algebra is the **Elias–Williamson calculus**, introduced in [**73**]. This approach also begins with an arbitrary Coxeter group (equipped with some additional data, called a "realization") and leads to an additive monoidal category EW. For Weyl groups of reductive groups in the generality considered in this section, it turns out that EW is equivalent to SBim. But even for affine Weyl groups (which we will encounter in Chapter 9), EW and SBim may behave differently. For a general theorem relating parity sheaves to the Elias–Williamson calculus, see [**194**, Chapter 10].

Exercises.

7.6.1. Prove Lemma 7.6.1. *Hint*: First consider the case where G is quasisimple and simply connected. You will have to carry out case-by-case computations, writing all roots in the basis of fundamental weights. To deduce the result for arbitrary reductive G, let G'_{sc} be the simply connected cover of the derived subgroup $G' \subset G$, let $T'_{\mathrm{sc}} \subset G'_{\mathrm{sc}}$ be its maximal torus, and consider the natural map $\Bbbk \otimes_{\mathbb{Z}} \mathbf{X}(T) \to \Bbbk \otimes_{\mathbb{Z}} \mathbf{X}(T'_{\mathrm{sc}})$.

7.6.2. Show directly from the definition (i.e., without invoking Theorem 7.6.11) that $\mathsf{B}_s \otimes_R \mathsf{B}_s \cong \mathsf{B}_s[1] \oplus \mathsf{B}_s[-1]$.

7.7. Additional exercises

Exercise 7.7.1. By the induction equivalence and Exercise 7.2.1, we have

$$D^{\mathrm{b}}_B(\mathcal{B}, \Bbbk) \cong D^{\mathrm{b}}_G(G \times^B G/B, \Bbbk) \cong D^{\mathrm{b}}_G(\mathcal{B} \times \mathcal{B}, \Bbbk). \tag{7.7.1}$$

Consider the three maps

$$\begin{tikzcd} & \mathcal{B} \times \mathcal{B} \times \mathcal{B} \arrow[dl, "\mathrm{pr}_{12}"'] \arrow[d, "\mathrm{pr}_{13}"'] \arrow[dr, "\mathrm{pr}_{23}"] & \\ \mathcal{B} \times \mathcal{B} & \mathcal{B} \times \mathcal{B} & \mathcal{B} \times \mathcal{B} \end{tikzcd}$$

where pr_{ij} is the projection onto the ith and jth copies of $\mathcal{B}$. Let

$$\diamond : D^{\mathrm{b}}_G(\mathcal{B} \times \mathcal{B}, \Bbbk) \times D^{\mathrm{b}}_G(\mathcal{B} \times \mathcal{B}, \Bbbk) \to D^{\mathrm{b}}_G(\mathcal{B} \times \mathcal{B}, \Bbbk)$$

be the functor given by

$$\mathcal{F} \diamond \mathcal{G} = \mathrm{pr}_{13*}(\mathrm{pr}_{12}^* \mathcal{F} \overset{L}{\otimes} \mathrm{pr}_{23}^* \mathcal{G})[-\dim \mathcal{B}].$$

In this exercise, you will prove that $\diamond$ corresponds to $\star$ under (7.7.1). (The functor $\diamond$ has the advantage that its definition makes sense even in the nonequivariant derived category. For an early proof (before the invention of equivariant derived categories) of Theorem 7.3.8 using $\diamond$, see [**228**].)

(a) Define maps

$$(G \times^B G/B) \times (G \times^B G/B) \xleftarrow{d} G \times^B G \times^B G/B \xrightarrow{f} G \times^B G/B$$

by $d(g_1, g_2, g_3B) = (g_1, g_2B, g_1g_2, g_3B)$ and $f(g_1, g_2, g_3B) = (g_1, g_2g_3B)$. Define a functor

$$\diamond' : D^{\mathrm{b}}_G(G \times^B G/B, \Bbbk) \times D^{\mathrm{b}}_G(G \times^B G/B, \Bbbk) \to D^{\mathrm{b}}_G(G \times^B G/B, \Bbbk)$$

by

$$\mathcal{F} \diamond' \mathcal{G} = f_* d^*(\mathcal{F} \boxtimes \mathcal{G})[-\dim G/B].$$

Show that $\diamond'$ corresponds to $\diamond$ under the second equivalence in (7.7.1).

(b) Show that the first equivalence in (7.7.1) is given by $\mathcal{F} \mapsto \underline{\Bbbk}_{\mathcal{B}} \tilde{\boxtimes} \mathcal{F}$.

(c) Show that for $\mathcal{F}, \mathcal{G} \in D^{\mathrm{b}}_B(\mathcal{B}, \Bbbk)$, we have

$$d^*((\underline{\Bbbk}_{\mathcal{B}} \tilde{\boxtimes} \mathcal{F}) \boxtimes (\underline{\Bbbk}_{\mathcal{B}} \tilde{\boxtimes} \mathcal{G})) \cong \underline{\Bbbk}_{\mathcal{B}} \tilde{\boxtimes} \mathcal{F} \tilde{\boxtimes} \mathcal{G}.$$

(d) Finish the proof by showing that $f_*(\underline{\Bbbk}_{\mathcal{B}} \tilde{\boxtimes} \mathcal{F} \tilde{\boxtimes} \mathcal{G}) \cong \underline{\Bbbk}_{\mathcal{B}} \tilde{\boxtimes} (\mathcal{F} \star \mathcal{G})$.

The next four exercises involve the objects Δ_w and ∇_w from Exercise 7.2.7. The Δ_w are called **standard perverse sheaves**, and the ∇_w are called **costandard perverse sheaves**. An object $\mathcal{F} \in \mathrm{Perv}_B(\mathcal{B}, \Bbbk)$ or $\mathrm{Perv}_{(B)}(\mathcal{B}, \Bbbk)$ is said to have a **standard filtration** if it admits a filtration all of whose subquotients are standard perverse sheaves. A **costandard filtration** is defined similarly.

Exercise 7.7.2. Let $\mathcal{F}, \mathcal{G} \in \mathrm{Perv}_{(B)}(\mathcal{B}, \Bbbk)$. Suppose that $\mathcal{F}$ has a standard filtration and that $\mathcal{G}$ has a costandard filtration. Show that for any B-stable open subset $U \subset \mathcal{B}$, the map

$$\mathrm{Hom}(\mathcal{F}, \mathcal{G}) \to \mathrm{Hom}(\mathcal{F}|_U, \mathcal{G}|_U)$$

is surjective.

Exercise 7.7.3. Let $w \in W$. Show that the functors

$$(-) \star \Delta_w : D^{\mathrm{b}}_B(\mathcal{B}, \Bbbk) \to D^{\mathrm{b}}_B(\mathcal{B}, \Bbbk),$$
$$(-) \star \nabla_w : D^{\mathrm{b}}_B(\mathcal{B}, \Bbbk) \to D^{\mathrm{b}}_B(\mathcal{B}, \Bbbk)$$

are, respectively, left t-exact and right t-exact.

For the remaining exercises, assume that $\Bbbk$ is a field. A perverse sheaf $\mathcal{F} \in \mathrm{Perv}_{(B)}(\mathcal{B}, \Bbbk)$ is said to be **tilting** if $j_w^! \mathcal{F}$ and $j_w^* \mathcal{F}$ are both perverse for all $w \in W$.

Exercise 7.7.4. In this exercise, you will classify the tilting perverse sheaves.

(a) Show that $\mathcal{F} \in \mathrm{Perv}_{(B)}(\mathcal{B}, \Bbbk)$ is tilting if and only if it admits both a standard filtration and a costandard filtration.

(b) Prove that the support of an indecomposable tilting perverse sheaf is the closure of a single B-orbit. Then prove that for each $w \in W$, there is at most one (up to isomorphism) indecomposable tilting perverse sheaf $\mathcal{T}_w$ whose support is $\overline{\mathcal{B}_w}$. Moreover, if $\mathcal{T}_w$ exists, then $\mathcal{T}_w|_{\mathcal{B}_w} \cong \underline{\Bbbk}_{\mathcal{B}_w}[\dim \mathcal{B}_w]$.

(c) Use Exercise 4.6.4 to show that for all $w \in W$, $\mathcal{T}_w$ exists.

(d) Let s be a simple reflection. Describe $\mathcal{T}_s$ explicitly. *Hint*: See Exercise 4.6.5.

Exercise 7.7.5. Let $G = \mathrm{SL}_2$.

(a) Classify the indecomposable objects in $\mathrm{Perv}_{(B)}(\mathcal{B}, \Bbbk)$. (In other words, do Exercise 4.6.6 if you have not already done it.)

(b) Show that if the characteristic of $\Bbbk$ is not 2, then the tilting perverse sheaf $\mathcal{T}_s$ is *not* B-equivariant. *Hint*: It might be useful to compare with Exercises 6.10.3 and 6.10.4.

(c) Show that if the characteristic of $\Bbbk$ is 2, then $\mathcal{T}_s$ is B-equivariant!

The following two exercises deal with $G = \mathrm{Sp}_4$.

EXERCISE 7.7.6. Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$ be the standard basis for $\mathbb{C}^4$, and equip $\mathbb{C}^4$ with the symplectic form given by

$$(\mathbf{e}_1, \mathbf{e}_4) = -(\mathbf{e}_4, \mathbf{e}_1) = 1, \qquad (\mathbf{e}_2, \mathbf{e}_3) = -(\mathbf{e}_3, \mathbf{e}_2) = 1,$$
$$(\mathbf{e}_i, \mathbf{e}_j) = 0 \text{ if } i + j \neq 5.$$

For any subspace $V \subset \mathbb{C}^4$, let $V^\perp$ denote its orthogonal complement with respect to this symplectic form. Recall that an **isotropic flag** F in $\mathbb{C}^4$ is a sequence of subspaces $0 \subset F_1 \subset F_2 \subset F_1^\perp \subset \mathbb{C}^4$, where $\dim F_i = i$ and $F_2 = F_2^\perp$. The **standard isotropic flag** is the flag F^{std} given by $F_1 = \langle \mathbf{e}_1 \rangle$ and $F_2 = \langle \mathbf{e}_1, \mathbf{e}_2 \rangle$. (It follows that $F_1^\perp = \langle \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3 \rangle$.) Let $B \subset \mathrm{Sp}_4$ be the stabilizer of the standard isotropic flag. Then there is a bijection

$$\mathscr{B} = \mathrm{Sp}_4/B \xrightarrow{\sim} \{\text{isotropic flags in } \mathbb{C}^4\}.$$

Let s and t be the simple reflections corresponding to the short and long simple roots, respectively.

(a) Show that

$$\overline{\mathscr{B}_s} = \{F \in \mathscr{B} \mid F_2 = F_2^{\mathrm{std}}\} \qquad \text{and} \qquad \overline{\mathscr{B}_t} = \{F \in \mathscr{B} \mid F_1 = F_1^{\mathrm{std}}\}.$$

(b) Show that the Bott–Samelson variety $\mathscr{B}(s,t,s)$ can be identified with

$$\{(F^{(1)}, F^{(2)}, F^{(3)}) \in \mathscr{B} \times \mathscr{B} \times \mathscr{B} \mid F_2^{(1)} = F_2^{\mathrm{std}},\ F_1^{(2)} = F_1^{(1)},\ F_2^{(3)} = F_2^{(2)}\}.$$

The map $m : \mathscr{B}(s,t,s) \to \overline{\mathscr{B}_{sts}}$ is given by $(F^{(1)}, F^{(2)}, F^{(3)}) \mapsto F^{(3)}$. Deduce that

$$\overline{\mathscr{B}_{sts}} = \{F \in \mathscr{B} \mid \dim F_2 \cap \langle \mathbf{e}_1, \mathbf{e}_2 \rangle \geq 1\}.$$

(c) Show that $m_* \underline{\Bbbk}[3]$ is given by

	$\mathscr{B}_{sts}$	$\mathscr{B}_{st}$	$\mathscr{B}_{ts}$	$\mathscr{B}_s$	$\mathscr{B}_t$	$\mathscr{B}_e$
-1				$\underline{\Bbbk}(-1)$		$\underline{\Bbbk}(-1)$
-2						
-3	$\underline{\Bbbk}$	$\underline{\Bbbk}$	$\underline{\Bbbk}$	$\underline{\Bbbk}$	$\underline{\Bbbk}$	$\underline{\Bbbk}$

EXERCISE 7.7.7. In this exercise, you will exhibit an example in which the p-canonical basis differs from the canonical basis. This phenomenon is related to the failure of the decomposition theorem seen in Exercise 3.10.8.

(a) Let $U \subset \overline{\mathscr{B}_{sts}}$ be the open subset given by

$$U = \{F \in \overline{\mathscr{B}_{sts}} \mid F_1 \not\subset \langle \mathbf{e}_1, \mathbf{e}_3, \mathbf{e}_4 \rangle \text{ and } F_2 \cap \langle \mathbf{e}_3, \mathbf{e}_4 \rangle = 0\}.$$

Let X be the variety of 2×2 nilpotent matrices. Show that $U \cong \mathbb{A}^1 \times X$. *Hint*: Let $F \in U$. Then F_1 is spanned by a (unique) vector of the form $(y, 1, b, d)$. Since $F_2 \supset F_1$ and $F_2 \cap \langle \mathbf{e}_3, \mathbf{e}_4 \rangle = 0$, F_2 must be spanned by $(y, 1, b, d)$ and a unique vector of the form $(1, 0, a, c)$. The fact that F_2 is isotropic implies that

$$d = a + cy,$$

and the fact that $\dim F_2 \cap \langle \mathbf{e}_1, \mathbf{e}_2 \rangle \geq 1$ implies that

$$\det \begin{bmatrix} 1 & 0 & 1 & y \\ 0 & 1 & 0 & 1 \\ 0 & 0 & a & b \\ 0 & 0 & c & a + cy \end{bmatrix} = 0 \qquad \text{or} \qquad a^2 + acy - bc = 0.$$

Show that the map $f : U \to \mathbb{A}^1 \times X$ given below is an isomorphism:

$$f(F) = \left(y, \begin{bmatrix} a & ay - b \\ c & -a \end{bmatrix} \right).$$

(b) Let $\tilde{U} = m^{-1}(U)$, and let $\tilde{X} = \{(x, L) \in X \times \mathbb{P}^1 \mid L \subset \ker x\}$. Show that there is an isomorphism $\tilde{U} \cong \mathbb{A}^1 \times \tilde{X}$ such $m|_{\tilde{U}} : \tilde{U} \to U$ is identified with $\mathrm{id}_{\mathbb{A}^1} \times p$, where $p : \tilde{X} \to X$ is the obvious projection map.

(c) Review Exercises 1.10.5 and 3.10.6(a). Let $\Bbbk$ be a field, and show that

$$\mathcal{E}_{sts} \cong \begin{cases} m_* \underline{\Bbbk}_{\mathcal{B}(s,t,s)}[3] & \text{if } \operatorname{char} \Bbbk = 2, \\ \underline{\Bbbk}_{\overline{\mathcal{B}_{sts}}} \cong \mathrm{IC}_{sts} & \text{if } \operatorname{char} \Bbbk \neq 2. \end{cases}$$

This is equivalent to showing that

$$m_* \underline{\Bbbk}_{\mathcal{B}(s,t,s)} \cong \begin{cases} \mathcal{E}_{sts} & \text{if } \operatorname{char} \Bbbk = 2, \\ \mathcal{E}_{sts} \oplus \mathcal{E}_s & \text{if } \operatorname{char} \Bbbk \neq 2. \end{cases}$$

Deduce that

$${}^p\underline{H}_{sts} = \begin{cases} \underline{H}_{sts} + \underline{H}_s & \text{if } p = 2, \\ \underline{H}_{sts} & \text{otherwise.} \end{cases}$$

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CHAPTER 8

Springer theory

In 1976, Springer [**226**, **227**] discovered a remarkable way to construct an action of the Weyl group W on the cohomology of certain closed subvarieties of the flag variety, now known as Springer fibers. Shortly thereafter, Lusztig [**157**] and Borho–MacPherson [**39**] showed how to reinterpret Springer's work in the language of perverse $\mathbb{Q}$-sheaves on the nilpotent cone $\mathcal{N}$. Much later, Juteau [**117**] extended this to the setting of perverse $\Bbbk$-sheaves on $\mathcal{N}$, with $\Bbbk$ a field. The main result is a bijection, known as the **Springer correspondence**, between irreducible $\Bbbk[W]$-modules and certain simple perverse $\Bbbk$-sheaves on $\mathcal{N}$. The first three sections of this chapter are devoted to establishing this result.

However, not all G-equivariant simple perverse sheaves on $\mathcal{N}$ appear in the Springer correspondence. The **generalized Springer correspondence**, due to Lusztig [**159**] for characteristic-0 coefficients, and to Achar–Henderson–Juteau–Riche [**7**, **9**, **10**] for general coefficients, accounts for the missing objects. In the characteristic-0 case, this result is a foundational step in Lusztig's theory of character sheaves [**155**]. In Sections 8.4 and 8.5, we will discuss some of the ingredients in the generalized Springer correspondence, although we will not give the full proof.

8.1. Nilpotent orbits and the Springer resolution

The nilpotent cone. Let G be a connected reductive complex algebraic group, and let $\mathfrak{g}$ be its Lie algebra. Recall that an element $x \in \mathfrak{g}$ is said to be **nilpotent** if it acts by a nilpotent operator on every finite-dimensional algebraic representation of G. If G is a closed subgroup of GL_n, then $x \in \mathfrak{gl}_n$ is nilpotent in the preceding sense if and only if it is a nilpotent matrix.

The **nilpotent cone**, denoted by $\mathcal{N}$ or $\mathcal{N}_G$, is the set of nilpotent elements in $\mathfrak{g}$. This is a closed subvariety of $\mathfrak{g}$ that is preserved by the adjoint action of G. In this section, we review a number of results on the geometry of $\mathcal{N}$.

Let $\mathrm{Z}(G)$ be the center of G. It is well known that the quotient map $G \to G/\mathrm{Z}(G)$ induces a G-equivariant isomorphism of varieties

$$\mathcal{N}_G \xrightarrow{\sim} \mathcal{N}_{G/\mathrm{Z}(G)}; \tag{8.1.1}$$

more generally, if H is any closed subgroup of $\mathrm{Z}(G)$, and then $\mathcal{N}_G \cong \mathcal{N}_{G/H}$. The following statement summarizes some basic facts about $\mathcal{N}$, due to Jacobson–Morozov [**112**, **182**], Malcev [**167**], and Kostant [**139**, **140**]. See [**52**] for an exposition and for additional references.

THEOREM 8.1.1. (1) *The variety $\mathcal{N}$ is irreducible.*
(2) *We have* $\dim \mathcal{N} = \dim G - \operatorname{rank} G$.
(3) *There are finitely many G-orbits on $\mathcal{N}$.*
(4) *There is a unique dense G-orbit $\mathcal{O}_{\mathrm{reg}} \subset \mathcal{N}$, called the* ***regular orbit****.*

(5) *The orbit* $\mathcal{O}_0 = \{0\}$ *is the unique closed* G*-orbit in* $\mathcal{N}$.

For $x \in \mathcal{N}$, let G^x be its stabilizer in G. We also introduce the notation

$$A_G(x) = G^x/(G^x)^\circ.$$

This is a finite group. If $\mathcal{O}$ is the orbit containing x, then $A_G(x)$ is closely related to $\pi_1(\mathcal{O}, x)$ (cf. Remark 6.2.14). By Proposition 6.2.13, we have

$$\mathrm{Loc}^{\mathrm{ft}}_G(\mathcal{O}, \Bbbk) \cong \Bbbk[A_G(x)]\text{-}\mathrm{mod}^{\mathrm{fg}}.$$

The main focus of this chapter is the study of G-equivariant perverse sheaves on $\mathcal{N}$ with field coefficients, especially when the field satisfies the following condition.

Definition 8.1.2. A field $\Bbbk$ is called a **Springer splitting field** for G if for every Levi subgroup $L \subset G$ and every nilpotent element $x \in \mathcal{N}_L$, $\Bbbk$ is a splitting field for the finite group $A_L(x)$.

It turns out that for GL_n, every field is a Springer splitting field, but for SL_n, a field is a Springer splitting field if and only if it contains all nth roots of unity. Explicit case-by-case conditions for $\Bbbk$ to be a Springer splitting field can be found in [**10**, Proposition 3.2].

Central characters. Let $\mathrm{Z}^\circ(G)$ be the identity component of $\mathrm{Z}(G)$. In view of (8.1.1), we can consider the functors

$$\mathrm{Perv}_{G/\mathrm{Z}(G)}(\mathcal{N}, \Bbbk) \xrightarrow{\mathrm{Infl}^{G/\mathrm{Z}^\circ(G)}_{G/\mathrm{Z}(G)}} \mathrm{Perv}_{G/\mathrm{Z}^\circ(G)}(\mathcal{N}, \Bbbk) \xrightarrow[\sim]{\mathrm{Infl}^{G}_{G/\mathrm{Z}^\circ(G)}} \mathrm{Perv}_G(\mathcal{N}, \Bbbk).$$

By Proposition 6.2.8, the second arrow above is an equivalence of categories, but the first arrow is in general only fully faithful. We can use the machinery of Section 6.3 to keep track of the failure of $\mathrm{Infl}^{G/\mathrm{Z}^\circ(G)}_{G/\mathrm{Z}(G)}$ to be essentially surjective. As in (6.3.5), there is a canonical surjective map

$$\pi_1(G/\mathrm{Z}(G), 1) \twoheadrightarrow \mathrm{Z}(G)/\mathrm{Z}^\circ(G). \tag{8.1.2}$$

We adapt Definition 6.3.13 to the setting of the nilpotent cone as follows.

Definition 8.1.3. Let $\mathcal{F} \in \mathrm{Perv}_G(\mathcal{N}, \Bbbk) \cong \mathrm{Perv}_{G/\mathrm{Z}^\circ(G)}(\mathcal{N}, \Bbbk)$, and let $\chi : \mathrm{Z}(G)/\mathrm{Z}^\circ(G) \to \Bbbk^\times$ be a character. The perverse sheaf $\mathcal{F}$ is said to have **central character** χ if $\mathrm{For}^G(\mathcal{F}) \in \mathrm{Perv}(\mathcal{N}, \Bbbk)$ admits a $(G/\mathrm{Z}(G), \chi)$-equivariant structure.

Of course, $\mathcal{F}$ admits the trivial central character if and only if it lies in the image of $\mathrm{Infl}^G_{G/\mathrm{Z}(G)}$.

Lemma 8.1.4. *Let* $\Bbbk$ *be a Springer splitting field for* G*. Then every simple perverse sheaf in* $\mathrm{Perv}_G(\mathcal{N}, \Bbbk)$ *admits a central character.*

Proof. Consider a simple perverse sheaf $\mathrm{IC}(\mathcal{O}, \mathcal{L})$, where $\mathcal{O}$ is a nilpotent orbit. Let $x \in \mathcal{O}$, and let E be the irreducible $\Bbbk[A_G(x)]$-module corresponding to $\mathcal{L}$. There is an obvious map $\mathrm{Z}(G)/\mathrm{Z}^\circ(G) \to A_G(x)$, and its image is contained in the center of $A_G(x)$. Since $\Bbbk$ is a splitting field for $A_G(x)$, Schur's lemma implies that elements of $\mathrm{Z}(G)/\mathrm{Z}^\circ(G)$ act on E by scalars. In other words, the representation E gives rise to a character

$$\chi_E : \mathrm{Z}(G)/\mathrm{Z}^\circ(G) \to \Bbbk^\times.$$

By Proposition 6.3.14, $\mathcal{L}$ is $(G/\mathrm{Z}(G), \chi_E)$-equivariant, and by Remark 6.3.5, so is $\mathrm{IC}(\mathcal{O}, \mathcal{L})$. $\square$

The Springer resolution and the Springer sheaf. Choose a Borel subgroup $B \subset G$ and a maximal torus $T \subset B$, and let $W = \mathrm{N}_G(T)/T$ be the Weyl group. As in Chapter 7, we set $\mathcal{B} = G/B$. Let $\mathfrak{b}$ and $\mathfrak{h}$ be the Lie algebras of B and T, respectively, and let $\mathfrak{u} = [\mathfrak{b}, \mathfrak{b}]$ be the nilradical of $\mathfrak{b}$. Note that $\mathfrak{u} \subset \mathcal{N}$.

DEFINITION 8.1.5. Let $\tilde{\mathcal{N}} = G \times^B \mathfrak{u}$. The map

$$\mu : \tilde{\mathcal{N}} \to \mathcal{N} \qquad \text{given by} \qquad \mu(g, x) = \mathrm{Ad}(g)(x)$$

is called the **Springer resolution**. For any $x \in \mathcal{N}$, the preimage $\mu^{-1}(x)$, also denoted by $\mathcal{B}_x$, is called the **Springer fiber** over x.

(This notation conflicts with the notation for Schubert cells from Chapter 7. However, as Schubert cells are rarely mentioned in this chapter—they occur only in the proofs of Lemmas 8.1.6 and 8.2.5—this conflict should not cause any ambiguity.)

In a minor abuse of language, the term "Springer resolution" is sometimes used to refer just to the space $\tilde{\mathcal{N}}$ (rather than the map $\mu : \tilde{\mathcal{N}} \to \mathcal{N}$). This space has another description that is sometimes useful: there is an isomorphism of varieties

$$\tilde{\mathcal{N}} \xrightarrow{\sim} \{(gB, x) \in \mathcal{B} \times \mathcal{N} \mid x \in \mathrm{Ad}(g)(\mathfrak{u})\} \tag{8.1.3}$$

given by $(g, x) \mapsto (gB, \mathrm{Ad}(g)(x))$. In terms of the right-hand side of (8.1.3), the map $\mu : \tilde{\mathcal{N}} \to \mathcal{N}$ corresponds to projection onto the second factor. The Springer fiber over $x \in \mathcal{N}$ can be identified as a closed subvariety of $\mathcal{B}$:

$$\mathcal{B}_x \cong \{gB \in \mathcal{B} \mid x \in \mathrm{Ad}(g)(\mathfrak{u})\} \subset \mathcal{B}.$$

LEMMA 8.1.6. *The Springer resolution $\mu : \tilde{\mathcal{N}} \to \mathcal{N}$ is semismall (with respect to the stratification of $\mathcal{N}$ by nilpotent orbits).*

PROOF. Let $Z = \tilde{\mathcal{N}} \times_{\mathcal{N}} \tilde{\mathcal{N}}$. (This space is known as the **Steinberg variety**.) Using (8.1.3), it is easy to see that Z can be described as follows:

$$Z \cong \{(g_1 B, g_2 B, x) \in \mathcal{B} \times \mathcal{B} \times \mathcal{N} \mid x \in \mathrm{Ad}(g_1)(\mathfrak{u}) \cap \mathrm{Ad}(g_2)(\mathfrak{u})\}.$$

By Exercise 3.8.1, it is enough to show that

$$\dim Z \le \dim \mathcal{N}. \tag{8.1.4}$$

For $w \in W$, let

$$Z_w = \{(g_1 B, g_2 B, x) \in Z \mid g_1^{-1} g_2 B \in \mathcal{B}_w\},$$

where $\mathcal{B}_w$ is the Schubert cell labelled by w. Then Z is the union of the locally closed subvarieties $(Z_w)_{w \in W}$. To prove (8.1.4), it is enough to show that

$$\dim Z_w \le \dim \mathcal{N} \tag{8.1.5}$$

for each $w \in W$. Given $w \in W$, choose a representative $\dot{w} \in \mathrm{N}_G(T) \subset G$. It is straightforward to check that there is an isomorphism of varieties

$$G \times^{B \cap \dot{w} B \dot{w}^{-1}} (\mathfrak{u} \cap \mathrm{Ad}(\dot{w})\mathfrak{u}) \to Z_w$$

given by $(g, x) \mapsto (gB, g\dot{w}B, \mathrm{Ad}(g)(x))$. (In fact, for our purposes, it is enough to check that this map is surjective.) We deduce that

$$\dim Z_w \le \dim G - \dim(B \cap \dot{w} B \dot{w}^{-1}) + \dim(\mathfrak{u} \cap \mathrm{Ad}(\dot{w})\mathfrak{u}).$$

The Lie algebra of $B \cap \dot{w} B \dot{w}^{-1}$ is $\mathfrak{h} \oplus (\mathfrak{u} \cap \mathrm{Ad}(\dot{w})\mathfrak{u})$, so the expression above simplifies to $\dim G - \dim \mathfrak{h} = \dim \mathcal{N}$. We have proved (8.1.5), and hence (8.1.4). $\square$

Remark 8.1.7. Let us spell out the content of the previous lemma, using Definition 3.8.1: if x is a point in a G-orbit $\mathcal{O} \subset \mathcal{N}$, then

$$\dim \mathcal{B}_x \leq \tfrac{1}{2}(\dim \mathcal{N} - \dim \mathcal{O}).$$

According to a result of Steinberg [**233**] (see also [**224**]), this inequality is actually an equality for all x.

8.2. The Springer sheaf

By Theorem 3.8.4, an immediate consequence of Lemma 8.1.6 is that the object $\mu_* \underline{\Bbbk}_{\tilde{\mathcal{N}}}[\dim \mathcal{N}]$ is a perverse sheaf.

Definition 8.2.1. The **Springer sheaf** for G, denoted by Spr or Spr_G, is the perverse sheaf given by

$$\mathrm{Spr} = \mu_* \underline{\Bbbk}_{\tilde{\mathcal{N}}}[\dim \mathcal{N}](\tfrac{1}{2}\dim \mathcal{N}) \in \mathrm{Perv}_G(\mathcal{N}, \Bbbk).$$

(Recall that $\dim \mathcal{N} = 2 \dim \mathfrak{u}$, so the Tate twist above is by an integer.) The goal of this section is to determine the endomorphism ring of the Springer sheaf. To do this, we will work with the space

$$\tilde{\mathfrak{g}} = G \times^B \mathfrak{b}$$

along with the map $\mu_{\tilde{\mathfrak{g}}} : \tilde{\mathfrak{g}} \to \mathfrak{g}$ defined by $\mu_{\tilde{\mathfrak{g}}}(g, x) = \mathrm{Ad}(g)(x)$. As in (8.1.3), we can also describe $\tilde{\mathfrak{g}}$ as

$$\tilde{\mathfrak{g}} \cong \{(gB, x) \in \mathcal{B} \times \mathfrak{g} \mid x \in \mathrm{Ad}(g)(\mathfrak{b})\},$$

and then $\mu_{\tilde{\mathfrak{g}}}$ is identified with projection onto the second coordinate.

Let $\mathfrak{g}_{\mathrm{rs}} \subset \mathfrak{g}$ be the set of regular semisimple elements of $\mathfrak{g}$, and let $\tilde{\mathfrak{g}}_{\mathrm{rs}} = \mu_{\tilde{\mathfrak{g}}}^{-1}(\mathfrak{g}_{\mathrm{rs}})$. To compute End(Spr), we will study the following diagram, known as the **main diagram of Springer theory**:

$$\begin{tikzcd} \tilde{\mathfrak{g}}_{\mathrm{rs}} \arrow[r, hook] \arrow[d, "\mu_{\mathrm{rs}}"'] & \tilde{\mathfrak{g}} \arrow[d, "\mu_{\tilde{\mathfrak{g}}}"'] & \tilde{\mathcal{N}} \arrow[l, hook'] \arrow[d, "\mu"] \\ \mathfrak{g}_{\mathrm{rs}} \arrow[r, hook] & \mathfrak{g} & \mathcal{N} \arrow[l, hook'] \end{tikzcd} \tag{8.2.1}$$

Remark 8.2.2. The space $\tilde{\mathfrak{g}}$ is sometimes called the **Grothendieck–Springer simultaneous resolution**. The explanation for this terminology is as follows. Let $\mathbb{C}[\mathfrak{g}]^G$ be the ring of G-invariants in the coordinate ring of $\mathfrak{g}$, and let $\mathfrak{g}/\!/G = \mathrm{Spec}\,\mathbb{C}[\mathfrak{g}]^G$. (This space is often called the **adjoint quotient** of $\mathfrak{g}$.) For $\lambda \in \mathfrak{g}/\!/G$, let $\mathfrak{g}_\lambda$ be its preimage under the natural map $\mathfrak{g} \to \mathfrak{g}/\!/G$. It turns out that for each λ, the map $\mu_{\tilde{\mathfrak{g}}} : \mu_{\tilde{\mathfrak{g}}}^{-1}(\mathfrak{g}_\lambda) \to \mathfrak{g}_\lambda$ resolves the singularities of $\mathfrak{g}_\lambda$ (although it may be generically finite-to-one). In other words, $\tilde{\mathfrak{g}}$ simultaneously resolves the singularities of all fibers of $\mathfrak{g} \to \mathfrak{g}/\!/G$.

In the special case $\lambda = 0$, the fiber $\mathfrak{g}_0$ is just the nilpotent cone $\mathcal{N}$, and its preimage $\mu_{\tilde{\mathfrak{g}}}^{-1}(\mathfrak{g}_0)$ is identifed with $\tilde{\mathcal{N}}$.

Study of the regular semisimple set. We begin by studying the leftmost column of (8.2.1).

Lemma 8.2.3. *The map $\mu_{\mathrm{rs}} : \tilde{\mathfrak{g}}_{\mathrm{rs}} \to \mathfrak{g}_{\mathrm{rs}}$ is a Galois covering map. Its Galois group is canonically isomorphic to W.*

PROOF. Let $\mathfrak{b}_{\mathrm{rs}} = \mathfrak{b} \cap \mathfrak{g}_{\mathrm{rs}}$ and $\mathfrak{h}_{\mathrm{rs}} = \mathfrak{h} \cap \mathfrak{g}_{\mathrm{rs}}$. We then have $\tilde{\mathfrak{g}}_{\mathrm{rs}} = G \times^B \mathfrak{b}_{\mathrm{rs}}$. The inclusion map $\mathfrak{h}_{\mathrm{rs}} \hookrightarrow \mathfrak{b}_{\mathrm{rs}}$ induces a map $G \times \mathfrak{h}_{\mathrm{rs}} \to G \times^B \mathfrak{b}_{\mathrm{rs}}$, and this factors through a map

$$G \times^T \mathfrak{h}_{\mathrm{rs}} \to G \times^B \mathfrak{b}_{\mathrm{rs}} = \tilde{\mathfrak{g}}_{\mathrm{rs}}. \tag{8.2.2}$$

It is straightforward to check that this is a bijection. Since $\tilde{\mathfrak{g}}_{\mathrm{rs}}$ is a normal variety, Zariski's main theorem (in the form found in, for instance, [**183**, Section III.9]) implies that (8.2.2) is an isomorphism of varieties.

Define an action of W on $G \times^T \mathfrak{h}_{\mathrm{rs}}$ as follows: for $w \in W$, we set $w \cdot (gT, x) = (g\dot{w}^{-1}T, \mathrm{Ad}(\dot{w})(x))$. This action is free, and μ_{rs} factors through the quotient to induce a map

$$(G \times^T \mathfrak{h}_{\mathrm{rs}})/W \to \mathfrak{g}_{\mathrm{rs}}. \tag{8.2.3}$$

To finish the proof of the lemma, we must show that this map is an isomorphism of varieties. Again, since $\mathfrak{g}_{\mathrm{rs}}$ is normal, it is enough to show that (8.2.3) is a bijection. This is equivalent to showing that for any $x \in \mathfrak{g}_{\mathrm{rs}}$, the preimage

$$\mu_{\mathrm{rs}}^{-1}(x) \cong \{gB \in \mathcal{B} \mid x \in \mathrm{Ad}(g)(\mathfrak{b})\}$$

consists of precisely $|W|$ many points. Since x is semisimple, it lies in some Cartan subalgebra of the Borel subalgebra $\mathrm{Ad}(g)(\mathfrak{b})$. But since x is regular, it lies in a unique Cartan subalgebra of $\mathfrak{g}$ (namely, its centralizer). It is well known that a given Cartan subalgebra is contained in exactly $|W|$ many Borel subalgebras, as desired. $\square$

In particular, Lemma 8.2.3 implies that there is a surjective map $\pi_1(\mathfrak{g}_{\mathrm{rs}}, x_0) \twoheadrightarrow W$ for any point $x_0 \in \mathfrak{g}_{\mathrm{rs}}$. Let

$$L_{\mathrm{rs}} : \Bbbk[W]\text{-}\mathrm{mod}^{\mathrm{fg}} \to \mathrm{Loc}^{\mathrm{ft}}(\mathfrak{g}_{\mathrm{rs}}, \Bbbk) \tag{8.2.4}$$

be the functor that assigns to $V \in \Bbbk[W]\text{-}\mathrm{mod}^{\mathrm{fg}}$ the local system that corresponds to it under Theorem 1.7.9.

LEMMA 8.2.4. *We have* $\mu_{\mathrm{rs}*}\underline{\Bbbk}_{\tilde{\mathfrak{g}}_{\mathrm{rs}}} \cong L_{\mathrm{rs}}(\Bbbk[W])$. *As a consequence, we have a canonical isomorphism* $\mathrm{End}(\mu_{\mathrm{rs}*}\underline{\Bbbk}_{\tilde{\mathfrak{g}}_{\mathrm{rs}}}) \cong \Bbbk[W]$.

PROOF. The first assertion follows from Proposition 1.7.16. The second assertion comes from the fact that $\mathrm{End}_{\Bbbk[W]}(\Bbbk[W]) \cong \Bbbk[W]$. $\square$

The following fact was first observed by Lusztig [**157**, Section 3].

LEMMA 8.2.5. *The map* $\mu_{\tilde{\mathfrak{g}}} : \tilde{\mathfrak{g}} \to \mathfrak{g}$ *is small with respect to* $\mathfrak{g}_{\mathrm{rs}} \subset \mathfrak{g}$.

PROOF. The proof is similar to that of Lemma 8.1.6. Let $Z' = \tilde{\mathfrak{g}} \times_{\mathfrak{g}} \tilde{\mathfrak{g}}$. This space can be identified as follows:

$$Z' \cong \{(g_1B, g_2B, x) \in \mathcal{B} \times \mathcal{B} \times \mathcal{N} \mid x \in \mathrm{Ad}(g_1)(\mathfrak{b}) \cap \mathrm{Ad}(g_2)(\mathfrak{b})\}.$$

Also, for each $w \in W$, we set

$$Z'_w = \{(g_1B, g_2B, x) \in Z' \mid g_1^{-1}g_2B \in \mathcal{B}_w\}.$$

As in Lemma 8.1.6, one can check that there is an isomorphism of varieties

$$G \times^{B \cap \dot{w}B\dot{w}^{-1}} (\mathfrak{b} \cap \mathrm{Ad}(\dot{w})\mathfrak{b}) \to Z'_w.$$

We deduce from this that $\dim Z'_w = \dim \mathfrak{g}$ for all w.

Next, let

$$Z'' = \{(g_1B, g_2B, x) \in Z' \mid x \notin \mathfrak{g}_{\mathrm{rs}}\} \qquad \text{and} \qquad Z''_w = Z'_w \cap Z'' \quad \text{for } w \in W.$$

Then Z'' is a closed subset of Z'. Let us estimate its dimension. We have

$$Z''_w \cong G \times^{B \cap \dot{w}B\dot{w}^{-1}} (\mathfrak{b} \cap \mathrm{Ad}(\dot{w})\mathfrak{b} \cap (\mathfrak{g} \smallsetminus \mathfrak{g}_{\mathrm{rs}})).$$

The key point now is that $\mathfrak{b} \cap \mathrm{Ad}(\dot{w})\mathfrak{b}$ always contains some regular semisimple elements (for instance, it at least contains $\mathfrak{h}_{\mathrm{rs}}$). Since $\mathfrak{g}_{\mathrm{rs}}$ is open in $\mathfrak{g}$, the set $\mathfrak{b} \cap \mathrm{Ad}(\dot{w})\mathfrak{b} \cap \mathfrak{g}_{\mathrm{rs}}$ is an open subset of a vector space. Its complement $\mathfrak{b} \cap \mathrm{Ad}(\dot{w})\mathfrak{b} \cap (\mathfrak{g} \smallsetminus \mathfrak{g}_{\mathrm{rs}})$ is therefore an affine variety whose dimension is strictly smaller than that of $\mathfrak{b} \cap \mathrm{Ad}(\dot{w})\mathfrak{b}$. We conclude that

$$\dim Z''_w < \dim Z'_w = \dim \mathfrak{g},$$

and hence that $\dim Z'' < \dim \mathfrak{g}$.

Now, choose a stratification $(Y_t)_{t \in \mathscr{T}}$ of $\mathfrak{g} \smallsetminus \mathfrak{g}_{\mathrm{rs}}$ such that the function $x \mapsto \dim \mu_{\tilde{\mathfrak{g}}}^{-1}(x)$ is constant on each stratum. (The existence of such a stratification follows from the fiber dimension theorem; see, for instance, [**97**, Exercise II.3.22(c)] or [**210**, Theorem 1.25].) For each $t \in \mathscr{T}$, let

$$Z'_t = \{(g_1B, g_2B, x) \in Z' \mid x \in Y_t\}.$$

Since Y_t does not meet $\mathfrak{g}_{\mathrm{rs}}$, Z'_t is contained in Z'', so we have $\dim Z'_t < \dim \mathfrak{g}$. On the other hand, the preimage of $x \in Y_t$ under the map $Z'_t \to Y_t$ is isomorphic to $\mu_{\tilde{\mathfrak{g}}}^{-1}(x) \times \mu_{\tilde{\mathfrak{g}}}^{-1}(x)$. Applying the fiber dimension theorem to $Z'_t \to Y_t$, we find that

$$2 \dim \mu_{\tilde{\mathfrak{g}}}^{-1}(x) = \dim Z'_t - \dim Y_t < \dim \mathfrak{g} - \dim Y_t$$

for all $x \in Y_t$. In view of this inequality and the fact that μ_{rs} is finite (Lemma 8.2.3), we see that $\mu_{\tilde{\mathfrak{g}}}$ is small. ☐

PROPOSITION 8.2.6. *There is a canonical isomorphism of perverse sheaves* $\mu_{\tilde{\mathfrak{g}}*}\underline{\Bbbk}_{\mathfrak{g}}[\dim \mathfrak{g}] \cong \mathrm{IC}(\mathfrak{g}_{\mathrm{rs}}, L_{\mathrm{rs}}(\Bbbk[W]))$. *As a consequence, we have*

$$\Bbbk[W] \xrightarrow{\sim} \mathrm{End}(\mu_{\tilde{\mathfrak{g}}*}\underline{\Bbbk}_{\mathfrak{g}}[\dim \mathfrak{g}]).$$

PROOF. The first assertion is obtained by combining Proposition 3.8.7 with Lemmas 8.2.4 and 8.2.5. The description of the ring $\mathrm{End}(\mu_{\tilde{\mathfrak{g}}*}\underline{\Bbbk}_{\mathfrak{g}}[\dim \mathfrak{g}])$ follows from Lemma 8.2.4 and the fact that IC is a fully faithful functor (Lemma 3.3.3). ☐

Endomorphisms of the Springer sheaf. Let us return to (8.2.1). Perhaps the most obvious approach to studying this diagram is base change. (Both squares in the diagram are cartesian.) This approach, developed in [**6**, **39**, **157**, **159**], is easy to set up, but the resulting maps of Hom-spaces are difficult to study.

Another approach is to use the Fourier–Laumon transform. This approach, developed in [**117**, **160**, **177**], is much less elementary to set up, but it has the advantage that it automatically gives rise to isomorphisms of Hom-groups (because it is an equivalence of categories). Below, we will carry out the Fourer–Laumon approach in full, and we will make brief comments on the base change approach.

The Fourier–Laumon transform requires us to work with $\mathbb{G}_{\mathrm{m}} \times G$-equivariant perverse sheaves on $\mathfrak{g}$ or $\mathcal{N}$, where $\mathbb{G}_{\mathrm{m}}$ acts by scaling. Since most of this chapter deals with G-equivariant perverse sheaves instead, let us begin by explaining how the two situations are related. Note that the forgetful functor

$$\mathrm{Perv}_{\mathbb{G}_{\mathrm{m}} \times G}(\mathcal{N}, \Bbbk) \to \mathrm{Perv}_G(\mathcal{N}, \Bbbk)$$

is fully faithful. (This can be seen by further forgetting to $\mathrm{Perv}(\mathcal{N}, \Bbbk)$, and using Proposition 6.2.15 twice, or else by applying Exercise 6.10.1.)

LEMMA 8.2.7. (1) *The orbits of* $\mathbb{G}_{\mathrm{m}} \times G$ *on* $\mathcal{N}$ *coincide with the* G*-orbits.*
(2) *The Springer sheaf* Spr *is* $\mathbb{G}_{\mathrm{m}} \times G$*-equivariant.*
(3) *Every simple* G*-equivariant perverse sheaf on* $\mathcal{N}$ *is* $\mathbb{G}_{\mathrm{m}} \times G$*-equivariant.*

PROOF. (1) We must show that for any $x \in \mathcal{N}$, its orbit under $\mathbb{G}_{\mathrm{m}}$ is contained in its orbit under G. Since there are only finitely many G-orbits, there is a unique orbit $\mathcal{O}$ such that $\mathcal{O} \cap \mathbb{G}_{\mathrm{m}} \cdot x$ is dense in $\mathbb{G}_{\mathrm{m}} \cdot x$. Without loss of generality, assume that $x \in \mathcal{O}$. Suppose there were a point $y \in \mathbb{G}_{\mathrm{m}} \cdot x \smallsetminus \mathcal{O}$, and let $\mathcal{O}' = G \cdot y$. Since y is in the closure of $\mathbb{G}_{\mathrm{m}} \cdot x \cap \mathcal{O}$, we see that $\mathcal{O}'$ is contained in the closure of $\mathcal{O}$, so $\dim \mathcal{O}' < \dim \mathcal{O}$. On the other hand, since the scaling action commutes with the G-action, we have $G^x = G^y$, and this implies that $\dim \mathcal{O} = \dim \mathcal{O}'$, a contradiction.

(2) If we let $\mathbb{G}_{\mathrm{m}}$ act on $\tilde{\mathcal{N}}$ by the scaling action on $\mathfrak{u}$, then $\mu : \tilde{\mathcal{N}} \to \mathcal{N}$ is $\mathbb{G}_{\mathrm{m}} \times G$-equivariant. The claim follows.

(3) It is a straightforward exercise to show that the natural map

$$A_G(x) = G^x/(G^x)^\circ \to (\mathbb{G}_{\mathrm{m}} \times G)^x/(\mathbb{G}_{\mathrm{m}} \times G)^{x,\circ}$$

is an isomorphism. In view of part (1), every simple G- or $\mathbb{G}_{\mathrm{m}} \times G$-equivariant perverse sheaf arises by applying the IC functor to a local system associated to an irreducible representation of one of these groups, and the claim follows. $\square$

Let us choose a G-equivariant isomorphism of vector spaces

$$\mathfrak{g} \xrightarrow{\sim} \mathfrak{g}^*. \tag{8.2.5}$$

This is equivalent to choosing a nondegenerate G-equivariant bilinear form on $\mathfrak{g}$. (Such a form always exists, but it is not unique. It is possible to avoid making this choice by working directly with regular semisimple elements of $\mathfrak{g}^*$.) Using (8.2.5), one can regard the Fourier–Laumon transform as a functor

$$\mathrm{Four}_{\mathfrak{g}} : D^{\mathrm{b}}_{\mathbb{G}_{\mathrm{m}} \times G}(\mathfrak{g}, \Bbbk) \to D^{\mathrm{b}}_{\mathbb{G}_{\mathrm{m}} \times G}(\mathfrak{g}, \Bbbk).$$

Here is the main result of the Fourier–Laumon transform approach to (8.2.1).

THEOREM 8.2.8. *Let* $h : \mathcal{N} \hookrightarrow \mathfrak{g}$ *be the inclusion map. There is a canonical isomorphism of perverse sheaves* $\mathrm{Four}_{\mathfrak{g}}(\mu_{\tilde{\mathfrak{g}}*}\underline{\Bbbk}_{\tilde{\mathfrak{g}}}[\dim \mathfrak{g}]) \cong h_*\mathrm{Spr}(-\dim \mathfrak{g})$*. As a consequence, there is an isomorphism*

$$\Bbbk[W] \xrightarrow{\sim} \mathrm{End}(\mathrm{Spr}).$$

PROOF. Regard $G \times^B \mathfrak{b}$ and $G \times^B \mathfrak{u}$ as subbundles of $G \times^B \mathfrak{g}$, and denote the inclusion maps by $h_{\mathfrak{b}}$ and $h_{\mathfrak{u}}$. Under any isomorphism (8.2.5), the annihilator of $\mathfrak{b}$ is $\mathfrak{u}$, so by Corollary 6.9.14, we have

$$\mathrm{Four}_{G \times^B \mathfrak{g}}(h_{\mathfrak{b}*}\underline{\Bbbk}_{\tilde{\mathfrak{g}}}[\dim \tilde{\mathfrak{g}}]) \cong h_{\mathfrak{u}*}\underline{\Bbbk}_{\tilde{\mathcal{N}}}[\dim \tilde{\mathcal{N}}](-\dim \mathfrak{b}). \tag{8.2.6}$$

Next, let $\mu_{G \times^B \mathfrak{g}} : G \times^B \mathfrak{g} \to \mathfrak{g}$ be the map given by $(g, x) \mapsto \mathrm{Ad}(g)(x)$. Observe that the square

$$\begin{tikzcd} G \times^B \mathfrak{g} \arrow[r, "\mu_{G \times^B \mathfrak{g}}"] \arrow[d] & \mathfrak{g} \arrow[d] \\ G/B \arrow[r] & \mathrm{pt} \end{tikzcd}$$

is cartesian. (In other words, $G \times^B \mathfrak{g}$ is actually a trivial vector bundle over G/B.) We are therefore in the situation of Proposition 6.9.15, which tells us that

$$\mu_{G\times^B\mathfrak{g},*}\mathrm{Four}_{G\times^B\mathfrak{g}}(h_{\mathfrak{b}*}\underline{\Bbbk}_{\tilde{\mathfrak{g}}}[\dim\tilde{\mathfrak{g}}]) \cong \mathrm{Four}_{\mathfrak{g}}(\mu_{G\times^B\mathfrak{g},*}h_{\mathfrak{b}*}\underline{\Bbbk}_{\tilde{\mathfrak{g}}}[\dim\tilde{\mathfrak{g}}]).$$

By (8.2.6), this becomes

$$\mu_{G\times^B\mathfrak{g},*}h_{\mathfrak{u}*}\underline{\Bbbk}_{\tilde{\mathcal{N}}}[\dim\tilde{\mathcal{N}}](-\dim\mathfrak{b}) \cong \mathrm{Four}_{\mathfrak{g}}(\mu_{G\times^B\mathfrak{g},*}h_{\mathfrak{b}*}\underline{\Bbbk}_{\tilde{\mathfrak{g}}}[\dim\tilde{\mathfrak{g}}]),$$

and hence $\mathrm{Four}_{\mathfrak{g}}(\mu_{\tilde{\mathfrak{g}}*}\underline{\Bbbk}_{\tilde{\mathfrak{g}}}[\dim\mathfrak{g}]) \cong h_*\mathrm{Spr}(-\dim\mathfrak{g})$. (The Tate twist comes from the fact that $-\dim\mathfrak{b} = \frac{1}{2}\dim\mathcal{N} - \dim\mathfrak{g}$.) The last assertion then follows from Proposition 8.2.6 and the fact that $\mathrm{Four}_{\mathfrak{g}}$ is an equivalence of categories. ☐

The next theorem is the main result of the base change approach to (8.2.1).

THEOREM 8.2.9. (1) *There is a canonical isomorphism*

$$(\mu_{\tilde{\mathfrak{g}}*}\underline{\Bbbk}_{\tilde{\mathfrak{g}}}[\dim\mathfrak{g}])|_{\mathcal{N}} \cong \mathrm{Spr}[\dim\mathfrak{h}](-\tfrac{1}{2}\dim\mathcal{N}).$$

(2) *The isomorphism in part* (1) *induces an isomorphism*

$$\Bbbk[W] \xrightarrow{\sim} \mathrm{End}(\mathrm{Spr}).$$

(3) *The map in part* (2) *differs from that in Theorem* 8.2.8 *by composition with the automorphism*

$$\Bbbk[W] \to \Bbbk[W] \qquad \textit{given by} \qquad w \mapsto (-1)^{\ell(w)}w.$$

Part (1) of Theorem 8.2.9 is obvious: apply proper base change to the right-hand square in (8.2.1). Here are some brief comments on the rest of this theorem:

- When $\Bbbk = \mathbb{C}$, the base-change approach to defining the map in part (2) was proposed in [**157**], and the fact that it is an isomorphism was proved in [**39**]. Part (3) of Theorem 8.2.9 was proved for $\Bbbk = \mathbb{C}$ in [**104**].
- When $\Bbbk$ is any field, part (2) of Theorem 8.2.9 can be deduced from [**13**].
- For arbitrary $\Bbbk$, parts (2) and (3) were proved in [**6**].

Exercises.

8.2.1. Let $G = \mathrm{GL}_2$.

(a) Show that if $\mathrm{char}\,\Bbbk \neq 2$, then Spr is a semisimple perverse sheaf, isomorphic to $\mathrm{IC}(\mathcal{O}_0, \underline{\Bbbk}) \oplus \mathrm{IC}(\mathcal{O}_{\mathrm{reg}}, \underline{\Bbbk})(1)$.

(b) Show that if $\mathrm{char}\,\Bbbk = 2$, then Spr is an indecomposable perverse sheaf, and that it has a unique composition series. Determine its Loewy layers.

Hint: See Exercises 1.10.5, 3.10.6(a), and 3.10.8.

8.3. The Springer correspondence

Identify End(Spr) with $\Bbbk[W]$ by Theorem 8.2.8. Then we have a functor

$$\mathrm{Hom}(\mathrm{Spr}, -) : \mathrm{Perv}_G(\mathcal{N}) \to \Bbbk[W]\text{-mod}.$$

(More precisely, the functor $\mathrm{Hom}(\mathrm{Spr}, -)$ naturally takes values in $\Bbbk[W]^{\mathrm{op}}$-mod, but of course, we can identify $\Bbbk[W]$ with its opposite ring by the map $w \mapsto w^{-1}$.) We would like to study what this functor does to (isomorphism classes of) simple objects. In this section, we will suppress all Tate twists.

The **Springer correspondence** (**by Fourier transform**) is the map ν defined in the following theorem. (See Remark 8.3.2 for more on this terminology.)

THEOREM 8.3.1. *Let $\Bbbk$ be a field.*

(1) *Let* $\mathrm{IC}(\mathcal{O}, \mathcal{L}) \in \mathrm{Perv}_G(\mathcal{N}, \Bbbk)$ *be a simple perverse sheaf. Then the* $\Bbbk[W]$*-module* $\mathrm{Hom}(\mathrm{Spr}, \mathrm{IC}(\mathcal{O}, \mathcal{L}))$ *is either irreducible or* 0.
(2) *Every irreducible* $\Bbbk[W]$*-module* V *arises as* $V \cong \mathrm{Hom}(\mathrm{Spr}, \mathrm{IC}(\mathcal{O}, \mathcal{L}))$ *for a unique pair* $(\mathcal{O}, \mathcal{L})$ *(up to isomorphism). The assignment* $V \mapsto \mathrm{IC}(\mathcal{O}, \mathcal{L})$ *defines an injective map*

$$\nu : \mathrm{Irr}(\Bbbk[W]) \to \mathrm{Irr}(\mathrm{Perv}_G(\mathcal{N}, \Bbbk)).$$

The map ν above is not surjective in general. A perverse sheaf that lies in its image is said to **occur in the Springer correspondence**.

PROOF. Let $h : \mathcal{N} \hookrightarrow \mathfrak{g}$ be the inclusion map, and let $\mathcal{F} = \mathrm{IC}(\mathcal{O}, \mathcal{L})$ be a simple perverse sheaf on $\mathcal{N}$. Suppose that $\mathrm{Hom}(\mathrm{Spr}, \mathcal{F}) \neq 0$. By Proposition 8.2.6 and Theorem 8.2.8, there is a nonzero map

$$\mathrm{IC}(\mathfrak{g}_{\mathrm{rs}}, L_{\mathrm{rs}}(\Bbbk[W])) \to \mathrm{Four}_{\mathfrak{g}}(h_*\mathcal{F}). \tag{8.3.1}$$

Moreover, $\mathrm{Four}_{\mathfrak{g}}(h_*\mathcal{F})$ is a simple perverse sheaf, so this map must be surjective. By Lemma 3.3.3, it cannot be supported on $\mathfrak{g} \smallsetminus \mathfrak{g}_{\mathrm{rs}}$. Restricting (8.3.1) to $\mathfrak{g}_{\mathrm{rs}}$, we obtain a surjective map of perverse sheaves

$$L_{\mathrm{rs}}(\Bbbk[W])[\dim \mathfrak{g}] \to \mathrm{Four}_{\mathfrak{g}}(h_*\mathcal{F})|_{\mathfrak{g}_{\mathrm{rs}}}.$$

By Proposition 3.4.1, $\mathrm{Four}_{\mathfrak{g}}(h_*\mathcal{F})|_{\mathfrak{g}_{\mathrm{rs}}}$ must be an irreducible shifted local system. Since this local system is a quotient of $L_{\mathrm{rs}}(\Bbbk[W])$, it must be of the form $L_{\mathrm{rs}}(V)$ for some irreducible quotient V of $\Bbbk[W]$. We have shown that

$$h_*\mathrm{IC}(\mathcal{O}, \mathcal{L}) \cong \mathrm{Four}_{\mathfrak{g}}(\mathrm{IC}(\mathfrak{g}_{\mathrm{rs}}, L_{\mathrm{rs}}(V))). \tag{8.3.2}$$

Moreover, we have

$$\begin{aligned} \mathrm{Hom}(\mathrm{Spr}, \mathcal{F}) &\cong \mathrm{Hom}(\mathrm{IC}(\mathfrak{g}_{\mathrm{rs}}, L_{\mathrm{rs}}(\Bbbk[W])), \mathrm{IC}(\mathfrak{g}_{\mathrm{rs}}, L_{\mathrm{rs}}(V))) \\ &\cong \mathrm{Hom}(L_{\mathrm{rs}}(\Bbbk[W]), L_{\mathrm{rs}}(V)) \cong \mathrm{Hom}_{\Bbbk[W]}(\Bbbk[W], V) \cong V. \end{aligned} \tag{8.3.3}$$

We have completed the proof of part (1).

For part (2), we would like to take (8.3.2) as the definition of ν for any irreducible $\Bbbk[W]$-module V. For this to make sense, we must check that the object $\mathrm{Four}_{\mathfrak{g}}(\mathrm{IC}(\mathfrak{g}_{\mathrm{rs}}, L_{\mathrm{rs}}(V)))$ is a simple perverse sheaf supported on $\mathcal{N}$. It is clearly simple, since $L_{\mathrm{rs}}(V)$ is irreducible and $\mathrm{Four}_{\mathfrak{g}}$ is an equivalence of categories. Furthermore, there is a surjective map of $\Bbbk[W]$-modules $\Bbbk[W] \to V$, and hence of the corresponding local systems on $\mathfrak{g}_{\mathrm{rs}}$. By Lemma 3.3.5, we get a surjective map $\mathrm{IC}(\mathfrak{g}_{\mathrm{rs}}, L_{\mathrm{rs}}(\Bbbk[W])) \to \mathrm{IC}(\mathfrak{g}_{\mathrm{rs}}, L_{\mathrm{rs}}(V))$. Therefore, $\mathrm{Four}_{\mathfrak{g}}(\mathrm{IC}(\mathfrak{g}_{\mathrm{rs}}, L_{\mathrm{rs}}(V)))$ is a quotient of $h_*\mathrm{Spr}$. By Proposition 3.1.10, it is supported on $\mathcal{N}$. Finally, it is clear from (8.3.2) that ν is injective. □

REMARK 8.3.2. In Theorem 8.3.1, we identified $\mathrm{End}(\mathrm{Spr})$ with $\Bbbk[W]$ using Theorem 8.2.8. If we had instead identified them using Theorem 8.2.9(2), we would have obtained a different map ν, called the **Springer correspondence by restriction**. By Theorem 8.2.9(3), the two versions of the Springer correspondence just differ by the sign character of W, so determining one is equivalent to determining the other.

In most of the literature on characteristic-0 Springer theory, the term "Springer correspondence" is used to mean the Springer correspondence by restriction, in part because that version interacts well with the classical W-action on $\mathbf{H}^\bullet(\mathcal{B}; \mathbb{Q})$. (See, for instance, the discussion in [**214**, Sections 5 and 6].) Thus, in most sources, the statements of Propositions 8.3.3 and 8.3.6 below are reversed.

However, when working with positive-characteristic coefficients, the Springer correspondence by Fourier transform is often more convenient. In this book, following [**116**] and other sources, the term "Springer correspondence" should be understood by default to mean the Springer correspondence by Fourier transform.

The Springer correspondence has been determined explicitly in all cases. For characteristic 0 coefficients, a summary can be found in [**51**, Section 13.3], based on results in [**15**, **105**, **212**, **213**]. For positive-characteristic coefficients, see [**119**] and the references therein, including [**9**, **116**]. Here is one important case.

PROPOSITION 8.3.3. *The Springer correspondence sends the trivial* $\Bbbk[W]$*-module to the skyscraper sheaf* $\mathrm{IC}(\mathcal{O}_0, \underline{\Bbbk})$.

PROOF. By (8.3.2), the Springer correspondence sends the trivial module to the Fourier–Laumon transform of $\mathrm{IC}(\mathfrak{g}_{\mathrm{rs}}, \underline{\Bbbk}) \cong \underline{\Bbbk}_{\mathfrak{g}}[\dim \mathfrak{g}]$. By Corollary 6.9.7, this is the skyscraper sheaf at $\mathcal{O}_0$. $\square$

The Springer correspondence with characteristic zero coefficients. For the remainder of this section, we work with $\Bbbk = \mathbb{Q}$. Since $\tilde{\mathcal{N}}$ is smooth and μ is proper, the decomposition theorem (along with Theorem 3.8.4) tells us that the Springer sheaf is a semisimple perverse sheaf: it can be written as

$$\mathrm{Spr} \cong \bigoplus_{\substack{\mathcal{O} \subset \mathcal{N} \text{ a nilpotent orbit} \\ \mathcal{L} \text{ an irreducible local system on } \mathcal{O}}} \mathrm{IC}(\mathcal{O}, \mathcal{L})^{\oplus m_{\mathcal{O},\mathcal{L}}},$$

where $m_{\mathcal{O},\mathcal{L}}$ is the multiplicity. Alternatively, we can encode these multiplicities by tensoring with a suitable vector space:

$$\mathrm{Spr} \cong \bigoplus_{\substack{\mathcal{O} \subset \mathcal{N} \text{ a nilpotent orbit} \\ \mathcal{L} \text{ an irreducible local system on } \mathcal{O}}} \mathrm{IC}(\mathcal{O}, \mathcal{L}) \otimes V_{\mathcal{O},\mathcal{L}}, \tag{8.3.4}$$

where $V_{\mathcal{O},\mathcal{L}}$ is a $\mathbb{Q}$-vector space with $\dim V_{\mathcal{O},\mathcal{L}} = m_{\mathcal{O},\mathcal{L}}$.

PROPOSITION 8.3.4. *In the decomposition* (8.3.4)*, each vector space* $V_{\mathcal{O},\mathcal{L}}$ *has the structure of a* $\mathbb{Q}[W]$*-module. Moreover, if* $V_{\mathcal{O},\mathcal{L}} \neq 0$*, it is an irreducible* $\mathbb{Q}[W]$*-module, and this module corresponds under the Springer correspondence to* $\mathrm{IC}(\mathcal{O}, \mathcal{L})$.

PROOF. From (8.3.4), we see that $\mathrm{Hom}(\mathrm{Spr}, \mathrm{IC}(\mathcal{O}, \mathcal{L})) \cong V^*_{\mathcal{O},\mathcal{L}}$. This shows that $V^*_{\mathcal{O},\mathcal{L}}$ (and hence $V_{\mathcal{O},\mathcal{L}}$) naturally has the structure of a $\mathbb{Q}[W]$-module. By Theorem 8.3.1, if $V^*_{\mathcal{O},\mathcal{L}}$ is nonzero, it must be irreducible, and the Springer correspondence sends it to $\mathrm{IC}(\mathcal{O}, \mathcal{L})$. Finally, $V^*_{\mathcal{O},\mathcal{L}}$ (like any finite group representation over $\mathbb{Q}$) admits a W-invariant positive-definite inner product, so it is isomorphic to its dual $V_{\mathcal{O},\mathcal{L}}$. $\square$

LEMMA 8.3.5. *Let* V *be an irreducible* $\mathbb{Q}[W]$*-module. We have*

$$\nu(V) = \mathrm{IC}(\mathfrak{g}_{\mathrm{rs}}, L_{\mathrm{rs}}(V \otimes \mathrm{sgn}))|_{\mathcal{N}}[\dim \mathcal{N} - \dim \mathfrak{g}],$$

where sgn *denotes the sign representation of* W.

PROOF. Since $\mathbb{Q}[W]$ decomposes as $\bigoplus_{V\in\mathrm{Irr}(\mathbb{Q}[W])} V\otimes V^*$, we have

$$\begin{aligned}(8.3.5)\quad &\mathrm{IC}(\mathfrak{g}_{\mathrm{rs}}, L_{\mathrm{rs}}(\mathbb{Q}[W]))|_{\mathcal{N}}[\dim\mathcal{N}-\dim\mathfrak{g}]\\ &\qquad\cong \bigoplus_{V\in\mathrm{Irr}(\mathbb{Q}[W])} \mathrm{IC}(\mathfrak{g}_{\mathrm{rs}}, L_{\mathrm{rs}}(V))|_{\mathcal{N}}[\dim\mathcal{N}-\dim\mathfrak{g}]\otimes V^*.\end{aligned}$$

Both sides are isomorphic to the semisimple perverse sheaf Spr, so each summand $\mathrm{IC}(\mathfrak{g}_{\mathrm{rs}}, L_{\mathrm{rs}}(V))|_{\mathcal{N}}[\dim\mathcal{N}-\dim\mathfrak{g}]$ is a semisimple perverse sheaf.

Recall from Theorem 8.2.9 that restricting $\mathrm{IC}(\mathfrak{g}_{\mathrm{rs}}, L_{\mathrm{rs}}(\mathbb{Q}[W]))$ to $\mathcal{N}$ induces an isomorphism of endomorphism rings. The same then holds for Hom-groups between direct summands. In particular, we have

$$\begin{aligned}&\mathrm{Hom}(\mathrm{Spr}, \mathrm{IC}(\mathfrak{g}_{\mathrm{rs}}, L_{\mathrm{rs}}(V))|_{\mathcal{N}}[\dim\mathcal{N}-\dim\mathfrak{g}])\\ &\qquad\cong \mathrm{Hom}(\mathrm{IC}(\mathfrak{g}_{\mathrm{rs}}, L_{\mathrm{rs}}(\mathbb{Q}[W])), \mathrm{IC}(\mathfrak{g}_{\mathrm{rs}}, L_{\mathrm{rs}}(V)))\cong \mathrm{Hom}(\mathbb{Q}[W], V)\cong V.\end{aligned}$$

This is an isomorphism of $\mathbb{Q}[W]$-modules if we identify End(Spr) with $\mathbb{Q}[W]$ using Theorem 8.2.9. If we switch to the identification from Theorems 8.2.8 and 8.3.1, we see that

$$\mathrm{Hom}(\mathrm{Spr}, \mathrm{IC}(\mathfrak{g}_{\mathrm{rs}}, L_{\mathrm{rs}}(V))|_{\mathcal{N}}[\dim\mathcal{N}-\dim\mathfrak{g}])\cong V\otimes\mathrm{sgn}.$$

Of course, every simple summand of $\mathrm{IC}(\mathfrak{g}_{\mathrm{rs}}, L_{\mathrm{rs}}(V))|_{\mathcal{N}}[\dim\mathcal{N}-\dim\mathfrak{g}]$ must occur in the Springer correspondence. Since $V\otimes\mathrm{sgn}$ is an irreducible $\mathbb{Q}[W]$-module, Theorem 8.3.1 implies that it is in fact a simple perverse sheaf. We have shown that $\nu(V\otimes\mathrm{sgn})\cong\mathrm{IC}(\mathfrak{g}_{\mathrm{rs}}, L_{\mathrm{rs}}(V))|_{\mathcal{N}}[\dim\mathcal{N}-\dim\mathfrak{g}]$, and the lemma follows. $\square$

As an immediate consequence of the formula in Lemma 8.3.5, we obtain the following fact (obtained in a slightly different form by Lusztig; see [**40**, Lemma 2.3]).

PROPOSITION 8.3.6. *The Springer correspondence sends the sign representation of* $\mathbb{Q}[W]$ *to the perverse sheaf* $\mathrm{IC}(\mathcal{O}_{\mathrm{reg}}, \underline{\mathbb{Q}})\cong\underline{\mathbb{Q}}_{\mathcal{N}}[\dim\mathcal{N}]$.

In particular, we obtain the following nontrivial topological fact about $\mathcal{N}$, first proved by Borho–MacPherson [**40**] using Lusztig's version of Proposition 8.3.6. (For the notion of rational smoothness, see Exercise 3.10.3.)

PROPOSITION 8.3.7. *The nilpotent cone* $\mathcal{N}$ *is rationally smooth.*

We conclude with a reformulation of the Springer correspondence in terms of the cohomology of Springer fibers. The following statement does not mention perverse sheaves; it is close to the version that originally appeared in [**226, 227**].

THEOREM 8.3.8. *For any* $x\in\mathcal{N}$*, the cohomology groups* $\mathbf{H}^i(\mathcal{B}_x;\mathbb{Q})$ *are equipped with a natural action of* $A_G(x)\times W$*, so that there is a canonical decomposition*

$$\mathbf{H}^i(\mathcal{B}_x;\mathbb{Q})\cong\bigoplus_{L\in\mathrm{Irr}(A_G(x))} L\otimes V^i_{x,L},$$

where each $V^i_{x,L}$ *is a* $\mathbb{Q}[W]$*-module. Moreover, for* $i=2\dim\mathcal{B}_x$*, we have:*

(1) *Each* $V^{2\dim\mathcal{B}_x}_{x,L}$ *is either irreducible or* 0.
(2) *Each irreducible* $\mathbb{Q}[W]$*-module occurs as* $V^{2\dim\mathcal{B}_x}_{x,L}$ *for a unique pair* (x, L) *(up to* G*-conjugacy).*

PROOF. By proper base change, it is easy to see that $\mathrm{Spr}_x \cong R\Gamma(\underline{\mathbb{Q}}_{\mathcal{B}_x})[\dim \mathcal{N}]$. Let $\mathcal{O}_1$ be the nilpotent orbit containing x. Using (8.3.4), we obtain

$$\mathbf{H}^i(\mathcal{B}_x; \mathbb{Q}) \cong \bigoplus_{\mathcal{O},\mathcal{L}} \mathsf{H}^{i-\dim \mathcal{N}}(\mathrm{IC}(\mathcal{O}, \mathcal{L})|_{\mathcal{O}_1})_x \otimes V_{\mathcal{O},\mathcal{L}}. \tag{8.3.6}$$

The sheaf $\mathsf{H}^{i-\dim \mathcal{N}}(\mathrm{IC}(\mathcal{O}, \mathcal{L})|_{\mathcal{O}_1})$ is a G-equivariant local system, so its stalk at x is naturally an $A_G(x)$-representation. We have thus constructed the $A_G(x) \times W$-action on $\mathsf{H}^i(\mathcal{B}_x, \underline{\mathbb{Q}})$.

Now let $i = 2\dim \mathcal{B}_x$. Recall from Remark 8.1.7 that $i = \dim \mathcal{N} - \dim \mathcal{O}_1$, so

$$\mathbf{H}^{2\dim \mathcal{B}_x}(\mathcal{B}_x; \mathbb{Q}) \cong \bigoplus_{\mathcal{O},\mathcal{L}} \mathsf{H}^{-\dim \mathcal{O}_1}(\mathrm{IC}(\mathcal{O}, \mathcal{L})|_{\mathcal{O}_1})_x \otimes V_{\mathcal{O},\mathcal{L}}.$$

If $\mathcal{O}_1$ is not contained in the closure of $\mathcal{O}$, then $\mathsf{H}^{-\dim \mathcal{O}_1}(\mathrm{IC}(\mathcal{O}, \mathcal{L})|_{\mathcal{O}_1})$ obviously vanishes. Otherwise, we see from, say, Lemma 3.3.11 that

$$\mathsf{H}^{-\dim \mathcal{O}_1}(\mathrm{IC}(\mathcal{O}, \mathcal{L})|_{\mathcal{O}_1}) \cong \begin{cases} \mathcal{L} & \text{if } \mathcal{O}_1 = \mathcal{O}, \\ 0 & \text{otherwise.} \end{cases}$$

To summarize, when $i = 2\dim \mathcal{B}_x$, the expression (8.3.6) simplifies to

$$\mathbf{H}^{2\dim \mathcal{B}_x}(\mathcal{B}_x; \mathbb{Q}) \cong \bigoplus_{\mathcal{L} \in \mathrm{Irr}(\mathrm{Loc}^{\mathrm{ft}}_G(\mathcal{O}_1))} \mathcal{L}_x \otimes V_{\mathcal{O},\mathcal{L}}.$$

Here, $\mathcal{L}_x$ is an irreducible $A_G(x)$-module. The last two assertions in the theorem now follow from Proposition 8.3.4. ☐

Exercises.

8.3.1. There is a natural permutation action of $A_G(x)$ on the set of irreducible components of $\mathcal{B}_x$. This gives rise to an action on $\mathbf{H}^{2\dim \mathcal{B}_x}(\mathcal{B}_x; \mathbb{Q})$ via the basis of dual fundamental classes (Definition 2.11.13). Show that this coincides with the $A_G(x)$-action from Theorem 8.3.8. Then deduce that for every orbit $\mathcal{O} \subset \mathcal{N}$, the perverse sheaf $\mathrm{IC}(\mathcal{O}, \underline{\mathbb{Q}})$ occurs in the Springer correspondence.

8.4. Parabolic induction and restriction

In this section, we will study functors that relate $\mathrm{Perv}_G(\mathcal{N}_G, \Bbbk)$ to the analogous category for a Levi subgroup of G. Let P be a parabolic subgroup containing B. Let L be the Levi factor of P containing T, and let U_P be the unipotent radical of P. Let $\mathfrak{p}$, $\mathfrak{l}$, and $\mathfrak{u}_P$ denote their Lie algebras, and identify $\mathfrak{l}$ with $\mathfrak{p}/\mathfrak{u}_P$. Let

$$\pi_P : \mathfrak{p} \to \mathfrak{l}$$

be the quotient map. The group P acts on $\mathfrak{p}$ and on $\mathfrak{u}_P$ by the adjoint action.

We can also make P act on $\mathfrak{l}$ by letting U_P act trivially and by letting L act by the adjoint action. Throughout this section, any mention of the P-action on $\mathfrak{l}$ or on $\mathcal{N}_L$ should be understood to be this action. With this action, the quotient map π_P is P-equivariant. (The inclusion map $\mathfrak{l} \to \mathfrak{p}$ is L-equivariant, but not P-equivariant in general.) We have

$$\mathcal{N}_L + \mathfrak{u}_P = \pi_P^{-1}(\mathcal{N}_L).$$

Similarly, if $\mathcal{O} \subset \mathcal{N}_L$ is an L-orbit, we have $\mathcal{O} + \mathfrak{u}_P = \pi_P^{-1}(\mathcal{O})$. The collection

$$\{\mathcal{O} + \mathfrak{u}_P \mid \mathcal{O} \subset \mathcal{N}_L \text{ a nilpotent } L\text{-orbit}\} \tag{8.4.1}$$

is a stratification of $\mathcal{N}_L + \mathfrak{u}_P$. (One can show that it is a P-equivariantly good stratification, in the sense of Exercise 6.5.2.) Let

$$D^{\mathrm{b}}_{P,\mathfrak{u}_P}(\mathcal{N}_L + \mathfrak{u}_P, \Bbbk) = \left\{ \mathcal{F} \in D^{\mathrm{b}}_P(\mathcal{N}_L + \mathfrak{u}_P, \Bbbk) \;\middle|\; \begin{array}{c} \mathcal{F} \text{ is constructible with} \\ \text{respect to (8.4.1)} \end{array} \right\}.$$

We will now study the following **induction diagram**:

$$\begin{tikzcd} \mathcal{N}_L & \mathcal{N}_L + \mathfrak{u}_P \arrow[l, "\pi_P"'] \arrow[r, "e_P"'] \arrow[rr, bend left, "i_P = \mu_P \circ e_P"] & G \times^P (\mathcal{N}_L + \mathfrak{u}_P) \arrow[r, "\mu_P"'] & \mathcal{N}_G \end{tikzcd} \tag{8.4.2}$$

Here, e_P is the usual inclusion map for an induction space, as in Theorem 6.5.10, and μ_P is the map given by $\mu_P(g,x) = \mathrm{Ad}(g)(x)$. Their composition is the inclusion map $i_P : \mathcal{N}_L + \mathfrak{u}_P \hookrightarrow \mathcal{N}_G$.

Recall from Theorem 6.5.10 that the functor

$$e_P^* \mathrm{For}_P^G[-\dim G/P] \cong e_P^! \mathrm{For}_P^G[\dim G/P](\dim G/P) : \\ D^{\mathrm{b}}_G(G \times^P (\mathcal{N}_L + \mathfrak{u}_P), \Bbbk) \to D^{\mathrm{b}}_P(\mathcal{N}_L + \mathfrak{u}_P, \Bbbk)$$

is an equivalence of categories. The **parabolic induction functor**, denoted by

$$\mathrm{ind}_{L\subset P}^G : D^{\mathrm{b}}_L(\mathcal{N}_L, \Bbbk) \to D^{\mathrm{b}}_G(\mathcal{N}_G, \Bbbk),$$

is the functor given by

$$\mathrm{ind}_{L\subset P}^G(\mathcal{F}) = \mu_{P*}(e_P^* \mathrm{For}_P^G[-\dim G/P])^{-1} \pi_P^\dagger \mathrm{Infl}_L^P(\mathcal{F})(\dim G/P). \tag{8.4.3}$$

Since the relative dimension of π_P is $\dim G/P$, this formula can be rewritten as

$$\mathrm{ind}_{L\subset P}^G(\mathcal{F}) \cong \mu_{P*}(e_P^* \mathrm{For}_P^G)^{-1} \pi_P^! \mathrm{Infl}_L^P(\mathcal{F}) \cong \mu_{P*}(e_P^! \mathrm{For}_P^G)^{-1} \pi_P^* \mathrm{Infl}_L^P(\mathcal{F}).$$

The **parabolic restriction** and **corestriction functors**, denoted by

$$\mathrm{res}_{L\subset P}^G, {}^{\backprime}\mathrm{res}_{L\subset P}^G : D^{\mathrm{b}}_L(\mathcal{N}_L, \Bbbk) \to D^{\mathrm{b}}_G(\mathcal{N}_G, \Bbbk),$$

respectively, are the functors given by

$$\mathrm{res}_{L\subset P}^G(\mathcal{F}) = \pi_{P!} e_P^* \mu_P^* \mathrm{For}_L^G(\mathcal{F}) \cong \pi_{P!} i_P^* \mathrm{For}_L^G(\mathcal{F}),$$
$${}^{\backprime}\mathrm{res}_{L\subset P}^G(\mathcal{F}) = \pi_{P*} e_P^! \mu_P^! \mathrm{For}_L^G(\mathcal{F}) \cong \pi_{P*} i_P^! \mathrm{For}_L^G(\mathcal{F}).$$

EXAMPLE 8.4.1. Consider a maximal torus T and a Borel subgroup B. The nilpotent cone $\mathcal{N}_T$ is a single point, and the induction diagram is

$$\mathrm{pt} \xleftarrow{a_{\mathfrak{u}}} \mathfrak{u} \xrightarrow{e_B} \tilde{\mathcal{N}} \xrightarrow{\mu} \mathcal{N}_G$$

From the definitions, we find that

$$\mathrm{ind}_{T\subset B}^G \underline{\Bbbk}_{\mathrm{pt}} \cong \mu_* \underline{\Bbbk}_{\tilde{\mathcal{N}}}[\dim \mathcal{N}_G](\tfrac{1}{2}\dim \mathcal{N}_G) \cong \mathrm{Spr}.$$

LEMMA 8.4.2. *For $\mathcal{F} \in D^{\mathrm{b}}_L(\mathcal{N}_L, \Bbbk)$, there are natural isomorphisms*

$$\mathrm{ind}_{L\subset P}^G(\mathcal{F}) \cong \mathrm{Av}_{P*}^G i_{P*} \pi_P^! \mathrm{Infl}_L^P(\mathcal{F}) \cong \mathrm{Av}_{P!}^G i_{P*} \pi_P^* \mathrm{Infl}_L^P(\mathcal{F}).$$

PROOF. Consider the following commutative diagram:

$$\begin{tikzcd} \mathcal{N}_L + \mathfrak{u}_P \arrow[r, "i_P"] \arrow[d, "e_P"'] & \mathcal{N}_G \arrow[d, "e_P'"] \arrow[dr, "\mu_P"] & \\ G \times^P (\mathcal{N}_L + \mathfrak{u}_P) \arrow[r, "i_P'"] \arrow[rr, bend right, "\mu_P"'] & G \times^P \mathcal{N}_G \arrow[r, "\bar{\sigma}"] & \mathcal{N}_G \end{tikzcd}$$

Here, e_P, e'_P, and i'_P are all inclusion maps, and $\bar{\sigma}$ is the "action" map, defined by the same formula as μ_P. The left-hand square is cartesian, and the horizontal maps are proper, so we have

$$(e'^*_P \circ \mathrm{For}^G_P) \circ i'_{P*} \cong i_{P*} \circ (e^*_P \circ \mathrm{For}^G_P).$$

By Theorem 6.5.10, $e'^*_P \circ \mathrm{For}^G_P$ and $e^*_P \circ \mathrm{For}^G_P$ are equivalences of categories, so

$$i'_{P*} \circ (e^*_P \circ \mathrm{For}^G_P)^{-1} \cong (e'^*_P \circ \mathrm{For}^G_P)^{-1} \circ i_{P*}.$$

Recall from the proof of Theorem 6.6.1 that $\mathrm{Av}^G_{P*} \cong \bar{\sigma}_* \circ (e'^*_P \circ \mathrm{For}^G_P)^{-1}$.

Let $\mathcal{F}' = \pi^!_P \mathrm{Infl}^P_L(\mathcal{F})$. From the discussion above, we have

$$\begin{aligned} \mathrm{ind}^G_{L\subset P}(\mathcal{F}) = \mu_{P*}(e^*_P \mathrm{For}^G_P)^{-1}(\mathcal{F}') &\cong \bar{\sigma}_* i'_{P*}(e^*_P \mathrm{For}^G_P)^{-1}(\mathcal{F}') \\ &\cong \bar{\sigma}_*(e'^*_P \circ \mathrm{For}^G_P)^{-1} i_{P*}(\mathcal{F}') \cong \mathrm{Av}^G_{P*} i_{P*} \pi^!_P \mathrm{Infl}^P_L(\mathcal{F}). \end{aligned}$$

The proof of the second formula in the statement of the lemma is similar. □

LEMMA 8.4.3. *The functor* $\mathrm{ind}^G_{L\subset P}$ *is right adjoint to* $\mathrm{res}^G_{L\subset P}$ *and left adjoint to* ${}^\backprime\mathrm{res}^G_{L\subset P}$.

PROOF. On the variety $\mathcal{N}_L$, Theorem 6.6.16 tells us that Infl^P_L is an equivalence of categories, with inverse For^P_L. Therefore, from the first formula in Lemma 8.4.2, $\mathrm{ind}^G_{L\subset P}$ is right adjoint to the functor

$$\mathrm{For}^P_L \circ \pi_{P!} \circ i^*_P \circ \mathrm{For}^G_P \cong \pi_{P!} \circ i^*_P \circ \mathrm{For}^G_L = \mathrm{res}^G_{L\subset P}.$$

The proof that $\mathrm{ind}^G_{L\subset P}$ is left adjoint to ${}^\backprime\mathrm{res}^G_{L\subset P}$ is similar. □

LEMMA 8.4.4. *Let $R \subset P \subset G$ be two parabolic subgroups containing B, and let $M \subset L$ be their respective Levi factors. For $\mathcal{F} \in D^{\mathrm{b}}_M(\mathcal{N}_M, \Bbbk)$, there is a natural isomorphism*

$$\mathrm{ind}^G_{L\subset P}\mathrm{ind}^L_{M\subset L\cap R}(\mathcal{F}) \cong \mathrm{ind}^G_{M\subset R}(\mathcal{F}).$$

For $\mathcal{G} \in D^{\mathrm{b}}_G(\mathcal{N}_G, \Bbbk)$, there are natural isomorphisms

$$\begin{aligned} \mathrm{res}^L_{M\subset L\cap R}\mathrm{res}^G_{L\subset P}(\mathcal{G}) &\cong \mathrm{res}^G_{M\subset R}(\mathcal{G}), \\ {}^\backprime\mathrm{res}^L_{M\subset L\cap R}{}^\backprime\mathrm{res}^G_{L\subset P}(\mathcal{G}) &\cong {}^\backprime\mathrm{res}^G_{M\subset R}(\mathcal{G}). \end{aligned}$$

PROOF. Consider the diagram

$$\begin{tikzcd} \mathcal{N}_M + \mathfrak{u}_R \arrow[r] \arrow[d] \arrow[rr, bend left, "i_R"] \arrow[dd, bend right, "\pi_R"'] & \mathcal{N}_L + \mathfrak{u}_P \arrow[r, "i_P"'] \arrow[d, "\pi_P"] & \mathcal{N}_G \\ \mathcal{N}_M + \mathfrak{u}_{L\cap R} \arrow[r, "i_{L\cap R}"] \arrow[d, "\pi_{L\cap R}"] & \mathcal{N}_L & \\ \mathcal{N}_M & & \end{tikzcd}$$

The square in the upper right is cartesian. The isomorphisms for the restriction and corestriction functors follow by proper base change. The isomorphism for the induction functor then follows by uniqueness of adjoints. □

EXAMPLE 8.4.5. Let $\mathrm{Spr}_L \in \mathrm{Perv}_L(\mathcal{N}_L, \Bbbk)$ and $\mathrm{Spr}_G \in \mathrm{Perv}_G(\mathcal{N}_G, \Bbbk)$ be the Springer sheaves for L and G, respectively. Combining Example 8.4.1 with transitivity of induction (Lemma 8.4.4), we see that there is a natural isomorphism

$$\mathrm{ind}_{L\subset P}^{G} \mathrm{Spr}_L \cong \mathrm{Spr}_G.$$

REMARK 8.4.6. Let G_1 and G_2 be two connected reductive groups, and identify $\mathcal{N}_{G_1\times G_2}$ with $\mathcal{N}_{G_1} \times \mathcal{N}_{G_2}$. Suppose we have Levi and parabolic subgroups $L_1 \subset P_1 \subset G_1$ and $L_2 \subset P_2 \subset G_2$. Since all the ingredients in the definition of the induction functor commute with $\boxtimes$, we have

$$\mathrm{ind}_{L_1\times L_2\subset P_1\times P_2}^{G_1\times G_2}(\mathcal{F}_1 \boxtimes \mathcal{F}_2) \cong (\mathrm{ind}_{L_1\subset P_1}^{G_1}\mathcal{F}_1) \boxtimes (\mathrm{ind}_{L_2\subset P_2}^{G_2}\mathcal{F}_2),$$

and likewise for res and ${}'\mathrm{res}$.

Since $\mathrm{Z}(G) \subset \mathrm{Z}(L)$, we have a natural map

$$\varphi : \mathrm{Z}(G)/\mathrm{Z}^\circ(G) \to \mathrm{Z}(L)/\mathrm{Z}^\circ(L). \tag{8.4.4}$$

LEMMA 8.4.7. *The following diagram commutes:*

$$\begin{tikzcd}[column sep=small]
 & \pi_1(G/\mathrm{Z}(G),1) \arrow[r, two heads] & \mathrm{Z}(G)/\mathrm{Z}^\circ(G) \arrow[dr, two heads, "\varphi"] & \\
\pi_1(P/\mathrm{Z}(G),1) \arrow[ur, two heads] \arrow[dr, "\sim"'] & & & \mathrm{Z}(L)/\mathrm{Z}^\circ(L) \\
 & \pi_1(L/\mathrm{Z}(G),1) \arrow[r, two heads] & \pi_1(L/\mathrm{Z}(L),1) \arrow[ur, two heads] &
\end{tikzcd}$$

Moreover, every map is surjective, and $\pi_1(P/\mathrm{Z}(G), 1) \to \pi_1(L/\mathrm{Z}(G), 1)$ *is an isomorphism.*

The unlabeled maps in this diagram are either the obvious maps of fundamental groups or the instances of (8.1.2).

PROOF SKETCH. The fact that φ is surjective can be proved by a study of the weight and root lattices of G and L; for a discussion, see [**30**, Corollaire 2.2].

The group $P/\mathrm{Z}(G)$ is the fiber of the fibration $G/\mathrm{Z}(G) \to G/P$. The surjectivity of $\pi_1(P/\mathrm{Z}(G), 1) \to \pi_1(G/\mathrm{Z}(G), 1)$ follows from the long exact sequence of homotopy groups (see [**98**, Theorem 4.41]), along with the well-known fact that the partial flag variety G/P is simply connected. Since $P/\mathrm{Z}(G) \cong L/\mathrm{Z}(G) \times U_P$, and since U_P is contractible, we have $\pi_1(P/\mathrm{Z}(G), 1) \cong \pi_1(L/\mathrm{Z}(G), 1)$. The commutativity of the diagram can be proved by studying the action of $\mathrm{Z}(G)/\mathrm{Z}^\circ(G)$ and $\mathrm{Z}(L)/\mathrm{Z}^\circ(L)$ by deck transformations on the various vertical maps (all of which are covering maps) in the following diagram:

$$\begin{tikzcd}
G/\mathrm{Z}^\circ(G) \arrow[dd] & L/\mathrm{Z}^\circ(G) \arrow[l] \arrow[d] & \\
 & L/(\mathrm{Z}^\circ(L)\cap \mathrm{Z}(G)) \arrow[r] \arrow[d] & L/\mathrm{Z}^\circ(L) \arrow[d] \\
G/\mathrm{Z}(G) & L/\mathrm{Z}(G) \arrow[l] \arrow[r] & L/\mathrm{Z}(L)
\end{tikzcd}$$

We omit further details. $\square$

PROPOSITION 8.4.8. *Let* $\chi : \mathrm{Z}(L)/\mathrm{Z}^\circ(L) \to \Bbbk^\times$ *be a character, and suppose* $\mathcal{F} \in \mathrm{Perv}_L(\mathcal{N}_L, \Bbbk)$ *has central character* χ. *Then* $\mathrm{ind}_{L\subset P}^{G}\mathcal{F}$ *has central character* $\chi \circ \varphi$, *where* φ *is as in* (8.4.4).

PROOF. The character χ determines characters of all the groups appearing in Lemma 8.4.7. In this proof, in a minor abuse of notation, we will simply denote all of these induced characters by χ. Note that $P/\mathrm{Z}(G)$ is a parabolic subgroup of $G/\mathrm{Z}(G)$ and that $L/\mathrm{Z}(G)$ is a Levi factor of $P/\mathrm{Z}(G)$. We identify

$$\mathcal{N}_{L/\mathrm{Z}(G)} = \mathcal{N}_L \qquad \text{and} \qquad (G/\mathrm{Z}(G)) \times^{P/\mathrm{Z}(G)} (\mathcal{N}_L + \mathfrak{u}_P) = G \times^P (\mathcal{N}_L + \mathfrak{u}_P).$$

Let us define a modified version ind_χ of $\mathrm{ind}^G_{L\subset P}$ as follows: in (8.4.3), replace Infl^P_L by the functor

$$\mathrm{Infl}^{P/\mathrm{Z}(G)}_{L/\mathrm{Z}(L)} : \mathrm{Perv}_{L/\mathrm{Z}(L),\chi}(\mathcal{N}_L, \Bbbk) \to \mathrm{Perv}_{P/\mathrm{Z}(G),\chi}(\mathcal{N}_L, \Bbbk)$$

from Exercise 6.3.1, and replace $(e^*_P \mathrm{For}^G_P[-\dim G/P])^{-1}$ by the functor

$$(e^*_{P/\mathrm{Z}(G)} \mathrm{For}^{G/\mathrm{Z}(G)}_{P/\mathrm{Z}(G)}[-\dim G/P])^{-1} : \\ \mathrm{Perv}_{P/\mathrm{Z}(G),\chi}(\mathcal{N}_L + \mathfrak{u}_P, \Bbbk) \xrightarrow{\sim} \mathrm{Perv}_{G/\mathrm{Z}(G),\chi}(G \times^P (\mathcal{N}_L + \mathfrak{u}_P), \Bbbk)$$

from Proposition 6.3.9. (This step depends on the fact that $\pi_1(P/\mathrm{Z}(G), 1) \to \pi_1(G/\mathrm{Z}(G), 1)$ is surjective.) By construction, ind_χ sends $(L/\mathrm{Z}(L), \chi)$-equivariant perverse sheaves to $(G/\mathrm{Z}(G), \chi)$-equivariant perverse sheaves. If $\mathcal{F}$ is an object of $\mathrm{Perv}_L(\mathcal{N}_L, \Bbbk)$ with central character χ, then it is easy to see that $\mathrm{ind}^G_{L\subset P}\mathcal{F} \cong \mathrm{ind}_\chi \mathcal{F}$, so we are done. ☐

PROPOSITION 8.4.9. *Assume that $\Bbbk$ is a Springer splitting field for G. Let $\chi : \mathrm{Z}(G)/\mathrm{Z}^\circ(G) \to \Bbbk^\times$ be a character, and let $\mathcal{F} \in \mathrm{Perv}_G(\mathcal{N}, \Bbbk)$ be a perverse sheaf with central character χ. If χ does not factor through $\varphi : \mathrm{Z}(G)/\mathrm{Z}^\circ(G) \to \mathrm{Z}(L)/\mathrm{Z}^\circ(L)$, then $\mathrm{res}^G_{L\subset P}\mathcal{F} = {}'\mathrm{res}^G_{L\subset P}\mathcal{F} = 0$.*

PROOF. Suppose ${}'\mathrm{res}^G_{L\subset P}\mathcal{F}$ is not zero, and choose a simple subobject $\mathcal{G} \subset {}'\mathrm{res}^G_{L\subset P}\mathcal{F}$. The inclusion map $\mathcal{G} \hookrightarrow {}'\mathrm{res}^G_{L\subset P}\mathcal{F}$ is a nonzero element of

$$\mathrm{Hom}(\mathcal{G}, {}'\mathrm{res}^G_{L\subset P}\mathcal{F}) \cong \mathrm{Hom}(\mathrm{ind}^G_{L\subset P}\mathcal{G}, \mathcal{F}). \tag{8.4.5}$$

By Lemma 8.1.4, $\mathcal{G}$ admits a central character $\psi : \mathrm{Z}(L)/\mathrm{Z}^\circ(L) \to \Bbbk^\times$. By Proposition 8.4.8, $\mathrm{ind}^G_{L\subset P}\mathcal{G}$ has central character $\psi \circ \varphi$. By assumption, we have $\chi \neq \psi \circ \varphi$. But then the nonvanishing of (8.4.5) contradicts Proposition 6.3.11(1). ☐

Exactness. Our next task is to show that the induction and restriction functors are t-exact, using the following result of Lusztig [**159**, Proposition 1.2(b)].

THEOREM 8.4.10. *Equip $G \times^P (\mathcal{N}_L + \mathfrak{u}_P)$ with the stratification*

$$\{G \times^P (\mathcal{O} + \mathfrak{u}_P) \mid \mathcal{O} \subset \mathcal{N}_L \text{ a nilpotent } L\text{-orbit}\}.$$

Then the map $\mu_P : G \times^P (\mathcal{N}_L + \mathfrak{u}_P) \to \mathcal{N}_G$ is stratified semismall.

The proof, which we omit, is broadly similar in structure to that of Lemma 8.1.6. We remark that [**159**] does not use the language of "stratified semismall maps"; instead, it proves the inequality below, which can easily be shown to be equivalent to Theorem 8.4.10: for $x \in \mathcal{O}$,

$$2 \dim \mu_P^{-1}(x) \cap (G \times^P (\mathcal{O}_L + \mathfrak{u}_P)) \leq (\dim \mathcal{N}_G - \dim \mathcal{O}) - (\dim \mathcal{N}_L - \dim \mathcal{O}_L).$$

THEOREM 8.4.11. *The functors $\mathrm{ind}^G_{L\subset P}$, $\mathrm{res}^G_{L\subset P}$, and ${}'\mathrm{res}^G_{L\subset P}$ are all t-exact for the perverse t-structure.*

Proof. Let $\mathcal{F} \in \mathrm{Perv}_L(\mathcal{N}_L, \Bbbk)$, and let

$$\mathcal{F}' = (e_P^* \mathrm{For}_P^G[-\dim G/P])^{-1} \pi_P^\dagger \mathrm{Infl}_L^P(\mathcal{F})(\dim G/P).$$

Each step in the definition of $\mathcal{F}'$ is t-exact (see Theorem 6.5.10 for the last step), so $\mathcal{F}'$ is a perverse sheaf on $G \times^P (\mathcal{N}_L + \mathfrak{u}_P)$. The object $\pi_P^\dagger \mathrm{Infl}_L^P(\mathcal{F})$ is constructible with respect to the stratification (8.4.1), so $\mathcal{F}'$ is constructible with respect to the stratification considered in Theorem 8.4.10. Thus, by Theorem 3.8.9, we see that $\mathrm{ind}_{L\subset P}^G(\mathcal{F}) = \mu_{P*}(\mathcal{F}')$ is perverse. In other words, $\mathrm{ind}_{L\subset P}^G$ is t-exact.

Any left adjoint of a t-exact functor is automatically right t-exact, and any right adjoint is left t-exact. Thus, $\mathrm{res}_{L\subset P}^G$ is right t-exact and ${}^{\backprime}\mathrm{res}_{L\subset P}^G$ is left t-exact.

Next, let $\bar{P}$ be the opposite parabolic to P (with respect to T), and let $\bar{\mathfrak{p}}$ be its Lie algebra. Of course, we could have carried out all our constructions using this parabolic instead. In particular, $\mathrm{res}_{L\subset \bar{P}}^G$ is right t-exact and ${}^{\backprime}\mathrm{res}_{L\subset \bar{P}}^G$ is left t-exact. We claim that there are natural isomorphisms

$$(8.4.6) \qquad {}^{\backprime}\mathrm{res}_{L\subset P}^G(\mathcal{F}) \cong \mathrm{res}_{L\subset \bar{P}}^G(\mathcal{F}), \qquad \mathrm{res}_{L\subset P}^G(\mathcal{F}) \cong {}^{\backprime}\mathrm{res}_{L\subset \bar{P}}^G(\mathcal{F}).$$

The rest of the theorem follows from this claim.

To prove (8.4.6), we use hyperbolic localization. Choose a cocharacter $\chi : \mathbb{G}_\mathrm{m} \to T$ whose pairing with all simple roots of L is 0 and whose pairing with every remaining simple root of G is strictly positive. Let $\mathbb{G}_\mathrm{m}$ act on $\mathfrak{g}$ by $\mathrm{Ad} \circ \chi$. Under this action, $\mathbb{G}_\mathrm{m}$ acts trivially on $\mathfrak{l}$; it acts with strictly positive weights on $\mathfrak{u}_P$, and with strictly negative weights on $\mathfrak{u}_{\bar{P}}$. Let us now focus our attention on the $\mathbb{G}_\mathrm{m}$-action on $\mathcal{N}_G$. In the notation of Definition 2.10.6, the fixed-point set is $\mathcal{N}_G^{\mathbb{G}_\mathrm{m}} = \mathcal{N}_G \cap \mathfrak{l} = \mathcal{N}_L$, and the attracting and repelling sets are given by

$$\mathcal{N}_L^+ = \mathcal{N}_L + \mathfrak{u}_P \qquad \text{and} \qquad \mathcal{N}_L^- = \mathcal{N}_L + \mathfrak{u}_{\bar{P}}.$$

Theorem 2.10.7 tells us that $\pi_{P*} i_P^!(\mathcal{F}) \cong \pi_{\bar{P}!} i_{\bar{P}}^*(\mathcal{F})$. This is the first isomorphism in (8.4.6). Replacing χ by its negative yields the other isomorphism. □

Induction and the Springer sheaf. By Example 8.4.5, the functor $\mathrm{ind}_{L\subset P}^G$ gives rise to a map

$$\mathrm{ind}_{L\subset P}^G : \mathrm{End}(\mathrm{Spr}_L) \to \mathrm{End}(\mathrm{Spr}_G).$$

We will conclude this section by describing this map explicitly in terms of the group rings of the Weyl groups of L and G, via the isomorphism of Theorem 8.2.9.

We will need to use the following variant of the induction diagram (8.4.2):

$$(8.4.7) \qquad \begin{tikzcd} \mathfrak{l} & \mathfrak{p} \arrow[l, hook', "\pi_P"'] \arrow[r, "e_P"'] \arrow[rr, bend left, "i_P = \mu_P \circ e_P"] & G \times^P \mathfrak{p} \arrow[r, "\mu_P"'] & \mathfrak{g} \end{tikzcd}$$

In a slight abuse of notation (justified by Lemma 8.4.12 below), we define functors

$$\mathrm{ind}_{L\subset P}^G : D_L^\mathrm{b}(\mathfrak{l}, \Bbbk) \to D_G^\mathrm{b}(\mathfrak{g}, \Bbbk),$$
$$\mathrm{res}_{L\subset P}^G, {}^{\backprime}\mathrm{res}_{L\subset P}^G : D_G^\mathrm{b}(\mathfrak{g}, \Bbbk) \to D_L^\mathrm{b}(\mathfrak{l}, \Bbbk)$$

by the same formulas as before:

$$\mathrm{ind}_{L\subset P}^G(\mathcal{F}) = \mu_{P*}(e_P^* \mathrm{For}_P^G[-\dim G/P])^{-1} \pi_P^\dagger \mathrm{Infl}_L^P(\mathcal{F})(\dim G/P),$$
$$\mathrm{res}_{L\subset P}^G(\mathcal{F}) = \pi_{P!} e_P^* \mu_P^* \mathrm{For}_L^G(\mathcal{F}) \cong \pi_{P!} i_P^* \mathrm{For}_L^G(\mathcal{F}),$$
$${}^{\backprime}\mathrm{res}_{L\subset P}^G(\mathcal{F}) = \pi_{P*} e_P^! \mu_P^! \mathrm{For}_L^G(\mathcal{F}) \cong \pi_{P*} i_P^! \mathrm{For}_L^G(\mathcal{F}).$$

Lemmas 8.4.2, 8.4.3, and 8.4.4 remain true for these versions of the induction and restriction functors, with the same proofs.

LEMMA 8.4.12. *Let* $h_L : \mathcal{N}_L \hookrightarrow \mathfrak{l}$ *and* $h_G : \mathcal{N}_G \hookrightarrow \mathfrak{g}$ *be the inclusion maps. Then, for* $\mathcal{F} \in D^{\mathrm{b}}_L(\mathfrak{l}, \Bbbk)$, *there is a natural isomorphism*

$$\mathrm{ind}^G_{L\subset P}(h_L^*\mathcal{F}) \cong h_G^*\mathrm{ind}^G_{L\subset P}\mathcal{F}.$$

Here, the functor $\mathrm{ind}^G_{L\subset P}$ on the left-hand side is defined using (8.4.2), while that on the right-hand side is defined using (8.4.7).

PROOF. Let us use the formula from Lemma 8.4.2 for the induction functors. The result follows by proper base change and Exercise 6.8.4, applied to the following diagram of cartesian squares:

$$\begin{tikzcd} \mathcal{N}_L \arrow[d, "h_L"'] & \mathcal{N}_L + \mathfrak{u}_P \arrow[l, "\pi_P"'] \arrow[r, "i_P"] \arrow[d] & \mathcal{N}_G \arrow[d, "h_G"] \\ \mathfrak{l} & \mathfrak{p} \arrow[l, "\pi_P"'] \arrow[r, "i_P"] & \mathfrak{g} \end{tikzcd}$$

□

EXAMPLE 8.4.13. Let us adapt Examples 8.4.1 and 8.4.5 to the setting considered in (8.4.7). When P is a Borel subgroup and L is a maximal torus, the last map in (8.4.7) is the map $\mu_{\tilde{\mathfrak{g}}} : \tilde{\mathfrak{g}} \to \mathfrak{g}$ from Section 8.2. In particular, we have

$$\mathrm{ind}^G_{T\subset B}\underline{\Bbbk}_{\mathfrak{h}}[\dim \mathfrak{h}] \cong \mu_{\tilde{\mathfrak{g}}*}\underline{\Bbbk}_{\tilde{\mathfrak{g}}}[\dim \mathfrak{g}](\tfrac{1}{2}\dim \mathcal{N}_G).$$

More generally, for a parabolic subgroup $P \subset G$ with Levi factor $L \subset G$, we have

$$\mathrm{ind}^G_{L\subset P}(\mu_{\tilde{\mathfrak{l}}*}\underline{\Bbbk}_{\tilde{\mathfrak{l}}}[\dim \tilde{\mathfrak{l}}](\tfrac{1}{2}\dim \mathcal{N}_L)) \cong \mu_{\tilde{\mathfrak{g}}*}\underline{\Bbbk}_{\tilde{\mathfrak{g}}}[\dim \mathfrak{g}](\tfrac{1}{2}\dim \mathcal{N}_G).$$

Here $\mu_{\tilde{\mathfrak{l}}} : \tilde{\mathfrak{l}} \to \mathfrak{l}$ is the Grothendieck–Springer simultaneous resolution for L.

THEOREM 8.4.14. *Let* W_L *denote the Weyl group of a Levi subgroup* L. *Consider the diagram*

$$\begin{tikzcd} \Bbbk[W_L] \arrow[r] \arrow[d, "\wr"] & \Bbbk[W] \arrow[d, "\wr"] \\ \mathrm{End}(\mathrm{Spr}_L) \arrow[r, "\mathrm{ind}^G_{L\subset P}"] & \mathrm{End}(\mathrm{Spr}_G) \end{tikzcd}$$

where the vertical maps come from Theorem 8.2.9, *and the top horizontal map comes from the inclusion* $W_L \hookrightarrow W$. *This diagram commutes.*

PROOF. The variety $\tilde{\mathfrak{l}} = L \times^{B\cap L} (\mathfrak{b} \cap \mathfrak{l})$ can be identified with $P \times^B (\mathfrak{b}/\mathfrak{u}_P)$. The latter description will be needed at various points in this proof.

Using the natural isomorphism of Lemma 8.4.12 and the observations of Example 8.4.13, we can form the commutative diagram

$$\begin{tikzcd} \mathrm{End}(\mu_{\tilde{\mathfrak{l}}*}\underline{\Bbbk}_{\tilde{\mathfrak{l}}}[\dim \mathfrak{l}]) \arrow[r, "\mathrm{ind}^G_{L\subset P}"] \arrow[d, "h_L^*"', "\wr"] & \mathrm{End}(\mu_{\tilde{\mathfrak{g}}*}\underline{\Bbbk}_{\tilde{\mathfrak{g}}}[\dim \mathfrak{g}]) \arrow[d, "h_G^*", "\wr"'] \\ \mathrm{End}(\mathrm{Spr}_L) \arrow[r, "\mathrm{ind}^G_{L\subset P}"] & \mathrm{End}(\mathrm{Spr}_G) \end{tikzcd} \tag{8.4.8}$$

According to Theorem 8.2.9, the vertical maps are isomorphisms.

Next, let $\mathfrak{l}_{\mathrm{rs}}$ be the set of regular semisimple elements in $\mathfrak{l}$, and let $\tilde{\mathfrak{l}}_{\mathrm{rs}} = \mu_{\tilde{\mathfrak{l}}}^{-1}(\mathfrak{l}_{\mathrm{rs}})$. (Note that $\mathfrak{l}_{\mathrm{rs}}$ is *not* the same as $\mathfrak{g}_{\mathrm{rs}} \cap \mathfrak{l}$: a regular element of $\mathfrak{l}$ may not be regular when regarded as an element of $\mathfrak{g}$.) Restricting $\mu_{\tilde{\mathfrak{l}}}$ and $\mu_{\tilde{\mathfrak{g}}}$, we obtain maps

$$\mu_{L,\mathrm{rs}} : \tilde{\mathfrak{l}}_{\mathrm{rs}} \to \mathfrak{l}_{\mathrm{rs}} \qquad \text{and} \qquad \mu_{G,\mathrm{rs}} : \tilde{\mathfrak{g}}_{\mathrm{rs}} \to \mathfrak{g}_{\mathrm{rs}}.$$

We next consider the diagram

$$\begin{tikzcd}
\mathrm{End}(\mu_{L,\mathrm{rs}*}\underline{\Bbbk}_{\tilde{\mathfrak{l}}_{\mathrm{rs}}}) \arrow[r, dashed] & \mathrm{End}(\mu_{G,\mathrm{rs}*}\underline{\Bbbk}_{\tilde{\mathfrak{g}}_{\mathrm{rs}}}) \\
\mathrm{End}(\mu_{\tilde{\mathfrak{l}}*}\underline{\Bbbk}_{\tilde{\mathfrak{l}}}[\dim \mathfrak{l}]) \arrow[u, "\wr"'] \arrow[r, "\mathrm{ind}_{L\subset P}^G"] & \mathrm{End}(\mu_{\tilde{\mathfrak{g}}*}\underline{\Bbbk}_{\tilde{\mathfrak{g}}}[\dim \mathfrak{g}]) \arrow[u, "\wr"]
\end{tikzcd} \tag{8.4.9}$$

where the vertical arrows come from the fact that the objects in the bottom row are intersection cohomology complexes (Proposition 8.2.6). There is a unique map that can be filled in across the top of this diagram to make it commute.

To determine this map, we must study the relationship between $\tilde{\mathfrak{l}}_{\mathrm{rs}}$ and $\tilde{\mathfrak{g}}_{\mathrm{rs}}$. Let

$$\mathfrak{b}^L_{\mathrm{rs}} = \mathfrak{b} \cap \pi_P^{-1}(\mathfrak{l}_{\mathrm{rs}}) \qquad \mathfrak{b}_{\mathrm{rs}} = \mathfrak{b} \cap \mathfrak{g}_{\mathrm{rs}}, \qquad \mathfrak{p}_{\mathrm{rs}} = \mathfrak{p} \cap \mathfrak{g}_{\mathrm{rs}}.$$

We claim that

$$\mathfrak{b}_{\mathrm{rs}} \subset \mathfrak{b}^L_{\mathrm{rs}}. \tag{8.4.10}$$

Indeed, recall that $\mathfrak{b}_{\mathrm{rs}} = \mathrm{Ad}(B)(\mathfrak{h}_{\mathrm{rs}})$. Since $\mathfrak{b}^L_{\mathrm{rs}}$ is stable under $\mathrm{Ad}(B)$, (8.4.10) follows from the observation that $\mathfrak{h}_{\mathrm{rs}} \subset \mathfrak{b}^L_{\mathrm{rs}}$.

Consider the diagram

$$\begin{tikzcd}
\tilde{\mathfrak{l}}_{\mathrm{rs}} \arrow[d, "W_L", "\mu_{L,\mathrm{rs}}"'] & P \times^B \mathfrak{b}^L_{\mathrm{rs}} \arrow[l, "\tilde{\pi}"'] \arrow[r, "\tilde{e}"] \arrow[d, "W_L"] & G \times^B \mathfrak{b}^L_{\mathrm{rs}} \arrow[d, "W_L"] & & \tilde{\mathfrak{g}}_{\mathrm{rs}} \arrow[ll, hook', "u"'] \arrow[d, "W_L"] \arrow[dr, "W", "\mu_{G,\mathrm{rs}}"'] & \\
\mathfrak{l}_{\mathrm{rs}} \arrow[d, hook] & \mathfrak{l}_{\mathrm{rs}} + \mathfrak{u}_P \arrow[l, "\pi_{\mathrm{rs}}"'] \arrow[r, "e_{\mathrm{rs}}"] \arrow[d, hook] & G \times^P (\mathfrak{l}_{\mathrm{rs}} + \mathfrak{u}_P) \arrow[dr] & & G \times^P \mathfrak{p}_{\mathrm{rs}} \arrow[ll, hook', "u'"'] \arrow[r, "\mu_{\mathfrak{p}_{\mathrm{rs}}}"] \arrow[dl] & \mathfrak{g}_{\mathrm{rs}} \arrow[d, hook] \\
\mathfrak{l} & \mathfrak{p} \arrow[l] \arrow[rr] & & G \times^P \mathfrak{p} \arrow[rr] & & \mathfrak{g}
\end{tikzcd} \tag{8.4.11}$$

Many of the maps here are Galois coverings with Galois group W_L; the diagonal map in the upper-right corner of the diagram is a Galois covering with Galois group W. The maps u and u' are open embeddings.

Let us compute the restriction to $\mathfrak{g}_{\mathrm{rs}}$ of $\mathrm{ind}^G_{L\subset P}(\mu_{\tilde{\mathfrak{l}}*}\underline{\Bbbk}_{\tilde{\mathfrak{l}}})$. For brevity, we omit the functors For^G_P and Infl^P_L from the notation below. By several instances of proper base change, we obtain a natural isomorphism

$$\begin{aligned}
\mu_{G,\mathrm{rs}*}\underline{\Bbbk}_{\tilde{\mathfrak{g}}_{\mathrm{rs}}} &\cong \left(\mathrm{ind}^G_{L\subset P}(\mu_{\tilde{\mathfrak{l}}*}\underline{\Bbbk}_{\tilde{\mathfrak{l}}})\right)|_{\mathfrak{g}_{\mathrm{rs}}} \\
&\cong \mu_{\mathfrak{p}_{\mathrm{rs}}*}(u')^*(e^!_{\mathrm{rs}})^{-1}\pi^*_{\mathrm{rs}}((\mu_{\tilde{\mathfrak{l}}*}\underline{\Bbbk}_{\tilde{\mathfrak{l}}})|_{\mathfrak{l}_{\mathrm{rs}}}) \cong \mu_{\mathfrak{p}_{\mathrm{rs}}*}(u')^*(e^!_{\mathrm{rs}})^{-1}\pi^*_{\mathrm{rs}}(\mu_{L,\mathrm{rs}*}\underline{\Bbbk}_{\tilde{\mathfrak{l}}_{\mathrm{rs}}}).
\end{aligned}$$

Thus, the unknown map along the top of (8.4.9) is induced by the functor

$$\mu_{\mathfrak{p}_{\mathrm{rs}}*}(u')^*(e^!_{\mathrm{rs}})^{-1}\pi^*_{\mathrm{rs}} : \mathrm{End}(\mu_{L,\mathrm{rs}*}\underline{\Bbbk}_{\tilde{\mathfrak{l}}_{\mathrm{rs}}}) \to \mathrm{End}(\mu_{G,\mathrm{rs}*}\underline{\Bbbk}_{\tilde{\mathfrak{g}}_{\mathrm{rs}}}). \tag{8.4.12}$$

Let $w \in W_L$, and let θ^L_w be the corresponding deck transformation of $\tilde{\mathfrak{l}}_{\mathrm{rs}} \to \mathfrak{l}_{\mathrm{rs}}$, as described in the proof of Lemma 8.2.3. As in Definition 1.7.21, this deck transformation gives rise to an internal monodromy operator

$$\mathrm{mon}_*(w) \in \mathrm{End}(\mu_{L,\mathrm{rs}*}\underline{\Bbbk}_{\tilde{\mathfrak{l}}_{\mathrm{rs}}}),$$

defined by applying $\mu_{L,\mathrm{rs}*}$ to the adjunction map $\underline{\Bbbk}_{\tilde{\mathfrak{l}}_{\mathrm{rs}}} \to \theta^L_{w*}\theta^{L*}_w \underline{\Bbbk}_{\tilde{\mathfrak{l}}_{\mathrm{rs}}}$.

The map θ^L_w also gives rise to deck transformations of each of the other vertical maps in the top part of (8.4.11):

(8.4.13)
$$\begin{tikzcd}
\tilde{\mathfrak{l}}_{\mathrm{rs}} \arrow[d, "\theta^L_w"'] & P \times^B \mathfrak{b}^L_{\mathrm{rs}} \arrow[l, "\tilde{\pi}"'] \arrow[r, "\tilde{e}"] \arrow[d, "\theta'_w"'] & G \times^B \mathfrak{b}^L_{\mathrm{rs}} \arrow[d, "\theta''_w"'] & \tilde{\mathfrak{g}}_{\mathrm{rs}} \arrow[l, hook', "u"'] \arrow[d, "\theta^G_w"] & \\
\tilde{\mathfrak{l}}_{\mathrm{rs}} \arrow[d, "W_L" description, "\mu_{L,\mathrm{rs}}"'] & P \times^B \mathfrak{b}^L_{\mathrm{rs}} \arrow[l, "\tilde{\pi}"'] \arrow[r, "\tilde{e}"] \arrow[d, "W_L" description] & G \times^B \mathfrak{b}^L_{\mathrm{rs}} \arrow[d, "W_L" description] & \tilde{\mathfrak{g}}_{\mathrm{rs}} \arrow[l, hook', "u"'] \arrow[d, "W_L" description] \arrow[dr, "W" description, "\mu_{G,\mathrm{rs}}"] & \\
\mathfrak{l}_{\mathrm{rs}} & \mathfrak{l}_{\mathrm{rs}} + \mathfrak{u}_P \arrow[l, "\pi_{\mathrm{rs}}"'] \arrow[r, "e_{\mathrm{rs}}"] & G \times^P (\mathfrak{l}_{\mathrm{rs}} + \mathfrak{u}_P) & G \times^P \mathfrak{p}_{\mathrm{rs}} \arrow[l, hook', "u'"'] \arrow[r, "\mu_{\mathfrak{p}_{\mathrm{rs}}}"] & \mathfrak{g}_{\mathrm{rs}}
\end{tikzcd}$$

By several applications of Proposition 1.6.4, we see that the sequence of functors in (8.4.12) takes the adjunction map $\mathrm{id} \to \theta^L_{w*}\theta^{L*}_w$ to the adjunction map $\mathrm{id} \to \theta^G_{w*}\theta^{G*}_w$. In other words,

(8.4.14)
$$\begin{tikzcd}
\mathrm{mon}_*(w) \arrow[r, mapsto] \arrow[d, phantom, "\cap" rotate=90] & \mathrm{mon}_*(w) \arrow[d, phantom, "\cap" rotate=90] \\
\mathrm{End}(\mu_{L,\mathrm{rs}*}\underline{\Bbbk}_{\tilde{\mathfrak{l}}_{\mathrm{rs}}}) \arrow[r, "\mu_{\mathfrak{p}_{\mathrm{rs}}*}(u')^*(e^!_{\mathrm{rs}})^{-1}\pi^*_{\mathrm{rs}}"] & \mathrm{End}(\mu_{G,\mathrm{rs}*}\underline{\Bbbk}_{\tilde{\mathfrak{g}}_{\mathrm{rs}}})
\end{tikzcd}$$

The theorem follows by combining (8.4.8), (8.4.9), (8.4.12), and (8.4.14). $\square$

Exercises.

8.4.1. Show that $\mathrm{ind}^G_{L\subset P}$ commutes with Verdier duality, and that there is a natural isomorphism $\mathbb{D} \circ \mathrm{res}^G_{L\subset P} \cong {}'\mathrm{res}^G_{L\subset P} \circ \mathbb{D}$.

8.4.2. Show that when $\Bbbk$ has characteristic 0, the functors $\mathrm{ind}^G_{L\subset P}$, $\mathrm{res}^G_{L\subset P}$, and ${}'\mathrm{res}^G_{L\subset P}$ all take semisimple complexes to semisimple complexes.

8.5. The generalized Springer correspondence

Recall that the Springer correspondence ν is, in general, not surjective. In this section, we briefly describe (without proof) how to extend ν to a map that accounts for all simple G-equivariant perverse sheaves on $\mathcal{N}$.

Definition 8.5.1. Let $\Bbbk$ be a field. A simple perverse sheaf $\mathrm{IC}(\mathcal{O}, \mathcal{L}) \in \mathrm{Perv}_G(\mathcal{N}_G, \Bbbk)$ is said to be **cuspidal** if for every proper parabolic subgroup $P \subsetneq G$ with Levi factor $L \subset P$, we have $\mathrm{res}^G_{L\subset P}\mathrm{IC}(\mathcal{O}, \mathcal{L}) = 0$.

Lemma 8.5.2. *Let $\Bbbk$ be a field, and let $\mathrm{IC}(\mathcal{O}, \mathcal{L}) \in \mathrm{Perv}_G(\mathcal{N}_G, \Bbbk)$ be a simple perverse sheaf. The following conditions are equivalent:*

(1) *The perverse sheaf $\mathrm{IC}(\mathcal{O}, \mathcal{L})$ is cuspidal.*
(2) *The perverse sheaf $\mathbb{D}\mathrm{IC}(\mathcal{O}, \mathcal{L})$ is cuspidal.*
(3) *For every proper parabolic subgroup $P \subsetneq G$ with Levi factor $L \subset P$, we have ${}'\mathrm{res}^G_{L\subset P}\mathrm{IC}(\mathcal{O}, \mathcal{L}) = 0$.*

Proof. We saw in the proof of Theorem 8.4.11 (see especially (8.4.6)) that the functor ${}'\mathrm{res}^G_{L\subset P}$ is isomorphic to $\mathrm{res}^G_{L\subset \bar{P}}$ for some other parabolic subgroup $\bar{P}$. It follows immediately that conditions (1) and (3) are equivalent. The equivalence of (1) and (2) then follows from Exercise 8.4.1. $\square$

PROPOSITION 8.5.3. *Every simple perverse sheaf* $\mathrm{IC}(\mathcal{O},\mathcal{L}) \in \mathrm{Perv}_G(\mathcal{N}_G,\Bbbk)$ *is a quotient of a perverse sheaf of the form* $\mathrm{ind}_{L\subset P}^G \mathrm{IC}(\mathcal{O}',\mathcal{L}')$*, where* $\mathrm{IC}(\mathcal{O}',\mathcal{L}') \in \mathrm{Perv}_L(\mathcal{N}_L,\Bbbk)$ *is a cuspidal simple perverse sheaf.*

PROOF. Note that $\mathrm{IC}(\mathcal{O},\mathcal{L}) \cong \mathrm{ind}_{G\subset G}^G \mathrm{IC}(\mathcal{O},\mathcal{L})$, so if $\mathrm{IC}(\mathcal{O},\mathcal{L})$ is itself cuspidal, the conclusion of the proposition holds trivially.

We proceed by induction on the semisimple rank of G. If G is a torus, then the unique simple perverse sheaf on $\mathcal{N}_G = \mathrm{pt}$ is trivially cuspidal (there are no proper parabolic subgroups), and we are done by the remark above. Assume now that the semisimple rank of G is at least 1. We may as well also assume that $\mathrm{IC}(\mathcal{O},\mathcal{L})$ is not cuspidal, so (by Lemma 8.5.2) there is some $L \subset P$ such that ${}'\mathrm{res}_{L\subset P}^G \mathrm{IC}(\mathcal{O},\mathcal{L}) \neq 0$. Let $\mathcal{F}$ be a simple subobject of ${}'\mathrm{res}_{L\subset P}^G \mathrm{IC}(\mathcal{O},\mathcal{L})$. By induction, there is a parabolic subgroup $R' \subset L$, a Levi subgroup $M \subset R'$, and a cuspidal perverse sheaf $\mathrm{IC}(\mathcal{O}',\mathcal{L}') \in \mathrm{Perv}_M(\mathcal{N}_M,\Bbbk)$ such that $\mathcal{F}$ is a quotient of $\mathrm{ind}_{M\subset R'}^L \mathrm{IC}(\mathcal{O}',\mathcal{L}')$. By construction, the composition

$$\mathrm{ind}_{M\subset R'}^L \mathrm{IC}(\mathcal{O}',\mathcal{L}') \twoheadrightarrow \mathcal{F} \hookrightarrow {}'\mathrm{res}_{L\subset P}^G \mathrm{IC}(\mathcal{O},\mathcal{L})$$

is nonzero. Now let $R = R'U_P$. This is a parabolic subgroup of G. The preceding remarks tell us that the following Hom-spaces are nonzero:

$$\begin{aligned} &\mathrm{Hom}(\mathrm{ind}_{M\subset R'}^L \mathrm{IC}(\mathcal{O}',\mathcal{L}'), {}'\mathrm{res}_{L\subset P}^G \mathrm{IC}(\mathcal{O},\mathcal{L})) \\ &\qquad \cong \mathrm{Hom}(\mathrm{ind}_{L\subset P}^G \mathrm{ind}_{M\subset R'}^L \mathrm{IC}(\mathcal{O}',\mathcal{L}'), \mathrm{IC}(\mathcal{O},\mathcal{L})) \\ &\qquad \cong \mathrm{Hom}(\mathrm{ind}_{M\subset R}^G \mathrm{IC}(\mathcal{O}',\mathcal{L}'), \mathrm{IC}(\mathcal{O},\mathcal{L})). \end{aligned}$$

Since $\mathrm{IC}(\mathcal{O},\mathcal{L})$ is simple, any nonzero map $\mathrm{ind}_{M\subset R}^G \mathrm{IC}(\mathcal{O}',\mathcal{L}') \to \mathrm{IC}(\mathcal{O},\mathcal{L})$ is surjective. In other words, $\mathrm{IC}(\mathcal{O},\mathcal{L})$ is a quotient of $\mathrm{ind}_{M\subset R}^G \mathrm{IC}(\mathcal{O}',\mathcal{L}')$. ☐

DEFINITION 8.5.4. A **cuspidal datum** for G is a triple $(L,\mathcal{O},\mathcal{L})$ where $L \subset G$ is a Levi subgroup, $\mathcal{O} \subset \mathcal{N}_L$ is a nilpotent L-orbit, and $\mathcal{L} \in \mathrm{Irr}(\mathrm{Loc}_L^{\mathrm{ft}}(\mathcal{O},\Bbbk))$ is (the isomorphism class of) an irreducible L-equivariant local system on $\mathcal{O}$ such that $\mathrm{IC}(\mathcal{O},\mathcal{L})$ is a cuspidal simple perverse sheaf. The set of cuspidal data for G is denoted by $\mathrm{Cusp}(G)$.

The **induction series** associated to a cuspidal datum $(L,\mathcal{O},\mathcal{L})$ is the subset

$$\mathrm{Irr}_{(L,\mathcal{O},\mathcal{L})}(\mathrm{Perv}_G(\mathcal{N}_G,\Bbbk)) \subset \mathrm{Irr}(\mathrm{Perv}_G(\mathcal{N}_G,\Bbbk))$$

defined by

$$\begin{aligned} &\mathrm{Irr}_{(L,\mathcal{O},\mathcal{L})}(\mathrm{Perv}_G(\mathcal{N}_G,\Bbbk)) \\ &\qquad = \left\{ \mathcal{F} \in \mathrm{Irr}(\mathrm{Perv}_G(\mathcal{N}_G,\Bbbk)) \;\middle|\; \begin{array}{c} \text{for some parabolic subgroup } P \subset G \\ \text{with Levi factor } L,\ \mathcal{F} \text{ is a} \\ \text{quotient of } \mathrm{ind}_{L\subset P}^G \mathrm{IC}(\mathcal{O},\mathcal{L}) \end{array} \right\}. \end{aligned}$$

The group G acts on the set of cuspidal data by conjugation, i.e., by

$$g\cdot(L,\mathcal{O},\mathcal{L}) \cong (gLg^{-1}, \mathrm{Ad}(g)(\mathcal{O}), \mathrm{Ad}(g^{-1})^*\mathcal{L}).$$

We will write $\mathrm{Cusp}(G)/G$ for the set of G-orbits on $\mathrm{Cusp}(G)$. Given a cuspidal datum $(L,\mathcal{O},\mathcal{L})$, we denote its orbit in $\mathrm{Cusp}(G)/G$ by $[L,\mathcal{O},\mathcal{L}]$. It is an exercise to check that any two G-conjugate cuspidal data have the same induction series. It thus makes sense to speak of the induction series associated to an element $[L,\mathcal{O},\mathcal{L}] \in \mathrm{Cusp}(G)/G$. This series is denoted by

$$\mathrm{Irr}_{[L,\mathcal{O},\mathcal{L}]}(\mathrm{Perv}_G(\mathcal{N}_G,\Bbbk)).$$

EXAMPLE 8.5.5. Consider the maximal torus T. The unique simple perverse sheaf $\mathrm{IC}(\mathcal{O}_0, \underline{\Bbbk})$ on $\mathcal{N}_T = \mathrm{pt}$ is trivially cuspidal (as there are no proper parabolic subgroups), so $(T, \mathcal{O}_0, \underline{\Bbbk})$ is a cuspidal datum. From Example 8.4.1 and Theorem 8.3.1, we see that the induction series

$$\mathrm{Irr}_{[T,\mathcal{O}_0,\underline{\Bbbk}]}(\mathrm{Perv}_G(\mathcal{N}_G, \Bbbk))$$

is precisely the set of isomorphism classes of simple perverse sheaves that occur in the Springer correspondence. This induction series is called the **principal series**.

Proposition 8.5.3 says that every simple perverse sheaf belongs to some induction series. The following result upgrades this.

THEOREM 8.5.6. *Every simple perverse sheaf in* $\mathrm{Perv}_G(\mathcal{N}_G, \Bbbk)$ *lies in the induction series of a unique cuspidal datum up to G-conjugacy. In other words, we have a disjoint union*

$$\mathrm{Irr}(\mathrm{Perv}_G(\mathcal{N}_G, \Bbbk)) = \bigsqcup_{[L,\mathcal{O},\mathcal{L}] \in \mathrm{Cusp}(G)/G} \mathrm{Irr}_{[L,\mathcal{O},\mathcal{L}]}(\mathrm{Perv}_G(\mathcal{N}_G, \Bbbk)).$$

When $\Bbbk = \mathbb{Q}$, this theorem is equivalent to [**159**, Proposition 6.3]. For general $\Bbbk$, this result is [**10**, Theorem 2.5].

We will not prove Theorem 8.5.6, but we will briefly indicate the main steps. Let $(L_1, \mathcal{O}_1, \mathcal{L}_1)$ and $(L_2, \mathcal{O}_2, \mathcal{L}_2)$ be two nonconjugate cuspidal data. The main task is to show that their induction series are disjoint. Let P_1 and P_2 be parabolic subgroups with Levi factors L_1 and L_2, respectively.

(1) One first shows that the set of isomorphism classes of simple subobjects of $\mathrm{ind}^G_{L_1 \subset P_1} \mathrm{IC}(\mathcal{O}_1, \mathcal{L}_1)$ coincides with the set of isomorphism classes of simple quotients. See [**9**, Remark 2.2].

(2) If $\mathrm{IC}(\mathcal{O}, \mathcal{L})$ is a simple perverse sheaf on $\mathcal{N}_G$ that belongs to the induction series of both cuspidal data, then it is a quotient of $\mathrm{ind}^G_{L_1 \subset P_1} \mathrm{IC}(\mathcal{O}_1, \mathcal{L}_1)$ and a subobject of $\mathrm{ind}^G_{L_2 \subset P_2} \mathrm{IC}(\mathcal{O}_2, \mathcal{L}_2)$, so the Hom-group

$$\begin{aligned} \mathrm{Hom}(\mathrm{ind}^G_{L_1 \subset P_1} \mathrm{IC}(\mathcal{O}_1, \mathcal{L}_1), \mathrm{ind}^G_{L_2 \subset P_2} \mathrm{IC}(\mathcal{O}_2, \mathcal{L}_2)) \\ \cong \mathrm{Hom}(\mathrm{IC}(\mathcal{O}_1, \mathcal{L}_1), {}'\mathrm{res}^G_{L_1 \subset P_1} \mathrm{ind}^G_{L_2 \subset P_2} \mathrm{IC}(\mathcal{O}_2, \mathcal{L}_2)) \end{aligned} \tag{8.5.1}$$

is nonzero.

(3) The hardest step is the analysis of the functor ${}'\mathrm{res}^G_{L_1 \subset P_1} \mathrm{ind}^G_{L_2 \subset P_2}$. There is a kind of "Mackey formula" (see [**10**, Theorem 2.2]) for this functor: for any $\mathcal{F} \in \mathrm{Perv}_{L_2}(\mathcal{N}_{L_2}, \Bbbk)$, the perverse sheaf ${}'\mathrm{res}^G_{L_1 \subset P_1} \mathrm{ind}^G_{L_2 \subset P_2}(\mathcal{F})$ admits a filtration whose subquotients are of the form

$$\mathrm{ind}^{L_1}_{L_1 \cap (gL_2g^{-1}) \subset L_1 \cap (gP_2g^{-1})}({}'\mathrm{res}^{gL_2g^{-1}}_{L_1 \cap (gL_2g^{-1}) \subset P_1 \cap (gL_2g^{-1})}(\mathrm{Ad}(g^{-1})^* \mathcal{F})).$$

(4) Now apply the preceding step to $\mathcal{F} = \mathrm{IC}(\mathcal{O}_2, \mathcal{L}_2)$. Because this perverse sheaf is cuspidal, the objects ${}'\mathrm{res}^{gL_2g^{-1}}_{L_1 \cap (gL_2g^{-1}) \subset P_1 \cap (gL_2g^{-1})}(\mathrm{Ad}(g^{-1})^* \mathcal{F})$ all vanish. It follows that ${}'\mathrm{res}^G_{L_1 \subset P_1} \mathrm{ind}^G_{L_2 \subset P_2} \mathrm{IC}(\mathcal{O}_2, \mathcal{L}_2) = 0$, contradicting the nonvanishing of (8.5.1).

The next theorem gives a concrete parametrization of the elements of an induction series.

THEOREM 8.5.7. *Assume that $\Bbbk$ is a Springer splitting field for G, and let $(L, \mathcal{O}, \mathcal{L})$ be a cuspidal datum for G. There is a bijection*

$$\nu_{(L,\mathcal{O},\mathcal{L})} : \mathrm{Irr}(\Bbbk[\mathrm{N}_G(L)/L]) \xrightarrow{\sim} \mathrm{Irr}_{(L,\mathcal{O},\mathcal{L})}(\mathrm{Perv}_G(\mathcal{N}_G, \Bbbk)).$$

Combining Theorems 8.5.6 and 8.5.7, we obtain a bijection

$$\nu : \bigsqcup_{[L,\mathcal{O},\mathcal{L}]\in \mathrm{Cusp}(G)/G} \mathrm{Irr}(\Bbbk[\mathrm{N}_G(L)/L]) \xrightarrow{\sim} \mathrm{Irr}(\mathrm{Perv}_G(\mathcal{N}_G, \Bbbk)).$$

This bijection is called the **generalized Springer correspondence**. By restricting this map to the part of its domain labelled by $[T, \mathcal{O}_0, \underline{\Bbbk}]$, we recover the ordinary Springer correspondence of Theorem 8.3.1.

We conclude with an empirical observation about cuspidal perverse sheaves.

THEOREM 8.5.8. *Let $\Bbbk$ be a Springer splitting field for G, and assume that its characteristic is good for G. For any character $\chi : \mathrm{Z}(G)/\mathrm{Z}^\circ(G) \to \Bbbk^\times$, there is at most one cuspidal simple perverse sheaf in* $\mathrm{Perv}_G(\mathcal{N}_G, \Bbbk)$ *with central character χ.*

When $\Bbbk = \mathbb{Q}$, this was proved in [**159**]; for the general case, see [**10**, Theorem 1.5]. The proof consists of classifying (or at least counting) the cuspidal simple perverse sheaves for each quasi-simple G, and then examining the list. A conceptual explanation seems to still be lacking, and the statement is false in bad characteristic.

A closely related observation from [**155**] is that when $\Bbbk = \mathbb{Q}$, cuspidal perverse sheaves are clean in the sense of Exercise 3.10.5. (See Exercise 8.5.1 below.) When $\mathrm{char}\,\Bbbk > 0$, many cuspidal perverse sheaves fail to be clean, but Mautner has conjectured (and it has been checked in some cases) that cleanness continues to hold in certain cases. See [**8**, Conjecture 1.2].

Exercises.

8.5.1. Let $\mathrm{IC}(\mathcal{O}, \mathcal{L}) \in \mathrm{Perv}_G(\mathcal{N}, \mathbb{Q})$ be a cuspidal simple perverse sheaf. Let $j : \mathcal{O} \hookrightarrow \mathcal{N}$ be the inclusion map. Using Theorem 8.5.8, prove that if $\mathrm{IC}(\mathcal{O}, \mathcal{L}) \in \mathrm{Perv}_G(\mathcal{N}, \mathbb{Q})$ is a cuspidal simple perverse sheaf, then

$$\mathrm{IC}(\mathcal{O}, \mathcal{L}) \cong j_!\mathcal{L}[\dim \mathcal{O}] \cong j_*\mathcal{L}[\dim \mathcal{O}],$$

where $j : \mathcal{O} \hookrightarrow \mathcal{N}$ is the inclusion map. In other words, $\mathrm{IC}(\mathcal{O}, \mathcal{L})$ is **clean**. *Hint*: To prove that $\mathrm{IC}(\mathcal{O}, \mathcal{L}) \cong j_!\mathcal{L}[\dim \mathcal{O}]$, first explain how to reduce the problem to the claim that

$$\mathrm{Hom}(\mathrm{IC}(\mathcal{O}, \mathcal{L}), \mathrm{IC}(\mathcal{O}', \mathcal{L}')[k]) = 0$$

for any $\mathcal{O}' \subset \overline{\mathcal{O}} \smallsetminus \mathcal{O}$ and any $\Bbbk \in \mathbb{Z}$. Thanks to Exercise 8.4.2, each such $\mathrm{IC}(\mathcal{O}', \mathcal{L}')$ is either cuspidal or a direct summand of an induced perverse sheaf.

8.6. Additional exercises

EXERCISE 8.6.1. In this exercise, you will establish a necessary condition for a perverse sheaf to be cuspidal, following [**159**, Proposition 2.8] and [**9**, Proposition 2.6]. Let $\mathcal{O} \subset \mathcal{N}_G$ be a nilpotent orbit, and let $\mathcal{L}$ be a local system on $\mathcal{O}$. Let $L \subset P$ be a Levi subgroup and a parabolic subgroup. Assume that $\mathcal{O} \cap \mathcal{N}_L \neq \varnothing$, and let $x \in \mathcal{O} \cap \mathcal{N}_L$.

(a) Show that $(x + \mathfrak{u}_P) \cap \overline{\mathcal{O}} = (x + \mathfrak{u}_P) \cap \mathcal{O}$.

(b) Show that the stalk of $\mathrm{res}^G_{L\subset P}\mathrm{IC}(\mathcal{O},\mathcal{L})$ at x is given by

$$\mathsf{H}^k(\mathrm{res}^G_{L\subset P}\mathrm{IC}(\mathcal{O},\mathcal{L}))_x \cong \mathbf{H}_{\mathrm{c}}^{k+\dim\mathcal{O}}((x+\mathfrak{u}_P)\cap\mathcal{O}, \mathcal{L}|_{(x+\mathfrak{u}_P)\cap\mathcal{O}}).$$

(c) Let $\mathcal{O}_L = L\cdot x$. Using Theorem 8.4.10, show that

$$\dim((x+\mathfrak{u}_P)\cap\mathcal{O}) \le \tfrac{1}{2}(\dim\mathcal{O} - \dim\mathcal{O}_L).$$

Hint: Estimate the dimension of the variety $\mu_P^{-1}(\mathcal{O})\cap(G\times^P(\mathcal{O}_L+\mathfrak{u}_P))$ in two different ways.

(d) Show that $P^x = L^x \ltimes U_P^x$. The same holds for the opposite parabolic $\bar{P} = LU_{\bar{P}}$. By the Bruhat decomposition, the multiplication map $U_{\bar{P}}\times P\to G$ is injective (and an open embedding). Deduce that

$$\dim U_P^x + \dim U_{\bar{P}}^x \le \dim G^x - \dim L^x.$$

(e) Show that $\dim U_P\cdot x = \frac{1}{2}(\dim\mathcal{O}-\dim\mathcal{O}_L)$. Since $(x+\mathfrak{u}_P)\cap\mathcal{O}$ is an affine variety, the orbit $U_P\cdot x$ is closed, by [**230**, Proposition 2.4.14]. Conclude that $U_P\cdot x$ is an irreducible component of $(x+\mathfrak{u}_P)\cap\mathcal{O}$.

(f) Show that $\mathcal{L}|_{U_P\cdot x}$ is a constant sheaf. Deduce that

$$\mathbf{H}_{\mathrm{c}}^{\dim\mathcal{O}-\dim\mathcal{O}_L}(U_P\cdot x, \mathcal{L}|_{U_P\cdot x}) \ne 0.$$

(g) Show that $\mathbf{H}_{\mathrm{c}}^{\dim\mathcal{O}-\dim\mathcal{O}_L}((x+\mathfrak{u}_P)\cap\mathcal{O}, \mathcal{L}|_{(x+\mathfrak{u}_P)\cap\mathcal{O}}) \ne 0$. Conclude that $\mathrm{res}^G_{L\subset P}\mathrm{IC}(\mathcal{O},\mathcal{L}) \ne 0$.

Thus, if $\mathrm{IC}(\mathcal{O},\mathcal{L})$ is cuspidal, then $\mathcal{O}$ must be a **distinguished nilpotent orbit**, i.e., an orbit that does not meet the nilpotent cone of any proper Levi subgroup.

EXERCISE 8.6.2. (Springer [**229**]) Show that every simple perverse sheaf in $\mathrm{Perv}_G(\mathcal{N},\mathbb{Q})$ comes from a mixed Hodge module. Use **Slodowy slices** (see [**215**, Sections 2.4–2.5]) to show that every simple object in $\mathrm{MHM}_G(\mathcal{N},\mathbb{Q})$ is pointwise pure.

EXERCISE 8.6.3. In this exercise, you will work in the mixed equivariant derived category $D^{\mathrm{b}}_G\mathrm{MHM}(\mathcal{N}_G,\mathbb{Q})$.

(a) Let $\mathcal{O}\subset\mathcal{N}_G$ be a nilpotent orbit, and let $\mathcal{L}$ and $\mathcal{L}'$ be two irreducible local systems on $\mathcal{O}$, regarded as mixed Hodge modules of weight 0. Show that $\underline{\mathrm{Hom}}^\bullet(\mathcal{L},\mathcal{L}')$ is pure and of Tate type.

(b) Show that $\underline{\mathrm{Hom}}^\bullet(\mathrm{Spr},\mathrm{Spr})$ is pure and of Tate type.

(c) Show that there is a (noncanonical) isomorphism

$$\underline{\mathrm{Hom}}(\mathrm{Spr},\mathrm{Spr}[k]) \cong \bigoplus_{\mathcal{O}\subset\mathcal{N}_G} \underline{\mathrm{Hom}}(j_{\mathcal{O}}^*\mathrm{Spr}, j_{\mathcal{O}}^!\mathrm{Spr}[k]),$$

where $j_{\mathcal{O}}:\mathcal{O}\hookrightarrow\mathcal{N}_G$ is the inclusion map of a nilpotent orbit.

(d) Show that the stalks of Spr are pure and of Tate type.

Thus, if $\mathrm{IC}(\mathcal{O},\mathcal{L})$ occurs in the Springer correspondence, then its stalks are pure and of Tate type. In particular, $\mathsf{H}^i(j_{\mathcal{O}}^*\mathrm{IC}(\mathcal{O},\mathcal{L}))$ vanishes for i odd. Since Spr is self-Verdier-dual, $\mathsf{H}^i(j_{\mathcal{O}}^!\mathrm{IC}(\mathcal{O},\mathcal{L}))$ also vanishes for i odd.

We have thus shown that when $\Bbbk=\mathbb{Q}$, the simple perverse sheaves on $\mathcal{N}_G$ that occur in the Springer correspondence are **parity sheaves**. It turns out that *all* simple perverse sheaves in $\mathrm{Perv}_G(\mathcal{N}_G,\mathbb{Q})$ are parity sheaves; see [**155**, Theorem 24.8]. For a discussion of parity sheaves on $\mathcal{N}_G$ when $\Bbbk\ne\mathbb{Q}$, see [**120**, Section 4.3].

EXERCISE 8.6.4. Let $G = \mathrm{SL}_2$, and recall that for $x \in \mathcal{O}_{\mathrm{reg}}$, we have $A_G(x) \cong \mathbb{Z}/2\mathbb{Z}$. Let $\mathcal{L}$ be the rank-1 local system on $\mathcal{O}_{\mathrm{reg}}$ corresponding to the representation of $A_G(x)$ in which the nontrivial element acts by -1. Show that $\mathrm{IC}(\mathcal{O}_{\mathrm{reg}}, \mathcal{L})$ is cuspidal. *Hint*: If $\operatorname{char} \Bbbk \neq 2$, this can be done by a central character argument, but the claim is true even when $\operatorname{char} \Bbbk = 2$.

Then show that for $G = \mathrm{GL}_2$, there is a simple cuspidal perverse sheaf in $\mathrm{Perv}(\mathcal{N}_G, \Bbbk)$ if and only if $\operatorname{char} \Bbbk = 2$.

EXERCISE 8.6.5. Let G be a group of type G_2. There are five nilpotent orbits in $\mathcal{N}$. Let $\mathcal{O}_{\mathrm{sreg}}$ be the **subregular orbit**, i.e., the unique dense orbit in the complement of $\mathcal{O}_{\mathrm{reg}}$. All orbits other than $\mathcal{O}_{\mathrm{sreg}}$ are simply connected; for $x \in \mathcal{O}_{\mathrm{sreg}}$, the group $\pi_1(\mathcal{O}, x) \cong A_G(x)$ is the symmetric group on three letters S_3. Thus, $|\mathrm{Irr}(\mathrm{Perv}_G(\mathcal{N}, \mathbb{Q}))| = 7$.

The Weyl group W is the dihedral group of order 12, and $|\mathrm{Irr}(\mathbb{Q}[W])| = 6$. There is therefore a unique simple perverse sheaf $\mathrm{IC}(\mathcal{O}_1, \mathcal{L}_1) \in \mathrm{Perv}_G(\mathcal{N}, \mathbb{Q})$ that does not occur in the Springer correspondence.

(a) Deduce from Exercise 8.3.1 that $\mathcal{O}_1 = \mathcal{O}_{\mathrm{sreg}}$.
(b) Show that $\mathrm{IC}(\mathcal{O}_{\mathrm{sreg}}, \mathcal{L}_1)$ is cuspidal and clean. *Hint*: You will have to be careful about what the proper Levi subgroups of G_2 are. By Exercise 8.6.4, $\mathrm{SL}_2 \times \mathbb{C}^\times$ has a cuspidal perverse sheaf, but GL_2 does not.
(c) It can be shown that for $x \in \mathcal{O}_{\mathrm{sreg}}$, the Springer fiber $\mathcal{B}_x$ has four irreducible components, each isomorphic to $\mathbb{P}^1$. The group $A_G(x)$ fixes one component and acts transitively on the other three. (See the example in [**216**, Section 6.3].) Use Exercise 8.3.1 to deduce that $\mathcal{L}_1$ corresponds to the sign representation of S_3.

EXERCISE 8.6.6. This problem deals with $G = \mathrm{SL}_n$. Recall that nilpotent orbits for SL_n are parametrized by partitions of n (see, for instance, [**52**, Proposition 3.1.7]). If $\lambda = [\lambda_1, \ldots, \lambda_k]$ is a partition of n, let $\mathcal{O}_\lambda$ be the corresponding orbit, and let $x_\lambda \in \mathcal{O}_\lambda$. According to [**52**, Section 6.1], we have

$$\pi_1(\mathcal{O}_\lambda, x) \cong A_{\mathrm{SL}_n}(x) \cong \mathbb{Z}/\gcd(\lambda_1, \ldots, \lambda_k)\mathbb{Z}.$$

We also have $\mathrm{Z}(G)/\mathrm{Z}^\circ(G) \cong \mathbb{Z}/n\mathbb{Z}$, and the natural map $\mathrm{Z}(G)/\mathrm{Z}^\circ(G) \to A_G(x)$ is always surjective.

(a) Let $\chi : \mathbb{Z}/n\mathbb{Z} \to \mathbb{C}^\times$ be a faithful character, and let $\mathcal{L}_\chi$ be the corresponding local system on $\mathcal{O}_{[n]} = \mathcal{O}_{\mathrm{reg}}$. Show that $\mathrm{IC}(\mathcal{O}_{\mathrm{reg}}, \mathcal{L}_\chi) \in \mathrm{Perv}_G(\mathcal{N}, \mathbb{C})$ is cuspidal and clean.
(b) Let $d > 1$ be a divisor of n, say $n = md$. Consider the Levi subgroup

$$L = (\underbrace{\mathrm{GL}_m \times \cdots \times \mathrm{GL}_m}_{d \text{ copies}}) \cap \mathrm{SL}_n.$$

Its nilpotent cone $\mathcal{N}_L$ can be identified with that of the group

$$L' = \underbrace{\mathrm{SL}_m \times \cdots \times \mathrm{SL}_m}_{d \text{ copies}}.$$

Let $\psi : \mathbb{Z}/m\mathbb{Z} \to \mathbb{C}^\times$ be a faithful character, and let $\mathcal{L}_\psi$ be the corresponding local system on the regular orbit for SL_m. By the previous part (and Remark 8.4.6), the perverse sheaf

$$\mathcal{F} = \mathrm{IC}(\mathcal{O}_{\mathrm{reg}}^{\mathrm{SL}_m}, \mathcal{L}_\psi) \boxtimes \cdots \boxtimes \mathrm{IC}(\mathcal{O}_{\mathrm{reg}}^{\mathrm{SL}_m}, \mathcal{L}_\psi)$$

is a cuspidal perverse sheaf for L'. Show that $\mathcal{F}$ is also L-equivariant. *Hint*: For x in the regular orbit in $\mathcal{N}_L$, describe the groups $A_{L'}(x)$ and $A_L(x)$, as well as the map $A_{L'}(x) \to A_L(x)$.

(c) Let L and $\mathcal{F}$ be as in part (b), and let P be a parabolic subgroup with Levi factor L. What is the central character of $\mathrm{ind}_{L\subset P}^{G}\mathcal{F}$?

(d) Let $\chi : \mathbb{Z}/n\mathbb{Z} \to \mathbb{C}^\times$ be a central character. Show that the number of simple perverse sheaves in $\mathrm{Perv}_G(\mathcal{N}, \mathbb{C})$ with central character χ is equal to the number of partitions of $|\ker\chi|$.

(e) Show that two simple perverse sheaves in $\mathrm{Perv}_G(\mathcal{N}, \mathbb{C})$ belong to the same induction series if and only if they have the same central character. As a consequence, the objects from part (a) are all the cuspidal simple perverse sheaves.

EXERCISE 8.6.7. Let $G = \mathrm{GL}_3$. Compute the table of stalks for all simple perverse sheaves in $\mathrm{Perv}_G(\mathcal{N}, \Bbbk)$. *Hint*: In the notation of the previous exercise, the three nilpotent orbits are $\mathcal{O}_{[3]}$, $\mathcal{O}_{[2,1]}$, and $\mathcal{O}_{[1,1,1,1]}$, but for GL_3, all the $A_G(x)$ groups are trivial. The objects $\mathrm{IC}(\mathcal{O}_{[1,1,1]}, \underline{\Bbbk})$ and $\mathrm{IC}(\mathcal{O}_{[2,1]}, \underline{\Bbbk})$ were described in Exercise 3.10.6(d). For hints on $\mathrm{IC}(\mathcal{O}_{[3]}, \underline{\Bbbk})$, see [**118**, Section 4].

$\mathrm{IC}(\mathcal{O}_{[3]}, \underline{\Bbbk})$, $\mathrm{char}\,\Bbbk = 2$:

	$\mathcal{O}_{[3]}$	$\mathcal{O}_{[2,1]}$	$\mathcal{O}_{[1,1,1]}$
-1			$\Bbbk(-3)$
-2			
-3			
-4			
-5			
-6	$\underline{\Bbbk}$	$\underline{\Bbbk}$	$\underline{\Bbbk}$

$\mathrm{IC}(\mathcal{O}_{[3]}, \underline{\Bbbk})$, $\mathrm{char}\,\Bbbk = 3$:

	$\mathcal{O}_{[3]}$	$\mathcal{O}_{[2,1]}$	$\mathcal{O}_{[1,1,1]}$
-2			$\Bbbk(-3)$
-3			$\Bbbk(-2)$
-4			
-5		$\underline{\Bbbk}(-1)$	$\Bbbk(-1)$
-6	$\underline{\Bbbk}$	$\underline{\Bbbk}$	$\underline{\Bbbk}$

$\mathrm{IC}(\mathcal{O}_{[3]}, \underline{\Bbbk})$, $\mathrm{char}\,\Bbbk \neq 2, 3$:

	$\mathcal{O}_{[3]}$	$\mathcal{O}_{[2,1]}$	$\mathcal{O}_{[1,1,1]}$
-6	$\underline{\Bbbk}$	$\underline{\Bbbk}$	$\underline{\Bbbk}$

EXERCISE 8.6.8. Let $G = \mathrm{SL}_4$. Compute the table of stalks for all simple perverse sheaves in $\mathrm{Perv}_G(\mathcal{N}, \mathbb{Q})$.

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CHAPTER 9

The geometric Satake equivalence

Let G be a (complex) connected reductive group, and let $\mathcal{G}r_G$ be its affine Grassmannian (see Section 9.1 for the definition). This space is equipped with a convolution product defined much like that in Chapter 7. Next, let $\check{G}_{\Bbbk}$ be the Langlands dual reductive group over $\Bbbk$, and let $\mathrm{Rep}(\check{G}_{\Bbbk})$ be the category of algebraic $\check{G}_{\Bbbk}$-representations that are finitely generated over $\Bbbk$. This chapter is concerned with a landmark 2007 result of Mirković–Vilonen [**179**], which states that there is an equivalence of tensor categories

$$(\mathrm{Perv}_{G(\mathbb{O})}(\mathcal{G}r_G, \Bbbk), \star) \xrightarrow{\sim} (\mathrm{Rep}(\check{G}_{\Bbbk}), \otimes).$$

This theorem, which had major antecedents due to Lusztig [**158**] and Ginzburg [**82**], can be seen as a categorification of a classical result of Satake relating the spherical Hecke algebra $\mathcal{H}_{\mathrm{sph}}$ to the Grothendieck ring of $\mathrm{Rep}(\check{G}_{\Bbbk})$. It is a cornerstone of the geometric Langlands program.

Sections 9.1–9.3 contain background material on sheaves and convolution on $\mathcal{G}r$ (and on the affine flag variety $\mathcal{F}\ell$). In Section 9.4, we discuss the classical Satake isomorphism, along with results of Lusztig [**158**] that show how information about $\mathrm{Rep}(\check{G}_{\mathbb{C}})$ is encoded in $\mathrm{Perv}_{G(\mathbb{O})}(\mathcal{G}r_G, \mathbb{C})$. The rest of the chapter outlines the proof of the monoidal equivalence above. Here are some highlights of the proof:

- The convolution product on $\mathcal{G}r$ (unlike those on $\mathcal{B}$ or $\mathcal{F}\ell$) is t-exact when $\Bbbk$ is a field.
- The convolution product on $\mathrm{Perv}_{G(\mathbb{O})}(\mathcal{G}r, \Bbbk)$ is commutative. This is perhaps the most difficult point in the proof.
- As a side effect of the proof, one obtains geometric bases for (dual) Weyl modules of $\check{G}_{\mathbb{Z}}$ in terms of so-called Mirković–Vilonen cycles.

We will give most details of the sheaf-theoretic aspects of the proof (especially in Sections 9.6–9.8), but there are a number of geometric and arithmetic aspects that are beyond the scope of this book. For a comprehensive exposition, see [**21**].

9.1. The affine flag variety and the affine Grassmannian

This section contains background material (mostly without proofs) on the affine Weyl group, the affine flag variety, and related objects.

The extended affine Weyl group. Let G be a connected complex reductive group. Choose a Borel subgroup $B \subset G$ and a maximal torus $T \subset B$. Let $\mathrm{N}_G(T)$ be the normalizer of T in G, and let $W = \mathrm{N}_G(T)/T$ be the Weyl group. Let Φ be the root system of G, and let $\Phi^+ \subset \Phi$ be the set of positive roots, i.e., the roots that occur in the Lie algebra of B. Let $\check{\mathbf{X}} = \check{\mathbf{X}}(T)$ be the coweight lattice of T, and let $\check{\mathbf{X}}^+ \subset \check{\mathbf{X}}$ be the set of dominant coweights (where "dominant" is defined with respect to the positive system $\Phi^+ \subset \Phi$). Next, let $\check{\Phi} \subset \check{\mathbf{X}}$ be the set of coroots, and let $\check{\Phi}^+$ be the set of positive coroots. Let

$$\rho = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha \qquad \text{and} \qquad \check{\rho} = \frac{1}{2} \sum_{\check{\alpha} \in \check{\Phi}^+} \check{\alpha}.$$

There is a partial order on $\check{\mathbf{X}}$ given by

$$\lambda \le_{\check{\mathbf{X}}} \mu \qquad \text{if} \qquad \mu - \lambda = \sum_{\check{\alpha} \in \check{\Phi}^+} n_{\check{\alpha}} \check{\alpha} \quad \text{for some integers } n_{\check{\alpha}} \ge 0. \tag{9.1.1}$$

Note that if two coweights lie in different cosets of the coroot lattice $\mathbb{Z}\check{\Phi} \subset \check{\mathbf{X}}$, then they are necessarily incomparable in this partial order.

The **affine Weyl group** is the group $W_{\mathrm{aff}} = W \ltimes \mathbb{Z}\check{\Phi}$, and the **extended affine Weyl group** is $W_{\mathrm{ext}} = W \ltimes \check{\mathbf{X}}$. The affine Weyl group is a Coxeter group (see [**111**, Section 1.4] or [**108**, Section 4.2]). The extended affine Weyl group may not be a Coxeter group, but it has many features in common with Coxeter groups. Notably, there is a **length function** $\ell : W_{\mathrm{ext}} \to \mathbb{Z}$. For $w \in W_{\mathrm{aff}} \subset W_{\mathrm{ext}}$, $\ell(w)$ is just the minimal length of an expression for w as a product of simple reflections. In general, for $v \in W$ and $\lambda \in \check{\mathbf{X}}$, $\ell(v \ltimes \lambda)$ is given by

$$\ell(v \ltimes \lambda) = \sum_{\alpha \in \Phi^+ \cap v^{-1}(\Phi^+)} |\langle \alpha, \lambda \rangle| + \sum_{\alpha \in \Phi^+ \cap v^{-1}(-\Phi^+)} |\langle \alpha, \lambda \rangle + 1|.$$

Unlike a Coxeter group, W_{ext} may have many elements of length 0. These form a subgroup, denoted by

$$\Omega = \{w \in W_{\mathrm{ext}} \mid \ell(w) = 0\}.$$

There are canonical isomorphisms

$$\Omega \xrightarrow{\sim} W_{\mathrm{ext}}/W_{\mathrm{aff}} \xleftarrow{\sim} \check{\mathbf{X}}/\mathbb{Z}\Phi, \tag{9.1.2}$$

$$W_{\mathrm{ext}} \cong \Omega \ltimes W_{\mathrm{aff}}. \tag{9.1.3}$$

Since W_{aff} is a Coxeter group, it is equipped with a **Bruhat order** as in [**108**, Section 5.9]. We extend this to a partial on W_{ext} using (9.1.3) as follows: for $\omega, \omega' \in \Omega$ and $w, w' \in W_{\mathrm{aff}}$, we declare that

$$(\omega \ltimes w) \le (\omega' \ltimes w') \quad \text{if} \quad \omega = \omega' \text{ and } w \le w' \text{ in the Bruhat order on } W_{\mathrm{aff}}.$$

In particular, elements in distinct cosets of W_{aff} are incomparable in the Bruhat order. The Bruhat order is partly compatible with the partial order defined in (9.1.1):

$$\lambda \le_{\check{\mathbf{X}}} \mu \quad \text{if and only if} \quad (1 \ltimes \lambda) \le (1 \ltimes \mu) \qquad \text{for } \lambda, \mu \in \check{\mathbf{X}}^+. \tag{9.1.4}$$

For a proof, see [**242**, Section 1.6]. (For nondominant coweights, the partial order $\le_{\check{\mathbf{X}}}$ and the Bruhat order are rather different.)

The affine flag variety and the affine Grassmannian. Let t be an indeterminate. Let $\mathbb{O} = \mathbb{C}[[t]]$ be the ring of formal power series in t, and let $\mathbb{K} = \mathbb{C}((t))$ be its fraction field, the field of formal Laurent series. In this chapter, we will work with the groups $G(\mathbb{K})$ and $G(\mathbb{O})$. If G is embedded as a closed subgroup of GL_n (given by some equations), then $G(\mathbb{K})$, resp. $G(\mathbb{O})$, can be thought of concretely as the group of invertible matrices with entries in $\mathbb{K}$, resp. $\mathbb{O}$, satisfying the same defining equations as G.

There is a group homomorphism $G(\mathbb{O}) \to G$ given by $t \mapsto 0$. Let $B^- \subset G$ be the opposite Borel subgroup to B, and let I be the preimage of B^- under this map:

$$I = \{g \in G(\mathbb{O}) \mid g|_{t=0} \in B^-\}.$$

This group is called an **Iwahori subgroup** of $G(\mathbb{K})$. Let us now describe double cosets for I and $G(\mathbb{O})$. For each $v \in W$, choose a representative $\dot{v} \in \mathrm{N}_G(T) \subset G$. Next, any coweight $\lambda : \mathbb{G}_{\mathrm{m}} \to T$ can be regarded as a point $t^\lambda \in T(\mathbb{K}) \subset G(\mathbb{K})$. For an element $w = v \ltimes \lambda \in W_{\mathrm{ext}}$, we put $\dot{w} = \dot{v}t^\lambda \in G(\mathbb{K})$. Then we have

$$G(\mathbb{K}) = \bigsqcup_{w \in W_{\mathrm{ext}}} I\dot{w}I = \bigsqcup_{\lambda \in \check{\mathbf{X}}} It^\lambda G(\mathbb{O}) = \bigsqcup_{\lambda \in \check{\mathbf{X}}^+} G(\mathbb{O})t^\lambda G(\mathbb{O}).$$

These decompositions, which are generalizations of the Bruhat decomposition, come from [**111**, Theorem 2.16 and Corollary 2.35]. Next, let

$$\mathcal{F}\ell = G(\mathbb{K})/I \qquad \text{and} \qquad \mathcal{G}r = G(\mathbb{K})/G(\mathbb{O}).$$

The set $\mathcal{F}\ell$ is called the **affine flag variety** of G, and $\mathcal{G}r$ is called the **affine Grassmannian**. Below, we will see how to equip these sets with a topology. There is an obvious projection map

$$\pi : \mathcal{F}\ell \to \mathcal{G}r$$

whose fibers can be identified with $G(\mathbb{O})/I \cong G/B^-$.

The main objects of study in this chapter are $G(\mathbb{O})$-equivariant perverse sheaves on $\mathcal{G}r$. (Sheaves on $\mathcal{F}\ell$ will also occur.) However, some work is required to make sense of these notions, because $\mathcal{F}\ell$ and $\mathcal{G}r$ are not algebraic varieties in the usual sense, and $G(\mathbb{O})$ and I are not algebraic groups. Instead, $\mathcal{F}\ell$ and $\mathcal{G}r$ are "ind-varieties," i.e., direct limits of algebraic varieties, while $G(\mathbb{O})$ and I are "pro-algebraic groups," i.e., inverse limits of algebraic groups.

Let us start with the groups $G(\mathbb{O})$ and I. For any positive integer $n \geq 1$, let $\mathbb{O}_n = \mathbb{C}[[t]]/(t^n)$. Then the group $G(\mathbb{O}_n)$ is a (finite-dimensional) complex algebraic group. If $m \geq n$, then the obvious quotient map $\mathbb{O}_m \to \mathbb{O}_n$ induces a homomorphism of algebraic groups

$$G(\mathbb{O}_m) \to G(\mathbb{O}_n).$$

This homomorphism is surjective and its kernel is unipotent. Similarly, let

$$I_n = \{g \in G(\mathbb{O}_n) \mid g|_{t=0} \in B^-\}.$$

This is a closed, connected, solvable subgroup of $G(\mathbb{O}_n)$. Again, for $m \geq n$, the obvious map

$$I_m \to I_n \tag{9.1.5}$$

is surjective and has a unipotent kernel.

Next, for any $w \in W_{\mathrm{ext}}$, let

$$\mathcal{F}\ell_w = I\dot{w}I/I \subset \mathcal{F}\ell \qquad \text{and} \qquad \overline{\mathcal{F}\ell_w} = \bigcup_{\substack{y \in W_{\mathrm{ext}} \\ y \leq w}} \mathcal{F}\ell_y.$$

We emphasize that for the moment, $\mathcal{F}\ell$ does not yet have a topology, and these are just sets. Of course, the notation $\overline{\mathcal{F}\ell_w}$ is meant to evoke "closure"; this will be justified in Proposition 9.1.1 below.

For $\lambda \in \check{\mathbf{X}}$, let $\mathbf{t}^\lambda$ denote the coset $t^\lambda G(\mathbb{O}) \in \mathcal{G}r$. For $\lambda \in \check{\mathbf{X}}^+$, let

$$\begin{aligned} \mathcal{G}r_\lambda &= G(\mathbb{O})t^\lambda G(\mathbb{O})/G(\mathbb{O}) \\ &= G(\mathbb{O}) \cdot \mathbf{t}^\lambda \subset \mathcal{G}r \end{aligned} \qquad \text{and} \qquad \overline{\mathcal{G}r_\lambda} = \bigcup_{\substack{\mu \in \check{\mathbf{X}}^+ \\ \mu \leq_{\check{\mathbf{X}}} \lambda}} \mathcal{G}r_\mu. \tag{9.1.6}$$

Again, for now, the description of $\overline{\mathcal{G}r_\lambda}$ is a definition. Finally, for any $\lambda \in \check{\mathbf{X}}$, let

$w_\lambda =$ the unique element of minimal length in the left coset $\lambda W \subset W_{\mathrm{ext}}$,

$w_\lambda^* =$ the unique element of maximal length in the left coset $\lambda W \subset W_{\mathrm{ext}}$.

According to [**111**, Proposition 1.25], we have

$$\ell(w_\lambda) = |\langle 2\rho, \lambda\rangle| - |\{\alpha \in \Phi^+ \mid \langle \alpha, \lambda\rangle > 0\}|.$$

We will sometimes consider the sets

$$I \cdot \mathbf{t}^\lambda = It^\lambda G(\mathbb{O})/G(\mathbb{O}) \subset \mathcal{G}r \qquad \text{and} \qquad \overline{I \cdot \mathbf{t}^\lambda} = \bigcup_{\substack{\mu \in \check{\mathbf{X}} \\ w_\mu^* \leq w_\lambda^*}} I \cdot \mathbf{t}^\mu.$$

These satisfy $I \cdot \mathbf{t}^\lambda = \pi(\mathcal{F}\ell_{w_\lambda})$ and $\overline{I \cdot \mathbf{t}^\lambda} = \pi(\overline{\mathcal{F}\ell_{w_\lambda}})$. (The latter can be proved using the characterization of the Bruhat order found in [**108**, Theorem 5.10]; alternatively, it is a geometric consequence of Proposition 9.1.3 below.) We also have

$$\mathcal{G}r_\lambda = \bigcup_{\mu \in W \cdot \lambda} I \cdot \mathbf{t}^\mu \qquad \text{for } \lambda \in \check{\mathbf{X}}^+.$$

For $\omega \in \Omega$, let

$$\mathcal{F}\ell^\omega = \bigcup_{y \in \omega W_{\mathrm{aff}}} \mathcal{F}\ell_y \qquad \text{and} \qquad \mathcal{G}r^\omega = \bigcup_{\substack{\mu \in \check{\mathbf{X}}^+ \\ \omega \text{ corresponds to } \mu + \mathbb{Z}\check{\Phi}}} \mathcal{G}r_\mu,$$

where in the latter, ω should correspond to $\mu + \mathbb{Z}\check{\Phi} \in \check{\mathbf{X}}/\mathbb{Z}\check{\Phi}$ under (9.1.2). In the special case where ω is the identity element $e \in \Omega$, we often instead write

$$\mathcal{F}\ell^\circ = \mathcal{F}\ell^e \qquad \text{and} \qquad \mathcal{G}r^\circ = \mathcal{G}r^e.$$

PROPOSITION 9.1.1. (1) *Each set $\overline{\mathcal{F}\ell_w}$ has the structure of an irreducible complex projective variety. Moreover, $\mathcal{F}\ell_w$ is a smooth, open, dense subset of $\overline{\mathcal{F}\ell_w}$, isomorphic to $\mathbb{A}^{\ell(w)}$.*

(2) *If $y \leq w$, the inclusion map $\overline{\mathcal{F}\ell_y} \hookrightarrow \overline{\mathcal{F}\ell_w}$ is a closed embedding.*

(3) *For each $w \in W_{\mathrm{ext}}$, there exists an integer $n \geq 1$ such that the action of I on $\overline{\mathcal{F}\ell_w}$ factors through the quotient $I \to I_n$. The induced action of I_n on $\overline{\mathcal{F}\ell_w}$ is algebraic.*

PROOF SKETCH. Assume first that G is semisimple and simply connected, so that $\check{\mathbf{X}} = \mathbb{Z}\check{\Phi}$. In this case, $W_{\mathrm{ext}} = W_{\mathrm{aff}}$ and $\mathcal{F}\ell = \mathcal{F}\ell^\circ$. The set $\mathcal{F}\ell$ can be identified with the flag variety of the affine Kac–Moody group corresponding to G, as explained in [**143**, Section 13.2]. The statements above then follow from the general theory of Kac–Moody flag varieties, developed in [**143**, Chapter VII].

For general G, the reasoning above can still be applied when $\overline{\mathcal{F}\ell_w} \subset \mathcal{F}\ell^\circ$, i.e., when $w \in W_{\mathrm{aff}}$. Transfer this structure to each $\mathcal{F}\ell^\omega$ via the bijection $\mathcal{F}\ell^\circ \xrightarrow{\sim} \mathcal{F}\ell^\omega$ given by multiplication on the left by $\dot{\omega}$. □

This proposition equips $\mathcal{Fl}$ with the structure of an **ind-variety**, i.e., an increasing union of algebraic varieties. It also determines a topology on all of $\mathcal{Fl}$: a subset $V \subset \mathcal{Fl}$ is said to be open if for all $w \in W_{\mathrm{ext}}$, $V \cap \overline{\mathcal{Fl}_w}$ is an (analytic) open subset of $\overline{\mathcal{Fl}_w}$. In this topology, $\overline{\mathcal{Fl}_w}$ is indeed the closure of $\mathcal{Fl}_w$, and the sets $\mathcal{Fl}^\omega$ are precisely the connected components of $\mathcal{Fl}$. Moreover, any finite union of I-orbit closures has the structure of a projective variety.

Let $Z \subset \mathcal{Fl}$ be a finite union of I-orbit closures. Choose an integer $n \geq 1$ such that the I-action on Z factors through $I \to I_n$, and then define

$$D^{\mathrm{b}}_I(Z, \Bbbk) = D^{\mathrm{b}}_{I_n}(Z, \Bbbk).$$

This definition is independent of n in the following sense: if $m \geq n$, then because (9.1.5) is surjective and has a unipotent kernel, Theorem 6.6.16 tells us that the inflation functor $D^{\mathrm{b}}_{I_n}(Z, \Bbbk) \to D^{\mathrm{b}}_{I_m}(Z, \Bbbk)$ is an equivalence of categories. Finally, define

$$D^{\mathrm{b}}_I(\mathcal{Fl}, \Bbbk) = \varinjlim_{\substack{Z \subset \mathcal{Fl} \text{ a finite} \\ \text{union of } I\text{-orbit closures}}} D^{\mathrm{b}}_I(Z, \Bbbk).$$

This limit can be thought of as just the union of the various $D^{\mathrm{b}}_I(Z, \Bbbk)$, where if $Z \subset Z'$, we identify $D^{\mathrm{b}}_I(Z, \Bbbk)$ with a full subcategory of $D^{\mathrm{b}}_I(Z', \Bbbk)$ via i_*, where $i : Z \hookrightarrow Z'$ is the inclusion map. In particular, by definition, the support of any object in $D^{\mathrm{b}}_I(\mathcal{Fl}, \Bbbk)$ is a finite union of I-orbit closures.

Proposition 9.1.2. (1) *Each set $\overline{\mathcal{Gr}_\lambda}$ has the structure of an irreducible complex projective variety. Moreover, $\mathcal{Gr}_\lambda$ is a smooth, open, dense subset of $\overline{\mathcal{Gr}_\lambda}$.*
(2) *If $\lambda \leq_{\check{\mathbf{X}}} \mu$, then the inclusion map $\overline{\mathcal{Gr}_\lambda} \hookrightarrow \overline{\mathcal{Gr}_\mu}$ is a closed embedding.*
(3) *For each $\lambda \in \check{\mathbf{X}}^+$, there exists an integer $n \geq 1$ such that the action of $G(\mathbb{O})$ on $\overline{\mathcal{Gr}_\lambda}$ factors through the quotient $G(\mathbb{O}) \to G(\mathbb{O}_n)$. The induced action of $G(\mathbb{O}_n)$ on $\overline{\mathcal{Gr}_\lambda}$ is algebraic.*

Like Proposition 9.1.1, this result can be deduced from the theory of partial flag varieties of Kac–Moody groups. There is also a variant involving I-orbits on $\mathcal{Gr}$. Comments similar to those after Proposition 9.1.1 apply to the topology of $\mathcal{Gr}$. In particular, the sets $\mathcal{Gr}^\omega$ are the connected components of $\mathcal{Gr}$, and one can define the equivariant derived categories

$$D^{\mathrm{b}}_I(\mathcal{Gr}, \Bbbk) \qquad \text{and} \qquad D^{\mathrm{b}}_{G(\mathbb{O})}(\mathcal{Gr}, \Bbbk).$$

As with $\mathcal{Fl}$, any object in either of these categories is, by definition, supported on a finite union of I- or $G(\mathbb{O})$-orbit closures. Similar considerations lead to the following result.

Proposition 9.1.3. (1) *Each $\mathcal{Gr}_\lambda$ is smooth and simply connected.*
(2) *The map $\pi : \mathcal{Fl} \to \mathcal{Gr}$ is smooth and proper, and its fibers are isomorphic to G/B.*
(3) *For $\lambda \in \check{\mathbf{X}}$, let δ_λ be the length of the shortest element $w \in W$ such that $w\lambda \in -\check{\mathbf{X}}^+$. Then $\dim I \cdot \mathbf{t}^\lambda = |\langle 2\rho, \lambda\rangle| - \delta_\lambda$. In particular, we have $\dim \mathcal{Gr}_\lambda = \langle 2\rho, \lambda\rangle$.*

Corollary 9.1.4. *Let $\omega \in \Omega$. For any two $G(\mathbb{O})$-orbits $\mathcal{Gr}_\lambda, \mathcal{Gr}_\mu \subset \mathcal{Gr}^\omega$, we have* $\dim \mathcal{Gr}_\lambda \equiv \dim \mathcal{Gr}_\mu \pmod 2$.

Proof. By Proposition 9.1.3, we have $\dim \mathcal{G}r_\lambda - \dim \mathcal{G}r_\mu = \langle 2\rho, \lambda - \mu\rangle$. Because $\mathcal{G}r_\lambda$ and $\mathcal{G}r_\mu$ are contained in the same component of $\mathcal{G}r$, $\lambda - \mu$ belongs to the coroot lattice $\mathbb{Z}\check{\Phi}$, so $\langle 2\rho, \lambda - \mu\rangle$ is an even integer. $\square$

For any $w \in W_{\mathrm{ext}}$ or any $\lambda \in \check{\mathbf{X}}^+$, we let $j_w : \mathcal{F}\ell_w \hookrightarrow \mathcal{F}\ell$ and $j_\lambda : \mathcal{G}r_\lambda \hookrightarrow \mathcal{G}r$ be the inclusion maps. As in Chapter 7, we use the abbreviated notation

$$\mathrm{IC}_w = \mathrm{IC}_w(\Bbbk) \qquad \text{and} \qquad \mathrm{IC}_\lambda = \mathrm{IC}_\lambda(\Bbbk)$$

in place of $\mathrm{IC}(\mathcal{F}\ell_w, \Bbbk)$ and $\mathrm{IC}(\mathcal{G}r_\lambda, \Bbbk)$, respectively.

Exercises.

9.1.1. Recall that a dominant coweight $\lambda \in \check{\mathbf{X}}^+$ is said to be **minuscule** if $\langle \alpha, \lambda\rangle \le 1$ for all $\alpha \in \Phi^+$. Show that $\mathcal{G}r_\lambda$ is closed in $\mathcal{G}r$ if and only if λ is minuscule. Then show that each connected component of $\mathcal{G}r$ contains a unique closed $G(\mathbb{O})$-orbit.

9.1.2. This exercise and the ones below give an alternative description of the affine Grassmannian and the affine flag variety for GL_n. By definition, a **lattice** in $\mathbb{K}^n$ is a free $\mathbb{O}$-submodule of rank n. Show that $\mathrm{GL}_n(\mathbb{K})$ acts transitively on the set of lattices in $\mathbb{K}^n$, and that the stabilizer of the lattice $\mathbb{O}^n \subset \mathbb{K}^n$ is $\mathrm{GL}_n(\mathbb{O})$. Thus, there is a bijection

$$\mathcal{G}r = \mathrm{GL}_n(\mathbb{K})/\mathrm{GL}_n(\mathbb{O}) \xleftrightarrow{\sim} \{\text{lattices in } \mathbb{K}^n\}. \tag{9.1.7}$$

Let $T \subset \mathrm{GL}_n$ be the maximal torus consisting of diagonal matrices. Identify the cocharacter lattice $\check{\mathbf{X}}$ for GL_n with $\mathbb{Z}^n$ as follows: for $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n$, the corresponding cocharacter $\mathbb{G}_{\mathrm{m}} \to T$ is the map

$$t \mapsto \begin{bmatrix} t^{\lambda_1} & & \\ & \ddots & \\ & & t^{\lambda_n} \end{bmatrix}.$$

(a) What lattice corresponds to $\mathbf{t}^{(\lambda_1,\ldots,\lambda_n)}$ under the bijection (9.1.7)?

(b) Identify the dominant coweights $\check{\mathbf{X}}^+$ with $\{(\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n \mid \lambda_1 \ge \cdots \ge \lambda_n\}$. Show that for every lattice $L \subset \mathbb{K}^n$, there is a unique dominant coweight $\lambda \in \check{\mathbf{X}}^+$ such that $L \in \mathrm{GL}_n(\mathbb{O}) \cdot \mathbf{t}^\lambda$.

9.1.3. Let $L \subset \mathbb{K}^n$ be a lattice. Choose an $\mathbb{O}$-basis $\{v_1, \ldots, v_n\}$ for L, and then consider the determinant of the matrix whose columns are $v_1, \ldots, v_n$: let

$$c = \det \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ v_1 & v_2 & \cdots & v_n \\ \downarrow & \downarrow & & \downarrow \end{bmatrix} \in \mathbb{K}^\times.$$

Let $\nu(L)$ be the smallest power of t that appears with nonzero coefficient in c.

(a) Show that $\nu(L)$ is independent of the choice of basis for L. This integer is called the **valuation** of L.

(b) Show that if $L' \subset L$ are both lattices in $\mathbb{K}^n$, then L/L' is a finite-dimensional complex vector space, and that $\dim_{\mathbb{C}}(L/L') = \nu(L) - \nu(L')$.

9.1.4. Choose integers $\nu, N \in \mathbb{Z}$ with $N \ge 0$. Let

$$\mathcal{G}r^{\nu,N} = \{\text{lattices } L \subset \mathbb{K}^n \text{ such that } \nu(L) = \nu \text{ and } t^N\mathbb{O}^n \subset L \subset t^{-N}\mathbb{O}^n\}.$$

There is a nilpotent endomorphism $\bar{t} : t^{-N}\mathbb{O}^n/t^N\mathbb{O}^n \to t^{-N}\mathbb{O}^n/t^N\mathbb{O}^n$ given by multiplication by t. Show that there is a bijection

$$\mathcal{G}r^{\nu,N} \overset{\sim}{\leftrightarrow} \left\{ \begin{array}{c} \bar{t}\text{-stable subspaces of } t^{-N}\mathbb{O}^n/t^N\mathbb{O}^n \\ \text{of dimension } nN-\nu \end{array} \right\}.$$

Let $\mathrm{Gr}(k,m)$ denote the ordinary (not affine) Grassmannian of k-dimensional subspaces of $\mathbb{C}^m$. Identify $t^{-N}\mathbb{O}^n/t^N\mathbb{O}^n$ with $\mathbb{C}^{2nN}$. Then the bijection above identifies $\mathcal{G}r^{\nu,N}$ with a subset of $\mathrm{Gr}(nN-\nu, 2nN)$. Show that this subset is closed in the Zariski topology. In this way, we equip $\mathcal{G}r^{\nu,N}$ with the structure of a complex projective variety. Then show that the inclusion map $\mathcal{G}r^{\nu,N} \hookrightarrow \mathcal{G}r^{\nu,N+1}$ is a closed embedding. In this way, we make the set

$$\mathcal{G}r^{\nu} = \bigcup_{N\geq 0} \mathcal{G}r^{\nu,N}$$

into an ind-variety.

9.1.5. Let $\lambda = (\lambda_1,\ldots,\lambda_n) \in \check{\mathbf{X}}^+$ be a dominant coweight. Let $\nu = \lambda_1+\cdots+\lambda_n$, and let N be an integer such that $N \geq \lambda_1 \geq \lambda_n \geq -N$. Show that

$$\mathcal{G}r_\lambda = \left\{ L \in \mathcal{G}r^{\nu,N} \;\middle|\; \text{for } -N \leq i \leq N,\ \dim \frac{L \cap t^i\mathbb{O}^n}{t^N\mathbb{O}^n} = \sum_{j=i}^{N-1} |\{j \mid \lambda_j \leq i\}| \right\}.$$

Similarly, show that the set $\overline{\mathcal{G}r_\lambda}$ (defined in (9.1.6)) is given by

$$\overline{\mathcal{G}r_\lambda} = \left\{ L \in \mathcal{G}r^{\nu,N} \;\middle|\; \text{for } -N \leq i \leq N,\ \dim \frac{L \cap t^i\mathbb{O}^n}{t^N\mathbb{O}^n} \geq \sum_{j=i}^{N-1} |\{j \mid \lambda_j \leq i\}| \right\}.$$

9.1.6. Show that $\overline{\mathcal{G}r_\lambda}$ is a closed subset of $\mathcal{G}r^{\nu,N}$ in the Zariski topology. In this way, we regard $\overline{\mathcal{G}r_\lambda}$ as a complex projective variety. Then prove the rest of Proposition 9.1.2. Also show that two lattices $L, L' \in \mathcal{G}r$ belong to the same connected component if and only if they have the same valuation.

9.1.7. Recall that the minuscule dominant coweights for GL_n are those of the form

$$\lambda = (\underbrace{a,\ldots,a,}_{i \text{ coordinates}}\ \underbrace{a-1,\ldots,a-1}_{n-i \text{ coordinates}})$$

for some $a \in \mathbb{Z}$ and some i with $1 \leq i \leq n$. For λ of this form, show that

$$\mathcal{G}r_\lambda = \overline{\mathcal{G}r_\lambda} \cong \mathrm{Gr}(n-i,n).$$

9.1.8. Show that the affine Grassmannian for $G = \mathrm{SL}_n$ can be identified with the set of lattices of valuation 0. In particular, show that this affine Grassmannian is connected. Give a description of the $\mathrm{SL}_n(\mathbb{O})$-orbits and their closures in the spirit of Exercise 9.1.5.

9.1.9. Two lattices $L, L' \subset \mathbb{K}^n$ are said to be **homothetic** if there is a nonzero scalar $c \in \mathbb{K}^\times$ such that $L' = cL$. Show that the affine Grassmannian for $G = \mathrm{PGL}_n$ can be identified with the set of homothety classes of lattices. Show, in addition, that this affine Grassmannian has exactly n connected components, corresponding to "valuation modulo n." Finally, give a description of the $\mathrm{PGL}_n(\mathbb{O})$-orbits and their closures in the spirit of Exercise 9.1.5.

9.1.10. This problem deals with $G = \mathrm{SL}_2$. Identify its coweight lattice $\check{\mathbf{X}}^+$ with the set of even integers $2\mathbb{Z}$. Determine the closure partial order on I-orbits in $\mathcal{G}r$. *Answer*: The closure partial order is a total order. Its Hasse diagram (increasing from left to right) is shown below:

$$\begin{array}{lccccccccccc}
\text{dimension:} & 0 & & 1 & & 2 & & 3 & & 4 & & \cdots \\
I\text{-orbits:} & I\cdot\mathbf{t}^0 & — & I\cdot\mathbf{t}^2 & — & I\cdot\mathbf{t}^{-2} & — & I\cdot\mathbf{t}^4 & — & I\cdot\mathbf{t}^{-4} & — & \cdots \\
\mathrm{SL}_2(\mathbb{O})\text{-orbits:} & \mathcal{G}r_0 & & \multicolumn{3}{c}{\mathcal{G}r_2} & & \multicolumn{3}{c}{\mathcal{G}r_4} & &
\end{array}$$

9.1.11. This problem deals with $G = \mathrm{PGL}_2$. Identify its coweight lattice $\check{\mathbf{X}}$ with $\mathbb{Z}$. Determine the closure partial order on I-orbits in $\mathcal{G}r$. *Answer*: The affine Grassmannian $\mathcal{G}r$ has two connected components. One component contains the I-orbits labelled by even coweights. It is described as in Exercise 9.1.10. The other component contains the I-orbits labelled by odd coweights. Its Hasse diagram is shown here:

$$\begin{array}{lccccccccccc}
\text{dimension:} & 0 & & 1 & & 2 & & 3 & & 4 & & \cdots \\
I\text{-orbits:} & I\cdot\mathbf{t}^1 & — & I\cdot\mathbf{t}^{-1} & — & I\cdot\mathbf{t}^3 & — & I\cdot\mathbf{t}^{-3} & — & I\cdot\mathbf{t}^5 & — & \cdots \\
\mathrm{PGL}_2(\mathbb{O})\text{-orbits:} & \multicolumn{3}{c}{\mathcal{G}r_1} & & \multicolumn{3}{c}{\mathcal{G}r_3} & & \multicolumn{2}{c}{\mathcal{G}r_5} & \cdots
\end{array}$$

9.2. Convolution

We will now adapt the convolution construction of Chapter 7 to the setting of $\mathcal{F}\ell$ and $\mathcal{G}r$. We begin by defining a functor

$$\star \text{ or } \star_{\mathcal{F}\ell} : D^{\mathrm{b}}_I(\mathcal{F}\ell, \Bbbk) \times D^{\mathrm{b}}_I(\mathcal{F}\ell, \Bbbk) \to D^{\mathrm{b}}_I(\mathcal{F}\ell, \Bbbk)$$

using the diagram

$$\mathcal{F}\ell \times \mathcal{F}\ell \xleftarrow{p} G(\mathbb{K}) \times \mathcal{F}\ell \xrightarrow{q} G(\mathbb{K}) \times^I \mathcal{F}\ell \xrightarrow{m} \mathcal{F}\ell. \tag{9.2.1}$$

Here, p and q are the obvious quotient maps, and m is the multiplication map. Let $I \times I$ act on $G(\mathbb{K}) \times \mathcal{F}\ell$ by $(b_1, b_2) \cdot (g_1, g_2B) = (b_1g_1b_2^{-1}, b_2g_2B)$. Then p is $I \times I$-equivariant, and q is the quotient map by the second copy of I.

Intuitively, the definition of convolution ought to be as follows: given $\mathcal{F}, \mathcal{G} \in D^{\mathrm{b}}_I(\mathcal{F}\ell, \Bbbk)$, we set

$$\mathcal{F} \mathbin{\tilde{\boxtimes}} \mathcal{G} = (q^*)^{-1}(p^*(\mathcal{F} \boxtimes \mathcal{G})) \in D^{\mathrm{b}}_I(G(\mathbb{K}) \times^I \mathcal{F}\ell, \Bbbk), \tag{9.2.2}$$

where $(q^*)^{-1}$ is the inverse of the quotient equivalence $q^* : D^{\mathrm{b}}_I(G(\mathbb{K}) \times^I \mathcal{F}\ell, \Bbbk) \xrightarrow{\sim} D^{\mathrm{b}}_{I\times I}(G(\mathbb{K}) \times \mathcal{F}\ell, \Bbbk)$. Then we set

$$\mathcal{F} \star_{\mathcal{F}\ell} \mathcal{G} = m_*(\mathcal{F} \mathbin{\tilde{\boxtimes}} \mathcal{G}). \tag{9.2.3}$$

But this does not quite make sense: we have not given $G(\mathbb{K})$ or $G(\mathbb{K}) \times^I \mathcal{F}\ell$ a topology, and we certainly have not discussed what $D^{\mathrm{b}}_{I\times I}(G(\mathbb{K}) \times \mathcal{F}\ell, \Bbbk)$ means.

To get around these issues, we must approximate the diagram (9.2.1) with honest varieties. Let $Z, Z' \subset \mathcal{F}\ell$ be finite unions of I-orbit closures such that $\mathcal{F}$ is supported on Z and $\mathcal{G}$ is supported on Z'. Let $n \geq 1$ be such that the I-action on Z' factors through $I \to I_n$, and let K be the kernel of $I \mapsto I_n$. Finally, let $\tilde{Z}$ be the preimage of Z under the quotient map $G(\mathbb{K})/K \to \mathcal{F}\ell$. Consider the diagram

$$Z \times Z' \xleftarrow{p} \tilde{Z} \times Z' \xrightarrow{q} \tilde{Z} \times^{I_n} Z' \xrightarrow{m} \mathcal{F}\ell. \tag{9.2.4}$$

The group I_n acts on $\tilde{Z}$ by multiplication on the right. It can be shown that $\tilde{Z}$ can be made into an algebraic variety in such a way that it becomes a principal I_n-variety, with $\tilde{Z}/I_n \cong Z$. This new diagram involves only finite-dimensional algebraic varieties and groups. The maps p and q are smooth, and $q^* : D^{\mathrm{b}}_I(\tilde{Z} \times^{I_n} Z', \Bbbk) \to D^{\mathrm{b}}_{I \times I_n}(\tilde{Z} \times Z', \Bbbk)$ is an equivalence of categories by Theorem 6.5.9. With this modified diagram, the formulas in (9.2.2) and (9.2.3) now make sense. It is left to the reader to check that the resulting definition of $\star_{\mathcal{Fl}}$ is independent of the choices of Z, Z', and I_n. From now on, we will suppress these technical details, and we work directly (and somewhat informally) with (9.2.1).

Note that the only role played by the I-equivariance of $\mathcal{F}$ in the definition of $\mathcal{F} \star \mathcal{G}$ is that it allows the functor $\star_{\mathcal{Fl}}$ to take values in $D^{\mathrm{b}}_I(\mathcal{Fl}, \Bbbk)$. If $\mathcal{F}$ instead began in, say, $D^{\mathrm{b}}_{\mathrm{c}}(\mathcal{Fl}, \Bbbk)$ or $D^{\mathrm{b}}_{G(\mathbb{O})}(\mathcal{Fl}, \Bbbk)$, then the formula (9.2.3) would still make sense, but it would take values in $D^{\mathrm{b}}_{\mathrm{c}}(\mathcal{Fl}, \Bbbk)$ or $D^{\mathrm{b}}_{G(\mathbb{O})}(\mathcal{Fl}, \Bbbk)$, accordingly.

With minor modifications, the discussion above shows how to define a functor

$$\star \text{ or } \star_{\mathcal{Gr}} : D^{\mathrm{b}}_{G(\mathbb{O})}(\mathcal{Gr}, \Bbbk) \times D^{\mathrm{b}}_{G(\mathbb{O})}(\mathcal{Gr}, \Bbbk) \to D^{\mathrm{b}}_{G(\mathbb{O})}(\mathcal{Gr}, \Bbbk)$$

using the diagram

$$\mathcal{Gr} \times \mathcal{Gr} \xleftarrow{p} G(\mathbb{K}) \times \mathcal{Gr} \xrightarrow{q} G(\mathbb{K}) \times^{G(\mathbb{O})} \mathcal{Gr} \xrightarrow{m} \mathcal{Gr}. \tag{9.2.5}$$

As above, this diagram also lets us define $\mathcal{F} \star_{\mathcal{Gr}} \mathcal{G}$ for $\mathcal{F}$ in $D^{\mathrm{b}}_{\mathrm{c}}(\mathcal{Gr}, \Bbbk)$ or $D^{\mathrm{b}}_I(\mathcal{Gr}, \Bbbk)$. Finally, there is also a functor

$$\star^{\mathcal{Fl}}_{\mathcal{Gr}} : D^{\mathrm{b}}_I(\mathcal{Fl}, \Bbbk) \times D^{\mathrm{b}}_I(\mathcal{Gr}, \Bbbk) \to D^{\mathrm{b}}_I(\mathcal{Gr}, \Bbbk)$$

defined using the diagram

$$\mathcal{Fl} \times \mathcal{Gr} \xleftarrow{p} G(\mathbb{K}) \times \mathcal{Gr} \xrightarrow{q} G(\mathbb{K}) \times^I \mathcal{Gr} \xrightarrow{m} \mathcal{Gr}.$$

If $\mathcal{F} \in D^{\mathrm{b}}_I(\mathcal{Fl}, \Bbbk)$ and $\mathcal{G} \in D^{\mathrm{b}}_{G(\mathbb{O})}(\mathcal{Gr}, \Bbbk)$, we may write $\mathcal{F} \star^{\mathcal{Fl}}_{\mathcal{Gr}} \mathcal{G}$ as a shorthand for $\mathcal{F} \star^{\mathcal{Fl}}_{\mathcal{Gr}} \mathrm{For}^{G(\mathbb{O})}_I(\mathcal{G})$, suppressing the forgetful functor from the notation.

Remark 9.2.1. As in Remark 7.2.2, there are variants of the convolution diagram involving subvarieties of $\mathcal{Fl}$ or $\mathcal{Gr}$. Let U and V be locally closed finite unions of I-orbits in $\mathcal{Fl}$. Let $\tilde{U}$ be the preimage of U under $G(\mathbb{K}) \to \mathcal{Fl}$, and let

$$U \mathbin{\tilde{\times}} V = \tilde{U} \times^I V.$$

It can be shown (using an appropriate variant of (9.2.4)) that $U \mathbin{\tilde{\times}} V$ has the structure of a quasiprojective complex variety. If U and V are both closed, then the convolution product of objects supported on U and V can be computed using $U \mathbin{\tilde{\times}} V$. Similar remarks apply to the convolution products $\star_{\mathcal{Gr}}$ and $\star^{\mathcal{Fl}}_{\mathcal{Gr}}$.

Some properties of convolution. The following three lemmas are very similar to Lemmas 7.2.3, 7.2.4, and 7.2.7. We omit their proofs.

Lemma 9.2.2. *For $\mathcal{F} \in D^{\mathrm{b}}_I(\mathcal{Fl}, \Bbbk)$, $\mathcal{G} \in D^{\mathrm{b}}_{G(\mathbb{O})}(\mathcal{Gr}, \Bbbk)$, and $\mathcal{H} \in D^{\mathrm{b}}_I(\mathcal{Gr}, \Bbbk)$, there are natural isomorphisms*

$$\mathrm{IC}_e \star_{\mathcal{Fl}} \mathcal{F} \cong \mathcal{F} \star_{\mathcal{Fl}} \mathrm{IC}_e \cong \mathcal{F}, \qquad \mathrm{IC}_0 \star_{\mathcal{Gr}} \mathcal{G} \cong \mathcal{G} \star_{\mathcal{Gr}} \mathrm{IC}_0 \cong \mathcal{G}, \qquad \mathrm{IC}_e \star^{\mathcal{Fl}}_{\mathcal{Gr}} \mathcal{H} \cong \mathcal{H}.$$

Lemma 9.2.3. *Suppose we have three objects in one of the following situations:*

(1) *$\mathcal{F}, \mathcal{G}, \mathcal{H} \in D^{\mathrm{b}}_I(\mathcal{Fl}, \Bbbk)$, or*
(2) *$\mathcal{F}, \mathcal{G}, \mathcal{H} \in D^{\mathrm{b}}_{G(\mathbb{O})}(\mathcal{Gr}, \Bbbk)$, or*

(3) $\mathcal{F}, \mathcal{G} \in D^{\mathrm{b}}_I(\mathcal{F}\ell, \Bbbk)$ *and* $\mathcal{H} \in D^{\mathrm{b}}_I(\mathcal{G}r, \Bbbk)$.

In all three cases, there is a natural isomorphism $(\mathcal{F} \star \mathcal{G}) \star \mathcal{H} \cong \mathcal{F} \star (\mathcal{G} \star \mathcal{H})$.

Lemma 9.2.4. *Suppose we have two objects in one of the following situations:*

(1) $\mathcal{F}, \mathcal{G} \in D^{\mathrm{b}}_I(\mathcal{F}\ell, \Bbbk)$, *or*
(2) $\mathcal{F}, \mathcal{G} \in D^{\mathrm{b}}_{G(\mathbb{O})}(\mathcal{G}r, \Bbbk)$, *or*
(3) $\mathcal{F} \in D^{\mathrm{b}}_I(\mathcal{F}\ell, \Bbbk)$ *and* $\mathcal{G} \in D^{\mathrm{b}}_I(\mathcal{G}r, \Bbbk)$.

In all three cases, there is a natural isomorphism $(\mathbb{D}\mathcal{F}) \star (\mathbb{D}\mathcal{G}) \cong \mathbb{D}(\mathcal{F} \star \mathcal{G})$.

Lemma 9.2.5. (1) *For* $\mathcal{F}, \mathcal{G} \in D^{\mathrm{b}}_I(\mathcal{F}\ell, \Bbbk)$, *there is a natural isomorphism*

$$\pi_*(\mathcal{F} \star_{\mathcal{F}\ell} \mathcal{G}) \cong \mathcal{F} \star^{\mathcal{F}\ell}_{\mathcal{G}r} (\pi_* \mathcal{G}).$$

(2) *For* $\mathcal{F} \in D^{\mathrm{b}}_I(\mathcal{F}\ell, \Bbbk)$ *and* $\mathcal{G} \in D^{\mathrm{b}}_I(\mathcal{G}r, \Bbbk)$, *there is a natural isomorphism*

$$\mathcal{F} \star_{\mathcal{F}\ell} (\pi^* \mathcal{G}) \cong \pi^*(\mathcal{F} \star^{\mathcal{F}\ell}_{\mathcal{G}r} \mathcal{G}).$$

(3) *For* $\mathcal{F} \in D^{\mathrm{b}}_I(\mathcal{G}r, \Bbbk)$ *and* $\mathcal{G} \in D^{\mathrm{b}}_{G(\mathbb{O})}(\mathcal{G}r, \Bbbk)$, *there is a natural isomorphism*

$$(\pi^* \mathcal{F}) \star^{\mathcal{F}\ell}_{\mathcal{G}r} \mathcal{G} \cong (\pi_* \pi^* \mathcal{F}) \star_{\mathcal{G}r} \mathcal{G}.$$

Proof. In the following diagram, all the vertical maps are proper and smooth, and every square is cartesian:

$$\begin{tikzcd}
\mathcal{F}\ell \times \mathcal{F}\ell \arrow[d, "\mathrm{id} \times \pi"'] & G(\mathbb{K}) \times \mathcal{F}\ell \arrow[l, "p_{\mathcal{F}\ell}"'] \arrow[r, "q_{\mathcal{F}\ell}"] \arrow[d, "\mathrm{id} \times \pi"] & G(\mathbb{K}) \times^I \mathcal{F}\ell \arrow[r, "m_{\mathcal{F}\ell}"] \arrow[d, "\tilde{\pi}"] & \mathcal{F}\ell \arrow[d, "\pi"] \\
\mathcal{F}\ell \times \mathcal{G}r & G(\mathbb{K}) \times \mathcal{G}r \arrow[l, "p^{\mathcal{F}\ell}_{\mathcal{G}r}"'] \arrow[r, "q^{\mathcal{F}\ell}_{\mathcal{G}r}"] & G(\mathbb{K}) \times^I \mathcal{G}r \arrow[r, "m^{\mathcal{F}\ell}_{\mathcal{G}r}"] & \mathcal{G}r
\end{tikzcd}$$

In particular, we have $(q^{\mathcal{F}\ell}_{\mathcal{G}r})^* \tilde{\pi}_* \cong (\mathrm{id} \times \pi)_* q^*_{\mathcal{F}\ell}$. Therefore, for I-equivariant descent along $q_{\mathcal{F}\ell}$ or $q^{\mathcal{F}\ell}_{\mathcal{G}r}$, we have $\tilde{\pi}_*(q^*_{\mathcal{F}\ell})^{-1} \cong ((q^{\mathcal{F}\ell}_{\mathcal{G}r})^*)^{-1}(\mathrm{id} \times \pi)_*$. For part (1), we have

$$\begin{aligned}
\pi_*(\mathcal{F} \star_{\mathcal{F}\ell} \mathcal{G}) &= \pi_*(m_{\mathcal{F}\ell})_*(q^*_{\mathcal{F}\ell})^{-1} p^*_{\mathcal{F}\ell}(\mathcal{F} \boxtimes \mathcal{G}) \\
&\cong (m^{\mathcal{F}\ell}_{\mathcal{G}r})_* \tilde{\pi}_* (q^*_{\mathcal{F}\ell})^{-1} p^*_{\mathcal{F}\ell}(\mathcal{F} \boxtimes \mathcal{G}) \\
&\cong (m^{\mathcal{F}\ell}_{\mathcal{G}r})_*((q^{\mathcal{F}\ell}_{\mathcal{G}r})^*)^{-1}(\mathrm{id} \times \pi)_* p^*_{\mathcal{F}\ell}(\mathcal{F} \boxtimes \mathcal{G}) \\
&\cong (m^{\mathcal{F}\ell}_{\mathcal{G}r})_*((q^{\mathcal{F}\ell}_{\mathcal{G}r})^*)^{-1}(p^{\mathcal{F}\ell}_{\mathcal{G}r})^*(\mathcal{F} \boxtimes \pi_* \mathcal{G}) \cong \mathcal{F} \star^{\mathcal{F}\ell}_{\mathcal{G}r} (\pi_* \mathcal{G}).
\end{aligned}$$

The proof of part (2) is very similar and is left to the reader.

We now turn to part (3). Let

$$X = \mathcal{F}\ell \times_{\mathcal{G}r} G(\mathbb{K}) = \{(gI, h) \in \mathcal{F}\ell \times G(\mathbb{K}) \mid gG(\mathbb{O}) = hG(\mathbb{O})\}.$$

Define $\bar{q} : X \times \mathcal{G}r \to G(\mathbb{K}) \times^I \mathcal{G}r$ by $\bar{q}(gI, h, kG(\mathbb{O})) = (g, g^{-1}hkG(\mathbb{O}))$, and consider the diagram

$$\begin{tikzcd}
\mathcal{F}\ell \times \mathcal{G}r \arrow[d, "\pi \times \mathrm{id}"'] & X \times \mathcal{G}r \arrow[l, "\mathrm{pr}_1"'] \arrow[r, "\bar{q}"] \arrow[d, "\mathrm{pr}_2"] & G(\mathbb{K}) \times^I \mathcal{G}r \arrow[d, "\bar{\pi}"] \arrow[dr, "m^{\mathcal{F}\ell}_{\mathcal{G}r}"] & \\
\mathcal{G}r \times \mathcal{G}r & G(\mathbb{K}) \times \mathcal{G}r \arrow[l, "p_{\mathcal{G}r}"'] \arrow[r, "q_{\mathcal{G}r}"] \arrow[ul, "p^{\mathcal{F}\ell}_{\mathcal{G}r}"] \arrow[ur, "q^{\mathcal{F}\ell}_{\mathcal{G}r}"'] & G(\mathbb{K}) \times^{G(\mathbb{O})} \mathcal{G}r \arrow[r, "m_{\mathcal{G}r}"] & \mathcal{G}r
\end{tikzcd}$$

The triangles incident to the bottom of the diagram are commutative, and both squares are cartesian. (However, the triangles involving $X \times \mathcal{G}r$ are *not* commutative.) Below, the notation $\tilde{\boxtimes}$ will be used in two distinct ways:

$$\begin{aligned} \mathcal{F} \mathbin{\tilde{\boxtimes}} \mathcal{G} &= (q^*_{\mathcal{G}r})^{-1} p^*_{\mathcal{G}r}(\mathcal{F} \boxtimes \mathcal{G}) && \in D^{\mathrm{b}}_I(G(\mathbb{K}) \times^{G(\mathbb{O})} \mathcal{G}r, \Bbbk), \\ (\pi^*\mathcal{F}) \mathbin{\tilde{\boxtimes}} \mathcal{G} &= ((q^{\mathcal{F}\ell}_{\mathcal{G}r})^*)^{-1} (p^{\mathcal{F}\ell}_{\mathcal{G}r})^*((\pi^*\mathcal{F}) \boxtimes \mathcal{G}) && \in D^{\mathrm{b}}_I(G(\mathbb{K}) \times^I \mathcal{G}r, \Bbbk). \end{aligned}$$

Since $p_{\mathcal{G}r} = (\pi \times \mathrm{id}) \circ p^{\mathcal{F}\ell}_{\mathcal{G}r}$, we have

$$q^*_{\mathcal{G}r}(\mathcal{F} \mathbin{\tilde{\boxtimes}} \mathcal{G}) \cong p^*_{\mathcal{G}r}(\mathcal{F} \boxtimes \mathcal{G}) \cong (p^{\mathcal{F}\ell}_{\mathcal{G}r})^*((\pi^*\mathcal{F}) \boxtimes \mathcal{G}) \cong (q^{\mathcal{F}\ell}_{\mathcal{G}r})^*((\pi^*\mathcal{F}) \mathbin{\tilde{\boxtimes}} \mathcal{G}).$$

On the other hand, since $q_{\mathcal{G}r} = \bar{\pi} \circ q^{\mathcal{F}\ell}_{\mathcal{G}r}$, we also have

$$q^*_{\mathcal{G}r}(\mathcal{F} \mathbin{\tilde{\boxtimes}} \mathcal{G}) \cong (q^{\mathcal{F}\ell}_{\mathcal{G}r})^* \bar{\pi}^*(\mathcal{F} \mathbin{\tilde{\boxtimes}} \mathcal{G}).$$

Apply equivariant descent to conclude that there is a natural isomorphism

$$(\pi^*\mathcal{F}) \mathbin{\tilde{\boxtimes}} \mathcal{G} \cong \bar{\pi}^*(\mathcal{F} \mathbin{\tilde{\boxtimes}} \mathcal{G}). \tag{9.2.6}$$

Next, using the proper base change several times, we obtain

$$\begin{aligned} q^*_{\mathcal{G}r}\bar{\pi}_*\bar{\pi}^*(\mathcal{F} \mathbin{\tilde{\boxtimes}} \mathcal{G}) &\cong \mathrm{pr}_{2*}\bar{q}^*\bar{\pi}^*(\mathcal{F} \mathbin{\tilde{\boxtimes}} \mathcal{G}) \cong \mathrm{pr}_{2*}\mathrm{pr}_2^* q^*_{\mathcal{G}r}(\mathcal{F} \mathbin{\tilde{\boxtimes}} \mathcal{G}) \\ &\cong \mathrm{pr}_{2*}\mathrm{pr}_2^* p^*_{\mathcal{G}r}(\mathcal{F} \boxtimes \mathcal{G}) \cong \mathrm{pr}_{2*}\mathrm{pr}_1^*((\pi^*\mathcal{F}) \boxtimes \mathcal{G}) \cong p^*_{\mathcal{G}r}((\pi_*\pi^*\mathcal{F}) \boxtimes \mathcal{G}). \end{aligned} \tag{9.2.7}$$

Using (9.2.6) and (9.2.7), we find that

$$\begin{aligned} (\pi_*\pi^*\mathcal{F}) \star_{\mathcal{G}r} \mathcal{G} &= (m_{\mathcal{G}r})_*(q^*_{\mathcal{G}r})^{-1} p^*_{\mathcal{G}r}((\pi_*\pi^*\mathcal{F}) \boxtimes \mathcal{G}) \\ &\cong (m_{\mathcal{G}r})_*\bar{\pi}_*\bar{\pi}^*(\mathcal{F} \mathbin{\tilde{\boxtimes}} \mathcal{G}) \cong (m^{\mathcal{F}\ell}_{\mathcal{G}r})_*((\pi^*\mathcal{F}) \mathbin{\tilde{\boxtimes}} \mathcal{G}) \cong (\pi^*\mathcal{F}) \star^{\mathcal{F}\ell}_{\mathcal{G}r} \mathcal{G}, \end{aligned}$$

as desired. $\square$

The following lemma is left as an exercise.

Lemma 9.2.6. *Let $\lambda, \mu \in \check{\mathbf{X}}^+$ be coweights such that $\langle 2\rho, \lambda\rangle = \langle 2\rho, \mu\rangle = 0$. Then $\mathrm{IC}_\lambda \star_{\mathcal{G}r} \mathrm{IC}_\mu \cong \mathrm{IC}_{\lambda+\mu}$.*

Exercises.

9.2.1. This exercise continues with the "lattice model" of the affine Grassmannian for GL_n, introduced in the exercises of Section 9.1. Let $L, L' \subset \mathbb{K}^n$ be lattices, and let $\lambda = (\lambda_1, \ldots, \lambda_n) \in \check{\mathbf{X}}^+$ be a dominant coweight for GL_n. Let N be an integer such that

$$t^N L' \subset L \subset t^{-N} L'. \tag{9.2.8}$$

We say that L is in **relative position** λ with respect to L' if for $-N \le i \le N$, we have

$$\dim \frac{L \cap t^i L'}{t^N L'} = \sum_{j=i}^{N-1} |\{j \mid \lambda_j \le i\}|.$$

Show that there always exists an N such that (9.2.8) holds. Then show that for any two lattices L, L', there exists a unique $\lambda \in \check{\mathbf{X}}^+$ such that L is in relative position λ with respect to L', and that this coweight is independent of the choice of N.

9.2.2. Let $\lambda, \mu \in \check{\mathbf{X}}^+$, and consider the space $\mathcal{G}r_\lambda \tilde{\times} \mathcal{G}r_\mu$ (see Remark 9.2.1). Show that there is an isomorphism

$$\mathcal{G}r_\lambda \tilde{\times} \mathcal{G}r_\mu \cong \left\{ (L', L) \,\middle|\, \begin{matrix} L' \in \mathcal{G}r_\lambda, \text{ and } L \text{ is in relative} \\ \text{position } \mu \text{ with respect to } L' \end{matrix} \right\}$$

such that under this identification, the maps $\mathrm{pr}_1 : \mathcal{G}r_\lambda \tilde{\times} \mathcal{G}r_\mu \to \mathcal{G}r$ and $m : \mathcal{G}r_\lambda \tilde{\times} \mathcal{G}r_\mu \to \mathcal{G}r$ are given by

$$\mathrm{pr}_1(L', L) = L' \qquad \text{and} \qquad m(L', L) = L.$$

9.3. Categorification of the affine and spherical Hecke algebras

In this section, we discuss analogues of Theorem 7.3.8 for $\mathcal{F}\ell$ and $\mathcal{G}r$.

The extended affine Hecke algebra. Let q be an indeterminate, and consider the ring $\mathbb{Z}[q^{\pm\frac{1}{2}}]$ of Laurent polynomials in $q^{\frac{1}{2}}$. The **extended affine Hecke algebra**, denoted by $\mathcal{H}_{\mathrm{ext}}$, is the $\mathbb{Z}[q^{\pm\frac{1}{2}}]$-algebra defined as follows. As a $\mathbb{Z}[q^{\pm\frac{1}{2}}]$-module, $\mathcal{H}_{\mathrm{ext}}$ is free with basis

$$\{T_w\}_{w \in W_{\mathrm{ext}}}.$$

The multiplication in $\mathcal{H}_{\mathrm{ext}}$ is determined by the following rules:

$$T_w T_v = T_{wv} \qquad \text{if } \ell(wv) = \ell(w) + \ell(v), \tag{9.3.1}$$

$$T_s^2 = (q-1)T_s + qT_1 \qquad \text{if } s \in S. \tag{9.3.2}$$

The **affine Hecke algebra**, denoted by $\mathcal{H}_{\mathrm{aff}}$, is the subalgebra of $\mathcal{H}_{\mathrm{ext}}$ generated by $\{T_w\}_{w \in W_{\mathrm{aff}}}$. The basic facts about the Hecke algebra from Chapter 7 carry over to $\mathcal{H}_{\mathrm{aff}}$ and $\mathcal{H}_{\mathrm{ext}}$ with little or no change. The ring $\mathcal{H}_{\mathrm{aff}}$ is generated as a $\mathbb{Z}[q^{\pm\frac{1}{2}}]$-algebra by $\{T_s\}_{s \in S_{\mathrm{aff}}}$, and $\mathcal{H}_{\mathrm{ext}}$ is generated by $\{T_s\}_{s \in S_{\mathrm{aff}}} \cup \{T_\omega\}_{\omega \in \Omega}$. Equation (7.1.6) holds for any $w \in W_{\mathrm{ext}}$ and any $s \in S_{\mathrm{aff}}$. There is a bar involution defined as in (7.1.7), and a canonical basis

$$\{\underline{H}_w\}_{w \in W_{\mathrm{ext}}}$$

that is characterized by Theorem 7.1.3.

The proofs of the next four statements can be copied verbatim from the corresponding statements in Chapter 7. (For Lemma 9.3.4, compare with Lemma 7.2.8.)

LEMMA 9.3.1. *Let $w \in W_{\mathrm{ext}}$. If $\ell(w)$ is even, then $\mathrm{IC}_w(\mathbb{Q})$ is both $*$-even and $!$-even. If $\ell(w)$ is odd, then $\mathrm{IC}_w(\mathbb{Q})$ is both $*$-odd and $!$-odd.*

PROPOSITION 9.3.2. *The category $\mathrm{Semis}_I(\mathcal{F}\ell, \mathbb{Q}) \subset D^{\mathrm{b}}_I(\mathcal{F}\ell, \mathbb{Q})$ is closed under the convolution product.*

THEOREM 9.3.3. *The map $\mathrm{ch} : \mathrm{K}_\oplus(\mathrm{Semis}_I(\mathcal{F}\ell, \mathbb{Q})) \to \mathcal{H}_{\mathrm{ext}}$ given by*

$$\mathrm{ch}([\mathcal{F}]) = \sum_{\substack{w \in W_{\mathrm{ext}} \\ i \in \mathbb{Z}}} (\mathrm{rank}\, \mathsf{H}^i(\mathcal{F}|_{\mathcal{B}_w})) q^{i/2} T_w$$

is an isomorphism of rings. It satisfies

$$\mathrm{ch}([\mathcal{F}[1]]) = q^{-1/2}\mathrm{ch}([\mathcal{F}]), \quad \mathrm{ch}([\mathbb{D}(\mathcal{F})]) = \overline{\mathrm{ch}([\mathcal{F}])}, \quad \mathrm{ch}([\mathrm{IC}_w(\mathbb{Q})]) = \underline{H}_w.$$

LEMMA 9.3.4. *Let w_0 be the longest element of W. There is a cartesian square*

$$\begin{array}{ccc} G(\mathbb{K}) \times^I \overline{\mathcal{F}\ell_{w_0}} & \xrightarrow{m} & \mathcal{F}\ell \\ \Big\downarrow \mathrm{pr}_1 & & \Big\downarrow \pi \\ \mathcal{F}\ell & \xrightarrow{\pi} & \mathcal{G}r \end{array}$$

As a consequence, for any $\mathcal{F} \in D^{\mathrm{b}}_I(\mathcal{F}\ell, \Bbbk)$, there is a natural isomorphism

$$\mathcal{F} \star \mathrm{IC}_{w_0} \cong \pi^* \pi_* \mathcal{F}[\ell(w_0)].$$

The next lemma could have been stated in Chapter 7. It will be needed below for the definition of the spherical Hecke algebra.

LEMMA 9.3.5. *We have*

$$\underline{H}_{w_0} = q^{-\ell(w_0)/2} \sum_{w \in W} T_w \qquad \text{and} \qquad \underline{H}^2_{w_0} = q^{-\ell(w_0)/2} \left(\sum_{w \in W} q^{\ell(w)} \right) \underline{H}_{w_0}.$$

PROOF. Since $\overline{\mathcal{F}\ell_{w_0}} \cong G/B^-$ is smooth of dimension $\ell(w_0)$, we have $\mathrm{IC}_{w_0}(\mathbb{Q}) \cong \underline{\mathbb{Q}}_{\overline{\mathcal{F}\ell_{w_0}}}[\ell(w_0)]$. The formula for $\underline{H}_{w_0}$ then follows from the definition of ch in Theorem 9.3.3. To compute $\underline{H}^2_{w_0}$, let us study $\mathrm{IC}_{w_0} \star_{\mathcal{F}\ell} \mathrm{IC}_{w_0}$, which, by Lemma 9.3.4, is isomorphic to $\pi^*\pi_*\mathrm{IC}_{w_0}$. Note that $\pi(\overline{\mathcal{F}\ell_{w_0}})$ is the singleton $\mathcal{G}r_0$, so

$$\pi_* \mathrm{IC}_{w_0} \cong \pi_* \underline{\mathbb{Q}}_{\overline{\mathcal{F}\ell_{w_0}}}[\ell(w_0)] \cong j_{0*} R\Gamma(\underline{\mathbb{Q}}_{G/B^-})[\ell(w_0)],$$

where $j_0 : \mathcal{G}r_0 \hookrightarrow \mathcal{G}r$ is the inclusion map. Since G/B^- is a projective variety with a stratification by affine spaces, it is well known (cf. Exercise 1.3.5) that

$$\mathbf{H}^i(G/B^-; \mathbb{Q}) \cong \begin{cases} \mathbb{Q}^{\text{number of strata of dimension } i/2}(-\frac{i}{2}) & \text{if } i \text{ is even,} \\ 0 & \text{if } i \text{ is odd.} \end{cases}$$

Of course, the number of strata in G/B^- of dimension j is precisely $|\{w \in W \mid \ell(w) = j\}|$. Since $\pi^{-1}(\mathcal{G}r_0) = \overline{\mathcal{F}\ell_{w_0}}$, we deduce that

$$\mathsf{H}^{i-2\ell(w_0)}(\pi^*\pi_*\mathrm{IC}_{w_0}[\ell(w_0)]) \cong \begin{cases} \underline{\mathbb{Q}}^{|\{w \in W \mid \ell(w) = i/2\}|}_{\overline{\mathcal{F}\ell_{w_0}}}(-\frac{i}{2}) & \text{if } i \text{ is even,} \\ 0 & \text{if } i \text{ is odd.} \end{cases}$$

By Lemma 9.3.4 and Proposition 9.3.2, we have

$$\begin{aligned} \mathrm{IC}_{w_0} \star_I \mathrm{IC}_{w_0} &\cong \bigoplus_{i \in \mathbb{Z}} \underline{\mathbb{Q}}_{\overline{\mathcal{F}\ell_{w_0}}}[2\ell(w_0) - 2i]^{\oplus(|\{w \in W \mid \ell(w) = i\}|)}(-i) \\ &\cong \bigoplus_{w \in W} \mathrm{IC}_{w_0}[\ell(w_0) - 2\ell(w)](-\ell(w)). \end{aligned}$$

By Theorem 9.3.3, this yields the desired formula for $\underline{H}^2_{w_0}$. ☐

The spherical Hecke algebra. To define the spherical Hecke algebra, we must (temporarily) enlarge the coefficient ring $\mathbb{Z}[q^{\pm\frac{1}{2}}]$ by inverting the element

$$W_q = \sum_{w \in W} q^{\ell(w)} \in \mathbb{Z}[q].$$

For any $\lambda \in \check{\mathbf{X}}^+$, let

$$K_\lambda = W_q^{-1} \sum_{w \in W\lambda W \subset W_{\mathrm{ext}}} T_w \in \mathbb{Z}[q^{\pm\frac{1}{2}}, W_q^{-1}] \otimes_{\mathbb{Z}[q^{\pm\frac{1}{2}}]} \mathcal{H}_{\mathrm{ext}}.$$

In particular, by Lemma 9.3.5, we have

$$K_0 = q^{\ell(w_0)/2} W_q^{-1} \underline{H}_{w_0} \qquad \text{and} \qquad K_0^2 = K_0.$$

We define the **spherical Hecke algebra** by

$$\mathcal{H}_{\mathrm{sph}} = \mathbb{Z}[q^{\pm\frac{1}{2}}]\text{-span of } \{K_\lambda\}_{\lambda \in \check{\mathbf{X}}^+}.$$

Of course, to justify calling this space an algebra, we must check that it is closed under multiplication.

LEMMA 9.3.6. *We have* $\mathcal{H}_{\mathrm{sph}} = K_0\mathcal{H}_{\mathrm{ext}} \cap \mathcal{H}_{\mathrm{ext}}K_0 = K_0\mathcal{H}_{\mathrm{ext}}K_0$. *As a consequence,* $\mathcal{H}_{\mathrm{sph}}$ *is a* $\mathbb{Z}[q^{\pm\frac{1}{2}}]$*-algebra, with multiplicative identity* K_0.

PROOF SKETCH. The fact that $K_0^2 = K_0$ implies that $K_0\mathcal{H}_{\mathrm{ext}} \cap \mathcal{H}_{\mathrm{ext}}K_0 = K_0\mathcal{H}_{\mathrm{ext}}K_0$ and that this set is a $\mathbb{Z}[q^{\pm\frac{1}{2}}]$-algebra with K_0 as its multiplicative identity. To identify $K_0\mathcal{H}_{\mathrm{ext}}K_0$ with $\mathcal{H}_{\mathrm{sph}}$, we need an alternative description of the former. Given $h \in \mathcal{H}_{\mathrm{ext}}$, denote its coefficients in the $\{T_w\}_{w \in W_{\mathrm{ext}}}$ basis by $a_w(h)$. In other words, we have

$$h = \sum_{w \in W_{\mathrm{ext}}} a_w(h) T_w.$$

One can show (by computing $K_0 T_w$ explicitly) that $K_0\mathcal{H}_{\mathrm{ext}}$ consists of elements whose coefficients in the $\{T_w\}_{w \in W_{\mathrm{ext}}}$ basis are constant on right cosets of $W \subset W_{\mathrm{ext}}$. That is,

$$K_0\mathcal{H}_{\mathrm{ext}} = \left\{ W_q^{-1} h \;\middle|\; \begin{array}{c} h \in \mathcal{H}_{\mathrm{ext}} \text{ and } a_w(h) = a_{vw}(h) \\ \text{for all } w \in W_{\mathrm{ext}} \text{ and } v \in W \end{array} \right\}.$$

Similarly, $\mathcal{H}_{\mathrm{ext}}K_0$ consists of elements whose coefficients are constant on left cosets, and

$$K_0\mathcal{H}_{\mathrm{ext}} \cap \mathcal{H}_{\mathrm{ext}}K_0 = \left\{ W_q^{-1} h \;\middle|\; \begin{array}{c} h \in \mathcal{H}_{\mathrm{ext}} \text{ and } a_w(h) = a_{vwv'}(h) \\ \text{for all } w \in W_{\mathrm{ext}} \text{ and } v, v' \in W \end{array} \right\}.$$

It is easy to see that $\{K_\lambda\}_{\lambda \in \check{\mathbf{X}}^+}$ is a $\mathbb{Z}[q^{\pm\frac{1}{2}}]$-basis for this set. $\square$

The spherical Hecke algebra inherits the bar involution from $\mathcal{H}_{\mathrm{ext}}$. The following theorem, analogous to Theorem 7.1.3, is a reformulation of results from [**158**, Section 6]. (The elements $\underline{S}_\lambda$ are denoted by C'_λ in [**158**].)

THEOREM 9.3.7. *There is a unique basis* $\{\underline{S}_\lambda\}_{\lambda \in \check{\mathbf{X}}^+}$ *for* $\mathcal{H}_{\mathrm{sph}}$ *with the following properties:*

(1) *For all* $\lambda \in \check{\mathbf{X}}^+$, *we have* $\overline{\underline{S}_\lambda} = \underline{S}_\lambda$.
(2) *If we write* $\underline{S}_\lambda$ *as*

$$\underline{S}_\lambda = q^{-\langle 2\rho, \lambda\rangle/2} \sum_{\mu \in \check{\mathbf{X}}^+} M_{\mu,\lambda} K_\mu,$$

then the coefficients $M_{\mu,\lambda} \in \mathbb{Z}[q^{\pm\frac{1}{2}}]$ *satisfy*
(a) $M_{\mu,\lambda} = 0$ *unless* $\mu \leq_{\check{\mathbf{X}}} \lambda$.
(b) $M_{\lambda,\lambda} = 1$.
(c) *If* $\mu <_{\check{\mathbf{X}}} \lambda$, *then* $M_{\mu,\lambda}$ *lies in* $\mathbb{Z}[q]$ *and has degree* $< \langle \rho, \lambda - \mu\rangle$.

The next two statements are straightforward consequences of their counterparts for $\mathcal{Fl}$. We omit their proofs.

LEMMA 9.3.8. *Let* $\lambda \in \check{\mathbf{X}}^+$. *If* $\langle 2\rho, \lambda\rangle$ *is even, then* $\mathrm{IC}_\lambda(\mathbb{Q})$ *is both* $*$*-even and* $!$*-even. If* $\langle 2\rho, \lambda\rangle$ *is odd, then* $\mathrm{IC}_\lambda(\mathbb{Q})$ *is both* $*$*-odd and* $!$*-odd.*

PROPOSITION 9.3.9. *The category* $\mathrm{Semis}_{G(\mathbb{O})}(\mathcal{Gr}, \mathbb{Q}) \subset D^{\mathrm{b}}_{G(\mathbb{O})}(\mathcal{Gr}, \mathbb{Q})$ *is closed under the convolution product.*

PROPOSITION 9.3.10. *The category* $\mathrm{Perv}_{G(\mathbb{O})}(\mathcal{Gr}, \mathbb{Q})$ *is semisimple.*

PROOF. Since $\mathrm{Perv}_{G(\mathbb{O})}(\mathcal{Gr}, \mathbb{Q})$ is a finite-length category whose simple objects are the $\mathrm{IC}_\lambda(\mathbb{Q})$, it is enough to show that for any $\lambda, \mu \in \check{\mathbf{X}}^+$, we have $\mathrm{Ext}^1(\mathrm{IC}_\lambda(\mathbb{Q}), \mathrm{IC}_\mu(\mathbb{Q})) = 0$, or, equivalently (see Proposition A.7.18), that

$$\mathrm{Hom}(\mathrm{IC}_\lambda(\mathbb{Q}), \mathrm{IC}_\mu(\mathbb{Q})[1]) = 0. \tag{9.3.3}$$

If $\mathcal{Gr}_\lambda$ and $\mathcal{Gr}_\mu$ lie in different connected components of $\mathcal{Gr}$, then (9.3.3) is obvious. If they belong to the same connected component, then $\dim \mathcal{Gr}_\lambda$ and $\dim \mathcal{Gr}_\mu$ have the same parity (see Corollary 9.1.4), so $\mathrm{IC}_\lambda(\mathbb{Q})$ and $\mathrm{IC}_\mu(\mathbb{Q})$ are either both even or both odd, by Lemma 9.3.8. In either case, (9.3.3) holds by Lemma 7.3.4. (That lemma was stated for $\mathcal{B}$, but the proof goes through verbatim for $\mathcal{Fl}$ or $\mathcal{Gr}$.) $\square$

We are now ready to prove the categorification theorem for $\mathcal{H}_{\mathrm{sph}}$.

THEOREM 9.3.11. *The map* $\mathrm{ch} : \mathrm{K}_\oplus(\mathrm{Semis}_{G(\mathbb{O})}(\mathcal{Gr}, \mathbb{Q})) \to \mathcal{H}_{\mathrm{sph}}$ *given by*

$$\mathrm{ch}([\mathcal{F}]) = \sum_{\substack{\lambda \in \check{\mathbf{X}}^+ \\ i \in \mathbb{Z}}} (\operatorname{rank} \mathsf{H}^i(\mathcal{F}|_{\mathcal{Gr}_\lambda})) q^{i/2} K_\lambda$$

is an isomorphism of rings. It satisfies

$$\mathrm{ch}([\mathcal{F}[1]]) = q^{-1/2}\mathrm{ch}([\mathcal{F}]), \quad \mathrm{ch}([\mathbb{D}(\mathcal{F})]) = \overline{\mathrm{ch}([\mathcal{F}])}, \quad \mathrm{ch}([\mathrm{IC}_\lambda(\mathbb{Q})]) = \underline{S}_\lambda.$$

PROOF. *Step 1. We have* $\mathrm{ch}([\mathcal{F}[1]]) = q^{-1/2}\mathrm{ch}([\mathcal{F}])$. *Moreover,* ch *is an isomorphism of abelian groups.* This is identical to Step 1 in the proof of Theorem 7.3.8.

Step 2. There is a commutative diagram

$$\begin{tikzcd} \mathrm{K}_\oplus(\mathrm{Semis}_{G(\mathbb{O})}(\mathcal{Gr}, \mathbb{Q})) \arrow[r, "\pi^*"] \arrow[d, "\mathrm{ch}_{\mathcal{Gr}}"'] & \mathrm{K}_\oplus(\mathrm{Semis}_I(\mathcal{Fl}, \mathbb{Q})) \arrow[d, "\mathrm{ch}_{\mathcal{Fl}}"] \\ \mathcal{H}_{\mathrm{sph}} \arrow[r, "\text{mult. by } W_q"] & \mathcal{H}_{\mathrm{ext}} \end{tikzcd}$$

This is a straightforward calculation from the definitions of $\mathrm{ch}_{\mathcal{Fl}}$ and $\mathrm{ch}_{\mathcal{Gr}}$.

Step 3. For any $\mathcal{F} \in \mathrm{Semis}_{G(\mathbb{O})}(\mathcal{Gr}, \mathbb{Q})$, *we have* $\mathrm{ch}([\pi_*\pi^*\mathcal{F}]) = W_q\mathrm{ch}([\mathcal{F}])$. There is no harm in multiplying both sides by W_q and the right side by K_0, so it is enough to check instead that

$$W_q\mathrm{ch}([\pi_*\pi^*\mathcal{F}]) = W_q^2\mathrm{ch}([\mathcal{F}])K_0.$$

In this equation, both sides are elements of $\mathcal{H}_{\mathrm{ext}}$. By Step 2 and the definition of K_0, this is in turn equivalent to the claim that

$$\mathrm{ch}_{\mathcal{Fl}}([\pi^*\pi_*\pi^*\mathcal{F}]) = q^{\ell(w_0)/2}\mathrm{ch}_{\mathcal{Fl}}([\pi^*\mathcal{F}])\underline{H}_{w_0}.$$

To prove this, we use Lemma 9.3.4, Lemma 9.3.5, and Theorem 9.3.3:

$$\begin{aligned}\mathrm{ch}_{\mathcal{Fl}}([\pi^*\pi_*\pi^*\mathcal{F}]) &= \mathrm{ch}_{\mathcal{Fl}}([\pi^*\mathcal{F} \star_{\mathcal{Fl}} \mathrm{IC}_{w_0}[-\ell(w_0)]]) \\ &= q^{\ell(w_0)/2}\mathrm{ch}_{\mathcal{Fl}}([\pi^*\mathcal{F}])\mathrm{ch}([\mathrm{IC}_{w_0}]) = q^{\ell(w_0)/2}\mathrm{ch}_{\mathcal{Fl}}([\pi^*\mathcal{F}])\underline{H}_{w_0}.\end{aligned}$$

Step 4. For $\mathcal{F}, \mathcal{G} \in \mathrm{Semis}_{G(\mathbb{O})}(\mathcal{Gr}, \mathbb{Q})$*, we have*

$$\mathrm{ch}([(\pi_*\pi^*\mathcal{F}) \star \mathcal{G}]) = W_q\mathrm{ch}([\mathcal{F} \star \mathcal{G}]).$$

For this assertion, we will use the fact that the convolution product $\star_{\mathcal{Gr}}$ equips $\mathrm{K}_\oplus(\mathrm{Semis}_{G(\mathbb{O})}(\mathcal{Gr}, \mathbb{Q}))$ with a ring structure (thanks to Proposition 9.3.9), even though we do not yet know whether ch is a ring homomorphism. The proof of Step 3 also shows that $[\pi_*\pi^*\mathcal{F}] = W_q[\mathcal{F}]$, so in $\mathrm{K}_\oplus(\mathrm{Semis}_{G(\mathbb{O})}(\mathcal{Gr}, \mathbb{Q}))$ we have

$$[(\pi_*\pi^*\mathcal{F}) \star \mathcal{G}] = [\pi_*\pi^*\mathcal{F}][\mathcal{G}] = W_q[\mathcal{F}][\mathcal{G}] = W_q[\mathcal{F} \star \mathcal{G}].$$

The claim follows.

Step 5. For $\mathcal{F}, \mathcal{G} \in \mathrm{Semis}_{G(\mathbb{O})}(\mathcal{Gr}, \mathbb{Q})$*, we have* $\mathrm{ch}([\mathcal{F} \star \mathcal{G}]) = \mathrm{ch}([\mathcal{F}])\mathrm{ch}([\mathcal{G}])$*. Therefore,* ch *is an isomorphism of rings.* It is enough to show instead that $W_q^2\mathrm{ch}([\mathcal{F}\star\mathcal{G}]) = W_q^2\mathrm{ch}([\mathcal{F}])\mathrm{ch}([\mathcal{G}])$. Using Step 2 and Step 4 along with Lemma 9.2.5 and Theorem 9.3.3, we have

$$\begin{aligned}W_q^2\mathrm{ch}([\mathcal{F} \star_{\mathcal{Gr}} \mathcal{G}]) &= W_q\mathrm{ch}([(\pi_*\pi^*\mathcal{F}) \star_{\mathcal{Gr}} \mathcal{G}]) \\ &= \mathrm{ch}_{\mathcal{Fl}}([\pi^*((\pi_*\pi^*\mathcal{F}) \star_{\mathcal{Gr}} \mathcal{G})]) \\ &= \mathrm{ch}_{\mathcal{Fl}}([\pi^*((\pi^*\mathcal{F}) \star_{\mathcal{Gr}}^{\mathcal{Fl}} \mathcal{G})]) \\ &= \mathrm{ch}_{\mathcal{Fl}}([(\pi^*\mathcal{F}) \star_{\mathcal{Fl}} (\pi^*\mathcal{G})]) \\ &= \mathrm{ch}_{\mathcal{Fl}}([\pi^*\mathcal{F}])\mathrm{ch}_{\mathcal{Fl}}([\pi^*\mathcal{G}]) \\ &= W_q^2\mathrm{ch}([\mathcal{F}])\mathrm{ch}([\mathcal{G}]).\end{aligned}$$

Step 6. We have $\mathrm{ch}([\mathbb{D}(\mathcal{F})]) = \overline{\mathrm{ch}([\mathcal{F}])}$*.* Again, we will instead prove that $W_q\mathrm{ch}([\mathbb{D}(\mathcal{F})]) = W_q\overline{\mathrm{ch}([\mathcal{F}])}$. Because π is smooth of relative dimension $\ell(w_0)$, we have $\pi^*(\mathbb{D}\mathcal{F}) \cong \mathbb{D}(\pi^!\mathcal{F}) \cong \mathbb{D}(\pi^*\mathcal{F}[2\ell(w_0)]) \cong \mathbb{D}(\pi^*\mathcal{F})[-2\ell(w_0)]$ (ignoring Tate twists). Therefore,

$$\begin{aligned}W_q\mathrm{ch}([\mathbb{D}(\mathcal{F})]) = \mathrm{ch}_{\mathcal{Fl}}([\pi^*(\mathbb{D}\mathcal{F})]) &= \mathrm{ch}_{\mathcal{Fl}}(\mathbb{D}(\pi^*\mathcal{F})[-2\ell(w_0)]) \\ &= q^{\ell(w_0)}\overline{\mathrm{ch}_{\mathcal{Fl}}([\pi^*\mathcal{F}])} = q^{\ell(w_0)}\overline{W_q\mathrm{ch}([\mathcal{F}])}.\end{aligned}$$

It is left to the reader to check that $W_q = q^{\ell(w_0)}\overline{W_q}$. This claim finishes the proof.

Step 7. We have $\mathrm{ch}([\mathrm{IC}_\lambda]) = \underline{S}_\lambda$. The proof that $\{\mathrm{ch}([\mathrm{IC}_\lambda])\}_{\lambda\in\check{\mathbf{X}}^+}$ satisfies the properties from Theorem 9.3.7 that uniquely characterize the basis $\{\underline{S}_\lambda\}_{\lambda\in\check{\mathbf{X}}^+}$ is essentially identical to Step 6 in the proof of Theorem 7.3.8. □

COROLLARY 9.3.12. *For* $\mu, \lambda \in \check{\mathbf{X}}^+$*, we have*

$$M_{\mu,\lambda} = q^{\langle\rho,\lambda\rangle}\sum_{i\in\mathbb{Z}}(\mathrm{rank}\,\mathsf{H}^i(\mathrm{IC}_\lambda(\mathbb{Q})|_{\mathcal{Gr}_\mu}))q^{i/2}.$$

9.4. The Satake isomorphism

This brief section contains an overview (without proofs) of a description of $\mathcal{H}_{\mathrm{sph}}$ due to Satake, and a number of results of Lusztig that give a geometric interpretation of this description. The statements in this section are not logically necessary for the rest of the chapter, but they do provide motivation.

Consider the group ring $\mathbb{Z}[\check{\mathbf{X}}]$ of $\check{\mathbf{X}}$. For $\lambda \in \check{\mathbf{X}}$, we write e^λ for the corresponding element of $\mathbb{Z}[\check{\mathbf{X}}]$. Then, of course, the set $\{e^\lambda\}_{\lambda\in\check{\mathbf{X}}}$ is a $\mathbb{Z}$-basis for $\mathbb{Z}[\check{\mathbf{X}}]$. If $\{\lambda_1, \lambda_2, \ldots, \lambda_k\}$ is a $\mathbb{Z}$-basis for the abelian group $\check{\mathbf{X}}$, then $\mathbb{Z}[\check{\mathbf{X}}]$ can be thought of as the ring of Laurent polynomials in the variables $e^{\lambda_1}, \ldots, e^{\lambda_k}$. In particular, $\mathbb{Z}[\check{\mathbf{X}}]$ is an integral domain. The action of W on $\check{\mathbf{X}}$ induces an action of W by ring automorphisms on $\mathbb{Z}[\check{\mathbf{X}}]$. For $\lambda \in \check{\mathbf{X}}^+$, let

$$c_\lambda = \sum_{\mu\in W\lambda} e^\mu.$$

It is straightforward to see that $\{c_\lambda\}_{\lambda\in\check{\mathbf{X}}^+}$ is a $\mathbb{Z}$-basis for the ring $\mathbb{Z}[\check{\mathbf{X}}]^W$ of W-invariants in $\mathbb{Z}[\check{\mathbf{X}}]$.

THEOREM 9.4.1 (Satake). *There is an isomorphism of $\mathbb{Z}[q^{\pm\frac{1}{2}}]$-algebras*

$$\gamma : \mathcal{H}_{\mathrm{sph}} \xrightarrow{\sim} \mathbb{Z}[q^{\pm\frac{1}{2}}][\check{\mathbf{X}}]^W.$$

The underlying idea of this statement goes back to [**202**], but the version proved there is in the language of "spherical functions on p-adic groups." For a version in terms of $\mathcal{H}_{\mathrm{sph}}$ as defined here, see [**171**, Théorème 4.4.8].

The definition of the map γ is rather complicated; see [**171**, Section 4.4.7] or the discussion preceding [**128**, Theorem 2.3]. Notably, γ does *not* send the standard basis $\{K_\lambda\}_{\lambda\in\check{\mathbf{X}}^+}$ to the obvious basis $\{c_\lambda\}_{\lambda\in\check{\mathbf{X}}^+}$. Instead, according to [**128**, Theorem 2.4] (see also [**165**]), we have

$$\gamma(K_\lambda) = \frac{q^{\langle\rho,\lambda\rangle}}{\sum\limits_{\substack{w\in W\\ w\lambda=\lambda}} q^{-\ell(w)}} \sum_{w\in W} e^{w\lambda} \prod_{\alpha\in\check{\Phi}^+} \frac{1-q^{-1}e^{-w\alpha}}{1-e^{-w\alpha}}.$$

We have already seen that the left-hand side of the isomorphism in Theorem 9.4.1 is categorified by $\mathrm{Semis}_{G(\mathbb{O})}(\mathcal{G}r, \mathbb{C})$. The right-hand side also has a natural, and much more elementary, categorification. Let $\check{G}_{\mathbb{C}}$ be the complex reductive algebraic group that is Langlands dual to G: in other words, it is the reductive group whose weight lattice is $\check{\mathbf{X}}$ and whose root system is $\check{\Phi}$. Let $\mathrm{Rep}(\check{G}_{\mathbb{C}})$ be the category of finite-dimensional algebraic $\check{G}_{\mathbb{C}}$-representations. For $\lambda \in \check{\mathbf{X}}^+$, let

$$L_\lambda(\mathbb{C}) = \text{the irreducible } \check{G}_{\mathbb{C}}\text{-representation of highest weight } \lambda.$$

For any $V \in \mathrm{Rep}(\check{G}_{\mathbb{C}})$ and any weight $\mu \in \check{\mathbf{X}}^+$, let V_μ be its μ-weight space. The tensor product on $\mathrm{Rep}(\check{G}_{\mathbb{C}})$ makes its Grothendieck group $\mathrm{K}_0(\mathrm{Rep}(\check{G}_{\mathbb{C}}))$ into a ring. It is a straightforward exercise to show that the map

$$\mathrm{ch} : \mathrm{K}_0(\mathrm{Rep}(\check{G}_{\mathbb{C}})) \to \mathbb{Z}[\check{\mathbf{X}}]^W \qquad \text{given by} \qquad \mathrm{ch}([V]) = \sum_{\mu\in\check{\mathbf{X}}} (\dim V_\mu) e^\mu$$

is an isomorphism of rings (cf. [**114**, Remark II.2.7]). As a consequence, by Proposition A.9.4, the set $\{\mathrm{ch}([L_\lambda(\mathbb{C})])\}_{\lambda\in\check{\mathbf{X}}^+}$ is a $\mathbb{Z}$-basis for $\mathbb{Z}[\check{\mathbf{X}}]^W$.

Thus: each side of the Satake isomorphism admits a categorification, and each categorification gives rise to a distinguished basis. In [**158**], Lusztig made the remarkable discovery that the two "categorical" bases coincide:

THEOREM 9.4.2 (Lusztig). *The Satake isomorphism $\gamma : \mathcal{H}_{\mathrm{sph}} \xrightarrow{\sim} \mathbb{Z}[q^{\pm\frac{1}{2}}][\check{\mathbf{X}}]^W$ satisfies $\gamma(\underline{S}_\lambda) = \mathrm{ch}([L_\lambda(\mathbb{C})])$.*

For a proof, see [**158**, Proposition 8.6]. (The version in [**158**] does not involve γ explicitly; instead, it is stated in terms of another map $\bar{\gamma} : \mathbb{Z}[q^{\pm\frac{1}{2}}][\check{\mathbf{X}}]^W \to \mathcal{H}_{\mathrm{sph}}$ that is defined using Bernstein's description of the center of $\mathcal{H}_{\mathrm{ext}}$ (see [**158**, Theorem 8.1]). However, it can be deduced from [**128**, Theorem 1.5] that $\bar{\gamma}$ is simply the inverse of the map γ from Theorem 9.4.1.)

One striking consequence of Theorem 9.4.2 is that the structure constants for $\mathcal{H}_{\mathrm{sph}}$ in the Kazhdan–Lusztig basis are *integers*, rather than more general elements of $\mathbb{Z}[q^{\pm\frac{1}{2}}]$. Specifically, let $\lambda, \lambda' \in \check{\mathbf{X}}^+$, and decompose the tensor product $L_\lambda(\mathbb{C}) \otimes L_{\lambda'}(\mathbb{C})$ into a direct sum of irreducible representations as

$$L_\lambda(\mathbb{C}) \otimes L_{\lambda'}(\mathbb{C}) \cong \bigoplus_{\mu \in \check{\mathbf{X}}^+} L_\mu(\mathbb{C})^{\oplus m^\mu_{\lambda,\lambda'}},$$

where the $m^\mu_{\lambda,\lambda'}$ are nonnegative integers. Then Theorem 9.4.2 implies that

$$\underline{S}_\lambda \underline{S}_{\lambda'} = \sum_{\mu \in \check{\mathbf{X}}^+} m^\mu_{\lambda,\lambda'} \underline{S}_\mu. \tag{9.4.1}$$

See Exercise 9.4.1 below for an important consequence of (9.4.1).

The second major result proved by Lusztig in [**158**] interprets the polynomials $M_{\mu,\lambda} \in \mathbb{Z}[q]$ from Theorem 9.3.7 in terms of representations of $\check{G}_{\mathbb{C}}$.

THEOREM 9.4.3. *For $\lambda, \mu \in \check{\mathbf{X}}^+$, we have $M_{\mu,\lambda}|_{q=1} = \dim L_\lambda(\mathbb{C})_\mu$.*

For a proof, see [**158**, Theorem 6.1]. Lusztig also conjectured an explicit combinatorial description of the $M_{\mu,\lambda}$ in terms of (a q-analogue of) the Kostant partition function. This conjecture was subsequently proved by Kato [**128**].

The following corollary is a kind of precursor to Theorem 9.9.4.

COROLLARY 9.4.4. *For $\lambda \in \check{\mathbf{X}}^+$, we have $\dim \mathbf{H}^\bullet(\mathcal{G}r, \mathrm{IC}_\lambda(\mathbb{C})) = \dim L_\lambda(\mathbb{C})$.*

PROOF SKETCH. Using the fact that each I-orbit $I \cdot \mathbf{t}^\mu$ is an affine space, one can show that for any $*$-even (or $*$-odd) object $\mathcal{F} \in D^{\mathrm{b}}_I(\mathcal{G}r, \mathbb{C})$, we have

$$\dim \mathbf{H}^\bullet(\mathcal{G}r, \mathcal{F}) = \sum_{\mu \in \check{\mathbf{X}}} \sum_{i \in \mathbb{Z}} \operatorname{rank} \mathsf{H}^i(\mathcal{F}|_{I \cdot \mathbf{t}^\mu}).$$

By Lemma 9.3.8, this formula holds for $\mathcal{F} = \mathrm{IC}_\lambda(\mathbb{C})$. By Theorem 9.4.3 and Corollary 9.3.12, we have

$$\sum_{i \in \mathbb{Z}} \operatorname{rank} \mathsf{H}^i(\mathrm{IC}_\lambda(\mathbb{C})|_{I \cdot \mathbf{t}^\mu}) = \dim L_\lambda(\mathbb{C})_\mu.$$

(This is immediate if μ is dominant, but Proposition 9.1.3 can be used to show that it holds for all $\mu \in \check{\mathbf{X}}$.) The result follows. □

Exercises.

9.4.1. Use (9.4.1) to show that $(-) \star_{\mathcal{G}r} (-) : D^{\mathrm{b}}_{G(\mathbb{O})}(\mathcal{G}r, \mathbb{C}) \times D^{\mathrm{b}}_{G(\mathbb{O})}(\mathcal{G}r, \mathbb{C}) \to D^{\mathrm{b}}_{G(\mathbb{O})}(\mathcal{G}r, \mathbb{C})$ is t-exact for the perverse t-structure in both variables. (A different proof of this will be given in Theorem 9.5.5 below.)

9.5. Exactness and commutativity

The convolution product on $\mathcal{G}r$ has two important properties that do not hold on $\mathcal{B}$ or $\mathcal{F}\ell$: it is right t-exact for the perverse t-structure, and at the abelian category level, it is commutative. We discuss these properties in this section.

Semi-infinite orbits and exactness. Let U^+ and U^- denote the unipotent radicals of the Borel subgroups B and B^-, respectively. In this section, we will use the **Iwasawa decompositions** of $G(\mathbb{K})$:

$$G(\mathbb{K}) = \bigsqcup_{\lambda \in \check{\mathbf{X}}^+} U^+(\mathbb{K}) t^\lambda G(\mathbb{O}) = \bigsqcup_{\lambda \in \check{\mathbf{X}}^+} U^-(\mathbb{K}) t^\lambda G(\mathbb{O}).$$

For a proof, see [**47**, Proposition 4.4.3]. Denote the orbits of $U^\pm(\mathbb{K})$ on $\mathcal{G}r$ by

$$S_\nu^+ = U^+(\mathbb{K}) \cdot \mathbf{t}^\nu \qquad \text{and} \qquad S_\nu^- = U^-(\mathbb{K}) \cdot \mathbf{t}^\nu, \qquad \text{for } \nu \in \check{\mathbf{X}}^+.$$

Then $\mathcal{G}r$ is the union of the $\{S_\nu^+\}_{\nu \in \check{\mathbf{X}}}$, as well as of the $\{S_\nu^-\}_{\nu \in \check{\mathbf{X}}}$. The $S_\nu^\pm$ are called **semi-infinite orbits**, because they are infinite-dimensional, but only "half as infinite-dimensional" as $\mathcal{G}r$ itself; see (9.5.1) below. For proofs of the following two statements, see [**179**, Proposition 3.1 and Theorem 3.2].

Lemma 9.5.1. *Let $\mathbb{G}_\mathrm{m}$ act on $\mathcal{G}r$ via the cocharacter $2\check{\rho} : \mathbb{G}_\mathrm{m} \to T$. For any $\nu \in \check{\mathbf{X}}$, we have*

$$S_\nu^+ = \{x \in \mathcal{G}r \mid \lim_{z \to 0} z \cdot x = \mathbf{t}^\nu\}, \qquad S_\nu^- = \{x \in \mathcal{G}r \mid \lim_{z \to \infty} z \cdot x = \mathbf{t}^\nu\}.$$

In particular, the set of fixed points for this $\mathbb{G}_\mathrm{m}$-action is $\{\mathbf{t}^\nu\}_{\nu \in \check{\mathbf{X}}}$.

Proposition 9.5.2. (1) *For $\nu \in \check{\mathbf{X}}$, we have*

$$\overline{S_\nu^+} = \bigcup_{\eta \le_{\check{\mathbf{X}}} \nu} S_\eta^+ \qquad \text{and} \qquad \overline{S_\nu^-} = \bigcup_{\eta \ge_{\check{\mathbf{X}}} \nu} S_\eta^-.$$

(2) *Let $\nu \in \check{\mathbf{X}}$ and $\lambda \in \check{\mathbf{X}}^+$. Then $S_\nu^\pm \cap \mathcal{G}r_\lambda$ is nonempty if and only if $\mathbf{t}^\nu \in \overline{\mathcal{G}r_\lambda}$. In this case, every irreducible component of $S_\nu^+ \cap \mathcal{G}r_\lambda$ has dimension $\langle \rho, \lambda + \nu \rangle$, and every irreducible component of $S_\nu^- \cap \mathcal{G}r_\lambda$ has dimension $\langle \rho, \lambda - \nu \rangle$.*

Note that if λ is very large compared to ν, we have

$$\frac{\dim S_\nu^\pm \cap \mathcal{G}r_\lambda}{\dim \mathcal{G}r_\lambda} = \frac{\langle \rho, \lambda \rangle \pm \langle \rho, \nu \rangle}{\langle 2\rho, \lambda \rangle} \approx \frac{1}{2}. \tag{9.5.1}$$

Remark 9.5.3. It follows from Proposition 9.5.2(1) that

$$\overline{S_\nu^+} \smallsetminus S_\nu^+ = \bigcup_{\check{\alpha} \in \check{\Pi}} \overline{S_{\nu - \check{\alpha}}^+},$$

where $\check{\Pi} \subset \check{\Phi}^+$ is the set of simple coroots. In particular, this shows that S_ν^+ is an open subset of $\overline{S_\nu^+}$, and hence that S_ν^+ is a locally closed subset of $\mathcal{G}r$. (Of course, orbits of an algebraic group on a finite-dimensional variety are always locally closed, but that general fact does not apply to $U^+(\mathbb{K})$-orbits on $\mathcal{G}r$.)

Proposition 9.5.4. *Equip $\mathcal{G}r$ with the stratification $(\mathcal{G}r_\lambda)_{\lambda \in \check{\mathbf{X}}^+}$, and equip $G(\mathbb{K}) \times^{G(\mathbb{O})} \mathcal{G}r$ with the stratification $(\mathcal{G}r_\lambda \tilde{\times} \mathcal{G}r_\mu)_{\lambda, \mu \in \check{\mathbf{X}}^+}$. Then the map $m : G(\mathbb{K}) \times^{G(\mathbb{O})} \mathcal{G}r \to \mathcal{G}r$ is stratified semismall.*

Proof. Let $\lambda, \mu, \nu \in \check{\mathbf{X}}^+$. Recall that $\dim \mathcal{G}r_\nu = \langle 2\rho, \nu \rangle$, and hence that $\dim \mathcal{G}r_\lambda \tilde{\times} \mathcal{G}r_\mu = \langle 2\rho, \lambda + \mu \rangle$. We must show that for any point $x \in \mathcal{G}r_\nu$, we have

$$\dim(m^{-1}(x) \cap (\mathcal{G}r_\lambda \tilde{\times} \mathcal{G}r_\mu)) \le \langle \rho, \lambda + \mu - \nu \rangle.$$

Since m is $G(\mathbb{O})$-equivariant, this dimension is independent of the choice of $x \in \mathcal{G}r_\nu$. From now on, we take $x = \mathbf{t}^{w_0\nu}$.

Let $X = m^{-1}(x) \cap \overline{\mathcal{G}r_\lambda \tilde{\times} \mathcal{G}r_\mu}$, and then consider its image $\mathrm{pr}_1(X)$ under the map $\mathrm{pr}_1 : G(\mathbb{K}) \times^{G(\mathbb{O})} \mathcal{G}r \to \mathcal{G}r$. The restriction of this map to X is proper, so $\mathrm{pr}_1(X)$ is a closed subvariety of $\overline{\mathcal{G}r_\lambda}$. It is easy to see that the map $X \to \mathrm{pr}_1(X)$ is a bijection, so it is enough to prove that $\dim \mathrm{pr}_1(X) \le \langle \rho, \lambda + \mu - \nu \rangle$. By Proposition 9.5.2, $\mathrm{pr}_1(X) \subset \overline{\mathcal{G}r_\lambda}$ meets finitely many semi-infinite orbits S^+_ϕ, so the problem further reduces to the claim that if $S^+_\phi \cap \mathrm{pr}_1(X)$ is nonempty, then

$$\dim(S^+_\phi \cap \mathrm{pr}_1(X)) \le \langle \rho, \lambda + \mu - \nu \rangle.$$

Since the point $x = \mathbf{t}^{w_0\nu}$ is fixed by the action of T on $\mathcal{G}r$, the varieties X and $\mathrm{pr}_1(X)$ are stable under T. Since $\mathrm{pr}_1(X)$ is a closed subset of $\overline{\mathcal{G}r_\lambda}$, Lemma 9.5.1 implies that if $S^+_\phi \cap \mathrm{pr}_1(X)$ is nonempty, then $\mathbf{t}^\phi \in \mathrm{pr}_1(X)$. The point in X corresponding to $\mathbf{t}^\phi \in \mathrm{pr}_1(X)$ must be of form $(t^\phi, t^{w_0\nu-\phi}G(\mathbb{O})) \in G(\mathbb{K}) \times^{G(\mathbb{O})} \mathcal{G}r$. Let $\psi = w_0\nu - \phi$, and rewrite this point as $(t^\phi, \mathbf{t}^\psi) \in \overline{\mathcal{G}r_\lambda \tilde{\times} \mathcal{G}r_\mu}$. To summarize, we have shown that if $S^+_\phi \cap \mathrm{pr}_1(X)$ is nonempty, then there is some $\psi \in \check{\mathbf{X}}$ such that

$$\phi + \psi = w_0\nu \qquad \text{and} \qquad \mathbf{t}^\psi \in \overline{\mathcal{G}r_\mu}.$$

Now, using Proposition 9.5.2 several times, if $S^+_\phi \cap \mathrm{pr}_1(X)$ is nonempty, then

$$\begin{aligned} \dim(S^+_\phi \cap \mathrm{pr}_1(X)) &\le \dim(S^+_\phi \cap \overline{\mathcal{G}r_\lambda}) = \dim(S^+_\phi \cap \mathcal{G}r_\lambda) \\ &\le \dim(S^+_\phi \cap \mathcal{G}r_\lambda) + \dim(S^+_\psi \cap \mathcal{G}r_\mu) \\ &= \langle \rho, \lambda + \phi + \mu + \psi \rangle = \langle \rho, \lambda + \mu + w_0\nu \rangle \\ &= \langle \rho, \lambda + \mu - \nu \rangle, \end{aligned}$$

where the last step follows from the fact that $\langle \rho, w_0\nu \rangle = \langle w_0\rho, \nu \rangle = -\langle \rho, \nu \rangle$. $\square$

THEOREM 9.5.5. *The functor* $\star : D^{\mathrm{b}}_{G(\mathbb{O})}(\mathcal{G}r, \Bbbk) \times D^{\mathrm{b}}_{G(\mathbb{O})}(\mathcal{G}r, \Bbbk) \to D^{\mathrm{b}}_{G(\mathbb{O})}(\mathcal{G}r, \Bbbk)$ *is right t-exact for the perverse t-structure. If $\Bbbk$ is a field, it is t-exact.*

PROOF. Let us unpack the definition of $\star$. Recall that we must replace the informal convolution diagram (9.2.5) by a modified diagram as in (9.2.4). The maps p and q in (9.2.4) are smooth and of the same relative dimension, say d, so by Proposition 3.6.1, both $p^*[d]$ and $q^*[d]$ are t-exact for the perverse t-structure. By Proposition 9.5.4 and Theorem 3.8.9, m_* is also t-exact. Finally, by Lemma 3.2.5, the functor $\boxtimes$ is right t-exact, and it is t-exact if $\Bbbk$ is a field. The theorem follows. $\square$

We now define a functor of abelian categories

$${}^p\star : \mathrm{Perv}_{G(\mathbb{O})}(\mathcal{G}r, \Bbbk) \times \mathrm{Perv}_{G(\mathbb{O})}(\mathcal{G}r, \Bbbk) \to \mathrm{Perv}_{G(\mathbb{O})}(\mathcal{G}r, \Bbbk)$$

by

$$\mathcal{F} \,{}^p\star\, \mathcal{G} = {}^p\mathsf{H}^0(\mathcal{F} \star \mathcal{G}).$$

Theorem 9.5.5 tells us that ${}^p\star$ is right exact; if $\Bbbk$ is a field, it is exact and agrees with $\mathcal{F} \star \mathcal{G}$. The associativity isomorphism for $\star_{\mathcal{G}r}$ from Lemma 9.2.2 induces an associativity isomorphism for ${}^p\star$, so the pair

$$(\mathrm{Perv}_{G(\mathbb{O})}(\mathcal{G}r, \Bbbk), {}^p\star)$$

is a monoidal category.

Commutativity. The rest of this section briefly outlines the proof (following an idea of Drinfeld) that ${}^p\star$ admits a commutativity constraint. This result requires a rather more sophisticated perspective on $\mathcal{G}r$ than that in Section 9.1: one must identify the affine Grassmannian with the "moduli space of principal G-bundles on the formal disc, with a trivialization on the punctured formal disc." This is beyond the scope of the present book. For an introduction to this topic, see [**246**].

This moduli space perspective makes it possible to define various "global" versions of the affine Grassmannian, known as **Beilinson–Drinfeld Grassmannians** after [**25**]. These are ind-schemes equipped with a map π to some variety X, such that the fibers of π are (products of) copies of the usual affine Grassmannian $\mathcal{G}r$. Here are some of the possible global versions:

(1) (Trivial family) Let $\mathbf{Gr}_{\mathbb{A}^1} = \mathcal{G}r \times \mathbb{A}^1$, and let $\pi : \mathbf{Gr}_{\mathbb{A}^1} \to \mathbb{A}^1$ be the projection onto the second factor.
(2) (Symmetric square) There is space $\pi : \mathbf{Gr}_{\mathbb{A}^2} \to \mathbb{A}^2$ such that

$$\pi^{-1}(x,y) \cong \begin{cases} \mathcal{G}r \times \mathcal{G}r & \text{if } x \neq y, \\ \mathcal{G}r & \text{if } x = y. \end{cases}$$

(3) (Convolution space) There is a space $\pi : \widetilde{\mathbf{Gr}}_{\mathbb{A}^2} \to \mathbb{A}^2$ such that

$$\pi^{-1}(x,y) \cong \begin{cases} \mathcal{G}r \times \mathcal{G}r & \text{if } x \neq y, \\ G(\mathbb{K}) \times^{G(\mathbb{O})} \mathcal{G}r & \text{if } x = y. \end{cases}$$

In fact, the description of fibers can be refined as follows. Let $\Delta : \mathbb{A}^1 \to \mathbb{A}^2$ be the diagonal embedding, given by $\Delta(x) = (x,x)$, and let

$$\mathbf{Gr}_\Delta = \mathbf{Gr}_{\mathbb{A}^2} \times_{\mathbb{A}^2} \mathbb{A}^1,$$

where the fiber product is taken with respect to Δ. Then there is a canonical isomorphism

$$\mathbf{Gr}_\Delta \cong \mathbf{Gr}_{\mathbb{A}^1}. \tag{9.5.2}$$

On the other hand, let $V = \mathbb{A}^2 \smallsetminus \Delta(\mathbb{A}^1)$, and let

$$\mathbf{Gr}_{\boldsymbol{\eta}} = \mathbf{Gr}_{\mathbb{A}^2} \times_{\mathbb{A}^2} V.$$

Then there are canonical isomorphisms

$$\mathbf{Gr}_{\boldsymbol{\eta}} \cong (\mathbf{Gr}_{\mathbb{A}^1} \times \mathbf{Gr}_{\mathbb{A}^1}) \times_{\mathbb{A}^2} V \cong \widetilde{\mathbf{Gr}}_{\mathbb{A}^2} \times_{\mathbb{A}^2} V. \tag{9.5.3}$$

Below, we will typically identify the two spaces in (9.5.2) with one another. We will likewise identify all three spaces in (9.5.3).

Finally, there is an ind-scheme $\mathbf{E} \to \mathbb{A}^2$ that fits into a diagram

$$\mathbf{Gr}_{\mathbb{A}^1} \times \mathbf{Gr}_{\mathbb{A}^1} \xleftarrow{\mathbf{p}} \mathbf{E} \xrightarrow{\mathbf{q}} \widetilde{\mathbf{Gr}}_{\mathbb{A}^2} \xrightarrow{\mathbf{m}} \mathbf{Gr}_{\mathbb{A}^2} \tag{9.5.4}$$

that serves as a "global version" of the convolution diagram (9.2.5). Specifically, over any point of the form $(x,x) \in \mathbb{A}^2$, the diagram (9.5.4) can be identified with (9.2.5). On the other hand, over a point $(x,y) \in \mathbb{A}^2$ with $x \neq y$, the diagram (9.5.4) can instead be identified with

$$\mathcal{G}r \times \mathcal{G}r \xleftarrow{p} G(\mathbb{K}) \times \mathcal{G}r \xrightarrow{p} \mathcal{G}r \times \mathcal{G}r \xrightarrow{\mathrm{id}} \mathcal{G}r \times \mathcal{G}r. \tag{9.5.5}$$

We will now define a global version of the convolution product. Let $\tau : \mathbf{Gr}_{\mathbb{A}^1} \to \mathcal{G}r$ be the projection map. For any $\mathcal{F} \in \mathrm{Perv}_{G(\mathbb{O})}(\mathcal{G}r, \Bbbk)$, we can consider the object

$$\tau^\dagger \mathcal{F} = \tau^* \mathcal{F}[1].$$

This is a perverse sheaf on $\mathbf{Gr}_{\mathbb{A}^1}$ that is equivariant for the action of a suitable global version of $G(\mathbb{O})$. For $\mathcal{F}, \mathcal{G} \in D^{\mathrm{b}}_{G(\mathbb{O})}(\mathcal{G}r, \Bbbk)$, let

$$\mathcal{F} \mathbin{\tilde{\boxtimes}}_{\mathbb{A}^2} \mathcal{G} = (\mathbf{q}^*)^{-1} \mathbf{p}^* (\tau^\dagger \mathcal{F} \boxtimes \tau^\dagger \mathcal{G}).$$

Of course, it takes some work to make sense of "equivariant descent along $\mathbf{q}$." We will not get into these difficulties here. We set

$$\mathcal{F} \star_{\mathbb{A}^2} \mathcal{G} = \mathbf{m}_* (\mathcal{F} \mathbin{\tilde{\boxtimes}}_{\mathbb{A}^2} \mathcal{G}).$$

For perverse sheaves $\mathcal{F}, \mathcal{G} \in \mathrm{Perv}_{G(\mathbb{O})}(\mathcal{G}r, \Bbbk)$, we also set

$$\mathcal{F} \mathbin{{}^p\tilde{\boxtimes}}_{\mathbb{A}^2} \mathcal{G} = {}^p\mathsf{H}^0(\mathcal{F} \mathbin{\tilde{\boxtimes}}_{\mathbb{A}^2} \mathcal{G}) \qquad \text{and} \qquad \mathcal{F} \mathbin{{}^p\star}_{\mathbb{A}^2} \mathcal{G} = {}^p\mathsf{H}^0(\mathcal{F} \star_{\mathbb{A}^2} \mathcal{G}).$$

For any $\lambda \in \check{\mathbf{X}}^+$, let $\mathbf{Gr}_\lambda = \mathcal{G}r_\lambda \times \mathbb{A}^1 \subset \mathbf{Gr}_{\mathbb{A}^1}$. The collection $(\mathbf{Gr}_\lambda \times \mathbf{Gr}_\mu)_{\lambda,\mu \in \check{\mathbf{X}}^+}$ forms a stratification of $\mathbf{Gr}_{\mathbb{A}^1} \times \mathbf{Gr}_{\mathbb{A}^1}$. For $\mathcal{F}, \mathcal{G} \in D^{\mathrm{b}}_{G(\mathbb{O})}(\mathcal{G}r, \Bbbk)$, the object $\tau^\dagger \mathcal{F} \boxtimes \tau^\dagger \mathcal{G}$ is constructible with respect to this stratification. Next, let

$$\mathbf{Gr}_\lambda \mathbin{\tilde{\times}} \mathbf{Gr}_\mu = \mathbf{q}(\mathbf{p}^{-1}(\mathbf{Gr}_\lambda \times \mathbf{Gr}_\mu)) \subset \widetilde{\mathbf{Gr}}_{\mathbb{A}^2}.$$

The collection $(\mathbf{Gr}_\lambda \mathbin{\tilde{\times}} \mathbf{Gr}_\mu)_{\lambda,\mu \in \check{\mathbf{X}}^+}$ forms a stratification of $\widetilde{\mathbf{Gr}}_{\mathbb{A}^2}$, and by construction, $\mathcal{F} \mathbin{\tilde{\boxtimes}}_{\mathbb{A}^2} \mathcal{G}$ is constructible with respect to this stratification.

LEMMA 9.5.6. *Consider the stratification* $(\mathbf{Gr}_\lambda \mathbin{\tilde{\times}} \mathbf{Gr}_\mu)_{\lambda,\mu \in \check{\mathbf{X}}^+}$ *of* $\widetilde{\mathbf{Gr}}_{\mathbb{A}^2}$.

(1) *The map* $\pi : \widetilde{\mathbf{Gr}}_{\mathbb{A}^2} \to \mathbb{A}^2$ *is a stratumwise locally trivial fibration with respect to this stratification.*
(2) *The map* $\mathbf{m} : \widetilde{\mathbf{Gr}}_{\mathbb{A}^2} \to \mathbf{Gr}_{\mathbb{A}^2}$ *is stratified small with respect to the open subset* $\mathbf{Gr}_{\boldsymbol{\eta}} \subset \mathbf{Gr}_{\mathbb{A}^2}$.

For the notion of a stratumwise locally trivial fibration, see Exercise 2.7.1.

PROOF SKETCH. For part (1), we must show that for any $\lambda, \mu \in \check{\mathbf{X}}^+$, the map $\pi|_{\mathbf{Gr}_\lambda \tilde{\times} \mathbf{Gr}_\mu} : \mathbf{Gr}_\lambda \mathbin{\tilde{\times}} \mathbf{Gr}_\mu \to \mathbb{A}^2$ is a locally trivial fibration. This map factors as

$$\begin{tikzcd} \mathbf{Gr}_\lambda \mathbin{\tilde{\times}} \mathbf{Gr}_\mu \arrow[r, "\mathrm{id} \tilde{\times} \pi_{\mathbb{A}^1}"'] \arrow[rr, bend left, "\pi"] & \mathbf{Gr}_\lambda \times \mathbb{A}^1 \arrow[r, "\pi_{\mathbb{A}^1} \times \mathrm{id}"'] & \mathbb{A}^2 \end{tikzcd}$$

where $\pi_{\mathbb{A}^1} : \mathbf{Gr}_{\mathbb{A}^1} \to \mathbb{A}^1$ is the natural map. Here, the first map is a locally trivial fibration with fibers isomorphic to $\mathcal{G}r_\mu$, and the second map is a trivial fibration with fibers isomorphic to $\mathcal{G}r_\lambda$.

For part (2), we have already seen in (9.5.5) that $\mathbf{m}$ is a bijection over $\mathbf{Gr}_{\boldsymbol{\eta}}$. Next, stratify $\mathbf{Gr}_\Delta \cong \mathbf{Gr}_{\mathbb{A}^1}$ by the collection $(\mathbf{Gr}_\nu)_{\nu \in \check{\mathbf{X}}^+}$. We must show that for any point $x \in \mathbf{Gr}_\nu$ and any $\lambda, \mu \in \check{\mathbf{X}}^+$, we have

$$\dim(\mathbf{m}^{-1}(x) \cap (\mathbf{Gr}_\lambda \mathbin{\tilde{\times}} \mathbf{Gr}_\mu)) < \tfrac{1}{2}(\dim(\mathbf{Gr}_\lambda \mathbin{\tilde{\times}} \mathbf{Gr}_\mu) - \dim \mathbf{Gr}_\nu). \tag{9.5.6}$$

The variety on the left-hand side can be identified with $m^{-1}(x) \cap (\mathcal{G}r_\lambda \mathbin{\tilde{\times}} \mathcal{G}r_\mu)$, where m is as in (9.2.5). Proposition 9.5.4 tells us that

$$\dim(\mathbf{m}^{-1}(x) \cap (\mathbf{Gr}_\lambda \mathbin{\tilde{\times}} \mathbf{Gr}_\mu)) \le \tfrac{1}{2}(\dim(\mathcal{G}r_\lambda \mathbin{\tilde{\times}} \mathcal{G}r_\mu) - \dim \mathcal{G}r_\nu).$$

Since $\dim(\mathbf{Gr}_\lambda \mathbin{\tilde\times} \mathbf{Gr}_\mu) = \dim(\mathcal{G}r_\lambda \mathbin{\tilde\times} \mathcal{G}r_\mu) + 2$ and $\dim \mathbf{Gr}_\nu = \dim \mathcal{G}r_\nu + 1$, this inequality implies (9.5.6). $\square$

The next lemma makes use of the identifications from (9.5.2) and (9.5.3).

LEMMA 9.5.7. *For $\mathcal{F}, \mathcal{G} \in \mathrm{Perv}_{G(\mathbb{O})}(\mathcal{G}r, \Bbbk)$, there are natural isomorphisms*

$$(\mathcal{F} \,{}^p{\star}_{\mathbb{A}^2}\, \mathcal{G})|_{\mathbf{Gr}_\Delta} \cong \tau^\dagger(\mathcal{F} \,{}^p{\star}\, \mathcal{G})[1] \qquad \textit{and} \qquad (\mathcal{F} \,{}^p{\star}_{\mathbb{A}^2}\, \mathcal{G})|_{\mathbf{Gr}_{\boldsymbol\eta}} \cong {}^p\mathsf{H}^0(\tau^\dagger\mathcal{F} \boxtimes \tau^\dagger\mathcal{G})|_{\mathbf{Gr}_{\boldsymbol\eta}}.$$

PROOF. Using the t-exactness of $\tau^\dagger$, these assertions follow easily by the base change of (9.5.4) to $\Delta(\mathbb{A}^1)$ and to V, respectively. $\square$

LEMMA 9.5.8. *Let $j : \mathbf{Gr}_{\boldsymbol\eta} \hookrightarrow \mathbf{Gr}_{\mathbb{A}^2}$ be the inclusion map. For any $\mathcal{F}, \mathcal{G} \in \mathrm{Perv}_{G(\mathbb{O})}(\mathcal{G}r, \Bbbk)$, there is a natural isomorphism*

$$\mathcal{F} \,{}^p{\star}_{\mathbb{A}^2}\, \mathcal{G} \cong j_{!*}({}^p\mathsf{H}^0(\tau^\dagger\mathcal{F} \boxtimes \tau^\dagger\mathcal{G})|_{\mathbf{Gr}_{\boldsymbol\eta}}).$$

PROOF. Using (9.5.3), we see that $(\mathcal{F} \,{}^p\tilde{\boxtimes}_{\mathbb{A}^2}\, \mathcal{G})|_{\mathbf{Gr}_{\boldsymbol\eta}} \cong {}^p\mathsf{H}^0(\tau^\dagger\mathcal{F} \boxtimes \tau^\dagger\mathcal{G})|_{\mathbf{Gr}_{\boldsymbol\eta}}$. The result then follows Lemma 9.5.6(2) and Proposition 3.8.10. $\square$

THEOREM 9.5.9. *For $\mathcal{F}, \mathcal{G} \in \mathrm{Perv}_{G(\mathbb{O})}(\mathcal{G}r, \Bbbk)$, there is a natural isomorphism*

$$\beta' = \beta'_{\mathcal{F},\mathcal{G}} : \mathcal{F} \,{}^p{\star}\, \mathcal{G} \cong \mathcal{G} \,{}^p{\star}\, \mathcal{F}$$

making $(\mathrm{Perv}_{G(\mathbb{O})}(\mathcal{G}r, \Bbbk), {}^p{\star})$ into a symmetric monoidal category.

We write "β'" because the notation β is reserved for a modified version of this map, to be defined in Remark 9.5.10.

PROOF SKETCH. Let $u : \mathbb{A}^2 \to \mathbb{A}^2$ be the map given by $u(x, y) = (y, x)$. Taking the base change of this map along $\pi : \mathbf{Gr}_{\mathbb{A}^2} \to \mathbb{A}^2$, we obtain an automorphism $s : \mathbf{Gr}_{\mathbb{A}^2} \to \mathbf{Gr}_{\mathbb{A}^2}$. This map preserves $\mathbf{Gr}_{\boldsymbol\eta}$, and in the commutative diagram

$$\begin{tikzcd}
(\mathbf{Gr}_{\mathbb{A}^1} \times \mathbf{Gr}_{\mathbb{A}^1}) \times_{\mathbb{A}^2} V \cong \mathbf{Gr}_{\boldsymbol\eta} \arrow[r, "j"] \arrow[d, "s_{\boldsymbol\eta} = s|_{\mathbf{Gr}_{\boldsymbol\eta}}"'] & \mathbf{Gr}_{\mathbb{A}^2} \arrow[d, "s"] \\
(\mathbf{Gr}_{\mathbb{A}^1} \times \mathbf{Gr}_{\mathbb{A}^1}) \times_{\mathbb{A}^2} V \cong \mathbf{Gr}_{\boldsymbol\eta} \arrow[r, "j"] & \mathbf{Gr}_{\mathbb{A}^2}
\end{tikzcd}$$

the map $s_{\boldsymbol\eta}$ swaps the two copies of $\mathbf{Gr}_{\mathbb{A}^1}$. We then have

$$\begin{aligned}
s^*(\mathcal{F} \,{}^p{\star}_{\mathbb{A}^2}\, \mathcal{G}) &\cong s^* j_{!*}({}^p\mathsf{H}^0(\tau^\dagger\mathcal{F} \boxtimes \tau^\dagger\mathcal{G})|_{\mathbf{Gr}_{\boldsymbol\eta}}) \cong j_{!*} s_{\boldsymbol\eta}^*({}^p\mathsf{H}^0(\tau^\dagger\mathcal{F} \boxtimes \tau^\dagger\mathcal{G})|_{\mathbf{Gr}_{\boldsymbol\eta}}) \\
&\cong j_{!*}({}^p\mathsf{H}^0(s_{\boldsymbol\eta}^*(\tau^\dagger\mathcal{F} \boxtimes \tau^\dagger\mathcal{G})|_{\mathbf{Gr}_{\boldsymbol\eta}})) \cong j_{!*}({}^p\mathsf{H}^0(\tau^\dagger\mathcal{G} \boxtimes \tau^\dagger\mathcal{F})|_{\mathbf{Gr}_{\boldsymbol\eta}}) \\
&\cong \mathcal{G} \,{}^p{\star}_{\mathbb{A}^2}\, \mathcal{F}.
\end{aligned} \tag{9.5.7}$$

Now, observe that $s|_{\mathbf{Gr}_\Delta} = \mathrm{id}_{\mathbf{Gr}_\Delta} : \mathbf{Gr}_\Delta \to \mathbf{Gr}_\Delta$. Therefore, by restricting (9.5.7) to $\mathbf{Gr}_\Delta$, we get a natural isomorphism $(\mathcal{F} \,{}^p{\star}_{\mathbb{A}^2}\, \mathcal{G})|_{\mathbf{Gr}_\Delta} \cong (\mathcal{G} \,{}^p{\star}_{\mathbb{A}^2}\, \mathcal{F})|_{\mathbf{Gr}_\Delta}$. By Lemma 9.5.7, this yields a natural isomorphism

$$\tau^\dagger(\mathcal{F} \,{}^p{\star}\, \mathcal{G}) \cong \tau^\dagger(\mathcal{G} \,{}^p{\star}\, \mathcal{F}). \tag{9.5.8}$$

Finally, since τ is a smooth morphism whose fibers are isomorphic to $\mathbb{A}^1$, Theorem 3.6.6 tells us that (9.5.8) is obtained by applying $\tau^\dagger$ to a natural isomorphism $\beta' : \mathcal{F} \,{}^p{\star}\, \mathcal{G} \xrightarrow{\sim} \mathcal{G} \,{}^p{\star}\, \mathcal{F}$.

It remains to show that β' satisfies the braiding axiom (see Definition A.2.3) and that $\beta' \circ \beta' = \mathrm{id}$. The latter can be deduced from the fact that $u \circ u = \mathrm{id}$. The braiding axiom can be proved by calculations similar to those above using the "symmetric cube" version of the affine Grassmannian, i.e., a global version of the affine Grassmannian over $\mathbb{A}^3$. We omit further details. $\square$

REMARK 9.5.10. Since the $G(\mathbb{O})$-orbits in a single connected component of $\mathcal{G}r$ all have the same parity (Corollary 9.1.4), any object $\mathcal{F} \in D^{\mathrm{b}}_{G(\mathbb{O})}(\mathcal{G}r, \Bbbk)$ admits a canonical decomposition

$$\mathcal{F} = \mathcal{F}_+ \oplus \mathcal{F}_-$$

where $\operatorname{supp} \mathcal{F}_+$ (resp. $\operatorname{supp} \mathcal{F}_-$) contains only $G(\mathbb{O})$-orbits of even (resp. odd) dimension. Now let $\mathcal{F}, \mathcal{G} \in \mathrm{Perv}_{G(\mathbb{O})}(\mathcal{G}r, \Bbbk)$, and form such a decomposition for both objects. The natural isomorphism β' from Theorem 9.5.9 is the sum of four terms:

$$\begin{array}{c}
\mathcal{F} \,{}^p\!\star\, \mathcal{G} = (\mathcal{F}_+ \,{}^p\!\star\, \mathcal{G}_+) \oplus (\mathcal{F}_+ \,{}^p\!\star\, \mathcal{G}_-) \oplus (\mathcal{F}_- \,{}^p\!\star\, \mathcal{G}_+) \oplus (\mathcal{F}_- \,{}^p\!\star\, \mathcal{G}_-) \\
\Big\downarrow {\scriptstyle \beta' = \beta'_{++} \oplus \beta'_{+-} \oplus \beta'_{-+} \oplus \beta'_{--}} \\
\mathcal{G} \,{}^p\!\star\, \mathcal{G} = (\mathcal{G}_+ \,{}^p\!\star\, \mathcal{F}_+) \oplus (\mathcal{G}_- \,{}^p\!\star\, \mathcal{F}_+) \oplus (\mathcal{G}_+ \,{}^p\!\star\, \mathcal{F}_-) \oplus (\mathcal{G}_- \,{}^p\!\star\, \mathcal{F}_-)
\end{array}$$

Define a map

$$\beta : \mathcal{F} \,{}^p\!\star\, \mathcal{G} \xrightarrow{\sim} \mathcal{G} \,{}^p\!\star\, \mathcal{F} \qquad \text{by} \qquad \beta = \beta'_{++} \oplus \beta'_{+-} \oplus \beta'_{-+} \oplus (-\beta'_{--}).$$

Then β still satisfies the braiding axiom, and we have $\beta \circ \beta = \mathrm{id}$. (These claims follow easily from the corresponding facts for β'.) In other words, β defines a new symmetric monoidal structure on $(\mathrm{Perv}_{G(\mathbb{O})}(\mathcal{G}r, \Bbbk), {}^p\!\star)$.

9.6. Weight functors

The goal of this section is to show that hypercohomology of perverse sheaves on $\mathcal{G}r$ admits a functorial direct-sum decomposition indexed by $\check{\mathbf{X}}$, where the summands are the functors

$$\mathrm{F}_\nu : \mathrm{Perv}_{G(\mathbb{O})}(\mathcal{G}r, \Bbbk) \to \Bbbk\text{-mod} \qquad \text{given by} \qquad \mathrm{F}_\nu(\mathcal{F}) = \mathbf{H}^{\langle 2\rho, \nu \rangle}_{\mathrm{c}}(S^+_\nu, \mathcal{F}|_{S^+_\nu}).$$

These functors are called **weight functors**.

Basic properties of weight functors. Using the following alternative description of the weight functors, we will show below that they are exact and faithful.

PROPOSITION 9.6.1. *For $\nu \in \check{\mathbf{X}}$, let $i^\pm_\nu : S^\pm_\nu \hookrightarrow \mathcal{G}r$ be the inclusion map. For any $\mathcal{F} \in D^{\mathrm{b}}_{G(\mathbb{O})}(\mathcal{G}r, \Bbbk)$, there is a natural isomorphism*

$$R\Gamma((i^-_\nu)^! \mathcal{F}) \xrightarrow{\sim} R\Gamma_{\mathrm{c}}((i^+_\nu)^* \mathcal{F}).$$

Moreover, if $\mathcal{F} \in \mathrm{Perv}_{G(\mathbb{O})}(\mathcal{G}r, \Bbbk)$, then the groups

$$\mathbf{H}^k(S^-_\nu, (i^-_\nu)^! \mathcal{F}) \cong \mathbf{H}^k_{\mathrm{c}}(S^+_\nu, (i^+_\nu)^* \mathcal{F})$$

vanish unless $k = \langle 2\rho, \nu \rangle$.

PROOF. Let $\mathbb{G}_{\mathrm{m}}$ act on $\mathcal{G}r$ by the cocharacter $2\check{\rho}$ as in Lemma 9.5.1. Since each connected component of the fixed-point set is a single point, the isomorphism $R\Gamma((i^-_\nu)^! \mathcal{F}) \xrightarrow{\sim} R\Gamma_{\mathrm{c}}((i^+_\nu)^* \mathcal{F})$ is just an application of Theorem 2.10.7. It follows immediately that $\mathbf{H}^k(S^-_\nu, (i^-_\nu)^! \mathcal{F}) \cong \mathbf{H}^k_{\mathrm{c}}(S^+_\nu, (i^+_\nu)^* \mathcal{F})$ for all k.

We now claim that

$$\text{(9.6.1)} \qquad \mathbf{H}^k_{\mathrm{c}}(S^+_\nu, (i^+_\nu)^* \mathcal{F}) = 0 \qquad \text{if } \mathcal{F} \in {}^pD^{\mathrm{b}}_{G(\mathbb{O})}(\mathcal{G}r, \Bbbk)^{\le 0} \text{ and } k > \langle 2\rho, \nu \rangle.$$

We proceed by induction on the number of $G(\mathbb{O})$-orbits in the support of $\mathcal{F}$. Let $\mathcal{G}r_\lambda$ be an orbit that is open in its support, and let $Z = \operatorname{supp}\mathcal{F} \smallsetminus \mathcal{G}r_\lambda$. Let $i : Z \hookrightarrow \mathcal{G}r$ be the inclusion map, and consider the distinguished triangle

$$j_{\lambda!}(\mathcal{F}|_{\mathcal{G}r_\lambda}) \to \mathcal{F} \to i_*i^*\mathcal{F} \to .$$

The analogue of (9.6.1) for $i_*i^*\mathcal{F}$ holds by induction. To prove (9.6.1) for $\mathcal{F}$, we must show that

$$\mathbf{H}^k_{\mathrm{c}}(S^+_\nu, (i^+_\nu)^* j_{\lambda!}(\mathcal{F}|_{\mathcal{G}r_\lambda})) = 0 \qquad \text{for } k > \langle 2\rho, \nu\rangle.$$

Let $u : S^+_\nu \cap \mathcal{G}r_\lambda \hookrightarrow S^+_\nu$ be the inclusion map. By proper base change, we have

$$\mathbf{H}^k_{\mathrm{c}}(S^+_\nu, (i^+_\nu)^* j_{\lambda!}(\mathcal{F}|_{\mathcal{G}r_\lambda})) \cong \mathbf{H}^k_{\mathrm{c}}(S^+_\nu, u_!(\mathcal{F}|_{S^+_\nu\cap\mathcal{G}r_\lambda})) \cong \mathbf{H}^k_{\mathrm{c}}(S^+_\nu \cap \mathcal{G}r_\lambda, \mathcal{F}|_{S^+_\nu\cap\mathcal{G}r_\lambda}).$$

Since $\mathcal{F} \in {}^pD^{\mathrm{b}}_{G(\mathbb{O})}(\mathcal{G}r, \Bbbk)^{\le 0}$, we have $\mathcal{F}|_{\mathcal{G}r_\lambda} \in D^{\mathrm{b}}_{G(\mathbb{O})}(\mathcal{G}r_\lambda, \Bbbk)^{\le -\dim \mathcal{G}r_\lambda}$, and hence

$$\mathcal{F}|_{S^+_\nu\cap\mathcal{G}r_\lambda} \in D^{\mathrm{b}}_{\mathrm{c}}(S^+_\nu \cap \mathcal{G}r_\lambda, \Bbbk)^{\le -\langle 2\rho,\lambda\rangle}$$

as well. By Proposition 9.5.2 and Theorem 2.7.4, we deduce that

$$\mathbf{H}^k_{\mathrm{c}}(S^+_\nu \cap \mathcal{G}r_\lambda, \mathcal{F}|_{S^+_\nu\cap\mathcal{G}r_\lambda}) = 0 \qquad \text{for } k > -\langle 2\rho, \lambda\rangle + 2\dim(S^+_\nu \cap \mathcal{G}r_\lambda) = \langle 2\rho, \nu\rangle,$$

as desired.

To finish the proof, we must show that

$$\mathbf{H}^k(S^-_\nu, (i^-_\nu)^!\mathcal{F}) = 0 \quad \text{if } \mathcal{F} \in {}^pD^{\mathrm{b}}_{G(\mathbb{O})}(\mathcal{G}r, \Bbbk)^{\ge 0} \text{ and } k < \langle 2\rho, \nu\rangle.$$

As above, an induction argument on the number of $G(\mathbb{O})$-orbits in the support of $\mathcal{F}$ lets us reduce to the case where $\mathcal{F} = j_{\lambda*}\mathcal{F}'$ for some some $\mathcal{F}' \in {}^pD^{\mathrm{b}}_{G(\mathbb{O})}(\mathcal{G}r_\lambda, \Bbbk)^{\ge 0}$. Now, $\mathcal{F}'$ must have locally constant cohomology sheaves. Another induction argument with truncation lets us further reduce to the case where $\mathcal{F}' = \mathcal{L}[m]$, where $\mathcal{L}$ is a local system, and $m \le \dim \mathcal{G}r_\lambda = \langle 2\rho, \lambda\rangle$. Let $v : S^-_\nu \cap \mathcal{G}r_\lambda \hookrightarrow S^-_\nu$ and $g : S^-_\nu \cap \mathcal{G}r_\lambda \hookrightarrow \mathcal{G}r_\lambda$ be the inclusion maps. By proper base change, we have

$$\begin{aligned}\mathbf{H}^k(S^-_\nu, (i^-_\nu)^!\mathcal{F}) &\cong \mathbf{H}^k(S^-_\nu, (i^-_\nu)^! j_{\lambda*}\mathcal{L}[m]) \\ &\cong \mathbf{H}^k(S^-_\nu, v_*g^!\mathcal{L}[m]) \cong \mathbf{H}^k(S^-_\nu \cap \mathcal{G}r_\lambda, g^!\mathcal{L}[m]).\end{aligned}$$

Note that $\dim \mathcal{G}r_\lambda - \dim(S^-_\nu \cap \mathcal{G}r_\lambda) = \langle \rho, \lambda + \nu\rangle$. By Corollary 2.2.14, we have

$$g^!\mathcal{L}[m] \in D^{\mathrm{b}}_{\mathrm{c}}(S^-_\nu \cap \mathcal{G}r_\lambda, \Bbbk)^{\ge 2\langle\rho,\lambda+\nu\rangle - m} \subset D^{\mathrm{b}}_{\mathrm{c}}(S^-_\nu \cap \mathcal{G}r_\lambda, \Bbbk)^{\ge\langle 2\rho,\nu\rangle}.$$

Since Γ is left exact, we conclude that $\mathbf{H}^k(S^-_\nu \cap \mathcal{G}r_\lambda, g^!\mathcal{L}[m]) = 0$ for $k < \langle 2\rho, \nu\rangle$. $\square$

REMARK 9.6.2. It follows from Proposition 9.6.1 that for any $\mathcal{F} \in D^{\mathrm{b}}_{G(\mathbb{O})}(\mathcal{G}r, \Bbbk)$ and any $n \in \mathbb{Z}$, we have

$$\mathrm{F}_\nu({}^p\mathsf{H}^n(\mathcal{F})) \cong \mathbf{H}^{\langle 2\rho,\nu\rangle+n}(S^-_\nu, (i^-_\nu)^!\mathcal{F}) \cong \mathbf{H}^{\langle 2\rho,\nu\rangle+n}_{\mathrm{c}}(S^+_\nu, (i^+_\nu)^*\mathcal{F}).$$

This more general version is sometimes useful.

The following statement is an immediate consequence of Proposition 9.6.1.

COROLLARY 9.6.3. *For each* $\nu \in \check{\mathbf{X}}$*, the weight functor* $\mathrm{F}_\nu : \mathrm{Perv}_{G(\mathbb{O})}(\mathcal{G}r, \Bbbk) \to$ $\Bbbk$-mod *is exact.*

In the next few lemmas, we record a few variants of Proposition 9.6.1. We will use the following notation: for a subset $Q \subset \check{\mathbf{X}}$, let

$$S^+_Q = \bigcup_{\nu\in Q} S^+_\nu.$$

LEMMA 9.6.4. *Let $Q \subset \check{\mathbf{X}}$ be a finite set of coweights, and assume that S_Q^+ is a locally closed subset of $\mathcal{G}r$. For any $\mathcal{F} \in \mathrm{Perv}_{G(\mathbb{O})}(\mathcal{G}r, \Bbbk)$, we have*

$$\mathbf{H}^i_{\mathrm{c}}(S_Q^+, \mathcal{F}|_{S_Q^+}) = 0 \quad \textit{unless} \quad \min\{\langle 2\rho, \eta\rangle \mid \eta \in Q\} \le i \le \max\{\langle 2\rho, \eta\rangle \mid \eta \in Q\}.$$

PROOF. If Q consists of a single coweight, this follows immediately from Proposition 9.6.1. Otherwise, let η be a maximal element of Q (with respect to $\le_{\check{\mathbf{X}}}$), and let $Q' = Q \smallsetminus \{\eta\}$. Let $j : S_\eta^+ \hookrightarrow S_Q^+$ and $i : S_{Q'}^+ \hookrightarrow S_Q^+$ be the inclusion maps. Proposition 9.5.2 (see also Remark 9.5.3) implies that j is an open embedding and that i is a closed embedding. Apply $R\Gamma_{\mathrm{c}}$ to the distinguished triangle $j_!(\mathcal{F}|_{S_\eta^+}) \to \mathcal{F}|_{S_Q^+} \to i_*(\mathcal{F}|_{S_{Q'}^+}) \to$ to obtain

$$R\Gamma_{\mathrm{c}}(\mathcal{F}|_{S_\eta^+}) \to R\Gamma_{\mathrm{c}}(\mathcal{F}|_{S_Q^+}) \to R\Gamma_{\mathrm{c}}(\mathcal{F}|_{S_{Q'}^+}) \to .$$

The result then follows by induction and the long exact sequence in cohomology. □

LEMMA 9.6.5. *Let $\nu_1, \dots, \nu_k \in \check{\mathbf{X}}$, and let $Z = \overline{S_{\nu_1}^+} \cup \dots \cup \overline{S_{\nu_k}^+}$. For $\mathcal{F} \in \mathrm{Perv}_{G(\mathbb{O})}(\mathcal{G}r, \Bbbk)$, we have*

$$\mathbf{H}^i_{\mathrm{c}}(Z, \mathcal{F}|_Z) = 0 \qquad \textit{for } i > \max\{\langle 2\rho, \nu_i\rangle\}.$$

PROOF. The support of $\mathcal{F}$ contains finitely points of the form $\mathbf{t}^\eta$. Let m be an integer such that $m < \langle 2\rho, \eta\rangle$ whenever $\mathbf{t}^\eta \in \operatorname{supp} \mathcal{F}$. Proposition 9.5.2 implies that if $\langle 2\rho, \sigma\rangle = m$, then $\overline{S_\sigma^+} \cap \operatorname{supp} \mathcal{F} = \varnothing$.

Let $\sigma_1, \dots, \sigma_r$ be the (finitely many) coweights satisfying the following two conditions: $\langle 2\rho, \sigma_i\rangle = m$, and $\sigma_i \le_{\check{\mathbf{X}}} \nu_j$ for some j. Let $Z' = \overline{S_{\sigma_1}^+} \cup \dots \cup \overline{S_{\sigma_r}^+}$. Our assumptions imply that $\mathcal{F}|_{Z'} = 0$. Next, let

$$Q = \{\eta \in \check{\mathbf{X}} \mid \eta \le_{\check{\mathbf{X}}} \nu_j \text{ for some } j, \text{ and } \langle 2\rho, \eta\rangle > m\}.$$

This is a finite set. The set S_Q^+ is open as a subset of Z; indeed, we have $S_Q^+ = Z \smallsetminus Z'$. Let $j : S_Q^+ \hookrightarrow Z$ and $i : Z' \hookrightarrow Z$ be the inclusion maps. In the distinguished triangle $j_!(\mathcal{F}|_{S_Q^+}) \to \mathcal{F}|_Z \to i_*(\mathcal{F}|_{Z'}) \to$, the last term vanishes, so the first two are isomorphic. Applying $R\Gamma_{\mathrm{c}}$, we deduce that

$$R\Gamma_{\mathrm{c}}(\mathcal{F}|_{S_Q^+}) \cong R\Gamma_{\mathrm{c}}(\mathcal{F}|_Z).$$

The result then follows from Lemma 9.6.4. □

LEMMA 9.6.6. *Let $\nu_1, \dots, \nu_k \in \check{\mathbf{X}}$, and assume that $\langle 2\rho, \nu_1\rangle = \dots = \langle 2\rho, \nu_k\rangle = m$. Let $Z = \overline{S_{\nu_1}^+} \cup \dots \cup \overline{S_{\nu_k}^+}$. For $\mathcal{F} \in \mathrm{Perv}_{G(\mathbb{O})}(\mathcal{G}r, \Bbbk)$, there is a natural isomorphism*

$$\bigoplus_{i=1}^k \mathrm{F}_{\nu_i}(\mathcal{F}) \xrightarrow{\sim} \mathbf{H}^m_{\mathrm{c}}(Z, \mathcal{F}|_Z).$$

PROOF. Let $Q = \{\nu_1, \dots, \nu_k\}$. Proposition 9.5.2 implies that S_Q^+ is just the disjoint union of $S_{\nu_1}^+, \dots, S_{\nu_k}^+$, so

$$\mathbf{H}^m_{\mathrm{c}}(S_Q^+, \mathcal{F}|_{S_Q^+}) \cong \bigoplus_{i=1}^k \mathbf{H}^m_{\mathrm{c}}(S_{\nu_i}^+, \mathcal{F}|_{S_{\nu_i}^+}) \cong \bigoplus_{i=1}^k \mathrm{F}_{\nu_i}(\mathcal{F}). \tag{9.6.2}$$

On the other hand, S^+_Q is an open subset of Z, and the complement $Z' = Z \smallsetminus S^+_Q$ is given by

$$Z' = \bigcup_{i=1}^{k} \bigcup_{\check\alpha \in \check\Pi} \overline{S^+_{\nu_i - \check\alpha}},$$

where $\check\Pi \subset \check\Phi^+$ is the set of simple coroots. Let $j : S^+_Q \hookrightarrow Z$ and $i : Z' \hookrightarrow Z$ be the inclusion maps. Apply $R\Gamma_c$ to $j_!(\mathcal{F}|_{S^+_Q}) \to \mathcal{F}|_Z \to i_*(\mathcal{F}|_{Z'}) \to$ to obtain the distinguished triangle

$$R\Gamma_c(\mathcal{F}|_{S^+_Q}) \to R\Gamma_c(\mathcal{F}|_Z) \to R\Gamma_c(\mathcal{F}|_{Z'}) \to .$$

By Lemma 9.6.4, the cohomology of the first term is concentrated in degree m, while that of the last term is concentrated in degrees $\le m-2$ (because $\langle 2\rho, \nu_i - \check\alpha\rangle = m-2$ for any $\check\alpha \in \check\Pi$). Thus, the long exact sequence in cohomology shows that the map

$$\mathbf{H}^m_c(S^+_Q, \mathcal{F}|_{S^+_Q}) \to \mathbf{H}^m_c(Z, \mathcal{F}|_Z)$$

is an isomorphism. Combining this with (9.6.2), we obtain the desired result. $\square$

Lemma 9.6.7. *Suppose $\mathcal{F} \in \mathrm{Perv}_{G(\mathbb{O})}(\mathcal{G}r, \Bbbk)$ is nonzero, and let $\mathcal{G}r_\lambda$ be a $G(\mathbb{O})$-orbit that is open in its support. Then $\mathrm{F}_\lambda(\mathcal{F})$ and $\mathrm{F}_{w_0\lambda}(\mathcal{F})$ are both nonzero.*

Proof. Since $\mathcal{G}r_\lambda$ is simply connected, $\mathcal{F}|_{\mathcal{G}r_\lambda}$ is a shifted constant sheaf: say

$$\mathcal{F}|_{\mathcal{G}r_\lambda} \cong \underline{M}_{\mathcal{G}r_\lambda}[\dim \mathcal{G}r_\lambda] = \underline{M}_{\mathcal{G}r_\lambda}[\langle 2\rho, \lambda\rangle]. \tag{9.6.3}$$

Proposition 9.5.2 implies that

$$S^-_\lambda \cap \operatorname{supp}\mathcal{F} = S^-_\lambda \cap \mathcal{G}r_\lambda \qquad \text{and} \qquad S^+_{w_0\lambda} \cap \operatorname{supp}\mathcal{F} = S^+_{w_0\lambda} \cap \mathcal{G}r_\lambda.$$

That proposition also tells us that $\dim(S^-_\lambda \cap \mathcal{G}r_\lambda) = \dim(S^+_{w_0\lambda} \cap \mathcal{G}r_\lambda) = 0$. In fact, Lemma 9.5.1 implies that $S^-_\lambda \cap \mathcal{G}r_\lambda = \{\mathbf{t}^\lambda\}$ and $S^+_{w_0\lambda} \cap \mathcal{G}r_\lambda = \{\mathbf{t}^{w_0\lambda}\}$.

Since $S^+_{w_0\lambda} \cap \mathcal{G}r_\lambda$ is a single point, using (9.6.3), we have

$$\mathrm{F}_{w_0\lambda}(\mathcal{F}) \cong \mathbf{H}^{-\langle 2\rho,\lambda\rangle}_c(\mathcal{F}|_{S^+_{w_0\lambda}}) \cong \mathbf{H}^{-\langle 2\rho,\lambda\rangle}(\mathcal{F}|_{S^+_{w_0\lambda}\cap\mathcal{G}r_\lambda}) \cong M.$$

Next, let $i : S^-_\lambda \cap \mathcal{G}r_\lambda \hookrightarrow \mathcal{G}r_\lambda$ be the inclusion map. By Theorem 2.2.13, we have

$$i^!(\mathcal{F}|_{\mathcal{G}r_\lambda}) \cong \underline{M}_{S^-_\lambda\cap\mathcal{G}r_\lambda}[-\dim \mathcal{G}r_\lambda](-\dim \mathcal{G}r_\lambda) \cong \underline{M}_{S^-_\lambda\cap\mathcal{G}r_\lambda}[-\langle 2\rho,\lambda\rangle](-\langle 2\rho,\lambda\rangle).$$

Since $S^-_\lambda \cap \mathcal{G}r_\lambda$ is a closed subset of S^-_λ, using Proposition 9.6.1, we have

$$\mathrm{F}_\lambda(\mathcal{F}) \cong \mathbf{H}^{\langle 2\rho,\lambda\rangle}(S^-_\lambda, (i^-_\lambda)^!\mathcal{F}) \cong \mathbf{H}^{\langle 2\rho,\lambda\rangle}(S^-_\lambda \cap \mathcal{G}r_\lambda, i^!(\mathcal{F}|_{\mathcal{G}r_\lambda})) \cong M(-\langle 2\rho,\lambda\rangle),$$

as desired. $\square$

Remark 9.6.8. Let $\mathcal{F} \in \mathrm{Perv}_{G(\mathbb{O})}(\mathcal{G}r, \Bbbk)$. If $\mathrm{F}_\nu(\mathcal{F}) \ne 0$, then we must have $\mathbf{t}^\nu \in \operatorname{supp}\mathcal{F}$. To see this, note that S^+_ν must meet the support of $\mathcal{F}$, so $S^+_\nu \cap \mathcal{G}r_\lambda \ne \varnothing$ for some $\mathcal{G}r_\lambda \subset \operatorname{supp}\mathcal{F}$. Then Proposition 9.5.2(2) tells us that $\mathbf{t}^\nu \in \overline{\mathcal{G}r_\lambda} \subset \operatorname{supp}\mathcal{F}$.

As an application of Lemma 9.6.7, we have the following.

Proposition 9.6.9. *For $\mathcal{F} \in D^{\mathrm{b}}_{G(\mathbb{O})}(\mathcal{G}r, \Bbbk)$, the following are equivalent:*

1. *The object $\mathcal{F}$ is perverse.*
2. *For all $\nu \in \check{\mathbf{X}}$, we have $\mathbf{H}^k(S^-_\nu, (i^-_\nu)^!\mathcal{F}) = 0$ unless $k = \langle 2\rho, \nu\rangle$.*
3. *For all $\nu \in \check{\mathbf{X}}$, we have $\mathbf{H}^k_c(S^+_\nu, (i^+_\nu)^*\mathcal{F}) = 0$ unless $k = \langle 2\rho, \nu\rangle$.*

PROOF. We have seen in Proposition 9.6.1 that condition (1) implies the other two. Suppose now that $\mathcal{F}$ is not perverse, so that there is some $n \neq 0$ such that ${}^p\mathsf{H}^n(\mathcal{F}) \neq 0$. By Lemma 9.6.7, there is some ν such that $\mathrm{F}_\nu({}^p\mathsf{H}^n(\mathcal{F})) \neq 0$. Remark 9.6.2 then tells us that conditions (2) and (3) also fail. $\square$

Main result. We are now ready to establish the desired decomposition of the hypercohomology functor.

THEOREM 9.6.10. *For $\mathcal{F} \in \mathrm{Perv}_{G(\mathbb{O})}(\mathcal{G}r, \Bbbk)$, there is a natural isomorphism*

$$\mathbf{H}^\bullet(\mathcal{G}r, \mathcal{F}) \cong \bigoplus_{\nu \in \check{\mathbf{X}}} \mathrm{F}_\nu(\mathcal{F}).$$

As a consequence, $\mathbf{H}^\bullet : \mathrm{Perv}_{G(\mathbb{O})}(\mathcal{G}r, \Bbbk) \to \Bbbk\text{-mod}^{\mathrm{fg}}$ is exact and faithful.

PROOF. Any perverse sheaf on $\mathcal{G}r$ can be written as a direct sum where each summand is supported on a single connected component of $\mathcal{G}r$. It is enough to construct the isomorphism for such a perverse sheaf. We therefore assume that $\mathcal{F}$ is supported on a single connected component. Let $m \in \mathbb{Z}$. We will prove that

$$\mathbf{H}^m(\mathcal{G}r, \mathcal{F}) \cong \bigoplus_{\substack{\nu \in \check{\mathbf{X}} \\ \langle 2\rho, \nu\rangle = m}} \mathrm{F}_\nu(\mathcal{F}). \tag{9.6.4}$$

Consider the (finite) set of coweights $P = \{\lambda \mid \mathbf{t}^\lambda \in \operatorname{supp} \mathcal{F}\}$. Because $\mathcal{F}$ is supported on a single connected component, all coweights in P belong to the same coset of the coroot lattice (see Section 9.1). Since P is finite, there exists a coweight μ such that $\lambda \leq_{\check{\mathbf{X}}} \mu$ for all $\lambda \in P$. Assume that $\langle 2\rho, \mu\rangle > m$. Proposition 9.5.2 then implies that $\operatorname{supp} \mathcal{F} \subset \overline{S_\mu^+}$. It follows that

$$R\Gamma(\mathcal{F}) \cong R\Gamma_{\mathrm{c}}(\mathcal{F}) \cong R\Gamma_{\mathrm{c}}(\mathcal{F}|_{\overline{S_\mu^+}}). \tag{9.6.5}$$

Let $\nu_1, \dots, \nu_k$ be the maximal coweights (with respect to $\leq_{\check{\mathbf{X}}}$) in the set $\{\nu \in \check{\mathbf{X}} \mid \nu \leq_{\check{\mathbf{X}}} \mu \text{ and } \langle 2\rho, \nu\rangle \leq m\}$. Recall that for any simple coroot $\check{\alpha}$, we have $\langle 2\rho, \check{\alpha}\rangle = 2$. Using this fact, one can deduce that

$$\langle 2\rho, \nu_1\rangle = \cdots = \langle 2\rho, \nu_k\rangle = \begin{cases} m & \text{if } m \equiv \langle 2\rho, \mu\rangle \pmod 2, \\ m-1 & \text{if } m \not\equiv \langle 2\rho, \mu\rangle \pmod 2. \end{cases}$$

Next, let $Q = \{\eta \in \check{\mathbf{X}} \mid \eta \leq_{\check{\mathbf{X}}} \mu \text{ and } \langle 2\rho, \eta\rangle > m\}$. This is a finite set, and we have

$$\min\{\langle 2\rho, \eta\rangle \mid \eta \in Q\} = \begin{cases} m+2 & \text{if } m \equiv \langle 2\rho, \mu\rangle \pmod 2, \\ m+1 & \text{if } m \not\equiv \langle 2\rho, \mu\rangle \pmod 2. \end{cases}$$

Let $Z = \overline{S_{\nu_1}^+} \cup \cdots \cup \overline{S_{\nu_k}^+}$. Let $j : S_Q^+ \hookrightarrow \overline{S_\mu^+}$ and $i : Z \hookrightarrow \overline{S_\mu^+}$ be the inclusion maps. We then have a distinguished triangle $j_!(\mathcal{F}|_{S_Q^+}) \to \mathcal{F}|_{\overline{S_\mu^+}} \to i_*(\mathcal{F}|_Z) \to$. Apply $R\Gamma_{\mathrm{c}}$ and use (9.6.5) to obtain a distinguished triangle

$$R\Gamma_{\mathrm{c}}(\mathcal{F}|_{S_Q^+}) \to R\Gamma(\mathcal{F}) \to R\Gamma_{\mathrm{c}}(\mathcal{F}|_Z) \to, \tag{9.6.6}$$

and then consider the long exact sequence in cohomology.

Suppose first that $m \not\equiv \langle 2\rho, \mu\rangle \pmod 2$. Then the first term of (9.6.6) is concentrated in degrees $\geq m+1$, and the third term in degrees $\leq m-1$. We conclude that $\mathbf{H}^m(\mathcal{G}r, \mathcal{F}) = 0$. On the other hand, if $\nu \in \check{\mathbf{X}}$ is such that $\langle 2\rho, \nu\rangle = m$, then ν must belong to a different coset of the coroot lattice than μ, so S_ν^+ is contained in a different component of $\mathcal{G}r$ from that on which $\mathcal{F}$ is supported. We conclude that $\mathcal{F}|_{S_\nu^+} = 0$. We have thus shown that both sides of (9.6.4) vanish.

Now suppose that $m \equiv \langle 2\rho, \mu\rangle \pmod 2$. This time, the first term of (9.6.6) is concentrated in degrees $\geq m+2$, so the long exact cohomology sequence shows that $\mathbf{H}^m(\mathcal{G}r, \mathcal{F}) \cong \mathbf{H}^m_{\mathrm{c}}(Z, \mathcal{F}|_Z)$. By Lemma 9.6.6, there is a natural isomorphism

$$\mathbf{H}^m(\mathcal{G}r, \mathcal{F}) \cong \bigoplus_{i=1}^{k} \mathrm{F}_{\nu_i}(\mathcal{F}). \tag{9.6.7}$$

If $\nu \in \check{\mathbf{X}}$ is another coweight satisfying $\langle 2\rho, \nu\rangle = m$ but not among $\nu_1, \ldots, \nu_k$, then we have $S_\nu^+ \not\subset \overline{S_\mu^+}$, and hence $\mathcal{F}|_{S_\nu^+} = 0$. In other words, all the summands on the right-hand side of (9.6.4) that do not appear in (9.6.7) actually vanish, so we have proved (9.6.4) in this case as well.

Finally, let us turn to the last assertion in the statement of the theorem. The exactness of $\mathbf{H}^\bullet$ follows from Corollary 9.6.3. Next, let $\phi : \mathcal{F} \to \mathcal{G}$ be a nonzero morphism in $\mathrm{Perv}_{G(\mathbb{O})}(\mathcal{G}r, \Bbbk)$. Since $\mathbf{H}^\bullet$ is already known to be exact, the image of $\mathbf{H}^\bullet(\phi)$ can be identified with $\mathbf{H}^\bullet(\operatorname{im}\phi)$, and the latter is nonzero by Lemma 9.6.7. We conclude that $\mathbf{H}^\bullet(\phi) \neq 0$, so $\mathbf{H}^\bullet$ is faithful. $\square$

The remainder of this section lists some consequences of Theorem 9.6.10.

COROLLARY 9.6.11. *If the support of $\mathcal{F} \in \mathrm{Perv}_{G(\mathbb{O})}(\mathcal{G}r, \Bbbk)$ contains only $G(\mathbb{O})$-orbits of even (resp. odd) dimension, then $\mathbf{H}^k(\mathcal{G}r, \mathcal{F})$ can be nonzero only when k is even (resp. odd).*

PROOF. Assume that the support of $\mathcal{F}$ contains only $G(\mathbb{O})$-orbits of even dimension. In view of (9.6.4), we must show that if $\mathrm{F}_\nu(\mathcal{F}) \neq 0$, then $\langle 2\rho, \nu\rangle$ is even.

If $\mathrm{F}_\nu(\mathcal{F}) \neq 0$, then by Remark 9.6.8, we have $\mathbf{t}^\nu \in \operatorname{supp}\mathcal{F}$. Let ν' be the unique dominant coweight in the W-orbit of ν, say $\nu' = w\nu$. Then $\mathcal{G}r_{\nu'} \subset \operatorname{supp}\mathcal{F}$, so by Proposition 9.1.3, $\dim \mathcal{G}r_{\nu'} = \langle 2\rho, \nu'\rangle$ is even. Next, $\nu' - \nu = \nu' - w\nu'$ lies in the coroot lattice $\mathbb{Z}\check{\Phi}$, so $\langle 2\rho, \nu' - \nu\rangle$ is even, and hence so is $\langle 2\rho, \nu\rangle$. $\square$

COROLLARY 9.6.12. *Let $\mathcal{F} \in \mathrm{Perv}_{G(\mathbb{O})}(\mathcal{G}r, \Bbbk)$. If $\mathbf{H}^\bullet(\mathcal{G}r, \mathcal{F})$ is flat over $\Bbbk$, then for any ring homomorphism $\varphi : \Bbbk \to \Bbbk'$, the object $\Bbbk' \otimes^L_\Bbbk \mathcal{F} \in D^{\mathrm{b}}_{G(\mathbb{O})}(\mathcal{G}r, \Bbbk')$ is perverse, and there are natural isomorphisms*

$$\mathrm{F}_\nu(\Bbbk' \overset{L}{\otimes}_\Bbbk \mathcal{F}) \cong \Bbbk' \otimes_\Bbbk \mathrm{F}_\nu(\mathcal{F}).$$

PROOF. Let $\nu \in \check{\mathbf{X}}$. Since $\mathrm{F}_\nu(\mathcal{F})$ is a direct summand of $\mathbf{H}^\bullet(\mathcal{G}r, \mathcal{F})$, it is flat over $\Bbbk$ by assumption, so

$$\Bbbk' \overset{L}{\otimes}_\Bbbk \mathrm{F}_\nu(\mathcal{F}) \cong \Bbbk' \otimes_\Bbbk \mathrm{F}_\nu(\mathcal{F}).$$

By Proposition 9.6.1, we have $R\Gamma_{\mathrm{c}}((i_\nu^+)^*\mathcal{F}) \cong \mathrm{F}_\nu(\mathcal{F})[-\langle 2\rho, \nu\rangle]$, so

$$\begin{aligned} R\Gamma_{\mathrm{c}}((i_\nu^+)^*(\Bbbk' \overset{L}{\otimes}_\Bbbk \mathcal{F})) &\cong \Bbbk' \overset{L}{\otimes}_\Bbbk R\Gamma_{\mathrm{c}}((i_\nu^+)^*\mathcal{F}) \\ &\cong \Bbbk' \overset{L}{\otimes}_\Bbbk \mathrm{F}_\nu(\mathcal{F})[-\langle 2\rho, \nu\rangle] \cong (\Bbbk' \otimes_\Bbbk \mathrm{F}_\nu(\mathcal{F}))[-\langle 2\rho, \nu\rangle]. \end{aligned}$$

We have shown that $\mathbf{H}^k_{\mathrm{c}}(S_\nu^+, (i_\nu^+)^*(\Bbbk' \otimes^L_\Bbbk \mathcal{F})) = 0$ unless $k = \langle 2\rho, \nu\rangle$, so by Proposition 9.6.9, $\Bbbk' \otimes^L_\Bbbk \mathcal{F}$ is perverse. $\square$

COROLLARY 9.6.13. *Let $\mathcal{F} \in \mathrm{Perv}_{G(\mathbb{O})}(\mathcal{G}r, \Bbbk)$. If $\mathbf{H}^\bullet(\mathcal{G}r, \mathcal{F})$ is projective over $\Bbbk$, then $\mathbb{D}\mathcal{F}$ is perverse, and there are natural isomorphisms*

$$\mathrm{F}_\nu(\mathbb{D}\mathcal{F}) \cong \mathrm{Hom}(\mathrm{F}_{w_0\nu}(\mathcal{F}), \Bbbk).$$

PROOF. Recall that the two unipotent groups $U^+, U^- \subset G$ are swapped by conjugation by $\dot{w}_0$ (a representative in $\mathrm{N}_G(T)$ of $w_0 \in W$). We therefore have

$$S^-_{w_0\nu} = U^-(\mathbb{K}) \cdot \mathbf{t}^{w_0\nu} = \dot{w}_0 U^+(\mathbb{K}) \dot{w}_0^{-1} \cdot \mathbf{t}^{w_0\nu} = \dot{w}_0 U^+(\mathbb{K}) \cdot \mathbf{t}^\nu = \dot{w}_0 \cdot S^+_\nu.$$

Let $\gamma : \mathcal{G}r \to \mathcal{G}r$ be the map given by the action of $\dot{w}_0 \in G$. We have shown that $S^-_{w_0\nu} = \gamma(S^+_\nu)$. Since γ is an automorphism, for any $\mathcal{G} \in D^{\mathrm{b}}_{G(\mathbb{O})}(\mathcal{G}r, \Bbbk)$, we have

$$R\Gamma_{\mathrm{c}}(\mathcal{G}|_{S^-_{w_0\nu}}) \cong R\Gamma_{\mathrm{c}}(\mathcal{G}|_{\gamma(S^+_\nu)}) \cong R\Gamma_{\mathrm{c}}((\gamma^*\mathcal{G})|_{S^+_\nu}).$$

Finally, since $\mathcal{G}$ is $G(\mathbb{O})$- (and hence G-) equivariant, there is a canonical isomorphism $\theta_{\dot{w}_0} : \gamma^*\mathcal{G} \xrightarrow{\sim} \mathcal{G}$. (See Lemma 6.4.6.) We conclude that

$$R\Gamma_{\mathrm{c}}(\mathcal{G}|_{S^+_\nu}) \cong R\Gamma_{\mathrm{c}}(\mathcal{G}|_{S^-_{w_0\nu}}).$$

Now let $\mathcal{F}$ be as in the statement above, and let $\mathcal{G} = \mathbb{D}\mathcal{F}$. We have

$$R\Gamma_{\mathrm{c}}((\mathbb{D}\mathcal{F})|_{S^+_\nu}) \cong R\Gamma_{\mathrm{c}}((i^-_{w_0\nu})^*\mathbb{D}\mathcal{F}) \cong \mathbb{D}(R\Gamma((i^-_{w_0\nu})^!\mathcal{F})).$$

By Proposition 9.6.1, we have $R\Gamma((i^-_{w_0\nu})^!\mathcal{F}) \cong \mathrm{F}_{w_0\nu}(\mathcal{F})[-\langle 2\rho, w_0\nu\rangle]$. Since this is a shift of a projective $\Bbbk$-module, its dual is given by

$$\mathbb{D}(R\Gamma((i^-_{w_0\nu})^!\mathcal{F})) \cong \mathrm{Hom}(\mathrm{F}_{w_0\nu}(\mathcal{F}), \Bbbk)[\langle 2\rho, w_0\nu\rangle] \cong \mathrm{Hom}(\mathrm{F}_{w_0\nu}(\mathcal{F}), \Bbbk)[-\langle 2\rho, \nu\rangle].$$

We have shown that $\mathbf{H}^k_{\mathrm{c}}(S^+_\nu, (i^+_\nu)^*(\mathbb{D}\mathcal{F})) = 0$ unless $k = \langle 2\rho, \nu\rangle$, so by Proposition 9.6.9, $\mathbb{D}\mathcal{F}$ is perverse. □

LEMMA 9.6.14. *Assume that $\Bbbk$ is a field. For $\mathcal{F} \in \mathrm{Perv}_{G(\mathbb{O})}(\mathcal{G}r, \Bbbk)$, we have $\dim \mathbf{H}^\bullet(\mathcal{G}r, \mathcal{F}) = 1$ if and only if $\mathcal{F} \cong \mathrm{IC}_\lambda$ for some coweight λ with $\langle 2\rho, \lambda\rangle = 0$.*

PROOF. If $\mathcal{F} \cong \mathrm{IC}_\lambda$ with $\langle 2\rho, \lambda\rangle = 0$, then $\mathcal{G}r_\lambda = \overline{\mathcal{G}r_\lambda}$ is a single point, and $\mathcal{F}$ is a skyscraper sheaf on this point. In this case, we clearly have $\dim \mathbf{H}^\bullet(\mathcal{G}r, \mathcal{F}) = 1$.

For the opposite implication, suppose that $\dim \mathbf{H}^\bullet(\mathcal{G}r, \mathcal{F}) = 1$. If there is more than one $G(\mathbb{O})$-orbit that is open in the support of $\mathcal{F}$, or if there is an open orbit $\mathcal{G}r_\lambda$ with $\lambda \neq w_0\lambda$, then Lemma 9.6.7 gives us more than one nonvanishing weight functor. Therefore, the support of $\mathcal{F}$ must be the closure of a single $G(\mathbb{O})$-orbit, say $\overline{\mathcal{G}r_\lambda}$, with $\lambda = w_0\lambda$. The latter condition implies that $\langle 2\rho, \lambda\rangle = 0$. That is, the support of $\mathcal{F}$ is the (closure of a) 0-dimensional $G(\mathbb{O})$-orbit. In other words, $\mathcal{F}$ is a skyscraper sheaf on $\mathcal{G}r_\lambda = \overline{\mathcal{G}r_\lambda}$. □

9.7. Standard sheaves and Mirković–Vilonen cycles

Let $Z \subset \mathcal{G}r$ be a closed union of finitely many $G(\mathbb{O})$-orbits. We will show in this section that the restriction of any weight functor F_ν to $\mathrm{Perv}_{G(\mathbb{O})}(Z, \Bbbk)$ is representable, and we will prove some structural properties of the representing object. These results are geometric counterparts of statements about "modules with a Weyl filtration" in representation theory.

Standard and costandard sheaves. For $\lambda \in \check{\mathbf{X}}^+$, let

$$\mathcal{I}_!(\lambda, \Bbbk) = {}^p\mathsf{H}^0(j_{\lambda!}\underline{\Bbbk}_{\mathcal{G}r_\lambda}[\dim \mathcal{G}r_\lambda]) \cong {}^p\tau^{\geq 0}(j_{\lambda!}\underline{\Bbbk}_{\mathcal{G}r_\lambda}[\dim \mathcal{G}r_\lambda]),$$
$$\mathcal{I}_*(\lambda, \Bbbk) = {}^p\mathsf{H}^0(j_{\lambda*}\underline{\Bbbk}_{\mathcal{G}r_\lambda}[\dim \mathcal{G}r_\lambda]) \cong {}^p\tau^{\leq 0}(j_{\lambda*}\underline{\Bbbk}_{\mathcal{G}r_\lambda}[\dim \mathcal{G}r_\lambda]).$$

We call $\mathcal{I}_!(\lambda, \Bbbk)$ a **standard (perverse) sheaf** and $\mathcal{I}_*(\lambda, \Bbbk)$ a **costandard (perverse) sheaf**. By construction, there are canonical maps

$$\mathcal{I}_!(\lambda, \Bbbk) \twoheadrightarrow \mathrm{IC}_\lambda(\Bbbk) \hookrightarrow \mathcal{I}_*(\lambda, \Bbbk).$$

According to Exercise 3.3.4, these objects can be characterized as follows.

LEMMA 9.7.1. *Let $\mathcal{F} \in \mathrm{Perv}_{G(\mathbb{O})}(\mathcal{G}r, \Bbbk)$. We have $\mathcal{F} \cong \mathcal{I}_!(\lambda, \Bbbk)$ if and only if the following conditions both hold:*

(1) *There is a surjective map $\mathcal{F} \twoheadrightarrow \mathrm{IC}_\lambda(\Bbbk)$ whose kernel is supported on $\overline{\mathcal{G}r_\lambda} \smallsetminus \mathcal{G}r_\lambda$.*
(2) *For any perverse sheaf $\mathcal{G}$ supported on $\overline{\mathcal{G}r_\lambda} \smallsetminus \mathcal{G}r_\lambda$, we have*

$$\mathrm{Hom}(\mathcal{F}, \mathcal{G}) = \mathrm{Ext}^1(\mathcal{F}, \mathcal{G}) = 0.$$

Similarly, we have $\mathcal{F} \cong \mathcal{I}_*(\lambda, \Bbbk)$ if and only if the following conditions both hold:

(1) *There is an injective map $\mathrm{IC}_\lambda(\Bbbk) \hookrightarrow \mathcal{F}$ whose cokernel is supported on $\overline{\mathcal{G}r_\lambda} \smallsetminus \mathcal{G}r_\lambda$.*
(2) *For any perverse sheaf $\mathcal{G}$ supported on $\overline{\mathcal{G}r_\lambda} \smallsetminus \mathcal{G}r_\lambda$, we have*

$$\mathrm{Hom}(\mathcal{G}, \mathcal{F}) = \mathrm{Ext}^1(\mathcal{G}, \mathcal{F}) = 0.$$

COROLLARY 9.7.2. *We have $\mathcal{I}_!(\lambda, \mathbb{Q}) \cong \mathrm{IC}_\lambda(\mathbb{Q}) \cong \mathcal{I}_*(\lambda, \mathbb{Q})$.*

PROOF. Since $\mathrm{Perv}_{G(\mathbb{O})}(\mathcal{G}r, \mathbb{Q})$ is semisimple (Proposition 9.3.10), $\mathrm{IC}_\lambda(\mathbb{Q})$ satisfies the conditions from Lemma 9.7.1 characterizing $\mathcal{I}_!(\lambda, \mathbb{Q})$ and $\mathcal{I}_*(\lambda, \mathbb{Q})$. □

LEMMA 9.7.3. *For any $\lambda \in \check{\mathbf{X}}^+$, restriction to $\mathcal{G}r_\lambda$ induces natural isomorphisms*

$$\mathrm{End}(\mathcal{I}_!(\lambda, \Bbbk)) \cong \mathrm{Hom}(\mathcal{I}_!(\lambda, \Bbbk), \mathcal{I}_*(\lambda, \Bbbk)) \cong \mathrm{End}(\mathcal{I}_*(\lambda, \Bbbk)) \cong \mathrm{End}(\underline{\Bbbk}_{\mathcal{G}r_\lambda}) \cong \Bbbk.$$

PROOF. By adjunction, we have

$$\begin{aligned}\mathrm{End}(\mathcal{I}_!(\lambda, \Bbbk)) &\cong \mathrm{Hom}({}^p\tau^{\geq 0}(j_{\lambda!}\underline{\Bbbk}_{\mathcal{G}r_\lambda}[\dim \mathcal{G}r_\lambda]), \mathcal{I}_!(\lambda, \Bbbk)) \\ &\cong \mathrm{Hom}(\underline{\Bbbk}_{\mathcal{G}r_\lambda}[\dim \mathcal{G}r_\lambda], j_\lambda^*\mathcal{I}_!(\lambda, \Bbbk)) \cong \mathrm{End}(\underline{\Bbbk}_{\mathcal{G}r_\lambda}).\end{aligned}$$

The proofs for $\mathrm{Hom}(\mathcal{I}_!(\lambda, \Bbbk), \mathcal{I}_*(\lambda, \Bbbk))$ and $\mathrm{End}(\mathcal{I}_*(\lambda, \Bbbk))$ are similar. □

PROPOSITION 9.7.4. *Let $\lambda \in \check{\mathbf{X}}^+$.*

(1) *There is a natural isomorphism*

$$\mathrm{F}_\nu(\mathcal{I}_!(\lambda, \Bbbk)) \cong \mathbf{H}_c^{\langle 2\rho, \lambda+\nu\rangle}(S_\nu^+ \cap \mathcal{G}r_\lambda; \Bbbk).$$

In particular, $\mathrm{F}_\nu(\mathcal{I}_!(\lambda, \Bbbk))(\langle \rho, \lambda + \nu\rangle)$ is a free $\Bbbk$-module with basis given by the dual fundamental classes of the irreducible components of $S_\nu^+ \cap \mathcal{G}r_\lambda$.

(2) *There is a natural isomorphism*

$$\mathrm{F}_\nu(\mathcal{I}_*(\lambda, \Bbbk)) \cong \mathbf{H}^{\mathrm{BM}}_{\langle 2\rho, \lambda-\nu\rangle}(S_\nu^- \cap \mathcal{G}r_\lambda; \Bbbk)(-\langle 2\rho, \lambda\rangle).$$

In particular, $\mathrm{F}_\nu(\mathcal{I}_*(\lambda, \Bbbk))(\langle \rho, \lambda + \nu\rangle)$ is a free $\Bbbk$-module with basis given by the fundamental classes of the irreducible components of $S_\nu^- \cap \mathcal{G}r_\lambda$.

PROOF. Using Remark 9.6.2, we find that

$$\begin{aligned}\mathrm{F}_\nu(\mathcal{I}_!(\lambda,\Bbbk)) &\cong \mathrm{F}_\nu({}^p\mathsf{H}^0(j_{\lambda!}\underline{\Bbbk}_{\mathcal{G}r_\lambda}[\dim \mathcal{G}r_\lambda])) \\ &\cong \mathbf{H}_c^{\langle 2\rho,\nu\rangle}(S_\nu^+,(i_\nu^*)j_{\lambda!}\underline{\Bbbk}_{\mathcal{G}r_\lambda}[\langle 2\rho,\lambda\rangle]) \cong \mathbf{H}_c^{\langle 2\rho,\lambda+\nu\rangle}(S_\nu^+,(i_\nu^*)j_{\lambda!}\underline{\Bbbk}_{\mathcal{G}r_\lambda}).\end{aligned}$$

Let $h : S_\nu^+ \cap \mathcal{G}r_\lambda \hookrightarrow S_\nu^+$ be the inclusion map. From the cartesian square

$$\begin{tikzcd} S_\nu^+ \cap \mathcal{G}r_\lambda \arrow[r] \arrow[d, "h"'] & \mathcal{G}r_\lambda \arrow[d, "j_\lambda"] \\ S_\nu^+ \arrow[r, "i_\nu^+"] & \mathcal{G}r \end{tikzcd}$$

we deduce that

$$\begin{aligned}\mathbf{H}_c^{\langle 2\rho,\lambda+\nu\rangle}(S_\nu^+,(i_\nu^*)j_{\lambda!}\underline{\Bbbk}_{\mathcal{G}r_\lambda}) &\cong \mathbf{H}_c^{2\langle \rho,\lambda+\nu\rangle}(S_\nu^+,h_!\underline{\Bbbk}_{S_\nu^+\cap\mathcal{G}r_\lambda}) \\ &\cong \mathbf{H}_c^{2\dim(S_\nu^+\cap\mathcal{G}r_\lambda)}(S_\nu^+\cap\mathcal{G}r_\lambda;\Bbbk).\end{aligned}$$

The rest of part (1) follows from Proposition 2.11.12.

The proof of part (2) is similar, but uses Proposition 2.11.11 instead. □

DEFINITION 9.7.5. Let $\lambda \in \check{\mathbf{X}}^+$ and $\nu \in \check{\mathbf{X}}$. An irreducible component of $\overline{S_\nu^+ \cap \mathcal{G}r_\lambda}$ is called a **Mirković–Vilonen cycle** or **MV cycle**.

LEMMA 9.7.6. *Let $\lambda \in \check{\mathbf{X}}^+$. The functors $\Bbbk\text{-mod}^{\mathrm{fg}} \to \mathrm{Perv}_{G(\mathbb{O})}(\mathcal{G}r,\Bbbk)$ given by*

$$M \mapsto {}^p\mathsf{H}^0(j_{\lambda!}\underline{M}_{\mathcal{G}r_\lambda}[\dim \mathcal{G}r_\lambda]) \qquad \textit{and} \qquad M \mapsto {}^p\mathsf{H}^0(j_{\lambda*}\underline{M}_{\mathcal{G}r_\lambda}[\dim \mathcal{G}r_\lambda])$$

are exact.

PROOF. In view of Theorem 9.6.10, it is enough to show that the functors

$$(9.7.1)\quad M \mapsto \mathrm{F}_\nu({}^p\mathsf{H}^0(j_{\lambda!}\underline{M}_{\mathcal{G}r_\lambda}[\dim \mathcal{G}r_\lambda])), \qquad M \mapsto \mathrm{F}_\nu({}^p\mathsf{H}^0(j_{\lambda*}\underline{M}_{\mathcal{G}r_\lambda}[\dim \mathcal{G}r_\lambda]))$$

are both exact for all $\nu \in \check{\mathbf{X}}$. One can repeat the calculations in the proof of Proposition 9.7.4 to show that

$$\begin{aligned}\mathrm{F}_\nu({}^p\mathsf{H}^0(j_{\lambda!}\underline{M}_{\mathcal{G}r_\lambda}[\dim \mathcal{G}r_\lambda])) &\cong \mathbf{H}_c^{\langle 2\rho,\lambda+\nu\rangle}(S_\nu^+\cap\mathcal{G}r_\lambda;M), \\ \mathrm{F}_\nu({}^p\mathsf{H}^0(j_{\lambda*}\underline{M}_{\mathcal{G}r_\lambda}[\dim \mathcal{G}r_\lambda])) &\cong \mathbf{H}^{\mathrm{BM}}_{\langle 2\rho,\lambda-\nu\rangle}(S_\nu^-\cap\mathcal{G}r_\lambda;M)(-\langle 2\rho,\lambda\rangle).\end{aligned}$$

Then, by Exercises 2.7.3 and 2.11.2, we obtain natural isomorphisms

$$\begin{aligned}\mathrm{F}_\nu({}^p\mathsf{H}^0(j_{\lambda!}\underline{M}_{\mathcal{G}r_\lambda}[\dim \mathcal{G}r_\lambda])) &\cong M\otimes \mathrm{F}_\nu(\mathcal{I}_!(\lambda,\Bbbk)), \\ \mathrm{F}_\nu({}^p\mathsf{H}^0(j_{\lambda*}\underline{M}_{\mathcal{G}r_\lambda}[\dim \mathcal{G}r_\lambda])) &\cong M\otimes \mathrm{F}_\nu(\mathcal{I}_*(\lambda,\Bbbk)).\end{aligned}$$

Since $\mathrm{F}_\nu(\mathcal{I}_!(\lambda,\Bbbk))$ and $\mathrm{F}_\nu(\mathcal{I}_*(\lambda,\Bbbk))$ are free (and hence flat) $\Bbbk$-modules, the functors (9.7.1) are exact. □

The next statement is obvious when $\Bbbk$ is a field.

PROPOSITION 9.7.7. *We have $\mathbb{D}\mathcal{I}_!(\lambda,\Bbbk) \cong \mathcal{I}_*(\lambda,\Bbbk)(\langle 2\rho,\lambda\rangle)$.*

PROOF. For brevity, let $n = \dim \mathcal{G}r_\lambda = \langle 2\rho,\lambda\rangle$. According to Corollary 9.6.13 (and Proposition 9.7.4), both $\mathbb{D}\mathcal{I}_!(\lambda,\Bbbk)$ and $\mathbb{D}\mathcal{I}_*(\lambda,\Bbbk)$ are perverse sheaves. Since $\mathcal{I}_!(\lambda,\Bbbk)|_{\mathcal{G}r_\lambda} \cong \mathcal{I}_*(\lambda,\Bbbk)|_{\mathcal{G}r_\lambda} \cong \underline{\Bbbk}_{\mathcal{G}r_\lambda}[n]$, their duals satisfy

$$(\mathbb{D}\mathcal{I}_!(\lambda,\Bbbk))|_{\mathcal{G}r_\lambda} \cong (\mathbb{D}\mathcal{I}_*(\lambda,\Bbbk))|_{\mathcal{G}r_\lambda} \cong \mathbb{D}(\underline{\Bbbk}_{\mathcal{G}r_\lambda}[n]) \cong \underline{\Bbbk}_{\mathcal{G}r_\lambda}[n](n).$$

There are natural adjunction maps $j_{\lambda!}\underline{\Bbbk}_{\mathcal{G}r_\lambda}[n](n) \to \mathbb{D}\mathcal{I}_*(\lambda,\Bbbk)$ and $\mathbb{D}\mathcal{I}_!(\lambda,\Bbbk) \to j_{\lambda*}\underline{\Bbbk}_{\mathcal{G}r_\lambda}[n](n)$. Applying ${}^p\mathsf{H}^0$, we obtain natural maps

$$\mathcal{I}_!(\lambda,\Bbbk)(n) \to \mathbb{D}\mathcal{I}_*(\lambda,\Bbbk) \qquad \text{and} \qquad \mathbb{D}\mathcal{I}_!(\lambda,\Bbbk) \to \mathcal{I}_*(\lambda,\Bbbk)(n). \tag{9.7.2}$$

Apply $\mathbb{D}$ and then (n) to obtain natural maps

$$\mathcal{I}_*(\lambda,\Bbbk)(n) \to \mathbb{D}\mathcal{I}_!(\lambda,\Bbbk) \qquad \text{and} \qquad \mathbb{D}\mathcal{I}_*(\lambda,\Bbbk) \to \mathcal{I}_!(\lambda,\Bbbk)(n). \tag{9.7.3}$$

All the maps in (9.7.2) and (9.7.3) become isomorphisms when restricted to $\mathcal{G}r_\lambda$, so the same holds for the four compositions

$$\mathcal{I}_!(\lambda,\Bbbk)(n) \to \mathbb{D}\mathcal{I}_*(\lambda,\Bbbk) \to \mathcal{I}_!(\lambda,\Bbbk)(n), \quad \mathbb{D}\mathcal{I}_*(\lambda,\Bbbk) \to \mathcal{I}_!(\lambda,\Bbbk)(n) \to \mathbb{D}\mathcal{I}_*(\lambda,\Bbbk),$$
$$\mathcal{I}_*(\lambda,\Bbbk)(n) \to \mathbb{D}\mathcal{I}_!(\lambda,\Bbbk) \to \mathcal{I}_*(\lambda,\Bbbk)(n), \quad \mathbb{D}\mathcal{I}_!(\lambda,\Bbbk) \to \mathcal{I}_*(\lambda,\Bbbk)(n) \to \mathbb{D}\mathcal{I}_!(\lambda,\Bbbk).$$

By Lemma 9.7.3 (and its Verdier dual), the four compositions above are themselves isomorphisms. It follows that the maps in (9.7.2) and (9.7.3) are isomorphisms. □

PROPOSITION 9.7.8. *We have*

$$\mathcal{I}_!(\lambda,\Bbbk) \cong \Bbbk \overset{L}{\otimes}_{\mathbb{Z}} \mathcal{I}_!(\lambda,\mathbb{Z}) \qquad \textit{and} \qquad \mathcal{I}_*(\lambda,\Bbbk) \cong \Bbbk \overset{L}{\otimes}_{\mathbb{Z}} \mathcal{I}_*(\lambda,\mathbb{Z}).$$

PROOF. We have $\Bbbk \otimes^L_{\mathbb{Z}} j_{\lambda!}\underline{\mathbb{Z}}_{\mathcal{G}r_\lambda}[\dim \mathcal{G}r_\lambda] \cong j_{\lambda!}\underline{\Bbbk}_{\mathcal{G}r_\lambda}[\dim \mathcal{G}r_\lambda]$. Since both of these objects live in ${}^pD^{\mathrm{b}}_{G(\mathbb{O})}(\mathcal{G}r,\Bbbk)^{\le 0}$, and since $\Bbbk\otimes^L_{\mathbb{Z}}(-)$ is right t-exact (see Lemma 3.2.4), we have

$${}^p\mathsf{H}^0(\Bbbk \overset{L}{\otimes}_{\mathbb{Z}} j_{\lambda!}\underline{\mathbb{Z}}_{\mathcal{G}r_\lambda}[\dim \mathcal{G}r_\lambda]) \cong {}^p\mathsf{H}^0(\Bbbk \overset{L}{\otimes}_{\mathbb{Z}} {}^p\mathsf{H}^0(j_{\lambda!}\underline{\mathbb{Z}}_{\mathcal{G}r_\lambda}[\dim \mathcal{G}r_\lambda])),$$

or, equivalently,

$$\mathcal{I}_!(\lambda,\Bbbk) \cong {}^p\mathsf{H}^0(\Bbbk \overset{L}{\otimes}_{\mathbb{Z}} \mathcal{I}_!(\lambda,\mathbb{Z})).$$

But Corollary 9.6.12 and Proposition 9.7.4 say that $\Bbbk\otimes^L_{\mathbb{Z}}\mathcal{I}_!(\lambda,\mathbb{Z})$ is already perverse, so we can drop the ${}^p\mathsf{H}^0$ on the right-hand side: $\mathcal{I}_!(\lambda,\Bbbk) \cong \Bbbk \otimes^L_{\mathbb{Z}} \mathcal{I}_!(\lambda,\mathbb{Z})$.

The claim for $\mathcal{I}_*(\lambda,\Bbbk)$ then follows from Proposition 9.7.7 and the fact that $\Bbbk \otimes^L_{\mathbb{Z}} (-)$ commutes with Verdier duality. □

COROLLARY 9.7.9. *We have* $\mathcal{I}_!(\lambda,\mathbb{Z}) \cong \mathrm{IC}_\lambda(\mathbb{Z})$.

PROOF. This is equivalent to the claim that the natural map $\mathcal{I}_!(\lambda,\mathbb{Z}) \to \mathcal{I}_*(\lambda,\mathbb{Z})$ is injective. Let $\mathcal{F}$ be the kernel of $\mathcal{I}_!(\lambda,\mathbb{Z}) \to \mathcal{I}_*(\lambda,\mathbb{Z})$. Since $\mathbb{Q}$ is a flat $\mathbb{Z}$-module, the functor $\mathbb{Q}\otimes^L_{\mathbb{Z}}(-)$ is t-exact (see Lemma 3.2.4), so $\mathbb{Q}\otimes^L_{\mathbb{Z}}\mathcal{F}$ is the kernel of the natural map

$$\mathbb{Q} \overset{L}{\otimes}_{\mathbb{Z}} \mathcal{I}_!(\lambda,\mathbb{Z}) \to \mathbb{Q} \overset{L}{\otimes}_{\mathbb{Z}} \mathcal{I}_*(\lambda,\mathbb{Z}).$$

By Corollary 9.7.2 and Proposition 9.7.8, this is an isomorphism. We conclude that

$$\mathbb{Q} \overset{L}{\otimes}_{\mathbb{Z}} \mathcal{F} = 0.$$

On the other hand, for all $\nu \in \check{\mathbf{X}}$, $\mathrm{F}_\nu(\mathcal{F})$ is a submodule of the free $\mathbb{Z}$-module $\mathrm{F}_\nu(\mathcal{I}_!(\lambda,\mathbb{Z}))$, and hence free. By Corollary 9.6.12, we have

$$0 = \mathrm{F}_\nu(\mathbb{Q} \overset{L}{\otimes}_{\mathbb{Z}} \mathcal{F})) \cong \mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{F}_\nu(\mathcal{F}),$$

so $\mathrm{F}_\nu(\mathcal{F}) = 0$ for all ν. Lemma 9.6.7 then tells us that $\mathcal{F} = 0$, as desired. □

Representing weight functors. The remainder of this section is devoted to the study of the objects $\mathcal{P}_Z(\nu,\Bbbk)$ defined in the following proposition.

PROPOSITION 9.7.10. *Let $Z \subset \mathcal{G}r$ be a closed union of finitely many $G(\mathbb{O})$-orbits, and let $\nu \in \check{\mathbf{X}}$. There is an object $\mathcal{P}_Z(\nu,\Bbbk) \in \mathrm{Perv}_{G(\mathbb{O})}(Z,\Bbbk)$ such that for any $\mathcal{F} \in \mathrm{Perv}_{G(\mathbb{O})}(Z,\Bbbk)$, there is a natural isomorphism*

$$\mathrm{Hom}(\mathcal{P}_Z(\nu,\Bbbk),\mathcal{F}) \cong \mathrm{F}_\nu(\mathcal{F}).$$

Moreover, $\mathcal{P}_Z(\nu,\Bbbk)$ is projective as an object of $\mathrm{Perv}_{G(\mathbb{O})}(Z,\Bbbk)$.

Of course, by Yoneda's lemma, $\mathcal{P}_Z(\nu,\Bbbk)$ is unique up to canonical isomorphism.

PROOF. Let $i : S_\nu^- \cap Z \hookrightarrow Z$ be the inclusion map. We have

$$\mathrm{F}_\nu(\mathcal{F}) \cong \mathbf{H}^{\langle 2\rho,\nu\rangle}(S_\nu^- \cap Z, i^!\mathcal{F}) \cong \mathrm{Hom}_{D^{\mathrm{b}}_{\mathrm{c}}(Z,\Bbbk)}(i_!\underline{\Bbbk}_{S_\nu^- \cap Z}[-\langle 2\rho,\nu\rangle],\mathcal{F}).$$

There is an implicit forgetful functor in this calculation. Let us make it explicit. Recall that there is some integer $n \geq 1$ such that the action of $G(\mathbb{O})$ on Z factors through $G(\mathbb{O}_n)$. By definition, we have

$$D^{\mathrm{b}}_{G(\mathbb{O})}(Z,\Bbbk) = D^{\mathrm{b}}_{G(\mathbb{O}_n)}(Z,\Bbbk),$$

and $\mathcal{F}$ lives in this equivariant derived category. Thus, we have

$$\begin{aligned} \mathrm{F}_\nu(\mathcal{F}) &\cong \mathrm{Hom}_{D^{\mathrm{b}}_{\mathrm{c}}(Z,\Bbbk)}(i_!\underline{\Bbbk}_{S_\nu^- \cap Z}[-\langle 2\rho,\nu\rangle], \mathrm{For}^{G(\mathbb{O}_n)}\mathcal{F}) \\ &\cong \mathrm{Hom}_{D^{\mathrm{b}}_{G(\mathbb{O})}(Z,\Bbbk)}(\mathrm{Av}_{1!}^{G(\mathbb{O}_n)}(i_!\underline{\Bbbk}_{S_\nu^- \cap Z}[-\langle 2\rho,\nu\rangle]),\mathcal{F}). \end{aligned} \tag{9.7.4}$$

If $\mathrm{Av}_{1!}^{G(\mathbb{O}_n)}(i_!\underline{\Bbbk}_{S_\nu^- \cap Z}) = 0$, then we are (trivially) done. From now on, we assume instead that this object is nonzero. Let m be the largest integer such that ${}^p\mathsf{H}^m(\mathrm{Av}_{1!}^{G(\mathbb{O}_n)}(i_!\underline{\Bbbk}_{S_\nu^- \cap Z}[-\langle 2\rho,\nu\rangle])) \neq 0$, and let

$$\begin{aligned} \mathcal{P}_Z(\nu,\Bbbk) &= {}^p\mathsf{H}^m(\mathrm{Av}_{1!}^{G(\mathbb{O}_n)}(i_!\underline{\Bbbk}_{S_\nu^- \cap Z}[-\langle 2\rho,\nu\rangle])) \\ &\cong {}^p\tau^{\geq m}(\mathrm{Av}_{1!}^{G(\mathbb{O}_n)}(i_!\underline{\Bbbk}_{S_\nu^- \cap Z}[-\langle 2\rho,\nu\rangle]))[m]. \end{aligned}$$

The truncation map shows that the following Hom-groups are nonzero:

$$\begin{aligned} &\mathrm{Hom}(\mathrm{Av}_{1!}^{G(\mathbb{O}_n)}(i_!\underline{\Bbbk}_{S_\nu^- \cap Z}[-\langle 2\rho,\nu\rangle]), \mathcal{P}_Z(\nu,\Bbbk)[-m]) \\ &\qquad \cong \mathrm{Hom}(i_!\underline{\Bbbk}_{S_\nu^- \cap Z}[-\langle 2\rho,\nu\rangle], \mathrm{For}^{G(\mathbb{O}_n)}(\mathcal{P}_Z(\nu,\Bbbk))[-m]) \\ &\qquad\qquad \cong \mathbf{H}^{\langle 2\rho,\nu\rangle - m}(S_\nu^- \cap Z, i^!\mathcal{P}_Z(\nu,\Bbbk)). \end{aligned}$$

But $\mathcal{P}_Z(\nu,\Bbbk)$ is a $G(\mathbb{O})$-equivariant perverse sheaf, so by Proposition 9.6.1, we must have $m = 0$, and hence $\mathrm{Av}_{1!}^{G(\mathbb{O}_n)}(i_!\underline{\Bbbk}_{S_\nu^- \cap Z})[-\langle 2\rho,\nu\rangle] \in {}^pD^{\mathrm{b}}_{G(\mathbb{O})}(Z,\Bbbk)^{\leq 0}$. By the adjunction property of truncation, (9.7.4) can be rewritten as

$$\mathrm{F}_\nu(\mathcal{F}) \cong \mathrm{Hom}(\mathcal{P}_Z(\nu,\Bbbk),\mathcal{F}),$$

as desired. Lastly, since F_ν is an exact functor, $\mathcal{P}_Z(\nu,\Bbbk)$ is projective. □

REMARK 9.7.11. For any $\nu \in \check{\mathbf{X}}$, there is a canonical element

$$\eta_\nu \in \mathrm{F}_\nu(\mathcal{P}_Z(\nu,\Bbbk)),$$

corresponding to $\mathrm{id} \in \mathrm{Hom}(\mathcal{P}_Z(\nu,\Bbbk),\mathcal{P}_Z(\nu,\Bbbk))$ via Proposition 9.7.10. By tracing through the construction in the proof of the preceding proposition, especially (9.7.4), one can show that η_ν comes from the unit map for a certain adjuction, and that the isomorphism

$$\mathrm{Hom}(\mathcal{P}_Z(\nu,\Bbbk),\mathcal{F}) \cong \mathrm{F}_\nu(\mathcal{F})$$

is given in general by $\phi \mapsto \mathrm{F}_\nu(\phi)(\eta_\nu)$.

LEMMA 9.7.12. *Let $Z \subset \mathcal{G}r$ be a closed union of finitely many $G(\mathbb{O})$-orbits, and let $\mathcal{G}r_\lambda$ be a $G(\mathbb{O})$-orbit that is open in Z. For any $\nu \in \check{\mathbf{X}}$, $\mathcal{P}_Z(\nu,\Bbbk)|_{\mathcal{G}r_\lambda}[-\dim \mathcal{G}r_\lambda]$ is canonically isomorphic to the constant sheaf with value* $\mathrm{Hom}(\mathrm{F}_\nu(\mathcal{I}_*(\lambda,\Bbbk)),\Bbbk)$.

PROOF. Our assumptions imply that $\mathcal{P}_Z(\nu,\Bbbk)|_{\mathcal{G}r_\lambda}$ is a shifted constant sheaf $\underline{Q}_{\mathcal{G}r_\lambda}[\dim \mathcal{G}r_\lambda]$ for some $\Bbbk$-module Q. The problem is to identify Q.

For any $\Bbbk$-module M, we have

$$\begin{aligned}\mathrm{Hom}(Q,M) &\cong \mathrm{Hom}(\underline{Q}_{\mathcal{G}r_\lambda},\underline{M}_{\mathcal{G}r_\lambda}) \cong \mathrm{Hom}(\mathcal{P}_Z(\nu,\Bbbk)|_{\mathcal{G}r_\lambda},\underline{M}_{\mathcal{G}r_\lambda}[\dim \mathcal{G}r_\lambda])\\ &\cong \mathrm{Hom}(\mathcal{P}_Z(\nu,\Bbbk), j_{\lambda *}\underline{M}_{\mathcal{G}r_\lambda}[\dim Gr_\lambda])\\ &\cong \mathrm{Hom}(\mathcal{P}_Z(\nu,\Bbbk), {}^p\tau^{\leq 0}(j_{\lambda *}\underline{M}_{\mathcal{G}r_\lambda}[\dim Gr_\lambda])).\end{aligned}$$

Since $\mathcal{P}_Z(\nu,\Bbbk)$ is projective in $\mathrm{Perv}_{G(\mathbb{O})}(Z,\Bbbk)$, the functor $\mathrm{Hom}(\mathcal{P}_Z(\nu,\Bbbk),-)$ is exact on perverse sheaves supported on Z. Combining this observation with Lemma 9.7.6, we conclude that the functor $M \mapsto \mathrm{Hom}(Q,M)$ is exact. Therefore, Q is a projective $\Bbbk$-module. Moreover, the calculation above shows that

$$\mathrm{Hom}(Q,\Bbbk) \cong \mathrm{Hom}(\mathcal{P}_Z(\nu,\Bbbk),\mathcal{I}_*(\lambda,\Bbbk)) \cong \mathrm{F}_\nu(\mathcal{I}_*(\lambda,\Bbbk)).$$

Since Q is projective, the evaluation map $Q \mapsto \mathrm{Hom}(\mathrm{Hom}(Q,\Bbbk),\Bbbk)$ is an isomorphism, and we conclude that $Q \cong \mathrm{Hom}(\mathrm{F}_\nu(\mathcal{I}_*(\lambda,\Bbbk)),\Bbbk)$, as desired. $\square$

LEMMA 9.7.13. *Let $Z' \subset Z \subset \mathcal{G}r$ be two closed unions of finitely many $G(\mathbb{O})$-orbits. For any $\nu \in \check{\mathbf{X}}$, we have*

$${}^p\mathsf{H}^0(\mathcal{P}_Z(\nu,\Bbbk)|_{Z'}) \cong \mathcal{P}_{Z'}(\nu,\Bbbk).$$

PROOF. Let $h : Z' \hookrightarrow Z$ be the inclusion map. For $\mathcal{F} \in \mathrm{Perv}_{G(\mathbb{O})}(Z',\Bbbk)$, we may regard $\mathcal{F}$ (or $h_*\mathcal{F}$) as an object of $\mathrm{Perv}_{G(\mathbb{O})}(\mathcal{G}r,\Bbbk)$ supported on Z'. Then

$$\mathrm{F}_\nu(\mathcal{F}) \cong \mathrm{F}_\nu(h_*\mathcal{F}) \cong \mathrm{Hom}(\mathcal{P}_Z(\nu,\Bbbk),h_*\mathcal{F}) \cong \mathrm{Hom}(h^*\mathcal{P}_Z(\nu,\Bbbk),\mathcal{F}).$$

Since $h^*\mathcal{P}_Z(\nu,\Bbbk) \in {}^pD^{\mathrm{b}}_{G(\mathbb{O})}(Z',\Bbbk)^{\leq 0}$, we deduce by truncation that

$$\mathrm{F}_\nu(\mathcal{F}) \cong \mathrm{Hom}({}^p\mathsf{H}^0(h^*\mathcal{P}_Z(\nu,\Bbbk)),\mathcal{F}).$$

But this is also naturally isomorphic to $\mathrm{Hom}(\mathcal{P}_{Z'}(\nu,\Bbbk),\mathcal{F})$, so by Yoneda's lemma, we have ${}^p\mathsf{H}^0(h^*\mathcal{P}_Z(\nu,\Bbbk)) \cong \mathcal{P}_{Z'}(\nu,\Bbbk)$. $\square$

THEOREM 9.7.14. *Let $Z \subset \mathcal{G}r$ be a closed union of finitely many $G(\mathbb{O})$-orbits.*

(1) *For all $\nu \in \check{\mathbf{X}}$, the perverse sheaf $\mathcal{P}_Z(\nu,\Bbbk)$ admits a filtration whose subquotients are of the form*

$$\mathrm{Hom}(\mathrm{F}_\nu(\mathcal{I}_*(\lambda,\Bbbk)),\Bbbk) \otimes \mathcal{I}_!(\lambda,\Bbbk).$$

(2) *For all $\nu \in \check{\mathbf{X}}$, we have that $\mathbf{H}^\bullet(\mathcal{G}r,\mathcal{P}_Z(\nu,\Bbbk))$ is a free $\Bbbk$-module.*

(3) *For all $\nu \in \check{\mathbf{X}}$, there is a natural isomorphism*

$$\mathcal{P}_Z(\nu,\Bbbk) \cong \Bbbk \overset{L}{\otimes}_{\mathbb{Z}} \mathcal{P}_Z(\nu,\mathbb{Z}).$$

Proof. Let us revisit the construction of $\mathcal{P}_Z(\nu,\Bbbk)$ from the proof of Proposition 9.7.10. Let $i : S_\nu^- \cap Z \hookrightarrow Z$ be the inclusion map. We saw in that proof that $\mathrm{Av}_{1!}^{G(\mathbb{O}_n)}(i_!\underline{\Bbbk}_{S_\nu^-\cap Z}[-\langle 2\rho,\nu\rangle])$ lies in ${}^pD^{\mathrm{b}}_{\mathrm{c}}(Z,\Bbbk)^{\le 0}$. The functor $\mathrm{Av}_{1!}^{G(\mathbb{O}_n)}$ commutes with the extension-of-scalars functor $\Bbbk \otimes^L_{\mathbb{Z}} (-)$, so we have

$$\mathrm{Av}_{1!}^{G(\mathbb{O}_n)}(i_!\underline{\Bbbk}_{S_\nu^-\cap Z}[-\langle 2\rho,\nu\rangle]) \cong \Bbbk \overset{L}{\otimes}_{\mathbb{Z}} \mathrm{Av}_{1!}^{G(\mathbb{O}_n)}(i_!\underline{\mathbb{Z}}_{S_\nu^-\cap Z}[-\langle 2\rho,\nu\rangle]).$$

Since $\Bbbk \otimes^L_{\mathbb{Z}} (-)$ is right t-exact (see Lemma 3.2.4), it follows that

$${}^p\mathsf{H}^0(\mathrm{Av}_{1!}^{G(\mathbb{O}_n)}(i_!\underline{\Bbbk}_{S_\nu^-\cap Z}[-\langle 2\rho,\nu\rangle])) \cong {}^p\mathsf{H}^0(\Bbbk \overset{L}{\otimes}_{\mathbb{Z}} {}^p\mathsf{H}^0(\mathrm{Av}_{1!}^{G(\mathbb{O}_n)}(i_!\underline{\mathbb{Z}}_{S_\nu^-\cap Z}[-\langle 2\rho,\nu\rangle]))),$$

or, equivalently,

$$\mathcal{P}_Z(\nu,\Bbbk) \cong {}^p\mathsf{H}^0(\Bbbk \overset{L}{\otimes}_{\mathbb{Z}} \mathcal{P}_Z(\nu,\mathbb{Z})).$$

Thus, part (3) is equivalent to the claim that $\Bbbk \otimes^L_{\mathbb{Z}} \mathcal{P}_Z(\nu,\mathbb{Z})$ is perverse.

We will now prove all three parts of the theorem simultaneously by induction on the number of $G(\mathbb{O})$-orbits in Z. If Z is empty, then $\mathcal{P}_Z(\nu,\Bbbk) = 0$, and there is nothing to prove. Otherwise, let $\mathcal{G}r_\lambda$ be a $G(\mathbb{O})$-orbit that is open in Z. Let $Z' = Z \smallsetminus \mathcal{G}r_\lambda$, and let $h : Z' \hookrightarrow Z$ be the inclusion map. Consider the natural distinguished triangle

$$j_{\lambda!}(\mathcal{P}_Z(\nu,\Bbbk)|_{\mathcal{G}r_\lambda}) \to \mathcal{P}_Z(\nu,\Bbbk) \to h_*(\mathcal{P}_Z(\nu,\Bbbk)|_{Z'}) \to .$$

The middle term is perverse, and the first and last terms lie in ${}^pD^{\mathrm{b}}_{G(\mathbb{O})}(\mathcal{G}r,\Bbbk)^{\le 0}$. Taking cohomology, we obtain the four-term exact sequence

$$\begin{aligned}0 \to {}^p\mathsf{H}^{-1}(h_*(\mathcal{P}_Z(\nu,\Bbbk)|_{Z'})) \to {}^p\mathsf{H}^0(j_{\lambda!}(\mathcal{P}_Z(\nu,\Bbbk)|_{\mathcal{G}r_\lambda}))&\\ \to \mathcal{P}_Z(\nu,\Bbbk) \to {}^p\mathsf{H}^0(h_*(\mathcal{P}_Z(\nu,\Bbbk)|_{Z'})) \to 0.&\end{aligned}$$

By Lemmas 9.7.12 and 9.7.13, we rewrite this sequence as

$$\begin{aligned}0 \to {}^p\mathsf{H}^{-1}(h_*(\mathcal{P}_Z(\nu,\Bbbk)|_{Z'})) \to \mathrm{Hom}(\mathrm{F}_\nu(\mathcal{I}_*(\lambda,\Bbbk)),\Bbbk) \otimes \mathcal{I}_!(\lambda,\Bbbk)&\\ \to \mathcal{P}_Z(\nu,\Bbbk) \to \mathcal{P}_{Z'}(\nu,\Bbbk) \to 0.&\end{aligned}\tag{9.7.5}$$

Let us now consider the special case where $\Bbbk = \mathbb{Z}$. By Corollary 9.7.9, the second term is a direct sum of copies of $\mathrm{IC}_\lambda(\mathbb{Z})$, and thus has no nonzero subobject supported on Z'. The first term of (9.7.5) must vanish, and we therefore have a short exact sequence

$$0 \to \mathrm{Hom}(\mathrm{F}_\nu(\mathcal{I}_*(\lambda,\mathbb{Z})),\mathbb{Z}) \otimes \mathcal{I}_!(\lambda,\mathbb{Z}) \to \mathcal{P}_Z(\nu,\mathbb{Z}) \to \mathcal{P}_{Z'}(\nu,\mathbb{Z}) \to 0.$$

Regard this as a distinguished triangle, and apply $\Bbbk \otimes^L_{\mathbb{Z}} (-)$. By Proposition 9.7.8 and the assumption that part (3) holds on Z', we obtain a distinguished triangle

$$\mathrm{Hom}(\mathrm{F}_\nu(\mathcal{I}_*(\lambda,\Bbbk)),\Bbbk) \otimes \mathcal{I}_!(\lambda,\Bbbk) \to \Bbbk \overset{L}{\otimes}_{\mathbb{Z}} \mathcal{P}_Z(\nu,\mathbb{Z}) \to \mathcal{P}_{Z'}(\nu,\Bbbk) \to .$$

But this shows that the middle term is also perverse, so by the observations at the beginning of the proof, we have proved part (3) of the theorem. The distinguished triangle above becomes a short exact sequence

$$0 \to \mathrm{Hom}(\mathrm{F}_\nu(\mathcal{I}_*(\lambda,\Bbbk)),\Bbbk) \otimes \mathcal{I}_!(\lambda,\Bbbk) \to \mathcal{P}_Z(\nu,\Bbbk) \to \mathcal{P}_{Z'}(\nu,\Bbbk) \to 0.$$

Part (1) follows by induction and this short exact sequence, and part (2) follows from part (1) and Proposition 9.7.4. $\square$

Corollary 9.7.15. *Every perverse sheaf $\mathcal{F} \in \mathrm{Perv}_{G(\mathbb{O})}(\mathcal{G}r, \Bbbk)$ is a quotient of a perverse sheaf $\tilde{\mathcal{F}} \in \mathrm{Perv}_{G(\mathbb{O})}(\mathcal{G}r, \Bbbk)$ with the property that $\mathbf{H}^\bullet(\mathcal{G}r, \tilde{\mathcal{F}})$ is a free $\Bbbk$-module. Moreover, $\tilde{\mathcal{F}}$ can be chosen to be a projective object in $\mathrm{Perv}_{G(\mathbb{O})}(Z, \Bbbk)$, where Z is some closed union of finitely many $G(\mathbb{O})$-orbits.*

Proof. Let $Z \subset \mathcal{G}r$ be a closed union of finitely many $G(\mathbb{O})$-orbits that contains the support of $\mathcal{F}$. Let $\nu \in \check{\mathbf{X}}$, and choose an element $m \in \mathrm{F}_\nu(\mathcal{F})$. Via Proposition 9.7.10, this corresponds to a map $\phi_m : \mathcal{P}_Z(\nu, \Bbbk) \to \mathcal{F}$. As explained in Remark 9.7.11, the image of $\mathrm{F}_\nu(\phi_m) : \mathrm{F}_\nu(\mathcal{P}_Z(\nu, \Bbbk)) \to \mathrm{F}_\nu(\mathcal{F})$ contains m.

Now choose a set of elements $m_1, \ldots, m_k$ that generate $\mathbf{H}^\bullet(\mathcal{G}r, \mathcal{F}) \cong \bigoplus_\nu \mathrm{F}_\nu(\mathcal{F})$ as a $\Bbbk$-module. Assume that each m_i is contained in some summand $\mathrm{F}_{\nu_i}(\mathcal{F})$. Each m_i thus determines a map $\phi_i : \mathcal{P}_Z(\nu_i, \Bbbk) \to \mathcal{F}$. Let $\tilde{\mathcal{F}} = \bigoplus_{i=1}^k \mathcal{P}_Z(\nu_i, \Bbbk)$, and let

$$\phi = \sum_{i=1}^k \phi_i : \tilde{\mathcal{F}} \to \mathcal{F}.$$

By the observation in the preceding paragraph, the image of $\mathrm{F}_{\nu_i}(\phi) : \mathrm{F}_{\nu_i}(\tilde{\mathcal{F}}) \to \mathrm{F}_{\nu_i}(\mathcal{F})$ contains m_i. It follows that

$$\mathbf{H}^\bullet(\mathcal{G}r, \phi) : \mathbf{H}^\bullet(\mathcal{G}r, \tilde{\mathcal{F}}) \to \mathbf{H}^\bullet(\mathcal{G}r, \mathcal{F})$$

is surjective. By Theorem 9.6.10, ϕ itself is surjective, and by Theorem 9.7.14, $\mathbf{H}^\bullet(\mathcal{G}r, \tilde{\mathcal{F}})$ is a free $\Bbbk$-module. $\square$

Exercises.

9.7.1. Let $Z \subset \mathcal{G}r$ be a closed union of finitely many $G(\mathbb{O})$-orbits, and suppose $\mathcal{G}r_\lambda \subset Z$. Show that there is a surjective map $\mathcal{P}_Z(\lambda, \Bbbk) \to \mathcal{I}_!(\lambda, \Bbbk)$ whose kernel admits a filtration with subquotients of the form

$$\mathrm{Hom}(\mathrm{F}_\lambda(\mathcal{I}_*(\mu, \Bbbk)), \Bbbk) \otimes \mathcal{I}_!(\mu, \Bbbk) \qquad \text{with } \mu >_{\check{\mathbf{X}}} \lambda.$$

9.7.2. Let $Z \subset \mathcal{G}r$ be a closed union of finitely many $G(\mathbb{O})$-orbits. Show that for any two orbits $\mathcal{G}r_\lambda, \mathcal{G}r_\mu \subset Z$, we have

$$\mathrm{Ext}^k_{\mathrm{Perv}_{G(\mathbb{O})}(Z,\Bbbk)}(\mathcal{I}_!(\lambda, \Bbbk), \mathcal{I}_*(\mu, \Bbbk)) \cong \begin{cases} \Bbbk & \text{if } \lambda = \mu \text{ and } k = 0, \\ 0 & \text{otherwise.} \end{cases}$$

9.7.3. Let $Z \subset \mathcal{G}r$ be a closed union of finitely many $G(\mathbb{O})$-orbits. Show that $D^{\mathrm{b}}\mathrm{Perv}_{G(\mathbb{O})}(Z, \Bbbk)$ is generated as a triangulated category by objects of the form $\mathcal{I}_!(\lambda, \Bbbk)$, and also by objects of the form $\mathcal{I}_*(\lambda, \Bbbk)$.

9.7.4. Let $Z \subset \mathcal{G}r$ be a closed union of finitely many $G(\mathbb{O})$-orbits. Show that $\mathrm{Perv}_{G(\mathbb{O})}(Z, \Bbbk)$ has finite projective dimension. *Hint*: This is equivalent to showing that $D^{\mathrm{b}}\mathrm{Perv}_{G(\mathbb{O})}(Z, \Bbbk)$ is generated by projective objects.

9.7.5. Let $Z' \subset Z \subset \mathcal{G}r$ be two closed unions of finitely many $G(\mathbb{O})$-orbits. Show that the obvious functor

$$D^{\mathrm{b}}\mathrm{Perv}_{G(\mathbb{O})}(Z', \Bbbk) \to D^{\mathrm{b}}\mathrm{Perv}_{G(\mathbb{O})}(Z, \Bbbk)$$

is fully faithful. Then show that

$$D^{\mathrm{b}}\mathrm{Perv}_{G(\mathbb{O})}(Z, \Bbbk) \to D^{\mathrm{b}}\mathrm{Perv}_{G(\mathbb{O})}(\mathcal{G}r, \Bbbk)$$

is fully faithful. *Hint*: Compare Ext-groups of standard and costandard sheaves in $\mathrm{Perv}_{G(\mathbb{O})}(Z', \Bbbk)$ and in $\mathrm{Perv}_{G(\mathbb{O})}(Z, \Bbbk)$.

9.8. Hypercohomology as a fiber functor

In this section, we show that the hypercohomology functor on $\mathrm{Perv}_{G(\mathbb{O})}(\mathcal{G}r, \Bbbk)$ can be made into a fiber functor, as defined below.

Definition 9.8.1. Let $(\mathscr{A}, \odot)$ be a symmetric monoidal $\Bbbk$-linear abelian category. A **fiber functor** for $(\mathscr{A}, \odot)$ is an exact, faithful symmetric monoidal functor $F : (\mathscr{A}, \odot) \to (\Bbbk\text{-mod}^{\mathrm{fg}}, \otimes)$.

Of course, we have already seen that $\mathbf{H}^\bullet(\mathcal{G}r, -)$ is exact and faithful, so the remaining task is to show that it is a symmetric monoidal functor. It will useful to introduce the following notation:

$$\mathrm{Perv}^{\Bbbk\text{-free}}_{G(\mathbb{O})}(\mathcal{G}r, \Bbbk) = \{\mathcal{F} \in \mathrm{Perv}_{G(\mathbb{O})}(\mathcal{G}r, \Bbbk) \mid \mathbf{H}^\bullet(\mathcal{G}r, \mathcal{F}) \text{ is a free } \Bbbk\text{-module}\}.$$

Lemma 9.8.2. *If $\mathcal{F}, \mathcal{G} \in \mathrm{Perv}^{\Bbbk\text{-free}}_{G(\mathbb{O})}(\mathcal{G}r, \Bbbk)$, then the objects $\mathcal{F} \boxtimes \mathcal{G}$, $\mathcal{F} \mathbin{\tilde{\boxtimes}} \mathcal{G}$, and $\mathcal{F} \star \mathcal{G}$ are all perverse.*

Proof. Once we prove that $\mathcal{F} \boxtimes \mathcal{G}$ is perverse, the claim for $\mathcal{F} \mathbin{\tilde{\boxtimes}} \mathcal{G}$ will follow immediately by construction, and the claim for $\mathcal{F} \star \mathcal{G}$ follows from the t-exactness of m_* (see the proof of Theorem 9.5.5).

For $\mathcal{F} \boxtimes \mathcal{G}$, let us regard $\mathcal{G}r \times \mathcal{G}r$ as the affine Grassmannian for the group $G \times G$, and apply the criterion from Proposition 9.6.9. Given a coweight $(\nu, \mu) \in \check{\mathbf{X}} \times \check{\mathbf{X}}$ for $G \times G$, the corresponding semi-infinite orbit $(U^+ \times U^+)(\mathbb{K}) \cdot \mathbf{t}^{(\nu,\mu)}$ can be identified with $S^+_\nu \times S^+_\mu$. We must prove that

$$\mathbf{H}^k_{\mathrm{c}}(S^+_\nu \times S^+_\mu, (\mathcal{F} \boxtimes \mathcal{G})|_{S^+_\nu \times S^+_\mu}) = 0 \qquad \text{unless } k = \langle 2\rho, \nu + \mu \rangle. \tag{9.8.1}$$

We have

$$R\Gamma_{\mathrm{c}}((\mathcal{F} \boxtimes \mathcal{G})|_{S^+_\nu \times S^+_\mu}) \cong R\Gamma_{\mathrm{c}}((\mathcal{F}|_{S^+_\nu}) \boxtimes (\mathcal{G}|_{S^+_\mu})) \cong R\Gamma_{\mathrm{c}}(\mathcal{F}|_{S^+_\nu}) \overset{L}{\otimes} R\Gamma_{\mathrm{c}}(\mathcal{G}|_{S^+_\mu}).$$

By Proposition 9.6.1, we have

$$R\Gamma_{\mathrm{c}}(\mathcal{F}|_{S^+_\nu}) \cong \mathrm{F}_\nu(\mathcal{F})[-\langle 2\rho, \nu \rangle] \qquad \text{and} \qquad R\Gamma_{\mathrm{c}}(\mathcal{G}|_{S^+_\mu}) \cong \mathrm{F}_\mu(\mathcal{G})[-\langle 2\rho, \mu \rangle].$$

Moreover, by Theorem 9.6.10, $\mathrm{F}_\nu(\mathcal{F})$ is a direct summand of $\mathbf{H}^\bullet(\mathcal{G}r, \mathcal{F})$, and hence a projective (and flat) $\Bbbk$-module. Therefore, the derived tensor product on the right-hand side below can be replaced by a non-derived tensor product:

$$R\Gamma_{\mathrm{c}}((\mathcal{F} \boxtimes \mathcal{G})|_{S^+_\nu \times S^+_\mu}) \cong (\mathrm{F}_\nu(\mathcal{F}) \otimes \mathrm{F}_\mu(\mathcal{G}))[-\langle 2\rho, \nu + \mu \rangle].$$

Then (9.8.1) follows immediately. □

Lemma 9.8.3. *For $\mathcal{F}, \mathcal{G} \in \mathrm{Perv}^{\Bbbk\text{-free}}_{G(\mathbb{O})}(\mathcal{G}r, \Bbbk)$, we have $\mathcal{F} \star \mathcal{G} \in \mathrm{Perv}^{\Bbbk\text{-free}}_{G(\mathbb{O})}(\mathcal{G}r, \Bbbk)$, and there is a natural isomorphism*

$$\phi : R\Gamma(\mathcal{F}) \overset{L}{\otimes} R\Gamma(\mathcal{G}) \xrightarrow{\sim} R\Gamma(\mathcal{F} \mathbin{{}^p\star} \mathcal{G})$$

making $R\Gamma : \mathrm{Perv}^{\Bbbk\text{-free}}_{G(\mathbb{O})}(\mathcal{G}r, \Bbbk) \to D^{\mathrm{b}}(\Bbbk\text{-mod}^{\mathrm{fg}})$ into a monoidal functor. Moreover, the following diagram commutes:

$$\begin{tikzcd}
R\Gamma(\mathcal{F}) \otimes^L R\Gamma(\mathcal{G}) \arrow[r, "\phi", "\sim"'] \arrow[d, "\sigma"', "\wr"] & R\Gamma(\mathcal{F} \mathbin{{}^p\star} \mathcal{G}) \arrow[d, "\wr"', "\beta'"] \\
R\Gamma(\mathcal{G}) \otimes^L R\Gamma(\mathcal{F}) \arrow[r, "\phi", "\sim"'] & R\Gamma(\mathcal{G} \mathbin{{}^p\star} \mathcal{F})
\end{tikzcd}$$

Here, β' comes from Theorem 9.5.9, and σ from Remark A.6.15.

Proof. We will work with both $\widetilde{\mathbf{Gr}}_{\mathbb{A}^2} \to \mathbb{A}^2$ and $\mathbf{Gr}_{\mathbb{A}^2} \to \mathbb{A}^2$. To distinguish between them, denote the former by $\tilde{\pi}$ and the latter by π.

Let us begin by constructing ϕ. Lemma 9.5.6(1) tells us that $\tilde{\pi}$ is a stratumwise locally trivial fibration. As in Exercise 2.7.1, this implies that $\tilde{\pi}_*(\mathcal{F}\, {}^p\tilde{\boxtimes}_{\mathbb{A}^2}\, \mathcal{G}) \in D^{\mathrm{b}}_{\mathrm{c}}(\mathbb{A}^2, \Bbbk)$ has locally constant (and hence constant) cohomology sheaves. Since

$$\pi_*(\mathcal{F}\, {}^p{\star}_{\mathbb{A}^2}\, \mathcal{G}) \cong \pi_* \mathbf{m}_*(\mathcal{F}\, {}^p\tilde{\boxtimes}_{\mathbb{A}^2}\, \mathcal{G}) \cong \tilde{\pi}_*(\mathcal{F}\, {}^p\tilde{\boxtimes}_{\mathbb{A}^2}\, \mathcal{G}),$$

we conclude that $\pi_*(\mathcal{F}\, {}^p{\star}_{\mathbb{A}^2}\, \mathcal{G}) \in D^{\mathrm{b}}_{\mathrm{c}}(\mathbb{A}^2, \Bbbk)$ has constant cohomology sheaves as well. Choose a point $(x, y) \in \mathbb{A}^2$. Since the restriction of π to the support of $\mathcal{F}\, {}^p{\star}_{\mathbb{A}^2}\, \mathcal{G}$ is proper, using Lemma 9.5.7, we have

$$\pi_*(\mathcal{F}\, {}^p{\star}_{\mathbb{A}^2}\, \mathcal{G})_{(x,y)} \cong R\Gamma((\mathcal{F}\, {}^p{\star}_{\mathbb{A}^2}\, \mathcal{G})|_{\pi^{-1}(x,y)}) \cong \begin{cases} R\Gamma({}^p\mathsf{H}^0(\mathcal{F} \boxtimes \mathcal{G}))[2] & \text{if } x \neq y, \\ R\Gamma(\mathcal{F}\, {}^p{\star}\, \mathcal{G})[2] & \text{if } x = y. \end{cases}$$

By Lemma 9.8.2, under our assumptions, $\mathcal{F} \boxtimes \mathcal{G}$ is already perverse, and we have

$$R\Gamma({}^p\mathsf{H}^0(\mathcal{F} \boxtimes \mathcal{G})) \cong R\Gamma(\mathcal{F} \boxtimes \mathcal{G}) \cong R\Gamma(\mathcal{F}) \overset{L}{\otimes} R\Gamma(\mathcal{G}).$$

On the other hand, since $\mathbb{A}^2$ is contractible, Corollary 1.8.11 gives us natural isomorphisms

$$\pi_*(\mathcal{F}\, {}^p{\star}_{\mathbb{A}^2}\, \mathcal{G})_{(x,y)} \xleftarrow{\sim} R\Gamma(\pi_*(\mathcal{F}\, {}^p{\star}_{\mathbb{A}^2}\, \mathcal{G})) \xrightarrow{\sim} \pi_*(\mathcal{F}\, {}^p{\star}_{\mathbb{A}^2}\, \mathcal{G})_{(x,x)}.$$

Combining this with the observations above, we obtain a natural isomorphism

$$\phi : R\Gamma(\mathcal{F}) \overset{L}{\otimes} R\Gamma(\mathcal{G}) \xrightarrow{\sim} R\Gamma(\mathcal{F}\, {}^p{\star}\, \mathcal{G}). \tag{9.8.2}$$

Since $\mathbf{H}^\bullet(\mathcal{G}r, \mathcal{F})$ and $\mathbf{H}^\bullet(\mathcal{G}r, \mathcal{G})$ are free (and hence flat) over $\Bbbk$, there is a natural isomorphism

$$\mathsf{H}^\bullet(R\Gamma(\mathcal{F}) \overset{L}{\otimes} R\Gamma(\mathcal{G})) \cong \mathsf{H}^\bullet(R\Gamma(\mathcal{F})) \otimes \mathsf{H}^\bullet(R\Gamma(\mathcal{F})), \tag{9.8.3}$$

by Lemma A.6.17. Therefore, applying $\mathsf{H}^\bullet$ to (9.8.2), we see that $\mathbf{H}^\bullet(\mathcal{G}r, \mathcal{F}\, {}^p{\star}\, \mathcal{G})$ is free over $\Bbbk$. In other words, $\mathrm{Perv}^{\Bbbk\text{-free}}_{G(\mathbb{O})}(\mathcal{G}r, \Bbbk)$ is closed under $\star$ (or ${}^p{\star}$), and thus has the structure of a monoidal category.

If we start with three perverse sheaves $\mathcal{F}, \mathcal{G}, \mathcal{H} \in \mathrm{Perv}^{\Bbbk\text{-free}}_{G(\mathbb{O})}(\mathcal{G}r, \Bbbk)$, one can carry out a similar calculation on $\mathbb{A}^3$ to show that (9.8.2) is compatible with the associativity of ${}^p{\star}$, i.e., that $R\Gamma$ is a monoidal functor.

Finally, it remains to show the compatibility with β' and σ. Recall that the definition of β' involves maps $u : \mathbb{A}^2 \to \mathbb{A}^2$ and $s : \mathbf{Gr}_{\mathbb{A}^2} \to \mathbf{Gr}_{\mathbb{A}^2}$ as in the proof of Theorem 9.5.9. Note that $\pi_* s^* \cong u^* \pi_*$ and that $(u^*\mathcal{L})_{(x,y)} \cong \mathcal{L}_{(y,x)}$. Using these observations, we build a commutative diagram

$$\begin{tikzcd}
\pi_*(\mathcal{F}\, {}^p{\star}_{\mathbb{A}^2}\, \mathcal{G})_{(y,x)} \arrow[d, "\wr"'] \arrow[ddd, bend right=60, "\sigma"'] \arrow[rr, bend left, "\phi"] & R\Gamma(\pi_*(\mathcal{F}\, {}^p{\star}_{\mathbb{A}^2}\, \mathcal{G})) \arrow[l, "\sim"'] \arrow[r, "\sim"] \arrow[d, "\wr"'] & \pi_*(\mathcal{F}\, {}^p{\star}_{\mathbb{A}^2}\, \mathcal{G})_{(x,x)} \arrow[d, "\wr"] \arrow[ddd, bend left=60, "\beta'"] \\
u^*\pi_*(\mathcal{F}\, {}^p{\star}_{\mathbb{A}^2}\, \mathcal{G})_{(x,y)} \arrow[d, "\wr"'] & R\Gamma(u^*\pi_*(\mathcal{F}\, {}^p{\star}_{\mathbb{A}^2}\, \mathcal{G})) \arrow[l, "\sim"'] \arrow[r, "\sim"] \arrow[d, "\wr"'] & u^*\pi_*(\mathcal{F}\, {}^p{\star}_{\mathbb{A}^2}\, \mathcal{G})_{(x,x)} \arrow[d, "\wr"] \\
\pi_* s^*(\mathcal{F}\, {}^p{\star}_{\mathbb{A}^2}\, \mathcal{G})_{(x,y)} \arrow[d, "\wr"', "(9.5.7)"] & R\Gamma(\pi_* s^*(\mathcal{F}\, {}^p{\star}_{\mathbb{A}^2}\, \mathcal{G})) \arrow[l, "\sim"'] \arrow[r, "\sim"] \arrow[d, "\wr"'] & \pi_* s^*(\mathcal{F}\, {}^p{\star}_{\mathbb{A}^2}\, \mathcal{G})_{(x,x)} \arrow[d, "\wr"] \\
\pi_*(\mathcal{G}\, {}^p{\star}_{\mathbb{A}^2}\, \mathcal{F})_{(x,y)} \arrow[rr, bend right, "\phi"'] & R\Gamma(\pi_*(\mathcal{G}\, {}^p{\star}_{\mathbb{A}^2}\, \mathcal{F})) \arrow[l, "\sim"'] \arrow[r, "\sim"] & \pi_*(\mathcal{G}\, {}^p{\star}_{\mathbb{A}^2}\, \mathcal{F})_{(x,x)}
\end{tikzcd}$$

To see that the leftmost column is identified with σ, observe that (9.5.7) involves the isomorphism

$$s_{\boldsymbol{\eta}}^*(\mathrm{pr}_1^*\tau^\dagger\mathcal{F} \overset{L}{\otimes} \mathrm{pr}_2^*\tau^\dagger\mathcal{G}) \cong \mathrm{pr}_2^*\tau^\dagger\mathcal{F} \overset{L}{\otimes} \mathrm{pr}_1^*\tau^\dagger\mathcal{G} \xrightarrow[\sim]{\text{Rmk. A.6.15}} \mathrm{pr}_1^*\tau^\dagger\mathcal{G} \overset{L}{\otimes} \mathrm{pr}_2^*\tau^\dagger\mathcal{F}. \qquad \square$$

Proposition 9.8.4. *For $\mathcal{F}, \mathcal{G} \in \mathrm{Perv}_{G(\mathbb{O})}(\mathcal{G}r, \Bbbk)$, there is a natural isomorphism*

$$\phi = \phi_{\mathcal{F},\mathcal{G}} : \mathbf{H}^\bullet(\mathcal{G}r, \mathcal{F}) \otimes \mathbf{H}^\bullet(\mathcal{G}r, \mathcal{G}) \xrightarrow{\sim} \mathbf{H}^\bullet(\mathcal{G}r, \mathcal{F} \mathbin{{}^p\!\star} \mathcal{G})$$

making $\mathbf{H}^\bullet(\mathcal{G}r, -) : \mathrm{Perv}_{G(\mathbb{O})}(\mathcal{G}r, \Bbbk) \to \Bbbk\text{-mod}^{\mathrm{fg}}$ into a monoidal functor. Furthermore, the following diagram commutes:

$$\begin{array}{ccc} \mathbf{H}^\bullet(\mathcal{G}r, \mathcal{F}) \otimes \mathbf{H}^\bullet(\mathcal{G}r, \mathcal{G}) & \xrightarrow[\sim]{\phi} & \mathbf{H}^\bullet(\mathcal{G}r, \mathcal{F} \mathbin{{}^p\!\star} \mathcal{G}) \\ \sigma \downarrow \wr & & \wr \downarrow \beta' \\ \mathbf{H}^\bullet(\mathcal{G}r, \mathcal{G}) \otimes \mathbf{H}^\bullet(\mathcal{G}r, \mathcal{F}) & \xrightarrow[\sim]{\phi} & \mathbf{H}^\bullet(\mathcal{G}r, \mathcal{G} \mathbin{{}^p\!\star} \mathcal{F}) \end{array} \tag{9.8.4}$$

Proof sketch. We consider various cases, depending on where $\mathcal{F}$ and $\mathcal{G}$ live.

Step 1. Both $\mathcal{F}$ and $\mathcal{G}$ lie in $\mathrm{Perv}_{G(\mathbb{O})}^{\Bbbk\text{-free}}(\mathcal{G}r, \Bbbk)$. In this case, the present proposition is obtained by simply applying $\mathsf{H}^\bullet$ to the statement of Lemma 9.8.3 (cf. (9.8.3)).

Step 2. We have $\mathcal{F} \in \mathrm{Perv}_{G(\mathbb{O})}^{\Bbbk\text{-free}}(\mathcal{G}r, \Bbbk)$ and $\mathcal{G} \in \mathrm{Perv}_{G(\mathbb{O})}(\mathcal{G}r, \Bbbk)$. By Corollary 9.7.15, one can find an exact sequence

$$\mathcal{P}_1 \to \mathcal{P}_0 \to \mathcal{G} \to 0 \tag{9.8.5}$$

where $\mathcal{P}_1, \mathcal{P}_0 \in \mathrm{Perv}_{G(\mathbb{O})}^{\Bbbk\text{-free}}(\mathcal{G}r, \Bbbk)$. Consider the diagram

$$\begin{array}{ccc} \mathbf{H}^\bullet(\mathcal{G}r, \mathcal{F}) \otimes \mathbf{H}^\bullet(\mathcal{G}r, \mathcal{P}_1) & \xrightarrow{\phi} & \mathbf{H}^\bullet(\mathcal{G}r, \mathcal{F} \star \mathcal{P}_1) \\ \downarrow & & \downarrow \\ \mathbf{H}^\bullet(\mathcal{G}r, \mathcal{F}) \otimes \mathbf{H}^\bullet(\mathcal{G}r, \mathcal{P}_0) & \xrightarrow{\phi} & \mathbf{H}^\bullet(\mathcal{G}r, \mathcal{F} \star \mathcal{P}_0) \\ \downarrow & & \downarrow \\ \mathbf{H}^\bullet(\mathcal{G}r, \mathcal{F}) \otimes \mathbf{H}^\bullet(\mathcal{G}r, \mathcal{G}) & \overset{\phi}{\dashrightarrow} & \mathbf{H}^\bullet(\mathcal{G}r, \mathcal{F} \mathbin{{}^p\!\star} \mathcal{G}) \\ \downarrow & & \downarrow \\ 0 & & 0 \end{array} \tag{9.8.6}$$

Using the exactness of $\mathbf{H}^\bullet(\mathcal{G}r, -)$, one can check that both columns are exact. The first two horizontal arrows come from Step 1. There is a unique way to fill in the dotted arrow to make this diagram commute, and this defines ϕ.

Step 3. Naturality and well-definedness. Let $\mathcal{G}' \in \mathrm{Perv}_{G(\mathbb{O})}(\mathcal{G}r, \Bbbk)$. Choose an exact sequence $\mathcal{P}_1' \to \mathcal{P}_0' \to \mathcal{G}' \to 0$ as in (9.8.5), and use it to define $\phi_{\mathcal{F},\mathcal{G}'}$. Let $Z \subset \mathcal{G}r$ be a closed union of finitely many $G(\mathbb{O})$-orbits that contains the supports of $\mathcal{G}$, $\mathcal{P}_0$, $\mathcal{P}_1$, $\mathcal{P}_0'$, and $\mathcal{P}_1'$. By Corollary 9.7.15, we can find another resolution $\mathcal{P}_1'' \to \mathcal{P}_0'' \to \mathcal{G} \to 0$, where $\mathcal{P}_1''$ and $\mathcal{P}_0''$ lie in $\mathrm{Perv}_{G(\mathbb{O})}^{\Bbbk\text{-free}}(\mathcal{G}r, \Bbbk)$ and, in addition, are projective as objects of $\mathrm{Perv}_{G(\mathbb{O})}(Z, \Bbbk)$. Using this projectivity, one can find horizontal maps making the following diagram commute:

$$
\begin{array}{ccccc}
\mathcal{P}_1 & \longleftarrow & \mathcal{P}_1'' & \longrightarrow & \mathcal{P}_1' \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{P}_0 & \longleftarrow & \mathcal{P}_0'' & \longrightarrow & \mathcal{P}_0' \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{G} & = \!= \!= & \mathcal{G} & \xrightarrow{\psi} & \mathcal{G}' \\
\downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0
\end{array}
\tag{9.8.7}
$$

Each column in (9.8.7) gives rise to a copy of (9.8.6). A diagram chase shows that the map $\phi_{\mathcal{F},\mathcal{G}}$ defined using the first column coincides with that defined using the middle column. Then a comparison of the constructions for the second and third columns shows that $\phi_{\mathcal{F},\mathcal{G}}$ is natural in $\mathcal{G}$, i.e., that

$$\mathbf{H}^\bullet(\mathcal{G}r, \mathrm{id}_{\mathcal{F}} \,{}^p\!\star\, \psi) \circ \phi_{\mathcal{F},\mathcal{G}} = \phi_{\mathcal{F},\mathcal{G}'} \circ (\mathrm{id}_{\mathbf{H}^\bullet(\mathcal{G}r,\mathcal{F})} \otimes \mathbf{H}^\bullet(\mathcal{G}r, \psi)).$$

In the special case where $\mathcal{G}' = \mathcal{G}$ and $\psi = \mathrm{id}$, this shows that $\phi_{\mathcal{F},\mathcal{G}}$ is independent of the choice of (9.8.5).

Step 4. Monoidality and compatibility with β' and σ. These properties of ϕ follow by a diagram chase from the case considered in Step 1.

Step 5. The case of general $\mathcal{F}$. Let us now drop the assumption that $\mathcal{F} \in \mathrm{Perv}^{\Bbbk\text{-free}}_{G(\mathbb{O})}(\mathcal{G}r, \Bbbk)$. By Corollary 9.7.15, one can find an exact sequence

$$\mathcal{Q}_1 \to \mathcal{Q}_0 \to \mathcal{F} \to 0$$

where $\mathcal{Q}_1, \mathcal{Q}_0 \in \mathrm{Perv}^{\Bbbk\text{-free}}_{G(\mathbb{O})}(\mathcal{G}r, \Bbbk)$. The definition of ϕ in this case and the verification of its properties follows a pattern very similar to that of Steps 2–4 above. We omit the details. $\square$

For the following theorem, which is the main result of this section, equip $(\mathrm{Perv}_{G(\mathbb{O})}(\mathcal{G}r, \Bbbk), {}^p\!\star)$ with the commutativity constraint β from Remark 9.5.10.

THEOREM 9.8.5. *The functor*

$$\mathbf{H}^\bullet(\mathcal{G}r, -) : (\mathrm{Perv}_{G(\mathbb{O})}(\mathcal{G}r, \Bbbk), {}^p\!\star) \to (\Bbbk\text{-mod}^{\mathrm{fg}}, \otimes)$$

is a fiber functor.

PROOF. If M and N are $\Bbbk$-modules, let $\tau : M \otimes N \to N \otimes M$ be the obvious isomorphism given by $\tau(x \otimes y) = y \otimes x$.

We have already seen in Theorem 9.6.10 that $\mathbf{H}^\bullet(\mathcal{G}r, -)$ is exact and faithful, and Proposition 9.8.4 tells us that it is monoidal. It remains to show that it is *symmetric* monoidal, i.e., that the diagram

$$
\begin{array}{ccc}
\mathbf{H}^\bullet(\mathcal{G}r, \mathcal{F}) \otimes \mathbf{H}^\bullet(\mathcal{G}r, \mathcal{G}) & \xrightarrow[\sim]{\phi} & \mathbf{H}^\bullet(\mathcal{G}r, \mathcal{F} \,{}^p\!\star\, \mathcal{G}) \\
{\scriptstyle \tau}\Big\downarrow{\scriptstyle \wr} & & {\scriptstyle \wr}\Big\downarrow{\scriptstyle \beta} \\
\mathbf{H}^\bullet(\mathcal{G}r, \mathcal{G}) \otimes \mathbf{H}^\bullet(\mathcal{G}r, \mathcal{F}) & \xrightarrow[\sim]{\phi} & \mathbf{H}^\bullet(\mathcal{G}r, \mathcal{G} \,{}^p\!\star\, \mathcal{F})
\end{array}
\tag{9.8.8}
$$

commutes. This diagram resembles (9.8.4), but they are not identical.

Let us compare the left-hand columns of (9.8.4) and (9.8.8). According to Remark A.6.15, σ sometimes differs from τ by a sign: it is given on

$$\mathbf{H}^i(\mathcal{G}r, \mathcal{F}) \otimes \mathbf{H}^j(\mathcal{G}r, \mathcal{G}) \xrightarrow{\sim} \mathbf{H}^j(\mathcal{G}r, \mathcal{G}) \otimes \mathbf{H}^i(\mathcal{G}r, \mathcal{F}) \quad \text{by} \quad x \otimes y \mapsto (-1)^{ij}(y \otimes x).$$

Let us decompose $\mathcal{F}$ and $\mathcal{G}$ as in Remark 9.5.10: we write

$$\mathcal{F} \cong \mathcal{F}_+ \oplus \mathcal{F}_- \qquad \text{and} \qquad \mathcal{G} \cong \mathcal{G}_+ \oplus \mathcal{G}_-,$$

where $\mathcal{F}_+$ and $\mathcal{G}_+$ (resp. $\mathcal{F}_-$ and $\mathcal{G}_-$) are supported only on even-dimensional (resp. odd-dimensional) $G(\mathbb{O})$-orbits. By Corollary 9.6.11, we can identify

$$\sigma_{\mathcal{F},\mathcal{G}} = (\tau_{\mathcal{F}_+,\mathcal{G}_+}) \oplus (\tau_{\mathcal{F}_+,\mathcal{G}_-}) \oplus (\tau_{\mathcal{F}_-,\mathcal{G}_+}) \oplus (-\tau_{\mathcal{F}_-,\mathcal{G}_-}).$$

Comparing this to the definition of β in Remark 9.5.10, we conclude that the commutativity of (9.8.4) implies that of (9.8.8), as desired. $\square$

9.9. The geometric Satake equivalence

In this section, we will give the statement of the main theorem, and discuss some of the ingredients of the proof. In the case where $\Bbbk$ is a field, one can invoke "Tannakian formalism" (which will be reviewed below); in the general case, ideas from Tannakian formalism still motivate the structure of the argument.

Definition 9.9.1. Let $\Bbbk$ be a field, and let $(\mathscr{A}, \odot)$ be a rigid symmetric monoidal $\Bbbk$-linear abelian category. If $(\mathscr{A}, \odot)$ admits a fiber functor F, then the triple $(\mathscr{A}, \odot, F)$ is said to be a **neutral Tannakian category**.

Note that the tensor functor F must take the unit object $\mathbf{1} \in \mathscr{A}$ to the unit object of $\Bbbk\text{-mod}^{\mathrm{fg}}$: that is, $F(\mathbf{1}) \cong \Bbbk$.

Given a $\Bbbk$-linear neutral Tannakian category $(\mathscr{A}, \odot, F)$, let

$$\mathrm{Aut}^{\odot}(F) : \{\text{commutative } \Bbbk\text{-algebras}\} \to \{\text{groups}\}$$

be the functor given by

$$\mathrm{Aut}^{\odot}(F)(R) = \begin{array}{c}\text{tensor automorphisms of the}\\ \text{functor } R \otimes_{\Bbbk} F(-) : \mathscr{A} \to R\text{-mod}^{\mathrm{fg}}.\end{array}$$

In other words, an element of the set $\mathrm{Aut}^{\odot}(F)(R)$ is a rule γ that assigns to each object $X \in \mathscr{A}$ an automorphism γ_X of $R \otimes_{\Bbbk} F(X)$ such that for all objects $X, Y \in \mathscr{A}$ and all morphisms $f : X \to Y$, the two diagrams

$$\begin{tikzcd} R \otimes_{\Bbbk} F(X) \arrow[r, "R\otimes_{\Bbbk} F(f)"] \arrow[d, "\gamma_X"'] & R \otimes_{\Bbbk} F(Y) \arrow[d, "\gamma_Y"] \\ R \otimes_{\Bbbk} F(X) \arrow[r, "R\otimes_{\Bbbk} F(f)"] & R \otimes_{\Bbbk} F(Y) \end{tikzcd} \qquad \begin{tikzcd} R \otimes_{\Bbbk} F(X \odot Y) \arrow[r, "\sim"] \arrow[d, "\gamma_{X\odot Y}"'] & (R \otimes_{\Bbbk} F(X)) \otimes_R (R \otimes_{\Bbbk} F(Y)) \arrow[d, "\gamma_X \otimes_R \gamma_Y"] \\ R \otimes_{\Bbbk} F(X \odot Y) \arrow[r, "\sim"] & (R \otimes_{\Bbbk} F(X)) \otimes_R (R \otimes_{\Bbbk} F(Y)) \end{tikzcd}$$

both commute, and such that $\gamma_{\mathbf{1}}$ is the identity map on $R \cong R \otimes_{\Bbbk} F(\mathbf{1})$.

The following statement is the main theorem about neutral Tannakian categories. For a proof, see [**62**, Theorem 2.11]. (The definition of "rigid" given in [**62**] looks rather different from the one given in this book in Definition A.2.5, but the two versions are indeed equivalent, as explained in [**133**].)

Theorem 9.9.2. *Let $\Bbbk$ be a field, and let $(\mathscr{A}, \odot, F)$ be a neutral Tannakian category with the property that* $\mathrm{End}(\mathbf{1}) \cong \Bbbk$. *Then the functor* $\mathrm{Aut}^{\odot}(F)$ *is represented by an affine group scheme H over $\Bbbk$. Moreover, up to canonical isomorphism, there is a unique equivalence of tensor categories*

$$\tilde{F} : (\mathscr{A}, \odot) \xrightarrow{\sim} (\mathrm{Rep}(H), \otimes)$$

such that the following diagram commutes:

$$\begin{tikzcd}
(\mathscr{A}, \odot) \arrow[rr, "\tilde{F}", "\sim"'] \arrow[dr, "F"'] & & (\mathrm{Rep}(H), \otimes) \arrow[dl] \\
& (\Bbbk\text{-mod}^{\mathrm{fg}}, \otimes) &
\end{tikzcd}$$

PROPOSITION 9.9.3. *Assume that* $\Bbbk$ *is a field. Then* $(\mathrm{Perv}_{G(\mathbb{O})}(\mathcal{G}r, \Bbbk), {}^p\!\star)$ *is a neutral Tannakian category.*

PROOF. We have already seen that $\mathrm{Perv}_{G(\mathbb{O})}(\mathcal{G}r, \Bbbk)$ is symmetric monoidal (Theorem 9.5.9) and that it admits a fiber functor (Theorem 9.8.5). It remains to show that it is rigid. According to [**62**, Proposition 1.20], any $\Bbbk$-linear symmetric monoidal abelian category $(\mathscr{A}, \odot)$ is rigid if it satisfies the following conditions:

(1) It admits a fiber functor $F : \mathscr{A} \to \Bbbk\text{-mod}^{\mathrm{fg}}$.
(2) For the unit object $\mathbf{1} \in \mathscr{A}$, we have $\mathrm{End}(\mathbf{1}) \cong \Bbbk$.
(3) For any $X \in \mathscr{A}$ with $\dim F(X) = 1$, there exists an object $X^{-1} \in \mathscr{A}$ such that $X \odot X^{-1} \cong \mathbf{1}$.

Let us apply this to $(\mathrm{Perv}_{G(\mathbb{O})}(\mathcal{G}r, \Bbbk), {}^p\!\star)$. We have constructed a fiber functor in Theorem 9.8.5, and we clearly have $\mathrm{End}(\mathrm{IC}_0) \cong \Bbbk$. Finally, let $\mathcal{F} \in \mathrm{Perv}_{G(\mathbb{O})}(\mathcal{G}r, \Bbbk)$ be an object such that $\dim \mathbf{H}^\bullet(\mathcal{G}r, \mathcal{F}) = 1$. By Lemma 9.6.14, we have $\mathcal{F} \cong \mathrm{IC}_\lambda$ where $\langle 2\rho, \lambda \rangle = 0$. The latter condition implies that $-\lambda$ is also a dominant coweight. We set $\mathcal{F}^{-1} = \mathrm{IC}_{-\lambda}$. By Lemma 9.2.6, we have $\mathcal{F} \,{}^p\!\star\, \mathcal{F}^{-1} \cong \mathrm{IC}_0$. □

Statement of the geometric Satake equivalence. Let $\check{G}_\Bbbk$ be the split reductive group scheme over $\Bbbk$ that is Langlands dual to G. This group comes with a maximal torus $\check{T}_\Bbbk$ whose character lattice is identified with $\mathbf{X}$. Let $\mathrm{Rep}(\check{G}_\Bbbk)$ be the category of algebraic $\check{G}_\Bbbk$-representations that are finitely generated over $\Bbbk$.

THEOREM 9.9.4 (The geometric Satake equivalence). *There is an equivalence of tensor categories*

$$\mathcal{S} : (\mathrm{Perv}_{G(\mathbb{O})}(\mathcal{G}r, \Bbbk), {}^p\!\star) \xrightarrow{\sim} (\mathrm{Rep}(\check{G}_\Bbbk), \otimes)$$

such that the following diagram commutes:

$$\begin{tikzcd}
(\mathrm{Perv}_{G(\mathbb{O})}(\mathcal{G}r, \Bbbk), {}^p\!\star) \arrow[rr, "\mathcal{S}", "\sim"'] \arrow[dr, "\mathbf{H}^\bullet(\mathcal{G}r, -)"'] & & (\mathrm{Rep}(\check{G}_\Bbbk), \otimes) \arrow[dl] \\
& (\Bbbk\text{-mod}^{\mathrm{fg}}, \otimes) &
\end{tikzcd}$$

We will not prove this theorem, but we will briefly discuss some of the ideas behind it. The proof involves two essentially separate tasks.

The first task is to show that there is *some* affine $\Bbbk$-group scheme H such that

$$(\mathrm{Perv}_{G(\mathbb{O})}(\mathcal{G}r, \Bbbk), {}^p\!\star) \cong (\mathrm{Rep}(H), \otimes). \tag{9.9.1}$$

When $\Bbbk$ is a field, this is an immediate consequence of Proposition 9.9.3 and Theorem 9.9.2. When $\Bbbk$ is not a field, Theorem 9.9.2 is not available. However, an examination of its proof shows that it could be carried out for general $\Bbbk$ if one knew that certain $\Bbbk$-modules arising at various intermediate steps were free. This can be ensured using Corollary 9.7.15. The details of the construction are carried out in [**179**, Section 11]. See also the exposition in [**21**, Section 13].

The second task is to identify the group H. We begin with the observation that the category $\mathrm{Rep}(\check{T}_\Bbbk)$ of algebraic $\check{T}_\Bbbk$-representations can be identified with the category of $\check{\mathbf{X}}$-graded finitely generated $\Bbbk$-modules. According to Theorem 9.6.10, for any $\mathcal{F} \in \mathrm{Perv}_{G(\mathbb{O})}(\mathcal{G}r, \Bbbk)$, the $\Bbbk$-module $\mathbf{H}^\bullet(\mathcal{G}r, \mathcal{F})$ comes with a canonical $\check{\mathbf{X}}$-grading, and so can be thought of as a functor

$$\mathrm{Perv}_{G(\mathbb{O})}(\mathcal{G}r, \Bbbk) \to \mathrm{Rep}(\check{T}_\Bbbk).$$

Combining this with (9.9.1), we obtain a functor $\mathrm{Rep}(H) \to \mathrm{Rep}(\check{T}_\Bbbk)$, and one can show that this is induced by a closed embedding $\check{T}_\Bbbk \hookrightarrow H$. This is an initial step towards showing that $\check{T}_\Bbbk$ is in fact a maximal torus of H. One must then show that H is a connected, reductive group scheme, and that its root datum is as expected. The argument is rather delicate and involves special focus on the case where $\Bbbk$ is a field. See [**179**, Section 12], and see [**21**, Section 14] for an expository account.

The following statement is a consequence of the construction (cf. Remark 9.6.8).

PROPOSITION 9.9.5. *Let $\mathcal{F} \in \mathrm{Perv}_{G(\mathbb{O})}(\mathcal{G}r, \Bbbk)$. For any $\nu \in \check{\mathbf{X}}$, the ν-weight space of $\mathcal{S}(\mathcal{F}) \in \mathrm{Rep}(\check{G}_\Bbbk)$ is identified with $\mathrm{F}_\nu(\mathcal{F})$. In particular, if $\mathcal{S}(\mathcal{F})$ has a nonzero ν-weight space, then we must have $\mathbf{t}^\nu \in \operatorname{supp} \mathcal{F}$.*

We conclude with a representation-theoretic interpretation of simple, standard, and costandard sheaves on $\mathcal{G}r$.

PROPOSITION 9.9.6. *Let $\Bbbk$ be a field, and let $\lambda \in \check{\mathbf{X}}^+$.*

(1) *$\mathcal{S}(\mathrm{IC}_\lambda(\Bbbk))$ is the irreducible $\check{G}_\Bbbk$-representation of highest weight λ.*
(2) *$\mathcal{S}(\mathcal{I}_!(\lambda, \Bbbk))$ is the Weyl module for $\check{G}_\Bbbk$ of highest weight λ.*
(3) *$\mathcal{S}(\mathcal{I}_*(\lambda, \Bbbk))$ is the dual Weyl module for $\check{G}_\Bbbk$ of highest weight λ.*

PROOF. By Proposition 9.9.5, the three $\check{G}_\Bbbk$-representations corresponding to $\mathrm{IC}_\lambda(\Bbbk)$, $\mathcal{I}_!(\lambda, \Bbbk)$, and $\mathcal{I}_*(\lambda, \Bbbk)$ must all have weights $\leq_{\check{\mathbf{X}}} \lambda$. Moreover, the proof of Lemma 9.6.7 shows in fact that the λ-weight space is nonzero. That is, all three of these representations have λ as their unique highest weight.

Since $\mathcal{S}$ is an equivalence of abelian categories, it must take the simple object $\mathrm{IC}_\lambda(\Bbbk)$ to a simple $\check{G}_\Bbbk$-representation, and hence to the irreducible representation L_λ of highest weight λ. Next, recall that the corresponding Weyl module V_λ is characterized by the following two properties:

(1) There is a surjective map $V_\lambda \to L_\lambda$ whose kernel has weights $<_{\check{\mathbf{X}}} \lambda$.
(2) For any representation M with weights $<_{\check{\mathbf{X}}} \lambda$, we have

$$\mathrm{Hom}(V_\lambda, M) = \mathrm{Ext}^1(V_\lambda, M) = 0.$$

These match the properties from Lemma 9.7.1 characterizing $\mathcal{I}_!(\lambda, \Bbbk)$, so we must have $\mathcal{S}(\mathcal{I}_!(\lambda, \Bbbk)) \cong V_\lambda$. Similar reasoning applies to $\mathcal{S}(\mathcal{I}_*(\lambda, \Bbbk))$. $\square$

9.10. Additional exercises

EXERCISE 9.10.1. Let $\Bbbk$ be a field. Show that for any $\mathcal{F} \in \mathrm{Perv}_{G(\mathbb{O})}(\mathcal{G}r, \Bbbk)$ and any $\mu \in \check{\mathbf{X}}^+$, we have

$$\dim \mathcal{S}(\mathcal{F})_\mu = \sum_{i \in \mathbb{Z}} (-1)^{i - \langle 2\rho, \mu \rangle} \operatorname{rank} \mathsf{H}^i(\mathcal{F}|_{\mathcal{G}r_\mu}),$$

where $\mathcal{S}(\mathcal{F})_\mu$ is the μ-weight space of the $\check{G}_\Bbbk$-representation $\mathcal{S}(\mathcal{F})$. *Hint*: When $\Bbbk = \mathbb{C}$ and $\mathcal{F}$ is simple, this is Theorem 9.4.3. Use that special case to prove the claim when $\mathcal{F}$ is a standard sheaf. Then show how to deduce the general case from the case of standard sheaves.

EXERCISE 9.10.2. Let $G = \mathrm{PGL}_2$, and identify $\check{\mathbf{X}}^+$ with the nonnegative integers $\mathbb{Z}_{\geq 0}$. Show that every $G(\mathbb{O})$-orbit in $\mathcal{G}r$ is rationally smooth, i.e., that

$$\mathrm{IC}_n(\mathbb{Q}) \cong \underline{\mathbb{Q}}_{\overline{\mathcal{G}r_n}}[n].$$

EXERCISE 9.10.3 (The Mirković–Vilonen conjecture for PGL_2). We continue to identify $\check{\mathbf{X}}^+$ for PGL_2 with $\mathbb{Z}_{\geq 0}$. In this exercise, you will prove that $\mathcal{I}_!(n, \mathbb{Z}) \cong \underline{\mathbb{Z}}_{\overline{\mathcal{G}r_n}}[n]$ for all $n \in \check{\mathbf{X}}^+$.

(a) Show that $\mathcal{G}r_1 = \overline{\mathcal{G}r_1} \cong \mathbb{P}^1$. Then, for any $n \geq 1$, give an explicit description in terms of lattices for the convolution space $\mathcal{G}r_1 \tilde{\times} \overline{\mathcal{G}r_n}$. Show that the convolution map

$$m : \mathcal{G}r_1 \tilde{\times} \overline{\mathcal{G}r_n} \to \mathcal{G}r \tag{9.10.1}$$

has image $\overline{\mathcal{G}r_{n+1}}$. Show also that it is an isomorphism over $\mathcal{G}r_{n+1}$ and that the preimage of $\overline{\mathcal{G}r_{n-1}}$ under (9.10.1) is isomorphic to $\mathbb{P}^1 \times \overline{\mathcal{G}r_{n-1}}$.

(b) Show by induction on n that $\underline{\mathbb{Z}}_{\overline{\mathcal{G}r_n}}[n]$ is a perverse sheaf.

(c) Show that $\mathcal{I}_!(n, \mathbb{Z}) \cong \underline{\mathbb{Z}}_{\overline{\mathcal{G}r_n}}[n]$.

The **Mirković–Vilonen conjecture** from [**179**, Conjecture 13.3] states that for any G, all stalks of $\mathcal{I}_!(\lambda, \mathbb{Z}) \cong \mathrm{IC}_\lambda(\mathbb{Z})$ are free abelian groups. (If this holds, their ranks are necessarily given by the polynomials $M_{\mu,\lambda}$ as in Corollary 9.3.12.) The conjecture turned out to be not quite correct: there are examples of torsion due to Juteau [**115**]. However, according to [**14, 173**], the stalks of $\mathcal{I}_!(\lambda, \mathbb{Z})$ have no p-torsion as long as p is a good prime for G.

EXERCISE 9.10.4. Use the geometric Satake equivalence to show that for $G = \mathrm{PGL}_2$, no $G(\mathbb{O})$-orbit closure $\overline{\mathcal{G}r_n}$ with $n \geq 2$ is smooth.

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CHAPTER 10

Quiver representations and quantum groups

In 1990, Ringel [**196**] discovered a remarkable way to construct (part of) a quantum group of simply laced type directly from its Dynkin diagram, using the "Hall algebra construction." Shortly thereafter, Lusztig observed that the Hall algebra can be understood in terms of functions on certain $\mathbb{F}_q$-varieties, called representation spaces. This hints at the possibility of upgrading Ringel's result via the sheaf–function correspondence. The main result of this chapter, due to Lusztig [**161**], states that the positive part of the quantum group is categorified by semisimple complexes on representation spaces. As an immediate consequence, one obtains a new basis for the positive part of the quantum group, called the **canonical basis**: it consists of elements corresponding to simple perverse sheaves on representation spaces.

This chapter, which is based mostly on [**161**], gives a brief look at a subject on which there is by now a vast literature. There are other approaches to canonical bases (and the closely related crystal bases) due to Kashiwara [**122**], Khovanov–Lauda [**134**, **135**] and Rouquier [**197**] (see also [**237**]), as well as a second, algebraic approach contained in [**161**] (cf. Exercise 10.5.1). Sources for further reading on this topic include [**48**, **103**].

10.1. Quiver representations

This section contains background material on quivers and their representations, on representation spaces, and on Gabriel's theorem.

Definition 10.1.1. A **quiver** is a quadruple $Q = (Q_0, Q_1, \mathrm{head}, \mathrm{tail})$, where Q_0 and Q_1 are finite sets, and $\mathrm{head}, \mathrm{tail} : Q_1 \to Q_0$ are functions. The set Q_0 is called the set of **vertices** of Q, and the set Q_1 is called the set of **edges**. An **oriented cycle** is a sequence of edges $e_1, \ldots, e_n \in Q_1$ such that $\mathrm{head}(e_i) = \mathrm{tail}(e_{i+1})$ for $i = 1, \ldots, n-1$, and $\mathrm{head}(e_n) = \mathrm{tail}(e_1)$.

A **representation** of Q over a field $\mathbb{F}$ is a rule V that assigns to each vertex $i \in Q_0$ a finite-dimensional vector space V_i, and to each edge $e \in Q_1$ a linear map $V_e : V_{\mathrm{tail}(e)} \to V_{\mathrm{head}(e)}$. The **dimension vector of** V is the function

$$\underline{\dim}\, V : Q_0 \to \mathbb{Z}_{\geq 0} \qquad \text{given by} \qquad (\underline{\dim}\, V)(i) = \dim V_i.$$

A **morphism** of quiver representations $\phi : V \to V'$ is a collection of maps $\phi_i : V_i \to V'_i$ for each $i \in Q_0$ such that for any edge e, the following diagram commutes:

$$\begin{tikzcd} V_{\mathrm{tail}(e)} \arrow[r, "\phi_{\mathrm{tail}(e)}"] \arrow[d, "V_e"'] & V'_{\mathrm{tail}(e)} \arrow[d, "V'_e"] \\ V_{\mathrm{head}(e)} \arrow[r, "\phi_{\mathrm{head}(e)}"] & V'_{\mathrm{head}(e)} \end{tikzcd}$$

The category of representations of a quiver Q over $\mathbb{F}$ is denoted by $\operatorname{Rep}(Q, \mathbb{F})$.

In practice, we often omit the functions head and tail from the notation, and just write $Q = (Q_0, Q_1)$. If $e \in Q_1$ is an edge with tail i and head j, we usually draw e as an arrow $i \to j$.

Note that we have built finite-dimensionality into the definition of a quiver representation. Here are some basic facts about quiver representations. Proofs can be found in numerous sources; see, for instance, [**66**, Chapters 1 and 2] or [**203**, Sections 1.3, 2.1, and 2.2].

PROPOSITION 10.1.2. *The category* $\operatorname{Rep}(Q, \mathbb{F})$ *is an abelian category. Every object has finite length.*

PROPOSITION 10.1.3. *Let* $Q = (Q_0, Q_1)$ *be a quiver. For each vertex* $i \in Q_0$*, there is a simple object* $S_i \in \operatorname{Rep}(Q, \mathbb{F})$ *given by*

$$(S_i)_j = \begin{cases} \mathbb{F} & \text{if } i = j \in Q_0, \\ 0 & \text{if } j \in Q_0 \text{ and } j \neq i, \end{cases} \qquad \text{and} \qquad (S_i)_e = 0 \quad \text{for all } e \in Q_1.$$

The simple objects S_i *are pairwise nonisomorphic. If* Q *has no oriented cycles, then every simple object in* $\operatorname{Rep}(Q, \mathbb{F})$ *is isomorphic to some* S_i.

PROPOSITION 10.1.4. *Let* Q *be a quiver with no oriented cycles.*

(1) *The category* $\operatorname{Rep}(Q, \mathbb{F})$ *has enough projectives. Any subobject of a projective object is projective.*
(2) *The category* $\operatorname{Rep}(Q, \mathbb{F})$ *has enough injectives. Any quotient of an injective object is injective.*
(3) *The category* $\operatorname{Rep}(Q, \mathbb{F})$ *has global dimension* ≤ 1.

The set of all possible dimension vectors, i.e., the set of all functions $\mathbf{v} : Q_0 \to \mathbb{Z}_{\geq 0}$, is denoted by $\mathbb{Z}_{\geq 0}^{Q_0}$. We will sometimes consider the larger set $\mathbb{Z}^{Q_0}$ consisting of all functions $\mathbf{v} : Q_0 \to \mathbb{Z}$.

DEFINITION 10.1.5. Let $Q = (Q_0, Q_1)$ be a quiver with no oriented cycles. The **Euler–Poincaré pairing** is the bilinear form $\langle -, - \rangle$ on the set $\mathbb{Z}^{Q_0}$ given by

$$\langle \mathbf{v}, \mathbf{w} \rangle = \sum_{i \in Q_0} \mathbf{v}_i \mathbf{w}_i - \sum_{(i \to j) \in Q_1} \mathbf{v}_i \mathbf{w}_j.$$

The following statement gives a representation-theoretic interpretation of this pairing. For a proof, see [**66**, Proposition 2.5.2] or [**203**, Proposition 8.4].

PROPOSITION 10.1.6. *Let* Q *be a quiver with no oriented cycles, and let* V *and* W *be representations of* Q *over* $\mathbb{F}$*. Then* $\operatorname{Hom}(V, W)$ *and* $\operatorname{Ext}^1(V, W)$ *are finite-dimensional* $\mathbb{F}$*-vector spaces, and*

$$\dim \operatorname{Hom}(V, W) - \dim \operatorname{Ext}^1(V, W) = \langle \underline{\dim}\, V, \underline{\dim}\, W \rangle.$$

For each vertex $i \in Q_0$, we let $\mathbf{e}_i$ be the dimension vector given by

$$\mathbf{e}_i = \underline{\dim}\, S_i \qquad \text{or} \qquad \mathbf{e}_i(j) = \delta_{ij}, \tag{10.1.1}$$

where δ_{ij} is the Kronecker delta. The set $\{\mathbf{e}_i\}_{i \in Q_0}$ forms a basis for the free abelian group $\mathbb{Z}^{Q_0}$. It is immediate from the definition that

$$\langle \mathbf{e}_i, \mathbf{e}_j \rangle = \delta_{ij} - |\{\text{edges } i \to j \text{ in } Q_1\}|. \tag{10.1.2}$$

For two simple representations S_i, S_j, we obviously have $\dim \operatorname{Hom}(S_i, S_j) = \delta_{ij}$, so as a special case of Proposition 10.1.6, we have

$$\dim \operatorname{Ext}^1(S_i, S_j) = |\{\text{edges } i \to j \text{ in } Q_1\}|. \tag{10.1.3}$$

Representation spaces. Let $Q = (Q_0, Q_1)$ be a quiver, and let $\mathbb{F}$ be a field. Let $\mathbf{v} : Q_0 \to \mathbb{Z}_{\geq 0}$ be a dimension vector. Let $\mathcal{E}_{\mathbf{v}}(Q)$ or $\mathcal{E}_{\mathbf{v}}$ be the variety given by

$$\mathcal{E}_{\mathbf{v}}(Q) = \mathcal{E}_{\mathbf{v}} = \prod_{(i \to j) \in Q_1} \operatorname{Hom}(\mathbb{F}^{\mathbf{v}_i}, \mathbb{F}^{\mathbf{v}_j}).$$

This variety is called the **representation space** of Q of dimension vector $\mathbf{v}$. Of course, it is just an affine space of dimension $\sum_{(i \to j) \in Q_1} \mathbf{v}_i \mathbf{v}_j$. Any point $x = (x_e)_{e \in Q_1}$ of $\mathcal{E}_{\mathbf{v}}$ determines a representation of Q (assigning to each vertex i the vector space $\mathbb{F}^{\mathbf{v}_i}$, and assigning to each edge e the linear map x_e).

It is easy to see that any representation of dimension vector $\mathbf{v}$ is isomorphic to one obtained from a point of $\mathcal{E}_{\mathbf{v}}$, but this point is usually not unique. To describe the choice involved, we introduce the group

$$G_{\mathbf{v}} = \prod_{i \in Q_0} \mathrm{GL}_{\mathbf{v}_i}(\mathbb{F}).$$

Let $G_{\mathbf{v}}$ act on $\mathcal{E}_{\mathbf{v}}$ by the formula

$$(g_i)_{i \in Q_0} \cdot (x_e)_{e \in Q_1} = \left(g_{\mathrm{head}(e)} \circ x_e \circ g^{-1}_{\mathrm{tail}(e)} \right)_{e \in Q_1}.$$

Two points $x, y \in \mathcal{E}_{\mathbf{v}}$ give rise to isomorphic quiver representations if and only if they lie in the same $G_{\mathbf{v}}$-orbit. Thus, there is a canonical bijection

$$\left\{ \begin{matrix} \text{isomorphism classes of representations} \\ \text{of dimension vector } \mathbf{v} \end{matrix} \right\} \overset{\sim}{\longleftrightarrow} \left\{ \begin{matrix} G_{\mathbf{v}}\text{-orbits} \\ \text{in } \mathcal{E}_{\mathbf{v}} \end{matrix} \right\}. \tag{10.1.4}$$

Below are a couple of lemmas on the geometry of $\mathcal{E}_{\mathbf{v}}$.

LEMMA 10.1.7. *Let $Q = (Q_0, Q_1)$ be a quiver, and let $\mathbf{v} : Q_0 \to \mathbb{Z}_{\geq 0}$ be a dimension vector. For any point $x \in \mathcal{E}_{\mathbf{v}}$, the stabilizer $G^x_{\mathbf{v}}$ is connected.*

PROOF. Let V be the quiver representation corresponding to x. Its endomorphism ring as an object of $\operatorname{Rep}(Q, \mathbb{F})$ is the vector space

$$\operatorname{End}(V) = \{ (\phi_i : \mathbb{F}^{\mathbf{v}_i} \to \mathbb{F}^{\mathbf{v}_i})_{i \in Q_0} \mid \phi_j x_e = x_e \phi_i \text{ for all } e = (i \to j) \in Q_1 \}.$$

The stabilizer $G^x_{\mathbf{v}}$ is then the (Zariski) open subset of $\operatorname{End}(V)$ consisting of points $(\phi_i)_{i \in Q_0}$ where every ϕ_i is invertible. The stabilizer is nonempty, so it is dense in the irreducible variety $\operatorname{End}(V)$, and hence connected. □

LEMMA 10.1.8. *Let $Q = (Q_0, Q_1)$ be a quiver, and let $\mathbf{v} : Q_0 \to \mathbb{Z}_{\geq 0}$ be a dimension vector.*

(1) *We have* $\dim G_{\mathbf{v}} - \dim \mathcal{E}_{\mathbf{v}} = \langle \mathbf{v}, \mathbf{v} \rangle$.
(2) *Let V be a representation of dimension vector $\mathbf{v}$, and let $\mathcal{O} \subset \mathcal{E}_{\mathbf{v}}$ be the corresponding $G_{\mathbf{v}}$-orbit. Then*

$$\dim \mathcal{E}_{\mathbf{v}} - \dim \mathcal{O} = \dim \operatorname{End}(V) - \langle \mathbf{v}, \mathbf{v} \rangle.$$

PROOF SKETCH. Part (1) is an easy calculation from the definitions. Part (2) follows from part (1) and the observation that for $x \in \mathcal{O}$, we have $\dim \mathcal{O} = \dim G_{\mathbf{v}} - \dim G^x_{\mathbf{v}} = \dim G_{\mathbf{v}} - \dim \operatorname{End}(V)$. □

Gabriel's theorem. We next discuss the classification of quivers of **finite type**, i.e., quivers Q such that $\mathrm{Rep}(Q, \mathbb{F})$ has finitely many isomorphism classes of indecomposable objects. The following result is due to Gabriel [**78**]. For a textbook-level treatment of it, see, for instance, [**66**, Theorem 4.4.13] or [**203**, Theorem 8.12].

THEOREM 10.1.9 (Gabriel's theorem). (1) *Let Q be a quiver. It is of finite type if and only if its underlying undirected graph is a simply-laced Dynkin diagram. In other words, each connected component of the underlying undirected graph must be one of the following:*

$$A_n : \quad \bullet - \bullet - \cdots - \bullet - \bullet$$

$$D_n : \quad \begin{array}{cccccccc} & & & & & & & \bullet \\ \bullet & - & \bullet & - & \cdots & - & \bullet & \\ & & & & & & & \bullet \end{array}$$

$$E_6 : \quad \begin{array}{ccccccccc} & & & & \bullet & & & & \\ & & & & | & & & & \\ \bullet & - & \bullet & - & \bullet & - & \bullet & - & \bullet \end{array}$$

$$E_7 : \quad \begin{array}{ccccccccccc} & & & & \bullet & & & & & & \\ & & & & | & & & & & & \\ \bullet & - & \bullet & - & \bullet & - & \bullet & - & \bullet & - & \bullet \end{array}$$

$$E_8 : \quad \begin{array}{ccccccccccccc} & & & & \bullet & & & & & & & & \\ & & & & | & & & & & & & & \\ \bullet & - & \bullet & - & \bullet & - & \bullet & - & \bullet & - & \bullet & - & \bullet \end{array}$$

(2) *Assume Q is of finite type, and let Φ be the corresponding root system. If $V \in \mathrm{Rep}(Q, \mathbb{F})$ is indecomposable, then $\underline{\dim}\, V \in \mathbb{Z}_{\geq 0}^{Q_0}$ is a positive root, and the map $V \mapsto \underline{\dim}\, V$ gives a bijection*

$$\left\{ \begin{array}{c} \textit{isomorphism classes of} \\ \textit{indecomposable representations of } Q \end{array} \right\} \xleftrightarrow{\sim} \Phi^+.$$

To make sense of part (2), we must explain how to regard roots as dimension vectors. With Φ as above, let $\Pi \subset \Phi$ be the set of simple roots, and let $\Phi^+ \subset \Phi$ be the corresponding set of positive roots. There is a canonical bijection

$$Q_0 \cong \Pi.$$

For $i \in Q_0$, we identify the corresponding simple root with the element $\mathbf{e}_i \in \mathbb{Z}_{\geq 0}^{Q_0}$ from (10.1.1). Since every root (resp. positive root) is a linear combination (resp. nonnegative linear combination) of simple roots, this identification extends to inclusions

$$\Phi \subset \mathbb{Z}^{Q_0}, \qquad \text{resp.} \qquad \Phi^+ \subset \mathbb{Z}_{\geq 0}^{Q_0}.$$

In the course of the proof of Theorem 10.1.9, one shows that if V is an indecomposable representation of a quiver of finite type, then

$$\dim \mathrm{Ext}^1(V, V) = 0. \tag{10.1.5}$$

For each $\gamma \in \Phi^+$, we let

$M(\gamma) =$ the indecomposable representation of Q with dimension vector γ.

List the elements of Φ^+ as $\gamma_1, \ldots, \gamma_N$. As a consequence of Theorem 10.1.9 and the fact that $\mathrm{Rep}(Q, \mathbb{F})$ is a Krull–Schmidt category (see Example A.8.6(1)), every representation V can be written as

$$V \cong M(\gamma_1)^{\oplus c_1} \oplus \cdots \oplus M(\gamma_N)^{\oplus c_N}$$

for a uniquely determined collection of nonnegative integers $c_1, \ldots, c_N$. We capture this observation with the following notion.

Definition 10.1.10. Let Q be a quiver of finite type, and let Φ be the corresponding root system. A function $\mathbf{c} : \Phi^+ \to \mathbb{Z}_{\geq 0}$ is called a **Kostant partition** of $\mathbf{v} : Q_0 \to \mathbb{Z}_{\geq 0}$ if

$$\sum_{\gamma \in \Phi^+} \mathbf{c}(\gamma)\gamma = \mathbf{v}.$$

The set of Kostant partitions of $\mathbf{v}$ is denoted by $\mathrm{KP}(\mathbf{v})$. The set of all Kostant partitions is denoted by

$$\mathrm{KP} = \bigcup_{\mathbf{v} \in \mathbb{Z}_{\geq 0}^{Q_0}} \mathrm{KP}(\mathbf{v}).$$

We typically write down Kostant partitions by listing their values as exponents:

$$\mathbf{c} = [\gamma_1^{\mathbf{c}(\gamma_1)}, \gamma_2^{\mathbf{c}(\gamma_2)}, \dots, \gamma_N^{\mathbf{c}(\gamma_N)}].$$

For brevity, we may omit roots on which $\mathbf{c}$ takes the value 0. For $\mathbf{c} \in \mathrm{KP}(\mathbf{v})$, let

$$M(\mathbf{c}) = \bigoplus_{\gamma \in \Phi^+} M(\gamma)^{\oplus \mathbf{c}(\gamma)}.$$

Gabriel's theorem implies that the assignment $\mathbf{c} \mapsto M(\mathbf{c})$ gives a bijection

$$\mathrm{KP}(\mathbf{v}) \xleftrightarrow{\sim} \left\{ \begin{matrix} \text{isomorphism classes of representations} \\ \text{of dimension vector } \mathbf{v} \end{matrix} \right\}.$$

By (10.1.4), the set $\mathrm{KP}(\mathbf{v})$ also parametrizes $G_{\mathbf{v}}$-orbits in $\mathscr{E}_{\mathbf{v}}$. Given $\mathbf{c} \in \mathrm{KP}(\mathbf{v})$, denote the corresponding orbit by

$$\mathscr{O}_{\mathbf{c}} \subset \mathscr{E}_{\mathbf{v}}.$$

Lemma 10.1.11. *Let $Q = (Q_0, Q_1)$ be a quiver of finite type, and let $\mathbf{v} : Q_0 \to \mathbb{Z}_{\geq 0}$ be a dimension vector. Let $\mathbf{c} : \Phi^+ \to \mathbb{Z}_{\geq 0}$ be a Kostant partition of $\mathbf{v}$ such that $\mathbf{c}(\gamma) \neq 0$ for exactly one positive root $\gamma \in \Phi^+$. Then $\mathscr{O}_{\mathbf{c}}$ is dense in $\mathscr{E}_{\mathbf{v}}$.*

Proof. By assumption, $M(\mathbf{c}) = M(\gamma)^{\oplus \mathbf{c}(\gamma)}$, so it follows from (10.1.5) that $\mathrm{Ext}^1(M(\mathbf{c}), M(\mathbf{c})) = 0$. Then Proposition 10.1.6 tells us that $\dim \mathrm{End}(M(\mathbf{c})) = \langle \mathbf{v}, \mathbf{v} \rangle$. By Lemma 10.1.8(2), we have $\dim \mathscr{E}_{\mathbf{v}} = \dim \mathscr{O}_{\mathbf{c}}$, so $\mathscr{O}_{\mathbf{c}}$ is dense in $\mathscr{E}_{\mathbf{v}}$. ☐

Adapted orders. The remainder of this section is devoted to Hom- and Ext-calculations for quivers of finite type, based on the following key fact.

Proposition 10.1.12. *Let Q be a quiver of finite type, and let Φ be the corresponding root system. Then Φ^+ admits an ordering $\gamma_1, \dots, \gamma_N$ such that*

$$\mathrm{Hom}(M(\gamma_m), M(\gamma_n)) = 0 \qquad \text{if } m > n.$$

Remark 10.1.13. A total order on Φ^+ like that in Proposition 10.1.12 is called an **adapted order** (for Q). We will not prove this proposition, but we will briefly discuss two approaches to it. One approach is related to the theory of "reflection functors" on $\mathrm{Rep}(Q, \mathbb{F})$, and to the combinatorics of the Weyl group associated to Φ. In this approach one first shows that the longest element w_0 admits a reduced expression that is "adapted" in a certain sense to the quiver Q, and then uses that reduced expression to define the order $\gamma_1, \dots, \gamma_N$. This approach is carried out in [**161**, Proposition 4.12(c)].

A second approach is based on Auslander–Reiten theory. Proposition 10.1.12 can be deduced as a corollary of the following claim: if $M_1, M_2, \ldots, M_n$ are distinct indecomposable representations of Q, then there is no diagram of morphisms

$$M_1 \to M_2 \to \cdots \to M_n \to M_1$$

in which all the maps are nonzero. This claim is equivalent to the assertion that the Auslander–Reiten quiver associated to $\mathrm{Rep}(Q, \mathbb{F})$ has no cycles. For a proof of this claim, see [**65**, Corollary 3.32] or [**66**, Section 6.6].

Corollary 10.1.14. *Let Q be a quiver of finite type. Choose an adapted order $\gamma_1, \ldots, \gamma_N$ on the corresponding set of positive roots Φ^+. Then*

$$\mathrm{Ext}^1(M(\gamma_m), M(\gamma_n)) = 0 \qquad \textit{if } m \leq n.$$

Proof. For $m = n$, this claim emerges from the proof of Theorem 10.1.9 (see (10.1.5)). Suppose now that $m < n$ and that we have a nonsplit short exact sequence

$$0 \to M(\gamma_n) \xrightarrow{f} M \xrightarrow{g} M(\gamma_m) \to 0. \tag{10.1.6}$$

Choose a decomposition $M \cong \bigoplus_{i=1}^k M(\gamma_{r_i})$ of M into indecomposable summands. For each i, there are induced maps $f_i : M(\gamma_n) \to M(\gamma_{r_i})$ and $g_i : M(\gamma_{r_i}) \to M(\gamma_m)$. There must be some i such that f_i and g_i are both nonzero; otherwise, (10.1.6) would split. But then, by the definition of an adapted order, we have $n \leq r_i \leq m$, a contradiction. ☐

Corollary 10.1.15. *Let Q be a quiver of finite type. Let $\gamma_1, \ldots, \gamma_N$ be an adapted order on the corresponding set of positive roots Φ^+. List the simple roots in this order as $\mathbf{e}_1, \ldots, \mathbf{e}_n$, and identify Q_0 with $\{1, \ldots, n\}$. If there is an edge $i \to j$, then $i > j$.*

Proof. If there is an edge $i \to j$, we see from (10.1.3) that $\mathrm{Ext}^1(S_i, S_j) \neq 0$, and then Corollary 10.1.14 implies that $i > j$. ☐

Lemma 10.1.16. *Let Q be a quiver of finite type. Let $\gamma_1, \ldots, \gamma_N$ be an adapted order on the corresponding set of positive roots Φ^+, and let $\mathbf{c} : \Phi^+ \to \mathbb{Z}_{\geq 0}$ be a Kostant partition. For $V \in \mathrm{Rep}(Q, \mathbb{F})$, the following conditions are equivalent:*

1. *V admits a decreasing filtration $V = V^1 \supset V^2 \supset \cdots \supset V^N \supset V^{N+1} = 0$ such that $V^t/V^{t+1} \cong M(\gamma_t)^{\oplus \mathbf{c}(\gamma_t)}$ for $t = 1, \ldots, N$.*
2. *V is isomorphic to $M(\mathbf{c})$.*

Moreover, if these conditions hold, then the filtration in part (1) *is unique.*

Proof. The lemma is trivial if $\mathbf{c} = 0$, so assume that $\mathbf{c}$ is nonzero, and let m be the largest integer such that $\mathbf{c}(\gamma_m) \neq 0$. Then, for any filtration as in part (1), we have $V^m \cong M(\gamma_m)^{\oplus \mathbf{c}(\gamma_m)}$, and $V^{m+1} = V^{m+2} = \cdots = V^{N+1} = 0$. Both directions of the proof proceed by induction on m.

Suppose first that V admits a filtration as in part (1). We must show that this filtration splits. Consider the short exact sequence

$$0 \to V^m \to V \to V/V^m \to 0. \tag{10.1.7}$$

By induction, the induced filtration on V/V^m splits; in other words, there is an isomorphism $V/V^m \cong \bigoplus_{t=1}^{m-1} M(\gamma_t)^{\oplus \mathbf{c}(\gamma_t)}$. Then Corollary 10.1.14 implies that $\mathrm{Ext}^1(V/V^m, V^m) = 0$, so (10.1.7) splits, as desired.

Conversely, if $V \cong M(\mathbf{c})$, then the existence of a filtration as in part (1) is obvious. Let us prove that it is unique. We claim that

$$V^m = \text{the sum of the images of all maps } M(\gamma_m) \to V.$$

It is obvious that V^m is contained in the right-hand side. For the opposite containment, suppose there were some map $M(\gamma_m) \to V$ whose image was not contained in V^m. It would give rise to a nonzero map $M(\gamma_m) \to V/V^m$. But the latter is an extension of copies of $M(\gamma_1), \ldots, M(\gamma_{m-1})$, so by the definition of an adapted order, there is no nonzero map $M(\gamma_m) \to V/V^m$.

We have shown that V^m is uniquely determined. By induction, the filtration on V/V^m is also uniquely determined. Lifting that filtration to V via (10.1.7), we find that $V^1, \ldots, V^{m-1}$ are uniquely determined as well. □

LEMMA 10.1.17. *Let Q be a quiver of finite type. Let $\gamma_1, \ldots, \gamma_N$ be an adapted order on the corresponding set of positive roots Φ^+. List the simple roots in this order as $\mathbf{e}_1, \ldots, \mathbf{e}_n$, and identify Q_0 with $\{1, \ldots, n\}$. Every representation V of dimension vector $\mathbf{v} : Q_0 \to \mathbb{Z}_{\geq 0}$ admits a unique filtration*

$$V = V^n \supset V^{n-1} \supset \cdots \supset V^1 \supset V^0 = 0$$

such that $V^i/V^{i-1} \cong S_i^{\mathbf{v}_i}$.

PROOF. It is clear that there is a unique collection of vector subspaces of V with the appropriate dimension vectors: we define V^i by setting

$$V^i_j = \begin{cases} V_j & \text{if } j \leq i, \\ 0 & \text{if } j > i. \end{cases}$$

To finish the proof, we must check that each V^i is actually a subrepresentation. This follows from the fact (Corollary 10.1.15) that there is no edge $(j \to k) \in Q_1$ with $j \leq i < k$. □

Exercises.

10.1.1. Let $0 \to M' \to M \to M'' \to 0$ be a nonsplit short exact sequence of quiver representations, and let $\mathbf{v} = \underline{\dim}\, M$. Show that the $G_{\mathbf{v}}$-orbit corresponding to $M' \oplus M''$ is contained in the closure of the $G_{\mathbf{v}}$-orbit corresponding to M.

10.2. Hall algebras and quantum groups

This section contains the definitions of the Hall algebra and of the (positive part of the) quantum group, as well as the statement of Ringel's theorem on the relationship between them.

Hall algebras. As in Section A.9, we denote by $\mathrm{Isom}(\mathrm{Rep}(Q, \mathbb{F}))$ the set of isomorphism classes in $\mathrm{Rep}(\mathbb{Q}, \mathbb{F})$. For $V \in \mathrm{Rep}(\mathbb{Q}, \mathbb{F})$, let $[V]$ denote its class in $\mathrm{Isom}(\mathrm{Rep}(Q, \mathbb{F}))$.

DEFINITION 10.2.1. Let Q be a quiver with no oriented cycles, and let q be a prime power. The **Hall algebra** of Q over $\mathbb{F}_q$, denoted by $\mathcal{H}_Q(\mathbb{F}_q)$, is the free $\mathbb{Z}[q^{\pm\frac{1}{2}}]$-module with basis $\mathrm{Isom}(\mathrm{Rep}(Q, \mathbb{F}_q))$, and with multiplication given by

$$[V_1][V_2] = q^{\frac{1}{2}\langle \underline{\dim}\, V_1, \underline{\dim}\, V_2 \rangle} \sum_{[W] \in \mathrm{Isom}(\mathrm{Rep}(Q, \mathbb{F}_q))} F^{[W]}_{[V_1],[V_2]} [W], \tag{10.2.1}$$

where

$$F^{[W]}_{[V_1],[V_2]} = \begin{array}{c}\text{the number of subrepresentations } W' \subset W \\ \text{such that } W' \cong V_2 \text{ and } W/W' \cong V_1.\end{array}$$

For the formula (10.2.1) to make sense, we must check that the number of isomorphism classes $[W]$ with $F^{[W]}_{[V_1],[V_2]} \neq 0$ is finite. If $F^{[W]}_{[V_1],[V_2]} \neq 0$, then there exists a short exact sequence

$$0 \to V_2 \to W \to V_1 \to 0, \tag{10.2.2}$$

and this determines an element in $\mathrm{Ext}^1(V_1, V_2)$. The latter is a finite-dimensional vector space over $\mathbb{F}_q$ (see Proposition 10.1.4), so in particular, it is a finite set. Since there are only finitely many isomorphism classes of short exact sequences as in (10.2.2), there are only finitely many $[W]$ with $F^{[W]}_{[V_1],[V_2]} \neq 0$.

LEMMA 10.2.2. *The Hall algebra $\mathcal{H}_Q(\mathbb{F}_q)$ is an associative ring.*

PROOF SKETCH. Given quiver representations V_1, V_2, V_3, and W, we define

$$F^{[W]}_{[V_1],[V_2],[V_3]} = \begin{array}{c}\text{the number of filtrations } W'' \subset W' \subset W \\ \text{such that } W'' \cong V_3,\ W'/W'' \cong V_2, \text{ and } W/W' \cong V_1.\end{array}$$

One can then show that both $([V_1][V_2])[V_3]$ and $[V_1]([V_2][V_3])$ are equal to

$$q^{\frac{1}{2}(\langle \underline{\dim} V_1, \underline{\dim} V_2\rangle + \langle \underline{\dim} V_1, \underline{\dim} V_3\rangle + \langle \underline{\dim} V_2, \underline{\dim} V_3\rangle)} \sum_{[W] \in \mathrm{Isom}(\mathrm{Rep}(Q,\mathbb{F}_q))} F^{[W]}_{[V_1],[V_2],[V_3]}[W].$$

Further details are left to the reader. □

The idea of the proof of Lemma 10.2.2 can be further generalized as follows: given any sequence of quiver representations $V_1, \ldots, V_k$, let

$$F^{[W]}_{[V_1],\ldots,[V_k]} = \begin{array}{c}\text{the number of decreasing filtrations} \\ W = W^1 \supset W^2 \supset \cdots \supset W^k \supset W^{k+1} = 0 \\ \text{such that } W^t/W^{t+1} \cong V_t \text{ for } t = 1, \ldots, k.\end{array}$$

One can then show that

$$[V_1] \cdots [V_k] = q^{\frac{1}{2}\sum_{1 \le s < t \le k} \langle \underline{\dim} V_s, \underline{\dim} V_t\rangle} \sum_{[W] \in \mathrm{Isom}(\mathrm{Rep}(Q,\mathbb{F}_q))} F^{[W]}_{[V_1],\ldots,[V_k]}[W]. \tag{10.2.3}$$

REMARK 10.2.3. When Q is a quiver of finite type, we saw in Section 10.1 that the isomorphism classes of representations are in bijection with Kostant partitions. In particular, the isomorphism classes have a parametrization that is independent of the field $\mathbb{F}_q$. One can ask whether the Hall algebra can be described in a way that is "independent of $\mathbb{F}_q$" in some sense.

An affirmative answer was obtained by Ringel in [**195**]. The main result of that paper can be stated as follows: there exists a family of polynomials

$$\mathbf{F}^{\mathbf{d}}_{\mathbf{c}_1,\mathbf{c}_2}(t) \in \mathbb{Z}[t], \qquad \mathbf{c}_1, \mathbf{c}_2, \mathbf{d} \in \mathrm{KP},$$

indexed by triples of Kostant partitions such that for any finite field $\mathbb{F}_q$, the coefficients in (10.2.1) are given by

$$F^{[M(\mathbf{d})]}_{[M(\mathbf{c}_1)],[M(\mathbf{c}_2)]} = \mathbf{F}^{\mathbf{d}}_{\mathbf{c}_1,\mathbf{c}_2}(q).$$

The polynomials $\mathbf{F}^{\mathbf{d}}_{\mathbf{c}_1,\mathbf{c}_2}(t)$ are called **Hall polynomials**.

One can then define a **generic Hall algebra** as follows: let v be an indeterminate, and let $\mathcal{H}_{Q,v}$ be the free $\mathbb{Z}[v, v^{-1}]$-module with a basis consisting of symbols $\{[M(\mathbf{c})] \mid \mathbf{c} \in \mathrm{KP}\}$. Make $\mathcal{H}_{Q,v}$ into an algebra by setting

$$[M(\mathbf{c}_1)][M(\mathbf{c}_2)] = v^{\langle \mathbf{v}_1, \mathbf{v}_2 \rangle} \sum_{\mathbf{d} \in \mathrm{KP}} \mathbf{F}^{\mathbf{d}}_{\mathbf{c}_1,\mathbf{c}_2}(v^2)[M(\mathbf{d})],$$

where $\mathbf{c}_1 \in \mathrm{KP}(\mathbf{v}_1)$ and $\mathbf{c}_2 \in \mathrm{KP}(\mathbf{v}_2)$. Then, for every finite field $\mathbb{F}_q$, we have $\mathcal{H}_Q(\mathbb{F}_q) \cong \mathcal{H}_{Q,v}/(v - q^{\frac{1}{2}})$. For additional remarks on the generic Hall algebra, see the discussion following Theorem 10.2.9.

The next few results deal with computations in the Hall algebra.

PROPOSITION 10.2.4. *Let $Q = (Q_0, Q_1)$ be a quiver with no oriented cycles.*

(1) *For any vertex $i \in Q_0$, we have*

$$[S_i]^n = q^{\frac{n(n-1)}{2}} \prod_{t=1}^{n} \frac{q^{t/2} - q^{-t/2}}{q^{1/2} - q^{-1/2}} [S_i^{\oplus n}].$$

(2) *Let $i, j \in Q_0$ be two vertices. If there is no edge $i \to j$ or $j \to i$ in Q_1, then*

$$[S_i][S_j] = [S_j][S_i].$$

If there is exactly one edge $i \to j$ or $j \to i$, then

$$[S_i^{\oplus 2}][S_j] - [S_i][S_j][S_i] + [S_j][S_i^{\oplus 2}] = 0.$$

PROOF SKETCH. We will briefly outline the proof of part (1). The proof of part (2) is left as an exercise. Recall that $\mathrm{Ext}^1(S_i, S_i) = 0$ (see (10.1.3)). Using this, it is easy to see that up to isomorphism, the only quiver representation V with a filtration $V = V^n \supset V^{n-1} \supset \cdots \supset V^1 \supset V^0 = 0$ with $V^t/V^{t-1} \cong S_i$ for all t is just $V \cong S_i^{\oplus n}$. The number of such filtrations is just the number of complete flags in an $\mathbb{F}_q$-vector space of dimension n, so by (10.2.3), we have

$$[S_i]^n = q^{\frac{n(n-1)}{4}} (\text{number of complete flags in } \mathbb{F}_q^n)[S_i^{\oplus n}].$$

It is left to the reader to check that the number of complete flags in $\mathbb{F}_q^n$ is given by

$$\prod_{t=1}^{n} \frac{q^t - 1}{q - 1},$$

and that the resulting expression for $[S_i]^n$ reduces to the one in the statement of the proposition. $\square$

PROPOSITION 10.2.5. *Let Q be a quiver of finite type, and let $\gamma_1, \ldots, \gamma_N$ be an adapted order on the corresponding set of positive roots Φ^+. For any function $\mathbf{c} : \Phi^+ \to \mathbb{Z}_{\geq 0}$, we have*

$$[M(\gamma_1)^{\oplus \mathbf{c}(\gamma_1)}][M(\gamma_2)^{\oplus \mathbf{c}(\gamma_2)}] \cdots [M(\gamma_N)^{\oplus \mathbf{c}(\gamma_N)}] = q^{\frac{1}{2} \sum_{1 \leq s < t \leq N} \langle \mathbf{c}(\gamma_s)\gamma_s, \mathbf{c}(\gamma_t)\gamma_t \rangle} [M(\mathbf{c})].$$

PROOF. This follows immediately from Lemma 10.1.16 and the description of multiplication given in (10.2.3). $\square$

PROPOSITION 10.2.6. *Let Q be a quiver of finite type. Let $\gamma_1, \ldots, \gamma_N$ be an adapted order on the corresponding set of positive roots Φ^+. List the simple roots* *in this order as* $\mathbf{e}_1, \dots, \mathbf{e}_n$, *and identify* Q_0 *with* $\{1, \dots, n\}$. *For any dimension vector* $\mathbf{v} : Q_0 \to \mathbb{Z}_{\geq 0}$, *we have*

$$[S_n^{\oplus \mathbf{v}_n}] \cdots [S_2^{\oplus \mathbf{v}_2}][S_1^{\oplus \mathbf{v}_1}] = q^{-\frac{1}{2}\sum_{(i\to j)\in Q_1} \mathbf{v}_i \mathbf{v}_j} \sum_{\mathbf{c} \in \mathrm{KP}(\mathbf{v})} [M(\mathbf{c})].$$

Proof. In view of Lemma 10.1.17 and the formula (10.2.3), we just have to check that the power of q on the right-hand side is correct. Using (10.1.2), this power is

$$\begin{aligned}\frac{1}{2} \sum_{n \geq i > j \geq 1} \langle \underline{\dim}\, S_i^{\oplus \mathbf{v}_i}, \underline{\dim}\, S_j^{\oplus \mathbf{v}_j} \rangle &= \frac{1}{2} \sum_{n \geq i > j \geq 1} \langle \mathbf{v}_i \mathbf{e}_i, \mathbf{v}_j \mathbf{e}_j \rangle \\ &= -\frac{1}{2} \sum_{n \geq i > j \geq 1} \mathbf{v}_i \mathbf{v}_j |\{\text{edges } i \to j\}| = -\frac{1}{2} \sum_{\substack{(i\to j)\in Q_1 \\ i > j}} \mathbf{v}_i \mathbf{v}_j.\end{aligned}$$

As in the proof of Lemma 10.1.17, if $i \leq j$, Corollary 10.1.14 implies that there are no edges $i \to j$, so the last expression is the same as $-\frac{1}{2}\sum_{(i\to j)\in Q_1} \mathbf{v}_i \mathbf{v}_j$. $\square$

Proposition 10.2.7. *Let Q be a quiver of finite type. The Hall algebra $\mathcal{H}_Q(\mathbb{F}_q)$ is generated as a $\mathbb{Z}[q^{\pm\frac{1}{2}}]$-algebra by $\{[S_i^{\oplus n}] \mid i \in Q_0,\ n \geq 1\}$.*

Proof. Let $\mathcal{H}'_Q(\mathbb{F}_q) \subset \mathcal{H}_Q(\mathbb{F}_q)$ be the $\mathbb{Z}[q^{\pm\frac{1}{2}}]$-subalgebra generated by elements of the form $[S_i^{\oplus n}]$. We must show that $[M(\mathbf{c})] \in \mathcal{H}'_Q(\mathbb{F}_q)$ for all $\mathbf{c} \in \mathrm{KP}$.

Equip the set of dimension vectors $\mathbb{Z}_{\geq 0}^{Q_0}$ with the following partial order: $\mathbf{v} \leq \mathbf{w}$ if $\mathbf{v}_i \leq \mathbf{w}_i$ for all $i \in Q_0$. We proceed by induction on the dimension vector $\mathbf{v}$ of which $\mathbf{c}$ is a Kostant partition.

The minimal nonzero dimension vectors are the simple roots $\mathbf{e}_i$. If $\mathbf{c} \in \mathrm{KP}(\mathbf{e}_i)$, then of course, $\mathbf{c} = [\mathbf{e}_i]$ and $[M(\mathbf{c})] = [S_i] \in \mathcal{H}'_Q(\mathbb{F}_q)$.

For the general case, suppose first that $\mathbf{c}$ takes nonzero values on at least two roots in Φ^+. Then Proposition 10.2.5 says that $[M(\mathbf{c})]$ is a product of at least two other non-identity elements $[M([\gamma_t^{\mathbf{c}(\gamma_t)}])]$. Each $[\gamma_t^{\mathbf{c}(\gamma_t)}]$ is a Kostant partition of a strictly smaller dimension vector, so $[M([\gamma_t^{\mathbf{c}(\gamma_t)}])] \in \mathcal{H}'_Q(\mathbb{F}_q)$ by induction, and hence $[M(\mathbf{c})] \in \mathcal{H}'_Q(\mathbb{F}_q)$.

Now suppose $\mathbf{c}$ takes a nonzero value on a single root, say $\mathbf{c} = [\gamma^n] \in \mathrm{KP}(n\gamma)$. Any other Kostant partition $\mathbf{c}' \in \mathrm{KP}(n\gamma)$ must take nonzero values on at least two roots: otherwise, if $\mathbf{c}' = [\beta^m]$ for some $\beta \in \Phi^+$, then $m\beta = n\gamma$. But no two distinct positive roots can be (rational) scalar multiples of one another: we must have $\beta = \gamma$ and $m = n$. In other words, every Kostant partition $\mathbf{c}' \in \mathrm{KP}(n\gamma)$ other than $[\gamma^n]$ is covered by the previous paragraph. Proposition 10.2.6 then lets us write $[M([\gamma^n])]$ as a linear combination of elements already known to lie in $\mathcal{H}'_Q(\mathbb{F}_q)$, so $[M([\gamma^n])] \in \mathcal{H}'_Q(\mathbb{F}_q)$ as well. $\square$

Quantum groups. Let v be an indeterminate. For an integer $m \geq 0$, we set

$$[m]_v = \frac{v^m - v^{-m}}{v - v^{-1}} = v^{m-1} + v^{m-3} + \cdots + v^{1-m} \in \mathbb{Z}[v, v^{-1}].$$

We also set

$$[m]_v! = [m]_v [m-1]_v \cdots [1]_v.$$

Let Φ be a simply laced root system, and choose a set of positive roots $\Phi^+ \subset \Phi$. Let Π be the set of vertices of the Dynkin diagram, and for $i \in \Pi$, let $\mathbf{e}_i \in \Phi^+$

be the corresponding simple root. The **strictly positive part of the quantum group** associated to Φ is the $\mathbb{Q}(v)$-algebra $\mathbf{U}^+ = \mathbf{U}^+(\Phi)$ generated by elements

$$\{E_i \mid i \in \Pi\},$$

subject to the relations

$$(10.2.4)\qquad \begin{aligned} E_iE_j - E_jE_i &= 0 \quad \text{if } \langle \mathbf{e}_i^\vee, \mathbf{e}_j\rangle = 0,\\ E_i^2E_j - (v+v^{-1})E_iE_jE_i + E_jE_i^2 &= 0 \quad \text{if } \langle \mathbf{e}_i^\vee, \mathbf{e}_j\rangle = -1. \end{aligned}$$

Next, for any nonnegative integer n, let

$$(10.2.5)\qquad E_i^{(n)} = \frac{E_i^n}{[n]_v!}.$$

Finally, let $\mathcal{U}_v^+ = \mathcal{U}_v^+(\Phi)$ be the $\mathbb{Z}[v,v^{-1}]$-subalgebra of $\mathbf{U}^+$ generated by the set

$$\{E_i^{(n)} \mid i \in \Pi,\ n \in \mathbb{Z}_{\geq 0}\}.$$

The algebra $\mathcal{U}_v^+$ is called the **Lusztig integral form** of $\mathbf{U}^+$, or the (strictly positive part of the) **quantum group with divided powers**.

There is a ring homomorphism

$$\overline{} : \mathbf{U}^+(\Phi) \to \mathbf{U}^+(\Phi) \qquad \text{given by} \qquad \overline{v} = v^{-1} \quad \text{and} \quad \overline{E_i} = E_i.$$

It is called the **bar involution**. Note that $\overline{[n]_v} = [n]_v$ for any nonnegative integer n. It follows that the bar involution preserves the subalgebra $\mathcal{U}_v^+(\Phi) \subset \mathbf{U}^+(\Phi)$.

In the following fundamental result, due to Lusztig, the "adapted order" on Φ^+ is an order obtained from a choice of reduced expression for the longest element of the Weyl group of Φ (cf. Remark 10.1.13).

THEOREM 10.2.8. *Let Φ be a simply-laced root system, and choose an adapted order $\gamma_1, \ldots, \gamma_N$ on Φ^+. There exists a collection of elements $\{E_\gamma \mid \gamma \in \Phi^+\} \subset \mathcal{U}_v^+(\Phi)$ such that the following two conditions hold:*

1. *If $\gamma = \mathbf{e}_i$ is a simple root, then $E_\gamma = E_i$.*
2. *For any Kostant partition $\mathbf{c} : \Phi^+ \to \mathbb{Z}_{\geq 0}$, let*

$$E_{\mathbf{c}} = \frac{E_{\gamma_1}^{\mathbf{c}(\gamma_1)}}{[\mathbf{c}(\gamma_1)]_v!}\frac{E_{\gamma_2}^{\mathbf{c}(\gamma_2)}}{[\mathbf{c}(\gamma_2)]_v!}\cdots\frac{E_{\gamma_N}^{\mathbf{c}(\gamma_N)}}{[\mathbf{c}(\gamma_N)]_v!}.$$

The set $\{E_{\mathbf{c}}\}_{\mathbf{c}\in\mathrm{KP}}$ is a $\mathbb{Z}[v,v^{-1}]$-basis for $\mathcal{U}_v^+(\Phi)$, and a $\mathbb{Q}(v)$-basis for $\mathbf{U}^+(\Phi)$.

For a proof of this theorem, see [**162**]. A basis of the kind described in this theorem is called a **PBW basis**, because of its resemblance to the basis for a universal enveloping algebra obtained from the Poincaré–Birkhoff–Witt theorem. Note that the basis described in this theorem is rather far from being unique: in addition to the choice of order on Φ^+, the construction of the elements E_γ for $\gamma \in \Phi^+ \smallsetminus \Pi$ also depends on choices.

We will not need the details of the construction of PBW bases, but we will need the following observations about them, from [**162**, Sections 1 and 4]:

1. The algebra $\mathcal{U}_v^+(\Phi)$ admits a grading $\mathcal{U}_v^+(\Phi) = \bigoplus_{\mathbf{v}\in\mathbb{Z}_{\geq 0}^{Q_0}} \mathcal{U}_v^+(\Phi)_{\mathbf{v}}$, where

$$(10.2.6)\quad \mathcal{U}_v^+(\Phi)_{\mathbf{v}} = \operatorname{span}\{E_{i_1}^{(n_1)}E_{i_2}^{(n_2)}\cdots E_{i_m}^{(n_m)} \mid n_1\mathbf{e}_{i_1} + n_2\mathbf{e}_{i_2} + \cdots + n_m\mathbf{e}_{i_m} = \mathbf{v}\}.$$

2. Each graded component $\mathcal{U}_v^+(\Phi)_{\mathbf{v}}$ of $\mathcal{U}_v^+(\Phi)$ is a free $\mathbb{Z}[v,v^{-1}]$-module of rank $|\mathrm{KP}(\mathbf{v})|$. If $\{E_{\mathbf{c}}\}_{\mathbf{c}\in\mathrm{KP}}$ is any PBW basis for $\mathcal{U}_v^+(\Phi)$, then $\{E_{\mathbf{c}}\}_{\mathbf{c}\in\mathrm{KP}(\mathbf{v})}$ is a $\mathbb{Z}[v,v^{-1}]$-basis for $\mathcal{U}_v^+(\Phi)_{\mathbf{v}}$.

THEOREM 10.2.9 (Ringel). *Let Q be a quiver of finite type, and let Φ be the corresponding root system. There is an isomorphism of algebras*

$$\psi : \mathcal{U}_v^+(\Phi)/(v - q^{\frac{1}{2}}) \xrightarrow{\sim} \mathcal{H}_Q(\mathbb{F}_q)$$

such that for each $i \in Q_0$, we have $\psi(E_i^{(n)}) = q^{\frac{n(n-1)}{2}}[S_i^{\oplus n}]$.

This result was proved (in a slightly different form) in [**196**]. For an exposition in terms that are close to the language of this book, see [**65**, Chapter 10]. Both of these proofs involve deducing Theorem 10.2.9 as a corollary of an isomorphism

$$\Psi : \mathcal{U}_v^+(\Phi) \xrightarrow{\sim} \mathcal{H}_{Q,v}, \tag{10.2.7}$$

where $\mathcal{H}_{Q,v}$ is the generic Hall algebra of Remark 10.2.3. Ringel's theorem was the main inspiration for Lusztig's geometric approach to the Hall algebra, which we will develop in the next section. Under this isomorphism, PBW basis elements in the quantum group go to scalar multiples of the natural basis in the Hall algebra. Specifically, we have (cf. [**65**, Theorem 11.25])

$$\Psi(E_{\mathbf{c}}) = v^{\dim \operatorname{End}(M(\mathbf{c})) - \dim M(\mathbf{c})}[M(\mathbf{c})]. \tag{10.2.8}$$

Exercises.

10.2.1. Let Q be a quiver of finite type. Let $\mathbf{v} : Q_0 \to \mathbb{Z}_{\geq 0}$ be a dimension vector, and let $\mathbf{c} \in \mathrm{KP}(\mathbf{v})$. Show that

$$\dim M(\mathbf{c}) - \dim \operatorname{End}(M(\mathbf{c})) - \dim \mathcal{O}_{\mathbf{c}} = \sum_{i \in Q_0} \mathbf{v}_i - \sum_{i \in Q_0} \mathbf{v}_i^2.$$

10.2.2. Let Q be a quiver of finite type, and let Φ be the corresponding root system. List the simple roots $\mathbf{e}_1, \ldots, \mathbf{e}_n$ in an order coming from an adapted order on Φ^+. Show that in the quantum group $\mathcal{U}_v^+(\Phi)$, for any dimension vector $\mathbf{v} : Q_0 \to Z_{\geq 0}$, we have

$$E_n^{(\mathbf{v}_n)} \cdots E_2^{(\mathbf{v}_2)} E_1^{(\mathbf{v}_1)} = v^{-\sum_{(i \to j) \in Q_1} \mathbf{v}_i \mathbf{v}_j} \sum_{\mathbf{c} \in \mathrm{KP}(\mathbf{v})} v^{\dim \mathcal{O}_{\mathbf{c}}} E_{\mathbf{c}}.$$

Hint: Transfer Proposition 10.2.6 through Theorem 10.2.9 (or rather the generic version in (10.2.7)) using (10.2.8) and Exercise 10.2.1.

10.3. Convolution

In this section, we consider the representation space $\mathcal{E}_{\mathbf{v}}$ over the complex numbers. We will study the equivariant derived category $D^{\mathrm{b}}_{G_{\mathbf{v}}}(\mathcal{E}_{\mathbf{v}}, \Bbbk)$.

We are mainly interested in the case where Q is of finite type and $\Bbbk = \mathbb{Q}$. We now introduce some notation specific to this context. For any orbit $\mathcal{O}_{\mathbf{c}} \subset \mathcal{E}_{\mathbf{v}}$ (where $\mathbf{c} \in \mathrm{KP}(\mathbf{v})$), let

$$j_{\mathbf{c}} : \mathcal{O}_{\mathbf{c}} \hookrightarrow \mathcal{E}_{\mathbf{v}}$$

be the inclusion map. By Lemma 10.1.7 and Proposition 6.2.13, every $G_{\mathbf{v}}$-equivariant local system on a $G_{\mathbf{v}}$-orbit is constant. For the rest of this chapter, we write

$$\mathrm{IC}_{\mathbf{c}} = \mathrm{IC}(\mathcal{O}_{\mathbf{c}}, \underline{\mathbb{Q}})(\tfrac{1}{2} \dim \mathcal{O}_{\mathbf{c}}). \tag{10.3.1}$$

(See (10.3.7) and (10.3.9) for variants.) Of course, $\dim \mathcal{O}_{\mathbf{c}}$ need not be even, so this formula may involve a half-integer Tate twist. For now, the reader who is ill at ease with this may simply ignore the Tate twists in all formulas. However, in

Sections 10.5 and 10.6, these Tate twists will be essential. See Remark 5.7.6 for an explanation of how to make sense of them.

Definition of the convolution product. Let Q be an arbitrary quiver, and let $\mathbf{v}, \mathbf{w} : Q_0 \to \mathbb{Z}_{\geq 0}$ be dimension vectors. We define a new pairing

$$\langle\!\langle \mathbf{v}, \mathbf{w} \rangle\!\rangle = \sum_{i \in Q_0} \mathbf{v}_i \mathbf{w}_i + \sum_{(i \to j) \in Q_1} \mathbf{v}_i \mathbf{w}_j,$$

which we call the **geometric pairing**. (It differs from the Euler–Poincaré pairing by a sign on the second term.) For each $i \in Q_0$, identify

$$\mathbb{C}^{\mathbf{v}_i + \mathbf{w}_i} \cong \mathbb{C}^{\mathbf{v}_i} \oplus \mathbb{C}^{\mathbf{w}_i}.$$

Then set

$$\begin{aligned} \mathcal{E}_{\mathbf{v},\mathbf{w}} &= \{x \in \mathcal{E}_{\mathbf{v}+\mathbf{w}} \mid \text{for all } e = (i \to j) \in Q_1,\ x_e(\mathbb{C}^{\mathbf{w}_i}) \subset \mathbb{C}^{\mathbf{w}_j}\} \\ &\cong \prod_{(i \to j) \in Q_1} \operatorname{Hom}(\mathbb{C}^{\mathbf{v}_i}, \mathbb{C}^{\mathbf{v}_j}) \times \operatorname{Hom}(\mathbb{C}^{\mathbf{w}_i}, \mathbb{C}^{\mathbf{w}_j}) \times \operatorname{Hom}(\mathbb{C}^{\mathbf{v}_i}, \mathbb{C}^{\mathbf{w}_j}) \\ &\cong \mathcal{E}_{\mathbf{v}} \times \mathcal{E}_{\mathbf{w}} \times \prod_{(i \to j) \in Q_1} \operatorname{Hom}(\mathbb{C}^{\mathbf{v}_i}, \mathbb{C}^{\mathbf{w}_j}). \end{aligned} \tag{10.3.2}$$

In other words, a point in $\mathcal{E}_{\mathbf{v},\mathbf{w}}$ is a quiver representation of dimension vector $\mathbf{v}+\mathbf{w}$ in which the subspaces $\mathbb{C}^{\mathbf{w}_i} \subset \mathbb{C}^{\mathbf{v}_i+\mathbf{w}_i}$ form a subrepresentation of dimension vector $\mathbf{w}$. Omitting the last factor in (10.3.2) gives a surjective linear map

$$p : \mathcal{E}_{\mathbf{v},\mathbf{w}} \to \mathcal{E}_{\mathbf{v}} \times \mathcal{E}_{\mathbf{w}}. \tag{10.3.3}$$

For each $i \in Q_0$, let

$$\begin{aligned} P^i_{\mathbf{v},\mathbf{w}} &= \{g \in \mathrm{GL}_{\mathbf{v}_i+\mathbf{w}_i}(\mathbb{C}) \mid g(\mathbb{C}^{\mathbf{w}_i}) \subset \mathbb{C}^{\mathbf{w}_i}\} \\ &\cong (\mathrm{GL}_{\mathbf{v}_i}(\mathbb{C}) \times \mathrm{GL}_{\mathbf{w}_i}(\mathbb{C})) \ltimes \operatorname{Hom}(\mathbb{C}^{\mathbf{v}_i}, \mathbb{C}^{\mathbf{w}_i}). \end{aligned}$$

Let

$$P_{\mathbf{v},\mathbf{w}} = \prod_{i \in Q_0} P^i_{\mathbf{v},\mathbf{w}}.$$

This group is a parabolic subgroup of the reductive group $G_{\mathbf{v}+\mathbf{w}}$. On the other hand, there is a quotient map

$$P_{\mathbf{v},\mathbf{w}} \twoheadrightarrow G_{\mathbf{v}} \times G_{\mathbf{w}}. \tag{10.3.4}$$

The kernel of this map is the unipotent radical of $P_{\mathbf{v},\mathbf{w}}$. The usual action of $G_{\mathbf{v}+\mathbf{w}}$ on $\mathcal{E}_{\mathbf{v}+\mathbf{w}}$ restricts to an action of $P_{\mathbf{v},\mathbf{w}}$ on $\mathcal{E}_{\mathbf{v},\mathbf{w}}$. If we make $P_{\mathbf{v},\mathbf{w}}$ act on $\mathcal{E}_{\mathbf{v}} \times \mathcal{E}_{\mathbf{w}}$ via the quotient map (10.3.4), then the linear map (10.3.3) is $P_{\mathbf{v},\mathbf{w}}$-equivariant.

We will study the following **convolution diagram** for representation spaces:

$$\mathcal{E}_{\mathbf{v}} \times \mathcal{E}_{\mathbf{w}} \xleftarrow{p} \mathcal{E}_{\mathbf{v},\mathbf{w}} \xrightarrow{i} G_{\mathbf{v}+\mathbf{w}} \times^{P_{\mathbf{v},\mathbf{w}}} \mathcal{E}_{\mathbf{v},\mathbf{w}} \xrightarrow{m} \mathcal{E}_{\mathbf{v}+\mathbf{w}}. \tag{10.3.5}$$

Here, i is the usual inclusion map for an induction space, as in Theorem 6.5.10, and m is given by $m(g, x) = g \cdot x$. For $\mathcal{F} \in D^{\mathrm{b}}_{G_{\mathbf{v}}}(\mathcal{E}_{\mathbf{v}}, \Bbbk)$ and $\mathcal{G} \in D^{\mathrm{b}}_{G_{\mathbf{w}}}(\mathcal{E}_{\mathbf{w}}, \Bbbk)$, define the **twisted external tensor product** $\mathcal{F} \mathbin{\tilde{\boxtimes}} \mathcal{G} \in D^{\mathrm{b}}_{G_{\mathbf{v}+\mathbf{w}}}(G_{\mathbf{v}+\mathbf{w}} \times^{P_{\mathbf{v},\mathbf{w}}} \mathcal{E}_{\mathbf{v},\mathbf{w}}, \Bbbk)$ by

$$\mathcal{F} \mathbin{\tilde{\boxtimes}} \mathcal{G} = (i^* \mathrm{For}^{G_{\mathbf{v}+\mathbf{w}}}_{P_{\mathbf{v},\mathbf{w}}})^{-1} p^* \mathrm{Infl}^{P_{\mathbf{v},\mathbf{w}}}_{G_{\mathbf{v}} \times G_{\mathbf{w}}} (\mathcal{F} \boxtimes \mathcal{G})[\langle\!\langle \mathbf{v}, \mathbf{w} \rangle\!\rangle](\tfrac{1}{2}\langle\!\langle \mathbf{v}, \mathbf{w} \rangle\!\rangle),$$

where $(i^* \mathrm{For}^{G_{\mathbf{v}+\mathbf{w}}}_{P_{\mathbf{v},\mathbf{w}}})^{-1}$ is the inverse of the equivalence from Theorem 6.5.10. Then set

$$\mathcal{F} \star \mathcal{G} = m_*(\mathcal{F} \mathbin{\tilde{\boxtimes}} \mathcal{G}).$$

This construction defines a functor

$$\star : D^{\mathrm{b}}_{G_{\mathbf{v}}}(\mathcal{E}_{\mathbf{v}}, \Bbbk) \times D^{\mathrm{b}}_{G_{\mathbf{w}}}(\mathcal{E}_{\mathbf{w}}, \Bbbk) \to D^{\mathrm{b}}_{G_{\mathbf{v+w}}}(\mathcal{E}_{\mathbf{v+w}}, \Bbbk).$$

REMARK 10.3.1. Here is a representation-theoretic interpretation of the fibers of the map m in (10.3.5). Let $x \in \mathcal{E}_{\mathbf{v+w}}$, and let V be the corresponding representation of Q. We claim that there is a bijection

$$m^{-1}(x) \xrightarrow{\sim} \left\{ \begin{matrix} \text{subrepresentations } V' \subset V \\ \text{of dimension vector } \mathbf{w} \end{matrix} \right\}. \tag{10.3.6}$$

Indeed, given a point $(g, y) \in m^{-1}(x)$, we see that the subspaces V' defined by $V'_i = g_i(\mathbb{C}^{\mathbf{w}_i})$ constitute a subrepresentation V of dimension vector $\mathbf{w}$. Two points $(g, y), (g', y') \in m^{-1}(x)$ determine the same subrepresentation if and only if $g_i^{-1} g'_i(\mathbb{C}^{\mathbf{w}_i}) = \mathbb{C}^{\mathbf{w}_i}$, i.e., if $g^{-1}g' \in P_{\mathbf{v},\mathbf{w}}$. But these conditions imply that $(g, y) = (g', y')$ in $G_{\mathbf{v+w}} \times^{P_{\mathbf{v},\mathbf{w}}} \mathcal{E}_{\mathbf{v},\mathbf{w}}$. We have constructed an injective map as in (10.3.6). Conversely, given any subrepresentation $V' \subset V$ with $\underline{\dim}\, V' = \mathbf{w}$, it is possible to change the basis of V so that for each $i \in Q_0$, V'_i is identified with $\mathbb{C}^{\mathbf{w}_i} \subset \mathbb{C}^{\mathbf{v}_i} \oplus \mathbb{C}^{\mathbf{w}_i}$. This shows that (10.3.6) is surjective.

Very similar reasoning shows that for orbits $\mathcal{O}_{\mathbf{c}} \subset \mathcal{E}_{\mathbf{v}}$ and $\mathcal{O}_{\mathbf{c}'} \subset \mathcal{E}_{\mathbf{w}}$, we have

$$m^{-1}(x) \cap (\mathcal{O}_{\mathbf{c}} \,\tilde{\times}\, \mathcal{O}_{\mathbf{c}'}) \xrightarrow{\sim} \left\{ \begin{matrix} \text{subrepresentations } V' \subset V \\ \text{such that } V' \cong M(\mathbf{c}') \text{ and } V/V' \cong M(\mathbf{c}) \end{matrix} \right\},$$

where $\mathcal{O}_{\mathbf{c}} \,\tilde{\times}\, \mathcal{O}_{\mathbf{c}'}$ is defined to be $G_{\mathbf{v+w}} \times^{P_{\mathbf{v},\mathbf{w}}} p^{-1}(\mathcal{O}_{\mathbf{c}} \times \mathcal{O}_{\mathbf{c}'})$. The proof of this bijection is left to the reader.

In light of the bijection (10.3.6), the variety $m^{-1}(x)$ is sometimes called a **quiver Grassmannian**.

The representation space $\mathcal{E}_0$ corresponding to the zero dimension vector is, of course, a single point. Denote the constant sheaf on this point by

$$\mathrm{IC}_{\varnothing} = \underline{\Bbbk}_{\mathcal{E}_0}. \tag{10.3.7}$$

(If Q has finite type, this is consistent with our notation for Kostant partitions; and if $\Bbbk = \mathbb{Q}$, this is consistent with (10.3.1).)

LEMMA 10.3.2. *For any $\mathcal{F} \in D^{\mathrm{b}}_{G_{\mathbf{v}}}(\mathcal{E}_{\mathbf{v}}, \Bbbk)$, there are natural isomorphisms*

$$\mathrm{IC}_{\varnothing} \star \mathcal{F} \cong \mathcal{F} \star \mathrm{IC}_{\varnothing} \cong \mathcal{F}.$$

PROOF. We clearly have natural isomorphisms $\mathrm{IC}_{\varnothing} \boxtimes \mathcal{F} \cong \mathcal{F} \boxtimes \mathrm{IC}_{\varnothing} \cong \mathcal{F}$ on $\mathcal{E}_{\mathbf{v}}$. Note that $P_{0,\mathbf{v}} = P_{\mathbf{v},0} = G_{\mathbf{v}}$, so when computing either $\mathrm{IC}_{\varnothing} \star \mathcal{F}$ or $\mathcal{F} \star \mathrm{IC}_{\varnothing}$, we see that the maps p, i, and m in the convolution diagram (10.3.5) are all isomorphisms. The result follows. $\square$

LEMMA 10.3.3. *For $\mathcal{F} \in D^{\mathrm{b}}_{G_{\mathbf{u}}}(\mathcal{E}_{\mathbf{u}}, \Bbbk)$, $\mathcal{G} \in D^{\mathrm{b}}_{G_{\mathbf{v}}}(\mathcal{E}_{\mathbf{v}}, \Bbbk)$, and $\mathcal{H} \in D^{\mathrm{b}}_{G_{\mathbf{w}}}(\mathcal{E}_{\mathbf{w}}, \Bbbk)$, there is a natural isomorphism $(\mathcal{F} \star \mathcal{G}) \star \mathcal{H} \cong \mathcal{F} \star (\mathcal{G} \star \mathcal{H})$.*

PROOF. Identify $\mathbb{C}^{\mathbf{u}_i + \mathbf{v}_i + \mathbf{w}_i}$ with $\mathbb{C}^{\mathbf{u}_i} \oplus \mathbb{C}^{\mathbf{v}_i} \oplus \mathbb{C}^{\mathbf{w}_i}$. Let

$$\mathcal{E}_{\mathbf{u},\mathbf{v},\mathbf{w}} = \left\{ x \in \mathcal{E}_{\mathbf{u+v+w}} \;\middle|\; \begin{matrix} \text{for all } e = (i \to j) \in Q_1, \text{ we have} \\ x_e(\mathbb{C}^{\mathbf{v}_i} \oplus \mathbb{C}^{\mathbf{w}_i}) \subset \mathbb{C}^{\mathbf{v}_j} \oplus \mathbb{C}^{\mathbf{w}_j} \text{ and } x_e(\mathbb{C}^{\mathbf{w}_i}) \subset \mathbb{C}^{\mathbf{w}_j} \end{matrix} \right\},$$

$$P_{\mathbf{u},\mathbf{v},\mathbf{w}} = \left\{ g \in G_{\mathbf{u+v+w}} \;\middle|\; \begin{matrix} \text{for all } i \in Q_0, \text{ we have} \\ g_i(\mathbb{C}^{\mathbf{v}_i} \oplus \mathbb{C}^{\mathbf{w}_i}) \subset \mathbb{C}^{\mathbf{v}_i} \oplus \mathbb{C}^{\mathbf{w}_i} \text{ and } g_i(\mathbb{C}^{\mathbf{w}_i}) \subset \mathbb{C}^{\mathbf{w}_i} \end{matrix} \right\}.$$

The proof of the lemma is based on the study of the following diagram, in which the maps are appropriate generalizations of those in (10.3.5):

$$\begin{tikzcd}
\mathcal{E}_{\mathbf{u}} \times \mathcal{E}_{\mathbf{v}} \times \mathcal{E}_{\mathbf{w}} & & & \\
\mathcal{E}_{\mathbf{u},\mathbf{v}} \times \mathcal{E}_{\mathbf{w}} \arrow[u] \arrow[d] & \mathcal{E}_{\mathbf{u},\mathbf{v},\mathbf{w}} \arrow[l] \arrow[ul, "\tilde{p}"'] \arrow[d] \arrow[dr, "\tilde{\imath}"] & & \\
(G_{\mathbf{u}+\mathbf{v}} \times^{P_{\mathbf{u},\mathbf{v}}} \mathcal{E}_{\mathbf{u},\mathbf{v}}) \times \mathcal{E}_{\mathbf{w}} \arrow[d] & P_{\mathbf{u}+\mathbf{v},\mathbf{w}} \times^{P_{\mathbf{u},\mathbf{v},\mathbf{w}}} \mathcal{E}_{\mathbf{u},\mathbf{v},\mathbf{w}} \arrow[l] \arrow[r] \arrow[d] & G_{\mathbf{u}+\mathbf{v}+\mathbf{w}} \times^{P_{\mathbf{u},\mathbf{v},\mathbf{w}}} \mathcal{E}_{\mathbf{u},\mathbf{v},\mathbf{w}} \arrow[d] \arrow[dr, "\tilde{m}"] & \\
\mathcal{E}_{\mathbf{u}+\mathbf{v}} \times \mathcal{E}_{\mathbf{w}} & \mathcal{E}_{\mathbf{u}+\mathbf{v},\mathbf{w}} \arrow[l] \arrow[r] & G_{\mathbf{u}+\mathbf{v}+\mathbf{w}} \times^{P_{\mathbf{u}+\mathbf{v},\mathbf{w}}} \mathcal{E}_{\mathbf{u}+\mathbf{v},\mathbf{w}} \arrow[r] & \mathcal{E}_{\mathbf{u}+\mathbf{v}+\mathbf{w}}
\end{tikzcd}$$

To make sense of the maps and spaces in the third row, it is useful to observe that there are canonical identifications

$$\begin{aligned}
(G_{\mathbf{u}+\mathbf{v}} \times^{P_{\mathbf{u},\mathbf{v}}} \mathcal{E}_{\mathbf{u},\mathbf{v}}) \times \mathcal{E}_{\mathbf{w}} &\cong P_{\mathbf{u}+\mathbf{v},\mathbf{w}} \times^{P_{\mathbf{u},\mathbf{v},\mathbf{w}}} (\mathcal{E}_{\mathbf{u},\mathbf{v}} \times \mathcal{E}_{\mathbf{w}}), \\
G_{\mathbf{u}+\mathbf{v}+\mathbf{w}} \times^{P_{\mathbf{u},\mathbf{v},\mathbf{w}}} \mathcal{E}_{\mathbf{u},\mathbf{v},\mathbf{w}} &\cong G_{\mathbf{u}+\mathbf{v}+\mathbf{w}} \times^{P_{\mathbf{u}+\mathbf{v},\mathbf{w}}} P_{\mathbf{u}+\mathbf{v},\mathbf{w}} \times^{P_{\mathbf{u},\mathbf{v},\mathbf{w}}} \mathcal{E}_{\mathbf{u},\mathbf{v},\mathbf{w}}.
\end{aligned}$$

The convolution product $(\mathcal{F} \star \mathcal{G}) \star \mathcal{H}$ is computed by going down the left-hand side and across the bottom of the diagram above. By several applications of base change and the induction theorem, one finds that there is a natural isomorphism

$$(\mathcal{F} \star \mathcal{G}) \star \mathcal{H} \cong \tilde{m}_*(\tilde{\imath}^* \mathrm{For}_{P_{\mathbf{u},\mathbf{v},\mathbf{w}}}^{G_{\mathbf{u}+\mathbf{v}+\mathbf{w}}})^{-1} \tilde{p}^* \mathrm{Infl}_{G_{\mathbf{u}} \times G_{\mathbf{v}} \times G_{\mathbf{w}}}^{P_{\mathbf{u},\mathbf{v},\mathbf{w}}} (\mathcal{F} \boxtimes \mathcal{G} \boxtimes \mathcal{H})[d](\tfrac{1}{2}d),$$

where $d = \langle\!\langle \mathbf{u}, \mathbf{v} \rangle\!\rangle + \langle\!\langle \mathbf{u}, \mathbf{w} \rangle\!\rangle + \langle\!\langle \mathbf{v}, \mathbf{w} \rangle\!\rangle$. A similar commutative diagram can be used to show that the right-hand side is also naturally isomorphic to $\mathcal{F} \star (\mathcal{G} \star \mathcal{H})$. $\square$

As in Chapters 7 and 9, the proof of Lemma 10.3.3 suggests a generalization of (10.3.5) for convolving several objects at once. Let $\mathbf{v}^1, \mathbf{v}^2, \dots, \mathbf{v}^k$ be dimension vectors, and suppose we are given objects

$$\mathcal{F}_1 \in D^{\mathrm{b}}_{G_{\mathbf{v}^1}}(\mathcal{E}_{\mathbf{v}^1}, \Bbbk), \quad \mathcal{F}_2 \in D^{\mathrm{b}}_{G_{\mathbf{v}^2}}(\mathcal{E}_{\mathbf{v}^2}, \Bbbk), \quad \dots, \quad \mathcal{F}_k \in D^{\mathrm{b}}_{G_{\mathbf{v}^k}}(\mathcal{E}_{\mathbf{v}^k}, \Bbbk).$$

Let $\mathbf{w} = \mathbf{v}^1 + \cdots + \mathbf{v}^k$, and identify

$$\mathbb{C}^{\mathbf{w}_i} = \mathbb{C}^{\mathbf{v}^1_i} \oplus \mathbb{C}^{\mathbf{v}^2_i} \oplus \cdots \oplus \mathbb{C}^{\mathbf{v}^k_i}.$$

For $r \in \{1, \dots, k\}$, we also set

$$\mathbb{C}^{\mathbf{v}^{\geq r}_i} = \mathbb{C}^{\mathbf{v}^r_i} \oplus \mathbb{C}^{\mathbf{v}^{r+1}_i} \oplus \cdots \oplus \mathbb{C}^{\mathbf{v}^k_i} \subset \mathbb{C}^{\mathbf{w}_i}.$$

Let

$$\begin{aligned}
\mathcal{E}_{\mathbf{v}^1,\dots,\mathbf{v}^k} &= \{x \in \mathcal{E}_{\mathbf{w}} \mid \text{for all } e = (i \to j) \in Q_1 \text{ and all } r,\ x_e(\mathbb{C}^{\mathbf{v}^{\geq r}_i}) \subset \mathbb{C}^{\mathbf{v}^{\geq r}_j}\}, \\
P_{\mathbf{v}^1,\dots,\mathbf{v}^k} &= \{g \in G_{\mathbf{w}} \mid \text{for all } i \in Q_0 \text{ and all } r,\ g_i(\mathbb{C}^{\mathbf{v}^{\geq r}_i}) \subset \mathbb{C}^{\mathbf{v}^{\geq r}_i}\}.
\end{aligned}$$

We can then consider the diagram

$$\mathcal{E}_{\mathbf{v}^1} \times \cdots \times \mathcal{E}_{\mathbf{v}^k} \xleftarrow{p} \mathcal{E}_{\mathbf{v}^1,\dots,\mathbf{v}^k} \xrightarrow{i} G_{\mathbf{w}} \times^{P_{\mathbf{v}^1,\dots,\mathbf{v}^k}} \mathcal{E}_{\mathbf{v}^1,\dots,\mathbf{v}^k} \xrightarrow{m} \mathcal{E}_{\mathbf{w}}. \tag{10.3.8}$$

Using the maps in this diagram, we have

$$\mathcal{F}_1 \mathbin{\tilde{\boxtimes}} \cdots \mathbin{\tilde{\boxtimes}} \mathcal{F}_k = (i^* \mathrm{For}_{P_{\mathbf{v}^1,\dots,\mathbf{v}^k}}^{G_{\mathbf{w}}})^{-1} p^* \mathrm{Infl}_{G_{\mathbf{v}^1} \times \cdots \times G_{\mathbf{v}^k}}^{P_{\mathbf{v}^1,\dots,\mathbf{v}^k}} (\mathcal{F}_1 \boxtimes \cdots \boxtimes \mathcal{F}_k)[d](\tfrac{1}{2}d),$$

where $d = \sum_{r<s} \langle\!\langle \mathbf{v}^r, \mathbf{v}^s \rangle\!\rangle$. We then have

$$\mathcal{F}_1 \star \cdots \star \mathcal{F}_k = m_*(\mathcal{F}_1 \mathbin{\tilde{\boxtimes}} \cdots \mathbin{\tilde{\boxtimes}} \mathcal{F}_k).$$

REMARK 10.3.4. A straightforward generalization of the ideas in Remark 10.3.1 leads to the following description of the fibers of the map m in (10.3.8): if $x \in \mathcal{E}_{\mathbf{w}}$ corresponds to a representation V of Q, then there is a bijection

$$m^{-1}(x) \xrightarrow{\sim} \left\{ \begin{matrix} \text{filtrations } V = V^1 \supset V^2 \supset \cdots \supset V^k \supset 0 \\ \text{such that } \underline{\dim} V^t = \mathbf{v}^t + \mathbf{v}^{t+1} + \cdots + \mathbf{v}^k \end{matrix} \right\}.$$

In analogy with the terminology from Remark 10.3.1, the variety $m^{-1}(x)$ here is sometimes called a **(partial) quiver flag variety**.

The next two statements are straightforward analogues of Lemma 7.2.7 and Proposition 7.2.6. We omit their proofs.

LEMMA 10.3.5. *For $\mathcal{F} \in D^{\mathrm{b}}_{G_{\mathbf{v}}}(\mathcal{E}_{\mathbf{v}}, \Bbbk)$ and $\mathcal{G} \in D^{\mathrm{b}}_{G_{\mathbf{w}}}(\mathcal{E}_{\mathbf{w}}, \Bbbk)$, there is a natural isomorphism $\mathbb{D}(\mathcal{F} \star \mathcal{G}) \cong \mathbb{D}(\mathcal{F}) \star \mathbb{D}(\mathcal{G})$.*

LEMMA 10.3.6. *If $\mathcal{F} \in \mathrm{Semis}_{G_{\mathbf{v}}}(\mathcal{E}_{\mathbf{v}}, \mathbb{Q})$ and $\mathcal{G} \in \mathrm{Semis}_{G_{\mathbf{w}}}(\mathcal{E}_{\mathbf{w}}, \mathbb{Q})$, then $\mathcal{F} \star \mathcal{G} \in \mathrm{Semis}_{G_{\mathbf{v}+\mathbf{w}}}(\mathcal{E}_{\mathbf{v}+\mathbf{w}}, \mathbb{Q})$.*

Geometric Hall algebra computations. The remainder of this section is devoted to geometric counterparts of the computations from Section 10.2. We assume from now on that Q has no oriented cycles. Thus, for any vertex $i \in Q_0$ and any $n \geq 1$, the space $\mathcal{E}_{n\mathbf{e}_i}$ is a single point. Denote the constant sheaf on this point by

$$\mathrm{IC}_{[\mathbf{e}_i^n]} = \underline{\Bbbk}_{\mathcal{E}_{n\mathbf{e}_i}}. \tag{10.3.9}$$

(If Q has finite type, this is consistent with our notation for Kostant partitions; and if $\Bbbk = \mathbb{Q}$, this is consistent with (10.3.1).)

LEMMA 10.3.7. *Assume that Q has no oriented cycles. For any $i \in Q_0$, we have*

$$\underbrace{\mathrm{IC}_{[\mathbf{e}_i]} \star \cdots \star \mathrm{IC}_{[\mathbf{e}_i]}}_{n \text{ factors}} \cong R\Gamma(\underline{\Bbbk}_{\mathrm{Fl}_n}) \boxtimes \mathrm{IC}_{[\mathbf{e}_i^n]}[\dim \mathrm{Fl}_n](\tfrac{1}{2} \dim \mathrm{Fl}_n),$$

where Fl_n denotes the variety of complete flags in $\mathbb{C}^n$.

PROOF. We will compute this product using (10.3.8). Since the dimension vectors $\mathbf{e}_i$ and $n\mathbf{e}_i$ are nonzero on only one vertex of Q, the spaces $\mathcal{E}_{\mathbf{e}_i}$ and $\mathcal{E}_{n\mathbf{e}_i}$ each consist of a single point. The group $G_{n\mathbf{e}_i}$ is just $\mathrm{GL}_n(\mathbb{C})$, and the subgroup $P_{\mathbf{e}_i,\ldots,\mathbf{e}_i}$ is the group of lower-triangular matrices. Identify $G_{n\mathbf{e}_i}/P_{\mathbf{e}_i,\ldots,\mathbf{e}_i}$ with Fl_n. Then diagram (10.3.8) becomes

$$\mathrm{pt} \xleftarrow{p} \mathrm{pt} \xrightarrow{i} \mathrm{Fl}_n \xrightarrow{m} \mathrm{pt}.$$

Of course, $i^* \mathrm{For}^{G_{n\mathbf{e}_i}}_{P_{\mathbf{e}_i,\ldots,\mathbf{e}_i}}(\underline{\Bbbk}_{\mathrm{Fl}_n}) \cong p^* \underline{\Bbbk}_{\mathrm{pt}}$. Using Lemma 10.4.2 below, we have

$$\sum_{1 \leq r < s \leq n} \langle\!\langle \mathbf{e}_i, \mathbf{e}_i \rangle\!\rangle = n(n-1)/2 = \dim \mathrm{Fl}_n.$$

Therefore, $\mathrm{IC}_{[\mathbf{e}_i]} \tilde{\boxtimes} \cdots \tilde{\boxtimes} \mathrm{IC}_{[\mathbf{e}_i]} \cong \underline{\Bbbk}_{\mathrm{Fl}_n}[\dim \mathrm{Fl}_n](\frac{1}{2} \dim \mathrm{Fl}_n)$, and the result follows. ☐

From now on, we focus on the case $\Bbbk = \mathbb{Q}$.

LEMMA 10.3.8. *Assume that Q has no oriented cycles, and let $i, j \in Q_0$.*

(1) *If there is no edge $i \to j$ or $j \to i$ in Q_1, then*

$$\mathrm{IC}_{[\mathbf{e}_i]} \star \mathrm{IC}_{[\mathbf{e}_j]} \cong \mathrm{IC}_{[\mathbf{e}_j]} \star \mathrm{IC}_{[\mathbf{e}_i]} \cong \mathrm{IC}_{[\mathbf{e}_i, \mathbf{e}_j]}.$$

(2) *If there is exactly one edge $i \to j$ or $j \to i$, then*

$$\mathrm{IC}_{[\mathbf{e}_i]} \star \mathrm{IC}_{[\mathbf{e}_j]} \star \mathrm{IC}_{[\mathbf{e}_i]} \cong (\mathrm{IC}_{[\mathbf{e}_i^2]} \star \mathrm{IC}_{[\mathbf{e}_j]}) \oplus (\mathrm{IC}_{[\mathbf{e}_j]} \star \mathrm{IC}_{[\mathbf{e}_i^2]}).$$

PROOF. If there is no edge between i and j, then $\mathcal{E}_{\mathbf{e}_i}$, $\mathcal{E}_{\mathbf{e}_j}$, and $\mathcal{E}_{\mathbf{e}_i+\mathbf{e}_j}$ are all one-point spaces, and $P_{\mathbf{e}_i,\mathbf{e}_j} = P_{\mathbf{e}_j,\mathbf{e}_i} = G_{\mathbf{e}_i+\mathbf{e}_j}$. All the maps in the convolution diagram (10.3.5) are isomorphisms, and the claim follows.

Suppose now that there is an edge $i \to j$. Let $\gamma = \mathbf{e}_i + \mathbf{e}_j$, and note that γ is a root. We will work with $\mathcal{E}_{\mathbf{e}_i+\mathbf{e}_j}$ and $\mathcal{E}_{2\mathbf{e}_i+\mathbf{e}_j}$. We have

$$\mathcal{E}_{\mathbf{e}_i+\mathbf{e}_j} \cong \mathbb{A}^1 \qquad \text{and} \qquad \mathcal{E}_{2\mathbf{e}_i+\mathbf{e}_j} \cong \mathbb{A}^2.$$

Each of these spaces consists of just two orbits:

$$\mathcal{E}_{\mathbf{e}_i+\mathbf{e}_j} = \mathcal{O}_{[\gamma]} \cup \mathcal{O}_{[\mathbf{e}_j,\mathbf{e}_i]} \qquad \text{and} \qquad \mathcal{E}_{2\mathbf{e}_i+\mathbf{e}_j} = \mathcal{O}_{[\gamma,\mathbf{e}_i]} \cup \mathcal{O}_{[\mathbf{e}_j,\mathbf{e}_i^2]}.$$

In each case, the first-named orbit is dense (and thus has a smooth closure) and the second one is 0-dimensional.

Let us compute $\mathrm{IC}_{[\mathbf{e}_i]} \star \mathrm{IC}_{[\mathbf{e}_j]}$. Observe that $\mathcal{E}_{\mathbf{e}_i,\mathbf{e}_j} = \mathcal{E}_{\mathbf{e}_i+\mathbf{e}_j} \cong \mathbb{A}^1$ and $P_{\mathbf{e}_i,\mathbf{e}_j} = G_\gamma$. In the convolution diagram (10.3.5), the maps i and m are isomorphisms, while p can be identified with the constant map $\mathbb{A}^1 \to \mathrm{pt}$. We deduce that

$$\mathrm{IC}_{[\mathbf{e}_i]} \star \mathrm{IC}_{[\mathbf{e}_j]} \cong \underline{\mathbb{Q}}_{\mathcal{E}_{\mathbf{e}_i+\mathbf{e}_j}}[1](\tfrac{1}{2}) \cong \mathrm{IC}_{[\gamma]}. \tag{10.3.10}$$

Next, consider $\mathrm{IC}_{[\gamma]} \star \mathrm{IC}_{[\mathbf{e}_i]}$. We have $\mathcal{E}_{\mathbf{e}_i+\mathbf{e}_j,\mathbf{e}_i} \cong \mathcal{E}_{\mathbf{e}_i+\mathbf{e}_j} \times \mathcal{E}_{\mathbf{e}_i} \cong \mathbb{A}^1 \times \mathrm{pt}$. On the other hand, $G_{2\mathbf{e}_i+\mathbf{e}_j} = \mathrm{GL}_2 \times \mathrm{GL}_1$, and $P_{\mathbf{e}_i+\mathbf{e}_j,\mathbf{e}_i} = B^- \times \mathrm{GL}_1$, where $B^- \subset \mathrm{GL}_2$ is the group of lower-triangular matrices. After unwinding the definitions, one can identify

$$G_{2\mathbf{e}_i+\mathbf{e}_j} \times^{P_{\mathbf{e}_i+\mathbf{e}_j,\mathbf{e}_i}} \mathcal{E}_{\mathbf{e}_i+\mathbf{e}_j,\mathbf{e}_i} \cong \mathrm{GL}_2 \times^{B^-} \mathbb{A}^1,$$

where B^- acts on $\mathbb{A}^1$ by $\left[\begin{smallmatrix} a & \\ c & d \end{smallmatrix}\right] \cdot x = a^{-1}x$. The map $m : G_{2\mathbf{e}_i+\mathbf{e}_j} \times^{P_{\gamma,\mathbf{e}_i}} \mathcal{E}_{\gamma,\mathbf{e}_i} \to \mathcal{E}_{2\mathbf{e}_i+\mathbf{e}_j}$ can then be identified with the map

$$m : \mathrm{GL}_2 \times^{B^-} \mathbb{A}^1 \to \mathbb{A}^2 \qquad \text{given by} \qquad m(\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right], x) = (\tfrac{dx}{ad-bc}, \tfrac{-bx}{ad-bc}).$$

From this formula, one can check that $m^{-1}(x,y)$ is a single point if $(x,y) \neq (0,0)$, and that $m^{-1}(0,0) \cong \mathrm{GL}_2/B^- \cong \mathbb{P}^1$. This lets us compute the table of stalks of $m_*\underline{\mathbb{Q}}[2](1)$:

(10.3.11) $m_*\underline{\mathbb{Q}}[2]$:

	$\mathcal{O}_{[\gamma,\mathbf{e}_i]}$	$\mathcal{O}_{[\mathbf{e}_j,\mathbf{e}_i^2]}$
0		$\underline{\mathbb{Q}}$
-1		
-2	$\underline{\mathbb{Q}}(1)$	$\underline{\mathbb{Q}}(1)$

Since $m_*\underline{\mathbb{Q}}[2](1) \cong \mathrm{IC}_{[\gamma]} \star \mathrm{IC}_{[\mathbf{e}_i]}$ is a semisimple complex, and since the stalks computed above agree with those of $\mathrm{IC}_{[\gamma,\mathbf{e}_i]} \oplus \mathrm{IC}_{[\mathbf{e}_j,\mathbf{e}_i^2]}$, these two objects must be isomorphic. We conclude that

$$\mathrm{IC}_{[\mathbf{e}_i]} \star \mathrm{IC}_{[\mathbf{e}_j]} \star \mathrm{IC}_{[\mathbf{e}_i]} \cong \mathrm{IC}_{[\gamma,\mathbf{e}_i]} \oplus \mathrm{IC}_{[\mathbf{e}_j,\mathbf{e}_i^2]}. \tag{10.3.12}$$

Next, consider $\mathrm{IC}_{[\mathbf{e}_j]} \star \mathrm{IC}_{[\mathbf{e}_i^2]}$. The space $\mathcal{E}_{\mathbf{e}_j,2\mathbf{e}_i}$ is a single point, and $P_{\mathbf{e}_j,2\mathbf{e}_i} = G_{2\mathbf{e}_i+\mathbf{e}_j}$. In the convolution diagram (10.3.5), the maps p and i are isomorphisms, and m is an embedding of a point into $\mathcal{E}_{2\mathbf{e}_i+\mathbf{e}_j}$. We conclude that

$$\mathrm{IC}_{[\mathbf{e}_j]} \star \mathrm{IC}_{[\mathbf{e}_i^2]} \cong \mathrm{IC}_{[\mathbf{e}_j,\mathbf{e}_i^2]}. \tag{10.3.13}$$

Finally, we must compute $\mathrm{IC}_{[\mathbf{e}_i^2]} \star \mathrm{IC}_{[\mathbf{e}_j]}$. We have $\mathcal{E}_{2\mathbf{e}_i,\mathbf{e}_j} = \mathcal{E}_{2\mathbf{e}_i+\mathbf{e}_j} \cong \mathbb{A}^2$ and $P_{2\mathbf{e}_i,\mathbf{e}_j} = G_{2\mathbf{e}_i+\mathbf{e}_j}$. Thus, in the convolution diagram (10.3.5), the maps i and m are isomorphisms, and p is the constant map $\mathbb{A}^2 \to \mathrm{pt}$. We have

$$\mathrm{IC}_{[\mathbf{e}_i^2]} \star \mathrm{IC}_{[\mathbf{e}_j]} \cong \underline{\mathbb{Q}}_{\mathcal{E}_{2\mathbf{e}_i+\mathbf{e}_j}}[2](1) \cong \mathrm{IC}_{[\gamma,\mathbf{e}_i]}. \tag{10.3.14}$$

In view of (10.3.12), (10.3.13), and (10.3.14), the lemma is now proved in the case of an edge $i \to j$.

It remains to consider the case where there is an edge $j \to i$. The calculations in this case are similar, but not identical, to those above. We briefly indicate the main steps. One can first show that

$$\mathrm{IC}_{[\mathbf{e}_j]} \star \mathrm{IC}_{[\mathbf{e}_i]} \cong \mathrm{IC}_{[\gamma]}.$$

Then $\mathcal{E}_{\mathbf{e}_i,\mathbf{e}_i+\mathbf{e}_j} \cong \mathcal{E}_{\mathbf{e}_i} \times \mathcal{E}_{\mathbf{e}_i+\mathbf{e}_j} \cong \mathrm{pt} \times \mathbb{A}^1$. A similar calculation to the one in (10.3.11) shows that

$$\mathrm{IC}_{[\mathbf{e}_i]} \star \mathrm{IC}_{[\mathbf{e}_j]} \star \mathrm{IC}_{[\mathbf{e}_i]} \cong \mathrm{IC}_{[\mathbf{e}_i,\gamma]} \oplus \mathrm{IC}_{[\mathbf{e}_i^2,\mathbf{e}_j]}.$$

On the other hand, it turns out that

$$\mathrm{IC}_{[\mathbf{e}_j]} \star \mathrm{IC}_{[\mathbf{e}_i^2]} \cong \mathrm{IC}_{[\mathbf{e}_i,\gamma]} \qquad \text{and} \qquad \mathrm{IC}_{[\mathbf{e}_i^2]} \star \mathrm{IC}_{[\mathbf{e}_j]} \cong \mathrm{IC}_{[\mathbf{e}_i^2,\mathbf{e}_j]}.$$

Further details in the case of an edge $j \to i$ are left to the reader. $\square$

PROPOSITION 10.3.9. *Let Q be a quiver of finite type. Let $\gamma_1, \ldots, \gamma_N$ be an adapted order on the corresponding set of positive roots Φ^+, and let $\mathbf{c} : \Phi^+ \to \mathbb{Z}_{\geq 0}$ be a Kostant partition.*

(1) *For $1 \leq i \leq N$, let $d_i = \dim \mathcal{O}_{[\gamma_i^{\mathbf{c}(\gamma_i)}]}$. We have*

$$j_{[\gamma_1^{\mathbf{c}(\gamma_1)}]!}\underline{\mathbb{Q}}[d_1](\tfrac{d_1}{2}) \star j_{[\gamma_2^{\mathbf{c}(\gamma_2)}]!}\underline{\mathbb{Q}}[d_2](\tfrac{d_2}{2}) \star \cdots \star j_{[\gamma_N^{\mathbf{c}(\gamma_N)}]!}\underline{\mathbb{Q}}[d_N](\tfrac{d_N}{2})$$
$$\cong j_{\mathbf{c}!}\underline{\mathbb{Q}}_{\mathcal{O}_\mathbf{c}}[\dim \mathcal{O}_\mathbf{c}](\tfrac{1}{2}\dim \mathcal{O}_\mathbf{c}).$$

(2) *The convolution product $\mathrm{IC}_{[\gamma_1^{\mathbf{c}(\gamma_1)}]} \star \mathrm{IC}_{[\gamma_2^{\mathbf{c}(\gamma_2)}]} \star \cdots \star \mathrm{IC}_{[\gamma_N^{\mathbf{c}(\gamma_N)}]}$ is supported on $\overline{\mathcal{O}_\mathbf{c}}$, and the multiplicity of $\mathrm{IC}_\mathbf{c}[n]$ as a summand in this object is*

$$\begin{cases} 1 & \text{if } n = 0, \\ 0 & \text{otherwise.} \end{cases}$$

PROOF. For $t \in \{1, \ldots, N\}$, let $\mathbf{v}^t = \mathbf{c}(\gamma_t)\gamma_t \in \mathbb{Z}_{\geq 0}^{Q_0}$. Then $\mathbf{v}^1, \ldots, \mathbf{v}^N$ are the dimension vectors of $M([\gamma_1^{\mathbf{c}(\gamma_1)}]), \ldots, M([\gamma_N^{\mathbf{c}(\gamma_N)}])$, respectively. Let $\mathbf{w} = \sum \mathbf{c}(\gamma_t)\gamma_t$, so that $\mathbf{c} \in \mathrm{KP}(\mathbf{w})$. For brevity, let $Z = G_\mathbf{w} \times^{P_{\mathbf{v}^1,\ldots,\mathbf{v}^N}} \mathcal{E}_{\mathbf{v}^1,\ldots,\mathbf{v}^N}$. To compute the convolution products in the proposition, we must consider the diagram

$$\mathcal{E}_{\mathbf{v}^1} \times \cdots \times \mathcal{E}_{\mathbf{v}^N} \xleftarrow{p} \mathcal{E}_{\mathbf{v}^1,\ldots,\mathbf{v}^N} \xrightarrow{i} Z \xrightarrow{m} \mathcal{E}_\mathbf{w}.$$

By Lemma 10.1.11, $\mathcal{O}_{[\gamma_i^{\mathbf{c}(\gamma_i)}]}$ is dense in $\mathcal{E}_{\mathbf{v}^i}$, so the set

$$U = G_\mathbf{w} \times^{P_{\mathbf{v}^1,\ldots,\mathbf{v}^N}} p^{-1}(\mathcal{O}_{[\gamma_1^{\mathbf{c}(\gamma_1)}]} \times \cdots \times \mathcal{O}_{[\gamma_N^{\mathbf{c}(\gamma_N)}]})$$

is a dense open subset of Z. Let us determine the orbits contained in $m(U)$. Given $x \in \mathcal{E}_\mathbf{w}$, let V be the corresponding representation of Q. Recall from Remark 10.3.4 that $m^{-1}(x)$ is in bijection with the set of decreasing filtrations

$$V = V^1 \supset V^2 \supset \cdots \supset V^N \supset V^{N+1} = 0 \tag{10.3.15}$$

such that $\underline{\dim}\, V^t = \mathbf{v}^t + \mathbf{v}^{t+1} + \cdots + \mathbf{v}^N$. Moreover, for $(g,y) \in m^{-1}(x)$, the orbit of the image $p(y) \in \mathcal{E}_{\mathbf{v}^1} \times \cdots \times \mathcal{E}_{\mathbf{v}^N}$ corresponds to the isomorphism classes of the subquotients $V^1/V^2, \ldots, V^{N-1}/V^N, V^N$.

Now take $x \in m(U)$. Thus, there exists a point $(g,y) \in m^{-1}(x) \cap U$. This means that $p(y)$ lies in $\mathcal{O}_{[\gamma_1^{\mathbf{c}(\gamma_1)}]} \times \cdots \times \mathcal{O}_{[\gamma_N^{\mathbf{c}(\gamma_N)}]}$, so we must have $V^t/V^{t+1} \cong M(\gamma_t)^{\oplus \mathbf{c}(\gamma_t)}$ for each t. We are in the setting of Lemma 10.1.16, which tells us that y (and hence x) lies in $\mathcal{O}_{\mathbf{c}}$. Since $m(U)$ is $G_{\mathbf{w}}$-invariant, we conclude that $m(U) = \mathcal{O}_{\mathbf{c}}$.

Finally, Lemma 10.1.16 also tells us that the filtration (10.3.15) is unique, so for any $x \in m(U)$, the set $m^{-1}(x) \cap U$ is a singleton. Thus, in the following commutative square, the left-hand vertical arrow is a bijection:

$$\begin{tikzcd} U \arrow[r, "h"] \arrow[d, "m|_U"'] & Z \arrow[d, "m"] \\ \mathcal{O}_{\mathbf{c}} \arrow[r, "j_{\mathbf{c}}"] & \mathcal{E}_{\mathbf{w}} \end{tikzcd}$$

We claim that this square is cartesian. It is enough to show that $m(Z \smallsetminus U)$ does not meet $\mathcal{O}_{\mathbf{c}}$. Since $\dim(Z \smallsetminus U) < \dim U = \dim \mathcal{O}_{\mathbf{c}}$, we have $\dim m(Z \smallsetminus U) < \dim \mathcal{O}_{\mathbf{c}}$. Since $m(Z \smallsetminus U)$ is stable under $G_{\mathbf{w}}$, it cannot meet $\mathcal{O}_{\mathbf{c}}$.

It is now clear from the definitions that

$$j_{[\gamma_1^{\mathbf{c}(\gamma_1)}]!}\underline{\mathbb{Q}}[d_1](\tfrac{d_1}{2}) \,\tilde{\boxtimes} \cdots \tilde{\boxtimes}\, j_{[\gamma_N^{\mathbf{c}(\gamma_N)}]!}\underline{\mathbb{Q}}[d_N](\tfrac{d_N}{2}) \cong h_!\underline{\mathbb{Q}}[d](\tfrac{d}{2}),$$

where $d = \dim U = \dim Z$. Part (1) follows.

For part (2), again using the fact that each $\mathcal{O}_{[\gamma_t^{\mathbf{c}(\gamma_t)}]}$ is dense in $\mathcal{E}_{\mathbf{v}^t}$, we have

$$\mathrm{IC}_{[\gamma_t^{\mathbf{c}(\gamma_t)}]} \cong \underline{\mathbb{Q}}_{\mathcal{E}_{\mathbf{v}^t}}[\dim \mathcal{E}_{\mathbf{v}^t}](\tfrac{1}{2}\dim \mathcal{E}_{\mathbf{v}^t}).$$

It follows that $\mathrm{IC}_{[\gamma_1^{\mathbf{c}(\gamma_1)}]} \,\tilde{\boxtimes} \cdots \tilde{\boxtimes}\, \mathrm{IC}_{[\gamma_N^{\mathbf{c}(\gamma_N)}]} \cong \underline{\mathbb{Q}}_Z[d](\frac{d}{2})$, and hence

$$\mathrm{IC}_{[\gamma_1^{\mathbf{c}(\gamma_1)}]} \star \cdots \star \mathrm{IC}_{[\gamma_N^{\mathbf{c}(\gamma_N)}]} \cong m_*\underline{\mathbb{Q}}_Z[d](\tfrac{d}{2}).$$

By proper base change, we have $(m_*\underline{\mathbb{Q}}_Z[d](\frac{d}{2}))|_{\mathcal{O}_{\mathbf{c}}} \cong \underline{\mathbb{Q}}_{\mathcal{O}_{\mathbf{c}}}[d](\frac{d}{2})$. Of course, the only simple perverse sheaf with support contained in $\overline{\mathcal{O}_{\mathbf{c}}}$ and nonzero restriction to $\mathcal{O}_{\mathbf{c}}$ is $\mathrm{IC}_{\mathbf{c}}$ itself, so the multiplicity of $\mathrm{IC}_{\mathbf{c}}[n]$ as a summand of $m_*\underline{\mathbb{Q}}_Z[d](\frac{d}{2})$ is as claimed. $\square$

PROPOSITION 10.3.10. *Let Q be a quiver of finite type, and let $\gamma_1, \ldots, \gamma_N$ be an adapted order on the corresponding set of positive roots Φ^+. List the simple roots in this order as $\mathbf{e}_1, \ldots, \mathbf{e}_n$. For any dimension vector $\mathbf{v} : Q_0 \to \mathbb{Z}_{\geq 0}$, we have*

$$\mathrm{IC}_{[\mathbf{e}_n^{\mathbf{v}_n}]} \star \cdots \star \mathrm{IC}_{[\mathbf{e}_2^{\mathbf{v}_2}]} \star \mathrm{IC}_{[\mathbf{e}_1^{\mathbf{v}_1}]} \cong \underline{\mathbb{Q}}_{\mathcal{E}_{\mathbf{v}}}[\dim \mathcal{E}_{\mathbf{v}}](\tfrac{1}{2}\dim \mathcal{E}_{\mathbf{v}}).$$

PROOF. The dimension vectors that arise in the convolution diagram (10.3.8) are $\mathbf{v}_n\mathbf{e}_n, \ldots, \mathbf{v}_1\mathbf{e}_1$. Recall from Corollary 10.1.15 that for any edge $i \to j$, we have $i > j$. It is easy to deduce from this that $\mathcal{E}_{\mathbf{v}_n\mathbf{e}_n,\ldots,\mathbf{v}_1\mathbf{e}_1} = \mathcal{E}_{\mathbf{v}}$ and that $P_{\mathbf{v}_n\mathbf{e}_n,\ldots,\mathbf{v}_1\mathbf{e}_1} = G_{\mathbf{v}}$. Thus, the maps i and m are isomorphisms. Since each $\mathcal{E}_{\mathbf{v}_t\mathbf{e}_t}$ is a single point, we have $\mathrm{IC}_{[\mathbf{e}_t^{\mathbf{v}_t}]} \cong \underline{\mathbb{Q}}_{\mathcal{E}_{\mathbf{v}_t\mathbf{e}_t}}$. The convolution product $\mathrm{IC}_{[\mathbf{e}_n^{\mathbf{v}_n}]} \star \cdots \star \mathrm{IC}_{[\mathbf{e}_1^{\mathbf{v}_1}]}$ is thus identified with

$$p^*(\underline{\mathbb{Q}}_{\mathcal{E}_{\mathbf{v}_n\mathbf{e}_n}} \boxtimes \cdots \boxtimes \underline{\mathbb{Q}}_{\mathcal{E}_{\mathbf{v}_1\mathbf{e}_1}})[\dim \mathcal{E}_{\mathbf{v}}](\tfrac{1}{2}\dim \mathcal{E}_{\mathbf{v}}) \cong \underline{\mathbb{Q}}_{\mathcal{E}_{\mathbf{v}}}[\dim \mathcal{E}_{\mathbf{v}}](\tfrac{1}{2}\dim \mathcal{E}_{\mathbf{v}}),$$

as desired. $\square$

PROPOSITION 10.3.11. *Let Q be a quiver of finite type, with corresponding root system Φ. For every Kostant partition $\mathbf{c} : \Phi^+ \to \mathbb{Z}_{\geq 0}$, there exists a sequence of elements $i_1, \ldots, i_m \in Q_0$ and integers $n_1, \ldots, n_m \geq 1$ such that the object*

$$\mathrm{IC}_{[\mathbf{e}_{i_1}^{n_1}]} \star \cdots \star \mathrm{IC}_{[\mathbf{e}_{i_m}^{n_m}]}$$

is supported on $\overline{\mathcal{O}_{\mathbf{c}}}$ and such that $\mathrm{IC}_{\mathbf{c}}$ occurs as a direct summand in this object with multiplicity 1.

PROOF. Recall from Lemma 10.1.11 that for any root $\gamma \in \Phi^+$ and any integer $c \geq 0$, the orbit $\mathcal{O}_{[\gamma^c]}$ is dense in $\mathcal{E}_{c\gamma}$, and hence that $\mathrm{IC}_{[\gamma^c]} \cong \underline{\mathbb{Q}}_{\mathcal{E}_{c\gamma}}[\dim \mathcal{E}_{c\gamma}]$. The result then follows by combining Propositions 10.3.9 and 10.3.10. □

Exercises.

10.3.1. The proof of Lemma 10.3.8 relies on the decomposition theorem to determine the object described in (10.3.11). Explain how to replace this by an Euler class argument, and then deduce that Lemma 10.3.8 is true for arbitrary $\Bbbk$.

10.4. Canonical bases for quantum groups

For the remainder of this chapter, all quivers will be of finite type.

The categorification theorem. We are now ready to prove the main result of this chapter. By Lemmas 10.3.2, 10.3.3, and 10.3.6, the convolution product endows the sum of split Grothendieck groups

$$\bigoplus_{\mathbf{v} \in \mathbb{Z}_{\geq 0}^{Q_0}} \mathrm{K}_{\oplus}(\mathrm{Semis}_{G_{\mathbf{v}}}(\mathcal{E}_{\mathbf{v}}, \mathbb{Q}))$$

with the structure of an associative ring.

THEOREM 10.4.1. *Let $Q = (Q_0, Q_1)$ be a quiver of finite type, and let Φ be the corresponding root system. There is an isomorphism of rings*

$$\mathrm{ch} : \bigoplus_{\mathbf{v} \in \mathbb{Z}_{\geq 0}^{Q_0}} \mathrm{K}_{\oplus}(\mathrm{Semis}_{G_{\mathbf{v}}}(\mathcal{E}_{\mathbf{v}}, \mathbb{Q})) \xrightarrow{\sim} \mathcal{U}_v^+(\Phi)$$

given by $\mathrm{ch}([\mathrm{IC}_{[\mathbf{e}_i^n]}]) = E_i^{(n)}$ for $i \in Q_0$ and $n \geq 1$. For any $\mathcal{F} \in \mathrm{Semis}_{G_{\mathbf{v}}}(\mathcal{E}_{\mathbf{v}}, \mathbb{Q})$, we have

$$\mathrm{ch}([\mathcal{F}[1]]) = v^{-1}\mathrm{ch}([\mathcal{F}]) \qquad \textit{and} \qquad \mathrm{ch}([\mathbb{D}\mathcal{F}]) = \overline{\mathrm{ch}([\mathcal{F}])}.$$

For another description of ch, see Theorem 10.5.7.

PROOF. For brevity, we write $\mathrm{K}_{\oplus}(Q) = \bigoplus_{\mathbf{v} \in \mathbb{Z}_{\geq 0}^{Q_0}} \mathrm{K}_{\oplus}(\mathrm{Semis}_{G_{\mathbf{v}}}(\mathcal{E}_{\mathbf{v}}, \mathbb{Q}))$. We will prove the theorem by constructing an isomorphism $\psi : \mathcal{U}_v^+(\Psi) \to \mathrm{K}_{\oplus}(Q)$ in the opposite direction. Make $\mathrm{K}_{\oplus}(Q)$ into a $\mathbb{Z}[v, v^{-1}]$-algebra by setting

$$v[\mathcal{F}] = [\mathcal{F}[-1]].$$

As a $\mathbb{Z}[v, v^{-1}]$-module, $\mathrm{K}_{\oplus}(Q)$ is free, with a basis given by the elements

$$\{[\mathrm{IC}_{\mathbf{c}}]\}_{\mathbf{c} \in \mathrm{KP}}. \tag{10.4.1}$$

By Lemma 10.3.7 and by Lemma 10.4.2 below, we have

$$[\mathrm{IC}_{[\mathbf{e}_i]}]^n = v^{\frac{n(n-1)}{2}}[n]_v![\mathrm{IC}_{[\mathbf{e}_i^n]}[n(n-1)/2]] = [n]_v![\mathrm{IC}_{[\mathbf{e}_i^n]}]. \tag{10.4.2}$$

Let us first define a map $\psi' : \mathbf{U}^+(\Phi) \to \mathbb{Q}(v) \otimes_{\mathbb{Z}[v,v^{-1}]} \mathrm{K}_\oplus(Q)$ by setting $\psi'(E_i) = [\mathrm{IC}_{[\mathbf{e}_i]}]$. For this map to be well defined, we just need to check that the $[\mathrm{IC}_{[\mathbf{e}_i]}]$ satisfy the relations (10.2.4). Using (10.4.2), we see that these relations are precisely the content of Lemma 10.3.8.

Since $\mathrm{K}_\oplus(Q)$ is free as a $\mathbb{Z}[v,v^{-1}]$-module, the natural map

$$\mathrm{K}_\oplus(Q) \to \mathbb{Q}(v) \otimes_{\mathbb{Z}[v,v^{-1}]} \mathrm{K}_\oplus(Q)$$

is injective. We henceforth regard $\mathrm{K}_\oplus(Q)$ as a subset of $\mathbb{Q}(v) \otimes_{\mathbb{Z}[v,v^{-1}]} \mathrm{K}_\oplus(Q)$.

The map ψ' restricts to a map from $\mathcal{U}_v^+(\Phi)$ to the $\mathbb{Z}[v,v^{-1}]$-subalgebra of $\mathbb{Q}(v) \otimes_{\mathbb{Z}[v,v^{-1}]} \mathrm{K}_\oplus(Q)$ generated by the elements of the form $\frac{1}{[n]_v!}[\mathrm{IC}_{[\mathbf{e}_i]}]^n$. According to (10.4.2), these elements actually lie in $\mathrm{K}_\oplus(Q)$. In other words, there is an induced map $\psi : \mathcal{U}_v^+(\Phi) \to \mathrm{K}_\oplus(Q)$ such that the diagram

$$\begin{tikzcd} \mathcal{U}_v^+(\Phi) \arrow[r, "\psi"] \arrow[d] & \mathrm{K}_\oplus(Q) \arrow[d] \\ \mathbf{U}^+(\Phi) \arrow[r, "\psi'"] & \mathbb{Q}(v) \otimes_{\mathbb{Z}[v,v^{-1}]} \mathrm{K}_\oplus(Q) \end{tikzcd} \tag{10.4.3}$$

commutes. The image of ψ is the subalgebra of $\mathrm{K}_\oplus(Q)$ generated by elements of the form $[\mathrm{IC}_{[\mathbf{e}_i^n]}]$. Proposition 10.3.11 implies that this subalgebra is in fact $\mathrm{K}_\oplus(Q)$ itself. That is, ψ is surjective. It follows that ψ' is surjective as well.

By construction, the maps ψ and ψ' are compatible with the grading on the quantum group defined in (10.2.6). That is, ψ' restricts to a map of $\mathbb{Q}(v)$-vector spaces

$$\psi'_{\mathbf{v}} : \mathbf{U}^+(\Phi)_{\mathbf{v}} \to \mathbb{Q}(v) \otimes_{\mathbb{Z}[v,v^{-1}]} \mathrm{K}_\oplus(\mathrm{Semis}_{G_{\mathbf{v}}}(\mathcal{E}_{\mathbf{v}}, \mathbb{Q})).$$

The set $\{[\mathrm{IC}_{\mathbf{c}}]\}_{\mathbf{c}\in\mathrm{KP}(\mathbf{v})}$ is a $\mathbb{Q}(v)$-basis for the latter. According to the discussion following (10.2.6), $\mathbf{U}^+(\Phi)_{\mathbf{v}}$ also has dimension $|\mathrm{KP}(\mathbf{v})|$. Since $\psi'_{\mathbf{v}}$ is a surjective map of $\mathbb{Q}(v)$-vector spaces of the same (finite) dimension, it is also injective. This holds for all $\mathbf{v}$, so ψ' itself is injective. Since the left-hand vertical map in (10.4.3) is injective, we conclude that ψ is injective as well, and hence an isomorphism.

Finally, Lemma 10.3.5 implies that $\mathbb{D}$ induces a ring automorphism of $\mathrm{K}_\oplus(Q)$. Since each $\mathrm{IC}_{\mathbf{e}_i}$ is the skyscraper sheaf $\underline{\mathbb{Q}}$ on a point, we clearly have $\mathbb{D}(\mathrm{IC}_{\mathbf{e}_i}) \cong \mathrm{IC}_{\mathbf{e}_i}$. Combining this with the observation that $\mathbb{D}(\mathcal{F}[1]) \cong (\mathbb{D}\mathcal{F})[-1]$, we conclude that the ring automorphism induced by $\mathbb{D}$ corresponds under ψ to the bar involution on $\mathcal{U}_v^+(\Phi)$. ☐

The proofs of Lemma 10.3.7 and Theorem 10.4.1 made use of the following facts about the topology of the flag variety Fl_n. For a proof, see [**41**, Proposition 21.17], or the discussion in the proof of [**98**, Theorem 4D.4].

Lemma 10.4.2. *We have* $\dim \mathrm{Fl}_n = n(n-1)/2$. *In addition,* $\mathbf{H}^k(\mathrm{Fl}_n, \Bbbk)$ *is a free* $\Bbbk$*-module for all* k*, and*

$$\sum_{k\geq 0} (\operatorname{rank} \mathbf{H}^k(\mathrm{Fl}_n, \Bbbk)) v^k = v^{\frac{n(n-1)}{2}} [n]_v!.$$

The canonical basis. In the setting of Theorem 10.4.1, for $\mathbf{c} \in \mathrm{KP}$, let $B_{\mathbf{c}} = \psi^{-1}([\mathrm{IC}_{\mathbf{c}}])$. Since (10.4.1) is a $\mathbb{Z}[v,v^{-1}]$-basis for $\bigoplus_{\mathbf{v}\in\mathbb{Z}_{\geq 0}^{Q_0}} \mathrm{K}_\oplus(\mathrm{Semis}_{G_{\mathbf{v}}}(\mathcal{E}_{\mathbf{v}}, \mathbb{Q}))$, the set

$$\{B_{\mathbf{c}}\}_{\mathbf{c}\in\mathrm{KP}} \tag{10.4.4}$$

is a $\mathbb{Z}[v,v^{-1}]$-basis for $\mathcal{U}_v^+(\Phi)$.

DEFINITION 10.4.3. The set (10.4.4) is called the **canonical basis** for the strictly positive part of the quantum group $\mathcal{U}_v^+(\Phi)$.

The terminology suggests that this basis is "independent of choices," but the statement of Theorem 10.4.1 involves the choice of the quiver Q, i.e., a choice of orientation for each edge of the Dynkin diagram of Φ. To show that the canonical basis is independent of these choices, we must compare the spaces $\mathcal{E}_{\mathbf{v}}(Q)$ and $\mathcal{E}_{\mathbf{v}}(Q')$, where Q and Q' are distinct quivers with the same underlying undirected graph. Of course, the key case is that in which Q and Q' differ in the orientation of a single edge.

Let us fix some notation to describe this situation. Given a quiver $Q = (Q_0, Q_1)$, fix an edge $h \in Q_1$. Let $Q' = (Q_0, Q_1')$, where we have

$$Q_1' = (Q_1 \smallsetminus \{h\}) \cup \{h'\}, \qquad \mathrm{head}(h') = \mathrm{tail}(h), \quad \mathrm{tail}(h') = \mathrm{head}(h).$$

It will also be convenient to refer to the quiver $\hat{Q} = (Q_0, \hat{Q}_1)$, where

$$\hat{Q}_1 = Q_1 \smallsetminus \{h\}.$$

We also let

$$\mathcal{E}_{\mathbf{v},h} = \mathrm{Hom}(\mathbb{C}^{\mathbf{v}_{\mathrm{tail}(h)}}, \mathbb{C}^{\mathbf{v}_{\mathrm{head}(h)}}) \qquad \text{and} \qquad \mathcal{E}_{\mathbf{v},h'} = \mathrm{Hom}(\mathbb{C}^{\mathbf{v}_{\mathrm{head}(h)}}, \mathbb{C}^{\mathbf{v}_{\mathrm{tail}(h)}}).$$

Note that the trace pairing

$$\mathcal{E}_{\mathbf{v},h} \times \mathcal{E}_{\mathbf{v},h'} \to \mathbb{C} \qquad \text{given by} \qquad (x, y) \mapsto \mathrm{tr}(xy)$$

identifies each of the vector spaces $\mathcal{E}_{\mathbf{v},h}$ and $\mathcal{E}_{\mathbf{v},h'}$ with the dual of the other. Let us denote the dual of a complex vector space V by V'. Then we have an identification

$$\mathcal{E}_{\mathbf{v},h}' = \mathcal{E}_{\mathbf{v},h'}. \tag{10.4.5}$$

We also have

$$\mathcal{E}_{\mathbf{v}}(Q) = \mathcal{E}_{\mathbf{v},h} \times \mathcal{E}_{\mathbf{v}}(\hat{Q}), \qquad \mathcal{E}_{\mathbf{v}}(Q') = \mathcal{E}_{\mathbf{v},h}' \times \mathcal{E}_{\mathbf{v}}(\hat{Q}).$$

These descriptions let us think of $\mathcal{E}_{\mathbf{v}}(Q)$ and $\mathcal{E}_{\mathbf{v}}(Q')$ as vector bundles over $\mathcal{E}_{\mathbf{v}}(\hat{Q})$, with fibers $\mathcal{E}_{\mathbf{v},h}$ and $\mathcal{E}_{\mathbf{v},h}'$, respectively.

To compare sheaves on $\mathcal{E}_{\mathbf{v}}(Q)$ and $\mathcal{E}_{\mathbf{v}}(Q')$, we will use the Fourier–Laumon transform. This requires introducing an additional $\mathbb{G}_{\mathrm{m}}$-action. Let $\mathbb{G}_{\mathrm{m}}$ act on $\mathcal{E}_{\mathbf{v},h}$ and on $\mathcal{E}_{\mathbf{v},h'}$ by scaling, and let it act trivially on $\mathcal{E}_{\mathbf{v}}(\hat{Q})$. The resulting action of $\mathbb{G}_{\mathrm{m}}$ on $\mathcal{E}_{\mathbf{v}}(Q)$ or on $\mathcal{E}_{\mathbf{v}}(Q')$ commutes with the action of $G_{\mathbf{v}}$. With this setup, there is a Fourier–Laumon transform functor

$$\mathrm{Four}_h : D^{\mathrm{b}}_{\mathbb{G}_{\mathrm{m}} \times G}(\mathcal{E}_{\mathbf{v}}(Q), \mathbb{Q}) \to D^{\mathrm{b}}_{\mathbb{G}_{\mathrm{m}} \times G}(\mathcal{E}_{\mathbf{v}}(Q'), \mathbb{Q}). \tag{10.4.6}$$

LEMMA 10.4.4. *The $\mathbb{G}_{\mathrm{m}} \times G_{\mathbf{v}}$-orbits on $\mathcal{E}_{\mathbf{v}}(Q)$ coincide with the $G_{\mathbf{v}}$-orbits. Every simple $G_{\mathbf{v}}$-equivariant perverse sheaf is also $\mathbb{G}_{\mathrm{m}} \times G_{\mathbf{v}}$-equivariant.*

PROOF. The first part is identical to Lemma 8.2.7(1). Since we have already seen that every $G_{\mathbf{v}}$-equivariant perverse sheaf arises as the IC object of the trivial local system on some orbit, the second assertion is obvious. $\square$

Suppose we have two dimension vectors $\mathbf{v}, \mathbf{w} : Q_0 \to \mathbb{Z}_{\geq 0}$. The $\mathbb{G}_{\mathrm{m}}$-action on $\mathcal{E}_{\mathbf{v}+\mathbf{w}}$ (involving only the edge h, as in the remarks preceding (10.4.6)) restricts to a $\mathbb{G}_{\mathrm{m}}$-action on $\mathcal{E}_{\mathbf{v},\mathbf{w}}$, and the latter induces a $\mathbb{G}_{\mathrm{m}}$-action on the space $G_{\mathbf{v}+\mathbf{w}} \times^{P_{\mathbf{v},\mathbf{w}}}$

$\mathcal{E}_{\mathbf{v},\mathbf{w}}$. All three maps in the convolution diagram (10.3.5) are $\mathbb{G}_{\mathrm{m}}$-equivariant, so the construction of the convolution product can be repeated to define a functor

$$\star : D^{\mathrm{b}}_{\mathbb{G}_{\mathrm{m}}\times G_{\mathbf{v}}}(\mathcal{E}_{\mathbf{v}}, \mathbb{Q}) \times D^{\mathrm{b}}_{\mathbb{G}_{\mathrm{m}}\times G_{\mathbf{v}}}(\mathcal{E}_{\mathbf{w}}, \mathbb{Q}) \to D^{\mathrm{b}}_{\mathbb{G}_{\mathrm{m}}\times G_{\mathbf{v}+\mathbf{w}}}(\mathcal{E}_{\mathbf{v}+\mathbf{w}}, \mathbb{Q}).$$

The convolution product commutes with the forgetful functor that forgets the $\mathbb{G}_{\mathrm{m}}$-action. Lemma 10.4.4 implies that the forgetful functor

$$\mathrm{For}^{\mathbb{G}_{\mathrm{m}}\times G_{\mathbf{v}}}_{G_{\mathbf{v}}} : \mathrm{Semis}_{\mathbb{G}_{\mathrm{m}}\times G_{\mathbf{v}}}(\mathcal{E}_{\mathbf{v}}, \mathbb{Q}) \to \mathrm{Semis}_{G_{\mathbf{v}}}(\mathcal{E}_{\mathbf{v}}, \mathbb{Q})$$

induces a bijection on the set of indecomposable objects, and also an isomorphism of split Grothendieck groups. Combining these observations with Theorem 10.4.1, we obtain an isomorphism of rings

$$\mathcal{U}_v^+(\Phi) \xrightarrow{\sim} \bigoplus_{\mathbf{v}\in\mathbb{Z}^{Q_0}_{\geq 0}} \mathrm{K}_{\oplus}(\mathrm{Semis}_{\mathbb{G}_{\mathrm{m}}\times G_{\mathbf{v}}}(\mathcal{E}_{\mathbf{v}}, \mathbb{Q})).$$

Of course, the preceding remarks apply to Q' as well.

PROPOSITION 10.4.5. *For $\mathcal{F} \in D^{\mathrm{b}}_{\mathbb{G}_{\mathrm{m}}\times G_{\mathbf{v}}}(\mathcal{E}_{\mathbf{v}}, \mathbb{Q})$ and $\mathcal{G} \in D^{\mathrm{b}}_{\mathbb{G}_{\mathrm{m}}\times G_{\mathbf{v}}}(\mathcal{E}_{\mathbf{w}}, \mathbb{Q})$, there is a natural isomorphism*

$$\mathrm{Four}_h(\mathcal{F}) \star \mathrm{Four}_h(\mathcal{G}) \cong \mathrm{Four}_h(\mathcal{F} \star \mathcal{G})(\tfrac{1}{2}d),$$

where $d = \mathbf{v}_{\mathrm{tail}(h)}\mathbf{w}_{\mathrm{head}(h)} + \mathbf{v}_{\mathrm{head}(h)}\mathbf{w}_{\mathrm{tail}(h)}$.

PROOF. Let $i = \mathrm{tail}(h)$ and $j = \mathrm{head}(h)$. Consider the vector spaces

$$\begin{aligned}
\mathcal{E}_{\mathbf{v},h} \times \mathcal{E}_{\mathbf{w},h} &= \mathrm{Hom}(\mathbb{C}^{\mathbf{v}_i}, \mathbb{C}^{\mathbf{v}_j}) \times \mathrm{Hom}(\mathbb{C}^{\mathbf{w}_i}, \mathbb{C}^{\mathbf{w}_j}),\\
\mathcal{E}_{\mathbf{v},\mathbf{w},h} &= \mathcal{E}_{\mathbf{v},h} \times \mathcal{E}_{\mathbf{w},h} \times \mathrm{Hom}(\mathbb{C}^{\mathbf{v}_i}, \mathbb{C}^{\mathbf{w}_j}),\\
\bar{\mathcal{E}}_{\mathbf{v},\mathbf{w},h} &= \mathcal{E}_{\mathbf{v},h} \times \mathcal{E}_{\mathbf{w},h} \times \mathrm{Hom}(\mathbb{C}^{\mathbf{w}_i}, \mathbb{C}^{\mathbf{v}_j}),\\
\mathcal{E}_{\mathbf{v}+\mathbf{w},h} &= \mathrm{Hom}(\mathbb{C}^{\mathbf{v}_i} \oplus \mathbb{C}^{\mathbf{w}_i}, \mathbb{C}^{\mathbf{v}_j} \oplus \mathbb{C}^{\mathbf{w}_j}).
\end{aligned}$$

These spaces fit into a commutative diagram

$$\begin{tikzcd}
\mathcal{E}_{\mathbf{v},h} \times \mathcal{E}_{\mathbf{w},h} \arrow[d, hook, "t_h"'] & \mathcal{E}_{\mathbf{v},\mathbf{w},h} \arrow[l, two heads, "p_h"'] \arrow[d, hook, "s_h"] \\
\bar{\mathcal{E}}_{\mathbf{v},\mathbf{w},h} & \mathcal{E}_{\mathbf{v}+\mathbf{w},h} \arrow[l, two heads, "q_h"']
\end{tikzcd} \tag{10.4.7}$$

where the horizontal maps p_h and q_h are quotient maps, and the vertical maps are inclusions of subspaces. Let us now take the vector space dual of this diagram. By (10.4.5), the duals of the upper-left and lower-right corners are the corresponding spaces for h'. On the other hand, the trace pairing on $\mathcal{E}_{\mathbf{v}+\mathbf{w},h} \times \mathcal{E}_{\mathbf{v}+\mathbf{w},h'}$ induces a nondegenerate pairing

$$\mathcal{E}_{\mathbf{v},\mathbf{w},h} \times \bar{\mathcal{E}}_{\mathbf{v},\mathbf{w},h'} \to \mathbb{C}$$

that gives rise to an identification $\mathcal{E}'_{\mathbf{v},\mathbf{w},h} = \bar{\mathcal{E}}_{\mathbf{v},\mathbf{w},h'}$. Indeed, the dual of (10.4.7) is the diagram

$$\begin{tikzcd}
\mathcal{E}'_{\mathbf{v},h} \times \mathcal{E}'_{\mathbf{w},h} \arrow[d, hook, "p'_h = t_{h'}"'] & \bar{\mathcal{E}}'_{\mathbf{v},\mathbf{w},h} = \mathcal{E}_{\mathbf{v},\mathbf{w},h'} \arrow[l, two heads, "t'_h = p_{h'}"'] \arrow[d, hook, "q'_h = s_{h'}"] \\
\mathcal{E}'_{\mathbf{v},\mathbf{w},h} = \bar{\mathcal{E}}_{\mathbf{v},\mathbf{w},h'} & \mathcal{E}'_{\mathbf{v}+\mathbf{w},h} \arrow[l, two heads, "s'_h = q_{h'}"']
\end{tikzcd} \tag{10.4.8}$$

As the labels show, passing to the dual has swapped the horizontal and vertical arrows. Using Proposition 6.9.13 and the fact that (10.4.8) is cartesian, we have

$$\begin{aligned}\mathrm{Four}(s_{h!}p_h^*\mathcal{H}) &\cong (s'_h)^* p'_{h!}\mathrm{Four}(\mathcal{H})[\mathbf{v}_j\mathbf{w}_i - \mathbf{v}_i\mathbf{w}_j](-\mathbf{v}_i\mathbf{w}_j) \\ &\cong s_{h'!}p_{h'}^*\mathrm{Four}(\mathcal{H})[\mathbf{v}_j\mathbf{w}_i - \mathbf{v}_i\mathbf{w}_j](-\mathbf{v}_i\mathbf{w}_j)\end{aligned}$$

for any $\mathcal{H} \in D^{\mathrm{b}}_{\mathbb{G}_{\mathrm{m}}\times G_{\mathbf{v}}}(\mathcal{E}_{\mathbf{v},h} \times \mathcal{E}_{\mathbf{w},h}, \mathbb{Q})$. The quivers Q and Q' determine different geometric pairings on $\mathbb{Z}^{Q_0}_{\geq 0}$, related by the rule $\mathbf{v}_j\mathbf{w}_i - \mathbf{v}_i\mathbf{w}_j = \langle\!\langle \mathbf{v}, \mathbf{w}\rangle\!\rangle_{Q'} - \langle\!\langle \mathbf{v}, \mathbf{w}\rangle\!\rangle_{Q}$. Let $d = \mathbf{v}_i\mathbf{w}_j + \mathbf{v}_j\mathbf{w}_i$. The isomorphism above can be rearranged to say

$$\begin{aligned}&\mathrm{Four}(s_{h!}p_h^*\mathcal{H})[\langle\!\langle \mathbf{v}, \mathbf{w}\rangle\!\rangle_{Q}](\tfrac{1}{2}\langle\!\langle \mathbf{v}, \mathbf{w}\rangle\!\rangle_{Q} + \tfrac{1}{2}d) \\ &\qquad\cong s_{h'!}p_{h'}^*\mathrm{Four}(\mathcal{H})[\langle\!\langle \mathbf{v}, \mathbf{w}\rangle\!\rangle_{Q'}](\tfrac{1}{2}\langle\!\langle \mathbf{v}, \mathbf{w}\rangle\!\rangle_{Q'}).\end{aligned} \tag{10.4.9}$$

To finish the proof, we need to embed the calculation above into the definition of the convolution product. Consider the diagram

(10.4.10)

$$\begin{tikzcd}
\boxed{\mathcal{E}_{\mathbf{v}}(\hat{Q}) \times \mathcal{E}_{\mathbf{w}}(\hat{Q}) \times \mathcal{E}_{\mathbf{v},h} \times \mathcal{E}_{\mathbf{w},h}} & \mathcal{E}_{\mathbf{v}}(\hat{Q}) \times \mathcal{E}_{\mathbf{w}}(\hat{Q}) \times \mathcal{E}_{\mathbf{v},\mathbf{w},h} \arrow[l] \arrow[r] & \mathcal{E}_{\mathbf{v}}(\hat{Q}) \times \mathcal{E}_{\mathbf{w}}(\hat{Q}) \times \mathcal{E}_{\mathbf{v}+\mathbf{w},h} \\
\mathcal{E}_{\mathbf{v},\mathbf{w}}(\hat{Q}) \times \mathcal{E}_{\mathbf{v},h} \times \mathcal{E}_{\mathbf{w},h} \arrow[u] \arrow[d] & \boxed{\mathcal{E}_{\mathbf{v},\mathbf{w}}(\hat{Q}) \times \mathcal{E}_{\mathbf{v},\mathbf{w},h}} \arrow[l] \arrow[r] \arrow[u] \arrow[d, "i"] \arrow[ul, "p"'] & \mathcal{E}_{\mathbf{v},\mathbf{w}}(\hat{Q}) \times \mathcal{E}_{\mathbf{v}+\mathbf{w},h} \arrow[u] \arrow[d] \\
G_{\mathbf{v}+\mathbf{w}} \times^{P_{\mathbf{v},\mathbf{w}}} (\mathcal{E}_{\mathbf{v},\mathbf{w}}(\hat{Q}) \times \mathcal{E}_{\mathbf{v},h} \times \mathcal{E}_{\mathbf{w},h}) \arrow[d] & \boxed{G_{\mathbf{v}+\mathbf{w}} \times^{P_{\mathbf{v},\mathbf{w}}} (\mathcal{E}_{\mathbf{v},\mathbf{w}}(\hat{Q}) \times \mathcal{E}_{\mathbf{v},\mathbf{w},h})} \arrow[l] \arrow[r] \arrow[d] \arrow[dr, "m"] & G_{\mathbf{v}+\mathbf{w}} \times^{P_{\mathbf{v},\mathbf{w}}} (\mathcal{E}_{\mathbf{v},\mathbf{w}}(\hat{Q}) \times \mathcal{E}_{\mathbf{v}+\mathbf{w},h}) \arrow[d] \\
\mathcal{E}_{\mathbf{v}+\mathbf{w}}(\hat{Q}) \times \mathcal{E}_{\mathbf{v},h} \times \mathcal{E}_{\mathbf{w},h} & \mathcal{E}_{\mathbf{v}+\mathbf{w}}(\hat{Q}) \times \mathcal{E}_{\mathbf{v},\mathbf{w},h} \arrow[l] \arrow[r] & \boxed{\mathcal{E}_{\mathbf{v}+\mathbf{w}}(\hat{Q}) \times \mathcal{E}_{\mathbf{v},\mathbf{w},h}}
\end{tikzcd}$$

Each row of this diagram consists of maps of vector bundles over a fixed base. In each case, the maps on the fibers of the vector bundles are given by

$$\mathcal{E}_{\mathbf{v},h} \times \mathcal{E}_{\mathbf{w},h} \xleftarrow{p_h} \mathcal{E}_{\mathbf{v},\mathbf{w},h} \xrightarrow{s_h} \mathcal{E}_{\mathbf{v}+\mathbf{w},h}.$$

Each vertical map in (10.4.10) is a base change for a vector bundle. In each column, the base changes come from the convolution diagram for $\hat{Q}$. Finally, the spaces enclosed in boxes are the ones that appear in the convolution diagram for Q.

There is analogous diagram for h', in which the spaces enclosed in boxes give the convolution diagram for Q'. We wish to compare these two large diagrams. The calculation (10.4.9) applies to each row of (10.4.10), and Proposition 6.9.15 tells us that the Fourier–Laumon transform commutes with pullback or pushforward along each vertical map. Together, these observations imply the desired result. $\square$

THEOREM 10.4.6. *Let Φ be a simply laced root system. The canonical basis of $\mathcal{U}_v^+(\Phi)$ is independent of the choice of quiver Q corresponding to Φ.*

PROOF. Let Q be a quiver whose underlying undirected graph is the Dynkin diagram of Φ, and let Q' be another quiver that differs from Q in the orientation of exactly one edge. It is enough to show that the canonical bases coming from Q

and Q' agree. Consider the diagram of ring homomorphisms

$$\begin{tikzcd} \bigoplus_{\mathbf{v}\in\mathbb{Z}_{\geq 0}^{Q_0}} \mathrm{K}_{\oplus}(\mathrm{Semis}_{\mathbb{G}_{\mathrm{m}}\times G_{\mathbf{v}}}(\mathcal{E}_{\mathbf{v}}(Q),\mathbb{Q})) \arrow[d, "\mathrm{Four}_h"'] \arrow[dr, "\mathrm{ch}_Q"] & \\ & \mathcal{U}_v^+(\Phi) \\ \bigoplus_{\mathbf{v}\in\mathbb{Z}_{\geq 0}^{Q_0}} \mathrm{K}_{\oplus}(\mathrm{Semis}_{\mathbb{G}_{\mathrm{m}}\times G_{\mathbf{v}}}(\mathcal{E}_{\mathbf{v}}(Q'),\mathbb{Q})) \arrow[ur, "\mathrm{ch}_{Q'}"'] & \end{tikzcd} \tag{10.4.11}$$

where both diagonal arrows come from Theorem 10.4.1. Recall that the canonical basis is obtained by transferring the basis of simple perverse sheaves across one of the diagonal arrows. Since the Fourier–Laumon transform takes simple perverse sheaves to simple perverse sheaves, to prove the theorem, it is enough to prove that the diagram (10.4.11) commutes. Since $\mathcal{U}_v^+(\Phi)$ is generated as a ring by the elements $E_i^{(n)}$, it is enough to show that

$$\mathrm{Four}_h(\mathrm{IC}_{[\mathbf{e}_i^n]}) \cong \mathrm{IC}_{[\mathbf{e}_i^n]}. \tag{10.4.12}$$

For the dimension vector $\mathbf{v} = n\mathbf{e}_i$, the vector spaces $\mathcal{E}_{\mathbf{v},h}$ and $\mathcal{E}'_{\mathbf{v},h}$ are both zero. The corresponding Fourier–Laumon transform is therefore just the identity functor, and (10.4.12) holds trivially. $\square$

10.5. Mixed Hodge modules and categorification

If we choose a PBW basis $\{E_{\mathbf{c}}\}_{\mathbf{c}\in\mathrm{KP}}$ for $\mathcal{U}_v^+(\Phi)$, then the canonical basis can be written in terms of it:

$$B_{\mathbf{c}} = \sum_{\mathbf{c}'\in\mathrm{KP}} P_{\mathbf{c}',\mathbf{c}} E_{\mathbf{c}'}.$$

Here, the elements $P_{\mathbf{c}',\mathbf{c}} \in \mathbb{Z}[v, v^{-1}]$ play a role analogous to that of Kazhdan–Lusztig polynomials in Chapter 7. The goal of this section is to give a sheaf-theoretic interpretation of these polynomials. Although the main result is about $\mathrm{Semis}_{G_{\mathbf{v}}}(\mathcal{E}_{\mathbf{v}},\mathbb{Q})$, the arguments make heavy use of the theory of mixed Hodge modules.

LEMMA 10.5.1. *Let $\mathcal{F} \in D^{\mathrm{b}}_{G_{\mathbf{v}}}\mathrm{MHM}(\mathcal{E}_{\mathbf{v}},\mathbb{Q})$ and $\mathcal{G} \in D^{\mathrm{b}}_{G_{\mathbf{w}}}\mathrm{MHM}(\mathcal{E}_{\mathbf{w}},\mathbb{Q})$. If $\mathcal{F}$ is pure of weight n and $\mathcal{G}$ is pure of weight m, then $\mathcal{F} \star \mathcal{G}$ is pure of weight $n+m$.*

PROOF. By Corollary 5.6.19, $\mathcal{F} \boxtimes \mathcal{G}$ is pure of weight $n+m$. Next, consider the steps in the convolution diagram (10.3.5). Since p is smooth and m is proper, p^* and m_* take pure objects to pure objects, by Corollary 5.6.20. Finally, consider the functor

$$i^*\mathrm{For}^{G_{\mathbf{v}+\mathbf{w}}}_{P_{\mathbf{v},\mathbf{w}}} \cong i^!\mathrm{For}^{G_{\mathbf{v}+\mathbf{w}}}_{P_{\mathbf{v},\mathbf{w}}}[2\dim G_{\mathbf{v}+\mathbf{w}}/P_{\mathbf{v},\mathbf{w}}](\dim G_{\mathbf{v}+\mathbf{w}}/P_{\mathbf{v},\mathbf{w}}).$$

(Here, the isomorphism comes from Theorem 6.5.10.) According to Theorem 5.6.18, the left-hand side preserves the property of having weights $\leq n+m$, while the right-hand side preserves the property of having weights $\geq n+m$. Therefore, both sides preserve weights, as do their inverses. $\square$

LEMMA 10.5.2. *If $\mathcal{F} \in D^{\mathrm{b}}_{G_{\mathbf{v}}}\mathrm{MHM}(\mathcal{E}_{\mathbf{v}},\mathbb{Q})$ is pure, then it is pointwise pure.*

Proof sketch. According to Proposition 5.7.3, it is enough to show that every $G_{\mathbf{v}}$-orbit admits a transverse slice with an attracting cocharacter. Let $\mathcal{O}_{\mathbf{c}} \subset \mathcal{E}_{\mathbf{v}}$ be a $G_{\mathbf{v}}$-orbit, and let $x = (x_e)_{e \in Q_1} \in \mathcal{O}_{\mathbf{c}}$.

Step 1. Description of the tangent space. Let $\mathfrak{g}_{\mathbf{v}} = \prod_{i \in Q_0} \mathfrak{gl}_{\mathbf{v}_i}(\mathbb{C})$ be the Lie algebra of $G_{\mathbf{v}}$, and let $\mathfrak{g}^x_{\mathbf{v}} \subset \mathfrak{g}_{\mathbf{v}}$ be the Lie algebra of the stabilizer $G^x_{\mathbf{v}}$. Then $\mathfrak{g}_{\mathbf{v}}/\mathfrak{g}^x_{\mathbf{v}}$ can be identified with the tangent space to $\mathcal{O}_{\mathbf{c}}$ at x. Explicitly, define a linear map $d_x : \mathfrak{g}_{\mathbf{v}} \to \mathcal{E}_{\mathbf{v}}$ by

$$d_x((\phi_i : \mathbb{C}^{\mathbf{v}_i} \to \mathbb{C}^{\mathbf{v}_i})_{i \in Q_0}) = (\phi_{\mathrm{head}(e)} \circ x_e - x_e \circ \phi_{\mathrm{tail}(e)})_{e \in Q_1}.$$

The kernel of d_x is precisely $\mathfrak{g}^x_{\mathbf{v}} = \{(\phi_i)_{i \in Q_0} \mid \phi_{\mathrm{head}(e)} x_e = x_e \phi_{\mathrm{tail}(e)} \text{ for all } e\}$, and $x + \mathrm{im}(d_x)$ is the tangent space to $\mathcal{O}_{\mathbf{c}}$ at x.

Step 2. Choice of a cocharacter. Choose an adapted order $\gamma_1, \ldots, \gamma_N$ on the set of positive roots, and then fix an isomorphism

$$V \cong M(\gamma_1)^{\oplus \mathbf{c}(\gamma_1)} \oplus \cdots \oplus M(\gamma_N)^{\oplus \mathbf{c}(\gamma_N)}. \tag{10.5.1}$$

Define a homomorphism $\lambda : \mathbb{G}_{\mathrm{m}} \to G_{\mathbf{v}}$ as follows: for $t \in \mathbb{G}_{\mathrm{m}}$, $\lambda(t)$ is the linear map $V \to V$ that acts on the summand $M(\gamma_i)^{\oplus \mathbf{c}(\gamma_i)}$ by the scalar t^{-i}. Each $\lambda(t)$ is actually an automorphism of V as a quiver representation, so λ takes values in the subgroup $G^x_{\mathbf{v}}$.

Via (10.5.1), write $\mathcal{E}_{\mathbf{v}}$ as a product as follows:

$$\begin{aligned}\mathcal{E}_{\mathbf{v}} = \prod_{1 \le i \le j \le N} &\mathrm{Hom}_{\mathbb{C}}(M(\gamma_i)^{\oplus \mathbf{c}(\gamma_i)}, M(\gamma_j)^{\oplus \mathbf{c}(\gamma_j)}) \\ &\times \prod_{1 \le j < i \le N} \mathrm{Hom}_{\mathbb{C}}(M(\gamma_i)^{\oplus \mathbf{c}(\gamma_i)}, M(\gamma_j)^{\oplus \mathbf{c}(\gamma_j)}).\end{aligned} \tag{10.5.2}$$

Then $\mathbb{G}_{\mathrm{m}}$ acts on the factor $\mathrm{Hom}_{\mathbb{C}}(M(\gamma_i)^{\oplus \mathbf{c}(\gamma_i)}, M(\gamma_j)^{\oplus \mathbf{c}(\gamma_j)})$ above by $t \mapsto t^{i-j}$.

Step 3. Existence of a transversal slice. One can show that the image of d_x is stable under the $\mathbb{G}_{\mathrm{m}}$-action described above, and that it contains the subspace $\prod_{1 \le i \le j \le N} \mathrm{Hom}_{\mathbb{C}}(M(\gamma_i)^{\oplus \mathbf{c}(\gamma_i)}, M(\gamma_j)^{\oplus \mathbf{c}(\gamma_j)})$. Therefore, there exists a $\mathbb{G}_{\mathrm{m}}$-stable linear complement $S \subset \mathcal{E}_{\mathbf{v}}$ to the image of d_x that is contained in the subspace $\prod_{1 \le j < i \le N} \mathrm{Hom}_{\mathbb{C}}(M(\gamma_i)^{\oplus \mathbf{c}(\gamma_i)}, M(\gamma_j)^{\oplus \mathbf{c}(\gamma_j)})$. In particular, the $\mathbb{G}_{\mathrm{m}}$-action on S is attracting, with a unique fixed point.

The set $x + S \subset \mathcal{E}_{\mathbf{v}}$ is a transverse slice to $\mathcal{O}_{\mathbf{c}}$ at x, and $\lambda : \mathbb{G}_{\mathrm{m}} \to G_{\mathbf{v}}$ is an attracting cocharacter for it. □

Lemma 10.5.3. *There is a ring homomorphism*

$$r : \bigoplus_{\mathbf{v} \in \mathbb{Z}^{Q_0}_{\ge 0}} \mathrm{K}_0(D^{\mathrm{b}}_{G_{\mathbf{v}}} \mathrm{MHM}(\mathcal{E}_{\mathbf{v}}, \mathbb{Q})) \to \bigoplus_{\mathbf{v} \in \mathbb{Z}^{Q_0}_{\ge 0}} \mathrm{K}_{\oplus}(\mathrm{Semis}_{G_{\mathbf{v}}}(\mathcal{E}_{\mathbf{v}}, \mathbb{Q}))$$

given by

$$r([\mathcal{F}]) = \sum_{i,j} (-1)^{i-j} v^j [\mathrm{rat}\, \mathrm{gr}^W_j\, {}^p\mathsf{H}^i(\mathcal{F})].$$

Here, $\mathrm{gr}^W_j\, {}^p\mathsf{H}^i(\mathcal{F})$ is the jth term of the associated graded object for the weight filtration on ${}^p\mathsf{H}^i(\mathcal{F})$. It is pure of weight j. Recall that $\mathrm{rat}\, \mathrm{gr}^W_j\, {}^p\mathsf{H}^i(\mathcal{F})$ is a semisimple perverse sheaf, so the formula at least makes sense.

PROOF. Let us first show that r is at least a well-defined map of abelian groups. By Proposition A.9.5, we may instead work at the level of the abelian category $\mathrm{MHM}_{G_{\mathbf{v}}}(\mathcal{E}_{\mathbf{v}}, \mathbb{Q})$, and show that the map

$$r : \mathrm{K}_0(\mathrm{MHM}_{G_{\mathbf{v}}}(\mathcal{E}_{\mathbf{v}}, \mathbb{Q})) \to \mathrm{K}_{\oplus}(\mathrm{Semis}_{G_{\mathbf{v}}}(\mathcal{E}_{\mathbf{v}}, \mathbb{Q}))$$

given by

$$r([\mathcal{F}]) = \sum_j (-1)^{-j} v^j [\mathrm{rat}\, \mathrm{gr}^W_j \mathcal{F}]$$

is well defined. Given a short exact sequence of mixed Hodge modules $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$, we must show that $r([\mathcal{F}]) = r([\mathcal{F}']) + r([\mathcal{F}''])$. For each j, we have

$$\mathrm{rat}\, \mathrm{gr}^W_j \mathcal{F} \cong \mathrm{rat}\, \mathrm{gr}^W_j \mathcal{F}' \oplus \mathrm{rat}\, \mathrm{gr}^W_j \mathcal{F}'',$$

and the claim follows.

Next, observe that if $\mathcal{F} \in D^{\mathrm{b}}_{G_{\mathbf{v}}} \mathrm{MHM}(\mathcal{E}_{\mathbf{v}}, \mathbb{Q})$ is pure of weight k, then ${}^p\mathsf{H}^i(\mathcal{F}) = \mathrm{gr}^W_{k+i} {}^p\mathsf{H}^i(\mathcal{F})$. Since $\mathrm{rat}\, \mathcal{F}$ is a semisimple complex, we conclude that

$$r([\mathcal{F}]) = (-1)^{-k} \sum_{i \in \mathbb{Z}} v^{k+i} [\mathrm{rat}\, {}^p\mathsf{H}^i(\mathcal{F})] = (-1)^{-k} v^k [\mathrm{rat}\, \mathcal{F}]. \tag{10.5.3}$$

It remains to show that r is a ring homomorphism, i.e., that

$$r([\mathcal{F} \star \mathcal{G}]) = r([\mathcal{F}]) r([\mathcal{G}]).$$

Since $\mathrm{K}_0(D^{\mathrm{b}}_{G_{\mathbf{v}}} \mathrm{MHM}(\mathcal{E}_{\mathbf{v}}, \mathbb{Q}))$ is spanned by the classes of pure objects, it is enough to prove this in the case where $\mathcal{F}$ and $\mathcal{G}$ are pure. Since $\mathcal{F} \star \mathcal{G}$ is also pure (Lemma 10.5.1), the claim follows from (10.5.3) and the fact that $\mathrm{rat}(\mathcal{F} \star \mathcal{G}) \cong (\mathrm{rat}\, \mathcal{F}) \star (\mathrm{rat}\, \mathcal{G})$. ☐

PROPOSITION 10.5.4. *There is an injective ring homomorphism*

$$\tilde{\psi} : \mathcal{U}^+_v(\Phi) \to \bigoplus_{\mathbf{v} \in \mathbb{Z}^{Q_0}_{\geq 0}} \mathrm{K}_0(D^{\mathrm{b}}_{G_{\mathbf{v}}} \mathrm{MHM}(\mathcal{E}_{\mathbf{v}}, \mathbb{Q}))$$

given by $\psi(E_i^{(n)}) = [\mathrm{IC}_{[\mathbf{e}_i^n]}]$ *for* $i \in Q_0$ *and* $n \geq 1$*. Moreover,* $r \circ \tilde{\psi}$ *is the inverse of the map* $\mathrm{ch} : \bigoplus_{\mathbf{v} \in \mathbb{Z}^{Q_0}_{\geq 0}} \mathrm{K}_{\oplus}(\mathrm{Semis}_{G_{\mathbf{v}}}(\mathcal{E}_{\mathbf{v}}, \mathbb{Q})) \to \mathcal{U}^+_v(\Phi)$ *from Theorem* 10.4.1*.*

PROOF. The construction of the map $\psi = \mathrm{ch}^{-1}$ in the proof of Theorem 10.4.1 just depends on the calculations in Lemmas 10.3.7 and 10.4.2. Both of these lemmas remain valid in the mixed Hodge module setting, so the construction can be repeated verbatim to produce the map $\tilde{\psi}$. It satisfies $r \circ \tilde{\psi} = \psi$ by construction, and since ψ is injective, $\tilde{\psi}$ is as well. ☐

The proof of the next lemma is left as an exercise.

LEMMA 10.5.5. *For any* $\mathcal{F} \in D^{\mathrm{b}}_{G_{\mathbf{v}}} \mathrm{MHM}(\mathcal{E}_{\mathbf{v}}, \mathbb{Q})$*, we have*

$$[\mathcal{F}] = \sum_{\mathbf{c} \in \mathrm{KP}(\mathbf{v})} \sum_{i \in \mathbb{Z}} [j_{\mathbf{c}!} {}^p\mathsf{H}^i(\mathcal{F}|_{\mathcal{O}_{\mathbf{c}}})[-i]].$$

PROPOSITION 10.5.6. *For any Kostant partition* $\mathbf{c} : \Phi^+ \to \mathbb{Z}_{\geq 0}$*, we have*

$$\tilde{\psi}(E_{\mathbf{c}}) = [j_{\mathbf{c}!} \underline{\mathbb{Q}}_{\mathcal{O}_{\mathbf{c}}} [\dim \mathcal{O}_{\mathbf{c}}](\tfrac{1}{2} \dim \mathcal{O}_{\mathbf{c}})].$$

PROOF. We proceed by induction on $\dim \mathcal{O}_{\mathbf{c}}$, and then on the number of roots γ such that $\mathbf{c}(\gamma) \neq 0$. The base case is that of the zero Kostant partition, which we have denoted elsewhere by $\mathbf{c} = \varnothing$. The corresponding PBW basis element is $E_\varnothing = 1$. The ring homomorphism ψ sends it to the unit element of $\bigoplus_{\mathbf{v} \in \mathbb{Z}_{\geq 0}^{Q_0}} \mathrm{K}_\oplus(\mathrm{Semis}_{G_\mathbf{v}}(\mathcal{E}_\mathbf{v}, \mathbb{Q}))$, given by

$$[\mathrm{IC}_\varnothing] = r([\mathrm{IC}_\varnothing]) = r([j_{\varnothing !}\underline{\mathbb{Q}}_{\mathcal{O}_\varnothing}]).$$

Next, consider the case of a general Kostant partition $\mathbf{c}$. From now on, let $\mathbf{v} : Q_0 \to \mathbb{Z}_{\geq 0}$ be the dimension vector such that $\mathbf{c} \in \mathrm{KP}(\mathbf{v})$. Suppose first that $\mathbf{c}(\gamma) \neq 0$ for exactly one positive root γ. List the simple roots $\mathbf{e}_1, \dots, \mathbf{e}_n$ in the order indicated in Proposition 10.3.10. Using that result together with Proposition 10.5.4 and Lemma 10.5.5, we have

$$\begin{aligned}\tilde{\psi}(E_n^{(\mathbf{v}_n)} \cdots E_2^{(\mathbf{v}_2)} E_1^{(\mathbf{v}_1)}) &= [\mathrm{IC}_{[\mathbf{e}_n^{\mathbf{v}_n}]} \star \cdots \star \mathrm{IC}_{[\mathbf{e}_2^{\mathbf{v}_2}]} \star \mathrm{IC}_{[\mathbf{e}_1^{\mathbf{v}_1}]}] \\ &= [\underline{\mathbb{Q}}_{\mathcal{E}_\mathbf{v}}[\dim \mathcal{E}_\mathbf{v}](\tfrac{1}{2}\dim \mathcal{E}_\mathbf{v})] = \textstyle\sum_{\mathbf{c}' \in \mathrm{KP}(\mathbf{v})} [j_{\mathbf{c}'!}\underline{\mathbb{Q}}_{\mathcal{O}_{\mathbf{c}'}}[\dim \mathcal{E}_\mathbf{v}](\tfrac{1}{2}\dim \mathcal{E}_\mathbf{v})].\end{aligned}$$

By Lemma 10.1.11, $\mathcal{O}_\mathbf{c}$ is the largest orbit in $\mathcal{E}_\mathbf{v}$, so by induction, the proposition holds for $\mathbf{c}' \in \mathrm{KP}(\mathbf{v})$, $\mathbf{c}' \neq \mathbf{c}$. Applying this to the calculation above, we find

$$\tilde{\psi}(E_n^{(\mathbf{v}_n)} \cdots E_1^{(\mathbf{v}_1)}) = \sum_{\substack{\mathbf{c}' \in \mathrm{KP}(\mathbf{v}) \\ \mathbf{c}' \neq \mathbf{c}}} v^{\mathcal{O}_{\mathbf{c}'} - \dim \mathcal{E}_\mathbf{v}} \tilde{\psi}(E_{\mathbf{c}'}) + [j_{\mathbf{c}!}\underline{\mathbb{Q}}_{\mathcal{O}_\mathbf{c}}[\dim \mathcal{O}_\mathbf{c}](\tfrac{1}{2}\dim \mathcal{O}_\mathbf{c})].$$

The claim follows by comparing this with the formula from Exercise 10.2.2.

Lastly, consider the case where $\mathbf{c}(\gamma) \neq 0$ for more than one positive root. By induction, we may assume that the result is known when $\mathbf{c}$ is replaced by $\mathbf{c}(\gamma)\gamma$ for any positive root γ. By definition, $E_\mathbf{c} = E_{\mathbf{c}(\gamma_1)\gamma_1} E_{\mathbf{c}(\gamma_2)\gamma_2} \cdots E_{\mathbf{c}(\gamma_N)\gamma_N}$. For $1 \leq i \leq N$, let $d_i = \dim \mathcal{O}_{[\gamma_i^{\mathbf{c}(\gamma_i)}]}$. By induction and Proposition 10.3.9(1), we have

$$\begin{aligned}\tilde{\psi}(E_\mathbf{c}) &= \tilde{\psi}(E_{\mathbf{c}(\gamma_1)\gamma_1} \cdots E_{\mathbf{c}(\gamma_N)\gamma_N}) \\ &= [j_{[\gamma_1^{\mathbf{c}(\gamma_1)}]!}\underline{\mathbb{Q}}[d_1](\tfrac{d_1}{2}) \star \cdots \star j_{[\gamma_N^{\mathbf{c}(\gamma_N)}]!}\underline{\mathbb{Q}}[d_N](\tfrac{d_N}{2})] \\ &= [j_{\mathbf{c}!}\underline{\mathbb{Q}}_{\mathcal{O}_\mathbf{c}}[\dim \mathcal{O}_\mathbf{c}](\tfrac{1}{2}\dim \mathcal{O}_\mathbf{c})]. \qquad \square\end{aligned}$$

We now come to the main result of this section. It gives an alternative perspective on Theorem 10.4.1.

THEOREM 10.5.7. *The ring isomorphism* $\mathrm{ch} : \bigoplus_{\mathbf{v} \in \mathbb{Z}_{\geq 0}^{Q_0}} \mathrm{K}_\oplus(\mathrm{Semis}_{G_\mathbf{v}}(\mathcal{E}_\mathbf{v}, \mathbb{Q})) \to \mathcal{U}_v^+(\Phi)$ *is given by*

$$\mathrm{ch}([\mathcal{F}]) = \sum_{\substack{\mathbf{c} \in \mathrm{KP} \\ i \in \mathbb{Z}}} (\mathrm{rank}\, \mathsf{H}^{i - \dim \mathcal{O}_\mathbf{c}}(\mathcal{F}|_{\mathcal{O}_\mathbf{c}})) v^i E_\mathbf{c}.$$

PROOF. Suppose $\tilde{\mathcal{F}} \in D^{\mathrm{b}}_{G_\mathbf{v}}\mathrm{MHM}(\mathcal{E}_\mathbf{v}, \mathbb{Q})$ is pure of weight 0. Let $\mathbf{c} \in \mathrm{KP}(\mathbf{v})$, and consider the mixed Hodge module

$$\mathcal{M}_\mathbf{c}^i = {}^p\mathsf{H}^i(\tilde{\mathcal{F}}|_{\mathcal{O}_\mathbf{c}}).$$

By Lemma 10.5.2, this object is pure of weight i. Since the underlying perverse sheaf $\mathrm{rat}\, \mathcal{M}_\mathbf{c}^i$ is a shifted constant sheaf, we must have

$$\mathcal{M}_\mathbf{c}^i \cong M_\mathbf{c}^i \boxtimes \underline{\mathbb{Q}}_{\mathcal{O}_\mathbf{c}}[\dim \mathcal{O}_\mathbf{c}](\tfrac{1}{2}\dim \mathcal{O}_\mathbf{c}),$$

where $M^i_{\mathbf{c}}$ is a pure Hodge structure of weight i on a point. Reasoning along the same lines shows that

$$j_{\mathbf{c}!}{}^p\mathsf{H}^i(\tilde{\mathcal{F}}|_{\mathcal{O}_{\mathbf{c}}}) \cong (M^i_{\mathbf{c}} \overset{L}{\otimes} \mathrm{IC}_{\varnothing}) \star j_{\mathbf{c}!}\underline{\mathbb{Q}}_{\mathcal{O}_{\mathbf{c}}}[\dim \mathcal{O}_{\mathbf{c}}](\tfrac{1}{2}\dim \mathcal{O}_{\mathbf{c}}). \tag{10.5.4}$$

Let d^i be the dimension of the underlying $\mathbb{Q}$-vector space of $M^i_{\mathbf{c}}$. In other words,

$$d^i = \operatorname{rank} \mathsf{H}^{i-\dim \mathcal{O}_{\mathbf{c}}}((\operatorname{rat} \tilde{\mathcal{F}})|_{\mathcal{O}_{\mathbf{c}}}).$$

We clearly have

$$\operatorname{rat} M^i_{\mathbf{c}} \cong \operatorname{rat} \underline{\mathbb{Q}}_{\mathrm{pt}}(-\tfrac{i}{2})^{\oplus d_i}.$$

Let us now pass from (10.5.4) to the Grothendieck group, and then apply the map r. From the above considerations and Proposition 10.5.4, we have

$$\begin{aligned} r([j_{\mathbf{c}!}{}^p\mathsf{H}^i(\tilde{\mathcal{F}}|_{\mathcal{O}_{\mathbf{c}}})]) &= (-1)^i d_i v^i r([\mathrm{IC}_{\varnothing}]) r([j_{\mathbf{c}!}\underline{\mathbb{Q}}_{\mathcal{O}_{\mathbf{c}}}[\dim \mathcal{O}_{\mathbf{c}}](\tfrac{1}{2}\dim \mathcal{O}_{\mathbf{c}})]) \\ &= (-1)^i (\operatorname{rank} \mathsf{H}^{i-\dim \mathcal{O}_{\mathbf{c}}}((\operatorname{rat} \tilde{\mathcal{F}})|_{\mathcal{O}_{\mathbf{c}}})) v^i \operatorname{ch}^{-1}(E_{\mathbf{c}}). \end{aligned}$$

Recall that since $\tilde{\mathcal{F}}$ is pure of weight 0, we have $r([\tilde{\mathcal{F}}]) = [\operatorname{rat} \tilde{\mathcal{F}}]$. (See (10.5.3).) Therefore, applying r to the equation from Lemma 10.5.5, we obtain

$$[\operatorname{rat} \tilde{\mathcal{F}}] = \sum_{\mathbf{c} \in \mathrm{KP}(\mathbf{v})} \sum_{i \in \mathbb{Z}} (\operatorname{rank} \mathsf{H}^{i-\dim \mathcal{O}_{\mathbf{c}}}((\operatorname{rat} \tilde{\mathcal{F}})|_{\mathcal{O}_{\mathbf{c}}})) v^i \operatorname{ch}^{-1}(E_{\mathbf{c}}).$$

Of course, every object $\mathcal{F} \in \mathrm{Semis}_{G_{\mathbf{v}}}(\mathcal{E}_{\mathbf{v}}, \mathbb{Q})$ arises as $\operatorname{rat} \tilde{\mathcal{F}}$ of some pure object $\tilde{\mathcal{F}} \in D^{\mathrm{b}}_{G_{\mathbf{v}}}\mathrm{MHM}(\mathcal{E}_{\mathbf{v}}, \mathbb{Q})$, so the equation above remains valid if we simply replace $\operatorname{rat} \tilde{\mathcal{F}}$ by $\mathcal{F}$. We have obtained the desired formula for ch. $\square$

COROLLARY 10.5.8. *For* $\mathbf{c}, \mathbf{c}' \in \mathrm{KP}$, *we have*

$$P_{\mathbf{c}',\mathbf{c}} = v^{\dim \mathcal{O}_{\mathbf{c}'}} \sum_{i \in \mathbb{Z}} (\operatorname{rank} \mathsf{H}^i(\mathrm{IC}_{\mathbf{c}}|_{\mathcal{O}_{\mathbf{c}'}})) v^i.$$

The following proposition records some properties of the polynomials $P_{\mathbf{c}',\mathbf{c}}$. This should be compared to Theorems 7.1.3 and 9.3.7.

PROPOSITION 10.5.9. *The polynomials* $P_{\mathbf{c}',\mathbf{c}} \in \mathbb{Z}[v, v^{-1}]$ *have the following properties:*

1. $P_{\mathbf{c}',\mathbf{c}} = 0$ *unless* $\mathcal{O}_{\mathbf{c}'} \subset \overline{\mathcal{O}_{\mathbf{c}}}$.
2. $P_{\mathbf{c},\mathbf{c}} = 1$.
3. *The coefficients of* $P_{\mathbf{c}',\mathbf{c}}$ *are nonnegative integers.*
4. *If* $\mathcal{O}_{\mathbf{c}'} \subset \overline{\mathcal{O}_{\mathbf{c}}} \smallsetminus \mathcal{O}_{\mathbf{c}}$, *then* $P_{\mathbf{c}',\mathbf{c}}$ *lies in*
$$v^{-1}\mathbb{Z}[v^{-1}] \cap v^{\dim \mathcal{O}_{\mathbf{c}'} - \dim \mathcal{O}_{\mathbf{c}}}\mathbb{Z}[v^2].$$

In particular, for any $\mathbf{c}, \mathbf{c}' \in \mathrm{KP}(\mathbf{v})$, *we have*

$$\mathsf{H}^i(\mathrm{IC}_{\mathbf{c}}|_{\mathcal{O}_{\mathbf{c}'}}) = 0 \qquad \textit{unless } i \equiv \dim \mathcal{O}_{\mathbf{c}} \pmod 2. \tag{10.5.5}$$

PROOF. The first three assertions are immediate from Corollary 10.5.8. If $\mathcal{O}_{\mathbf{c}'} \subset \overline{\mathcal{O}_{\mathbf{c}}} \smallsetminus \mathcal{O}_{\mathbf{c}}$, then $\mathsf{H}^i(\mathrm{IC}_{\mathbf{c}}|_{\mathcal{O}_{\mathbf{c}'}}) = 0$ unless $-\dim \mathcal{O}_{\mathbf{c}} \le i < -\dim \mathcal{O}_{\mathbf{c}'}$. It follows that

$$P_{\mathbf{c}',\mathbf{c}} \in v^{-1}\mathbb{Z}[v^{-1}] \cap v^{\dim \mathcal{O}_{\mathbf{c}'} - \dim \mathcal{O}_{\mathbf{c}}}\mathbb{Z}[v].$$

But part (4) contains the stronger assertion that

$$P_{\mathbf{c}',\mathbf{c}} \in v^{\dim \mathcal{O}_{\mathbf{c}'} - \dim \mathcal{O}_{\mathbf{c}}}\mathbb{Z}[v^2].$$

This is equivalent to (10.5.5), and the latter is what remains to be proved.

Let $\mathcal{F}$ be an object of the form $\mathrm{IC}_{[\mathbf{e}_{i_1}^{n_1}]} \star \cdots \star \mathrm{IC}_{[\mathbf{e}_{i_m}^{n_m}]}$ as in Proposition 10.3.11, so that $\mathrm{IC}_{\mathbf{c}}$ is a direct summand of $\mathcal{F}$. Of course, it is enough to show that

$$\mathsf{H}^i(\mathcal{F}|_{\mathcal{O}_{\mathbf{c}'}}) = 0 \qquad \text{unless } i \equiv \dim \mathcal{O}_{\mathbf{c}} \pmod 2. \tag{10.5.6}$$

Define a polynomial $f_{\mathbf{c}'} \in \mathbb{Z}[v, v^{-1}]$ by

$$\begin{aligned} f_{\mathbf{c}'}(v) &= v^{\sum_{i=1}^m n_i(n_i-1)+\dim \mathcal{O}_{\mathbf{c}'} - \dim M(\mathbf{c}') + \dim \operatorname{End}(M(\mathbf{c}'))} \sum_{i\in\mathbb{Z}} \operatorname{rank} \mathsf{H}^i(\mathcal{F}|_{\mathcal{O}_{\mathbf{c}'}}) v^i \\ &= v^{\sum_{i=1}^m n_i(n_i-1)+\sum_{i\in Q_0}(\mathbf{v}_i^2-\mathbf{v}_i)} \sum_{i\in\mathbb{Z}} \operatorname{rank} \mathsf{H}^i(\mathcal{F}|_{\mathcal{O}_{\mathbf{c}'}}) v^i. \end{aligned}$$

Here, the second expression comes from Exercise 10.2.1. By Theorem 10.5.7, we have

$$\begin{aligned} E_{i_1}^{(n_1)} \cdots E_{i_m}^{(n_m)} &= \operatorname{ch}([\mathcal{F}]) \\ &= v^{-\sum_{i=1}^m n_i(n_i-1)} \sum_{\mathbf{c}'\in \mathrm{KP}(\mathbf{v})} f_{\mathbf{c}'}(v) v^{\dim M(\mathbf{c}') - \dim \operatorname{End}(M(\mathbf{c}'))} E_{\mathbf{c}'}. \end{aligned}$$

Now pass through Theorem 10.2.9. Using (10.2.8), we have

$$[S_{i_1}^{\oplus n_1}] \cdots [S_{i_m}^{\oplus i_m}] = \sum_{\mathbf{c}'\in \mathrm{KP}(\mathbf{v})} f_{\mathbf{c}'}(q^{\frac{1}{2}})[M(\mathbf{c}')].$$

The left-hand side can also be computed using the definition of the Hall algebra. It involves a factor of

$$q^{d/2} \qquad \text{where } d = \sum_{1\le i<j\le m} \langle n_i \mathbf{e}_i, n_j \mathbf{e}_j \rangle.$$

Indeed, we see that for all prime powers q, we have

$$\begin{cases} f_{\mathbf{c}'}(q^{\frac{1}{2}}) \in \mathbb{Z} & \text{if } d \equiv 0 \pmod 2, \\ q^{\frac{1}{2}} f_{\mathbf{c}'}(q^{\frac{1}{2}}) \in \mathbb{Z} & \text{if } d \equiv 1 \pmod 2). \end{cases}$$

An elementary argument shows that $f_{\mathbf{c}'}(v)$ involves either only even powers of v or only odd powers of v, depending on the parity of d (and independent of $\mathbf{c}'$). It follows that $\mathsf{H}^i(\mathcal{F}|_{\mathcal{O}_{\mathbf{c}'}})$ can be nonzero only for i of a fixed parity. Since $\mathrm{IC}_{\mathbf{c}}$ is a direct summand of $\mathcal{F}$, $\mathsf{H}^{-\dim \mathcal{O}_{\mathbf{c}}}(\mathcal{F}|_{\mathcal{O}_{\mathbf{c}}})$ is nonzero, so (10.5.6) holds. $\square$

Exercises.

10.5.1. Show that the canonical basis $\{B_{\mathbf{c}}\}_{\mathbf{c}\in\mathrm{KP}}$ is the unique $\mathbb{Z}[v,v^{-1}]$-basis of $\mathcal{U}_v^+(\Phi)$ with the following properties:

(a) For all $\mathbf{c} \in \mathrm{KP}$, we have $\overline{B_{\mathbf{c}}} = B_{\mathbf{c}}$.

(b) In the expression

$$B_{\mathbf{c}} = \sum_{\mathbf{c}'\in\mathrm{KP}} P_{\mathbf{c}',\mathbf{c}} E_{\mathbf{c}'},$$

the coefficients $P_{\mathbf{c}',\mathbf{c}} \in \mathbb{Z}[v,v^{-1}]$ satisfy

(i) $P_{\mathbf{c}',\mathbf{c}} = 0$ unless $\mathcal{O}_{\mathbf{c}'} \subset \overline{\mathcal{O}_{\mathbf{c}}}$.

(ii) $P_{\mathbf{c},\mathbf{c}} = 1$.

(iii) If $\mathcal{O}_{\mathbf{c}'} \subset \overline{\mathcal{O}_{\mathbf{c}}} \smallsetminus \mathcal{O}_{\mathbf{c}}$, then $P_{\mathbf{c}',\mathbf{c}}$ lies in $v^{-1}\mathbb{Z}[v^{-1}] \cap v^{\dim \mathcal{O}_{\mathbf{c}'} - \dim \mathcal{O}_{\mathbf{c}}}\mathbb{Z}[v]$.

In other words, the canonical basis is uniquely characterized by a weakened version of Proposition 10.5.9.

10.6. Mixed ℓ-adic sheaves and the Hall algebra

In this section, we work with the space $\mathcal{E}_{\mathbf{v}}$ over a finite field $\mathbb{F}_q$. Let

$$\mathcal{H}'_Q(\mathbb{F}_q)_{\mathbf{v}} = \{G_{\mathbf{v}}\text{-invariant functions } \mathcal{E}_{\mathbf{v}}(\mathbb{F}_q) \to \mathbb{Z}[q^{\pm\frac{1}{2}}]\},$$

and then let

$$\mathcal{H}'_Q(\mathbb{F}_q) = \bigoplus_{\mathbf{v}:Q_0\to\mathbb{Z}_{\geq 0}} \mathcal{H}'_Q(\mathbb{F}_q)_{\mathbf{v}}.$$

We will make $\mathcal{H}'_Q(\mathbb{F}_q)$ into a ring using an $\mathbb{F}_q$-version of the convolution diagram:

$$\mathcal{E}_{\mathbf{v}}(\mathbb{F}_q) \times \mathcal{E}_{\mathbf{w}}(\mathbb{F}_q) \overset{p}{\leftarrow} \mathcal{E}_{\mathbf{v},\mathbf{w}}(\mathbb{F}_q) \overset{i}{\to} G_{\mathbf{v}+\mathbf{w}}(\mathbb{F}_q) \times^{P_{\mathbf{v},\mathbf{w}}(\mathbb{F}_q)} \mathcal{E}_{\mathbf{v},\mathbf{w}}(\mathbb{F}_q) \overset{m}{\to} \mathcal{E}_{\mathbf{v}+\mathbf{w}}(\mathbb{F}_q).$$

Given $f_1 \in \mathcal{H}'_Q(\mathbb{F}_q)_{\mathbf{v}}$ and $f_2 \in \mathcal{H}'_Q(\mathbb{F}_q)_{\mathbf{w}}$, define a function

$$f_1 \times f_2 : \mathcal{E}_{\mathbf{v}}(\mathbb{F}_q) \times \mathcal{E}_{\mathbf{w}}(\mathbb{F}_q) \to \mathbb{Z}[q^{\pm\frac{1}{2}}] \quad \text{by} \quad (f_1 \times f_2)(x_1, x_2) = f_1(x_1)f_2(x_2).$$

Next, observe that there is a unique $G_{\mathbf{v}}(\mathbb{F}_q)$-invariant function

$$f_1 \mathbin{\tilde{\times}} f_2 : G_{\mathbf{v}+\mathbf{w}}(\mathbb{F}_q) \times^{P_{\mathbf{v},\mathbf{w}}(\mathbb{F}_q)} \mathcal{E}_{\mathbf{v},\mathbf{w}}(\mathbb{F}_q) \to \mathbb{Z}[q^{\pm\frac{1}{2}}]$$

such that $(f_1 \mathbin{\tilde{\times}} f_2) \circ q = (f_1 \times f_2) \circ p$. We define the **convolution product** of f_1 and f_2 to be the function

$$f_1 \star f_2 \in \mathcal{H}'_Q(\mathbb{F}_q)_{\mathbf{v}+\mathbf{w}} \qquad \text{given by} \qquad f_1 \star f_2 = q^{-\langle\!\langle \mathbf{v},\mathbf{w}\rangle\!\rangle/2} \int_m (f_1 \mathbin{\tilde{\times}} f_2),$$

where the integral is defined by $(\int_m f)(x) = \sum_{y\in m^{-1}(x)} f(y)$.

LEMMA 10.6.1. *The convolution product makes $\mathcal{H}'_Q(\mathbb{F}_q)$ into an associative algebra. Moreover, there is an isomorphism of rings*

$$\mathcal{H}_Q(\mathbb{F}_q) \xrightarrow{\sim} \mathcal{H}'_Q(\mathbb{F}_q)$$

that is characterized by the property that for any Kostant partition $\mathbf{c} \in \mathrm{KP}(\mathbf{v})$, the basis element $[M(\mathbf{c})] \in \mathcal{H}_Q(\mathbb{F}_q)_{\mathbf{v}}$ is sent to the function

$$x \mapsto \begin{cases} q^{\frac{1}{2}\sum_{i\in Q_0}(\mathbf{v}_i - \mathbf{v}_i^2)} & \text{if } x \in \mathcal{O}_{\mathbf{c}}, \\ 0 & \text{otherwise.} \end{cases}$$

PROOF. Let $e_{\mathbf{c}}$ be the characteristic function of $\mathcal{O}_{\mathbf{c}}$, i.e., the function given by

$$e_{\mathbf{c}}(x) = \begin{cases} 1 & \text{if } x \in \mathcal{O}_{\mathbf{c}}, \\ 0 & \text{otherwise.} \end{cases}$$

The map $\theta : \mathcal{H}_Q(\mathbb{F}_q) \xrightarrow{\sim} \mathcal{H}'_Q(\mathbb{F}_q)$ described in the statement of the lemma is given by $\theta([M(\mathbf{c})]) = q^{\frac{1}{2}\sum_{i\in Q_0}(\mathbf{v}_i - \mathbf{v}_i^2)} e_{\mathbf{c}}$. This map is at least an isomorphism of $\mathbb{Z}[q^{\pm\frac{1}{2}}]$-modules. To prove the lemma, it is enough to show that for all $\mathbf{c} \in \mathrm{KP}(\mathbf{v})$ and $\mathbf{c}' \in \mathrm{KP}(\mathbf{w})$, we have

$$\theta([M(\mathbf{c})]) \star \theta([M(\mathbf{c}')]) = \theta([M(\mathbf{c})][M(\mathbf{c}')]). \tag{10.6.1}$$

The value of $\int_m (e_{\mathbf{c}} \mathbin{\tilde{\times}} e_{\mathbf{c}'})$ at a point $x \in \mathcal{E}_{\mathbf{v}+\mathbf{w}}(\mathbb{F}_q)$ is the number of points in $m^{-1}(x) \cap (\mathcal{O}_{\mathbf{c}} \mathbin{\tilde{\times}} \mathcal{O}_{\mathbf{c}'})(\mathbb{F}_q)$. This set has been described in Remark 10.3.1: its cardinality is none other than the Hall number $F^{[V]}_{[M(\mathbf{c})],[M(\mathbf{c}')]}$, where V is the quiver representation corresponding to x. We thus have

$$e_{\mathbf{c}} \star e_{\mathbf{c}'} = q^{-\langle\!\langle \mathbf{v},\mathbf{w}\rangle\!\rangle/2} \sum_{\mathbf{d}\in \mathrm{KP}(\mathbf{v}+\mathbf{w})} F^{[M(\mathbf{d})]}_{[M(\mathbf{c}_1)],[M(\mathbf{c}_2)]} e_{\mathbf{d}}.$$

Comparing this with the definition of the Hall algebra, we see that (10.6.1) reduces to the claim that

$$\sum_{i\in Q_0}(\mathbf{v}_i - \mathbf{v}_i^2) + \sum_{i\in Q_0}(\mathbf{w}_i - \mathbf{w}_i^2) - \langle\!\langle \mathbf{v},\mathbf{w}\rangle\!\rangle = \sum_{i\in Q_0}(\mathbf{v}_i + \mathbf{w}_i - (\mathbf{v}_i+\mathbf{w}_i)^2) + \langle \mathbf{v},\mathbf{w}\rangle.$$

This is immediate from the definitions. □

From now on, we identify $\mathcal{H}'_Q(\mathbb{F}_q)$ with $\mathcal{H}_Q(\mathbb{F}_q)$ via Lemma 10.6.1. In particular, for any $\mathcal{F} \in D^{\mathrm{b}}_{\mathrm{m},G_{\mathbf{v}}}(\mathcal{E}_{\mathbf{v}},\overline{\mathbb{Q}}_\ell)$, we can regard the characteristic function $\chi_{\mathcal{F}} : \mathcal{E}_{\mathbf{v}}(\mathbb{F}_q) \to \overline{\mathbb{Q}}_\ell$ (see Definition 5.3.11) as an element of $\overline{\mathbb{Q}}_\ell \otimes_{\mathbb{Z}[q^{\pm\frac{1}{2}}]} \mathcal{H}_Q(\mathbb{F}_q)$. Explicitly, if we choose some point $x_{\mathbf{c}} \in \mathcal{O}_{\mathbf{c}}(\mathbb{F}_q)$ for each $\mathbf{c} \in \mathrm{KP}(\mathbf{v})$, then for $\mathcal{F} \in D^{\mathrm{b}}_{\mathrm{m},G_{\mathbf{v}}}(\mathcal{E}_{\mathbf{v}},\overline{\mathbb{Q}}_\ell)$, we have

$$\chi_{\mathcal{F}} = q^{\frac{1}{2}\sum_{i\in Q_0}(\mathbf{v}_i^2-\mathbf{v}_i)} \sum_{\mathbf{c}\in\mathrm{KP}(\mathbf{v})}\sum_{i\in\mathbb{Z}}(-1)^i \operatorname{tr}(\mathsf{Fr}_q, \mathsf{H}^i(\mathcal{F}_{x_{\mathbf{c}}}))[M(\mathbf{c})]. \tag{10.6.2}$$

Proposition 10.6.2. *The characteristic function map*

$$\chi : \bigoplus_{\mathbf{v}:Q_0\to\mathbb{Z}_{\ge 0}} \mathrm{K}_0(D^{\mathrm{b}}_{\mathrm{m},G_{\mathbf{v}}}(\mathcal{E}_{\mathbf{v}},\overline{\mathbb{Q}}_\ell)) \to \overline{\mathbb{Q}}_\ell \otimes_{\mathbb{Z}[q^{\pm\frac{1}{2}}]} \mathcal{H}_Q(\mathbb{F}_q)$$

is a ring homomorphism.

Proof. This amounts to the observation that the definitions of the two kinds of convolution product (for sheaves and for functions) correspond under the sheaf–function correspondence (Theorem 5.3.13). □

The proofs of the next two lemmas are identical to those of their mixed Hodge module counterparts. We omit the details.

Lemma 10.6.3. *Let $\mathcal{F} \in D^{\mathrm{b}}_{\mathrm{m},G_{\mathbf{v}}}(\mathcal{E}_{\mathbf{v}},\overline{\mathbb{Q}}_\ell)$ and $\mathcal{G} \in D^{\mathrm{b}}_{\mathrm{m},G_{\mathbf{w}}}(\mathcal{E}_{\mathbf{w}},\overline{\mathbb{Q}}_\ell)$. If $\mathcal{F}$ is pure of weight n and $\mathcal{G}$ is pure of weight m, then $\mathcal{F}\star\mathcal{G}$ is pure of weight $n+m$.*

Lemma 10.6.4. *If $\mathcal{F} \in D^{\mathrm{b}}_{\mathrm{m},G_{\mathbf{v}}}(\mathcal{E}_{\mathbf{v}},\overline{\mathbb{Q}}_\ell)$ is pure, then it is pointwise pure.*

Proposition 10.6.5. *Let $\mathbf{c} \in \mathrm{KP}(\mathbf{v})$.*

(1) *For $x \in \mathcal{E}_{\mathbf{v}}(\mathbb{F}_q)$, all eigenvalues of Fr_q on $\mathsf{H}^i((\mathrm{IC}_{\mathbf{c}})_x)$ are equal to $q^{i/2}$.*
(2) *We have $\mathsf{H}^i(\mathrm{IC}_{\mathbf{c}}) = 0$ if $i \not\equiv \dim\mathcal{O}_{\mathbf{c}} \pmod 2$.*

Proof. Let $\mathcal{F}$ denote an object of the form $\mathrm{IC}_{[\mathbf{e}_{i_1}^{n_1}]} \star\cdots\star \mathrm{IC}_{[\mathbf{e}_{i_m}^{n_m}]}$ as in Proposition 10.3.11, so that $\mathrm{IC}_{\mathbf{c}}$ is a direct summand of $\mathrm{egf}(\mathcal{F})$. Let us compute the characteristic function of $\mathcal{F}$ using Proposition 10.6.2 and the Hall algebra. Each $\mathrm{IC}_{[\mathbf{e}_{i_j}^{n_j}]}$ is a skyscraper sheaf on a 0-dimensional orbit $\mathcal{O}_{[\mathbf{e}_{i_j}^{n_j}]}$. The corresponding quiver representation is $S_{i_j}^{\oplus n_j}$. By (10.6.2), its characteristic function is given by

$$\chi_{\mathrm{IC}_{[\mathbf{e}_{i_j}^{n_j}]}} = q^{(n_j^2-n_j)/2}[S_{i_j}^{\oplus n_j}].$$

It follows that

$$\chi_{\mathcal{F}} = q^{\frac{1}{2}\sum_{j=1}^m(n_j^2-n_j)}[S_{i_1}^{\oplus n_1}]\cdots[S_{i_m}^{\oplus n_m}].$$

This product can be further expanded in terms of Hall polynomials. For brevity, if $\mathbf{c}' \in \mathrm{KP}(\mathbf{v})$, we introduce the notation

$$g_{\mathbf{c}'}(t) = \mathbf{F}^{\mathbf{c}'}_{[\mathbf{e}_{i_1}^{n_1}],[\mathbf{e}_{i_2}^{n_2}],\ldots,[\mathbf{e}_{i_m}^{n_m}]}(t) \in \mathbb{Z}[t].$$

We also let

$$d = \sum_{j=1}^{m}(n_j^2 - n_j) + \sum_{1\le j<k\le m} \langle n_j \mathbf{e}_j, n_k \mathbf{e}_k\rangle.$$

We then have

$$\chi_{\mathcal{F}} = q^{d/2} \sum_{\mathbf{c}'\in \mathrm{KP}(\mathbf{v})} g_{\mathbf{c}'}(q)[M(\mathbf{c}')].$$

More generally, if we work instead over $\mathbb{F}_{q^n}$, we have

$$\chi_{\mathcal{F},\mathbb{F}_{q^n}} = q^{nd/2} \sum_{\mathbf{c}'\in \mathrm{KP}(\mathbf{v})} g_{\mathbf{c}'}(q^n)[M(\mathbf{c}')]. \tag{10.6.3}$$

Now choose a point $x \in \mathcal{O}_{\mathbf{c}'}(\mathbb{F}_q)$. By Lemma 10.6.4, the stalk $\mathcal{F}_x \in D^{\mathrm{b}}_{\mathrm{m}}(\operatorname{Spec}\mathbb{F}_q, \overline{\mathbb{Q}}_\ell)$ is pure. In view of (10.6.3), Exercise 5.4.1 tells us that the eigenvalues of Fr_{q^n} on $\mathsf{H}^i(\mathcal{F}_x)$ are equal to $q^{in/2}$. Since $\mathrm{IC}_{\mathbf{c}}$ is a direct summand of $\operatorname{egf}(\mathcal{F})$, part (1) of the proposition follows.

Exercise 5.4.1 also says that

$$v^{d+\sum_{i\in Q_0}(\mathbf{v}_i - \mathbf{v}_i^2)} g_{\mathbf{c}'}(v^2) = \sum_{i\in\mathbb{Z}} (-1)^i \dim \mathsf{H}^i(\mathcal{F}_x) v^i.$$

The left-hand side involves either only odd powers of v or only even powers of v, depending on the parity of $d + \sum_{i\in Q_0}(\mathbf{v}_i - \mathbf{v}_i^2)$ (and independent of $\mathbf{c}'$). It follows that $\dim \mathsf{H}^i(\mathcal{F}_x)$ can be nonzero only for i of a fixed parity. Since $\mathrm{IC}_{\mathbf{c}}$ is a direct summand of $\operatorname{egf}(\mathcal{F})$, we have $\mathsf{H}^{-\dim \mathcal{O}_{\mathbf{c}}}(\mathcal{F}_x) \neq 0$ for $x \in \mathcal{O}_{\mathbf{c}}$, so $\mathsf{H}^i(\mathcal{F}) = 0$ if $i \not\equiv \dim \mathcal{O}_{\mathbf{c}} \pmod 2$. Part (2) of the proposition follows. ☐

PROPOSITION 10.6.6. *For* $\mathbf{c} \in \mathrm{KP}(\mathbf{v})$, *the characteristic function of* $\mathrm{IC}_{\mathbf{c}}$ *is*

$$\chi_{\mathrm{IC}_{\mathbf{c}}} = (-1)^{\dim \mathcal{O}_{\mathbf{c}}} q^{\frac12 \sum_{i\in Q_0}(\mathbf{v}_i^2 - \mathbf{v}_i)} \sum_{\mathbf{c}'\in \mathrm{KP}(\mathbf{v})} q^{-\frac12 \dim \mathcal{O}_{\mathbf{c}'}} P_{\mathbf{c}',\mathbf{c}}(q^{\frac12})[M(\mathbf{c}')].$$

PROOF SKETCH. Combining (10.6.2) and Proposition 10.6.5, we see that this characteristic function is given by

$$\begin{aligned}\chi_{\mathrm{IC}_{\mathbf{c}}} &= q^{\frac12 \sum_{i\in Q_0}(\mathbf{v}_i^2 - \mathbf{v}_i)} \sum_{\substack{\mathbf{c}'\in \mathrm{KP}(\mathbf{v}),\ i\in\mathbb{Z} \\ i\equiv \dim \mathcal{O}_{\mathbf{c}} \pmod 2}} (-1)^i q^{i/2} \operatorname{rank} \mathsf{H}^i(\mathrm{IC}_{\mathbf{c}}|_{\mathcal{O}_{\mathbf{c}'}})[M(\mathbf{c}')] \\ &= (-1)^{\dim \mathcal{O}_{\mathbf{c}}} q^{\frac12 \sum_{i\in Q_0}(\mathbf{v}_i^2 - \mathbf{v}_i)} \sum_{\substack{\mathbf{c}'\in \mathrm{KP}(\mathbf{v}),\ i\in\mathbb{Z} \\ i\equiv \dim \mathcal{O}_{\mathbf{c}} \pmod 2}} q^{i/2} \operatorname{rank} \mathsf{H}^i(\mathrm{IC}_{\mathbf{c}}|_{\mathcal{O}_{\mathbf{c}'}})[M(\mathbf{c}')].\end{aligned}$$

In view of Corollary 10.5.8, to finish the proof, we must show that the $\mathbb{F}_q$ version of $\mathsf{H}^i(\mathrm{IC}_{\mathbf{c}}|_{\mathcal{O}_{\mathbf{c}'}})$ has the same rank as the $\mathbb{C}$ version. It is possible to carry out such a comparison using the powerful general machinery explained in [**24**, Chapitre 6]. An easier alternative argument is as follows: define an element $B'_{\mathbf{c}} \in \mathcal{U}_v^+(\Phi)$ by

$$B'_{\mathbf{c}} = \sum_{\mathbf{c}'\in \mathrm{KP}} v^{\dim \mathcal{O}_{\mathbf{c}'}} \sum_{i\in\mathbb{Z}} (\operatorname{rank} \mathsf{H}^i(\mathrm{IC}^{\mathbb{F}_q}_{\mathbf{c}}|_{\mathcal{O}_{\mathbf{c}'}})) v^i E_{\mathbf{c}'}. \tag{10.6.4}$$

(We write the superscript $\mathbb{F}_q$ to emphasize that we are working with ℓ-adic sheaves on the $\mathbb{F}_q$-version of $\mathcal{E}_{\mathbf{v}}$.) Then one can check that $\{B'_{\mathbf{c}}\}_{\mathbf{c}\in \mathrm{KP}}$ is a basis for $\mathcal{U}_v^+(\Phi)$ satisfying the conditions in Exercise 10.5.1, so it coincides with the canonical basis $\{B_{\mathbf{c}}\}_{\mathbf{c}\in \mathrm{KP}}$. In particular, the coefficient of $E_{\mathbf{c}'}$ in (10.6.4) is $P_{\mathbf{c}',\mathbf{c}}$. $\square$

10.7. Additional exercises

The following exercises outline a sheaf-theoretic approach to computing the canonical basis for $\mathcal{U}_v^+(\mathfrak{sl}_3)$. Let Q be the quiver

$$\underset{\alpha}{\bullet} \longrightarrow \underset{\beta}{\bullet}$$

and let Φ be the corresponding root system. The positive roots are $\{\alpha, \beta, \alpha+\beta\}$.

Exercise 10.7.1. Show that $\beta, \alpha+\beta, \alpha$ is an adapted order.

Exercise 10.7.2. Let us write dimension vectors $\mathbf{v} : Q_0 \to \mathbb{Z}_{\geq 0}$ as ordered pairs (a, b), where $a = \mathbf{v}_\alpha$ and $b = \mathbf{v}_\beta$. Every Kostant partition $\mathbf{c} \in \mathrm{KP}(\mathbf{v})$ can be written as a triple $(a-r, r, b-r)$, where

$$\mathbf{c}(\alpha) = a - r, \qquad \mathbf{c}(\alpha+\beta) = r, \qquad \mathbf{c}(\beta) = b - r.$$

We have $\mathcal{E}_{(a,b)} = \mathrm{Hom}(\mathbb{C}^a, \mathbb{C}^b)$.

(a) Show that $\mathcal{O}_{(a-r,r,b-r)} \subset \mathcal{E}_{(a,b)}$ is the set of linear maps $\mathbb{C}^a \to \mathbb{C}^b$ of rank r.
(b) Show that $\mathcal{O}_{(a-r,r,b-r)} \subset \overline{\mathcal{O}_{(a-s,s,b-s)}}$ if and only if $r \leq s$.
(c) Show that $\dim \mathcal{O}_{(a-r,r,b-r)} = r(a+b-r)$.

Exercise 10.7.3. Suppose $0 \leq r \leq a \leq b$, and consider the dimension vectors

$$\mathbf{v}_1 = (r, 0), \qquad \mathbf{v}_2 = (0, b), \qquad \mathbf{v}_3 = (a-r, 0),$$

as well as $\mathbf{w} = \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 = (a, b)$. Consider the convolution map

$$m : G_{\mathbf{w}} \times^{P_{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3}} \mathcal{E}_{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3} \to \mathcal{E}_{\mathbf{w}}.$$

Show that the image of m is $\overline{\mathcal{O}}_{(a-r,r,b-r)}$. Then show that if $x \in \mathcal{O}_{a-s,s,b-s}$ for some $s \leq r$, we have

$$m^{-1}(x) \cong \mathrm{Gr}(a-r, a-s),$$

where $\mathrm{Gr}(a-r, a-s)$ is the Grassmannian of $(a-r)$-dimensional subspaces of $\mathbb{C}^{a-s}$.

If $0 \leq r \leq b \leq a$, carry out a similar computation using the dimension vectors

$$\mathbf{v}_1 = (0, b-r), \qquad \mathbf{v}_2 = (a, 0), \qquad \mathbf{v}_3 = (0, r).$$

Exercise 10.7.4. Show that the map

$$m : G_{\mathbf{w}} \times^{P_{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3}} \mathcal{E}_{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3} \to \overline{\mathcal{O}_{(a-r,r,b-r)}}$$

from the previous exercise is small with respect to $\mathcal{O}_{(a-r,r,b-r)}$. Deduce that

$$\mathrm{IC}_{(a-r,r,b-r)} \cong \begin{cases} \mathrm{IC}_{[\alpha^r]} \star \mathrm{IC}_{[\beta^b]} \star \mathrm{IC}_{[\alpha^{a-r}]} & \text{if } a \leq b, \\ \mathrm{IC}_{[\beta^{b-r}]} \star \mathrm{IC}_{[\alpha^a]} \star \mathrm{IC}_{[\beta^r]} & \text{if } b \leq a. \end{cases}$$

Finally, conclude that the canonical basis elements of $\mathcal{U}_v^+(\mathfrak{sl}_3)$ are given by

$$B_{(a-r,r,b-r)} = \begin{cases} E_\alpha^{(r)} E_\beta^{(b)} E_\alpha^{(a-r)} & \text{if } a \leq b, \\ E_\beta^{(b-r)} E_\alpha^{(a)} E_\beta^{(r)} & \text{if } b \leq a. \end{cases}$$

(Compare with [**161**, Example 3.4].)

Exercise 10.7.5. Compute the polynomials $P_{\mathbf{c}',\mathbf{c}}$.

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APPENDIX A

Category theory and homological algebra

This appendix collects a number of definitions and theorems that are needed in the main body of the book. Almost all proofs are omitted (although more details are given in some of the later sections).

A.1. Categories and functors

Categories. Following [**2**, **79**, **126**] and other sources, we work in **Tarski–Grothendieck set theory**, which includes the axiom that for every set x, there exists a Grothendieck universe $\mathscr{U}$ such that $x \in \mathscr{U}$. We always work inside some fixed uncountable Grothendieck universe $\mathscr{U}$. A set is said to be **$\mathscr{U}$-small**, or simply **small**, if it is in bijection with some element of $\mathscr{U}$.

All rings and modules in this book should implicitly be understood to be small.

Definition A.1.1. A **category** $\mathscr{C}$ consists of the following:

(1) a set $\mathrm{Ob}(\mathscr{C})$, called the set of **objects**;
(2) for any two objects $X, Y \in \mathrm{Ob}(\mathscr{C})$, a small set denoted by $\mathrm{Hom}(X, Y)$ or $\mathrm{Hom}_{\mathscr{C}}(X, Y)$, called the set of **morphisms from** X **to** Y; and
(3) for any three objects $X, Y, Z \in \mathrm{Ob}(\mathscr{C})$, a map

$$\circ : \mathrm{Hom}(Y, Z) \times \mathrm{Hom}(X, Y) \to \mathrm{Hom}(X, Z),$$

called the **composition** map.

These data are required to satisfy the following conditions:

(1) (Associativity) If $f \in \mathrm{Hom}(X, Y)$, $g \in \mathrm{Hom}(Y, Z)$, and $h \in \mathrm{Hom}(Z, W)$, then

$$h \circ (g \circ f) = (h \circ g) \circ f.$$

(2) (Identity) For every object X, there exists a morphism $\mathrm{id}_X \in \mathrm{Hom}(X, X)$, called the **identity morphism** of X, such that for all objects Y and all morphisms $f \in \mathrm{Hom}(X, Y)$ and $g \in \mathrm{Hom}(Y, X)$, we have

$$f \circ \mathrm{id}_X = f \qquad \text{and} \qquad \mathrm{id}_X \circ g = g.$$

We often write $f : X \to Y$ instead of $f \in \mathrm{Hom}(X, Y)$ to indicate that f is a morphism from X to Y. In this situation, the object X is called the **domain** of f, and Y is called the **codomain**. In the main body of this book, we usually omit the notation "Ob," and simply write $X \in \mathscr{C}$ to mean that X is an object of $\mathscr{C}$.

Example A.1.2. Let $\Bbbk$ be a commutative ring. We denote by $\Bbbk$-mod the category whose objects are (small) $\Bbbk$-modules. For $X, Y \in \Bbbk$-mod, the set of morphisms $\mathrm{Hom}_{\Bbbk\text{-mod}}(X, Y)$ is the set $\mathrm{Hom}_{\Bbbk}(X, Y)$ of $\Bbbk$-module homomorphisms $X \to Y$. (Because X and Y are themselves small, $\mathrm{Hom}_{\Bbbk}(X, Y)$ is again a small set.)

DEFINITION A.1.3. Let $\mathscr{C}$ be a category. Its **opposite category**, denoted by $\mathscr{C}^{\mathrm{op}}$, is the category whose objects are given by $\mathrm{Ob}(\mathscr{C}^{\mathrm{op}}) = \mathrm{Ob}(\mathscr{C})$, and whose morphisms are given by $\mathrm{Hom}_{\mathscr{C}^{\mathrm{op}}}(X,Y) = \mathrm{Hom}_{\mathscr{C}}(Y,X)$ (for any $X,Y \in \mathrm{Ob}(\mathscr{C}^{\mathrm{op}})$). If $f : X \to Y$ and $g : Y \to Z$ are morphisms in $\mathscr{C}^{\mathrm{op}}$, their composition is given by

$$g \circ_{\mathscr{C}^{\mathrm{op}}} f = f \circ_{\mathscr{C}} g.$$

Here are some properties that morphisms may have.

DEFINITION A.1.4. Let $f : X \to Y$ be a morphism in a category $\mathscr{C}$. We say that f is a **monomorphism** if for all morphisms $g_1, g_2 : Z \to X$,

$$f \circ g_1 = f \circ g_2 \qquad \text{implies} \qquad g_1 = g_2.$$

It is an **epimorphism** if for all morphisms $g_1, g_2 : Y \to Z$,

$$g_1 \circ f = g_2 \circ f \qquad \text{implies} \qquad g_1 = g_2.$$

Finally, f is an **isomorphism** if there exists a morphism $g : Y \to X$ such that $f \circ g = \mathrm{id}_Y$ and $g \circ f = \mathrm{id}_X$. In this case, the morphism g is called the **inverse** of f and is often denoted by $f^{-1} : Y \to X$. We say that X and Y are **isomorphic**, and we write $X \cong Y$ if there exists an isomorphism $f : X \to Y$.

DEFINITION A.1.5. Let $\mathscr{C}$ be a category. A **full subcategory** of $\mathscr{C}$ is a category $\mathscr{C}'$ such that the following conditions hold:

(1) The set $\mathrm{Ob}(\mathscr{C}')$ is a subset of $\mathrm{Ob}(\mathscr{C})$.
(2) For all $X, Y \in \mathrm{Ob}(\mathscr{C}')$, we have $\mathrm{Hom}_{\mathscr{C}'}(X,Y) = \mathrm{Hom}_{\mathscr{C}}(X,Y)$.

A **strictly full subcategory** is a full subcategory $\mathscr{C}' \subset \mathscr{C}$ with the following additional property:

(3) If $X \in \mathrm{Ob}(\mathscr{C})$ is isomorphic to an object of $\mathscr{C}'$, then $X \in \mathrm{Ob}(\mathscr{C}')$.

EXAMPLE A.1.6. Let $\Bbbk$ be a ring, and let $\Bbbk\text{-mod}^{\mathrm{fg}}$ be the category whose objects are finitely generated (small) $\Bbbk$-modules, and whose morphisms are $\Bbbk$-module homomorphisms. Then $\Bbbk\text{-mod}^{\mathrm{fg}}$ is a strictly full subcategory of $\Bbbk$-mod.

Functors. The following notion is used to relate categories to one another.

DEFINITION A.1.7. Let $\mathscr{C}$ and $\mathscr{D}$ be categories. A **functor** $F : \mathscr{C} \to \mathscr{D}$ is a rule that:

(1) assigns to each object $X \in \mathscr{C}$ an object $F(X) \in \mathscr{D}$, and
(2) assigns to each morphism $f : X \to Y$ in $\mathscr{C}$ a morphism $F(f) : F(X) \to F(Y)$ in $\mathscr{D}$.

This rule is required to satisfy the following conditions:

(1) For every object $X \in \mathscr{C}$, we have $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$.
(2) For any two morphisms $f : X \to Y$ and $g : Y \to Z$ in $\mathscr{C}$, we have

$$F(g \circ f) = F(g) \circ F(f).$$

A functor $\mathscr{C}^{\mathrm{op}} \to \mathscr{D}$ is sometimes called a **contravariant functor** from $\mathscr{C}$ to $\mathscr{D}$.

Given a functor $F : \mathscr{C} \to \mathscr{D}$ and two objects $X, Y \in \mathscr{C}$, the assignment $f \mapsto F(f)$ gives us a map

$$F : \mathrm{Hom}_{\mathscr{C}}(X,Y) \to \mathrm{Hom}_{\mathscr{D}}(F(X), F(Y)). \tag{A.1.1}$$

DEFINITION A.1.8. Let $F : \mathscr{C} \to \mathscr{D}$ be a functor. It is said to be:

(1) **faithful** if for any two objects $X, Y \in \mathscr{C}$, the map (A.1.1) is injective.
(2) **full** if for any two objects $X, Y \in \mathscr{C}$, the map (A.1.1) is surjective.
(3) **fully faithful** if it is both full and faithful.

DEFINITION A.1.9. The **essential image** of a functor $F : \mathscr{C} \to \mathscr{D}$, denoted by $F(\mathscr{C})$, is the strictly full subcategory of $\mathscr{D}$ consisting of objects $Y \in \mathscr{D}$ such that $Y \cong F(X)$ for some object $X \in \mathscr{C}$. The functor F is said to be **essentially surjective** if its essential image is all of $\mathscr{D}$.

DEFINITION A.1.10. Let $F, G : \mathscr{C} \to \mathscr{D}$ be functors. A **natural transformation** $\theta : F \to G$, also called a **morphism of functors**, is a rule that assigns to every object $X \in \mathscr{C}$ a morphism $\theta_X : F(X) \to G(X)$ such that for all morphisms $f : X \to Y$ in $\mathscr{C}$, the following diagram in $\mathscr{D}$ commutes:

$$\begin{tikzcd} F(X) \arrow[r, "F(f)"] \arrow[d, "\theta_X"'] & F(Y) \arrow[d, "\theta_Y"] \\ G(X) \arrow[r, "G(f)"] & G(Y) \end{tikzcd}$$

The natural transformation θ is said to be a **natural isomorphism** if for all objects $X \in \mathscr{C}$, the morphism θ_X is an isomorphism in $\mathscr{D}$.

DEFINITION A.1.11. A functor $F : \mathscr{C} \to \mathscr{D}$ is said to be an **equivalence of categories** if there exists a functor $G : \mathscr{D} \to \mathscr{C}$ along with two natural isomorphisms

$$F \circ G \cong \mathrm{id}_{\mathscr{D}} \qquad \text{and} \qquad G \circ F \cong \mathrm{id}_{\mathscr{C}}.$$

In this case, the functor G is called an **inverse** to F.

The inverse of an equivalence of categories is *not* unique in general: instead, it is only unique up to a natural isomorphism. Nevertheless, if F is an equivalence of categories, one often writes F^{-1} to denote some inverse functor.

REMARK A.1.12. In the context of Definition A.1.11, if we in fact have $F \circ G = \mathrm{id}_{\mathscr{D}}$ and $G \circ F = \mathrm{id}_{\mathscr{C}}$, then F is said to be an **isomorphism of categories**. But this is rarely a useful notion; most equivalences of categories that arise in practice are not isomorphisms.

PROPOSITION A.1.13. *A functor $F : \mathscr{C} \to \mathscr{D}$ is an equivalence of categories if and only if it is fully faithful and essentially surjective.*

PROPOSITION A.1.14 (Yoneda's lemma). *Let X and Y be objects of $\mathscr{C}$. There are natural bijections*

$$\mathrm{Hom}(X, Y) \xrightarrow{\sim} \{\textit{natural transformations } \mathrm{Hom}(Y, -) \to \mathrm{Hom}(X, -)\},$$
$$\mathrm{Hom}(X, Y) \xrightarrow{\sim} \{\textit{natural transformations } \mathrm{Hom}(-, X) \to \mathrm{Hom}(-, Y)\}.$$

As a consequence, the following conditions are equivalent:

(1) *The objects X and Y are isomorphic.*
(2) *The functors* $\mathrm{Hom}(X, -)$ *and* $\mathrm{Hom}(Y, -)$ *are naturally isomorphic.*
(3) *The functors* $\mathrm{Hom}(-, X)$ *and* $\mathrm{Hom}(-, Y)$ *are naturally isomorphic.*

Definition A.1.15. Let $\mathscr{C}$ and $\mathscr{D}$ be categories, and let $F : \mathscr{C} \to \mathscr{D}$ and $G : \mathscr{D} \to \mathscr{C}$ be functors. We say that (F, G) is an **adjoint pair** if for $X \in \mathscr{C}$ and $Y \in \mathscr{D}$, there is a natural isomorphism

$$\theta_{X,Y} : \operatorname{Hom}_{\mathscr{D}}(F(X), Y) \xrightarrow{\sim} \operatorname{Hom}_{\mathscr{C}}(X, G(Y)). \tag{A.1.2}$$

We also say that F is **left adjoint** to G, or that G is **right adjoint** to F.

When we say that (A.1.2) is natural, we actually mean two conditions: it is "natural in X" and "natural in Y." Explicitly, these two naturality conditions say that if $f : X' \to X$ is a morphism in $\mathscr{C}$, and if $\phi : F(X) \to Y$ and $g : Y \to Y'$ are morphisms in $\mathscr{D}$, then

$$\begin{aligned} \theta_{X',Y}(\phi \circ F(f)) &= \theta_{X,Y}(\phi) \circ f : X' \to G(Y), \\ G(g) \circ \theta_{X,Y}(\phi) &= \theta_{X,Y'}(g \circ \phi) : X \to G(Y'). \end{aligned} \tag{A.1.3}$$

Proposition A.1.16. *Let $F : \mathscr{C} \to \mathscr{D}$ and $G : \mathscr{D} \to \mathscr{C}$ be functors. The following conditions are equivalent:*

(1) *The pair (F, G) is an adjoint pair.*
(2) *There are natural transformations*

$$\eta : \mathrm{id}_{\mathscr{C}} \to G \circ F \qquad \textit{and} \qquad \epsilon : F \circ G \to \mathrm{id}_{\mathscr{D}}$$

such that for $X \in \mathscr{C}$ and $Y \in \mathscr{D}$, we have

$$\epsilon_{F(X)} \circ F(\eta_X) = \mathrm{id}_{F(X)} \qquad \textit{and} \qquad G(\epsilon_Y) \circ \eta_{G(Y)} = \mathrm{id}_{G(Y)}. \tag{A.1.4}$$

The natural map $\eta : \mathrm{id}_{\mathscr{C}} \to GF$ is called the **unit** of the adjunction, and $\epsilon : FG \to \mathrm{id}_{\mathscr{D}}$ is called the **counit**. (Informally, they are both simply called **adjunction maps**.) The conditions in (A.1.4) are called the **zig-zag equations**.

Proof sketch. Assume first that (F, G) is an adjoint pair. Define $\eta_X = \theta_{X,F(X)}(\mathrm{id}_{F(X)})$ and $\epsilon_Y = \theta^{-1}_{G(Y),Y}(\mathrm{id}_{G(Y)})$. Using (A.1.3), one can show that the zig-zag equations hold.

Conversely, given $\eta : \mathrm{id}_{\mathscr{C}} \to GF$ and $\epsilon : FG \to \mathrm{id}_{\mathscr{D}}$ satisfying (A.1.4), define $\theta_{X,Y} : \operatorname{Hom}(F(X), Y) \to \operatorname{Hom}(X, G(Y))$ by $\theta_{X,Y}(\phi) = G(\phi) \circ \eta_X$. The zig-zag equations imply that $\theta_{X,Y}$ is an isomorphism (with inverse given by $\psi \mapsto \epsilon_Y \circ F(\psi)$) and that the naturality equations (A.1.3) hold. $\square$

Objects defined via representable functors. Let F be a functor from $\mathscr{C}$ to the category of (small) sets. If F is isomorphic to a functor of the form $\operatorname{Hom}_{\mathscr{C}}(X, -)$, then Yoneda's lemma implies that the object X is determined up to unique isomorphism. In this situation, one says that F is a **representable functor** and that X is the object that **represents** it. (Similarly, a functor from $\mathscr{C}^{\mathrm{op}}$ to the category of sets is **representable** if it is isomorphic to a functor of the form $\operatorname{Hom}_{\mathscr{C}}(-, X)$.)

In view of this, one can try to define an object by writing down the functor it represents. An object defined in this way may not exist (the functor may not be representable), but if it does exist, it is unique up to unique isomorphism. The next few definitions are of this form.

Definition A.1.17. An **initial object** in a category $\mathscr{C}$ is an object $I \in \mathscr{C}$ such that for all $X \in \mathscr{C}$, the set $\operatorname{Hom}(I, X)$ contains exactly one element. A **terminal object** is an object $T \in \mathscr{C}$ such that for all $X \in \mathscr{C}$, the set $\operatorname{Hom}(X, T)$ contains exactly one element. A **zero object** is an object that is both initial and terminal.

If $\mathscr{C}$ has a zero object 0, then for any two objects $X, Y \in \mathscr{C}$, the **zero morphism** $0 : X \to Y$ is defined to be the composition $X \to 0 \to Y$.

DEFINITION A.1.18. Let $\mathscr{C}$ be a category, and let $(X_\alpha)_{\alpha \in I}$ be a collection of objects in $\mathscr{C}$ indexed by some set I. Their **product** is an object $\prod_\alpha X_\alpha$ together with morphisms $\mathrm{pr}_\beta : \prod_\alpha X_\alpha \to X_\beta$ for each $\beta \in I$ such that for any object $Z \in \mathscr{C}$, the map $f \mapsto (\mathrm{pr}_\alpha \circ f)_{\alpha \in I}$ induces a bijection

$$\mathrm{Hom}(Z, \prod_\alpha X_\alpha) \xrightarrow{\sim} \prod_\alpha \mathrm{Hom}(Z, X_\alpha). \tag{A.1.5}$$

Similarly, their **coproduct** is an object $\coprod_\alpha X_\alpha$ together with morphisms $\mathrm{in}_\beta : X_\beta \to \coprod_\alpha X_\alpha$ for each $\beta \in I$ such that for any object $Z \in \mathscr{C}$, the map $f \mapsto (f \circ \mathrm{in}_\alpha)_{\alpha \in I}$ induces a bijection

$$\mathrm{Hom}(\coprod_\alpha X_\alpha, Z) \xrightarrow{\sim} \prod_\alpha \mathrm{Hom}(X_\alpha, Z). \tag{A.1.6}$$

Products and coproducts are special cases of limits, which we will discuss below.

DEFINITION A.1.19. Let $\mathscr{C}$ be a category, and let $(I, \leq)$ be a set with a preorder. A **direct system** in $\mathscr{C}$ indexed by $(I, \leq)$ is a collection of objects $(X_\alpha)_{\alpha \in I}$ together with a collection of morphisms $f_{\alpha\beta} : X_\alpha \to X_\beta$ for each pair $\alpha \leq \beta$ such that the following conditions hold:

$$\begin{aligned} f_{\alpha\alpha} &= \mathrm{id}_{X_\alpha} && \text{for all } \alpha \in I, \text{ and} \\ f_{\alpha\gamma} &= f_{\beta\gamma} \circ f_{\alpha\beta} && \text{if } \alpha \leq \beta \leq \gamma. \end{aligned}$$

An **inverse system** is collection of objects $(X_\alpha)_{\alpha \in I}$ together with a collection of morphisms $f_{\alpha\beta} : X_\beta \to X_\alpha$ for each pair $\alpha \leq \beta$, satisfying similar conditions to those above. (Equivalently, an inverse system in $\mathscr{C}$ is a direct system in $\mathscr{C}^{\mathrm{op}}$.)

Before defining direct and inverse limits in general, let us consider the special case of (small) sets. If $((X_\alpha)_{\alpha \in I}, (f_{\alpha\beta})_{\alpha \leq \beta})$ is an inverse system of sets, their **inverse limit** is defined by

$$\varprojlim_{(I, \leq)} X_\alpha = \left\{ (x_\alpha) \in \prod_{\alpha \in I} X_\alpha \;\middle|\; f_{\alpha\beta}(x_\beta) = x_\alpha \text{ whenever } \alpha \leq \beta \right\} \subset \prod_{\alpha \in I} X_\alpha.$$

DEFINITION A.1.20. Let $(I, \leq)$ be a preordered set, and let $((X_\alpha)_{\alpha \in I}, (f_{\alpha\beta})_{\alpha \leq \beta})$ be a direct system of objects in $\mathscr{C}$. The **direct limit**, also called the **colimit**, of this system is an object $\varinjlim X_\alpha$ together with maps $\phi_\beta : X_\beta \to \varinjlim X_\alpha$ for each $\beta \in I$ such that for any object $Z \in \mathscr{C}$, the map $f \mapsto (f \circ \phi_\alpha)_{\alpha \in I}$ induces a bijection

$$\mathrm{Hom}(\varinjlim X_\alpha, Z) \xrightarrow{\sim} \varprojlim \mathrm{Hom}(X_\alpha, Z).$$

Similarly, let $((X_\alpha)_{\alpha \in I}, (f_{\alpha\beta})_{\alpha \leq \beta})$ be an inverse system of objects in $\mathscr{C}$. The **inverse limit**, also called the **limit**, of this system is an object $\varprojlim X_\alpha$ together with maps $\psi_\beta : \varprojlim X_\alpha \to X_\beta$ for each $\beta \in I$ such that for any object $Z \in \mathscr{C}$, the map $f \mapsto (\psi_\alpha \circ f)_{\alpha \in I}$ induces a bijection

$$\mathrm{Hom}(Z, \varprojlim X_\alpha) \xrightarrow{\sim} \varprojlim \mathrm{Hom}(Z, X_\alpha).$$

EXAMPLE A.1.21. Products and coproducts are examples of inverse and direct limits, respectively, with respect to a preorder in which no two distinct elements are comparable. Initial and terminal objects are examples of inverse and direct limits, respectively, with respect to the empty preordered set.

EXAMPLE A.1.22. Here is a concrete description of direct limits in the category of small sets or in the category of modules over some ring. Assume for simplicity that $(I, \leq)$ is **filtered**, i.e., for any two elements $\alpha, \beta \in I$, there exists a $\gamma \in I$ with $\alpha \leq \gamma$ and $\beta \leq \gamma$. If $((X_\alpha)_{\alpha \in I}, (f_{\alpha\beta})_{\alpha \leq \beta})$ is a direct system of sets or modules, one can show that

$$\varinjlim_{(I,\leq)} X_\alpha \cong \{(\alpha, x) \mid \alpha \in I,\, x \in X_\alpha\}/\sim,$$

where the equivalence relation $\sim$ is defined as follows: $(\alpha, x) \sim (\beta, y)$ if there exists an element $\gamma \in I$ with $\alpha \leq \gamma$, $\beta \leq \gamma$, and $f_{\alpha\gamma}(x) = f_{\beta\gamma}(y)$.

A.2. Monoidal categories

This brief section contains a list of definitions related to the following concept, following [**164**].

DEFINITION A.2.1. Let $\mathscr{C}$ be a category. A **monoidal category** structure on $\mathscr{C}$ consists of the following data:

- a functor $\odot : \mathscr{C} \times \mathscr{C} \to \mathscr{C}$,
- an object $\mathbf{1} \in \mathscr{C}$,
- for any three objects $X, Y, Z \in \mathscr{C}$, a natural isomorphism $\alpha_{X,Y,Z} : (X \odot Y) \odot Z \xrightarrow{\sim} X \odot (Y \odot Z)$, called the **associator**,
- for any object $X \in \mathscr{C}$, two natural isomorphisms $\lambda_X : \mathbf{1} \odot X \xrightarrow{\sim} X$ and $\rho_X : X \odot \mathbf{1} \xrightarrow{\sim} X$, called the **left** and **right unitors**.

These data are required to satisfy the following conditions:

(1) (Pentagon axiom) For all $X, Y, Z, W \in \mathscr{C}$, the following diagram commutes:

$$\begin{tikzcd}
((X \odot Y) \odot Z) \odot W \arrow[r, "\alpha_{X,Y,Z} \odot \mathrm{id}_W"] \arrow[d, "\alpha_{X \odot Y, Z, W}"'] & (X \odot (Y \odot Z)) \odot W \arrow[r, "\alpha_{X, Y \odot Z, W}"] & X \odot ((Y \odot Z) \odot W) \arrow[d, "\mathrm{id}_X \odot \alpha_{Y,Z,W}"] \\
(X \odot Y) \odot (Z \odot W) \arrow[rr, "\alpha_{X,Y,Z \odot W}"] & & X \odot (Y \odot (Z \odot W))
\end{tikzcd}$$

(2) (Triangle axiom) For all $X, Y \in \mathscr{C}$, the following diagram commutes:

$$\begin{tikzcd}
(X \odot \mathbf{1}) \odot Y \arrow[rr, "\alpha_{X,\mathbf{1},Y}"] \arrow[dr, "\rho_X \odot \mathrm{id}_Y"'] & & X \odot (\mathbf{1} \odot Y) \arrow[dl, "\mathrm{id}_X \odot \lambda_Y"] \\
& X \odot Y &
\end{tikzcd}$$

The associator is sometimes called the **associativity constraint**. The definition in [**164**] contains an additional axiom, requiring that the morphisms $\lambda_{\mathbf{1}}, \rho_{\mathbf{1}} : \mathbf{1} \odot \mathbf{1} \to \mathbf{1}$ be equal, but according to [**132**], this follows from the pentagon and triangle axioms.

We often informally write $(\mathscr{C}, \odot)$ for a monoidal category, regarding the object $\mathbf{1}$ and the three natural transformations α, λ, and ρ as implicit.

DEFINITION A.2.2. Let $(\mathscr{C}, \odot)$ and $(\mathscr{C}', \odot')$ be two monoidal categories. A **monoidal functor** $F : (\mathscr{C}, \odot) \to (\mathscr{C}', \odot')$ is a functor $F : \mathscr{C} \to \mathscr{C}'$ together with the following data:

- for any two objects $X, Y \in \mathscr{C}$, a natural transformation $\phi_{X,Y} : (FX) \odot' (FY) \to F(X \odot Y)$,
- a morphism $\eta : \mathbf{1}_{\mathscr{C}'} \to F(\mathbf{1}_{\mathscr{C}})$,

such that the following **coherence diagrams** commute:

$$\begin{tikzcd}
((FX)\odot'(FY))\odot'(FZ) \arrow[r, "\phi_{X,Y}\odot'\mathrm{id}_{FZ}"] \arrow[d, "\alpha_{FX,FY,FZ}"'] & F(X\odot Y)\odot'(FZ) \arrow[r, "\phi_{X\odot Y,Z}"] & F((X\odot Y)\odot Z) \arrow[d, "F(\alpha_{X,Y,Z})"] \\
(FX)\odot'((FY)\odot'(FZ)) \arrow[r, "\mathrm{id}_{FX}\odot'\phi_{Y,Z}"] & (FX)\odot' F(Y\odot Z) \arrow[r, "\phi_{X,Y\odot Z}"] & F(X\odot(Y\odot Z))
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbf{1}_{\mathscr{C}'}\odot'(FY) \arrow[r, "\lambda_{FY}"] \arrow[d, "\eta\odot'\mathrm{id}_{FY}"'] & FY \\
F(\mathbf{1}_{\mathscr{C}})\odot'(FY) \arrow[r, "\phi_{\mathbf{1}_{\mathscr{C}},Y}"] & F(\mathbf{1}_{\mathscr{C}}\odot Y) \arrow[u, "F(\lambda_Y)"']
\end{tikzcd}
\qquad
\begin{tikzcd}
(FY)\odot'\mathbf{1}_{\mathscr{C}'} \arrow[r, "\rho_{FY}"] \arrow[d, "\mathrm{id}_{FY}\odot'\eta"'] & FY \\
(FY)\odot' F(\mathbf{1}_{\mathscr{C}}) \arrow[r, "\phi_{Y,\mathbf{1}_{\mathscr{C}}}"] & F(Y\odot\mathbf{1}_{\mathscr{C}}) \arrow[u, "F(\rho_Y)"']
\end{tikzcd}$$

If $\phi_{X,Y}$ and η are isomorphisms, then F is said to be a **strong monoidal functor**.

DEFINITION A.2.3. A **symmetric monoidal category** consists of a monoidal category $(\mathscr{C},\odot)$ together with a natural isomorphism $\sigma_{X,Y} : X\odot Y \xrightarrow{\sim} Y\odot X$, called the **commutator**, such that following conditions hold:

(1) (Braiding axiom) For all $X, Y, Z \in \mathscr{C}$, the following diagram commutes:

$$\begin{tikzcd}
(X\odot Y)\odot Z \arrow[r, "\sigma_{X,Y}\odot\mathrm{id}_Z"] \arrow[d, "\alpha_{X,Y,Z}"'] & (Y\odot X)\odot Z \arrow[r, "\alpha_{Y,X,Z}"] & Y\odot(X\odot Z) \arrow[d, "\mathrm{id}_Y\odot\sigma_{X,Z}"] \\
X\odot(Y\odot Z) \arrow[r, "\sigma_{X,Y\odot Z}"] & (Y\odot Z)\odot X \arrow[r, "\alpha_{Y,Z,X}"] & Y\odot(Z\odot X)
\end{tikzcd}$$

(2) (Involutivity axiom) For all $X, Y \in \mathscr{C}$, the following diagram commutes:

$$\begin{tikzcd}
 & Y\odot X \arrow[dr, "\sigma_{Y,X}"] & \\
X\odot Y \arrow[ur, "\sigma_{X,Y}"] \arrow[rr, "\mathrm{id}_{X\odot Y}"] & & X\odot Y
\end{tikzcd}$$

The commutator is sometimes called the **commutativity constraint**. Given a natural isomorphism $\sigma_{X,Y} : X\odot Y \xrightarrow{\sim} Y\odot X$, one can define another natural map $\bar{\sigma}_{X,Y} : X\odot Y \xrightarrow{\sim} Y\odot X$ by $\bar{\sigma}_{X,Y} = (\sigma_{Y,X})^{-1}$. The involutivity axiom above says precisely that $\sigma = \bar{\sigma}$. If we drop the involutivity axiom, but then require both σ and $\bar{\sigma}$ to obey the braiding axiom, we obtain the notion of a **braided monoidal category**.

DEFINITION A.2.4. Let $(\mathscr{C},\odot)$ and $(\mathscr{C}',\odot')$ be two symmetric monoidal categories. A **symmetric monoidal functor** $F : (\mathscr{C},\odot)\to(\mathscr{C}',\odot')$ is a monoidal functor with the additional property that the following diagram commutes:

$$\begin{tikzcd}
(FX)\odot'(FY) \arrow[r, "\sigma_{FX,FY}"] \arrow[d, "\phi_{X,Y}"] & (FY)\odot'(FX) \arrow[d, "\phi_{Y,X}"] \\
F(X\odot Y) \arrow[r, "F(\sigma_{X,Y})"] & F(Y\odot X)
\end{tikzcd}$$

DEFINITION A.2.5. Let $(\mathscr{C},\odot)$ be a symmetric monoidal category. A **dual** for an object $X\in\mathscr{C}$ is an object $X^*\in\mathscr{C}$ together with two morphisms $\eta : \mathbf{1}\to X\odot X^*$ and $\epsilon : X^*\odot X\to\mathbf{1}$ such that both of the following compositions are identity maps:

$$X \xrightarrow{\lambda^{-1}} \mathbf{1}\odot X \xrightarrow{\eta\odot\mathrm{id}} (X\odot X^*)\odot X \xrightarrow{\alpha} X\odot(X^*\odot X) \xrightarrow{\mathrm{id}\odot\epsilon} X\odot\mathbf{1} \xrightarrow{\rho} X,$$

$$X^* \xrightarrow{\rho^{-1}} X^*\odot\mathbf{1} \xrightarrow{\mathrm{id}\odot\eta} X^*\odot(X\odot X^*) \xrightarrow{\alpha^{-1}} (X^*\odot X)\odot X^* \xrightarrow{\epsilon\odot\mathrm{id}} \mathbf{1}\odot X^* \xrightarrow{\lambda} X^*.$$

The category $(\mathscr{C},\odot)$ is said to be **rigid** if every object admits a dual.

Note that this definition does not explicitly involve the commutator. In a (not necessarily symmetric) monoidal category, the conditions in Definition A.2.5 define **right duals** and **right rigid** categories. Similar conditions in terms of maps $\mathbf{1} \to X^* \odot X$ and $X \odot X^* \to \mathbf{1}$ give the notions of **left duals** and **left rigid** categories. In the symmetric case, the right and left versions coincide.

A.3. Additive and abelian categories

Additive categories. Essentially all categories we encounter in this book are of the following kind.

DEFINITION A.3.1. A category $\mathscr{A}$ is said to be **additive** if the following conditions hold:

(1) For any two objects $X, Y \in \mathscr{A}$, the set $\mathrm{Hom}(X,Y)$ of morphisms $X \to Y$ is equipped with the structure of an abelian group. Moreover, for any three objects $X, Y, Z \in \mathscr{A}$, the composition map
$$\circ : \mathrm{Hom}(Y,Z) \times \mathrm{Hom}(X,Y) \to \mathrm{Hom}(X,Z)$$
is bilinear.
(2) The category has a zero object.
(3) The coproduct (also called the **direct sum**) of any two objects exists.

A functor $F : \mathscr{A} \to \mathscr{B}$ of additive categories is said to be an **additive functor** if the map (A.1.1) is a homomorphism of abelian groups.

Next, let $\Bbbk$ be a commutative ring. The category $\mathscr{A}$ is said to be **$\Bbbk$-linear** if the following stronger variant of axiom (1) holds:

(1′) For any two objects $X, Y \in \mathscr{A}$, the set $\mathrm{Hom}(X,Y)$ of morphisms $X \to Y$ is equipped with the structure of a $\Bbbk$-module. Moreover, for any three objects $X, Y, Z \in \mathscr{A}$, the composition map
$$\circ : \mathrm{Hom}(Y,Z) \times \mathrm{Hom}(X,Y) \to \mathrm{Hom}(X,Z)$$
is $\Bbbk$-bilinear.

An additive functor $F : \mathscr{A} \to \mathscr{B}$ of $\Bbbk$-linear additive categories is said to be **$\Bbbk$-linear** if the map (A.1.1) is a homomorphism of $\Bbbk$-modules.

The direct sum of two objects X and Y in additive category is usually denoted by $X \oplus Y$, instead of $X \amalg Y$. Of course, this object comes with maps $\mathrm{in}_1 : X \to X \oplus Y$ and $\mathrm{in}_2 : Y \to X \oplus Y$. It turns out that the direct sum is automatically also a product: indeed, the maps $\mathrm{pr}_1 : X \oplus Y \to X$ and $\mathrm{pr}_2 : X \oplus Y \to Y$ are the maps corresponding to
$$(\mathrm{id}_X, 0) \in \mathrm{Hom}(X,X) \times \mathrm{Hom}(Y,X) \quad \text{and} \quad (0, \mathrm{id}_Y) \in \mathrm{Hom}(X,Y) \times \mathrm{Hom}(Y,Y),$$
respectively, under (A.1.6). The product and coproduct maps satisfy
$$\begin{matrix} \mathrm{pr}_1 \circ \mathrm{in}_1 = \mathrm{id}_X, & \mathrm{pr}_2 \circ \mathrm{in}_1 = 0, \\ \mathrm{pr}_2 \circ \mathrm{in}_2 = \mathrm{id}_Y, & \mathrm{pr}_1 \circ \mathrm{in}_2 = 0, \end{matrix} \qquad \mathrm{in}_1 \circ \mathrm{pr}_1 + \mathrm{in}_2 \circ \mathrm{pr}_2 = \mathrm{id}_{X \oplus Y}. \tag{A.3.1}$$
If $F : \mathscr{A} \to \mathscr{B}$ is an additive functor, it can be shown that for any two objects $X, Y \in \mathscr{A}$, there is a natural isomorphism
$$F(X \oplus Y) \cong F(X) \oplus F(Y).$$
The third axiom in Definition A.3.1 implies, more generally, that the direct sum of any finite collection of objects exists and coincides with their product. In contrast,

infinite direct sums and infinite products do not necessarily exist; and when they do exist, they need not coincide.

Definition A.3.2. Let $f : X \to Y$ be a morphism in an additive category. The **kernel** of f, denoted by $\ker f$, is the inverse limit of the inverse system

$$\begin{tikzcd} X \arrow[dr, "f"] & \\ & Y \\ 0 \arrow[ur] & \end{tikzcd}$$

The **cokernel** of f, denoted by $\operatorname{cok} f$, is the direct limit of the direct system

$$\begin{tikzcd} & Y \\ X \arrow[ur, "f"] \arrow[dr] & \\ & 0 \end{tikzcd}$$

As with (co)limits in general, kernels and cokernels do not necessarily exist. Here is a more concrete description: the kernel of $f : X \to Y$ is an object $\ker f$ together with a map $k : \ker f \to X$ such that $f \circ k = 0$, and such that for any other object Z, there is a bijection

$$\operatorname{Hom}(Z, \ker f) \xrightarrow[\sim]{g \mapsto k \circ g} \{h : Z \to X \mid f \circ h = 0\}. \tag{A.3.2}$$

Similarly, the cokernel consists of an object $\operatorname{cok} f$ with a map $q : Y \to \operatorname{cok} f$ such that $q \circ f = 0$, and such that for any other object Z, there is a bijection

$$\operatorname{Hom}(\operatorname{cok} f, Z) \xrightarrow[\sim]{g \mapsto g \circ q} \{h : Y \to Z \mid h \circ f = 0\}. \tag{A.3.3}$$

The following observations about kernels and cokernels are sometimes useful.

Lemma A.3.3. *In an additive category, any kernel is a monomorphism, and any cokernel is an epimorphism.*

Lemma A.3.4. *Let $f : X \to Y$ be a morphism in an additive category.*

(1) *The map $0 \to X$ is the kernel of f if and only if f is a monomorphism.*
(2) *The map $Y \to 0$ is the cokernel of f if and only if f is an epimorphism.*

Definition A.3.5. Let $\mathscr{A}$ be an additive category in which every morphism has a kernel and a cokernel, and let $f : X \to Y$ be a morphism in $\mathscr{A}$. The **coimage** of f, denoted by $\operatorname{coim} f$, is given by $\operatorname{coim} f = \operatorname{cok}(\ker f)$. The **image** of f, denoted by $\operatorname{im} f$, is given by $\operatorname{im} f = \ker(\operatorname{cok} f)$.

Using Lemma A.3.3 and the bijections (A.3.2) and (A.3.3), one can show that if f has both an image and a coimage, then there is a unique morphism $s : \operatorname{coim} f \to \operatorname{im} f$ that makes the following diagram commute:

$$\begin{tikzcd} \ker f \arrow[r] & X \arrow[r, "f"] \arrow[d] & Y \arrow[r] & \operatorname{cok} f \\ & \operatorname{coim} f \arrow[r, dashed, "\exists! s"] & \operatorname{im} f \arrow[u] & \end{tikzcd}$$

Definition A.3.6. An additive category $\mathscr{A}$ is said to be **abelian** if it satisfies the following conditions:

(1) Every morphism has a kernel and a cokernel.
(2) For every morphism $f : X \to Y$, the canonical map $s : \operatorname{coim} f \to \operatorname{im} f$ is an isomorphism.

The following alternative characterization of abelian categories is often useful.

THEOREM A.3.7. *An additive category $\mathscr{A}$ is abelian if and only if it satisfies the following conditions:*

(1) *Every morphism has a kernel and a cokernel.*
(2) *Every monomorphism is the kernel of its cokernel, and every epimorphism is the cokernel of its kernel.*

It is easy to see that Definition A.3.6 implies the conditions in Theorem A.3.7. A proof of the opposite implication can be found in [**76**].

LEMMA A.3.8. *In an abelian category, a morphism is an isomorphism if and only if it is both a monomorphism and an epimorphism.*

The category $\Bbbk$-mod is a key example of an abelian category. We will adopt some terminology from this category for use in general abelian categories, as follows:

- A monomorphism is also called an **injective morphism**. If $f : Y \to X$ is injective, we sometimes refer to Y as a **subobject** of X. In situations where the morphism f is obvious, we may write $Y \subset X$.
- An epimorphism is also called a **surjective morphism**. If $g : X \to Z$ is surjective, we sometimes refer to Z as a **quotient** of X. In particular, if g is the cokernel of an injective map $f : Y \to X$, we sometimes write $Z = X/Y$.

For any two morphisms $f_1 : Y_1 \to X$ and $f_2 : Y_2 \to X$, we can form their sum $f_1 + f_2 : Y_1 \oplus Y_2 \to X$. Suppose now that f_1 and f_2 are injective, so that Y_1 and Y_2 are subobjects of X. Their **intersection** is given by

$$Y_1 \cap Y_2 = \ker(f_1 - f_2)$$

and their **sum** is given by

$$Y_1 + Y_2 = \operatorname{im}(f_1 + f_2).$$

Exact sequences. The following notion is of central importance in the study of abelian categories.

DEFINITION A.3.9. Let $f : X \to Y$ and $g : Y \to Z$ be morphisms in an abelian category. The sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ is said to be **exact at** Y if $g \circ f = 0$ and if the induced map $\operatorname{im} f \to \ker g$ is an isomorphism. A **short exact sequence** is a sequence

$$0 \to Y \to E \to X \to 0$$

that is exact at Y, E, and X. The short exact sequence above is also called an **extension of** X **by** Y. Two extensions of X by Y are said to be **equivalent** if they occur as the rows of a commutative diagram of the form

$$\begin{tikzcd}
0 \arrow[r] & Y \arrow[r] \arrow[d, equal] & E \arrow[r] \arrow[d, "u", "\wr"'] & X \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & Y \arrow[r] & E' \arrow[r] & X \arrow[r] & 0
\end{tikzcd}$$

More generally, a sequence of objects and morphisms $X_0 \to X_1 \to X_2 \to \cdots \to X_n$ is said to be **exact** if it is exact at $X_1, X_2, \ldots, X_{n-1}$.

The following lemma characterizes a particular equivalence class of extensions.

LEMMA A.3.10. *The following conditions on a short exact sequence* $0 \to Y \xrightarrow{f} E \xrightarrow{g} X \to 0$ *are equivalent:*

(1) *There is a map* $r : E \to Y$ *such that* $r \circ f = \mathrm{id}_Y$.
(2) *There is a map* $s : X \to E$ *such that* $g \circ s = \mathrm{id}_X$.
(3) *There is an isomorphism* $u : E \to Y \oplus X$ *such that the following diagram commutes:*

$$\begin{tikzcd} 0 \arrow[r] & Y \arrow[r, "f"] \arrow[d, "\mathrm{id}"'] & E \arrow[r, "g"] \arrow[d, "u"] & X \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\ 0 \arrow[r] & Y \arrow[r, "\mathrm{in}_1"] & Y \oplus X \arrow[r, "\mathrm{pr}_2"] & X \arrow[r] & 0 \end{tikzcd}$$

DEFINITION A.3.11. An extension or short exact sequence satisfying the equivalent conditions of Lemma A.3.10 is said to be **split**.

LEMMA A.3.12 (Five lemma). *Suppose we have the following commutative diagram in an abelian category:*

$$\begin{tikzcd} A \arrow[r] \arrow[d, "a"'] & B \arrow[r] \arrow[d, "b"'] & C \arrow[r] \arrow[d, "c"] & D \arrow[r] \arrow[d, "d"] & E \arrow[d, "e"] \\ A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & D' \arrow[r] & E' \end{tikzcd}$$

Assume that the rows are exact and that a is an epimorphism, e is a monomorphism, and b and d are isomorphisms. Then c is an isomorphism.

LEMMA A.3.13 (Snake lemma). *Suppose we have the following commutative diagram in an abelian category:*

$$\begin{tikzcd} & A \arrow[r] \arrow[d, "a"'] & B \arrow[r] \arrow[d, "b"] & C \arrow[r] \arrow[d, "c"] & 0 \\ 0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' & \end{tikzcd}$$

Assume that the rows are exact. Then there is a natural map $\delta : \ker c \to \operatorname{cok} a$ *such that the following sequence is exact:*

$$\ker a \to \ker b \to kerc \xrightarrow{\delta} \operatorname{cok} a \to \operatorname{cok} b \to \operatorname{cok} c.$$

REMARK A.3.14. The traditional proofs of Lemma A.3.12 and Lemma A.3.13 for abelian groups or for modules over a ring is by a "diagram chase": one follows elements of various modules around the maps in the diagrams. At first glance, this approach cannot be used for other abelian categories, since the word "element" has no meaning in general.

One possibility is to argue directly with formal properties of kernels and cokernels. This is very tedious.

A second approach is to first prove the **Freyd–Mitchell embedding theorem**, which says that for every abelian category $\mathscr{A}$, there exists a ring R and an exact, fully faithful functor $F : \mathscr{A} \to R$-mod. Using this functor, one can prove the five lemma in $\mathscr{A}$ by a diagram-chase argument in R-mod.

A third approach, explained in [**81**, Exercises II.5.1–II.5.5], is to develop the notion of "pseudoelements." A **pseudoelement** of an object X in an abelian category $\mathscr{A}$ is an equivalence class of pairs (Y, h), where Y is an on object in $\mathscr{A}$ and $h : Y \to X$ is a morphism. The equivalence relation is as follows: we set

$(Y, h) \sim (Y', h')$ if there is an object Z and epimorphisms $u : Z \to Y$, $u' : Z \to Y'$ such that $h \circ u = h' \circ u'$. According to [**81**, Exercise II.5.5], pseudoelements obey the usual "diagram chase rules," so by using pseudoelements instead of elements, one is justified in carrying out diagram-chase arguments in any abelian category. (Note, however, that pseudoelements are *not* the same as elements in a category like R-mod.)

Definition A.3.15. Let $\mathscr{A}$ be an abelian category, and let $\mathscr{A}' \subset \mathscr{A}$ be a strictly full additive subcategory. It is said to be a **Serre subcategory** if for every short exact sequence $0 \to X \to Y \to Z \to 0$ in $\mathscr{A}$, we have $Y \in \mathscr{A}'$ if and only if both X and Z lie in $\mathscr{A}'$. Equivalently, a Serre subcategory is a full additive subcategory that is closed under subobjects, quotients, and extensions.

Definition A.3.16. Let $\mathscr{A}$ and $\mathscr{B}$ be abelian categories, and let $F : \mathscr{A} \to \mathscr{B}$ be an additive functor. The functor F is said to be **left exact** if for every exact sequence $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathscr{A}$, the sequence $0 \to F(X) \xrightarrow{F(f)} F(Y) \xrightarrow{F(g)} F(Z)$ is exact. The functor F is said to be **right exact** if for every exact sequence $X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ in $\mathscr{A}$, the sequence $F(X) \xrightarrow{F(f)} F(Y) \xrightarrow{F(g)} F(Z) \to 0$ is exact. The functor F is **exact** if it is both left exact and right exact.

Lemma A.3.17. *Let $\mathscr{A}$ be a $\Bbbk$-linear abelian category. For any object $X \in \mathscr{A}$, the functors*

$$\mathrm{Hom}(X, -) : \mathscr{A} \to \Bbbk\text{-mod} \qquad \textit{and} \qquad \mathrm{Hom}(-, X) : \mathscr{A}^{\mathrm{op}} \to \Bbbk\text{-mod}$$

are both left exact.

Definition A.3.18. An object P in an abelian category is said to be **projective** if for any surjective map $f : X \to Y$ and any morphism $g : P \to Y$, there exists a map $g' : P \to X$ such that $f \circ g' = g$.

Similarly, an object I is said to be **injective** if for any injective map $f : X \to Y$ and any morphism $h : X \to I$, there exists a map $h' : Y \to I$ such that $h' \circ f = h$.

This definition is summarized by the following diagrams:

$$\begin{tikzcd} & P \arrow[dl, dashed, "\exists g'"'] \arrow[d, "g"] \\ X \arrow[r, two heads, "f"'] & Y \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, hook, "f"] \arrow[d, "h"'] & Y \arrow[dl, dashed, "\exists h'"] \\ I & \end{tikzcd}$$

Lemma A.3.19. *Let $\mathscr{A}$ be a $\Bbbk$-linear abelian category.*

(1) *An object $P \in \mathscr{A}$ is projective if and only if* $\mathrm{Hom}(P, -) : \mathscr{A} \to \Bbbk\text{-mod}$ *is an exact functor.*
(2) *An object $I \in \mathscr{A}$ is injective if and only if* $\mathrm{Hom}(-, I) : \mathscr{A}^{\mathrm{op}} \to \Bbbk\text{-mod}$ *is an exact functor.*

Finiteness conditions on abelian categories. The following definitions give some conditions on (sequences of) subobjects in an abelian category.

Definition A.3.20. Let $\mathscr{A}$ be an abelian category. An object $X \in \mathscr{A}$ is said to be **noetherian** if for every ascending chain of subobjects

$$X_1 \subset X_2 \subset \cdots$$

of X, there is an integer N such that for all $k \geq N$, the injective map $X_k \hookrightarrow X_{k+1}$ is an isomorphism. The category $\mathscr{A}$ is said to be **noetherian** if every object is noetherian.

An object $X \in \mathscr{A}$ is said to be **artinian** if for every descending chain of subobjects

$$X_1 \supset X_2 \supset \cdots$$

of X, there is an integer N such that for all $k \geq N$, the injective map $X_{k+1} \hookrightarrow X_k$ is an isomorphism. The category $\mathscr{A}$ is said to be **artinian** if every object is artinian.

Definition A.3.21. An object X in an abelian category is said to be **simple** if it is not the zero object and if it satisfies either of the following equivalent conditions:

(1) Any nonzero injective map $Y \to X$ is an isomorphism.
(2) Any nonzero surjective map $X \to Y$ is an isomorphism.

Definition A.3.22. Let X be an object in an abelian category $\mathscr{A}$. A **composition series** of length k for X is a sequence of subobjects

$$0 = X_0 \subset X_1 \subset X_2 \subset \cdots \subset X_k = X$$

such that each quotient X_i/X_{i-1} (for $i = 1, \ldots, k$) is a simple object. The simple objects X_i/X_{i-1} are called the **composition factors** of X.

The integer k is called the **length** of X. For any simple object $S \in \mathscr{A}$, the **multiplicity** of S in X, denoted by $[X : S]$, is the integer given by

$$[X : S] = |\{i \in \{1, \ldots, k\} \mid X_i/X_{i-1} \cong S\}|.$$

The following theorem says that the last two notions are well defined.

Theorem A.3.23 (Jordan–Hölder theorem). *Let $\mathscr{A}$ be an abelian category, and let X be an object in $\mathscr{A}$ that admits a composition series. Then the length of X and the multiplicities of composition factors are independent of the choice of composition series.*

Definition A.3.24. Let $\mathscr{A}$ be an abelian category. An object $X \in \mathscr{A}$ is said to be of **finite length** if it admits a composition series. The category $\mathscr{A}$ is said to be a **finite-length category** if every object is of finite length.

Lemma A.3.25. *An abelian category $\mathscr{A}$ is a finite-length category if and only if it is both noetherian and artinian.*

Definition A.3.26. Let $\mathscr{A}$ be an abelian category. An object $X \in \mathscr{A}$ is said to be **semisimple** if there is an isomorphism $X \cong X_1 \oplus \cdots \oplus X_k$, where $X_1, \ldots, X_k$ are simple objects. The category $\mathscr{A}$ is said to be **semisimple** if every object is semisimple.

Definition A.3.27. Suppose $X \in \mathscr{A}$ has finite length. Its **socle** is defined to be its unique maximal semisimple subobject, and its **radical**, denoted by $\mathrm{rad}(X)$, is the unique minimal subobject such that $X/\mathrm{rad}(X)$ is semisimple. The quotient $X/\mathrm{rad}(X)$ is called the **head** or **cosocle** of X. The **(lower) Loewy series** of X is the sequence of subobjects

$$0 = X_0 \subset X_1 \subset X_2 \subset \cdots \subset X_k = X$$

determined by the property that X_i/X_{i-1} is the socle of X/X_{i-1}. The quotients X_i/X_{i-1} are called the **Loewy layers** of X, and the integer k is its **Loewy length**.

Exercises.

A.3.1. Let $\mathscr{A}$ be an abelian category, and let $\mathscr{A}' \subset \mathscr{A}$ be a full additive subcategory that is closed under taking kernels and cokernels. Show that $\mathscr{A}'$ is an abelian category and that the inclusion functor $\mathscr{A}' \to \mathscr{A}$ is exact. In particular, this applies to Serre subcategories.

A.3.2. Let $0 \to X \to Y \to Z \to 0$ be a short exact sequence in a finite-length abelian category. Show that the length of Y is equal to the sum of the lengths of X and Z.

A.3.3. Let X be an object in a finite-length abelian category, and let $T : X \to X$ be an endomorphism. Show that the following conditions are equivalent:

(a) T is injective.
(b) T is surjective.
(c) T is an automorphism.

A.4. Triangulated categories

This section covers basic properties of the following fundamental notion, due to Verdier [**241**].

Definition A.4.1. Let $\mathscr{T}$ be an additive category equipped with an automorphism $[1] : \mathscr{T} \to \mathscr{T}$ (called the **shift functor**) and a collection of diagrams

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1],$$

called **distinguished triangles**. The category $\mathscr{T}$ is called a **triangulated category** if the following axioms hold:

(1) Any morphism $f : X \to Y$ can be completed to a distinguished triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$. For any object X, the diagram $X \xrightarrow{\mathrm{id}} X \to 0 \to X[1]$ is a distinguished triangle. If we have a commutative diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "p"', "\wr"] & Y \arrow[r] \arrow[d, "q"', "\wr"] & Z \arrow[r] \arrow[d, "r"', "\wr"] & X[1] \arrow[d, "p[1]"', "\wr"] \\ X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & X'[1] \end{tikzcd}$$

in which the top row is a distinguished triangle and the vertical maps are isomorphisms, then the bottom row is also a distinguished triangle.

(2) (Rotation) The diagram $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$ is a distinguished triangle if and only if $Y \xrightarrow{g} Z \xrightarrow{h} X[1] \xrightarrow{-f[1]} Y[1]$ is a distinguished triangle.

(3) (Completion) Given a commutative diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "p"'] & Y \arrow[r] \arrow[d, "q"'] & Z \arrow[r] & X[1] \\ X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & X'[1] \end{tikzcd}$$

in which the rows are distinguished triangles, there is a morphism $r : Z \to Z'$ making the following diagram commute:

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "p"'] & Y \arrow[r] \arrow[d, "q"'] & Z \arrow[r] \arrow[d, "r"'] & X[1] \arrow[d, "p[1]"] \\
X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & X'[1]
\end{tikzcd}$$

(4) (Octahedral) Suppose we have three distinguished triangles

$$X \xrightarrow{a} Y \xrightarrow{a'} A \xrightarrow{a''} X[1],$$
$$Y \xrightarrow{c} Z \xrightarrow{c'} C \xrightarrow{c''} Y[1],$$
$$X \xrightarrow{b} Z \xrightarrow{b'} B \xrightarrow{b''} X[1],$$

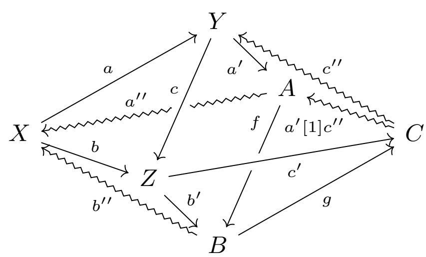
with $b = c \circ a$. There are morphisms $f : A \to B$ and $g : B \to C$ such that

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{a'[1] \circ c''} A[1]$$

is a distinguished triangle, and such that

$$b'' \circ f = a'', \qquad g \circ b' = c', \qquad b' \circ c = f \circ a', \qquad a[1] \circ b'' = c'' \circ g.$$

The situation described in axiom (4) can be drawn as shown below, where an arrow like $P \rightsquigarrow Q$ should be understood as denoting a morphism $P \to Q[1]$:



Here, each face is either a distinguished triangle or a commutative triangle, and the two large squares involving Y and B are commutative. Alternatively, one can draw an octahedral diagram in two pieces, called the "upper cap" and the "lower cap":

$$\begin{tikzcd}
A \arrow[dd, rightsquigarrow, "a''"'] & & C \arrow[ll, rightsquigarrow, "a'[1]c''"'] \\
& Y \arrow[ul, "a'"] \arrow[ur, rightsquigarrow, "c''"'] \arrow[dr, "c"] & \\
X \arrow[ur, "a"] \arrow[rr, "b"] & & Z \arrow[uu, "c'"']
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[dd, rightsquigarrow, "a''"'] \arrow[dr, "f"] & & C \arrow[ll, rightsquigarrow, "a'[1]c''"'] \\
& B \arrow[ur, "g"] \arrow[dl, rightsquigarrow, "b''"'] & \\
X \arrow[rr, "b"] & & Z \arrow[uu, "c'"'] \arrow[ul, "b'"']
\end{tikzcd}$$

Remark A.4.2. The axioms in Definition A.4.1 are somewhat redundant: it is shown in [**174**, Section 5] that one can replace the "if and only if" in axiom (2) by an implication in one direction only, and that one can omit axiom (3) altogether.

In many ways, distinguished triangles in triangulated categories play an analogous role to that of short exact sequences in abelian categories. The next few lemmas express some of these parallels.

Lemma A.4.3. (1) *In a distinguished triangle* $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h}$, *we have* $g \circ f = 0$, $h \circ g = 0$, *and* $-f[1] \circ h = 0$.

(2) *Suppose we have a commutative diagram*

$$\begin{tikzcd} X \arrow[r] \arrow[d, "p"'] & Y \arrow[r] \arrow[d, "q"'] & Z \arrow[r] \arrow[d, "r"'] & X[1] \arrow[d, "p[1]"'] \\ X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & X'[1] \end{tikzcd}$$

in which the rows are distinguished triangles. If any two of the maps p, q, and r are isomorphisms, then the third is an isomorphism as well.

(3) *In a distinguished triangle $X \xrightarrow{f} Y \to Z \to X[1]$, f is an isomorphism if and only if $Z = 0$.*

Part (2) implies that if $X \xrightarrow{f} Y \to Z \to X[1]$ and $X \xrightarrow{f} Y \to Z' \to X[1]$ are both distinguished triangles, then $Z \cong Z'$.

Definition A.4.4. A **cone** of a morphism $f : X \to Y$ is any member of the isomorphism class of objects Z such that there is a distinguished triangle $X \xrightarrow{f} Y \to Z \to X[1]$.

Lemma A.4.5. *The following conditions on a distinguished triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$ are equivalent:*

(1) *There is a map $r : Y \to X$ such that $r \circ f = \mathrm{id}_X$.*
(2) *There is a map $s : Z \to Y$ such that $g \circ s = \mathrm{id}_Z$.*
(3) *We have $h = 0$.*
(4) *There is an isomorphism $u : Y \to X \oplus Z$ such that the following diagram commutes:*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\mathrm{id}"'] & Y \arrow[r, "g"] \arrow[d, "u"'] & Z \arrow[r, "h"] \arrow[d, "\mathrm{id}"'] & X[1] \arrow[d, "\mathrm{id}"] \\ X \arrow[r, "\mathrm{in}_1"] & X \oplus Z \arrow[r, "\mathrm{pr}_2"] & Z \arrow[r, "0"] & X[1] \end{tikzcd}$$

Definition A.4.6. A distinguished triangle satisfying the equivalent conditions of Lemma A.4.5 is said to be **split**.

The next definition is reminiscent of Definition A.3.9.

Definition A.4.7. Let $\mathscr{T}$ be a triangulated category, and let $X, Y \in \mathscr{T}$. An **extension** of X by Y is an object E such that there exists a distinguished triangle of the form $Y \to E \to X \to Y[1]$. A full subcategory $\mathscr{C} \subset \mathscr{T}$ is said to be **stable under extensions** if for any two objects $X, Y \in \mathscr{C}$, every extension of X by Y also lies in $\mathscr{C}$.

Definition A.4.8. Let $\mathscr{T}$ be a triangulated category, and let $\mathscr{A}$ be an abelian category. An additive functor $F : \mathscr{T} \to \mathscr{A}$ is called a **cohomological functor** if for every distinguished triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h}$, the sequence

$$\cdots \to F(Z[-1]) \xrightarrow{F(-h[-1])} F(X) \xrightarrow{F(f)} F(Y) \xrightarrow{F(g)} F(Z)$$
$$\xrightarrow{F(h)} F(X[1]) \xrightarrow{F(-f[1])} F(Y[1]) \xrightarrow{F(-g[1])} F(Z[1]) \xrightarrow{F(-h[1])} F(X[2]) \to \cdots$$

is a long exact sequence in $\mathscr{A}$.

LEMMA A.4.9. *Let $\mathscr{T}$ be a $\Bbbk$-linear triangulated category. For any object $X \in \mathscr{T}$, the functors* $\operatorname{Hom}(X, -) : \mathscr{T} \to \Bbbk\text{-mod}$ *and* $\operatorname{Hom}(-, X) : \mathscr{T}^{\mathrm{op}} \to \Bbbk\text{-mod}$ *are cohomological.*

LEMMA A.4.10. *Suppose we have two distinguished triangles and a map q as shown below.*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, dashed, "p"'] & Y \arrow[r, "g"] \arrow[d, "q"'] & Z \arrow[r, "h"] \arrow[d, dashed, "r"'] & X[1] \arrow[d, dashed, "p[1]"] \\ X' \arrow[r, "f'"] & Y' \arrow[r, "g'"] & Z' \arrow[r, "h'"] & X'[1] \end{tikzcd}$$

If $g' \circ q \circ f = 0$, then there exist morphisms $p : X \to X'$ and $r : Z \to Z'$ that make this into a commutative diagram.

Moreover, if $\operatorname{Hom}(X, Z'[-1]) = 0$, *then p is uniquely determined by the condition that $f' \circ p = q \circ f$, and r is uniquely determined by the condition that $r \circ g = g' \circ q$.*

PROOF. From Lemma A.4.9, consider the following exact sequence:

$$\cdots \to \operatorname{Hom}(X, Z'[-1]) \to \operatorname{Hom}(X, X') \to \operatorname{Hom}(X, Y') \to \operatorname{Hom}(X, Z') \to \cdots .$$

By assumption, the element $q \circ f \in \operatorname{Hom}(X, Y')$ lies in the kernel of the map $\operatorname{Hom}(X, Y') \to \operatorname{Hom}(X, Z')$, so it is the image of an element $p \in \operatorname{Hom}(X, X')$. That is, there is a map $p : X \to X'$ such that $f' \circ p = q \circ f$. The existence of r follows by axiom (3).

If $\operatorname{Hom}(X, Z'[-1]) = 0$, then we see from the exact sequence above that p is uniquely determined by its image in $\operatorname{Hom}(X, Y')$. Similar reasoning shows that r is also uniquely determined. $\square$

COROLLARY A.4.11. *Given two morphisms $f : X \to Y$ and $g : Y \to Z$, if* $\operatorname{Hom}(X, Z[-1]) = 0$, *then there is at most one morphism $h : Z \to X[1]$ such that $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$ is a distinguished triangle.*

PROOF. Suppose we had two morphisms $h, h' : Z \to X[1]$ that both gave distinguished triangles. Consider the following diagram:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, dashed, "p"'] & Y \arrow[r, "g"] \arrow[d, equal] & Z \arrow[r, "h"] \arrow[d, dashed, "r"'] & X[1] \arrow[d, dashed, "p[1]"] \\ X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \arrow[r, "h'"] & X[1] \end{tikzcd}$$

Lemma A.4.10 tells us that there are unique morphisms p and r making this commute, and that they are determined just by requiring the left and middle squares to commute. Thus, we must have $p = \mathrm{id}_X$ and $r = \mathrm{id}_Z$. But then the commutativity of the rightmost square above shows us that $h = h'$. $\square$

DEFINITION A.4.12. Let $\mathscr{T}$ and $\mathscr{T}'$ be two triangulated categories. An additive functor $F : \mathscr{T} \to \mathscr{T}'$ is called a **triangulated functor** if:

(1) For all $X \in \mathscr{T}$, there is a natural isomorphism $\phi_X : F(X[1]) \to F(X)[1]$.

(2) For any distinguished triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$ in $\mathscr{T}$, the diagram

$$F(X) \xrightarrow{F(f)} F(Y) \xrightarrow{F(g)} F(Z) \xrightarrow{\phi_X \circ F(h)} F(X)[1]$$

is a distinguished triangle in $\mathscr{T}'$.

LEMMA A.4.13. *Let $\mathscr{T}$ and $\mathscr{T}'$ be two triangulated categories, and let $F : \mathscr{T} \to \mathscr{T}'$ and $G : \mathscr{T}' \to \mathscr{T}$ be additive functors. Suppose that F is left adjoint to G. Then F is triangulated if and only if G is triangulated.*

For a proof, see [**186**, Lemma 5.3.6].

Generators. The structure of a triangulated category sometimes provides shortcuts to showing that a functor is fully faithful or that it is an equivalence. We explain this in Propositions A.4.16 and A.4.17 below.

DEFINITION A.4.14. A full subcategory $\mathscr{S} \subset \mathscr{T}$ is called a **full triangulated subcategory** if it satisfies the following two conditions:

(1) For any object $X \in \mathscr{T}$, we have $X \in \mathscr{S}$ if and only if $X[1] \in \mathscr{S}$.
(2) If $f : X \to Y$ is a morphism in $\mathscr{S}$, then $\mathscr{S}$ contains a cone of f.

Under these conditions, $\mathscr{S}$ is a triangulated category in its own right (its distinguished triangles are just the distinguished triangles of $\mathscr{T}$ whose objects lie in $\mathscr{S}$), and the inclusion functor $\mathscr{S} \to \mathscr{T}$ is a triangulated functor.

DEFINITION A.4.15. Let $\mathscr{Q} \subset \mathscr{T}$ be a set of objects. We say that $\mathscr{Q}$ **generates** $\mathscr{T}$ if the smallest strictly full triangulated subcategory of $\mathscr{T}$ containing all the objects of $\mathscr{Q}$ is $\mathscr{T}$ itself.

PROPOSITION A.4.16. *Let $F : \mathscr{T} \to \mathscr{T}'$ be a triangulated functor, and let $\mathscr{Q} \subset \mathscr{T}$ be a set of objects that generates $\mathscr{T}$. If the map*

$$\mathrm{Hom}_{\mathscr{T}}(X, Y[n]) \to \mathrm{Hom}_{\mathscr{T}'}(F(X), F(Y[n]))$$

is an isomorphism for all $X, Y \in \mathscr{Q}$, then F is fully faithful.

PROOF. Let $\mathscr{T}_1$ be the strictly full subcategory of $\mathscr{T}$ whose objects are isomorphic to $X[n]$ for some $X \in \mathscr{Q}$ and $n \in \mathbb{Z}$. Then, we inductively define $\mathscr{T}_2 \subset \mathscr{T}_3 \subset \cdots$ as follows: $\mathscr{T}_n$ is the strictly full subcategory of $\mathscr{T}$ containing all objects of $\mathscr{T}_{n-1}$ and all cones of morphisms in $\mathscr{T}_{n-1}$. Then $\bigcup_{n \ge 1} \mathscr{T}_n$ is the smallest strictly full triangulated subcategory of $\mathscr{T}$ containing $\mathscr{Q}$. Since $\mathscr{Q}$ generates $\mathscr{T}$, we have $\bigcup_{n \ge 1} \mathscr{T}_n = \mathscr{T}$.

We will show by induction on n that

$$\mathrm{Hom}_{\mathscr{T}}(X, Y) \to \mathrm{Hom}_{\mathscr{T}'}(F(X), F(Y)) \tag{A.4.1}$$

is an isomorphism for all $X, Y \in \mathscr{T}_n$. The proposition follows from this claim.

The claim for $n = 1$ is true by assumption. Suppose it is known for $n = k$, and let $X \in \mathscr{T}_{k+1}$ and $Y \in \mathscr{T}_k$. If X happens to lie in $\mathscr{T}_k$, then (A.4.1) is already known to be an isomorphism. Otherwise, there is a distinguished triangle $X' \to X'' \to X \to$ with $X', X'' \in \mathscr{T}_k$. By Lemma A.4.9, the rows in the following diagram are long exact sequences:

$$\begin{tikzcd}
\mathrm{Hom}(X''[1], Y) \arrow[r] \arrow[d] & \mathrm{Hom}(X'[1], Y) \arrow[r] \arrow[d] & \mathrm{Hom}(X, Y) \arrow[r] \arrow[d] & \mathrm{Hom}(X'', Y) \arrow[r] \arrow[d] & \mathrm{Hom}(X', Y) \arrow[d] \\
\mathrm{Hom}(FX''[1], FY) \arrow[r] & \mathrm{Hom}(FX'[1], FY) \arrow[r] & \mathrm{Hom}(FX, FY) \arrow[r] & \mathrm{Hom}(FX'', FY) \arrow[r] & \mathrm{Hom}(FX', FY)
\end{tikzcd}$$

The first, second, fourth, and fifth vertical maps are isomorphisms by induction, so by the five lemma, the middle one is as well. In other words, (A.4.1) is an isomorphism when $X \in \mathscr{T}_{k+1}$ and $Y \in \mathscr{T}_k$. A similar five-lemma argument shows that it is also an isomorphism when Y is any object of $\mathscr{T}_{k+1}$. $\square$

PROPOSITION A.4.17. *Let $F : \mathscr{T} \to \mathscr{T}'$ be a fully faithful triangulated functor. If the image of F contains a set of objects that generates $\mathscr{T}'$, then F is an equivalence of categories.*

PROOF. Let $\mathscr{Q} \subset \mathscr{T}'$ be a set of objects that generates $\mathscr{T}'$ and is contained in the image of F. Define a sequence of strictly full subcategories $\mathscr{T}'_1 \subset \mathscr{T}'_2 \subset \cdots$ of $\mathscr{T}'$ as in the proof of Proposition A.4.16. We have $\bigcup_{n\geq 1} \mathscr{T}'_n = \mathscr{T}'$.

Because F is fully faithful, if X and Y are (isomorphic to) objects in its image, then the cone of any morphism $X \to Y$ is also (isomorphic to) an object its image. Thus, if all objects of $\mathscr{T}'_{n-1}$ are in the image of F, then so are all objects of $\mathscr{T}'_n$.

It is clear that all objects of $\mathscr{T}'_1$ lie in the image of F, so by induction, all objects of $\bigcup_{n\geq 1} \mathscr{T}'_n$ lie in the image. In other words, F is essentially surjective, and hence an equivalence of categories. □

Localization. Let $\mathscr{C}$ be an additive category, and let $\mathscr{S}$ be a set of morphisms in $\mathscr{C}$ that is closed under composition. The **localization** of $\mathscr{C}$ at $\mathscr{S}$ is an additive category $\mathscr{C}_{\mathscr{S}}$ together with an additive functor $L : \mathscr{C} \to \mathscr{C}_{\mathscr{S}}$ with the following properties:

(1) The functor L sends every morphism in $\mathscr{S}$ to an isomorphism.
(2) The pair $(\mathscr{C}_{\mathscr{S}}, L)$ is universal with respect to the preceding property. That is, if $F : \mathscr{C} \to \mathscr{C}'$ is any additive functor that sends all morphisms in $\mathscr{S}$ to isomorphisms, then there is a functor $\bar{F} : \mathscr{C}_{\mathscr{S}} \to \mathscr{C}'$ and a natural isomorphism $\epsilon : \bar{F} \circ L \xrightarrow{\sim} F$. Moreover, the pair $(\bar{F}, \epsilon)$ is unique up to isomorphism.

The construction of localizations in general is rather complicated, but for certain sets of morphisms in a triangulated category (which is the only case we will need), there is a short description due to Verdier [**241**, Section 2], which we now recall.

DEFINITION A.4.18. Let $\mathscr{T}$ be a triangulated category, and let $\mathscr{S}$ be a set of morphisms in $\mathscr{T}$. The set $\mathscr{S}$ is called a **Verdier localizing system** if there exists a strictly full triangulated subcategory $\mathscr{T}' \subset \mathscr{T}$ such that a morphism $f : X \to Y$ belongs to $\mathscr{S}$ if and only if any cone of f belongs to $\mathscr{T}'$.

It is immediate from the definition that if $\mathscr{S}$ is a Verdier localizing system, then it contains all isomorphisms (and in particular, all identity morphisms). One can show using the octahedral axiom that any Verdier localizing system is closed under composition.

Given a Verdier localizing system $\mathscr{S}$ in a triangulated category $\mathscr{T}$, we will now define a category $\mathscr{T}_{\mathscr{S}}$. The objects of $\mathscr{T}_{\mathscr{S}}$ are the same as those of $\mathscr{T}$. For objects $X, Y \in \mathscr{T}_{\mathscr{S}}$, the set of morphisms $\mathrm{Hom}_{\mathscr{T}_{\mathscr{S}}}(X, Y)$ is the set of equivalence classes of diagrams (sometimes called **roof diagrams**)

$$\begin{tikzcd} & Z \arrow[dl, "s"'] \arrow[dr, "f"] & \\ X & & Y \end{tikzcd}$$

where $s \in \mathscr{S}$ and $f \in \mathrm{Hom}_{\mathscr{T}}(Z, Y)$, and where the equivalence relation is as follows: two diagrams $X \xleftarrow{s} Z \xrightarrow{f} Y$ and $X \xleftarrow{s'} Z' \xrightarrow{f'} Y$ are equivalent if there is a commutative diagram in $\mathscr{T}$

$$\begin{tikzcd}
& Z \arrow[dl, "s"'] \arrow[dr, "f"] & \\
X & Z'' \arrow[l, "u"'] \arrow[u] \arrow[d] & Y \\
& Z' \arrow[ul, "s'"] \arrow[ur, "f'"'] &
\end{tikzcd}$$

with $u \in S$. For an explanation of how to compose morphisms in $\mathscr{T}_{\mathscr{S}}$, see [**125**, Definition 1.6.2] or [**81**, Lemma III.2.8]. The claim that $\mathscr{T}_{\mathscr{S}}$ is a well-defined category requires proof; see [**241**, Théorème 2.2.2] or [**125**, Proposition 1.6.7]. Once this is known, it makes sense to define a functor $L : \mathscr{T} \to \mathscr{T}_{\mathscr{S}}$ as follows: L sends every object to itself, and it sends a morphism $f : X \to Y$ to the equivalence class of the roof diagram $X \xleftarrow{\mathrm{id}_X} X \xrightarrow{f} Y$.

Theorem A.4.19 (Verdier). *Let $\mathscr{T}$ be a triangulated category, and let $\mathscr{S}$ be a Verdier localizing system. Then the pair $(\mathscr{T}_{\mathscr{S}}, L : \mathscr{T} \to \mathscr{T}_{\mathscr{S}})$ defined above is the localization of $\mathscr{T}$ at $\mathscr{S}$. Moreover, the category $\mathscr{T}_{\mathscr{S}}$ inherits from $\mathscr{T}$ the structure of a triangulated category, and the functor $L : \mathscr{T} \to \mathscr{T}_{\mathscr{S}}$ is triangulated.*

Definition A.4.20. Let $\mathscr{T}$ be a triangulated category, and let $\mathscr{S}$ be a Verdier localizing system. The category $\mathscr{T}_{\mathscr{S}}$ is called the **Verdier localization** of $\mathscr{T}$ with respect to $\mathscr{S}$.

Let us briefly discuss the triangulated structure on $\mathscr{T}_{\mathscr{S}}$. It is straightforward to define the shift functor $[1] : \mathscr{T}_{\mathscr{S}} \to \mathscr{T}_{\mathscr{S}}$ (using the observation that a Verdier localizing system must be stable under $[1]$). We then define a distinguished triangle in $\mathscr{T}_{\mathscr{S}}$ to be a diagram $X \to Y \to Z \to X[1]$ that is isomorphic to the image under L of a distinguished triangle in $\mathscr{T}$.

Let $\mathscr{T}' \subset \mathscr{T}$ be the full triangulated category corresponding to $\mathscr{S}$ as in Definition A.4.18. One can show that for any object $X \in \mathscr{T}$, we have $L(X) \cong 0$ if and only if $X \in \mathscr{T}'$. Thus, $\mathscr{T}'$ can be thought of as the "kernel" of L. For this reason, the Verdier localization $\mathscr{T}_{\mathscr{S}}$ is sometimes called the **Verdier quotient**, and it may be denoted by $\mathscr{T}/\mathscr{T}'$.

A.5. Chain complexes and the derived category

Chain complexes. In this section, we explain how to build triangulated categories from additive and abelian ones, starting with the following notion.

Definition A.5.1. Let $\mathscr{A}$ be an additive category. A **chain complex** over $\mathscr{A}$ is a sequence of objects and morphisms in $\mathscr{A}$,

$$\cdots \xrightarrow{d^{-3}} A^{-2} \xrightarrow{d^{-2}} A^{-1} \xrightarrow{d^{-1}} A^0 \xrightarrow{d^0} A^1 \xrightarrow{d^1} A^2 \xrightarrow{d^2} \cdots$$

such that $d^i \circ d^{i-1} = 0$ for all $i \in \mathbb{Z}$. The d^i are called **differentials**. A **chain map** $f : A \to B$ is a collection of maps $(f^i : A^i \to B^i)_{i \in \mathbb{Z}}$ such that $d^i_B \circ f^i = f^{i+1} \circ d^i_A$ for all i. The category of chain complexes over $\mathscr{A}$ is denoted by $Ch(\mathscr{A})$.

Two chain maps $f, g : A \to B$ are said to be **homotopic** if there is a collection of morphisms $(s^i : A^i \to B^{i-1})_{i \in \mathbb{Z}}$ such that

$$f^i - g^i = d^{i-1}_B \circ s^i + s^{i+1} \circ d^i_A.$$

The **homotopy category** of $\mathscr{A}$, denoted by $K(\mathscr{A})$, is the category whose objects are the same as those of $Ch(\mathscr{A})$, but whose morphisms are homotopy classes of chain maps.

Both $Ch(\mathscr{A})$ and $K(\mathscr{A})$ are additive categories. If $\mathscr{A}$ is an abelian category, one can show that $Ch(\mathscr{A})$ is also an abelian category.

DEFINITION A.5.2. Given a chain complex $A = (A^i, d_A^i)_{i\in\mathbb{Z}}$, let $A[1]$ be the chain complex given by $(A[1])^i = A^{i+1}$, with differentials

$$d^i_{A[1]} = -d_A^{i+1} : (A[1])^i \to (A[1])^{i+1}.$$

For a chain map $f : A \to B$, let $f[1] : A[1] \to B[1]$ be the chain map given by $(f[1])^i = f^{i+1}$. This defines the **shift functor** $[1] : Ch(\mathscr{A}) \to Ch(\mathscr{A})$.

DEFINITION A.5.3. Let $f : A \to B$ be a chain map. The **chain-map cone** of f, denoted by $\mathrm{chcone}(f)$, is the chain complex whose terms are given by $\mathrm{chcone}(f)^i = A^{i+1} \oplus B^i$, and whose differentials are given by

$$d^i_{\mathrm{chcone}(f)} = \begin{bmatrix} -d_A^{i+1} & \\ f^{i+1} & d_B^i \end{bmatrix} : A^{i+1} \oplus B^i \to A^{i+2} \oplus B^{i+1}.$$

There are obvious chain maps $\mathrm{in}_2 : B \to \mathrm{chcone}(f)$ and $\mathrm{pr}_1 : \mathrm{chcone}(f) \to A[1]$.

DEFINITION A.5.4. A diagram $X \xrightarrow{f} Y \to Z \to X[1]$ in $K(\mathscr{A})$ is called a **distinguished triangle** if for some chain map $\tilde{f} : X \to Y$ in the homotopy class f, there is a homotopy equivalence $u : Z \to \mathrm{chcone}(\tilde{f})$ such that the following diagram commutes:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\mathrm{id}"'] & Y \arrow[r] \arrow[d, "\mathrm{id}"'] & Z \arrow[r] \arrow[d, "u"'] & X[1] \arrow[d, "\mathrm{id}"] \\
X \arrow[r, "f"] & Y \arrow[r, "\mathrm{in}_2"] & \mathrm{chcone}(\tilde{f}) \arrow[r, "\mathrm{pr}_1"] & X[1]
\end{tikzcd}$$

THEOREM A.5.5. *With respect to the shift functor and distinguished triangles indicated above, $K(\mathscr{A})$ is a triangulated category.*

Proofs of this statement can be found in [**67**, Proposition 1.2.4], [**81**, Theorem IV.1.9], or [**125**, Proposition 1.4.4].

DEFINITION A.5.6. A chain complex (A^i, d^i) is said to be **bounded above** (resp. **bounded below**, **bounded**) if there is a positive integer N such that $A^i = 0$ for $i > N$ (resp. $i < -N$, $|i| > N$). The full subcategory of $Ch(\mathscr{A})$ consisting of bounded-above (resp. bounded-below, bounded) complexes is denoted by $Ch^-(\mathscr{A})$ (resp. $Ch^+(\mathscr{A})$, $Ch^{\mathrm{b}}(\mathscr{A})$). Similarly, the full subcategory of $K(\mathscr{A})$ consisting of bounded-above (resp. bounded-below, bounded) complexes is denoted by $K^-(\mathscr{A})$ (resp. $K^+(\mathscr{A})$, $K^{\mathrm{b}}(\mathscr{A})$).

Cohomology. Above, $\mathscr{A}$ was any additive category. When $\mathscr{A}$ is abelian, we have an additional operation available.

DEFINITION A.5.7. Let $\mathscr{A}$ be an abelian category. Given a chain complex $A = (A^i, d^i)_{i\in\mathbb{Z}}$ in $Ch(\mathscr{A})$, its nth **cohomology object**, denoted by $\mathsf{H}^n(A)$, is the object of $\mathscr{A}$ given by

$$\mathsf{H}^n(A) = \ker(d^n : A^n \to A^{n+1}) / \operatorname{im}(d^{n-1} : A^{n-1} \to A^n).$$

This defines a functor $\mathsf{H}^n : Ch(\mathscr{A}) \to \mathscr{A}$.

A chain complex A is **acyclic** if $\mathsf{H}^n(A) = 0$ for all $n \in \mathbb{Z}$. A chain map $f : A \to B$ is called a **quasi-isomorphism** if $\mathsf{H}^n(f) : \mathsf{H}^n(A) \to \mathsf{H}^n(B)$ is an isomorphism for all $n \in \mathbb{Z}$.

Homotopic chain maps induce equal maps in cohomology, so the functor $\mathsf{H}^n : Ch(\mathscr{A}) \to \mathscr{A}$ gives rise to a functor $\mathsf{H}^n : K(\mathscr{A}) \to \mathscr{A}$. A morphism $f : A \to B$ in $K(\mathscr{A})$ is called a **quasi-isomorphism** if any (equivalently, every) chain map in the homotopy class f is a quasi-isomorphism.

The following two lemmas are applications of Lemma A.3.13.

LEMMA A.5.8 (Snake lemma for chain complexes). *Let $\mathscr{A}$ be an abelian category, and let $0 \to A \to B \to C \to 0$ be a short exact sequence in $Ch(\mathscr{A})$. For each $i \in \mathbb{Z}$, there is a natural map $\delta^i : \mathsf{H}^i(C) \to \mathsf{H}^{i+1}(A)$ such that*

$$\cdots \to \mathsf{H}^i(A) \to \mathsf{H}^i(B) \to \mathsf{H}^i(C) \xrightarrow{\delta^i} \mathsf{H}^{i+1}(A) \to \cdots$$

is a long exact sequence.

LEMMA A.5.9. *Let $\mathscr{A}$ be an abelian category. The functor $\mathsf{H}^0 : K(\mathscr{A}) \to \mathscr{A}$ is a cohomological functor.*

COROLLARY A.5.10. *Let $\mathscr{A}$ be an abelian category.*

(1) *The full subcategory of $K(\mathscr{A})$ consisting of acyclic complexes is a full triangulated subcategory.*
(2) *Let $X \xrightarrow{f} Y \to Z \to$ be a distinguished triangle in $K(\mathscr{A})$. Then f is a quasi-isomorphism if and only if Z is acyclic.*

Corollary A.5.10 implies that the set of quasi-isomorphisms in $K(\mathscr{A})$ is a Verdier localizing system.

DEFINITION A.5.11. The **derived category** $D(\mathscr{A})$ of an abelian category $\mathscr{A}$ is the Verdier localization of $K(\mathscr{A})$ at the set of quasi-isomorphisms. The localization functor is denoted by $\mathrm{Qis} : K(\mathscr{A}) \to D(\mathscr{A})$.

By Theorem A.4.19, $D(\mathscr{A})$ inherits from $K(\mathscr{A})$ the structure of a triangulated category. Since $\mathsf{H}^n : K(\mathscr{A}) \to \mathscr{A}$ sends quasi-isomorphisms to isomorphisms, the defining property of localization tells us that it induces a (cohomological) functor

$$\mathsf{H}^n : D(\mathscr{A}) \to \mathscr{A}.$$

We use this to define bounded variants of $D(\mathscr{A})$: let $D^-(\mathscr{A})$ (resp. $D^+(\mathscr{A})$, $D^{\mathrm{b}}(\mathscr{A})$) be the strictly full triangulated subcategory of $D(\mathscr{A})$ consisting of objects X with the property that there exists a positive integer N such that $\mathsf{H}^i(X) = 0$ for $i > N$ (resp. $i < N$, $|i| > N$). Below, we collect some basic properties of derived categories.

LEMMA A.5.12 (Fancy snake lemma). *Let $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ be a short exact sequence in $Ch(\mathscr{A})$. There is a morphism $h : Z \to X[1]$ in $D(\mathscr{A})$ such that $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$ is a distinguished triangle.*

Note that Lemma A.5.8 can be recovered from the fancy snake lemma by taking the long exact cohomology sequence of the distinguished triangle.

PROOF SKETCH. Define a map $\theta : \mathrm{chcone}(f) \to Z$ as follows: let $\theta^i : X^{i+1} \oplus Y^i \to Z^i$ be given by $\theta^i = \begin{bmatrix} 0 & g^i \end{bmatrix}$. It is a routine exercise to show that θ is a quasi-isomorphism. The diagram

$$X \xrightarrow{f} Y \xrightarrow{g=\theta\circ\mathrm{in}_2} Z \xrightarrow{\mathrm{pr}_1\circ\theta^{-1}} X[1]$$

is a distinguished triangle. We set $h = \mathrm{pr}_1 \circ \theta^{-1}$. $\square$

Truncation. Given an integer n, we define full subcategories

$$D(\mathscr{A})^{\leq n} = \{X \in D(\mathscr{A}) \mid \mathsf{H}^i(X) = 0 \text{ for } i > n\},$$
$$D(\mathscr{A})^{\geq n} = \{X \in D(\mathscr{A}) \mid \mathsf{H}^i(X) = 0 \text{ for } i < n\}.$$

Note that these subcategories are *not* triangulated (they are not stable under [1]). The following notion is useful for studying these categories.

Definition A.5.13. The **truncation functors**

$$\tau^{\leq n} : Ch(\mathscr{A}) \to Ch(\mathscr{A}) \qquad \text{and} \qquad \tau^{\geq n} : Ch(\mathscr{A}) \to Ch(\mathscr{A})$$

are defined by the following formulas:

$$(\tau^{\leq n}X)^i = \begin{cases} 0 & \text{if } i > n, \\ \ker d^i & \text{if } i = n, \\ X^i & \text{if } i < n, \end{cases} \qquad (\tau^{\geq n}X)^i = \begin{cases} X^i & \text{if } i > n, \\ X^i / \operatorname{im} d^{i-1} & \text{if } i = n, \\ 0 & \text{if } i < n. \end{cases}$$

The same notation is used for the induced functors on $K(\mathscr{A})$ or $D(\mathscr{A})$.

It is left to the reader to check that the truncation functors do indeed give rise to well-defined functors on $K(\mathscr{A})$ and on $D(\mathscr{A})$. The fact that they take quasi-isomorphisms to quasi-isomorphisms follows from the observation that

$$\mathsf{H}^i(\tau^{\leq n}X) \cong \begin{cases} \mathsf{H}^i(X) & \text{if } i \leq n, \\ 0 & \text{if } i > n, \end{cases} \qquad \mathsf{H}^i(\tau^{\geq n}X) \cong \begin{cases} 0 & \text{if } i < n, \\ \mathsf{H}^i(X) & \text{if } i \geq n. \end{cases}$$

By construction, we have natural maps

$$\tau^{\leq n}X \to X \qquad \text{and} \qquad X \to \tau^{\geq n}X.$$

The former is a quasi-isomorphism if and only if $X \in D(\mathscr{A})^{\leq n}$, and the latter is a quasi-isomorphism if and only if $X \in D(\mathscr{A})^{\geq n}$. The following two statements are applications of this observation.

Lemma A.5.14. *Let $\mathscr{A}$ be an abelian category.*

(1) *If $X \in D(\mathscr{A})^{\leq n}$ and $Y \in D(\mathscr{A})^{\geq n+1}$, then* $\operatorname{Hom}(X, Y) = 0$.
(2) *The natural functor $\mathscr{A} \to D(\mathscr{A})$ is fully faithful, and its essential image is $D(\mathscr{A})^{\leq 0} \cap D(\mathscr{A})^{\geq 0}$.*

Lemma A.5.15. *For any $X \in D(\mathscr{A})$ and any $n \in \mathbb{Z}$, there is a unique natural map $\delta : \tau^{\geq n+1}X \to \tau^{\leq n}X[1]$ that gives rise to a natural distinguished triangle*

$$\tau^{\leq n}X \to X \to \tau^{\geq n+1}X \xrightarrow{\delta} .$$

Proof sketch. In $Ch(\mathscr{A})$, we have an obvious injective map $\tau^{\leq n}X \to X$, and an obvious surjective map $X \to \tau^{\geq n+1}X$. The latter factors through a map

$$X/\tau^{\leq n}X \to \tau^{\geq n+1}X. \tag{A.5.1}$$

In fact, this is a quasi-isomorphism.

The fancy snake lemma gives us a distinguished triangle $\tau^{\leq n}X \to X \to X/\tau^{\leq n}X \to$. Replace the third term with $\tau^{\geq n+1}X$ by (A.5.1). The resulting map $\delta : \tau^{\geq n+1}X \to (\tau^{\leq n}X)[1]$ is unique by Corollary A.4.11. Finally, Lemma A.4.10 implies that δ is natural. $\square$

Let X and Y be objects in an abelian category $\mathscr{A}$. For $n \in \mathbb{Z}$, the nth **Ext group** of X and Y, denoted by $\mathrm{Ext}^n_{\mathscr{A}}(X, Y)$ or $\mathrm{Ext}^n(X, Y)$, is given by

$$\mathrm{Ext}^n_{\mathscr{A}}(X, Y) = \mathrm{Hom}_{D(\mathscr{A})}(X, Y[n]).$$

Lemma A.5.14 implies that $\mathrm{Ext}^n(X, Y) = 0$ for $n < 0$, and that $\mathrm{Ext}^0(X, Y) \cong \mathrm{Hom}_{\mathscr{A}}(X, Y)$. We have the following concrete interpretation of $\mathrm{Ext}^1(X, Y)$.

PROPOSITION A.5.16. *Let $\mathscr{A}$ be an abelian category, and let $X, Y \in \mathscr{A}$. There is a natural bijection*

$$\mathrm{Ext}^1_{\mathscr{A}}(X, Y) \cong \left\{ \begin{array}{c} \textit{equivalence classes of} \\ \textit{extensions of } X \textit{ by } Y \end{array} \right\}.$$

For a sketch of the proof, see Proposition A.7.11 below. The content of the next lemma is sometimes called the **effaceability** of higher Ext-groups.

LEMMA A.5.17. *Let $\mathscr{A}$ be an abelian category. Let $X, Y \in \mathscr{A}$, and let $n \geq 1$. For any $f \in \mathrm{Ext}^n(X, Y)$, there exists a surjective map $p : X' \to X$ such that $f \circ p = 0$. There also exists an injective map $i : Y \to Y'$ such that $i[n] \circ f = 0$.*

PROOF. Complete $f : X \to Y[n]$ to a distinguished triangle, and rotate it to obtain

$$Y[n-1] \to Z \to X \xrightarrow{f} Y[n].$$

From the long exact sequence in cohomology, we see that $\mathsf{H}^i(Z) = 0$ unless $i = 0$ or $i = -n + 1$. The natural maps

$$Z \leftarrow \tau^{\leq 0} Z \to \tau^{\geq -n+1} \tau^{\leq 0} Z$$

are then both quasi-isomorphisms. Replace Z by $\tau^{\geq -n+1}\tau^{\leq 0} Z$. In other words, assume that Z is a chain complex with $Z^i = 0$ for $i < -n + 1$ or $i > 0$.

The long exact cohomology sequence gives a surjective map $\mathsf{H}^0(Z) \to X$ and an injective map $Y \to \mathsf{H}^{-n+1}(Z)$. (Both are isomorphisms if $n > 1$.) Our assumptions on the terms of the chain complex Z imply that $\mathsf{H}^0(Z) \cong Z^0/(\mathrm{im}(d^{-1} : Z^{-1} \to Z^0))$. In particular, there is a surjective map $p : Z^0 \to X$, defined to be the composition $Z^0 \to \mathsf{H}^0(Z) \to X$. This map also factors as $Z^0 \to Z \to X$, so Lemma A.4.3(1) implies that $f \circ p = 0$. Similar reasoning gives an injective map $i : Y \to Z^{-n+1}$ such that $i[n] \circ f = 0$. ☐

Exercises.

A.5.1. Prove that $K(\mathscr{A})$ is the localization of $Ch(\mathscr{A})$ at the set of homotopy equivalences.

A.5.2. Prove that every homotopy equivalence is a quasi-isomorphism. Deduce that $D(\mathscr{A})$ is the localization of $Ch(\mathscr{A})$ at the set of quasi-isomorphisms.

A.6. Derived functors

Statement of the problem. Any additive functor of additive categories $F : \mathscr{A} \to \mathscr{B}$ induces a functor $F : Ch(\mathscr{A}) \to Ch(\mathscr{B})$, given by applying F term-by-term to each term of a chain complex. This new functor sends homotopic maps to homotopic maps, so there is a further induced functor

$$F : K(\mathscr{A}) \to K(\mathscr{B}). \tag{A.6.1}$$

This functor clearly commutes with [1] and sends chain-map cones to chain-map cones, so (A.6.1) is in fact a triangulated functor.

Suppose now that $\mathscr{A}$ and $\mathscr{B}$ are abelian categories. If F is exact, then for a chain complex $X \in Ch(\mathscr{A})$, there is a natural isomorphism

$$F(\mathsf{H}^n(X)) \cong \mathsf{H}^n(F(X)).$$

As a consequence, F sends quasi-isomorphisms to quasi-isomorphisms, and so it gives rise to a functor of derived categories. Explicitly, if F is exact, there is a triangulated functor $\bar{F} : D(\mathscr{A}) \to D(\mathscr{B})$ along with a natural isomorphism

$$\mathrm{Qis} \circ F \cong \bar{F} \circ \mathrm{Qis}, \tag{A.6.2}$$

i.e., such that the following diagram commutes:

$$\begin{tikzcd} K(\mathscr{A}) \arrow[r, "\mathrm{Qis}"] \arrow[d, "F"'] & D(\mathscr{A}) \arrow[d, "\bar{F}"] \\ K(\mathscr{B}) \arrow[r, "\mathrm{Qis}"] & D(\mathscr{B}) \end{tikzcd}$$

When F is *not* exact, there might not be any natural isomorphism like (A.6.2). However, there may still be a natural transformation in one direction or the other. This idea is captured by the following definition.

DEFINITION A.6.1. Let $F : K(\mathscr{A}) \to K(\mathscr{B})$ be a triangulated functor. A **right derived functor** of F is a triangulated functor $RF : D(\mathscr{A}) \to D(\mathscr{B})$ together with a natural transformation

$$\epsilon : \mathrm{Qis} \circ F \to RF \circ \mathrm{Qis}$$

that is universal in the following sense: if $G : D(\mathscr{A}) \to D(\mathscr{B})$ is any other triangulated functor with a natural transformation $\phi : \mathrm{Qis} \circ F \to G \circ \mathrm{Qis}$, there is a unique natural transformation $\tilde{\phi} : RF \to G$ such that $\phi = (\tilde{\phi}\mathrm{Qis}) \circ \epsilon$.

Similarly, a **left derived functor** of F is a triangulated functor $LF : D(\mathscr{A}) \to D(\mathscr{B})$ together with a natural transformation

$$\eta : LF \circ \mathrm{Qis} \to \mathrm{Qis} \circ F$$

that is universal in the following sense: if $G : D(\mathscr{A}) \to D(\mathscr{B})$ is any other triangulated functor with a natural transformation $\phi : G \circ \mathrm{Qis} \to \mathrm{Qis} \circ F$, there is a unique natural transformation $\tilde{\phi} : G \to LF$ such that $\phi = \eta \circ (\tilde{\phi}\mathrm{Qis})$.

These definitions also make sense if one replaces the full derived categories $D(\mathscr{A})$ and $D(\mathscr{B})$ by, say, the bounded-below or bounded versions. Most derived functors we encounter will involve some boundedness condition.

Existence. To construct derived functors of nonexact functors, we need to find subcategories on which they behave "as though they were exact." This is made precise in Lemma A.6.4 below.

DEFINITION A.6.2. Let $F : \mathscr{A} \to \mathscr{B}$ be a left exact functor of abelian categories. A full subcategory $\mathscr{Q} \subset \mathscr{A}$ is said to be a **right adapted class** for F if it satisfies the following conditions:

(1) For any object $A \in \mathscr{A}$, there is an injective map $A \to X$ with $X \in \mathscr{Q}$.
(2) If $0 \to X' \to X \to X'' \to 0$ is a short exact sequence with $X', X \in \mathscr{Q}$, then $X'' \in \mathscr{Q}$.

(3) For any short exact sequence $0 \to X' \to X \to X'' \to 0$ with $X' \in \mathscr{Q}$, the sequence $0 \to F(X') \to F(X) \to F(X'') \to 0$ is exact.

Similarly, given a right exact functor $F : \mathscr{A} \to \mathscr{B}$, a full subcategory $\mathscr{Q} \subset \mathscr{A}$ is said to be a **left adapted class** for F if it satisfies the following conditions:

(1) For any object $A \in \mathscr{A}$, there is a surjective map $X \to A$ with $X \in \mathscr{Q}$.
(2) If $0 \to X' \to X \to X'' \to 0$ is a short exact sequence with $X, X'' \in \mathscr{Q}$, then $X' \in \mathscr{Q}$.
(3) For any short exact sequence $0 \to X' \to X \to X'' \to 0$ with $X'' \in \mathscr{Q}$, the sequence $0 \to F(X') \to F(X) \to F(X'') \to 0$ is exact.

We often say that $\mathscr{Q}$ is **large enough** to mean that condition (1) in (either part of) Definition A.6.2 holds. In particular, an abelian category has **enough injectives** if every object admits an injective map to an injective object, and it has **enough projectives** if every object admits a surjective map from a projective object. If $\mathscr{A}$ has enough injectives, then the injectives form an adapted class for every left exact functor. Similarly, if $\mathscr{A}$ has enough projectives, they form an adapted class for every right exact functor.

LEMMA A.6.3. *Let $\mathscr{A}$ be an abelian category.*

(1) *If $\mathscr{Q} \subset \mathscr{A}$ is a right adapted class, then for every object $X \in Ch^+(\mathscr{A})$, there is a quasi-isomorphism $q : X \to Q$ with $Q \in Ch^+(\mathscr{Q})$.*
(2) *If $\mathscr{Q} \subset \mathscr{A}$ is a left adapted class, then for every object $X \in Ch^-(\mathscr{A})$, there is a quasi-isomorphism $q : Q \to X$ with $Q \in Ch^-(\mathscr{Q})$.*

LEMMA A.6.4. *Let $\mathscr{A}$ and $\mathscr{B}$ be abelian categories.*

(1) *Let $F : \mathscr{A} \to \mathscr{B}$ be a left exact functor, and let $\mathscr{Q}$ be a right adapted class for F. If $X \in Ch^+(\mathscr{Q})$ is acyclic, then $F(X)$ is acyclic. If $f : X \to Y$ is a quasi-isomorphism in $Ch^+(\mathscr{Q})$, then $F(f)$ is a quasi-isomorphism.*
(2) *Let $F : \mathscr{A} \to \mathscr{B}$ be a right exact functor, and let $\mathscr{Q}$ be a left adapted class for F. If $X \in Ch^-(\mathscr{Q})$ is acyclic, then $F(X)$ is acyclic. If $f : X \to Y$ is a quasi-isomorphism in $Ch^-(\mathscr{Q})$, then $F(f)$ is a quasi-isomorphism.*

THEOREM A.6.5. *Let $\mathscr{A}$ and $\mathscr{B}$ be abelian categories.*

(1) *If $F : \mathscr{A} \to \mathscr{B}$ is a left exact functor that admits a right adapted class, then it admits a right derived functor $RF : D^+(\mathscr{A}) \to D^+(\mathscr{B})$.*
(2) *If $F : \mathscr{A} \to \mathscr{B}$ is a right exact functor that admits a left adapted class, then it admits a left derived functor $LF : D^-(\mathscr{A}) \to D^-(\mathscr{B})$.*

PROOF SKETCH. We will briefly indicate the main steps in the construction of a right derived functor. Let $F : \mathscr{A} \to \mathscr{B}$ be a left exact functor, and let $\mathscr{Q}$ be a right adapted class for F.

Step 1. Definition of RF on objects. For each object $X \in Ch^+(\mathscr{A})$, use Lemma A.6.3 to choose, once and for all, a quasi-isomorphism $q_X : X \to Q_X$ with $Q_X \in Ch^+(\mathscr{Q})$. Define $RF(X) = F(Q_X)$.

Step 2. Definition of RF on morphisms. Given a morphism $f : X \to Y$ in $D^+(\mathscr{A})$, let $\tilde{f} = q_Y \circ f \circ q_X^{-1} : Q_X \to Q_Y$. One can show that $\tilde{f}$ is represented by a roof diagram $Q_X \xleftarrow{s} Z \xrightarrow{h} Q_Y$ with $Z \in Ch^+(\mathscr{Q})$. We then define $RF(f) : RF(X) \to RF(Y)$ to be the morphism given by the roof diagram $F(Q_X) \xleftarrow{F(s)}$

$F(Z) \xrightarrow{F(h)} F(Q_Y)$. (By Lemma A.6.4, $F(s)$ is a quasi-isomorphism, so this is a valid roof diagram.)

Step 3. Definition of the natural transformation $\epsilon : \mathrm{Qis} \circ F \to RF \circ \mathrm{Qis}$. For $X \in K^+(\mathscr{A})$, let ϵ_X be the map

$$\mathrm{Qis}(F(X)) \xrightarrow{\mathrm{Qis}(F(q_X))} \mathrm{Qis}(F(Q_X)) = RF(\mathrm{Qis}(X)).$$

One can then check that the pair (RF, ϵ) has the required universal property. □

Next, suppose we have two left exact functors $F : \mathscr{A} \to \mathscr{B}$ and $G : \mathscr{B} \to \mathscr{C}$, and suppose the right derived functors RF, RG, and $R(G \circ F)$ all exist. The universal property of $R(G \circ F)$ gives us a canonical natural transformation

$$R(G \circ F) \to RG \circ RF. \tag{A.6.3}$$

Under some circumstances, this map is an isomorphism.

Proposition A.6.6. *Let $F : \mathscr{A} \to \mathscr{B}$ and $G : \mathscr{B} \to \mathscr{C}$ be left exact functors with adapted classes $\mathscr{Q} \subset \mathscr{A}$ and $\mathscr{S} \subset \mathscr{B}$, respectively, such that*

$$F(\mathscr{Q}) \subset \mathscr{S}.$$

Then the natural map $R(G \circ F) \to RG \circ RF$ is an isomorphism.

Of course, there is an analogous statement for right exact functors.

Definition A.6.7. Let $\mathscr{A}$ and $\mathscr{B}$ be abelian categories.

(1) Let $F : \mathscr{A} \to \mathscr{B}$ be a left exact functor that admits a right derived functor. An object $X \in \mathscr{A}$ is said to be **F-acyclic** if $\mathsf{H}^i(RF(X)) = 0$ for all $i > 0$.
(2) Let $F : \mathscr{A} \to \mathscr{B}$ be a right exact functor that admits a left derived functor. An object $X \in \mathscr{A}$ is said to be **F-acyclic** if $\mathsf{H}^i(LF(X)) = 0$ for all $i < 0$.

Remark A.6.8. Let $F : \mathscr{A} \to \mathscr{B}$ be a left exact functor that admits a right derived functor, and let $\mathrm{Acyc}(F)$ be the set of F-acyclic objects in $\mathscr{A}$. If $\mathrm{Acyc}(F)$ is large enough, then it is easy to see that it forms an adapted class. On the other hand, the construction of RF in Theorem A.6.5 shows that any member of any right adapted class is F-acyclic. Thus, if F admits any right adapted class, then $\mathrm{Acyc}(F)$ is large enough, and it is in fact the largest right adapted class for F. (Of course, similar comments apply to right exact functors.)

Finite cohomological dimension. Note that Lemma A.6.3 involves only chain complexes that are bounded on one side; there are technical obstacles to constructing such quasi-isomorphisms for unbounded complexes in general. For a general discussion of this problem, see [**223**]. However, this problem has a straightforward solution for functors satisfying the following condition.

Definition A.6.9. Let $\mathscr{A}$ and $\mathscr{B}$ be abelian categories.

(1) Let $F : \mathscr{A} \to \mathscr{B}$ be a left exact functor that has a right adapted class. Then F is said to have **cohomological dimension** $\leq d$ if for all $X \in \mathscr{A}$, we have $RF(X) \in D^+(\mathscr{B})^{\leq d}$.
(2) Let $F : \mathscr{A} \to \mathscr{B}$ be a right exact functor that has a left adapted class. Then F is said to have **cohomological dimension** $\leq d$ if for all $X \in \mathscr{A}$, we have $LF(X) \in D^-(\mathscr{B})^{\geq -d}$.

For functors of finite cohomological dimension, it is possible to prove analogues of Lemmas A.6.3 and A.6.4 for unbounded complexes, or for complexes that are bounded on the "wrong" side. Then the proof of Theorem A.6.5 can be repeated to give the following result.

THEOREM A.6.10. *Let $\mathscr{A}$ and $\mathscr{B}$ be abelian categories.*

(1) *If $F : \mathscr{A} \to \mathscr{B}$ is a left exact functor that admits a right adapted class and has finite cohomological dimension, then it admits a right derived functor $RF : D^-(\mathscr{A}) \to D^-(\mathscr{B})$.*
(2) *If $F : \mathscr{A} \to \mathscr{B}$ is a right exact functor that admits a left adapted class and has finite cohomological dimension, then it admits a left derived functor $LF : D^+(\mathscr{A}) \to D^+(\mathscr{B})$.*

The following calculation can be extracted from the proofs of Theorems A.6.5 and A.6.10.

PROPOSITION A.6.11. *Let $\mathscr{A}$ and $\mathscr{B}$ be abelian categories.*

(1) *Let $F : \mathscr{A} \to \mathscr{B}$ be a left exact functor that admits a right adapted class. Then $RF(D^+(\mathscr{A})^{\geq n}) \subset D^+(\mathscr{B})^{\geq n}$, and there is a natural isomorphism*
$$\mathsf{H}^n(RF(X)) \cong F(\mathsf{H}^n(X)) \qquad \textit{for any } X \in D^+(\mathscr{A})^{\geq n}.$$
If F has cohomological dimension $\leq d$, then we have $RF(D^-(\mathscr{A})^{\leq n}) \subset D^-(\mathscr{B})^{\leq n+d}$, and there is a natural isomorphism
$$\mathsf{H}^{n+d}(RF(X)) \cong \mathsf{H}^d(RF(\mathsf{H}^n(X))) \qquad \textit{for any } X \in D^-(\mathscr{A})^{\leq n}.$$
(2) *Let $F : \mathscr{A} \to \mathscr{B}$ be a right exact functor that admits a left adapted class. Then $LF(D^-(\mathscr{A})^{\leq n}) \subset D^-(\mathscr{B})^{\leq n}$, and there is a natural isomorphism*
$$\mathsf{H}^n(LF(X)) \cong F(\mathsf{H}^n(X)) \qquad \textit{for any } X \in D^-(\mathscr{A})^{\leq n}.$$
If F has cohomological dimension $\leq d$, then we have $LF(D^+(\mathscr{A})^{\geq n}) \subset D^+(\mathscr{B})^{\geq n-d}$, and there is a natural isomorphism
$$\mathsf{H}^{n-d}(LF(X)) \cong \mathsf{H}^{-d}(LF(\mathsf{H}^n(X))) \qquad \textit{for any } X \in D^+(\mathscr{A})^{\geq n}.$$

We have the following variant of Definition A.6.9.

DEFINITION A.6.12. Let $\mathscr{A}$ be a $\Bbbk$-linear abelian category. An object $X \in \mathscr{A}$ is said to have **projective dimension** $\leq d$ if the functor $\mathrm{Hom}_{\mathscr{A}}(X, -) : \mathscr{A} \to \Bbbk\text{-mod}$ has cohomological dimension $\leq d$. It is said to have **injective dimension** $\leq d$ if $\mathrm{Hom}_{\mathscr{A}}(-, X) : \mathscr{A}^{\mathrm{op}} \to \Bbbk\text{-mod}$ has cohomological dimension $\leq d$.

The category $\mathscr{A}$ has **global dimension** $\leq d$ if every object has projective dimension $\leq d$, or equivalently if every object has injective dimension $\leq d$.

In the special case where $\mathscr{A} = \Bbbk\text{-mod}$, one usually speaks of the "global dimension of $\Bbbk$" (rather than of $\Bbbk$-mod), and denotes it by $\mathrm{gldim}\,\Bbbk$.

Derived Hom and tensor product. We conclude this section by discussing two examples of "derived functors of two variables." For a discussion of this notion in a more general context, see [**125**, Section 1.10].

Let $\Bbbk$ be a commutative ring, and let $\mathscr{A}$ be a $\Bbbk$-linear abelian category. Consider the functor $\mathrm{Hom} : \mathscr{A}^{\mathrm{op}} \times \mathscr{A} \to \Bbbk\text{-mod}$. We extend this to a functor
$$\mathrm{chHom} : Ch^-(\mathscr{A})^{\mathrm{op}} \times Ch^+(\mathscr{A}) \to Ch^+(\mathscr{A})$$

as follows: for $A \in Ch^-(\mathscr{A})$ and $B \in Ch^+(\mathscr{B})$, $\mathrm{chHom}(A,B)$ is the chain complex given by

$$\mathrm{chHom}(A,B)^n = \bigoplus_{j-i=n} \mathrm{Hom}(A^i, B^j),$$

with differential given by

$$d(f) = d_B \circ f + (-1)^{j-i+1} f \circ d_A \qquad \text{for } f \in \mathrm{Hom}(A^i, B^j).$$

The **derived Hom functor** is a functor $R\mathrm{Hom} : D^-(\mathscr{A})^{\mathrm{op}} \times D^+(\mathscr{A}) \to D^+(\mathscr{A})$ that is triangulated in both variables, together with a natural transformation

$$\epsilon : \mathrm{Qis}(\mathrm{chHom}(A,B)) \to R\mathrm{Hom}(\mathrm{Qis}(A), \mathrm{Qis}(B))$$

satisfying an appropriate universal property.

Proposition A.6.13. *Assume that $\mathscr{A}$ has either enough projectives or enough injectives. Then the functor $R\mathrm{Hom} : D^-(\mathscr{A})^{\mathrm{op}} \times D^+(\mathscr{A}) \to D^+(\Bbbk\text{-mod})$ exists. Furthermore:*

(1) *If $A \in D^-(\mathscr{A})^{\leq n}$ and $B \in D^+(\mathscr{A})^{\geq m}$, then $R\mathrm{Hom}(A,B)$ belongs to $D^+(\Bbbk\text{-mod})^{\geq m-n}$, and there is a natural isomorphism*

$$\mathsf{H}^{m-n}(R\mathrm{Hom}(A,B)) \cong \mathrm{Hom}(\mathsf{H}^n(A), \mathsf{H}^m(B)).$$

(2) *For $A \in D^-(\mathscr{A})$ and $B \in D^+(\mathscr{A})$, there is a natural isomorphism*

$$\mathrm{Hom}_{D(\mathscr{A})}(A,B) \cong \mathsf{H}^0 R\mathrm{Hom}(A,B).$$

As a special case of Proposition A.6.13(2), we have

$$\mathrm{Ext}^n_{\mathscr{A}}(A,B) \cong \mathsf{H}^n(R\mathrm{Hom}(A,B)) \qquad \text{for } A, B \in \mathscr{A}.$$

Similarly, we extend $\otimes : \Bbbk\text{-mod} \times \Bbbk\text{-mod} \to \Bbbk\text{-mod}$ to a functor

$$\overset{\mathrm{ch}}{\otimes} : Ch^-(\Bbbk\text{-mod}) \times Ch^-(\Bbbk\text{-mod}) \to Ch^-(\Bbbk\text{-mod})$$

as follows: for $A, B \in Ch^-(\Bbbk\text{-mod})$, $A \otimes^{\mathrm{ch}} B$ is the chain complex given by

$$(A \overset{\mathrm{ch}}{\otimes} B)^n = \bigoplus_{i+j=n} A^i \otimes B^j$$

and with differential given by

$$d(a \otimes b) = d_A(a) \otimes b + (-1)^i a \otimes d_B(b) \qquad \text{for } a \otimes b \in A^i \otimes B^j. \tag{A.6.4}$$

The **derived tensor product** is a functor

$$\overset{L}{\otimes} : D^-(\Bbbk\text{-mod}) \times D^-(\Bbbk\text{-mod}) \to D^-(\Bbbk\text{-mod})$$

that is triangulated in both variables, together with a natural transformation

$$\eta : \mathrm{Qis}(A) \overset{L}{\otimes} \mathrm{Qis}(B) \to \mathrm{Qis}(A \overset{\mathrm{ch}}{\otimes} B)$$

satisfying an appropriate universal property.

Proposition A.6.14. *The derived tensor product functor $\otimes^L : D^-(\Bbbk\text{-mod}) \times D^-(\Bbbk\text{-mod}) \to D^-(\Bbbk\text{-mod})$ exists. Moreover, if $A \in D^-(\Bbbk\text{-mod})^{\leq n}$ and $B \in D^-(\Bbbk\text{-mod})^{\leq m}$, then $A \otimes^L B \in D^-(\Bbbk\text{-mod})^{\leq m+n}$, and there is a natural isomorphism $\mathsf{H}^{n+m}(A \otimes^L B) \cong \mathsf{H}^n(A) \otimes \mathsf{H}^m(B)$.*

Let A and B be $\Bbbk$-modules. For $n \in \mathbb{Z}$, the nth **Tor group** of A and B, denoted by $\mathrm{Tor}_n^{\Bbbk}(A, B)$ or $\mathrm{Tor}_n(A, B)$, is given by

$$\mathrm{Tor}_n^{\Bbbk}(A, B) = \mathsf{H}^{-n}(A \overset{L}{\otimes} B).$$

By Proposition A.6.14, we have $\mathrm{Tor}_n(A, B) = 0$ for $n < 0$, and $\mathrm{Tor}_0(A, B) \cong A \otimes B$.

Remark A.6.15. The category $\Bbbk$-mod is equipped with an obvious commutativity constraint: namely, the isomorphism $\sigma_0 : A \otimes B \xrightarrow{\sim} B \otimes A$ given by $\sigma_0(a \otimes b) = b \otimes a$. The category $D^-(\Bbbk\text{-mod})$ also admits a commutativity constraint, but we must take the signs in (A.6.4) into account. For $A, B \in Ch^-(\Bbbk\text{-mod})$, define a map $\sigma : A \otimes^{\mathrm{ch}} B \xrightarrow{\sim} B \otimes^{\mathrm{ch}} A$ by

$$\sigma(a \otimes b) = (-1)^{ij} b \otimes a \qquad \text{for } a \otimes b \in A^i \otimes B^j.$$

It is left as an exercise to check that σ is indeed an isomorphism of chain complexes, and that it induces an isomorphism $\sigma : A \otimes^L B \xrightarrow{\sim} B \otimes^L A$ in $D^-(\Bbbk\text{-mod})$.

For rings of finite global dimension, we have the following counterpart of Theorem A.6.10.

Proposition A.6.16. *Let $\Bbbk$ be a commutative ring of global dimension $\le d$. Then the derived tensor product $\otimes^L : D^+(\Bbbk\text{-mod}) \times D^+(\Bbbk\text{-mod}) \to D^+(\Bbbk\text{-mod})$ exists. Moreover, if $A \in D^+(\Bbbk\text{-mod})^{\ge n}$ and $B \in D^+(\Bbbk\text{-mod})^{\ge m}$, then $A \otimes^L B \in D^+(\Bbbk\text{-mod})^{\ge n+m-d}$, and there is a natural isomorphism*

$$\mathsf{H}^{n+m-d}(A \overset{L}{\otimes} B) \cong \mathrm{Tor}_d^{\Bbbk}(\mathsf{H}^n(A), \mathsf{H}^m(B)).$$

Lemma A.6.17. *Let $A, B \in D^-(\Bbbk\text{-mod})$. For any n, there is a natural map*

$$\bigoplus_{i+j=n} \mathsf{H}^i(A) \otimes \mathsf{H}^j(B) \to \mathsf{H}^n(A \overset{L}{\otimes} B).$$

If $\mathsf{H}^i(A)$ is flat for all i, then this map is an isomorphism.

For instance, if $\Bbbk$ is a field, this map is always an isomorphism.

Proposition A.6.18. *Let $\Bbbk$ be a commutative ring of global dimension ≤ 1. For any $M \in \Bbbk\text{-mod}$ and any $X \in D(\Bbbk\text{-mod})$, there is a short exact sequence*

$$0 \to M \otimes \mathsf{H}^n(X) \to \mathsf{H}^n(M \overset{L}{\otimes} X) \to \mathrm{Tor}_1^{\Bbbk}(M, \mathsf{H}^{n+1}(X)) \to 0.$$

Proof. Apply the functor $M \otimes^L (-)$ to the distinguished triangle $\tau^{\le n} X \to X \to \tau^{\ge n+1} X \to$ from Lemma A.5.15, and then take the long exact sequence in cohomology to obtain

$$\cdots \to \mathsf{H}^{n-1}(M \overset{L}{\otimes} \tau^{\ge n+1} X) \to \mathsf{H}^n(M \overset{L}{\otimes} \tau^{\le n} X)$$
$$\to \mathsf{H}^n(M \overset{L}{\otimes} X) \to \mathsf{H}^n(M \overset{L}{\otimes} \tau^{\ge n+1} X) \to \mathsf{H}^{n+1}(M \overset{L}{\otimes} \tau^{\le n} X) \to \cdots.$$

Propositions A.6.14 and A.6.16 tell us that the first and last terms vanish, and identify the second and fourth terms with $M \otimes \mathsf{H}^n(X)$ and $\mathrm{Tor}_1(M, \mathsf{H}^{n+1}(X))$, respectively. $\square$

The proof of the next proposition is very similar.

PROPOSITION A.6.19. *Let $\Bbbk$ be a commutative ring of global dimension ≤ 1. For any $M \in \Bbbk\text{-mod}$ and any $X \in D^-(\Bbbk\text{-mod})$, there is a short exact sequence*

$$0 \to \mathrm{Ext}^1_{\Bbbk}(\mathsf{H}^{n+1}(X), M) \to \mathsf{H}^{-n}(R\mathrm{Hom}(X, M)) \to \mathrm{Hom}_{\Bbbk}(\mathsf{H}^n(X), M) \to 0.$$

Exercises.

A.6.1. Let $\Bbbk$ be a ring of global dimension ≤ 1. Show that for any $A, B \in D^-(\Bbbk\text{-mod})$, there is a natural short exact sequence

$$0 \to \bigoplus_{i+j=n} \mathsf{H}^i(A) \otimes \mathsf{H}^j(B) \to \mathsf{H}^n(A \overset{L}{\otimes} B) \to \bigoplus_{i+j=n+1} \mathrm{Tor}_1(\mathsf{H}^i(A), \mathsf{H}^j(B)) \to 0.$$

A.7. t-structures

In this section, we describe machinery for extracting an abelian category from a triangulated category.

DEFINITION A.7.1. Let $\mathscr{T}$ be a triangulated category, and let $(\mathscr{T}^{\leq 0}, \mathscr{T}^{\geq 0})$ be a pair of strictly full subcategories. For $n \in \mathbb{Z}$, let

$$\mathscr{T}^{\leq n} = \mathscr{T}^{\leq 0}[-n] \qquad \text{and} \qquad \mathscr{T}^{\geq n} = \mathscr{T}^{\geq 0}[-n].$$

The pair $(\mathscr{T}^{\leq 0}, \mathscr{T}^{\geq 0})$ is called a **t-structure** on $\mathscr{T}$ if the following axioms hold:

(1) We have $\mathscr{T}^{\leq -1} \subset \mathscr{T}^{\leq 0}$ and $\mathscr{T}^{\geq -1} \supset \mathscr{T}^{\geq 0}$.
(2) If $X \in \mathscr{T}^{\leq -1}$ and $Y \in \mathscr{T}^{\geq 0}$, then $\mathrm{Hom}(X, Y) = 0$.
(3) For any $X \in \mathscr{T}$, there is a distinguished triangle $A \to X \to B \to$ with $A \in \mathscr{T}^{\leq -1}$ and $B \in \mathscr{T}^{\geq 0}$.

The t-structure is said to be **bounded below**, resp. **bounded above**, if for all $X \in \mathscr{T}$, there is an integer n such that $X \in \mathscr{T}^{\geq n}$, resp. $X \in \mathscr{T}^{\leq n}$. It is **bounded** if it is both bounded below and bounded above. Finally, it is **nondegenerate** if $\bigcap_{n \in \mathbb{Z}} \mathscr{T}^{\leq n}$ and $\bigcap_{n \in \mathbb{Z}} \mathscr{T}^{\geq n}$ both contain only the zero object.

EXAMPLE A.7.2. Let $\mathscr{A}$ be an abelian category. By Lemmas A.5.14 and A.5.15, the full subcategories of $D(\mathscr{A})$ given by

$$D(\mathscr{A})^{\leq 0} = \{X \mid \mathsf{H}^i(X) = 0 \text{ for } i > 0\},$$
$$D(\mathscr{A})^{\geq 0} = \{X \mid \mathsf{H}^i(X) = 0 \text{ for } i < 0\}$$

form a t-structure on $D(\mathscr{A})$, called the **natural t-structure**.

In fact, Lemma A.5.15 proved a substantially stronger statement than the third axiom from Definition A.7.1. We will generalize this to arbitrary t-structures in Proposition A.7.4 below.

LEMMA A.7.3. *Let $(\mathscr{T}^{\leq 0}, \mathscr{T}^{\geq 0})$ be a t-structure on $\mathscr{T}$.*

(1) *We have $X \in \mathscr{T}^{\leq n}$ if and only if $\mathrm{Hom}(X, Y) = 0$ for all $Y \in \mathscr{T}^{\geq n+1}$.*
(2) *We have $X \in \mathscr{T}^{\geq n}$ if and only if $\mathrm{Hom}(Y, X) = 0$ for all $Y \in \mathscr{T}^{\leq n-1}$.*
(3) *The categories $\mathscr{T}^{\leq n}$ and $\mathscr{T}^{\geq n}$ are stable under extensions.*

PROPOSITION A.7.4 (Truncation). *Let $(\mathscr{T}^{\leq 0}, \mathscr{T}^{\geq 0})$ be a t-structure on $\mathscr{T}$.*

(1) *The inclusion $\mathscr{T}^{\leq n} \hookrightarrow \mathscr{T}$ admits a right adjoint ${}^t\tau^{\leq n} : \mathscr{T} \to \mathscr{T}^{\leq n}$.*
(2) *The inclusion $\mathscr{T}^{\geq n} \hookrightarrow \mathscr{T}$ admits a left adjoint ${}^t\tau^{\geq n} : \mathscr{T} \to \mathscr{T}^{\geq n}$.*

(3) *There is a unique natural transformation* $\delta : {}^t\tau^{\geq n+1} \to {}^t\tau^{\leq n}[1]$ *such that for any* $X \in \mathscr{T}$, *the diagram*

$$ {}^t\tau^{\leq n}X \to X \to {}^t\tau^{\geq n+1}X \xrightarrow{\delta} {}^t\tau^{\leq n}X[1] \tag{A.7.1}$$

is a distinguished triangle. Any distinguished triangle $A \to X \to B \to$ *with* $A \in \mathscr{T}^{\leq n}$ *and* $B \in \mathscr{T}^{\geq n+1}$ *is canonically isomorphic to this one.*

Proof sketch. We start with part (3): one can use Lemma A.4.10 to show that any distinguished triangle satisfying the condition in Definition A.7.1(3) is canonically isomorphic to any other. Fix one such triangle for each object X, and define $\tau^{\leq 0}X$ and $\tau^{\geq 1}X$ to be its first and third terms. Use Lemma A.4.10 again to show that $\tau^{\leq 0}$ and $\tau^{\geq 1}$ are in fact functors. Finally, use Definition A.7.1(2) to show that they have the required adjunction properties. ☐

Lemma A.7.5. *Let* $(\mathscr{T}^{\leq 0}, \mathscr{T}^{\geq 0})$ *be a* t*-structure on* $\mathscr{T}$. *For any two integers* $a \leq b$, *there are natural isomorphisms*

$$ {}^t\tau^{\leq a} \circ {}^t\tau^{\leq b} \xrightarrow{\sim} {}^t\tau^{\leq a}, \qquad {}^t\tau^{\geq b} \xrightarrow{\sim} {}^t\tau^{\geq b} \circ {}^t\tau^{\geq a}, \qquad {}^t\tau^{\geq a} \circ {}^t\tau^{\leq b} \xrightarrow{\sim} {}^t\tau^{\leq b} \circ {}^t\tau^{\geq a}.$$

Definition A.7.6. Let $(\mathscr{T}^{\leq 0}, \mathscr{T}^{\geq 0})$ be a t-structure on $\mathscr{T}$, and let $\mathscr{C} = \mathscr{T}^{\leq 0} \cap \mathscr{T}^{\geq 0}$. This category is called the **heart** of the t-structure. The zeroth **t-cohomology** functor, also called the zeroth **cohomology with respect to the t-structure**, is the functor

$$ {}^t\mathsf{H}^0 = {}^t\tau^{\leq 0} \circ {}^t\tau^{\geq 0} : \mathscr{T} \to \mathscr{C}.$$

More generally, for any integer n, we define the nth t-cohomology functor ${}^t\mathsf{H}^n : \mathscr{T} \to \mathscr{C}$ by ${}^t\mathsf{H}^n(X) = {}^t\mathsf{H}^0(X[n])$.

Lemma A.7.7. *Let* $(\mathscr{T}^{\leq 0}, \mathscr{T}^{\geq 0})$ *be a* t*-structure on* $\mathscr{T}$, *and let* $\mathscr{C} = \mathscr{T}^{\leq 0} \cap \mathscr{T}^{\geq 0}$ *be its heart. Let* $f : X \to Y$ *be a morphism in* $\mathscr{C}$. *Extend it to a distinguished triangle* $X \to Y \to Z \to$.

(1) *We have* $Z \in \mathscr{T}^{\geq -1} \cap \mathscr{T}^{\leq 0}$.
(2) *The natural map* ${}^t\mathsf{H}^{-1}(Z) \to X$ *is a kernel of* f *in* $\mathscr{C}$, *and the natural map* $Y \to {}^t\mathsf{H}^0(Z)$ *is a cokernel of* f *in* $\mathscr{C}$.

The main result of this section is the following.

Theorem A.7.8. *Let* $(\mathscr{T}^{\leq 0}, \mathscr{T}^{\geq 0})$ *be a* t*-structure on* $\mathscr{T}$. *Its heart* $\mathscr{C} = \mathscr{T}^{\leq 0} \cap \mathscr{T}^{\geq 0}$ *is an abelian category.*

Proof. We have already seen in Lemma A.7.7 that every morphism in $\mathscr{C}$ has a kernel and a cokernel. Now let $f : X \to Y$ be a morphism in $\mathscr{C}$. Complete this morphism to a distinguished triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$. By Lemma A.3.4, one sees that f is a monomorphism (resp. an epimorphism) if and only if $Z \in \mathscr{C}$ (resp. $Z[-1] \in \mathscr{C}$). Thus, if f is a monomorphism, g belongs to $\mathscr{C}$, and Lemma A.7.7 tells us that f is its kernel. Similarly, if g is an epimorphism, then $h[-1]$ belongs to $\mathscr{C}$, and f is its cokernel. By Theorem A.3.7, $\mathscr{C}$ is an abelian category. ☐

The next statement follows from the description of kernels and cokernels above.

Corollary A.7.9. *Let* $\mathscr{C}$ *be the heart of a* t*-structure on* $\mathscr{T}$, *and let*

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

be two morphisms in $\mathscr{C}$. *The following conditions are equivalent:*

(1) *The sequence* $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ *is a short exact sequence.*
(2) *There exists a morphism* $h : Z \to X[1]$ *in* $\mathscr{T}$ *such that* $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h}$ *is a distinguished triangle.*

Moreover, if these conditions hold, then h *is unique.*

The following statement generalizes Lemma A.5.9.

LEMMA A.7.10. *Let* $\mathscr{C}$ *be the heart of a* t*-structure on* $\mathscr{T}$*. The functor* ${}^t\mathsf{H}^0 : \mathscr{T} \to \mathscr{C}$ *is a cohomological functor.*

PROPOSITION A.7.11. *Let* $\mathscr{C}$ *be the heart of a* t*-structure on* $\mathscr{T}$*, and let* $X, Y \in \mathscr{C}$*. There is a natural bijection*

$$\mathrm{Hom}(X, Y[1]) \cong \left\{ \begin{array}{c} \textit{equivalence classes of} \\ \textit{extensions of } X \textit{ by } Y \end{array} \right\}.$$

See Definition A.3.9 for the notion of extensions in an abelian category. In the case where $\mathscr{T} = D(\mathscr{C})$ and where the t-structure under consideration is the natural t-structure, this statement is the same as Proposition A.5.16.

PROOF SKETCH. Let $\mathcal{E}(X, Y)$ denote the set of equivalence classes of extensions of X by Y. Define a map $\theta : \mathrm{Hom}(X, Y[1]) \to \mathcal{E}(X, Y)$ as follows: given $u : X \to Y[1]$, complete u to a distinguished triangle, and then rotate to obtain a diagram

$$Y \to E \to X \xrightarrow{u} Y[1]. \tag{A.7.2}$$

Since $\mathscr{C}$ is stable under extensions (see Lemma A.7.3(3)), we have $E \in \mathscr{C}$, and then Corollary A.7.9 tells us that $0 \to Y \to E \to X \to 0$ is a short exact sequence, i.e., an extension of X by Y. Let $\theta(u)$ be the equivalence class of this extension.

Conversely, define a map $\psi : \mathcal{E}(X, Y) \to \mathrm{Hom}(X, Y[1])$ as follows: given an extension ξ, say $0 \to Y \to E \to X \to 0$, Corollary A.7.9 gives us a distinguished triangle of the form (A.7.2). Let $\psi(\xi) = u$. Using Lemma A.4.10, one can check that $\psi(\xi)$ only depends on the equivalence class of ξ. We see from the construction that θ and ξ are inverse to one another; in particular, they are both bijections.

We omit the proof that θ is natural. □

Comparing Propositions A.5.16 and A.7.11, we obtain the following result.

COROLLARY A.7.12. *Let* $\mathscr{C}$ *be the heart of a* t*-structure on* $\mathscr{T}$*. For* $X, Y \in \mathscr{C}$*, there is a natural isomorphism* $\mathrm{Ext}^1_{\mathscr{C}}(X, Y) \cong \mathrm{Hom}(X, Y[1])$*.*

DEFINITION A.7.13. Let $\mathscr{T}_1$ and $\mathscr{T}_2$ be triangulated categories equipped with t-structures $(\mathscr{T}_1^{\le 0}, \mathscr{T}_1^{\ge 0})$ and $(\mathscr{T}_2^{\le 0}, \mathscr{T}_2^{\ge 0})$, respectively. A triangulated functor $F : \mathscr{T}_1 \to \mathscr{T}_2$ is said to be **left t-exact** if $F(\mathscr{T}_1^{\ge 0}) \subset \mathscr{T}_2^{\ge 0}$, and **right t-exact** if $F(\mathscr{T}_1^{\le 0}) \subset \mathscr{T}_2^{\le 0}$. It is said to be **$t$-exact** if it is both left and right t-exact.

According to Proposition A.6.11, the right derived functor of a left exact functor is left t-exact (with respect to the natural t-structure), and the left derived functor of a right exact functor is right t-exact. The following statement is a kind of converse to this observation.

LEMMA A.7.14. *Let* $\mathscr{T}_1$ *and* $\mathscr{T}_2$ *be triangulated categories equipped with* t*-structures, and let* $\mathscr{C}_1$ *and* $\mathscr{C}_2$ *denote their hearts, respectively. Let* $F : \mathscr{T}_1 \to \mathscr{T}_2$ *be a triangulated functor.*

(1) *If* F *is left* t*-exact, then the functor* ${}^t\mathsf{H}^0 \circ F : \mathscr{C}_1 \to \mathscr{C}_2$ *is left exact.*
(2) *If* F *is right* t*-exact, then the functor* ${}^t\mathsf{H}^0 \circ F : \mathscr{C}_1 \to \mathscr{C}_2$ *is right exact.*

Remark A.7.15. Suppose $F : \mathscr{T}_1 \to \mathscr{T}_2$ is a left t-exact functor of triangulated categories equipped with t-structures. In analogy with Definition A.6.9, one sometimes says that "F has cohomological dimension $\le d$" to mean that $F(\mathscr{T}_1^{\le 0}) \subset \mathscr{T}_2^{\le d}$. In this situation, we have (cf. Proposition A.6.11)

$$ {}^t\mathsf{H}^{n+d}(F(X)) \cong {}^t\mathsf{H}^d(F({}^t\mathsf{H}^n(X))) \qquad \text{for any } X \in \mathscr{T}_1^{\le n}. $$

A similar statement holds for right t-exact functors.

Realization functors. Let $\mathscr{T}$ be a triangulated functor equipped with a bounded t-structure, and let $\mathscr{C}$ be its heart. Then one may wish to compare the derived category $D^{\mathrm{b}}(\mathscr{C})$ with our original triangulated category $\mathscr{T}$. To do this, we must impose a technical condition on $\mathscr{T}$: we require that it admit a **filtered version**. See [**23**, Definition A.1] for the definition of this notion. We will not need the details of this condition; we simply remark that if $\mathscr{T}$ is a full triangulated subcategory of the derived category of an abelian category or of the homotopy category of an additive category, then it does admit a filtered version. In other words, the following theorem applies to all triangulated categories that arise "in practice."

Theorem A.7.16. *Let $\mathscr{T}$ be a triangulated category that admits a filtered version. Let $\mathscr{C}$ be the heart of a bounded t-structure on $\mathscr{T}$. Then there is a t-exact, triangulated functor*

$$ \mathrm{real} : D^{\mathrm{b}}(\mathscr{C}) \to \mathscr{T} $$

whose restriction to $\mathscr{C} \subset \mathrm{D}^{\mathrm{b}}\mathscr{C}$ is the inclusion functor $\mathscr{C} \hookrightarrow \mathscr{T}$.

For a proof, see [**23**, Appendix] or [**24**, Section 3.1]. The functor given by this theorem is called a **realization functor**. Because it is t-exact and restricts to the identity functor on $\mathscr{C}$, we have a natural isomorphism

$$ \mathsf{H}^i(X) \cong {}^t\mathsf{H}^i(\mathrm{real}(X)) $$

for all $X \in D^{\mathrm{b}}(\mathscr{C})$. Nevertheless, the realization functor is not, in general, an equivalence of categories. Below we give a criterion for it to be an equivalence.

Definition A.7.17. Let $\mathscr{T}$ be a triangulated category, and let $\mathscr{C} \subset \mathscr{T}$ be the heart of a t-structure. Let $X, Y \in \mathscr{C}$, and let $n > 0$. A morphism $f : X \to Y[n]$ is said to be **effaceable** if there are morphisms $p : X' \to X$ and $i : Y \to Y'$ in $\mathscr{C}$ such that p is surjective, i is injective, and $i[n] \circ f \circ p : X' \to Y'[n]$ is the zero morphism.

Thus, Lemma A.5.17 says that for the natural t-structure, all morphisms $f : X \to Y[n]$ are effaceable.

Proposition A.7.18. *Let $\mathscr{T}$ be a triangulated category that admits a filtered version. Let $\mathscr{C}$ be the heart of a t-structure on $\mathscr{T}$, and let $\mathscr{C}' \subset \mathscr{C}$ be a Serre subcategory. For $X, Y \in \mathscr{C}'$, consider the map*

$$ \mathrm{Ext}^i_{\mathscr{C}'}(X, Y) \to \mathrm{Hom}_{\mathscr{T}}(X, Y[i]) \tag{A.7.3} $$

induced by the composition $D^{\mathrm{b}}(\mathscr{C}') \to D^{\mathrm{b}}(\mathscr{C}) \xrightarrow{\mathrm{real}} \mathscr{T}$.

(1) *The map* (A.7.3) *is an isomorphism for $i = 0$ and $i = 1$.*
(2) *If* (A.7.3) *is an isomorphism for all X and Y when $i = 0, 1, \ldots, n-1$, then it is injective when $i = n$, and its image is the set of effaceable morphisms.*

PROOF. For brevity, let ρ denote the composition $D^{\mathrm{b}}(\mathscr{C}') \to D^{\mathrm{b}}(\mathscr{C}) \xrightarrow{\mathrm{real}} \mathscr{T}$. The fact that (A.7.3) is an isomorphism for $i = 0$ is just a restatement of the fact that $\rho|_{\mathscr{C}'}$ is the inclusion functor. For $i = 1$, consider the diagram

$$\begin{tikzcd} \left\{ \begin{matrix} \text{equivalence classes of} \\ \text{extensions in } \mathscr{C}' \text{ of } X \text{ by } Y \end{matrix} \right\} \arrow[r, equal] & \left\{ \begin{matrix} \text{equivalence classes of} \\ \text{extensions in } \mathscr{C} \text{ of } X \text{ by } Y \end{matrix} \right\} \\ \mathrm{Ext}^1_{\mathscr{C}'}(X,Y) \arrow[u] \arrow[r, "\rho"] & \mathrm{Hom}_{\mathscr{T}}(X, Y[1]) \arrow[u] \end{tikzcd} \tag{A.7.4}$$

The equality along the top comes from the fact that $\mathscr{C}'$ is a Serre subcategory of $\mathscr{C}$. Propositions A.5.16 and A.7.11 say that the two vertical arrows are bijections. These arrows are both defined by taking the cone of a morphism $X \to Y[1]$ to form a short exact sequence. Since real sends the cone of a morphism $f \in \mathrm{Ext}^1(X,Y)$ to the cone of $\rho(f) \in \mathrm{Hom}(X, Y[1])$, the diagram (A.7.4) commutes, and (A.7.3) is an isomorphism for $i = 1$.

Now suppose (A.7.3) is an isomorphism for $i = 0, 1, \ldots, n-1$. Let $f \in \mathrm{Ext}^n_{\mathscr{C}'}(X,Y)$. By Lemma A.5.17, there exists a surjective map $p : X' \to X$ such that $f \circ p = 0$. Let $K = \ker p$. The short exact sequence $0 \to K \to X' \to X \to 0$ gives rise to a commutative diagram

$$\begin{tikzcd} \mathrm{Ext}^{n-1}_{\mathscr{C}'}(X',Y) \arrow[r] \arrow[d, "\wr"'] & \mathrm{Ext}^{n-1}_{\mathscr{C}'}(K,Y) \arrow[r] \arrow[d, "\wr"'] & \mathrm{Ext}^{n}_{\mathscr{C}'}(X,Y) \arrow[r] \arrow[d] & \mathrm{Ext}^{n}_{\mathscr{C}'}(X',Y) \arrow[d] \\ \mathrm{Hom}_{\mathscr{T}}(X', Y[n-1]) \arrow[r] & \mathrm{Hom}_{\mathscr{T}}(K, Y[n-1]) \arrow[r] & \mathrm{Hom}_{\mathscr{T}}(X, Y[n]) \arrow[r] & \mathrm{Hom}_{\mathscr{T}}(X', Y[n]) \end{tikzcd} \tag{A.7.5}$$

with exact rows. Our setup implies that the first two vertical maps are isomorphisms and that f is in the kernel of $\mathrm{Ext}^n(X,Y) \to \mathrm{Ext}^n(X',Y)$. A diagram chase shows that if $\rho(f) = 0$, then $f = 0$, so (A.7.3) is injective for $i = n$.

It is clear from Lemma A.5.17 that every morphism in the image of (A.7.3) is effaceable. Suppose now that $g \in \mathrm{Hom}_{\mathscr{T}}(X, Y[n])$ is effaceable: let $p : X' \to X$ and $i : Y \to Y'$ be maps with p surjective, i injective, and $i[n] \circ g \circ p = 0$. We must show that g lies in the image of (A.7.3).

Let us first treat the special case where $Y' = Y$ and $i = \mathrm{id}_Y$, so that $g \circ p = 0$. In this case, another diagram chase using (A.7.5) proves the claim.

Let $g' = i[n] \circ g : X \to Y'[n]$. The special case above applies to g': there is a map $\tilde{g}' \in \mathrm{Ext}^n_{\mathscr{C}'}(X, Y')$ such that $\rho(\tilde{g}') = g'$. Let $Q = \operatorname{cok} i$, and consider the diagram

$$\begin{tikzcd} \mathrm{Ext}^{n-1}_{\mathscr{C}'}(X,Q) \arrow[r] \arrow[d, "\wr"'] & \mathrm{Ext}^{n}_{\mathscr{C}'}(X,Y) \arrow[r] \arrow[d] & \mathrm{Ext}^{n}_{\mathscr{C}'}(X,Y') \arrow[r] \arrow[d] & \mathrm{Ext}^{n}_{\mathscr{C}'}(X,Q) \arrow[d] \\ \mathrm{Hom}_{\mathscr{T}}(X, Q[n-1]) \arrow[r] & \mathrm{Hom}_{\mathscr{T}}(X, Y[n]) \arrow[r] & \mathrm{Hom}_{\mathscr{T}}(X, Y'[n]) \arrow[r] & \mathrm{Hom}_{\mathscr{T}}(X, Q[n]) \end{tikzcd}$$

Here, the first vertical is an isomorphism, and the other three are injective. A diagram chase shows that there exists a $\tilde{g} \in \mathrm{Ext}^n_{\mathscr{C}'}(X,Y)$ such that $\rho(\tilde{g}) = g$. $\square$

COROLLARY A.7.19. *Let $\mathscr{T}$ be a triangulated category that admits a filtered version, and let $\mathscr{C}$ be the heart of a bounded t-structure on $\mathscr{T}$. The following conditions are equivalent:*

(1) *The functor* $\mathrm{real} : D^{\mathrm{b}}(\mathscr{C}) \to \mathscr{T}$ *is an equivalence of categories.*

(2) *For all $X, Y \in \mathscr{C}$ and all $n > 0$, every morphism $X \to Y[n]$ in $\mathscr{T}$ is effaceable.*

PROOF. If the realization functor is an equivalence of categories, then the second condition holds by Lemma A.5.17. For the other direction, if every morphism $f : X \to Y[n]$ is effaceable, then Proposition A.7.18 says that (A.7.3) is always an isomorphism. Since $\mathscr{C}$ generates both $D^{\mathrm{b}}(\mathscr{C})$ and $\mathscr{T}$ (Exercise A.7.2), Proposition A.4.16 tells us that the realization functor is fully faithful, and then Proposition A.4.17 tells us that it is an equivalence of categories. $\square$

Here is an application of the preceding corollary.

PROPOSITION A.7.20. *Let $\Bbbk$ be a commutative ring, and let $D^{\mathrm{b}}_{\mathrm{fg}}(\Bbbk\text{-mod})$ be the full triangulated subcategory of $D^{\mathrm{b}}(\Bbbk\text{-mod})$ consisting of chain complexes X such that each $\mathsf{H}^i(X)$ is a finitely generated $\Bbbk$-module. Then the obvious functor*

$$D^{\mathrm{b}}(\Bbbk\text{-mod}^{\mathrm{fg}}) \to D^{\mathrm{b}}_{\mathrm{fg}}(\Bbbk\text{-mod})$$

is an equivalence of categories.

PROOF. Let $X, Y \in \Bbbk\text{-mod}^{\mathrm{fg}}$, and let $f : X \to Y[n]$ be a morphism in $D^{\mathrm{b}}_{\mathrm{fg}}(\Bbbk\text{-mod})$. In the larger category $D^{\mathrm{b}}(\Bbbk\text{-mod})$, we can apply Lemma A.5.17 to find a surjective map $p : X' \to X$ such that $f \circ p = 0$, but X' may not be finitely generated. Since X is finitely generated, there is a finite set of elements $\{x_1, \ldots, x_n\} \subset X'$ such that $\{p(x_1), \ldots, p(x_n)\}$ generates X. Let $X'' \subset X'$ be the submodule generated by $\{x_1, \ldots, x_n\}$, and let $q = p|_{X''} : X'' \to X$. Then q is surjective, and $f \circ q = 0$. Since X'' is finitely generated, this shows that f is effaceable in $D^{\mathrm{b}}_{\mathrm{fg}}(\Bbbk\text{-mod})$. The result follows by Corollary A.7.19. $\square$

Finally, the following lemma will sometimes be useful for checking whether a given morphism is effaceable.

LEMMA A.7.21. *Let $\mathscr{T}$ and $\mathscr{T}'$ be two triangulated categories equipped with t-structures, and let $\mathscr{C}$ and $\mathscr{C}'$ be their hearts. Let $\rho : \mathscr{T} \to \mathscr{T}'$ be a t-exact triangulated functor such that $\rho|_{\mathscr{C}} : \mathscr{C} \to \mathscr{C}'$ is fully faithful and such that $\rho(\mathscr{C})$ is a Serre subcategory of $\mathscr{C}'$. Let $X, Y \in \mathscr{C}$, and let $f : \rho(X) \to \rho(Y)[n]$ be a morphism in $\mathscr{T}'$. The following conditions are equivalent:*

(1) *The morphism f is in the image of the map $\rho : \mathrm{Hom}_{\mathscr{T}}(X, Y[n]) \to \mathrm{Hom}_{\mathscr{T}'}(\rho(X), \rho(Y)[n])$.*
(2) *There is an object $Z \in \mathscr{T}$ such that the cone of f is isomorphic to $\rho(Z)$.*

PROOF. (1) $\implies$ (2). If $f = \rho(g)$ for some morphism $g : X \to Y[n]$ in $\mathscr{T}$, then of course the cone of f is obtained by applying ρ to the cone of g.

(2) $\implies$ (1). If $n < 0$, then $f = 0$, so condition (1) holds trivially, and if $n = 0$, then condition (1) follows from the assumption that $\rho|_{\mathscr{C}} : \mathscr{C} \to \mathscr{C}'$ is fully faithful. If $n = 1$, then according to Proposition A.7.18, f can be regarded as an element of $\mathrm{Ext}^1_{\mathscr{C}'}(\rho(X), \rho(Y))$; it determines an extension of $\rho(X)$ by $\rho(Y)$. To prove condition (1), we must show that this extension lies in the image of ρ. This holds by the assumption that $\rho(\mathscr{C})$ is a Serre subcategory of $\mathscr{C}'$.

Assume now that $n \geq 2$, and let $Z' = Z[-1]$. We have a distinguished triangle

$$\rho(Y)[n-1] \to \rho(Z') \to \rho(X) \xrightarrow{f}$$

in $\mathscr{T}'$. On the other hand, we can apply truncation functors in $\mathscr{T}$ to Z', and then apply ρ to obtain a distinguished triangle

$$\rho(\tau^{\leq -1}Z') \to \rho(Z') \to \rho(\tau^{\geq 0}Z') \xrightarrow{\delta} .$$

By the uniqueness in Proposition A.7.4(3), the two distinguished triangles above are canonically isomorphic: there is a commutative diagram

$$\begin{tikzcd}
\rho(Y)[n-1] \arrow[r] \arrow[d, "v[-1]"'] & \rho(Z') \arrow[r] \arrow[d, equal] & \rho(X) \arrow[r, "f"] \arrow[d, "u"'] & \rho(Y)[n] \arrow[d, "v"] \\
\rho(\tau^{\leq -1}Z') \arrow[r] & \rho(Z') \arrow[r] & \rho(\tau^{\geq 0}Z') \arrow[r] & \rho(\tau^{\leq -1}Z')[1]
\end{tikzcd}$$

in which the vertical maps are isomorphisms. In particular, $\tau^{\geq 0}Z' \in \mathscr{C}'$, and $\tau^{\leq -1}Z' \in \mathscr{C}'[n-1]$. Since ρ is fully faithful on $\mathscr{C}$, the maps u and v must be in its image: say $u = \rho(u')$ and $v = \rho(v')$. We conclude that $f = \rho((v')^{-1} \circ \delta \circ u')$, as desired. □

Exercises.

A.7.1. Show that a bounded t-structure is automatically nondegenerate.

A.7.2. Let $\mathscr{C}$ be the heart of a bounded t-structure on $\mathscr{T}$. Show that $\mathscr{C}$ generates $\mathscr{T}$ as a triangulated category.

A.7.3. Consider the ring $R = \mathbb{C}[x]/(x^2)$. Let $\mathscr{A}$ be the category of finitely generated R-modules. Call a chain complex *perfect* if it is quasi-isomorphic to a bounded complex of projective R-modules. Let $\mathscr{T}$ be the full triangulated subcategory of $D^{\mathrm{b}}(\mathscr{A})$ consisting of perfect complexes, and let

$$\mathscr{T}^{\leq 0} = \mathscr{T} \cap D^{\mathrm{b}}(\mathscr{A})^{\leq 0}, \qquad \mathscr{T}^{\geq 0} = \mathscr{T} \cap D^{\mathrm{b}}(\mathscr{A})^{\geq 0}.$$

Is $(\mathscr{T}^{\leq 0}, \mathscr{T}^{\geq 0})$ a t-structure on $\mathscr{T}$? (*Hint:* No.)

A.7.4. The next few exercises deal with the following notion: A **recollement diagram** consists of three triangulated categories and six triangulated functors

$$\begin{tikzcd}
\mathscr{T}' \arrow[r, "\iota"] & \mathscr{T} \arrow[l, bend right, "\iota^{\mathrm{L}}"'] \arrow[l, bend left, "\iota^{\mathrm{R}}"] \arrow[r, "\Pi"] & \mathscr{T}'' \arrow[l, bend right, "\Pi^{\mathrm{L}}"'] \arrow[l, bend left, "\Pi^{\mathrm{R}}"]
\end{tikzcd}$$

that satisfy the following axioms:

(a) The functor ι^{L} is left adjoint to ι, and ι^{R} is right adjoint to ι.
(b) The functor Π^{L} is left adjoint to Π, and Π^{R} is right adjoint to Π.
(c) We have $\Pi \circ \iota = 0$.
(d) For any object $X \in \mathscr{T}$, there are distinguished triangles

$$\iota\iota^{\mathrm{R}}(X) \to X \to \Pi^{\mathrm{R}}\Pi(X) \to,$$
$$\Pi^{\mathrm{L}}\Pi(X) \to X \to \iota\iota^{\mathrm{L}}\Pi(X) \to,$$

where the first two morphisms in each triangle are adjunction maps.
(e) The functors ι, Π^{L}, and Π^{R} are fully faithful.

Show that in the definition of a recollement diagram, axiom (e) is equivalent to the following condition:

(e′) The adjunction maps $\iota^{\mathrm{L}}\iota \to \mathrm{id}$, $\mathrm{id} \to \iota^{\mathrm{R}}\iota$, $\Pi\Pi^{\mathrm{R}} \to \mathrm{id}$, and $\mathrm{id} \to \Pi\Pi^{\mathrm{L}}$ are all isomorphisms.

Then show that there is a natural transformation $\mu : \Pi^{\mathrm{R}} \to \Pi^{\mathrm{L}}$ such that $\Pi(\mu)$ is an isomorphism.

A.7.5. Let $\mathscr{S}$ be the set of morphisms in $\mathscr{T}$ whose cone lies in the essential image of ι. Note that $\mathscr{S}$ is a Verdier localizing system. Show that Π induces an equivalence of categories $\mathscr{T}_{\mathscr{S}} \xrightarrow{\sim} \mathscr{T}''$.

A.7.6. Suppose $\mathscr{T}'$ is equipped with a t-structure $(\mathscr{T}'^{\le 0}, \mathscr{T}'^{\ge 0})$ and that $\mathscr{T}''$ is equipped with a t-structure $(\mathscr{T}''^{\le 0}, \mathscr{T}''^{\ge 0})$. Show that there is a unique t-structure $(\mathscr{T}^{\le 0}, \mathscr{T}^{\ge 0})$ such that ι and Π are t-exact. Namely,

$$\mathscr{T}^{\le 0} = \{X \in \mathscr{T} \mid \iota^{\mathrm{L}}(X) \in \mathscr{T}'^{\le 0} \text{ and } \Pi(X) \in \mathscr{T}''^{\le 0}\},$$
$$\mathscr{T}^{\ge 0} = \{X \in \mathscr{T} \mid \iota^{\mathrm{R}}(X) \in \mathscr{T}'^{\ge 0} \text{ and } \Pi(X) \in \mathscr{T}''^{\ge 0}\}.$$

Then show that ι^{R} and Π^{R} are left t-exact, and that ι^{L} and Π^{L} are right t-exact. (This t-structure on $\mathscr{T}$ is said to be obtained by **recollement** or **gluing** from those on $\mathscr{T}'$ and $\mathscr{T}''$.)

A.7.7. Consider the three t-structures on $\mathscr{T}'$, $\mathscr{T}''$, and $\mathscr{T}$ as in the preceding exercise, and let $\mathscr{C}'$, $\mathscr{C}''$, and $\mathscr{C}$ denote their hearts, respectively.

(a) Show that if A is a simple object in $\mathscr{C}'$, then $\iota(A)$ is a simple object in $\mathscr{C}$.

(b) For any object $B \in \mathscr{C}$, let $\Pi^0(B)$ denote the image of the natural map ${}^t\mathsf{H}^0(\mu_B) : {}^t\mathsf{H}^0(\Pi^{\mathrm{R}}(B)) \to {}^t\mathsf{H}^0(\Pi^{\mathrm{L}}(B))$. Show that if B is simple, then $\Pi^0(B)$ is simple.

(c) Show that every simple object in $\mathscr{C}$ arises from one of the two constructions above.

A.8. Karoubian and Krull–Schmidt categories

In this section, we study how an object in an additive category may decompose into a direct sum of other objects.

Definition A.8.1. Let $\mathscr{C}$ be an additive category. Let X be an object in $\mathscr{C}$, and let $e \in \mathrm{End}(X)$ be an idempotent, i.e., a morphism such that $e \circ e = e$. This idempotent is said to **split** if there exists an object Y and morphisms $p : X \to Y$, $i : Y \to X$ in $\mathscr{C}$ such that $i \circ p = e$ and $p \circ i = \mathrm{id}_Y$. The category $\mathscr{C}$ is said to be **Karoubian** if all idempotents in all endomorphism rings split.

Recall that a collection of idempotent elements $\{e_1, \ldots, e_k\}$ in a ring is said to be **orthogonal** if $e_i e_j = 0$ for $i \ne j$, and **complete** if $e_1 + \cdots + e_k = 1$. Now let X be an object in a (not necessarily Karoubian) additive category, and suppose that

$$X \cong Y_1 \oplus Y_2 \oplus \cdots \oplus Y_k. \tag{A.8.1}$$

This decomposition comes with maps $\mathrm{pr}_i : X \to Y_i$ and $\mathrm{in}_i : Y_i \to X$ that satisfy a generalization of (A.3.1). If we let $e_i = \mathrm{in}_i \circ \mathrm{pr}_i \in \mathrm{End}(X)$, then each e_i is a split idempotent, and $\{e_1, \ldots, e_k\}$ is a complete set of orthogonal idempotents.

Conversely, given a complete set of orthogonal idempotents $\{e_1, \ldots, e_k\}$ in $\mathrm{End}(X)$, we can recover the right-hand side of (A.8.1) if each e_i splits. Thus, in a Karoubian category, any idempotent in $\mathrm{End}(X)$ corresponds to a direct summand of X, and any complete set of orthogonal idempotents comes from a direct-sum decomposition like that in (A.8.1).

LEMMA A.8.2. *Every abelian category is Karoubian.*

PROOF SKETCH. Let $e : X \to X$ be an idempotent. Let Y be its image, and let $p : X \to Y$ and $i : Y \to X$ be the natural maps. By construction, we have $i \circ p = e$, and a short calculation shows that $p \circ i = \mathrm{id}_Y$. □

THEOREM A.8.3. *Every triangulated category with a bounded t-structure is Karoubian.*

For a proof, see [**153**].

DEFINITION A.8.4. An additive category $\mathscr{C}$ is said to be a **Krull–Schmidt category** if every object X is isomorphic to a direct sum $X_1 \oplus \cdots \oplus X_k$, where each $\mathrm{End}(X_i)$ is a local ring.

PROPOSITION A.8.5. *For an additive category $\mathscr{C}$, the following conditions are equivalent:*

(1) *$\mathscr{C}$ is Krull–Schmidt.*
(2) *$\mathscr{C}$ is Karoubian, and the endomorphism ring of every object is a semiperfect ring.*

For the notion of a **semiperfect ring**, see [**144**, Section 23]. For a proof of Proposition A.8.5, see [**142**, Corollary 4.4].

EXAMPLE A.8.6. Suppose $\Bbbk$ is either a field or a commutative noetherian complete local ring. According to [**144**, Example 23.3], every finitely generated $\Bbbk$-algebra is semiperfect. Combining this observation with Lemma A.8.2, Theorem A.8.3, and Proposition A.8.5, we obtain the following key examples.

(1) Let $\mathscr{C}$ be a $\Bbbk$-linear abelian category in which all Hom-groups are finitely generated over $\Bbbk$. Then $\mathscr{C}$ is Krull–Schmidt.
(2) Let $\mathscr{T}$ be a $\Bbbk$-linear triangulated category that admits a bounded t-structure, and in which all Hom-groups are finitely generated over $\Bbbk$. Then $\mathscr{T}$ is Krull–Schmidt.

THEOREM A.8.7 (Krull–Schmidt theorem). *Let $\mathscr{C}$ be a Krull–Schmidt category, and let X be an object in $\mathscr{C}$. Then there is a unique decomposition (up to isomorphism and reordering of the summands)*

$$X \cong X_1 \oplus \cdots \oplus X_k$$

where $X_1, \ldots, X_k$ are objects with local endomorphism rings.

Of course, the existence of such a decomposition is part of the definition of a Krull–Schmidt category; the content of this theorem is the uniqueness. For a proof, see [**142**, Theorem 4.2]. For the next statement, recall that an object is **indecomposable** if it is not isomorphic to a direct sum of two nonzero objects.

COROLLARY A.8.8. *Let $\mathscr{C}$ be a Krull–Schmidt category. An object $X \in \mathscr{C}$ is indecomposable if and only if* $\mathrm{End}(X)$ *is a local ring.*

PROOF. If X is indecomposable, then Definition A.8.4 implies that $\mathrm{End}(X)$ is a local ring. Conversely, if $\mathrm{End}(X)$ is local, the uniqueness of the decomposition in Theorem A.8.7 implies that X is indecomposable. □

Generalized eigenspace decomposition. We conclude this section with an adaptation of the notion of "generalized eigenspaces" from linear algebra to the setting of Krull–Schmidt categories over a field. In the following statement, x is an indeterminate, and $\Bbbk[x]$ is the polynomial ring over $\Bbbk$ in one variable.

PROPOSITION A.8.9. *Let $\Bbbk$ be a field, and let $\mathscr{C}$ be a $\Bbbk$-linear Krull–Schmidt category in which all* Hom*-groups are finite dimensional. Let $X \in \mathscr{C}$, and let $\phi : X \to X$ be an endomorphism of X. There is a canonical decomposition*

$$X \cong \bigoplus_{\substack{\alpha(x)\in\Bbbk[x] \\ \textit{irreducible monic}}} X_\alpha$$

(with finitely many nonzero X_α) such that ϕ is the direct sum of a collection of endomorphisms $\phi_\alpha : X_\alpha \to X_\alpha$ with the following properties:

(1) *For any irreducible monic polynomial $\alpha(x) \in \Bbbk[x]$, the morphism $\alpha(\phi_\alpha) : X_\alpha \to X_\alpha$ is nilpotent.*
(2) *For two distinct irreducible monic polynomials $\alpha(x), \beta(x) \in \Bbbk[x]$, the morphism $\beta(\phi_\alpha) : X_\alpha \to X_\alpha$ is an isomorphism.*

By analogy with linear algebra, the object X_α is called the **generalized α-eigenspace** of $\phi : X \to X$. If $\Bbbk$ is algebraically closed, then every irreducible monic polynomial in $\Bbbk[x]$ is of the form $x - \lambda$ for some scalar $\lambda \in \Bbbk$, so the direct sum in the proposition above can be indexed by scalars (called **eigenvalues**) instead.

PROOF. Let $\Bbbk[\phi]$ denote the $\Bbbk$-subalgebra of $\operatorname{End}(X)$ generated by ϕ. There is an obvious surjective ring homomorphism $\Bbbk[x] \to \Bbbk[\phi]$ given by $x \mapsto \phi$. Since $\Bbbk[\phi]$ is finite dimensional, this map has a nontrivial kernel, generated by some monic polynomial $m_\phi \in \Bbbk[x]$. (This is called the **minimal polynomial** of ϕ.) This polynomial has a factorization

$$m_\phi = \alpha_1^{a_1} \cdots \alpha_k^{a_k},$$

where $\alpha_1, \ldots, \alpha_k \in \Bbbk[x]$ are distinct irreducible monic polynomials. Let $s_i = m_\phi/\alpha_i^{a_i}$. The gcd of $s_1, \ldots, s_k$ is 1, so there exist polynomials $b_1, \ldots, b_k \in \Bbbk[x]$ such that

$$b_1 s_1 + \cdots + b_k s_k = 1.$$

Let $e_i = b_i(\phi)s_i(\phi) \in \operatorname{End}(X)$. The equation above implies that

$$e_1 + \cdots + e_k = \mathrm{id}_X. \tag{A.8.2}$$

Note that if $i \neq j$, then $s_i s_j$ is divisible by m_ϕ. Since $m_\phi(\phi) = 0$, we have $s_i(\phi)s_j(\phi) = 0$, and hence

$$e_i e_j = 0 \qquad \text{if } i \neq j. \tag{A.8.3}$$

Now multiply both sides of (A.8.2) by e_i. In view of (A.8.3), we deduce that

$$e_i^2 = e_i.$$

To summarize, we have shown that $e_1, \ldots, e_k$ is a complete set of orthogonal idempotents in $\operatorname{End}(X)$. It gives rise to a decomposition

$$X = X_{\alpha_1} \oplus \cdots \oplus X_{\alpha_k}.$$

Let $\mathrm{pr}_i : X \to X_{\alpha_i}$ and $\mathrm{in}_i : X_{\alpha_i} \to X$ be the projection and inclusion maps, respectively, so that $e_i = \mathrm{in}_i \mathrm{pr}_i$. Define a map $\phi_{\alpha_i} : X_{\alpha_i} \to X_{\alpha_i}$ by $\phi_{\alpha_i} = \mathrm{pr}_i \phi \mathrm{in}_i$. Since ϕ commutes with e_i, we see that

$$\mathrm{in}_i \phi_{\alpha_i} \mathrm{pr}_i = e_i \phi e_i = e_i \phi = \phi e_i.$$

By (A.8.2), we have $\phi = \mathrm{in}_1 \phi_{\alpha_1} \mathrm{pr}_1 + \cdots + \mathrm{in}_k \phi_{\alpha_k} \mathrm{pr}_k$, or in other words,

$$\phi = \phi_{\alpha_1} \oplus \cdots \oplus \phi_{\alpha_k}.$$

Let us now show that $\alpha_i(\phi_{\alpha_i})$ is nilpotent. This is equivalent to showing that $\alpha_i(\mathrm{in}_i \phi_{\alpha_i} \mathrm{pr}_i) = \alpha_i(\phi) e_i$ is nilpotent. For the latter, we have

$$(\alpha_i(\phi) e_i)^{a_i} = \alpha_i(\phi)^{a_i} e_i = \alpha_i(\phi)^{a_i} b_i(\phi) s_i(\phi) = b_i(\phi) m_\phi(\phi) = 0.$$

Finally, let $\beta(x) \in \Bbbk[x]$ be an irreducible monic polynomial different from α_i. It remains to show that $\beta(\phi_{\alpha_i})$ is an isomorphism. Since β and $\alpha_i^{a_i}$ are relatively prime, there are polynomials $c, d \in \Bbbk[x]$ such that

$$c\beta + d\alpha_i^{a_i} = 1.$$

We have seen above that $\alpha_i(\phi_{\alpha_i})^{a_i} = 0$, so this equation implies that $c(\phi_{\alpha_i})\beta(\phi_{\alpha_i}) = \mathrm{id}_{X_{\alpha_i}}$. Thus, $\beta(\phi_{\alpha_i})$ is invertible. □

REMARK A.8.10. The last part of Proposition A.8.9 implies that $\phi_\alpha : X_\alpha \to X_\alpha$ is an automorphism of X_α unless $\alpha(x) = x$, in which case ϕ_α is nilpotent. From this observation, one can deduce that ϕ itself is an automorphism if and only if $X_x = 0$.

The following lemma is a kind of naturality property for the eigenspace decomposition from Proposition A.8.9.

LEMMA A.8.11. *Let $\mathscr{C}$ be a category satisfying the assumptions of Proposition* A.8.9, *and let X and Y be objects of $\mathscr{C}$ equipped with endomorphisms $\phi_X : X \to X$ and $\phi_Y : Y \to Y$. If $\theta : X \to Y$ is a morphism satisfying*

$$\theta \circ \phi_X = \phi_Y \circ \theta,$$

then θ is compatible with the eigenspace decompositions of X and Y. In other words, θ is the direct sum of a collection of morphisms $\theta_\alpha : X_\alpha \to Y_\alpha$ that satisfy

$$\theta_\alpha \circ \phi_{X,\alpha} = \phi_{Y,\alpha} \circ \theta_\alpha,$$

where α runs over all irreducible monic polynomials in $\Bbbk[x]$.

PROOF. After forming the eigenspace decompositions of X and Y, θ can be written as a matrix $(\theta_{\alpha\beta} : X_\alpha \to Y_\beta)_{\alpha,\beta}$, where α and β run over irreducible monic polynomials. The assumption that $\theta\phi_X = \phi_Y \theta$ implies that

$$\theta_{\alpha\beta} \circ \phi_{X,\alpha} = \phi_{Y,\beta} \circ \theta_{\alpha\beta}$$

for any two irreducible monic polynomials α and β. This in turn implies that

$$\theta_{\alpha\beta} \circ \alpha(\phi_{X,\alpha})^N = \alpha(\phi_{Y,\beta})^N \circ \theta_{\alpha\beta}$$

for any integer $N > 0$. If N is large enough, the left-hand side vanishes. But if $\alpha \neq \beta$, then $\alpha(\phi_{Y,\beta})$ is invertible. We conclude that $\theta_{\alpha\beta} = 0$ if $\alpha \neq \beta$. In other words, θ is the direct sum of morphisms $\theta_{\alpha\alpha} : X_\alpha \to Y_\alpha$, as desired. □

A.9. Grothendieck groups

For a category $\mathscr{C}$, let $\mathrm{Isom}(\mathscr{C})$ denote the set of isomorphism classes of objects of $\mathscr{C}$. In this section, it will be convenient to assume that $\mathrm{Isom}(\mathscr{C})$ is a small set (by enlarging the universe $\mathscr{U}$ if necessary—see Section A.1). We may then consider the (small) free abelian group $\mathbb{Z}[\mathrm{Isom}(\mathscr{C})]$. For an object $X \in \mathscr{C}$, let $[X]$ denote its isomorphism class, regarded as an element of $\mathrm{Isom}(\mathscr{C})$ or of $\mathbb{Z}[\mathrm{Isom}(\mathscr{C})]$.

Definition A.9.1. Let $\mathscr{C}$ be an additive category. The **split Grothendieck group** of $\mathscr{C}$, denoted by $\mathrm{K}_\oplus(\mathscr{C})$, is the abelian group given by

$$\mathrm{K}_\oplus(\mathscr{C}) = \mathbb{Z}[\mathrm{Isom}(\mathscr{C})]/\operatorname{span}\{[X \oplus Y] - [X] - [Y] \mid X, Y \in \mathscr{C}\}.$$

Proposition A.9.2. *Let $\mathscr{C}$ be a Krull–Schmidt category, and let $\mathrm{Ind}(\mathscr{C})$ be the set of isomorphism classes of indecomposable objects in $\mathscr{C}$. There is a canonical isomorphism*

$$\mathrm{K}_\oplus(\mathscr{C}) \cong \mathbb{Z}[\mathrm{Ind}(\mathscr{C})].$$

Proof. Define a map $\tilde{p} : \mathbb{Z}[\mathrm{Isom}(\mathscr{C})] \to \mathbb{Z}[\mathrm{Ind}(\mathscr{C})]$ by setting

$$\tilde{p}([X]) = [X_1] + \cdots + [X_k] \qquad \text{if} \qquad X \cong X_1 \oplus \cdots \oplus X_k.$$

Theorem A.8.7 implies that $\tilde{p}$ is well defined, and that it factors through a map $p : \mathrm{K}_\oplus(\mathscr{C}) \to \mathbb{Z}[\mathrm{Ind}(\mathscr{C})]$. On the other hand, the inclusion map $\mathrm{Ind}(\mathscr{C}) \hookrightarrow \mathrm{Isom}(\mathscr{C})$ gives rise to an obvious map $q : \mathbb{Z}[\mathrm{Ind}(\mathscr{C})] \to \mathrm{K}_\oplus(\mathscr{C})$. It is straightforward to check that p and q are inverse to one another. $\square$

Definition A.9.3. The **Grothendieck group** of an abelian category $\mathscr{C}$ is the abelian group given by

$$\mathrm{K}_0(\mathscr{C}) = \mathbb{Z}[\mathrm{Isom}(\mathscr{C})] \Big/ \operatorname{span}\left\{ [Z] - [X] - [Y] \,\middle|\, \begin{array}{c}\text{there is a short exact sequence} \\ 0 \to X \to Z \to Y \to 0\end{array} \right\}.$$

Similarly, the **Grothendieck group** of a triangulated category $\mathscr{T}$ is given by

$$\mathrm{K}_0(\mathscr{T}) = \mathbb{Z}[\mathrm{Isom}(\mathscr{T})] \Big/ \operatorname{span}\left\{ [Z] - [X] - [Y] \,\middle|\, \begin{array}{c}\text{there is a distinguished triangle} \\ X \to Z \to Y \to\end{array} \right\}.$$

The proofs of the next two statements are very similar to that of Proposition A.9.2, and will be omitted.

Proposition A.9.4. *Let $\mathscr{C}$ be a finite-length abelian category, and let $\mathrm{Irr}(\mathscr{C})$ be the set of isomorphism classes of simple objects in $\mathscr{C}$. There is a canonical isomorphism*

$$\mathrm{K}_\oplus(\mathscr{C}) \cong \mathbb{Z}[\mathrm{Irr}(\mathscr{C})].$$

Proposition A.9.5. *Let $\mathscr{T}$ be a triangulated category, and let $\mathscr{C}$ be the heart of a bounded t-structure on $\mathscr{T}$. Then there is a canonical isomorphism*

$$\mathrm{K}_0(\mathscr{T}) \cong \mathrm{K}_0(\mathscr{C}).$$

As a consequence, if $\mathscr{C}$ is a finite-length category, there is a canonical isomorphism

$$\mathrm{K}_0(\mathscr{T}) \cong \mathbb{Z}[\mathrm{Irr}(\mathscr{C})].$$

A.10. Duality for rings of finite global dimension

It is well known that if M is a finite-dimensional vector space over a field $\Bbbk$, then the "evalution map"

$$\mathrm{ev} : M \to \mathrm{Hom}(\mathrm{Hom}(M, \Bbbk), \Bbbk)$$

given by $\mathrm{ev}(m)(f) = f(m)$ is an isomorphism. For modules over general commutative rings, this statement often fails. But if the ring is not too bad, it turns out that a version of this statement still holds at the derived level.

This will require some tools from commutative algebra. Let $\Bbbk$ be a noetherian commutative ring. For any $\Bbbk$-module M, let $\mathrm{ann}(M) \subset \Bbbk$ denote its annihilator. Recall that for any prime ideal $\mathfrak{p} \subset \Bbbk$, the localization functor $\Bbbk\text{-mod}^{\mathrm{fg}} \to \Bbbk_{\mathfrak{p}}\text{-mod}^{\mathrm{fg}}$ is exact ([**19**, Proposition 3.3]) and takes finitely generated projective modules to finitely generated projective modules ([**172**, Theorem 7.12]). For $M, N \in \Bbbk\text{-mod}^{\mathrm{fg}}$, there is a natural isomorphism

$$\mathrm{Hom}_{\Bbbk}(M, N)_{\mathfrak{p}} \cong \mathrm{Hom}_{\Bbbk_{\mathfrak{p}}}(M_{\mathfrak{p}}, N_{\mathfrak{p}}). \tag{A.10.1}$$

(See [**172**, Corollary 7.11].) Also for $M \in \Bbbk\text{-mod}^{\mathrm{fg}}$, there is a natural isomorphism

$$\mathrm{ann}(M_{\mathfrak{p}}) \cong \mathrm{ann}(M)_{\mathfrak{p}}. \tag{A.10.2}$$

(See [**19**, Proposition 3.14].) If M is nonzero, then there exists a prime ideal (and in fact a maximal ideal) $\mathfrak{p}$ such that $M_{\mathfrak{p}} \neq 0$ ([**19**, Proposition 3.8]).

LEMMA A.10.1. *Let $\Bbbk$ be a noetherian commutative ring. For any finitely generated projective $\Bbbk$-module M, the evaluation map* $\mathrm{ev} : M \to \mathrm{Hom}(\mathrm{Hom}(M, \Bbbk), \Bbbk)$ *is an isomorphism.*

PROOF SKETCH. In the special case where M is a free module, one can simply repeat the usual proof for vector spaces. The general case follows from the naturality of ev and the observation that any finitely generated projective module is a direct summand of a finite-rank free module. □

It is easy to see that if M is a finitely generated module over a noetherian commutative ring $\Bbbk$, then $\mathrm{Hom}(M, \Bbbk)$ is again finitely generated, so the functor $\mathbb{D}$ defined in the following theorem makes sense.

THEOREM A.10.2. *Let $\Bbbk$ be a noetherian commutative ring of finite global dimension, and let*

$$\mathbb{D} : D^{\mathrm{b}}(\Bbbk\text{-mod}^{\mathrm{fg}})^{\mathrm{op}} \to D^{\mathrm{b}}(\Bbbk\text{-mod}^{\mathrm{fg}})$$

be the functor given by $\mathbb{D}(X) = R\mathrm{Hom}(X, \Bbbk)$. For any $X \in D^{\mathrm{b}}(\Bbbk\text{-mod}^{\mathrm{fg}})$, there is a natural isomorphism

$$X \xrightarrow{\sim} \mathbb{D}(\mathbb{D}(X)).$$

PROOF. We first claim that if M is a finitely generated projective $\Bbbk$-module, then $\mathrm{Hom}(M, \Bbbk)$ is also projective. In the special case where M is free, this is clear. Since finitely generated projective modules are the same as direct summands of finite-rank free modules, our claim follows.

Now let $X \in D^{\mathrm{b}}(\Bbbk\text{-mod}^{\mathrm{fg}})$. To compute $\mathbb{D}(X)$, choose a quasi-isomorphism $q : M \to X$, where M is a bounded complex of finitely generated projective modules. Then $\mathbb{D}(X) \cong \mathrm{chHom}(M, \Bbbk)$. By the previous paragraph, this is again a chain complex of finitely generated projective modules, so we have $\mathbb{D}(\mathbb{D}(X)) \cong \mathrm{chHom}(\mathrm{chHom}(M, \Bbbk), \Bbbk)$. By Lemma A.10.1, the evaluation map

$$M \to \mathrm{chHom}(\mathrm{chHom}(M, \Bbbk), \Bbbk)$$

is an isomorphism of chain complexes, so we obtain our desired isomorphism $X \xrightarrow{\sim} \mathbb{D}(\mathbb{D}(X))$. □

When $\Bbbk$ is a field, $\mathbb{D}$ preserves the heart of the natural t-structure, but for general $\Bbbk$, $\mathbb{D}(\Bbbk\text{-mod}^{\mathrm{fg}})$ will be the heart of some new, nontrivial t-structure on $D^{\mathrm{b}}(\Bbbk\text{-mod}^{\mathrm{fg}})$. A natural problem is that of describing this t-structure. We will give a partial solution (describing the "negative" part of this t-structure) in Proposition A.10.6 below, in terms of the following notion.

Definition A.10.3. Let $\Bbbk$ be a noetherian commutative ring, and let M be a finitely generated $\Bbbk$-module. The **grade** of M is defined by

$$\operatorname{grade} M = \begin{cases} \operatorname{depth}(\operatorname{ann}(M), \Bbbk) & \text{if } M \neq 0, \\ \infty & \text{otherwise.} \end{cases}$$

Of course, if $\Bbbk$ is a field, then every nonzero finite-dimensional $\Bbbk$-vector space has grade 0. The following theorem gives another interpretation of the grade of a module. For a proof, see [**172**, Theorem 16.6] or [**72**, Proposition 18.4].

Theorem A.10.4 (Rees). *Let $\Bbbk$ be a noetherian commutative ring. Let M be a nonzero finitely generated $\Bbbk$-module, and let $k = \operatorname{grade} M$. Then $\operatorname{Ext}^i(M, \Bbbk) = 0$ for $i = 0, 1, \ldots, k-1$, and $\operatorname{Ext}^k(M, \Bbbk) \neq 0$.*

Corollary A.10.5. *Let $\Bbbk$ be a noetherian commutative ring, and let M be a nonzero finitely generated $\Bbbk$-module. Then $\operatorname{grade} M \leq \operatorname{gldim} \Bbbk$.*

Proposition A.10.6. *Let $\Bbbk$ be a noetherian commutative ring of finite global dimension. Then*

$$D^{\mathrm{b}}(\Bbbk\text{-mod}^{\mathrm{fg}})^{\geq 0} = \{X \in D^{\mathrm{b}}(\Bbbk\text{-mod}^{\mathrm{fg}}) \mid \textit{for all } i,\ \operatorname{grade} \mathsf{H}^i(\mathbb{D}(X)) \geq i\}.$$

Equivalently, we have

$$\mathbb{D}(D^{\mathrm{b}}(\Bbbk\text{-mod}^{\mathrm{fg}})^{\geq 0}) = \{X \in D^{\mathrm{b}}(\Bbbk\text{-mod}^{\mathrm{fg}}) \mid \textit{for all } i,\ \operatorname{grade} \mathsf{H}^i(X) \geq i\}.$$

The proof of this proposition will occupy the rest of this section. We will need a number of commutative algebra lemmas, starting with the following generalization of (A.10.1):

Lemma A.10.7. *Let $\Bbbk$ be a noetherian commutative ring, and let $\mathfrak{p} \subset \Bbbk$ be a prime ideal. For $X \in D^-(\Bbbk\text{-mod})$ and $Y \in D^+(\Bbbk\text{-mod})$, there is a natural isomorphism*

$$R\mathrm{Hom}(X, Y)_{\mathfrak{p}} \cong R\mathrm{Hom}(X_{\mathfrak{p}}, Y_{\mathfrak{p}}).$$

Proof. Choose a projective resolution $q : P \to X$. Then the left-hand side is given by $\mathrm{chHom}(P, Y)_{\mathfrak{p}}$. Since localization is exact and takes projectives to projectives, $q_{\mathfrak{p}} : P_{\mathfrak{p}} \to X_{\mathfrak{p}}$ is a projective resolution of $X_{\mathfrak{p}}$. The right-hand side is then given by $\mathrm{chHom}(P_{\mathfrak{p}}, Y_{\mathfrak{p}})$. Finally, (A.10.1) implies that there is a natural isomorphism $\mathrm{chHom}(P, Y)_{\mathfrak{p}} \cong \mathrm{chHom}(P_{\mathfrak{p}}, Y_{\mathfrak{p}})$. □

Serre's characterization of regular local rings ([**172**, Theorem 19.2]) says that a noetherian local ring is regular if and only if it has finite global dimension. This statement extends to nonlocal rings in the following way.

LEMMA A.10.8. *Let $\Bbbk$ be a noetherian ring of finite global dimension. For every prime ideal $\mathfrak{p} \subset \Bbbk$, the localization $\Bbbk_{\mathfrak{p}}$ is a regular local ring with* $\operatorname{gldim} \Bbbk_{\mathfrak{p}} = \operatorname{ht}(\mathfrak{p})$.

PROOF. By assumption, the $\Bbbk$-module $\Bbbk/\mathfrak{p}$ admits a finite projective resolution. Since localization is exact and takes projectives to projectives, it follows that $\Bbbk_{\mathfrak{p}}/\mathfrak{p}\Bbbk_{\mathfrak{p}}$ admits a finite projective resolution as a $\Bbbk_{\mathfrak{p}}$-module. It follows from [**172**, Section 19, Lemma 1] that $\Bbbk_{\mathfrak{p}}$ has finite global dimension, and then by Serre's theorem ([**172**, Theorem 19.2]) that $\Bbbk_{\mathfrak{p}}$ is regular, and that its global dimension is equal to its Krull dimension. The latter is the height of $\mathfrak{p}$. □

LEMMA A.10.9. *Let $\Bbbk$ be a noetherian commutative ring, and let M be a finitely generated $\Bbbk$-module. For any prime ideal $\mathfrak{p} \subset \Bbbk$, we have* $\operatorname{grade} M_{\mathfrak{p}} \geq \operatorname{grade} M$. *Moreover, there exists a maximal ideal $\mathfrak{p}$ such that* $\operatorname{grade} M_{\mathfrak{p}} = \operatorname{grade} M$.

PROOF. Using (A.10.2), we see that this statement is equivalent to the assertion that $\operatorname{depth}(\operatorname{ann}(M)_{\mathfrak{p}}, \Bbbk_{\mathfrak{p}}) \geq \operatorname{depth}(\operatorname{ann}(M), \Bbbk)$, with equality for some maximal ideal $\mathfrak{p}$. This holds by [**72**, Lemma 18.1]. □

LEMMA A.10.10. *Let $\Bbbk$ be a noetherian commutative ring of finite global dimension, and let M be a finitely generated $\Bbbk$-module. Then*

$$\operatorname{grade} M = \begin{cases} \inf\{\operatorname{ht}(\mathfrak{p}) : \mathfrak{p} \subset \Bbbk \textit{ a prime ideal with } M_{\mathfrak{p}} \neq 0\} & \textit{if } M \neq 0, \\ \infty & \textit{if } M = 0. \end{cases}$$

PROOF. Assume that $M \neq 0$. If $M_{\mathfrak{p}} \neq 0$, then by Lemma A.10.9, we have $\operatorname{grade} M \leq \operatorname{grade} M_{\mathfrak{p}}$. By [**72**, Proposition 18.2], $\operatorname{grade} M_{\mathfrak{p}}$ is bounded above by the Krull dimension of $\Bbbk_{\mathfrak{p}}$, which is the height of $\mathfrak{p}$. In other words, $\operatorname{grade} M \leq \operatorname{ht}(\mathfrak{p})$.

We must exhibit a prime ideal for which equality holds. Choose a maximal ideal $\mathfrak{m} \subset \Bbbk$ such that $\operatorname{grade} M_{\mathfrak{m}} = \operatorname{grade} M$. Since $\Bbbk_{\mathfrak{m}}$ is a regular local ring (Lemma A.10.8), by [**72**, Theorem 18.7], we have

$$\operatorname{depth}(\operatorname{ann}(M_{\mathfrak{m}}), \Bbbk_{\mathfrak{m}}) = \inf\{\operatorname{ht}(\mathfrak{q}) : \mathfrak{q} \subset \Bbbk_{\mathfrak{m}} \text{ a prime ideal containing } \operatorname{ann}(M_{\mathfrak{m}})\}.$$

Choose a prime ideal $\mathfrak{q} \subset \Bbbk_{\mathfrak{m}}$ such that $\operatorname{depth}(\operatorname{ann}(M_{\mathfrak{m}}), \Bbbk_{\mathfrak{m}}) = \operatorname{ht}(\mathfrak{q})$. Then $\mathfrak{q}$ is the localization of some prime ideal $\mathfrak{q}' \subset \mathfrak{m} \subset \Bbbk$. We conclude that $\operatorname{grade} M = \operatorname{ht}(\mathfrak{q}')$. Because $\mathfrak{q}$ contains $\operatorname{ann}(M_{\mathfrak{m}})$, we have $M_{\mathfrak{q}'} = (M_{\mathfrak{m}})_{\mathfrak{q}} \neq 0$. □

LEMMA A.10.11. *Let $\Bbbk$ be a noetherian commutative ring of finite global dimension, and let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence of finitely generated $\Bbbk$-modules. We have*

$$\operatorname{grade} M = \min\{\operatorname{grade} M', \operatorname{grade} M''\}.$$

PROOF. We have an exact sequence $0 \to M'_{\mathfrak{p}} \to M_{\mathfrak{p}} \to M''_{\mathfrak{p}} \to 0$ for each prime ideal $\mathfrak{p}$. If either $M'_{\mathfrak{p}}$ or $M''_{\mathfrak{p}}$ is nonzero, then $M_{\mathfrak{p}}$ is nonzero as well. If we choose $\mathfrak{p}$ to be such that $\operatorname{grade} M' = \operatorname{ht}(\mathfrak{p})$, then we see that $\operatorname{grade} M \leq \operatorname{grade} M'$. The same reasoning shows that $\operatorname{grade} M \leq \operatorname{grade} M''$, so $\operatorname{grade} M \leq \min\{\operatorname{grade} M', \operatorname{grade} M''\}$. If this inequality were strict, there would be a prime ideal $\mathfrak{p}$ with $M_{\mathfrak{p}} \neq 0$ and $\operatorname{grade} M = \operatorname{ht}(\mathfrak{p})$, but with $M'_{\mathfrak{p}} = M''_{\mathfrak{p}} = 0$, which is absurd. □

Corollary A.10.12. *Let $\Bbbk$ be a noetherian commutative ring of finite global dimension, and let $X' \to X \to X'' \to$ be a distinguished triangle in $D^{\mathrm{b}}(\Bbbk\text{-mod}^{\mathrm{fg}})$. Suppose there is an integer n such that*

$$\operatorname{grade} \mathsf{H}^k(X') \geq k+n \qquad \textit{and} \qquad \operatorname{grade} \mathsf{H}^k(X'') \geq k+n$$

for all k. Then $\operatorname{grade} \mathsf{H}^k(X) \geq k+n$ for all k as well.

Proof. Consider the exact sequence

$$\mathsf{H}^k(X') \xrightarrow{r} \mathsf{H}^k(X) \xrightarrow{s} \mathsf{H}^k(X'').$$

Let $M \subset \mathsf{H}^k(X)$ be the image of r (or the kernel of s), and let $N \subset \mathsf{H}^k(X'')$ be the image of s. Since M is a quotient of $\mathsf{H}^k(X')$, and N is a submodule of $\mathsf{H}^k(X'')$, Lemma A.10.11 implies that $\operatorname{grade} M \geq k+n$ and $\operatorname{grade} N \geq k+n$. Apply that same lemma to the short exact sequence $0 \to M \to \mathsf{H}^k(X) \to N \to 0$ to conclude that $\operatorname{grade} \mathsf{H}^k(X) \geq k+n$. $\square$

Proof of Proposition A.10.6. For brevity, let $D^{\geq 0} = D^{\mathrm{b}}(\Bbbk\text{-mod}^{\mathrm{fg}})^{\geq 0}$ and

$$D' = \{X \in D^{\mathrm{b}}(\Bbbk\text{-mod}^{\mathrm{fg}}) \mid \text{for all } i, \operatorname{grade} \mathsf{H}^i(X) \geq i\}.$$

It is enough to show that $\mathbb{D}(D^{\geq 0}) \subset D'$ and $\mathbb{D}(D') \subset D^{\geq 0}$. This is achieved in Steps 2 and 3 below.

Step 1. For any $M \in \Bbbk\text{-mod}^{\mathrm{fg}}$, we have $\operatorname{grade} \operatorname{Ext}^k_{\Bbbk}(M, \Bbbk) \geq k$. Let $\mathfrak{p} \subset \Bbbk$ be a prime ideal such that $\operatorname{Ext}^k_{\Bbbk}(M, \Bbbk)_{\mathfrak{p}} \neq 0$. By Lemma A.10.10, it is enough to show that $\operatorname{ht}(\mathfrak{p}) \geq k$. By Lemma A.10.7 and Lemma A.10.8, we have $\operatorname{Ext}^k_{\Bbbk_{\mathfrak{p}}}(M_{\mathfrak{p}}, \Bbbk_{\mathfrak{p}}) \neq 0$, which implies that $k \leq \operatorname{gldim} \Bbbk_{\mathfrak{p}} = \operatorname{ht}(\mathfrak{p})$, as desired.

Step 2. If $X \in D^{\geq 0}$, then $\mathbb{D}(X) \in D'$. We proceed by induction on the largest integer i such that $\mathsf{H}^i(X) \neq 0$. If $i = 0$, then $X \in \Bbbk\text{-mod}^{\mathrm{fg}}$, and Step 1 tells us that $\mathbb{D}(X) \in D'$. Otherwise, apply $\mathbb{D}$ to a truncation distinguished triangle to obtain

$$\mathbb{D}(\tau^{\geq i} X) \to \mathbb{D}(X) \to \mathbb{D}(\tau^{\leq i-1} X) \to .$$

The last term is in D' by induction. Corollary A.10.12 implies that D' is stable under extensions, so it is enough to prove that $\mathbb{D}(\tau^{\geq i} X) \in D'$. Our assumptions imply that $\tau^{\geq i} X \cong \mathsf{H}^i(X)[-i]$, so $\mathbb{D}(\tau^{\geq i} X) \cong \mathbb{D}(\mathsf{H}^i(X))[i]$. We have already seen that $\mathbb{D}(\mathsf{H}^i(X)) \in D'$. It is clear from the definition of D' that it is preserved by $[i]$ for $i \geq 0$, so $\mathbb{D}(\tau^{\geq i} X) \in D'$, as desired.

Step 3. If $X \in D'$, then $\mathbb{D}(X) \in D^{\geq 0}$. Again, let i be the largest integer such that $\mathsf{H}^i(X) \neq 0$. If $i \leq 0$, then $X \in D^{\mathrm{b}}(\Bbbk\text{-mod}^{\mathrm{fg}})^{\leq 0}$, and it is clear (from Proposition A.6.13) that $\mathbb{D}(X) \in D^{\geq 0}$. Otherwise, apply $\mathbb{D}$ to a suitable truncation distinguished triangle to obtain

$$\mathbb{D}(\tau^{\geq i} X) \to \mathbb{D}(X) \to \mathbb{D}(\tau^{\leq i-1} X) \to .$$

The last term is in $D^{\geq 0}$ by induction. It is enough to prove that the first term is as well. As above, we have $\tau^{\geq i} X \cong \mathsf{H}^i(X)[-i]$. Since $\operatorname{grade} \mathsf{H}^i(X) \geq i$, Theorem A.10.4 tells us that

$$\mathsf{H}^k(R\mathrm{Hom}(\mathsf{H}^i(X), \Bbbk)) \cong \operatorname{Ext}^k(\mathsf{H}^i(X), \Bbbk) = 0 \qquad \text{if } k < i.$$

In other words, $\mathbb{D}(\mathsf{H}^i(X)) \in D^{\mathrm{b}}(\Bbbk\text{-mod}^{\mathrm{fg}})^{\geq i}$, so $\mathbb{D}(\mathsf{H}^i(X)[-i]) \cong \mathbb{D}(\mathsf{H}^i(X))[i]$ belongs to $D^{\geq 0}$, as desired. $\square$

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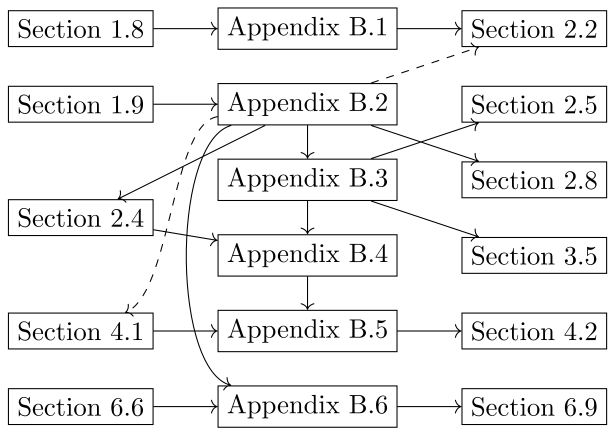
APPENDIX B

Calculations on $\mathbb{C}^n$

This appendix contains a number of results on sheaves and sheaf functors on (subsets of) $\mathbb{C}^n$. In addition to being illustrative examples in their own right, these results are needed for various proofs in the main body of the text. These logical relationships are summarized in the following figure.



Here, a solid arrow indicates that results in the "source" section are invoked in proofs in the "target" section, while a dotted arrow means that they are invoked only in examples.

B.1. Holomorphic maps and cohomology

According to Remark 1.3.8, any open embedding $j : U \hookrightarrow X$ gives rise to an induced map $j_\sharp : \mathbf{H}^k_c(U;\Bbbk) \to \mathbf{H}^k_c(X;\Bbbk)$. The goal of this section is to study the map $j_\sharp$ (or a "relative" variant) when $X = \mathbb{C}^n$, and j is holomorphic.

Lemma B.1.1. *For any complex linear automorphism $f : \mathbb{C}^n \to \mathbb{C}^n$, the induced map $f_\sharp : \mathbf{H}^{2n}_c(\mathbb{C}^n;\Bbbk) \to \mathbf{H}^{2n}_c(\mathbb{C}^n;\Bbbk)$ is the identity map.*

Proof. We rely on the well-known fact that the topological group $\mathrm{GL}(n,\mathbb{C})$ is path-connected. Choose a path $\gamma : [0,1] \to \mathrm{GL}(n,\mathbb{C})$ such that $\gamma(0) = \mathrm{id}$ and $\gamma(1) = f$. Let $g : [0,1] \times \mathbb{C}^n \to \mathbb{C}^n$ be the map $g(t,x) = \gamma(t) \cdot x$. Then g is a homotopy between f and the identity map that satisfies the assumptions of Lemma 1.8.12. That lemma implies that $f_\sharp$ is the identity map. $\square$

Lemma B.1.2. *Let $j : U \hookrightarrow \mathbb{C}^n$ be the inclusion of an open subset that is homeomorphic to $\mathbb{C}^n$. Then the map $j_\sharp : \mathbf{H}^{2n}_c(U;\Bbbk) \to \mathbf{H}^{2n}_c(\mathbb{C}^n;\Bbbk)$ is an isomorphism.*

Proof. Consider first the special case where U is an open ball with respect to the Euclidean metric on $\mathbb{C}^n$. It is an elementary exercise to explicitly write down a homotopy $f : [0,1] \times U \to \mathbb{C}^n$ between j and some homeomorphism $U \xrightarrow{\sim} \mathbb{C}^n$ that, in addition, satisfies the hypotheses of Lemma 1.8.12. That lemma then implies that $j_\sharp$ is an isomorphism.

We now return to the general case: let U be any open subset that is homeomorphic to $\mathbb{C}^n$. Choose an open ball $U' \subset U$ with respect to the Euclidean metric. Let $u : U' \hookrightarrow U$ be the inclusion map, and consider the maps

$$\mathbf{H}^{2n}_{\mathrm{c}}(U';\Bbbk) \xrightarrow{u_\sharp} \mathbf{H}^{2n}_{\mathrm{c}}(U;\Bbbk) \xrightarrow{j_\sharp} \mathbf{H}^{2n}_{\mathrm{c}}(\mathbb{C}^n;\Bbbk).$$

By the special case considered above, the composition $j_\sharp \circ u_\sharp$ is an isomorphism, and hence $j_\sharp$ is at least surjective. On the other hand, $\mathbf{H}^{2n}_{\mathrm{c}}(U;\Bbbk)$ and $\mathbf{H}^{2n}_{\mathrm{c}}(\mathbb{C}^n;\Bbbk)$ are both free $\Bbbk$-modules of rank 1 (by Exercise 1.8.3). Any surjective map of free $\Bbbk$-modules of rank 1 is an isomorphism. ☐

Lemma B.1.3. *Let $j : U \hookrightarrow \mathbb{C}^n$ be the inclusion of an open subset that is homeomorphic to $\mathbb{C}^n$, and let $f : U \to \mathbb{C}^n$ be a holomorphic map that restricts to a biholomorphic equivalence $U \xrightarrow{\sim} f(U)$. The induced maps*

$$j_\sharp, f_\sharp : \mathbf{H}^{2n}_{\mathrm{c}}(U;\Bbbk) \to \mathbf{H}^{2n}_{\mathrm{c}}(\mathbb{C}^n;\Bbbk)$$

are isomorphisms and are equal to one another.

Proof. Choose a point $x_0 \in U$, and let $y_0 = f(x_0)$. Consider the differential df_{x_0}: this is a linear map $\mathbb{C}^n \to \mathbb{C}^n$, and it is invertible because f restricts to a biholomorphism. Let $g : \mathbb{C} \times U \to \mathbb{C}^n$ be the holomorphic map $g(t,x) = (1-t)df_{x_0}(x) + tf(x)$. For any $t \in \mathbb{C}$, we let $g_t = g(t,-) : U \to \mathbb{C}^n$, and we define $\tilde{g} : \mathbb{C} \times U \to \mathbb{C} \times \mathbb{C}^n$ to be the map $\tilde{g}(t,x) = (t, g(t,x))$. Its differential $d\tilde{g}$ at a point $(t,x) \in \mathbb{C} \times U$ is the linear map $\mathbb{C} \times \mathbb{C}^n \to \mathbb{C} \times \mathbb{C}^n$ given by

$$d\tilde{g}_{t,x} = \begin{bmatrix} 1 & 0 \\ -df_{x_0}(x) + f(x) & (1-t)df_{x_0}(x) + tdf_x(x) \end{bmatrix}.$$

In particular, $\det d\tilde{g}_{t,x_0} = \det df_{x_0} \neq 0$, so by the inverse function theorem, there is a neighborhood V of (t,x_0) on which $\tilde{g}$ restricts to a biholomorphic map. Now let us take $t \in \mathbb{R}$. Then one can find an interval $[a,b] \subset \mathbb{R}$ containing t and an open set $B \subset U$ containing x_0 such that $[a,b] \times B \subset V$. The map $\tilde{g}$ restricts to an open embedding $[a,b] \times B \to [a,b] \times \mathbb{C}^n$.

By repeating this construction several times, one can find a collection of sets

$$[a_1,b_1] \times B_1, \quad [a_2,b_2] \times B_2, \quad \ldots, \quad [a_k,b_k] \times B_k$$

such that the union of the intervals $[a_i,b_i]$ covers $[0,1]$. Let $B \subset B_1 \cap \cdots \cap B_k$ be a Euclidean ball containing x_0. Then $\tilde{g}$ restricts to an open embedding $[0,1] \times B \to [0,1] \times \mathbb{C}^n$. We can now apply Lemma 1.8.12 to conclude that $(g_t)_\sharp : \mathbf{H}^{2n}_{\mathrm{c}}(B;\Bbbk) \to \mathbf{H}^{2n}_{\mathrm{c}}(\mathbb{C}^n;\Bbbk)$ is independent of t for $t \in [0,1]$. Let $h : B \to U$ be the inclusion map, and consider the following diagram:

$$\begin{tikzcd}
\mathbf{H}^{2n}_{\mathrm{c}}(B;\Bbbk) \arrow[r, "h_\sharp"] \arrow[rrr, bend left, "(g_0)_\sharp = (df_{x_0})_\sharp j_\sharp h_\sharp"] \arrow[rrr, bend right, "(g_1)_\sharp = f_\sharp h_\sharp"'] & \mathbf{H}^{2n}_{\mathrm{c}}(U;\Bbbk) \arrow[r, "j_\sharp"] \arrow[rr, bend right, "f_\sharp"'] & \mathbf{H}^{2n}_{\mathrm{c}}(\mathbb{C}^n;\Bbbk) \arrow[r, "(df_{x_0})_\sharp"] & \mathbf{H}^{2n}_{\mathrm{c}}(\mathbb{C}^n;\Bbbk)
\end{tikzcd}$$

Each cohomology group in this picture is a free $\Bbbk$-module of rank one. Lemma B.1.2 tells us that $j_\sharp$ and $j_\sharp \circ h_\sharp$ are both isomorphisms, so $h_\sharp$ is itself an isomorphism. Therefore, the fact that $(g_0)_\sharp = (g_1)_\sharp$ implies that $f_\sharp = (df_{x_0})_\sharp j_\sharp$. Finally, df_{x_0} is a linear map, so by Lemma B.1.1, $(df_{x_0})_\sharp$ is the identity map. We conclude that $f_\sharp = j_\sharp$, as desired. $\square$

The next lemma is a "relative" version of Lemma B.1.3.

LEMMA B.1.4. *Let Y be complex analytic set, and let $k : V \hookrightarrow Y$ be the inclusion of an open subset. Let $U \subset \mathbb{C}^n$ be an open subset. Suppose $f : V \times U \to Y \times \mathbb{C}^n$ is a holomorphic map that restricts to a biholomorphism with its image and that commutes with projection to the first factor. In other words, the diagram*

$$\begin{tikzcd}
V \times U \arrow[r, "f"] \arrow[d, "\mathrm{pr}_1"'] & Y \times \mathbb{C}^n \arrow[d, "\mathrm{pr}_1"] \\
V \arrow[r, "k"] & Y
\end{tikzcd}$$

commutes. On the other hand, let $j : V \times U \to Y \times \mathbb{C}^n$ be the inclusion map. The induced maps of cohomology sheaves

$$j_\sharp, f_\sharp : \mathsf{H}^{2n}(\mathrm{pr}_{1!}\underline{\Bbbk}_{V\times U}) \to \mathsf{H}^{2n}(k^!\mathrm{pr}_{1!}\underline{\Bbbk}_{Y\times\mathbb{C}^n})$$

are isomorphisms and are equal to one another.

PROOF. It is enough to show that $j_\sharp$ and $f_\sharp$ induce equal isomorphisms on the stalk at any point $y \in V$. Consider the diagram

$$\begin{tikzcd}
& V \times U \arrow[rr, "f"] \arrow[dd, "\mathrm{pr}_1"' near start] & & Y \times \mathbb{C}^n \arrow[dd, "\mathrm{pr}_1"] \\
\{y\} \times U \arrow[ur] \arrow[rr, "f_y"' near start] \arrow[dd, "\mathrm{a}_U"'] & & \{y\} \times \mathbb{C}^n \arrow[ur] \arrow[dd, "\mathrm{a}_{\mathbb{C}^n}" near start] & \\
& V \arrow[rr, "k" near end] & & Y \\
\{y\} \arrow[ur] \arrow[rr, equal] & & \{y\} \arrow[ur] &
\end{tikzcd}$$

By several applications of the proper base change theorem, we obtain the following commutative diagram of $\Bbbk$-modules:

$$\begin{tikzcd}
\mathsf{H}^{2n}((\mathrm{pr}_{1!}\underline{\Bbbk}_{V\times U})_y) \arrow[r, "(f_\sharp)_y"] \arrow[d, "\wr"'] & \mathsf{H}^{2n}((k^!\mathrm{pr}_{1!}\underline{\Bbbk}_{Y\times\mathbb{C}^n})_y) \arrow[d, "\wr"] \\
\mathsf{H}^{2n}(\mathrm{a}_{U!}(\underline{\Bbbk}_{V\times U}|_{\{y\}\times U})) \arrow[r] \arrow[d, "\wr"'] & \mathsf{H}^{2n}(\mathrm{a}_{\mathbb{C}^n!}(\underline{\Bbbk}_{Y\times\mathbb{C}^n}|_{\{y\}\times\mathbb{C}^n})) \arrow[d, "\wr"] \\
\mathbf{H}^{2n}_{\mathrm{c}}(U;\Bbbk) \arrow[r, "(f_y)_\sharp"] & \mathbf{H}^{2n}_{\mathrm{c}}(\mathbb{C}^n;\Bbbk)
\end{tikzcd}$$

By Lemma B.1.3, the maps $(j_y)_\sharp$ and $(f_y)_\sharp$ are isomorphisms and are equal to one another, so the same holds for $(f_\sharp)_y$ and $(j_\sharp)_y$. $\square$

B.2. Constructible sheaves on $\mathbb{C}^n$, I

In this section, we develop a vast generalization of Exercise 1.10.3. Let k and n be integers with $1 \le k \le n$, and let $U \subset \mathbb{C}^n$ be the open set

$$U = \{(x_1, \ldots, x_n) \in \mathbb{C}^n \mid x_i \neq 0 \text{ if } 1 \le i \le k\} = (\mathbb{C}^\times)^k \times \mathbb{C}^{n-k}.$$

Let $j : U \hookrightarrow \mathbb{C}^n$ be the inclusion map. The goal of this section is to give an "algebraic" description of $j_*\mathcal{L}$, where $\mathcal{L}$ is a local system on U. The answer will be in terms of local systems on certain subsets $X_I \subset \mathbb{C}^n$ (defined below).

Notation. For $1 \le i \le k$, let

$$Z_i = \{(x_1, \ldots, x_n) \in \mathbb{C}^n \mid x_i = 0\}.$$

We also let $Z = Z_1 \cup \cdots \cup Z_k$. This is the complement of U. For any subset $I \subset \{1, \ldots, k\}$, let

$$X_I = \left\{ (x_1, \ldots, x_n) \in \mathbb{C}^n \;\middle|\; \begin{matrix} x_i = 0 \text{ if } 1 \le i \le k \text{ and } i \in I, \text{ and} \\ x_i \ne 0 \text{ if } 1 \le i \le k \text{ and } i \notin I. \end{matrix} \right\}.$$

There is no condition on x_i when $k < i \le n$. The subsets $\{X_I\}$ are called **strata** (cf. Section 2.3). The unique closed stratum $X_{\{1,\ldots,k\}}$ is contractible. We set

$$X_{\subset I} = \bigcup_{K \subset I} X_K = \{(x_1, \ldots, x_n) \mid x_i \ne 0 \text{ if } 1 \le i \le k \text{ and } i \notin I\}.$$

Note that $U = X_\varnothing = X_{\subset\varnothing}$. For each $I \subset \{1, \ldots, k\}$, we let

$$j_I : X_I \hookrightarrow \mathbb{C}^n, \qquad h_I : U \hookrightarrow X_{\subset I}, \qquad i_I : X_I \hookrightarrow X_{\subset I}$$

be the inclusion maps. Let $p_I : X_{\subset I} \to X_I$ be the map given by

$$\text{(B.2.1)} \qquad p_I(x_1, \ldots, x_n) = (y_1, \ldots, y_n) \qquad \text{where} \qquad y_i = \begin{cases} 0 & \text{if } i \in I, \\ x_i & \text{if } i \notin I. \end{cases}$$

Note that $p_I \circ i_I = \mathrm{id}_{X_I}$. It will be convenient to introduce the notation $\dot{p}_I = p_I \circ h_I : U \to X_I$.

Choose a base point $u_0 \in U$, and let $R = \Bbbk[\pi_1(U, u_0)]$. More generally, let $R_I = \Bbbk[\pi_1(X_I, p_I(u_0))]$. The map $\dot{p}_I$ induces a group homomorphism $\pi_1(U, u_0) \to \pi_1(X_I, p_I(u_0))$, and hence a ring homomorphism $R \to R_I$. In this way, we can regard R_I as an R-algebra.

To make this a bit more explicit, recall that the fundamental group $\pi_1(U, u_0)$ is a free abelian group on generators $T_1, \ldots, T_k$, where T_i is a loop based at u_0 that goes around the hyperplane $x_i = 0$. We therefore have

$$R \cong \Bbbk[T_1^{\pm 1}, \ldots, T_k^{\pm 1}].$$

The path $\dot{p}_I \circ T_i$ is a loop in X_I that still goes around that hyperplane if $i \notin I$, but is null-homotopic if $i \in I$. In other words,

$$R_I \cong \Bbbk[T_1^{\pm 1}, \ldots, T_k^{\pm 1}]/(\{T_i - 1 \mid i \in I\}).$$

Since each X_I has a contractible universal cover, Theorems 1.7.9 and 1.9.7 give us the following equivalences of categories:

$$\text{(B.2.2)} \qquad D^{\mathrm{b}}_{\mathrm{loc}}(X_I, \Bbbk) \xleftarrow{\sim} D^{\mathrm{b}}\mathrm{Loc}(X_I, \Bbbk) \xrightarrow{\sim} D^{\mathrm{b}}(R_I\text{-mod}).$$

Remark B.2.1. Given a point $x = (x_1, \ldots, x_n) \in X_I$, define a neighborhood $D = D_1 \times \cdots \times D_n$ of x by setting

$$D_i = \begin{cases} \mathbb{C} & \text{if } i \in I, \text{ or if } k < i \le n, \\ \text{a disc in } \mathbb{C} \text{ around } x_i \text{ not containing } 0 & \text{if } 1 \le i \le k \text{ and } i \notin I. \end{cases}$$

Then D is contained in $X_{\subset I}$ and homeomorphic to $\mathbb{C}^n$. Under a suitable identification $D \cong \mathbb{C}^n$, we have that $Z_i \cap D$ is a coordinate hyperplane if $i \in I$, and empty if $i \notin I$. In other words, the space D together with the collection of coordinate hyperplanes $\{Z_i \cap D\}_{i\in I}$ matches the setup at the beginning of this section (but with the integer k replaced by $|I|$). The sets $X_J \cap D$ with $J \subset I$ are the strata of D, and $X_I \cap D$ is the unique closed stratum. This observation can sometimes be used to reduce a claim about all strata X_I to the case of the closed stratum.

Stalks of push-forwards. We now aim to describe objects of the form $j_I^* j_* \mathcal{F}$.

LEMMA B.2.2. *For any $\mathcal{F} \in D^+_{\mathrm{loc}}(U, \Bbbk)$ and any $I \subset \{1, \dots, k\}$, there is a natural isomorphism $\dot{p}_{I*}\mathcal{F} \xrightarrow{\sim} j_I^* j_* \mathcal{F}$.*

PROOF. Let $u : X_{\subset I} \hookrightarrow \mathbb{C}^n$ be the inclusion map so that $j = u \circ h_I$ and $j_I = u \circ i_I$. We therefore have natural isomorphisms

$$i_I^* h_{I*}\mathcal{F} \cong i_I^* u^* u_* h_{I*}\mathcal{F} \cong j_I^* j_* \mathcal{F}. \tag{B.2.3}$$

We define a natural map $\dot{p}_{I*}\mathcal{F} \to j_I^* j_* \mathcal{F}$ by composing (B.2.3) with the following sequence of maps:

$$\dot{p}_{I*}\mathcal{F} \xrightarrow{\sim} p_{I*} h_{I*}\mathcal{F} \to p_{I*} i_{I*} i_I^* h_{I*}\mathcal{F} \xrightarrow{\sim} i_I^* h_{I*}\mathcal{F}. \tag{B.2.4}$$

We must show that (B.2.4) is an isomorphism.

Step 1. The case where $k = n$ and $I = \{1, \dots, n\}$. In this case, $X_I = \{0\}$, $\dot{p}_I$ can be identified with $\mathrm{a}_U : U \to \mathrm{pt}$, and the map (B.2.4) can be rewritten as

$$R\Gamma(\mathcal{F}) \to (j_*\mathcal{F})_0. \tag{B.2.5}$$

By Lemma 1.2.10, we have

$$\mathsf{H}^k((j_*\mathcal{F})_0) \cong \varinjlim_{V \ni 0} \mathbf{H}^k(V, (j_*\mathcal{F})|_V).$$

Let j_V be the inclusion map $V \cap U \to V$. By Proposition 1.2.16, we have

$$\mathsf{H}^k((j_*\mathcal{F})_0) \cong \varinjlim_{V \ni 0} \mathbf{H}^k(V, j_{V*}(\mathcal{F}|_{V\cap U})) \cong \varinjlim_{V \ni 0} \mathbf{H}^k(V \cap U, \mathcal{F}|_{V\cap U}). \tag{B.2.6}$$

Now, suppose V is a small ball around 0 with respect to the Euclidean metric on $\mathbb{C}^n$. Then the inclusion $V \cap U \hookrightarrow U$ is a homotopy equivalence. Since $\underline{\Bbbk}_U$ and $\mathcal{F}$ both lie in $D^+_{\mathrm{loc}}(U, \Bbbk)$, Theorem 1.8.10 tells us that restriction induces an isomorphism $\mathrm{Hom}(\underline{\Bbbk}_U, \mathcal{F}[k]) \xrightarrow{\sim} \mathrm{Hom}(\underline{\Bbbk}_{V\cap U}, \mathcal{F}|_{V\cap U}[k])$, and hence an isomorphism

$$\mathbf{H}^k(U, \mathcal{F}) \to \mathbf{H}^k(V \cap U, \mathcal{F}|_{V\cap U}).$$

Since the point $0 \in \mathbb{C}^r$ has a basis of neighborhoods consisting of such Euclidean balls, we conclude that there is a natural isomorphism

$$\mathbf{H}^k(U, \mathcal{F}) \xrightarrow{\sim} \mathsf{H}^k((j_*\mathcal{F})_0),$$

and hence an isomorphism $R\Gamma(\mathcal{F}) \to (j_*\mathcal{F})_0$, as desired.

Step 2. The general case. It is enough to show that the maps in (B.2.4) induce isomorphisms on stalks at any point $y \in X_I$. Let $U_y = \dot{p}_I^{-1}(y)$ and $X_y = p_I^{-1}(y)$, and consider the following two diagrams of cartesian squares:

$$\begin{tikzcd}
\{y\} \arrow[r, "i_y"'] \arrow[d, hook] \arrow[rr, bend left, "\mathrm{id}"] & X_y \arrow[r, "\mathrm{a}_{X_y}"'] \arrow[d] & \{y\} \arrow[d] \\
X_I \arrow[r, "i_I"] \arrow[rr, bend right, "\mathrm{id}"'] & X_{\subset I} \arrow[r, "p_I"] & X_I
\end{tikzcd}
\qquad
\begin{tikzcd}
U_y \arrow[r, "j_y"'] \arrow[d, hook] \arrow[rr, bend left, "\mathrm{a}_{U_y}"] & X_y \arrow[r, "\mathrm{a}_{X_y}"'] \arrow[d] & \{y\} \arrow[d] \\
U \arrow[r, "h_I"] \arrow[rr, bend right, "\dot{p}_I"'] & X_{\subset I} \arrow[r, "p_I"] & X_I
\end{tikzcd}$$

Using these, we build the following diagram in $D^+(\mathrm{pt}, \Bbbk)$, in which the various maps come from adjunction, base change, or composition:

$$\begin{tikzcd}[column sep=small]
(\dot{p}_{I*}\mathcal{F})_y \arrow[r, "\text{comp.}", "\sim"'] \arrow[ddd, "\text{b.ch.}"', "\wr"] & (p_{I*}h_{I*}\mathcal{F})_y \arrow[r, "\text{adj.}"] \arrow[d, "\text{b.ch.}"] & (p_{I*}i_{I*}i_I^*h_{I*}\mathcal{F})_y \arrow[r, "\text{comp.}", "\sim"'] \arrow[d, "\text{b.ch.}"] \arrow[dr, phantom, "\text{Prop. 1.6.3}"] & (i_I^*h_{I*}\mathcal{F})_y \arrow[d, "\text{comp.}", "\wr"'] \\
{} \arrow[r, phantom, "\text{Prop. 1.6.3}"] & R\Gamma((h_{I*}\mathcal{F})|_{X_y}) \arrow[r, "\text{adj.}"] \arrow[dd, "\text{b.ch.}"] \arrow[dr, "\text{adj.}"'] & R\Gamma((i_{I*}i_I^*h_{I*}\mathcal{F})|_{X_y}) \arrow[r, "\text{b.ch.}", "\sim"'] \arrow[d, phantom, "\text{Prop. 1.6.4}"] & R\Gamma(i_{y*}((i_I^*h_{I*}\mathcal{F})_y)) \arrow[dl, "\sim"', "\text{comp.}"] \\
 & & R\Gamma(i_{y*}i_y^*((h_{I*}\mathcal{F})|_{X_y})) \arrow[d, "\text{b.ch.}", "\wr"'] & \\
R\Gamma(\mathcal{F}|_{U_y}) \arrow[r, "\text{comp.}", "\sim"'] & R\Gamma(j_{y*}(\mathcal{F}|_{U_y})) \arrow[r, "\text{adj.}"] & R\Gamma(i_{y*}i_y^*j_{y*}(\mathcal{F}|_{U_y})) \arrow[r, "\text{comp.}", "\sim"'] & i_y^*j_{y*}(\mathcal{F}|_{U_y})
\end{tikzcd}$$

Here, each subdiagram commutes either by naturality or by the indicated result from Section 1.6. For now, many of the base change and adjunction maps in this diagram are not known to be isomorphisms. However, because U can be identified with $U_y \times X_I$, the leftmost vertical arrow is an isomorphism by Theorem 1.9.5. Similar reasoning applies to the bottom right vertical arrow, after identifying $X_{\subset I}$ with $X_y \times X_I$.

Let $r = |I|$. There is an obvious identification of X_y with $\mathbb{C}^r$, and this identification sends U_y to $(\mathbb{C}^\times)^r$ and y to 0. With these identifications, we see that the composition of the maps along the bottom of the large commutative diagram above is an instance of the map (B.2.5) treated in Step 1. It follows that the composition of the maps along the top of the diagram is also an isomorphism, and hence that (B.2.4) is an isomorphism. $\square$

The main result of this section is the following.

PROPOSITION B.2.3. *For any $\mathcal{F} \in D^{\mathrm{b}}_{\mathrm{loc}}(U, \Bbbk)$ and any $I \subset \{1, \ldots, k\}$, the object $j_I^* j_* \mathcal{F}$ lies in $D^{\mathrm{b}}_{\mathrm{loc}}(X_I, \Bbbk)$. Moreover, there is a commutative diagram*

$$\begin{tikzcd}
D^{\mathrm{b}}_{\mathrm{loc}}(U, \Bbbk) \arrow[r, "\sim"] \arrow[d, "\mathcal{F} \mapsto j_I^* j_* \mathcal{F}"'] & D^{\mathrm{b}}(R\text{-mod}) \arrow[d, "R\mathrm{Hom}_R(R_I, -)"] \\
D^{\mathrm{b}}_{\mathrm{loc}}(X_I, \Bbbk) \arrow[r, "\sim"] & D^{\mathrm{b}}(R_I\text{-mod})
\end{tikzcd} \tag{B.2.7}$$

PROOF. By Lemma B.2.2, we may instead study the functor $\mathcal{F} \mapsto \dot{p}_{I*}\mathcal{F}$.

Step 1. The functor $R\mathrm{Hom}_R(R_I, -) : D^+(R\text{-mod}) \to D^+(R_I\text{-mod})$ *sends objects in* $D^{\mathrm{b}}(R\text{-mod})$ *to* $D^{\mathrm{b}}(R_I\text{-mod})$. This follows from the fact that as an R-module, R_I admits a free resolution of finite length: see, for instance, [**72**, Corollary 17.5].

Step 2. The functor $\dot{p}_I^* : D^{\mathrm{b}}_{\mathrm{loc}}(X_I, \Bbbk) \to D^{\mathrm{b}}_{\mathrm{loc}}(U, \Bbbk)$ *admits a right adjoint* $F : D^{\mathrm{b}}_{\mathrm{loc}}(U, \Bbbk) \to D^{\mathrm{b}}_{\mathrm{loc}}(X_I, \Bbbk)$. By Proposition 1.7.10 combined with the equivalences in (B.2.2), we have a commutative diagram

$$\begin{tikzcd}
D^{\mathrm{b}}_{\mathrm{loc}}(X_I, \Bbbk) \arrow[r, "\sim"] \arrow[d, "\dot{p}_I^*"'] & D^{\mathrm{b}}(R_I\text{-mod}) \arrow[d, "\mathrm{Res}_R^{R_I}"] \\
D^{\mathrm{b}}_{\mathrm{loc}}(U, \Bbbk) \arrow[r, "\sim"] & D^{\mathrm{b}}(R\text{-mod})
\end{tikzcd} \tag{B.2.8}$$

As a functor of abelian categories, $\mathrm{Res}_R^{R_I}$ is left adjoint to $\mathrm{Hom}_R(R_I, -)$. Using Step 1, a routine argument shows that at the level of bounded derived categories, $\mathrm{Res}_R^{R_I}$ is left adjoint to $R\mathrm{Hom}_R(R_I, -)$. Via the horizontal equivalences in (B.2.8), $R\mathrm{Hom}_R(R_I, -)$ corresponds to some functor $F : D^{\mathrm{b}}_{\mathrm{loc}}(U, \Bbbk) \to D^{\mathrm{b}}_{\mathrm{loc}}(X_I, \Bbbk)$.

By construction, there is a commutative diagram similar to (B.2.7), but in which the left-hand vertical arrow is replaced by F. Thus, the following claim completes the proof.

Step 3. For $\mathcal{F} \in D^{\mathrm{b}}_{\mathrm{loc}}(U, \Bbbk)$, there is a natural isomorphism $F(\mathcal{F}) \cong \dot{p}_{I*}\mathcal{F}$. Theorem 1.9.5 at least tells us that $\dot{p}_{I*}\mathcal{F} \in D^{+}_{\mathrm{loc}}(X_I, \Bbbk)$. For $\mathcal{G} \in D^{\mathrm{b}}_{\mathrm{loc}}(X_I, \Bbbk)$, we have natural isomorphisms

$$\mathrm{Hom}(\mathcal{G}, F(\mathcal{F})) \cong \mathrm{Hom}(\dot{p}_I^*\mathcal{G}, \mathcal{F}) \cong \mathrm{Hom}(\mathcal{G}, \dot{p}_{I*}\mathcal{F}).$$

If we already knew that $\dot{p}_{I*}\mathcal{F}$ lay in $D^{\mathrm{b}}_{\mathrm{loc}}(X_I, \Bbbk)$, then we would be done by Yoneda's lemma. Instead, we have to work a little bit harder. Let n be an integer such that $F(\mathcal{F}) \in D^{\mathrm{b}}_{\mathrm{loc}}(X_I, \Bbbk)^{\leq n}$. For $\mathcal{G} \in D^{\mathrm{b}}_{\mathrm{loc}}(X_I, \Bbbk)^{\leq n}$, we have natural isomorphisms

$$\mathrm{Hom}(\mathcal{G}, F(\mathcal{F})) \cong \mathrm{Hom}(\mathcal{G}, \dot{p}_{I*}\mathcal{F}) \cong \mathrm{Hom}(\mathcal{G}, \tau^{\leq n}\dot{p}_{I*}\mathcal{F}),$$

and then Yoneda's lemma tells us that $F(\mathcal{F}) \cong \tau^{\leq n}\dot{p}_{I*}\mathcal{F}$. But this holds for all sufficiently large n, so $\dot{p}_{I*}\mathcal{F}$ is in fact bounded. ☐

Remark B.2.4. Suppose $\Bbbk$ has finite global dimension. Then the rings R_I also have finite global dimension, and for $M, N \in D^{\mathrm{b}}(R_I\text{-mod})$, the object $M \otimes^L_{\Bbbk} N$ still belongs to $D^{\mathrm{b}}(R_I\text{-mod})$. For $M, N \in D^{\mathrm{b}}(R\text{-mod})$, there is a natural map

$$R\mathrm{Hom}_R(R_I, M) \overset{L}{\otimes}_{\Bbbk} R\mathrm{Hom}_R(R_I, N) \to R\mathrm{Hom}_R(R_I, M \overset{L}{\otimes}_{\Bbbk} N).$$

One can check that under the construction carried out in the proof of Proposition B.2.3, this corresponds to the natural map

$$j_I^* j_* \mathcal{F} \overset{L}{\otimes} j_I^* j_* \mathcal{G} \to j_I^* j_* (\mathcal{F} \overset{L}{\otimes} \mathcal{G})$$

coming from Proposition 1.4.5 and Remark 1.4.12.

Corollary B.2.5. *If $\mathcal{L}$ is a local system on U, then each $\mathsf{H}^i(j_I^* j_* \mathcal{L})$ is locally constant and vanishes unless $0 \leq i \leq |I|$. If, in addition, $\Bbbk$ is noetherian and $\mathcal{L}$ is of finite type, then each $\mathsf{H}^i(j_I^* j_* \mathcal{L})$ is a local system of finite type.*

Proof. This corollary is essentially an amplification of Step 1 in the proof of Proposition B.2.3. Let M be the R-module corresponding to $\mathcal{L}$ via Theorem 1.7.9. Proposition B.2.3 lets us translate the problem into certain vanishing and finiteness claims about

$$\mathsf{H}^i(R\mathrm{Hom}_R(R_I, M)) \cong \mathrm{Ext}^i_R(R_I, M).$$

The Koszul complex construction [**72**, Corollary 17.5] shows that as an R-module, R_I admits a free resolution of length $|I|$. It follows immediately that $\mathrm{Ext}^i_R(R_I, M) = 0$ unless $0 \leq i \leq |I|$. Moreover, the resolution may be chosen such that its terms have finite rank, so that each $\mathrm{Ext}^i_R(R_I, M)$ is a subquotient of some $\mathrm{Hom}_R(R^{\oplus j}, M) \cong M^{\oplus j}$. If $\Bbbk$ is noetherian and M is finitely generated over $\Bbbk$, then it follows that each $\mathrm{Ext}^i_R(\Bbbk, M)$ is also finitely generated over $\Bbbk$, as desired. ☐

A vanishing result. We conclude this section with the following computation.

Lemma B.2.6. *Let $I \subset \{1, \ldots, k\}$ be a nonempty subset. For $\mathcal{F} \in D^{+}_{\mathrm{loc}}(U, \Bbbk)$, we have $p_{I!} h_{I*} \mathcal{F} = 0$. As a consequence, $R\Gamma_{\mathrm{c}}(h_{I*}\mathcal{F}) = 0$.*

Proof. Assume for simplicity that $I = \{1, \ldots, j\}$ for some integer j with $1 \leq j \leq k$. Let $V = X_{\subset I} \smallsetminus X_I$, and let $h : U \hookrightarrow V$ and $v : V \hookrightarrow X_{\subset I}$ be the inclusion maps. Of course, we have $h_{I*}\mathcal{F} \cong v_* h_* \mathcal{F}$. Note that v and i_I are complementary open and closed embeddings, so there is a distinguished triangle $v_!v^*(h_{I*}\mathcal{F}) \to h_{I*}\mathcal{F} \to i_{I*}i_I^*(h_{I*}\mathcal{F}) \to$. Note also that $v^*h_{I*}\mathcal{F} \cong h_*\mathcal{F}$ and that $p_{I*}i_{I*}$ is the identity functor on $D^+(X_I, \Bbbk)$. Applying p_{I*}, we obtain a diagram

$$\begin{tikzcd}
p_{I*}v_!h_*\mathcal{F} \arrow[r] & p_{I*}h_{I*}\mathcal{F} \arrow[r] & p_{I*}i_{I*}i_I^*(h_{I*}\mathcal{F}) \arrow[r] & {} \\
& \dot{p}_{I*}\mathcal{F} \arrow[u, "\wr"', leftrightarrow] & j_I^*j_*\mathcal{F} \arrow[u, "\wr"', leftrightarrow] &
\end{tikzcd}$$

The second map in this triangle is an isomorphism by Lemma B.2.2, so

$$p_{I*}v_!h_*\mathcal{F} = 0. \tag{B.2.9}$$

This expression involves a composition of a !-push-forward with a $*$-push-forward, a situation in which Proposition 1.2.8 is useless. Lemma B.2.7 below gets us out of this difficulty: together with (B.2.9), it implies that

$$p_{I!}v_*s^*h_*\mathcal{F} = 0,$$

for some suitable homeomorphism $s : V \to V$ that preserves U. Let $s_U = s|_U : U \to U$. We then have $s^*h_*\mathcal{F} \cong h_*(s_U^*\mathcal{F})$, and hence

$$p_{I!}v_*h_*(s_U^*\mathcal{F}) \cong p_{I!}h_{I*}(s_U^*\mathcal{F}) = 0. \tag{B.2.10}$$

Since s_U is a homeomorphism, s_U^* is an equivalence of categories; every object $\mathcal{F} \in D^+_{\mathrm{loc}}(U, \Bbbk)$ can be written as $s_U^*\mathcal{F}'$ for some $\mathcal{F}' \in D^+_{\mathrm{loc}}(U, \Bbbk)$. Thus, (B.2.10) implies that $p_{I!}h_{I*}\mathcal{F} = 0$ for all $\mathcal{F} \in D^+_{\mathrm{loc}}(U, \Bbbk)$. ☐

LEMMA B.2.7. *Let $I \subset \{1, \ldots, n\}$ be a nonempty subset. Let $V = X_{\subset I} \smallsetminus X_I$, and let $v : V \hookrightarrow X_{\subset I}$ be the inclusion map. There is a homeomorphism $s : V \to V$ with the following properties:*

1. *For any proper subset $K \subset I$, we have $s(X_K) = X_K$.*
2. *For $\mathcal{G} \in D^+(V, \Bbbk)$, there is a natural isomorphism $p_{I*}v_!\mathcal{G} \cong p_{I!}v_*(s^*\mathcal{G})$.*

PROOF. Assume for simplicity that $I = \{1, \ldots, j\}$ for some integer j with $1 \le j \le k$, so that

$$X_{\subset I} = \mathbb{C}^j \times (\mathbb{C}^\times)^{k-j} \times \mathbb{C}^{n-k} \qquad \text{and} \qquad X_I = \{0\} \times (\mathbb{C}^\times)^{k-j} \times \mathbb{C}^{n-k}.$$

Consider the $2j$-sphere S^{2j}, which we identify with the one-point compactification of $\mathbb{C}^j$: $S^{2j} = \mathbb{C}^j \cup \{\infty\}$. Let $u : \mathbb{C}^j \hookrightarrow S^{2j}$ be the inclusion map. There is also another open embedding $\bar{u} : \mathbb{C}^j \hookrightarrow S^{2j}$, given by $\bar{u}(z) = z/\|z\|^2$, where $\|\cdot\|$ is the Euclidean norm on $\mathbb{C}^j$. (This formula should be understood to mean $\bar{u}(0) = \infty$.) Next, let $K = S^{2j} \times (\mathbb{C}^\times)^{k-j} \times \times \mathbb{C}^{n-k}$, and define maps $w, \bar{w} : X_{\subset I} \to K$ by

$$w = u \times \mathrm{id}_{(\mathbb{C}^\times)^{k-j}} \times \mathrm{id}_{\mathbb{C}^{n-k}}, \qquad \bar{w} = \bar{u} \times \mathrm{id}_{(\mathbb{C}^\times)^{k-j}} \times \mathrm{id}_{\mathbb{C}^{n-k}}.$$

Define $s : V \to V$ by the formula

$$s(z_1, \ldots, z_j, z_{j+1}, \ldots, z_n) = \left(\frac{(z_1, \ldots, z_j)}{|z_1|^2 + \cdots + |z_j|^2}, z_{j+1}, \ldots, z_n \right).$$

(Note that on V, one cannot have $z_1 = \cdots = z_j = 0$, so the denominator in this formula is always nonzero.) It is clear from the definition that s preserves each

$X_K \subset V$. Moreover, the following diagram is commutative and cartesian:

$$\begin{array}{ccc} V & \xrightarrow{v} & X_{\subset I} \\ {\scriptstyle v\circ s}\downarrow & & \downarrow{\scriptstyle w} \\ X_{\subset I} & \xrightarrow{\bar{w}} & K \end{array}$$

Let $\mathcal{G} \in D^+(V, \Bbbk)$. We claim that

$$w_* v_! \mathcal{G} \cong \bar{w}_! v_* (s^* \mathcal{G}). \tag{B.2.11}$$

Let us first show that both sides have the same pullback along $\bar{w}$. We clearly have $\bar{w}^* \bar{w}_! v_*(s^*\mathcal{G}) \cong v_*(s^*\mathcal{G})$. On the other hand, using Proposition 1.2.16 and the fact that s is an involution, we have

$$\bar{w}^* w_* v_! \mathcal{G} \cong (v \circ s)_* v^* v_! \mathcal{G} \cong v_* s_* \mathcal{G} \cong v_* s^* \mathcal{G}.$$

Now, the right-hand side of (B.2.11) is the extension by zero of an object on $\bar{w}(X_{\subset I}) \subset K$, so to prove (B.2.11), it suffices to show that the restriction of the left-hand to the complement of $\bar{w}(X_{\subset I})$, i.e. to $\{0\} \times (\mathbb{C}^\times)^{k-j} \times \mathbb{C}^{n-k}$, vanishes. Since w is an open embedding and $\{0\} \times (\mathbb{C}^\times)^{k-j} \times \mathbb{C}^{n-k} = w(X_I)$, we have

$$(w_* v_! \mathcal{G})|_{\{0\}\times(\mathbb{C}^\times)^{k-j}\times\mathbb{C}^{n-k}} \cong (v_! \mathcal{G})|_{X_I} = 0.$$

This completes the proof of (B.2.11).

Let $\bar{p}_I : K \to X_I$ be the map defined by the same formula as p_I. We have $\bar{p}_I \circ w = \bar{p}_I \circ \bar{w} = p_I$. Note that $\bar{p}_I$ is a proper map, so that $\bar{p}_{I!} = \bar{p}_{I*}$. Apply $\bar{p}_{I*}$ to the left-hand side of (B.2.11), and apply $\bar{p}_{I!}$ to the right-hand side. We then obtain that $p_{I*} v_! \mathcal{G} \cong p_{I!} v_*(s^*\mathcal{G})$, as desired. ☐

B.3. Local systems on open subsets of $\mathbb{C}$

This section contains results on the hypercohomology of local systems on Zariski open subsets of $\mathbb{C}$. These results will be needed for the base case of an induction argument in Section 2.6.

LEMMA B.3.1. *Let $I \subset \mathbb{R}$ be an interval, and let $j : I^\circ \hookrightarrow I$ be the inclusion of its interior. For any $\Bbbk$-module M, we have $j_* \underline{M}_{I^\circ} \cong \underline{M}_I$.*

Here, an **interval** means any subset of one of the following forms:

$$(a,b), \quad (a,b], \quad [a,b), \quad [a,b], \quad (-\infty,b), \quad (-\infty,b], \quad (a,\infty), \quad [a,\infty), \quad \mathbb{R}.$$

In particular, I and I° are connected, and $I \smallsetminus I^\circ$ consists of at most two points.

PROOF. The underived version of this statement, ${}^\circ j_* \underline{M}_{I^\circ} \cong \underline{M}_I$, holds by Example 1.7.6. It remains to show that $\mathsf{H}^i(j_* \underline{M}_{I^\circ}) = 0$ for $i > 0$. We must show that every stalk $\mathsf{H}^i(j_* \underline{M}_{I^\circ})_x$ vanishes. By Lemma 1.2.10, we have

$$\mathsf{H}^i(j_* \underline{M}_{I^\circ})_x \cong \varinjlim_{U \ni x} \mathbf{H}^i(U, (j_* \underline{M}_{I^\circ})|_U) \cong \varinjlim_{U \ni x} \mathbf{H}^i(U \cap I^\circ, \underline{M}_{U \cap I^\circ}).$$

Every point $x \in I$ admits a basis of neighborhoods U such that $U \cap I^\circ$ is contractible, so the cohomology groups $\mathbf{H}^i(U \cap I^\circ; M)$ vanish for $i > 0$, as desired. ☐

LEMMA B.3.2. *Let $B \subset \mathbb{C}$ be a closed Euclidean ball, and let $h : U \hookrightarrow \mathbb{C}$ be the inclusion of the complementary open subset. For any local system $\mathcal{L}$ on U, we have $R\Gamma(h_! \mathcal{L}) = 0$.*

PROOF. Assume without loss of generality that B is centered at $0 \in \mathbb{C}$ and has radius $r \geq 0$. Define a map $f : \mathbb{C} \to \mathbb{C}$ by

$$f(z) = \begin{cases} \frac{|z|-r}{|z|}z & \text{if } |z| \geq r, \\ 0 & \text{if } |z| \leq r. \end{cases}$$

Then $f(B) = \{0\}$ and f restricts to a homeomorphism $f' : U \xrightarrow{\sim} \mathbb{C}^\times$. Let $j : \mathbb{C}^\times \hookrightarrow \mathbb{C}$ be the inclusion map. Consider the commutative square

$$\begin{tikzcd} U \arrow[r, "h"] \arrow[d, "f'"', "\wr"] & \mathbb{C} \arrow[d, "f"] \\ \mathbb{C}^\times \arrow[r, "j"] & \mathbb{C} \end{tikzcd}$$

Since f is proper, we have

$$R\Gamma(h_!\mathcal{L}) \cong R\Gamma(f_*h_!\mathcal{L}) \cong R\Gamma(f_!h_!\mathcal{L}) \cong R\Gamma(j_!f'_!\mathcal{L}).$$

Since f' is a homeomorphism, $f'_!\mathcal{L}$ is a local system on $\mathbb{C}^\times$. Let $\mathcal{L}' = f'_!\mathcal{L}$, and let $i : \{0\} \hookrightarrow \mathbb{C}$ be the inclusion map. Apply $R\Gamma$ to the distinguished triangle $j_!\mathcal{L}' \to j_*\mathcal{L}' \to i_*((j_*\mathcal{L}')_0) \to$ to obtain

$$R\Gamma(j_!\mathcal{L}') \to R\Gamma(\mathcal{L}') \to (j_*\mathcal{L}')_0 \to .$$

According to Proposition B.2.3 (or its proof; see (B.2.5)), the map $R\Gamma(\mathcal{L}') \to (j_*\mathcal{L}')_0$ is an isomorphism, so $R\Gamma(j_!\mathcal{L}') = 0$, as desired. $\square$

LEMMA B.3.3. *Let B be a closed Euclidean ball in $\mathbb{C}$, and let $j : B^\circ \hookrightarrow B$ be the inclusion of its interior. Let $S \subset B^\circ$ be a finite set, and let $\mathcal{F} \in \mathrm{Sh}(B, \Bbbk)$ be such that $\mathcal{F}|_{B \smallsetminus S}$ is locally constant. Then $\mathcal{F} \cong j_*(\mathcal{F}|_{B^\circ})$. In particular, the natural map*

$$R\Gamma(\mathcal{F}) \to R\Gamma(\mathcal{F}|_{B^\circ})$$

is an isomorphism.

PROOF. We wish to show that the adjunction map $\mathcal{F} \to j_*j^*\mathcal{F}$ is an isomorphism. It is enough to show that for any point $x \in B$, there is a neighborhood U such that $\mathcal{F}|_U \to (j_*j^*\mathcal{F})|_U$ is an isomorphism. For $x \in B^\circ$, this is obvious: we can just take $U = B^\circ$. For $x \in B \smallsetminus B^\circ$, there exists a neighborhood U that is homeomorphic to $(0,1) \times (0,1]$ and that does not meet S. Moreover, this homeomorphism identifies $U \cap B^\circ$ with $(0,1) \times (0,1)$. Then $\mathcal{F}|_U$ is a constant sheaf, say $\underline{M}_U$. Let $h : U \cap B^\circ \hookrightarrow U$ be the inclusion map. We must show that $h_*\underline{M}_{U \cap B^\circ} \cong \underline{M}_U$. Proposition 1.9.2 reduces this problem to computing the push-forward of $\underline{M}_{(0,1)}$ along the inclusion $(0,1) \hookrightarrow (0,1]$, and the latter has been shown to be a constant sheaf in Lemma B.3.1. $\square$

PROPOSITION B.3.4. *Let $S \subset \mathbb{C}$ be a finite, nonempty set, and let $j : U \hookrightarrow \mathbb{C}$ be the inclusion of the complementary open subset. Let $\mathcal{L}$ be a local system on U.*

(1) *For $k \neq 1$, we have $\mathbf{H}^k(\mathbb{C}, j_!\mathcal{L}) = 0$.*
(2) *The $\Bbbk$-module $\mathbf{H}^1(\mathbb{C}, j_!\mathcal{L})$ admits a filtration of length $|S| - 1$, each of whose subquotients is isomorphic to a stalk of $\mathcal{L}$. In particular, if $\mathcal{L}$ is of finite type, then $\mathbf{H}^1(\mathbb{C}, j_!\mathcal{L})$ is a finitely generated $\Bbbk$-module.*

REMARK B.3.5. The second part of this proposition implies, in particular, that if $|S| \geq 2$, then $\mathbf{H}^1(\mathbb{C}, j_!\mathcal{L}) \neq 0$. If $\Bbbk$ is noetherian and has finite global dimension, and if $\mathcal{L}$ is of finite type, then an iterated application of Lemma A.10.11 shows that

$$\operatorname{grade} \mathbf{H}^1(\mathbb{C}, j_!\mathcal{L}) = \begin{cases} \infty & \text{if } |S| = 1 \text{ or } \mathcal{L} = 0, \\ \operatorname{grade} \mathcal{L} & \text{otherwise.} \end{cases}$$

PROOF. We proceed by induction on the number of points in S. If $|S| = 1$, then $R\Gamma(j_!\mathcal{L}) = 0$ by Lemma B.3.2.

Now suppose $|S| > 1$. Choose a point $z_0 \in \mathbb{C}$ such that there is a unique element of S whose distance from z_0 is maximal. Call that point s_0. Let B be the closed Euclidean ball centered at z_0 and with radius $|s_0 - z_0|$. Let B° be the interior of B. Then B° contains $|S| - 1$ points of S. The boundary $\partial B = B \smallsetminus B^\circ$ contains exactly one point of S, and the complement of B contains no points of S.

Let $h : \mathbb{C} \smallsetminus B \hookrightarrow \mathbb{C}$ and $i : B \hookrightarrow \mathbb{C}$ be the inclusion maps. Since $\mathbb{C} \smallsetminus B \subset U$, we have $(j_!\mathcal{L})|_{\mathbb{C}\smallsetminus B} \cong \mathcal{L}|_{\mathbb{C}\smallsetminus B}$. Apply $R\Gamma$ to the distinguished triangle

$$h_!(\mathcal{L}|_{\mathbb{C}\smallsetminus B}) \to j_!\mathcal{L} \to i_*((j_!\mathcal{L})|_B) \to .$$

We have $R\Gamma(h_!(\mathcal{L}|_{\mathbb{C}\smallsetminus B})) = 0$ by Lemma B.3.2, so

$$R\Gamma(j_!\mathcal{L}) \cong R\Gamma((j_!\mathcal{L})|_B).$$

Next, let $k : B^\circ \hookrightarrow B$ be the inclusion map, and consider the natural map

$$(j_!\mathcal{L})|_B \to k_*((j_!\mathcal{L})|_{B^\circ}). \tag{B.3.1}$$

This map obviously induces an isomorphism of stalks at any point of B°. The proof of Lemma B.3.3 can be repeated to show that it also induces an isomorphism of stalks at any point of ∂B other than s_0, so the cone of (B.3.1) is supported at the point s_0. Since $(j_!\mathcal{L})_{s_0} = 0$, we have a distinguished triangle

$$(j_!\mathcal{L})|_B \to k_*((j_!\mathcal{L})|_{B^\circ}) \to t_*(k_*((j_!\mathcal{L})|_{B^\circ})_{s_0}) \to, \tag{B.3.2}$$

where $t : \{s_0\} \hookrightarrow B$ is the inclusion map. If we let $v : U \cap B^\circ \hookrightarrow B^\circ$ be the inclusion map, then the middle term above can be rewritten as $k_*v_!(\mathcal{L}|_{U\cap B^\circ})$.

Examining the proof of Lemma B.3.3 again, we see that $k_*((j_!\mathcal{L})|_{B^\circ})$ restricts to a constant sheaf on a small neighborhood of any point of ∂B. So the stalk $k_*((j_!\mathcal{L})|_{B^\circ})_{s_0}$ is isomorphic to the stalk $((j_!\mathcal{L})|_{B^\circ})_x \cong \mathcal{L}_x$ at some nearby point $x \in B^\circ \smallsetminus S$. Applying $R\Gamma$ to (B.3.2), we obtain

$$R\Gamma(j_!\mathcal{L}) \to R\Gamma(v_!(\mathcal{L}|_{U\cap B^\circ})) \to \mathcal{L}_x \to .$$

Take the long exact sequence in cohomology, we obtain

$$\cdots \to \mathsf{H}^{k-1}(\mathcal{L}_x) \to \mathbf{H}^k(\mathbb{C}, j_!\mathcal{L}) \to \mathbf{H}^k(B^\circ, v_!(\mathcal{L}|_{U\cap B^\circ})) \to \cdots .$$

The first term obviously vanishes unless $k = 1$. For the last term, note that B° is homeomorphic to $\mathbb{C}$, and that $U \cap B^\circ$ is the complement of $|S| - 1$ points. So by induction, $\mathbf{H}^k(B^\circ, v_!(\mathcal{L}|_{U\cap B^\circ}))$ vanishes for $k \neq 1$, and when $k = 1$, it is an $(|S|-2)$-fold iterated extension of stalks of $\mathcal{L}$. The desired properties of $\mathbf{H}^k(\mathbb{C}, j_!\mathcal{L})$ then follow. ☐

LEMMA B.3.6. *Let $S \subset \mathbb{C}$ be a finite, nonempty set. Let $U = \mathbb{C} \smallsetminus S$, and let $\mathcal{L}$ be a nonzero local system on U. For any point $z \in U$, there is a surjective map $\mathbf{H}^1(U \smallsetminus \{z\}, \mathcal{L}|_{U\smallsetminus\{z\}}) \twoheadrightarrow \mathcal{L}_z(-1)$. In particular, $\mathbf{H}^1(U \smallsetminus \{z\}, \mathcal{L}|_{U\smallsetminus\{z\}}) \neq 0$. On the other hand, $\mathbf{H}^k(U \smallsetminus \{z\}, \mathcal{L}|_{U\smallsetminus\{z\}}) = 0$ for all $k \geq 2$.*

PROOF. Let $V = U \smallsetminus \{z\}$, and let $j : V \hookrightarrow U$ and $i : \{z\} \hookrightarrow U$ be the inclusion maps. We claim that there is a distinguished triangle of the form

$$\mathcal{L} \to j_*(\mathcal{L}|_V) \to i_*(\mathcal{L}_z)[-1](-1) \to .$$

In fact, this is a truncation distinguished triangle; it is produced by applying Exercise 1.10.1 in a neighborhood of z that is homeomorphic to $\mathbb{C}$. Now apply $R\Gamma$ and take cohomology to produce the sequence

$$\cdots \to \mathbf{H}^1(V, \mathcal{L}|_V) \to \mathsf{H}^1(\mathcal{L}_z[-1](-1)) \to \mathbf{H}^2(U, \mathcal{L}) \to \cdots . \tag{B.3.3}$$

The middle term is identified with $\mathcal{L}_z(-1)$. Next, using Proposition B.3.7 below, together with Theorem 1.9.7, we have

$$\mathbf{H}^2(U, \mathcal{L}) \cong \mathrm{Hom}(\underline{\Bbbk}_U, \mathcal{L}[2]) \cong \mathrm{Ext}^2_{\mathrm{Loc}(U,\Bbbk)}(\underline{\Bbbk}_U, \mathcal{L}),$$

and then by Theorem 1.7.9, this is in turn isomorphic to an Ext^2-group in the category of modules for the free group. But it is well known that the free group has cohomological dimension 1 (see, for instance [**46**, Example I.4.3] and the discussion after [**46**, Proposition VIII.2.2]), so this Ext^2-group vanishes. The same reasoning shows that $\mathbf{H}^k(V, \mathcal{L}|_V) = 0$ for all $k \geq 2$.

It follows that the first map in (B.3.3) is surjective, as desired. $\square$

The following result is a standard fact in the theory of covering spaces.

PROPOSITION B.3.7. *Let $S \subset \mathbb{C}$ be a finite, nonempty set, and let $U = \mathbb{C} \subset S$. Then the fundamental group $\pi_1(U, u_0)$ (for any point $u_0 \in U$) is a free group on $|S|$ generators. Moreover, the universal cover of U is contractible.*

B.4. Constructible sheaves on $\mathbb{C}^n$, II

We return to the setting of Section B.2. Add the assumption that $\Bbbk$ is noetherian and of finite global dimension. Broadly speaking, the goal of this section is to prove analogues of various statements from Section B.2 in which the roles of $*$- and !-push-forwards have been swapped. In particular, we will obtain an algebraic description of $j_!\mathcal{L}$, where $\mathcal{L}$ is a local system on U.

We will make use of terminology and results from Section 2.4. In particular, we observe that the collection of subsets $\{X_I\}$ of $\mathbb{C}^n$ is an example of a normal crossings stratification.

PROPOSITION B.4.1. *For $\mathcal{F} \in D^+(X_{\subset I}, \Bbbk)$, there is a natural map $i_I^!\mathcal{F} \to p_{I!}\mathcal{F}$. If $\mathcal{F}$ is weakly constructible with respect to the normal crossings stratification, then this map is an isomorphism.*

PROOF. Let $V = X_{\subset I} \smallsetminus X_I$, and let $v : V \hookrightarrow X_{\subset I}$ be the inclusion map. Apply $p_{I!}$ to the distinguished triangle $i_{I*}i_I^!\mathcal{F} \to \mathcal{F} \to v_*v^*\mathcal{F} \to$: since $p_I \circ i_I = \mathrm{id}_{X_I}$, the result is a natural distinguished triangle

$$i_I^!\mathcal{F} \to p_{I!}\mathcal{F} \to p_{I!}v_*(v^*\mathcal{F}) \to .$$

The first map in this triangle is the map we wish to study. To show that it is an isomorphism when $\mathcal{F}$ is weakly constructible, we must show that

$$p_{I!}v_*(v^*\mathcal{F}) = 0.$$

We proceed by induction on the number of strata in the support of $\mathcal{F}$. If its support consists of a single stratum, that stratum must be the unique closed stratum X_I. In this case, $v^*\mathcal{F} = 0$, and the claim holds trivially. Otherwise, let X_J be a stratum that is open in the support of $\mathcal{F}$. Let $Z' = \operatorname{supp} \mathcal{F} \smallsetminus X_J$, and let $u : X_J \hookrightarrow X_{\subset I}$ and $z : Z' \hookrightarrow X_{\subset I}$ be the inclusion maps. We then have a distinguished triangle

$$z_* z^! \mathcal{F} \to \mathcal{F} \to u_* u^* \mathcal{F} \to . \tag{B.4.1}$$

By Lemma 2.3.22 and Remark 2.4.3, the first term above is still weakly constructible. It has smaller support than $\mathcal{F}$, so $p_{I!} v_* (v^* z_* z^! \mathcal{F}) = 0$ by induction. To study the last term, we introduce some more notation. Let $X' = \overline{X_J} \cap X_{\subset I}$, and let $h' : X_J \hookrightarrow X'$ be the inclusion map. We also let $p' = p_I|_{X'} : X' \to X_I$.

Let $\mathcal{G} \in D^+_{\mathrm{loc}}(X_J, \Bbbk)$. Since $X_J \subset V$, there is a natural isomorphism $v_* v^* u_* \mathcal{G} \cong u_* \mathcal{G}$. Similarly, since $X_J \subset X'$, we have $p_{I!} u_* \mathcal{G} \cong p'_!((u_* \mathcal{G})|_{X'})$ and $(u_* \mathcal{G})|_{X'} \cong h'_* \mathcal{G}$. Combining these observations for $\mathcal{G} = u^* \mathcal{F}$, we obtain a natural isomorphism

$$p_{I!} v_* v^* (u_* u^* \mathcal{F}) \cong p'_! h'_* (\mathcal{F}|_{X_J}).$$

The last expression matches the setup of Lemma B.2.6, so it vanishes. We have shown that $p_{I!} v_* v^*$ kills the first and last terms of (B.4.1), so it must kill the middle term as well, as desired. ☐

Proposition B.4.2. *For $\mathcal{F} \in D^+(X_{\subset I}, \Bbbk)$, there is a natural map $p_{I*} \mathcal{F} \to i_I^* \mathcal{F}$. If $\mathcal{F}$ is weakly constructible with respect to the normal crossings stratification, then this map is an isomorphism.*

Proof. Define V and $v : V \hookrightarrow X_{\subset I}$ as in the proof of Proposition B.4.1. Apply p_{I*} to the distinguished triangle $v_! v^* \mathcal{F} \to \mathcal{F} \to i_{I*} i_I^* \mathcal{F} \to$ to obtain the distinguished triangle

$$p_{I*} v_! (v^* \mathcal{F}) \to p_{I*} \mathcal{F} \to i_I^* \mathcal{F} \to .$$

The second map in this triangle is our desired map. Suppose now that $\mathcal{F}$ is weakly constructible with respect to the normal crossings stratification. According to Lemma B.2.7, we have $p_{I*} v_! (v^* \mathcal{F}) \cong p_{I!} v_* (s^* v^* \mathcal{F})$, where $s : V \to V$ is a certain homeomorphism that preserves the strata of the normal crossings stratification. In particular, $s^* v^* \mathcal{F}$ is also weakly constructible with respect to that stratification, as is $v_*(s^* v^* \mathcal{F})$. Then Proposition B.4.1 tells us that

$$p_{I*} v_! (v^* \mathcal{F}) \cong p_{I!} v_* (s^* v^* \mathcal{F}) \cong i_I^! v_* (s^* v^* \mathcal{F}) = 0.$$

We conclude that $p_{I*} \mathcal{F} \to i_I^* \mathcal{F}$ is an isomorphism, as desired. ☐

Corollary B.4.3. *Let $I \subset \{1, \dots, k\}$ be a nonempty subset. For any $\mathcal{F} \in D^+_{\mathrm{loc}}(U, \Bbbk)$, we have $p_{I*} h_{I!} \mathcal{F} = 0$. As a consequence, $R\Gamma(h_{I!} \mathcal{F}) = 0$.*

Lemma B.4.4. *Let I and J be disjoint subsets of $\{1, \dots, k\}$, and consider the diagram*

$$\begin{tikzcd} U \arrow[r, "h_I"] \arrow[d, "h_J"'] & X_{\subset I} \arrow[d, "u_I"] \\ X_{\subset J} \arrow[r, "u_J"] & \mathbb{C}^n \end{tikzcd}$$

For any $\mathcal{F} \in D^+_{\mathrm{loc}}(U, \Bbbk)$, there is a natural isomorphism $u_{J!} h_{J*} \mathcal{F} \cong u_{I*} h_{I!} \mathcal{F}$.

Proof. The square above is cartesian, and all four maps are open embeddings. We therefore have

$$u_J^* u_{I*} h_{I!} \mathcal{F} \cong h_{J*} h_I^* h_{I!} \mathcal{F} \cong h_{J*} \mathcal{F}.$$

By adjunction, we obtain a natural map $u_{J!}h_{J*}\mathcal{F} \to u_{I*}h_{I!}\mathcal{F}$ that is an isomorphism over $X_{\subset J}$. To show that it is an isomorphism, we must show that the restriction of $u_{I*}h_{I!}\mathcal{F}$ to the complement of $X_{\subset J}$ is zero. We may work one stratum at a time: let $K \subset \{1, \ldots, n\}$ be a subset not contained in J. We must show that $j_K^* u_{I*}h_{I!}\mathcal{F} = 0$. Note that

$$j_K^* u_{I*}h_{I!}\mathcal{F} \cong i_K^*((u_{I*}h_{I!}\mathcal{F})|_{X_{\subset K}}).$$

Consider the diagram of cartesian squares

$$\begin{tikzcd}
U \arrow[r, "h_{I\cap K}"] \arrow[d, equal] & X_{\subset I\cap K} \arrow[r, "u'_I"] \arrow[d] & X_{\subset K} \arrow[d] \\
U \arrow[r, "h_I"] & X_{\subset I} \arrow[r, "u_I"] & \mathbb{C}^n
\end{tikzcd}$$

By base change, we have

$$(u_{I*}h_{I!}\mathcal{F})|_{X_{\subset K}} \cong u'_{I*}(h_{I\cap K})_!\mathcal{F},$$

and then by Proposition B.4.2, we have

$$i_K^*((u_{I*}h_{I!}\mathcal{F})|_{X_{\subset K}}) \cong i_K^* u'_{I*}(h_{I\cap K})_!\mathcal{F} \cong p_{K*}u'_{I*}(h_{I\cap K})_!\mathcal{F}.$$

Let $q = p_K|_{X_{I\cap K}} : X_{I\cap K} \to X_K$. Then $q \circ p_{I\cap K} = p_K \circ u'_I$. Using this observation and Proposition B.4.2 again, the expression above can be rewritten as

$$p_{K*}u'_{I*}h'_{I!}\mathcal{F} \cong q_*(p_{I\cap K})_*(h_{I\cap K})_!\mathcal{F} \cong q_* i_{I\cap K}^*(h_{I\cap K})_!\mathcal{F}.$$

The last expression clearly vanishes, as desired. $\square$

Lemma B.4.5. *Assume $k = 2$. Let $v_1 : X_1 \hookrightarrow Z_1$ and $z_1 : Z_1 \hookrightarrow \mathbb{C}^n$ be the inclusion maps. For any $\mathcal{F} \in D^{\mathrm{b}}_{\mathrm{loc}}(U, \Bbbk)$, there are natural isomorphisms $z_1^* j_*\mathcal{F} \cong v_{1*}i_1^* h_{1*}\mathcal{F}$ and $z_1^! j_!\mathcal{F} \cong v_{1!}i_1^! h_{1!}\mathcal{F}$.*

Proof. We will prove the first isomorphism; the second one is similar. Let $s_1 : X_{12} \hookrightarrow Z_1$ be the inclusion map, and consider the following diagram:

$$\begin{tikzcd}
U \arrow[r, "h_1"] \arrow[d, "h_2"'] \arrow[dr, "j"] & X_{\subset\{1\}} \arrow[d, "u_1"] & X_1 \arrow[l, "i_1"'] \arrow[d, "v_1"] & \\
X_{\subset\{2\}} \arrow[r, "u_2"] & \mathbb{C}^n & Z_1 \arrow[l, "z_1"'] & X_{12} \arrow[l, "s_1"'] \arrow[ll, bend left, "i_{12}"]
\end{tikzcd}$$

We have a distinguished triangle

$$s_{1*}s_1^! z_1^* j_*\mathcal{F} \to z_1^* j_*\mathcal{F} \to v_{1*}v_1^* z_1^* j_*\mathcal{F} \to .$$

The third term can be identified with $v_{1*}i_1^* h_{1*}\mathcal{F}$. To prove the lemma, we must show that the following object vanishes:

$$s_1^! z_1^* j_*\mathcal{F} \cong i_{12}^! z_{1*}z_1^* j_*\mathcal{F} \cong p_{12!}z_{1*}z_1^* j_*\mathcal{F}.$$

(The second isomorphism above comes from Proposition B.4.1.) Since $X_{12} \cong \mathbb{C}^{n-2}$ is smooth and contractible, the functor $\mathrm{a}_{X_{12}}^! \cong \mathrm{a}_{X_{12}}^*[2n-4](n-2) : D^{\mathrm{b}}(\mathrm{pt}, \Bbbk) \to D^{\mathrm{b}}_{\mathrm{loc}}(X_{12}, \Bbbk)$ is an equivalence of categories (by Theorems 1.8.10 and 2.2.9), and hence so is its left adjoint $R\Gamma_{\mathrm{c}}$. In other words, it is enough to prove the vanishing of

$$R\Gamma_{\mathrm{c}}(p_{12!}z_{1*}z_1^* j_*\mathcal{F}) \cong R\Gamma_{\mathrm{c}}(z_1^* j_*\mathcal{F}).$$

Consider the distinguished triangle $u_{2!}u_2^*j_*\mathcal{F} \to j_*\mathcal{F} \to z_{1*}z_1^*j_*\mathcal{F} \to$. Rewrite the first term as $u_{2!}h_{2*}\mathcal{F}$, and then apply $R\Gamma_{\mathrm{c}}$ to obtain a distinguished triangle

$$R\Gamma_{\mathrm{c}}(h_{2*}\mathcal{F}) \to R\Gamma_{\mathrm{c}}(j_*\mathcal{F}) \to R\Gamma_{\mathrm{c}}(z_1^*j^*\mathcal{F}) \to .$$

The first two terms both vanish by Lemma B.2.6, so we are done. $\square$

LEMMA B.4.6. *For any $\mathcal{F} \in D^+_{\mathrm{loc}}(U, \Bbbk)$ and any $I \subset \{1, \dots, k\}$, there is a natural isomorphism $j_I^! j_! \mathcal{F} \to \dot{p}_{I!}\mathcal{F}$.*

PROOF. There are natural isomorphisms $j_I^! j_! \mathcal{F} \cong i_I^! h_{I!}\mathcal{F}$ and $\dot{p}_{I!}\mathcal{F} \cong p_{I!}h_{I!}\mathcal{F}$, and Proposition B.4.1 gives us a natural isomorphism $i_I^! h_{I!}\mathcal{F} \to p_{I!}h_{I!}\mathcal{F}$. $\square$

Let $R = \Bbbk[\pi_1(U, u_0)]$ and $R_I = \Bbbk[\pi_1(X_I, p_I(u_0)]$ be as in Section B.2. Recall that the map p_I induces a ring homomorphism $R \to R_I$.

PROPOSITION B.4.7. *For any $\mathcal{F} \in D^{\mathrm{b}}_{\mathrm{loc}}(U, \Bbbk)$ and any $I \subset \{1, \dots, k\}$, the object $j_I^! j_! \mathcal{F}$ lies in $D^{\mathrm{b}}_{\mathrm{loc}}(X_I, \Bbbk)$. Moreover, there is a commutative diagram*

$$\begin{tikzcd}
D^{\mathrm{b}}_{\mathrm{loc}}(U, \Bbbk) \arrow[r, "\sim"] \arrow[d, "\mathcal{F} \mapsto j_I^! j_! \mathcal{F}"'] & D^{\mathrm{b}}(R\text{-mod}) \arrow[d, "R_I \otimes^L_R (-)[-2|I|](-|I|)"] \\
D^{\mathrm{b}}_{\mathrm{loc}}(X_I, \Bbbk) \arrow[r, "\sim"] & D^{\mathrm{b}}(R_I\text{-mod})
\end{tikzcd}$$

PROOF. For the first assertion, see Remark 2.4.3. By Lemma B.4.6, we may instead study $\dot{p}_{I!}\mathcal{F}$, which is left adjoint to $\dot{p}_I^! : D^+_{\mathrm{loc}}(X_I, \Bbbk) \to D^+_{\mathrm{loc}}(U, \Bbbk)$. The map $\dot{p}_I : U \to X_I$ is smooth of relative dimension $|I|$. By Theorem 2.2.9, we have $\dot{p}_I^! \cong \dot{p}_I^*[2|I|](|I|)$. In the language of R- and R_I-modules, Proposition 1.7.10 implies that $\dot{p}_I^!$ corresponds to the functor

$$\mathrm{Res}_R^{R_I}[2|I|](|I|) : D^{\mathrm{b}}(R\text{-mod}) \to D^{\mathrm{b}}(R_I\text{-mod}).$$

Therefore, $\dot{p}_{I!}$ corresponds to the the left adjoint of this functor, which is given by $R_I \otimes^L_R (-)[-2|I|](-|I|)$. $\square$

B.5. Nearby cycles on $\mathbb{C}^n$

We continue with the settting of Section B.2: we have subsets U and $Z = Z_1 \cup \cdots \cup Z_k$ of $\mathbb{C}^n$, and we have a base point $u_0 = (u_1, \dots, u_n) \in U$. Let $f : \mathbb{C}^n \to \mathbb{C}$ be a continuous function, and let $X_{\mathbf{s}} = f^{-1}(0)$ and $X_{\boldsymbol{\eta}} = f^{-1}(\mathbb{C}^\times)$. Assume that

$$X_{\mathbf{s}} = Z_1 \cup \cdots \cup Z_j \text{ for some integer } j \text{ with } 1 \le j \le k. \tag{B.5.1}$$

It follows that

$$X_{\boldsymbol{\eta}} = X_{\subset\{j+1,\dots,k\}}.$$

In this section, we will study the nearby cycles functor Ψ_f. More specifically, our goal to is compute $\Psi_f(h_*\mathcal{L})$ and $\Psi_f(h_!\mathcal{L})$, where

$$h : U \hookrightarrow X_{\boldsymbol{\eta}}$$

is the inclusion map and $\mathcal{L}$ is a local system on U. For this to have a reasonable answer, we must impose some conditions on the homomorphism $(f|_U)_\sharp : \pi_1(U, u_0) \to \pi_1(\mathbb{C}^\times, f(u_0))$. Denote its kernel and image by

$$\begin{aligned} H_f &= \ker(f|_U)_\sharp \quad &\lhd \pi_1(U, u_0), \\ \pi_f &= \mathrm{im}(f|_U)_\sharp \quad &\lhd \pi_1(\mathbb{C}^\times, f(u_0)). \end{aligned}$$

We assume throughout that

(B.5.2) The subgroup $\pi_f \lhd \pi_1(\mathbb{C}^\times, f(u_0))$ is nontrivial.

(Later, we will impose a stronger assumption on f; see (B.5.6).)

Study of the covering map over U. Let $\tilde{U} = \exp_X^{-1}(U)$. The next two lemmas involve the following diagram of cartesian squares:

$$\begin{tikzcd} \tilde{U} \arrow[r, "\tilde{h}"] \arrow[d, "\exp_U"'] & \tilde{X}_{\boldsymbol{\eta}} \arrow[r] \arrow[d, "\exp_X"'] & \mathbb{C} \arrow[d, "\exp"] \\ U \arrow[r, "h"] & X_{\boldsymbol{\eta}} \arrow[r] & \mathbb{C}^\times \end{tikzcd}$$

LEMMA B.5.1. *For $\mathcal{F} \in D^+_{\mathrm{loc}}(U, \Bbbk)$, we have natural isomorphisms*

$$\exp_{X*} \exp_X^*(h_*\mathcal{F}) \cong h_* \exp_{U*} \exp_U^* \mathcal{F},$$
$$\exp_{X*} \exp_X^*(h_!\mathcal{F}) \cong h_! \exp_{U*} \exp_U^* \mathcal{F}.$$

PROOF. Since $\exp_X$ is a covering map (and hence a topological submersion), Theorem 1.9.3 tells us that

$$\exp_{X*} \exp_X^*(h_*\mathcal{F}) \cong \exp_{X*} \tilde{h}_* \exp_U^* \mathcal{F} \cong h_* \exp_{U*} \exp_U^* \mathcal{F}.$$

The statement involving $h_!$ takes a bit more work. We begin by observing that $h^* \exp_{X*} \exp_X^*(h_!\mathcal{F}) \cong \exp_{U*} \exp_U^* \mathcal{F}$, so by adjunction, there is a natural map

$$h_! \exp_{U*} \exp_U^* \mathcal{F} \to \exp_{X*} \exp_X^*(h_!\mathcal{F}).$$

To show that this is an isomorphism, it is enough to show that the right-hand side has vanishing stalks at all points of $X_{\boldsymbol{\eta}} \smallsetminus U$. Let $x \in X_{\boldsymbol{\eta}} \smallsetminus U$. We have

$$\begin{aligned} \mathsf{H}^i((\exp_{X*} \exp_X^* h_!\mathcal{F})_x) &\cong \varinjlim_{B \ni x} \mathbf{H}^i(B, (\exp_{X*} \exp_X^* h_!\mathcal{F})|_B) \\ &\cong \varinjlim_{B \ni x} \mathbf{H}^i(\exp_X^{-1}(B), (\exp_X^* h_!\mathcal{F})|_{\exp_X^{-1}(B)}). \end{aligned}$$

When B is small enough, it is evenly covered; that is, $\exp_X^{-1}(B)$ is the disjoint union of a collection of open subsets $(B^{(y)})_{y \in \exp_X^{-1}(x)}$, each of which is homeomorphic to B via $\exp_X$. In this situation, by Exercise 1.2.6, we have

$$\mathbf{H}^i(\exp_X^{-1}(B), (\exp_X^* h_!\mathcal{F})|_{\exp_X^{-1}(B)}) \cong \prod_{y \in \exp_X^{-1}(x)} \mathbf{H}^i(B, (h_!\mathcal{F})|_B),$$

and hence

$$\mathsf{H}^i((\exp_{X*} \exp_X^* h_!\mathcal{F})_x) \cong \varinjlim_{\substack{B \ni x \\ B \text{ evenly covered}}} \prod_{y \in \exp_X^{-1}(x)} \mathbf{H}^i(B, (h_!\mathcal{F})|_B). \tag{B.5.3}$$

The point x belongs to some stratum X_I, where $I \cap \{1, \ldots, j\} \neq \varnothing$. The limit above may be computed using a basis of neighborhoods B of x in $X_{\subset I}$ that are obtained by the construction in Remark B.2.1. Each such neighborhood B is homeomorphic to $\mathbb{C}^n$ and matches the setup from Section B.2. Let $h_B = h|_{B \cap U} : B \cap U \hookrightarrow B$. Then $R\Gamma((h_!\mathcal{F})|_B) \cong R\Gamma(h_{B!}(\mathcal{F}|_{B \cap U}))$ vanishes by Corollary B.4.3. We conclude that (B.5.3) vanishes, as desired. $\square$

Condition (B.5.2) is, of course, equivalent to requiring that π_f have finite index in $\pi_1(\mathbb{C}^\times, f(u_0))$. Denote this index by

$$N = [\pi_1(\mathbb{C}^\times, f(u_0)) : \pi_f].$$

LEMMA B.5.2. (1) *The space $\tilde{U}$ has exactly N connected components.*

(2) *All connected components of $\tilde{U}$ are homeomorphic to one another.*

(3) *If U' is a connected component of $\tilde{U}$, then for any point $\tilde{u}_0 \in \exp_U^{-1}(u_0) \cap U'$, the fundamental group $\pi_1(U', \tilde{u}_0)$ is canonically identified with H_f.*

PROOF. Let γ_0 be a generator of the free abelian group $\pi_1(\mathbb{C}^\times, f(u_0))$, and let $\gamma = \gamma_0^N$. Then, of course, π_f is generated by γ. Let $\tilde{\gamma} \in \pi_1(U, u_0)$ be an element such that $f_\sharp(\tilde{\gamma}) = \gamma$.

Consider the set $\exp_U^{-1}(u_0)$, which is in bijection with $\exp^{-1}(1) = 2\pi i\mathbb{Z}$. To make this bijection explicit, choose a complex number z_0 such that $e^{z_0} = f(u_0)$. Then

$$\exp_U^{-1}(u_0) = \{(u_0, z_0 + 2\pi i n) \in U \times_{\mathbb{C}^\times} \mathbb{C} \mid n \in \mathbb{Z}\}.$$

Define an equivalence relation $\sim$ on $\exp_U^{-1}(u_0)$ by declaring two points to be equivalent if they belong to the same connected component of $\tilde{U}$. To count the connected components of $\tilde{U}$, it is enough to show that there are exactly N equivalence classes for $\sim$. In fact, we will show that $(u_0, z_0) \sim (u_0, z_0 + 2\pi i n)$ if and only if $n \in N\mathbb{Z}$.

In a slight abuse of notation, let us treat $\tilde{\gamma} : [0,1] \to U$ as an actual loop (rather than a homotopy class of loops), and suppose that $\gamma = f \circ \tilde{\gamma}$. Let $n \in N\mathbb{Z}$; say $n = mN$. Lift the loop γ^m to $\mathbb{C}$: let $\nu : [0,1] \to \mathbb{C}$ be the unique path such that $\nu(0) = z_0$ and $\exp \circ \nu = \gamma^m$. Because γ^m is homotopic to γ_0^{mN}, we must have $\nu(1) = z_0 + 2\pi i m N$. Now let $\tilde{\nu} : [0,1] \to \tilde{U}$ be the path given by $\tilde{\nu}(t) = (\tilde{\gamma}^m(t), \nu(t))$. This path joins (u_0, z_0) to $(u_0, z_0 + 2\pi i m N)$, so these two points are in the same equivalence class.

Now suppose $n \notin N\mathbb{Z}$. The space $\tilde{U}$ is locally path-connected (because U is), so each connected component is path-connected. If $(u_0, z_0) \sim (u_0, z_0 + 2\pi i n)$, then there is a path $\xi : [0,1] \to \tilde{U}$ with $\xi(0) = (u_0, z_0)$ and $\xi(1) = (u_0, z_0 + 2\pi i n)$. Then $\exp_U \circ \xi : [0,1] \to U$ is a loop, i.e., an element of $\pi_1(U, u_0)$, whose image in $\pi_1(\mathbb{C}^\times, f(u_0))$ is homotopic to γ_0^n. But this contradicts the fact that the image of $\pi_1(U, u_0)$ is generated by γ_0^m, so $(u_0, z_0) \not\sim (u_0, z_0 + 2\pi i n)$.

Next, for any $n \in \mathbb{Z}$, there is a deck transformation of $\tilde{U} \to U$ given by $(u, z) \mapsto (u, z + 2\pi i n)$. Such deck transformations can send a given connected component to any other connected component, so all the connected components are homeomorphic.

Finally, the identification of $\pi_1(U', \tilde{u}_0)$ with H_f is an exercise in the theory of covering spaces. We omit further details. $\square$

Let $I \subset \{1, \ldots, k\}$, and let M be an R_I-module equipped with a compatible action of π_f (that is, π_f acts on M by R_I-module automorphisms). For instance, in the case $I = \varnothing$, M may be of the form

$$M = \mathrm{Hom}_{\Bbbk[H_f]}(R, \mathrm{Res}^R_{\Bbbk[H_f]} M')$$

for some R-module M'. (The action of $\pi_f \cong \pi_1(U, u_0)/H_f$ on this module is given by the $\mathrm{Int}_{\mathrm{r}}$ construction from Section 1.7.) One may then consider

$$\Bbbk[\pi_1(\mathbb{C}^\times, f(u_0))] \otimes_{\Bbbk[\pi_f]} M \qquad \text{and} \qquad \mathrm{Hom}_{\Bbbk[\pi_f]}(\Bbbk[\pi_1(\mathbb{C}^\times, f(u_0))], M).$$

These are R_I-modules equipped with a compatible action of $\pi_1(\mathbb{C}^\times, u_0)$. In fact, they are isomorphic to each other: one can construct an isomorphism

$$\Bbbk[\pi_1(\mathbb{C}^\times, f(u_0))] \otimes_{\Bbbk[\pi_f]} M \xrightarrow{\sim} \mathrm{Hom}_{\Bbbk[\pi_f]}(\Bbbk[\pi_1(\mathbb{C}^\times, f(u_0))], M)$$

by choosing some set of coset representatives for π_f in $\pi_1(\mathbb{C}^\times, f(u_0))$.

Any element $\gamma \in \pi_1(\mathbb{C}^\times, f(u_0))$ gives rise to a deck transformation of $\exp : \mathbb{C} \to \mathbb{C}^\times$, and then, by base change, to a deck transformation of $\exp_U : \tilde{U} \to U$. For $\mathcal{L} \in \mathrm{Loc}(U, \Bbbk)$, as in Proposition 1.7.20, there are induced maps

$$\mathrm{mon}_!(\gamma) : \exp_{U!} \exp_U^* \mathcal{L} \to \exp_{U!} \exp_U^* \mathcal{L},$$
$$\mathrm{mon}_*(\gamma) : \exp_{U*} \exp_U^* \mathcal{L} \to \exp_{U*} \exp_U^* \mathcal{L}.$$

Since U may be disconnected, Proposition 1.7.20 does not quite apply as written to our situation, but a minor modification of its proof yields the following statement (whose proof we omit).

LEMMA B.5.3. *Let $\mathcal{L} \in \mathrm{Loc}(U, \Bbbk)$, and suppose $\mathcal{L}$ corresponds to $M \in R$-mod. Let $\gamma \in \pi_1(\mathbb{C}^\times, f(u_0))$.*

(1) *The map $\mathrm{mon}_*(\gamma) : \exp_{U*} \exp_U^* \mathcal{L} \to \exp_{U*} \exp_U^* \mathcal{L}$ corresponds to the action of γ on*

$$\mathrm{Hom}_{\Bbbk[\pi_f]}(\Bbbk[\pi_1(\mathbb{C}^\times, f(u_0))], \mathrm{Hom}_{\Bbbk[H_f]}(R, \mathrm{Res}^R_{\Bbbk[H_f]} M)).$$

(2) *The map $\mathrm{mon}_!(\gamma) : \exp_{U!} \exp_U^* \mathcal{L} \to \exp_{U!} \exp_U^* \mathcal{L}$ corresponds to the action of γ on*

$$\Bbbk[\pi_1(\mathbb{C}^\times, f(u_0))] \otimes_{\Bbbk[\pi_f]} (R \otimes_{\Bbbk[H_f]} \mathrm{Res}^R_{\Bbbk[H_f]} M).$$

Computation of the nearby cycles functor. We are now ready to describe the nearby cycles functor in terms similar to Propositions B.2.3 and B.4.7.

PROPOSITION B.5.4. *Let $f : \mathbb{C}^n \to \mathbb{C}$ be a function that satisfies* (B.5.1) *and* (B.5.2). *Let $I \subset \{1, \ldots, k\}$ be a subset that contains at least one element from $\{1, \ldots, j\}$.*

(1) *For any $\mathcal{F} \in D^{\mathrm{b}}_{\mathrm{loc}}(U, \Bbbk)$, the object $j_I^* \Psi_f(h_* \mathcal{F})$ lies in $D^{\mathrm{b}}_{\mathrm{loc}}(X_I, \Bbbk)$. Moreover, there is a commutative diagram*

$$\begin{tikzcd}
D^{\mathrm{b}}_{\mathrm{loc}}(U, \Bbbk) \arrow[r, "\sim"] \arrow[d, "\mathcal{F} \mapsto j_I^* \Psi_f(h_* \mathcal{F})"'] & D^{\mathrm{b}}(R\text{-mod}) \arrow[d, "M \mapsto R\mathrm{Hom}_{\Bbbk[\pi_f]}(\Bbbk[\pi_1(\mathbb{C}^\times, f(u_0))], R\mathrm{Hom}_{\Bbbk[H_f]}(R_I, \mathrm{Res}^R_{\Bbbk[H_f]} M))[-1]"] \\
D^{\mathrm{b}}_{\mathrm{loc}}(X_I, \Bbbk) \arrow[r, "\sim"] & D^{\mathrm{b}}(R_I\text{-mod})
\end{tikzcd}$$

(2) *For any $\mathcal{F} \in D^{\mathrm{b}}_{\mathrm{loc}}(U, \Bbbk)$, the object $j_I^! \Psi_f(h_! \mathcal{F})$ lies in $D^{\mathrm{b}}_{\mathrm{loc}}(X_I, \Bbbk)$. Moreover, there is a commutative diagram*

$$\begin{tikzcd}
D^{\mathrm{b}}_{\mathrm{loc}}(U, \Bbbk) \arrow[r, "\sim"] \arrow[d, "\mathcal{F} \mapsto j_I^! \Psi_f(h_! \mathcal{F})"'] & D^{\mathrm{b}}(R\text{-mod}) \arrow[d, "M \mapsto \Bbbk[\pi_1(\mathbb{C}^\times, f(u_0))] \otimes^L_{\Bbbk[\pi_f]} R_I \otimes^L_R R\mathrm{Hom}_{\Bbbk[H_f]}(R, \mathrm{Res}^R_{\Bbbk[H_f]} M)[-2|I|](-|I|)"] \\
D^{\mathrm{b}}_{\mathrm{loc}}(X_I, \Bbbk) \arrow[r, "\sim"] & D^{\mathrm{b}}(R_I\text{-mod})
\end{tikzcd}$$

PROOF. Let $\mathcal{F}' = \exp_{U*} \exp_U^* \mathcal{F}$. This is an object in $D^{\mathrm{b}}_{\mathrm{loc}}(U, \Bbbk)$; as in the statement of Lemma B.5.3, it corresponds under (B.2.2) to

$$\begin{aligned}(\text{B.5.4}) \quad & \Bbbk[\pi_1(\mathbb{C}^\times, f(u_0))] \overset{L}{\otimes}_{\Bbbk[\pi_f]} R\mathrm{Hom}_{\Bbbk[H_f]}(R, \mathrm{Res}^R_{\Bbbk[H_f]} M) \\ & \cong R\mathrm{Hom}_{\Bbbk[\pi_f]}(\Bbbk[\pi_1(\mathbb{C}^\times, f(u_0))], R\mathrm{Hom}_{\Bbbk[H_f]}(R, \mathrm{Res}^R_{\Bbbk[H_f]} M))\end{aligned}$$

in $D^{\mathrm{b}}(R\text{-mod})$. (The use of derived functors here is justified by Remark 1.9.8.)

For part (1) of the proposition, we have

$$(\text{B.5.5}) \quad j_I^* \Psi_f(h_* \mathcal{F}) \cong j_I^* j_{\boldsymbol{\eta}*} \exp_{X*} \exp_X^* h_* \mathcal{F}[-1] \cong j_I^* j_{\boldsymbol{\eta}*} h_* \mathcal{F}'[-1] \cong j_I^* j_* \mathcal{F}'[-1].$$

By Proposition B.2.3 together with (B.5.4), the object $j_I^* j_* \mathcal{F}'[-1]$ corresponds under (B.2.2) to

$$\begin{aligned} & R\mathrm{Hom}_R(R_I, R\mathrm{Hom}_{\Bbbk[\pi_f]}(\Bbbk[\pi_1(\mathbb{C}^\times, f(u_0))], R\mathrm{Hom}_{\Bbbk[H_f]}(R, \mathrm{Res}^R_{\Bbbk[H_f]} M)))[-1] \\ & \cong R\mathrm{Hom}_{\Bbbk[\pi_f]}(R_I \overset{L}{\otimes}_R \Bbbk[\pi_1(\mathbb{C}^\times, f(u_0))], R\mathrm{Hom}_{\Bbbk[H_f]}(R, \mathrm{Res}^R_{\Bbbk[H_f]} M))[-1] \\ & \cong R\mathrm{Hom}_{\Bbbk[H_f]}(R_I \overset{L}{\otimes}_R R \overset{L}{\otimes}_{\Bbbk[\pi_f]} \Bbbk[\pi_1(\mathbb{C}^\times, f(u_0))], \mathrm{Res}^R_{\Bbbk[H_f]} M)[-1] \\ & \cong R\mathrm{Hom}_{\Bbbk[\pi_f]}(\Bbbk[\pi_1(\mathbb{C}^\times, f(u_0))], R\mathrm{Hom}_{\Bbbk[H_f]}(R_I, \mathrm{Res}^R_{\Bbbk[H_f]} M))[-1].\end{aligned}$$

For part (2), by Corollary 1.3.12, we may write $\Psi_f(\mathcal{G}) \cong i^!_{\mathbf{s}} j_{\boldsymbol{\eta}!}(\exp_{X*} \exp_X^* \mathcal{G})$. Combining this observation with Lemma B.5.1, we have

$$j_I^! \Psi_f(h_! \mathcal{F}) \cong j_I^! j_{\boldsymbol{\eta}!} \exp_{X*} \exp_X^* h_! \mathcal{F} \cong j_I^! j_{\boldsymbol{\eta}!} h_! \mathcal{F}' \cong j_I^! j_! \mathcal{F}'.$$

The rest of the proof is a calculation similar to that above, using Proposition B.4.7 instead of Proposition B.2.3. We omit further details. ☐

EXAMPLE B.5.5. Suppose $j = k = 1$, so that $U = X_{\boldsymbol{\eta}}$, and h is the identity map. Assume that $f : \mathbb{C}^n \to \mathbb{C}$ satisfies (B.5.1) and (B.5.2). Then H_f is the trivial group and $R_{\{1\}} \cong \Bbbk$, so the formulas in Proposition B.5.4 simplify somewhat: if $\mathcal{L} \in \mathrm{Loc}(U, \Bbbk)$ corresponds to M, then $\Psi_f(\mathcal{L})[1]$ is the constant sheaf with value

$$\Bbbk[\pi_1(\mathbb{C}^\times, f(u_0))] \otimes_{\Bbbk[\pi_f]} M \cong \mathrm{Hom}_{\Bbbk[\pi_f]}(\Bbbk[\pi_1(\mathbb{C}^\times, f(u_0))], M).$$

In particular, any stalk of $\Psi_f(\mathcal{L})[1]$ is isomorphic to a $[\pi_1(\mathbb{C}^\times, f(u_0)) : \pi_f]$-fold direct sum of copies of a stalk of $\mathcal{L}$. This example is a generalization of Example 4.1.4.

EXAMPLE B.5.6. Suppose $j = k = 2$. As in the previous example, we have $U = X_{\boldsymbol{\eta}}$, and h is the identity map. Let $v : X_1 \hookrightarrow Z_1$ be the inclusion map. Combining (B.5.5) with Lemma B.4.5, we see that

$$\Psi_f(\mathcal{L})|_{Z_1} \cong v_*(\Psi_f(\mathcal{L})|_{X_1}),$$

and in particular, $\Psi_f(\mathcal{L})|_{X_{12}} \cong (v_*(\Psi_f(\mathcal{L})|_{X_1}))|_{X_{12}}$. Let $p = p_{12}|_{Z_1} : Z_1 \to X_{12}$. By Proposition B.4.2, we have $(v_*(\Psi_f(\mathcal{L})|_{X_1}))|_{X_{12}} \cong p_* v_*(\Psi_f(\mathcal{L})|_{X_1})$, and hence

$$R\Gamma(\Psi_f(\mathcal{L})|_{X_{12}}) \cong R\Gamma(\Psi_f(\mathcal{L})|_{X_1}).$$

LEMMA B.5.7. *Let $J = \{1, \ldots, j\}$. Let $V = X_{\mathbf{s}} \cap X_{\subset J}$, and let $v : V \hookrightarrow X_{\mathbf{s}}$ be the inclusion map. Let $f' = f|_{X_{\subset J}} : X_{\subset J} \to \mathbb{C}$. For any $\mathcal{F} \in D^{\mathrm{b}}_{\mathrm{loc}}(U, \Bbbk)$, we have natural isomorphisms*

$$\Psi_f(h_* \mathcal{F}) \cong v_* \Psi_{f'}(\mathcal{F}), \qquad \Psi_f(h_! \mathcal{F}) \cong v_! \Psi_{f'}(\mathcal{F}).$$

PROOF. Let $\mathcal{F}' = \exp_{U*} \exp_U^* \mathcal{F}$. Calculations similar to (B.5.5) show that

$$\Psi_f(h_*\mathcal{F}) \cong i_{\mathbf{s}}^! j_{\boldsymbol{\eta}!} h_* \mathcal{F}', \qquad \Psi_f(h_!\mathcal{F}) \cong i_{\mathbf{s}}^* j_{\boldsymbol{\eta}*} h_! \mathcal{F}'[-1].$$

Now let $I = \{j+1, \ldots, k\}$, and consider the commutative diagram

$$\begin{tikzcd} U \arrow[r, "j'_{\boldsymbol{\eta}}"] \arrow[d, "h"'] & X_{\subset J} \arrow[d, "u_J"] & V \arrow[l, "i'_{\mathbf{s}}"'] \arrow[d, "v"] \\ X_{\boldsymbol{\eta}} = X_{\subset I} \arrow[r, "j_{\boldsymbol{\eta}}"] & \mathbb{C}^n & X_{\mathbf{s}} \arrow[l, "i_{\mathbf{s}}"'] \end{tikzcd}$$

where all the maps are inclusion maps. Using Lemma B.4.4 and proper base change, we find that

$$\Psi_f(h_*\mathcal{F}) \cong i_{\mathbf{s}}^! u_{J*} j'_{\boldsymbol{\eta}!} \mathcal{F}' \cong v_* (i'_{\mathbf{s}})^! j_{\boldsymbol{\eta}!} \mathcal{F}' \cong v_* \Psi_{f'}(\mathcal{F}),$$
$$\Psi_f(h_!\mathcal{F}) \cong i_{\mathbf{s}}^* u_{J!} j'_{\boldsymbol{\eta}*} \mathcal{F}'[-1] \cong v_! (i'_{\mathbf{s}})^* j_{\boldsymbol{\eta}*} \mathcal{F}'[-1] \cong v_! \Psi_{f'}(\mathcal{F}),$$

as desired. $\square$

Constructibility. We now turn to the question of whether Ψ_f takes constructible complexes to constructible complexes. To make progress on this question, we must impose some additional conditions on f. For $1 \le i \le k$, let $f_i : \mathbb{C}^\times \to \mathbb{C}^\times$ be the function given by

$$f_i(z) = f(u_1, \ldots, u_{i-1}, z, u_{i+1}, \ldots, u_n).$$

Most results in this section involve the following additional assumption:

(B.5.6) For $1 \le i \le j$, the map $f_{i\sharp} : \pi_1(\mathbb{C}^\times, u_i) \to \pi_1(\mathbb{C}^\times, f(u_0))$ is nontrivial.

It is obvious that (B.5.6) implies (B.5.2).

REMARK B.5.8. Let $f : \mathbb{C}^n \to \mathbb{C}$ be a function satisfying (B.5.1) and (B.5.6). Choose a point x in some stratum X_I, and then consider a neighborhood D of x as in Remark B.2.1. It is easy to see from the definitions that $f|_D : D \to \mathbb{C}$ again satisfies (the appropriate analogues of) conditions (B.5.1) and (B.5.6).

LEMMA B.5.9. *Suppose $f : \mathbb{C}^n \to \mathbb{C}$ satisfies* (B.5.1)*. If f is holomorphic, then it satisfies* (B.5.6) *as well.*

PROOF. Let $\gamma : [0,1] \to \mathbb{C}$ be a differentiable simple closed curve based at u_i, and whose interior Ω contains the point 0. Thus, γ gives a generator of $\pi_1(\mathbb{C}^\times, u_i)$. To finish the proof, it is enough to show that if $1 \le i \le j$, then $f_i \circ \gamma : [0,1] \to \mathbb{C}^\times$ has nonzero winding number. This is an exercise in complex analysis: the winding number is given by

$$\frac{1}{2\pi i} \oint_{f_i \circ \gamma} \frac{dz}{z} = \frac{1}{2\pi i} \oint_\gamma \frac{f_i'(z)\, dz}{f_i(z)}.$$

But the argument principle tells us that this expression computes the number of zeroes (with multiplicity) of f_i in Ω, and this is nonzero. $\square$

REMARK B.5.10. Here is a minor variant of Lemma B.5.9. Suppose that for $i = 1, \ldots, n$, we have homeomorphisms $b_i : \mathbb{C} \to D_i$, where each D_i is a disc in $\mathbb{C}$, and such that $b_i(0) = 0$. These can be assembled into a homeomorphism $b = (b_1, \ldots, b_n) : \mathbb{C}^n \xrightarrow{\sim} D$, where D is the polydisc $D_1 \times \cdots \times D_n$. Let

$$g = f \circ b^{-1} : D \to \mathbb{C}.$$

Assume that g is holomorphic and that f satisfies (B.5.1). Then f will in general not be holomorphic, but the reasoning of Lemma B.5.9 can be carried out with g instead to show that f still satisfies (B.5.6).

LEMMA B.5.11. *Suppose $f : \mathbb{C}^n \to \mathbb{C}$ satisfies* (B.5.1) *and* (B.5.6)*. If $I \subset \{1, \dots, k\}$ contains at least one element from $\{1, \dots, j\}$, then R_I is finitely generated as a $\Bbbk[H_f]$-module.*

PROOF. Recall that $\pi_1(U, u_0)$ is a free abelian group on generators $T_1, \dots, T_k$. Let $\gamma_0, \gamma \in \pi_1(\mathbb{C}^\times, f(u_0))$ and $\tilde{\gamma} \in \pi_1(U, u_0)$ be as in the proof of Lemma B.5.2. For each generator T_i, there is an integer m_i such that $f_\sharp(T_i) = \gamma^{m_i}$. By replacing T_i by its inverse if necessary, we may assume that $m_i \geq 0$ for all i. The group H_f is generated by the set $\{T_i \tilde{\gamma}^{-m_i} : 1 \leq i \leq k\}$.

Note that $f_\sharp(T_i)$ generates the image of $f_{i\sharp} : \pi_1(\mathbb{C}^\times, u_i) \to \pi_1(\mathbb{C}^\times, f(u_0))$. Thus, assumption (B.5.6) implies that $m_i > 0$ if $1 \leq i \leq j$.

Observe that R is generated as a $\Bbbk[H_f]$-module by the set $S = \{\tilde{\gamma}^m\}_{m \in \mathbb{Z}}$. Recall from the discussion in Section B.2 that the ring map $\dot{p}_{I\sharp} : R \to R_I$ is surjective. Therefore, it is enough to show that $\dot{p}_{I\sharp}(S)$ is contained in the $\Bbbk[H_f]$-span of a finite set. Let $i \in I$ be such that $1 \leq i \leq j$. We saw in Section B.2 that $\dot{p}_{I\sharp}(T_i) = 1$. Since $T_i^{-1}\tilde{\gamma}^{m_i}$ lies in H_f, we conclude that

$$\dot{p}_{I\sharp}(\tilde{\gamma}^{m_i}) = \dot{p}_{I\sharp}(T_i^{-1}\tilde{\gamma}^{m_i})\dot{p}_{I\sharp}(T_i) \in \Bbbk[H_f] \cdot 1.$$

Since $m_i > 0$, we deduce that $\dot{p}_{I\sharp}(S)$ is contained in the $\Bbbk[H_f]$-span of the finite set $\{1, \dot{p}_{I\sharp}(\tilde{\gamma}), \dots, \dot{p}_{I\sharp}(\tilde{\gamma}^{m_i - 1})\}$. ☐

We are now ready to prove the following constructibility result.

PROPOSITION B.5.12. *Let $f : \mathbb{C}^n \to \mathbb{C}$ be a function that satisfies* (B.5.1) *and* (B.5.6)*. Then, for any $\mathcal{L} \in \mathrm{Loc}^{\mathrm{ft}}(U, \Bbbk)$, the objects $\Psi_f(h_*\mathcal{L})$ and $\Psi_f(h_!\mathcal{L})$ are constructible with respect to the stratification $(X_I)_{I \cap \{1,\dots,j\} \neq \varnothing}$ of $X_{\mathbf{s}}$.*

PROOF. Lemma B.5.7 implies that $\Psi_f(h_!\mathcal{L}) \cong v_!(\Psi_f(h_*\mathcal{L})|_V)$, so it is enough to prove that $\Psi_f(h_*\mathcal{L})$ is constructible. Specifically, we must show that for any stratum $X_I \subset X_{\mathbf{s}}$, the sheaf $\mathsf{H}^i(\Psi_f(h_*\mathcal{L})|_{X_I})$ is a local system of finite type, and that it vanishes for all but finitely many i.

Suppose $\mathcal{L}$ corresponds to the R-module M under (B.2.2). By assumption, M is finitely generated over $\Bbbk$. By Proposition B.5.4, $\mathsf{H}^i(\Psi_f(h_*\mathcal{L})|_{X_I})$ corresponds to

$$\mathsf{H}^i(R\mathrm{Hom}_{\Bbbk[\pi_f]}(\Bbbk[\pi_1(\mathbb{C}^\times, f(u_0))], R\mathrm{Hom}_{\Bbbk[H_f]}(R_I, \mathrm{Res}^R_{\Bbbk[H_f]} M))[-1]).$$

We must show that this R_I-module is finitely generated over $\Bbbk$ and that it vanishes for all but finitely many i.

The functor $\mathrm{Hom}_{\Bbbk[\pi_f]}(\Bbbk[\pi_1(\mathbb{C}^\times, f(u_0))], -)$ is exact and preserves the property of being finitely generated over $\Bbbk$ (because $\Bbbk[\pi_1(\mathbb{C}^\times, f(u_0))]$ is free of finite rank over $\Bbbk[\pi_f]$). So we may instead study

$$\mathsf{H}^i(R\mathrm{Hom}_{\Bbbk[H_f]}(R_I, \mathrm{Res}^R_{\Bbbk[H_f]} M)). \tag{B.5.7}$$

The fact that this vanishes for all but finitely many i follows immediately from the fact that $\Bbbk[H_f]$ has finite global dimension. On the other hand, since $\Bbbk[H_f]$ is noetherian and R_I is finitely generated as a $\Bbbk[H_f]$-module (by Lemma B.5.11), it admits a resolution by finitely generated projective $\Bbbk[H_f]$-modules. If P is a finitely generated projective $\Bbbk[H_f]$-module, then $\mathrm{Hom}_{\Bbbk[H_f]}(P, M)$ is finitely generated over

$\Bbbk$, because M is. It follows that (B.5.7) is finitely generated over $\Bbbk$ for all i, as desired. $\square$

Sheaf functor compatibilities. We conclude this section with a study of the compatibility of the nearby cycles functor with extension of scalars and with Verdier duality. The main technical tool is an alternative formula for a special case of Proposition B.5.4(2).

LEMMA B.5.13. *Let $I_0 = \{1, \ldots, k\}$, so that $R_{I_0} \cong \Bbbk$. For $M \in D^{\mathrm{b}}(R\text{-mod})$, there is a natural isomorphism*

$$R_{I_0} \overset{L}{\otimes}_R R\mathrm{Hom}_{\Bbbk[H_f]}(R, \mathrm{Res}^R_{\Bbbk[H_f]} M) \cong R_{I_0} \overset{L}{\otimes}_{\Bbbk[H_f]} (\mathrm{Res}^R_{\Bbbk[H_f]} M)[1](1).$$

PROOF. In this proof, for brevity, we simply write $\Bbbk$ instead of R_{I_0}, and we omit $\mathrm{Res}^R_{\Bbbk[H_f]}$ from the notation. Let γ be a generator of $\pi_f \subset \pi_1(\mathbb{C}^\times, f(u_0))$, and let $\tilde{\gamma} \in \pi_1(U, u_0)$ be an element in the preimage of γ under $(f|_U)_\sharp : \pi_1(U, u_0) \to \pi_1(\mathbb{C}^\times, f(u_0))$. In analogy with (4.1.3), there is a short exact sequence

$$0 \to \Bbbk[\gamma, \gamma^{-1}](1) \xrightarrow{d} \Bbbk[\gamma, \gamma^{-1}] \to \Bbbk \to 0,$$

where the map d is given by multiplication by $1 - \gamma$. We can rewrite this as

$$0 \to R \otimes_{\Bbbk[H_f]} \Bbbk(1) \xrightarrow{d} R \otimes_{\Bbbk[H_f]} \Bbbk \to \Bbbk \to 0. \tag{B.5.8}$$

Next, as in the proof of Lemma B.5.11, R is a free $\Bbbk[H_f]$-module with basis $\{\tilde{\gamma}^m\}_{m \in \mathbb{Z}}$. We can thus form a short exact sequence of $\Bbbk[H_f]$-modules

$$0 \to R(1) \xrightarrow{1-\tilde{\gamma}} R \to \Bbbk[H_f] \to 0, \tag{B.5.9}$$

where the map $R \to \Bbbk[H_f]$ sends every basis element $\tilde{\gamma}^m$ to 1. Then (B.5.8) is obtained by applying $(-) \otimes_{\Bbbk[H_f]} \Bbbk$ to (B.5.9).

Let P be the chain complex

$$\cdots \to 0 \to R \xrightarrow{1-\tilde{\gamma}} R(-1) \to 0 \to \cdots,$$

in degrees 0 and 1. Then (B.5.9) shows that there is a quasi-isomorphism

$$P \to \Bbbk[H_f][-1](-1). \tag{B.5.10}$$

Let $\mathrm{Proj}(R)$ denote the additive category of projective R-modules. Define a functor $F : K^{\mathrm{b}}\mathrm{Proj}(R) \to D^{\mathrm{b}}(\Bbbk\text{-mod})$ by

$$F(M) = \Bbbk \overset{\mathrm{ch}}{\otimes}_{\Bbbk[H_f]} \mathrm{chHom}_{\Bbbk[H_f]}(P, M).$$

Note that R has finite global dimension (because $\Bbbk$ does), and hence $K^{\mathrm{b}}\mathrm{Proj}(R) \cong D^{\mathrm{b}}(R\text{-mod})$. Thus, F can be regarded as a functor $D^{\mathrm{b}}(R\text{-mod}) \to D^{\mathrm{b}}(\Bbbk\text{-mod})$. We will use this functor as an intermediary between the two functors in the statement of the proposition.

Step 1. There is a canonical identification

$$F(M) \cong (P \overset{\mathrm{ch}}{\otimes}_{\Bbbk[H_f]} \Bbbk) \overset{\mathrm{ch}}{\otimes}_R \mathrm{chHom}_{\Bbbk[H_f]}(R, M)[1](1).$$

This claim follows from the observation that when M is a single projective R-module, we have the following commutative square:

$$\begin{tikzcd}
\Bbbk \otimes_{\Bbbk[H_f]} \mathrm{Hom}_{\Bbbk[H_f]}(R(-1), M) \arrow[r, "1-\tilde{\gamma}"] \arrow[d, "\wr"'] & \Bbbk \otimes_{\Bbbk[H_f]} \mathrm{Hom}_{\Bbbk[H_f]}(R, M) \arrow[d, "\wr"] \\
(R(1) \otimes_{\Bbbk[H_f]} \Bbbk) \otimes_R \mathrm{Hom}_{\Bbbk[H_f]}(R, M) \arrow[r, "1-\tilde{\gamma}"] & (R \otimes_{\Bbbk[H_f]} \Bbbk) \otimes_R \mathrm{Hom}_{\Bbbk[H_f]}(R, M)
\end{tikzcd}$$

Here, the top row is the definition of $F(M)$ and the bottom row is $(P \otimes^{\mathrm{ch}}_{\Bbbk[H_f]} \Bbbk) \otimes^{\mathrm{ch}}_R \mathrm{chHom}_{\Bbbk[H_f]}(R, M)[1](1)$.

Step 2. The terms of $\mathrm{chHom}_{\Bbbk[H_f]}(R, M)$ *or* $\mathrm{chHom}_{\Bbbk[H_f]}(P, M)$ *are flat over* $\Bbbk[H_f]$. It is enough to prove that if N is a free R-module, then $\mathrm{Hom}_{\Bbbk[H_f]}(R, N)$ is flat over $\Bbbk[H_f]$. Since R is free as a $\Bbbk[H_f]$-module, N is also free (and hence flat), and $\mathrm{Hom}_{\Bbbk[H_f]}(R, N)$ is isomorphic as a $\Bbbk[H_f]$-module to the product of infinitely many copies of N. Since $\Bbbk[H_f]$ is noetherian, Chase's theorem (see [**145**, Theorem 4.47]) says that the product of infinitely many copies of N is also flat over $\Bbbk[H_f]$.

Step 3. There is a natural isomorphism $F(M) \cong \Bbbk \otimes^L_{\Bbbk[H_f]} M[1](1)$. Since the terms of P are free over $\Bbbk[H_f]$, (B.5.10) implies that we have quasi-isomorphisms

$$\mathrm{chHom}_{\Bbbk[H_f]}(P, M) \to \mathrm{chHom}_{\Bbbk[H_f]}(\Bbbk[H_f][-1](-1), M) \cong M[1](1).$$

Thus, Step 2 tells us that $\mathrm{chHom}_{\Bbbk[H_f]}(P, M)$ is a flat resolution of $M[1](1)$. The claim follows.

Step 4. There is a natural isomorphism $F(M) \cong \Bbbk \otimes^L_R R\mathrm{Hom}_{\Bbbk[H_f]}(R, M)$. Since R is free over $\Bbbk[H_f]$, we have $\mathrm{chHom}_{\Bbbk[H_f]}(R, M) \cong R\mathrm{Hom}_{\Bbbk[H_f]}(R, M)$. Unfortunately, the terms of this chain complex may not be flat over R. Nevertheless, the universal property of the derived functor $\otimes^L$ gives rise to the vertical arrows in the following commutative diagram of distinguished triangles:

$$\begin{tikzcd}
(R(1) \underset{\Bbbk[H_f]}{\otimes} \Bbbk) \overset{L}{\underset{R}{\otimes}} R\mathrm{Hom}(R, M) \arrow[r] \arrow[d] & (R \underset{\Bbbk[H_f]}{\otimes} \Bbbk) \overset{L}{\underset{R}{\otimes}} R\mathrm{Hom}(R, M) \arrow[r] \arrow[d] & \Bbbk \overset{L}{\underset{R}{\otimes}} R\mathrm{Hom}(R, M) \arrow[r] \arrow[d] & {} \\
(R(1) \overset{\mathrm{ch}}{\underset{\Bbbk[H_f]}{\otimes}} \Bbbk) \overset{\mathrm{ch}}{\underset{R}{\otimes}} \mathrm{chHom}(R, M) \arrow[r] & (R \overset{\mathrm{ch}}{\underset{\Bbbk[H_f]}{\otimes}} \Bbbk) \overset{\mathrm{ch}}{\underset{R}{\otimes}} \mathrm{chHom}(R, M) \arrow[r] & (P[1](1) \overset{\mathrm{ch}}{\underset{\Bbbk[H_f]}{\otimes}} \Bbbk) \overset{\mathrm{ch}}{\underset{R}{\otimes}} \mathrm{chHom}(R, M) \arrow[r] & {}
\end{tikzcd}$$

By Step 2, the terms of $\mathrm{chHom}_{\Bbbk[H_f]}(R, M) \cong R \otimes^{\mathrm{ch}}_R \mathrm{chHom}_{\Bbbk[H_f]}(R, M)$ are flat over $\Bbbk[H_f]$, and so the first and second vertical maps are isomorphisms in the derived category. It follows that the third vertical map is as well. $\square$

Combining Lemma B.5.13 with Proposition B.5.4(2) yields the following result.

COROLLARY B.5.14. *Let* $I_0 = \{1, \ldots, k\}$. *If* $\mathcal{L} \in \mathrm{Loc}(U, \Bbbk)$ *corresponds to* $M \in R\text{-mod}$, *then* $j^!_{I_0} \Psi_f(h_! \mathcal{L})$ *corresponds under* (B.2.2) *to*

$$\Bbbk[\pi_1(\mathbb{C}^\times, f(u_0))] \overset{L}{\otimes}_{\Bbbk[\pi_f]} R_{I_0} \overset{L}{\otimes}_{\Bbbk[H_f]} \mathrm{Res}^R_{\Bbbk[H_f]} M[1-2k](1-k).$$

PROPOSITION B.5.15. *For any* $\mathcal{F} \in D^{\mathrm{b}}_{\mathrm{locf}}(U, \Bbbk)$, *the natural map* $\Psi_f(\mathbb{D}(h_! \mathcal{F})) \to \mathbb{D}(\Psi_f(h_! \mathcal{F}))(1)$ *is an isomorphism.*

The natural map in this statement comes from Corollary 4.1.9.

PROOF. Let $M \in D^{\mathrm{b}}(R\text{-mod})$ be the object corresponding to $\mathcal{F}$ under (B.2.2). Let $\mathcal{F}^\vee = R\mathscr{H}om(\mathcal{F}, \underline{\Bbbk}_U) \cong (\mathbb{D}\mathcal{F})[-2n](-n)$. By Proposition 1.7.11 (and Remark 1.9.8), $\mathcal{F}^\vee$ corresponds to $M^\vee = R\mathrm{Hom}_\Bbbk(M, \Bbbk)$ in $D^{\mathrm{b}}(R\text{-mod})$.

It is enough to show that $j_I^*\Psi_f(\mathbb{D}(h_!\mathcal{F})) \to j_I^*\mathbb{D}(\Psi_f(h_!\mathcal{F}))(1)$ is an isomorphism for each stratum $X_I \subset X_{\mathbf{s}}$. We have $\mathbb{D}(h_!\mathcal{F}) \cong h_*\mathcal{F}^\vee[2n](n)$, so by Proposition B.5.4, $j_I^*\Psi_f(\mathbb{D}(h_!\mathcal{F}))$ corresponds under (B.2.2) to

$$R\mathrm{Hom}_{\Bbbk[\pi_f]}(\Bbbk[\pi_1(\mathbb{C}^\times, f(u_0))], R\mathrm{Hom}_{\Bbbk[H_f]}(R_I, M^\vee))[2n-1](n). \tag{B.5.11}$$

Let us first consider the case of the closed stratum: assume that $I = I_0 = \{1, \dots, k\}$. We have

$$\begin{aligned} R\mathrm{Hom}_{\Bbbk[H_f]}(R_{I_0}, M^\vee)[2n-1](n) &\cong R\mathrm{Hom}_{\Bbbk[H_f]}(R_{I_0}, R\mathrm{Hom}_\Bbbk(M,\Bbbk))[2n-1](n) \\ &\cong R\mathrm{Hom}_\Bbbk(R_{I_0} \otimes_{\Bbbk[H_f]} M[1-2k](1-k), \Bbbk)[2n-2k](n-k+1). \end{aligned}$$

It follows that (B.5.11) is naturally isomorphic to
(B.5.12)

$$R\mathrm{Hom}_\Bbbk(\Bbbk[\pi_1(\mathbb{C}^\times, f(u_0))] \overset{L}{\underset{\Bbbk[\pi_f]}{\otimes}} R_{I_0} \overset{L}{\underset{\Bbbk[H_f]}{\otimes}} M[1-2k](1-k), \Bbbk[2n-2k](n-k+1)).$$

Recall that $\dim X_{I_0} = n-k$, so $\omega_{X_{I_0}} \cong \underline{\Bbbk}_{X_{I_0}}[2n-2k](n-k)$. Using Corollary B.5.14, we see that (B.5.12) corresponds to

$$\begin{aligned} &R\mathscr{H}om(j_{I_0}^!\Psi_f(h_!\mathcal{F}), \underline{\Bbbk}_{X_{I_0}}[2n-2k](n-k))(1) \\ &\qquad\qquad \cong \mathbb{D}(j_{I_0}^!\Psi_f(h_!\mathcal{F}))(1) \cong j_{I_0}^*\mathbb{D}(\Psi_f(h_!\mathcal{F}))(1), \end{aligned}$$

as desired.

For $I \neq I_0$, instead of treating the whole stratum at once, it is enough to show that every point in X_I has an analytic neighborhood over which $j_I^*\Psi_f(\mathbb{D}(h_!\mathcal{F})) \to j_I^*\mathbb{D}(\Psi_f(h_!\mathcal{F}))(1)$ is an isomorphism. By choosing this analytic neighborhood as in Remark B.2.1, we reduce the problem for an arbitrary stratum to the case of a closed stratum, which we have already dealt with above. □

PROPOSITION B.5.16. *Let $\varphi : \Bbbk \to \Bbbk'$ be a ring homomorphism. For any $\mathcal{F} \in D^{\mathrm{b}}_{\mathrm{locf}}(U, \Bbbk)$, the natural map $\Bbbk' \otimes^L_\Bbbk \Psi_f(h_!\mathcal{F}) \to \Psi_f(\Bbbk' \otimes^L_\Bbbk h_!\mathcal{F})$ is an isomorphism.*

PROOF. By Exercise 2.3.1, it is enough to show that $j_I^!(\Bbbk' \otimes^L_\Bbbk \Psi_f(h_!\mathcal{F})) \to j_I^!\Psi_f(\Bbbk' \otimes^L_\Bbbk h_!\mathcal{F})$ is an isomorphism for each stratum $X_I \subset X_{\mathbf{s}}$. Moreover, as in the proof of Proposition B.5.15, Remark B.2.1 lets us reduce to the case of the closed stratum X_{I_0}, corresponding to $I_0 = \{1, \dots, k\}$.

Let $R' = \Bbbk' \otimes_\Bbbk R$, and let $R'_{I_0} = \Bbbk' \otimes_\Bbbk R_{I_0}$. By Corollary B.5.14, our problem comes down to checking that the natural map

$$\begin{aligned} &\Bbbk' \overset{L}{\otimes}_\Bbbk (\Bbbk[\pi_1(\mathbb{C}^\times, f(u_0))] \overset{L}{\otimes}_{\Bbbk[\pi_f]} R_{I_0} \overset{L}{\otimes}_{\Bbbk[H_f]} \mathrm{Res}^R_{\Bbbk[H_f]} M) \\ &\qquad \to \Bbbk'[\pi_1(\mathbb{C}^\times, f(u_0))] \overset{L}{\otimes}_{\Bbbk'[\pi_f]} R'_{I_0} \overset{L}{\otimes}_{\Bbbk'[H_f]} \mathrm{Res}^{R'}_{\Bbbk'[H_f]}(\Bbbk' \overset{L}{\otimes}_\Bbbk M) \end{aligned}$$

is an isomorphism. This claim is clear. □

B.6. Equivariant sheaves on $\mathbb{C}$

Let $u : \mathbb{A}^1 \smallsetminus \{0\} \hookrightarrow \mathbb{A}^1$ be the inclusion map, and let

$$\mathcal{J} = u_*\underline{\Bbbk}_{\mathbb{A}^1 \smallsetminus \{0\}}[1](1),$$

regarded as an object in the equivariant derived category $D^{\mathrm{b}}_{\mathbb{G}_{\mathrm{m}}}(\mathbb{A}^1, \Bbbk)$, where $\mathbb{G}_{\mathrm{m}}$ acts on $\mathbb{A}^1$ by scaling. In this section, we carry out some computations involving this object. The most significant result is its "multiplicativity" property, as expressed in Proposition B.6.3. The results of this section are used in Section 6.9.

LEMMA B.6.1. *We have* $R\Gamma_{\mathrm{c}}(\mathcal{J}) = 0$.

PROOF. This follows from Lemma B.2.6. $\square$

LEMMA B.6.2. *Let* $s : \{0\} \hookrightarrow \mathbb{A}^1$ *be the inclusion map. There are canonical isomorphisms* $s^*\mathcal{J} \cong \mathrm{Av}_{1*}^{\mathbb{G}_{\mathrm{m}}} \underline{\Bbbk}_{\mathrm{pt}}[1](1) \cong \mathrm{Av}_{1!}^{\mathbb{G}_{\mathrm{m}}} \underline{\Bbbk}_{\mathrm{pt}}$.

PROOF. Consider the constant map $\mathrm{a}_{\mathbb{A}^1} : \mathbb{A}^1 \to \mathrm{pt}$. By Theorem 2.10.3, we have a natural isomorphism

$$\mathrm{a}_{\mathbb{A}^1 *}\mathcal{J} \xrightarrow{\sim} s^*\mathcal{J}.$$

The left-hand side simplifies to just $(\mathrm{a}_{\mathbb{A}^1 \smallsetminus \{0\}})_* \underline{\Bbbk}_{\mathbb{A}^1 \smallsetminus \{0\}}[1](1)$. Now identify $\mathbb{A}^1 \smallsetminus \{0\}$ with $\mathbb{G}_{\mathrm{m}} \times^1 \mathrm{pt}$, and let $i : \mathrm{pt} \hookrightarrow \mathbb{G}_{\mathrm{m}} \times^1 \mathrm{pt}$ be the inclusion map. We obviously have $\underline{\Bbbk}_{\mathrm{pt}}[1](1) \cong i^*\mathrm{For}_1^{\mathbb{G}_{\mathrm{m}}} \underline{\Bbbk}_{\mathbb{A}^1 \smallsetminus \{0\}}[1](1)$, so from the definition of $\mathrm{Av}_{1*}^{\mathbb{G}_{\mathrm{m}}}$ (see the proof of Theorem 6.6.1), we deduce the first claim in the statement of the lemma.

For the second claim, using the fact that $i^!\mathrm{For}_1^{\mathbb{G}_{\mathrm{m}}} \underline{\Bbbk}_{\mathbb{A}^1 \smallsetminus \{0\}} \cong \underline{\Bbbk}_{\mathrm{pt}}[-2](-1)$, we find that

$$\mathrm{Av}_{1!}^{\mathbb{G}_{\mathrm{m}}} \underline{\Bbbk}_{\mathrm{pt}} \cong (\mathrm{a}_{\mathbb{A}^1 \smallsetminus \{0\}})_! \underline{\Bbbk}_{\mathbb{A}^1 \smallsetminus \{0\}}[2](1) \cong \mathrm{a}_{\mathbb{A}^1 !} u_! \underline{\Bbbk}_{\mathbb{A}^1 \smallsetminus \{0\}}[2](1).$$

Now apply $\mathrm{a}_{\mathbb{A}^1 !}$ to the distinguished triangle

$$u_! \underline{\Bbbk}_{\mathbb{A}^1 \smallsetminus \{0\}}[1](1) \to \mathcal{J} \to s_*s^*\mathcal{J} \to .$$

Since $\mathrm{a}_{\mathbb{A}^1 !}(\mathcal{J}) = 0$ (see Lemma B.6.1), we deduce that

$$s^*\mathcal{J} \cong \mathrm{a}_{\mathbb{A}^1 !} s_* s^* \mathcal{J} \cong \mathrm{a}_{\mathbb{A}^1 !} u_! \underline{\Bbbk}_{\mathbb{A}^1 \smallsetminus \{0\}}[2](1) \cong \mathrm{Av}_{1!}^{\mathbb{G}_{\mathrm{m}}} \underline{\Bbbk}_{\mathrm{pt}}. \qquad \square$$

PROPOSITION B.6.3. *Let* $\alpha : \mathbb{A}^2 \to \mathbb{A}^1$ *be the addition map* $\alpha(x, y) = x + y$. *Let* $\mathbb{G}_{\mathrm{m}}^{\Delta}$ *denote the subgroup* $\{(z, z) \mid z \in \mathbb{G}_{\mathrm{m}}\} \subset \mathbb{G}_{\mathrm{m}}^2$. *Then, in* $D^{\mathrm{b}}_{\mathbb{G}_{\mathrm{m}}^2}(\mathbb{A}^2, \Bbbk)$, *there is a canonical isomorphism* $\mathrm{Av}_{\mathbb{G}_{\mathrm{m}}^{\Delta}!}^{\mathbb{G}_{\mathrm{m}}^2} \alpha^*\mathcal{J} \cong \mathcal{J} \boxtimes \mathcal{J}$.

PROOF. Let $\delta : \mathbb{G}_{\mathrm{m}}^{\Delta} \hookrightarrow \mathbb{G}_{\mathrm{m}}^2$ be the inclusion map. To reduce clutter, in this proof, we will write For_δ and $\mathrm{Av}_{\delta !}$ instead of $\mathrm{For}_{\mathbb{G}_{\mathrm{m}}^{\Delta}}^{\mathbb{G}_{\mathrm{m}}^2}$ and $\mathrm{Av}_{\mathbb{G}_{\mathrm{m}}^{\Delta}!}^{\mathbb{G}_{\mathrm{m}}^2}$, respectively. The proof will involve converting the question into one about constructible sheaves on $\mathbb{A}^3$ and $\mathbb{A}^2$.

Step 1. Notation for subvarieties of affine space. Let $X = \mathbb{A}^3$. A point of X will typically be written as (z, x, y). Let $X^\circ = \mathbb{G}_{\mathrm{m}} \times \mathbb{A}^2 = \{(z, x, y) \in X \mid z \neq 0\}$. Let $f : X \to \mathbb{A}^1$ be the map $f(z, x, y) = x + zy$. Let $W = f^{-1}(\mathbb{A}^1 \smallsetminus \{0\})$, and let $W^\circ = X^\circ \cap W$. Let $q : X \to \mathbb{A}^2$ be the projection onto the last two coordinates: that is, $q(z, x, y) = (x, y)$. We define several other maps between these spaces as shown in the following diagram:

$$\begin{tikzcd}
W^\circ \arrow[rr, "h^\circ"] \arrow[d, "g^\circ"'] & & W \arrow[r, "f|_W"] \arrow[d, "g"'] & \mathbb{A}^1 \smallsetminus \{0\} \arrow[d, "u"] \\
X^\circ \arrow[rr, "h"] \arrow[dr, "q^\circ = q \circ h"'] & & X = \mathbb{A}^3 \arrow[r, "f"] \arrow[dl, "q"] & \mathbb{A}^1 \\
& \mathbb{A}^2 & &
\end{tikzcd} \tag{B.6.1}$$

Here, g, h, g°, h°, and u are all inclusion maps. Note that f is a smooth morphism, so by smooth base change, we have

$$(f^*\mathcal{J})|_{X^\circ} \cong h^* f^* u_* \underline{\Bbbk}_{\mathbb{A}^1 \smallsetminus \{0\}}[1](1) \cong g^\circ_* \underline{\Bbbk}_{W^\circ}[1](1). \tag{B.6.2}$$

Let us also establish some notation related to $\mathbb{A}^2$. Let $Y = \mathbb{A}^2$. Let $Z_1 = \{0\} \times \mathbb{A}^1$ and $Z_2 = \mathbb{A}^1 \times \{0\}$, both regarded as subsets of Y, and let $Z = Z_1 \cup Z_2$. Let $V = Y \smallsetminus Z$, and let

$$j : V \hookrightarrow Y, \quad i_1 : Z_1 \hookrightarrow Y, \quad i_2 : Z_2 \hookrightarrow Y, \quad i : Z \hookrightarrow Y$$

be the inclusion maps. It is clear from the definition that

$$j_* \underline{\Bbbk}_V[2](2) \cong \mathcal{J} \boxtimes \mathcal{J}. \tag{B.6.3}$$

Step 2. We have $\mathrm{Av}_{\delta !}\alpha^*\mathcal{J} \cong q^\circ_! g^\circ_* \underline{\Bbbk}_{W^\circ}[3](2)$. Let $b : \mathbb{G}_\mathrm{m}^2 \times^{\mathbb{G}_\mathrm{m}^\Delta} \mathbb{A}^2 \to X^\circ$ be the map given by $b(z_1, z_2, x, y) = (z_1 z_2^{-1}, z_1 x, z_2 y)$. It is easily checked that b is an isomorphism of varieties. Consider the diagram

$$\begin{tikzcd}
\mathbb{A}^2 \arrow[r, "i"] & \mathbb{G}_\mathrm{m}^2 \times^{\mathbb{G}_\mathrm{m}^\Delta} \mathbb{A}^2 \arrow[rr, "b"] \arrow[dr, "\bar\sigma"'] & & X^\circ \arrow[r, "f \circ h"] \arrow[dl, "q^\circ"] & \mathbb{A}^1 \\
& & \mathbb{A}^2 & &
\end{tikzcd}$$

where i and $\bar\sigma$ are as in the proof of Theorem 6.6.1. Let $\mathbb{G}_\mathrm{m}^2$ act on X by $(z_1, z_2) \cdot (z, x, y) = (z_1 z_2^{-1} z, z_1 x, z_2 y)$, and on $\mathbb{A}^1$ by $(z_1, z_2) \cdot x = z_1 x$. Then b and f are $\mathbb{G}_\mathrm{m}^2$-equivariant.

By Theorem 6.5.10, if $\mathcal{G} \in D^\mathrm{b}_{\mathbb{G}_\mathrm{m}^2}(\mathbb{G}_\mathrm{m}^2 \times^{\mathbb{G}_\mathrm{m}^\Delta} \mathbb{A}^2, \Bbbk)$ satisfies $i^*\mathrm{For}_\delta(\mathcal{G}) \cong \mathcal{F}$, then $i^!\mathrm{For}_\delta(\mathcal{G}) \cong \mathcal{F}[-2](-1)$. It follows that $\mathrm{Av}_{\delta !}(\mathcal{F}) \cong \bar\sigma_! \mathcal{G}[2](1) \cong q^\circ_! b_! \mathcal{G}[2](1)$.

Now take $\mathcal{G} = b^*((f^*\mathcal{J})|_{X^\circ})$. Note that $f \circ h \circ b \circ i = \alpha : \mathbb{A}^2 \to \mathbb{A}^1$. It follows that $i^*\mathrm{For}_\delta(\mathcal{G}) \cong \alpha^*\mathcal{J}$. The previous paragraph tells us that $\mathrm{Av}_{\delta !}(\alpha^*\mathcal{J}) \cong q^\circ_! b_! b^*((f^*\mathcal{J})|_{X^\circ})[2](1)$. Since b is an isomorphism of varieties, this simplifies to $q^\circ_!((f^*\mathcal{J})|_{X^\circ})[2](1)$, and then the claim follows from (B.6.2).

Step 3. We have $(q^\circ_! g^\circ_* \underline{\Bbbk}_{W^\circ})|_V \cong \underline{\Bbbk}_V[-1]$. For each space and map in the left-hand part of (B.6.1), denote the fiber product with $V \hookrightarrow Y$ by a subscript "V." We obtain a diagram

$$\begin{tikzcd}
W^\circ_V \arrow[rr, "h^\circ_V"] \arrow[d, "g^\circ_V"'] & & W_V \arrow[d, "g_V"'] \\
X^\circ_V = \mathbb{G}_\mathrm{m}^3 \arrow[rr, "h_V"] \arrow[dr, "q^\circ_V"'] & & X_V = \mathbb{A}^1 \times \mathbb{G}_\mathrm{m}^2 \arrow[dl, "q_V"] \\
& V = \mathbb{G}_\mathrm{m}^2 &
\end{tikzcd}$$

The claim we wish to prove is equivalent to the assertion that $q^\circ_{V!} g^\circ_{V*} \underline{\Bbbk}_{W^\circ_V} \cong \underline{\Bbbk}_V[-1]$, or, rephrasing again, to the claim that

$$q_{V!} h_{V!} h_V^* g_{V*} \underline{\Bbbk}_{W_V} \cong \underline{\Bbbk}_V[-1]. \tag{B.6.4}$$

Note that q_V admits a right inverse $s : V \to X_V$, given by $s(x, y) = (0, x, y)$. This map is a closed embedding, complementary to $h_V : X^\circ_V \to X_V$. There is therefore a natural distinguished triangle

$$h_{V!} h_V^* g_{V*} \underline{\Bbbk}_{W_V} \to g_{V*} \underline{\Bbbk}_{W_V} \to s_* s^* g_{V*} \underline{\Bbbk}_{W_V} \to .$$

The image of s is contained in W_V, so $s^* g_{V*}\underline{\Bbbk}_{W_V}$ simplifies to $\underline{\Bbbk}_V$. Our distinguished triangle becomes

$$h_{V!}h_V^* g_{V*}\underline{\Bbbk}_{W_V} \to g_{V*}\underline{\Bbbk}_{W_V} \to s_*\underline{\Bbbk}_V \to .$$

Now apply $q_{V!}$. To prove (B.6.4), we must show that

$$q_{V!}g_{V*}\underline{\Bbbk}_{W_V} = 0. \tag{B.6.5}$$

Consider the diagram

$$\begin{tikzcd}
\mathbb{G}_\mathrm{m} \times V \arrow[r, "{(z,x,y)\mapsto(z-xy^{-1},x,y)}"] \arrow[d, "u\times\mathrm{id}"'] & W_V \arrow[d, "g_V"] \\
\mathbb{A}^1 \times V \arrow[r, "{(z,x,y)\mapsto(z-xy^{-1},x,y)}"] \arrow[dr, "\mathrm{pr}_2"'] & X_V \arrow[d, "q_V"] \\
& V
\end{tikzcd}$$

It is easy to check that this diagram commutes and that the horizonal maps are isomorphisms of varieties. This diagram lets us identify $q_{V!}g_{V*}\underline{\Bbbk}_{W_V}$ with $R\Gamma_\mathrm{c}(u_*\underline{\Bbbk}_{\mathbb{G}_\mathrm{m}}) \boxtimes \underline{\Bbbk}_V$. Since $R\Gamma_\mathrm{c}(u_*\underline{\Bbbk}_{\mathbb{G}_\mathrm{m}}) = 0$ (cf. Lemma B.6.1), we see that (B.6.5) holds, and hence so does (B.6.4).

Step 4. We have $i_1^! q_!^\circ g_*^\circ \underline{\Bbbk}_{W^\circ} = 0$ and $i_2^! q_!^\circ g_*^\circ \underline{\Bbbk}_{W^\circ} = 0$. We will prove the first assertion; the second is similar. Let $p_1 : \mathbb{A}^2 \to Z_1$ be the map given by $p_1(x,y) = (0,y)$. Note that Z_1 is the fixed-point set for the action of the first copy of $\mathbb{G}_\mathrm{m} \subset \mathbb{G}_\mathrm{m}^2$. Since $q_!^\circ g_*^\circ \underline{\Bbbk}_{W^\circ}$ is $\mathbb{G}_\mathrm{m}^2$-equivariant, Theorem 2.10.3 tells us that there is a natural isomorphism

$$i_1^! q_!^\circ g_*^\circ \underline{\Bbbk}_{W^\circ} \xrightarrow{\sim} p_{1!} q_!^\circ g_*^\circ \underline{\Bbbk}_{W^\circ}.$$

Consider the diagram

$$\begin{tikzcd}
\mathbb{G}_\mathrm{m} \times \mathbb{G}_\mathrm{m} \times \mathbb{A}^1 \arrow[r, "{(z,t,y)\mapsto(z,t-zy,y)}"] \arrow[d, "\mathrm{id}\times u\times\mathrm{id}"] & W^\circ \arrow[d, "g^\circ"] \\
\mathbb{G}_\mathrm{m} \times \mathbb{A}^1 \times \mathbb{A}^1 \arrow[r, "{(z,t,y)\mapsto(z,t-zy,y)}"] \arrow[dr, "\mathrm{pr}_3"'] & X^\circ \arrow[d, "{p_1\circ q^\circ:(z,x,y)\mapsto y}"] \\
& Z_1 = \mathbb{A}^1
\end{tikzcd}$$

Again, this diagram commutes, and the horizontal maps are isomorphisms of varieties, so we can identify $p_{1!}q_!^\circ g_*^\circ \underline{\Bbbk}_{W^\circ}$ with $R\Gamma_\mathrm{c}(\underline{\Bbbk}_{\mathbb{G}_\mathrm{m}}) \boxtimes R\Gamma_\mathrm{c}(u_*\underline{\Bbbk}_{\mathbb{G}_\mathrm{m}}) \boxtimes \underline{\Bbbk}_{\mathbb{A}^1}$. As in Step 3, we have $R\Gamma_\mathrm{c}(u_*\underline{\Bbbk}_{\mathbb{G}_\mathrm{m}}) = 0$, and the claim follows.

Step 5. We have $i^! q_!^\circ g_*^\circ \underline{\Bbbk}_{W^\circ} = 0$. Let $Y_2 = Z_2 \smallsetminus \{(0,0)\}$. As a subset of Z, Y_2 is open, and the complement of $Z_1 \subset Z$. Let $k : Z_1 \hookrightarrow Z$ and $y : Y_2 \hookrightarrow Z$ be the inclusion maps. For any $\mathcal{G} \in D^\mathrm{b}_{\mathbb{G}_\mathrm{m}^2}(Y, \Bbbk)$, we have a distinguished triangle $k_*k^!i^!\mathcal{G} \to i^!\mathcal{G} \to y_*y^*i^!\mathcal{G} \to$. This can rewritten as

$$k_*i_1^!\mathcal{G} \to i^!\mathcal{G} \to y_*((i_2^!\mathcal{G})|_{Y_2}) \to .$$

For $\mathcal{G} = q_!^\circ g_*^\circ \underline{\Bbbk}_{W^\circ}$, the first and last terms vanish by Step 4, so the middle term does as well.

Step 6. Conclusion of the proof. Steps 3 and 5 tell us that $q_!^\circ g_*^\circ \underline{\Bbbk}_{W^\circ}[3](2) \cong j_*\underline{\Bbbk}_V[2](2)$. The result then follows from Step 2 and (B.6.3). ☐

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Quick reference

QR1. Triangulated categories

QR1.1 (Distinguished triangles)

In a distinguished triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h}$:

- f is an isomorphism $\iff Z = 0$.
- The triangle splits $\iff h = 0$. In this case, $Y \cong X \oplus Z$.

QR1.2 (Cohomological functors)

Cohomological functors take distinguished triangles to long exact sequences. E.g.:

- $\mathrm{Hom}(X, -)$ and $\mathrm{Hom}(-, X)$.
- $\mathsf{H}^0 : D(\mathscr{A}) \to \mathscr{A}$.

QR1.3 (Hom and Ext)

The inclusion functor $\mathscr{A} \to D(\mathscr{A})$ is fully faithful. For $X, Y \in \mathscr{A}$, we have

$$\mathrm{Ext}^n_{\mathscr{A}}(X, Y) = \mathrm{Hom}_{D(\mathscr{A})}(X, Y[n]).$$

If $\mathscr{A}$ has enough projectives or injectives, $R\mathrm{Hom}$ is defined, and for $X, Y \in D^+(\mathscr{A})$,

$$\mathrm{Hom}(X, Y[n]) \cong \mathsf{H}^n(R\mathrm{Hom}(X, Y)).$$

QR1.4 (Zero objects)

For $X \in D(\mathscr{A})$, $X = 0 \iff \mathsf{H}^i(X) = 0 \ \forall i$. However, a morphism $f : X \to Y$ can be nonzero even if $\mathsf{H}^i(f) = 0 \ \forall i$.

QR1.5 (Short exact sequences)

A sequence $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ in $\mathscr{A}$ is exact $\iff \exists h : Z \to X[1]$ in $D(\mathscr{A})$ such that $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h}$ is a disting'd triangle.

QR1.6 (Truncation)

For any $X \in D(\mathscr{A})$ and any $n \in \mathbb{Z}$, there is a functorial distinguished triangle

$$\tau^{\leq n} X \to X \to \tau^{\geq n+1} X \to$$

such that

$$\mathsf{H}^i(\tau^{\leq n} X) \cong \begin{cases} \mathsf{H}^i(X) \\ 0 \end{cases} \qquad \mathsf{H}^i(\tau^{\geq n+1} X) \cong \begin{cases} 0 & i \leq n, \\ \mathsf{H}^i(X) & i > n. \end{cases}$$

QR1.7 (t-structures)

$(\mathscr{T}^{\leq 0}, \mathscr{T}^{\geq 0})$ a nondegenerate t-structure on $\mathscr{T}$. Its heart $\mathscr{C} = \mathscr{T}^{\leq 0} \cap \mathscr{T}^{\geq 0}$ is an abelian category, and there are functors ${}^t\tau^{\leq n}$, ${}^t\tau^{\geq n}$, ${}^t\mathsf{H}^n$ that behave as in QR1.4–QR1.6.

${}^t\mathsf{H}^0 : \mathscr{T} \to \mathscr{C}$ is a cohomological functor.

For $X, Y \in \mathscr{C}$, there is a natural map

$$\mathrm{Ext}^n_{\mathscr{C}}(X, Y) \to \mathrm{Hom}_{\mathscr{T}}(X, Y[n]).$$

It is an isomorphism for $n \leq 1$.

QR2. Sheaves

Assumptions: all spaces are locally compact, locally contractible, and of finite c-soft dimension. The ring $\Bbbk$ is noetherian and of finite global dimension.

QR2.1 (The six operations)

$$f^*,\ f_*,\ f_!,\ f^!,\ \otimes^L,\ R\mathscr{H}om.$$

Basic facts and special cases:

- $R\Gamma = \mathrm{a}_{X*}$, $\mathbf{H}^k(X, \mathcal{F}) = \mathsf{H}^k R\Gamma(\mathcal{F})$. $R\Gamma_{\mathrm{c}} = \mathrm{a}_{X!}$, $\mathbf{H}^k_{\mathrm{c}}(X, \mathcal{F}) = \mathsf{H}^k R\Gamma_{\mathrm{c}}(\mathcal{F})$.
- Constant sheaves: $f^*\underline{M} \cong \underline{M}$.
- Dualizing complex: $\omega_X = \mathrm{a}_X^! \Bbbk_{\mathrm{pt}}$.
- Stalks: $\mathsf{H}^k(\mathcal{F}_x) \cong \lim_{\to} \mathbf{H}^k(U, \mathcal{F}|_U)$.
- $\mathbf{H}^k(X, \mathcal{F}) \cong \mathrm{Hom}(\underline{\Bbbk}_X, \mathcal{F}[k])$.
- f proper (e.g. closed emb.) $\implies f_! = f_*$.
- f local homeomorphism (e.g. open embedding or a covering map) $\implies f^! = f^*$.
- f a locally closed embedding $\implies$ $f_! =$ extn. by 0, $f^! =$ restr. w. supps.

QR2.2 (Singular cohomology)

$$\mathbf{H}^k_{\mathrm{sing}}(X;M) \cong \mathbf{H}^k(X,\underline{M}_X),$$
$$\mathbf{H}^k_{\mathrm{sing,c}}(X;M) \cong \mathbf{H}^k_{\mathrm{c}}(X,\underline{M}_X).$$

Also: $\mathbf{H}^\bullet_{\mathrm{sing}}(X;\Bbbk) \cong \bigoplus_k \mathrm{Hom}(\underline{\Bbbk}_X,\underline{\Bbbk}_X[k])$ is a ring isomorphism.

QR2.3 (Open–closed triangles)

Let $j : U \hookrightarrow X$ and $i : Z \hookrightarrow X$ be complementary open and closed embeddings. Then there are natural distinguished triangles

$$j_!j^*\mathcal{F} \to \mathcal{F} \to i_*i^*\mathcal{F} \to,$$
$$i_*i^!\mathcal{F} \to \mathcal{F} \to j_*j^*\mathcal{F} \to .$$

Cor. i_* is fully faithful, so

$$D^+(Z,\Bbbk) \cong \{\mathcal{F} \in D^+(X,\Bbbk) \mid \operatorname{supp}\mathcal{F} \subset Z\}.$$

QR2.4 (Adjunction theorems)

$$\mathrm{Hom}(f^*\mathcal{F},\mathcal{G}) \cong \mathrm{Hom}(\mathcal{F},f_*\mathcal{G}),$$
$$\mathrm{Hom}(f_!\mathcal{F},\mathcal{G}) \cong \mathrm{Hom}(\mathcal{F},f^!\mathcal{G}).$$
$$\mathrm{Hom}(\mathcal{F} \overset{L}{\otimes} \mathcal{G},\mathcal{H}) \cong \mathrm{Hom}(\mathcal{F},R\mathscr{H}om(\mathcal{G},\mathcal{H})).$$

There are also $R\mathscr{H}om$ versions of these.

QR2.5 (Base change theorems)

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array} \text{ cartesian} \Longrightarrow \begin{array}{c} g^*f_! \cong f'_!(g')^* \\ f^!g_* \cong g'_*(f')^! \end{array}$$

There is a natural map $g^*f_*\mathcal{F} \to f'_*(g')^*\mathcal{F}$. It is an isomorphism in some special cases:

- g an open embedding.
- g a smooth morphism of varieties.
- f proper.
- f a loc. triv. fibration, $\mathcal{F} \in D^+_{\mathrm{loc}}(X,\Bbbk)$.

QR2.6 (Sheaf functor relations)

Composition rules: for $X \xrightarrow{f} Y \xrightarrow{g} Z$

$$(g\circ f)^* \cong f^*g^* \qquad (g\circ f)_* \cong g_*f_*,$$
$$(g\circ f)^! \cong f^!g^! \qquad (g\circ f)_! \cong g_!f_!.$$

Projection formula:

$$f_!\mathcal{F} \otimes^L \mathcal{G} \cong f_!(\mathcal{F} \otimes^L f^*\mathcal{G}),$$

Other formulas:

$$f^*(\mathcal{F} \otimes^L \mathcal{G}) \cong f^*\mathcal{F} \otimes^L f^*\mathcal{G},$$
$$R\mathrm{Hom}(\mathcal{F},\mathcal{G}) \cong R\Gamma(R\mathscr{H}om(\mathcal{F},\mathcal{G})),$$
$$f^!R\mathscr{H}om(\mathcal{F},\mathcal{G}) \cong R\mathscr{H}om(f^*\mathcal{F},f^!\mathcal{G}).$$

QR2.7 (Local systems)

There is an equivalence of categories

$$\mathrm{Loc}(X,\Bbbk) \cong \Bbbk[\pi_1(X,x_0)]\text{-mod},$$

compatible with $\otimes$, $\mathscr{H}om$, and f^* (and also with f_* and $f_!$ for covering maps).

If X has a contractible universal cover,

$$D^{\mathrm{b}}\mathrm{Loc}(X,\Bbbk) \cong D^{\mathrm{b}}_{\mathrm{loc}}(X,\Bbbk).$$

QR3. Constructible sheaves

From now on: all spaces are complex algebraic varieties, and all maps are morphisms of varieties.

QR3.1 (Tate twist)

Let $\Bbbk(1) = \mathrm{Hom}_{\Bbbk}(\mathbf{H}^1_{\mathrm{c}}(\mathbb{C};\Bbbk),\Bbbk)$ and $\mathcal{F}(n) = \mathcal{F} \otimes^L \underline{\Bbbk}(1)^{\otimes n}$. This operation commutes with all sheaf functors.

(One can choose an isom. $\Bbbk \cong \Bbbk(1)$ and ignore Tate twists, except in the context of mixed sheaves.)

QR3.2 (Smooth pullback)

Let $f : X \to Y$ be a smooth morphism of relative dimension d. Then $f^! \cong f^*[2d](d)$.

Cor. Poincaré duality (see QR3.6).

QR3.3 (Constructibility)

The six operations f^*, f_*, $f_!$, $f^!$, $\otimes^L$, and $R\mathscr{H}om$ all send constructible complexes to constructible complexes.

QR3.4 (Cohomology bounds)

Let $\mathcal{F}$ be a constructible sheaf on X.

- $\mathbf{H}^i(X,\mathcal{F}) = 0$ for $i > 2\dim X$.
- (Artin's vanishing theorem) X affine $\Longrightarrow$ $\mathbf{H}^i(X,\mathcal{F}) = 0$ for $i > \dim X$.

QR3.5 (Verdier duality)

For constructible complexes, the functor $\mathbb{D} = R\mathscr{H}om(-,\omega_X)$ is an involution:

$$\mathcal{F} \xrightarrow{\sim} \mathbb{D}(\mathbb{D}(\mathcal{F})).$$

Further properties:

$$f_*\mathbb{D} \cong \mathbb{D}f_!, \qquad f^!\mathbb{D} \cong \mathbb{D}f^*,$$
$$f_!\mathbb{D} \cong \mathbb{D}f_*, \qquad f^*\mathbb{D} \cong \mathbb{D}f^!,$$
$$R\mathscr{H}om(\mathcal{F},\mathcal{G}) \cong \mathbb{D}(\mathcal{F} \otimes^L \mathbb{D}\mathcal{G}) \cong R\mathscr{H}om(\mathbb{D}\mathcal{G},\mathbb{D}\mathcal{F}).$$

QR3.6 (Poincaré duality)

If X is a smooth variety of dimension n, then $\omega_X \cong \underline{\Bbbk}_X[2n](n)$. As a corollary,

$$R\Gamma_c(\underline{\Bbbk}_X)[2n](n) \cong R\mathrm{Hom}(R\Gamma(\underline{\Bbbk}_X), \Bbbk).$$

If $\mathcal{L}$ is a locally free local system of finite rank, and $\mathcal{L}^\vee$ is its dual, then

$$\mathbb{D}(\mathcal{L}) \cong \mathcal{L}^\vee[2n](n).$$

QR3.7 (Compatibilities)

For constructible complexes, $\boxtimes$ and extension of scalars $\Bbbk' \otimes^L_{\Bbbk} (-)$ commute with all six operations, and with $\mathbb{D}$.

QR3.8 ($\mathbb{G}_m$-localization)

Let X be a variety with an attracting $\mathbb{G}_m$-action, $i : X^{\mathbb{G}_m} \hookrightarrow X$ the inclusion map, and $p : X \to X^{\mathbb{G}_m}$ the limit map.

If $\mathcal{F} \in D^b_c(X)$ is weakly $\mathbb{G}_m$-equivariant,

$$i^!\mathcal{F} \cong p_!\mathcal{F}, \qquad p_*\mathcal{F} \cong i^*\mathcal{F}.$$

This is a special case of **hyperbolic localization**.

QR4. Perverse sheaves

QR4.1 (Perverse t-structure)

${}^pD^b_c(X, \Bbbk)^{\leq 0}$ consists of objects whose table of stalks lives "below the diagonal," i.e.

$$\mathsf{H}^i(\mathcal{F}|_{X_s}) = 0 \qquad \text{for all } i > -\dim X_s,$$

$\mathrm{Perv}(X, \Bbbk)$ = heart of this t-structure. All perverse sheaves are noetherian.

For $\Bbbk$ a field:

- All perverse sheaves have finite length.
- ${}^pD^b_c(X, \Bbbk)^{\geq 0} = \mathbb{D}({}^pD^b_c(X, \Bbbk)^{\leq 0})$.

QR4.2 (IC objects)

$\mathrm{IC}(Y, \mathcal{L})$ is the unique perverse sheaf that

- is supported on $\overline{Y}$,
- has $\mathrm{IC}(Y, \mathcal{L})|_Y \cong \mathcal{L}[\dim Y]$,
- has no subobject or quotient on $\overline{Y} \smallsetminus Y$.

Every perverse sheaf has a finite filtration by IC objects.

For $\Bbbk$ a field:

- Simple perverse sheaves are of the form $\mathrm{IC}(Y, \mathcal{L})$, $\mathcal{L}$ an irred. local system.
- $\mathbb{D}\mathrm{IC}(Y, \mathcal{L}) \cong \mathrm{IC}(Y, \mathcal{L}^\vee)(\dim Y)$.

QR4.3 (Smooth varieties)

If $\overline{Y}$ is smooth, and $\mathcal{L}$ is a local system on $\overline{Y}$, then $\mathrm{IC}(Y, \mathcal{L}|_Y) \cong \mathcal{L}[\dim Y]$. Special case: if $\overline{Y}$ is smooth, then

$$\mathrm{IC}(Y, \underline{\Bbbk}_Y) \cong \underline{\Bbbk}_{\overline{Y}}[\dim Y].$$

QR4.4 (Affine morphisms)

For $f : X \to Y$ an affine morphism, f_* is right t-exact and $f_!$ is left t-exact.

Cor. If $f : X \to Y$ is the inclusion map of a locally closed affine subvariety, then f_* and $f_!$ are t-exact.

QR4.5 (Smooth pullback)

Let $f : X \to Y$ be a smooth morphism of relative dimension d, and let $f^\dagger = f^*[d] = f^![-d](-d)$. Then $f^\dagger$ is t-exact. If f is surjective and has connected fibers, then

$$f^\dagger : \mathrm{Perv}(Y, \Bbbk) \to \mathrm{Perv}(X, \Bbbk)$$

is fully faithful, and its image is closed under subobjects and quotients. Generalization: if f is surjective, $\mathrm{Perv}(Y, \Bbbk)$ is equivalent to the category of **descent data** for f.

QR4.6 (Semismall maps)

A map $f : X \to Y$ is **semismall** if there is a stratification $(Y_t)_{t \in \mathscr{T}}$ such that $\forall y \in Y_t$,

$$\dim f^{-1}(y) \leq \tfrac{1}{2}(\dim X - \dim Y_t).$$

If f is proper, semismall and X is smooth, then $f_*\underline{\Bbbk}_X[\dim X]$ is perverse. Variants:

- If f is proper, **small**, and X is smooth, then $f_*\underline{\Bbbk}_X[\dim X]$ is an IC.
- If f is proper and **stratified semismall** with respect to $(X_s)_{s \in \mathscr{S}}$, then $f_* : D^b_{\mathscr{S}}(X, \Bbbk) \to D^b_{\mathscr{T}}(Y, \Bbbk)$ is t-exact.

QR4.7 (Decomposition theorem)

Let $\Bbbk = \mathbb{Q}$ (or another field of char. 0). A **semisimple complex** is a direct sum of shifts of simple perverse sheaves.

If $f : X \to Y$ is proper, and $\mathcal{F}$ is a semisimple complex, then so is $f_*\mathcal{F}$. Special cases:

- If $f : X \to Y$ is proper and X is smooth, $f_*\underline{\mathbb{Q}}_X$ is a semisimple complex.
- If $f : X \to Y$ is proper and semismall, and X is smooth, then $f_*\underline{\mathbb{Q}}_X[\dim X]$ is a semisimple perverse sheaf.

QR5. Mixed sheaves

The statements below apply to:

- $D^{\mathrm{b}}_{\mathrm{m}}(X_0, \overline{\mathbb{Q}}_\ell)$ for X_0 a variety over $\mathbb{F}_q$.
- $D^{\mathrm{b}}\mathrm{MHM}(X, \mathbb{Q})$ for X a complex variety.

QR5.1 (General properties of weights)
$D^{\mathrm{b}}_{\mathrm{m}}(X_0, \overline{\mathbb{Q}}_\ell)$ or $D^{\mathrm{b}}\mathrm{MHM}(X, \mathbb{Q})$ has subcategories $D^{\mathrm{b}}_{\leq w}$, $D^{\mathrm{b}}_{\geq w}$ for $w \in \mathbb{Z}$ satisfying

$$D^{\mathrm{b}}_{\leq w}[1] = D^{\mathrm{b}}_{\leq w+1}, \quad D^{\mathrm{b}}_{\geq w}[1] = D^{\mathrm{b}}_{\geq w+1},$$
$$D^{\mathrm{b}}_{\leq w} \cap D^{\mathrm{b}}_{\geq v} = 0 \quad \text{for } w < v.$$

Objects in $D^{\mathrm{b}}_{\leq w} \cap D^{\mathrm{b}}_{\geq w}$ are called **pure**. Further properties:

- f^* and $f_!$ preserve $D^{\mathrm{b}}_{\leq w}$.
- $f^!$ and f_* preserve $D^{\mathrm{b}}_{\geq w}$.
- $\otimes^L$ sends $D^{\mathrm{b}}_{\leq w} \times D^{\mathrm{b}}_{\leq v}$ to $D^{\mathrm{b}}_{\leq v+w}$.
- $R\mathscr{H}om$ sends $D^{\mathrm{b}}_{\leq w} \times D^{\mathrm{b}}_{\geq v}$ to $D^{\mathrm{b}}_{\geq v-w}$.
- $\mathbb{D}$ exchanges $D^{\mathrm{b}}_{\leq w}$ and $D^{\mathrm{b}}_{\geq -w}$.

If f is proper, $f_* = f_!$ preserves purity.

QR5.2 (Purity and weight filtration)
Any $\mathcal{F} \in \mathrm{Perv}_{\mathrm{m}}(X_0, \overline{\mathbb{Q}}_\ell)$ or $\mathrm{MHM}(X, \mathbb{Q})$ carries a canonical filtration $W_\bullet\mathcal{F}$ such that $W_k\mathcal{F}/W_{k-1}\mathcal{F}$ is pure of weight k.

- All simple objects are pure.
- All morphisms are strictly compatible with the weight filtration.

QR5.3 (Semisimplicity)
Let $\mathcal{F} \in D^{\mathrm{b}}_{\mathrm{m}}(X_0, \overline{\mathbb{Q}}_\ell)$ or $D^{\mathrm{b}}\mathrm{MHM}(X, \mathbb{Q})$. If $\mathcal{F}$ is pure, then $\mathrm{egf}(\mathcal{F})$ or $\mathrm{rat}(\mathcal{F})$ is a semisimple complex.

QR6. Equivariant sheaves

QR6.1 (The six operations)
For X a G-variety, $D^{\mathrm{b}}_G(X, \Bbbk)$ is a triangulated category with a functor

$$\mathrm{For} : D^{\mathrm{b}}_G(X, \Bbbk) \to D^{\mathrm{b}}(X, \Bbbk)$$

that commutes with the six operations.

QR6.2 (Change of groups)
For $H \subset G$, X a G-variety, have

$$\mathrm{For}^G_H : D^{\mathrm{b}}_G(X, \Bbbk) \to D^{\mathrm{b}}_H(X, \Bbbk),$$
$$\mathrm{Av}^G_{H!}, \mathrm{Av}^G_{H*} : D^{\mathrm{b}}_H(X, \Bbbk) \to D^{\mathrm{b}}_G(X, \Bbbk).$$

For $K \lhd G$, X a G/K-variety, have

$$\mathrm{Infl}^G_{G/K} : D^{\mathrm{b}}_{G/K}(X, \Bbbk) \to D^{\mathrm{b}}_G(X, \Bbbk),$$
$$\mathrm{Inv}_{K!}, \mathrm{Inv}_{K*} : D^{\pm}_G(X, \Bbbk) \to D^{\pm}_{G/K}(X, \Bbbk).$$

For^G_H and $\mathrm{Infl}^G_{G/K}$ commute with six ops.

QR6.3 (Connected groups)
If G is connected, then

$$\mathrm{For} : \mathrm{Perv}_G(X, \Bbbk) \to \mathrm{Perv}(X, \Bbbk).$$

is fully faithful, and its image is closed under subobjects and quotients.

QR6.4 (Quotient equivalence)
Let $H \lhd G$. If X is a G-variety and a principal H-variety, then

$$D^{\mathrm{b}}_{G/H}(X/H, \Bbbk) \cong D^{\mathrm{b}}_G(X, \Bbbk).$$

QR6.5 (Induction equivalence)
For $H \subset G$ and X an H-variety,

$$D^{\mathrm{b}}_G(G \times^H X, \Bbbk) \cong D^{\mathrm{b}}_H(X, \Bbbk).$$

QR6.6 (Unipotent groups)
If $G = H \ltimes U$ where U is unipotent, then for any G-variety X,

$$\mathrm{For}^G_H : D^{\mathrm{b}}_G(X, \Bbbk) \to D^{\mathrm{b}}_H(X, \Bbbk)$$

is fully faithful. If U acts trivially on X, this is an equivalence, with inverse

$$\mathrm{Infl}^G_H : D^{\mathrm{b}}_H(X, \Bbbk) \to D^{\mathrm{b}}_G(X, \Bbbk).$$

QR6.7 (Special cases)
For $X = \mathrm{pt}$, we have

$$\mathrm{Perv}_G(\mathrm{pt}, \Bbbk) \cong \Bbbk[G/G^\circ]\text{-}\mathrm{mod}^{\mathrm{fg}}.$$

If G is a finite group and $\Bbbk$ is a field,

$$D^{\mathrm{b}}\mathrm{Perv}_G(X, \Bbbk) \cong D^{\mathrm{b}}_G(X, \Bbbk).$$

QR6.8 (Equivariant cohomology)
For $\mathcal{F} \in D^{\mathrm{b}}_G(X, \Bbbk)$, the **equivariant hypercohomology** is defined by

$$\mathbf{H}^k_G(X, \mathcal{F}) = \mathrm{Hom}_{D^{\mathrm{b}}_G(\mathrm{pt},\Bbbk)}(\underline{\Bbbk}_{\mathrm{pt}}, a_{X*}\mathcal{F}[k])$$
$$\cong \mathrm{Hom}_{D^{\mathrm{b}}_G(X,\Bbbk)}(\underline{\Bbbk}_X, \mathcal{F}[k]).$$

This is a graded module over $\mathbf{H}^\bullet_G(X; \Bbbk)$ and over $\mathbf{H}^\bullet_G(\mathrm{pt}; \Bbbk)$.

For a torus T with character lattice $\mathbf{X}(T)$,

$$\mathbf{H}^\bullet_T(\mathrm{pt}; \Bbbk) \cong \mathrm{Sym}(\Bbbk(-1) \otimes_{\mathbb{Z}} \mathbf{X}(T)).$$

For a connected reductive group G whose torsion primes are invertible in $\Bbbk$,

$$\mathbf{H}^\bullet_G(\mathrm{pt}; \Bbbk) \cong (\mathbf{H}^\bullet_T(\mathrm{pt}; \Bbbk))^W.$$

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