Éléments de géométrie algébrique

A. Grothendieck and J. Dieudonné
Publications mathématiques de l'I.H.É.S

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What this is

This whole chapter is written by the translators.

This is a community translation of Alexander Grothendieck's and Jean Dieudonné's *Éléments de géométrie algébrique* (EGA). As it is a work in progress by multiple people, there will probably be a few mistakes—if you spot any then please do let us know[1].
To contribute, please visit

https://github.com/ryankeleti/ega.

On est désolés, Grothendieck.

In defense of a translation

From Wikipedia[2]:

> In January 2010, Grothendieck wrote the letter "Déclaration d'intention de non-publication" to Luc Illusie, claiming that all materials published in his absence have been published without his permission. He asks that none of his work be reproduced in whole or in part and that copies of this work be removed from libraries.[3] [...] This order may have been reversed later in 2010.[4]

It is a matter of often heated contention as to whether or not any translation of Grothendieck's work should take place, given his extremely explicit views on the matter. By no means do we mean to argue that somehow Grothendieck's wishes should be invalidated or ignored, nor do we wish to somehow twist his earlier words around in order to justify what we have done: we fully accept that he himself would probably have branded this project "an abomination". With this in mind, it remains to explain why we have gone ahead anyway.

First, and possibly foremost, it does not make sense (to us) for an individual to own the rights to knowledge. Arguments can be made about how the EGA is the product of years and years of intense work by Grothendieck, and so *this* is something that he 'owns' and has full control over. Indeed, it is true that there are almost innumerably many sentences in these works that only Grothendieck himself could have engineered, but, in translation, we have never improved anything, but only (regrettably, but almost certainly) worsened. The work in these pages is that of Grothendieck; we have been not much more than typesetters and eager readers. However, there is some important point to be made about the fact that Grothendieck collaborated and worked with many other incredibly proficient mathematicians during the writing of this treatise; although it is impossible to pinpoint which parts exactly others may have contributed (and by no means do we wish to imply that any of this work is derivative or fraudulent in any way whatsoever—EGA was written *by Grothendieck*) it seems fair that, in some amount, there are bits of the EGAs that 'belong' to a broader collection of minds.

It is a very good idea here to repeat the oft-quoted aphorism: "the work here is not ours, but any mistakes are"—it is very understandable for an author to not want their name on something that they have not themselves written, or, at the very least, read. This may be, in part, a reason for Grothendieck's wishes, but that is pure speculation. Even so, we include this above disclaimer.

Secondly, then, we note that the French version of EGA is still entirely readily accessible. Anybody reading these copies who is not a native French speaker, will probably be translating at least some part of EGA into English in their head, or into their notebooks, as they read. This document is just the product of a few people doing exactly that, but then passing on their efforts to make things just that little bit easier for anyone else who follows.

[1] https://github.com/ryankeleti/ega/issues

[2] https://en.wikipedia.org/wiki/Alexander_Grothendieck#Retirement_into_reclusion_and_death

[3] Grothendieck's letter. *Secret Blogging Seminar.* 9 February 2010. Retrieved 3 September 2019.

[4] Réédition des SGA. Archived from the original on 29 June 2016. Retrieved 12 November 2013.

Lastly, to quote another adage, "the guilty person is often the loudest". If it seems like we are over-eager to defend ourselves because we know that we are somehow in the wrong, it is because we are, at least partially. Working on this translation has meant going against Grothendieck's explicit requests, and for that we are sorry. We only hope that the freedom of knowledge is an excusable defense.

Notes from the translators

Grothendieck's writing style in EGA is quite particular, most notably for its long sentence structure. As translators, we have tried to give the best possible approximation of this style in English, resisting the temptation to "streamline" things in places where the language is more dense than usual.

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Any translations about which we are not entirely sure will be marked with a (?).

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Whenever a note is made by the translators, it will be prefaced by "[Trans.]".

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Along the margins we have provided the page numbers corresponding to the original text, as published by *Publications mathématiques de l'I.H.É.S.*, where the EGA were published as the following volumes:[5]

- EGA I (*tome 4, 1960*)
- EGA II (*tome 8, 1961*)
- EGA III, part 1 (*tome 11, 1961*)
- EGA III, part 2 (*tome 17, 1963*)
- EGA IV, part 1 (*tome 20, 1964*)
- EGA IV, part 2 (*tome 24, 1965*)
- EGA IV, part 3 (*tome 28, 1966*)
- EGA IV, part 4 (*tome 32, 1967*).

Due to EGA being a collection of volumes (one non-preliminary chapter, or part of a chapter, per volume), the page numbers reset at every new chapter. In addition, the preliminary section is stretched out over multiple volumes. To combat this, we label the pages as

$$\mathbf{X} \mid p,$$

referring to Chapter X, page p. For EGA III and IV, which are split across multiple chapters, we label the pages as

$$\mathbf{X}\text{-}n \mid p,$$

referring to Chapter X, part n, page p. In the case of the preliminaries (which are often collectively referred to as EGA 0), the preliminaries from volume Y are denoted as $\mathbf{0_Y}$.

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Later volumes (EGA II, III, and IV) include errata for earlier chapters. Where possible, we have used these to 'update' our translation, and entirely replace whatever mistakes might have been in the original copies of EGA I and II. If the change is minor (e.g. 'intersection' replacing 'inter-section') then we will not mention it; if it is anything more fundamental (e.g. X' replacing X) then we will include some margin note on the relevant line detailing the location of the erratum (e.g. $\mathbf{Err}_{\mathrm{II}}$ to denote that the correction is listed in the Errata section of EGA II).

Mathematical warnings

EGA uses *prescheme* for what is now usually called a scheme, and *scheme* for what is now usually called a separated scheme.

In some cases, we (the translators) have changed "$\to$" to "$\mapsto$" where appropriate.

[5]PDFs of which can be found online, hosted by the *Grothendieck circle*.

Introduction

To Oscar Zariski and André Weil.

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This memoir, and the many others will undoubtedly follow, are intended to form a treatise on the foundations of algebraic geometry. They do not, in principle, presume any particular knowledge of the subject, and it has even been recognised that such knowledge, despite its obvious advantages, could sometimes (because of the much-too-narrow interpretation—through the birational point of view—that it usually implies) be a hindrance to the one who wants to become familiar with the point of view and techniques presented here. However, we assume that the reader has a good knowledge of the following topics:

(a) *Commutative algebra*, as it is laid out, for example, in the volumes (in progress of being written) of the *Éléments* of N. Bourbaki (and, pending the publication of these volumes, in Samuel–Zariski [**SZ60**] and Samuel [**Sam53b**, **Sam53a**]).
(b) *Homological algebra*, for which we refer to Cartan–Eilenberg [**CE56**] (cited as (M)) and Godement [**God58**] (cited as (G)), as well as the recent article by A. Grothendieck [**Gro57**] (cited as (T)).
(c) *Sheaf theory*, where our main references will be (G) and (T); this theory provides the essential language for interpreting, in "geometric" terms, the essential notions of commutative algebra, and for "globalizing" them.
(d) Finally, it will be useful for the reader to have some familiarity with *functorial language*, which will be constantly used in this treatise, and for which the reader may consult (M), (G), and especially (T); the principles of this language and the main results of the general theory of functors will be described in more detail in a book currently in preparation by the authors of this treatise.

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It is not the place, in this introduction, to give a more or less summary description from the "schemes" point of view in algebraic geometry, nor the long list of reasons which made its adoption necessary, and in particular the systematic acceptance of nilpotent elements in the local rings of "manifolds" that we consider (which necessarily shifts the idea of rational maps into the background, in favor of those of regular maps or "morphisms"). To be precise, this treatise aims to systematically develop the language of schemes, and will demonstrate, we hope, its necessity. Although it would be easy to do so, we will not try to give here an "intuitive" **I** | 6
introduction to the notions developed in Chapter I. For the reader who would like to have a glimpse of the preliminary study of the subject matter of this treatise, we refer them to the conference by A. Grothendieck at the International Congress of Mathematicians in Edinburgh in 1958 [**Gro58**], and the exposé [**Gro**] of the same author. The work [**Ser55a**] (cited as (FAC)) of J.-P. Serre can also be considered as an intermediary exposition between the classical point of view and the schemes point of view in algebraic geometry, and, as such, its reading may be an excellent preparation for the reading of our *Éléments*.

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We give below the general outline planned for this treatise, subject to later modifications, especially concerning the later chapters.

Chapter I. — The language of schemes.
— II. — Elementary global study of some classes of morphisms.
— III. — Cohomology of algebraic coherent sheaves. Applications.
— IV. — Local study of morphisms.
— V. — Elementary procedures of constructing schemes.
— VI. — Descent. General method of constructing schemes.
— VII. — Group schemes, principal fibre bundles.
— VIII. — Differential study of fibre bundles.
— IX. — The fundamental group.
— X. — Residues and duality.
— XI. — Theories of intersection, Chern classes, Riemann–Roch theorem.
— XII. — Abelian schemes and Picard schemes.
— XIII. — Weil cohomology.

In principle, all chapters are considered open to changes, and supplementary sections could always be added later; such sections would appear in separate fascicles in order to minimize the inconveniences accompanying whatever mode of publication adopted. When the writing of such a section is foreseen or in progress during the publication of a chapter, it will be mentioned in the summary of the chapter in question, even if, owing to certain orders of urgency, its actual publication clearly ought to have been later. For the convenience of the reader, we give in "Chapter 0" the necessary tools in commutative algebra, homological algebra, and sheaf theory, that will be used throughout this treatise, that are more or less well known but for which it was not possible to give convenient references. It is recommended for the reader to not read Chapter 0 except whilst reading the actual treatise, when the results to which we refer seem unfamiliar. Besides, we think that in this way, **I** | 7
the reading of this treatise could be a good method for the beginner to familiarize themselves with commutative algebra and homological algebra, whose study, when not accompanied with tangible applications, is considered tedious, or even depressing, by many.

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It is outside of our capabilities to give a historic overview, or even a summary thereof, of the ideas and results described herein. The text will contain only those references considered particularly useful for comprehension, and we indicate the origin of only the most important results. Formally, at least, the subjects discussed in our work are reasonably new, which explains the scarcity of references made to the fathers of algebraic geometry from the 19th to the beginning of the 20th century, whose works we know only by hear-say. It is suitable, however, to say some words here about the works which have most directly influenced the authors and contributed to the development of scheme-theoretic point of view. We absolutely must mention the fundamental work (FAC) of J.-P. Serre first, which has served as an introduction to algebraic geometry for more that one young student (the author of this treatise being one), deterred by the dryness of the classic *Foundations* of A. Weil [**Wei46**]. It is there that it is shown, for the first time, that the "Zariski topology" of an "abstract" algebraic variety is perfectly suited to applying certain techniques from algebraic topology, and notably to be able to define a cohomology theory. Further, the definition of an algebraic variety given therein is that which translates most naturally to the idea that we develop here[6]. Serre himself had incidentally noted that the cohomology theory of affine algebraic varieties could be translated without difficulty by replacing the affine algebras over a field by arbitrary commutative rings. Chapters I and II of this treatise, and the first two paragraphs of Chapter III, can thus be considered, for the most part, as easy translations, to this bigger framework, of the principal results of (FAC) and a later article of the same author [**Ser57**]. We have also vastly profited from the *Séminaire de géométrie algébrique* de C. Chevalley [**CC**]; in particular, the systematic usage of "constructible sets" introduced by him has turned out to be extremely useful in the theory of schemes (cf. Chapter IV). We have also borrowed from him the study of morphisms from the point of view of dimension (Chapter IV), that translates with negligible change **I** | 8
to the framework of schemes. It also merits noting that the idea of "schemes of local rings", introduced by Chevalley, naturally lends itself to being extended to algebraic geometry (not having, however, all the flexibility

[6] Just as J.-P. Serre informed us, it is right to note that the idea of defining the structure of a manifold by the data of a sheaf of rings is due to H. Cartan, who took this idea as the starting point of his theory of analytic spaces. Of course, just as in algebraic geometry, it would be important in "analytic geometry" to give the allow the use of nilpotent elements in local rings of analytic spaces. This extension of the definition of H. Cartan and J.-P. Serre has recently been broached by H. Grauert [**Gra60**], and there is room to hope that a systematic report of analytic geometry in this setting will soon see the light of day. It is also evident that the ideas and techniques developed in this treatise retain a sense of analytic geometry, even though one must expect more considerable technical difficulties in this latter theory. We can foresee that algebraic geometry, by the simplicity of its methods, will be able to serve as a sort of formal model for future developments in the theory of analytic spaces.

and generality that we intend to give it here); for the connections between this idea and our theory, see Chapter I, §8. One such extension has been developed by M. Nagata in a series of memoirs [**Nag58a**], which contain many special results concerning algebraic geometry over Dedekind rings[7].

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It goes without saying that a book on algebraic geometry, and especially a book dealing with the fundamentals, is of course influenced, if only by proxy, by mathematicians such as O. Zariski and A. Weil. In particular, the *Théorie des fonctions holomorphes* by Zariski [**Zar51**], reasonably flexible thanks to the cohomological methods and an existence theorem (Chapter III, §§4 and 5), is (along with the method of descent described in Chapter VI) one of the principal tools used in this treatise, and it seems to us one of the most powerful at our disposal in algebraic geometry.

The general technique in which it is employed can be sketched as follows (a typical example of which will be given in Chapter XI, in the study of the fundamental group). We have a proper morphism (Chapter II) $f : X \to Y$ of algebraic varieties (or, more generally, of schemes) that we wish to study on the neighborhood of a point $y \in Y$, with the aim of resolving a problem P relative to a neighborhood of y. We proceed step by step:

- 1st We can suppose that Y is affine, so that X becomes a scheme defined on the affine ring A of Y, and we can even replace A by the local ring of y. This reduction is always easy in practice (Chapter V) and brings us to the case where A is a *local* ring.
- 2nd We study the problem in question when A is a local *Artinian* ring. So that the problem P still makes sense when A is not assumed to be integral, we sometimes have to reformulate P, and it appears that we often obtain a better understanding of the problem in doing so, in an "infinitesimal" way.
- 3rd The theory of formal schemes (Chapter III, §§3, 4, and 5) lets us pass from the case of an Artinian ring to a *complete local ring*.
- 4th Finally, if A is an arbitrary local ring, considering "multiform (?) sections" of suitable schemes over X, approximating a given "formal" section (Chapter IV), will let us pass, by extension of scalars, to the **I** | 9
completion of A, from a known result (about the schemed induced by X by extension of scalars to the completion of A) to an analogous result for a finite simple (e.g. unramified) extension of A.

This sketch shows the importance of the systematic study of schemes defined over an Artinian ring A. The point of view of Serre in his formulation of the theory of local class fields, and the recent works of Greenberg, seem to suggest that such a study could be undertaken by functorially attaching, to some such scheme X, a scheme X' over the residue field k of A (assumed perfect) of dimension equal (in nice cases) to $n \dim X$, where n is the height of A.

As for the influence of A. Weil, it suffices to say that it is the need to develop the tools necessary to formulate, with full generality, the definition of "Weil cohomology", and to tackle the proof[8] of all the formal properties necessary to establish the famous conjectures in Diophantine geometry [**Wei49**], that has been one of the principal motivations for the writing of this treatise, as well as the desire to find the natural setting of the usual ideas and methods of algebraic geometry, and to give the authors the chance to understand said ideas and methods.

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Finally, we believe it useful to warn the reader that they, as did all the authors themselves, will almost certainly have difficulty before becoming accustomed to the language of schemes, and to convince themselves that the usual constructions that suggest geometric intuition can be translated, in essentially only one sensible way, to this language. As in many parts of modern mathematics, the first intuition seems further and further away, in appearance, from the correct language needed to express the mathematics in question with complete precision and the desired level of generality. In practice, the psychological difficulty comes from the need to replicate some familiar set-theoretic constructions to a category that is already quite different from that of sets (the category of preschemes, or the category of preschemes over a given prescheme): Cartesian products, group laws, ring laws, module laws, fibre bundles, principal homogeneous fibre bundles, etc. It will most likely be difficult for the mathematician, in the future, to shy away from this new effort of abstraction (maybe rather negligible, on the whole, in comparison with that supplied by our fathers) to familiarize themselves with the theory of sets.

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[7] Among the works that come close to our point of view of algebraic geometry, we pick out the work of E. Kähler [**Käh58**] and a recent note of Chow and Igusa [**CI58**], which go back over certain results of (FAC) in the context of Nagata–Chevalley theory, as well as giving a Künneth formula.

[8] To avoid any misunderstanding, we point out that this task has barely been undertaken at the moment of writing this introduction, and still hasn't led to the proof of the Weil conjectures.

The references are given following the numerical system; for example, in **III**, 4.9.3, the **III** indicates the volume, the 4 the chapter, the 9 the section, and the 3 the paragraph.[9] If we reference a volume from within itself then we omit the volume number. **I** | 10

[Trans] Page 10 in the original is left blank.

[9] *[Trans] This is not a direct translation of the original, but instead uses the language more familiar to modern book (and LaTeX document) layouts.*

CHAPTER 0

Preliminaries (EGA 0)

§1. Rings of fractions

1.0. Rings and Algebras. $\mathbf{0_I}$ | 11

(1.0.1). All the rings considered in this treatise will have a *unit element*; all the modules over such a ring will be assumed to be *unitary*; the ring homomorphisms will always be assumed to *send the unit element to the unit element*; unless otherwise stated, a subring of a ring A will be assumed to *contain the unit element of* A. We will focus in particular on *commutative* rings, and when we speak of a ring without specifying any details, it will be implied that it is commutative. If A is a not-necessarily-commutative ring, by A-module we will we mean a *left* module unless stated otherwise.

(1.0.2). Let A and B be not-necessarily-commutative rings and $\varphi : A \to B$ a homomorphism. Any left (resp. right) B-module M can be provided with a left (resp. right) A-module structure by $a \cdot m = \varphi(a) \cdot m$ (resp. $m \cdot a = m \cdot \varphi(a)$); when it will be necessary to distinguish M as an A-module or a B-module, we will denote by $M_{[\varphi]}$ the left (resp. right) A-module defined as such. If L is an A-module, then a homomorphism $u : L \to M_{[\varphi]}$ is a homomorphism of abelian groups such that $u(a \cdot x) = \varphi(a) \cdot u(x)$ for $a \in A$, $x \in L$; we will also say that it is a *φ-homomorphism* $L \to M$, and that the pair (φ, u) (or, by abuse of language, u) is a *di-homomorphism* from (A, L) to (B, M). The pairs (A, L) consisting of a ring A and an A-module L thus form a *category* whose morphisms are di-homomorphisms.

(1.0.3). Under the hypotheses of (1.0.2), if $\mathfrak{J}$ is a left (resp. right) ideal of A, we denote by $B\mathfrak{J}$ (resp. $\mathfrak{J}B$) the left (resp. right) ideal $B\varphi(\mathfrak{J})$ (resp. $\varphi(\mathfrak{J})B$) of B generated by $\varphi(\mathfrak{J})$; it is also the image of the canonical homomorphism $B \otimes_A \mathfrak{J} \to B$ (resp. $\mathfrak{J} \otimes_A B \to B$) of left (resp. right) B-modules.

(1.0.4). If A is a (commutative) ring, and B a not-necessarily-commutative ring, then the data of a structure of an *A-algebra* on B is equivalent to the data of a ring homomorphism $\varphi : A \to B$ such that $\varphi(A)$ is contained in the center of B. For all ideals $\mathfrak{J}$ of A, $\mathfrak{J}B = B\mathfrak{J}$ is then a two-sided ideal of B, and for every B-module M, $\mathfrak{J}M$ is then a B-module equal to $(B\mathfrak{J})M$.

(1.0.5). We will not dwell much on the notions of *modules of finite type* and (commutative) *algebras of finite type*; to say that an A-module M is of finite type means that there exists an exact sequence $A^p \to M \to 0$. We say that $\mathbf{0_I}$ | 12
an A-module M admits a *finite presentation* if it is isomorphic to the cokernel of a homomorphism $A^p \to A^q$, or, in other words, if there exists an exact sequence $A^p \to A^q \to M \to 0$. We note that for a *Noetherian* ring A, every A-module of finite type admits a finite presentation.

Let us recall that an A-algebra B is said to be *integral* over A if every element in B is a root in B of a monic polynomial with coefficients in A; equivalently, if every element of B is contained in a subalgebra of B which is an *A-module of finite type*. When this is so, and if B is commutative, the subalgebra of B generated by a finite subset of B is an A-module of finite type; for a commutative algebra B to be integral and of finite type over A, it is necessary and sufficient that B be an A-module of finite type; we also say that B is an *integral A-algebra of finite type* (or simply *finite*, if there is no chance of confusion). It should be noted that in these definitions, it is not assumed that the homomorphism $A \to B$ defining the A-algebra structure is injective.

(1.0.6). An *integral* ring (or an *integral domain*) is a ring in which the product of a finite family of elements $\neq 0$ is $\neq 0$; equivalently, in such a ring, we have $0 \neq 1$, and the product of two elements $\neq 0$ is $\neq 0$. A *prime* ideal of a ring A is an ideal $\mathfrak{p}$ such that $A/\mathfrak{p}$ is integral; this implies that $\mathfrak{p} \neq A$. For a ring A to have at least one prime ideal, it is necessary and sufficient that $A \neq \{0\}$.

(1.0.7). A *local* ring is a ring A in which there exists a unique maximal ideal, which is thus the complement of the invertible elements, and contains all the ideals $\neq A$. If A and B are local rings, and $\mathfrak{m}$ and $\mathfrak{n}$ their respective maximal ideals, then we say that a homomorphism $\varphi : A \to B$ is *local* if $\varphi(\mathfrak{m}) \subset \mathfrak{n}$ (or, equivalently, if $\varphi^{-1}(\mathfrak{n}) = \mathfrak{m}$). By passing to quotients, such a homomorphism then defines a monomorphism from the residue field $A/\mathfrak{m}$ to the residue field $B/\mathfrak{n}$. The composition of any two local homomorphisms is a local homomorphism.

1.1. Radical of an ideal. Nilradical and radical of a ring.

(1.1.1). Let $\mathfrak{a}$ be an ideal of a ring A; the *radical* of $\mathfrak{a}$, denoted by $\mathfrak{r}(\mathfrak{a})$, is the set of $x \in A$ such that $x^n \in \mathfrak{a}$ for an integer $n > 0$; it is an ideal containing $\mathfrak{a}$. We have $\mathfrak{r}(\mathfrak{r}(\mathfrak{a})) = \mathfrak{r}(\mathfrak{a})$; the relation $\mathfrak{a} \subset \mathfrak{b}$ implies $\mathfrak{r}(\mathfrak{a}) \subset \mathfrak{r}(\mathfrak{b})$; the radical of a finite intersection of ideals is the intersection of their radicals. If φ is a homomorphism from another ring A' to A, then we have $\mathfrak{r}(\varphi^{-1}(\mathfrak{a})) = \varphi^{-1}(\mathfrak{r}(\mathfrak{a}))$ for any ideal $\mathfrak{a} \subset A$. For an ideal to be the radical of an ideal, it is necessary and sufficient that it be an intersection of prime ideals. The radical of an ideal $\mathfrak{a}$ is the intersection of the *minimal* prime ideals which contain $\mathfrak{a}$; if A is Noetherian, there are finitely many of these minimal prime ideals.

The radical of the ideal (0) is also called the *nilradical* of A; it is the set $\mathfrak{N}$ of the nilpotent elements of A. We say that the ring A is *reduced* if $\mathfrak{N} = (0)$; for every ring A, the quotient $A/\mathfrak{N}$ of A by its nilradical is a reduced ring.

(1.1.2). Recall that the *nilradical* $\mathfrak{N}(A)$ of a (not-necessarily-commutative) ring A is the intersection of the maximal left ideals of A (and also the intersection of maximal right ideals). The nilradical of $A/\mathfrak{N}(A)$ is (0).

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1.2. Modules and rings of fractions.

(1.2.1). We say that a subset S of a ring A is *multiplicative* if $1 \in S$ and the product of two elements of S is in S. The examples which will be the most important in what follows are: 1st, the set S_f of powers f^n $(n \geqslant 0)$ of an element $f \in A$; and 2nd, the complement $A - \mathfrak{p}$ of a *prime* ideal $\mathfrak{p}$ of A.

(1.2.2). Let S be a multiplicative subset of a ring A, and M an A-module; on the set $M \times S$, the relation between pairs (m_1, s_1) and (m_2, s_2):

"there exists an $s \in S$ such that $s(s_1 m_2 - s_2 m_1) = 0$"

is an equivalence relation. We denote by $S^{-1}M$ the quotient set of $M \times S$ by this relation, and by m/s the canonical image of the pair (m, s) in $S^{-1}M$; we call $i_M^S : m \mapsto m/1$ (also denoted i^S) the *canonical* map from M to $S^{-1}M$. This map is, in general, neither injective nor surjective; its kernel is the set of $m \in M$ such that there exists an $s \in S$ for which $sm = 0$.

On $S^{-1}M$ we define an additive group law by setting

$$(m_1/s_1) + (m_2/s_2) = (s_2 m_1 + s_1 m_2)/(s_1 s_2)$$

(one can check that it is independent of the choice of representative of the elements of $S^{-1}M$, which are equivalence classes). On $S^{-1}A$ we further define a multiplicative law by setting $(a_1/s_1)(a_2/s_2) = (a_1 a_2)/(s_1 s_2)$, and finally an exterior law on $S^{-1}M$, acted on by the set of elements of $S^{-1}A$, by setting $(a/s)(m/s') = (am)/(ss')$. It can then be shown that $S^{-1}A$ is endowed with a ring structure (called *the ring of fractions of* A *with denominators in* S) and $S^{-1}M$ with the structure of an $S^{-1}A$-module (called *the module of fractions of* M *with denominators in* S); for all $s \in S$, $s/1$ is invertible in $S^{-1}A$, its inverse being $1/s$. The canonical map i_A^S (resp. i_M^S) is a ring homomorphism (resp. a homomorphism of A-modules, $S^{-1}M$ being considered as an A-module by means of the homomorphism $i_A^S : A \to S^{-1}A$).

(1.2.3). If $S_f = \{f^n\}_{n \geqslant 0}$ for a $f \in A$, we write A_f and M_f instead of $S_f^{-1}A$ and $S_f^{-1}M$; when A_f is considered as algebra over A, we can write $A_f = A[1/f]$. A_f is isomorphic to the quotient algebra $A[T]/(fT - 1)A[T]$. When $f = 1$, A_f and M_f are canonically identified with A and M; if f is nilpotent, then A_f and M_f are 0.

When $S = A - \mathfrak{p}$, with $\mathfrak{p}$ a prime ideal of A, we write $A_\mathfrak{p}$ and $M_\mathfrak{p}$ instead of $S^{-1}A$ and $S^{-1}M$; $A_\mathfrak{p}$ is a *local ring* whose maximal ideal $\mathfrak{q}$ is generated by $i_A^S(\mathfrak{p})$, and we have $(i_A^S)^{-1}(\mathfrak{q}) = \mathfrak{p}$; by passing to quotients, i_A^S gives a monomorphism from the integral ring $A/\mathfrak{p}$ to the field $A_\mathfrak{p}/\mathfrak{q}$, which can be identified with the field of fractions of $A/\mathfrak{p}$.

(1.2.4). The ring of fractions $S^{-1}A$ and the canonical homomorphism i_A^S are a solution to a *universal mapping problem*: any homomorphism u from A to a ring B such that $u(S)$ is composed of *invertible* elements in B factors uniquely as

$$u : A \xrightarrow{i_A^S} S^{-1}A \xrightarrow{u^*} B$$

where u^* is a ring homomorphism. Under the same hypotheses, let M be an A-module, N a B-module, and $v : M \to N$ a homomorphism of A-modules (for the B-module structure on N defined by $u : A \to B$); then v factors uniquely as **$\mathbf{0_I}$ | 14**

$$v : M \xrightarrow{i_M^S} S^{-1}M \xrightarrow{v^*} N$$

where v^* is a homomorphism of $S^{-1}A$-modules (for the $S^{-1}A$-module structure on N defined by u^*).

(1.2.5). We define a canonical isomorphism $S^{-1}A \otimes_A M \simeq S^{-1}M$ of $S^{-1}A$-modules, sending the element $(a/s) \otimes m$ to the element $(am)/s$, with the inverse isomorphism sending m/s to $(1/s) \otimes m$.

(1.2.6). For every ideal $\mathfrak{a}'$ of $S^{-1}A$, $\mathfrak{a} = (i_A^S)^{-1}(\mathfrak{a}')$ is an ideal of A, and $\mathfrak{a}'$ is the ideal of $S^{-1}A$ generated by $i_A^S(\mathfrak{a})$, which can be identified with $S^{-1}\mathfrak{a}$ (1.3.2). The map $\mathfrak{p}' \mapsto (i_A^S)^{-1}(\mathfrak{p}')$ is an isomorphism, for the structure given by ordering, from the set of *prime* ideals of $S^{-1}A$ to the set of prime ideals $\mathfrak{p}$ of A such that $\mathfrak{p} \cap S = \varnothing$. In addition, the local rings $A_\mathfrak{p}$ and $(S^{-1}A)_{S^{-1}\mathfrak{p}}$ are canonically isomorphic (1.5.1).

(1.2.7). When A is an *integral* ring, for which K denotes its field of fractions, the canonical map $i_A^S : A \to S^{-1}A$ is injective for any multiplicative subset S not containing 0, and $S^{-1}A$ is then canonically identified with a subring of K containing A. In particular, for every prime ideal $\mathfrak{p}$ of A, $A_\mathfrak{p}$ is a local ring containing A, with maximal ideal $\mathfrak{p}A_\mathfrak{p}$, and we have $\mathfrak{p}A_\mathfrak{p} \cap A = \mathfrak{p}$.

(1.2.8). If A is a *reduced* ring (1.1.1), so is $S^{-1}A$: indeed, if $(x/s)^n = 0$ for $x \in A$, $s \in S$, then this means that there exists an $s' \in S$ such that $s'x^n = 0$, hence $(s'x)^n = 0$, which, by hypothesis, implies $s'x = 0$, so $x/s = 0$.

1.3. Functorial properties.

(1.3.1). Let M and N be A-modules, and u an A-homomorphism $M \to N$. If S is a multiplicative subset of A, we define a $S^{-1}A$-homomorphism $S^{-1}M \to S^{-1}N$, denoted by $S^{-1}u$, by setting $S^{-1}u(m/s) = u(m)/s$; if $S^{-1}M$ and $S^{-1}N$ are canonically identified with $S^{-1}A \otimes_A M$ and $S^{-1}A \otimes_A N$ (1.2.5), then $S^{-1}u$ is considered as $1 \otimes u$. If P is a third A-module, and v an A-homomorphism $N \to P$, we have $S^{-1}(v \circ u) = (S^{-1}v) \circ (S^{-1}u)$; in other words, $S^{-1}M$ is a *covariant functor in* M, from the category of A-modules to that of $S^{-1}A$-modules (A and S being fixed).

(1.3.2). The functor $S^{-1}M$ is *exact*; in other words, if the sequence

$$M \xrightarrow{u} N \xrightarrow{v} P$$

is exact, then so is the sequence

$$S^{-1}M \xrightarrow{S^{-1}u} S^{-1}N \xrightarrow{S^{-1}v} S^{-1}P.$$

In particular, if $u : M \to N$ is injective (resp. surjective), the same is true for $S^{-1}u$; if N and P are submodules of M, $S^{-1}N$ and $S^{-1}P$ are canonically identified with submodules of $S^{-1}M$, and we have **0_I** | 15

$$S^{-1}(N+P) = S^{-1}N + S^{-1}P \text{ and } S^{-1}(N \cap P) = (S^{-1}N) \cap (S^{-1}P).$$

(1.3.3). Let $(M_\alpha, \varphi_{\beta\alpha})$ be an inductive system of A-modules; then $(S^{-1}M_\alpha, S^{-1}\varphi_{\beta\alpha})$ is an inductive system of $S^{-1}A$-modules. Expressing the $S^{-1}M_\alpha$ and $S^{-1}\varphi_{\beta\alpha}$ as tensor products ((1.2.5) and (1.3.1)), it follows from the permutability of the tensor product and inductive limit operations that we have a canonical isomorphism

$$S^{-1} \varinjlim M_\alpha \simeq \varinjlim S^{-1}M_\alpha$$

which is we can further express by saying that the functor $S^{-1}M$ (in M) *commutes with inductive limits.*

(1.3.4). Let M and N be A-modules; there is a canonical *functorial* (in M and N) isomorphism

$$(S^{-1}M) \otimes_{S^{-1}A} (S^{-1}N) \simeq S^{-1}(M \otimes_A N)$$

which sends $(m/s) \otimes (n/t)$ to $(m \otimes n)/st$.

(1.3.5). We also have a *functorial* (in M and N) homomorphism

$$S^{-1}\operatorname{Hom}_A(M,N) \longrightarrow \operatorname{Hom}_{S^{-1}A}(S^{-1}M, S^{-1}N)$$

which sends u/s to the homomorphism $m/t \mapsto u(m)/st$. When M has a finite presentation, the above homomorphism is an *isomorphism*: it is immediate when M is of the form A^r, and we pass to the general case by starting with the exact sequence $A^p \to A^q \to M \to 0$ and using the exactness of the functor $S^{-1}M$ and the left-exactness of the functor $\operatorname{Hom}_A(M,N)$ in M. Note that this is always the case when A is *Noetherian* and the A-module M is *of finite type.*

1.4. Change of multiplicative subset.

(1.4.1). Let S and T be multiplicative subsets of a ring A such that $S \subset T$; there exists a canonical homomorphism $\rho_A^{T,S}$ (or simply $\rho^{T,S}$) from $S^{-1}A$ to $T^{-1}A$, sending the element denoted a/s of $S^{-1}A$ to the element denoted a/s in $T^{-1}A$; we have $i_A^T = \rho_A^{T,S} \circ i_A^S$. For every A-module M, there exists, in the same way, an $S^{-1}A$-linear map from $S^{-1}M$ to $T^{-1}M$ (the latter considered as an $S^{-1}A$-module by the homomorphism $\rho_A^{T,S}$), which sends the element m/s of $S^{-1}M$ to the element m/s of $T^{-1}M$; we denote this map by $\rho_M^{T,S}$, or simply $\rho^{T,S}$, and we still have $i_M^T = \rho_M^{T,S} \circ i_M^S$; by the canonical identification (1.2.5), $\rho_M^{T,S}$ is identified with $\rho_A^{T,S} \otimes 1$. The homomorphism $\rho_M^{T,S}$ is a *functorial morphism* (or natural transformation) from the functor $S^{-1}M$ to the functor $T^{-1}M$, in other words, the diagram

$$\begin{tikzcd} S^{-1}M \arrow[r, "S^{-1}u"] \arrow[d, "\rho_M^{T,S}"'] & S^{-1}N \arrow[d, "\rho_N^{T,S}"] \\ T^{-1}M \arrow[r, "T^{-1}u"] & T^{-1}N \end{tikzcd}$$

is commutative, for every homomorphism $u : M \to N$; $T^{-1}u$ is entirely determined by $S^{-1}u$, since, for $m \in M$ **0_{I}** | 16
and $t \in T$, we have

$$(T^{-1}u)(m/t) = (t/1)^{-1}\rho^{T,S}((S^{-1}u)(m/1)).$$

(1.4.2). With the same notation, for A-modules M and N, the diagrams (cf. (1.3.4) and (1.3.5))

$$\begin{tikzcd} (S^{-1}M) \otimes_{S^{-1}A} (S^{-1}N) \arrow[r, "\sim"] \arrow[d] & S^{-1}(M \otimes_A N) \arrow[d] \\ (T^{-1}M) \otimes_{T^{-1}A} (T^{-1}N) \arrow[r, "\sim"] & T^{-1}(M \otimes_A N) \end{tikzcd}$$

$$\begin{tikzcd} S^{-1}\operatorname{Hom}_A(M,N) \arrow[r] \arrow[d] & \operatorname{Hom}_{S^{-1}A}(S^{-1}M, S^{-1}N) \arrow[d] \\ T^{-1}\operatorname{Hom}_A(M,N) \arrow[r] & \operatorname{Hom}_{T^{-1}A}(T^{-1}M, T^{-1}N) \end{tikzcd}$$

are commutative.

(1.4.3). There is an important case, in which the homomorphism $\rho^{T,S}$ is *bijective*, when we then know that every element of T is a divisor of an element of S; we then identify the modules $S^{-1}M$ and $T^{-1}M$ via $\rho^{T,S}$. We say that S is *saturated* if every divisor in A of an element of S is in S; by replacing S with the set T of all the divisors of the elements of S (a set which is multiplicative and saturated), we see that we can always, if we wish, consider only modules of fractions $S^{-1}M$, where S is saturated.

(1.4.4). If S, T, and U are three multiplicative subsets of A such that $S \subset T \subset U$, then we have

$$\rho^{U,S} = \rho^{U,T} \circ \rho^{T,S}.$$

(1.4.5). Consider an *increasing filtered family* (S_α) of multiplicative subsets of A (we write $\alpha \leqslant \beta$ for $S_\alpha \subset S_\beta$), and let S be the multiplicative subset $\bigcup_\alpha S_\alpha$; let us put $\rho_{\beta\alpha} = \rho_A^{S_\beta,S_\alpha}$ for $\alpha \leqslant \beta$; according to (1.4.4), the homomorphisms $\rho_{\beta\alpha}$ define a ring A' as the *inductive limit* of the inductive system of rings $(S_\alpha^{-1}A, \rho_{\beta\alpha})$. Let ρ_α be the canonical map $S_\alpha^{-1}A \to A'$, and let $\varphi_\alpha = \rho_A^{S,S_\alpha}$; as $\varphi_\alpha = \varphi_\beta \circ \rho_{\beta\alpha}$ for $\alpha \leqslant \beta$ according to (1.4.4), we can uniquely define a homomorphism $\varphi : A' \to S^{-1}A$ such that the diagram

$$\begin{tikzcd} & S_\alpha^{-1}A \arrow[d, "\rho_{\beta\alpha}"] \arrow[ddl, "\rho_\alpha"'] \arrow[ddr, "\varphi_\alpha"] & \\ & S_\beta^{-1}A \arrow[dl, "\rho_\beta"] \arrow[dr, "\varphi_\beta"'] & \\ A' \arrow[rr, "\varphi"] & & S^{-1}A \end{tikzcd} \qquad (\alpha \leqslant \beta)$$

is commutative. In fact, φ is an *isomorphism*; it is indeed immediate by construction that φ is surjective. On the other hand, if $\rho_\alpha(a/s_\alpha) \in A'$ is such that $\varphi(\rho_\alpha(a/s_\alpha)) = 0$, then this means that $a/s_\alpha = 0$ in $S^{-1}A$, that is to say that there exists an $s \in S$ such that $sa = 0$; but there is a $\beta \geqslant \alpha$ such that $s \in S_\beta$, and consequently, as

$\rho_\alpha(a/s_\alpha) = \rho_\beta(sa/ss_\alpha) = 0$, we find that φ is injective. The case for an A-module M is treated likewise, and we have thus defined canonical isomorphisms

$$\varinjlim S_\alpha^{-1}A \simeq (\varinjlim S_\alpha)^{-1}A, \ \varinjlim S_\alpha^{-1}M \simeq (\varinjlim S_\alpha)^{-1}M,$$

the second being *functorial* in M.

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(1.4.6). Let S_1 and S_2 be multiplicative subsets of A; then S_1S_2 is also a multiplicative subset of A. Let us denote by S_2' the canonical image of S_2 in the ring $S_1^{-1}A$, which is a multiplicative subset of this ring. For every A-module M there is then a functorial isomorphism

$$S_2'^{-1}(S_1^{-1}M) \simeq (S_1S_2)^{-1}M$$

which sends $(m/s_1)/(s_2/1)$ to the element $m/(s_1s_2)$.

1.5. Change of ring.

(1.5.1). Let A and A' be rings, φ a homomorphism $A' \to A$, and S (resp. S') a multiplicative subset of A (resp. A'), such that $\varphi(S') \subset S$; the composition homomorphism $A' \xrightarrow{\varphi} A \to S^{-1}A$ factors as

$$A' \longrightarrow S'^{-1}A' \xrightarrow{\varphi^{S'}} S^{-1}A,$$

by (1.2.4); where $\varphi^{S'}(a'/s') = \varphi(a')/\varphi(s')$. If $A = \varphi(A')$ and $S = \varphi(S')$, then $\varphi^{S'}$ is *surjective.* If $A' = A$ and φ is the identity, then $\varphi^{S'}$ is exactly the homomorphism $\rho_A^{S,S'}$ defined in (1.4.1).

(1.5.2). Under the hypotheses of (1.5.1), let M be an A-module. There exists a canonical functorial morphism

$$\sigma : S'^{-1}(M_{[\varphi]}) \longrightarrow (S^{-1}M)_{[\varphi^{S'}]}$$

of $S'^{-1}A'$-modules, sending each element m/s' of $S'^{-1}(M_{[\varphi]})$ to the element $m/\varphi(s')$ of $(S^{-1}M)_{[\varphi^{S'}]}$; indeed, we immediately see that this definition does not depend on the representative m/s' of the element in question. When $S = \varphi(S')$, the homomorphism σ is *bijective.* When $A' = A$ and φ is the identity, σ is none other than the homomorphism $\rho_M^{S,S'}$ defined in (1.4.1).

When, in particular, we take $M = A$ the homomorphism φ defines an A'-algebra structure on A; $S'^{-1}(A_{[\varphi]})$ is then endowed with a ring structure, with which it can be identified with $(\varphi(S'))^{-1}A$, and the homomorphism $\sigma : S'^{-1}(A_{[\varphi]}) \to S^{-1}A$ is a homomorphism of $S'^{-1}A'$-algebras.

(1.5.3). Let M and N be A-modules; by composing the homomorphisms defined in (1.3.4) and (1.5.2), we obtain a homomorphism

$$(S^{-1}M \otimes_{S^{-1}A} S^{-1}N)_{[\varphi^{S'}]} \longleftarrow S'^{-1}((M \otimes A)_{[\varphi]})$$

which is an isomorphism when $\varphi(S') = S$. Similarly, by composing the homomorphisms in (1.3.5) and (1.5.2), we obtain a homomorphism

$$S'^{-1}((\mathrm{Hom}_A(M,N))_{[\varphi]}) \longrightarrow (\mathrm{Hom}_{S^{-1}A}(S^{-1}M, S^{-1}N))_{[\varphi^{S'}]}$$

which is an isomorphism when $\varphi(S') = S$ and M admits a finite presentation.

(1.5.4). Let us now consider an A'-module N', and form the tensor product $N' \otimes_{A'} A_{[\varphi]}$, which can be considered as an A-module by setting $a \cdot (n' \otimes b) = n' \otimes (ab)$. There is a functorial isomorphism of $S^{-1}A$-modules

$$\tau : (S'^{-1}N') \otimes_{S'^{-1}A'} (S^{-1}A)_{[\varphi^{S'}]} \simeq S^{-1}(N' \otimes_{A'} A_{[\varphi]})$$

which sends the element $(n'/s') \otimes (a/s)$ to the element $(n' \otimes a)/(\varphi(s')s)$; indeed, we can show that when we replace n'/s' (resp. a/s) by another expression of the same element, $(n' \otimes a)/(\varphi(s')s)$ does not change; on the other hand, we can define a homomorphism inverse to τ by sending $(n' \otimes a)/s$ to the element $(n'/1) \otimes (a/s)$: we use the fact that $S^{-1}(N' \otimes_{A'} A_{[\varphi]})$ is canonically isomorphic to $(N' \otimes_{A'} A_{[\varphi]}) \otimes_A S^{-1}A$ (1.2.5), so also to $N' \otimes_{A'} (S^{-1}A)_{[\psi]}$, where we denote by ψ the composite homomorphism $a' \mapsto \varphi(a')/1$ from A' to $S^{-1}A$. $\mathbf{0_I}$ | 18

(1.5.5). If M' and N' are A'-modules, then by composing the isomorphisms (1.3.4) and (1.5.4), we obtain an isomorphism

$$S'^{-1}M \otimes_{S'^{-1}A'} S'^{-1}N' \otimes_{S'^{-1}A'} S^{-1}A \simeq S^{-1}(M' \otimes_{A'} N' \otimes_{A'} A).$$

Likewise, if M' admits a finite presentation, we have by (1.3.5) and (1.5.4) an isomorphism

$$\mathrm{Hom}_{S'^{-1}A'}(S'^{-1}M', S'^{-1}N') \otimes_{S'^{-1}A'} S^{-1}A \simeq S^{-1}(\mathrm{Hom}_{A'}(M',N') \otimes_{A'} A).$$

(1.5.6). Under the hypotheses of (1.5.1), let T (resp. T') be another multiplicative subset of A (resp. A') such that $S \subset T$ (resp. $S' \subset T'$) and $\varphi(T') \subset T$. Then the diagram

$$\begin{tikzcd} S'^{-1}A' \arrow[r, "\varphi^{S'}"] \arrow[d, "\rho^{T',S'}"'] & S^{-1}A \arrow[d, "\rho^{T,S}"] \\ T'^{-1}A' \arrow[r, "\varphi^{T'}"] & T^{-1}A \end{tikzcd}$$

is commutative. If M is an A-module, then the diagram

$$\begin{tikzcd} S'^{-1}(M_{[\varphi]}) \arrow[r, "\sigma"] \arrow[d, "\rho^{T',S'}"'] & (S^{-1}M)_{[\varphi^{S'}]} \arrow[d, "\rho^{T,S}"] \\ T'^{-1}(M_{[\varphi]}) \arrow[r, "\sigma"] & (T^{-1}M)_{[\varphi^{T'}]} \end{tikzcd}$$

is commutative. Finally, if N' is an A'-module, then the diagram

$$\begin{tikzcd} (S'^{-1}N') \otimes_{S'^{-1}A'} (S^{-1}A)_{[\varphi^{S'}]} \arrow[r, "\sim", "\tau"'] \arrow[d] & S^{-1}(N' \otimes_{A'} A_{[\varphi]}) \arrow[d, "\rho^{T,S}"] \\ (T'^{-1}N') \otimes_{T'^{-1}A'} (T^{-1}A)_{[\varphi^{T'}]} \arrow[r, "\sim", "\tau"'] & T^{-1}(N' \otimes_{A'} A_{[\varphi]}) \end{tikzcd}$$

is commutative, the left vertical arrow obtained by applying $\rho_{N'}^{T',S'}$ to $S'^{-1}N'$ and $\rho_A^{T,S}$ to $S^{-1}A$. $\mathbf{0_I}$ | 19

(1.5.7). Let A'' be a third ring, $\varphi' : A'' \to A'$ a ring homomorphism, and S'' a multiplicative subset of A'' such that $\varphi'(S'') \subset S'$. Let $\varphi'' = \varphi \circ \varphi'$; then we have

$$\varphi''^{S''} = \varphi^{S'} \circ \varphi'^{S''}.$$

Let M be an A-module; evidently we have $M_{[\varphi'']} = (M_{[\varphi]})_{[\varphi']}$; if σ' and σ'' are the homomorphisms defined by φ' and φ'' in the same way as how σ is defined in (1.5.2) by φ, then we have the transitivity formula

$$\sigma'' = \sigma \circ \sigma'.$$

Finally, let N'' be an A''-module; the A-module $N'' \otimes_{A''} A_{[\varphi'']}$ is canonically identified with $(N'' \otimes_{A''} A'_{[\varphi']}) \otimes_{A'} A_{[\varphi]}$, and likewise the $S^{-1}A$-module $(S''^{-1}N'') \otimes_{S''^{-1}A''} (S^{-1}A)_{[\varphi''^{S''}]}$ is canonically identified with $((S''^{-1}N'') \otimes_{S''^{-1}A''} (S'^{-1}A')_{[\varphi'^{S''}]}) \otimes_{S'^{-1}A'} (S^{-1}A)_{[\varphi^{S'}]}$. With these identifications, if τ' and τ'' are the isomorphisms defined by φ' and φ'' in the same way as how τ is defined in (1.5.4) by φ, then we have the transitivity formula

$$\tau'' = \tau \circ (\tau' \otimes 1).$$

(1.5.8). Let A be a subring of a ring B; for every *minimal* prime ideal $\mathfrak{p}$ of A, there exists a minimal prime ideal $\mathfrak{q}$ of B such that $\mathfrak{p} = A \cap \mathfrak{q}$. Indeed, $A_{\mathfrak{p}}$ is a subring of $B_{\mathfrak{p}}$ (1.3.2) and has *a single prime ideal* $\mathfrak{p}'$ (1.2.6); since $B_{\mathfrak{p}}$ is not 0, it has at least one prime ideal $\mathfrak{q}'$ and we necessarily have $\mathfrak{q}' \cap A_{\mathfrak{p}} = \mathfrak{p}'$; the prime ideal $\mathfrak{q}_1$ of B, the inverse image of $\mathfrak{q}'$, is thus such that $\mathfrak{q}_1 \cap A = \mathfrak{p}$, and *a fortiori* we have $\mathfrak{q} \cap A = \mathfrak{p}$ for every minimal prime ideal $\mathfrak{q}$ of B contained in $\mathfrak{q}_1$.

1.6. Identification of the module M_f as an inductive limit.

(1.6.1). Let M be an A-module and f an element of A. Consider a sequence (M_n) of A-modules, all identical to M, and for each pair of integers $m \leqslant n$, let φ_{nm} be the homomorphism $z \mapsto f^{n-m}z$ from M_m to M_n; it is immediate that $((M_n), (\varphi_{nm}))$ is an *inductive system* of A-modules; let $N = \varinjlim M_n$ be the inductive limit of this system. We define a canonical *functorial* A-isomorphism from N to M_f. For this, let us note that, for all n, $\theta_n : z \mapsto z/f^n$ is an A-homomorphism from $M = M_n$ to M_f, and it follows from the definitions that we have $\theta_n \circ \varphi_{nm} = \theta_m$ for $m \leqslant n$. As a result, there exists an A-homomorphism $\theta : N \to M_f$ such that, if φ_n denotes the canonical homomorphism $M_n \to N$, then we have $\theta_n = \theta \circ \varphi_n$ for all n. Since, by hypothesis, every element of M_f is of the form z/f^n for at least one n, it is clear that θ is surjective. On the other hand, if $\theta(\varphi_n(z)) = 0$, or, in other words, if $z/f^n = 0$, then there exists an integer $k > 0$ such that $f^k z = 0$, so $\varphi_{n+k,n}(z) = 0$, which gives $\varphi_n(z) = 0$. We can therefore identify M_f with $\varinjlim M_n$ via θ.

(1.6.2). Now write $M_{f,n}$, φ^f_{nm}, and φ^f_n instead of M_n, φ_{nm}, and φ_n. Let g be another element of A. Since f^n divides $f^n g^n$, we have a functorial homomorphism

$$\rho_{fg,f} : M_f \longrightarrow M_{fg} \quad ((1.4.1) \text{ and } (1.4.3));$$

if we identify M_f and M_{fg} with $\varinjlim M_{f,n}$ and $\varinjlim M_{fg,n}$ respectively, then $\rho_{fg,f}$ identifies with the *inductive limit* of the maps $\rho^n_{fg,f} : M_{f,n} \to M_{fg,n}$, defined by $\rho^n_{fg,f}(z) = g^n z$. Indeed, this follows immediately from the commutativity of the diagram **0**$_{\mathbf{I}}$ | 20

$$\begin{tikzcd} M_{f,n} \arrow[r, "\rho^n_{fg,f}"] \arrow[d, "\varphi^f_n"'] & M_{fg,n} \arrow[d, "\varphi^{fg}_n"] \\ M_f \arrow[r, "\rho_{fg,f}"] & M_{fg}. \end{tikzcd}$$

1.7. Support of a module.

(1.7.1). Given an A-module M, we define the *support* of M, denoted by $\mathrm{Supp}(M)$, to be the set of prime ideals $\mathfrak{p}$ of A such that $M_{\mathfrak{p}} \neq 0$. For it to be the case that $M = 0$, it is necessary and sufficient that $\mathrm{Supp}(M) = \varnothing$, because if $M_{\mathfrak{p}} = 0$ for all $\mathfrak{p}$, then the annihilator of an element $x \in M$ cannot be contained in any prime ideal of A, and so is the whole of A.

(1.7.2). If $0 \to N \to M \to P \to 0$ is an exact sequence of A-modules, then we have

$$\mathrm{Supp}(M) = \mathrm{Supp}(N) \cup \mathrm{Supp}(P)$$

because, for every prime ideal $\mathfrak{p}$ of A, the sequence $0 \to N_{\mathfrak{p}} \to M_{\mathfrak{p}} \to P_{\mathfrak{p}} \to 0$ is exact (1.3.2) and in order that $M_{\mathfrak{p}} = 0$, it is necessary and sufficient that $N_{\mathfrak{p}} = P_{\mathfrak{p}} = 0$.

(1.7.3). If M is the sum of a family (M_λ) of submodules, then $M_{\mathfrak{p}}$ is the sum of the $(M_\lambda)_{\mathfrak{p}}$ for every prime ideal $\mathfrak{p}$ of A ((1.3.3) and (1.3.2)), so $\mathrm{Supp}(M) = \bigcup_\lambda \mathrm{Supp}(M_\lambda)$.

(1.7.4). If M is an A-module *of finite type*, then $\mathrm{Supp}(M)$ is the set of prime ideals *containing the annihilator of* M. Indeed, if M is cyclic and generated by x, then to say that $M_{\mathfrak{p}} = 0$ is to say that that there exists an $s \notin \mathfrak{p}$ such that $s \cdot x = 0$, and thus that $\mathfrak{p}$ does not contain the annihilator of x. Now if M admits a finite system $(x_i)_{1 \leqslant i \leqslant n}$ of generators, and if $\mathfrak{a}_i$ is the annihilator of x_i, then it follows from (1.7.3) that $\mathrm{Supp}(M)$ is the set of the $\mathfrak{p}$ containing one of the $\mathfrak{a}_i$, or equivalently, the set of the $\mathfrak{p}$ containing $\mathfrak{a} = \bigcap_i \mathfrak{a}_i$, which is the annihilator of M.

(1.7.5). If M and N are two A-modules *of finite type*, then we have

$$\mathrm{Supp}(M \otimes_A N) = \mathrm{Supp}(M) \cap \mathrm{Supp}(N).$$

It is a question of seeing that, if $\mathfrak{p}$ is a prime ideal of A, then the condition $M_{\mathfrak{p}} \otimes_{A_{\mathfrak{p}}} N_{\mathfrak{p}} \neq 0$ is equivalent to "$M_{\mathfrak{p}} \neq 0$ and $N_{\mathfrak{p}} \neq 0$" (taking (1.3.4) into account). In other words, it is a question of seeing that, if P and Q are modules of finite type over a *local* ring $B \neq 0$, then $P \otimes_B Q \neq 0$. Let $\mathfrak{m}$ be the maximal ideal of B. By Nakayama's Lemma, the vector spaces $P/\mathfrak{m}P$ and $Q/\mathfrak{m}Q$ are not 0, and so it is the same for the tensor product $(P/\mathfrak{m}P) \otimes_{B/\mathfrak{m}} (Q/\mathfrak{m}Q) = (P \otimes_B Q) \otimes_B (B/\mathfrak{m})$, whence the conclusion.

In particular, if M is an A-module of finite type, and $\mathfrak{a}$ an ideal of A, then $\mathrm{Supp}(M/\mathfrak{a}M)$ is the set of prime ideals containing both $\mathfrak{a}$ and the annihilator $\mathfrak{n}$ of M (1.7.4), that is, the set of prime ideals containing $\mathfrak{a} + \mathfrak{n}$.

§2. Irreducible spaces. Noetherian spaces

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2.1. Irreducible spaces.

(2.1.1). We say that a topological space X is *irreducible* if it is nonempty and if it is not a union of two distinct closed subspaces of X. It is equivalent to say that $X \neq \varnothing$ and the intersection of two nonempty open sets (and consequently of a finite number of open sets) of X is nonempty, or that every nonempty open set is everywhere dense, or that any closed set is *rare*[1], or, lastly, that all open sets of X are *connected*.

(2.1.2). For a subspace Y of a topological space X to be irreducible, it is necessary and sufficient that its closure $\overline{Y}$ be irreducible. In particular, any subspace which is the closure $\overline{\{x\}}$ of a singleton is irreducible; we will express the relation $y \in \overline{\{x\}}$ (equivalent to $\overline{\{y\}} \subset \overline{\{x\}}$) by saying that y is a *specialization of* x or that x is a *generalization of* y. When there exists, in an irreducible space X, a point x such that $X = \overline{\{x\}}$, we will say that x is a *generic point of* X. Any nonempty open subset of X then contains x, and any subspace containing x has x as a generic point.

[1][Trans] also known as *nowhere dense*.

(2.1.3). Recall that a *Kolmogoroff space* is a topological space X satisfying the axiom of separation:

(T_0) If $x \neq y$ are any two points of X, there is an open set containing one of the points x and y, but not the other.

If an irreducible Kolmogoroff space admits a generic point, it admits *exactly* one, since a nonempty open set contains any generic point.

Recall that a topological space X is said to be *quasi-compact* if, from any collection of open sets of X, one can extract a finite cover of X (or, equivalently, if any decreasing filtered family of nonempty closed sets has a nonempty intersection). If X is a quasi-compact space, then any nonempty closed subset A of X contains a *minimal* nonempty closed set M, because the set of nonempty closed subsets of A is inductive under the relation $\supset$; if, in addition, X is a Kolmogoroff space, M is necessarily a single point (or, as we say by abuse of language, is a *closed point*).

(2.1.4). In an irreducible space X, every nonempty open subspace U is irreducible, and if X admits a generic point x, x is also a generic point of U.

To prove this, let (U_α) be a cover (whose set of indices is nonempty) of a topological space X, consisting of nonempty open sets; if X is irreducible, it is necessary and sufficient that U_α is irreducible for all α, and that $U_\alpha \cap U_\beta \neq \varnothing$ for any α, β. The condition is clearly necessary; to see that it is sufficient, it suffices to prove that if V is a nonempty open subset of X, then $V \cap U_\alpha$ is nonempty for all α, since then $V \cap U_\alpha$ is dense in U_α for all α, and consequently V is dense in X. Now there is at least one index γ such that $V \cap U_\gamma \neq \varnothing$, so $V \cap U_\gamma$ is dense in U_γ, and as for all α, $U_\alpha \cap V_\alpha \neq \varnothing$, we also have $V \cap U_\alpha \cap U_\gamma \neq \varnothing$.

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(2.1.5). Let X be an irreducible space, and f a continuous map from X into a topological space Y. Then $f(X)$ is irreducible, and if x is a generic point of X, then $f(x)$ is a generic point of $f(X)$ and hence also of $\overline{f(X)}$. In particular, if, in addition, Y is irreducible and with a single generic point y, then for $f(X)$ to be everywhere dense, it is necessary and sufficient that $f(x) = y$.

(2.1.6). Any irreducible subspace of a topological space X is contained in a maximal irreducible subspace, which is necessarily closed. Maximal irreducible subspaces of X are called the *irreducible components* of X. If Z_1 and Z_2 are two irreducible components distinct from the space X, then $Z_1 \cap Z_2$ is a closed *rare* set in each of the subspaces Z_1, Z_2; in particular, if an irreducible component of X admits a generic point (2.1.2), such a point cannot belong to any other irreducible component. If X has only a *finite* number of irreducible components Z_i ($1 \leqslant i \leqslant n$), and if, for each i, we put $U_i = \bigcup_{j \neq i} Z_j$, then the U_i are open, irreducible, disjoint, and their union is **Err**$_{\mathrm{II}}$ dense in X. Let U be an open subset of a topological space X. If Z is an irreducible subset of X that intersects U, then $Z \cap U$ is open and dense in Z, thus irreducible; conversely, for any irreducible closed subset Y of U, the closure $\overline{Y}$ of Y in X is irreducible and $\overline{Y} \cap U = Y$. We conclude that there is a *bijective correspondence* between the irreducible components of U and the irreducible components of X which intersect U.

(2.1.7). If a topological space X is a union of a *finite* number of irreducible closed subspaces Y_i, then the irreducible components of X are the maximal elements of the set of the Y_i, because if Z is an irreducible closed subset of X, then Z is the union of the $Z \cap Y_i$, from which one sees that Z must be contained in one of the Y_i. Let Y be a subspace of a topological space X, and suppose that Y has only a finite number of irreducible components Y_i, ($1 \leqslant i \leqslant n$); then the closures $\overline{Y_i}$ in X are the irreducible components of Y.

(2.1.8). Let Y be an irreducible space admitting a single generic point y. Let X be a topological space, and f a continuous map from X to Y. Then, for any irreducible component Z of X intersecting $f^{-1}(y)$, $f(Z)$ is dense in Y. The converse is not necessarily true; however, if Z has a generic point z, and if $f(Z)$ is dense in Y, then we must have $f(z) = y$ (2.1.5); in addition, $Z \cap f^{-1}(y)$ is then the closure of $\{z\}$ in $f^{-1}(y)$ and is therefore irreducible, and as an irreducible subset of $f^{-1}(y)$ containing z is necessarily contained in Z (2.1.6), z is a generic point of $Z \cap f^{-1}(y)$. As any irreducible component of $f^{-1}(y)$ is contained in an irreducible component of X, we see that, if any irreducible component Z of X intersecting $f^{-1}(y)$ admits a generic point, then there is a *bijective correspondence* between all these components and all the irreducible components $Z \cap f^{-1}(y)$ of $f^{-1}(y)$, the generic points of Z being identical to those of $Z \cap f^{-1}(y)$.

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2.2. Noetherian spaces.

(2.2.1). We say that a topological space X is *Noetherian* if the set of open subsets of X satisfies the *maximal* condition, or, equivalently, if the set of closed subsets of X satisfies the *minimal* condition. We say that X is *locally Noetherian* if each $x \in X$ admits a neighborhood which is a Noetherian subspace.

(2.2.2). Let E be an ordered set satisfying the *minimal* condition, and let $\mathbf{P}$ be a property of the elements of E subject to the following condition: if $a \in E$ is such that for any $x < a$, $\mathbf{P}(x)$ is true, then $\mathbf{P}(a)$ is true. Under these conditions, $\mathbf{P}(x)$ *is true for all* $x \in E$ ("principle of Noetherian recurrence"). Indeed, let F be the set of $x \in E$ for which $\mathbf{P}(x)$ is false; if F were not empty, it would have a minimal element a, and as then $\mathbf{P}(x)$ is true for all $x < a$, $\mathbf{P}(a)$ would be true, which is a contradiction.

We will apply this principle in particular when E is a *set of closed subsets of a Noetherian space.*

(2.2.3). Any subspace of a Noetherian space is Noetherian. Conversely, any topological space that is a finite union of Noetherian subspaces is Noetherian.

(2.2.4). Any Noetherian space is quasi-compact; conversely, any topological space in which all open sets are quasi-compact is Noetherian.

(2.2.5). A Noetherian space has only a *finite* number of irreducible components, as we see by Noetherian recurrence.

§3. Supplement on sheaves

3.1. Sheaves with values in a category.

(3.1.1). Let C be a category, $(A_\alpha)_{\alpha\in I}$, $(A_{\alpha\beta})_{(\alpha,\beta)\in I\times I}$ two families of objects of C such that $A_{\beta\alpha} = A_{\alpha\beta}$, and $(\rho_{\alpha\beta})_{(\alpha,\beta)\in I\times I}$ a family of morphisms $\rho_{\alpha\beta} : A_\alpha \to A_{\alpha\beta}$. We say that a pair consisting of an object A of C and a family of morphisms $\rho_\alpha : A \to A_\alpha$ is a *solution to the universal problem* defined by the data of the families (A_α), $(A_{\alpha\beta})$, and $(\rho_{\alpha\beta})$ if, for every object B of C, the map which sends $f \in \operatorname{Hom}(B,A)$ to the family $(\rho_\alpha \circ f) \in \prod_\alpha \operatorname{Hom}(B,A_\alpha)$ is a *bijection* of $\operatorname{Hom}(B,A)$ to the set of all (f_α) such that $\rho_{\alpha\beta}\circ f_\alpha = \rho_{\beta\alpha}\circ f_\beta$ for any pair of indices (α,β). If such a solution exists, it is unique up to an isomorphism.

(3.1.2). We will not recall the definition of a *presheaf* $U \mapsto \mathscr{F}(U)$ on a topological space X with values in a category C (G, I, 1.9); we say that such a presheaf is a *sheaf with values in* C if it satisfies the following axiom:

(F) *For any covering* (U_α) *of an open set* U *of* X *by open sets* U_α *contained in* U*, if we denote by* ρ_α (resp. $\rho_{\alpha\beta}$) *the restriction morphism*

$$\mathscr{F}(U) \longrightarrow \mathscr{F}(U_\alpha) \quad (\text{resp. } \mathscr{F}(U_\alpha) \longrightarrow \mathscr{F}(U_\alpha \cap U_\beta)),$$

the pair formed by $\mathscr{F}(U)$ *and the family* (ρ_α) *are a solution to the universal problem for* $(\mathscr{F}(U_\alpha))$, $(\mathscr{F}(U_\alpha \cap U_\beta))$, **$0_{\mathrm{I}}$ | 24**
and $(\rho_{\alpha\beta})$ *in* (3.1.1)[2].

Equivalently, we can say that, for each object T of C, that the family $U \mapsto \operatorname{Hom}(T,\mathscr{F}(U))$ is a *sheaf of sets.*

(3.1.3). Assume that C is the category defined by a "type of structure with morphisms" Σ, the objects of C being the sets with structures of type Σ and morphisms those of Σ. Suppose that the category C also satisfies the following condition:

(E) If $(A,(\rho_\alpha))$ is a solution of a universal mapping problem *in the category* C for families (A_α), $(A_{\alpha\beta})$, $(\rho_{\alpha\beta})$, then it is also a solution of the universal mapping problem for the same families *in the category of sets* (that is, when we consider A, A_α, and $A_{\alpha\beta}$ as sets, ρ_α and $\rho_{\alpha\beta}$ as functions)[3].

Under these conditions, the condition (F) gives that, when considered as a presheaf *of sets*, $U \mapsto \mathscr{F}(U)$ is a *sheaf.* In addition, for a map $u : T \to \mathscr{F}(U)$ to be a morphism of C, it is necessary and sufficient, according to (F), that each map $\rho_\alpha \circ u$ is a morphism $T \to \mathscr{F}(U_\alpha)$, which means that the structure of type Σ on $\mathscr{F}(U)$ is the *initial structure* for the morphisms ρ_α. Conversely, suppose a presheaf $U \mapsto \mathscr{F}(U)$ on X, with values in C, is a *sheaf of sets* and satisfies the previous condition; it is then clear that it satisfies (F), so it is a *sheaf with values in* C.

(3.1.4). When Σ is a type of a group or ring structure, the fact that the presheaf $U \mapsto \mathscr{F}(U)$ with values in C is a sheaf of *sets* implies *ipso facto* that it is a sheaf with values in C (in other words, a sheaf of groups or rings within the meaning of (G))[4]. But it is not the same when, for example, C is the category of *topological rings* (with morphisms as continuous homomorphisms): a sheaf with values in C is a sheaf of rings $U \mapsto \mathscr{F}(U)$ such that for any open U and any covering of U by open sets $U_\alpha \subset U$, the topology of the ring $\mathscr{F}(U)$ is to be *the least fine* making the homomorpisms $\mathscr{F}(U) \to \mathscr{F}(U_\alpha)$ continuous. We will say in this case that $U \mapsto \mathscr{F}(U)$, considered as a sheaf of

[2] This is a special case of the more general notion of a (non-filtered) *projective limit* (*see* (T, I, 1.8) and the book in preparation announced in the introduction).

[3] It can be proved that it also means that the canonical functor $\mathsf{C} \to \mathsf{Set}$ *commutes with projective limits* (not necessarily filtered).

[4] This is because in the category C, any morphism that is a *bijection* (as a map of sets) is an *isomorphism.* This is no longer true when C is the category of topological spaces, for example.

rings (without a topology), is *underlying* the sheaf of topological rings $U \mapsto \mathscr{F}(U)$. Morphisms $u_V : \mathscr{F}(V) \to \mathscr{G}(V)$ (V an arbitrary open subset of X) of sheaves of topological rings are therefore homomorphisms of the underlying sheaves of rings, such that u_V is *continuous* for all open $V \subset X$; to distinguish them from any homomorphisms of the sheaves of the underlying rings, we will call them continuous homomorphisms of sheaves of topological rings. We have similar definitions and conventions for sheaves of topological spaces or topological groups.

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(3.1.5). It is clear that for any category C, if there is a presheaf (respectively a sheaf) $\mathscr{F}$ on X with values in C and U is an open set of X, the $\mathscr{F}(V)$ for open $V \subset U$ constitute a presheaf (or a sheaf) with values in C, which we call the presheaf (or sheaf) *induced* by $\mathscr{F}$ on U and denote it by $\mathscr{F}|U$.

For any morphism $u : \mathscr{F} \to \mathscr{G}$ of presheaves on X with values in C, we denote by $u|U$ the morphism $\mathscr{F}|U \to \mathscr{G}|U$ consisting of the u_V for $V \subset U$.

(3.1.6). Suppose now that the category C admits *inductive limits* (T, 1.8); then, for any presheaf (and in particular any sheaf) $\mathscr{F}$ on X with values in C and each $x \in X$, we can define the *stalk* $\mathscr{F}_x$ as the object of C defined by the inductive limit of the $\mathscr{F}(U)$ with respect to the filtered set (for $\supset$) of the open neighborhoods U of x in X, and the morphisms $\rho_U^V : \mathscr{F}(V) \to \mathscr{F}(U)$. If $u : \mathscr{F} \to \mathscr{G}$ is a morphism of presheaves with values in C, we define for each $x \in X$ the morphism $u_x : \mathscr{F}_x \to \mathscr{G}_x$ as the inductive limit of $u_U : \mathscr{F}(U) \to \mathscr{G}(U)$ with respect to all open neighborhoods of x; we thus define $\mathscr{F}_x$ as a covariant functor in $\mathscr{F}$, with values in C, for all $x \in X$.

When C is further defined by a kind of structure with morphisms Σ, we call *sections over* U of a *sheaf* $\mathscr{F}$ with values in C the elements of $\mathscr{F}(U)$, and we write $\Gamma(U, \mathscr{F})$ instead of $\mathscr{F}(U)$; for $s \in \Gamma(U, \mathscr{F})$, V an open set contained in U, we write $s|V$ instead of $\rho_V^U(s)$; for all $x \in U$, the canonical image of s in $\mathscr{F}_x$ is the *germ* of s at the point x, denoted by s_x (*we will never replace the notation $s(x)$ in this sense*, this notation being reserved for another notion relating to sheaves which will be considered in this treatise (5.5.1)).

If then $u : \mathscr{F} \to \mathscr{G}$ is a morphism of sheaves with values in C, we will write $u(s)$ instead of $u_V(s)$ for all $s \in \Gamma(V, \mathscr{F})$. **$\mathrm{Err}_{\mathrm{II}}$**

If $\mathscr{F}$ is a sheaf of commutative groups, or rings, or modules, we say that the set of $x \in X$ such that $\mathscr{F}_x \neq \{0\}$ is the *support* of $\mathscr{F}$, denoted $\mathrm{Supp}(\mathscr{F})$; this set is not necessarily closed in X.

When C is defined by a type of structure with morphisms, *we systematically refrain from using the point of view of "étalé spaces"* in terms of relating to sheaves with values in C; in other words, we will never consider a sheaf as a topological space (nor even as the whole union of its stalks), and we will not consider also a morphism $u : \mathscr{F} \to \mathscr{G}$ of such sheaves on X as a continuous map of topological spaces.

3.2. Presheaves on an open basis.

(3.2.1). We will restrict to the following categories C admitting *projective limits* (generalized, that is, corresponding to not necessarily filtered preordered sets, cf. (T, 1.8)). Let X be a topological space, $\mathfrak{B}$ an open basis for the topology of X. We will call a *presheaf on* $\mathfrak{B}$, *with values in* C, a family of objects $\mathscr{F}(U) \in$ C, corresponding to each $U \in \mathfrak{B}$, and a family of morphisms $\rho_U^V : \mathscr{F}(V) \to \mathscr{F}(U)$ defined for any pair (U, V) of elements of $\mathfrak{B}$ such that $U \subset V$, with the conditions $\rho_U^U =$ identity and $\rho_U^W = \rho_U^V \circ \rho_V^W$ if U, V, W in $\mathfrak{B}$ are such that $U \subset V \subset W$. We can **0_{I} | 26**
associate a *presheaf with values in* C: $U \mapsto \mathscr{F}(U)$ in the ordinary sense, taking for all open U, $\mathscr{F}'(U) = \varprojlim \mathscr{F}(V)$, where V runs through the ordered set (for $\subset$, *not filtered* in general) of $V \in \mathfrak{B}$ sets such that $V \subset U$, since the (V) form a projective system for the ρ_V^W ($V \subset W \subset U$, $V \in \mathfrak{B}$, $W \in \mathfrak{B}$). Indeed, if U, U' are two open sets of X such that $U \subset U'$, we define $\rho'^{U'}_U$ as the projective limit (for $V \subset U$) of the canonical morphisms $\mathscr{F}'(U') \to \mathscr{F}(V)$, in other words the unique morphism $\mathscr{F}'(U') \to \mathscr{F}'(U)$, which, when composed with the canonical morphisms $\mathscr{F}'(U) \to \mathscr{F}(V)$, gives the canonical morphisms $\mathscr{F}'(U') \to \mathscr{F}(V)$; the verification of the transitivity of $\rho'^{U'}_U$ is then immediate. Moreover, if $U \in \mathfrak{B}$, the canonical morphism $\mathscr{F}'(U) \to \mathscr{F}(U)$ is an isomorphism, allowing us to identify these two objects[5].

(3.2.2). For the presheaf $\mathscr{F}'$ thus defined to be a *sheaf*, it is necessary and sufficient that the presheaf $\mathscr{F}$ on $\mathfrak{B}$ satisfies the condition:

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[5]If X is a *Noetherian* space, we can still define $\mathscr{F}'(U)$ and show that it is a presheaf (in the ordinary sense) when one supposes only that C admits projective limits for *finite* projective systems. Indeed, if U is any open set of X, there is a *finite* covering (V_i) of U consisting of sets of $\mathfrak{B}$; for every couple (i, j) of indices, let (V_{ijk}) be a finite covering of $V_i \cap V_j$ formed by sets of $\mathfrak{B}$. Let I be the set of i and triples (i, j, k), ordered only by the relations $i > (i, j, k)$, $j > (i, j, k)$; we then take $\mathscr{F}'(U)$ to be the projective limit of the system of $\mathscr{F}(V_i)$ and $\mathscr{F}(V_{ijk})$; it is easy to verify that this does not depend on the coverings (V_i) and (V_{ijk}) and that $U \mapsto \mathscr{F}'(U)$ is a presheaf.

(F_0) *For any covering (U_α) of $U \in \mathfrak{B}$ by sets $U_\alpha \in \mathfrak{B}$ contained in U, and for any object $T \in \mathsf{C}$, the map which sends $f \in \mathrm{Hom}(T, \mathscr{F}(U))$ to the family $(\rho^U_{U_\alpha} \circ f) \in \Pi_\alpha \mathrm{Hom}(T, \mathscr{F}(U_\alpha))$ is a bijection from $\mathrm{Hom}(T, \mathscr{F}(U))$ to the set of all (f_α) such that $\rho^{U_\alpha}_V \circ f_\alpha = \rho^{U_\beta}_V \circ f_\beta$ for any pair of indices (α, β) and any $V \in \mathfrak{B}$ such that $V \subset U_\alpha \cap U_\beta$*[6].

The condition is obviously necessary. To show that it is sufficient, consider first a second basis $\mathfrak{B}'$ of the topology of X, *contained in* $\mathfrak{B}$, and show that if $\mathscr{F}''$ denotes the presheaf induced by the subfamily $(\mathscr{F}(V))_{V\in\mathfrak{B}'}$, $\mathscr{F}''$ is *canonically isomorphic* to $\mathscr{F}'$. Indeed, first the projective limit (for $V \in \mathfrak{B}'$, $V \subset U$) of the canonical morphisms $\mathscr{F}'(U) \to \mathscr{F}(V)$ is a morphism $\mathscr{F}'(U) \to \mathscr{F}''(U)$ for all open U. If $U \in \mathfrak{B}$, this morphism is an isomorphism, because by hypothesis the canonical morphisms $\mathscr{F}''(U) \to \mathscr{F}(V)$ for $V \in \mathfrak{B}'$, $V \subset U$, factorize as $\mathscr{F}''(U) \to \mathscr{F}(U) \to \mathscr{F}(V)$, and it is immediate to see that the composition of morphisms $\mathscr{F}(U) \to \mathscr{F}''(U)$ and $\mathscr{F}''(U) \to \mathscr{F}(U)$ thus defined are the identities. This being so, for all open U, the morphisms $\mathscr{F}''(U) \to \mathscr{F}''(W) = \mathscr{F}(W)$ for $W \in \mathfrak{B}$ and $W \subset U$ satisfy the conditions characterizing the projective limit of $\mathscr{F}(W)$ ($W \in \mathfrak{B}$, $W \subset U$), which proves our assertion given the uniqueness of a projective limit up to isomorphism.

This being so, let U be any open set of X, (U_α) a covering of U by the open sets contained in U, and $\mathfrak{B}'$ the subfamily of $\mathfrak{B}$ formed by the sets of $\mathfrak{B}$ contained in at least one U_α; it is clear that $\mathfrak{B}'$ is still a basis of the **0_{I}** | 27
topology of U, so $\mathscr{F}'(U)$ (resp. $\mathscr{F}''(U_\alpha)$) is the projective limit of $\mathscr{F}(V)$ for $V \in \mathfrak{B}'$ and $V \subset U$ (resp., $V \subset U_\alpha$), the axiom (F) is then immediately verified by virtue of the definition of the projective limit.

When (F_0) is satisfied, we will say by abuse of language that the presheaf $\mathscr{F}$ on the basis $\mathfrak{B}$ is a sheaf.

(3.2.3). Let $\mathscr{F}$, $\mathscr{G}$ be two presheaves on a basis $\mathfrak{B}$, with values in C; we define a *morphism* $u : \mathscr{F} \to \mathscr{G}$ as a family $(u_V)_{V\in\mathfrak{B}}$ of morphisms $u_V : \mathscr{F}(V) \to \mathscr{G}(V)$ satisfying the usual compatibility conditions with the restriction morphisms ρ^W_V. With the notation of (3.2.1), we have a morphism $u' : \mathscr{F}' \to \mathscr{G}'$ of (ordinary) presheaves by taking for u'_U the projective limit of the u_V for $V \in \mathfrak{B}$ and $V \subset U$; the verification of the compatibility conditions with the $\rho'^{U'}_U$ follows from the functorial properties of the projective limit.

(3.2.4). If the category C admits inductive limits, and if $\mathscr{F}$ is a presheaf on the basis $\mathfrak{B}$, with values in C, for each $x \in X$ the neighborhoods of x belonging to $\mathfrak{B}$ form a cofinal set (for $\supset$) in the set of neighborhoods of x, therefore, if $\mathscr{F}'$ is the (ordinary) presheaf corresponding to $\mathscr{F}$, the stalk $\mathscr{F}'_x$ is equal to $\varinjlim_{\mathfrak{B}} \mathscr{F}(V)$ over the set of $V \in \mathfrak{B}$ containing x. If $u : \mathscr{F} \to \mathscr{G}$ is morphism of presheaves on $\mathfrak{B}$ with values in C, $u' : \mathscr{F}' \to \mathscr{G}'$ the corresponding morphism of ordinary presheaves, u'_x is likewise the inductive limit of the morphisms $u_V : \mathscr{F}(V) \to \mathscr{G}(V)$ for $V \in \mathfrak{B}$, $x \in V$.

(3.2.5). We return to the general conditions of (3.2.1). If $\mathscr{F}$ is an ordinary *sheaf* with values in C, $\mathscr{F}_1$ the sheaf *on* $\mathfrak{B}$ obtained by the restriction of $\mathscr{F}$ to $\mathfrak{B}$, then the ordinary sheaf $\mathscr{F}'_1$ obtained from $\mathscr{F}_1$ by the procedure of (3.2.1) is canonically isomorphic to $\mathscr{F}$, by virtue of the condition (F) and the uniqueness properties of the projective limit. We identify the ordinary sheaf $\mathscr{F}$ with $\mathscr{F}'_1$.

If $\mathscr{G}$ is a second (ordinary) sheaf on X with values in C, and $u : \mathscr{F} \to \mathscr{G}$ a morphism, the preceding remark shows that the data of the $u_V : \mathscr{F}(V) \to \mathscr{G}(V)$ *for only the* $V \in \mathfrak{B}$ completely determines u; conversely, it is sufficient, the u_V being given for $V \in \mathfrak{B}$, to verify the commutative diagram with the restriction morphisms ρ^W_V for $V \in \mathfrak{B}$, $W \in \mathfrak{B}$, and $V \subset W$, for there to exist a morphism u' and a unique $\mathscr{F}$ in $\mathscr{G}$ such that $u'_V = u_V$ for each $V \in \mathfrak{B}$ (3.2.3).

(3.2.6). Suppose that C admits projective limits. Then the category of *sheaves on X with values in* C admits *projective limits*; if $(\mathscr{F}_\lambda)$ is a projective system of sheaves on X with values in C, the $\mathscr{F}(U) = \varprojlim_\lambda \mathscr{F}_\lambda(U)$ indeed define a presheaf with values in C, and the verification of the axiom (F) follows from the transitivity of projective limits; the fact that $\mathscr{F}$ is then the projective limit of the $\mathscr{F}_\lambda$ is immediate.

When C is the category of sets, for each projective system $(\mathscr{H}_\lambda)$ such that $\mathscr{H}_\lambda$ is a *subsheaf* of $\mathscr{F}_\lambda$ for each λ, **0_{I}** | 28
$\varprojlim_\lambda \mathscr{H}_\lambda$ canonically identifies with a *subsheaf* of $\varprojlim_\lambda \mathscr{F}_\lambda$. If C is the category of abelian groups, the covariant functor $\varprojlim_\lambda \mathscr{F}_\lambda$ is *additive* and *left exact.*

3.3. Gluing sheaves.

(3.3.1). Suppose still that the category C admits (generalized) projective limits. Let X be a topological space, $\mathfrak{U} = (U_\lambda)_{\lambda\in L}$ an open cover of X, and for each $\lambda \in L$, let $\mathscr{F}_\lambda$ be a sheaf on U_λ, with values in C; for each pair of indices (λ, μ), suppose that we are given an *isomorphism* $\theta_{\lambda\mu} : \mathscr{F}_\mu|(U_\lambda \cap U_\mu) \simeq \mathscr{F}|(U_\lambda \cap U_\mu)$; in addition, suppose that for each triple (λ, μ, ν), if we denote by $\theta'_{\lambda\mu}$, $\theta'_{\mu\nu}$, $\theta'_{\lambda\nu}$ the restrictions of $\theta_{\lambda\mu}$, $\theta_{\mu\nu}$, $\theta_{\lambda\nu}$ to $U_\lambda \cap U_\mu \cap U_\nu$, then we

[6] It also means that the pair formed by $\mathscr{F}(U)$ and the $\rho_\alpha = \rho^U_{U_\alpha}$ is a *solution to the universal problem* defined in (3.1.1) by the data of $A_\alpha = \mathscr{F}(U_\alpha)$, $A_{\alpha\beta} = \Pi\mathscr{F}(V)$ (for $V \in \mathfrak{B}$ such that $V \subset U_\alpha \cap U_\beta$) and $\rho_{\alpha\beta} = (\rho''_V) : \mathscr{F}(U_\alpha) \to \Pi\mathscr{F}(V)$ defined by the condition that for $V \in \mathfrak{B}$, $V' \in \mathfrak{B}$, $W \in \mathfrak{B}$, $V \cup V' \subset U_\alpha \cap U_\beta$, $W \subset V \cap V'$, $\rho^V_W \circ \rho''_V = \rho^{V'}_W \circ \rho''_{V'}$.

have $\theta'_{\lambda\nu} = \theta'_{\lambda\mu} \circ \theta'_{\mu\nu}$ (*gluing condition* for the $\theta_{\lambda\mu}$). Then there exists a sheaf $\mathscr{F}$ on X, with values in $\mathtt{C}$, and for each λ an isomorphism $\eta_\lambda : \mathscr{F}|U_\lambda \simeq \mathscr{F}_\lambda$ such that, for each pair (λ, μ), if we denote by η'_λ and η'_μ the restrictions of η_λ and η_μ to $U_\lambda \cap U_\mu$, then we have $\theta_{\lambda\mu} = \eta'_\lambda \circ \eta'^{-1}_\mu$; in addition, $\mathscr{F}$ and the η_λ are determined up to unique isomorphism by these conditions. The uniqueness indeed follows immediately from (3.2.5). To establish the existence of $\mathscr{F}$, denote by $\mathfrak{B}$ the open basis consisting of the open sets contained in at least one U_λ, and for each $U \in \mathfrak{B}$, choose (by the Hilbert function τ) one of the $\mathscr{F}_\lambda(U)$ for one of the λ such that $U \subset U_\lambda$; if we denote this object by $\mathscr{F}(U)$, the ρ^V_U for $U \subset V$, $U \in \mathfrak{B}$, $V \in \mathfrak{B}$ are defined in an evident way (by means of the $\theta_{\lambda\mu}$), and the transitivity conditions is a consequence of the gluing condition; in addition, the verification of (F_0) is immediate, so the presheaf on $\mathfrak{B}$ thus clearly defines a sheaf, and we deduce by the general procedure (3.2.1) an (ordinary) sheaf still denoted $\mathscr{F}$ and which answers the question. We say that $\mathscr{F}$ is obtained by *gluing the $\mathscr{F}_\lambda$ by means of the $\theta_{\lambda\mu}$* and we usually identify the $\mathscr{F}_\lambda$ and $\mathscr{F}|U_\lambda$ by means of the η_λ.

It is clear that each sheaf $\mathscr{F}$ on X with values in $\mathtt{C}$ can be considered as being obtained by the gluing of the sheaves $\mathscr{F}_\lambda = \mathscr{F}|U_\lambda$ (where (U_λ) is an arbitrary open cover of X), by means of the isomorphisms $\theta_{\lambda\mu}$ reduced to the identity.

(3.3.2). With the same notation, let $\mathscr{G}_\lambda$ be a second sheaf on U_λ (for each $\lambda \in L$) with values in $\mathtt{C}$, and for each pair (λ, μ) let us be given an isomorphism $\omega_{\lambda\mu} : \mathscr{G}_\mu|(U_\lambda \cap U_\mu) \simeq \mathscr{G}_\lambda|(U_\lambda \cap U_\mu)$, these isomorphisms satisfying the gluing condition. Finally, suppose that we are given for each λ a morphism $u_\lambda : \mathscr{F}_\lambda \to \mathscr{G}_\lambda$, and that the diagrams

(3.3.2.1)
$$\begin{tikzcd} \mathscr{F}_\mu|(U_\lambda \cap U_\mu) \arrow[r, "u_\mu"] \arrow[d] & \mathscr{G}_\mu|(U_\lambda \cap U_\mu) \arrow[d] \\ \mathscr{F}_\lambda|(U_\lambda \cap U_\mu) \arrow[r, "u_\lambda"] & \mathscr{G}_\lambda|(U_\lambda \cap U_\mu) \end{tikzcd}$$

are commutative. Then, if $\mathscr{G}$ is obtained by gluing the $\mathscr{G}_\lambda$ by means of the $\omega_{\lambda\mu}$, there exists a unique morphism $u : \mathscr{F} \to \mathscr{G}$ such that the diagrams $\mathbf{0_I}$ | 29

$$\begin{tikzcd} \mathscr{F}|U_\lambda \arrow[r, "u|U_\lambda"] \arrow[d] & \mathscr{G}|U_\lambda \arrow[d] \\ \mathscr{F}_\lambda \arrow[r, "u_\lambda"] & \mathscr{G}_\lambda \end{tikzcd}$$

are commutative; this follows immediately from (3.2.3). The correspondence between the family (u_λ) and u is in a functorial bijection with the subset of $\Pi_\lambda \operatorname{Hom}(\mathscr{F}_\lambda, \mathscr{G}_\lambda)$ satisfying the conditions (3.3.2.1) on $\operatorname{Hom}(\mathscr{F}, \mathscr{G})$.

(3.3.3). With the notation of (3.3.1), let V be an open set of X; it is immediate that the restrictions to $V \cap U_\lambda \cap U_\mu$ of the $\theta_{\lambda\mu}$ satisfy the gluing condition for the induced sheaves $\mathscr{F}_\lambda|(V \cap U_\lambda)$ and that the sheaves on V obtained by gluing the latter identifies canonically with $\mathscr{F}|V$.

3.4. Direct images of presheaves.

(3.4.1). Let X, Y be two topological spaces, $\psi : X \to Y$ a continuous map. Let $\mathscr{F}$ be a presheaf on X with values in a category $\mathtt{C}$; for each open $U \subset Y$, let $\mathscr{G}(U) = \mathscr{F}(\psi^{-1}(U))$, and if U, V are two open subsets of Y such that $U \subset V$, let ρ^V_U be the morphism $\mathscr{F}(\psi^{-1}(V)) \to \mathscr{F}(\psi^{-1}(U))$; it is immediate that the $\mathscr{G}(U)$ and the ρ^V_U define a *presheaf* on Y with values in $\mathtt{C}$, that we call the *direct image of $\mathscr{F}$ by ψ* and we denote it by $\psi_*(\mathscr{F})$. If $\mathscr{F}$ is a sheaf, we immediately verify the axiom (F) for the presheaf $\mathscr{G}$, so $\psi_*(\mathscr{F})$ is a sheaf.

(3.4.2). Let $\mathscr{F}_1$, $\mathscr{F}_2$ be two presheaves of X with values in $\mathtt{C}$, and let $u : \mathscr{F}_1 \to \mathscr{F}_2$ be a morphism. When U varies over the set of open subsets of Y, the family of morphisms $u_{\psi^{-1}(U)} : \mathscr{F}_1(\psi^{-1}(U)) \to \mathscr{F}_2(\psi^{-1}(U))$ satisfies the compatibility conditions with the restriction morphisms, and as a result defines a morphism $\psi_*(u) : \psi_*(\mathscr{F}_1) \to \psi_*(\mathscr{F}_2)$. If $v : \mathscr{F}_2 \to \mathscr{F}_3$ is a morphism from $\mathscr{F}_2$ to a third presheaf on X with values in $\mathtt{C}$, we have $\psi_*(v \circ u) = \psi_*(v) \circ \psi_*(u)$; in other words, $\psi_*(\mathscr{F})$ is a *covariant functor* in $\mathscr{F}$, from the category of presheaves (resp. sheaves) on X with values in $\mathtt{C}$, to that of presheaves (resp. sheaves) on Y with values in $\mathtt{C}$.

(3.4.3). Let Z be a third topological space, $\psi' : Y \to Z$ a continuous map, and let $\psi'' = \psi' \circ \psi$. It is clear that we have $\psi''_*(\mathscr{F}) = \psi'_*(\psi_*(\mathscr{F}))$ for each presheaf $\mathscr{F}$ on X with values in $\mathtt{C}$; in addition, for each morphism $u : \mathscr{F} \to \mathscr{G}$ of such presheaves, we have $\psi''_*(u) = \psi'_*(\psi_*(u))$. In other words, ψ''_* is the *composition* of the functors ψ'_* and ψ_*, and this can be written as
$$(\psi' \circ \psi)_* = \psi'_* \circ \psi_*.$$

In addition, for each open set U of Y, the image under the restriction $\psi|\psi^{-1}(U)$ of the induced presheaf $\mathscr{F}|\psi^{-1}(U)$ is none other than the induced presheaf $\psi_*(\mathscr{F})|U$.

(3.4.4). Suppose that the category C admits inductive limits, and let $\mathscr{F}$ be a presheaf on X with values in C; for all $x \in X$, the morphisms $\Gamma(\psi^{-1}(U), \mathscr{F}) \to \mathscr{F}_x$ (U an open neighborhood of $\psi(x)$ in Y) form an inductive limit, which gives by passing to the limit a morphism $\psi_x : (\psi_*(\mathscr{F}))_{\psi(x)} \to \mathscr{F}_x$ of the stalks; in general, these $\mathbf{0_I}$ | 30
morphisms are *neither injective or surjective.* It is functorial; indeed, if $u : \mathscr{F}_1 \to \mathscr{F}_2$ is a morphism of presheaves on X with values in C, the diagram

$$\begin{tikzcd} (\psi_*(\mathscr{F}_1))_{\psi(x)} \arrow[r, "\psi_x"] \arrow[d, "(\psi_*(u))_{\psi(x)}"'] & (\mathscr{F}_1)_x \arrow[d, "u_x"] \\ (\psi_*(\mathscr{F}_2))_{\psi(x)} \arrow[r, "\psi_x"] & (\mathscr{F}_2)_x \end{tikzcd}$$

is commutative. If Z is a third topological space, $\psi' : Y \to Z$ a continuous map, and $\psi'' = \psi' \circ \psi$, then we have $\psi''_x = \psi_x \circ \psi'_{\psi(x)}$ for $x \in X$.

(3.4.5). Under the hypotheses of (3.4.4), suppose in addition that ψ is a *homeomorphism* from X to the subspace $\psi(X)$ of Y. Then, for each $x \in X$, ψ_x is an *isomorphism.* This applies in particular to the canonical injection j of a subset X of Y into Y.

(3.4.6). Suppose that C be the category of groups, or of rings, etc. If $\mathscr{F}$ is a sheaf on X with values in C, of support S, and if $y \notin \overline{\psi(S)}$, then it follows from the definition of $\psi_*(\mathscr{F})$ that $(\psi_*(\mathscr{F}))_y = \{0\}$, or in other words, that the support of $\psi_*(\mathscr{F})$ is contained in $\overline{\psi(S)}$; but it is not necessarily contained in $\psi(S)$. Under the same hypotheses, if j is the canonical injection of a subset X of Y into Y, the sheaf $j_*(\mathscr{F})$ induces $\mathscr{F}$ on X; if moreover X is *closed* in Y, $j_*(\mathscr{F})$ is the sheaf on Y which induces $\mathscr{F}$ on X and 0 on $Y - X$ (G, II, 2.9.2), but it is in general distinct from the latter when we suppose that X is locally closed but not closed.

3.5. Inverse images of presheaves.

(3.5.1). Under the hypotheses of (3.4.1), if $\mathscr{F}$ (resp. $\mathscr{G}$) is a presheaf on X (resp. Y) with values in C, then each morphism $u : \mathscr{G} \to \psi_*(\mathscr{F})$ of presheaves on Y is called a *φ-morphism* from $\mathscr{G}$ to $\mathscr{F}$, and we denote it also by $\mathscr{G} \to \mathscr{F}$. We denote also by $\mathrm{Hom}_\varphi(\mathscr{G}, \mathscr{F})$ the set of $\mathrm{Hom}_Y(\mathscr{G}, \psi_*(\mathscr{F}))$ the ψ-morphisms from $\mathscr{G}$ to $\mathscr{F}$. For each pair (U, V), where U is an open set of X, V an open set of Y such that $\psi(U) \subset V$, we have a morphism $u_{U,V} : \mathscr{G}(U) \to \mathscr{F}(U)$ by composing the restriction morphism $\mathscr{F}(\psi^{-1}(V)) \to \mathscr{F}(U)$ and the morphism $u_V : \mathscr{G}(V) \to \psi_*(\mathscr{F})(V) = \mathscr{F}(\psi^{-1}(V))$; it is immediate that these morphisms render commutative the diagrams

$$\begin{tikzcd} \mathscr{G}(V) \arrow[r, "u_{U,V}"] \arrow[d] & \mathscr{F}(U) \arrow[d] \\ \mathscr{G}(V') \arrow[r, "u_{U',V'}"] & \mathscr{F}(U') \end{tikzcd} \tag{3.5.1.1}$$

for $U' \subset U$, $V' \subset V$, $\psi(U') \subset V'$. Conversely, the data of a family $(u_{U,V})$ of morphisms rendering commutative the diagrams (3.5.1.1) define a ψ-morphism u, since it suffices to take $u_V = u_{\psi^{-1}(V),V}$. $\mathbf{0_I}$ | 31

If the category C admits (generalized) projective limits, and if $\mathfrak{B}$, $\mathfrak{B}'$ are bases for the topologies of X and Y respectively, to define a ψ-morphism u of *sheaves*, we can restrict to giving the $u_{U,V}$ for $U \in \mathfrak{B}$, $V \in \mathfrak{B}'$, and $\psi(U) \subset V$, satisfying the compatibility conditions of (3.5.1.1) for U, U' in $\mathfrak{B}$ and V, V' in $\mathfrak{B}'$; it indeed suffices to define u_W, for each open $W \subset Y$, as the projective limit of the $u_{U,V}$ for $V \in \mathfrak{B}'$ and $V \subset W$, $U \in \mathfrak{B}$ and $\psi(U) \subset V$.

When the category C admits inductive limits, we have, for each $x \in X$, a morphism $\mathscr{G}(V) \to \mathscr{F}(\psi^{-1}(V)) \to \mathscr{F}_x$, for each open neighborhood V of $\psi(x)$ in Y, and these morphisms form an inductive system which gives by passing to the limit a morphism $\mathscr{G}_{\psi(x)} \to \mathscr{F}_x$.

(3.5.2). Under the hypotheses of (3.4.3), let $\mathscr{F}$, $\mathscr{G}$, $\mathscr{H}$ be presheaves with values in C on X, Y, Z respectively, and let $u : \mathscr{G} \to \psi_*(\mathscr{F})$, $v : \mathscr{H} \to \psi'_*(\mathscr{G})$ be a ψ-morphism and a ψ'-morphism respectively. We obtain a ψ''-morphism $w : \mathscr{H} \xrightarrow{v} \psi'_*(\mathscr{G}) \xrightarrow{\psi'_*(u)} \psi'_*(\psi_*(\mathscr{F})) = \psi''_*(\mathscr{F})$, that we call, by definition, the *composition* of u and v. We can therefore consider the pairs $(X, \mathscr{F})$ consisting of a topological space X and a presheaf $\mathscr{F}$ on X (with values in C) as forming a *category*, the morphisms being the pairs $(\psi, \theta) : (X, \mathscr{F}) \to (Y, \mathscr{G})$ consisting of a continuous map $\psi : X \to Y$ and of a ψ-morphism $\theta : \mathscr{G} \to \mathscr{F}$.

(3.5.3). Let $\psi : X \to Y$ be a continuous map, $\mathscr{G}$ a *presheaf* on Y with values in C. We call the *inverse image of $\mathscr{G}$ under ψ* the pair $(\mathscr{G}', \rho)$, where $\mathscr{G}'$ is a *sheaf* on X with values in C, and $\rho : \mathscr{G} \to \mathscr{G}'$ a ψ-morphism (in other words a homomorphism $\mathscr{G} \to \psi_*(\mathscr{G}'))$ such that, for each *sheaf* $\mathscr{F}$ on X with values in C, the map

$$\mathrm{Hom}_X(\mathscr{G}', \mathscr{F}) \longrightarrow \mathrm{Hom}_\psi(\mathscr{G}, \mathscr{F}) \longrightarrow \mathrm{Hom}_Y(\mathscr{G}, \psi_*(\mathscr{F})) \tag{3.5.3.1}$$

sending v to $\psi_*(v) \circ \rho$, is a *bijection*; this map, being functorial in $\mathscr{F}$, then defines an isomorphism of functors in $\mathscr{F}$. The pair $(\mathscr{G}', \rho)$ is the solution of a universal problem, and we say it is *determined up to unique isomorphism* when it exists. We then write $\mathscr{G}' = \psi^*(\mathscr{G})$, $\rho = \rho_{\mathscr{G}}$, and by abuse of language, we say that $\psi^*(\mathscr{G})$ is *the inverse image sheaf* of $\mathscr{G}$ under ψ, and we agree that $\psi^*(\mathscr{G})$ is considered as equipped with a *canonical ψ-morphism* $\rho_{\mathscr{G}} : \mathscr{G} \to \psi^*(\mathscr{G})$, that is to say the *canonical homomorphism* of presheaves on Y:

(3.5.3.2)
$$\rho_{\mathscr{G}} : \mathscr{G} \longrightarrow \psi_*(\psi^*(\mathscr{G})).$$

For each homomorphism $v : \psi^*(\mathscr{G}) \to \mathscr{F}$ (where $\mathscr{F}$ is a sheaf on X with values in C), we put $v^\flat = \psi_*(v) \circ \rho_{\mathscr{G}} : \mathscr{G} \to \psi_*(\mathscr{F})$. By definition, *each* morphism of presheaves $u : \mathscr{G} \to \psi_*(\mathscr{F})$ is of the form $v^\flat$ for a unique v, which we will denote $u^\sharp$. In other words, each morphism $u : \mathscr{G} \to \psi_*(\mathscr{F})$ of presheaves factorizes in a unique way as

(3.5.3.3)
$$u : \mathscr{G} \xrightarrow{\rho_{\mathscr{G}}} \psi_*(\psi^*(\mathscr{G})) \xrightarrow{\psi_*(u^\sharp)} \psi_*(\mathscr{F}).$$ $\mathbf{0_I}$ | 32

(3.5.4). Suppose now that the category C be such[7] that *each* presheaf $\mathscr{F}$ on Y with values in C admits an inverse image under ψ, and we denote it by $\psi^*(\mathscr{G})$.

We will see that we can define $\psi^*(\mathscr{G})$ as a *covariant functor* in $\mathscr{G}$, from the category of presheaves on Y with values in C, to that of sheaves on X with values in C, in such a way that the isomorphism $v \mapsto v^\flat$ is an *isomorphism of bifunctors*

(3.5.4.1)
$$\operatorname{Hom}_X(\psi^*(\mathscr{G}), \mathscr{F}) \simeq \operatorname{Hom}_Y(\mathscr{G}, \psi_*(\mathscr{F}))$$

in $\mathscr{G}$ and $\mathscr{F}$.

Indeed, for each morphism $w : \mathscr{G}_1 \to \mathscr{G}_2$ of presheaves on Y with values in C, consider the composite morphism $\mathscr{G}_1 \xrightarrow{w} \mathscr{G}_2 \xrightarrow{\rho_{\mathscr{G}_2}} \psi_*(\psi^*(\mathscr{G}_2))$; to it corresponds a morphism $(\rho_{\mathscr{G}_2} \circ w)^\sharp : \psi^*(\mathscr{G}_1) \to \psi^*(\mathscr{G}_2)$, that we denote by $\psi^*(w)$. We therefore have, according to (3.5.3.3),

(3.5.4.2)
$$\psi_*(\psi^*(w)) \circ \rho_{\mathscr{G}_1} = \rho_{\mathscr{G}_2} \circ w.$$

For each morphism $u : \mathscr{G}_2 \to \psi_*(\mathscr{F})$, where $\mathscr{F}$ is a sheaf on X with values in C, we have, according to (3.5.3.3), (3.5.4.2), and the definition of $u^\flat$, that

$$(u^\sharp \circ \psi^*(w))^\flat = \psi_*(u^\sharp) \circ \psi_*(\psi^*(w)) \circ \rho_{\mathscr{G}_1} = \psi_*(u^\sharp) \circ \rho_{\mathscr{G}_2} \circ w = u \circ w$$

where again

(3.5.4.3)
$$(u \circ w)^\sharp = u^\sharp \circ \psi^*(w).$$

If we take in particular for u a morphism $\mathscr{G}_2 \xrightarrow{w'} \mathscr{G}_3 \xrightarrow{\rho_{\mathscr{G}_3}} \psi_*(\psi^*(\mathscr{G}_3))$, it becomes $\psi^*(w' \circ w) = (\rho_{\mathscr{G}_3} \circ w' \circ w)^\sharp = (\rho_{\mathscr{G}_3} \circ w')^\sharp \circ \psi^*(w) = \psi^*(w') \circ \psi^*(w)$, hence our assertion.

Finally, for each sheaf $\mathscr{F}$ on X with values in C, let $i_{\mathscr{F}}$ be the identity morphism of $\psi_*(\mathscr{F})$ and denote by

$$\sigma_{\mathscr{F}} : \psi^*(\psi_*(\mathscr{F}) \longrightarrow \mathscr{F}$$

the morphism $(i_{\mathscr{F}})^\sharp$; the formula (3.5.4.3) gives in particular the factorization

(3.5.4.4)
$$u^\sharp : \psi^*(\mathscr{G}) \xrightarrow{\psi^*(u)} \psi^*(\psi_*(\mathscr{F})) \xrightarrow{\sigma_{\mathscr{F}}} \mathscr{F}$$

for each morphism $u : \mathscr{G} \to \psi_*(\mathscr{F})$. We say that the morphism $\sigma_{\mathscr{F}}$ is *canonical.*

(3.5.5). Let $\psi' : Y \to Z$ be a continuous map, and suppose that each presheaf $\mathscr{H}$ on Z with values in C admits an inverse image $\psi'^*(\mathscr{H})$ under ψ'. Then (with the hypotheses of (3.5.4)) each presheaf $\mathscr{H}$ on Z with values in C admits an inverse image under $\psi'' = \psi' \circ \psi$ and we have a canonical functorial isomorphism

(3.5.5.1)
$$\psi''^*(\mathscr{H}) \simeq \psi^*(\psi'^*(\mathscr{H})).$$

This indeed follows immediately from the definitions, taking into account that $\psi''_* = \psi'_* \circ \psi_*$. In addition, if $\mathbf{0_I}$ | 33
$u : \mathscr{G} \to \psi_*(\mathscr{F})$ is a ψ-morphism, $v : \mathscr{H} \to \psi'_*(\mathscr{G})$ a ψ'-morphism, and $w = \psi'_*(u) \circ v$ their composition (3.5.2), then we have immediately that $w^\sharp$ is the composite morphism

$$w^\sharp : \psi^*(\psi'^*(\mathscr{H})) \xrightarrow{\psi^*(v^\sharp)} \psi^*(\mathscr{G}) \xrightarrow{u^\sharp} \mathscr{F}.$$

(3.5.6). We take in particular for ψ the identity map $1_X : X \to X$. Then if the inverse image under ψ of a presheaf $\mathscr{F}$ on X with values in C exists, we say that this inverse image is the *sheaf associated to the presheaf $\mathscr{F}$*. Each morphism $u : \mathscr{F} \to \mathscr{F}'$ from $\mathscr{F}$ to a *sheaf* $\mathscr{F}'$ with values in C factorizes in a unique way as $\mathscr{F} \xrightarrow{\rho_{\mathscr{F}}} 1_X^*(\mathscr{F}) \xrightarrow{u^\sharp} \mathscr{F}'$.

[7]In the book mentioned in the introduction, we will give very general conditions on the category C ensuring the existence of inverse images of presheaves with values in C.

3.6. Simple and locally simple sheaves.

(3.6.1). We say that a *presheaf* $\mathscr{F}$ on X, with values in $\mathtt{C}$, is *constant* if the canonical morphisms $\mathscr{F}(X)\to\mathscr{F}(U)$ are *isomorphisms* for each nonempty open $U\subset X$; we note that $\mathscr{F}$ is not necessarily a sheaf. We say that a *sheaf* is *simple* if it is the associated sheaf (3.5.6) of a constant presheaf. We say that a sheaf $\mathscr{F}$ is *locally simple* if each $x\in X$ admits an open neighborhood U such that $\mathscr{F}|U$ is simple.

(3.6.2). Suppose that X is *irreducible* (2.1.1); then the following properties are equivalent:

(a) *$\mathscr{F}$ is a constant presheaf on X;*
(b) *$\mathscr{F}$ is a simple sheaf on X;*
(c) *$\mathscr{F}$ is a locally simple sheaf on X.*

Indeed, let $\mathscr{F}$ be a constant presheaf on X; if U, V are two nonempty open sets in X, then $U\cap V$ is nonempty, so $\mathscr{F}(X)\to\mathscr{F}(U)\to\mathscr{F}(U\cap V)$ and $\mathscr{F}(X)\to\mathscr{F}(U)$ are isomorphisms, and similarly both $\mathscr{F}(U)\to\mathscr{F}(U\cap V)$ and $\mathscr{F}(V)\to\mathscr{F}(U\cap V)$ are isomorphisms. We therefore conclude immediately that the axiom (F) of (3.1.2) is clearly satisfied, $\mathscr{F}$ is isomorphic to its associated sheaf, and as a result (a) implies (b).

Now let (U_α) be an open cover of X by nonempty open sets and let $\mathscr{F}$ be a sheaf on X such that $\mathscr{F}|U_\alpha$ is simple for each α; as U_α is irreducible, $\mathscr{F}|U_\alpha$ is a constant presheaf according to the above. As $U_\alpha\cap U_\beta$ is not empty, $\mathscr{F}(U_\alpha)\to\mathscr{F}(U_\alpha\cap U_\beta)$ and $\mathscr{F}(U_\beta)\to\mathscr{F}(U_\alpha\cap U_\beta)$ are isomorphisms, hence we have a canonical isomorphism $\theta_{\alpha\beta}:\mathscr{F}(U_\alpha)\to\mathscr{F}(U_\beta)$ for each pair of indices. But then if we apply the condition (F) for $U=X$, we see that for each index α_0, $\mathscr{F}(U_{\alpha_0})$ and the $\theta_{\alpha_0\alpha}$ are solutions to the universal problem, which (according to the uniqueness) implies that $\mathscr{F}(X)\to\mathscr{F}(U_{\alpha_0})$ is an isomorphism, and hence proves that (c) implies (a).

3.7. Inverse images of presheaves of groups or rings. $\mathbf{0_I}$ | 34

(3.7.1). We will show that when we take $\mathtt{C}$ to be the category of sets, the inverse image under ψ for each presheaf $\mathscr{G}$ with values in $\mathtt{C}$ *always exists* (the notation and hypotheses on X, Y, ψ being that of (3.5.3)). Indeed, for each open $U\subset X$, define $\mathscr{G}'(U)$ as follows: an element s' of $\mathscr{G}'(U)$ is a family $(s'_x)_{x\in U}$, where $s'_x\in\mathscr{G}_{\psi(x)}$ for each $x\in U$, and where, for each $x\in U$, the following condition is satisfied: there exists an open neighborhood V of $\psi(x)$ in Y, a neighborhood $W\subset\psi^{-1}(V)\cap U$ of x, and an element $s\in\mathscr{G}(V)$ such that $s'_z=s_{\psi(x)}$ for all $z\in W$. We verify immediately that $U\mapsto\mathscr{G}'(U)$ clearly satisfies the axioms of a *sheaf*.

Now let $\mathscr{F}$ be a sheaf of sets on X, and let $u:\mathscr{G}\to\psi_*(\mathscr{F})$, $v:\mathscr{G}'\to\mathscr{F}$ be morphisms. We define $u^\sharp$ and $v^\flat$ in the following manner: if s' is a section of $\mathscr{G}'$ over a neighborhood U of $x\in X$ and if V is an open neighborhood of $\psi(x)$ and $s\in\mathscr{G}(V)$ such that we have $s'_z=s_{\psi(x)}$ for z in a neighborhood of x contained in $\psi^{-1}(V)\cap U$, we take $u^\sharp_x(s'_x)=u_{\psi(x)}(s_{\psi(x)})$. Similarly, if $s\in\mathscr{G}(V)$ (V open in Y), $v^\flat(s)$ is the section of $\mathscr{F}$ over $\psi^{-1}(V)$, the image under v of the section s' of $\mathscr{G}'$ such that $s'_x=s_{\psi(x)}$ for all $x\in\psi^{-1}(V)$. In addition, the canonical homomorphism (3.5.3) $\rho:\mathscr{G}\to\psi_*(\psi^*(\mathscr{G}))$ is defined in the following manner: for each open $V\subset Y$ and each section $s\in\Gamma(V,\mathscr{G})$, $\rho(s)$ is the section $(s_{\psi(x)})_{x\in\psi^{-1}(V)}$ of $\psi^*(\mathscr{G})$ over $\psi^{-1}(V)$. The verification of the relations $(u^\sharp)^\flat=u$, $(v^\flat)^\sharp=v$, and $v^\flat=\psi_*(v)\circ\rho$ is immediate, and proves our assertion.

We check that, if $w:\mathscr{G}_1\to\mathscr{G}_2$ is a homomorphism of sheaves of sets on Y, $\psi^*(w)$ is expressed in the following manner: if $s'=(s'_x)_{x\in U}$ is a section of $\psi^*(\mathscr{G}_1)$ over an open set U of X, then $(\psi^*(w))(s')$ is the family $(w_{\psi(x)}(s'_x))_{x\in U}$. Finally, it is immediate that for each open set V of Y, the inverse image of $\mathscr{G}|V$ under the restriction of ψ to $\psi^{-1}(V)$ is identical to the induced sheaf $\psi^*(\mathscr{G})|\psi^{-1}(V)$.

When ψ is the identity 1_X, we recover the definition of a sheaf of sets associated to a presheaf (G, II, 1.2). The above considerations apply without change when $\mathtt{C}$ is the category of groups or of rings (not necessarily commutative).

When X is any subset of a topological space Y, and j the canonical injection $X\to Y$, for each sheaf $\mathscr{G}$ on Y with values in a category $\mathtt{C}$, we call the *induced* sheaf of X by $\mathscr{G}$ the inverse image $j^*(\mathscr{G})$ (whenever it exists); for the sheaves of sets (or of groups, or of rings) we recover the usual definition (G, II, 1.5).

(3.7.2). Keeping the notation and hypotheses of (3.5.3), suppose that $\mathscr{G}$ is a *sheaf* of groups (resp. of rings) on Y. The definition of sections of $\psi^*(\mathscr{G})$ (3.7.1) shows (taking into account (3.4.4)) that the homomorphism of stalks $\psi_x\circ\rho_{\psi(x)}:\mathscr{G}_{\psi(x)}\to(\psi^*(\mathscr{G}))_x$ is a *functorial isomorphism in* $\mathscr{G}$, that identifies the two stalks; with this identification, $u^\sharp_x$ is identical to the homomorphism defined in (3.5.1), and in particular, we have $\operatorname{Supp}(\psi^*(\mathscr{G}))=\psi^{-1}(\operatorname{Supp}(\mathscr{G}))$.

An immediate consequence of this result is that *the functor* $\psi^*(\mathscr{G})$ *is exact in* $\mathscr{G}$ on the abelian category of sheaves of abelian groups.

3.8. Sheaves on pseudo-discrete spaces. $\mathbf{0_I}$ | 35

(3.8.1). Let X be a topological space whose topology admits a basis $\mathfrak{B}$ consisting of open *quasi-compact* subsets. Let $\mathscr{F}$ be a *sheaf of sets* on X; if we equip each of the $\mathscr{F}(U)$ with the *discrete* topology, $U\mapsto\mathscr{F}(U)$ is a *presheaf* *of topological spaces.* We will see that there exists a *sheaf of topological spaces* $\mathscr{F}'$ *associated to* $\mathscr{F}$ (3.5.6) such that $\Gamma(U, \mathscr{F}')$ is the discrete space $\mathscr{F}(U)$ for each open *quasi-compact* subsets U. It will suffice to show that the presheaf $U \mapsto \mathscr{F}(U)$ of discrete topological spaces *on* $\mathfrak{B}$ satisfy the condition (F_0) of (3.2.2), and more generally that if U is an open quasi-compact subset and if (U_α) is a cover of U by sets of $\mathfrak{B}$, then the least fine topology $\mathscr{T}$ on $\Gamma(U, \mathscr{F})$ renders continuous the maps $\Gamma(U, \mathscr{F}) \to \Gamma(U_\alpha, \mathscr{F})$ is the *discrete* topology. There exists a finite number of indices α_i such that $U = \bigcup_i U_{\alpha_i}$. Let $s \in \Gamma(U, \mathscr{F})$ and let s_i be its image in $\Gamma(U_{\alpha_i}, \mathscr{F})$; the intersection of the inverse images of the sets $\{s_i\}$ is by definition a neighborhood of s for $\mathscr{T}$; but since $\mathscr{F}$ is a sheaf of sets and the U_{α_i} cover U, this intersection is reduced to s, hence our assertion.

We note that if U is an open non quasi-compact subset of X, the topological space $\Gamma(U, \mathscr{F}')$ still has $\Gamma(U, \mathscr{F})$ as the underlying set, but the topology is not discrete in general: it is the least fine rendering commutative the maps $\Gamma(U, \mathscr{F}) \to \Gamma(V, \mathscr{F})$, for $V \in \mathfrak{B}$ and $V \subset U$ (the $\Gamma(V, \mathscr{F})$ being discrete).

The above considerations apply without modification to sheaves of groups or of rings (not necessarily commutative), and associate to them sheaves of *topological groups* or *topological rings*, respectively. To summarize, we say that the sheaf $\mathscr{F}'$ is the *pseudo-discrete* sheaf of *spaces* (resp. *groups*, *rings*) associated to a sheaf of sets (resp. groups, rings) $\mathscr{F}$.

(3.8.2). Let $\mathscr{F}$, $\mathscr{G}$ be two sheaves of sets (resp. groups, rings) on X, $u : \mathscr{F} \to \mathscr{G}$ a homomorphism. Then u is thus a *continuous* homomorphism $\mathscr{F}' \to \mathscr{G}'$, if we denote by $\mathscr{F}'$ and $\mathscr{G}'$ the pseudo-discrete sheaves associated to $\mathscr{F}$ and $\mathscr{G}$; this follows in effect from (3.2.5).

(3.8.3). Let $\mathscr{F}$ be a sheaf of sets, $\mathscr{H}$ a subsheaf of $\mathscr{F}$, $\mathscr{F}'$ and $\mathscr{H}'$ the pseudo-discrete sheaves associated to $\mathscr{F}$ and $\mathscr{H}$ respectively. Then, for each open $U \subset X$, $\Gamma(U, \mathscr{H}')$ is *closed* in $\Gamma(U, \mathscr{F}')$: indeed, it is the intersection of the inverse images of the $\Gamma(V, \mathscr{H})$ (for $V \in \mathfrak{B}$, $V \subset U$) under the continuous maps $\Gamma(U, \mathscr{F}) \to \Gamma(V, \mathscr{F})$, and $\Gamma(V, \mathscr{H})$ is closed in the discrete space $\Gamma(V, \mathscr{F})$.

§4. Ringed spaces

4.1. Ringed spaces, sheaves of $\mathscr{A}$-modules, $\mathscr{A}$-algebras.

(4.1.1). A *ringed space* (resp. topologically ringed space) is a pair $(X, \mathscr{A})$ consisting of a topological space X and a sheaf of (not necessarily commutative) rings (resp. of a sheaf of topological rings) $\mathscr{A}$ on X; we say that X is the *underlying* topological space of the ringed space $(X, \mathscr{A})$, and $\mathscr{A}$ the *structure sheaf.* The latter is denoted $\mathscr{O}_X$, and its stalk at a point $x \in X$ is denoted $\mathscr{O}_{X,x}$ or simply $\mathscr{O}_x$ when there is no chance of confusion.

We denote by 1 or e the *unit section* of $\mathscr{O}_X$ over X (the unit element of $\Gamma(X, \mathscr{O}_X)$).

As in this treatise we will have to consider in particular sheaves of *commutative* rings, it will be understood, when we speak of a ringed space $(X, \mathscr{A})$ without specification, that $\mathscr{A}$ is a sheaf of commutative rings.

The ringed spaces with not-necessarily-commutative structure sheaves (resp. the topologically ringed spaces) form a *category*, where we define a *morphism* $(X, \mathscr{A}) \to (Y, \mathscr{B})$ as a couple $(\psi, \theta) = \Psi$ consisting of a continuous map $\psi : X \to Y$ and a ψ*-morphism* $\theta : \mathscr{B} \to \mathscr{A}$ (3.5.1) of sheaves of rings (resp. of sheaves of topological rings); **Err**$_{\mathrm{II}}$ the *composition* of a second morphism $\Psi' = (\psi', \theta') : (Y, \mathscr{B}) \to (Z, \mathscr{C})$ and of Ψ, denoted $\Psi'' = \Psi' \circ \Psi$, is the morphism (ψ'', θ'') where $\psi'' = \psi' \circ \psi$, and θ'' is the composition of θ and θ' (equal to $\psi'_*(\theta) \circ \theta'$, cf. (3.5.2)). For ringed spaces, remember that we then have $\theta''^\sharp = \theta^\sharp \circ \psi^*(\theta'^\sharp)$ (3.5.5); therefore if $\theta'^\sharp$ and $\theta^\sharp$ are *injective* (resp. *surjective*), then the same is true of $\theta''^\sharp$, taking into account that $\psi_x \circ \rho_{\psi(x)}$ is an isomorphism for all $x \in X$ (3.7.2). We verify immediately, thanks to the above, that when ψ is an *injective* continuous map and when $\theta^\sharp$ is a *surjective* homomophism of sheaves of rings, the morphism (ψ, θ) is a *monomorphism* (T, 1.1) in the category of ringed spaces.

By abuse of language, we will often replace ψ by Ψ in notation, for example in writing $\Psi^{-1}(U)$ in place of $\psi^{-1}(U)$ for a subset U of Y, when the is no risk of confusion.

(4.1.2). For each subset M of X, the pair $(M, \mathscr{A}|M)$ is evidently a ringed space, said to be *induced* on M by the ringed space $(X, \mathscr{A})$ (and is still called the *restriction* of $(X, \mathscr{A})$ to M). If j is the canonical injection $M \to X$ and ω is the identity map of $\mathscr{A}|M$, $(j, \omega^\flat)$ is a monomorphism $(M, \mathscr{A}|M) \to (X, \mathscr{A})$ of ringed spaces, called the *canonical injection.* The composition of a morphism $\Psi : (X, \mathscr{A}) \to (Y, \mathscr{B})$ and this injection is called the *restriction* of Ψ to M.

(4.1.3). We will not revisit the definitions of $\mathscr{A}$*-modules* or *algebraic sheaves* on a ringed space $(X, \mathscr{A})$ (G, II, 2.2); when $\mathscr{A}$ is a sheaf of not necessarily commutative rings, by $\mathscr{A}$-module we will always mean "left $\mathscr{A}$-module" unless expressly stated otherwise. The $\mathscr{A}$-submodules of $\mathscr{A}$ will be called *sheaves of ideals* (left, right, or two-sided) in $\mathscr{A}$ or $\mathscr{A}$*-ideals.*

When $\mathscr{A}$ is a sheaf of commutative rings, and in the definition of $\mathscr{A}$-modules, we replace everywhere the *module* structure by that of an *algebra*, we obtain the definition of an *$\mathscr{A}$-algebra* on X. It is the same to say that an $\mathscr{A}$-algebra (not necessarily commutative) is a $\mathscr{A}$-module $\mathscr{C}$, given with a homomorphism of $\mathscr{A}$-modules $\varphi : \mathscr{C} \otimes_{\mathscr{A}} \mathscr{C} \to \mathscr{C}$ and a section e over X, such that: 1st the diagram

$$\begin{tikzcd} \mathscr{C} \otimes_{\mathscr{A}} \mathscr{C} \otimes_{\mathscr{A}} \mathscr{C} \arrow[r, "\varphi\otimes 1"] \arrow[d, "1\otimes\varphi"'] & \mathscr{C} \otimes_{\mathscr{A}} \mathscr{C} \arrow[d, "\varphi"] \\ \mathscr{C} \otimes_{\mathscr{A}} \mathscr{C} \arrow[r, "\varphi"] & \mathscr{C} \end{tikzcd}$$

is commutative; 2nd for each open $U \subset X$ and each section $s \in \Gamma(U, \mathscr{C})$, we have $\varphi((e|U) \otimes s) = \varphi(s \otimes (e|U)) = s$. We say that $\mathscr{C}$ is a commutative $\mathscr{A}$-algebra if the diagram

$$\begin{tikzcd} \mathscr{C} \otimes_{\mathscr{A}} \mathscr{C} \arrow[rr, "\sigma"] \arrow[dr, "\varphi"'] & & \mathscr{C} \otimes_{\mathscr{A}} \mathscr{C} \arrow[dl, "\varphi"] \\ & \mathscr{C} & \end{tikzcd}$$

is commutative, σ denoting the canonical symmetry (twist) map of the tensor product $\mathscr{C} \otimes_{\mathscr{A}} \mathscr{C}$.

The homomorphisms of $\mathscr{A}$-algebras are also defined as the homomorphisms of $\mathscr{A}$-modules in (G, II, 2.2), but naturally no longer form an abelian group.

If $\mathscr{M}$ is an $\mathscr{A}$-submodule of an $\mathscr{A}$-algebra $\mathscr{C}$, the *$\mathscr{A}$-subalgebra of $\mathscr{C}$ generated by $\mathscr{M}$* is the sum of the images of the homomorphisms $\bigotimes^n \mathscr{M} \to \mathscr{C}$ (for each $n \geqslant 0$). This is also the sheaf associated to the presheaf $U \mapsto \mathscr{B}(U)$ of algebras, $\mathscr{B}(U)$ being the subalgebra of $\Gamma(U, \mathscr{C})$ generated by the submodule $\Gamma(U, \mathscr{M})$.

(4.1.4). We say that a sheaf of rings $\mathscr{A}$ on a topological space X is *reduced at a point x in X* if the stalk $\mathscr{A}_x$ is a *reduced* ring (1.1.1); we say that $\mathscr{A}$ is *reduced* if it is reduced at all points of X. Recall that a ring A is called *regular* if each of the local rings $A_{\mathfrak{p}}$ (where $\mathfrak{p}$ varies over the set of prime ideals of A) is a regular local ring; we will say that a sheaf of rings $\mathscr{A}$ on X is *regular at a point x* (resp. *regular*) if the stalk $\mathscr{A}_x$ is a regular ring (resp. if $\mathscr{A}$ is regular at each point). Finally, we will say that a sheaf of rings $\mathscr{A}$ on X is *normal at a point x* (resp. *normal*) if the stalk $\mathscr{A}_x$ is an *integral and integrally closed* ring (resp. if $\mathscr{A}$ is normal at each point). We will say that a ringed space $(X, \mathscr{A})$ has any of these preceeding properties if the sheaf of rings $\mathscr{A}$ has that property.

A *graded* sheaf of rings $\mathscr{A}$ is by definition a sheaf of rings that is the direct sum (G, II, 2,7) of a family $(\mathscr{A}_n)_{n\in\mathbf{Z}}$ of sheaves of abelian groups with the conditions $\mathscr{A}_m\mathscr{A}_n \subset \mathscr{A}_{m+n}$; a *graded $\mathscr{A}$-module* is an $\mathscr{A}$-module $\mathscr{F}$ that is the direct sum of a family $(\mathscr{F}_n)_{n\in\mathbf{Z}}$ of sheaves of abelian groups, satisfying the conditions $\mathscr{A}_m\mathscr{F}_n \subset \mathscr{F}_{m+n}$. It is equivalent to say that $(\mathscr{A}_m)_x(\mathscr{A}_n)_x \subset (\mathscr{A}_{m+n})_x$ (resp. $(\mathscr{A}_m)_x(\mathscr{F}_n)_x \subset (\mathscr{F}_{m+n})_x$) for each point x.

(4.1.5). Given a ringed space $(X, \mathscr{A})$ (not necessarily commutative), we will not recall here the definitions of the bifunctors $\mathscr{F} \otimes_{\mathscr{A}} \mathscr{G}$, $\mathscr{H}om_{\mathscr{A}}(\mathscr{F}, \mathscr{F})$, and $\operatorname{Hom}_{\mathscr{A}}(\mathscr{F}, \mathscr{G})$ (G, II, 2.8 and 2.2) in the categories of left or right (depending on the case) $\mathscr{A}$-modules, with values in the category of sheaves of abelian groups (or more generally of $\mathscr{C}$-modules, if $\mathscr{C}$ is the center of $\mathscr{A}$). The stalk $(\mathscr{F} \otimes_{\mathscr{A}} \mathscr{G})_x$ for each point $x \in X$ canonically identifies with $\mathscr{F}_x \otimes_{\mathscr{A}_x} \mathscr{G}_x$ and we define a canonical and functorial homomorphism $(\mathscr{H}om_{\mathscr{A}}(\mathscr{F}, \mathscr{G}))_x \to \operatorname{Hom}_{\mathscr{A}_x}(\mathscr{F}_x, \mathscr{G}_x)$ which is in general neither injective nor surjective. The bifunctors considered above are additive and in particular, commute with finite direct limits; $\mathscr{F} \otimes_{\mathscr{A}} \mathscr{G}$ is right exact in $\mathscr{F}$ and in $\mathscr{G}$, commutes with inductive limits, and $\mathscr{A} \otimes_{\mathscr{A}} \mathscr{G}$ (resp. $\mathscr{F} \otimes_{\mathscr{A}} \mathscr{A}$) canonically identifies with $\mathscr{G}$ (resp. $\mathscr{F}$). The functors $\mathscr{H}om_{\mathscr{A}}(\mathscr{F}, \mathscr{G})$ and $\operatorname{Hom}_{\mathscr{A}}(\mathscr{F}, \mathscr{G})$ are *left exact* in $\mathscr{F}$ and $\mathscr{G}$; more precisely, if we have an exact sequence of the form $0 \to \mathscr{G}' \to \mathscr{G} \to \mathscr{G}''$, the sequence

$$0 \longrightarrow \mathscr{H}om_{\mathscr{A}}(\mathscr{F}, \mathscr{G}') \longrightarrow \mathscr{H}om_{\mathscr{A}}(\mathscr{F}, \mathscr{G}) \longrightarrow \mathscr{H}om_{\mathscr{A}}(\mathscr{F}, \mathscr{G}'')$$

is exact, and if we have an exact sequence of the form $\mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$, the sequence

$$0 \longrightarrow \mathscr{H}om_{\mathscr{A}}(\mathscr{F}'', \mathscr{G}) \longrightarrow \mathscr{H}om_{\mathscr{A}}(\mathscr{F}, \mathscr{G}) \longrightarrow \mathscr{H}om_{\mathscr{A}}(\mathscr{F}', \mathscr{G})$$

is exact, with the analogous properties for the functor Hom. In addiiton, $\mathscr{H}om_{\mathscr{A}}(\mathscr{A}, \mathscr{G})$ canonically identifies with $\mathscr{G}$; finally, for each open $U \subset X$, we have

$$\Gamma(U, \mathscr{H}om_{\mathscr{A}}(\mathscr{F}, \mathscr{G}) = \operatorname{Hom}_{\mathscr{A}|U}(\mathscr{F}|U, \mathscr{G}|U).$$

For each left (resp. right) $\mathscr{A}$-module, we define the *dual* of $\mathscr{F}$ and denote it by $\mathscr{F}^\vee$ the right (resp. left) $\mathscr{A}$-module $\mathscr{H}om_{\mathscr{A}}(\mathscr{F}, \mathscr{A})$.

Finally, if $\mathscr{A}$ is a sheaf of commutative rings, $\mathscr{F}$ an $\mathscr{A}$-module, $U \mapsto \wedge^p\Gamma(U, \mathscr{F})$ is a presheaf whose associated sheaf is an $\mathscr{A}$-module denoted $\wedge^p\mathscr{F}$ and is called the *p-th exterior power of $\mathscr{F}$*; we verify easily that the canonical map of the presheaf $U \mapsto \wedge^p \Gamma(U, \mathscr{F})$ to the associated sheaf $\wedge^p \mathscr{F}$ is *injective*, and for each $x \in X$, $(\wedge^p \mathscr{F})_x = \wedge^p(\mathscr{F}_x)$. It is clear that $\wedge^p \mathscr{F}$ is a covariant functor in $\mathscr{F}$.

(4.1.6). Suppose that $\mathscr{A}$ is a sheaf of not-necessarily-commutative rings, $\mathscr{J}$ a left sheaf of ideals of $\mathscr{A}$, $\mathscr{F}$ an left $\mathscr{A}$-module; we then denote by $\mathscr{J}\mathscr{F}$ the $\mathscr{A}$-submodule of $\mathscr{F}$, the image of $\mathscr{J} \otimes_{\mathbf{Z}} \mathscr{F}$ (where $\mathbf{Z}$ is the sheaf associated to the constant presheaf $U \mapsto \mathbf{Z}$) under the canonical map $\mathscr{J} \otimes_{\mathbf{Z}} \mathscr{F} \to \mathscr{F}$; it is clear that for each $x \in X$, we have $(\mathscr{J}\mathscr{F})_x = \mathscr{J}_x \mathscr{F}_x$. When $\mathscr{A}$ is commutative, $\mathscr{J}\mathscr{F}$ is also the canonical image of $\mathscr{J} \otimes_{\mathscr{A}} \mathscr{F} \to \mathscr{F}$. It is immediate that $\mathscr{J}\mathscr{F}$ is also the $\mathscr{A}$-module associated to the presheaf $U \mapsto \Gamma(U, \mathscr{J})\Gamma(U, \mathscr{F})$. If $\mathscr{J}_1$, $\mathscr{J}_2$ are two left sheaves of ideals of $\mathscr{A}$, we have $\mathscr{J}_1(\mathscr{J}_2\mathscr{F}) = (\mathscr{J}_1\mathscr{J}_2)\mathscr{F}$.

(4.1.7). Let $(X_\lambda, \mathscr{A}_\lambda)_{\lambda \in L}$ be a family of ringed spaces; for each couple (λ, μ), suppose we are given an open subset $V_{\lambda\mu}$ of X_λ, and an isomorphism of ringed spaces $\varphi_{\lambda\mu} : (V_{\mu\lambda}, \mathscr{A}_\mu | V_{\lambda\mu}) \simeq (V_{\lambda\mu}, \mathscr{A}_\lambda | V_{\lambda\mu})$, with $V_{\lambda\lambda} = X_\lambda$, $\varphi_{\lambda\lambda}$ being the identity. Furthermore, suppose that, for each triple (λ, μ, ν), if we denote by $\varphi'_{\mu\lambda}$ the restriction of $\varphi_{\mu\lambda}$ to $V_{\lambda\mu} \cap V_{\lambda\nu}$, $\varphi'_{\mu\lambda}$ is an isomorphism from $(V_{\lambda\mu} \cap V_{\lambda\nu}, \mathscr{A}_\lambda | (V_{\lambda\mu} \cap V_{\lambda\nu}))$ to $(V_{\mu\nu} \cap V_{\mu\lambda}, \mathscr{A}_\mu | (V_{\mu\nu} \cap V_{\mu\lambda}))$ and that we have $\varphi'_{\lambda\nu} = \varphi'_{\lambda\mu} \circ \varphi'_{\mu\nu}$ (*gluing condition* for the $\varphi_{\lambda\mu}$). We can first consider the topological space obtained by gluing (by means of the $\varphi_{\lambda\mu}$) of the X_λ along the $V_{\lambda\mu}$; if we identify X_λ with the corresponding open subset X'_λ in X, **$0_{\mathbf{I}}$** | 39
the hypotheses imply that the three sets $V_{\lambda\mu} \cap V_{\lambda\nu}$, $V_{\mu\nu} \cap V_{\mu\lambda}$, $V_{\nu\lambda} \cap V_{\nu\mu}$ identify with $X'_\lambda \cap X'_\mu \cap X'_\nu$. We can also transport to X'_λ the ringed space structure of X_λ, and if $\mathscr{A}'_\lambda$ are the transported sheaves of rings corresponding to the $\mathscr{A}_\lambda$, the $\mathscr{A}'_\lambda$ satisfy the gluing condition (3.3.1) and therefore define a sheaf of rings $\mathscr{A}$ on X; we say that $(X, \mathscr{A})$ is the ringed space obtained by *gluing the* $(X_\lambda, \mathscr{A}_\lambda)$ *along the* $V_{\lambda\mu}$, by means of the $\varphi_{\lambda\mu}$.

4.2. Direct image of an $\mathscr{A}$-module.

(4.2.1). Let $(X, \mathscr{A})$, $(Y, \mathscr{B})$ be two ringed spaces, $\Psi = (\psi, \theta)$ a morphism $(X, \mathscr{A}) \to (Y, \mathscr{B})$; $\psi_*(\mathscr{A})$ is then a sheaf of rings on Y, and θ a homomorphism $\mathscr{B} \to \psi_*(\mathscr{A})$ of sheaves of rings. Then let $\mathscr{F}$ be an $\mathscr{A}$-module; the direct image $\psi_*(\mathscr{F})$ is a sheaf of abelian groups on Y. In addition, for each open $U \subset Y$,

$$\Gamma(U, \psi_*(\mathscr{F})) = \Gamma(\psi^{-1}(U), \mathscr{F})$$

is equipped with the structure of a module over the ring $\Gamma(U, \psi_*(\mathscr{A})) = \Gamma(\psi^{-1}(U), \mathscr{A})$; the bilinear maps which define these structures are compatible with the restriction operations, defining on $\psi_*(\mathscr{F})$ the structure of a $\psi_*(\mathscr{A})$-module. The homomorphism $\theta : \mathscr{B} \to \psi_*(\mathscr{A})$ then defines also on $\psi_*(\mathscr{F})$ a *$\mathscr{B}$-module* structure; we say that this $\mathscr{B}$-module is the *direct image of $\mathscr{F}$ under the morphism* Ψ, and we denote it by $\Psi_*(\mathscr{F})$. If $\mathscr{F}_1$, $\mathscr{F}_2$ are two $\mathscr{A}$-modules over X and u an $\mathscr{A}$-homomorphism $\mathscr{F}_1 \to \mathscr{F}_2$, it is immediate (by considering the sections over the open subsets of Y) that $\psi_*(u)$ is a $\psi_*(\mathscr{A})$-homomorphism $\psi_*(\mathscr{F}_1) \to \psi_*(\mathscr{F}_2)$, and *a fortiori* a $\mathscr{B}$-homomorphism $\Psi_*(\mathscr{F}_1) \to \Psi_*(\mathscr{F}_2)$; as a $\mathscr{B}$-homomorphism, we denote it by $\Psi_*(u)$. So we see that Ψ_* is a *covariant functor* from the category of $\mathscr{A}$-modules to that of $\mathscr{B}$-modules. In addition, it is immediate that this functor is *left exact* (G, II, 2.12).

On $\psi_*(\mathscr{A})$, the structure of a $\mathscr{B}$-module and the structure of a sheaf of rings define a $\mathscr{B}$-algebra structure; we denote by $\Psi_*(\mathscr{A})$ this $\mathscr{B}$-algebra.

(4.2.2). Let $\mathscr{M}$, $\mathscr{N}$ be two $\mathscr{A}$-modules. For each open set U of Y, we have a canonical map

$$\Gamma(\psi^{-1}(U), \mathscr{M}) \times \Gamma(\psi^{-1}(U), \mathscr{N}) \longrightarrow \Gamma(\psi^{-1}(U), \mathscr{M} \otimes_{\mathscr{A}} \mathscr{N})$$

which is bilinear over the ring $\Gamma(\psi^{-1}(U), \mathscr{A}) = \Gamma(U, \psi_*(\mathscr{A}))$, and *a fortiori* over $\Gamma(U, \mathscr{B})$; it therefore defines a homomorphism

$$\Gamma(U, \Psi_*(\mathscr{M})) \otimes_{\Gamma(U, \mathscr{B})} \Gamma(U, \Psi_*(\mathscr{N})) \longrightarrow \Gamma(U, \Psi_*(\mathscr{M} \otimes_{\mathscr{A}} \mathscr{N}))$$

and as we check immediately that these homomorphisms are compatible with the restriction operations, they give a canonical functorial homomorphism of $\mathscr{B}$-modules

$$\Psi_*(\mathscr{M}) \otimes_{\mathscr{B}} \Psi_*(\mathscr{N}) \longrightarrow \Psi_*(\mathscr{M} \otimes_{\mathscr{A}} \mathscr{N}) \tag{4.2.2.1}$$

which is in general neither injective nor surjective. If $\mathscr{P}$ is a third $\mathscr{A}$-module, we check immediately that the **$0_{\mathbf{I}}$** | 40
diagram

$$\begin{tikzcd} \Psi_*(\mathscr{M}) \otimes_{\mathscr{B}} \Psi_*(\mathscr{N}) \otimes_{\mathscr{B}} \Psi_*(\mathscr{P}) \arrow[r] \arrow[d] & \Psi_*(\mathscr{M} \otimes_{\mathscr{A}} \mathscr{N}) \otimes_{\mathscr{B}} \Psi_*(\mathscr{P}) \arrow[d] \\ \Psi_*(\mathscr{M}) \otimes_{\mathscr{B}} \Psi_*(\mathscr{N} \otimes_{\mathscr{A}} \mathscr{P}) \arrow[r] & \Psi_*(\mathscr{M} \otimes_{\mathscr{A}} \mathscr{N} \otimes_{\mathscr{A}} \mathscr{P}) \end{tikzcd} \tag{4.2.2.2}$$

is commutative.

(4.2.3). Let $\mathscr{M}$, $\mathscr{N}$ be two $\mathscr{A}$-modules. For each open $U \subset Y$, we have by definition that $\Gamma(\psi^{-1}(U), \mathscr{H}om_{\mathscr{A}}(\mathscr{M}, \mathscr{N})) = \mathrm{Hom}_{\mathscr{A}|V}(\mathscr{M}|V, \mathscr{N}|V)$, where we put $V = \psi^{-1}(U)$; the map $u \mapsto \Psi_*(u)$ is a homomorphism

$$\mathrm{Hom}_{\mathscr{A}|V}(\mathscr{M}|V, \mathscr{N}|V) \longrightarrow \mathrm{Hom}_{\mathscr{B}|U}(\Psi_*(\mathscr{M})|U, \Psi_*(\mathscr{N})|U)$$

on the $\Gamma(U, \mathscr{B})$-module structures; these homomorphisms are compatible with the restriction operations, hence they define a canonical functorial homomorphism of $\mathscr{B}$-modules

$$\Psi_*(\mathscr{H}om_{\mathscr{A}}(\mathscr{M}, \mathscr{N})) \longrightarrow \mathscr{H}om_{\mathscr{B}}(\Psi_*(\mathscr{M}), \Psi_*(\mathscr{N})). \tag{4.2.3.1}$$

(4.2.4). If $\mathscr{C}$ is an $\mathscr{A}$-algebra, the composite homomorphism

$$\Psi_*(\mathscr{C}) \otimes_{\mathscr{B}} \Psi_*(\mathscr{C}) \longrightarrow \Psi_*(\mathscr{C} \otimes_{\mathscr{A}} \mathscr{C}) \longrightarrow \Psi_*(\mathscr{C})$$

defines on $\Psi_*(\mathscr{C})$ the structure of a *$\mathscr{B}$-algebra*, as a result of (4.2.2.2). We see similarly that if $\mathscr{M}$ is a $\mathscr{C}$-module, $\Psi_*(\mathscr{M})$ is canonically equipped with the structure of a $\Psi_*(\mathscr{C})$-module.

(4.2.5). Consider in particular the case where X is a *closed* subspace of Y and where ψ is the canonical injection $j : X \to Y$. If $\mathscr{B}' = \mathscr{B}|X = j^*(\mathscr{B})$ is the restriction of the sheaf of rings $\mathscr{B}$ to X, an $\mathscr{A}$-module $\mathscr{M}$ can be considered as a $\mathscr{B}'$-module by means of the homomorphism $\theta^\sharp : \mathscr{B}' \to \mathscr{A}$; then $\Psi_*(\mathscr{M})$ is the $\mathscr{B}$-module which induces $\mathscr{M}$ on X and 0 elsewhere. If $\mathscr{N}$ is a second $\mathscr{A}$-module, $\Psi_*(\mathscr{M}) \otimes_{\mathscr{B}} \Psi_*(\mathscr{N})$ canonically identifies with $\Psi_*(\mathscr{M} \otimes_{\mathscr{B}'} \mathscr{N})$ and $\mathscr{H}om_{\mathscr{B}}(\Psi_*(\mathscr{M}), \Psi_*(\mathscr{N}))$ with $\Psi_*(\mathscr{H}om_{\mathscr{B}'}(\mathscr{M}, \mathscr{N}))$.

(4.2.6). Let $(Z, \mathscr{C})$ be a third ringed space, $\Psi' = (\psi', \theta')$ a morphism $(Y, \mathscr{B}) \to (Z, \mathscr{C})$; if Ψ'' is the composite morphism $\Psi' \circ \Psi$, it is clear that we have $\Psi''_* = \Psi'_* \circ \Psi_*$.

4.3. Inverse image of an $\mathscr{A}$-module.

(4.3.1). The hypotheses and notation being the same as (4.2.1), let $\mathscr{G}$ be a $\mathscr{B}$-module and $\psi^*(\mathscr{G})$ the inverse image (3.7.1) which is therefore a sheaf of abelian groups on X. The definition of sections of $\psi^*(\mathscr{G})$ and of $\psi^*(\mathscr{B})$ (3.7.1) shows that $\psi^*(\mathscr{G})$ is canonically equipped with a $\psi^*(\mathscr{B})$-module structure. On the other hand, the homomorphism $\theta^\sharp : \psi^*(\mathscr{B}) \to \mathscr{A}$ endows $\mathscr{A}$ with the a $\psi^*(\mathscr{B})$-module structure, which we denote by $\mathscr{A}_{[\theta]}$ when necessary to avoid confusion; the tensor product $\psi^*(\mathscr{G}) \otimes_{\psi^*(\mathscr{B})} \mathscr{A}_{[\theta]}$ is then equipped with an $\mathscr{A}$-module structure. We say that this $\mathscr{A}$-module is *the inverse image of $\mathscr{G}$ under the morphism* Ψ and we denote it by $\Psi^*(\mathscr{G})$. $\mathbf{0_I}$ | 41
If $\mathscr{G}_1$, $\mathscr{G}_2$ are two $\mathscr{B}$-modules over Y, v a $\mathscr{B}$-homomorphism $\mathscr{G}_1 \to \mathscr{G}_2$, then $\psi^*(v)$, as we check immediately, is a $\psi^*(\mathscr{B})$-homomorphism from $\psi^*(\mathscr{G}_1)$ to $\psi^*(\mathscr{G}_2)$; as a result $\psi^*(v) \otimes 1$ is an $\mathscr{A}$-homomorphism $\Psi^*(\mathscr{G}_1) \to \Psi^*(\mathscr{G}_2)$, which we denote by $\Psi^*(v)$. So we define Ψ^* as a *covariant functor* from the category of $\mathscr{B}$-modules to that of $\mathscr{A}$-modules. Here, this functor (contrary to ψ^*) is no longer exact in general, but only *right exact*, the tensorization by $\mathscr{A}$ being a right exact functor to the category of $\psi^*(\mathscr{B})$-modules.

For each $x \in X$, we have $(\Psi^*(\mathscr{G}))_x = \mathscr{G}_{\psi(x)} \otimes_{\mathscr{B}_{\psi(x)}} \mathscr{A}_x$, according to (3.7.2). The support of $\Psi^*(\mathscr{G})$ is thus contained in $\psi^{-1}(\mathrm{Supp}(\mathscr{G}))$.

(4.3.2). Let $(\mathscr{G}_\lambda)$ be an inductive system of $\mathscr{B}$-modules, and let $\mathscr{G} = \varinjlim \mathscr{G}_\lambda$ be its inductive limit. The canonical homomorphisms $\mathscr{G}_\lambda \to \mathscr{G}$ define the $\psi^*(\mathscr{B})$-homomorphisms $\psi^*(\mathscr{G}_\lambda) \to \psi^*(\mathscr{G})$, which give a canonical homomorphism $\varinjlim \psi^*(\mathscr{G}_\lambda) \to \psi^*(\mathscr{G})$. As the stalk at a point of an inductive limit of sheaves is the inductive limit of the stalks at the same point (G, II, 1.11), the preceding canonical homomorphism is *bijective* (3.7.2). In addition, the tensor product commutes with inductive limits of sheaves, and we thus have a *canonical functorial isomorphism* $\varinjlim \Psi^*(\mathscr{G}_\lambda) \simeq \Psi^*(\varinjlim \mathscr{G}_\lambda)$ of $\mathscr{A}$-modules.

On the other hand, for a finite direct sum $\bigoplus_i \mathscr{G}_i$ of $\mathscr{B}$-modules, it is clear that $\psi^*(\bigoplus_i \mathscr{G}_i) = \bigoplus_i \psi^*(\mathscr{G}_i)$, therefore, by tensoring with $\mathscr{A}_{[\theta]}$,

$$\Psi^*\Big(\bigoplus_i \mathscr{G}_i\Big) = \bigoplus_i \Psi^*(\mathscr{G}_i). \tag{4.3.2.1}$$

By passing to the inductive limit, we deduce, in light of the above, that the above equality is still true for *any* direct sum.

(4.3.3). Let $\mathscr{G}_1$, $\mathscr{G}_2$ be two $\mathscr{B}$-modules; from the definition of the inverse images of sheaves of abelian groups (3.7.1), we obtain immediately a canonical homomorphism $\psi^*(\mathscr{G}_1) \otimes_{\psi^*(\mathscr{B})} \psi^*(\mathscr{G}_2) \to \psi^*(\mathscr{G}_1 \otimes_{\mathscr{B}} \mathscr{G}_2)$ of $\psi^*(\mathscr{B})$-modules, and the stalk at a point of a tensor product of sheaves being the tensor product of the stalks at this point (G, II, 2.8), we deduce from (3.7.2) that the above homomorphism is in fact a *isomorphism.* By tensoring with $\mathscr{A}$, we obtain a *canonical functorial isomorphism*

$$\Psi^*(\mathscr{G}_1) \otimes_{\mathscr{A}} \Psi^*(\mathscr{G}_2) \simeq \Psi^*(\mathscr{G}_1 \otimes_{\mathscr{B}} \mathscr{G}_2). \tag{4.3.3.1}$$

(4.3.4). Let $\mathscr{C}$ be a $\mathscr{B}$-algebra; the data of the algebra structure on $\mathscr{C}$ is the same as the data of a $\mathscr{B}$-homomorphism $\mathscr{C} \otimes_{\mathscr{B}} \mathscr{C} \to \mathscr{C}$ satisfying the associativity and commutativity conditions (conditions which are checked stalk-wise); the above isomorphism allows us to consider this homomorphism as a homomorphism of $\mathscr{A}$-modules $\Psi^*(\mathscr{C}) \otimes_{\mathscr{A}} \Psi^*(\mathscr{C}) \to \Psi^*(\mathscr{C})$ satisfying the same conditions, so $\Psi^*(\mathscr{C})$ is thus equipped with an $\mathscr{A}$-algebra structure. In particular, it follows immediately from the definitions that the $\mathscr{A}$-algebra $\Psi^*(\mathscr{B})$ is *equal to* $\mathscr{A}$ (up to a canonical isomorphism).

Similarly, if $\mathscr{M}$ is a $\mathscr{C}$-module, the data of this module structure is the same as that of a $\mathscr{B}$-homomorphism $\mathbf{0_I}$ | 42
$\mathscr{C} \otimes_{\mathscr{B}} \mathscr{M} \to \mathscr{M}$ satisfying the associativity condition; hence we give a $\Psi^*(\mathscr{C})$-module structure on $\Psi^*(\mathscr{M})$.

(4.3.5). Let $\mathscr{J}$ be a sheaf of ideals of $\mathscr{B}$; as the functor ψ^* is exact, the $\psi^*(\mathscr{B})$-module $\psi^*(\mathscr{J})$ canonically identifies with a sheaf of ideals of $\psi^*(\mathscr{B})$; the canonical injection $\psi^*(\mathscr{J}) \to \psi^*(\mathscr{B})$ then gives a homomorphism of $\mathscr{A}$-modules $\Psi^*(\mathscr{J}) = \psi^*(\mathscr{J}) \otimes_{\psi^*(\mathscr{B})} \mathscr{A}_{[\theta]} \to \mathscr{A}$; we denote by $\Psi^*(\mathscr{J})\mathscr{A}$, or $\mathscr{J}\mathscr{A}$ if there is no fear of confusion, the image of $\Psi^*(\mathscr{J})$ under this homomorphism. So we have by definition $\mathscr{J}\mathscr{A} = \theta^\sharp(\psi^*(\mathscr{J}))\mathscr{A}$ and in particular, for each $x \in X$, $(\mathscr{J}\mathscr{A})_x = \theta_x^\sharp(\mathscr{J}_{\psi(x)})\mathscr{A}_x$, taking into account the canonical identification between the stalks of $\psi^*(\mathscr{J})$ and those of $\mathscr{J}$ (3.7.2). If $\mathscr{J}_1$, $\mathscr{J}_2$ are two sheaves of ideals of $\mathscr{B}$, then we have $(\mathscr{J}_1\mathscr{J}_2)\mathscr{A} = \mathscr{J}_1(\mathscr{J}_2\mathscr{A}) = (\mathscr{J}_1\mathscr{A})(\mathscr{J}_2\mathscr{A})$.

If $\mathscr{F}$ is an $\mathscr{A}$-module, we set $\mathscr{J}\mathscr{F} = (\mathscr{J}\mathscr{A})\mathscr{F}$.

(4.3.6). Let $(Z, \mathscr{C})$ be a third ringed space, $\Psi' = (\psi', \theta')$ a morphism $(Y, \mathscr{B}) \to (Z, \mathscr{C})$; if Ψ'' is the composite morphism $\Psi' \circ \Psi$, it follows from the definition (4.3.1) and from (4.3.3.1) that we have $\Psi''^* = \Psi^* \circ \Psi'^*$.

4.4. Relation between direct and inverse images.

(4.4.1). The hypotheses and notation being the same as in (4.2.1), let $\mathscr{G}$ be a $\mathscr{B}$-module. By definition, a homomorphism $u : \mathscr{G} \to \Psi_*(\mathscr{F})$ of $\mathscr{B}$-modules is still called a Ψ*-morphisms from* $\mathscr{G}$ *to* $\mathscr{F}$, or simply a *homomorphism from* $\mathscr{G}$ *to* $\mathscr{F}$ and we write it as $u : \mathscr{G} \to \mathscr{F}$ when no confusion will occur. To give such a homomorphism is the same as giving, for each pair (U, V) where U is an open set of X, V an open set of Y such that $\psi(U) \subset V$, a *homomorphism* $u_{U,V} : \Gamma(V, \mathscr{G}) \to \Gamma(U, \mathscr{F})$ *of* $\Gamma(V, \mathscr{B})$*-modules*, $\Gamma(U, \mathscr{F})$ being considered as a $\Gamma(V, \mathscr{B})$-module by means of the ring homomorphism $\theta_{U,V} : \Gamma(V, \mathscr{B}) \to \Gamma(U, \mathscr{A})$; the $u_{U,V}$ must in addition render commutative the diagrams (3.5.1.1). It suffices, moreover, to define u by the data of the $u_{U,V}$ when U (resp. V) varies over a basis $\mathfrak{B}$ (resp. $\mathfrak{B}'$) for the topology of X (resp. Y) and to check the commutativity of (3.5.1.1) for these restrictions.

(4.4.2). Under the hypotheses of (4.2.1) and (4.2.6), let $\mathscr{H}$ be a $\mathscr{C}$-module, $v : \mathscr{H} \to \Psi'_*(\mathscr{G})$ a Ψ'-morphism; then $w : \mathscr{H} \xrightarrow{v} \Psi'_*(\mathscr{G}) \xrightarrow{\Psi'_*(u)} \Psi'_*(\Psi_*(\mathscr{F}))$ is a Ψ''-morphism which we call the *composition* of u and v.

(4.4.3). We will now see that we can define a canonical *isomorphism* of *bifunctors* in $\mathscr{F}$ and $\mathscr{G}$

$$\mathrm{Hom}_{\mathscr{A}}(\Psi^*(\mathscr{G}), \mathscr{F}) \simeq \mathrm{Hom}_{\mathscr{B}}(\mathscr{G}, \Psi_*(\mathscr{F})) \tag{4.4.3.1}$$

which we denote by $v \mapsto v_\theta^\flat$ (or simply $v \mapsto v^\flat$ if there is no chance of confusion); we denote by $u \mapsto u_\theta^\sharp$, or $u \mapsto u^\sharp$, the inverse isomorphism. This definition is the following: by composing $v : \Psi^*(\mathscr{G}) \to \mathscr{F}$ with the canonical map $\psi^*(\mathscr{G}) \to \Psi^*(\mathscr{G})$, we obtain a homomorphism of sheaves of groups $v' : \psi^*(\mathscr{G}) \to \mathscr{F}$, which is also a homomorphism of $\psi^*(\mathscr{B})$-modules. We obtain (3.7.1) a homomorphism $v'^\flat : \mathscr{G} \to \psi_*(\mathscr{F}) = \Psi_*(\mathscr{F})$, which is also a homomorphism of $\mathscr{B}$-modules as we check easily; we take $v_\theta^\flat = v'^\flat$. Similarly, for $u : \mathscr{G} \to \Psi_*(\mathscr{F})$, which $\mathbf{0_I}$ | 43
is a homomorphism of $\mathscr{B}$-modules, we obtain (3.7.1) a homomorphism $u^\sharp : \psi^*(\mathscr{G}) \to \mathscr{F}$ of $\psi^*(\mathscr{B})$-modules, hence by tensoring with $\mathscr{A}$ we have a homomorphism of $\mathscr{A}$-modules $\Psi^*(\mathscr{G}) \to \mathscr{F}$, which we denote by $u_\theta^\sharp$. It is immediate to check that $(u_\theta^\sharp)_\theta^\flat = u$ and $(v_\theta^\flat)_\theta^\sharp = v$, so we have established the functorial nature in $\mathscr{F}$ of the isomorphism $v \mapsto v_\theta^\flat$. The functorial nature in $\mathscr{G}$ of $u \mapsto u_\theta^\sharp$ is then formally shown as in (3.5.4) (reasoning that would also prove the functorial nature of Ψ^* established in (4.3.1) directly).

If we take for v the identity homomorphism of $\Psi^*(\mathscr{B})$, $v_\theta^\flat$ is a homomorphism

$$\rho_{\mathscr{G}} : \mathscr{G} \longrightarrow \Psi_*(\Psi^*(\mathscr{G})); \tag{4.4.3.2}$$

if we take for u the identity homomorphism of $\Psi_*(\mathscr{F})$, $u_\theta^\sharp$ is a homomorphism

$$\sigma_{\mathscr{F}} : \Psi^*(\Psi_*(\mathscr{F})) \longrightarrow \mathscr{F}; \tag{4.4.3.3}$$

these homomorphisms will be called *canonical*. They are in general neither injective or surjective. We have canonical factorizations analogous to (3.5.3.3) and (3.5.4.4).

We note that if s is a section of $\mathscr{G}$ over an open set V of Y, $\rho_{\mathscr{G}}(s)$ is the section $s' \otimes 1$ of $\Psi^*)(\mathscr{G})$ over $\psi^{-1}(V)$, s' being such that $s'_x = s_{\psi(x)}$ for all $x \in \psi^{-1}(V)$. We also note that if $u : \mathscr{G} \to \psi_*(\mathscr{F})$ is a homomorphism, it defines for all $x \in X$ a homomorphism $u_x : \mathscr{G}_{\psi(x)} \to \mathscr{F}_x$ on the stalks, obtained by composing $(u^\sharp)_x : (\Psi^*(\mathscr{G}))_x \to \mathscr{F}_x$

and the canonical homomorphism $s_x \mapsto s_x \otimes 1$ from $\mathscr{G}_{\psi(x)}$ to $(\Psi^*(\mathscr{G}))_x = \mathscr{G}_{\psi(x)} \otimes_{\mathscr{B}_{\psi(x)}} \mathscr{A}_x$. The homomorphism u_x is obtained also by passing to the inductive limit relative to the homomorphisms $\Gamma(V, \mathscr{G}) \xrightarrow{u} \Gamma(\psi^{-1}(V), \mathscr{F}) \to \mathscr{F}_x$, where V varies over the neighborhoods of $\psi(x)$.

(4.4.4). Let $\mathscr{F}_1$, $\mathscr{F}_2$ be $\mathscr{A}$-modules, $\mathscr{G}_1$, $\mathscr{G}_2$ be $\mathscr{B}$-modules, u_i $(i = 1, 2)$ a homomorphism from $\mathscr{G}_i$ to $\mathscr{F}_i$. We denote by $u_1 \otimes u_2$ the homomorphism $u : \mathscr{G}_1 \otimes_{\mathscr{B}} \mathscr{G}_2 \to \mathscr{F}_1 \otimes_{\mathscr{A}} \mathscr{F}_2$ such that $u^\sharp = (u_1)^\sharp \otimes (u_2)^\sharp$ (taking into account (4.3.3.1)); we check that u is also the composition $\mathscr{G}_1 \otimes_{\mathscr{B}} \mathscr{G}_2 \to \Psi_*(\mathscr{F}_1) \otimes_{\mathscr{B}} \Psi_*(\mathscr{F}_2) \to \Psi_*(\mathscr{F}_1 \otimes_{\mathscr{A}} \mathscr{F}_2)$, where the first arrow is the ordinary tensor product $u_1 \otimes_{\mathscr{B}} u_2$ and the second is the canonical homomorphism (4.2.2.1).

(4.4.5). Let $(\mathscr{G}_\lambda)_{\lambda \in L}$ be an inductive system of $\mathscr{B}$-modules, and, for each $\lambda \in L$, let u_λ be a homomorphism $\mathscr{G}_\lambda \to \Psi_*(\mathscr{F})$, form an inductive limit; we put $\mathscr{G} = \varinjlim \mathscr{G}_\lambda$ and $u = \varinjlim u_\lambda$; then the $(u_\lambda)^\sharp$ form an inductive system of homomorphisms $\Psi^*(\mathscr{G}_\lambda) \to \mathscr{F}$, and the inductive limit of this system is none other than $u^\sharp$.

(4.4.6). Let $\mathscr{M}$, $\mathscr{N}$ be two $\mathscr{B}$-modules, V an open set of Y, $U = \psi^{-1}(V)$; the map $v \mapsto \Psi^*(v)$ is a homomorphism

$$\mathrm{Hom}_{\mathscr{B}|V}(\mathscr{M}|V, \mathscr{N}|V) \longrightarrow \mathrm{Hom}_{\mathscr{A}|U}(\Psi^*(\mathscr{M})|U, \Psi^*(\mathscr{N})|U)$$

for the $\Gamma(V, \mathscr{B})$-module structures ($\mathrm{Hom}_{\mathscr{A}|U}(\Psi^*(\mathscr{M})|U, \Psi^*(\mathscr{N})|U)$ is normaly equipped with the a $\Gamma(U, \psi^*(\mathscr{B}))$-module structure, and thanks to the canonical homomorphism (3.7.2) $\Gamma(V, \mathscr{B}) \to \Gamma(U, \psi^*(\mathscr{B}))$, it is also a $\Gamma(V, \mathscr{B})$-module). We see immediately that these homomorphisms are compatible with the restriction morphisms, and as a result define a canonical functorial homomorphism $\mathbf{0_I} \mid 44$

$$\gamma : \mathscr{H}om_{\mathscr{B}}(\mathscr{M}, \mathscr{N}) \longrightarrow \Psi_*(\mathscr{H}om_{\mathscr{A}}(\Psi^*(\mathscr{M}), \Psi^*(\mathscr{N}));$$

it also corresponds to this homomorphism the homomorphism

$$\gamma^\sharp : \Psi^*(\mathscr{H}om_{\mathscr{B}}(\mathscr{M}, \mathscr{N})) \longrightarrow \mathscr{H}om_{\mathscr{A}}(\Psi^*(\mathscr{M}), \Psi^*(\mathscr{N}))$$

and these canonical morphisms are functorial in $\mathscr{M}$ and $\mathscr{N}$.

(4.4.7). Suppose that $\mathscr{F}$ (resp. $\mathscr{G}$) is an $\mathscr{A}$-algebra (resp. a $\mathscr{B}$-algebra). If $u : \mathscr{G} \to \Psi_*(\mathscr{F})$ is a homomorphism of $\mathscr{B}$-algebras, $u^\sharp$ is a homomorphism $\Psi^*(\mathscr{G}) \to \mathscr{F}$ of $\mathscr{A}$-algebras; this follows from the commutativity of the diagram

$$\begin{tikzcd}
\mathscr{G} \otimes_{\mathscr{B}} \mathscr{G} \arrow[r] \arrow[d] & \mathscr{G} \arrow[d, "u"] \\
\Psi_*(\mathscr{F} \otimes_{\mathscr{A}} \mathscr{F}) \arrow[r] & \Psi_*(\mathscr{F})
\end{tikzcd}$$

and from (4.4.4). Similarly, if $v : \Psi^*(\mathscr{G}) \to \mathscr{F}$ is a homomorphism of $\mathscr{A}$-algebras, $v^\flat : \mathscr{G} \to \Psi_*(\mathscr{F})$ is a homomorphism of $\mathscr{B}$-algebras.

(4.4.8). Let $(Z, \mathscr{C})$ be a third ringed space, $\Psi' = (\psi', \theta')$ a morphism $(Y, \mathscr{B}) \to (Z, \mathscr{C})$, and $\Psi'' : (X, \mathscr{A}) \to (Z, \mathscr{C})$ the composite morphism $\Psi' \circ \Psi$. Let $\mathscr{H}$ be a $\mathscr{C}$-module, u' a homomorphsim from $\mathscr{H}$ to $\mathscr{G}$; the composition $v'' = v \circ v'$ is by definition the homomorphism from $\mathscr{H}$ to $\mathscr{F}$ defined by $\mathscr{H} \xrightarrow{v'} \Psi'_*(\mathscr{G}) \xrightarrow{\Psi'_*(v)} \Psi'_*(\Psi_*(\mathscr{F}))$; we check that $v''^\sharp$ is the homomorphism

$$\Psi^*(\Psi'^*(\mathscr{H})) \xrightarrow{\Psi^*(v'^\sharp)} \Psi^*(\mathscr{G}) \xrightarrow{v^\sharp} \mathscr{F}.$$

§5. Quasi-coherent and coherent sheaves

5.1. Quasi-coherent sheaves.

(5.1.1). Let $(X, \mathscr{O}_X)$ be a ringed space, $\mathscr{F}$ an $\mathscr{O}_X$-module. The data of a homomorphism $u : \mathscr{O}_X \to \mathscr{F}$ of $\mathscr{O}_X$-modules is equivalent to that of the section $s = u(1) \in \Gamma(X, \mathscr{F})$. Indeed, when s is given, for each section $t \in \Gamma(U, \mathscr{O}_X)$, we necessarily have $u(t) = t \cdot (s|U)$; we say that u is *defined by the section* s. If now I is any set of indices, consider the direct sum sheaf $\mathscr{O}_X^{(I)}$, and for each $i \in I$, let h_i be the canonical injection of the i-th factor into $\mathscr{O}_X^{(I)}$; we know that $u \mapsto (u \circ h_i)$ is an isomorphism from $\mathrm{Hom}_{\mathscr{O}_X}(\mathscr{O}_X^{(I)}, \mathscr{F})$ to the product $(\mathrm{Hom}_{\mathscr{O}_X}(\mathscr{O}_X, \mathscr{F}))^I$. So there is a canonical one-to-one correspondence between the homomorphisms $u : \mathscr{O}_X^{(I)} \to \mathscr{F}$ and the *families of sections* $(s_i)_{i \in I}$ *of* $\mathscr{F}$ *over* X. The homomorphism u corresponding to (s_i) sends an element $(a_i) \in (\Gamma(U, \mathscr{O}_X))^{(I)}$ to $\sum_{i \in I} a_i \cdot (s_i|U)$.

We say that $\mathscr{F}$ is *generated by the family* (s_i) if the homomorphism $\mathscr{O}_X^{(I)} \to \mathscr{F}$ defined for each family is $\mathbf{0_I} \mid 45$ *surjective* (in other words, if, for each $x \in X$, $\mathscr{F}_x$ is an $\mathscr{O}_x$-module generated by the $(s_i)_x$). We say that $\mathscr{F}$ is *generated by its sections over* X if it is generated by the family of all these sections (or by a subfamily), in other words, if there exists a surjective homomorphism $\mathscr{O}_X^{(I)} \to \mathscr{F}$ for a suitable I.

We note that a $\mathscr{O}_X$-module $\mathscr{F}$ can be such that there exists a point $x_0 \in X$ for which $\mathscr{F}|U$ is not generated by its sections over U, *regardless of the choice of neighborhood U of x_0*: it suffices to take $X = \mathbf{R}$, for $\mathscr{O}_X$ the simple sheaf $\mathbf{Z}$, for $\mathscr{F}$ the algebraic subsheaf of $\mathscr{O}_X$ such that $\mathscr{F}_0 = \{0\}$, $\mathscr{F}_x = \mathbf{Z}$ for $x \neq 0$, and finally $x_0 = 0$: the only section of $\mathscr{F}|U$ over U is 0 for a neighborhood U of 0.

(5.1.2). Let $f : X \to Y$ be a morphism of ringed spaces. If $\mathscr{F}$ is a $\mathscr{O}_X$-module generated by its sections over X, then the canonical homomorphism $f^*(f_*(\mathscr{F})) \to \mathscr{F}$ (4.4.3.3) is *surjective*; indeed, with the notation of (5.1.1), $s_i \otimes 1$ is a section of $f^*(f_*(\mathscr{F}))$ over X, and its image in $\mathscr{F}$ is s_i. The example in (5.1.1) where f is the identity shows that the inverse of this proposition is false in general.

If $\mathscr{G}$ is an $\mathscr{O}_Y$-module generated by its sections over Y, then $f^*(\mathscr{G})$ is generated by its sections over X, since f^* is a right exact functor. **Err$_{\text{II}}$**

(5.1.3). We say that an $\mathscr{O}_X$-module $\mathscr{F}$ is *quasi-coherent* if for each $x \in X$ there is an open neighborhood U of x such that $\mathscr{F}|U$ is isomorphic to the *cokernel* of a homomorphism of the form $\mathscr{O}_X^{(I)}|U \to \mathscr{O}_X^{(J)}|U$, where I and J are sets of arbitrary indices. It is clear that $\mathscr{O}_X$ is itself a quasi-coherent $\mathscr{O}_X$-module, and that any direct sum of quasi-coherent $\mathscr{O}_X$-modules is again a quasi-coherent $\mathscr{O}_X$-module. We say that an *$\mathscr{O}_X$-algebra $\mathscr{A}$* is *quasi-coherent* if it is quasi-coherent as an $\mathscr{O}_X$-module.

(5.1.4). Let $f : X \to Y$ be a morphism of ringed spaces. If $\mathscr{G}$ is a quasi-coherent $\mathscr{O}_Y$-module, then $f^*(\mathscr{G})$ is a quasi-coherent $\mathscr{O}_X$-module. Indeed, for each $x \in X$, there is an open neighborhood V of $f(x)$ in Y such that $\mathscr{G}|V$ is the cokernel of a homomorphism $\mathscr{O}_Y^{(I)}|V \to \mathscr{O}_Y^{(J)}|V$. If $U = f^{-1}(V)$, and if f_U is the restriction of f to U, then we have $f^*(\mathscr{G})|U = f_U^*(\mathscr{G}|V)$; as f_U^* is right exact and commutes with direct sums, $f_U^*(\mathscr{G}|V)$ is the cokernel of a homomorphism $\mathscr{O}_X^{(I)}|U \to \mathscr{O}_X^{(J)}|U$.

5.2. Sheaves of finite type.

(5.2.1). We say that an $\mathscr{O}_X$-module $\mathscr{F}$ is *of finite type* if for each $x \in X$ there exists an open neighborhood U of x such that $\mathscr{F}|U$ is generated by a *finite* family of sections over U, or if it is isomorphic to a sheaf quotient of a sheaf of the form $(\mathscr{O}_X|U)^p$ where p is finite. Each sheaf quotient of a sheaf of finite type is again a sheaf of finite type, as well as each finite direct sum and each finite tensor product of sheaves of finite type. An $\mathscr{O}_X$-module of finite type is not necessarily quasi-coherent, as we can see for the $\mathscr{O}_X$-module $\mathscr{O}_X/\mathscr{F}$, where $\mathscr{F}$ is the example in (5.1.1). If $\mathscr{F}$ is of finite type, then $\mathscr{F}_x$ is a $\mathscr{O}_x$-module of finite type for each $x \in X$, but the example in (5.1.1) shows that this condition is necessary but not sufficient in general.

(5.2.2). Let $\mathscr{F}$ be an $\mathscr{O}_X$-module *of finite type.* If s_i $(1 \leqslant i \leqslant n)$ are the sections of $\mathscr{F}$ over an open neighborhood U of a point $x \in X$ and the $(s_i)_x$ generate $\mathscr{F}_x$, then there exists an open neighborhood $V \subset U$ of x such that the $(s_i)_y$ generate $\mathscr{F}_y$ for all $y \in Y$ (FAC, I, 2, 12, prop. 1). In particular, we conclude that the support of $\mathscr{F}$ is *closed.* **0$_{\mathbf{I}}$ | 46**

Similarly, if $u : \mathscr{F} \to \mathscr{G}$ is a homomorphism such that $u_x = 0$, then there exists a neighborhood U of x such that $u_y = 0$ for all $y \in U$.

(5.2.3). Suppose that X is *quasi-compact*, and let $\mathscr{F}$ and $\mathscr{G}$ be two $\mathscr{O}_X$-modules such that $\mathscr{G}$ is *of finite type*, $u : \mathscr{F} \to \mathscr{G}$ a *surjective* homomorphism. In addition, suppose that $\mathscr{F}$ is the inductive limit of an inductive system $(\mathscr{F}_\lambda)$ of $\mathscr{O}_X$-modules. Then there exists an index μ such that the homomorphism $\mathscr{F}_\mu \to \mathscr{G}$ is *surjective.* Indeed, for each $x \in X$, there exists a finite system of sections s_i of $\mathscr{G}$ over an open neighborhood $U(x)$ of x such that the $(s_i)_y$ generate $\mathscr{G}_y$ for all $y \in U(x)$; there is then an open neighborhood $V(x) \subset U(x)$ of x and n sections t_i of $\mathscr{F}$ over $V(x)$ such that $s_i|V(s) = u(t_i)$ for all i; we can also suppose that the t_i are the canonical images of sections of a similar sheaf $\mathscr{F}_{\lambda(x)}$ over $V(x)$. We then cover X with a finite number of neighborhoods $V(x_k)$, and let μ be the maximal index of the $\lambda(x_k)$; it is clear that this index gives the answer.

Suppose still that X is quasi-compact, and let $\mathscr{F}$ be an $\mathscr{O}_X$-module of finite type generated by its sections over X (5.1.1); then $\mathscr{F}$ is generated by a *finite* subfamily of these sections: indeed, it suffices to cover X by a finite number of open neighborhoods U_k such that, for each k, there is a finite number of sections s_{ik} of $\mathscr{F}$ over X whose restrictions to U_k generate $\mathscr{F}|U_k$; it is clear that the s_{ik} then generate $\mathscr{F}$.

(5.2.4). Let $f : X \to Y$ be a morphism of ringed spaces. If $\mathscr{G}$ is an $\mathscr{O}_Y$-module of finite type, then $f^*(\mathscr{G})$ is an $\mathscr{O}_X$-module of finite type. Indeed, for each $x \in X$, there is an open neighborhood V of $f(x)$ in Y and a surjective homomorphism $v : \mathscr{O}_Y^p|V \to \mathscr{G}|V$. If $U = f^{-1}(V)$ and if f_U is the restriction of f to U, then we have $f^*(\mathscr{G})|U = f_U^*(\mathscr{G}|V)$; since f_U^* is right exact (4.3.1) and commutes with direct sums (4.3.2), $f_U^*(v)$ is a surjective **Err$_{\text{II}}$** homomorphism $\mathscr{O}_X^p|U \to f^*(\mathscr{G})|U$.

(5.2.5). We say that an $\mathscr{O}_X$-module $\mathscr{F}$ *admits a finite presentation* if for each $x \in X$ there exists an open neighborhood U of x such that $\mathscr{F}|U$ is isomorphic to a *cokernel of a $(\mathscr{O}_X|U)$-homomorphism* $\mathscr{O}_X^p|U \to \mathscr{O}_X^q|U$, p and q being two integers >0. Such an $\mathcal{O}_X$-module is therefore of finite type and quasi-coherent. If $f : X \to Y$ is a morphism of ringed spaces, and if $\mathcal{G}$ is an $\mathcal{O}_Y$-module admitting a finite presentation, then $f^*(\mathcal{G})$ admits a finite presentation, as shown in the argument of (5.1.4).

(5.2.6). Let $\mathcal{F}$ be an $\mathcal{O}_X$-module admitting a finite presentation (5.2.5); then, for each $\mathcal{O}_X$-module $\mathcal{H}$, the canonical functorial homomorphism

$$(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{H}))_x \longrightarrow \mathrm{Hom}_{\mathcal{O}_x}(\mathcal{F}_x,\mathcal{H}_x)$$

is *bijective* (T, 4.1.1).

(5.2.7). Let $\mathcal{F}$ and $\mathcal{G}$ be two $\mathcal{O}_X$-modules admitting a finite presentation. If for some $x \in X$, $\mathcal{F}_x$ and $\mathcal{G}_x$ are *isomorphic* as $\mathcal{O}_x$-modules, then there exists an open neighborhood U of x such that $\mathcal{F}|U$ and $\mathcal{G}|U$ are *isomorphic*. Indeed, if $\varphi : \mathcal{F}_x \to \mathcal{G}_x$ and $\psi : \mathcal{G}_x \to \mathcal{F}_x$ are an isomorphism and its inverse isomorphism, then there exists, according to (5.2.6), an open neighborhood V of x and a section u (resp. v) of $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{G})$ (resp. $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{G},\mathcal{F})$) over V such that $u_x = \varphi$ (resp. $v_x = \psi$). As $(u\circ v)_x$ and $(v\circ u)_x$ are the identity automorphisms, $\mathbf{0_I}$ | 47
there exists an open neighborhood $U \subset V$ of x such that $(u\circ v)|U$ and $(v\circ u)|U$ are the identity automorphisms, hence the proposition.

5.3. Coherent sheaves.

(5.3.1). We say that an $\mathcal{O}_X$-module $\mathcal{F}$ is *coherent* if it satisfies the two following conditions:

(a) $\mathcal{F}$ is of finite type.
(b) for each open $U \subset X$, integer $n > 0$, and homomorphism $u : \mathcal{O}_X^n|U \to \mathcal{F}|U$, the kernel of u is of finite type.

We note that these two conditions are of a *local* nature.

For most of the proofs of the properties of coherent sheaves in what follows, cf. (FAC, I, 2).

(5.3.2). Each coherent $\mathcal{O}_X$-module admits a finite presentation (5.2.5); the inverse is not necessarily true, since $\mathcal{O}_X$ itself is not necessarily a coherent $\mathcal{O}_X$-module.

Each $\mathcal{O}_X$-submodule *of finite type* of a coherent $\mathcal{O}_X$-module is coherent; each *finite* direct sum of coherent $\mathcal{O}_X$-modules is a coherent $\mathcal{O}_X$-module.

(5.3.3). If $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ is an exact sequence of $\mathcal{O}_X$-modules and if two of these $\mathcal{O}_X$-modules are coherent, then so is the third.

(5.3.4). If $\mathcal{F}$ and $\mathcal{G}$ are two coherent $\mathcal{O}_X$-modules, $u : \mathcal{F} \to \mathcal{G}$ a homomorphism, then $\mathrm{Im}(u)$, $\mathrm{Ker}(u)$, and $\mathrm{Coker}(u)$ are coherent $\mathcal{O}_X$-modules. In particular, if $\mathcal{F}$ and $\mathcal{G}$ are $\mathcal{O}_X$-submodules of a coherent $\mathcal{O}_X$-module, then $\mathcal{F}+\mathcal{G}$ and $\mathcal{F}\cap\mathcal{G}$ are coherent.

If $\mathcal{A} \to \mathcal{B} \to \mathcal{C} \to \mathcal{D} \to \mathcal{E}$ is an exact sequence of $\mathcal{O}_X$-modules, and if $\mathcal{A}$, $\mathcal{B}$, $\mathcal{D}$, $\mathcal{E}$ are coherent, then $\mathcal{C}$ is coherent. $\mathbf{Err_{II}}$

(5.3.5). If $\mathcal{F}$ and $\mathcal{G}$ are two coherent $\mathcal{O}_X$-modules, then so are $\mathcal{F}\otimes_{\mathcal{O}_X}\mathcal{G}$ are $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{G})$.

(5.3.6). Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module, $\mathcal{J}$ a coherent sheaf of ideals of $\mathcal{O}_X$. Then the $\mathcal{O}_X$-module $\mathcal{J}\mathcal{F}$ is coherent, as the image of $\mathcal{J}\otimes_{\mathcal{O}_X}\mathcal{F}$ under the canonical homomorphism $\mathcal{J}\otimes_{\mathcal{O}_X}\mathcal{F}\to\mathcal{F}$ ((5.3.4) and (5.3.5)).

(5.3.7). We say that an $\mathcal{O}_X$-algebra $\mathcal{A}$ is *coherent* if it is coherent as an $\mathcal{O}_X$-module. In particular, $\mathcal{O}_X$ is a *coherent sheaf of rings* if and only if for each open $U \subset X$ and each homomorphism of the form $u : \mathcal{O}_X^p|U \to \mathcal{O}_X|U$, the kernel of u is an $(\mathcal{O}_X|U)$-module of finite type.

If $\mathcal{O}_X$ is a coherent sheaf of rings, then each $\mathcal{O}_X$-module $\mathcal{F}$ admitting a finite presentation (5.2.5) is coherent, according to (5.3.4).

The *annihilator* of an $\mathcal{O}_X$-module $\mathcal{F}$ is the kernel $\mathcal{J}$ of the canonical homomorphism $\mathcal{O}_X \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{F})$ which sends each section $s \in \Gamma(U,\mathcal{O}_X)$ to the multiplication by s map in $\mathrm{Hom}(\mathcal{F}|U,\mathcal{F}|U)$; if $\mathcal{O}_X$ is coherent and if $\mathcal{F}$ is a coherent $\mathcal{O}_X$-module, then $\mathcal{J}$ is coherent ((5.3.4) and (5.3.5)) and for each $x \in X$, $\mathcal{J}_x$ is the annihilator of $\mathcal{F}_x$ (5.2.6).

$\mathbf{0_I}$ | 48

(5.3.8). Suppose that $\mathcal{O}_X$ is coherent; let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module, x a point of X, M a submodule of finite type of $\mathcal{F}_x$; then there exists an open neighborhood U of x and a coherent $(\mathcal{O}_X|U)$-submodule $\mathcal{G}$ of $\mathcal{F}|U$ such that $\mathcal{G}_x = M$ (T, 4.1, Lemma 1).

This result, along with the properties of the $\mathcal{O}_X$-submodules of a coherent $\mathcal{O}_X$-module, impose the necessary conditions on the rings $\mathcal{O}_x$ such that $\mathcal{O}_X$ is coherent. For example (5.3.4), the intersection of two ideals of finite type of $\mathcal{O}_x$ must still be an ideal of finite type.

(5.3.9). Suppose that $\mathscr{O}_X$ is coherent, and let M be an $\mathscr{O}_x$-module admitting a finite presentation, therefore isomorphic to a cokernel of a homomorphism $\varphi : \mathscr{O}_x^p \to \mathscr{O}_x^q$; then there exists an open neighborhood U of X and a coherent $(\mathscr{O}_X|U)$-module $\mathscr{F}$ such that $\mathscr{F}_x$ is isomorphic to M. Indeed, according to (5.2.6), there exists a section u of $\mathscr{H}om_{\mathscr{O}_X}(\mathscr{O}_X^p, \mathscr{O}_X^q)$ over an open neighborhood U of x such that $u_x = \varphi$; the cokernel $\mathscr{F}$ of the homomorphism $u : \mathscr{O}_X^p|U \to \mathscr{O}_X^q|U$ gives the answer (5.3.4).

(5.3.10). Suppose that $\mathscr{O}_X$ is coherent, and let $\mathscr{J}$ be a coherent sheaf of ideals of $\mathscr{O}_X$. For a $(\mathscr{O}_X/\mathscr{J})$-module $\mathscr{F}$ to be coherent, it is necessary and sufficient for it to be coherent as a $\mathscr{O}_X$-module. In particular, $\mathscr{O}_X/\mathscr{J}$ is a coherent sheaf of rings.

(5.3.11). Let $f : X \to Y$ be a morphism of ringed spaces, and suppose that $\mathscr{O}_X$ is coherent; then, for each coherent $\mathscr{O}_Y$-module $\mathscr{G}$, $f^*(\mathscr{G})$ is a coherent $\mathscr{O}_X$-module. Indeed, with the notation of (5.2.4), we can assume that $\mathscr{G}|V$ is the cokernel of a homomorphism $v : \mathscr{O}_Y^q|V \to \mathscr{O}_Y^p|V$; as f_U^* is right exact, $f^*(\mathscr{G})|U = f_U^*(\mathscr{G}|V)$ is the cokernel of the homomorphism $f_U^*(v) : \mathscr{O}_X^q|U \to \mathscr{O}_X^p|U$, hence our assertion.

(5.3.12). Let Y be a closed subset of X, $j : Y \to X$ the canonical injection, $\mathscr{O}_Y$ a sheaf of rings on Y, and set $\mathscr{O}_X = j_*(\mathscr{O}_Y)$. For a $\mathscr{O}_Y$-module $\mathscr{G}$ to be of finite type (resp. quasi-coherent, coherent), it is necessary and sufficient for $j_*(\mathscr{G})$ to be an $\mathscr{O}_X$-module of finite type (resp. quasi-coherent, coherent).

5.4. Locally free sheaves.

(5.4.1). Let X be a ringed space. We say that an $\mathscr{O}_X$-module $\mathscr{F}$ is *locally free* if for each $x \in X$ there exists an open neighborhood U of x such that $\mathscr{F}|U$ is isomorphic to a $(\mathscr{O}_X|U)$-module of the form $\mathscr{O}_X^{(I)}|U$, where I can depend on U. If for each U, I is finite, then we say that $\mathscr{F}$ is *of finite rank*; if for each U, I has the same finite number of elements n, we say that $\mathscr{F}$ is *of rank n*. A locally free $\mathscr{O}_X$-module of rank 1 is called *invertible* (cf. (5.4.3)). If $\mathscr{F}$ is a locally free $\mathscr{O}_X$-module of finite rank, then for each $x \in X$, $\mathscr{F}_x$ is a free $\mathscr{O}_x$-module of finite rank $n(x)$, and there exists a neighborhood U of x such that $\mathscr{F}|U$, is of rank $n(x)$; if X is connected, then $n(x)$ is *constant.*

It is clear that each locally free sheaf is quasi-coherent, and if $\mathscr{O}_X$ is a coherent sheaf of rings, then each locally free $\mathscr{O}_X$-module of finite rank is coherent.

If $\mathscr{L}$ is locally free, then $\mathscr{L} \otimes_{\mathscr{O}_X} \mathscr{F}$ is an *exact* functor in $\mathscr{F}$ to the category of $\mathscr{O}_X$-modules.

We will mostly consider locally free $\mathscr{O}_X$-modules of finite rank, and when we speak of locally free sheaves without specifying, it will be understood that they are of *finite rank*. $\mathbf{0_I}$ | 49

Suppose that $\mathscr{O}_X$ is *coherent*, and let $\mathscr{F}$ be a *coherent* $\mathscr{O}_X$-module. Then, if at a point $x \in X$, $\mathscr{F}_x$ is an $\mathscr{O}_x$-module *free of rank n*, there exists a neighborhood U of x such that $\mathscr{F}|U$ is *locally free of rank n*; in fact, $\mathscr{F}_x$ is then isomorphic to $\mathscr{O}_x^n$, and the proposition follows from (5.2.7). **Err$_{\mathrm{II}}$**

(5.4.2). If $\mathscr{L}$, $\mathscr{F}$ are two $\mathscr{O}_X$-modules, we have a canonical functorial homomorphism

$$\mathscr{L}^\vee \otimes_{\mathscr{O}_X} \mathscr{F} = \mathscr{H}om_{\mathscr{O}_X}(\mathscr{L}, \mathscr{O}_X) \otimes_{\mathscr{O}_X} \mathscr{F} \longrightarrow \mathscr{H}om_{\mathscr{O}_X}(\mathscr{L}, \mathscr{F}) \tag{5.4.2.1}$$

defined in the following way: for each open set U, send any pair (u, t), where $u \in \Gamma(U, \mathscr{H}om_{\mathscr{O}_X}(\mathscr{L}, \mathscr{O}_X)) = \operatorname{Hom}(\mathscr{L}|U, \mathscr{O}_X|U)$ and $t \in \Gamma(U, \mathscr{F})$, to the element of $\operatorname{Hom}(\mathscr{L}|U, \mathscr{F}|U)$ which, for each $x \in U$, sends $s_x \in \mathscr{L}_x$ to the element $u_x(s_x)t_x$ of $\mathscr{F}_x$. If $\mathscr{L}$ is *locally free of finite rank*, then this homomorphism is *bijective*; the property being local, we can in fact reduce to the case where $\mathscr{L} = \mathscr{O}_X^n$; as for each $\mathscr{O}_X$-module $\mathscr{G}$, $\mathscr{H}om_{\mathscr{O}_X}(\mathscr{O}_X^n, \mathscr{G})$ is canonically isomorphic to $\mathscr{G}^n$, we have reduced to the case $\mathscr{L} = \mathscr{O}_X$, which is immediate.

(5.4.3). If $\mathscr{L}$ is invertible, then so is its dual $\mathscr{L}^\vee = \mathscr{H}om_{\mathscr{O}_X}(\mathscr{L}, \mathscr{O}_X)$, since we can immediately reduce (as the question is local) to the case $\mathscr{L} = \mathscr{O}_X$. In addition, we have a canonical isomorphism

$$\mathscr{H}om_{\mathscr{O}_X}(\mathscr{L}, \mathscr{O}_X) \otimes_{\mathscr{O}_X} \mathscr{L} \simeq \mathscr{O}_X \tag{5.4.3.1}$$

as, according to (5.3.2), it suffices to define a canonical isomorphism $\mathscr{H}om_{\mathscr{O}_X}(\mathscr{L}, \mathscr{L}) \simeq \mathscr{O}_X$. For *each* $\mathscr{O}_X$-module $\mathscr{F}$, we have a canonical homomorphism $\mathscr{O}_X \simeq \mathscr{H}om_{\mathscr{O}_X}(\mathscr{F}, \mathscr{F})$ (5.3.7). It remains to prove that if $\mathscr{F} = \mathscr{L}$ is invertible, then this homomorphism is bijective, and as the question is local, it reduces to the case $\mathscr{L} = \mathscr{O}_X$, which is immediate.

Due to the above, we put $\mathscr{L}^{-1} = \mathscr{H}om_{\mathscr{O}_X}(\mathscr{L}, \mathscr{O}_X)$, and we say that $\mathscr{L}^{-1}$ is the *inverse* of $\mathscr{L}$. The terminology "invertible sheaf" can be justified in the following way when X is a point and $\mathscr{O}_X$ is a *local* ring A with maximal ideal $\mathfrak{m}$; if M and M' are two A-modules (M being of finite type) such that $M \otimes_A M'$ is isomorphic to A, then as $(A/\mathfrak{m}) \otimes_A (M \otimes_A M')$ identifies with $(M/\mathfrak{m}M) \otimes_{A/\mathfrak{m}} (M'/\mathfrak{m}M')$, this latter tensor product of vector spaces over the field $A/\mathfrak{m}$ is isomorphic to $A/\mathfrak{m}$, which requires $M/\mathfrak{m}M$ and $M'/\mathfrak{m}M'$ to be of dimension 1. For each element $z \in M$ not in $\mathfrak{m}M$, we have $M = Az + \mathfrak{m}M$, which implies that $M = Az$ according to Nakayama's Lemma, M being of finite type. Moreover, as the annihilator of z kills $M \otimes_A M'$, which is isomorphic to A, this annihilator is $\{0\}$, and as a result M is *isomorphic to* A. In the general case, this shows that $\mathscr{L}$ is an $\mathscr{O}_X$-module of finite type, such that there exists an $\mathcal{O}_X$-module $\mathcal{F}$ for which $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{F}$ is isomorphic to $\mathcal{O}_X$, and if in addition the rings $\mathcal{O}_x$ are local rings, then $\mathcal{L}_x$ is an $\mathcal{O}_x$-module isomorphic to $\mathcal{O}_x$ for each $x \in X$. If $\mathcal{O}_X$ and $\mathcal{L}$ are assumed to be *coherent*, then we conclude that $\mathcal{L}$ is invertible according to (5.2.7).

(5.4.4). If $\mathcal{L}$ and $\mathcal{L}'$ are two invertible $\mathcal{O}_X$-modules, then so is $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{L}'$, since the question is local, we can assume that $\mathcal{L} = \mathcal{O}_X$, and the result is then trivial. For each integer $n \geqslant 1$, we denote by $\mathcal{L}^{\otimes n}$ the tensor product of n copies of the sheaf $\mathcal{L}$; we set by convention $\mathcal{L}^{\otimes 0} = \mathcal{O}_X$, and for $n \geqslant 1$, $\mathcal{L}^{\otimes(-n)} = (\mathcal{L}^{-1})^{\otimes n}$. With these notation, there is then a *canonical functorial isomorphism* $\mathbf{0_I}$ | 50

$$\mathcal{L}^{\otimes m} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n} \simeq \mathcal{L}^{\otimes(n+m)} \tag{5.4.4.1}$$

for any rational integers m and n: indeed, by definition, we immediately reduce to the case where $m = -1$, $n = 1$, and the isomorphism in question is then that defined in (5.4.3).

(5.4.5). Let $f : Y \to X$ be a morphism of ringed spaces. If $\mathcal{L}$ is a locally free (resp. invertible) $\mathcal{O}_X$-module, then $f^*(\mathcal{L})$ is a locally free (resp. invertible) $\mathcal{O}_Y$-module: this follows immediately from that the inverse images of two locally isomorphic $\mathcal{O}_X$-modules are locally isomorphic, that f^* commutes with finite direct sums, and that $f^*(\mathcal{O}_X) = \mathcal{O}_Y$ (4.3.4). In addition, we know that we have a canonical functorial homomorphism $f^*(\mathcal{L}^\vee) \to (f^*(\mathcal{L}))^\vee$ (4.4.6), and when $\mathcal{L}$ is locally free, this homomorphism is *bijective*: indeed, we again reduce to the case where $\mathcal{L} = \mathcal{O}_X$ which is trivial. We conclude that if $\mathcal{L}$ is invertible, then $f^*(\mathcal{L}^{\otimes n})$ canonically identifies with $(f^*(\mathcal{L}))^{\otimes n}$ for each rational integer n.

(5.4.6). Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module; we denote by $\Gamma_*(X, \mathcal{L})$ or simply $\Gamma_*(\mathcal{L})$ the abelian group direct sum $\bigoplus_{n \in \mathbf{Z}} \Gamma(X, \mathcal{L}^{\otimes n})$; we equip it with the structure of a *graded ring*, by corresponding to a pair (s_n, s_m), where $s_n \in \Gamma(X, \mathcal{L}^{\otimes n})$, $s_m \in \Gamma(X, \mathcal{L}^{\otimes m})$, the section of $\mathcal{L}^{\otimes(n+m)}$ over X which corresponds canonically (5.4.4.1) to the section $s_n \otimes s_m$ of $\mathcal{L}^{\otimes n} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes m}$; the associativity of this multiplication is verified in an immediate way. It is clear that $\Gamma_*(X, \mathcal{L})$ is a covariant functor in $\mathcal{L}$, with values in the category of graded rings.

If now $\mathcal{F}$ is any $\mathcal{O}_X$-module, then we set

$$\Gamma_*(\mathcal{L}, \mathcal{F}) = \bigoplus_{n \in \mathbf{Z}} \Gamma(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}).$$

We equip this abelian group with the structure of a *graded module* over the graded ring $\Gamma_*(\mathcal{L})$ in the following way: to a pair (s_n, u_m), where $s_n \in \Gamma(X, \mathcal{L}^{\otimes n})$ and $u_m \in \Gamma(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes m})$, we associate the section of $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes(m+n)}$ which canonically corresponds (5.4.4.1) to $s_n \otimes u_m$; the verification of the module axioms are immediate. For X and $\mathcal{L}$ fixed, $\Gamma_*(\mathcal{L}, \mathcal{F})$ is a covariant functor in $\mathcal{F}$ with values in the category of graded $\Gamma_*(\mathcal{L})$-modules; for X and $\mathcal{F}$ fixed, it is a covariant functor in $\mathcal{L}$ with values in the category of abelian groups.

If $f : Y \to X$ is a morphism of ringed spaces, the canonical homomorphism (4.4.3.2) $\rho : \mathcal{L}^{\otimes n} \to f_*(f^*(\mathcal{L}^{\otimes n}))$ defines a homomorphism of abelian groups $\Gamma(X, \mathcal{L}^{\otimes n}) \to \Gamma(Y, f^*(\mathcal{L}^{\otimes n}))$, and as $f^*(\mathcal{L}^{\otimes n}) = (f^*(\mathcal{L}))^{\otimes n}$), it follows from the definitions of the canonical homomorphisms (4.4.3.2) and (5.4.4.1) that the above homomorphisms define a *functorial homomorphism of graded rings* $\Gamma_*(\mathcal{L}) \to \Gamma_*(f^*(\mathcal{L}))$. The same canonical homomorphism (4.4.3) similarly defines a homomorphism of abelian groups $\Gamma(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}) \to \Gamma(Y, f^*(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}))$, and as

$$f^*(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}) = f^*(\mathcal{F}) \otimes_{\mathcal{O}_Y} (f^*(\mathcal{L}))^{\otimes n} \quad (4.3.3.1),$$

these homomorphism (for n variable) define a *di-homomorphism of graded modules* $\Gamma_*(\mathcal{L}, \mathcal{F}) \to \Gamma_*(f^*(\mathcal{L}), f^*(\mathcal{F}))$. $\mathbf{0_I}$ | 51

(5.4.7). One can show that there exists a *set* $\mathfrak{M}$ (also denoted $\mathfrak{M}(X)$) of invertible $\mathcal{O}_X$-modules such that each invertible $\mathcal{O}_X$-module is isomorphic to a unique element of $\mathfrak{M}$ [8]; we define on $\mathfrak{M}$ a composition law by sending two elements $\mathcal{L}$ and $\mathcal{L}'$ of $\mathfrak{M}$ to the unique element of $\mathfrak{M}$ isomorphic to $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{L}'$. With this composition law, $\mathfrak{M}$ *is a group isomorphic to the cohomology group* $\mathrm{H}^1(X, \mathcal{O}_X^*)$, where $\mathcal{O}_X^*$ is the subsheaf of $\mathcal{O}_X$ such that $\Gamma(U, \mathcal{O}_X^*)$ is the group of invertible elements of the ring $\Gamma(U, \mathcal{O}_X)$ for each open $U \subset X$ ($\mathcal{O}_X^*$ is therefore a sheaf of *multiplicative* abelian groups).

We will note that for all open $U \subset X$, the group of sections $\Gamma(U, \mathcal{O}_X^*)$ canonically identifies with the *automorphism group* of the $(\mathcal{O}_X|U)$-module $\mathcal{O}_X|U$, the identification sending a section ε of $\mathcal{O}_X^*$ over U to the automorphism u of $\mathcal{O}_X|U$ such that $u_x(s_x) = \varepsilon_x s_x$ for all $x \in X$ and all $s_x \in \mathcal{O}_x$. Then let $\mathfrak{U} = (U_\lambda)$ be an open cover of X; the data, for each pair of indices (λ, μ), of an automorphism $\theta_{\lambda\mu}$ of $\mathcal{O}_X|(U_\lambda \cap U_\mu)$ is the same as giving a 1-*cochain* of the cover $\mathfrak{U}$, with values in $\mathcal{O}_X^*$, and say that the $\theta_{\lambda\mu}$ satisfy the gluing condition (3.3.1), meaning that the corresponding cochain is a *cocycle*. Similarly, the data, for each λ, of an automorphism ω_λ of $\mathcal{O}_X|U_\lambda$ is the same as the data of a 0-cochain of the cover $\mathfrak{U}$, with values in $\mathcal{O}_X^*$, and its *coboundary* corresponds to the family of automorphisms $(\omega_\lambda|U_\lambda \cap U_\mu) \circ (\omega_\mu|U_\lambda \cap U_\mu)^{-1}$. We can send each 1-cocycle of $\mathfrak{U}$ with values in $\mathcal{O}_X^*$

[8] See the book in preparation cited in the introduction.

to the element of $\mathfrak{M}$ isomorphic to an invertible $\mathscr{O}_X$-module obtained by gluing with respect to the family of automorphisms $(\theta_{\lambda\mu})$ corresponding to this cocycle, and to two cohomologous coycles correspond two equal elements of $\mathfrak{M}$ (3.3.2); in other words, we thus define a map $\varphi_{\mathfrak{U}} : \mathrm{H}^1(\mathfrak{U}, \mathscr{O}_X^*) \to \mathfrak{M}$. In addition, if $\mathfrak{V}$ is a second open cover of X, finer than $\mathfrak{U}$, then the diagram

$$\begin{tikzcd} \mathrm{H}^1(\mathfrak{U}, \mathscr{O}_X^*) \arrow[dd] \arrow[dr, "\varphi_{\mathfrak{U}}"] & \\ & \mathfrak{M} \\ \mathrm{H}^1(\mathfrak{V}, \mathscr{O}_X^*) \arrow[ur, "\varphi_{\mathfrak{V}}"'] & \end{tikzcd}$$

where the vertical arrow is the canonical homomorphism (G, II, 5.7), is commutative, as a result of (3.3.3). By passing to the inductive limit, we therefore obtain a map $\mathrm{H}^1(X, \mathscr{O}_X^*) \to \mathfrak{M}$, the Čech cohomology group $\check{\mathrm{H}}^1(X, \mathscr{O}_X^*)$ identifying as we know with the first cohomology group $\mathrm{H}^1(X, \mathscr{O}_X^*)$ (G, II, 5.9, Cor. of Thm. 5.9.1). This map is *surjective*: indeed, by definition, for each invertible $\mathscr{O}_X$-module $\mathscr{L}$, there is an open cover (U_λ) of X such that $\mathscr{L}$ is obtained by gluing the sheaves $\mathscr{O}_X|U_\lambda$ (3.3.1). It is also *injective*, since it suffices to prove for the maps $\mathrm{H}^1(\mathfrak{U}, \mathscr{O}_X) \to \mathfrak{M}$, and this follows from (3.3.2). It remains to show that the bijection thus defined is a group **$0_{\mathbf{I}}$** | 52
homomorphism. Given two invertible $\mathscr{O}_X$-modules $\mathscr{L}$ and $\mathscr{L}'$, there is an open cover (U_λ) such that $\mathscr{L}|U_\lambda$ and $\mathscr{L}'|U_\lambda$ are isomorphic to $\mathscr{O}_X|U_\lambda$ for each λ; so there is for each index λ an element a_λ (resp. a'_λ) of $\Gamma(U_\lambda, \mathscr{L})$ (resp. $\Gamma(U_\lambda, \mathscr{L}')$) such that the elements of $\Gamma(U_\lambda, \mathscr{L})$ (resp. $\Gamma(U_\lambda, \mathscr{L}')$) are the $s_\lambda \cdot a_\lambda$ (resp. $s_\lambda \cdot a'_\lambda$), where s_λ varies over $\Gamma(U_\lambda, \mathscr{O}_X)$. The corresponding cocycles $(\varepsilon_{\lambda\mu})$, $(\varepsilon'_{\lambda\mu})$ are such that $s_\lambda \cdot a_\lambda = s_\mu \cdot a_\mu$ (resp. $s_\lambda \cdot a'_\lambda = s_\mu \cdot a'_\mu$) over $U_\lambda \cap U_\mu$ is equivalent to $s_\lambda = \varepsilon_{\lambda\mu} s_\mu$ (resp. $s_\lambda = \varepsilon'_{\lambda\mu} s_\mu$) over $U_\lambda \cap U_\mu$. As the sections of $\mathscr{L} \otimes_{\mathscr{O}_X} \mathscr{L}'$ over U_λ are the finite sums of the $s_\lambda s'_\lambda \cdot (a_\lambda \otimes a'_\lambda)$ where s_λ and s'_λ vary over $\Gamma(U_\lambda, \mathscr{O}_X)$, it is clear that the cocycle $(\varepsilon_{\lambda\mu}, \varepsilon'_{\lambda\mu})$ corresponds to $\mathscr{L} \otimes_{\mathscr{O}_X} \mathscr{L}'$, which finishes the proof. [9]

(5.4.8). Let $f = (\psi, \omega)$ be a morphism $Y \to X$ of ringed spaces. The functor $f^*(\mathscr{L})$ to the category of free $\mathscr{O}_X$-modules defines a map (which we still denote f^* by abuse of language) from the set $\mathfrak{M}(X)$ to the set $\mathfrak{M}(Y)$. Second, we have a canonical homomorphism (T, 3.2.2)

$$\mathrm{H}^1(X, \mathscr{O}_X^*) \longrightarrow \mathrm{H}^1(Y, \mathscr{O}_Y^*). \tag{5.4.8.1}$$

When we canonically identify (5.4.7) $\mathfrak{M}(X)$ and $\mathrm{H}^1(X, \mathscr{O}_X^*)$ (resp. $\mathfrak{M}(Y)$ and $\mathrm{H}^1(Y, \mathscr{O}_Y^*)$), the homomorphism (5.4.8.1) *identifies with the map* f^*. Indeed, if $\mathscr{L}$ comes from a cocycle $(\varepsilon_{\lambda\mu})$ corresponding to an open cover (U_λ) of X, then it suffices to show that $f^*(\mathscr{L})$ comes from a cocycle whose cohomology class is the image under (5.4.8.1) of $(\varepsilon_{\lambda\mu})$. If $\theta_{\lambda\mu}$ is the automorphism of $\mathscr{O}_X|(U_\lambda \cap U_\mu)$ which corresponds to $\varepsilon_{\lambda\mu}$, then it is clear that $f^*(\mathscr{L})$ is obtained by gluing the $\mathscr{O}_Y|\psi^{-1}(U_\lambda)$ by means of the automorphisms $f^*(\theta_{\lambda\mu})$, and it then suffices to check that these latter automorphisms corresponds to the cocycle $(\omega^\sharp(\varepsilon_{\lambda\mu}))$, which follows immediately from the definitions (we can identify $\varepsilon_{\lambda\mu}$ with its canonical image under ρ (3.7.2), a section of $\psi^*(\mathscr{O}_X^*)$ over $\psi^{-1}(U_\lambda \cap U_\mu)$).

(5.4.9). Let $\mathscr{E}$ and $\mathscr{F}$ be two $\mathscr{O}_X$-modules, $\mathscr{F}$ assumed to be *locally free*, and let $\mathscr{G}$ be an *$\mathscr{O}_X$-module extension of $\mathscr{F}$ by $\mathscr{E}$*, in other words there exists an exact sequence $0 \to \mathscr{E} \xrightarrow{i} \mathscr{G} \xrightarrow{p} \mathscr{F} \to 0$. Then, for each $x \in X$, there exists an open neighborhood U of x such that $\mathscr{G}|U$ is isomorphic to the *direct sum* $\mathscr{E}|U \oplus \mathscr{F}|U$. We can reduce to the case where $\mathscr{F} = \mathscr{O}_X^n$; let e_i $(1 \leqslant i \leqslant n)$ be the canonical sections (5.5.5) of $\mathscr{O}_X^n$; there then exists an open neighborhood U of x and n sections s_i of $\mathscr{G}$ over U such that $p(s_i|U) = e_i|U$ for $1 \leqslant i \leqslant n$. That being so, let f be the homomorphism $\mathscr{F}|U \to \mathscr{G}|U$ defined by the sections $s_i|U$ (5.1.1). It is immediate that for each open $V \subset U$, and each section $s \in \Gamma(V, \mathscr{G})$ we have $s - f(p(s)) \in \Gamma(V, \mathscr{E})$, hence our assertion.

(5.4.10). Let $f : X \to Y$ be a morphism of ringed spaces, $\mathscr{F}$ an $\mathscr{O}_X$-module, and $\mathscr{L}$ a locally free $\mathscr{O}_Y$-module of finite rank. Then there exists a canonical isomorphism

$$f_*(\mathscr{F}) \otimes_{\mathscr{O}_Y} \mathscr{L} \simeq f_*(\mathscr{F} \otimes_{\mathscr{O}_X} f^*(\mathscr{L})) \tag{5.4.10.1}$$ **$0_{\mathbf{I}}$** | 53

Indeed, for each $\mathscr{O}_Y$-module $\mathscr{L}$, we have a canonical homomorphism

$$f_*(\mathscr{F}) \otimes_{\mathscr{O}_Y} \mathscr{L} \xrightarrow{1 \otimes \rho} f_*(\mathscr{F}) \otimes_{\mathscr{O}_Y} f_*(f^*(\mathscr{L})) \xrightarrow{\alpha} f_*(\mathscr{F} \otimes_{\mathscr{O}_X} f^*(\mathscr{L})),$$

[9] For a general form of this result, see the book cited in the note on p. 51.

ρ the homomorphism (4.4.3.2) and α the homomorphism (4.2.2.1). To show that when $\mathscr{L}$ is locally free, this homomorphism is bijective, it suffices, the question being local, to consider the case where $\mathscr{L}=\mathscr{O}_X^n$; in addition, f_* and f^* commute with finite direct sums, so we can assume $n=1$, and in this case the proposition follows immediately from the definitions and from the relation $f^*(\mathscr{O}_Y)=\mathscr{O}_X$.

5.5. Sheaves on a locally ringed space.

(5.5.1). We say that a ringed space $(X,\mathscr{O}_X)$ is a *locally ringed space* if for each $x\in X$, $\mathscr{O}_x$ is a local ring; these ringed spaces will be by far the most frequent ringed spaces that we will consider in this work. We then denote by $\mathfrak{m}_x$ the *maximal ideal* of $\mathscr{O}_x$, by $k(x)$ the *residue field* $\mathscr{O}_x/\mathfrak{m}_x$; for each $\mathscr{O}_X$-module $\mathscr{F}$, each open set U of X, each point $x\in U$, and each section $f\in\Gamma(U,\mathscr{F})$, we denote by $f(x)$ the *class* of the germ $f_x\in\mathscr{F}_x$ mod. $\mathfrak{m}_x\mathscr{F}_x$, and we say that this is the *value* of f at the point x. The relation $f(x)=0$ then means that $f_x\in\mathfrak{m}_x\mathscr{F}_x$; when this is so, we say (by abuse of language) that f *is zero at* x. We will take care not to confuse this relation with $f_x=0$.

(5.5.2). Let X be a locally ringed space, $\mathscr{L}$ an invertible $\mathscr{O}_X$-module, and f a section of $\mathscr{L}$ over X. There is then an *equivalence* between the three following properties for a point $x\in X$:

(a) f_x *is a generator of* $\mathscr{L}_x$;
(b) $f_x\notin\mathfrak{m}_x\mathscr{L}_x$ (in other words, $f(x)\neq 0$);
(c) *there exists a section* g *of* $\mathscr{L}^{-1}$ *over an open neighborhood* V *of* x *such that the canonical image of* $f\otimes g$ *in* $\Gamma(V,\mathscr{O}_X)$ (5.4.3) *is the unit section.*

Indeed, the question being local, we can reduce to the case where $\mathscr{L}=\mathscr{O}_X$; the equivalence of (a) and (b) are then evident, and it is clear that (c) implies (b). Conversely, if $f_x\notin\mathfrak{m}_x$, then f_x is invertible in $\mathscr{O}_x$, say $f_xg_x=1_x$. By definition of germs of sections, this means that there exists a neighborhood V of x and a section g of $\mathscr{O}_X$ over V such that $fg=1$ in V, hence (c).

It follows immediately from the condition (c) that the set X_f of x satisfying the equivalent conditions (a), (b), (c) is *open* in X; following the terminology introduced in (5.5.1), this is the set of the x for which f *does not vanish.*

(5.5.3). Under the hypotheses of (5.5.2), let $\mathscr{L}'$ be a second invertible $\mathscr{O}_X$-module; then, if $f\in\Gamma(X,\mathscr{L})$, $g\in\Gamma(X,\mathscr{L}')$, we have

$$X_f\cap X_g=X_{f\otimes g}.$$

We can in fact reduce immediately to the case where $\mathscr{L}=\mathscr{L}'=\mathscr{O}_X$ (the question being local); as $f\otimes g$ then canonically identifies with the product fg, the proposition is evident. **$\mathbf{0_I}$** | 54

(5.5.4). Let $\mathscr{F}$ be a locally free $\mathscr{O}_X$ of rank n; it is immediate that $\wedge^p\mathscr{F}$ is a locally free $\mathscr{O}_X$-module of rank $\binom{n}{p}$ if $p\leqslant n$ and 0 if $p>n$, since the question is local and we can reduce to the case where $\mathscr{F}=\mathscr{O}_X^n$; in addition, for each $x\in X$, $(\wedge^p\mathscr{F})_x/\mathfrak{m}_x(\wedge^p\mathscr{F})_x$ is a vector space of dimension $\binom{n}{p}$ over $k(x)$, which canonically identifies with $\wedge^p(\mathscr{F}_x/\mathfrak{m}_x\mathscr{F}_x)$. Let $s_1,\ldots,s_p$ be the sections of $\mathscr{F}$ over an open subset U of X, and let $s=s_1\wedge\cdots\wedge s_p$, which is a section of $\wedge^p\mathscr{F}$ over U (4.1.5); we have $s(x)=s_1(x)\wedge\cdots\wedge s_p(x)$, and as a result, we say that the $s_1(x),\ldots,s_p(x)$ are *linearly dependent* means $s(x)=0$. We conclude that the *set of the* $x\in X$ *such that* $s_1(x),\ldots,s_p(x)$ *are linearly independent is open in* X: it suffices in fact, by reducing to the case where $\mathscr{F}=\mathscr{O}_X^n$, to apply (5.5.2) to the section image of s under one of the projections of $\wedge^p\mathscr{F}=\mathscr{O}_X^{\binom{n}{p}}$ to the $\binom{n}{p}$ factors.

In particular, if $s_1,\ldots,s_n$ are n sections of $\mathscr{F}$ over U such that $s_1(x),\ldots,s_n(x)$ are linearly independent for each point $x\in U$, then the homomorphism $u:\mathscr{O}_X^n|U\to\mathscr{F}|U$ defined by the s_i (5.1.1) is an *isomorphism*: indeed, we can restrict to the case where $\mathscr{F}=\mathscr{O}_X^n$ and where we canonically identify $\wedge^n\mathscr{F}$ and $\mathscr{O}_X$; $s=s_1\wedge\cdots\wedge s_n$ is then an *invertible* section of $\mathscr{O}_X$ over U, and we define an inverse homomorphism for u by means of the Cramer formulas.

(5.5.5). Let $\mathscr{E}$ and $\mathscr{F}$ be two locally free $\mathscr{O}_X$-modules (of finite rank), and let $u:\mathscr{E}\to\mathscr{F}$ be a homomorphism. For there to exist a neighborhood U of $x\in X$ such that $u|U$ is *injective* and that $\mathscr{F}|U$ is *the direct sum of the* $u(\mathscr{E})|U$ *and of a locally free* $(\mathscr{O}_X|U)$*-submodule* $\mathscr{G}$, it is necessary and sufficient that $u_x:\mathscr{E}_x\to\mathscr{F}_x$ gives, by passing to quotients, an *injective* homomorphism of vector spaces $\mathscr{E}_x/\mathfrak{m}_x\mathscr{E}_x\to\mathscr{F}_x/\mathfrak{m}_x\mathscr{F}_x$. The condition is indeed *necessary*, since $\mathscr{F}_x$ is then the direct sum of the free $\mathscr{O}_x$-modules $u_x(\mathscr{E}_x)$ and $\mathscr{G}_x$, so $\mathscr{F}_x/\mathfrak{m}_x\mathscr{F}_x$ is the direct sum of $u_x(\mathscr{E}_x)/\mathfrak{m}_xu_x(\mathscr{E}_x)$ and of $\mathscr{G}_x/\mathfrak{m}_x\mathscr{G}_x$. The condition is *sufficient*, since we can reduce to the case where $\mathscr{E}=\mathscr{O}_X^m$; let $s_1,\ldots,s_m$ be the images under u of the sections e_i of $\mathscr{O}_X^m$ such that $(e_i)_y$ is equal to the i-th element of the canonical basis of $\mathscr{O}_y^m$ for each $y\in Y$ (*canonical sections* of $\mathscr{O}_X^m$); by hypothesis, the $s_1(x),\ldots,s_m(x)$ are linearly independent, so if $\mathscr{F}$ is of rank n, then there exist $n-m$ sections $s_{m+1},\ldots,s_n$ of $\mathscr{F}$ over a neighborhood V of x such that the $s_i(x)$ $(1\leqslant i\leqslant n)$ form a basis for $\mathscr{F}_x/\mathfrak{m}_x\mathscr{F}_x$. There then exists (5.5.4) a neighborhood $U\subset V$ of x such that the $s_i(y)$

$(1 \leqslant i \leqslant n)$ form a basis for $\mathscr{F}_y/\mathfrak{m}_y\mathscr{F}_y$ for each $y \in V$, and we conclude (5.5.4) that there is an isomorphism from $\mathscr{F}|U$ to $\mathscr{O}_X^n|U$, sending the $s_i|U$ $(1 \leqslant i \leqslant m)$ to the $e_i|U$, which finishes the proof.

§6. Flatness

(6.0). The notion of flatness is due to J.-P. Serre [**Ser56**]; in the following, we omit the proofs of the results which are presented in the *Algèbre commutative* of N. Bourbaki, to which we refer the reader. We assume that all rings are commutative.[10] $\mathbf{0_I}$ | 55

If M, N are two A-modules, M' (resp. N') a submodule of M (resp. N), we denote by $\operatorname{Im}(M' \otimes_A N')$ the submodule of $M \otimes_A N$, the image under the canonical map $M' \otimes_A N' \to M \otimes_A N$.

6.1. Flat modules.

(6.1.1). Let M be an A-module. The following conditions are equivalent:

(a) The functor $M \otimes_A N$ is exact in N on the category of A-modules;
(b) $\operatorname{Tor}_i^A(M,N) = 0$ for each $i > 0$ and for each A-module N;
(c) $\operatorname{Tor}_1^A(M,N) = 0$ for each A-module N.

When M satisfies these conditions, we say that M is a *flat A-module.* It is clear that each free A-module is flat.

For M to be a flat A-module, it suffices that for each ideal $\mathfrak{J}$ of A, *of finite type*, the canonical map $M \otimes_A \mathfrak{J} \to M \otimes_A A = M$ is *injective.*

(6.1.2). Each inductive limit of flat A-modules is a flat A-module. For a direct sum $\bigoplus_{\lambda\in L} M_\lambda$ of A-modules to be a flat A-modules, it is necessary and sufficient that each of the A-modules M_λ is flat. In particular, every projective A-module is flat.

Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of A-modules, such that M'' is *flat.* Then, for each A-module N, the sequence

$$0 \longrightarrow M' \otimes_A N \longrightarrow M \otimes_A N \longrightarrow M'' \otimes_A N \longrightarrow 0$$

is exact. In addition, for M to be flat, is it necessary and sufficient that M' is (but it can be that M and M' are flat without $M'' = M/M'$ being so).

(6.1.3). Let M be a flat A-module, N any A-module; for two submodules N' N'' of N, we then have

$$\operatorname{Im}(M \otimes_A (N' + N'')) = \operatorname{Im}(M \otimes_A N') + \operatorname{Im}(M \otimes_A N''),$$

$$\operatorname{Im}(M \otimes_A (N' \cap N'')) = \operatorname{Im}(M \otimes_A N') \cap \operatorname{Im}(M \otimes_A N'')$$

(images taken in $M \otimes_A N$).

(6.1.4). Let M and N be two A-modules, M' (resp. N') a submodule of M (resp. N), and suppose that one of the modules M/M', N/N' is flat. Then we have $\operatorname{Im}(M' \otimes_A N') = \operatorname{Im}(M' \otimes_A N) \cap (M \otimes_A N')$ (images in $M \otimes_A N$). In particular, if $\mathfrak{J}$ is an ideal of A and if M/M' is flat, then we have $\mathfrak{J}M' = M' \cap \mathfrak{J}M$.

6.2. Change of ring. When an additive group M is equipped with multiple modules structures relative to the rings A, B, ..., we say that M is flat as an A-module, B-module, ..., we sometimes also say that M is *A-flat*, *B-flat*, ...

(6.2.1). Let A and B be two rings, M an A-module, N an (A,B)-bimodule. If M is flat and if N is B-flat, then $M \otimes_A N$ is B-flat. In particular, if M and N are two flat A-modules, then $M \otimes_A N$ is a flat A-module. If B is an A-algebra and if M is a flat A-module, then the B-module $M_{(B)} = M \otimes_A B$ is flat. Finally, if B is an A-algebra which $\mathbf{0_I}$ | 56
is flat as an A-module, and if N is a flat B-module, then N is also A-flat.

(6.2.2). Let A be a ring, B an A-algebra which is flat as an A-module. Let M, N be two A-modules, such that M admits a finite presentation; then the canonical homomorphism

$$\operatorname{Hom}_A(M,N) \otimes_A B \longrightarrow \operatorname{Hom}_B(M \otimes_A B, N \otimes_A B) \tag{6.2.2.1}$$

(sending $u \otimes b$ to the homomorphism $m \otimes b' \mapsto u(m) \otimes b'b$) is an isomorphism.

(6.2.3). Let $(A_\lambda, \varphi_{\mu\lambda})$ be a filtered inductive system of rings; let $A = \varinjlim A_\lambda$. On the other hand, for each λ, let M_λ be an A_λ-module, and for $\lambda \leqslant \mu$ let $\theta_{\mu\lambda} : M_\lambda \to M_\mu$ be a $\varphi_{\mu\lambda}$-homomorphism, such that $(M_\lambda, \theta_{\mu\lambda})$ is an inductive system; $M = \varinjlim M_\lambda$ is then an A-module. This being so, if for each λ, M_λ is a *flat* A_λ-module, then M is a *flat* A-module. Indeed, let $\mathfrak{J}$ be an ideal *of finite type* of A; by definition of the inductive limit, there exists an index λ

[10] See the exposé cited of N. Bourbaki for the generalization from most of the results to the noncommutative case.

and an ideal $\mathfrak{J}_\lambda$ of A_λ such that $\mathfrak{J} = \mathfrak{J}_\lambda A$. If we put $\mathfrak{J}'_\mu = \mathfrak{J}_\lambda A_\mu$ for $\mu \geqslant \lambda$, we also have $\mathfrak{J} = \varinjlim \mathfrak{J}'_\mu$ (where μ varies over the indices $\geqslant \lambda$), hence (the functor $\varinjlim$ being exact and commuting with tensor products)

$$M \otimes_A \mathfrak{J} = \varinjlim (M_\mu \otimes_{A_\mu} \mathfrak{J}'_\mu) = \varinjlim \mathfrak{J}'_\mu M_\mu = \mathfrak{J} M.$$

6.3. Local nature of flatness.

(6.3.1). If A is a ring, S a multiplicative subset of A, $S^{-1}A$ is a *flat A-module.* Indeed, for each A-module N, $N \otimes_A S^{-1}A$ identifies with $S^{-1}N$ (1.2.5) and we know (1.3.2) that $S^{-1}N$ is an exact functor in N.

If now M is a flat A-module, $S^{-1}M = M \otimes_A S^{-1}A$ is a flat $S^{-1}A$-module (6.2.1), so it is also A-flat according to the above and from (6.2.1). In particular, if P is an $S^{-1}A$-module, we can consider it as an A-module isomorphic to $S^{-1}P$; for P to be A-flat, it is necessary and sufficient that it is $S^{-1}A$-flat.

(6.3.2). Let A be a ring, B an A-algebra, and T a multiplicative subset of B. If P is a B-module which is *A-flat*, $T^{-1}P$ is *A-flat.* Indeed, for each A-module N, we have $(T^{-1}P) \otimes_A N = (T^{-1}B \otimes_B P) \otimes_A N = T^{-1}B \otimes_B (P \otimes_A N) = T^{-1}(P \otimes_A N)$; $T^{-1}(P \otimes_A N)$ is an exact functor in N, being the composition of the two exact functors $P \otimes_A N$ (in N) and $T^{-1}Q$ (in Q). If S is a multiplicative subset of A such that its image in B is *contained in* T, then $T^{-1}P$ is equal to $S^{-1}(T^{-1}P)$, so it is also $S^{-1}A$-flat according to (6.3.1).

(6.3.3). Let $\varphi : A \to B$ be a ring homomorphism, M a B-module. The following properties are equivalent:

(a) M is a flat A-module.
(b) For each maximal ideal $\mathfrak{n}$ of B, $M_\mathfrak{n}$ is a flat A-module.
(c) For each maximal ideal $\mathfrak{n}$ of B, by setting $\mathfrak{m} = \varphi^{-1}(\mathfrak{n})$, $M_\mathfrak{n}$ is a flat $A_\mathfrak{m}$-module.

Indeed, as $M_\mathfrak{n} = (M_\mathfrak{n})_\mathfrak{m}$, the equivalence of (b) and (c) follows from (6.3.1), and the fact that (a) implies (b) is a particular case of (6.3.2). It remains to see that (b) implies (a), that is to say, that for each injective $\mathbf{0_I}$ | 57
homomorhism $u : N' \to N$ of A-modules, the homomorphism $v = 1 \otimes u : M \otimes_A N' \to M \otimes_A N$ is injective. We have that v is also a homomorphism of B-modules, and we know that for it to be injective, it suffices that for each maximal ideal $\mathfrak{n}$ of B, $v_\mathfrak{n} : (M \otimes_A N')_\mathfrak{n} \to (M \otimes_A N)_\mathfrak{n}$ is injective. But as

$$(M \otimes_A N)_\mathfrak{n} = B_\mathfrak{n} \otimes_B (M \otimes_A N) = M_\mathfrak{n} \otimes_A N,$$

$v_\mathfrak{n}$ is none other that the homomorphism $1 \otimes u : M_\mathfrak{n} \otimes_A N' \to M_\mathfrak{n} \otimes_A N$, which is injective since $M_\mathfrak{n}$ is A-flat.

In particular (by taking $B = A$), for an A-module M to be flat, it is necessary and sufficient that $M_\mathfrak{m}$ is $A_\mathfrak{m}$-flat for each maximal ideal $\mathfrak{m}$ of A.

(6.3.4). Let M be an A-module; if M is flat, and if $f \in A$ does not divide 0 in A, f does not kill any element $\neq 0$ in M, since the homomorphism $m \mapsto f \cdot m$ is expressed as $1 \otimes u$, where u is the multiplication $a \mapsto f \cdot a$ on A and M is identified with $M \otimes_A A$; if u is injective, it is the same for $1 \otimes u$ since M is flat. In particular, if A is *integral*, M is *torsion-free.*

Conversely, suppose that A is integral, M is torsion-free, and suppose that for each maximal ideal $\mathfrak{m}$ of A, $A_\mathfrak{m}$ is a *discrete valuation ring*; then M is *A-flat.* Indeed, it suffices (6.3.3) to prove that $M_\mathfrak{m}$ is $A_\mathfrak{m}$-flat, and we can therefore suppose that A is already a discrete valuation ring. But as M is the inductive limit of its submodules of finite type, and these latter submodules are torsion-free, we can in addition reduce to the case where M is of finite type (6.1.2). The proposition follows in this case from that M is a free A-module.

In particular, if A is an *integral* ring, $\varphi : A \to B$ a ring homomorphism making B a *flat* A-module and $\neq \{0\}$, φ is necessarily *injective.* Conversely, if B is integral, A a subring of B, and if for each maximal ideal $\mathfrak{m}$ of A, $A_\mathfrak{m}$ is a discrete valuation ring, then B is *A-flat.*

6.4. Faithfully flat modules.

(6.4.1). For an A-module M, the following four properties are equivalent:

(a) For a sequence $N' \to N \to N''$ of A-modules to be exact, it is necessary and sufficient that the sequence $M \otimes_A N' \to M \otimes_A N \to M \otimes_A N''$ is exact;
(b) M is flat for each A-module N, the relation $M \otimes_A N = 0$ implies $N = 0$;
(c) M is flat for each homomorphism $v : N \to N'$ of A-modules, the relation $1_M \otimes v = 0$, 1_M being the identity automorphism of M;
(d) M is flat for each maximal ideal $\mathfrak{m}$ of A, $\mathfrak{m}M \neq M$.

When M satisfies these conditions, we say that M is a *faithfully flat* A-module; M is then necessarily a *faithful* module. In addition, if $u : N \to N'$ is a homomorphism of A-modules, then for u to be injective (resp. surjective, bijective), it is necessary and sufficient that $1 \otimes u : M \otimes_A N \to M \otimes_A N'$ is so.

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(6.4.2). A free module $\neq \{0\}$ is faithfully flat; it is the same for the direct sum of a flat module and a faithfully flat module. If S is a multiplicative subset of A, then $S^{-1}A$ is a faithfully flat A-module if S consists of invertible elements (so $S^{-1}A = A$).

(6.4.3). Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of A-modules; if M' and M'' are flat, and if one of the two is faithfully flat, then M is also faithfully flat.

(6.4.4). Let A and B be two rings, M an A-module, N an (A, B)-bimodule. If M is faithfully flat and if N is a faithfully flat B-module, then $M \otimes_A N$ is a faithfully flat B-module. In particular, if M and N are two faithfully flat A-modules, then so is $M \otimes_A N$. If B is an A-algebra and if M is a faithfully flat A-module, the B-module $M_{(B)}$ is faithfully flat.

(6.4.5). If M is a faithfully flat A-modules and if S is a multiplicative subset of A, $S^{-1}M$ is a faithfully flat $S^{-1}A$-module, since $S^{-1}M = M \otimes_A (S^{-1}A)$ (6.4.4). Conversely, if for each maximal ideal $\mathfrak{m}$ of A, $M_\mathfrak{m}$ is a faithfully flat $A_\mathfrak{m}$-module, then M is a faithfully flat A-module, since M is A-flat (6.3.3), and we have

$$M_\mathfrak{m}/\mathfrak{m}M_\mathfrak{m} = (M \otimes_A A_\mathfrak{m}) \otimes_{A_\mathfrak{m}} (A_\mathfrak{m}/\mathfrak{m}A_\mathfrak{m}) = M \otimes_A (A/\mathfrak{m}) = M/\mathfrak{m}M,$$

so the hypotheses imply that $M/\mathfrak{m}M \neq 0$ for each maximal ideal $\mathfrak{m}$ of A, which proves our assertion (6.4.1).

6.5. Restriction of scalars.

(6.5.1). Let A be a ring, $\varphi : A \to B$ a ring homomorphism making B an A-algebra. Suppose that there exists a B-module N which is a *faithfully flat* A-module. Then, for each A-module M, the homomorphism $x \mapsto 1 \otimes x$ from M to $B \otimes_A M = M_{(B)}$ is *injective*. In particular, φ is injective; for each ideal $\mathfrak{a}$ of A, we have $\varphi^{-1}(\mathfrak{a}B) = \mathfrak{a}$; for each maximal (resp. prime) ideal $\mathfrak{m}$ of A, there exists a maximal (resp. prime) ideal $\mathfrak{n}$ of B such that $\varphi^{-1}(\mathfrak{n}) = \mathfrak{m}$.

(6.5.2). When the conditions of (6.5.1) are satisfied, we identify A with the subring of B by φ and more generally, for each A-module M, we identify M with an A-submodule of $M_{(B)}$. We note that if B is also *Noetherian*, then so is A, since the map $\mathfrak{a} \mapsto \mathfrak{a}B$ is an increasing injection from the set of ideals of A to the set of ideals of B; the existence of an infinite strictly increasing sequence of ideals of A thus implies the existence of an analogous sequence of ideals of B.

6.6. Faithfully flat rings.

(6.6.1). Let $\varphi : A \to B$ be a ring homomorphism making B an A-algebra. The following five properties are equivalent:

(a) B is a faithfully flat A-module (in other words, $M_{(B)}$ is an *exact* and *faithful* functor in M).
(b) The homomorphism φ is injective and the A-module $B/\varphi(A)$ is flat. $\mathbf{0_I}$ | 59
(c) The A-module B is flat (in other words, the functor $M_{(B)}$ is *exact*), and for each A-module M. the homomorphism $x \mapsto 1 \otimes x$ from M to $M_{(B)}$ is injective.
(d) The A-module B is flat and for each ideal $\mathfrak{a}$ of A, we have $\varphi^{-1}(\mathfrak{a}B) = \mathfrak{a}$.
(e) The A-module B is flat and for each maximal ideal $\mathfrak{m}$ of A, there exists a maximal ideal $\mathfrak{n}$ of B such that $\varphi^{-1}(\mathfrak{n}) = \mathfrak{m}$.

When these conditions are satisfied, we identify A with a subring of B.

(6.6.2). Let A be a *local* ring, $\mathfrak{m}$ its maximal ideal, and B an A-algebra such that $\mathfrak{m}B \neq B$ (which is so when for example B is a local ring and $A \to B$ is a *local* homomorphism). If B is a *flat* A-module, B is a *faithfully flat* A-module. Indeed, this follows from (6.4.1, (d)). Under the indicated conditions, we thus see that if B is Noetherian, then so too is A (6.5.2). $\mathbf{Err_{II}}$

(6.6.3). Let B be an A-algebra which is a faithfully flat A-module. For each A-module M and each A-submodule M' of M, we have (by identifying M with an A-submodule of $M_{(B)}$) $M' = M \cap M'_{(B)}$. For M to be a flat (resp. faithfully flat) A-module, it is necessary and sufficient that $M_{(B)}$ is a flat (resp. faithfully flat) B-module.

(6.6.4). Let B be an A-algebra, N a faithfully flat B-module. For B to be a flat (resp. faithfully flat) A-module, it is necessary and sufficient that N is.

In particular, let C be a B-algebra; if the ring C is faithfully flat over B and B is faithfully flat over A, then C is faithfully flat over A; if C is faithfully flat over B and over A, then B is faithfully flat over A.

6.7. Flat morphisms of ringed spaces.

(6.7.1). Let $f : X \to Y$ be a morphism of ringed spaces, and let $\mathscr{F}$ be a $\mathscr{O}_X$-module. We say that $\mathscr{F}$ is *f-flat* (or *Y-flat* when there is no chance of confusion with f) *at a point* $x \in X$ if $\mathscr{F}_x$ is a flat $\mathscr{O}_{f(x)}$-module; we say that $\mathscr{F}$ is *f-flat over* $y \in Y$ if $\mathscr{F}$ is f-flat for all the points $x \in f^{-1}(y)$; we say that $\mathscr{F}$ is *f-flat* if $\mathscr{F}$ is f-flat at all the points of X. We say that the morphism f is *flat at* $x \in X$ (resp. *flat over* $y \in Y$, resp. *flat*) if $\mathscr{O}_X$ is f-flat at x (resp. f-flat over y, resp. f-flat). If f is a flat morphism, we then say that X is *flat over* Y, or *Y-flat*. $\mathbf{Err}_{\mathrm{II}}$

(6.7.2). With the notation of (6.7.1), if $\mathscr{F}$ is f-flat at x, for each open neighborhood U of $y = f(x)$, the functor $(f^*(\mathscr{G}) \otimes_{\mathscr{O}_X} \mathscr{F})_x$ in $\mathscr{G}$ is *exact* on the category of $(\mathscr{O}_Y|U)$-modules; indeed, this stalk canonically identifies with $\mathscr{G}_y \otimes_{\mathscr{O}_y} \mathscr{F}_x$, and our assertion follows from the definition. In particular, if f is a *flat* morphism, the functor f^* is *exact* on the category of $\mathscr{O}_Y$-modules.

(6.7.3). Conversely, suppose the sheaf of rings $\mathscr{O}_Y$ is *coherent*, and suppose that for *each* open neighborhood U of y, the functor $(f^*(\mathscr{G}) \otimes_{\mathscr{O}_X} \mathscr{F})_x$ is exact in $\mathscr{G}$ on the category of *coherent* $(\mathscr{O}_Y|U)$-modules. Then $\mathscr{F}$ is *f-flat at x*. In fact, it suffices to prove that for each ideal of finite type $\mathfrak{J}$ of $\mathscr{O}_y$, the canonical homomorphism $\mathfrak{J} \otimes_{\mathscr{O}_y} \mathscr{F}_x \to \mathscr{F}_x$ is injective (6.1.1). We know (5.3.8) that there then exists an open neighborhood U of y and a coherent sheaf of ideals $\mathscr{J}$ of $\mathscr{O}_Y|U$ such that $\mathscr{J}_y = \mathfrak{J}$, hence the conclusion. $\mathbf{0_I}$ | 60

(6.7.4). The results of (6.1) for flat modules are immediately translated into propositions for sheaves with are f-flat at a point:

If $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ is an exact sequence of $\mathscr{O}_X$-modules and if $\mathscr{F}''$ is f-flat at a point $x \in X$, then, for each open neighborhood U of $y = f(x)$ and each $(\mathscr{O}_Y|U)$-module $\mathscr{G}$, the sequence

$$0 \longrightarrow (f^*(\mathscr{G}) \otimes_{\mathscr{O}_X} \mathscr{F}')_x \longrightarrow (f^*(\mathscr{G}) \otimes_{\mathscr{O}_X} \mathscr{F})_x \longrightarrow (f^*(\mathscr{G}) \otimes_{\mathscr{O}_X} \mathscr{F}'')_x \longrightarrow 0$$

is exact. For $\mathscr{F}$ to be f-flat at x, it is necessary and sufficient that $\mathscr{F}'$ is. We have similar statements for the corresponding notions of a f-flat $\mathscr{O}_X$-modules over $y \in Y$, or of a f-flat $\mathscr{O}_X$-module.

(6.7.5). Let $f : X \to Y$, $g : Y \to Z$ be two morphisms of ringed spaces; let $x \in X$, $y = f(x)$, and $\mathscr{F}$ be an $\mathscr{O}_X$-module. If $\mathscr{F}$ is f-flat at the point x and if the morphism g is flat at the point y, then $\mathscr{F}$ is $(g \circ f)$-flat at x (6.2.1). In particular, if f and g are flat morphisms, then $g \circ f$ is flat.

(6.7.6). Let X, Y be two ringed spaces, $f : X \to Y$ a *flat* morphism. Then the canonical homomorphism of bifunctors (4.4.6)

$$f^*(\mathscr{H}om_{\mathscr{O}_Y}(\mathscr{F}, \mathscr{G})) \longrightarrow \mathscr{H}om_{\mathscr{O}_X}(f^*(\mathscr{F}), f^*(\mathscr{G})) \tag{6.7.6.1}$$

is an *isomorphism* when $\mathscr{F}$ admits a *finite presentation* (5.2.5).

Indeed, the question being local, we can assume that there exists an exact sequence $\mathscr{O}_Y^m \to \mathscr{O}_Y^n \to \mathscr{F} \to 0$. The two sides of (6.7.6.1) are right exact functors in $\mathscr{F}$ according to the hypotheses on f; we then have reduced to proving the proposition in the case where $\mathscr{F} = \mathscr{O}_Y$, in which the result is trivial.

(6.7.8). We say that a morphism $f : X \to Y$ of ringed spaces is *faithfully flat* if f is *surjective* and if, for each $x \in X$, $\mathscr{O}_x$ is a *faithfully flat* $\mathscr{O}_{f(x)}$-module. When X and Y are locally ringed spaces (5.5.1), it is equivalent to say that the morphism f is *surjective* and *flat* (6.6.2). When f is faithfully flat, f^* is an *exact* and *faithful* functor on the category of $\mathscr{O}_Y$-modules (6.6.1, a), and for an $\mathscr{O}_Y$-module $\mathscr{G}$ to be Y-flat, it is necessary and sufficient that $f^*(\mathscr{G})$ is (6.6.3).

§7. Adic rings

7.1. Admissible rings.

(7.1.1). Recall that in a topological ring A (not necessarily separated), we say that an element x is *topologically nilpotent* if 0 is *a* limit of the sequence $(x^n)_{n \geqslant 0}$. We say that a topological ring A is *linearly topologized* if there exists a fundamental system of neighborhoods of 0 in A of (necessarily *open*) *ideals*.

Definition (7.1.2). — In a linearly topologized ring A, we say that an ideal $\mathfrak{J}$ is an *ideal of definition* if $\mathfrak{J}$ is open and if, for each neighborhood V of 0, there exists a integer $n > 0$ such that $\mathfrak{J}^n \subset V$ *(which we express, by abuse of language, by saying that the sequence* $(\mathfrak{J}^n)$ *tends to* 0*)*. We say that a linearly topologized ring A is *preadmissible* if there exists in A an ideal of definition; we say that A is *admissible* if it is preadmissible and if in addition it is separated and complete. $\mathbf{0_I}$ | 61

It is clear that if $\mathfrak{J}$ is an ideal of definition, $\mathfrak{L}$ an open ideal of A, then $\mathfrak{J} \cap \mathfrak{L}$ is also an ideal of definition; the ideals of definition of a preadmissible ring A thus form a *fundamental system of neighborhoods of* 0.

Lemma (7.1.3). — *Let A be a linearly topologized ring.*

(i) *For $x \in A$ to be topologically nilpotent, it is necessary and sufficient that for each open ideal $\mathfrak{J}$ of A, the canonical image of x in $A/\mathfrak{J}$ is nilpotent. The set $\mathfrak{T}$ of topologically nilpotent elements of A is an ideal.*
(ii) *Suppose that in addition A is preadmissible, and let $\mathfrak{J}$ be an ideal of definition for A. For $x \in A$ to be topologically nilpotent, it is necessary and sufficient that its canonical image in $A/\mathfrak{J}$ is nilpotent; the ideal $\mathfrak{T}$ is the inverse image of the nilradical of $A/\mathfrak{J}$ and is thus open.*

Proof. (i) follows immediately from the definitions. To prove (ii), it suffices to note that for each neighborhood V of 0 in A, there exists an $n > 0$ such that $\mathfrak{J}^n \subset V$; if $x \in A$ is such that $x^m \in \mathfrak{J}$, we have $x^{mq} \in V$ for $q \geqslant n$, so x is topologically nilpotent. □

Proposition (7.1.4). — *Let A be a preadmissible ring, $\mathfrak{J}$ an ideal of definition for A.*

(i) *For an ideal $\mathfrak{J}'$ of A to be contained in an ideal of definition, it is necessary and sufficient that there exists an integer $n > 0$ such that $\mathfrak{J}'^n \subset \mathfrak{J}$.*
(ii) *For an $x \in A$ to be contained in an ideal of definition, it is necessary and sufficient that it is topologically nilpotent.*

Proof.

(i) If $\mathfrak{J}'^n \subset \mathfrak{J}$, then for each open neighborhood V of 0 in A, there exists an m such that $\mathfrak{J}^m \subset V$, thus $\mathfrak{J}'^{mn} \subset V$.
(ii) The condition is evidently necessary; it is sufficient, since if it satisfied, then there exists an n such that $x^n \in \mathfrak{J}$, so $\mathfrak{J}' = \mathfrak{J} + Ax$ is an ideal of definition, because it is open, and $\mathfrak{J}'^n \subset \mathfrak{J}$.

□

Corollary (7.1.5). — *In a preadmissible ring A, an open prime ideal contains all the ideals of definition.*

Corollary (7.1.6). — *The notation and hypotheses being that of (7.1.4), the following properties of an ideal $\mathfrak{J}_0$ of A are equivalent:*

(a) *$\mathfrak{J}_0$ is the largest ideal of definition of A;*
(b) *$\mathfrak{J}_0$ is a maximal ideal of definition;*
(c) *$\mathfrak{J}_0$ is an ideal of definition such that the ring $A/\mathfrak{J}_0$ is reduced.*

For there to exist an ideal $\mathfrak{J}_0$ to have these properties, it is necessary and sufficient that the nilradical of $A/\mathfrak{J}$ to be nilpotent; $\mathfrak{J}_0$ is then equal to the ideal $\mathfrak{T}$ of topologically nilpotent elements of A.

Proof. It is clear that (a) implies (b), and (b) implies (c) according to (7.1.4, ii), and (7.1.3, ii); for the same reason, (c) implies (a). The latter assertion follows from (7.1.4, i) and (7.1.3, ii). □

When $\mathfrak{T}/\mathfrak{J}$, the nilradical of $A/\mathfrak{J}$, is nilpotent, and we denote by A_{red} the (reduced) quotient ring $A/\mathfrak{T}$. $\mathbf{0_I}$ | 62

Corollary (7.1.7). — *A preadmissible Noetherian ring admits a largest ideal of definition.*

Corollary (7.1.8). — *If a preadmissible ring A is such that, for an ideal of definition $\mathfrak{J}$, the powers $\mathfrak{J}^n$ ($n > 0$) form a fundamental system of neighborhoods of 0, it is the same for the powers $\mathfrak{J}'^n$ for each ideal of definition $\mathfrak{J}'$ of A.*

Definition (7.1.9). — We say that a preadmissible ring A is *preadic* if there exists an ideal of definition $\mathfrak{J}$ for A such that the $\mathfrak{J}^n$ form a fundamental system of neighborhoods of 0 in A *(or equivalently, such that the $\mathfrak{J}^n$ are* open*)*. We call a ring *adic* if it is a separated and complete preadic ring.

If $\mathfrak{J}$ is an ideal of definition for a preadic (resp. adic) ring A, we say that A is a $\mathfrak{J}$*-preadic* (resp. $\mathfrak{J}$*-adic*) ring, and that its topology is the $\mathfrak{J}$*-preadic* (resp. $\mathfrak{J}$*-adic*) topology. More generally, if M is an A-module, the topology on M having for a fundamental system of neighborhoods of 0 the submodules $\mathfrak{J}^n M$ is called the $\mathfrak{J}$*-preadic* (resp. $\mathfrak{J}$*-adic*) topology. According to (7.1.8), these topologies are independent of the choice of the ideal of definition $\mathfrak{J}$.

Proposition (7.1.10). — *Let A be an admissible ring, $\mathfrak{J}$ an ideal of definition for A. Then $\mathfrak{J}$ is contained in the radical of A.*

This statement is equivalent to any of the following corollaries:

Corollary (7.1.11). — *For each $x \in \mathfrak{J}$, $1 + x$ is invertible in A.*

Corollary (7.1.12). — *For $f \in A$ to be invertible in A, it is necessary and sufficient that its canonical image in $A/\mathfrak{J}$ is invertible in $A/\mathfrak{J}$.*

Corollary (7.1.13). — *For each A-module M of finite type, the relation $M = \mathfrak{J}M$ (equivalent to $M \otimes_A (A/\mathfrak{J}) = 0$) implies that $M = 0$.*

Corollary (7.1.14). — *Let $u : M \to N$ be a homomorphism of A-modules, N being of finite type; for u to be surjective, it is necessary and sufficient that $u \otimes 1 : M \otimes_A (A/\mathfrak{J}) \to N \otimes_A (A/\mathfrak{J})$ is.*

Proof. The equivalence of (7.1.10) and (7.1.11) follows from Bourbaki, *Alg.*, chap. VIII, §6, no. 3, th. 1, and the equivalence of (7.1.10) and (7.1.10) and (7.1.13) follows from *loc. cit.*, th. 2; the fact that (7.1.10) implies (7.1.14) follows from *loc. cit.*, cor. 4 of the prop. 6; on the other hand, (7.1.14) implies (7.1.13) by applying the zero homomorphism. Finally, (7.1.10) implies that if f is invertible in $A/\mathfrak{J}$, then f is not contained in any maximal ideal of A, thus f is invertible in A, in other words, (7.1.10) implies (7.1.12); conversely, (7.1.12) implies (7.1.11).

It therefore remains to prove (7.1.11). Now as A is separated and complete, and the sequence $(\mathfrak{J}^n)$ tends to 0, it is immediate that the series $\sum_{n=0}^{\infty}(-1)^n x^n$ is convergent in A, and that if y is its sum, then we have $y(1+x)=1$. □

7.2. Adic rings and projective limits.

(7.2.1). Each projective limit of *discrete* rings is evidently a linearly topologized ring, separated and compact. Conversely, let A be a linearly topologized ring, and let $(\mathfrak{J}_\lambda)$ be a fundamental system of open neighborhoods of 0 in A consisting of ideals. The canonical maps $\varphi_\lambda : A \to A/\mathfrak{J}_\lambda$ form a projective system of continuous representations and therefore define a continuous representation $\varphi : A \to \varprojlim A/\mathfrak{J}_\lambda$; if A is *separated*, then φ is a topological isomorphism from A to an everywhere dense subring of $\varprojlim A/\mathfrak{J}_\lambda$; if in addition A is *complete*, then φ is a topological isomorphism from A to $\varprojlim A/\mathfrak{J}_\lambda$. $\mathbf{0_I}$ | 63

Lemma (7.2.2). — *For a linearly topologized ring to be admissible, it is necessary and sufficient that it is isomorphic to a projective limit $A = \varprojlim A_\lambda$, where $(A_\lambda, \mu_{\lambda\mu})$ is a projective limit of discrete rings having for the set of indices a filtered ordered (by $\leqslant$) L which admits a smallest element denoted 0 and satisfies the following conditions: 1st. the $u_\lambda : A \to A_\lambda$ are sujective; 2nd. the kernel $\mathfrak{J}_\lambda$ of $u_{0\lambda} : A_\lambda \to A_0$ is nilpotent. When this is so, the kernel $\mathfrak{J}$ of $u_0 : A \to A_0$ is equal to $\varprojlim \mathfrak{J}_\lambda$.*

Proof. The necessity of the condition follows from (7.2.1), by choosing $(\mathfrak{J}_\lambda)$ to be a fundamental system of neighborhoods of 0 consisting of ideals of definitions contained in an ideal of definition $\mathfrak{J}_0$ and by applying (7.1.4, i). The converse follows from the definition of the projective limit and from (7.2.1), and the latter assertion is immediate. □

(7.2.3). Let A be an *admissible* topological ring, $\mathfrak{J}$ an ideal of A contained in an ideal of definition (in other words (7.1.4) such that $(\mathfrak{J}^n)$ tends to 0); we can consider on A the ring topology having for a fundamental system of neighborhoods of 0 the powers $\mathfrak{J}^n$ $(n > 0)$; we call again this the $\mathfrak{J}$-*preadic* topology. The hypothesis that A is admissible implies that $\bigcup_n \mathfrak{J}^n = 0$, therefore the $\mathfrak{J}$-preadic topology on A is *separated*; let $\widehat{A} = \varprojlim A/\mathfrak{J}^n$ be the completion of A for this topology (where the $A/\mathfrak{J}^n$ are equipped with the discrete topology), and denote by u the (not necessarily continuous) ring homomorphism $A \to \widehat{A}$, the projective limit of the sequence of homomorphisms $u_n : A \to A/\mathfrak{J}^n$. On the other hand, the $\mathfrak{J}$-preadic topology on A is finer than the given topology $\mathscr{T}$ on A; as A is separated and complete for $\mathscr{T}$, we can extend by continuity the identity map of A (equipped with the $\mathfrak{J}$-preadic topology) to A equipped with $\mathscr{T}$; this gives a continuous representation $v : \widehat{A} \to A$.

Proposition (7.2.4). — *If A is an admissible ring and $\mathfrak{J}$ is contained in an ideal of definition of A, then A is separated and complete for the $\mathfrak{J}$-preadic topology.*

Proof. With the notation of (7.2.3), it is immediate that $v \circ u$ is the identity map of A. On the other hand, $u_n \circ v : \widehat{A} \to A/\mathfrak{J}^n$ is the extension by continuity (for the $\mathfrak{J}$-preadic topology on A and the discrete topology on $A/\mathfrak{J}^n$) of the canonical map u_n; in other words, it is the canonical map from $\widehat{A} = \varprojlim_k A/\mathfrak{J}^k$ to $A/\mathfrak{J}^n$; $u \circ v$ is therefore the projective limit of this sequence of maps, which is by definition the identity map of $\widehat{A}$; this proves the proposition. □

Corollary (7.2.5). — *Under the hypotheses of (7.2.3), the following conditions are equivalent:*

(a) *the homomorphism u is continuous;* $\mathbf{0_I}$ | 64
(b) *the homomorphism v is bicontinuous;*
(c) *A is a $\mathfrak{J}$-adic ring.*

Corollary (7.2.6). — *Let A be an admissible ring A, $\mathfrak{J}$ an ideal of definition for A. For A to be Noetherian, it is necessary and sufficient for $A/\mathfrak{J}$ to be Noetherian and for $\mathfrak{J}/\mathfrak{J}^2$ to be an $A/\mathfrak{J}$-module of finite type.*

These conditions are evidently necessary. Conversely, suppose the conditions are satisfied; as according to (7.2.4) A is complete for the $\mathfrak{J}$-preadic topology, for it to be Noetherian, it is necessary and sufficient that the associated graded ring $\mathrm{grad}(A)$ (for the filtration on the $\mathfrak{J}^n$) is ([**CC**, p .18–07, th. 4]). Now, let $a_1, \ldots, a_n$ be the elements of $\mathfrak{J}$ whose classes mod. $\mathfrak{J}^2$ are the generators of $\mathfrak{J}/\mathfrak{J}^2$ as a $A/\mathfrak{J}$-module. It is immediate by induction that the classes mod. $\mathfrak{J}^{m+1}$ of the monomials of total degree m in the a_i $(1 \leqslant i \leqslant n)$ form a system of generators for the $A/\mathfrak{J}$-module $\mathfrak{J}^m/\mathfrak{J}^{m+1}$. We conclude that $\mathrm{grad}(A)$ is a ring isomorphic to a quotient of $(A/\mathfrak{J})[T_1, \ldots, T_n]$ (T_i indeterminates), which finishes the proof.

Proposition (7.2.7). — *Let (A_i, u_{ij}) be a projective system $(i \in \mathbf{N})$ of discrete rings, and for each integer i, let $\mathfrak{J}_i$ be the kernel in A_i of the homomorphism $u_{0i} : A_i \to A_0$. We suppose that:*

(a) *For $i \leqslant j$, u_{ij} is surjective and its kernel is $\mathfrak{J}_j^{i+1}$* (therefore A_i is isomorphic to $A_j/\mathfrak{J}_j^{i+1}$).

(b) *$\mathfrak{J}_1/\mathfrak{J}_1^2$ $(= \mathfrak{J}_1)$ is a module of finite type over $A_0 = A_1/\mathfrak{J}_1$.*

Let $A = \varprojlim_i A_i$, and for each integer $n \geqslant 0$, let u_n be the canonical homomorphism $A \to A_n$, and let $\mathfrak{J}^{(n+1)} \subset A$ be its kernel. Then we have these conditions:

(i) *A is an adic ring, having $\mathfrak{J} = \mathfrak{J}^{(1)}$ for an ideal of definition.*

(ii) *We have $\mathfrak{J}^{(n)} = \mathfrak{J}^n$ for each $n \geqslant 1$.*

(iii) *$\mathfrak{J}/\mathfrak{J}^2$ is isomorphic to $\mathfrak{J}_1 = \mathfrak{J}_1/\mathfrak{J}_1^2$, and as a result is a module of finite type over $A_0 = A/\mathfrak{J}$.*

Proof. The hypothesis of surjectivity on the u_{ij} implies that u_n is surjective; in addition, the hypothesis (a) implies that $\mathfrak{J}_j^{j+1} = 0$, therefore A is an admissible ring (7.2.2); by definition, the $\mathfrak{J}^{(n)}$ form a fundamental system of neighborhoods of 0 in A, so (ii) implies (i). In addition, we have $\mathfrak{J} = \varprojlim_i \mathfrak{J}_i$ and the maps $\mathfrak{J} \to \mathfrak{J}_i$ are surjective, so (ii) implies (iii), and it remains to prove (ii). By definition, $\mathfrak{J}^{(n)}$ consists of the elements $(x_k)_{k\geqslant 0}$ of A such that $x_k = 0$ for $k < n$, therefore $\mathfrak{J}^{(n)}\mathfrak{J}^{(m)} \subset \mathfrak{J}^{(n+m)}$, in other words the $\mathfrak{J}^{(n)}$ form a *filtration* of A. On the other hand, $\mathfrak{J}^{(n)}/\mathfrak{J}^{(n+1)}$ is isomorphic to the projection from $\mathfrak{J}^{(n)}$ to A_n; as $\mathfrak{J}^{(n)} = \varprojlim_{i>n} \mathfrak{J}_i^n$, this projection is none other than $\mathfrak{J}_n^n$, which is a module over $A_0 = A_n/\mathfrak{J}_n$. Now let $a_j = (a_{jk})_{k\geqslant 0}$ be r elements of $\mathfrak{J} = \mathfrak{J}^{(1)}$ such that $a_{11}, \ldots, a_{r1}$ form a system of generators for $\mathfrak{J}_1$ over A_0; we will see that the set S_n of monomials of total degree n and the a_j generate the ideal $\mathfrak{J}^{(n)}$ of A. As $\mathfrak{J}_i^{i+1} = 0$, it is clear first of all that $S_n \subset \mathfrak{J}^{(n)}$; since A is complete for the filtration $(\mathfrak{J}^{(m)})$, it suffices to prove that the set $\overline{S}_n$ of classes mod. $\mathfrak{J}^{(n+1)}$ of elements of S_n generate the graded module $\mathrm{grad}(\mathfrak{J}^{(n)})$ over the graded ring $\mathrm{grad}(A)$ for the above filtration ([**CC**, p. 18–06, lemme]); according to the definition of the multiplication on $\mathrm{grad}(A)$, it suffices to prove that for each m, $\overline{S}_m$ is a system of generators **0_I** | 65
for the A_0-module $\mathfrak{J}^{(m)}/\mathfrak{J}^{(m+1)}$, or that $\mathfrak{J}_m^m$ is generated by the monomials of degree m in the a_{jm} $(1 \leqslant j \leqslant r)$. For this, it remains to show that $\mathfrak{J}_m$ is generated (as an A_m-module) by the monomials of degree $\leqslant m$ relative to a_{jm}; the proposition being evident by definition for $m = 1$, we argue by induction on m, and let $\mathfrak{J}'_m$ be the A_m-submodule of $\mathfrak{J}_m$ generated by these monomials. The relation $\mathfrak{J}_{m-1} = \mathfrak{J}_m/\mathfrak{J}_m^m$ and the induction hypothesis prove that $\mathfrak{J}_m = \mathfrak{J}'_m + \mathfrak{J}_m^m$, hence, since $\mathfrak{J}_m^{m+1} = 0$, we have $\mathfrak{J}_m^m = {\mathfrak{J}'_m}^m$, and finally $\mathfrak{J}_m = \mathfrak{J}'_m$. □

Corollary (7.2.8). — *Under the conditions of Proposition (7.2.7), for A to be Noetherian, it is necessary and sufficient that A_0 is.*

Proof. This follows immediately from Corollary (7.2.6). □

Proposition (7.2.9). — *Suppose the hypotheses of Proposition (7.2.7): for each integer i, let M_i be an A_i-module, and for $i \leqslant j$, let $v_{ij} : M_j \to M_i$ be a u_{ij}-homomorphism, such that (M_i, v_{ij}) is a projective system. In addition, suppose that M_0 is an A_0-module of finite type and that the v_{ij} are surjective with kernel $\mathfrak{J}_j^{i+1}M_j$. Then $M = \varprojlim M_i$ is an A-module of finite type, and the kernel of the surjective u_n-homomorphism $v_n : M \to M_n$ is $\mathfrak{J}^{n+1}M$* (such that M_n identifies with $M/\mathfrak{J}^{n+1}M = M \otimes_A (A/\mathfrak{J}^{n+1})$).

Proof. Let $z_h = (z_{hk})_{k\geqslant 0}$ be a system of s elements of M such that the z_{h0} $(1 \leqslant h \leqslant s)$ forms a system of generators for M_0; we will show that the z_h generate the A-module M. The A-module M is separated and complete for the filtration by the $M^{(n)}$, where $M^{(n)}$ is the set of $y = (y_k)_{k\geqslant 0}$ in M such that $y_k = 0$ for $k < n$; it is clear that we have $\mathfrak{J}^{(n)}M \subset M^{(n)}$ and that $M^{(n)}/M^{(n+1)} = \mathfrak{J}_n^n M_n$. We therefore have reduced to showing that the classes of the z_h modulo $M^{(0)}$ generate the graded module $\mathrm{grad}(M)$ (by the above filtration) over the graded ring $\mathrm{grad}(A)$ [**CC**, p. 18–06, lemme]; for this, we observe that it suffices to prove that the z_{hn} $(1 \leqslant h \leqslant s)$ generate the A_n-module M_n. We argue by induction on n, the proposition being evident by definition for $n = 0$; the relation $M_{n-1} = M_n/\mathfrak{J}_n^n M_n$ and the induction hypothesis show that if M'_n is the submodule of M_n generated by the z_{hn}, we have that $M_n = M'_n + \mathfrak{J}_n^n M_n$, and as $\mathfrak{J}_n$ is nilpotent, this implies that $M_n = M'_n$. Similarly, passing to the associated graded modules shows that the canonical map from $\mathfrak{J}^{(n)}$ to $M^{(n)}$ is surjective (thus bijection), in other words that $\mathfrak{J}^{(n)}M = \mathfrak{J}^n M$ is the kernel of $M \to M_{n-1}$. □ **Err**$_{\mathrm{II}}$

Corollary (7.2.10). — *Let (N_i, w_{ij}) be a second projective system of A_i-modules satisfying the conditions of Proposition (7.2.9), and let $N = \varprojlim N_i$. There is a bijective correspondence between the projective systems (h_i) of A_i-homomorphisms $h_i : M_i \to N_i$ and the homomorphisms of A-modules $h : M \to N$ (which is necessarily continuous for the $\mathfrak{J}$-adic topologies).*

Proof. It is clear that if $h : M \to N$ is an A-homomorphism, then we have $h(\mathfrak{J}^n M) \subset \mathfrak{J}^n N$, hence the continuity of h; by passing to quotients, there corresponds to h a projective system of A_i-homomorphisms $h_i : M_i \to N_i$, whose projective limit is h, hence the corollary. □

Remark (7.2.11). — Let A be an adic ring with an ideal of definition $\mathfrak{J}$ such that $\mathfrak{J}/\mathfrak{J}^2$ is an $A/\mathfrak{J}$-module of finite type; it is clear that the $A_i = A/\mathfrak{J}^{i+1}$ satisfy the conditions of Proposition (7.2.7); as A is the projective limit **0**$_{\mathrm{I}}$ | 66
of the A_i, we see that Proposition (7.2.7) gives the description of *all* the adic rings of the type considered (and in particular of all the *adic Noetherian* rings).

Example (7.2.12). — Let B be a ring, $\mathfrak{J}$ an ideal of B such that $\mathfrak{J}/\mathfrak{J}^2$ is a module of finite type over $B/\mathfrak{J}$ (or over B, equivalently); set $A = \varprojlim_n B/\mathfrak{J}^{n+1}$; A is the separated completion of B equipped with the $\mathfrak{J}$-preadic topology. If $A_n = B/\mathfrak{J}^{n+1}$, then it is immediate that the A_n satisfy the conditions of Proposition (7.2.7); therefore A is an adic ring and if $\overline{\mathfrak{J}}$ is the closure in A of the canonical image of $\mathfrak{J}$, then $\overline{\mathfrak{J}}$ is an ideal of definition for A, $\overline{\mathfrak{J}^n}$ is the closure of the canonical image of $\mathfrak{J}^n$, $A/\overline{\mathfrak{J}^n}$ identifies with $B/\mathfrak{J}^n$ and $\overline{\mathfrak{J}}/\overline{\mathfrak{J}}^2$ is isomorphic to $\mathfrak{J}/\mathfrak{J}^2$ as an $A/\overline{\mathfrak{J}}$-module. Similarly, if N is such that $N/\mathfrak{J}N$ is a B-module of finite type, and if we set $M_i = N/\mathfrak{J}^{i+1}N$, then $M = \varprojlim M_i$ is an A-module of finite type, isomorphic to the separated completion of N for the $\mathfrak{J}$-preadic topology, and $\overline{\mathfrak{J}^n}M$ identifies with the closure of the canonical image of $\mathfrak{J}^n N$, and $M/\overline{\mathfrak{J}^n}M$ identifies with $N/\mathfrak{J}^n N$.

7.3. Preadic Noetherian rings.

(7.3.1). Let A be a ring, $\mathfrak{J}$ an ideal of A, and M an A-module; we denote by $\widehat{A} = \varprojlim_n A/\mathfrak{J}^n$ (resp. $\widehat{M} = \varprojlim_n M/\mathfrak{J}^n M$) the separated completion of A (resp. M) for the $\mathfrak{J}$-preadic topology. Let $M' \xrightarrow{u} M \xrightarrow{v} M'' \to 0$ be an exact sequence of A-modules; as $M/\mathfrak{J}^n M = M \otimes_A (A/\mathfrak{J}^n)$, the sequence

$$M'/\mathfrak{J}^n M' \xrightarrow{u_n} M/\mathfrak{J}^n M \xrightarrow{v_n} M''/\mathfrak{J}^n M'' \longrightarrow 0$$

is exact for each n. In addition, as $v(\mathfrak{J}^n M) = \mathfrak{J}^n v(M) = \mathfrak{J}^n M''$, $\widehat{v} = \varprojlim v_n$ is surjective (Bourbaki, *Top. gén.*, Chap. IX, 2nd ed., p. 60, Cor. 2). On the other hand, if $z = (z_k)$ is an element of the kernel of $\widehat{v}$, then for each integer k, there exists a $z'_k \in M'/\mathfrak{J}^k M'$ such that $u_k(z'_k) = z_k$; we conclude that there exists a $z' = (z'_n) \in \widehat{M'}$ such that the first k components of $\widehat{u}(z')$ coincide with the z; in other words, the image of $\widehat{M'}$ under $\widehat{u}$ is *dense* in the kernel of $\widehat{v}$.

If we suppose that A is *Noetherian*, then so is $\widehat{A}$, according to (7.2.12), $\mathfrak{J}/\mathfrak{J}^2$ is then an A-module of finite type. In addition, we have the following theorem.

Theorem (7.3.2). — (Krull's Theorem). *Let A be a* Noetherian *ring, $\mathfrak{J}$ an ideal of A, M an A-module* of finite type, *and M' a submodule of M; then the induced topology on M' by the $\mathfrak{J}$-preadic topology of M is identical to the $\mathfrak{J}$-preadic topology of M'.*

This follows immediately from

Lemma (7.3.2.1). — (Artin–Rees Lemma). *Under the hypotheses of (7.3.2), there exists an integer p such that, for $n \geqslant p$, we have*

$$M' \cap \mathfrak{J}^n M = \mathfrak{J}^{n-p}(M' \cap \mathfrak{J}^p M).$$

For the proof, see [**CC**, p. 2–04]. **0**$_{\mathrm{I}}$ | 67

Corollary (7.3.3). — *Under the hypotheses of (7.3.2), the canonical map $M \otimes_A \widehat{A} \to \widehat{M}$ is bijective, and the functor $M \otimes_A \widehat{A}$ is exact in M on the category of A-modules of finite type; as a result, the separated $\mathfrak{J}$-adic completion $\widehat{A}$ is a flat A-module* (6.1.1).

Proof. We first note that $\widehat{M}$ is an *exact* functor in M on the category of A-modules of finite type. Indeed, let $0 \to M' \xrightarrow{u} M \xrightarrow{v} M'' \to 0$ be an exact sequence; we have seen that $\widehat{v} : \widehat{M} \to \widehat{M''}$ is surjective (7.3.1); on the other hand, if i is the canonical homomorphism $M \to \widehat{M}$, it follows from Krull's Theorem (7.3.2) that the closure of $i(u(M'))$ in $\widehat{M}$ identifies with the separated completion of M' for the $\mathfrak{J}$-preadic topology; thus $\widehat{u}$ is injective, and according to (7.3.1), the image of $\widehat{u}$ is equal to the kernel of $\widehat{v}$.

This being so, the canonical map $M \otimes_A \widehat{A} \to \widehat{M}$ is obtained by passing to the projective limit of the maps $M \otimes_A \widehat{A} \to M \otimes_A (A/\mathfrak{J}^n) = M/\mathfrak{J}^n M$. It is clear that this map is bijective when M is of the form A^p. If M is an A-module of finite type, then we have an exact sequence $A^p \to A^q \to M \to 0$, hence, by virtue of the *right* exactness of the functors $M \otimes_A \widehat{A}$ and $\widehat{M}$ (in M) on the category of A-modules of finite type, we have the commutative diagram

$$\begin{tikzcd}
A^p \otimes_A \widehat{A} \arrow[r] \arrow[d] & A^q \otimes_A \widehat{A} \arrow[r] \arrow[d] & M \otimes_A \widehat{A} \arrow[r] \arrow[d] & 0 \\
\widehat{A^p} \arrow[r] & \widehat{A^q} \arrow[r] & \widehat{M} \arrow[r] & 0,
\end{tikzcd}$$

where the two rows are exact and the first two vertical arrows are isomorphisms; this immediately finishes the proof. □

Corollary (7.3.4). — *Let A be a Noetherian ring, $\mathfrak{J}$ an ideal of A, M and N two A-modules of finite type; we have the canonical functorial isomorphisms*

$$(M \otimes_A N)^\wedge \simeq \widehat{M} \otimes_{\widehat{A}} \widehat{N}, \ (\operatorname{Hom}_A(M,N))^\wedge \simeq \operatorname{Hom}_{\widehat{A}}(\widehat{M},\widehat{N}).$$

Proof. This follows from Corollary (7.3.3), (6.2.1), and (6.2.2). □

Corollary (7.3.5). — *Let A be a Noetherian ring, $\mathfrak{J}$ an ideal of A. The following conditions are equivalent:*

(a) *$\mathfrak{J}$ is contained in the radical of A.*
(b) *$\widehat{A}$ is a faithfully flat A-module* (6.4.1).
(c) *Each A-module of finite type is separated for the $\mathfrak{J}$-preadic topology.*
(d) *Each submodule of an A-module of finite type is closed for the $\mathfrak{J}$-preadic topology.*

Proof. As $\widehat{A}$ is a flat A-module, the conditions (b) and (c) are equivalent, since (b) is equivalent to saying that if M is an A-module of finite type, then the canonical map $M \to \widehat{M} = M \otimes_A \widehat{A}$ is injective (6.6.1, c). It is immediate that (c) implies (d), since if N is a submodule of an A-module M of finite type, then M/N is separated for the $\mathfrak{J}$-preadic topology, so N is closed in M. We show that (d) implies (a): if $\mathfrak{m}$ is a maximal ideal of A, then $\mathfrak{m}$ is closed in A for the $\mathfrak{J}$-preadic topology, so $\mathfrak{m} = \bigcap_{p \geqslant 0}(\mathfrak{m} + \mathfrak{J}^p)$, and as $\mathfrak{m} + \mathfrak{J}^p$ is necessarily equal to A or to $\mathfrak{m}$, we have that $\mathfrak{m} + \mathfrak{J}^p = \mathfrak{m}$ for large enough p, hence $\mathfrak{J}^p \subset \mathfrak{m}$, and $\mathfrak{J} \subset \mathfrak{m}$ when $\mathfrak{m}$ is prime. Finally, (a) implies (b): indeed, let P be the closure of $\{0\}$ in an A-module M of finite type, for the $\mathfrak{J}$-preadic topology; according to Krull's Theorem (7.3.2), the topology induced on P by the $\mathfrak{J}$-preadic topology of M is the $\mathfrak{J}$-preadic topology of P, so $\mathfrak{J}P = P$; as P is of finite type, it follows from Nakayama's Lemma that $P = 0$ ($\mathfrak{J}$ being contained in the radical of A). □ $\mathbf{0_I}$ | 68

We note that the conditions of Corollary (7.3.5) are satisfied when A is a *local Noetherian ring* and $\mathfrak{J} \neq A$ is any ideal of A.

Corollary (7.3.6). — *If A is a $\mathfrak{J}$-preadic Noetherian ring, then each A-module of finite type is separated and complete for the $\mathfrak{J}$-preadic topology.*

Proof. As we then have $\widehat{A} = A$, this follows immediately from Corollary (7.3.3). □

We conclude that Proposition (7.2.9) gives the description of *all* the modules of finite type over an adic Noetherian ring.

Corollary (7.3.7). — *Under the hypotheses of* (7.3.2), *the kernel of the canonical map $M \to \widehat{M} = M \otimes_A \widehat{A}$ is the set of the $x \in M$ killed by an element of $1 + \mathfrak{J}$.*

Proof. For each $x \in M$ in this kernel, it is necessary and sufficient that the separated completion of the submodule Ax is 0 (by Krull's Theorem (7.3.2)), in other words, that $x \in \mathfrak{J}x$. □

7.4. Quasi-finite modules over local rings.

Definition (7.4.1). — Given a local ring A, with maximal ideal $\mathfrak{m}$, we say that an A-module M is quasi-finite (over A) if $M/\mathfrak{m}M$ is of finite rank over the residue field $k = A/\mathfrak{m}$.

When A is *Noetherian*, the separated completion $\widehat{M}$ of M for the $\mathfrak{m}$-preadic topology is then an *$\widehat{A}$-module of finite type*; indeed, as $\mathfrak{m}/\mathfrak{m}^2$ is then an A-module of finite type, this follows from Example (7.2.12) and from the hypothesis on $M/\mathfrak{m}M$.

In particular, if we suppose that in addition A is *complete* and M is *separated* for the $\mathfrak{m}$-preadic topology (in other words, $\bigcap_n \mathfrak{m}^n M = 0$), then M is also an *A-module of finite type*: indeed, $\widehat{M}$ is then an A-module of finite type, and as M identifies with a submodule of $\widehat{M}$, M is also of finite type (and is indeed identical to its completion according to Corollary (7.3.6)).

Proposition (7.4.2). — *Let A, B be two local rings, $\mathfrak{m}$, $\mathfrak{n}$ their maximal ideals, and suppose that B is Noetherian. Let $\varphi : A \to B$ be a local homomorphism, M a B-module of finite type. If M is a quasi-finite A-module, then the $\mathfrak{m}$-preadic and $\mathfrak{n}$-preadic topologies on M are identical, thus separated.*

Proof. We note that by hypothesis $M/\mathfrak{m}M$ is of *finite length* as an A-module, thus also *a fortiori* as a B-module. We conclude that $\mathfrak{n}$ is the *unique prime ideal* of B containing the annihilator of $M/\mathfrak{m}M$: indeed, we immediately reduce (according to (1.7.4) and (1.7.2)) to the case where $M/\mathfrak{m}M$ is *simple*, thus necessarily isomorphic to $B/\mathfrak{n}$, and our assertion is evident in this case. On the other hand, as M is a B-module of finite type, the prime ideals which contain the annihilator of $M/\mathfrak{m}M$ are those which contain $\mathfrak{m}B + \mathfrak{b}$, where we denote by $\mathfrak{b}$ the annihilator of the B-module M (1.7.5). As B is Noetherian, we conclude ([**Sam53b**, p. 127, Cor. 4]) that $\mathfrak{m}B + \mathfrak{b}$ is an ideal of definition for B, in other words there exists a $k > 0$ such that $\mathfrak{n}^k \subset \mathfrak{m}B + \mathfrak{b} \subset \mathfrak{n}$; as a result, for each $h > 0$, we have $\mathbf{0_I}$ | 69

$$\mathfrak{n}^{hk} \subset (\mathfrak{m}B + \mathfrak{b})^h M = \mathfrak{m}^h M \subset \mathfrak{n}^h M,$$

which proves that the $\mathfrak{m}$-preadic and $\mathfrak{n}$-preadic topologies on M are the same; the second is separated according to Corollary (7.3.5). □

Corollary (7.4.3). — *Under the hypotheses of Proposition (7.2.4), if in addition A is Noetherian and complete for the $\mathfrak{m}$-preadic topology, then M is an A-module of finite type.*

Proof. Indeed, M is then separated for the $\mathfrak{m}$-preadic topology, and our assertion follows from the remark after Definition (7.4.1). □

(7.4.4). The most important case of Proposition (7.4.2) is when B is a quasi-finite A-module, i.e., $B/\mathfrak{m}B$ is an *algebra of finite rank over* $k = A/\mathfrak{m}$; furthermore, this condition can be broken down into the combination of the following:

(i) $\mathfrak{m}B$ *is an ideal of definition for* B;
(ii) $B/\mathfrak{n}$ *is an extension of finite rank of the field* $A/\mathfrak{m}$.

When this is so, every B-module of finite type is evidently a quasi-finite A-module.

Corollary (7.4.5). — *Under the hypotheses of Proposition (7.4.2), if $\mathfrak{b}$ is the annihilator of the B-module M, then $B/\mathfrak{b}$ is a quasi-finite A-module.*

Proof. Suppose $M \neq 0$ (otherwise the corollary is evident). We can consider M as a module over the local Noetherian ring $B/\mathfrak{b}$; its annihilator then being 0, the proof of Proposition (7.4.2) shows that $\mathfrak{m}(B/\mathfrak{b})$ is an ideal of definition for $B/\mathfrak{b}$. On the other hand, $M/\mathfrak{n}M$ is a vector space of finite rank over $A/\mathfrak{m}$, being a quotient of $M/\mathfrak{m}M$, which is by hypothesis of finite rank over $A/\mathfrak{m}$; as $M \neq 0$, we have $M \neq \mathfrak{n}M$ by virtue of Nakayama's Lemma; as $M/\mathfrak{n}M$ is a vector space $\neq 0$ over $B/\mathfrak{n}$, the fact that it is of finite rank over $A/\mathfrak{m}$ implies that $B/\mathfrak{n}$ is also of finite rank over $A/\mathfrak{m}$; the conclusion follows from (7.4.4) applied to the ring $B/\mathfrak{b}$. □

7.5. Rings of restricted formal series.

(7.5.1). Let A be a topological ring, linearly topologized, separated and complete; let $(\mathfrak{J}_\lambda)$ be a fundamental system of neighborhoods of 0 in A consisting of (open) ideals, such that A canonically identifies with $\varprojlim A/\mathfrak{J}_\lambda$ (7.2.1). For each λ, let $B_\lambda = (A/\mathfrak{J}_\lambda)[T_1, \ldots, T_r]$, where the T_i are indeterminates; it is clear that the B_λ form a projective system of discrete rings. We set $\varprojlim B_\lambda = A\{T_1, \ldots, T_r\}$, and we will see that this topological ring is independent of the fundamental system of ideals $(\mathfrak{J}_\lambda)$ considered. More precisely, let A' be the subring of the ring of formal series $A[[T_1, \ldots, T_r]]$ consisting of formal series $\sum_\alpha c_\alpha T^\alpha$ (with $\alpha = (\alpha_1, \ldots, \alpha_r) \in \mathbf{N}^r$) such that $\lim c_\alpha = 0$ (according to the filter by compliments of finite subsets of $\mathbf{N}^r$); we say that these series are the *restricted* formal series in the T_i, with coefficients in A. For each neighborhood V of 0 in A, let V' be the set of $x = \sum_\alpha c_\alpha T^\alpha \in A'$ such that $c_\alpha \in V$ for all α. We verify immediately that the V' form a fundamental system of neighborhoods of 0 defining on A' a *separated* ring topology; we will canonically define a *topological isomorphism* from the ring $A\{T_1, \ldots, T_r\}$ to A'. For each $\alpha \in \mathbf{N}^r$ and each λ, let $\varphi_{\lambda,\alpha}$ be the map from $(A/\mathfrak{J}_\lambda)[T_1, \ldots, T_r]$ to $A/\mathfrak{J}\lambda$ which sends each polynomial in the first ring to coefficient of T^α in that polynomial. It is clear that the $\varphi_{\lambda,\alpha}$ form a projective system of homomorphisms of $A/\mathfrak{J}_\lambda$-modules, so their projective limit is a continuous homomorphism $\varphi_\alpha : A\{T_1, \ldots, T_r\} \to A$; we will see that, for each $y \in A\{T_1, \ldots, T_r\}$, the formal series $\sum_\alpha \varphi_\alpha(y) T^\alpha$ is *restricted*. Indeed, if y_λ is the component of y in B_λ, and if we denote by H_λ the finite set of the $\alpha \in \mathbf{N}^r$ for which the coefficients of the polynomial y_λ are nonzero, then we have $\varphi_{\lambda,\alpha}(y_\mu) \in \mathfrak{J}_\lambda$ for $\mathfrak{J}_\mu \subset \mathfrak{J}_\lambda$ and $\alpha \notin H_\lambda$, and by passing to the limit, $\varphi_\alpha(y) \in \mathfrak{J}_\lambda$ for $\alpha \notin H_\lambda$. We thus define a ring homomorphism $\varphi : A\{T_1, \ldots, T_r\} \to A'$ by setting $\mathbf{0_I}$ | 70

$\varphi(y)=\sum_\alpha \varphi_\alpha(y)T^\alpha$, and it is immediate that φ is continuous. Conversely, if θ_λ is the canonical homomorphism $A \to A/\mathfrak{J}_\lambda$, then for each element $z=\sum_\alpha c_\alpha T^\alpha \in A'$ and each λ, there are only a finite number of indices α such that $\theta_\lambda(c_\alpha) \neq 0$, and as a result $\psi_\lambda(z)=\sum_\alpha \theta_\lambda(c_\alpha)T^\alpha$ is in B_λ; the ψ_λ are continuous and form a projective system of homomorphisms whose projective limit is a continuous homomorphism $\psi : A' \to A\{T_1,\ldots,T_r\}$; is remains to verify that $\varphi\circ\psi$ and $\psi\circ\varphi$ are the identity automorphisms, which is immediate.

(7.5.2). We identify $A\{T_1,\ldots,T_r\}$ with the ring A' of restricted formal series by means of the isomorphisms defined in (7.5.1). The canonical isomorphisms

$$((A/\mathfrak{J}_\lambda)[T_1,\ldots,T_r])[T_{r+1},\ldots,T_s] \simeq (A/\mathfrak{J}_\lambda)[T_1,\ldots,T_s]$$

define, by passing to the projective limit, a canonical isomorphism

$$(A\{T_1,\ldots,T_r\})\{T_{r+1},\ldots,T_s\} \simeq A\{T_1,\ldots,T_s\}.$$

(7.5.3). For every continuous homomorphism $u : A \to B$ from A to a linearly topologized ring B, separated and complete, and each system $(b_1,\ldots,b_r)$ of r elements of B, there exists *a unique continuous homomorphism* $\overline{u} : A\{T_1,\ldots,T_r\} \to B$, such that $\overline{u}(a)=u(a)$ for all $a\in A$ and $\overline{u}(T_j)=b_j$ for $1 \leqslant j \leqslant r$. It suffices to set

$$\overline{u}\Big(\sum_\alpha c_\alpha T^\alpha\Big)=\sum_\alpha u(c_\alpha)b_1^{\alpha_1}\cdots b_r^{\alpha_r};$$

the verification of the fact that the family $(u(c_\alpha)b_1^{\alpha_1}\cdots b_r^{\alpha_r})$ is summable in B and that $\overline{u}$ is continuous are immediate and left to the reader. We note that this property (for arbitrary B and b_j) *characterize* the topological ring $A\{T_1,\ldots,T_r\}$ up to unique isomorphism.

Proposition (7.5.4). —

(i) *If A is an admissible ring, then so is $A'=A\{T_1,\ldots,T_r\}$.*
(ii) *Let A be an adic ring, $\mathfrak{J}$ an ideal of definition for A such that $\mathfrak{J}/\mathfrak{J}^2$ is of finite type over $A/\mathfrak{J}$. If we set $\mathfrak{J}'=\mathfrak{J}A'$,* **$0_{\mathrm{I}}$ | 71** *then A' is also a $\mathfrak{J}'$-adic ring, and $\mathfrak{J}'/\mathfrak{J}'^2$ is of finite type over $A'/\mathfrak{J}'$. If in addition A is Noetherian, then so is A'.*

Proof.

(i) If $\mathfrak{J}$ is an ideal of A, $\mathfrak{J}'$ the ideal of A' consisting of the $\sum_\alpha c_\alpha T^\alpha$ such that $c_\alpha \in \mathfrak{J}$ for all α, then $(\mathfrak{J}')^n \subset (\mathfrak{J}^n)'$; if $\mathfrak{J}$ is an ideal of definition for A, then $\mathfrak{J}'$ is also an ideal of definition for A'.
(ii) Set $A_i=A/\mathfrak{J}^{i+1}$, and for $i \leqslant j$, let u_{ij} be the canonical homomorphism $A/\mathfrak{J}^{j+1} \to A/\mathfrak{J}^{i+1}$; set $A'_i=A_i[T_1,\ldots,T_r]$, and let u'_{ij} be the homomorphism $A'_j \to A'_i$ $(i \leqslant j)$ obtained by applying u_{ij} to the coefficients of the polynomials in A'_j. We will show that the projective system (A'_i,u'_{ij}) satisfies the conditions of Proposition (7.2.7); as $\mathfrak{J}'$ is the kernel of $A' \to A'_0$, this proves the first assertion of (ii). It is clear that the u'_{ij} are surjective; the kernel $\mathfrak{J}'_i$ of u_{0i} is the set of polynomials in $A_i[T_1,\ldots,T_r]$ whose coefficients are in $\mathfrak{J}/\mathfrak{J}^{i+1}$; in particular, $\mathfrak{J}'_1$ is the set of polynomials in $A_1[T_1,\ldots,T_r]$ whose coefficients are in $\mathfrak{J}/\mathfrak{J}^2$. As $\mathfrak{J}/\mathfrak{J}^2$ is of finite type over $A_1=A/\mathfrak{J}^2$, we see that $\mathfrak{J}'_1/{\mathfrak{J}'_1}^2$ is a module of finite type over A'_1 (or equivalently, over $A'_0=A'_1/\mathfrak{J}'_1$). We will now show that the kernel of u_{ij} is ${\mathfrak{J}'_j}^{i+1}$. It is evident that ${\mathfrak{J}'_j}^{i+1}$ is contained in this kernel. On the other hand, let $a_1,\ldots,a_m$ be the elements of $\mathfrak{J}$ whose classes mod $\mathfrak{J}^2$ generate $\mathfrak{J}/\mathfrak{J}^2$; we verify immediately that the classes mod $\mathfrak{J}^{j+1}$ of monomials of degree $\leqslant j$ in the a_k $(1 \leqslant k \leqslant m)$ generate $\mathfrak{J}/\mathfrak{J}^{j+1}$, and the classes of monomials of degree $> i$ and $\leqslant j$ generate $\mathfrak{J}^{i+1}/\mathfrak{J}^{j+1}$; a monomial in the T_k having such an element for a coefficient is thus a product of $i+1$ elements of $\mathfrak{J}'_i$, which establishes our assertion. Finally, if A is Noetherian, then so is $A'/\mathfrak{J}'=(A/\mathfrak{J})[T_1,\ldots,T_r]$, hence A' is Notherian (7.2.8).

□

Proposition (7.5.5). — *Let A be a Notherian $\mathfrak{J}$-adic ring, B an admissible topological ring, $\varphi : A \to B$ a continuous homomorphism, making B and A-algebra. The following conditions are equivalent:*

(a) *B is Noetherian and $\mathfrak{J}B$-adic, and $B/\mathfrak{J}B$ is an algebra of finite type over $A/\mathfrak{J}$.*
(b) *B is topologically A-isomorphic to $\varprojlim B_n$, where $B_n=B_m/\mathfrak{J}^{n+1}B_m$ for $m \geqslant n$, and B_1 is an algebra of finite type over $A_1=A/\mathfrak{J}^2$.*
(c) *B is topologically A-isomorphic to a quotient of an algebra of the form $A\{T_1,\ldots,T_r\}$ by an ideal (necessarily closed according to Corollary (7.3.6) and Proposition (7.5.4, ii)).*

Proof. As A is Noetherian, so is $A' = A\{T_1, \dots, T_r\}$ (7.5.4), so (c) implies that B is Noetherian; as $\mathfrak{J}' = \mathfrak{J}A'$ is an open neighborhood of 0 in A' such that the $\mathfrak{J}'^n$ form a fundamental system of neighborhoods of 0, the images $\mathfrak{J}^n B$ of the $\mathfrak{J}'^n$ form a fundamental system of neighborhoods of 0 in B, and as B is separated and complete, B is a $\mathfrak{J}B$-adic ring. Finally, $B/\mathfrak{J}B$ is an algebra (over $(A/\mathfrak{J})$ quotient of $A'/\mathfrak{J}A' = (A/\mathfrak{J})[T_1, \dots, T_r]$, so it is of finite type, which proves that (c) implies (a).

If B is $\mathfrak{J}B$-adic and Noetherian, then B is isomorphic to $\varprojlim B_n$, where $B_n = B/\mathfrak{J}^{n+1}B$ (7.2.11), and $\mathfrak{J}B/\mathfrak{J}^2B$ is a module of finite type over $B/\mathfrak{J}B$. Let $(a_j)_{1\leqslant j\leqslant s}$ be a system of generators for the $B/\mathfrak{J}B$-module $\mathfrak{J}B/\mathfrak{J}^2B$, and let $(c_i)_{1\leqslant i\leqslant r}$ be a system of elements of $B/\mathfrak{J}^2B$ such that the classes mod $\mathfrak{J}B/\mathfrak{J}^2B$ form a system of generators for the $A/\mathfrak{J}$-algebra $B/\mathfrak{J}B$; we see immediately that the $c_i a_j$ form a system of generators for the $A/\mathfrak{J}^2$-algebra $B/\mathfrak{J}^2B$, hence (a) implies (b). $\mathbf{0_I}$ | 72

It remains to prove that (b) implies (c). The hypotheses imply that B_1 is a Noetherian ring, and as $B_1 = B_2/\mathfrak{J}^2B_2$, we have $\mathfrak{J}^2B_1 = 0$, hence $\mathfrak{J}B_1 = \mathfrak{J}B_1/\mathfrak{J}^2B_1$ is a B_0-module of finite type. The conditions of Proposition (7.2.7) are thus satisfied by the projective system (B_n) and B is a $\mathfrak{J}B$-adic ring. Let $(c_i)_{1\leqslant i\leqslant r}$ be a finite system of elements of B whose classes mod $\mathfrak{J}B$ generate the $A/\mathfrak{J}$-algebra $B/\mathfrak{J}B$, and whose linear combinations with coefficients in $\mathfrak{J}$ are such that their classes mod $\mathfrak{J}^2B$ generate the B_0-module $\mathfrak{J}B/\mathfrak{J}^2B$. There exists a continuous A-homomorphism u from $A' = A\{T_1, \dots, T_r\}$ to B which reduces to φ on A and is such that $u(T_i) = c_i$ for $1 \leqslant i \leqslant r$ (7.5.3); if we prove that u is *surjective*, then (c) will be established, since from $u(A') = B$ we deduce that $u(\mathfrak{J}^nA') = \mathfrak{J}^nB$, in other words that u is a strict morphism of topological rings and B is this isomorphic to a quotient of A' by a closed ideal. As B is complete for the $\mathfrak{J}B$-adic topology, it suffices ([**CC**, p. 18–07]) to show that the homomorphism $\mathrm{grad}(A') \to \mathrm{grad}(B)$, induced canonically by u for the $\mathfrak{J}$-adic filtrations on A' and B, is surjective. But by definition, the homomorphisms $A'/\mathfrak{J}A' \to B/\mathfrak{J}B$ and $\mathfrak{J}A'/\mathfrak{J}^2A' \to \mathfrak{J}B/\mathfrak{J}^2B$ induced by u are surjective; by induction on n, we immediately deduce that so is $\mathfrak{J}A'/\mathfrak{J}^nA' \to \mathfrak{J}B/\mathfrak{J}^nB$, and *a fortiori* so is $\mathfrak{J}^nA'/\mathfrak{J}^{n+1}A' \to \mathfrak{J}^nB/\mathfrak{J}^{n+1}B$, which finishes the proof. □

7.6. Completed rings of fractions.

(7.6.1). Let A be a linearly topologized ring, $(\mathfrak{J}_\lambda)$ a fundamental system of neighborhoods of 0 in A consisting of ideals, S a multiplicative subset of A. Let u_λ be the canonical homomorphism $A \to A_\lambda = A/\mathfrak{J}_\lambda$, and for $\mathfrak{J}_\mu \subset \mathfrak{J}_\lambda$, let $u_{\lambda\mu}$ be the canonical homomorphism $A_\mu \to A_\lambda$. Set $S_\lambda = u_\lambda(S)$, so that $u_{\lambda\mu}(S_\mu) = S_\lambda$. The $u_{\lambda\mu}$ canonically induce surjective homomorphisms $S_\mu^{-1}A_\mu \to S_\lambda^{-1}A_\lambda$, for which these rings form a projective system; we denote by $A\{S^{-1}\}$ the projective limit of this system. This definition does not depend on the fundamental system of neighborhoods $(\mathfrak{J}_\lambda)$ chosen; indeed:

Proposition (7.6.2). — *The ring $A\{S^{-1}\}$ is topologically isomorphic to the separated completion of the ring $S^{-1}A$ for the topology which has a fundamental system of neighborhoods of 0 consisting of the $S^{-1}\mathfrak{J}_\lambda$.*

Proof. If v_λ is the canonical homomorphism $S^{-1}A \to S_\lambda^{-1}A_\lambda$ induced by u_λ, then the kernel of v_λ is surjective, hence the proposition (7.2.1). □

Corollary (7.6.3). — *If S' is the canonical image of S in the separated completion $\widehat{A}$ of A, then $A\{S^{-1}\}$ canonically identifies with $\widehat{A}\{S'^{-1}\}$.*

We note that if A is separated and complete, then it is not necessarily the same for $S^{-1}A$ with the topology defined by the $S^{-1}\mathfrak{J}_\lambda$, as we see for example by taking S to be the set of the f^n $(n \geqslant 0)$, where f is topologically nilpotent but not nilpotent: indeed, $S^{-1}A$ is not 0 and on the other hand, for each λ there exists an n such that $f^n \in \mathfrak{J}_\lambda$, so $1 = f^n/f^n \in S^{-1}\mathfrak{J}_\lambda$ and $S^{-1}\mathfrak{J}_\lambda = S^{-1}A$. $\mathbf{0_I}$ | 73

Corollary (7.6.4). — *If, in A, 0 does not belong to S, then the ring $A\{S^{-1}\}$ is not 0.*

Proof. Indeed, 0 does not belong to $\{1\}$ in the ring $S^{-1}A$; otherwise, we would have that $1 \in S^{-1}\mathfrak{J}_\lambda$ for each open ideal $\mathfrak{J}_\lambda$ of A, and it follows that $\mathfrak{J}_\lambda \cap S \neq \varnothing$ for all λ, contradicting the hypothesis. □

(7.6.5). We say that $A\{S^{-1}\}$ is the *completed ring of fractions* of A with denominators in S. With the above notation, it is clear that the inverse image of $S^{-1}\mathfrak{J}_\lambda$ in A contains $\mathfrak{J}_\lambda$, hence the canonical map $A \to S^{-1}A$ is continuous, and if we compose it with the canonical map $S^{-1}A \to A\{S^{-1}\}$, we obtain a canonical continuous homomorphism $A \to A\{S^{-1}\}$, the projective limit of the homomorphisms $A \to S_\lambda^{-1}A_\lambda$.

(7.6.6). The couple consisting of $A\{S^{-1}\}$ and the canonical homomorphism $A \to A\{S^{-1}\}$ are characterized by the following *universal property*: every continuous homomorphism u from A to a linearly topologized ring B, separated and complete, such that $u(S)$ consists of the invertible elements of B, uniquely factorizes as $A \to A\{S^{-1}\} \xrightarrow{u'} B$, where u' is continuous. Indeed, u uniquely factorizes as $A \to S^{-1}A \xrightarrow{v'} B$; as for each open ideal $\mathfrak{K}$ of B we have that $u^{-1}(\mathfrak{K})$ contains a $\mathfrak{J}_\lambda$, $v'^{-1}(\mathfrak{K})$ necessarily contains $S^{-1}\mathfrak{J}_\lambda$, so v' is continuous; since B is separated and complete, v' uniquely factorizes as $S^{-1}A \to A\{S^{-1}\} \xrightarrow{u'} B$, where u' is continuous; hence our assertion.

(7.6.7). Let B be a second linearly topologized ring, T a multiplicative subset of B, $\varphi : A \to B$ a continuous homomorphism such that $\varphi(S) \subset T$. According to the above, the continuous homomorphism $A \xrightarrow{\varphi} B \to B\{T^{-1}\}$ uniquely factorizes as $A \to A\{S^{-1}\} \xrightarrow{\varphi'} B\{T^{-1}\}$, where φ' is continuous. In particular, if $B = A$ and if φ is the identity, we see that for $S \subset T$ we have a continuous homomorphism $\rho^{T,S} : A\{S^{-1}\} \to A\{T^{-1}\}$ obtained by passing to the separated completion from $S^{-1}A \to T^{-1}A$; if U is a third multiplicative subset of A such that $S \subset T \subset U$, then we have $\rho^{U,S} = \rho^{U,T} \circ \rho^{T,S}$.

(7.6.8). Let S_1, S_2 be two multiplicative subsets of A, and let S_2' be the canonical image of S_2 in $A\{S_1^{-1}\}$; we then have a canonical topological isomorphism $A\{(S_1S_2)^{-1}\} \simeq A\{S_1^{-1}\}\{S_2'^{-1}\}$, as we see from the canonical isomorphism $(S_1S_2)^{-1}A \simeq S_2''^{-1}(S_1^{-1}A)$ (where S_2'' is the canonical image of S_2 in $S_1^{-1}A$), which is bicontinuous.

(7.6.9). Let $\mathfrak{a}$ be an *open* ideal of A; we can assume that $\mathfrak{J}_\lambda \subset \mathfrak{a}$ for all λ, and as a result $S^{-1}\mathfrak{J}_\lambda \subset S^{-1}\mathfrak{a}$ in the ring $S^{-1}A$, in other words, $S^{-1}\mathfrak{a}$ is an *open* ideal of $S^{-1}A$; we denote by $\mathfrak{a}\{S^{-1}\}$ its separated completion, equal to $\varprojlim(S^{-1}\mathfrak{a}/S^{-1}\mathfrak{J}_\lambda)$, which is an *open* ideal of $A\{S^{-1}\}$, isomorphic to the closure of the canonical image of $S^{-1}\mathfrak{a}$. In addition, *the discrete ring* $A\{S^{-1}\}/\mathfrak{a}\{S^{-1}\}$ *is canonically isomorphic to* $S^{-1}A/S^{-1}\mathfrak{a} = S^{-1}(A/\mathfrak{a})$. Conversely, if $\mathfrak{a}'$ is an open ideal of $A\{S^{-1}\}$, then $\mathfrak{a}'$ contains an ideal of the form $\mathfrak{J}_\lambda\{S^{-1}\}$ which is the inverse image of an ideal of $S^{-1}A/S^{-1}\mathfrak{J}_\lambda$, which is necessarily (1.2.6) of the form $S^{-1}\mathfrak{a}$, where $\mathfrak{a} \supset \mathfrak{J}_\lambda$. We conclude that $\mathfrak{a}' = \mathfrak{a}\{S^{-1}\}$. In particular (1.2.6):

Proposition (7.6.10). — *The map* $\mathfrak{p} \mapsto \mathfrak{p}\{S^{-1}\}$ *is an increasing bijection from the set of* open prime *ideals* $\mathfrak{p}$ *of* A *such that* $\mathfrak{p} \cap S = \varnothing$ *to the set of* open prime *ideals of* $A\{S^{-1}\}$*; in addition, the field of fractions of* $A\{S^{-1}\}/\mathfrak{p}\{S^{-1}\}$ *is* $\mathbf{0_I}$ | 74
canonically isomorphic to that of $A/\mathfrak{p}$.

Proposition (7.6.11). —

(i) *If* A *is an admissible ring, then so is* $A' = A\{S^{-1}\}$*, and for every ideal of definition* $\mathfrak{J}$ *for* A*,* $\mathfrak{J}' = \mathfrak{J}\{S^{-1}\}$ *is an ideal of definition for* A'.

(ii) *Let* A *be an adic ring,* $\mathfrak{J}$ *an ideal of definition for* A *such that* $\mathfrak{J}/\mathfrak{J}^2$ *is of finite type over* $A/\mathfrak{J}$*; then* A' *is a* $\mathfrak{J}'$*-adic ring and* $\mathfrak{J}'/\mathfrak{J}'^2$ *is of finite type over* $A'/\mathfrak{J}'$*. If in addition* A *is Noetherian, then so is* A'.

Proof.

(i) If $\mathfrak{J}$ is an ideal of definition for A, then it is clear that $S^{-1}\mathfrak{J}$ is an ideal of definition for the topological ring $S^{-1}A$, since we have $(S^{-1}\mathfrak{J})^n = S^{-1}\mathfrak{J}^n$. Let A'' be the separated ring associated to $S^{-1}A$, $\mathfrak{J}''$ the image of $S^{-1}\mathfrak{J}$ in A''; the image of $S^{-1}\mathfrak{J}^n$ is $\mathfrak{J}''^n$, so $\mathfrak{J}''^n$ tends to 0 in A''; as $\mathfrak{J}'$ is the closure of $\mathfrak{J}''$ in A', $\mathfrak{J}'^n$ is contained in the closure of $\mathfrak{J}''^n$, hence tends to 0 in A'.

(ii) Set $A_i = A/\mathfrak{J}^{i+1}$, and for $i \leqslant j$, let u_{ij} be the canonical homomorphism $A/\mathfrak{J}^{j+1} \to A/\mathfrak{J}^{i+1}$; let S_i be the canonical image of S in A_i, and set $A_i' = S_i^{-1}A_i$; finally, let $u_{ij}' : A_j' \to A_i'$ be the homomorphism canonically induced by u_{ij}. We show that the projective system (A_i', u_{ij}') satisfies the conditions of Proposition (7.2.7): it is clear that the u_{ij}' are surjective; on the other hand, the kernel of u_{ij}' is $S_j^{-1}(\mathfrak{J}^{i+1}/\mathfrak{J}^{j+1})$ (1.3.2), equal to $\mathfrak{J}_j'^{i+1}$, where $\mathfrak{J}_j' = S_j^{-1}(\mathfrak{J}/\mathfrak{J}^{j+1})$; finally, $\mathfrak{J}_1'/\mathfrak{J}_1'^2 = S_1^{-1}(\mathfrak{J}/\mathfrak{J}^2)$, and as $\mathfrak{J}/\mathfrak{J}^2$ is of finite type over $A/\mathfrak{J}^2$, $\mathfrak{J}_1'/\mathfrak{J}_1'^2$ is of finite type over A_1'. Finally, if A is Noetherian, then so is $A_0' = S_0^{-1}(A/\mathfrak{J})$, which finishes the proof (7.2.8).

□

Corollary (7.6.12). — *Under the hypotheses of Proposition* (7.6.11, ii)*, we have* $(\mathfrak{J}\{S^{-1}\})^n = \mathfrak{J}^n\{S^{-1}\}$.

Proof. This follows from Proposition (7.2.7) and the proof of Proposition (7.6.11). □

Proposition (7.6.13). — *Let* A *be an adic Noetherian ring,* S *a multiplicative subset of* A*; then* $A\{S^{-1}\}$ *is a flat* A*-module.*

Proof. If $\mathfrak{J}$ is an ideal of definition for A, then $A\{S^{-1}\}$ is the separated completion of the Noetherian ring $S^{-1}A$ equipped with the $S{-}1\mathfrak{J}$-preadic topology; as a result (7.3.3) $A\{S^{-1}\}$ is a flat $S^{-1}A$-module; as $S^{-1}A$ is a flat A-module (6.3.1), the proposition follows from the transitivity of flatness (6.2.1). □

Corollary (7.6.14). — *Under the hypotheses of Proposition* (7.6.13)*, let* $S' \subset S$ *be a second multiplicative subset of* A*; then* $A\{S^{-1}\}$ *is a flat* $A\{S'^{-1}\}$*-module.*

Proof. By (7.6.8), $A\{S^{-1}\}$ canonically identifies with $A\{S'^{-1}\}\{S_0^{-1}\}$, where S_0 is the canonical image of S in $A\{S'^{-1}\}$, and $A\{S'^{-1}\}$ is Noetherian (7.6.11). □

(7.6.15). For each element f of a linearly topologized ring A, we denote by $A_{\{f\}}$ the completed ring of fractions $A\{S_f^{-1}\}$, where S_f is the multiplicative set of the f^n ($n \geqslant 0$); for each open ideal $\mathfrak{a}$ of A, we write $\mathfrak{a}_{\{f\}}$ for $\mathfrak{a}\{S_f^{-1}\}$. If g is a second element of A, then we have a canonical continuous homomorphism $A_{\{f\}} \to A_{\{fg\}}$ (7.6.7). When f varies over a multiplicative subset S of A, the $A_{\{f\}}$ form a filtered inductive system with the above homomorphisms; we set $A_{\{S\}} = \varinjlim_{f\in S} A_{\{f\}}$. For every $f \in S$, we have a homomorphism $A_{\{f\}} \to A\{S^{-1}\}$ (7.6.7), and these homomorphisms form an inductive system; by passing to the inductive limit, they thus define a canonical homomorphism $A_{\{S\}} \to A\{S^{-1}\}$. $\mathbf{0_I}$ | 75

Proposition (7.6.16). — *If A is a Noetherian ring, then $A\{S^{-1}\}$ is a flat module over $A_{\{S\}}$.*

Proof. By (7.6.14), $A\{S^{-1}\}$ is flat for each of the rings $A_{\{f\}}$ for $f \in S$, and the conclusion follows from (6.2.3). □

Proposition (7.6.17). — *Let $\mathfrak{p}$ be an open prime ideal in an admissible ring A, and let $S = A - \mathfrak{p}$. Then the rings $A\{S^{-1}\}$ and $A_{\{S\}}$ are local rings, the canonical homomorphism $A_{\{S\}} \to A\{S^{-1}\}$ is local, and the residue fields of $A_{\{S\}}$ and $A\{S^{-1}\}$ are canonically isomorphic to the field of fractions of $A/\mathfrak{p}$.*

Proof. Let $\mathfrak{J} \subset \mathfrak{p}$ be an ideal of definition for A; we have $S^{-1}\mathfrak{J} \subset S^{-1}\mathfrak{p} = \mathfrak{p}A_{\mathfrak{p}}$, so $A_{\mathfrak{p}}/S^{-1}\mathfrak{J}$ is a local ring; we conclude from Corollary (7.1.12), (7.6.9), and Proposition (7.6.11, i) that $A\{S^{-1}\}$ is a local ring. Set $\mathfrak{m} = \varinjlim_{f\in S} \mathfrak{p}_{\{f\}}$, which is an ideal in $A_{\{S\}}$; we will see that each element in $A_{\{S\}}$ not in $\mathfrak{m}$ is invertible. Indeed, such an element is the image in $A_{\{S\}}$ of an element $z \in A_{\{f\}}$ not in $\mathfrak{p}_{\{f\}}$, for an $f \in S$; its canonical image z_0 in $A_{\{f\}}/\mathfrak{J}_{\{f\}} = S_f^{-1}(A/\mathfrak{J})$ therefore is not in $S_f^{-1}(\mathfrak{p}/\mathfrak{J})$ (7.6.9), which means that $z_0 = \overline{x}/\overline{f}^k$, where $x \notin \mathfrak{p}$ and $\overline{x}$, $\overline{f}$ are the classes of x, f mod $\mathfrak{J}$. As $x \in S$, we have $g = xf \in S$, and in $S_g^{-1}A$, the canonical image $y_0 = x^{k+1}/g^k$ of $x/f^k \in S_f^{-1}A$ admits an inverse $x^{k-1}f^{2k}/g^k$. This implies *a fortiori* that the image of y_0 in $S_g^{-1}A/S_g^{-1}\mathfrak{J}$ is invertible, so ((7.6.9) and Corollary (7.1.12)) the canonical image y of z in $A_{\{g\}}$ is invertible; the image of z in $A_{\{S\}}$ (equal to that of y) is as a result invertible. We thus see that $A_{\{S\}}$ a local ring with maximal ideal $\mathfrak{m}$; in addition, the image of $\mathfrak{p}_{\{f\}}$ in $A\{S^{-1}\}$ is contained in the maximal ideal $\mathfrak{p}\{S^{-1}\}$ of this ring; *a fortiori*, the image of $\mathfrak{m}$ in $A\{S^{-1}\}$ is contained in $\mathfrak{p}\{S^{-1}\}$, so the canonical homomorphism $A_{\{S\}} \to A\{S^{-1}\}$ is local. Finally, as each element of $A\{S^{-1}\}/\mathfrak{p}\{S^{-1}\}$ is the image of an element in the ring $S_f^{-1}A$ for a suitable $f \in S$, the homomorphism $A_{\{S\}} \to A\{S^{-1}\}/\mathfrak{p}\{S^{-1}\}$ is surjective, and gives an isomorphism of the residue fields by passing to quotients. □

Corollary (7.6.18). — *Under the hypotheses of Proposition (7.6.17), if we suppose also that A is an adic Noetherian ring, then the local rings $A\{S^{-1}\}$ and $A_{\{S\}}$ are Noetherian, and $A\{S^{-1}\}$ is a faithfully flat $A_{\{S\}}$-module.*

Proof. We know from before (7.6.11, ii) that $A\{S^{-1}\}$ is Noetherian and $A_{\{S\}}$-flat (7.6.16); as the homomorphism $A_{\{S\}} \to A\{S^{-1}\}$ is local, we conclude that $A\{S^{-1}\}$ is a faithfully flat $A_{\{S\}}$-module (6.6.2), and as a result that $A_{\{S\}}$ is Noetherian (6.5.2). □

7.7. Completed tensor products.

(7.7.1). Let A be a linearly topologized ring, M, N two linearly topologized A-modules. Let $\mathfrak{J}$, V, W be open neighborhoods of 0 in A, M, N respectively, which are A-modules, and such that $\mathfrak{J}M \subset V$, $\mathfrak{J}N \subset W$, so that M/V and N/W can be considered as $A/\mathfrak{J}$-modules. When $\mathfrak{J}$, V, W vary over the systems of open neighborhoods satisfying these properties, it is immediate that the modules $(M/V)\otimes_{A/\mathfrak{J}}(N/W)$ form a projective system of modules over the projective system of rings $A/\mathfrak{J}$; by passing to the projective limit, we thus obtain a module over the separated completion $\widehat{A}$ of A, which we call the *completed tensor product* of M and N and denote by $(M\otimes_A N)^\wedge$. If we have that M/V is canonically isomorphic to $\widehat{M}/\overline{V}$, where $\widehat{M}$ is the separated completion of M and $\overline{V}$ the closure in $\widehat{M}$ of the image of V, then we see that the completed tensor product $(M\otimes_A N)^\wedge$ canonically identifies with $(\widehat{M}\otimes_{\widehat{A}}\widehat{N})^\wedge$, which we denote by $\widehat{M}\widehat{\otimes}_{\widehat{A}}\widehat{N}$. $\mathbf{0_I}$ | 76

(7.7.2). With the above notation, the tensor products $(M/V)\otimes_A(N/W)$ and $(M/V)\otimes_{A/\mathfrak{J}}(N/W)$ identify canonically; they identify with $(M\otimes_A N)/(\mathrm{Im}(V\otimes_A N)+\mathrm{Im}(M\otimes_A W))$. We conclude that $(M\otimes_A N)^\wedge$ is *the separated completion of the A-module $M\otimes_A N$, equipped with the topology for which the submodules*

$$\mathrm{Im}(V\otimes_A N)+\mathrm{Im}(M\otimes_A W)$$

form a fundamental system of neighborhoods of 0 (V and W varying over the set of open submodules of M and N respectively); we say for brevity that this topology is the *tensor product* of the given topologies on M and N.

(7.7.3). Let M', N' be two linearly topologized A-modules, $u : M \to M'$, $v : N \to N'$ two continuous homomorphisms; it is immediate that $u \otimes v$ is continuous for the tensor product topologies on $M \otimes_A N$ and $M' \otimes_A N'$ respectively; by passing to the separated completions, we obtain a continuous homomorphism $(M \otimes_A N)^\wedge \to (M' \otimes_A N')^\wedge$, which we denote by $u \widehat{\otimes} v$; $(M \otimes_A N)^\wedge$ is thus a *bifunctor* in M and N on the category of linearly topologized A-modules.

(7.7.4). We similarly define the completed tensor product of any finite number of linearly topologized A-modules; it is immediate that this product has the usual properties of associativity and commutativity.

(7.7.5). If B, C are two linearly topologized A-algebras, then the tensor product topology on $B \otimes_A C$ has for a fundamental system of neighborhoods of 0 the *ideals* $\operatorname{Im}(\mathfrak{K} \otimes_A C) + \operatorname{Im}(B \otimes_A \mathfrak{L})$ of the algebra $B \otimes_A C$, $\mathfrak{K}$ (resp. $\mathfrak{L}$) varying over the set of open ideals of B (resp. C). As a result, $(B \otimes_A C)^\wedge$ is equipped with the structure of a *topological* $\widehat{A}$*-algebra*, the projective limit of the projective system of $A/\mathfrak{J}$-algebras $(B/\mathfrak{K}) \otimes_{A/\mathfrak{J}} (C/\mathfrak{L})$ ($\mathfrak{J}$ the open ideal of A such that $\mathfrak{J}B \subset \mathfrak{K}$, $\mathfrak{J}C \subset \mathfrak{L}$; it always exists). We say that this algebra is the *completed tensor product* of the algebras B and C.

(7.7.6). The A-algebra homomorphisms $b \mapsto b \otimes 1$, $c \mapsto 1 \otimes c$ from B and C to $B \otimes_A C$ are continuous when we equip the latter algebra with the tensor product topology; by composing with the canonical homomorphism from $B \otimes_A C$ to its separated completion, they give *canonical homomorphisms* $\rho : B \to (B \otimes_A C)^\wedge$, $\sigma : C \to (B \otimes_A C)^\wedge$. The algebra $(B \otimes_A C)^\wedge$ and the homomorphisms ρ and σ have in addition the following *universal property*: for every linearly topologized A-algebra D, separated and complete, and each pair of continuous A-homomorphisms $u : B \to D$, $v : C \to D$, there exists a unique continuous A-homomorphism $w : (B \otimes_A C)^\wedge \to D$ such that $u = w \circ \rho$ and $v = w \circ \sigma$. Indeed, there already exists a unique A-homomorphism $w_0 : B \otimes_A C \to D$ such that $u(b) = w_0(b \otimes 1)$ and $v(c) = w_0(1 \otimes c)$, and it remains to prove that w_0 is *continuous*, since it then gives a continuous homomorphism **0_{I}** | 77
w by passing to the separated completion. If $\mathfrak{M}$ is an open ideal of D, then there exists by hypothesis open ideals $\mathfrak{K} \subset B$, $\mathfrak{L} \subset C$ such that $u(\mathfrak{K}) \subset \mathfrak{M}$, $v(\mathfrak{L}) \subset \mathfrak{M}$; the image under w_0 of $\operatorname{Im}(\mathfrak{K} \otimes_A C) + \operatorname{Im}(B \otimes_A \mathfrak{L})$ is again contained in $\mathfrak{M}$, hence our assertion.

Proposition (7.7.7). — *If B and C are two preadmissible A-algebras, then $(B \otimes_A C)^\wedge$ is admissible, and if $\mathfrak{K}$ (resp. $\mathfrak{L}$) is an ideal of definition for B (resp. C), then the closure in $(B \otimes_A C)^\wedge$ of the canonical image of $\mathfrak{H} = \operatorname{Im}(\mathfrak{K} \otimes_A C) + \operatorname{Im}(B \otimes_A \mathfrak{L})$ is an ideal of definition.*

Proof. It suffices to show that $\mathfrak{H}^n$ tends to 0 for the tensor product topology, which follows immediately from the inclusion

$$\mathfrak{H}^{2n} \subset \operatorname{Im}(\mathfrak{K}^n \otimes_A C) + \operatorname{Im}(B \otimes_A \mathfrak{L}^n).$$

□

Proposition (7.7.8). — *Let A be a preadic ring, $\mathfrak{J}$ an ideal of defintion for A, M an A-module* of finite type, *equipped with the $\mathfrak{J}$-preadic topology. For every topological adic Noetherian A-algebra B, $B \otimes_A M$ identifies with the completed tensor product* $(B \otimes_A M)^\wedge$.

Proof. If $\mathfrak{K}$ is an ideal of definition for B, there exists by hypothesis an integer m such that $\mathfrak{J}^m B \subset \mathfrak{K}$, so $\operatorname{Im}(B \otimes_A \mathfrak{J}^{nm} M) = \operatorname{Im}(\mathfrak{J}^{nm} B \otimes_A M) \subset \operatorname{Im}(\mathfrak{K}^n B \otimes_A M) = \mathfrak{K}^n (B \otimes_A M)$; we conclude that over $B \otimes_A M$, the tensor products of the topologies of B and M is the $\mathfrak{K}$-preadic topology. As $B \otimes_A M$ is a B-module of finite type, the proposition follows from Corollary (7.3.6). □

7.8. Topologies on modules of homomorphisms.

(7.8.1). Let A be a *Noetherian $\mathfrak{J}$-adic* ring, M and N two A-modules of finite type, equipped with the $\mathfrak{J}$-preadic topology; we know (7.3.6) that they are separated and complete; in addition, every A-homomorphism $M \to N$ is automatically continuous, and the A-module $\operatorname{Hom}_A(M, N)$ is of finite type. For every integer $i \geqslant 0$, set $A_i = A/\mathfrak{J}^{i+1}$, $M_i = M/\mathfrak{J}^{i+1}M$, $N_i = N/\mathfrak{J}^{i+1}N$; for $i \leqslant j$, every homomorphism $u_j : M_j \to N_j$ maps $\mathfrak{J}^{i+1}M_j$ to $\mathfrak{J}^{i+1}N_j$, thus giving by passage to quotients a homomorphism $u_i : M_i \to N_i$, which defines a canonial homomorphism $\operatorname{Hom}_{A_j}(M_j, N_j) \to \operatorname{Hom}_{A_i}(M_i, N_i)$; in addition, the $\operatorname{Hom}_{A_i}(M_i, N_i)$ form a *projective system* for these homomorphisms, and it follows from Corollary (7.2.10) that there is a canonical isomorphism $\varphi : \operatorname{Hom}_A(M, N) \to \varprojlim_i \operatorname{Hom}_{A_i}(M_i, N_i)$. In addition:

Proposition (7.8.2). — *If M and N are modules of finite type over a $\mathfrak{J}$-adic Noetherian ring A, then the submodules* $\operatorname{Hom}_A(M, \mathfrak{J}^{i+1}N)$ *form a fundamental system of neighborhoods of* 0 *in* $\operatorname{Hom}_A(M, N)$ *for the $\mathfrak{J}$-adic topology, and the canonical isomorphism* $\varphi : \operatorname{Hom}_A(M, N) \to \varprojlim_i \operatorname{Hom}_{A_i}(M_i, N_i)$ *is a topological isomorphism.*

Proof. We can consider M as the quotient of a free A-module L of finite type, and as a result identify $\operatorname{Hom}_A(M,N)$ as a submodule of $\operatorname{Hom}_A(L,N)$; in this identification, $\operatorname{Hom}_A(M,\mathfrak{J}^{i+1}N)$ is the intersection of $\operatorname{Hom}_A(M,N)$ and $\operatorname{Hom}_A(L,\mathfrak{J}^{i+1}N)$; as the induced topology on $\operatorname{Hom}_A(M,N)$ by the $\mathfrak{J}$-adic topology of $\operatorname{Hom}_A(L,N)$ is the $\mathfrak{J}$-adic (7.3.2), we have reduced to proving the first assertion for $M=L=A^m$; but then $\operatorname{Hom}_A(L,N)=N^m$, $\mathbf{0_I}$ | 78
$\operatorname{Hom}_A(L,\mathfrak{J}^{i+1}N)=(\mathfrak{J}^{i+1}N)^m=\mathfrak{J}^{i+1}N^m$ and the result is evident. To establish the second assertion, we note that the image of $\operatorname{Hom}_A(M,\mathfrak{J}^{i+1}N)$ in $\operatorname{Hom}_{A_j}(M_j,N_j)$ is zero for $j\leqslant i$, hence φ is continuous; conversely, the inverse image in $\operatorname{Hom}_A(M,N)$ of 0 of $\operatorname{Hom}_{A_i}(M_i,N_i)$ is $\operatorname{Hom}_A(M,\mathfrak{J}^{i+1}N)$, so φ is bicontinuous. □

If we only suppose that A is a *Noetherian $\mathfrak{J}$-preadic* ring, M and N two A-modules of finite type, separated for the $\mathfrak{J}$-preadic topology, then the following proof shows that the first assertion of Propositon (7.8.2) remains valid, and that φ is a topological isomorphism from $\operatorname{Hom}_A(M,N)$ to a submodule of $\varprojlim_i \operatorname{Hom}_{A_i}(M_i,N_i)$.

Proposition (7.8.3). — *Under the hypotheses of Proposition (7.8.2), the set of injective (resp. surjective, bijective) homomorphisms from M to N is an open subset of* $\operatorname{Hom}_A(M,N)$.

Proof. According to Corollaries (7.3.5) and (7.1.14), for u to be injective, it is necessary and sufficient that the corresponding homomorphism $u_0 : M/\mathfrak{J}M \to N/\mathfrak{J}N$ is, and the set of surjective homomorphisms from M to N is thus the inverse image under the continuous map $\operatorname{Hom}_A(M,N)\to\operatorname{Hom}_{A_0}(M_0,N_0)$ of a subset of a discrete space. We now show that the set of injective homomorphisms is open; let v be such a homomorphism and set $M'=v(M)$; by the Artin–Rees Lemma (7.3.2.1), there exists an integer $k\geqslant 0$ such that $M'\cap\mathfrak{J}^{m+k}N\subset\mathfrak{J}^mM'$ for all $m>0$; we will see that for all $w\in\mathfrak{J}^{k+1}\operatorname{Hom}_A(M,N)$, $u=v+w$ is injective, which will finish the proof. Indeed, let $x\in M$ be such that $u(x)=0$; we prove that for every $i\geqslant 0$ the relation $x\in\mathfrak{J}^iM$ implies that $x\in\mathfrak{J}^{i+1}M$; this follows from $x\in\bigcap_{i\geqslant 0}\mathfrak{J}^iM=(0)$. Indeed, we then have $w(x)\in\mathfrak{J}^{i+k+1}N$, and as a result $w(x)=-v(x)\in M'$, so $v(x)\in M'\cap\mathfrak{J}^{i+k+1}N\subset\mathfrak{J}^{i+1}M'$, and as v is an isomorphism from M to M', $x\in\mathfrak{J}^{i+1}M$; q.e.d. □

§8. Representable functors $\mathbf{0_{III}}$ | 5

8.1. Representable functors.

(8.1.1). We denote by $\mathtt{Set}$ the category of sets. Let $\mathtt{C}$ be a category; for two objects X, Y of $\mathtt{C}$, we set $h_X(Y)=\operatorname{Hom}(Y,X)$; for each morphism $u:Y\to Y'$ in $\mathtt{C}$, we denote by $h_X(u)$ the map $v\mapsto vu$ from $\operatorname{Hom}(Y',X)$ to $\operatorname{Hom}(Y,X)$. It is immediate that with these definitions, $h_X:\mathtt{C}\to\mathtt{Set}$ is a *contravariant functor*, i.e., an object of the category $\mathtt{Hom}(\mathtt{C}^{\mathrm{op}},\mathtt{Set})$, of covariant functors from the category $\mathtt{C}^{\mathrm{op}}$ (the dual of the category $\mathtt{C}$) to the category $\mathtt{Set}$ (T, 1.7, (d) and [**Car**]).

(8.1.2). Now let $w:X\to X'$ be a morphism in $\mathtt{C}$; for each $Y\in\mathtt{C}$ and each $v\in\operatorname{Hom}(Y,X)=h_X(Y)$, we have $wv\in\operatorname{Hom}(Y,X')=h_{X'}(Y)$; we denote by $h_w(Y)$ the map $v\mapsto wv$ from $h_X(Y)$ to $h_{X'}(Y)$. It is immediate that for each morphism $u:Y\to Y'$ in $\mathtt{C}$, the diagram

$$\begin{tikzcd} h_X(Y') \arrow[r, "h_X(u)"] \arrow[d, "h_w(Y')"'] & h_X(Y) \arrow[d, "h_w(Y)"] \\ h_{X'}(Y') \arrow[r, "h_{X'}(u)"] & h_{X'}(Y) \end{tikzcd}$$

is commutative; in other words, h_w is a *natural transformation (or functorial morphism)* $h_X\to h_{X'}$ (T, 1.2), also a morphism in the category $\mathtt{Hom}(\mathtt{C}^{\mathrm{op}},\mathtt{Set})$ (T, 1.7, (d)). The definitions of h_X and of h_w therefore constitute the definition of a *canonical covariant functor*

(8.1.2.1) $$h:\mathtt{C}\longrightarrow\mathtt{Hom}(\mathtt{C}^{\mathrm{op}},\mathtt{Set}),\quad X\longmapsto h_X.$$

(8.1.3). Let X be an object in $\mathtt{C}$, F a contravariant functor from $\mathtt{C}$ to $\mathtt{Set}$ (an object of $\mathtt{Hom}(\mathtt{C}^{\mathrm{op}},\mathtt{Set})$). Let $g:h_X\to F$ be a *natural transformation*: for all $Y\in\mathtt{C}$, $g(Y)$ is thus a map $h_X(Y)\to F(Y)$ such that for each $\mathbf{0_{III}}$ | 6
morphism $u:Y\to Y'$ in $\mathtt{C}$, the diagram

(8.1.3.1) $$\begin{tikzcd} h_X(Y') \arrow[r, "h_X(u)"] \arrow[d, "g(Y')"'] & h_X(Y) \arrow[d, "g(Y)"] \\ F(Y') \arrow[r, "F(u)"] & F(Y) \end{tikzcd}$$

is commutative. In particular, we have a map $g(X):h_X(X)=\operatorname{Hom}(X,X)\to F(X)$, hence an element

(8.1.3.2) $$\alpha(g)=(g(X))(1_X)\in F(X)$$

and as a result a canonical map

(8.1.3.3) $$\alpha : \mathrm{Hom}(h_X, F) \longrightarrow F(X).$$

Conversely, consider an element $\xi \in F(X)$; for each morphism $v : Y \to X$ in $\mathtt{C}$, $F(v)$ is a map $F(X) \to F(Y)$; consider the map

(8.1.3.4) $$v \longmapsto (F(v))(\xi)$$

from $h_X(Y)$ to $F(Y)$; if we denote by $(\beta(\xi))(Y)$ this map,

(8.1.3.5) $$\beta(\xi) : h_X \longrightarrow F$$

is a *natural transformation*, since for each morphism $u : Y \to Y'$ in $\mathtt{C}$ we have $(F(vu))(\xi) = (F(v) \circ F(u))(\xi)$, which makes (8.1.3.1) commutative for $g = \beta(\xi)$. We have thus defined a canonical map

(8.1.3.6) $$\beta : F(X) \longrightarrow \mathrm{Hom}(h_X, F).$$

Proposition (8.1.4). — *The maps α and β are the inverse bijections of each other.*

Proof. We calculate $\alpha(\beta(\xi))$ for $\xi \in F(X)$; for each $Y \in \mathtt{C}$, $(\beta(\xi))(Y)$ is a map $g_1(Y) : v \mapsto (F(v))(\xi)$ from $h_X(Y)$ to $F(Y)$. We thus have

$$\alpha(\beta(\xi)) = (g_1(X))(1_X) = (F(1_X))(\xi) = 1_{F(X)}(\xi) = \xi.$$

We now calculate $\beta(\alpha(g))$ for $g \in \mathrm{Hom}(h_X, F)$; for each $Y \in \mathtt{C}$, $(\beta(\alpha(g)))(Y)$ is the map $v \mapsto (F(v))((g(X))(1_X))$; according to the commutativity of (8.1.3.1), this map is none other than $v \mapsto (g(Y))((h_X(v))(1_X)) = (g(Y))(v)$ by definition of $h_X(v)$, in other words, it is equal to $g(Y)$, which finishes the proof. □

(8.1.5). Recall that a *subcategory* $\mathtt{C}'$ of a category $\mathtt{C}$ is defined by the condition that its objects are objects of $\mathtt{C}$, and that if X', Y' are two objects of $\mathtt{C}'$, then the set $\mathrm{Hom}_{\mathtt{C}'}(X', Y')$ of morphisms $X' \to Y'$ *in* $\mathtt{C}'$ is a subset of the set $\mathrm{Hom}_{\mathtt{C}}(X', Y')$ of morphisms $X' \to Y'$ *in* $\mathtt{C}$, the canonical map of "composition of morphisms"

$$\mathrm{Hom}_{\mathtt{C}'}(X', Y') \times \mathrm{Hom}_{\mathtt{C}'}(Y', Z') \longrightarrow \mathrm{Hom}_{\mathtt{C}'}(X', Z')$$

being the restriction of the canonical map **0_{III}** | 7

$$\mathrm{Hom}_{\mathtt{C}}(X', Y') \times \mathrm{Hom}_{\mathtt{C}}(Y', Z') \longrightarrow \mathrm{Hom}_{\mathtt{C}}(X', Z').$$

We say that $\mathtt{C}'$ is a *full* subcategory of $\mathtt{C}$ if $\mathrm{Hom}_{\mathtt{C}'}(X', Y') = \mathrm{Hom}_{\mathtt{C}}(X', Y')$ for every pair of objects in $\mathtt{C}'$. The subcategory $\mathtt{C}''$ of $\mathtt{C}$ consisting of the objects of $\mathtt{C}$ isomorphic to objects of $\mathtt{C}'$ is then again a full subcategory of $\mathtt{C}$, *equivalent* (T, 1.2) to $\mathtt{C}'$ as we verify easily.

A covariant functor $F : \mathtt{C}_1 \to \mathtt{C}_2$ is called *fully faithful* if for every pair of objects X_1, Y_1 of $\mathtt{C}_1$, the map $u \mapsto F(u)$ from $\mathrm{Hom}(X_1, Y_1)$ to $\mathrm{Hom}(F(X_1), F(Y_1))$ is *bijective*; this implies that the subcategory $F(\mathtt{C}_1)$ of $\mathtt{C}_2$ is *full*. In addition, if two objects X_1, X_1' have the same image X_2, then there exists a unique isomorphism $u : X_1 \to X_1'$ such that $F(u) = 1_{X_1}$. For each object X_2 of $F(\mathtt{C}_1)$, let $G(X_2)$ be one of the objects X_1 of $\mathtt{C}_1$ such that $F(X_1) = X_2$ (G is defined by means of the axiom of choice); for each morphism $v : X_2 \to Y_2$ in $F(\mathtt{C}_1)$, $G(v)$ will be the unique morphism $u : G(X_2) \to G(Y_2)$ such that $F(u) = v$; G is then a *functor* from $F(\mathtt{C}_1)$ to $\mathtt{C}_1$; FG is the identity functor on $F(\mathtt{C}_1)$, and the above shows that there exists an isomorphism of functors $\varphi : 1_{\mathtt{C}_1} \to GF$ such that F, G, φ, and the identity $1_{F(\mathtt{C}_1)} \to FG$ defines an *equivalence* between the category $\mathtt{C}_1$ and the full subcategory $F(\mathtt{C}_1)$ of $\mathtt{C}_2$ (T, 1.2).

(8.1.6). We apply Proposition (8.1.4) to the case where F is $h_{X'}$, X' being any object of $\mathtt{C}$; the map β : $\mathrm{Hom}(X, X') \to \mathrm{Hom}(h_X, h_{X'})$ is none other than the map $w \mapsto h_w$ defined in (8.1.2); this map being *bijective*, we see with the terminology of (8.1.5) that:

Proposition (8.1.7). — *The canonical functor $h : \mathtt{C} \to \mathtt{Hom}(\mathtt{C}^{\mathrm{op}}, \mathtt{Set})$ is fully faithful.*

(8.1.8). Let F be a contravariant functor from $\mathtt{C}$ to $\mathtt{Set}$; we say that F is *representable* if there exists an object $X \in \mathtt{C}$ such that F is *isomorphic* to h_X; it follows from Proposition (8.1.7) that the data of an $X \in \mathtt{C}$ and an isomorphism of functors $g : h_X \to F$ determines X up to unique isomorphism. Proposition (8.1.7) then implies that h defines an *equivalence* between $\mathtt{C}$ and the full subcategory of $\mathtt{Hom}(\mathtt{C}^{\mathrm{op}}, \mathtt{Set})$ consisting of the *contravariant representable functors*. It follows from Proposition (8.1.4) that the data of a natural transformation $g : h_X \to F$ is equivalent to that of an element $\xi \in F(X)$; to say that g is an *isomorphism* is equivalent to the following condition on ξ: *for every object Y of $\mathtt{C}$ the map $v \mapsto (F(v))(\xi)$ from* $\mathrm{Hom}(Y, X)$ *to* $F(Y)$ *is bijective.* When ξ satsifies this condition, we say that the pair (X, ξ) *represents* the representable functor F, By abuse of language, we also say that the object $X \in \mathtt{C}$ represents F if there exists a $\xi \in F(X)$ such that (X, ξ) represents F, in other words if h_X is isomorphic to F.

Let F, F' be two contravariant representable functors from $\mathtt{C}$ to $\mathtt{Set}$, $h_X \to F$ and $h_{X'} \to F'$ two isomorphisms of functors. Then it follows from (8.1.6) that there is a canonical bijective correspondence between $\operatorname{Hom}(X,X')$ and the set $\operatorname{Hom}(F,F')$ of natural transformations $F \to F'$.

(8.1.9). *Example I. Projective limits.* The notion of a contravariant representable functor covers in particular the "dual" notion of the usual notion of a "solution to a universal problem". More generally, we will see that the **0_{III} | 8** notion of the *projective limit* is a special case of the notion of a representable functor. Recall that in a category $\mathtt{C}$, we define a *projective system* by the data of a preordered set I, a family $(A_\alpha)_{\alpha\in I}$ of objects of $\mathtt{C}$, and for every pair of indices (α,β) such that $\alpha \leqslant \beta$, a morphism $u_{\alpha\beta} : A_\beta \to A_\alpha$. A *projective limit* of this system in $\mathtt{C}$ consists of an object B of $\mathtt{C}$ (denoted $\varprojlim A_\alpha$), and for each $\alpha \in I$, a morphism $u_\alpha : B \to A_\alpha$ such that: 1st. $u_\alpha = u_{\alpha\beta}u_\beta$ for $\alpha \leqslant \beta$; 2nd. for every object X of $\mathtt{C}$ and every family $(v_\alpha)_{\alpha\in I}$ of morphisms $v_\alpha : X \to A_\alpha$ such that $v_\alpha = u_{\alpha\beta}v_\beta$ for $\alpha \leqslant \beta$, there exists a unique morphism $v : X \to B$ (denoted $\varprojlim v_\alpha$) such that $v_\alpha = u_\alpha v$ for all $\alpha \in I$ (T, 1.8). This can be interpreted in the following way: the $u_{\alpha\beta}$ canonically define maps

$$\overline{u}_{\alpha\beta} : \operatorname{Hom}(X,A_\beta) \longrightarrow \operatorname{Hom}(X,A_\alpha)$$

which define a *projective system* of sets $(\operatorname{Hom}(X,A_\alpha),\overline{u}_{\alpha\beta})$, and (v_α) is by definition an element of the set $\varprojlim \operatorname{Hom}(X,A_\alpha)$; it is clear that $X \mapsto \varprojlim \operatorname{Hom}(X,A_\alpha)$ is a *contravariant functor* from $\mathtt{C}$ to $\mathtt{Set}$, and the existence of the projective limit B is equivalent to saying that $(v_\alpha) \mapsto \varprojlim v_\alpha$ is an *isomorphism* of functors in X

$$\varprojlim \operatorname{Hom}(X,A_\alpha) \simeq \operatorname{Hom}(X,B), \tag{8.1.9.1}$$

in other words, that the functor $X \mapsto \varprojlim \operatorname{Hom}(X,A_\alpha)$ is *representable*.

(8.1.10). *Example II. Final objects.* Let $\mathtt{C}$ be a category, $\{a\}$ a singleton set. Consider the contravariant functor $F : \mathtt{C} \to \mathtt{Set}$ which sends every object X of $\mathtt{C}$ to the set $\{a\}$, and every morphism $X \to X'$ in $\mathtt{C}$ to the unique map $\{a\} \to \{a\}$. To say that this functor is *representable* means that there exists an object $e \in \mathtt{C}$ such that for every $Y \in \mathtt{C}$, $\operatorname{Hom}(Y,e) = h_e(Y)$ is a *singleton set*; we say that e is an *final object* of $\mathtt{C}$, and it is clear that two final objects of $\mathtt{C}$ are isomorphic (which allows us to define, in general with the axiom of choice, *one* final object of $\mathtt{C}$ which we then denote $e_{\mathtt{C}}$). For example, in the category $\mathtt{Set}$, the final objects are the singleton sets; in the category of *augmented algebras* over a field K (where the morphisms are the algebra homomorphisms compatible with the augmentation), K is a final object; in the category of *S-preschemes* (**I**, 2.5.1), S is a final object.

(8.1.11). For two objects X and Y of a category $\mathtt{C}$, set $h'_X(Y) = \operatorname{Hom}(X,Y)$, and for every morphism $u : Y \to Y'$, let $h_X(u)$ be the map $v \mapsto vu$ from $\operatorname{Hom}(X,Y)$ to $\operatorname{Hom}(X,Y')$; h'_X is then a *covariant functor* $\mathtt{C} \to \mathtt{Set}$, so we deduce as in (8.1.2) the definition of a canonical covariant functor $h' : \mathtt{C}^{\mathrm{op}} \to \mathtt{Hom}(\mathtt{C},\mathtt{Set})$; a *covariant* functor F from $\mathtt{C}$ to $\mathtt{Set}$, in other words an object of $\mathtt{Hom}(\mathtt{C},\mathtt{Set})$, is then *representable* if there exists an object $X \in \mathtt{C}$ (necessarily unique up to unique isomorphism) such that F is *isomorphic to* h'_X; we leave it to the reader to develop the "dual" notions of the above, which this time cover the notion of an *inductive limit*, and in particular the usual notion of a "solution to a universal problem".

8.2. Algebraic structures in categories.

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(8.2.1). Given two contravariant functors F and F' from $\mathtt{C}$ to $\mathtt{Set}$, recall that for every object $Y \in \mathtt{C}$, we set $(F \times F')(Y) = F(Y) \times F'(Y)$, and for every morphism $u : Y \to Y'$ in $\mathtt{C}$, we set $(F \times F')(u) = F(u) \times F'(u)$, which is the map $(t,t') \mapsto (F(u)(t), F'(u)(t'))$ from $F(Y') \times F'(Y')$ to $F(Y) \times F'(Y)$; $F \times F' : \mathtt{C} \to \mathtt{Set}$ is thus a *contravariant functor* (which is none other than the *product* of the objects F and F' in the category $\mathtt{Hom}(\mathtt{C}^{\mathrm{op}},\mathtt{Set})$). Given an object $X \in \mathtt{C}$, we call an *internal composition law* on X a *natural transformation*

$$\gamma_X : h_X \times h_X \longrightarrow h_X. \tag{8.2.1.1}$$

In other words (T, 1.2), for every object $Y \in \mathtt{C}$, $\gamma_X(Y)$ is a map $h_X(Y) \times h_X(Y) \to h_X(Y)$ (thus by definition an *internal composition law* on the set $h_X(Y)$) with the condition that for every morphism $u : Y \to Y'$ in $\mathtt{C}$, the diagram

$$\begin{tikzcd}
h_X(Y') \times h_X(Y') \arrow[r, "h_X(u)\times h_X(u)"] \arrow[d, "\gamma_X(Y')"'] & h_X(Y) \times h_X(Y) \arrow[d, "\gamma_X(Y)"] \\
h_X(Y') \arrow[r, "h_X(u)"] & h_X(Y)
\end{tikzcd}$$

is commutative; this implies that for the composition laws $\gamma_X(Y)$ and $\gamma_X(Y')$, $h_X(u)$ is a *homomorphism* from $h_X(Y')$ to $h_X(Y)$.

In a similar way, given two objects Z and X of C, we call an *external composition law on X, with Z as its domain of operators* a natural transformation

$$\omega_{X,Z} : h_Z \times h_X \longrightarrow h_X. \tag{8.2.1.2}$$

We see as above that for every $Y \in \mathsf{C}$, $\omega_{X,Z}(Y)$ is an external composition law on $h_X(Y)$, with $h_Z(Y)$ as its domain of operators and such that for every morphism $u : Y \to Y'$, $h_X(u)$ and $h_Z(u)$ form a *di-homomorphism* from $(h_Z(Y'), h_X(Y'))$ to $(h_Z(Y), h_X(Y))$.

(8.2.2). Let X' be a second object of C, and suppose we are given an internal composition law $\gamma_{X'}$ on X'; we say that a morphism $w : X \to X'$ in C is a *homomorphism* for the composition laws if for every $Y \in \mathsf{C}$, $h_w(Y) : h_X(Y) \to h_{X'}(Y)$ is a *homomorphism* for the composition laws $\gamma_X(Y)$ and $\gamma_{X'}(Y)$. If X'' is a third object of C $\mathbf{0_{III}}$ | 10
equipped with an internal composition law $\gamma_{X''}$ and $w' : X' \to X''$ is a morphism in C which is a homomorphism for $\gamma_{X'}$ and $\gamma_{X''}$, then it is clear that the morphism $w'w : X \to X''$ is a homomorphism for the composition laws γ_X and $\gamma_{X''}$. An isomorphism $w : X \simeq X'$ in C is called an *isomorphism for the composition laws γ_X and $\gamma_{X'}$* if w is a homomorphism for these composition laws, and if its inverse morphism w^{-1} is a homomorphism for the composition laws $\gamma_{X'}$ and γ_X.

We define in a similar way the *di-homomorphisms* for pairs of objects of C equipped with external composition laws.

(8.2.3). When an internal composition law γ_X on an object $X \in \mathsf{C}$ is such that $\gamma_X(Y)$ is a *group law* on $h_X(Y)$ for *every* $Y \in \mathsf{C}$, we say that X, equipped with this law, is a C-*group* or a *group object in* C. We similarly define C-*rings*, C-*modules*, etc.

(8.2.4). Suppose that the *product* $X \times X$ of an object $X \in \mathsf{C}$ by itself exists in C; by definition, we then have $h_{X\times X} = h_X \times h_X$ up to canonical isomorphism, since it is a particular case of the projective limit (8.1.9); an internal composition law on X can thus be considered as a functorial morphism $\gamma_X : h_{X\times X} \to h_X$, and thus canonically determine (8.1.6) an element $c_X \in \operatorname{Hom}(X \times X, X)$ such that $h_{c_X} = \gamma_X$; in this case, the data of an internal composition law on X is equivalent to the data of a morphism $X \times X \to X$; when C is the category $\mathtt{Set}$, we recover the classical notion of an internal composition law on a set. We have an analogous result for an external composition law when the product $Z \times X$ exists in C.

(8.2.5). With the above notation, suppose that in addition $X \times X \times X$ exists in C; the characterization of the product as an object representing a functor (8.1.9) implies the existence of canonical isomorphisms

$$(X \times X) \times X \simeq X \times X \times X \simeq X \times (X \times X);$$

if we canonically identity $X \times X \times X$ with $(X \times X) \times X$, then the map $\gamma_X(Y) \times 1_{h_X(Y)}$ identifies with $h_{c_X \times 1_X}(Y)$ for all $Y \in \mathsf{C}$. As a result, it is equivalent to say that for every $Y \in \mathsf{C}$, the internal law $\gamma_X(Y)$ is associative, or that the diagram of maps

$$\begin{tikzcd}
h_X(Y) \times h_X(Y) \times h_X(Y) \arrow[r, "\gamma_X(Y)\times 1"] \arrow[d, "1\times\gamma_X(Y)"'] & h_X(Y) \times h_X(Y) \arrow[d, "\gamma_X(Y)"] \\
h_X(Y) \times h_X(Y) \arrow[r, "\gamma_X(Y)"] & h_X(Y)
\end{tikzcd}$$

is commutative, or that the diagram of morphisms $\mathbf{0_{III}}$ | 11

$$\begin{tikzcd}
X \times X \times X \arrow[r, "c_X \times 1_X"] \arrow[d, "1_X \times c_X"'] & X \times X \arrow[d, "c_X"] \\
X \times X \arrow[r, "c_X"] & X
\end{tikzcd}$$

is commutative.

(8.2.6). Under the hypotheses of (8.2.5), if we want to express, for every $Y \in \mathsf{C}$, the internal law $\gamma_X(Y)$ as a *group law*, then it is first necessary that it is associative, and second that there exists a map $\alpha_X(Y) : h_X(Y) \to h_X(Y)$ having the properties of the *inverse* operation of a group; as for every morphism $u : Y \to Y'$ in C, we have seen that $h_X(u)$ must be a group homomorphism $h_X(Y') \to h_X(Y)$, we first see that $\alpha_X : h_X \to h_X$ must be a *natural transformation.* On the other hand, one can express the chacteristic properties of the inverse $s \mapsto s^{-1}$ of a group G without involving the identity element: it suffices to check that the two composite maps

$$(s,t) \longmapsto (s,s^{-1},t) \longmapsto (s,s^{-1}t) \longmapsto s(s^{-1}t),$$

$$(s,t) \longmapsto (s,s^{-1},t) \longmapsto (s,ts^{-1}) \longmapsto (ts^{-1})s$$

are equal to the second projection $(s,t) \mapsto t$ from $G \times G$ to G. By (8.1.3), we have $\alpha_X = h_{a_X}$, where $a_X \in \mathrm{Hom}(X,X)$; the first condition above then expresses that the composite morphism

$$X \times X \xrightarrow{(1_X,a_X)\times 1_X} X \times X \times X \xrightarrow{1_X \times c_X} X \times X \xrightarrow{c_X} X$$

is the second projection $X \times X \to X$ in C, and the second condition is similar.

(8.2.7). Now suppose that there exists a *final object* e (8.1.10) in C. Let us always assume that $\gamma_X(Y)$ is a group law on $h_X(Y)$ for every $Y \in \mathsf{C}$, and denote by $\eta_X(Y)$ the identity element of $\gamma_X(Y)$. As, for every morphism $u : Y \to Y'$ in C, $h_X(u)$ is a group homomorphism, we have $\eta_X(Y) = (h_X(u))(\eta_X(Y'))$; taking in particular $Y' = e$, in which case u is the unique element ε of $\mathrm{Hom}(Y,e)$, we see that the element $\eta_X(e)$ completely determines $\eta_X(Y)$ for every $Y \in \mathsf{C}$. Set $e_X = \eta_X(X)$, the identity element of the group $h_X(X) = \mathrm{Hom}(X,X)$; the commutativity of the diagram

$$\begin{tikzcd} h_X(e) \arrow[r, "h_X(\varepsilon)"] \arrow[d, "h_{e_X}(e)"'] & h_X(Y) \arrow[d, "h_{e_X}(Y)"] \\ h_X(e) \arrow[r, "h_X(\varepsilon)"] & h_X(Y) \end{tikzcd}$$

(cf. (8.1.2)) shows that, on the set $h_X(Y)$, the map $h_{e_X}(Y)$ is none other than $s \mapsto \eta_X(Y)$ sending every element **0**$_{\mathbf{III}}$ | 12
to the identity element. We then verify that the fact that $\eta_X(Y)$ is the identity element of $\gamma_X(Y)$ for every $Y \in \mathsf{C}$ is equivalent to saying that the composite morphism

$$X \xrightarrow{(1_X,1_X)} X \times X \xrightarrow{1_X \times e_X} X \times X \xrightarrow{c_X} X,$$

and the analog in which we swap 1_X and e_X, are both *equal* to 1_X.

(8.2.8). One could of course easily extend the examples of algebraic structures in categories. The example of groups was treated with enough detail, but latter on we will usually leave it to the reader to develop analogous notions for the examples of algebraic structures we will encounter.

§9. Constructible sets

9.1. Constructible sets.

Definition (9.1.1). — We say that a continuous map $f : X \to Y$ is *quasi-compact* if for every quasi-compact open subset U of Y, $f^{-1}(U)$ is quasi-compact. We say that a subset Z of a topological space X is *retrocompact* in X if the canonical injection $Z \to X$ is quasi-compact, in other words, if for every quasi-compact open subset U of X, $U \cap Z$ is quasi-compact.

A *closed* subset of X is retrocompact in X, but a quasi-compact subset of X is not necessarily retrocompact in X. If X is quasi-compact, every retrocompact open subset of X is quasi-compact. It is clear that every *finite* union of retrocompact sets in X is retrocompact in X, as every finite union of quasi-compact sets is quasi-compact. Every finite intersection of retrocompact open sets in X is a retrocompact open set in X. In a *locally Noetherian* space X, every quasi-compact set is a Noetherian subspace, and as a result *every subset* of X is retrocompact in X.

Definition (9.1.2). — Given a topological space X, we say that a subset of X is *constructible* if it belongs to the smallest set of subsets $\mathfrak{F}$ of X containing all the retrocompact open subsets of $\mathfrak{F}$ and is stable under finite intersections and complements (which implies that $\mathfrak{F}$ is also stable under finite unions).

Proposition (9.1.3). — *For a subset of X to be constructible, it is necessary and sufficient for it to be a finite union of sets of the form $U \cap \complement V$, where U and V are retrocompact open sets in X.*

Proof. It is clear that the condition is sufficient. To see that it is necessary, consider the set $\mathfrak{G}$ of finite unions of sets of the form $U \cap \complement V$, where U and V are retrocompact open sets in X; it suffices to see that every complement of a set in $\mathfrak{G}$ is in $\mathfrak{G}$. Let $Z = \bigcup_{i\in I}(U_i \cap \complement V_i)$, where I is finite, U_i and V_i retrocompact open sets in X; we have $\complement Z = \bigcap_{i\in I}(V_i \cup \complement U_i)$, so Z is a finite union of sets which are intersections of a certain number of the V_i and of a certain number of the $\complement U_i$, thus of the form $V \cap \complement U$, where U is the union of a certain number of $\mathbf{0_{III}}$ | 13
the U_i and V is the intersection of a certain number of the V_i; but we have noted above that finite unions and intersections of retrocompact open sets in X are retrocompact open sets in X, hence the conclusion. □

Corollary (9.1.4). — *Every constructible subset of X is retrocompact in X.*

Proof. It suffices to show that if U and V are retrocompact open sets in X, then $U \cap \complement V$ is retrocompact in X; if W is a quasi-compact open set in X, then $W \cap U \cap \complement V$ is closed in the quasi-compact space $W \cap U$, hence it is quasi-compact. □

In particular:

Corollary (9.1.5). — *For an open subset U of X to be constructible, it is necessary and sufficient for it to be retrocompact in X. For a closed subset F of X to be constructible, it is necessary and sufficient for the open set $\complement F$ to be retrocompact.*

(9.1.6). An important case is when every quasi-compact open subset of X is retrocompact, in other words, when the intersection of two quasi-compact open subsets of X is quasi-compact (cf. (**I**, 5.5.6)). When X is also quasi-compact, this implies that the retrocompact open subsets of X are identical to the quasi-compact open subsets of X, and the constructible subsets of X are finite unions of sets of the form $U \cap \complement V$, where U and V are quasi-compact open sets.

Corollary (9.1.7). — *For a subset of a Noetherian space to be constructible, it is necessary and sufficient for it to be a finite union of locally closed subsets of X.*

Proposition (9.1.8). — *Let X be a topological space, U an open subset of X.*

(i) *If T is a constructible subset of X, then $T \cap U$ is a constructible subset of U.*
(ii) *In addition, suppose that U is retrocompact in X. For a subset Z of U to be constructible in X, it is necessary and sufficient for it to be constructible in U.*

Proof.

(i) Using Proposition (9.1.3), we reduce to showing that if T is a retrocompact open set in X, then $T \cap U$ is a retrocompact open set in U, in other words, for every quasi-compact open $W \subset U$, $T \cap U \cap W = T \cap W$ is quasi-compact, which immediately follows from the hypothesis.
(ii) The condition is necessary by (i), so it remains to show that it is sufficient. By Proposition (9.1.3), it suffices to consider the case where Z is a retrocompact open set *in* U, because it will then follow that $U - Z$ is constructible in X, and if Z and Z' are two retrocompact opens *in* U, then $Z \cap (U - Z')$ will be constructible in X. If W is a quasi-compact open set in X and Z a retrocompact open set in U, then we have $Z \cap W = Z \cap (W \cap U)$, and by hypothesis $W \cap U$ is a quasi-compact open set in U; so $W \cap Z$ is quasi-compact, and as a result Z is a retrocompact open set in X, and *a fortiori* constructible in X.

□

Corollary (9.1.9). — *Let X be a topological space, $(U_i)_{i\in I}$ a finite cover of X by retrocompact open sets in X. For a subset Z of X to be constructible in X, it is necessary and sufficient for $Z \cap U_i$ to be constructible in U_i for all $i \in I$.*

(9.1.10). In particular, suppose that X is quasi-compact and every point of X admits a fundamental system of $\mathbf{0_{III}}$ | 14
retrocompact open neighborhoods in X (and *a fortiori* quasi-compact); then the condition for a subset Z of X to be constructible in X is of a *local* nature, in other words, it is necessary and sufficient that for evert $x \in X$, there exists an open neighborhood V of x such that $V \cap Z$ is constructible in V. Indeed, if this condition is satisfied, then there exists for every $x \in X$ an open neighborhood V of x which is *retrocompact in* X and such that $V \cap Z$ is constructible in V, by the hypotheses on X and by Proposition (9.1.8, i); it then suffices to cover X by a finite number of these neighborhoods and to apply Corollary (9.1.9).

Definition (9.1.11). — Let X be a topological space. We say that a subset T of X is *locally constructible* in X if for every $x \in X$ there exists an open neighborhood V of x such that $T \cap V$ is constructible in V.

It follows from Proposition (9.1.8, i) that if V is such that $V \cap T$ is constructible in V, then for every open $W \subset V$, $W \cap T$ is constructible in W. If T is locally constructible in X, then for every open set U in X, $T \cap U$ is locally constructible in U, as a result of the above remark. The same remark shows that the set of locally constructible subsets of X is stable under finite unions and finite intersections; on ther other hand, it is clear that it is also stabble under taking complements.

Proposition (9.1.12). — *Let X be a topological space. Every constructible set in X is locally constructible in X. The converse is true if X is quasi-compact and if its topology admits a basis formed by the retrocompact sets in X.*

Proof. The first assertion follows from Definition (9.1.11) and the second from (9.1.10). □

Corollary (9.1.13). — *Let X be a topological space whose topology admits a basis formed by the retrocompact sets in X. Then every locally constructible subset T of X is retrocompact in X.*

Proof. Let U be a quasi-compact open set in X; $T \cap U$ is locally constructible in U, hence constructible in U by Proposition (9.1.12), and as a result quasi-compact by Corollary (9.1.4). □

9.2. Constructible subsets of Noetherian spaces.

(9.2.1). We have seen (9.1.7) that in a Noetherian space X, the constructible subsets of X are the *finite unions of locally closed subsets of X.*

The inverse image of a constructible set in X by a continuous map from a Noetherian space X' to X is constructible in X'. If Y is a constructible subset of a Noetherian space X, then the subsets of Y are constructible as subspaces of Y and are identical to those which are constructible as subspaces of X.

Proposition (9.2.2). — *Let X be an irreducible Noetherian space, E a constructible subset of X. For E to be everywhere dense in X, it is necessary and sufficient for E to contain a nonempty open subset of X.*

Proof. The condition is evidently sufficient, as every nonempty open set is dense in X. Conversely, let $E = \bigcup_{i=1}^{n}(U_i \cap F_i)$ be a constructible subset of X, the U_i being nonempty open sets and the F_i closed in X; we then have $\overline{E} \subset \bigcup_i F_i$. As a result, if $\overline{E} = X$, then X is equal to one of the F_i, hence $E \supset U_i$, which finishes the proof. □

When X admits a generic point x (**0**, 2.1.2), the condition of Proposition (9.2.2) is equivalent to the relation $x \in E$.

Proposition (9.2.3). — *Let X be a Noetherian space. For a subset E of X to be constructible, it is necessary and sufficient that, for every irreducible closed subset Y of X, $E \cap Y$ is rare in Y or contains a nonempty open subset of Y.*

Proof. The necessity of the condition follows from the fact that $E \cap Y$ must be a constructible subset of Y and from Proposition (9.2.2), since a nondense subset of Y is necessarily rare in the irreducible space Y](**0**,)0.2.1.1. To prove that the condition is sufficient, apply the principle of Noetherian induction (**0**, 2.2.2) to the set $\mathfrak{F}$ of closed subsets Y of X such that $Y \cap E$ is constructible (relative to Y or relative to X, which are equivalent): we can thus assume that for every closed subset $Y \neq X$ of X, $E \cap Y$ is constructible. First suppose that X is not irreducible, and let X_i $(1 \leqslant i \leqslant m)$ are its irreducible components, necessarily of finite number (**0**, 2.2.5); by hypothesis the $E \cap X_i$ are constructible, hence their union E is as well. Suppose now that X is irreducible; then by hypothesis, if E is rare, then $\overline{E} \neq X$ and $E = E \cap \overline{E}$ is constructible; if E contains a nonempty open subset U of X, then it is the union of U and $E \cap (X - U)$; but $X - U$ is a closed set distinct from X, so $E \cap (X - U)$ is constructible; as a result, E is itself constructible, which finishes the proof. □

Corollary (9.2.4). — *Let X be a Noetherian space, (E_α) an increasing filtered family of constructible subsets of X, such that*

(1st) *X is the union of the family (E_α).*

(2nd) *Every irreducible closed subset of X is contained in the closure of one of the E_α.*

Then there exists an index α such that $X = E_\alpha$.

When every irreducible closed subset of X admits a generic point, the hypothesis (1st) *can be omitted.*

Proof. We apply the principle of Noetherian induction (**0**, 2.2.2) to the set $\mathfrak{M}$ of closed subsets of X contained in at least one of the E_α; we can thus suppose that every closed subset $Y \neq X$ of X is contained in one of the E_α. The proposition is evident if X is not irreducible, because each of the irreducible components X_i of X $(1 \leqslant i \leqslant m)$ is contained in an E_{α_i}, and there exists an E_α containing all of the E_{α_i}. Now suppose that X is irreducible. By hypothesis, there exists a β such that $X = \overline{E_\beta}$, so (9.2.2) E_β contains a nonempty open subset U of X. But then the closed set $X - U$ is contained in an E_γ, and it suffices to take an E_α containing E_β and E_γ. When every irreducible closed subset Y of X admits a generic point y, there exists α such that $y \in E_\alpha$, so $Y = \overline{\{y\}} \subset \overline{E_\alpha}$, and **$0_{\mathrm{III}}$** | 16
condition (2nd) is therefore a consequence of (1st). □

Proposition (9.2.5). — *Let X be a Noetherian space, x a point of X, and E a constructible subset of X. For E to be a neighborhood of x, it is necessary and sufficient that for every irreducible closed subset Y of X containing x, $E \cap Y$ is dense in Y (if there exists a generic point y of Y, this also implies (9.2.2) that $y \in E$).*

Proof. The condition is evidently necessary; we will prove that it is sufficient. Applying the principle of Noetherian induction to the set $\mathfrak{M}$ of closed subsets Y of X containing x and such that $E \cap Y$ is a neighborhood of x in Y, we can assume that every closed subset $Y \neq X$ of X containing x belongs to $\mathfrak{M}$. If X is not irreducible, then each of the irreducible components X_i of X containing x are distinct from X, hence $E \cap X_i$ is a neighborhood of x with respect to X_i; as a result, E is a neighborhood of x in the union of the irreducible components of X containing x, and as this union is a neighborhood of x in X, so is E. If X is irreducible, then E is dense in X by hypothesis, so it contains a nonempty open subset U of X (9.2.2); the proposition is then evident if $x \in U$; otherwise, x is by hypothesis inside $E \cap (X - U)$ with respect to $X - U$, so the closure of $X - E$ in X does not contain x, and the complement of this closure is a neighborhood of x contained in E, which finishes the proof. □

Corollary (9.2.6). — *Let X be a Noetherian space, E a subset of X. For E to be an open set in X, it is necessary and sufficient that for every irreducible closed subset Y of X intersecting E, $E \cap Y$ contains a nonempty open subset of Y.*

Proof. The condition is evidently necessary; conversely, if it is satisfied, then it implies that E is constructible by Proposition (9.2.3). In addition, Proposition (9.2.5) shows that E is then a neighborhood of each of its points, hence the conclusion. □

9.3. Constructible functions.

Definition (9.3.1). — Let h be a map from a topological spacee X to a set T. We say that h is *constructible* if $h^{-1}(t)$ is constructible for every $t \in T$, and empty except for finitely many values of t; then for every subset S of T, $h^{-1}(S)$ is constructible. We say that h is *locally constructible* if every $x \in X$ has an open neighborhood V such that $h|V$ is constructible.

Every constructible function is locally constructible; the converse is true when X is quasi-compact and admits a basis formed by the retrocompact open sets in X (in particular, when X is Noetherian).

Proposition (9.3.2). — *Let h be a map from a Noetherian space X to a set T. For h to be constructible, it is necessary and sufficient that for every irreducible closed subset Y of X, there exists a nonempty subset U of Y, open relative to Y, in which h is constant.*

Proof. The condition is necessary: indeed, by hypothesis, h does not take finitely many values t_i on Y, and each of the sets $h^{-1}(t_i) \cap Y$ is constructible in Y (9.2.1); as they can not all be rare subsets of the space Y, at least one of them contains a nonempty open set (9.2.3). **0**$_{\mathbf{III}}$ | 17

To see that the condition is sufficient, we apply the principle of Noetherian induction on the set $\mathfrak{M}$ of closed subsets Y of X such that the restriction $h|Y$ is constructible; we can thus assume that for every closed subset $Y \neq X$ of X, $h|Y$ is constructible. If X is not irreducible, then the restriction of h to each of the (finitely many) irreducible components X_i of X is constructible, and it then follows immediately from Definition (9.3.1) that h is constructible. If X is irreducible, then there exists by hypothesis a nonempty open subset U of X on which h is constant; on the other hand, the restriction of h to $X - U$ is constructible by hypothesis, and it follows immediately that h is constructible. □

Corollary (9.3.3). — *Let X be a Noetherian space in which every irreducible closed subset admits a generic point. If h is a map from X to a set T such that, for every $t \in T$, $h^{-1}(t)$ is constructible, then h is constructible.*

Proof. If Y is an irreducible closed subset of X and y its generic point, then $Y \cap h^{-1}(h(y))$ is constructible and contains y, hence (9.2.2) this set contains a nonempty open subset of Y, and it suffices to apply Proposition (9.3.2). □

Proposition (9.3.4). — *Let X be a Noetherian space in which every irreducible closed subset admits a generic point, h a constructible map from X to an ordered set. For h to be upper semi-continuous on X, it is necessary and sufficient that for every $x \in X$ and every specialization (**0**, 2.1.2) x' of x, we have $h(x') \leqslant h(x)$.*

Proof. The function h does not take a finite number of values; therefore, to say that it is upper semi-continuous means that for every $x \in X$, the set E of the $y \in X$ such that $h(y) \leqslant h(x)$ is a neighborhood of x. By hypothesis, E is a constructible subset of X; on the other hand,to say that an irreducible closed subset Y of X contains x means that its generic point y is a specialization of x; the conclusion then follows from Proposition (9.2.5). □

§10. Supplement on flat modules

For any proofs missing in (10.1) and (10.2), we refer the reader to Bourbaki, *Alg. comm.*, chap. II and III.

10.1. Relations between flat modules and free modules.

(10.1.1). Let A be a ring, $\mathfrak{J}$ an ideal of A, and M an A-module; for every integer $p \geqslant 0$, we have a canonical homomorphism of $(A/\mathfrak{J})$-modules

(10.1.1.1) $$\varphi_p : (M/\mathfrak{J}M) \otimes_{A/\mathfrak{J}} (\mathfrak{J}^p/\mathfrak{J}^{p+1}) \longrightarrow \mathfrak{J}^p M/\mathfrak{J}^{p+1}M,$$

which is evidently *surjective*. We denote by $\mathrm{gr}(A) = \oplus_{p\geqslant 0}\mathfrak{J}^p/\mathfrak{J}^{p+1}$ the graded ring associated to A filtered by the $\mathfrak{J}^p$, and by $\mathrm{gr}(M) = \oplus_{p\geqslant 0}\mathfrak{J}^p M/\mathfrak{J}^{p+1}M$ the graded $\mathrm{gr}(A)$-module associated to M filtered by the $\mathfrak{J}^p M$; we then have $\mathrm{gr}_p(A) = \mathfrak{J}^p/\mathfrak{J}^{p+1}$, and $\mathrm{gr}_p(M) = \mathfrak{J}^p M/\mathfrak{J}^{p+1}M$; the φ define a *surjective* homomorphism of graded $\mathrm{gr}(A)$-modules

(10.1.1.2) $$\varphi : \mathrm{gr}_0(M) \otimes_{\mathrm{gr}_0(A)} \mathrm{gr}(A) \longrightarrow \mathrm{gr}(M).$$

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(10.1.2). Suppose that *one* of the following hypotheses is satisfied:

(i) $\mathfrak{J}$ is nilpotent;
(ii) A is Noetherian, $\mathfrak{J}$ is contained in the radical of A, and M is of finite type.

Then the following properties are equivalent.

(a) M is a free A-module.
(b) $M/\mathfrak{J}M = N \otimes_A (A/\mathfrak{J})$ is a free $(A/\mathfrak{J})$-module, and $\mathrm{Tor}_1^A(M, A/\mathfrak{J}) = 0$.
(c) $M/\mathfrak{J}M$ is a free $(A/\mathfrak{J})$-module, and the canonical homomorphism (10.1.1.2) is injective (and thus bijective).

(10.1.3). Suppose that $A/\mathfrak{J}$ is a *field* (in other words, that $\mathfrak{J}$ is maximal), and that one of the hypotheses, (i) and (ii), of (10.1.2) is satisfied. Then the following properties are equivalent.

(a) M is a free A-module.
(b) M is a projective A-module.
(c) M is a flat A-module.
(d) $\mathrm{Tor}_1^A(M, A/\mathfrak{J}) = 0$.
(e) The canonical homomorphism (10.1.1.2) is bijective.

This result can be applied, in particular, to the following two cases:

(i) M is an *arbitrary* module, over a local ring A whose maximal ideal $\mathfrak{J}$ is *nilpotent* (for example, a local Artinian ring);
(ii) M is a module *of finite type* over a *local Noetherian* ring.

10.2. Local flatness criteria.

(10.2.1). With the hypotheses and notation of (10.1.1), consider the following conditions.

(a) M is a flat A-module.
(b) $M/\mathfrak{J}M$ is a flat $(A/\mathfrak{J})$-module, and $\mathrm{Tor}_1^A(M, A/\mathfrak{J}) = 0$.
(c) $M/\mathfrak{J}M$ is a flat $(A/\mathfrak{J})$-module, and the canonical homomorphism (10.1.1.2) is bijective.
(d) For all $n \geqslant 1$, $M/\mathfrak{J}^n M$ is a flat $(A/\mathfrak{J}^n)$-module.

Then we have the implications

$$\text{(a)} \implies \text{(b)} \implies \text{(c)} \implies \text{(d)},$$

and, if $\mathfrak{J}$ is *nilpotent*, then the four conditions are *equivalent*. This is also the case if A is Noetherian and M is *ideally separated*, that is to say, for every ideal $\mathfrak{a}$ of A, the A-module $\mathfrak{a} \otimes_A M$ is *separated* for the $\mathfrak{J}$-preadic topology.

(10.2.2). Let A be a Noetherian ring, B a commutative Noetherian A-algebra, $\mathfrak{J}$ an ideal of A such that $\mathfrak{J}B$ is contained in the radical of B, and M a B-*module* of finite type. Then, when M is considered as an A-*module*, the four conditions of (10.2.1) are *equivalent*. This result applies first and foremost in the case where A and B are $\mathbf{0}_{\mathbf{III\text{-}1}}$ | 10
local Noetherian rings, with the homomorphism $A \to B$ being a *local* homomorphism. More specifically, if $\mathfrak{J}$ is then the *maximal* ideal of A, we can, in conditions *(b)* and *(c)*, remove the hypothesis that $M/\mathfrak{J}M$ is flat, since it is automatically satisfied, and condition *(d)* implies that the modules $M/\mathfrak{J}^n M$ are *free* over the $A/\mathfrak{J}^n$.

(10.2.3). With the hypotheses on A, B, $\mathfrak{J}$, and M from the start of (10.2.2), let $\widehat{A}$ be the separated completion of A for the $\mathfrak{J}$-preadic topology, and $\widehat{M}$ the separated completion of M for the $\mathfrak{J}B$-preadic topology. Then, for M to be a flat A-module, it is necessary and sufficient for $\widehat{M}$ to be a flat $\widehat{A}$-module.

(10.2.4). Let $\rho : A \to B$ be a local homomorphism of local Noetherian rings, k the residue field of A, and M and N both B-modules of finite type, with N assumed to be A-*flat*. Let $u : M \to N$ be a B-homomorphism. Then the following conditions are equivalent.

(a) u is injective, and $\operatorname{Coker}(u)$ is a flat A-module.
(b) $u \otimes 1 : M \otimes_A k \to N \otimes_A k$ is injective.

(10.2.5). Let $\rho : A \to B$ and $\sigma : B \to C$ be local homomorphisms of local Noetherian rings, k the residue field of A, and M a C-module of finite type. Suppose that B is a *flat* A-module. Then the following conditions are equivalent.

(a) M is a flat B-module.
(b) M is a flat A-module, and $M \otimes_A k$ is a flat $(B \otimes_A k)$-module.

Proposition (10.2.6). — *Let A and B be local Noetherian rings, $\rho : A \to B$ a local homomorphism, $\mathfrak{J}$ an ideal of B contained in the maximal ideal, and M a B-module of finite type. Suppose that, for all $n \geqslant 0$, $M_n = M/\mathfrak{J}^{n+1}M$ is a flat A-module. Then M is a flat A-module.*

Proof. We have to prove that, for every injective homomorphism $u : N' \to N$ of A-modules of finite type, $v = 1 \otimes u : M \otimes_A N' \to M \otimes_A N$ is injective. But $M \otimes_A N'$ and $M \otimes_A N$ are B-modules of finite type, and thus separated for the $\mathfrak{J}$-preadic topology ($\mathbf{0_I}$, 7.3.5); it thus suffices to prove that the homomorphism $\widehat{v} : \widehat{M \otimes_A N'} \to \widehat{M \otimes_A N}$ of the separated completions is injective. But $\widehat{v} = \varprojlim v_n$, where v_n is the homomorphism $1 \otimes u : M_n \otimes_A N' \to M_n \otimes_A N$; since, by hypothesis, M_n is A-flat, v_n is injective for all n, and thus so too is v, because the functor $\varprojlim$ is left exact. □

Corollary (10.2.7). — *Let A be a Noetherian ring, B a local Noetherian ring, $\rho : A \to B$ a homomorphism, f an element of the maximal ideal of B, and M a B-module of finite type. Suppose that the homothety $f_M : x \to fx$ on M is injective, and that M/fM is a flat A-module. Then M is a flat A-module.*

Proof. Let $M_i = f^i M$ for $i \geqslant 0$; since f_M is injective, M_i/M_{i+1} is isomorphic to M/fM, and thus A-flat for all $i \geqslant 0$; the exact sequence
$$0 \longrightarrow M_i/M_{i+1} \longrightarrow M/M_{i+1} \longrightarrow M/M_i \longrightarrow 0$$
gives us, by induction on i, that M/M_i is A-*flat* for all $i \geqslant 0$ ($\mathbf{0_I}$, 6.1.2); we can thus apply (10.2.6). We can also argue directly as follows: for every A-module N of finite type, $M \otimes_A N$ is a B-module of finite type; since f belongs to the radical $\mathfrak{n}$ of B, the (f)-adic topology on $M \otimes_A N$ is finer than the $\mathfrak{n}$-adic topology, and we know $\mathbf{0_{III\text{-}1}}$ | 20
that the latter is *separated* ($\mathbf{0_I}$,**0.7.3.5**,) Now, since M/M_i is A-flat, we have that
$$f^i(M \otimes_A N) = \operatorname{Im}(M_i \otimes_A N \longrightarrow M \otimes_A N) = \operatorname{Ker}(M \otimes_A N \longrightarrow (M/M_i) \otimes_A N)$$
by ($\mathbf{0_I}$, 6.1.2). So let N be an A-module of finite type, and N' a submodule of N, with canonical injection $j : N' \to N$; in the commutative diagram
$$\begin{tikzcd} M \otimes_A N' \arrow[r] \arrow[d, "1_M \otimes j"'] & (M/M_i) \otimes_A N' \arrow[d, "1_{M/M_i} \otimes j"] \\ M \otimes_A N \arrow[r] & (M/M_i) \otimes_A N \end{tikzcd}$$
$1_{M/M_i} \otimes j$ is injective, because M/M_i is A-flat; we thus conclude that
$$\operatorname{Ker}(M \otimes_A N' \longrightarrow M \otimes_A N) \subset \operatorname{Ker}(M \otimes_A N' \longrightarrow (M/M_i) \otimes_A N')$$
for any i; since the intersection (over i) of the latter kernel is 0, as we saw above, so too is the intersection (over i) of the former, and so M is A-flat. □

Proposition (10.2.8). — *Let A be a reduced Noetherian ring, and M an A-module of finite type. Suppose that, for every A-algebra B (which is then a discrete valuation ring), $M \otimes_A B$ is a flat B-module (and thus free* (10.1.3)*). Then M is a flat A-module.*

Proof. We know that, for M to be flat, it is necessary and suffices for $M_\mathfrak{m}$ to be a flat $A_\mathfrak{m}$-module for every maximal ideal $\mathfrak{m}$ of A ($\mathbf{0_I}$, 6.3.3); we can thus restrict to the case where A is *local* ($\mathbf{0_I}$, 1.2.8). So let $\mathfrak{m}$ be the maximal ideal of A, $\mathfrak{p}_i$ $(1 \leqslant i \leqslant r)$ the minimal prime ideals of A, and k the residue field A/fkm. We know (**II**, 7.1.7) that there exists, for each i, a discrete valuation ring B_i that has the same field of fractions K_i as the integral ring $A/\mathfrak{p}_i$, and that, further, dominates $A/\mathfrak{p}_i$. Let $M_i = M \otimes_A B_i$. By hypothesis, M_i is free over B_i, and so, denoting by k_i the residue field of B_i, we have

(10.2.8.1) $$\operatorname{rg}_{k_i}(M_i \otimes_{B_i} k_i) = \operatorname{rg}_{K_i}(M_i \otimes_{B_i} K_i).$$

But it is clear that the composite homomorphism $A \to A/\mathfrak{p}_i \to B_i$ is local, and so k is an extension of k_i, and that we have $M_i \otimes_{B_i} k_i = M \otimes_A k_i = (M \otimes_A k) \otimes_k k_i$, and also that $M_i \otimes_{B_i} K_i = M \otimes_A K_i$. Equation (10.2.8.1) thus implies that

$$\operatorname{rg}_k(M \otimes_A k) = \operatorname{rg}_{K_i}(M \otimes_A K_i) \quad \text{for } 1 \leqslant i \leqslant r$$

and since A is reduced, we know that this condition implies that M is a *free* A-module (Bourbaki, *Alg. comm.*, chap. II, § 3, n° 2, prop. 7). □

10.3. Existence of flat extensions of local rings.

Proposition (10.3.1). — *Let A be a local Noetherian ring, with maximal ideal $\mathfrak{J}$, and residue field $k = A/\mathfrak{J}$. Let K be a field extension of k. Then there exists a local homomorphism from A to a local Noetherian ring B, such that $B/\mathfrak{J}B$ is k-isomorphic to K, and such that B is a flat A-module.*

The rest of this section is devoted to proving this proposition, step-by-step.

(10.3.1.1). First suppose that $K = k(T)$, where T is an indeterminate. In the ring of polynomials $A' = A[T]$, consider the prime ideal $\mathfrak{J}' = \mathfrak{J}A$, consisting of the polynomials that have coefficients in the ideal $\mathfrak{J}$; it is clear $\mathbf{0_{III\text{-}1}}$ | 21
that $A'/\mathfrak{J}'$ is canonically isomorphic to $k[T]$. We will show that the ring of fractions $B = A'_{\mathfrak{J}}$ is that for which we are searching (that is, a ring which satisfies the conditions of the conclusion of the proposition); it is clearly a local Noetherian ring, with maximal ideal $\mathfrak{L} = \mathfrak{J}B$. Further, $B/\mathfrak{L} = (A'/\mathfrak{J}')_{\mathfrak{J}'} = (k[T])_{\mathfrak{J}'}$ is exactly the field of fractions K of $k[T]$. Finally, B is a flat A'-module, and A' is a free A-module, so B is a flat A-module ($\mathbf{0_I}$, 6.2.1).

(10.3.1.2). Now suppose that $K = k(t) = k[t]$, where t is algebraic over k; let $f \in k[T]$ be the minimal polynomial of t; there exists a monic polynomial $F \in A[T]$ whose canonical image in $k[T]$ is f. So let $A' = A[T]$, and let $\mathfrak{J}'$ be the ideal $\mathfrak{J}A' + (F)$ in A'. We will see that the quotient ring $B = A'/(F)$ is that for which we are searching. First of all, it is clear that B is a *free* A-module, and thus flat. The ring $A'/\mathfrak{J}'$ is isomorphic to

$$(A'/\mathfrak{J}A')/\big((\mathfrak{J}A' + (F))/\mathfrak{J}A'\big) = k[T]/(f) = K;$$

the image $\mathfrak{L}$ of $\mathfrak{J}'$ in B is thus maximal, and we evidently have that $\mathfrak{L} = \mathfrak{J}B$. Finally, B is a semi-local ring, because it is an A-module of finite type (Bourbaki, *Alg. comm.*, chap. IV, § 2, n° 5, cor. 3 of prop. 9), and its maximal ideals are in bijective correspondence with those of $B/\mathfrak{J}B$ ([**SZ60**, vol. I, p. 259]); the previous arguments then prove that B is a local ring.

Lemma (10.3.1.3). — *Let $(A_\lambda, f_{\mu\lambda})$ be a filtered inductive system of local rings, such that the $f_{\mu\lambda}$ are local homomorphisms; let $\mathfrak{m}_\lambda$ be the maximal ideal of A_λ, and let $K_\lambda = A_\lambda/\mathfrak{m}_\lambda$. Then $A' = \varinjlim A_\lambda$ is a local ring, with maximal ideal $\mathfrak{m} = \varinjlim \mathfrak{m}_\lambda$, and residue field $K = \varinjlim K_\lambda$. Further, if $\mathfrak{m}_\mu = \mathfrak{m}_\lambda A_\mu$ with $\lambda < \mu$, then we have $\mathfrak{m}' = \mathfrak{m}_\lambda A'$ for all λ. If, further, for $\lambda < \mu$, A_μ is a flat A_λ-module, and if all the A_λ are Noetherian, then A' is a flat Noetherian A_λ-modules for all λ.*

Proof. Since, by hypothesis, $(f_\mu \lambda)(\mathfrak{m}_\lambda) \subset \mathfrak{m}_\mu$ for $\lambda < \mu$, the $\mathfrak{m}_\lambda$ form an inductive system, and its limit $\mathfrak{m}'$ is evidently an ideal of A'. Further, if $x' \notin \mathfrak{m}'$, there exists a λ such that $x' = f_\lambda(x_\lambda)$ for some $x_\lambda \in A_\lambda$ (where $f_\lambda : A_\lambda \to A'$ denotes the canonical homomorphism); because $x' \notin \mathfrak{m}'$, we necessarily have that $x_\lambda \notin \mathfrak{m}_\lambda$, and so x_λ admits an inverse y_λ in A_λ, and $y' = f_\lambda(y_\lambda)$ is the inverse of x' in A', which proves that A' is a local ring with maximal ideal $\mathfrak{m}'$; the claim about K follows immediately from the fact that $\varinjlim$ is an exact functor. The hypothesis that $\mathfrak{m}_\mu = \mathfrak{m}_\lambda A_\mu$ implies that the canonical map $\mathfrak{m}_\lambda \otimes_{A_\lambda} A_\mu \to \mathfrak{m}_\mu$ is surjective; the equality $\mathfrak{m}' = \mathfrak{m}_\lambda A'$ then follows from, again, the fact that the functor $\varinjlim$ is exact and commutes with the tensor product.

Now suppose that, for $\lambda < \mu$, we have $\mathfrak{m}_\mu = \mathfrak{m}_\lambda A_\mu$, and that A_μ is a flat A_λ-module. Then A' is a flat A_λ-module for all λ, by ($\mathbf{0_I}$, 6.2.3); since A' and A_λ are local rings, and since $\mathfrak{m}' = \mathfrak{m}_\lambda A'$, A' is even a *faithfully flat* A_λ-module ($\mathbf{0_I}$, 6.6.2). Finally, suppose further that the A_λ are *Noetherian*; the $\mathfrak{m}_\lambda$-adic topologies are then separated ($\mathbf{0_I}$, 7.3.5); we now show that, from this, it follows that the $\mathfrak{m}'$-adic topology on A' is *separated.* Indeed, if $x' \in A'$ belongs to all the $\mathfrak{m}'^n$ ($n \geqslant 0$), then it is the image of some $x_\mu \in A_\mu$ for a specific index μ, and since the inverse image in A_μ of $\mathfrak{m}'^n = \mathfrak{m}_\mu^n A'$ is $\mathfrak{m}_\mu^n$ ($\mathbf{0_I}$, 6.6.1), x_μ belongs to all the $\mathfrak{m}_\mu^n$, so $x_\mu = 0$, by hypothesis, and so $x' = 0$. Denote by $\widehat{A'}$ the completion of A' for the $\mathfrak{m}'$-adic topology; the above shows that we have $A' \subset \widehat{A'}$. We will now show that $\widehat{A'}$ is *Noetherian* and A_λ*-flat* for all λ; from this, it will follow that $\widehat{A'}$ is A'-flat ($\mathbf{0_I}$, 6.2.3), and $\mathbf{0_{III\text{-}1}}$ | 22
since $\mathfrak{m}'\widehat{A'} \neq \widehat{A'}$, that $\widehat{A'}$ is a faithfully flat A'-module ($\mathbf{0_I}$, 6.6.2), whence the final conclusion that A' is *Noetherian* ($\mathbf{0_I}$, 6.5.2), which will finish the proof of the lemma.

We have $\widehat{A'} = \varprojlim_n A'/\mathfrak{m}'^n$; by the fact that A' is A_λ-flat, we have that

$$\mathfrak{m}'^n/\mathfrak{m}'^{n+1} = (\mathfrak{m}_\lambda^n/\mathfrak{m}_\lambda^{n+1}) \otimes_{A_\lambda} A' = (\mathfrak{m}_\lambda^n/\mathfrak{m}_\lambda^{n+1}) \otimes_{K_\lambda} (K_\lambda \otimes_{A_\lambda} A') = (\mathfrak{m}_\lambda^n/\mathfrak{m}_\lambda^{n+1}) \otimes_{K_\lambda} K;$$

since $\mathfrak{m}_\lambda^n/\mathfrak{m}_\lambda^{n+1}$ is a K_λ-vector space of finite dimension, $\mathfrak{m}_\lambda'^n/\mathfrak{m}_\lambda'^{n+1}$ is a K-vector space of finite dimension for all $n \geqslant 0$. It thus follows from $(\mathbf{0_I}, 7.2.12)$ and $(\mathbf{0_I}, 7.2.8)$ that $\widehat{A'}$ is *Noetherian*. We further know that the maximal ideal of $\widehat{A'}$ is $\mathfrak{m}'A'$, and that $\widehat{A'}/\mathfrak{m}'^n\widehat{A'}$ is isomorphic to $A'/\mathfrak{m}'^n$; since $A'/\mathfrak{m}'^n = (A_\lambda/\mathfrak{m}_\lambda^n) \otimes_{A_\lambda} A'$, we see that $A'/\mathfrak{m}'^n$ is a flat $(A_\lambda/\mathfrak{m}_\lambda^n)$-module $(\mathbf{0_I}, 6.2.1)$; criterion (10.2.2) is thus applicable to the Noetherian A_λ-algebra $\widehat{A'}$, and shows that $\widehat{A'}$ is A_λ-*flat*. □

(10.3.1.4). We now treat the general case. There exists an ordinal γ and, for every ordinal $\lambda \leqslant \gamma$, a subfield k_λ of K that contains k, such that (i) for all $\lambda < \gamma$, $k_{\lambda+1}$ is an extension of k_λ generated by a single element; (ii) for every limit ordinal μ, $k_\mu = \bigcup_{\lambda<\mu} k_\lambda$; and (iii) $K = k_\gamma$. In fact, it suffices to consider a bijection $\xi \mapsto t_\xi$ from the set of ordinals $\xi \leqslant \beta$ (for some suitable β) to K, and to define k_λ by transfinite induction (for $\lambda \leqslant \beta$) as the union of the k_μ for $\mu < \lambda$ if λ is a limit ordinal, and as $k_\nu(t_\xi)$ if $\lambda = \nu+1$, where ξ is the smallest ordinal such that $t_\xi \notin k_\nu$; γ is then, by definition, the smallest ordinal $\leqslant \beta$ such that $k_\gamma = K$.

With this in mind, we will define, by transfinite induction, a family of local Noetherian rings A_λ for $\lambda \leqslant \gamma$, and local homomorphisms $f_{\mu\lambda} : A_\lambda \to A_\mu$ for $\lambda \leqslant \mu$, satisfying the following conditions:

(i) $(A_\lambda, f_{\mu\lambda})$ is an inductive system, and $A_0 = A$;
(ii) for all λ, we have a k-isomorphism $A_\lambda/\mathfrak{J}A_\lambda \simeq k_\lambda$;
(iii) for $\lambda \leqslant \mu$, A_μ is a flat A_λ-module.

So suppose that the A_λ and the $f_{\mu\lambda}$ are defined for $\lambda < \mu < \xi$, and suppose, first of all, that $\xi = \zeta + 1$, so that $k_\xi = k_\zeta(t)$. If t is transcendental over k_ζ, we define A_ζ, following the procedure of (10.3.1.1), to be equal to $(A_\zeta[t])_{\mathfrak{J}A_\zeta[t]}$; the canonical map is $f_{\zeta\xi}$, and, for $\lambda < \zeta$, we take $f_{\xi\lambda} = f_{\xi\zeta} \circ f_{\zeta\lambda}$; the verification of conditions (i) to (iii) is then immediate, given that what we have shown in (10.3.1.1). So now suppose that t is algebraic, and let h be its minimal polynomial in $k_\zeta[T]$, and H a monic polynomial in $A_\zeta[T]$ whose image in $k_\zeta[T]$ is h; we then take A_ξ to be equal to $A_\zeta[T](H)$, with the $f_{\xi\lambda}$ being defined as berofe; the verification of conditions (i) to (iii) then follows from what we have shown in (10.3.1.2).

Now suppose that ξ has no predecessor; we then take A_ξ to be the inductive limit of the inductive system of local rings $(A_\lambda, f_{\mu\lambda})$ for $\lambda < \xi$; we define $f_{\xi\lambda}$ as the canonical map for $\lambda < \xi$. The fact that A_ξ is local and Noetherian, that the $f_{\xi\lambda}$ are local homomorphism, and that conditions (i) to (iii) are satisfied for $\lambda \leqslant \xi$ then $\mathbf{0_{III\text{-}1}}$ | 23
follows from the induction hypothesis, and from Lemma (10.3.1.3). With this construction, it is clear that the ring $B = A_\gamma$ satisfies the conditions of (10.3.1). □

We note that, by (10.2.1, c), we have a canonical isomorphism

$$\operatorname{gr}(A) \otimes_k K \xrightarrow{\sim} \operatorname{gr}(B). \tag{10.3.1.5}$$

We can also replace B by its $\mathfrak{J}B$-adic completion $\widehat{B}$ without changing the conclusions of (10.3.1), because $\widehat{B}$ is a flat B-module $(\mathbf{0_I}, 7.3.3)$, and thus a flat A-module $(\mathbf{0_I}, 6.2.1)$.

We have also shown the following:

Corollary (10.3.2). — *If K is an extension of finite degree, then we can assume that B is a finite A-algebra.*

§11. Supplement on homological algebra

11.1. Review of spectral sequences.

(11.1.1). In the following, we use a more general notion of a spectral sequence than that defined in (T, 2.4); keeping the notations of (T, 2.4), we call a *spectral sequence* in an abelian category C a system E consisting of the following parts:

(a) A family (E_r^{pq}) of objects of C defined for $p, q \in \mathbf{Z}$ and $r \geqslant 2$.
(b) A family of morphisms $d_r^{pq} : E_r^{pq} \to E_r^{p+r,q-r+1}$ such that $d_r^{p+r,q-r+1} d_r^{pq} = 0$. We set $Z_{r+1}(E_r^{pq}) = \operatorname{Ker}(d_r^{pq})$ and $B_{r+1}(E_r^{pq}) = \operatorname{Im}(d_r^{p+r,q-r+1})$, so that
$$B_{r+1}(E_r^{pq}) \subset Z_{r+1}(E_r^{pq}) \subset E_r^{pq}.$$
(c) A family of isomorphisms $\alpha_r^{pq} : Z_{r+1}(E_r^{pq})/B_{r+1}(E_r^{pq}) \simeq E_{r+1}^{pq}$.
We then define for $k \geqslant r+1$, by induction on k, the subobjects $B_k(E_r^{pq})$ and $Z_k(E_r^{pq})$ as the inverse images, under the canonical morphism $E_r^{pq} \to E_r^{pq}/B_{r+1}(E_r^{pq})$ of the subobjects of this quotient identified via α_r^{pq} with the subobjects $B_k(E_{r+1}^{pq})$ and $Z_k(E_{r+1}^{pq})$ respectively. It is clear that we then have, up to isomorphism,

$$Z_k(E_r^{pq})/B_k(E_r^{pq}) = E_k^{pq} \text{ for } k \geqslant r+1, \tag{11.1.1.1}$$

and, if we set $B_r(E_r^{pq}) = 0$ and $Z_r(E_r^{pq}) = E_r^{pq}$, then we have the inclusion relations

(11.1.1.2)
$$0 = B_r(E_r^{pq}) \subset B_{r+1}(E_r^{pq}) \subset B_{r+2}(E_r^{pq}) \subset \cdots \subset Z_{r+2}(E_r^{pq}) \subset Z_{r+1}(E_r^{pq}) \subset Z_r(E_r^{pq}) = E_r^{pq}.$$

The other parts of the data of E are then:

(d) Two subobjects $B_\infty(E_2^{pq})$ and $Z_\infty(E_2^{pq})$ of E_2^{pq} such that we have $B_\infty(E_2^{pq}) \subset Z_\infty(E_2^{pq})$ and, for every $k \geqslant 2$,
$$B_k(E_2^{pq}) \subset B_\infty(E_2^{pq}) \text{ and } Z_\infty(E_2^{pq}) \subset Z_k(E_2^{pq}).$$
We set

(11.1.1.3)
$$E_\infty^{pq} = Z_\infty(E_2^{pq})/B_\infty(E_2^{pq}).$$

(e) A family (E^n) of objects of $\mathtt{C}$, each equipped with a *decreasing filtration* $(F^p(E^n))_{p\in\mathbf{Z}}$. As usual, we denote by $\mathrm{gr}(E^n)$ the graded object associated to the filtered object E^n, the direct sum of the $\mathrm{gr}_p(E^n) = F^p(E^n)/F^{p+1}(E^n)$. $\mathbf{0}_{\mathbf{III}}$ | 24

(f) For every pair $(p,q) \in \mathbf{Z}\times\mathbf{Z}$, an isomorphism $\beta^{pq} : E_\infty^{pq} \simeq \mathrm{gr}_p(E^{p+q})$.

The family (E^n), without the filtrations, is called the *abutment* (or *limit*) of the spectral sequence E.

Suppose that the category $\mathtt{C}$ admits infinite direct sums, or that for every $r \geqslant 2$ and every $n \in \mathbf{Z}$, there are finitely many pairs (p,q) such that $p+q=n$ and $E_r^{pq} \neq 0$ (it suffices for it to hold for $r=2$). Then the $E_r^{(n)} = \sum_{p+q=n} E_r^{pq}$ are defined, and we if denote by $d_r^{(n)}$ the morphism $E_r^{(n)} \to E_r^{(n+1)}$ whose restriction to E_r^{pq} is d_r^{pq} for every pair (p,q) such that $p+q=n$, then $d_r^{(n+1)} \circ d_r^{(n)} = 0$, in other words, $(E_r^{(n)})_{n\in\mathbf{Z}}$ is a *complex* $E_r^{(\bullet)}$ in $\mathtt{C}$, with differentials of degree $+1$, and it follows from (c) that $\mathrm{H}^n(E_n^{(\bullet)})$ is *isomorphic to* $E_{r+1}^{(n)}$ for every $r \geqslant 2$.

(11.1.2). A *morphism* $u : E \to E'$ from a spectral sequence E to a spectral sequence $E' = (E_r'^{pq}, E'^n)$ consists of systems of morphisms $u_r^{pq} : E_r^{pq} \to E_r'^{pq}$ and $u^n : E^n \to E'^n$, the u^n compatible with the filtrations on E^n and E'^n, and the diagrams

$$\begin{tikzcd}
E_r^{pq} \arrow[r, "d_r^{pq}"] \arrow[d, "u_r^{pq}"'] & E_r^{p+r,q-r+1} \arrow[d, "u_r^{p+r,q-r+1}"] \\
E_r'^{pq} \arrow[r, "d_r'^{pq}"] & E_r''^{p+r,q-r+1}
\end{tikzcd}$$

being commutative; in addition, by passing to quotients, u_r^{pq} gives a morphism $\overline{u}_r^{pq} : Z_{r+1}(Z_r^{pq})/B_{r+1}(E_r^{pq}) \to Z_{r+1}(E_r'^{pq})/B_{r+1}(E_r'^{pq})$ and we must have $\alpha_r'^{pq} \circ \overline{u}_r^{pq} = u_{r+1}^{pq} \circ \alpha_r^{pq}$; finally, we must have $u_2^{pq}(B_\infty(E_2^{pq})) \subset B_\infty(E_2'^{pq})$ and $u_2^{pq}(Z_\infty(E_2^{pq})) \subset Z_\infty(E_2'^{pq})$; by passing to quotients, u_2^{pq} then gives a morphism $u_\infty'^{pq} : E_\infty^{pq} \to E_\infty'^{pq}$, and the diagram

$$\begin{tikzcd}
E_\infty^{pq} \arrow[r, "u_\infty'^{pq}"] \arrow[d, "\beta^{pq}"'] & E_\infty'^{pq} \arrow[d, "\beta'^{pq}"] \\
\mathrm{gr}_p(E^{p+q}) \arrow[r, "\mathrm{gr}_p(u^{p+q})"] & \mathrm{gr}_p(E'^{p+q})
\end{tikzcd}$$

must be commutative.

The above definitions show, by induction on r, that if the u_2^{pq} are *isomorphisms*, then so are the u_r^{pq} for $r \geqslant 2$; *if in addition we know that* $u_2^{pq}(B_\infty(E_2^{pq})) = B_\infty(E_2'^{pq})$ *and* $u_2^{pq}(Z_\infty(E_2^{pq})) = Z_\infty(E_2'^{pq})$ *and the* u^n *are isomorphisms*, then we can conclude that u is an *isomorphism*.

(11.1.3). Recall that if $(F^p(X))_{p\in\mathbf{Z}}$ is a (decreasing) *filtration* of an object $X \in \mathtt{C}$, then we say that this filtration is *separated* if $\inf(F^p(X)) = 0$, *discrete* if there exists a p such that $F^p(X) = 0$, *exhaustive* (or *coseparated*) if $\sup(F^p(X)) = X$, *codiscrete* if there exists a p such that $F^p(X) = X$.

We say that a spectral sequence $E = (E_r^{pq}, E^n)$ is *weakly convergent* if we have $B_\infty(E_2^{pq}) = \sup_k(B_k(E_2^{pq}))$ and $Z_\infty(E_2^{pq}) = \inf_k(Z_k(E_2^{pq}))$ (in other words, the objects of $B_\infty(E_2^{pq})$ and $Z_\infty(E_2^{pq})$ are determined from the data of (a) and (c) of the spectral sequence E). We say that the spectral sequence E is *regular* if it is weakly convergent and if in addition:

(1st) For every pair (p,q), the decreasing sequence $(Z_k(E_2^{pq}))_{k\geqslant 2}$ is *stable*; the hypothesis that E is weakly convergent then implies that $Z_\infty(E_2^{pq}) = Z_k(E_2^{pq})$ for k large enough (depending on p and q).

(2nd) For every n, the filtration $(F^p(E^n))_{p\in\mathbf{Z}}$ of E^n is *discrete* and *exhaustive*.

We say that the spectral sequence E is *coregular* if it is weakly convergent and if in addition:

(3rd) For every pair (p,q), the increasing sequence $(B_k(E_2^{pq}))_{k\geqslant 2}$ is *stable*, which implies that $B_\infty(E_2^{pq}) = B_k(E_2^{pq})$, and as a result, $E_\infty^{pq} = \inf E_k^{pq}$.

(4th) For every n, the filtration of E^n is *codiscrete.*

Finally, we say that E is *biregular* if it is both regular and coregular, in other words if we have the following conditions:

(a) For every pair (p,q), the sequences $(B_k(E_2^{pq}))_{k\geqslant 2}$ and $(Z_k(E_2^{pq}))_{k\geqslant 2}$ are *stable* and we have $B_\infty(E_2^{pq}) = B_k(E_2^{pq})$ and $Z_\infty(E_2^{pq}) = Z_k(E_2^{pq})$ for k large enough (which implies that $E_\infty^{pq} = E_k^{pq}$).

(b) For every n, the filatration $(F^p(E^n))_{p\in\mathbf{Z}}$ is *discrete* and *codiscrete* (which we also call *finite*).

The spectral sequences defined in (T, 2.4) are thus biregular spectral sequences.

(11.1.4). Suppose that in the category C, filtered inductive limits exist and the functor $\varinjlim$ is *exact* (which is equivalent to saying that the axiom (AB 5) of (T, 1.5) is satisfied (cf. T, 1.8)). The condition that the filtration $(F^p(X))_{p\in\mathbf{Z}}$ of an object $X\in\mathsf{C}$ is exhaustive is then expressed as $\varinjlim_{p\to-\infty} F^p(X) = X$. If a spectral sequence E is weakly convergent, then we have $B_\infty(E_2^{pq}) = \varinjlim_{k\to\infty} B_k(E_2^{pq})$; if in addition $u : E \to E'$ is a morphism from E to a weakly convergent spectral sequence E' in C, then we have $u_2^{pq}(B_\infty(E_2^{pq})) = B_\infty(E_2'^{pq})$, by the exactness of $\varinjlim$. In addition:

Proposition (11.1.5). — *Let* C *be an abelian category in which filtered inductive limits are exact,* E *and* E' *two regular spectral sequences in* C*,* $u : E \to E'$ *a morphism of spectral sequences. If the* u_2^{pq} *are isomorphisms, then so is* u*.*

Proof. We already know (11.1.2) that the u_r^{pq} are isomorphisms and that

$$u_2^{pq}(B_\infty(E_2^{pq})) = B_\infty(E_2'^{pq});$$

the hypothesis that E and E' are regular also implies that $u_2^{pq}(Z_\infty(E_2^{pq})) = Z_\infty(E_2'^{pq})$, and as u_2^{pq} is an isomorphism, $\mathbf{0}_{\mathbf{III}}$ | 26
so is $u_\infty'^{pq}$; we thus conclude that $\mathrm{gr}_p(u^{p+q})$ is also an isomorphism. But as the filtrations of the E^n and the E'^n are discrete and exhaustive, this implies that the u^n are also isomorphisms (Bourbaki, *Alg. comm.*, chap. III, §2, n°8, th. 1). □

(11.1.6). It follows from (11.1.1.2) and the definition (11.1.1.3) that if, for a spectral sequence E, we have $E_r^{pq} = 0$, then we have $E_k^{pq} = 0$ for $k \geqslant r$ and $E_\infty^{pq} = 0$. We say that a spectral sequence *degenerates* if there exists an integer $r \geqslant 2$ and, for every integer $n \in \mathbf{Z}$, an integer $q(n)$ such that $E_r^{n-q,q} = 0$ *for every* $q \neq q(n)$. We first deduce from the previous remark that we also have $E_k^{n-q,q} = 0$ for $k \geqslant r$ (including $k=\infty$) and $q \neq q(n)$. In addition, the definition of E_{r+1}^{pq} shows that we have $E_{r+1}^{n-q(n),q(n)} = E_r^{n-q(n),q(n)}$; if E is *weakly convergent*, then we also have $E_\infty^{n-q(n),q(n)} = E_r^{n-q(n),q(n)}$; in other words, for every $n\in\mathbf{Z}$, $\mathrm{gr}_p(E^n) = 0$ for $p \neq q(n)$ and $\mathrm{gr}_{q(n)}(E^n) = E_r^{n-q(n),q(n)}$. If in addition the filtration of E^n is *discrete* and *exhaustive*, then the spectral sequence E is *regular*, and we have $E^n = E_r^{n-q(n),q(n)}$ up to unique isomorphism.

(11.1.7). Suppose that filtered inductive limits exist and are exact in the category C, and let $(E_\lambda, u_{\mu\lambda})$ be an inductive system (over a filtered set of indices) of spectral seqeneces in C. Then the *inductive limit* of this inductive system exists in the additive category of spectral sequences of objects of C: to see this, it suffices to define E_r^{pq}, d_r^{pq}, α_r^{pq}, $B_\infty(E_2^{pq})$, $Z_\infty(E_2^{pq})$, E^n, $F^p(E^n)$, and β^{pq} as the respective inductive limits of the $E_{r,\lambda}^{pq}$, $d_{r,\lambda}^{pq}$, $\alpha_{r,\lambda}^{pq}$, $B_\infty(E_{2,\lambda}^{pq})$, $Z_\infty(E_{2,\lambda}^{pq})$, E_λ^n, $F^p(E_\lambda^n)$, and β_λ^{pq}; the verification of the conditions of (11.1.1) follows from the exactness of the functor $\varinjlim$ on C.

Remark (11.1.8). — Suppose that the category C is the category of A-modules over a *Noetherian* ring A (resp. a ring A). Then the definitions of (11.1.1) show that if, for a given r, the E_r^{pq} are A-modules *of finite type* (resp. *of finite length*), then so are each of the modules E_s^{pq} for $s\geqslant r$, hence so is E_∞^{pq}. If in addition the filtration of the abutment/limit (E^n) is *discrete* or *codiscrete* for all n, then we conclude that each of the E^n is also an A-module *of finite type* (resp. *of finite length*).

(11.1.9). We will have to consider conditions which ensure that a spectral sequence E is biregular is a "uniform" way in $p+q=n$. We will then use the following lemma:

Lemma (11.1.10). — *Let* (E_r^{pq}) *be a family of objects of* C *related by the data of* (a), (b), *and* (c) *of* (11.1.1). *For a fixed integer* n*, the following properties are equivalent:*

(a) *There exists an integer* $r(n)$ *such that for* $r\geqslant r(n)$*,* $p+q=n$ *or* $p+q=n-1$*, the morphisms* d_r^{pq} *are zero.*

(b) *There exists an integer* $r(n)$ *such that for* $p+q=n$ *or* $p+q=n+1$*, we have* $B_r(E_2^{pq}) = B_s(E_2^{pq})$ *for* $s\geqslant r\geqslant r(n)$*.*

(c) *There exists an integer* $r(n)$ *such that for* $p+q=n$ *or* $p+q=n-1$, *we have* $Z_r(E_2^{pq})=Z_s(E_2^{pq})$ *for* $s\geqslant r\geqslant r(n)$.

(d) *There exists an integer* $r(n)$ *such that for* $p+q=n$, *we have* $B_r(E_2^{pq})=B_s(E_2^{pq})$ *and* $Z_r(E_2^{pq})=Z_s(E_2^{pq})$ *for* $s\geqslant r\geqslant r(n)$.

Proof. According to the conditions (a), (b), and (c) of (11.1.1), we have that saying $Z_{r+1}(E_2^{pq})=Z_r(E_2^{pq})$ is equivalent to saying that $d_r^{pq}=0$ and that saying $B_r(E_2^{p+r,q-r+1})=B_{r+1}(E_2^{p+r,q-r+1})$ is equivalent to saying that $d_r^{pq}=0$; the lemma immediately follows from this remark. □

11.2. The spectral sequence of a filtered complex.

(11.2.1). Given an abelian category C, we will agree to denote by notations such as $K^\bullet$ the *complexes* $(K^i)_{i\in\mathbf{Z}}$ of objects of C whose differential is of degree $+1$, and by the notations such as $K_\bullet$ the complexes $(K_i)_{i\in\mathbf{Z}}$ of objects of C whose differential is of degree -1. To each complex $K^\bullet=(K^i)$ whose differential d is of degree $+1$, we can associate a complex $K'_\bullet=(K'_i)$ by setting $K'_i=K^{-i}$, the differential $K'_i\to K'_{i-1}$ being the operator $d:K^{-i}\to K^{-i-1}$; and *vice versa*, which, depending on the circumstances, will allow one to consider either one of the types of complexes and translate any result from one type into results for the other. We similarly denote by notations such as $K^{\bullet\bullet}=(K^{ij})$ (resp. $K_{\bullet\bullet}=(K_{ij})$) the *bicomplexes* (or *double complexes*) of objects of C in which the *two* differntials are of degree $+1$ (resp. -1); we can still pass from one type to the other by changing the signs of the indices, and we have similar notations and remarks for any multicomplexes. The notation $K^\bullet$ and $K_\bullet$ will also be used for $\mathbf{Z}$-*graded objects* of C, which are not necessarily complexes (they can be considered as such for the *zero* differentials); for example, we write $\mathrm{H}^\bullet(K^\bullet)=(\mathrm{H}^i(K^\bullet))_{i\in\mathbf{Z}}$ for the *cohomology* of a complex $K^\bullet$ whose differential is of degree $+1$, and $H_\bullet(K_\bullet)=(\mathrm{H}_i(K_\bullet))_{i\in\mathbf{Z}}$ for the *homology* of a complex $K_\bullet$ whose differential is of degree -1; when we pass from $K^\bullet$ to $K'_\bullet$ by the method described above, we have $\mathrm{H}_i(K'_\bullet)=\mathrm{H}^{-i}(K^\bullet)$.

Recall in this case that for a complex $K^\bullet$ (resp. $K_\bullet$), we will write in general $Z^i(K^\bullet)=\operatorname{Ker}(K^i\to K^{i+1})$ ("object of cocycles") and $B^i(K^\bullet)=\operatorname{Im}(K^{i-1}\to K^i)$ ("object of coboundaries") (resp. $Z_i(K_\bullet)=\operatorname{Ker}(K_i\to K_{i-1})$ ("object of cycles") and $B_i(K_\bullet)=\operatorname{Im}(K_{i+1}\to K_i)$ ("object of boundaries")) so that $\mathrm{H}^i(K^\bullet)=Z^i(K^\bullet)/B^i(K^\bullet)$ (resp. $\mathrm{H}_i(K_\bullet)=Z_i(K_\bullet)/B_i(K_\bullet)$).

If $K^\bullet=(K^i)$ (resp. $K_\bullet=(K_i)$) is a complex in C and $T:\mathsf{C}\to\mathsf{C}'$ a functor from C to an abelian category C', then we denote by $T(K^\bullet)$ (resp. $T(K_\bullet)$) the complex $(T(K^i))$ (resp. $(T(K_i))$) in C'.

We will not review the definition of the ∂-*functors* (T, 2.1), except to note that we *also* say ∂-functor in place of ∂^*-functor when the morphism ∂ decreases the degree of a unit, the context clarifying this point if there is cause for confusion.

Finally, we say that a *graded object* $(A_i)_{i\in\mathbf{Z}}$ of C is *bounded below* (resp. *above*) if there exists an i_0 such that $A_i=0$ for $i<i_0$ (resp. $i>i_0$).

(11.2.2). Let $K^\bullet$ be a complex in C whose differential d is of degree $+1$, and suppose it is equipped with a *filtration* $F(K^\bullet)=(F^p(K^\bullet))_{p\in\mathbf{Z}}$ consisting of *graded* subobjects of $K^\bullet$, in other words, $F^p(K^\bullet)=(K^i\cap F^p(K^\bullet))_{i\in\mathbf{Z}}$; **0_III** | 28
in addition, we assume that $d(F^p(K^\bullet))\subset F^p(K^\bullet)$ for every $p\in\mathbf{Z}$. Let us quickly recall how one *functorially* defines a spectral sequence $E(K^\bullet)$ from $K^\bullet$ (M, XV, 4 and G, I, 4.3). For $r\geqslant 2$, the canonical morphism $F^p(K^\bullet)/F^{p+r}(K^\bullet)\to F^p(K^\bullet)/F^{p+1}(K^\bullet)$ defines a morphism in cohomology

$$\mathrm{H}^{p+q}(F^p(K^\bullet)/F^{p+r}(K^\bullet))\longrightarrow \mathrm{H}^{p+q}(F^p(K^\bullet)/F^{p+1}(K^\bullet)).$$

We denote by $Z_r^{pq}(K^\bullet)$ the image of this morphism. Similarly, from the exact sequence

$$0\longrightarrow F^p(K^\bullet)/F^{p+1}(K^\bullet)\longrightarrow F^{p-r+1}(K^\bullet)/F^{p+1}(K^\bullet)\longrightarrow F^{p-r+1}(K^\bullet)/F^p(K^\bullet)\longrightarrow 0,$$

we deduce from the exact sequence in cohomology a morphism

$$\mathrm{H}^{p+q-1}(F^{p-r+1}(K^\bullet)/F^p(K^\bullet))\longrightarrow \mathrm{H}^{p+q}(F^p(K^\bullet)/F^{p+1}(K^\bullet)),$$

and we denote by $B_r^{pq}(K^\bullet)$ the image of this morphism; we show that $B_r^{pq}(K^\bullet)\subset Z_r^{pq}(K^\bullet)$ and we take $E_r^{pq}(K^\bullet)=Z_r^{pq}(K^\bullet)/B_r^{pq}(K^\bullet)$; we will not specify the definition of the d_r^{pq} or the α_r^{pq}.

We note here that all the $Z_r^{pq}(K^\bullet)$ and $B_r^{pq}(K^\bullet)$, for p and q fixed, are subobjects of the same object $\mathrm{H}^{p+q}(F^p(K^\bullet)/F^{p+1}(K^\bullet))$, which we denote by $Z_1^{pq}(K^\bullet)$; we set $B_1^{pq}(K^\bullet)=0$, so that the above definitions of $Z_r^{pq}(K^\bullet)$ and $B_r^{pq}(K^\bullet)$ also work for $r=1$; we still set $E_1^{pq}(K^\bullet)=Z_1^{pq}(K^\bullet)$. We define d_1^{pq} and α_1^{pq} such that the conditions of (11.1.1) are satisfied for $r=1$. On the other hand, we define the subobjects $Z_\infty^{pq}(K^\bullet)$ as the image of the morphism

$$\mathrm{H}^{p+q}(F^p(K^\bullet))\longrightarrow \mathrm{H}^{p+q}(F^p(K^\bullet)/F^{p+1}(K^\bullet))=E_1^{pq}(K^\bullet),$$

and $B_\infty^{pq}(K^\bullet)$ as the image of the morphism

$$\mathrm{H}^{p+q-1}(K^\bullet/F^p(K^\bullet))\longrightarrow \mathrm{H}^{p+q}(F^p(K^\bullet)/F^{p+1}(K^\bullet))=E_1^{pq}(K^\bullet),$$

induced as above from the exact sequence in cohomology. We set $Z_\infty(E_2^{pq}(K^\bullet))$ and $B_\infty(E_2^{pq}(K^\bullet))$ to be the canonical images of $E_2^{pq}(K^\bullet)$ in $Z_\infty^{pq}(K^\bullet)$ and $B_\infty^{pq}(K^\bullet)$.

Finally, we denote by $F^p(\mathrm{H}^n(K^\bullet))$ the image in $\mathrm{H}^n(K^\bullet)$ of the morphism $\mathrm{H}^n(F^p(K^\bullet)) \to \mathrm{H}^n(K^\bullet)$ induced from the canonical injection $F^p(K^\bullet) \to K^\bullet$; by the exact sequence in cohomology, this is also the kernel of the morphism $\mathrm{H}^n(K^\bullet) \to \mathrm{H}^n(K^\bullet/F^p(K^\bullet))$. This defines a filtration on $E^n(K^\bullet) = \mathrm{H}^n(K^\bullet)$; we will not give here the definition of the isomorphisms β^{pq}.

(11.2.3). The *functorial* nature of $E(K^\bullet)$ is understood in the following way: given two *filtered* complexes $K^\bullet$ and $K'^\bullet$ in C and a morphism of complexes $u : K^\bullet \to K'^\bullet$ that is *compatible with the filtrations*, we induce in an evident way the morphisms u_r^{pq} (for $r \geqslant 1$) and u^n, and we show that these morphisms are compatible with the d_r^{pq}, α_r^{pq}, and β^{pq} in the sense of (11.1.2), and thus given a well-defined morphism $E(u) : E(K^\bullet) \to E(K'^\bullet)$ of spectral sequences. In addition, we show that if u and v are morphisms $K^\bullet \to K'^\bullet$ of the above type, *homotopic in degree* $\leqslant k$, then $u_r^{pq} = v_r^{pq}$ for $r > k$ and $u^n = v^n$ for all n (M, XV, 3.1). $\mathbf{0}_{\mathbf{III}}$ | 29

(11.2.4). Suppose that filtered inductive limits in C are exact. Then if the filtration $(F^p(K^\bullet))$ of $K^\bullet$ is *exhaustive*, then so is the filatration $(F^p(\mathrm{H}^n(K^\bullet)))$ for all n, since by hypothesis we have $K^\bullet = \varinjlim_{p\to-\infty} F^p(K^\bullet)$ and since the hypothesis on C implies that cohomology commutes with inductive limits. In addition, for the same reason, we have $B_\infty(E_2^{pq}(K^\bullet)) = \sup_k B_k(E_2^{pq}(K^\bullet))$. We say that the filtration $(F^p(K^\bullet))$ of $K^\bullet$ is *regular* if for every n there exists an integer $u(n)$ such that $\mathrm{H}^n(F^p(K^\bullet)) = 0$ for $p > u(n)$. This is particularly the case when the filtration of $K^\bullet$ is *discrete*. When the filtration of $K^\bullet$ is regular and exhaustive, and filtered inductive limits are exact in C, we have (M, XV, 4) that the spectral sequence $E(K^\bullet)$ is *regular*.

11.3. The spectral sequences of a bicomplex.

(11.3.1). With regard the conventions for bicomplexes, we follow those of (T, 2.4) rather than those of (M), the two differentials d' and d'' (of degree $+1$) of such a bicomplex $K^{\bullet\bullet} = (K^{ij})$ being thus assumed to be *permutable*. Suppose that *one* of the following two conditions is satisifed: 1st. *Infinite direct sums* exist in C; 2nd. For all $n \in \mathbf{Z}$, there is only a *finite* number of pairs (p, q) such that $p + q = n$ and $K^{pq} \neq 0$. Then, the bicomplex $K^{\bullet\bullet}$ defines a (simple) *complex* $(K'^n)_{n\in\mathbf{Z}}$ with $K'^n = \sum_{i+j=n} K^{ij}$, the differential d (of degree $+1$) of this complex being given by $dx = d'x + (-1)^i d''x$ for $x \in K^{ij}$. When we later speak of the spectral sequence of a (simple) *complex* that is *defined by a bicomplex* $K^{\bullet\bullet}$, it will always be understood that of the above conditions is satisfied. We adopt the analogous conventions for multicomplexes.

We denote by $K^{i,\bullet}$ (resp. $K^{\bullet,j}$) the simple complex $(K^{ij})_{j\in\mathbf{Z}}$ (resp. $(K^{ij})_{i\in\mathbf{Z}}$), by $Z_{\mathrm{II}}^p(K^{i,\bullet})$, $B_{\mathrm{II}}^p(K^{i,\bullet})$, $\mathrm{H}_{\mathrm{II}}^p(K^{i,\bullet})$ (resp. $Z_{\mathrm{I}}^p(K^{\bullet,j})$, $B_{\mathrm{I}}^p(K^{\bullet,j})$, $\mathrm{H}_{\mathrm{I}}^p(K^{\bullet,j})$) its p objects of cocycles, of coboundaries, and of cohomology, respectively; the differential $d' : K^{i,\bullet} \to K^{i+1,\bullet}$ is a morphism of complexes, which thus gives an operator on the cocycles, coboundaries, and cohomology,

$$\begin{aligned} d' &: Z_{\mathrm{II}}^p(K^{i,\bullet}) \longrightarrow Z_{\mathrm{II}}^p(K^{i+1,\bullet}), \\ d' &: B_{\mathrm{II}}^p(K^{i,\bullet}) \longrightarrow B_{\mathrm{II}}^p(K^{i+1,\bullet}), \\ d' &: \mathrm{H}_{\mathrm{II}}^p(K^{i,\bullet}) \longrightarrow \mathrm{H}_{\mathrm{II}}^p(K^{i+1,\bullet}), \end{aligned}$$

and it is clear that for these operators, $(Z_{\mathrm{II}}^p(K^{i,\bullet}))_{i\in\mathbf{Z}}$, $(B_{\mathrm{II}}^p(K^{i,\bullet}))_{i\in\mathbf{Z}}$, and $(\mathrm{H}_{\mathrm{II}}^p(K^{i,\bullet}))_{i\in\mathbf{Z}}$ are complexes; we denote the complex $(\mathrm{H}_{\mathrm{II}}^p(K^{i,\bullet}))_{i\in\mathbf{Z}}$ by $\mathrm{H}_{\mathrm{II}}^p(K^{\bullet\bullet})$, its q objects of cocycles, coboundaries, and cohomology by $Z_{\mathrm{I}}^q(\mathrm{H}_{\mathrm{II}}^p(K^{\bullet\bullet}))$, $B_{\mathrm{I}}^q(\mathrm{H}_{\mathrm{II}}^p(K^{\bullet\bullet}))$, and $\mathrm{H}_{\mathrm{I}}^q(\mathrm{H}_{\mathrm{II}}^p(K^{\bullet\bullet}))$. We similarly define the complexes $\mathrm{H}_{\mathrm{I}}^p(K^{\bullet\bullet})$ and their cohomology objects $\mathrm{H}_{\mathrm{II}}^q(\mathrm{H}_{\mathrm{I}}^p(K^{\bullet\bullet}))$. Recall on the other hand that $\mathrm{H}^n(K^{\bullet\bullet})$ denotes the n object of the cohomology of the *(simple)* complex defined by $K^{\bullet\bullet}$.

(11.3.2). On the complex defined by a bicomplex $K^{\bullet\bullet}$, we can consider two canonical filtrations $(F_{\mathrm{I}}^p(K^{\bullet\bullet}))$ and $(F_{\mathrm{II}}^p(K^{\bullet\bullet}))$ given by

$$F_{\mathrm{I}}^p(K^{\bullet\bullet}) = \left(\sum_{i+j=n, i\geqslant p} K^{ij}\right)_{n\in\mathbf{Z}} \quad \text{and} \quad F_{\mathrm{II}}^p(K^{\bullet\bullet}) = \left(\sum_{i+j=n, j\geqslant p} K^{ij}\right)_{n\in\mathbf{Z}}, \tag{11.3.2.1}$$

which, by definition, are graded subobjects of the (simple) complex define by $K^{\bullet\bullet}$, and thus make this complex a filtered complex; moreover, is is clear that these filtrations are *exhaustive* and *separated*. $\mathbf{0}_{\mathbf{III}}$ | 30

There corresponds to each of these filtrations a spectral sequence (11.2.2); we denote by $'E(K^{\bullet\bullet})$ and $''E(K^{\bullet\bullet})$ the spectral sequences corresponding to $(F_{\mathrm{I}}^p(K^{\bullet\bullet}))$ and $(F_{\mathrm{II}}^p(K^{\bullet\bullet}))$ respectively, called the *spectral sequence of the bicomplex* $K^{\bullet\bullet}$, and both having as their abutment the cohomology $(\mathrm{H}^n(K^{\bullet\bullet}))$. We show in addition (M, XV, 6) that we have

$$'E_2^{pq}(K^{\bullet\bullet}) = \mathrm{H}_{\mathrm{I}}^p(\mathrm{H}_{\mathrm{II}}^p(K^{\bullet\bullet})), \quad ''E_2^{pq}(K^{\bullet\bullet})) = \mathrm{H}_{\mathrm{II}}^p(\mathrm{H}_{\mathrm{I}}^p(K^{\bullet\bullet})). \tag{11.3.2.2}$$

Every morphism $u : K^{\bullet\bullet} \to K'^{\bullet\bullet}$ of bicomplexes is *ipso facto* compatible with the filtrations of the same type of $K^{\bullet\bullet}$ and $K'^{\bullet\bullet}$, thus define a morphism for each of the two spectral sequences; in addition, two *homotopic* morphisms define a homotopy *of order* $\leqslant 1$ of the corresponding filtered (simple) complexes, thus the *same* morphism for each of the two spectral sequences (M, XV, 6.1).

Proposition (11.3.3). — *Let $K^{\bullet\bullet} = (K^{ij})$ be a bicomplex in an abelian category* C.

(i) *If there exist i_0 and j_0 such that $K^{ij} = 0$ for $i < i_0$ or $j < j_0$ (resp. $i > i_0$ or $j > j_0$), then the two spectral sequences $'E(K^{\bullet\bullet})$ and $''E(K^{\bullet\bullet})$ are biregular.*

(ii) *If there exist i_0 and i_1 such that $K^{ij} = 0$ for $i < i_0$ or $i > i_1$ (resp. if there exist j_0 and j_1 such that $K^{ij} = 0$ for $j < j_0$ or $j > j_1$), then the two spectral sequences $'E(K^{\bullet\bullet})$ and $''E(K^{\bullet\bullet})$ are biregular.*

(iii) *If there exists i_0 such that $K^{ij} = 0$ for $i > i_0$ (resp. if there exists j_0 such that $K^{ij} = 0$ for $j < j_0$), then the spectral sequence $'E(K^{\bullet\bullet})$ is regular.*

(iv) *If there exists i_0 such that $K^{ij} = 0$ for $i < i_0$ (resp. if there exists j_0 such that $K^{ij} = 0$ for $j > j_0$), then the spectral sequence $''E(K^{\bullet\bullet})$ is regular.*

Proof. The proposition follows immediately from the definitions (11.1.3) and from (11.2.4), □

§12. Supplement on sheaf cohomology

§13. Projective limits in homological algebra

§14. Combinatorial dimension of a topological space

Summary

IV-1 | 5

Almost all of the preceding sections have been focused on the exposition of ideas of commutative algebra that will be used throughout Chapter IV. Even though a large amount of these ideas already appear in multiple works ([**CC**], [**Sam53a**], [**SZ60**], [**Ser55b**], [**Nag62**]), we thought that it would be more practical for the reader to have a coherent, vaguely independent exposition. Together with §§5, 6, and 7 of Chapter IV (where we use the language of schemes), these sections constitute, in the middle of our treatise, a miniature special treatise, somewhat independent of Chapters I to III, and one that aims to present, in a coherent manner, the properties of rings that "behave well" relative to operations such as completion, or integral closure, by systematically associating these properties to more general ideas.[11]

$\mathbf{0}_{\mathbf{IV}\text{-}\mathbf{1}}$ | 6

14.1. Combinatorial dimension of a topological space.

(14.1.1). Let I be an ordered set; a *chain* of elements of I is, by definition, a strictly-increasing finite sequence $i_0 < i_1 < \cdots < i_n$ of elements of I ($n \geqslant 0$); by definition, the *length* of this chain is n. If X is a topological space, the set of its *irreducible closed* subsets is ordered by inclusion, and so we have the notion of a *chain* of irreducible closed subsets of X.

Definition (14.1.2). — Let X be a topological space. We define the combinatorial dimension of X (or simply the dimension of X, if there is no risk of confusion), denoted by $\operatorname{dimc}(X)$ (or simply $\dim(X)$), to be the upper bound of lengths of chains of irreducible closed subsets of X. For all $x \in X$, we define the combinatorial dimension of X at x (or simply the dimension of X at x), denoted by $\dim_x(X)$, to be the number $\inf_U(\dim(U))$, where U varies over the open neighbourhoods of x in X.

[11] The majority of properties which we discuss were discovered by Chevalley, Zariski, Nagata, and Serre. The method used here was first developed in the autumn of 1961, in a course taught at Harvard University by A. Grothendieck.

It follows from this definition that we have

$$\dim(\varnothing) = -\infty$$

(the upper bound in $\overline{\mathbf{R}}$ of the empty set being $-\infty$). If (X_α) is the family of irreducible components of X, then we have

(14.1.2.1) $$\dim(X) = \sup_\alpha \dim(X_\alpha),$$

because every chain of irreducible closed subsets of X is, by definition, contained in some irreducible component of X, and, conversely, the irreducible components are closed in X, so every irreducible closed subset of an X_α is a irreducible closed subset of X.

Definition (14.1.3). — We say that a topological space X is equidimensional if all its irreducible components have the same dimension (which is thus equal to $\dim(X)$, by (14.1.2.1)).

Proposition (14.1.4). —

(i) *For every closed subset Y of a topological space X, we have* $\dim(Y) \leqslant \dim(X)$.

(ii) *If a topological space X is a finite union of closed subsets X_i, then we have* $\dim(X) = \sup_i \dim(X_i)$.

Proof. For every irreducible closed subset Z of Y, the closure $\overline{Z}$ of Z in X is irreducible ($\mathbf{0_I}$, 2.1.2), and $\overline{Z} \cap Y = Z$, whence (i). Now, if $X = \bigcup_{i=1}^n X_i$, where the X_i are closed, then every irreducible closed subset of X is contained in one of the X_i ($\mathbf{0_I}$, 2.1.1), and so every chain of irreducible closed subsets of X is contained in one of the X_i, whence (ii). □

From (14.1.4, i), we see that, for all $x \in X$, we can also write

(14.1.4.1) $$\dim_x(X) = \lim_U \dim(U),$$

where the limit is taken over the downward-directed set of open neighbourhoods of x in X. $\mathbf{0_{IV\text{-}1}}$ | 7

Corollary (14.1.5). — *Let X be a topological space, x a point of X, U a neighbourhood of x, and Y_i $(1 \leqslant i \leqslant n)$ closed subsets of U such that, for all i, $x \in Y_i$, and such that U is the union of the Y_i. Then we have*

(14.1.5.1) $$\dim_x(X) = \sup_i(\dim_x(Y_i)).$$

Proof. It follows from (14.1.4, ii) that we have $\dim_x(X) = \inf_V(\sup_i(\dim(Y_i \cap V)))$, where V ranges over the set of open neighbourhoods of x that are contained in U; similarly, we have $\dim_x(Y_i) = \inf_V(\dim(Y_i \cap V))$ for all i. The corollary is thus evident if

$$\sup_i(\dim_x(Y_i)) = +\infty;$$

if this were not the case, then there would be an open neighbourhood $V_0 \subset U$ of x such that $\dim(Y_i \cap V) = \dim_x(Y_i)$ for $1 \leqslant i \leqslant n$ and for all $V \subset V_0$, whence the conclusion. □

Proposition (14.1.6). — *For every topological space X, we have* $\dim(X) = \sup_{x\in X} \dim_x(X)$.

Proof. It follows from Definition (14.1.2) and Proposition (14.1.4) that $\dim_x(X) \leqslant \dim(X)$ for all $x \in X$. Now, let $Z_0 \subset Z_1 \subset \ldots \subset Z_n$ be a chain of irreducible closed subsets of X, and let $x \in Z_0$; for every open subset $U \subset X$ that contains x, $U \cap Z_i$ is irreducible ($\mathbf{0_I}$, 2.1.6) and closed in U, and since we have $\overline{U \cap Z_i} = Z_i$ in X, the $U \cap Z_i$ are pairwise distinct; thus $\dim(U) \geqslant n$, which finishes the proof. □

Corollary (14.1.7). — *If (X_α) is an open, or closed and locally finite, cover of X, then* $\dim(X) = \sup_\alpha(\dim(X_\alpha))$.

Proof. If X_α is a neighbourhood of $x \in X$, then $\dim_x(X) \leqslant \dim(X_\alpha)$, whence the claim for open covers. On the other hand, if the X_α are closed, and U is a neighbourhood of $x \in X$ which meets only finitely many of the X_α, then

$$\dim_x(X) \leqslant \dim(U) = \sup_\alpha(\dim(U \cap X_\alpha)) \leqslant \sup_\alpha(\dim(X_\alpha))$$

by (14.1.4), whence the other claim. □

Corollary (14.1.8). — *Let X be a Noetherian Kolmogoroff space* ($\mathbf{0_I}$, 2.1.3), *and F the set of closed points of X. Then* $\dim(X) = \sup_{x\in F} \dim_x(X)$.

Proof. With the notation from the proof of (14.1.6), it suffices to note that there exists a closed point in Z_0 ($\mathbf{0_I}$, 2.1.3). □

Proposition (14.1.9). — *Let X be a nonempty Noetherian Kolmogoroff space. To have* $\dim(X)=0$, *it is necessary and sufficient for X to be finite and discrete.*

Proof. If a space X is separated (and *a fortiori* if X is a discrete space), then all the irreducible closed subsets of X are single points, and so $\dim(X)=0$. Conversely, suppose that X is a Noetherian Kolmogoroff space such that $\dim(X)=0$; since every irreducible component of X contains a closed point ($\mathbf{0_I}$, 2.1.3), it must be exactly this single point. Since X has only a finite number of irreducible components, it is thus finite and discrete. □

Corollary (14.1.10). — *Let X be a Noetherian Kolmogoroff space. For a point $x\in X$ to be isolated, it is necessary and sufficient to have* $\dim_x(X)=0$.

Proof. The condition is clearly necessary (without any hypotheses on X). It is also sufficient, because it implies that $\dim(U)=0$ for any open neighbourhood U of x, and since U is a Noetherian Kolmogoroff space, U is finite and discrete. □ $\mathbf{0_{IV\text{-}1}}$ | 8

Proposition (14.1.11). — *The function $x\mapsto\dim_x(X)$ is upper semi-continuous on X.*

Proof. It is clear that this function is upper semi-continuous at every point where its value is $+\infty$. So suppose that $\dim_x(X)=n<+\infty$; then Equation (14.1.4.1) shows that there exists an open neighbourhood U_0 of x such that $\dim(U)=n$ for every open neighbourhood $U\subset U_0$ of x. So, for all $y\in U_0$ and every open neighbourhood $V\subset U_0$ of y, we have $\dim(V)\leqslant\dim(U_0)=n$ (14.1.4); we thus deduce from (14.1.4.1) that $\dim_y(X)\leqslant n$. □

Remark (14.1.12). — If X and Y are topological spaces, and $f:X\to Y$ a continuous map, then it can be the case that $\dim(f(X))>\dim(X)$; we obtain such an example by taking X to be a discrete space with 2 points, a and b, and Y to be the set $\{a,b\}$ endowed with the topology for which[12] the closed sets are $\varnothing$, $\{a\}$, and $\{a,b\}$; if $f:X\to Y$ is the identity map, then $\dim(Y)=1$ and $\dim(X)=0$. We note that Y is the spectrum of a discrete valuation ring A, of which a is the unique closed point, and b the generic point; if K and k are the field of fractions and the residue field of A (respectively), then X is the spectrum of the ring $k\times K$, and f is the continuous map corresponding to the homomorphism $(\varphi,\psi):A\to k\times K$, where $\varphi:A\to k$ and $\psi:A\to K$ are the canonical homomorphism (cf. (**IV**, 5.4.3)).

14.2. Codimension of a closed subset.

Definition (14.2.1). — Given an irreducible closed subset Y of a topological space X, we define the combinatorial codimension (or simply codimension) of Y in X, denoted by $\operatorname{codim}(Y,X)$, to be the upper bound of the lengths of chains of irreducible closed subsets of X of which Y is the smallest element. If Y is an arbitrary closed subset of X, then we define the codimension of Y in X, again denoted by $\operatorname{codim}(Y,X)$, to be the lower bound of the codimensions in X of the irreducible components of Y. We say that X is equicodimensional if all the minimal irreducible closed subsets of X has the same codimension in X.

It follows from this definition that $\operatorname{codim}(\varnothing,X)=+\infty$, since the lower bound of the empty set of $\overline{\mathbf{R}}$ is $+\infty$. If Y is closed in X, and if (X_α) (resp. (Y_α)) is the family of irreducible components of X (resp. Y), then every Y_β is contained in some X_α, and, more generally, every chain of irreducible closed subsets of X of which Y_β is the smallest element is formed of subsets of some X_α; we thus have

(14.2.1.1) $$\operatorname{codim}(Y,X)=\inf_\beta(\sup_\alpha(\operatorname{codim}(Y_\beta,X_\alpha))),$$

where, for every β, α ranges over the set of indices such that $Y_\beta\subset X_\alpha$.

Proposition (14.2.2). — *Let X be a topological space.*

(i) *If Φ is the set of irreducible closed subsets of X, then*

(14.2.2.1) $$\dim(X)=\sup_{Y\in\Phi}(\operatorname{codim}(Y,X)).$$

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(ii) *For every nonempty closed subset Y of X, we have*

(14.2.2.2) $$\dim(Y)+\operatorname{codim}(Y,X)\leqslant\dim(X).$$

(iii) *If Y, Z, and T are closed subsets of X such that $Y\subset Z\subset T$, then*

(14.2.2.3) $$\operatorname{codim}(Y,Z)+\operatorname{codim}(Z,T)\leqslant\operatorname{codim}(Y,T).$$

[12] *[Trans.] This is now often referred to as the* Sierpiński space, *or the* connected two-point set.

(iv) *For a closed subset Y of X to be such that* $\operatorname{codim}(Y,X)=0$, *it is necessary and sufficient for Y to contain an irreducible component of X.*

Proof. Claims (i) and (iv) are immediate consequences of Definition (14.2.1). To show (ii), we can restrict to the case where Y is irreducible, and then the equation follows from Definitions (14.1.1) and (14.2.1). Finally, to show (iii), we can, by Definition (14.2.1), first restrict to the case where Y is irreducible; then $\operatorname{codim}(Y,Z)=\sup_\alpha(\operatorname{codim}(Y,Z_\alpha))$ for the irreducible components Z_α of Z that contain Y; it is clear that $\operatorname{codim}(Y,T)\geqslant\operatorname{codim}(Y,Z)$, so the inequality is true if $\operatorname{codim}(Y,Z)=+\infty$; but if this were not the case, then there would exist some α such that $\operatorname{codim}(Y,Z)=\operatorname{codim}(Y,Z_\alpha)$, and by (14.2.1), we can restrict to the case where Z itself is irreducible; but then the inequality in (14.2.2.3) is an evident consequence of Definition (14.2.1). □

Proposition (14.2.3). — *Let X be a topological space, and Y a closed subset of X. For every open subset U of X, we have*

$$\operatorname{codim}(Y\cap U,U)\geqslant\operatorname{codim}(Y,X). \tag{14.2.3.1}$$

Furthermore, for this inequality (14.2.3.1) to be an equality, it is necessary and sufficient to have $\operatorname{codim}(Y,X)=\inf_\alpha(\operatorname{codim}(Y_\alpha,X))$, *where (Y_α) is the family of irreducible components of Y that meet U.*

Proof. We know ($\mathbf{0_I}$, 2.1.6) that $Z\mapsto\overline{Z}$ is a bijection from the set of irreducible closed subsets of U to the set of irreducible closed subsets of X that meet U, and, in particular, induces a correspondence between the irreducible components of $Y\cap U$ and the irreducible components of Y that meet U; if Y_α is one of the latter such components, then we have $\operatorname{codim}(Y_\alpha,X)=\operatorname{codim}(Y_\alpha\cap U,U)$, and the proposition then follows from Definition (14.2.1). □

Definition (14.2.4). — Let X be a topological space, Y a closed subset of X, and x a point of X. We define the codimension of Y in X at the point x, denoted by $\operatorname{codim}_x(Y,X)$, to be the number $\sup_U(\operatorname{codim}(Y\cap U,U))$, where U ranges over the set of open neighbourhoods of x in X.

By (14.2.3), we can also write

$$\operatorname{codim}_x(Y,X)=\lim_U(\operatorname{codim}(Y\cap U,U)), \tag{14.2.4.1}$$

where the limit is taken over the downward-directed set of open neighbourhoods of x in X. We note that we have

$$\operatorname{codim}_x(Y,X)=+\infty\ \text{if}\ x\in X-Y.$$

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Proposition (14.2.5). — *If $(Y_i)_{1\leqslant i\leqslant n}$ is a finite family of closed subsets of a topological space X, and Y is the union of this family, then*

$$\operatorname{codim}(Y,X)=\inf_i(\operatorname{codim}(Y_i,X)). \tag{14.2.5.1}$$

Proof. Every irreducible component of one of the Y_i is contained in an irreducible component of Y, and, conversely, every irreducible component of Y is also an irreducible component of ome of the Y_i ($\mathbf{0_I}$, 2.1.1); the conclusion then follows from Definition (14.2.1) and the inequality in (14.2.2.3). □

Corollary. — *Let X be a topological space, and Y a locally-Noetherian closed subspace of X.*

(i) *For all $x\in X$, there exists only a finite number of irreducible components Y_i $(1\leqslant i\leqslant n)$ of Y that contain x, and we have* $\operatorname{codim}_x(Y,X)=\inf_i(\operatorname{codim}(Y_i,X))$.
(ii) *The function* $x\mapsto\operatorname{codim}_x(Y,X)$ *is lower semi-continuous on X.*

Proof. By hypothesis, there exists an open neighbourhood U_0 of x in X such that $Y\cap U_0$ is Noetherian, and so U_0 has only a finite number of irreducible components, which are the intersections of U_0 with the irreducible components of Y; *a fortiori* there are only a finite number of irreducible components Y_i $(1\leqslant i\leqslant n)$ of Y that contain x, and we can, by replacing U_0 with an open neighbourhood $U\subset U_0$ of x that doesn't meet any of the Y_j that do not contain x, assume that the $Y_i\cap U$ are the irreducible components of $Y\cap U$; for every open neighbourhood $V\subset U$ of x in X, the $Y_i\cap V$ are thus the irreducible components of $Y\cap V$, and (14.2.3) then shows that $\operatorname{codim}(Y_i,X)=\operatorname{codim}(Y_i\cap V,V)$, which proves (i). Further, for every $x'\in U$, the irreducible components of Y that contain x' are certain Y_i, and so $\operatorname{codim}_{x'}(Y,X)\geqslant\operatorname{codim}_x(Y,X)$, which proves (ii). □

14.3. The chain condition.

(14.3.1). In a topological space X, we say that a chain $Z_0 \subset Z_1 \subset \cdots \subset Z_n$ of irreducible closed subsets if *saturated* if there does not exist an irreducible closed subset Z', distinct from each of the Z_i, such that $Z_k \subset Z' \subset Z_{k+1}$ for any k.

Proposition (14.3.2). — *Let X be a topological space such that, for any two irreducible closed subsets Y and Z of X with $Y \subset Z$, we have* $\operatorname{codim}(Y,Z) < +\infty$. *The following two conditions are equivalent.*

(a) *Any two saturated chains of closed irreducible subsets of X that have the same first and last elements as one another have the same length.*
(b) *If Y, Z, and T are irreducible closed subsets of X such that $Y \subset Z \subset T$, then*

$$\operatorname{codim}(Y,T) = \operatorname{codim}(Y,Z) + \operatorname{codim}(Z,T). \tag{14.3.2.1}$$

Proof. It is immediate that (a) implies (b). Conversely, suppose that (b) is satisfied, and we will show that if we have two saturated chains with the same first and last elements as one another, of lengths m and $n \leqslant m$ (respectively), then $m = n$. We proceed by induction on n, with the proposition being clear for $n = 1$. So suppose that $1 < n < m$, and let $Z_0 \subset Z_1 \subset \cdots \subset Z_n$ be a saturated chain such that there exists another saturated chain, $\mathbf{0}_{\mathbf{IV}\text{-}1}$ | 11
with first element Z_0 and last element Z_n, of length m. Since $\operatorname{codim}(Z_0,Z_n) \geqslant m > n$, and $\operatorname{codim}(Z_0,Z_1) = 1$, it follows from *(b)* that $\operatorname{codim}(Z_1,Z_n) = \operatorname{codim}(Z_0,Z_n) - 1 > n-1$, which contradicts our induction hypothesis. □

When the conditions of (14.3.2) are satisfied, we say that X satisfies the *chain condition*, or that it is a *catenary space*. It is clear that every closed subspace of a catenary space is catenary.

Proposition (14.3.3). — *Let X be a Noetherian Kolmogoroff space of finite dimension. The following conditions are equivalent.*

(a) *Any two maximal chains of irreducible closed subsets of X have the same length.*
(b) *X is equidimensional, equicodimensional, and catenary.*
(c) *X is equidimensional, and, for any irreducible closed subsets Y and Z of X with $Y \subset Z$, we have*

$$\dim(Z) = \dim(Y) + \operatorname{codim}(Y,Z). \tag{14.3.3.1}$$

(d) *X is equicodimensional, and, for any irreducible closed subsets Y and Z of X with $Y \subset Z$, we have*

$$\operatorname{codim}(Y,Z) = \operatorname{codim}(Y,Z) + \operatorname{codim}(Z,X). \tag{14.3.3.2}$$

Proof. The hypotheses on X imply that the first and last elements of a maximal chain of irreducible closed subsets of X are necessarily a closed point and an irreducible component of X (respectively) ($\mathbf{0}_{\mathbf{I}}$, 2.1.3); further, every saturated chain with first element Y and last element Z (thus $Y \subset Z$) is contained in a maximal chain whose elements differ from those of the given chain, or are contained in Y, or contain Z. These remarks immediately establish the equivalence between (a) and (b), and also show that if (a) is satisfied, then we have, for every irreducible closed subset Y fo X,

$$\dim(Y) + \operatorname{codim}(Y,X) = \dim(X); \tag{14.3.3.3}$$

from (14.3.2.1), we immediately deduce (14.3.3.1) and (14.3.3.2) from (14.3.3.3). Conversely, (14.3.3.1) implies (14.3.2.1), and so (14.3.3.1) implies the chain condition, by (14.3.2); further, by applying (14.3.3.1) to the case where Y is a single closed point x of X, and Z is an irreducible component of X, we get that $\operatorname{codim}(\{x\},X) = \dim(Z)$; we thus conclude that (c) implies (b). Similarly, (14.3.3.2) implies (14.3.2.1), and thus the chain condition; further, with the same choice of Y and Z as above, (14.3.3.2) again implies that $\operatorname{codim}(\{x\},X) = \dim(Z)$, and so (since every irreducible component of X contains a closed point, by ($\mathbf{0}_{\mathbf{I}}$, 2.1.3)), (d) implies (b). □

We say that a Noetherian Kolmogoroff space is *biequidimensional* if it is of *finite dimension* and if it verifies any of the (equivalent) conditions of (14.3.3).

Corollary (14.3.4). — *Let X be a biequidimensional Noetherian Kolmogoroff space; then, for every closed point x of X, and every irreducible component Z of X, we have*

$$\dim(X) = \dim(Z) = \operatorname{codim}(\{x\},X) = \dim_x(X). \tag{14.3.4.1}$$

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Proof. The last equality follows from the fact that, if $Y_0 = \{x\} \subset Y_1 \subset \cdots \subset Y_m$ is a maximal chain of irreducible closed subsets of X, and U an open neighbourhood of x, then the $U \cap Y_i$ are pairwise disjoint irreducible closed subsets of U (because $\overline{U \cap Y_i} = Y_i$), whence $\dim(U) = \dim(X)$, by (14.1.4). □

Corollary (14.3.5). — *Let X be a Noetherian Kolmogoroff space; if X is biequidimensional, then so is every union of irreducible components of X, and every irreducible closed subset of X. In addition, for every closed subset Y of X, we have*

(14.3.5.1) $$\dim(Y) + \operatorname{codim}(Y,X) = \dim(X).$$

Proof. Every chain of irreducible closed subsets of X is contained in an irreducible component of X, and so the first claim follows immediately from (14.3.3). Further, if X' is an irreducible closed subset of X, then X' trivially satisfies the conditions of (14.3.3, c), whence the second claim.

Finally, to show (14.3.5.1), note that we have seen, in the proof of (14.3.3), that this equation is true whenever Y is irreducible; if Y_i $(1 \leqslant i \leqslant m)$ are the irreducible components of Y, then the Y_i for which $\dim(Y_i)$ is the largest are also those for which $\operatorname{codim}(Y_i,X)$ is the smallest; so (14.3.5.1) follows from the definitions of $\dim(Y)$ and $\operatorname{codim}(Y,X)$. □

Remark (14.3.6). — The reader will note that the proof of (14.3.2) applies to any ordered set, and the fact that we are working with the example of a set of irreducible closed subsets of a topological space is not used at all in the proof. It is the same in the proof of (14.3.3), which holds, more generally, for any ordered set E such that, for all $x \in E$, there exists some $z \leqslant x$ which is *minimal* in E, and such that the length of chains of elements of E is bounded.

CHAPTER I

The language of schemes (EGA I)

Summary

Sections §§1–8 do little more than develop a language, which will be used in what follows. It should be noted, however, that, in accordance with the general spirit of this treatise, §§7–8 will be used less than the others, and in a less essential way; we have speak of Chevalley's schemes only to make the link with the language of Chevalley [**CC**] and Nagata [**Nag58a**]. Section §9 gives definitions and results concerning quasi-coherent sheaves, some of which are no longer limited to a translation into a "geometric" language of known notions of commutative algebra, but are already of a global nature; they will be indispensable, in the following chapters, for the global study of morphisms. Finally, §10 introduces a generalization of the notion of schemes, which will be used as an intermediary in Chapter III to formulate and prove, in a convenient way, the fundamental results of the cohomological study of the proper morphisms; moreover, it should be noted that the notion of formal schemes seems indispensable in expressing certain facts about the "theory of modules" (classification problems of algebraic varieties). The results of §10 will not be used before §3 of Chapter III, and it is recommended to omit their reading until then. **I** | 79

§1. Affine schemes

1.1. The prime spectrum of a ring.

(1.1.1). *Notation.* Let A be a (commutative) ring, and M an A-module. In this chapter and the following, we will constantly use the following notation: **I** | 80

- $\operatorname{Spec}(A) =$ *set of prime ideals* of A, also called the *prime spectrum* of A; for $x \in X = \operatorname{Spec}(A)$, it will often be convenient to write $\mathfrak{j}_x$ instead of x. For $\operatorname{Spec}(A)$ to be *empty*, it is necessary and sufficient for the ring A to be 0.
- $A_x = A_{\mathfrak{j}_x} =$ *(local) ring of fractions* $S^{-1}A$, where $S = A - \mathfrak{j}_x$.
- $\mathfrak{m}_x = \mathfrak{j}_x A_{\mathfrak{j}_x} =$ *maximal ideal of* A_x.
- $k(x) = A_x/\mathfrak{m}_x =$ *residue field of* A_x, canonically isomorphic to the field of fractions of the integral ring $A/\mathfrak{j}_x$, with which we identify it.
- $f(x) =$ *class of* f *mod.* $\mathfrak{j}_x$ in $A/\mathfrak{j}_x \subset k(x)$, for $f \in A$ and $x \in X$. We also say that $f(x)$ is the *value* of f at a point $x \in \operatorname{Spec}(A)$; the equations $f(x) = 0$ and $f \in \mathfrak{j}_x$ are *equivalent.*
- $M_x = M \otimes_A A_x =$ *module of fractions with denominators in* $A - \mathfrak{j}_x$.
- $\mathfrak{r}(E) =$ *radical of the ideal of* A *generated by a subset* E *of* A.
- $V(E) =$ *set of* $x \in X$ *such that* $E \subset \mathfrak{j}_x$ (or the set of $x \in X$ such that $f(x) = 0$ for all $f \in E$), for $E \subset A$. So we have

(1.1.1.1) $$\mathfrak{r}(E) = \bigcap_{x \in V(E)} \mathfrak{j}_x.$$

- $V(f) = V(\{f\})$ for $f \in A$.
- $D(f) = X - V(f) =$ *set of* $x \in X$ *where* $f(x) \neq 0$.

Proposition (1.1.2). — *We have the following properties:*

(i) $V(0) = X$, $V(1) = \varnothing$.
(ii) *The relation* $E \subset E'$ *implies* $V(E) \supset V(E')$.
(iii) *For each family* (E_λ) *of subsets of* A, $V(\bigcup_\lambda E_\lambda) = V(\sum_\lambda E_\lambda) = \bigcap_\lambda V(E_\lambda)$.
(iv) $V(EE') = V(E) \cup V(E')$.
(v) $V(E) = V(\mathfrak{r}(E))$.

Proof. The properties (i), (ii), (iii) are trivial, and (v) follows from (ii) and from equation (1.1.1.1). It is evident that $V(EE') \supset V(E) \cap V(E')$; conversely, if $x \notin V(E)$ and $x \notin V(E')$, then there exists $f \in E$ and $f' \in E'$ such that $f(x) \neq 0$ and $f'(x) \neq 0$ in $k(x)$, hence $f(x)f'(x) \neq 0$, i.e., $x \notin V(EE')$, which proves (iv). □

Proposition (1.1.2) shows, among other things, that sets of the form $V(E)$ (where E varies over the subsets of A) are the *closed sets* of a topology on X, which we will call the *spectral topology* [1]; unless expressly stated otherwise, we always assume that $X = \operatorname{Spec}(A)$ is equipped with the spectral topology.

(1.1.3). For each subset Y of X, we denote by $\mathfrak{j}(Y)$ the set of $f \in A$ such that $f(y) = 0$ for all $y \in Y$; equivalently, $\mathfrak{j}(Y)$ is the intersection of the prime ideals $\mathfrak{j}_y$ for $y \in Y$. It is clear that the relation $Y \subset Y'$ implies that $\mathfrak{j}(Y) \supset \mathfrak{j}(Y')$ and that we have **I | 81**

$$\mathfrak{j}\left(\bigcup_\lambda Y_\lambda\right) = \bigcap_\lambda \mathfrak{j}(Y_\lambda) \tag{1.1.3.1}$$

for each family (Y_λ) of subsets of X. Finally we have

$$\mathfrak{j}(\{x\}) = \mathfrak{j}_x. \tag{1.1.3.2}$$

Proposition (1.1.4). —

(i) *For each subset* E *of* A, *we have* $\mathfrak{j}(V(E)) = \mathfrak{r}(E)$.
(ii) *For each subset* Y *of* X, $V(\mathfrak{j}(Y)) = \overline{Y}$, *the closure of* Y *in* X.

Proof. (i) is an immediate consequence of the definitions and (1.1.1.1); on the other hand, $V(\mathfrak{j}(Y))$ is closed and contains Y; conversely, if $Y \subset V(E)$, we have $f(y) = 0$ for $f \in E$ and all $y \in Y$, so $E \subset \mathfrak{j}(Y)$, $V(E) \supset V(\mathfrak{j}(Y))$, which proves (ii). □

Corollary (1.1.5). — *The closed subsets of* $X = \operatorname{Spec}(A)$ *and the ideals of* A *equal to their radicals (in other words, those that are the intersection of prime ideals) correspond bijectively by the inclusion-reversing maps* $Y \mapsto \mathfrak{j}(Y)$, $\mathfrak{a} \mapsto V(\mathfrak{a})$; *the union* $Y_1 \cup Y_2$ *of two closed subsets corresponds to* $\mathfrak{j}(Y_1) \cap \mathfrak{j}(Y_2)$, *and the intersection of any family* (Y_λ) *of closed subsets corresponds to the radical of the sum of the* $\mathfrak{j}(Y_\lambda)$.

Corollary (1.1.6). — *If* A *is a Noetherian ring,* $X = \operatorname{Spec}(A)$ *is a Noetherian space.*

Note that the converse of this corollary is false, as shown by any non-Noetherian integral ring with a single prime ideal $\neq \{0\}$ (for example a nondiscrete valuation ring of rank 1).

As an example of ring A whose spectrum is not a Noetherian space, one can consider the ring $\mathscr{C}(Y)$ of continuous real functions on an infinite compact space Y; we know that, as a set, Y corresponds to the set of maximal ideals of A, and it is easy to see that the topology induced on Y by that of $X = \operatorname{Spec}(A)$ is the original topology of Y. Since Y is not a Noetherian space, the same is true for X.

Corollary (1.1.7). — *For each* $x \in X$, *the closure of* $\{x\}$ *is the set of* $y \in X$ *such that* $\mathfrak{j}_x \subset \mathfrak{j}_y$. *For* $\{x\}$ *to be closed, it is necessary and sufficient that* $\mathfrak{j}_x$ *is maximal.*

Corollary (1.1.8). — *The space* $X = \operatorname{Spec}(A)$ *is a Kolmogoroff space.*

Proof. If x and y are two distinct points of X, we have either $\mathfrak{j}_x \not\subset \mathfrak{j}_y$ or $\mathfrak{j}_y \not\subset \mathfrak{j}_x$, so one of the points x, y does not belong to the closure of the other. □

[1] The introduction of this topology in algebraic geometry is due to Zariski. So this topology is usually called the "Zariski topology" on X.

(1.1.9). According to Proposition (1.1.2, iv), for two elements f, g of A, we have

$$D(fg) = D(f) \cap D(g). \tag{1.1.9.1}$$

Note also that the equality $D(f) = D(g)$ means, according to Proposition (1.1.4, i) and Proposition (1.1.2, v), that $\mathfrak{r}(f) = \mathfrak{r}(g)$, or that the minimal prime ideals containing (f) and (g) are the same; in particular, it is also the case when $f = ug$, where u is invertible.

Proposition (1.1.10). — I | 82

(i) *When f ranges over A, the sets $D(f)$ forms a basis for the topology of X.*
(ii) *For every $f \in A$, $D(f)$ is quasi-compact. In particular, $X = D(1)$ is quasi-compact.*

Proof.

(i) Let U be an open set in X; by definition, we have $U = X - V(E)$ where E is a subset of A, and $V(E) = \bigcap_{f\in E} V(f)$, hence $U = \bigcup_{f\in E} D(f)$.
(ii) By (i), it suffices to prove that, if $(f_\lambda)_{\lambda\in L}$ is a family of elements of A such that $D(f) \subset \bigcup_{\lambda\in L} D(f_\lambda)$, then there exists a finite subset J of L such that $D(f) \subset \bigcup_{\lambda\in J} D(f_\lambda)$. Let $\mathfrak{a}$ be the ideal of A generated by the f_λ; we have, by hypothesis, that $V(f) \supset V(\mathfrak{a})$, so $\mathfrak{r}(f) \subset \mathfrak{r}(\mathfrak{a})$; since $f \in \mathfrak{r}(f)$, there exists an integer $n \geqslant 0$ such that $f^n \in \mathfrak{a}$. But then f^n belongs to the ideal $\mathfrak{b}$ generated by the finite subfamily $(f_\lambda)_{\lambda\in J}$, and we have $V(f) = V(f^n) \supset V(\mathfrak{b}) = \bigcap_{\lambda\in J} V(f_\lambda)$, that is to say, $D(f) \supset \bigcup_{\lambda\in J} D(f_\lambda)$.

□

Proposition (1.1.11). — *For each ideal $\mathfrak{a}$ of A,* $\operatorname{Spec}(A/\mathfrak{a})$ *is canonically identified with the closed subspace $V(\mathfrak{a})$ of* $\operatorname{Spec}(A)$.

Proof. We know there is a canonical bijective correspondence (respecting the inclusion order structure) between ideals (resp. prime ideals) of $A/\mathfrak{a}$ and ideals (resp. prime ideals) of A containing $\mathfrak{a}$. □

Recall that the set $\mathfrak{N}$ of nilpotent elements of A (the *nilradical* of A) is an ideal equal to $\mathfrak{r}(0)$, the intersection of all the prime ideals of A (**0**, 1.1.1).

Corollary (1.1.12). — *The topological spaces* $\operatorname{Spec}(A)$ *and* $\operatorname{Spec}(A/\mathfrak{N})$ *are canonically homeomorphic.*

Proposition (1.1.13). — *For $X = \operatorname{Spec}(A)$ to be irreducible* (**0**, 2.1.1)*, it is necessary and sufficient that the ring $A/\mathfrak{N}$ is integral (or, equivalently, that the ideal $\mathfrak{N}$ is prime).*

Proof. By virtue of Corollary (1.1.12), we can restrict to the case where $\mathfrak{N} = 0$. If X is reducible, then there exist two distinct closed subsets Y_1 and Y_2 of X such that $X = Y_1 \cup Y_2$, so $\mathfrak{j}(X) = \mathfrak{j}(Y_1) \cap \mathfrak{j}(Y_2) = 0$, since the ideals $\mathfrak{j}(Y_1)$ and $\mathfrak{j}(Y_2)$ are distinct from (0) (1.1.5); so A is not integral. Conversely, if there are elements $f \neq 0$, $g \neq 0$ of A such that $fg = 0$, we have $V(f) \neq X$, $V(g) \neq X$ (since the intersection of all the prime ideals of A is (0)), and $X = V(fg) = V(f) \cup V(g)$. □

Corollary (1.1.14). —

(i) *In the bijective correspondence between closed subsets of $X = \operatorname{Spec}(A)$ and ideals of A equal to their radicals, the irreducible closed subsets of X correspond to the prime ideals of A. In particular, the irreducible components of X correspond to the minimal prime ideals of A.*
(ii) *The map $x \mapsto \overline{\{x\}}$ establishes a bijective correspondence between X and the set of closed irreducible subsets of X (*in other words, *all closed irreducible subsets of X admit exactly one generic point).*

Proof. (i) follows immediately from (1.1.13) and (1.1.11); and for proving (ii), we can, by (1.1.11), restrict to the case where X is irreducible; then, according to Proposition (1.1.13), there exists a smaller prime ideal $\mathfrak{N}$ in A, which corresponds to the generic point of X; in addition, X admits at most one generic point since it is a Kolmogoroff space ((1.1.8) and (**0**, 2.1.3)). □ I | 83

Proposition (1.1.15). — *If $\mathfrak{J}$ is an ideal in A containing the radical $\mathfrak{N}(A)$, the only neighborhood of $V(\mathfrak{J})$ in $X = \operatorname{Spec}(A)$ is the whole space X.*

Proof. Each maximal ideal of A belongs, by definition, to $V(\mathfrak{J})$. As each ideal $\mathfrak{a} \neq A$ of A is contained in a maximal ideal, we have $V(\mathfrak{a}) \cap V(\mathfrak{J}) \neq 0$, whence the proposition. □

1.2. Functorial properties of prime spectra of rings.

(1.2.1). Let A, A' be two rings, and

$$\varphi : A' \longrightarrow A$$

a homomorphism of rings. For each prime ideal $x = \mathfrak{j}_x \in \operatorname{Spec}(A) = X$, the ring $A'/\varphi^{-1}(\mathfrak{j}_x)$ is canonically isomorphic to a subring of $A/\mathfrak{j}_x$, and so it is integral, or, in other words, $\varphi^{-1}(\mathfrak{j}_x)$ is a prime ideal of A'; we denote it by ${}^a\varphi(x)$, and we have thus defined a map

$${}^a\varphi : X = \operatorname{Spec}(A) \longrightarrow X' = \operatorname{Spec}(A')$$

(also denoted $\operatorname{Spec}(\varphi)$), that we call the map *associated* to the homomorphism φ. We denote by φ^x the injective homomorphism from $A'/\varphi^{-1}(\mathfrak{j}_x)$ to $A/\mathfrak{j}_x$ induced by φ by passing to quotients, as well as its canonical extension to a monomorphism of fields

$$\varphi^x : k({}^a\varphi(x)) \longrightarrow k(x);$$

for each $f' \in A'$, we therefore have, by definition, **Err**$_{\mathrm{II}}$

$$\varphi^x(f'({}^a\varphi(x))) = (\varphi(f'))(x) \qquad (x \in X). \tag{1.2.1.1}$$

Proposition (1.2.2). —

(i) *For each subset E' of A', we have*

$${}^a\varphi^{-1}(V(E')) = V(\varphi(E')), \tag{1.2.2.1}$$

and in particular, for each $f' \in A'$,

$${}^a\varphi^{-1}(D(f')) = D(\varphi(f')). \tag{1.2.2.2}$$

(ii) *For each ideal $\mathfrak{a}$ of A, we have*

$$\overline{{}^a\varphi(V(\mathfrak{a}))} = V(\varphi^{-1}(\mathfrak{a})). \tag{1.2.2.3}$$

Proof. The relation ${}^a\varphi(x) \in V(E')$ is, by definition, equivalent to $E' \subset \varphi^{-1}(\mathfrak{j}_x)$, so $\varphi(E') \subset \mathfrak{j}_x$, and finally $x \in V(\varphi(E'))$, hence (i). To prove (ii), we can suppose that $\mathfrak{a}$ is equal to its radical, since $V(\mathfrak{r}(\mathfrak{a})) = V(\mathfrak{a})$ (1.1.2, v) and $\varphi^{-1}(\mathfrak{r}(\mathfrak{a})) = \mathfrak{r}(\varphi^{-1}(\mathfrak{a}))$; the relation $f' \in \mathfrak{a}'$ is, by definition, equivalent to $f'(x') = 0$ for each $x \in {}^a\varphi(Y)$, so, by Equation (1.2.1.1), it is also equivalent to $\varphi(f')(x) = 0$ for each $x \in Y$, or to $\varphi(f') \in \mathfrak{j}(Y) = \mathfrak{a}$, since $\mathfrak{a}$ is equal to its radical; hence (ii). □

Corollary (1.2.3). — *The map ${}^a\varphi$ is continuous.*

We remark that, if A'' is a third ring, and φ' a homomorphism $A'' \to A'$, then we have ${}^a(\varphi' \circ \varphi) = {}^a\varphi \circ {}^a\varphi'$; this result, with Corollary (1.2.3), says that $\operatorname{Spec}(A)$ is a *contravariant functor* in A, from the category of rings to that of topological spaces. **I | 84**

Corollary (1.2.4). — *Suppose that φ is such that every $f \in A$ can be written as $f = h\varphi(f')$, where h is invertible in A (*which, in particular, is the case when φ is *surjective). Then ${}^a\varphi$ is a homeomorphism from X to ${}^a\varphi(X)$.*

Proof. We show that for each subset $E \subset A$, there exists a subset E' of A' such that $V(E) = V(\varphi(E'))$; according to the (T_0) axiom (1.1.8) and the formula (1.2.2.1), this implies first of all that ${}^a\varphi$ is injective, and then, by (1.2.2.1), that ${}^a\varphi$ is a homeomorphism. But it suffices, for each $f \in E$, to take $f' \in A'$ such that $h\varphi(f') = f$ with h invertible in A; the set E' of these elements f' is exactly what we are searching for. □

(1.2.5). In particular, when φ is the canonical homomorphism from A to a ring quotient $A/\mathfrak{a}$, we again get (1.1.12), and ${}^a\varphi$ is the *canonical injection* of $V(\mathfrak{a})$, identified with $\operatorname{Spec}(A/\mathfrak{a})$, into $X = \operatorname{Spec}(A)$.

Another particular case of (1.2.4):

Corollary (1.2.6). — *If S is a multiplicative subset of A, the spectrum* $\operatorname{Spec}(S^{-1}A)$ *is canonically identified (with its topology) with the subspace of $X =$* $\operatorname{Spec}(A)$ *consisting of the x such that $\mathfrak{j}_x \cap S = \varnothing$.*

Proof. We know by (**0**, 1.2.6) that the prime ideals of $S^{-1}A$ are the ideals $S^{-1}\mathfrak{j}_x$ such that $\mathfrak{j}_x \cap S = \varnothing$, and that we have $\mathfrak{j}_x = (i_A^S)^{-1}(S^{-1}\mathfrak{j}_x)$. It then suffices to apply Corollary (1.2.4) to the i_A^S. □

Corollary (1.2.7). — *For ${}^a\varphi(X)$ to be dense in X', it is necessary and sufficient for each element of the kernel* $\operatorname{Ker}\varphi$ *to be nilpotent.*

Proof. Applying Equation (1.2.2.3) to the ideal $\mathfrak{a} = (0)$, we have $\widetilde{{}^a\varphi(X)} = V(\operatorname{Ker}\varphi)$, and for $V(\operatorname{Ker}\varphi) = X'$ **Err**$_{\mathrm{II}}$
to hold, it is necessary and sufficient for $\operatorname{Ker}\varphi$ to be contained in all the prime ideals of A', or, equivalently, in the nilradical $\mathfrak{r}'$ of A'. □ **Err**$_{\mathrm{II}}$

1.3. Sheaf associated to a module.

(1.3.1). Let A be a commutative ring, M an A-module, f an element of A, and S_f the multiplicative set consisting of the f^n, where $n \geqslant 0$. Recall that we set $A_f = S_f^{-1}A$, $M_f = S_f^{-1}M$. If S'_f is the saturated multiplicative subset of A consisting of the $g \in A$ which divide an element of S_f, we know that A_f and M_f are canonically identified with $S_f'^{-1}A$ and $S_f'^{-1}M$ (**0**, 1.4.3).

Lemma (1.3.2). — *The following conditions are equivalent:*

(a) $g \in S'_f$;
(b) $S'_g \subset S'_f$;
(c) $f \in \mathfrak{r}(g)$;
(d) $\mathfrak{r}(f) \subset \mathfrak{r}(g)$;
(e) $V(g) \subset V(f)$;
(f) $D(f) \subset D(g)$.

Proof. This follows immediately from the definitions and (1.1.5). □

(1.3.3). If $D(f) = D(g)$, then Lemma (1.3.2, b) shows that $M_f = M_g$. More generally, if $D(f) \supset D(g)$, then $S'_f \subset S'_g$, and we know (**0**, 1.4.1) that there exists a canonical functorial homomorphism

$$\rho_{g,f} : M_f \longrightarrow M_g,$$

and if $D(f) \supset D(g) \supset D(h)$, we have (**0**, 1.4.4)

(1.3.3.1) $$\rho_{h,g} \circ \rho_{g,f} = \rho_{h,f}.$$ I | 85

When f ranges over the elements of $A - \mathfrak{j}_x$ (for a given x in $X = \operatorname{Spec}(A)$), the sets S'_f constitute an increasing filtered set of subsets of $A - \mathfrak{j}_x$, since for elements f and g of $A - \mathfrak{j}_x$, S'_f and S'_g are contained in S'_{fg}; since the union of the S'_f over $f \in A - \mathfrak{j}_x$ is $A - \mathfrak{j}_x$, we conclude (**0**, 1.4.5) that the A_x-module M_x is canonically identified with the *inductive limit* $\varinjlim M_f$, relative to the family of homomorphisms $(\rho_{g,f})$. We denote by

$$\rho_x^f : M_f \longrightarrow M_x$$

the canonical homomorphism for $f \in A - \mathfrak{j}_x$ (or, equivalently, $x \in D(f)$).

Definition (1.3.4). — We define the structure sheaf of the prime spectrum $X = \operatorname{Spec}(A)$ (resp. the sheaf associated to the A-module M), denoted by $\widetilde{A}$ or $\mathscr{O}_X$ (resp. $\widetilde{M}$) as the sheaf of rings (resp. the $\widetilde{A}$-module) associated to the presheaf $D(f) \mapsto A_f$ (resp. $D(f) \mapsto M_f$), defined on the basis $\mathfrak{B}$ of X consisting of the $D(f)$ for $f \in A$ ((1.1.10), (**0**, 3.2.1), and (**0**, 3.5.6)).

We saw (**0**, 3.2.4) that the stalk $\widetilde{A}_x$ (resp. $\widetilde{M}_x$) *can be identified with the ring* A_x (resp. *the* A_x*-module* M_x); we denote by

$$\theta_f : A_f \longrightarrow \Gamma(D(f), \widetilde{A})$$
$$(\text{resp. } \theta_f : M_f \longrightarrow \Gamma(D(f), \widetilde{M})),$$

the canonical map, so that, for all $x \in D(f)$ and all $\xi \in M_f$, we have

(1.3.4.1) $$(\theta_f(\xi))_x = \rho_x^f(\xi).$$

Proposition (1.3.5). — *$\widetilde{M}$ is an exact functor, covariant in M, from the category of A-modules to the category of $\widetilde{A}$-modules.*

Proof. Indeed, let M, N be two A-modules, and u a homomorphism $M \to N$; for each $f \in A$, u corresponds canonically to a homomorphism u_f from the A_f-module M_f to the A_f-module N_f, and the diagram (for $D(g) \subset D(f)$)

$$\begin{tikzcd} M_f \arrow[r, "u_f"] \arrow[d, "\rho_{g,f}"'] & N_f \arrow[d, "\rho_{g,f}"] \\ M_g \arrow[r, "u_g"] & N_g \end{tikzcd}$$

is commutative (1.4.1); these homomorphisms then define a homomorphism of $\widetilde{A}$-modules $\widetilde{u} : \widetilde{M} \to \widetilde{N}$ (**0**, 3.2.3). In addition, for each $x \in X$, $\widetilde{u}_x$ is the inductive limit of the u_f for $x \in D(f)$ $(f \in A)$, and as a result (**0**, 1.4.5), if we canonically identify $\widetilde{M}_x$ and $\widetilde{N}_x$ with M_x and N_x respectively, then $\widetilde{u}_x$ is identified with the homomorphism

u_x canonically induced by u. If P is a third A-module, v a homomorphism $N \to P$, and $w = v \circ u$, it is immediate that $w_x = v_x \circ u_x$, so $\widetilde{w} = \widetilde{v} \circ \widetilde{u}$. We have therefore clearly defined a *covariant (in M) functor* $\widetilde{M}$, from the category of A-modules to that of $\widetilde{A}$-modules. *This functor is exact*, since, for each $x \in X$, M_x is an exact functor in M (**0**, 1.3.2); in addition, we have $\operatorname{Supp}(M) = \operatorname{Supp}(\widetilde{M})$, by definition ((**0**, 1.7.1) and (**0**, 3.1.6)). ☐

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Proposition (1.3.6). — *For each $f \in A$, the open subset $D(f) \subset X$ is canonically identified with the prime spectrum* $\operatorname{Spec}(A_f)$*, and the sheaf $\widetilde{M_f}$ associated to the A_f-module M_f is canonically identified with the restriction $\widetilde{M}|D(f)$.*

Proof. The first assertion is a particular case of (1.2.6). In addition, if $g \in A$ is such that $D(g) \subset D(f)$, then M_g is canonically identified with the module of fractions of M_f whose denominators are the powers of the canonical image of g in A_f (**0**, 1.4.6). The canonical identification of $\widetilde{M_f}$ with $\widetilde{M}|D(f)$ then follows from the definitions. ☐

Theorem (1.3.7). — *For each A-module M and each $f \in A$, the homomorphism*

$$\theta_f : M_f \longrightarrow \Gamma(D(f), \widetilde{M})$$

is bijective (in other words, the presheaf $D(f) \mapsto M_f$ is a *sheaf*). *In particular, M can be identified with $\Gamma(X, \widetilde{M})$ via* θ_1.

Proof. We note that, if $M = A$, then θ_f is a homomorphism of structure rings; Theorem (1.3.7) then implies that, if we identify the rings A_f and $\Gamma(D(f), \widetilde{A})$ via θ_f, the homomorphism $\theta_f : M_f \to \Gamma(D(f), \widetilde{M})$ is an isomorphism of *modules.*

We show first that θ_f is *injective.* Indeed, if $\xi \in M_f$ is such that $\theta_f(\xi) = 0$, then, for each prime ideal $\mathfrak{p}$ of A_f, there exists $h \notin \mathfrak{p}$ such that $h\xi = 0$; as the annihilator of ξ is not contained in any prime ideal of A_f, each A_f integral, and so $\xi = 0$.

It remains to show that θ_f is *surjective*; we can restrict to the case where $f = 1$ (the general case then following by "localizing", using (1.3.6)). Now let s be a section of $\widetilde{M}$ over X; according to (1.3.4) and (1.1.10, ii), there exists a *finite* cover $(D(f_i))_{i \in I}$ of X ($f_i \in A$) such that, for each $i \in I$, the restriction $s_i = s|D(f_i)$ is of the form $\theta_{f_i}(\xi_i)$, where $\xi_i \in M_{f_i}$. If i, j are indices of I, and if the restrictions of s_i and s_j to $D(f_i) \cap D(f_j) = D(f_i f_j)$ are equal, then it follows, by definition of M, that

(1.3.7.1) $$\rho_{f_i f_j, f_i}(\xi_i) = \rho_{f_i f_j, f_j}(\xi_j).$$

By definition, we can write, for each $i \in I$, $\xi_i = z_i / f_i^{n_i}$, where $z_i \in M$, and. since I is finite, by multiplying each z_i by a power of f_i, we can assume that all the n_i are equal to one single n. Then, by definition, (1.3.7.1) implies that there exists an integer $m_{ij} \geqslant 0$ such that $(f_i f_j)^{m_{ij}}(f_j^n z_i - f_i^n z_j) = 0$, and we can moreover suppose that the m_{ij} are equal to the one single m; then replacing the z_i by $f_i^m z_i$, it remains to prove the case where $m = 0$, in other words, the case where we have

(1.3.7.2) $$f_j^n z_i = f_i^n z_j$$

for any i, j. We have $D(f_i^n) = D(f_i)$, and since the $D(f_i)$ form a cover of X, the ideal generated by the f_i^n is A; in other words, there exist elements $g_i \in A$ such that $\sum_i g_i f_i^n = 1$. Then consider the element $z = \sum_i g_i z_i$ of M; in (1.3.7.2), we have $f_i^n z = \sum_j g_j f_i^n z_j = (\sum_j g_j f_j^n) z_i = z_i$, where, by definition, $\xi_i = z/1$ in M_{f_i}. We conclude that the s_i are the restrictions to $D(f_i)$ of $\theta_1(z)$, which proves that $s = \theta_1(z)$ and finishes the proof. ☐ I | 87

Corollary (1.3.8). — *Let M and N be A-modules; the canonical homomorphism $u \mapsto \widetilde{u}$ from* $\operatorname{Hom}_A(M, N)$ *to* $\operatorname{Hom}_{\widetilde{A}}(\widetilde{M}, \widetilde{N})$ *is bijective. In particular, the equations $M = 0$ and $\widetilde{M} = 0$ are equivalent.*

Proof. Consider the canonical homomorphism $v \mapsto \Gamma(v)$ from $\operatorname{Hom}_{\widetilde{A}}(\widetilde{M}, \widetilde{N})$ to $\operatorname{Hom}_{\Gamma(\widetilde{A})}(\Gamma(\widetilde{M}), \Gamma(\widetilde{N}))$; the latter module is canonically identified with $\operatorname{Hom}_A(M, N)$, by Theorem (1.3.7). It remains to show that $u \mapsto \widetilde{u}$ and $v \mapsto \Gamma(v)$ are inverses of each other; it is evident that $\Gamma(\widetilde{u}) = u$ by definition of $\widetilde{u}$; on the other hand, if we let $u = \Gamma(v)$ for $v \in \operatorname{Hom}_{\widetilde{A}}(\widetilde{M}, \widetilde{N})$, then the map $w : \Gamma(D(f), \widetilde{M}) \to \Gamma(D(f), \widetilde{N})$ canonically induced from v is such that the diagram

$$\begin{tikzcd} M \arrow[r, "u"] \arrow[d, "\rho_{f,1}"'] & N \arrow[d, "\rho_{f,1}"] \\ M_f \arrow[r, "w"] & N_f \end{tikzcd}$$

is commutative; so we necessarily have that $w = u_f$ for all $f \in A$ (**0**, 1.2.4), which shows that $\widetilde{\Gamma(v)} = v$. ☐

Corollary (1.3.9). —

(i) *Let u be a homomorphism from an A-module M to an A-module N; then the sheaves associated to* $\operatorname{Ker} u$, $\operatorname{Im} u$, *and* $\operatorname{Coker} u$, *are* $\operatorname{Ker}\widetilde{u}$, $\operatorname{Im}\widetilde{u}$, *and* $\operatorname{Coker}\widetilde{u}$ *(respectively). In particular, for $\widetilde{u}$ to be injective (resp. surjective, bijective), it is necessary and sufficient for u to be so too.*

(ii) *If M is an inductive limit (resp. direct sum) of a family of A-modules (M_λ), then $\widetilde{M}$ is the inductive limit (resp. direct sum) of the family $(\widetilde{M_\lambda})$, via a canonical isomorphism.*

Proof.

(i) It suffices to apply the fact that $\widetilde{M}$ is an exact functor in M (1.3.5) to the two exact sequences of A-modules
$$0 \longrightarrow \operatorname{Ker} u \longrightarrow M \longrightarrow \operatorname{Im} u \longrightarrow 0,$$
$$0 \longrightarrow \operatorname{Im} u \longrightarrow N \longrightarrow \operatorname{Coker} u \longrightarrow 0.$$
The second claim then follows from Theorem (1.3.7).

(ii) Let $(M_\lambda, g_{\mu\lambda})$ be an inductive system of A-modules, with inductive limit M, and let g_λ be the canonical homomorphism $M_\lambda \to M$. Since we have $\widetilde{g_{\nu\mu}} \circ \widetilde{g_{\mu\lambda}} = \widetilde{g_{\nu\lambda}}$ and $\widetilde{g_\lambda} = \widetilde{g_\mu} \circ \widetilde{g_{\mu\lambda}}$ for $\lambda \leqslant \mu \leqslant \nu$, it follows that $(\widetilde{M_\lambda}, \widetilde{g_{\mu\lambda}})$ is an inductive system of sheaves on X, and if we denote by h_λ the canonical homomorphism $\widetilde{M_\lambda} \to \varinjlim \widetilde{M_\lambda}$, then there is a unique homomorphism $v : \varinjlim \widetilde{M_\lambda} \to \widetilde{M}$ such that $v \circ h_\lambda = \widetilde{g_\lambda}$. To see that v is bijective, it suffices to check, for each $x \in X$, that v_x is a bijection from $(\varinjlim \widetilde{M_\lambda})_x$ to $\widetilde{M}_x$; but $\widetilde{M}_x = M_x$, and
$$(\varinjlim \widetilde{M_\lambda})_x = \varinjlim (\widetilde{M_\lambda})_x = \varinjlim (M_\lambda)_x = M_x \quad (\mathbf{0},\ 1.3.3).$$
Conversely, it follows from the definitions that $(\widetilde{g_\lambda})_x$ and (h_λ) are both equal to the canonical map from $(M_\lambda)_x$ to M_x; since $(\widetilde{g_\lambda})_x = v_x \circ (h_\lambda)_x$, v_x is the identity. Finally, if M is the direct sum of two **I | 88** A-modules N and P, it is immediate that $\widetilde{M} = \widetilde{N} \oplus \widetilde{P}$; each direct sum being the inductive limit of finite direct sums, the claims of (ii) are thus proved.

□

We note that Corollary (1.3.8) proves that the sheaves isomorphic to the associated sheaves of A-modules form an *abelian category* (T, I, 1.4).

We also note that Corollary (1.3.9) implies that, if M is an A-module *of finite type* (that is to say, there exists a surjective homomorphism $A^n \to M$) then there exists a surjective homomorphism $\widetilde{A^n} \to \widetilde{M}$, or, in other words, the $\widetilde{A}$-module $\widetilde{M}$ is *generated by a finite family of sections over X* ($\mathbf{0}$, 5.1.1), and vice versa.

(1.3.10). If N is a submodule of an A-module M, the canonical injection $j : N \to M$ gives, by (1.3.9), an injective homomorphism $\widetilde{N} \to \widetilde{M}$, which allows us to canonically identify $\widetilde{N}$ with an $\widetilde{A}$-*submodule* of $\widetilde{M}$; we will always assume that we have made this identification. If N and P are submodules of M, then we have

$$(N+P)^{\sim} = \widetilde{N} + \widetilde{P}, \tag{1.3.10.1}$$

$$(N \cap P)^{\sim} = \widetilde{N} \cap \widetilde{P}, \tag{1.3.10.2}$$

since $N+P$ and $N \cap P$ are the image of the canonical homomorphism $N \oplus P \to M$ and the kernel of the canonical homomorphism $M \to (M/N) \oplus (M/P)$ (respectively), and it suffices to apply (1.3.9).

We conclude from (1.3.10.1) and (1.3.10.2) that, if $\widetilde{N} = \widetilde{P}$, then we have $N = P$.

Corollary (1.3.11). — *On the category of sheaves isomorphic to the associated sheaves of A-modules, the functor Γ is exact.*

Proof. Let $\widetilde{M} \xrightarrow{\widetilde{u}} \widetilde{N} \xrightarrow{\widetilde{v}} \widetilde{P}$ be an exact sequence corresponding to two homomorphisms $u : M \to N$ and $v : N \to P$ of A-modules. If $Q = \operatorname{Im} u$ and $R = \operatorname{Ker} v$, we have $\widetilde{Q} = \operatorname{Im}\widetilde{u} = \operatorname{Ker}\widetilde{v} = \widetilde{R}$ (Corollary (1.3.9)), hence $Q = R$. □

Corollary (1.3.12). — *Let M and N be A-modules.*

(i) *The sheaf associated to $M \otimes_A N$ is canonically identified with $\widetilde{M} \otimes_{\widetilde{A}} \widetilde{N}$.*

(ii) *If, in addition, M admits a finite presentation, then the sheaf associated to* $\operatorname{Hom}_A(M,N)$ *is canonically identified with $\mathscr{H}om_{\widetilde{A}}(\widetilde{M}, \widetilde{N})$.*

Proof.

(i) The sheaf $\mathscr{F} = \widetilde{M} \otimes_{\widetilde{A}} \widetilde{N}$ is associated to the presheaf

$$U \longmapsto \mathscr{F}(U) = \Gamma(U, \widetilde{M}) \otimes_{\Gamma(U,\widetilde{A})} \Gamma(U, \widetilde{N}),$$

with U varying over the basis (1.1.10, i) of X consisting of the $D(f)$, where $f \in A$. We know that $\mathscr{F}(D(f))$ is canonically identified with $M_f \otimes_{A_f} N_f$, by (1.3.7) and (1.3.6). Moreover, we know that the A_f-module $M_f \otimes_{A_f} N_f$ is canonically isomorphic to $(M \otimes_A N)_f$ (**0**, 1.3.4), which is itself canonically isomorphic to $\Gamma(D(f), (M \otimes_A N)^\sim)$ (Theorem (1.3.7) and Proposition (1.3.6)). In addition, we see immediately that the canonical isomorphisms

$$\mathscr{F}(D(f)) \simeq \Gamma(D(f), (M \otimes_A N)^\sim)$$

thus obtained satisfy the compatibility conditions with respect to the restriction operations (**0**, 1.4.2), **I** | 89
so they define a canonical functorial isomorphism

$$\widetilde{M} \otimes_{\widetilde{A}} \widetilde{N} \simeq (M \otimes_A N)^\sim.$$

(ii) The sheaf $\mathscr{G} = \mathscr{H}om_{\widetilde{A}}(\widetilde{M}, \widetilde{N})$ is associated to the presheaf

$$U \longmapsto \mathscr{G}(U) = \operatorname{Hom}_{\widetilde{A}|U}(\widetilde{M}|U, \widetilde{N}|U),$$

with U varying over the basis of X consisting of the $D(f)$. We know that $\mathscr{G}(D(f))$ is canonically identified with $\operatorname{Hom}_{A_f}(M_f, N_f)$ (Proposition (1.3.6) and Corollary (1.3.8)), which, according to the hypotheses on M, is canonically identified with $(\operatorname{Hom}_A(M,N))_f$ (**0**, 1.3.5). Finally, $(\operatorname{Hom}_A(M,N))_f$ is canonically identified with $\Gamma(D(f), (\operatorname{Hom}_A(M,N))^\sim)$ (Proposition (1.3.6) and Theorem (1.3.7)), and the canonical isomorphisms $\mathscr{G}(D(f)) \simeq \Gamma(D(f), (\operatorname{Hom}_A(M,N))^\sim)$ thus obtained are compatible with the restriction operations (**0**, 1.4.2); they thus define a canonical isomorphism $\mathscr{H}om_{\widetilde{A}}(\widetilde{M}, \widetilde{N}) \simeq (\operatorname{Hom}_A(M,N))^\sim$.

□

(1.3.13). Now let B be a (commutative) A-algebra; this can be understood by saying that B is an A-module such that we have some given element $e \in B$ and an A-homomorphism $\varphi : B \otimes_A B \to B$, so that *(a)* the diagrams

$$\begin{tikzcd} B \otimes_A B \otimes_A B \arrow[r, "\varphi\otimes 1"] \arrow[d, "1\otimes\varphi"'] & B \otimes_A B \arrow[d, "\varphi"] \\ B \otimes_A B \arrow[r, "\varphi"] & B \end{tikzcd} \qquad \begin{tikzcd} B \otimes_A B \arrow[rr, "\sigma"] \arrow[dr, "\varphi"'] & & B \otimes_A B \arrow[dl, "\varphi"] \\ & B & \end{tikzcd}$$

(σ being the canonical symmetry map) are commutative; and *(b)* $\varphi(e \otimes x) = \varphi(x \otimes e) = x$. By Corollary (1.3.12), the homomorphism $\widetilde{\varphi} : \widetilde{B} \otimes_{\widetilde{A}} \widetilde{B} \to \widetilde{B}$ of $\widetilde{A}$-modules satisfies the analogous conditions, and so it defines an $\widetilde{A}$*-algebra* structure on $\widetilde{B}$. In a similar way, the data of a B-module N is the same as the data of an A-module N and an A-homomorphism $\psi : B \otimes_A N \to N$ such that the diagram

$$\begin{tikzcd} B \otimes_A B \otimes_A N \arrow[r, "\varphi\otimes 1"] \arrow[d, "1\otimes\psi"'] & B \otimes_A N \arrow[d, "\psi"] \\ B \otimes_A N \arrow[r, "\psi"] & N \end{tikzcd}$$

is commutative and $\psi(e \otimes n) = n$; the homomorphism $\widetilde{\psi} : \widetilde{B} \otimes_{\widetilde{A}} \widetilde{N} \to \widetilde{N}$ satisfies the analogous condition, and so defines a $\widetilde{B}$*-module* structure on $\widetilde{N}$.

In a similar way, we see that if $u : B \to B'$ (resp. $v : N \to N'$) is a homomorphism of A-algebras (resp. of B-modules), then $\widetilde{u}$ (resp. $\widetilde{v}$) is a homomorphism of $\widetilde{A}$-algebras (resp. of $\widetilde{B}$-modules), and $\operatorname{Ker}\widetilde{u}$ is a $\widetilde{B}$-ideal (resp. $\operatorname{Ker}\widetilde{v}$, $\operatorname{Coker}\widetilde{v}$, and $\operatorname{Im}\widetilde{v}$ are $\widetilde{B}$-modules). If N is a B-module, then $\widetilde{N}$ is a $\widetilde{B}$-module of finite type if and only if N is a B-module of finite type (**0**, 5.2.3). **I** | 90

If M, N are B-modules, then the $\widetilde{B}$-module $\widetilde{M} \otimes_{\widetilde{B}} \widetilde{N}$ is canonically identified with $(M \otimes_B N)^\sim$; similarly $\mathscr{H}om_{\widetilde{B}}(\widetilde{M}, \widetilde{N})$ is canonically identified with $(\operatorname{Hom}_B(M,N))^\sim$ whenever M admits a finite presentation; the proofs are similar to those for Corollary (1.3.12).

If $\mathfrak{J}$ is an ideal of B, and N is a B-module, then we have $(\mathfrak{J}N)^\sim = \widetilde{\mathfrak{J}} \cdot \widetilde{N}$.

Finally, if B is an A-algebra *graded* by the A-submodules B_n ($n \in \mathbf{Z}$), then the $\widetilde{A}$-algebra $\widetilde{B}$, the direct sum of the $\widetilde{A}$-modules $\widetilde{B_n}$ (1.3.9), is graded by these $\widetilde{A}$-submodules, the axiom of graded algebras saying that the image of the homomorphism $B_m \otimes B_n \to B$ is contained in B_{m+n}. Similarly, if M is a B-module graded by the submodules M_n, then $\widetilde{M}$ is a $\widetilde{B}$-module graded by the $\widetilde{M_n}$.

(1.3.14). If B is an A-algebra, and M a submodule of B, then the $\widetilde{A}$-subalgebra of $\widetilde{B}$ generated by $\widetilde{M}$ (**0**, 4.1.3) is the $\widetilde{A}$-subalgebra $\widetilde{C}$, where we denote by C the subalgebra of B generated by M. Indeed, C is the direct sum of the submodules of B which are the images of the homomorphisms $\bigotimes^n M \to B$ ($n \geqslant 0$), so it suffices to apply (1.3.9) and (1.3.12).

1.4. Quasi-coherent sheaves over a prime spectrum.

Theorem (1.4.1). — *Let X be the prime spectrum of a ring A, V a quasi-compact open subset of X, and $\mathscr{F}$ an $(\mathscr{O}_X|V)$-module. The following four conditions are equivalent.*

(a) *There exists an A-module M such that $\mathscr{F}$ is isomorphic to $\widetilde{M}|V$.*
(b) *There exists a finite open cover (V_i) of V by sets of the form $D(f_i)$ ($f_i \in A$) contained in V, such that, for each i, $\mathscr{F}|V_i$ is isomorphic to a sheaf of the form $\widetilde{M_i}$, where M_i is an A_{f_i}-module.*
(c) *The sheaf $\mathscr{F}$ is quasi-coherent* (**0**, 5.1.3).
(d) *The two following properties are satisfied:*
   (d1) *For each $f \in A$ such that $D(f) \subset V$, and for each section $s \in \Gamma(D(f), \mathscr{F})$, there exists an integer $n \geqslant 0$ such that $f^n s$ extends to a section of $\mathscr{F}$ over V.*
   (d2) *For each $f \in A$ such that $D(f) \subset V$ and for each section $t \in \Gamma(V, \mathscr{F})$ such that the restriction of t to $D(f)$ is 0, there exists an integer $n \geqslant 0$ such that $f^n t = 0$.*

(In the statement of the conditions (d1) and (d2), we have tacitly identified A and $\Gamma(\widetilde{A})$ using Theorem (1.3.7)).

Proof. The fact that (a) implies (b) is an immediate consequence of Proposition (1.3.6) and the fact that the $D(f_i)$ form a basis for the topology of X (1.1.10). As each A-module is isomorphic to the cokernel of a homomorphism of the form $A^{(I)} \to A^{(J)}$, (1.3.6) implies that each sheaf associated to an A-module is quasi-coherent; so (b) implies (c). Conversely, if $\mathscr{F}$ is quasi-coherent, each $x \in V$ has a neighborhood of the form $D(f) \subset V$ such that $\mathscr{F}|D(f)$ is isomorphic to the cokernel of a homomorphism $\widetilde{A_f}^{(I)} \to \widetilde{A_f}^{(J)}$, so also to the sheaf $\widetilde{N}$ associated to the module N, the cokernel of the corresponding homomorphism $A_f^{(I)} \to A_f^{(J)}$ (Corollaries (1.3.8) and (1.3.9)); since V is quasi-compact, it is clear that (c) implies (b). I | 91

To prove that (b) implies (d1) and (d2), we first assume that $V = D(g)$ for some $g \in A$, and that $\mathscr{F}$ is isomorphic to the sheaf $\widetilde{N}$ associated to an A_g-module N; by replacing X with V and A with A_g (1.3.6), we can reduce to the case where $g = 1$. Then $\Gamma(D(f), \widetilde{N})$ and N_f are canonically identified with one another (Proposition (1.3.6) and Theorem (1.3.7)), so a section $s \in \Gamma(D(f), \widetilde{N})$ is identified with an element of the form z/f^n, where $z \in N$; the section $f^n s$ is identified with the element $z/1$ of N_f and, as a result, is the restriction to $D(f)$ of the section of $\widetilde{N}$ over X that is identified with the element $z \in N$; hence (d1) in this case. Similarly, $t \in \Gamma(X, \widetilde{N})$ is identified with an element $z' \in N$, the restriction of t to $D(f)$ is identified with the image $z'/1$ of z' in N_f, and to say that this image is zero means that there exists some $n \geqslant 0$ such that $f^n z' = 0$ in N, or, equivalently, $f^n t = 0$.

To finish the proof, that (b) implies (d1) and (d2), it suffices to establish the following lemma.

Lemma (1.4.1.1). — *Suppose that V is the finite union of sets of the form $D(g_i)$, and that all of the sheaves $\mathscr{F}|D(g_i)$ and $\mathscr{F}|(D(g_i) \cap D(g_j)) = \mathscr{F}|D(g_i g_j)$ satisfy* (d1) *and* (d2)*; then $\mathscr{F}$ has the following two properties:*

(d'1) *For each $f \in A$ and for each section $s \in \Gamma(D(f) \cap V, \mathscr{F})$, there exists an integer $n \geqslant 0$ such that $f^n s$ extends to a section of $\mathscr{F}$ over V.*
(d'2) *For each $f \in A$ and for each section $t \in \Gamma(V, \mathscr{F})$ such that the restriction of t to $D(f) \cap V$ is 0, there exists an integer $n \geqslant 0$ such that $f^n t = 0$.*

We first prove (d′2): since $D(f) \cap D(g_i) = D(f g_i)$, there exists, for each i, an integer n_i such that the restriction of $(f g_i)^{n_i} t$ to $D(g_i)$ is zero: since the image of g_i in A_{g_i} is invertible, the restriction of $f^{n_i} t$ to $D(g_i)$ is also zero; taking n to be the largest of the n_i, we have proved (d′2).

To show (d′1), we apply (d1) to the sheaf $\mathscr{F}|D(g_i)$: there exists an integer $n_i \geqslant 0$ and a section s_i' of $\mathscr{F}$ over $D(g_i)$ extending the restriction of $(f g_i)^{n_i} s$ to $D(f g_i)$; since the image of g_i in A_{g_i} is invertible, there is a section s_i of $\mathscr{F}$ over $D(g_i)$ such that $s_i' = g_i^{n_i} s_i$, and s_i extends the restriction of $f^{n_i} s$ to $D(f g_i)$; in addition we can suppose that all the n_i are equal to a single integer n. By construction, the restriction of $s_i - s_j$ to $D(f) \cap D(g_i) \cap D(g_j) = D(f g_i g_j)$ is zero; by (d2) applied to the sheaf $\mathscr{F}|D(g_i g_j)$, there exists an integer $m_{ij} \geqslant 0$ such that the restriction to $D(g_i g_j)$ of $(f g_i g_j)^{m_{ij}}(s_i - s_j)$ is zero; since the image of $g_i g_j$ in $A_{g_i g_j}$ is invertible, the restriction of $f^{m_{ij}}(s_i - s_j)$ to $D(g_i g_j)$ is zero. We can then assume that all the m_{ij} are equal to a single integer m, and so there exists a section $s' \in \Gamma(V, \mathscr{F})$ extending the $f^m s_i$; as a result, this section extends $f^{n+m} s$, hence we have proved (d′1).

It remains to show that (d1) and (d2) imply (a). We first show that (d1) and (d2) imply that these conditions are satisfied for each sheaf $\mathscr{F}|D(g)$, where $g \in A$ is such that $D(g) \subset V$. It is evident for (d1); on the other hand, if $t \in \Gamma(D(g), \mathscr{F})$ is such that its restriction to $D(f) \subset D(g)$ is zero, there exists, by (d1), an integer $m \geqslant 0$ such that $g^m t$ extends to a section s of $\mathscr{F}$ over V; applying (d2), we see that there exists an integer $n \geqslant 0$ such that $f^n g^m t = 0$, and as the image of g in A_g is invertible, $f^n t = 0$. I | 92

That being so, since V is quasi-compact, Lemma (1.4.1.1) proves that the conditions (d′1) and (d′2) are satisfied. Consider then the A-module $M = \Gamma(V, \mathscr{F})$, and define a homomorphism of $\widetilde{A}$-modules $u : \widetilde{M} \to j_*(\mathscr{F})$, where j is the canonical injection $V \to X$. Since the $D(f)$ form a basis for the topology of X, it suffices, for each $f \in A$, to define a homomorphism $u_f : M_f \to \Gamma(D(f), j_*(\mathscr{F})) = \Gamma(D(f) \cap V, \mathscr{F})$, with the usual compatibility conditions (**0**, 3.2.5). Since the canonical image of f in A_f is invertible, the restriction homomorphism $M = \Gamma(V, \mathscr{F}) \to \Gamma(D(f) \cap V, \mathscr{F})$ factors as $M \to M_f \xrightarrow{u_f} \Gamma(D(f) \cap V, \mathscr{F})$ (**0**, 1.2.4), and the verification of these compatibility conditions for $D(g) \subset D(f)$ is immediate. This being so, we show that the condition (d′1) (resp. (d′2)) implies that each of the u_f are surjective (resp. injective), which proves that u is *bijective*, and as a result that $\mathscr{F}$ is the restriction to V of an $\widetilde{A}$-module isomorphic to $\widetilde{M}$. If $s \in \Gamma(D(f) \cap V, \mathscr{F})$, there exists, by (d′1), an integer $n \geqslant 0$ such that $f^n s$ extends to a section $z \in M$; we then have $u_f(z/f^n) = s$, so u_f is surjective. Similarly, if $z \in M$ is such that $u_f(z/1) = 0$, this means that the restriction to $D(f) \cap V$ of the section z is zero; according to (d′2), there exists an integer $n \geqslant 0$ such that $f^n z = 0$, hence $z/1 = 0$ in M_f, and so u_f is injective. □

Corollary (1.4.2). — *Each quasi-coherent sheaf over a quasi-compact open subset of X is induced by a quasi-coherent sheaf on X.*

Corollary (1.4.3). — *Every quasi-coherent $\mathscr{O}_X$-algebra over $X = \operatorname{Spec}(A)$ is isomorphic to an $\mathscr{O}_X$-algebra of the form $\widetilde{B}$, where B is an algebra over A; every quasi-coherent $\widetilde{B}$-module is isomorphic to a $\widetilde{B}$-module of the form $\widetilde{N}$, where N is a B-module.*

Proof. Indeed, a quasi-coherent $\mathscr{O}_X$-algebra is a quasi-coherent $\mathscr{O}_X$-module, and therefore of the form $\widetilde{B}$, where B is an A-module; the fact that B is an A-algebra follows from the characterization of the structure of an $\mathscr{O}_X$-algebra using the homomorphism $\widetilde{B} \otimes_{\widetilde{A}} \widetilde{B} \to \widetilde{B}$ of $\widetilde{A}$-modules, as well as Corollary (1.3.12). If $\mathscr{G}$ is a quasi-coherent $\widetilde{B}$-module, it suffices to show, in a similar way, that it is also a quasi-coherent $\widetilde{A}$-module to conclude the proof; since the question is local, we can, by restricting to an open subset of X of the form $D(f)$, assume that $\mathscr{G}$ is the cokernel of a homomorphism $\widetilde{B}^{(I)} \to \widetilde{B}^{(J)}$ of $\widetilde{B}$-modules (and *a fortiori* of $\widetilde{A}$-modules); the proposition then follows from Corollaries (1.3.8) and (1.3.9). □

1.5. Coherent sheaves over a prime spectrum.

Theorem (1.5.1). — *Let A be a* Noetherian *ring, $X = \operatorname{Spec}(A)$ its prime spectrum, V an open subset of X, and $\mathscr{F}$ an $(\mathscr{O}_X|V)$-module. The following conditions are equivalent.*

(a) *$\mathscr{F}$ is coherent.*
(b) *$\mathscr{F}$ is of finite type and quasi-coherent.*
(c) *There exists an A-module M of finite type such that $\mathscr{F}$ is isomorphic to the sheaf $\widetilde{M}|V$.* I | 93

Proof. (a) trivially implies (b). To see that (b) implies (c), note that, since V is quasi-compact (**0**, 2.2.3), we have previously seen that $\mathscr{F}$ is isomorphic to a sheaf $\widetilde{N}|V$, where N is an A-module (1.4.1). We have $N = \varinjlim M_\lambda$, where M_λ run over the set of A-submodules of N of finite type, hence (1.3.9) $\mathscr{F} = \widetilde{N}|V = \varinjlim \widetilde{M_\lambda}|V$; but since $\mathscr{F}$ is of finite type, and V is quasi-compact, there exists an index λ such that $\mathscr{F} = \widetilde{M_\lambda}|V$ (**0**, 5.2.3).

Finally, we show that (c) implies (a). It is clear that $\mathscr{F}$ is then of finite type ((1.3.6) and (1.3.9)); in addition, the question being local, we can restrict to the case where $V = D(f)$, $f \in A$. Since A_f is Noetherian, we see that it suffices to prove that the kernel of a homomorphism $\widetilde{A^n} \to \widetilde{M}$, where M is an A-module, is of finite type. But such a homomorphism is of the form $\widetilde{u}$, where u is a homomorphism $A^n \to M$ (1.3.8), and if $P = \operatorname{Ker} u$ then we have $\widetilde{P} = \operatorname{Ker} \widetilde{u}$ (1.3.9). Since A is Noetherian, P is of finite type, which finishes the proof. □

Corollary (1.5.2). — *Under the hypotheses of* (1.5.1)*, the sheaf $\mathscr{O}_X$ is a quasi-coherent sheaf of rings.*

Corollary (1.5.3). — *Under the hypotheses of* (1.5.1)*, every coherent sheaf over an open subset of X is induced by a coherent sheaf on X.*

Corollary (1.5.4). — *Under the hypotheses of* (1.5.1)*, every quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$ is the inductive limit of the coherent $\mathscr{O}_X$-submodules of $\mathscr{F}$.*

Proof. Indeed, $\mathscr{F} = \widetilde{M}$, where M is an A-module, and M is the inductive limit of its submodules of finite type; we conclude the proof by appealing to (1.3.9) and (1.5.1). □

1.6. Functorial properties of quasi-coherent sheaves over a prime spectrum.

(1.6.1). Let A, A' be rings,
$$\varphi : A' \longrightarrow A$$
a homomorphism, and
$${}^a\varphi : X = \operatorname{Spec}(A) \longrightarrow X' = \operatorname{Spec}(A')$$
the continuous map associated to φ (1.2.1). We will define a *canonical homomorphism*
$$\widetilde{\varphi} : \mathcal{O}_{X'} \longrightarrow {}^a\varphi_*(\mathcal{O}_X)$$
of sheaves of rings. For each $f' \in A'$, we put $f = \varphi(f')$; we have ${}^a\varphi^{-1}(D(f')) = D(f)$ (1.2.2.2). The rings $\Gamma(D(f'),\widetilde{A'})$ and $\Gamma(D(f),\widetilde{A})$ are identified with $A'_{f'}$ and A_f (respectively) ((1.3.6) and (1.3.7)). The homomorphism φ canonically defines a homomorphism $\varphi_{f'} : A'_{f'} \to A_f$ (**0**, 1.5.1), in other words, we have a homomorphism of rings
$$\Gamma(D(f),\widetilde{A'}) \longrightarrow \Gamma({}^a\varphi^{-1}(D(f')),\widetilde{A}) = \Gamma(D(f'),{}^a\varphi_*(\widetilde{A})).$$
In addition, these homomorphisms satisfy the usual compatibility conditions: for $D(f') \supset D(g')$, the diagram **I** | 94
$$\begin{tikzcd}
\Gamma(D(f'),\widetilde{A'}) \arrow[r] \arrow[d] & \Gamma(D(f'),{}^a\varphi_*(\widetilde{A})) \arrow[d] \\
\Gamma(D(g'),\widetilde{A'}) \arrow[r] & \Gamma(D(g'),{}^a\varphi_*(\widetilde{A})
\end{tikzcd}$$
is commutative (**0**, 1.5.1); we have thus defined a homomorphism of $\mathcal{O}_{X'}$-algebras, as the $D(f')$ form a basis for the topology of X' (**0**, 3.2.3). The pair $\Phi = ({}^a\varphi,\widetilde{\varphi})$ is thus a *morphism* of ringed spaces
$$\Phi : (X, \mathcal{O}_X) \longrightarrow (X', \mathcal{O}_{X'}),$$
(**0**, 4.1.1).

We also note that, if we put $x' = {}^a\varphi(x)$, then the homomorphism $\widetilde{\varphi}_x^\sharp$ (**0**, 3.7.1) is exactly the homomorphism
$$\varphi_x : A'_{x'} \longrightarrow A_x$$
canonically induced by $\varphi : A' \to A$ (**0**, 1.5.1). Indeed, each $z' \in A'_{x'}$ can be written as g'/f', where f', g' are in A' and $f' \notin \mathfrak{j}_{x'}$; $D(f')$ is then a neighborhood of x' in X', and the homomorphism $\Gamma(D(f'),\widetilde{A'}) \to \Gamma({}^a\varphi^{-1}(D(f')),\widetilde{A})$ induced by $\widetilde{\varphi}$ is exactly $\varphi_{f'}$; by considering the section $s' \in \Gamma(D(f'),\widetilde{A'})$ corresponding to $g'/f' \in A'_{f'}$, we obtain $\widetilde{\varphi}_x^\sharp(z') = \varphi(g')/\varphi(f')$ in A_x.

Example (1.6.2). — Let S be a multiplicative subset of A, and φ the canonical homomorphism $A \to S^{-1}A$; we have already seen (1.2.6) that ${}^a\varphi$ is a *homeomorphism* from $Y = \operatorname{Spec}(S^{-1}A)$ to the subspace of $X = \operatorname{Spec}(A)$ consisting of the x such that $\mathfrak{j}_x \cap S = \varnothing$. In addition, for each x in this subspace, which is thus of the form ${}^a\varphi(y)$ with $y \in Y$, the homomorphism $\widetilde{\varphi}_y^\sharp : \mathcal{O}_x \to \mathcal{O}_y$ is *bijective* (**0**, 1.2.6); in other words, $\mathcal{O}_Y$ is identified with the sheaf on Y induced by $\mathcal{O}_X$.

Proposition (1.6.3). — *For every A-module M, there exists a canonical functorial isomorphism from the $\mathcal{O}_{X'}$-module $(M_{[\varphi]})^\sim$ to the direct image $\Phi_*(\widetilde{M})$.*

Proof. For purposes of abbreviation, we write $M' = M_{[\varphi]}$, and for each $f' \in A'$, we put $f = \varphi(f')$. The modules of sections $\Gamma(D(f'),\widetilde{M'})$ and $\Gamma(D(f),\widetilde{M})$ are identified, respectively, with the modules $M'_{f'}$ and M_f (over $A'_{f'}$ and A_f, respectively); in addition, the $A'_{f'}$-module $(M_f)_{[\varphi_{f'}]}$ is canonically isomorphic to $M'_{f'}$ (**0**, 1.5.2). We thus have a functorial isomorphism of $\Gamma(D(f'),\widetilde{A'})$-modules: $\Gamma(D(f'),\widetilde{M'}) \simeq \Gamma({}^a\varphi^{-1}(D(f')),\widetilde{M})_{[\varphi_{f'}]}$ and these isomorphisms satisfy the usual compatibility conditions with the restrictions (**0**, 1.5.6), thus defining the desired functorial isomorphism. We note that, in a precise way, if $u : M_1 \to M_2$ is a homomorphism of A-modules, it can be considered as a homomorphism $(M_1)_{[\varphi]} \to (M_2)_{[\varphi]}$ of A'-modules; if we denote this homomorphism by $u_{[\varphi]}$, then $\Phi_*(\widetilde{u})$ is identified with $(u_{[\varphi]})^\sim$. ☐

This proof also shows that, for each *A-algebra* B, the canonical functorial isomorphism $(B_{[\varphi]})^\sim \simeq \Phi_*(\widetilde{B})$ is an isomorphism of *$\mathcal{O}_{X'}$-algebras*; if M is a B-module, the canonical functorial isomorphism $(M_{[\varphi]})^\sim \simeq \Phi_*(\widetilde{M})$ is an isomorphism of $\Phi_*(\widetilde{B})$-modules. **I** | 95

Corollary (1.6.4). — *The direct image functor Φ_* is exact on the category of quasi-coherent $\mathcal{O}_X$-modules.*

Proof. Indeed, it is clear that $M_{[\varphi]}$ is an exact functor in M and $\widetilde{M'}$ is an exact functor in M' (1.3.5). ☐

Proposition (1.6.5). — *Let N' be an A'-module, and N the A-module $N' \otimes_{A'} A_{[\varphi]}$; then there exists a canonical functorial isomorphism from the $\mathscr{O}_X$-module $\Phi^*(\widetilde{N'})$ to $\widetilde{N}$.*

Proof. We first remark that $j : z' \mapsto z' \otimes 1$ is an A'-homomorphism from N' to $N_{[\varphi]}$: indeed, by definition, for $f' \in A'$, we have $(f'z') \otimes 1 = z' \otimes \varphi(f') = \varphi(f')(z' \otimes 1)$. We have (1.3.8) a homomorphism $\widetilde{j} : \widetilde{N'} \to (N_{[\varphi]})^\sim$ of $\mathscr{O}_{X'}$-modules, and, thanks to (1.6.3), we can consider $\widetilde{j}$ as mapping $\widetilde{N'}$ to $\Phi_*(\widetilde{N})$. There canonically corresponds to this homomorphism $\widetilde{j}$ a homomorphism $h = \widetilde{j}^\sharp$ from $\Phi^*(\widetilde{N'})$ to $\widetilde{N}$ (**0**, 4.4.3); we will see that, for each stalk, h_x is *bijective.* Put $x' = {}^a\varphi(x)$ and let $f' \in A'$ be such that $x' \in D(f')$; let $f = \varphi(f')$. The ring $\Gamma(D(f), \widetilde{A})$ is identified with A_f, the modules $\Gamma(D(f), \widetilde{N})$ and $\Gamma(D(f'), \widetilde{N'})$ with N_f and $N'_{f'}$ (respectively); let $s \in \Gamma(D(f'), \widetilde{N'})$, identified with n'/f'^p $(n' \in N')$, and s be its image under $\widetilde{j}$ in $\Gamma(D(f), \widetilde{N})$; s is identified with $(n' \otimes 1)/f^p$. On the other hand, let $t \in \Gamma(D(f), \widetilde{A})$, identified with g/f^q $(g \in A)$; then, by definition, we have $h_x(s'_x \otimes t_x) = t_x \cdot s_x$ (**0**, 4.4.3). But we can canonically identify N_f with $N'_{f'} \otimes_{A'_{f'}} (A_f)_{[\varphi_{f'}]}$ (**0**, 1.5.4); s then corresponds to the element $(n'/f'^p) \otimes 1$, and the section $y \mapsto t_y \cdot s_y$ with $(n'/f'^p) \otimes (g/f^q)$. The compatibility diagram of (**0**, 1.5.6) show that h_x is exactly the canonical isomorphism

$$N'_{x'} \otimes_{A'_{x'}} (A_x)_{[\varphi_{x'}]} \simeq N_x = (N' \otimes_{A'} A_{[\varphi]})_x. \tag{1.6.5.1}$$

In addition, let $v : N'_1 \to N'_2$ be a homomorphism of A'-modules; since $\widetilde{v}_{x'} = v_{x'}$ for each $x' \in X'$, it follows immediately from the above that $\Phi^*(\widetilde{v})$ is canonically identified with $(v \otimes 1)^\sim$, which finishes the proof of (1.6.5). □

If B' is an A'-algebra, the canonical isomorphism from $\Phi^*(\widetilde{B'})$ to $(B' \otimes_{A'} A_{[\varphi]})^\sim$ is an isomorphism of $\mathscr{O}_X$-algebras; if, in addition, N' is a B'-module, then the canonical isomorphism from $\Phi^*(\widetilde{N'})$ to $(N' \otimes_{A'} A_{[\varphi]})^\sim$ is an isomorphism of $\Phi^*(\widetilde{B'})$-modules.

Corollary (1.6.6). — *The sections of $\Phi^*(\widetilde{N'})$, the canonical images of the sections s', where s' varies over the A'-module $\Gamma(\widetilde{N'})$, generate the A-module $\Gamma(\Phi^*(N'))$.*

Proof. Indeed, these images are identified with the elements $z' \otimes 1$ of N, when we identify N' and N with $\Gamma(\widetilde{N'})$ and $\Gamma(\widetilde{N})$ (respectively) (1.3.7), and z' varies over N'. □

(1.6.7). In the proof of (1.6.5), we had proved in passing that the canonical map (**0**, 4.4.3.2) $\rho : \widetilde{N'} \to \Phi_*(\Phi^*(\widetilde{N'}))$ is exactly the homomorphism $\widetilde{j}$, where $j : N' \to N' \otimes_{A'} A_{[\varphi]}$ is the homomorphism $z' \mapsto z' \otimes 1$. Similarly, the canonical map (**0**, 4.4.3.3) $\sigma : \Phi^*(\Phi_*(\widetilde{M})) \to \widetilde{M}$ is exactly $\widetilde{p}$, where $p : M_{[\varphi]} \otimes_{A'} A_{[\varphi]} \to M$ is the canonical homomorphism, which sends each tensor product $z \otimes a$ $(z \in M,\ a \in A)$ to $a \cdot z$; this follows immediately from the definitions ((**0**, 3.7.1), (**0**, 4.4.3), and (1.3.7)). **I** | 96

We conclude ((**0**, 4.4.3) and (**0**, 3.5.4.4)) that if $v : N' \to M_{[\varphi]}$ is an A'-homomorphism, then $\widetilde{v}^\sharp = (v \otimes 1)^\sim$.

(1.6.8). Let N'_1 and N'_2 be A'-modules, and assume N'_1 admits a *finite presentation*; it then follows from (1.6.7) and (1.3.12, ii) that the canonical homomorphism (**0**, 4.4.6)

$$\Phi^*(\mathscr{H}om_{\widetilde{A'}}(\widetilde{N'_1}, \widetilde{N'_2})) \longrightarrow \mathscr{H}om_{\widetilde{A}}(\Phi^*(\widetilde{N'_1}), \Phi^*(\widetilde{N'_2}))$$

is exactly $\widetilde{\gamma}$, where γ denotes the canonical homomorphism of A-modules $\operatorname{Hom}_{A'}(N'_1, N'_2) \otimes_{A'} A \to \operatorname{Hom}_A(N'_1 \otimes_{A'} A, N'_2 \otimes_{A'} A)$.

(1.6.9). Let $\mathfrak{J}'$ be an ideal of A', and M an A-module; since, by definition, $\widetilde{\mathfrak{J}}'\widetilde{M}$ is the image of the canonical homomorphism $\Phi^*(\widetilde{\mathfrak{J}}') \otimes_{\widetilde{A}} \widetilde{M} \to \widetilde{M}$, it follows from Proposition (1.6.5) and Corollary (1.3.12, i) that $\widetilde{\mathfrak{J}}'\widetilde{M}$ canonically identifies with $(\mathfrak{J}'M)^\sim$; in particular, $\Phi^*(\widetilde{\mathfrak{J}}')\widetilde{A}$ is identified with $(\mathfrak{J}'A)^\sim$, and, taking the right exactness of the functor Φ^* into account, the $\widetilde{A}$-algebra $\Phi^*((A'/\mathfrak{J}')^\sim)$ is identified with $(A/\mathfrak{J}'A)^\sim$.

(1.6.10). Let A'' be a third ring, φ' a homomorphism $A'' \to A'$, and write $\varphi'' = \varphi \circ \varphi'$. It follows immediately from the definitions that ${}^a\varphi'' = ({}^a\varphi') \circ ({}^a\varphi)$, and $\widetilde{\varphi''} = \widetilde{\varphi} \circ \widetilde{\varphi'}$ (**0**, 1.5.7). We conclude that $\Phi'' = \Phi' \circ \Phi$; in other words, $(\operatorname{Spec}(A), \widetilde{A})$ is a *functor* from the category of rings to that of ringed spaces.

1.7. Characterization of morphisms of affine schemes.

Definition (1.7.1). — We say that a ringed space $(X, \mathscr{O}_X)$ is an *affine scheme* if it is isomorphic to a ringed space of the form $(\operatorname{Spec}(A), \widetilde{A})$, where A is a ring; we then say that $\Gamma(X, \mathscr{O}_X)$, which is canonically identified with the ring A (1.3.7), is the ring of the affine scheme $(X, \mathscr{O}_X)$, and we denote it by $A(X)$ when there is no chance of confusion.

By abuse of language, when we speak of an *affine scheme* $\mathrm{Spec}(A)$; it will always be the ringed space $(\mathrm{Spec}(A), \widetilde{A})$.

(1.7.2). Let A and B be rings, and $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ the affine schemes corresponding to the prime spectra $X = \mathrm{Spec}(A)$, $Y = \mathrm{Spec}(B)$. We have seen (1.6.1) that each ring homomorphism $\varphi : B \to A$ corresponds to a morphism $\Phi = ({}^a\varphi, \widetilde{\varphi}) = \mathrm{Spec}(\varphi) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$. We note that φ is entirely determined by Φ, since we have, by definition, $\varphi = \Gamma(\widetilde{\varphi}) : \Gamma(\widetilde{B}) \to \Gamma({}^a\varphi_*(\widetilde{A}) = \Gamma(\widetilde{A})$.

Theorem (1.7.3). — [2] *Let $(X, \mathcal{O}_X)$ $(Y, \mathcal{O}_Y)$ be affine schemes. For a morphism of ringed spaces $(\psi, \theta) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ to be of the form $({}^a\varphi, \widetilde{\varphi})$, where φ is a homomorphism of rings $A(Y) \to A(X)$, it is necessary and sufficient that, for each $x \in X$, $\theta_x^\sharp$ is a local homomorphism: $\mathcal{O}_{\psi(x)} \to \mathcal{O}_x$.* **I** | 97

Proof. Let $A = A(X)$, $B = A(Y)$. The condition is necessary, since we saw (1.6.1) that $\widetilde{\varphi}_x^\sharp$ is the homomorphism from $B_{{}^a\varphi(x)}$ to A_x canonically induced by φ, and, by definition, of ${}^a\varphi(x) = \varphi^{-1}(\mathfrak{j}_x)$, this homomorphism is local.

We now prove that the condition is sufficient. By definition, θ is a homomorphism $\mathcal{O}_Y \to \psi_*(\mathcal{O}_X)$, and we canonically obtain a ring homomorphism

$$\varphi = \Gamma(\theta) : B = \Gamma(Y, \mathcal{O}_Y) \longrightarrow \Gamma(Y, \psi_*(\mathcal{O}_X)) = \Gamma(X, \mathcal{O}_X) = A.$$

The hypotheses on $\theta_x^\sharp$ mean that this homomorphism induces, by passing to quotients, a monomorphism θ^x from the residue field $k(\psi(x))$ to the residue field $k(x)$, such that, for each section $f \in \Gamma(Y, \mathcal{O}_Y) = B$, we have $\theta^x(f(\psi(x))) = \varphi(f)(x)$. The relation $f(\psi(x)) = 0$ is therefore equivalent to $\varphi(f)(x) = 0$, which means that $\mathfrak{j}_{\psi(x)} = \mathfrak{j}_{{}^a\varphi(x)}$, and we now write $\psi(x) = {}^a\varphi(x)$ for each $x \in X$, or $\psi = {}^a\varphi$. We also know that the diagram

$$\begin{tikzcd}
B = \Gamma(Y, \mathcal{O}_Y) \arrow[r, "\varphi"] \arrow[d] & \Gamma(X, \mathcal{O}_X) = A \arrow[d] \\
B_{\psi(x)} \arrow[r, "\theta_x^\sharp"] & A_x
\end{tikzcd}$$

is commutative (**0**, 3.7.2), which means that $\theta_x^\sharp$ is equal to the homomorphism $\varphi_x : B_{\psi(x)} \to A_x$ canonically induced by φ (**0**, 1.5.1). As the data of the $\theta_x^\sharp$ completely characterize $\theta^\sharp$, and as a result θ (**0**, 3.7.1), we conclude that we have $\theta = \widetilde{\varphi}$, by the definition of $\widetilde{\varphi}$ (1.6.1). □

We say that a morphism (ψ, θ) of ringed spaces satisfying the condition of (1.7.3) is a *morphism of affine schemes.*

Corollary (1.7.4). — *If $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ are affine schemes, there exists a canonical isomorphism from the set of morphisms of affine schemes* $\mathrm{Hom}((X, \mathcal{O}_X), (Y, \mathcal{O}_Y))$ *to the set of ring homomorphisms from B to A, where $A = \Gamma(\mathcal{O}_X)$ and $B = \Gamma(\mathcal{O}_Y)$.*

Furthermore, we can say that the functors $(\mathrm{Spec}(A), \widetilde{A})$ in A and $\Gamma(X, \mathcal{O}_X)$ in $(X, \mathcal{O}_X)$ define an *equivalence* between the category of commutative rings and the opposite category of affine schemes (T, I, 1.2).

Corollary (1.7.5). — *If $\varphi : B \to A$ is surjective, then the corresponding morphism $({}^a\varphi, \widetilde{\varphi})$ is a monomorphism of ringed spaces* (cf. (4.1.7)).

Proof. Indeed, we know that ${}^a\varphi$ is injective (1.2.5), and, since φ is surjective, for each $x \in X$, $\varphi_x^\sharp : B_{{}^a\varphi(x)} \to A_x$, which is induced by φ by passing to rings of fractions, is also surjective (**0**, 1.5.1); hence the conclusion (**0**, 4.1.1). □

1.8. Morphisms from locally ringed spaces to affine schemes. Due to a remark by J. Tate, the statements of Theorem (1.7.3) and Proposition (2.2.4) can be generalized as follows:[3] **II** | 217

Proposition (1.8.1). — *Let $(S, \mathcal{O}_S)$ be an affine scheme, and $(X, \mathcal{O}_X)$ a locally ringed space. Then there is a canonical bijection from the set of ring homomorphisms $\Gamma(S, \mathcal{O}_S) \to \Gamma(X, \mathcal{O}_X)$ to the set of morphisms of ringed spaces (ψ, θ) :* **II** | 218 *$(X, \mathcal{O}_X) \to (S, \mathcal{O}_S)$ such that, for each $x \in X$, $\theta_x^\sharp$ is a local homomorphism $\mathcal{O}_{\psi(x)} \to \mathcal{O}_x$.*

Proof. We note first that if $(X, \mathcal{O}_X)$ and $(S, \mathcal{O}_S)$ are any two ringed spaces, then a morphism (ψ, θ) from $(X, \mathcal{O}_X)$ to $(S, \mathcal{O}_S)$ canonically defines a ring homomorphism $\Gamma(\theta) : \Gamma(S, \mathcal{O}_S) \to \Gamma(X, \mathcal{O}_X)$, hence a first map

$$(1.8.1.1) \qquad \rho : \mathrm{Hom}((X, \mathcal{O}_X), (S, \mathcal{O}_S)) \longrightarrow \mathrm{Hom}(\Gamma(S, \mathcal{O}_S), \Gamma(X, \mathcal{O}_X)).$$

[2] *[Trans.] See (1.8) and the footnote there.*

[3] *[Trans.] The following section (I.1.8) was added in the errata of EGA II, hence the temporary change in page numbers, which refer to EGA II.*

Conversely, under the stated hypotheses, we set $A = \Gamma(S, \mathcal{O}_S)$, and consider a ring homomorphism $\varphi : A \to \Gamma(X, \mathcal{O}_X)$. For each $x \in X$, it is clear that the set of the $f \in A$ such that $\varphi(f)(x) = 0$ is a *prime ideal* of A, since $\mathcal{O}_x/\mathfrak{m}_x = k(x)$ is a field; it is therefore an element of $S = \operatorname{Spec}(A)$, which we denote by ${}^a\varphi(x)$. In addition, for each $f \in A$, we have, by definition (**0**, 5.5.2), that ${}^a\varphi^{-1}(D(f)) = X_f$, which proves that ${}^a\varphi$ is a *continuous map* $X \to S$. We then define a homomorphism

$$\widetilde{\varphi} : \mathcal{O}_S \longrightarrow {}^a\varphi_*(\mathcal{O}_X)$$

of $\mathcal{O}_S$-modules; for each $f \in A$, we have $\Gamma(D(f), \mathcal{O}_S) = A_f$ (1.3.6); for each $s \in A$, we associate to $s/f \in A_f$ the element $(\varphi(s)|X_f)(\varphi(f)|X_f)^{-1}$ of $\Gamma(X_f, \mathcal{O}_X) = \Gamma(D(f), {}^a\varphi_*(\mathcal{O}_X))$, and we immediately see (by passing from $D(f)$ to $D(fg)$) that this is a well-defined homomorphism of $\mathcal{O}_S$-modules, hence a morphism $({}^a\varphi, \widetilde{\varphi})$ of ringed spaces. In addition, with the same notation, and setting $y = {}^a\varphi(x)$ for brevity, we immediately see (**0**, 3.7.1) that we have $\widetilde{\varphi}_x^\sharp(s_y/f_y) = (\varphi(s)_x)(\varphi(f)_x)^{-1}$; since the relation $s_y \in \mathfrak{m}_y$ is, by definition, equivalent to $\varphi(s)_x \in \mathfrak{m}_x$, we see that $\widetilde{\varphi}_x^\sharp$ is a *local* homomorphism $\mathcal{O}_y \to \mathcal{O}_x$, and we have thus defined a second map $\sigma : \operatorname{Hom}(\Gamma(S, \mathcal{O}_S), \Gamma(X, \mathcal{O}_X)) \to \mathfrak{L}$, where $\mathfrak{L}$ is the set of the morphisms $(\psi, \theta) : (X, \mathcal{O}_X) \to (S, \mathcal{O}_S)$ such that $\theta_x^\sharp$ is local for each $x \in X$. It remains to prove that σ and ρ (restricted to $\mathfrak{L}$) are inverses of each other; the definition of $\widetilde{\varphi}$ immediately shows that $\Gamma(\widetilde{\varphi}) = \varphi$, and, as a result, that $\rho \circ \sigma$ is the identity. To see that $\sigma \circ \rho$ is the identity, start with a morphism $(\psi, \theta) \in \mathfrak{L}$ and let $\varphi = \Gamma(\theta)$; the hypotheses on $\theta_x^\sharp$ mean that this morphism induces, by passing to quotients, a monomorphism $\theta^x : k(\psi(x)) \to k(x)$ such that for each section $f \in A = \Gamma(S, \mathcal{O}_S)$, we have $\theta^x(f(\psi(x))) = \varphi(f)(x)$; the equation $f(\varphi(x)) = 0$ is therefore equivalent to $\varphi(f)(x) = 0$, which proves that ${}^a\varphi = \psi$. On the other hand, the definitions imply that the diagram

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[d] & \Gamma(X, \mathcal{O}_X) \arrow[d] \\ A_{\psi(x)} \arrow[r, "\theta_x^\sharp"] & \mathcal{O}_x \end{tikzcd}$$

is commutative, and it is the same for the analogous diagram where $\theta_x^\sharp$ is replaced by $\widetilde{\varphi}_x^\sharp$, hence $\widetilde{\varphi}_x^\sharp = \theta_x^\sharp$ (**0**, 1.2.4), and, as a result, $\widetilde{\varphi} = \theta$. □

(1.8.2). When $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ are *locally* ringed spaces, we will consider the morphisms $(\psi, \theta) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ such that, for each $x \in X$, $\theta_x^\sharp$ is a *local* homomorphism $\mathcal{O}_{\psi(x)} \to \mathcal{O}_x$. Henceforth when we speak of a *morphism of locally ringed spaces*, it will always be a morphism like the above; with this definition of morphisms, it is clear that the locally ringed spaces form a *category*; for any two objects X and Y of this category, $\operatorname{Hom}(X, Y)$ thus denotes the set of morphisms of locally ringed spaces from X to Y (the set denoted $\mathfrak{L}$ in (1.8.1)); when we consider the set of *morphisms of ringed spaces* from X to Y, we will denote it by $\operatorname{Hom}_{\mathrm{rs}}(X, Y)$ to avoid any confusion. The map (1.8.1.1) is then written as **II | 219**

(1.8.2.1) $$\rho : \operatorname{Hom}_{\mathrm{rs}}(X, Y) \longrightarrow \operatorname{Hom}(\Gamma(Y, \mathcal{O}_Y), \Gamma(X, \mathcal{O}_X))$$

and its restriction

(1.8.2.2) $$\rho' : \operatorname{Hom}(X, Y) \longrightarrow \operatorname{Hom}(\Gamma(Y, \mathcal{O}_Y), \Gamma(X, \mathcal{O}_X))$$

is a *functorial* map in X and Y on the category of locally ringed spaces.

Corollary (1.8.3). — *Let $(Y, \mathcal{O}_Y)$ be a locally ringed space. For Y to be an affine scheme, it is necessary and sufficient that, for each locally ringed space $(X, \mathcal{O}_X)$, the map (1.8.2.2) be bijective.*

Proof. Proposition (1.8.1) shows that the condition is necessary. Conversely, if we suppose that the condition is satisfied, and if we put $A = \Gamma(Y, \mathcal{O}_Y)$, then it follows from the hypotheses and from (1.8.1) that the functors $X \mapsto \operatorname{Hom}(X, Y)$ and $X \mapsto \operatorname{Hom}(X, \operatorname{Spec}(A))$, from the category of locally ringed spaces to that of sets, are *isomorphic*. We know that this implies the existence of a canonical isomorphism $X \to \operatorname{Spec}(A)$ (cf. **0**, 8). □

(1.8.4). Let $S = \operatorname{Spec}(A)$ be an affine scheme; denote by (S', A') the ringed space whose underlying space is *a point* and the structure sheaf A' is the (necessarily simple) sheaf on S' defined by the ring A. Let $\pi : S \to S'$ be the unique map from S to S'; on the other hand, we note that, for each open subset U of S, we have a canonical map $\Gamma(S', A') = \Gamma(S, \mathcal{O}_S) \to \Gamma(U, \mathcal{O}_S)$ which thus defines a π*-morphism* $\iota : A' \to \mathcal{O}_S$ of sheaves of rings. We have thus canonically defined a *morphism of ringed spaces* $i = (\pi, \iota) : (S, \mathcal{O}_S) \to (S', A')$. For each A-module M, we denote by M' the simple sheaf on S' defined by M, which is evidently an A'-module. It is clear that $i_*(\widetilde{M}) = M'$ (1.3.7).

Lemma (1.8.5). — *With the notation of (1.8.4), for each A-module M, the canonical functorial $\mathcal{O}_S$-homomorphism* (**0**, 4.3.3)

$$i^*(i_*(\widetilde{M})) \longrightarrow \widetilde{M} \tag{1.8.5.1}$$

is an isomorphism.

Proof. Indeed, the two parts of (1.8.5.1) are right exact (the functor $M \mapsto i_*(\widetilde{M})$ evidently being exact) and commute with direct sums; by considering M as the cokernel of a homomorphism $A^{(I)} \to A^{(J)}$, we can reduce to proving the lemma for the case where $M = A$, and it is evident in this case. □

Corollary (1.8.6). — *Let $(X, \mathcal{O}_X)$ be a ringed space, and $u : X \to S$ a morphism of ringed spaces. For each A-module M, we have (with the notation of (1.8.4)) a canonical functorial isomorphism of $\mathcal{O}_X$-modules* II | 220

$$u^*(\widetilde{M}) \simeq u^*(i^*(M')). \tag{1.8.6.1}$$

Corollary (1.8.7). — *Under the hypotheses of (1.8.6), we have, for each A-module M and each $\mathcal{O}_X$-module $\mathscr{F}$, a canonical isomorphism, functorial in M and $\mathscr{F}$,*

$$\operatorname{Hom}_{\mathcal{O}_S}(\widetilde{M}, u_*(\mathscr{F})) \simeq \operatorname{Hom}_A(M, \Gamma(X, \mathscr{F})). \tag{1.8.7.1}$$

Proof. We have, according to (**0**, 4.4.3) and Lemma (1.8.5), a canonical isomorphism of bifunctors

$$\operatorname{Hom}_{\mathcal{O}_S}(\widetilde{M}, u_*(\mathscr{F})) \simeq \operatorname{Hom}_{A'}(M', i_*(u_*(\mathscr{F})))$$

and it is clear that the right hand side is exactly $\operatorname{Hom}_A(M, \Gamma(X, \mathscr{F}))$. We note that the canonical homomorphism (1.8.7.1) sends each $\mathcal{O}_S$-homomorphism $h : \widetilde{M} \to u_*(\mathscr{F})$ (in other words, each u-morphism $\widetilde{M} \to \mathscr{F}$) to the A-homomorphism $\Gamma(h) : M \to \Gamma(S, u_*(\mathscr{F})) = \Gamma(X, \mathscr{F})$. □

(1.8.8). With the notation of (1.8.4), it is clear (**0**, 4.1.1) that each morphism of ringed spaces $(\psi, \theta) : X \to S'$ is equivalent to the data of a ring homomorphism $A \to \Gamma(X, \mathcal{O}_X)$. We can thus interpret Proposition (1.8.1) as defining a canonical bijection $\operatorname{Hom}(X, S) \simeq \operatorname{Hom}(X, S')$ (where we understand that the right-hand side is the collection of morphisms of ringed spaces, since in general A is not a local ring). More generally, if X and Y are locally ringed spaces, and if (Y', A') is the ringed space whose underlying space is a point and whose sheaf of rings A' is the simple sheaf defined by the ring $\Gamma(Y, \mathcal{O}_Y)$, we can interpret (1.8.2.1) as a map

$$\rho : \operatorname{Hom}_{\mathrm{rs}}(X, Y) \longrightarrow \operatorname{Hom}(X, Y'). \tag{1.8.8.1}$$

The result of Corollary (1.8.3) is interpreted by saying that affine schemes are characterized among locally ringed spaces as those for which the restriction of ρ to $\operatorname{Hom}(X, Y)$:

$$\rho' : \operatorname{Hom}(X, Y) \longrightarrow \operatorname{Hom}(X, Y') \tag{1.8.8.2}$$

is *bijective* for *every* locally ringed space X. In the following chapter, we generalize this definition, which allows us to associate to *any* ringed space Z (and not only to a ringed space whose underlying space is a point) a locally ringed space which we will call $\operatorname{Spec}(Z)$; this will be the starting point for a "relative" theory of preschemes over any ringed space, extending the results of Chapter I.

(1.8.9). We can consider the pairs $(X, \mathscr{F})$ consisting of a locally ringed space X and an $\mathcal{O}_X$-module $\mathscr{F}$ as forming a category, a *morphism* in this category being a pair (u, h) consisting of a morphism of locally ringed spaces $u : X \to Y$ and a u-morphism $h : \mathscr{G} \to \mathscr{F}$ of modules; these morphisms (for $(X, \mathscr{F})$ and $(Y, \mathscr{G})$ fixed) form a set II | 221 which we denote by $\operatorname{Hom}((X, \mathscr{F}), (Y, \mathscr{G}))$; the map $(u, h) \mapsto (\rho'(u), \Gamma(h))$ is a canonical map

$$\operatorname{Hom}((X, \mathscr{F}), (Y, \mathscr{G})) \longrightarrow \operatorname{Hom}((\Gamma(Y, \mathcal{O}_Y), \Gamma(Y, \mathscr{G})), (\Gamma(X, \mathcal{O}_X), \Gamma(X, \mathscr{F}))) \tag{1.8.9.1}$$

functorial in $(X, \mathscr{F})$ and $(Y, \mathscr{G})$, the right-hand side being the set of di-homomorphisms corresponding to the rings and modules considered (**0**, 1.0.2).

Corollary (1.8.10). — *Let Y be a locally ringed space, and $\mathscr{G}$ an $\mathcal{O}_Y$-module. For Y to be an affine scheme and $\mathscr{G}$ to be a quasi-coherent $\mathcal{O}_Y$-module, it is necessary and sufficient that, for each pair $(X, \mathscr{F})$ consisting of a locally ringed space X and an $\mathcal{O}_X$-module $\mathscr{F}$, the canonical map (1.8.9.1) be bijective.*

We leave the proof, which is modelled on that of (1.8.3), using (1.8.1) and (1.8.7), to the reader.

Remark (1.8.11). — The statements (1.7.3), (1.7.4), and (2.2.4) are particular cases of (1.8.1), as well as the definition in (1.6.1); similarly, (2.2.5) follows from (1.8.7). Corollary (1.8.7) also implies (1.6.3) (and, as a result, (1.6.4)) as a particular case, since if X is an affine scheme and $\Gamma(X, \mathscr{F}) = N$, then the functors $M \mapsto \operatorname{Hom}_{\mathcal{O}_S}(\widetilde{M}, u_*(\widetilde{N}))$ and $M \mapsto \operatorname{Hom}_{\mathcal{O}_S}(\widetilde{M}, (N_{[\varphi]})^\sim)$ (where $\varphi : A \to \Gamma(X, \mathcal{O}_X)$ corresponds to u) are isomorphic, by Corollaries (1.8.7) and (1.3.8). Finally, (1.6.5) (and, as a result, (1.6.6)) follow from (1.8.6), and the fact that, for each $f \in A$, the A_f-modules $N' \otimes_{A'} A_f$ and $(N' \otimes_{A'} A)_f$ (with the notation of (1.6.5)) are canonically isomorphic.

§2. Preschemes and morphisms of preschemes

2.1. Definition of preschemes.

(2.1.1). Given a ringed space $(X, \mathcal{O}_X)$, we say that an open subset V of X is an *affine open* subset if the ringed space $(V, \mathcal{O}_X|V)$ is an affine scheme (1.7.1).

Definition (2.1.2). — We define a prescheme to be a ringed space $(X, \mathcal{O}_X)$ such that every point of X admits an affine open neighborhood.

Proposition (2.1.3). — *If $(X, \mathcal{O}_X)$ is a prescheme, then its affine open subsets form a basis for the topology of X.* I | 98

Proof. If V is an arbitrary open neighborhood of $x \in X$, then there exists by hypothesis an open neighborhood W of x such that $(W, \mathcal{O}_X|W)$ is an affine scheme; we write A to mean its ring. In the space W, $V \cap W$ is an open neighborhood of x; so there exists some $f \in A$ such that $D(f)$ is an open neighborhood of x contained inside $V \cap W$ (1.1.10, i). The ringed space $(D(f), \mathcal{O}_X|D(f))$ is thus an affine scheme, isomorphic to A_f (1.3.6), whence the proposition. □

Proposition (2.1.4). — *The underlying space of a prescheme is a Kolmogoroff space.*

Proof. If x and y are two distinct points of a prescheme X, then it is clear that there exists an open neighborhood of one of these points that does not contain the other if x and y are not in the same affine open subset; and if they are in the same affine open subset, this is a result of (1.1.8). □

Proposition (2.1.5). — *If $(X, \mathcal{O}_X)$ is a prescheme, then every closed irreducible subset of X admits exactly one generic point, and the map $x \mapsto \overline{\{x\}}$ is thus a bijection of X onto its set of closed irreducible subsets.*

Proof. If Y is a closed irreducible subset of X and $y \in Y$, and if U is an affine open neighborhood of y in X, then $U \cap Y$ is dense in Y, and also irreducible ((**0**, 2.1.1) and (**0**, 2.1.4)); thus, by Corollary (1.1.14), $U \cap Y$ is the closure in U of a point x, and so $Y \subset \overline{U}$ is the closure of x in X. The uniqueness of the generic point of X is a result of Proposition (2.1.4) and of (**0**, 2.1.3). □

(2.1.6). If Y is a closed irreducible subset of X, and y its generic point, then the local ring $\mathcal{O}_y$ (also written $\mathcal{O}_{X/Y}$) is called the *local ring of X along Y*, or the *local ring of Y in X*.

If X itself is irreducible and x its generic point then we say that $\mathcal{O}_x$ is the *ring of rational functions on X* (cf. §7).

Proposition (2.1.7). — *If $(X, \mathcal{O}_X)$ is a prescheme, then the ringed space $(U, \mathcal{O}_X|U)$ is a prescheme for every open subset U.*

Proof. This follows directly from Definition (2.1.2) and Proposition (2.1.3). □

We say that $(U, \mathcal{O}_X|U)$ is the prescheme *induced* on U by $(X, \mathcal{O}_X)$, or the *restriction* of $(X, \mathcal{O}_X)$ to U.

(2.1.8). We say that a prescheme $(X, \mathcal{O}_X)$ is *irreducible* (resp. *connected*) if the underlying space X is irreducible (resp. connected). We say that a prescheme is *integral* if it is *irreducible and reduced* (cf. (5.1.4)). We say that a prescheme $(X, \mathcal{O}_X)$ is *locally integral* if every $x \in X$ admits an open neighborhood U such that the prescheme induced on U by $(X, \mathcal{O}_X)$ is integral.

2.2. Morphisms of preschemes.

Definition (2.2.1). — Given two preschemes, $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$, we define a morphism (of preschemes) from $(X, \mathcal{O}_X)$ to $(Y, \mathcal{O}_Y)$ to be a morphism of ringed spaces (ψ, θ) such that, for all $x \in X$, $\theta_x^\sharp$ is a local homomorphism $\mathcal{O}_{\psi(x)} \to \mathcal{O}_x$.

By passing to quotients, the map $\mathcal{O}_{\psi(x)} \to \mathcal{O}_x$ gives us a monomorphism $\theta^x : k(\psi(x)) \to k(x)$, which lets us consider $k(x)$ as an *extension* of the field $k(\psi(x))$. I | 99

(2.2.2). The composition (ψ'', θ'') of two morphisms (ψ, θ), (ψ', θ') of preschemes is also a morphism of preschemes, since it is given by the formula $\theta''^\sharp = \theta^\sharp \circ \psi^*(\theta'^\sharp)$ (**0**, 3.5.5). From this we conclude that preschemes form a *category*; using the usual notation, we will write $\mathrm{Hom}(X, Y)$ to mean the set of morphisms from a prescheme X to a prescheme Y.

Example (2.2.3). — If U is an open subset of X, then the canonical injection (**0**, 4.1.2) of the induced prescheme $(U, \mathcal{O}_X|U)$ into $(X, \mathcal{O}_X)$ is a morphism of preschemes; it is further a *monomorphism* of ringed spaces (and *a fortiori* a monomorphism of preschemes), which follows rapidly from (**0**, 4.1.1).

Proposition (2.2.4). — [4] *Let $(X, \mathcal{O}_X)$ be a prescheme, and $(S, \mathcal{O}_S)$ an affine scheme associated to a ring A. Then there exists a canonical bijective correspondence between morphisms of preschemes from $(X, \mathcal{O}_X)$ to $(S, \mathcal{O}_S)$ and ring homomorphisms from A to $\Gamma(X, \mathcal{O}_X)$.*

Proof. First note that, if $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ are two arbitrary ringed spaces, a morphism (ψ, θ) from $(X, \mathcal{O}_X)$ to $(Y, \mathcal{O}_Y)$ canonically defines a ring homomorphism $\Gamma(\theta) : \Gamma(Y, \mathcal{O}_Y) \to \Gamma(Y, \psi_*(\mathcal{O}_X)) = \Gamma(X, \mathcal{O}_X)$. In the case that we consider, everything boils down to showing that any homomorphism $\varphi : A \to \Gamma(X, \mathcal{O}_X)$ is of the form $\Gamma(\theta)$ for exactly one θ. Now, by hypothesis, there is a covering (V_α) of X by affine open subsets; by composing φ with the restriction homomorphism $\Gamma(X, \mathcal{O}_X) \to \Gamma(V_\alpha, \mathcal{O}_X|V_\alpha)$, we obtain a homomorphism $\varphi_\alpha : A \to \Gamma(V_\alpha, \mathcal{O}_X|V_\alpha)$ that corresponds to a unique morphism $(\psi_\alpha, \theta_\alpha)$ from the prescheme $(V_\alpha, \mathcal{O}_X|V_\alpha)$ to $(S, \mathcal{O}_S)$, by Theorem (1.7.3). Furthermore, for each pair of indices (α, β), each point of $V_\alpha \cap V_\beta$ admits an affine open neighborhood W contained inside $V_\alpha \cap V_\beta$ (2.1.3); it is clear that, by composing φ_α and φ_β with the restriction homomorphisms to W, we obtain the same homomorphism $\Gamma(S, \mathcal{O}_S) \to \Gamma(W, \mathcal{O}_X|W)$, so, with the equation $(\theta_\alpha^\sharp)_x = (\varphi_\alpha)_x$ for all $x \in V_\alpha$ and all α (1.6.1), the restriction to W of the morphisms $(\psi_\alpha, \theta_\alpha)$ and $(\psi_\beta, \theta_\beta)$ coincide. From this we conclude that there is a morphism $(\psi, \theta) : (X, \mathcal{O}_X) \to (S, \mathcal{O}_S)$ of ringed spaces, and only one such that its restriction to each V_α is $(\psi_\alpha, \theta_\alpha)$, and it is clear that this morphism is a morphism of preschemes and such that $\Gamma(\theta) = \varphi$.

Let $u : A \to \Gamma(X, \mathcal{O}_X)$ be a ring homomorphism, and $v = (\psi, \theta)$ the corresponding morphism $(X, \mathcal{O}_X) \to (S, \mathcal{O}_S)$. For each $f \in A$, we have that

$$\psi^{-1}(D(f)) = X_{u(f)} \tag{2.2.4.1}$$

with the notation of (**0**, 5.5.2) relative to the locally free sheaf $\mathcal{O}_X$. In fact, it suffices to verify this formula when X itself is affine, and then this is nothing but (1.2.2.2). □

Proposition (2.2.5). — *Under the hypotheses of Proposition (2.2.4), let $\varphi : A \to \Gamma(X, \mathcal{O}_X)$ be a ring homomorphism, $f : (X, \mathcal{O}_X) \to (S, \mathcal{O}_S)$ the corresponding morphism of preschemes, $\mathcal{G}$ (resp. $\mathcal{F}$) an $\mathcal{O}_X$-module (resp. $\mathcal{O}_S$-module), and $M = \Gamma(S, \mathcal{F})$. Then there exists a canonical bijective correspondence between f-morphisms $\mathcal{F} \to \mathcal{G}$ (**0**, 4.4.1) and A-homomorphisms $M \to (\Gamma(X, \mathcal{G}))_{[\varphi]}$.* **I | 100**

Proof. Reasoning as in Proposition (2.2.4), we reduce to the case where X is affine, and the proposition then follows from Proposition (1.6.3) and from Corollary (1.3.8). □

(2.2.6). We say that a morphism of preschemes $(\psi, \theta) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is *open* (resp. *closed*) if, for all open subsets U of X (resp. all closed subsets F of X), $\psi(U)$ is open (resp. $\psi(F)$ is closed) in Y. We say that (ψ, θ) is *dominant* if $\psi(X)$ is dense in Y, and *surjective* if ψ is surjective. We note that these conditions rely only on the continuous map ψ.

Proposition (2.2.7). — *Let*

$$f = (\psi, \theta) : (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$$

and

$$g = (\psi', \theta') : (Y, \mathcal{O}_Y) \longrightarrow (Z, \mathcal{O}_Z)$$

be morphisms of preschemes.

(i) *If f and g are both open (resp. closed, dominant, surjective), then so is $g \circ f$.*
(ii) *If f is surjective and $g \circ f$ closed, then g is closed.*
(iii) *If $g \circ f$ is surjective, then g is surjective.*

Proof. Claims (i) and (iii) are evident. Write $g \circ f = (\psi'', \theta'')$. If F is closed in Y then $\psi^{-1}(F)$ is closed in X, so $\psi''(\psi^{-1}(F))$ is closed in Z; but since ψ is surjective, $\psi(\psi^{-1}(F)) = F$, so $\psi''(\psi^{-1}(F)) = \psi'(F)$, which proves (ii). □

Proposition (2.2.8). — *Let $f = (\psi, \theta)$ be a morphism $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$, and (U_α) an open cover of Y. For f to be open (resp. closed, surjective, dominant), it is necessary and sufficient for its restriction to each induced prescheme $(\psi^{-1}(U_\alpha), \mathcal{O}_X|\psi^{-1}(U_\alpha))$, considered as a morphism of preschemes from this induced prescheme to the induced prescheme $(U_\alpha, \mathcal{O}_Y|U_\alpha)$ to be open (resp. closed, surjective, dominant).*

[4] *[Trans.] See (1.8) and the footnote there.*

Proof. The proposition follows immediately from the definitions, taking into account the fact that a subset F of Y is closed (resp. open, dense) in Y if and only if each of the $F \cap U_\alpha$ are closed (resp. open, dense) in U_α. □

(2.2.9). Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be two preschemes; suppose that X (resp. Y) has a finite number of irreducible components X_i (resp. Y_i) $(1 \leqslant i \leqslant n)$; let ξ_i (resp. η_i) be the generic point of X_i (resp. Y_i) (2.1.5). We say that a morphism

$$f = (\psi, \theta) : (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$$

is *birational* if, for all i, $\psi^{-1}(\eta_i) = \{\xi_i\}$ and $\theta^\sharp_{\xi_i} : \mathcal{O}_{\eta_i} \to \mathcal{O}_{\xi_i}$ is an *isomorphism*. It is clear that a birational morphism is dominant (**0**, 2.1.8), and thus it is surjective if it is also closed.

Notation (2.2.10). — In all that follows, when there is no risk of confusion, we *suppress* the structure sheaf (resp. the morphism of structure sheaves) from the notation of a prescheme (resp. morphism of preschemes). If U is an open subset of the underlying space X of a prescheme, then whenever we speak of U as a prescheme we always mean the induced prescheme on U.

2.3. Gluing preschemes. I | 101

(2.3.1). It follows from Definition (2.1.2) that every ringed space obtained by *gluing* preschemes (**0**, 4.1.7) is again a prescheme. In particular, although every prescheme admits, by definition, a cover by affine open sets, we see that every prescheme can actually be obtained by *gluing affine schemes.*

Example (2.3.2). — Let K be a field, $B = K[s]$ and $C = K[t]$ polynomial rings in one indeterminate over K, and define $X_1 = \operatorname{Spec}(B)$ and $X_2 = \operatorname{Spec}(C)$, which are isomorphic affine schemes. In X_1 (resp. X_2), let U_{12} (resp. U_{21}) be the affine open $D(s)$ (resp. $D(t)$) where the ring B_s (resp. C_t) is formed of rational fractions of the form $f(s)/s^m$ (resp. $g(t)/t^n$) with $f \in B$ (resp. $g \in C$). Let u_{12} be the isomorphism of preschemes $U_{21} \to U_{12}$ corresponding (2.2.4) to the isomorphism from B_s to C_t that, to $f(s)/s^m$, associates the rational fraction $f(1/t)/(1/t^m)$. We can glue X_1 and X_2 along U_{12} and U_{21} by using u_{12}, because there is clearly no gluing condition. We later show that the prescheme X obtained in this manner is a particular case of a general method of construction (**II**, 2.4.3). Here we show only that X *is not an affine scheme*; this will follow from the fact that the ring $\Gamma(X, \mathcal{O}_X)$ is *isomorphic* to K, and so its spectrum reduces to a point. Indeed, a section of $\mathcal{O}_X$ over X has a restriction over X_1 (resp. X_2), identified with an affine open of X, that is a polynomial $f(s)$ (resp. $g(t)$), and it follows from the definitions that we should have $g(t) = f(1/t)$, which is only possible if $f = g \in K$.

2.4. Local schemes.

(2.4.1). We say that an affine scheme is a *local scheme* if it is the affine scheme associated to a local ring A; there then exists, in $X = \operatorname{Spec}(A)$, a single *closed point* $a \in X$, and for all other $b \in X$ we have that $a \in \overline{\{b\}}$ (1.1.7).

For all preschemes Y and points $y \in Y$, the local scheme $\operatorname{Spec}(\mathcal{O}_y)$ is called the *local scheme of Y at the point y*. Let V be an affine open subset of Y containing y, and B the ring of the affine scheme V; then $\mathcal{O}_y$ is canonically identified with B_y (1.3.4), and the canonical homomorphism $B \to B_y$ thus corresponds (1.6.1) to a morphism of preschemes $\operatorname{Spec}(\mathcal{O}_y) \to V$. If we compose this morphism with the canonical injection $V \to Y$, then we obtain a morphism $\operatorname{Spec}(\mathcal{O}_y) \to Y$ which is *independent* of the affine open subset V (containing y) that we chose: indeed, if V' is some other affine open subset containing y, then there exists a third affine open subset W that contains y and is such that $W \subset V \cap V'$ (2.1.3); we can thus assume that $V \subset V'$, and then if B' is the ring of V', so everything relies on remarking that the diagram

$$\begin{tikzcd} B' \arrow[rr] \arrow[dr] & & B \arrow[dl] \\ & \mathcal{O}_y & \end{tikzcd}$$

is commutative (**0**, 1.5.1). The morphism

$$\operatorname{Spec}(\mathcal{O}_y) \longrightarrow Y$$

thus defined is said to be *canonical*. I | 102

Proposition (2.4.2). — *Let $(Y, \mathcal{O}_Y)$ be a prescheme; for all $y \in Y$, let (ψ, θ) be the canonical morphism* $(\operatorname{Spec}(\mathcal{O}_y), \widetilde{\mathcal{O}_y}) \to (Y, \mathcal{O}_Y)$. *Then ψ is a homeomorphism from* $\operatorname{Spec}(\mathcal{O}_y)$ *to the subspace S_y of Y given by the z such that* $y \in \overline{\{z\}}$ (or, equivalently, the *generalizations* of y (**0**, 2.1.2)); *furthermore, if $z = \psi(\mathfrak{p})$, then $\theta^\sharp_z : \mathcal{O}_z \to (\mathcal{O}_y)_\mathfrak{p}$ is an isomorphism; (ψ, θ) is thus a monomorphism of ringed spaces.*

Proof. Since the unique closed point a of $\operatorname{Spec}(\mathcal{O}_y)$ is contained in the closure of any point of this space, and since $\psi(a) = y$, the image of $\operatorname{Spec}(\mathcal{O}_y)$ under the continuous map ψ is contained in S_y. Since S_y is contained in every affine open containing y, one can consider just the case where Y is an affine scheme; but then this proposition follows from (1.6.2). □

We see (2.1.5) that there is a bijective correspondence between $\operatorname{Spec}(\mathcal{O}_y)$ *and the set of closed irreducible subsets of Y containing y.*

Corollary (2.4.3). — *For $y \in Y$ to be the generic point of an irreducible component of Y, it is necessary and sufficient for the only prime ideal of the local ring $\mathcal{O}_y$ to be its maximal ideal (*in other words, for $\mathcal{O}_y$ to be of *dimension zero).*

Proposition (2.4.4). — *Let $(X, \mathcal{O}_X)$ be a local scheme of some ring A, $\mathfrak{a}$ its unique closed point, and $(Y, \mathcal{O}_Y)$ a prescheme. Every morphism $u = (\psi, \theta) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ then factors uniquely as $X \to \operatorname{Spec}(\mathcal{O}_{\psi(a)}) \to Y$, where the second arrow denotes the canonical morphism, and the first corresponds to a local homomorphism $\mathcal{O}_{\psi(a)} \to A$. This establishes a canonical bijective correspondence between the set of morphisms $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ and the set of local homomorphisms $\mathcal{O}_y \to A$ for ($y \in Y$).*

Indeed, for all $x \in X$, we have that $a \in \overline{\{x\}}$, so $\psi(a) \in \overline{\{\psi(x)\}}$, which shows that $\psi(X)$ is contained in every affine open subset that contains $\psi(a)$. So it suffices to consider the case where $(Y, \mathcal{O}_Y)$ is an affine scheme of ring B, and then we have that $u = ({}^a\varphi, \tilde{\varphi})$, where $\varphi \in \operatorname{Hom}(B, A)$ (1.7.3). Further, we have that $\varphi^{-1}(\mathfrak{j}_a) = \mathfrak{j}_{\psi(a)}$, and hence that the image under φ of any element of $B - \mathfrak{j}_{\psi(a)}$ is invertible in the local ring A; the factorization in the result follows from the universal property of the ring of fractions (**0**, 1.2.4). Conversely, to each local homomorphism $\mathcal{O}_y \to A$ there is a unique corresponding morphism $(\psi, \theta) : X \to \operatorname{Spec}(\mathcal{O}_y)$ such that $\psi(a) = y$ (1.7.3), and, by composing with the canonical morphism $\operatorname{Spec}(\mathcal{O}_y) \to Y$, we obtain a morphism $X \to Y$, which proves the proposition.

(2.4.5). The affine schemes whose ring is a field K have an underlying space that is just a point. If A is a local ring with maximal ideal $\mathfrak{m}$, then each local homomorphism $A \to K$ has kernel equal to $\mathfrak{m}$, and so factors as $A \to A/\mathfrak{m} \to K$, where the second arrow is a monomorphism. The morphisms $\operatorname{Spec}(K) \to \operatorname{Spec}(A)$ thus correspond bijectively to monomorphisms of fields $A/\mathfrak{m} \to K$.

Let $(Y, \mathcal{O}_Y)$ be a prescheme; for each $y \in Y$ and each ideal $\mathfrak{a}_y$ of $\mathcal{O}_y$, the canonical homomorphism $\mathcal{O}_y \to \mathcal{O}_y/\mathfrak{a}_y$ defines a morphism $\operatorname{Spec}(\mathcal{O}_y/\mathfrak{a}_y) \to \operatorname{Spec}(\mathcal{O}_y)$; if we compose this with the canonical morphism $\operatorname{Spec}(\mathcal{O}_y) \to Y$, then we obtain a morphism $\operatorname{Spec}(\mathcal{O}_y/\mathfrak{a}_y) \to Y$, again said to be *canonical*. For $\mathfrak{a}_y = \mathfrak{m}_y$, this says that $\mathcal{O}_y/\mathfrak{a}_y = k(y)$, and so Proposition (2.4.4) says that:

Corollary (2.4.6). — *Let $(X, \mathcal{O}_X)$ be a local scheme whose ring K is a field, ξ the unique point of X, and $(Y, \mathcal{O}_Y)$ a prescheme. Then each morphism $u : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ factors uniquely as $X \to \operatorname{Spec}(k(\psi(\xi))) \to Y$, where the second arrow denotes the canonical morphism, and the first corresponds to a monomorphism $k(\psi(\xi)) \to K$. This establishes a canonical bijective correspondence between the set of morphisms $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ and the set of monomorphisms $k(y) \to K$ (for $y \in Y$).* **I | 103**

Corollary (2.4.7). — *For all $y \in Y$, every canonical morphism* $\operatorname{Spec}(\mathcal{O}_y/\mathfrak{a}_y) \to Y$ *is a monomorphism of ringed spaces.*

Proof. We have already seen this when $\mathfrak{a}_y = 0$ (2.4.2), and it suffices to apply Corollary (1.7.5). □

Remark. — 2.4.8 Let X be a local scheme, and a its unique closed point. Since every affine open subset containing a is necessarily equal to the whole of X, every *invertible* $\mathcal{O}_X$-module (**0**, 5.4.1) is necessarily *isomorphic to* $\mathcal{O}_X$ (or, as we say, again, *trivial*). This property does not hold in general for an arbitrary affine scheme $\operatorname{Spec}(A)$; we will see in Chapter V that if A is a normal ring then this is true when A is a unique factorisation domain.

2.5. Preschemes over a prescheme.

Definition (2.5.1). — Given a prescheme S, we say that the data of a prescheme X and a morphism of preschemes $\varphi : X \to S$ defines a prescheme X *over the prescheme* S, or an S*-prescheme*; we say that S is the *base prescheme* of the S-prescheme X. The morphism φ is called the *structure morphism* of the S-prescheme X. When S is an affine scheme of ring A, we also say that X endowed with φ is a prescheme *over the ring* A (or an A*-prescheme*).

It follows from (2.2.4) that the data of a prescheme over a ring A is equivalent to the data of a prescheme $(X, \mathcal{O}_X)$ whose structure sheaf $\mathcal{O}_X$ is a sheaf of A-*algebras. An arbitrary prescheme can always be considered as a* $\mathbf{Z}$*-prescheme in a unique way.*

If $\varphi : X \to S$ is the structure morphism of an S-prescheme X, we say that a point $x \in X$ is *over a point* $s \in S$ if $\varphi(x) = s$. We say that X *dominates* S if φ is a dominant morphism (2.2.6).

(2.5.2). Let X and Y be S-preschemes; we say that a morphism of preschemes $u : X \to Y$ is a *morphism of preschemes over* S (or an S*-morphism*) if the diagram

$$\begin{tikzcd} X \arrow[rr, "u"] \arrow[dr] & & Y \arrow[dl] \\ & S & \end{tikzcd}$$

(where the diagonal arrows are the structure morphisms) is commutative: this ensures that, for all $s \in S$ and $x \in X$ over s, $u(x)$ also lies over s.

It follows immediately from this definition that the composition of any two S-morphisms is an S-morphism; S-preschemes thus form a *category*.

We denote by $\mathrm{Hom}_S(X,Y)$ the set of S-morphisms from an S-prescheme X to an S-prescheme Y; the identity morphism of an S-prescheme X is denoted by 1_X.

When S is an affine scheme of ring A, we will also say A*-morphism* instead of S-morphism.

(2.5.3). If X is an S-prescheme, and $v : X' \to X$ a morphism of preschemes, then the composition $X' \to X \to S$ endows X' with the structure of an S-prescheme; in particular, every prescheme induced by an open set U of X can be considered as an S-prescheme by the canonical injection. **I | 104**

If $u : X \to Y$ is an S-morphism of S-preschemes, then the restriction of u to any prescheme induced by an open subset U of X is also an S-morphism $U \to Y$. Conversely, let (U_α) be an open cover of X, and for each α let $u_\alpha : U_\alpha \to Y$ be an S-morphism; if, for all pairs of indices (α,β), the restrictions of u_α and u_β to $U_\alpha \cap U_\beta$ agree, then there exists an S-morphism $X \to Y$, and exactly one such that the restriction to each U_α is u_α.

If $u : X \to Y$ is an S-morphism such that $u(X) \subset V$, where V is an open subset of Y, then u, considered as a morphism from X to V, is also an S-morphism.

(2.5.4). Let $S' \to S$ be a morphism of preschemes; for all S'-preschemes, the composition $X \to S' \to S$ endows X with the structure of an S-prescheme. Conversely, suppose that S' is the induced prescheme of an open subset of S; let X be an S-prescheme and suppose that the structure morphism $f : X \to S$ is such that $f(X) \subset S'$; then we can consider X as an S'-prescheme. In this latter case, if Y is another S-prescheme whose structure morphism sends the underlying space to S', then every S-morphism from X to Y is also an S'-morphism.

(2.5.5). If X is an S-prescheme, with structure morphism $\varphi : X \to S$, we define an S*-section of* X to be an S-morphism from S to X, that is to say a morphism of preschemes $\psi : S \to X$ such that $\varphi \circ \psi$ is the identity on S. We denote by $\Gamma(X/S)$ the set of S-sections of X.

§3. Products of preschemes

3.1. Sums of preschemes. Let (X_α) be any family of preschemes; let X be a topological space which is the *sum* of the underlying spaces X_α; X is then the union of pairwise disjoint open subspaces X'_α, and for each α there is a homomorphism φ_α from X_α to X'_α. If we equip each of the X'_α with the sheaf $(\varphi_\alpha)_*(\mathscr{O}_{X_\alpha})$, it is clear that X becomes a prescheme, which we call the *sum* of the family of preschemes (X_α) and which we denote $\coprod_\alpha X_\alpha$. If Y is a prescheme, then the map $f \mapsto (f \circ \varphi_\alpha)$ is a *bijection* from the set $\mathrm{Hom}(X,Y)$ to the product set $\prod_\alpha \mathrm{Hom}(X_\alpha,Y)$. In particular, if the X_α are S-preschemes, with structure morphisms ψ_α, then X is an S-prescheme by the unique morphism $\psi : X \to S$ such that $\psi \circ \varphi_\alpha = \psi_\alpha$ for each α. The sum of two preschemes X and Y is denoted by $X \sqcup Y$. It is immediate that, if $X = \mathrm{Spec}(A)$ and $Y = \mathrm{Spec}(B)$, then $X \sqcup Y$ is canonically identified with $\mathrm{Spec}(A \times B)$.

3.2. Products of preschemes.

Definition (3.2.1). — Given S-preschemes X and Y, we say that a triple (Z, p_1, p_2), consisting of an S-prescheme Z, and S-morphisms $p_1 : Z \to X$ and $p_2 : Z \to Y$, is a product of the S-preschemes X and Y, if, for each S- **I | 105**
prescheme T, the map $f \mapsto (p_1 \circ f, p_2 \circ f)$ is a bijection from the set of S-morphisms from T to Z, to the set of pairs consisting of an S-morphism $T \to X$ and an S-morphism $T \to Y$ (in other words, a bijection

$$\mathrm{Hom}_S(T,Z) \simeq \mathrm{Hom}_S(T,X) \times \mathrm{Hom}_S(T,Y)).$$

This is the general notion of a *product* of two objects in a category, applied to the category of S-preschemes (T, I, 1.1); in particular, a product of two S-preschemes is *unique* up to a unique S-isomorphism. Because of this uniqueness, most of the time we will denote a product of two S-preschemes X and Y by $X \times_S Y$ (or simply $X \times Y$, when there is no chance of confusion), with the morphisms p_1 and p_2 (the *canonical projections* of $X \times_S Y$ to X and to Y, respectively) being suppressed in the notation. If $g : T \to X$ and $h : T \to Y$ are S-morphisms, we denote by $(g,h)_S$, or simply (g,h), the S-morphism $f : T \to X \times_S Y$ such that $p_1 \circ f = g$, $p_2 \circ f = h$. If X' and Y' ar two S-preschemes, p_1' and p_2' the canonical projections of $X' \times_S Y'$ (assumed to exist), and $u : X' \to X$ and $v : Y' \to Y$ S-morphisms, then we write $u \times_S v$ (or simply $u \times v$) for the S-morphism $(u \circ p_1', v \circ p_2')_S$ from $X' \times_S Y'$ to $X \times_S Y$.

When S is an affine scheme given by some ring A, we often replace S by A is the above notation.

Proposition (3.2.2). — *Let X, Y, and S be affine schemes, given by rings B, C, and A (respectively). Let* $Z = \operatorname{Spec}(B \otimes_A C)$, *and let p_1 and p_2 be the S-morphisms corresponding (2.2.4) to the canonical A-homomorphisms $u : b \mapsto b \otimes 1$ and $v : c \mapsto 1 \otimes c$ (respectively) from B and C to $B \otimes_A C$; then (Z, p_1, p_2) is a product of X and Y.*

Proof. According to (2.2.4), it suffices to check that, if, to each A-homomorphism $f : B \otimes_A C \to L$ (where L is an A-algebra), we associate the pair $(f \circ u, f \circ v)$, then this defines a bijection $\operatorname{Hom}_A(B \otimes_A C, L) \simeq \operatorname{Hom}_A(B, L) \times \operatorname{Hom}_A(C, L)$,[5] which follows immediately from the definitions and the fact that $b \otimes c = (b \otimes 1)(1 \otimes c)$. ☐

Corollary (3.2.3). — *Let T be an affine scheme given by some ring D, and $\alpha = ({}^a\rho, \widetilde{\rho})$ (resp. $\beta = ({}^a\sigma, \widetilde{\sigma})$) an S-morphism $T \to X$ (resp. $T \to Y$), where ρ (resp. σ) is an A-homomorphism from B (resp. C) to D; then $(\alpha, \beta)_S = ({}^a\tau, \widetilde{\tau})$, where τ is the homomorphism $B \otimes_A C \to D$ such that $\tau(b \otimes c) = \rho(b)\sigma(c)$.*

Proposition (3.2.4). — *Let $f : S' \to S$ be a* monomorphism *of preschemes* (T, I, 1.1), *and let X and Y be S'-preschemes, also considered as S-preschemes via f. Every product of the S-preschemes X and Y is then a product of the S'-preschemes X and Y, and vice versa.*

Proof. Let $\varphi : X \to S'$ and $\psi : Y \to S'$ be the structure morphisms. If T is an S-prescheme, and $u : T \to X$ and $v : T \to Y$ are S-morphisms, then we have, by definition, that $f \circ \varphi \circ u = f \circ \psi \circ v = \theta$, the structure morphism of T; the hypotheses on f imply that $\varphi \circ u = \psi \circ v = \theta'$, and so we can consider T as an S'-prescheme with structure morphism θ', and u and v as S'-morphisms. The conclusion of the proposition follows immediately, taking (3.2.1) into account. ☐

Corollary (3.2.5). — *Let X and Y be S-preschemes, with structure morphisms $\varphi : X \to S$ and $\psi : Y \to S$, and let S' be an open subset of S such that $\varphi(X) \subset S'$ and $\psi(Y) \subset S'$. Every product of the S-preschemes X and Y is then also a product of the S'-preschemes X and Y, and conversely.*

It suffices to apply (3.2.4) to the canonical injection $S' \to S$. **I** | 106

Theorem (3.2.6). — *Given S-preschemes X and Y, there exists a product $X \times_S Y$.*

The proof proceeds in several steps.

Lemma (3.2.6.1). — *Let (Z, p, q) be a product of X and Y, and U and V open subsets of X and Y, respectively. If we let $W = p^{-1}(U) \cap q^{-1}(V)$, then the triple consisting of W and the restrictions of p and q to W* (considered as the morphisms $W \to U$ and $W \to V$, respectively) *is a product of U and V.*

Indeed, if T is an S-prescheme, then we can identify the S-morphisms $T \to W$ with the S-morphisms $T \to Z$ mapping T to W. Then, if $g : T \to U$ and $h : T \to V$ are any two S-morphisms, we can consider them as S-morphisms from T to X and to Y, respectively, and, by hypothesis, there is then a unique S-morphism $f : T \to Z$ such that $g = p \circ f$ and $h = q \circ f$. Since $p(f(Y)) \subset U$, $q(f(T)) \subset V$, we have

$$f(T) \subset p^{-1}(U) \cap q^{-1}(V) = W,$$

hence our claim.

Lemma (3.2.6.2). — *Let Z be an S-prescheme, $p : Z \to X$ and $q : Z \to Y$ both S-morphisms, (U_α) an open cover of X, and (V_λ) an open cover of Y. Suppose that, for each pair (α, λ), the S-prescheme $W_{\alpha\lambda} = p^{-1}(U_\alpha) \cap q^{-1}(V_\lambda)$ and the restrictions of p and q to $W_{\alpha\lambda}$ form a product of U_α and V_λ. Then (Z, p, q) is a product of X and Y.*

We first show that, if f_1 and f_2 are S-morphisms $T \to Z$, then the equations $p \circ f_1 = p \circ f_2$ and $q \circ f_1 = q \circ f_2$ imply that $f_1 = f_2$. Indeed, Z is the union of the $W_{\alpha\lambda}$, so the $f_1^{-1}(W_{\alpha\lambda})$ form an open cover of T, and similarly for $f_2^{-1}(W_{\alpha\lambda})$. In addition, we have

$$f_1^{-1}(W_{\alpha\lambda}) = f_1^{-1}(p^{-1}(U_\alpha)) \cap f_1^{-1}(q^{-1}(V_\lambda)) = f_2^{-1}(p^{-1}(U_\alpha)) \cap f_2^{-1}(q^{-1}(V_\lambda)) = f_2^{-1}(W_{\alpha\lambda})$$

by hypothesis, and it thus reduces to noting that the the restrictions of f_1 and f_2 to $f_1^{-1}(W_{\alpha\lambda}) = f_2^{-1}(W_{\alpha\lambda})$ are identical for each pair of indices. But since these restrictions can be considered as S-morphisms from $f_1^{-1}(W_{\alpha\lambda})$ to $W_{\alpha\lambda}$, our claim follows from the hypotheses and Definition (3.2.1).

[5] The notation Hom_A denotes here the set of homomorphisms of *A-algebras*.

Suppose now that we are given S-morphisms $g : T \to X$ and $h : T \to Y$. Let $T_{\alpha\lambda} = g^{-1}(U_\alpha) \cap h^{-1}(V_\lambda)$; then the $T_{\alpha\lambda}$ form an open cover of T. By hypothesis, there exists an S-morphism $f_{\alpha\lambda}$ such that $p \circ f_{\alpha\lambda}$ and $q \circ f_{\alpha\lambda}$ are the restrictions of g and h to $T_{\alpha\lambda}$ (respectively). Now, we will show that the restrictions of $f_{\alpha\lambda}$ and $f_{\beta\mu}$ to $T_{\alpha\lambda} \cap T_{\beta\mu}$ coincide, which will finish the proof of Lemma (3.2.6.2). The images of $T_{\alpha\lambda} \cap T_{\beta\mu}$ under $f_{\alpha\lambda}$ and $f_{\beta\mu}$ are contained in $W_{\alpha\lambda} \cap W_{\beta\mu}$ by definition. Since

$$W_{\alpha\lambda} \cap W_{\beta\mu} = p^{-1}(U_\alpha \cap U_\beta) \cap q^{-1}(V_\lambda \cap V_\mu),$$

it follows from Lemma (3.2.6.1) that $W_{\alpha\lambda} \cap W_{\beta\mu}$ and the restrictions to this prescheme of p and q form a *product* of $U_\alpha \cap U_\beta$ and $V_\lambda \cap V_\mu$. Since $p \circ f_{\alpha\lambda}$ and $p \circ f_{\beta\mu}$ coincide on $T_{\alpha\lambda} \cap T_{\beta\mu}$ and similarly for $q \circ f_{\alpha\lambda}$ and $q \circ f_{\beta\mu}$, we see that $f_{\alpha\lambda}$ and $f_{\beta\mu}$ coincide on $T_{\alpha\lambda} \cap T_{\beta\mu}$.

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Lemma (3.2.6.3). — *Let (U_α) be an open cover of X, (V_λ) an open cover of Y, and suppose that, for each pair (α, λ), there exists a product of U_α and V_λ; then there exists a product of X and Y.*

Applying Lemma (3.2.6.1) to the open sets $U_\alpha \cap U_\beta$ and $V_\lambda \cap V_\mu$, we see that there exists a product of S-preschemes induced, respectively, by X and Y on these open sets; in addition, the uniqueness of the product shows that, if we set $i = (\alpha, \lambda)$ and $j = (\beta, \mu)$, then there is a canonical isomorphism h_{ij} (resp. h_{ji}) from this product to an S-prescheme W_{ij} (resp. W_{ji}) induced by $U_\alpha \times_S V_\lambda$ (resp. $U_\beta \times_S V_\mu$) on an open set; then $f_{ij} = h_{ij} \circ h_{ji}^{-1}$ is an isomorphism from W_{ji} to W_{ij}. In addition, for a third pair $k = (\gamma, \nu)$, we have $f_{ik} = f_{ij} \circ f_{jk}$ on $W_{ki} \cap W_{kj}$, by applying Lemma (3.2.6.1) to the open sets $U_\alpha \cap U_\beta \cap U_\gamma$ and $V_\lambda \cap V_\mu \cap V_\nu$ in U_β and V_μ, respectively. It follows that we have a prescheme Z, an open cover (Z_i) of the underlying space of Z, and, for each i, an isomorphism g_i from the induced prescheme Z_i to the prescheme $U_\alpha \times_S V_\lambda$, so that, for each pair (i, j), we have $f_{ij} = g_i \circ g_j^{-1}$ (2.3.1); in addition, we have $g_i(Z_i \cap Z_j) = W_{ij}$. If p_i, q_i, and θ_i are the projections and the structure morphism of the S-prescheme $U_\alpha \times_S V_\lambda$ (respectively), we immediately see that $p_i \circ g_i = p_j \circ g_j$ on $Z_i \cap Z_j$, and similarly for the two other morphisms. We can thus define the morphisms of preschemes $p : Z \to X$ (resp. $q : Z \to Y$, $\theta : Z \to S$) by the condition that p (resp. q, θ) coincide with $p_i \circ g_i$ (resp. $q_i \circ g_i$, $\theta_i \circ g_i$) on each of the Z_i; Z, equipped with θ, is then an S-prescheme. We now show that $Z_i' = p^{-1}(U_\alpha) \cap q^{-1}(V_\lambda)$ is equal to Z_i. For each index $j = (\beta, \mu)$, we have $Z_j \cap Z_i' = g_j^{-1}(p_j^{-1}(U_\alpha) \cap q_j^{-1}(V_\lambda))$. We have, by Lemma (3.2.6.1),

$$p_j^{-1}(U_\alpha) \cap q_j^{-1}(V_\lambda) = p_j^{-1}(U_\alpha \cap U_\beta) \cap q_j^{-1}(V_\lambda \cap V_\mu);$$

with the restrictions of p_j and q_j to $p_j^{-1}(U_\alpha) \cap q_j^{-1}(V_\lambda)$ defining, on this S-prescheme, the structure of a product of $U_\alpha \cap U_\beta$ and $V_\lambda \cap V_\mu$; but the uniqueness of the product then implies that $p_j^{-1}(U_\alpha) \cap q_j^{-1}(V_\lambda) = W_{ji}$. As a result, we have $Z_j \cap Z_i' = Z_j \cap Z_i$ for each j, hence $Z_i' = Z_i$. We then deduce from Lemma (3.2.6.2) that (Z, p, q) is a product of X and Y.

Lemma (3.2.6.4). — *Let $\varphi : X \to S$ and $\psi : Y \to S$ be the structure morphisms of X and Y, (S_i) an open cover of S, and let $X_i = \varphi^{-1}(S_i)$, $Y_i = \psi^{-1}(S_i)$. If each of the products $X_i \times_S Y_i$ exists, then $X \times_S Y$ exists.*

According to Lemma (3.2.6.3), everything follows from proving that the products $X_i \times_S Y_i$ exists for any i and j. Set $X_{ij} = X_i \cap X_j = \varphi^{-1}(S_i \cap S_j)$, $Y_{ij} = Y_i \cap Y_j = \psi^{-1}(S_i \cap S_j)$; by Lemma (3.2.6.1), the product $Z_{ij} = X_{ij} \times_S Y_{ij}$ exists. We now note that, if T is an S-prescheme, and if $g : T \to X_i$ and $h : T \to Y_j$ are S-morphisms, then we necessarily have that $\varphi(g(T)) = \psi(h(T)) \subset S_i \cap S_j$ by the definition of an S-morphism, and thus that $g(T) \subset X_{ij}$ and $h(T) \subset Y_{ij}$; it is then immediate that Z_{ij} is the product of X_i and Y_j.

(3.2.6.5). We can now complete the proof of Theorem (3.2.6). If S is an *affine scheme*, then there are covers (U_α) and (V_λ) of X and Y (respectively) consisting of affine open subsets; since $U_\alpha \times_S V_\lambda$ exists, by (3.2.2), $X \times_S Y$ exists similarly, by Lemma (3.2.6.3). If S is any prescheme, then there is a cover (S_i) of S consisting of affine open subsets. If $\varphi : X \to S$ and $\psi : Y \to S$ are the structure morphisms, and if we set $X_i = \varphi^{-1}(S_i)$ and $Y_i = \psi^{-1}(S_i)$, then the products $X_i \times_{S_i} Y_i$ exist, by the above; but then the products $X_i \times_S Y_i$ also exist (3.2.5), therefore $X \times_S Y$ I | 108
exists similarly, by Lemma (3.2.6.4). □

Corollary (3.2.7). — *Let $Z = X \times_S Y$ be the product of two S-preschemes, p and q the projections from Z to X and to Y (respectively),and φ (resp. ψ) the structure morphism of X (resp. Y). Let S' be an open subset of S, and U (resp. V) an open subset of X (resp. Y) contained in $\varphi^{-1}(S')$ (resp. $\psi^{-1}(S')$). Then the product $U \times_{S'} V$ is canonically identified with the prescheme induced on Z by $p^{-1}(U) \cap q^{-1}(V)$ (considered as an S'-prescheme). In addition, if $f : T \to X$ and $g : T \to Y$ are S-morphisms such that $f(T) \subset U$ and $g(T) \subset V$, then the S'-morphism $(f, g)_{S'}$ can be identified with the restriction of $(f, g)_S$ to $p^{-1}(U) \cap q^{-1}(V)$.*

Proof. This follows from Corollary (3.2.5) and Lemma (3.2.6.1). □

(3.2.8). Let (X_α) and (Y_λ) be families of S-preschemes, and X (resp. Y) the sum of the family (X_α) (resp. (Y_λ)) (3.1). Then $X \times_S Y$ can be identified with the *sum* of the family $(X_\alpha \times_S Y_\lambda)$; this follows immediately from Lemma (3.2.6.3).

(3.2.9). [6] It follows from (1.8.1) that we can state (3.2.2) in the following manner: $Z = \operatorname{Spec}(B \otimes_A C)$ is not only a product of $X = \operatorname{Spec}(B)$ and $Y = \operatorname{Spec}(C)$ in the category of *S-preschemes*, but also in the category of *locally ringed spaces over S* (with a definition of S-morphisms modelled on that of (2.5.2)). The proof of (3.2.6) also proves that, for any two S-preschemes X and Y, the prescheme $X \times_S Y$ is not only the product of X and Y in the category of S-preschemes, but also in the category of locally ringed spaces over the prescheme S. **II** | 221

3.3. Formal properties of the product; change of the base prescheme.

(3.3.1). The reader will notice that all the properties stated in this section, except (3.3.13) and (3.3.15), are true without modification in any category, whenever the products involved in the statements exist (since it is clear that the notions of an S-object and of an S-morphism can be defined exactly as in (2.5) for any object S of the category).

(3.3.2). First of all, $X \times_S Y$ is a *covariant bifunctor* in X and Y on the category of S-preschemes: it suffices in fact to note that the diagram

$$\begin{tikzcd} X \times Y \arrow[r, "f\times 1"] \arrow[d] & X' \times Y \arrow[r, "f'\times 1"] \arrow[d] & X'' \times Y \arrow[d] \\ X \arrow[r, "f"] & X' \arrow[r, "f'"] & X'' \end{tikzcd}$$

is commutative.

Proposition (3.3.3). — *For each S-prescheme X, the first (resp. second) projection from $X \times_S S$ (resp. $S \times_S X$) is a functorial isomorphism from $X \times_S S$ (resp. $S \times_S X$) to X, whose inverse isomorphism is $(1_X, \varphi)_S$ (resp. $(\varphi, 1_X)_S$), where we denote by φ the structure morphism $X \to S$; we can therefore write, up to a canonical isomorphism,*

$$X \times_S S = S \times_S X = X.$$

Proof. It suffices to prove that the triple $(X, 1_X, \varphi)$ is a product of X and S. If T is an S-prescheme, then the only S-morphism from T to S is necessarily the structure morphism $\psi : T \to S$. If f is an S-morphism from T to X, we necessarily have $\psi = \varphi \circ f$, hence our claim. □

Corollary (3.3.4). — *Let X and Y be S-preschemes, with structure morphisms $\varphi : X \to S$ and $\psi : Y \to S$. If we canonically identify X with $X \times_S S$, and Y with $S \times_S Y$, then the projections $X \times_S Y \to X$ and $X \times_S Y \to Y$ are identified with $1_X \times \psi$ and $\varphi \times 1_Y$ (respectively).*

The proof is immediate and is left to the reader.

(3.3.5). We can define, in a manner similar to (3.2), the product of a finite number n of S-preschemes, and the existence of these products follows from (3.2.6) by induction on n, and by noting that $(X_1 \times_S X_2 \times_S \cdots \times_S X_{n-1}) \times_S X_n$ satisfies the definition of a product. The uniqueness of the product implies, as in any category, its *commutativity* and *associativity* properties. If, for example, p_1, p_2, and p_3 denote the projections from $X_1 \times_S X_2 \times_S X_3$, and if we identify this prescheme with $(X_1 \times_S X_2) \times_S X_3$, then the projection to $X_1 \times_S X_2$ is identified with $(p_1, p_2)_S$. **I** | 109

(3.3.6). Let S and S' be preschemes, and $\varphi : S' \to S$ a morphism, which lets us consider S' as an S-prescheme. For each S-prescheme X, consider the product $X \times_S S'$, and let p and π' be the projections to X and to S' (respectively). Equipped with π', this product is an S'-prescheme; when we consider it as such, we denote it by $X_{(S')}$ or $X_{(\varphi)}$, and we say that this is the prescheme obtained by *base change* (or *a change of base*) from S to S' by means of the morphism φ, or the *inverse image* of X by φ. We note that, if π is the structure morphism of X, and θ the structure morphism of $X \times_S S'$, considered as an S-prescheme, then the diagram

$$\begin{tikzcd} X \arrow[d, "\pi"'] & X_{(S')} \arrow[l, "p"'] \arrow[d, "\pi'"] \arrow[dl, "\theta"'] \\ S & S' \arrow[l, "\varphi"] \end{tikzcd}$$

is commutative.

[6] *[Trans.] (3.2.9) is from the errata of EGA II, on page 221, whence the change in page numbering.*

(3.3.7). With the notation of (3.3.6), for each S-morphism $f : X \to Y$, we denote by $f_{(S')}$ the S'-morphism $f \times_S 1 : X_{(S')} \to Y_{(S')}$, and we say that $f_{(S')}$ is the *base change* (or *inverse image*) of f by φ. Therefore $X_{(S')}$ is a *covariant functor* in X, from the category of S-preschemes to that of S'-preschemes.

(3.3.8). The prescheme $X_{(S')}$ can be considered as a solution to a *universal mapping problem*: each S'-prescheme T is also an S-prescheme via φ; each S-morphism $g : T \to X$ is then uniquely written as $g = p \circ f$, where f is an S'-morphism $T \to X_{(S')}$, as follows from the definition of the product applied to the S-morphisms f and $\psi : T \to S'$ (the structure morphism of T).

Proposition (3.3.9). — ("Transitivity of base change"). *Let S'' be a prescheme, and $\varphi' : S'' \to S$ a morphism. For each S-prescheme X, there exists a canonical functorial isomorphism from the S''-prescheme $(X_{(\varphi)})_{(\varphi')}$ to the S''-prescheme $X_{(\varphi\circ\varphi')}$.*

Proof. Let T be a S''-prescheme, ψ its structure morphism, and g an S-morphism from T to X (T being considered as an S-prescheme with structure morphism $\varphi \circ \varphi' \circ \psi$). Since T is also an S'-prescheme with structure morphism $\varphi' \circ \psi$, we can write $g = p \circ g'$, where g' is an S'-morphism $T \to X_{(\varphi)}$, and then $g' = p' \circ g''$, where g'' is an S''-morphism $T \to (X_{(\varphi)})_{(\varphi')}$:

$$\begin{tikzcd} X \arrow[d, "\pi"'] & X_{(\varphi)} \arrow[l, "p"'] \arrow[d, "\pi'"'] & (X_{(\varphi)})_{(\varphi')} \arrow[l, "p'"'] \arrow[d, "\pi''"] \\ S & S \arrow[l, "\varphi"'] & S''. \arrow[l, "\varphi'"'] \end{tikzcd}$$

So the result follows by the uniqueness of the solution to a universal mapping problem. □ **I | 110**

This result can be written as the equality (up to a canonical isomorphism) $(X_{(S')})_{(S'')} = X_{(S'')}$ (if there is no chance of confusion), or also as

(3.3.9.1) $$(X \times_S S') \times_{S'} S'' = X \times_S S'';$$

the functorial nature of the isomorphism defined in (3.3.9) can similarly be expressed by the transitivity formula for base change morphisms

(3.3.9.2) $$(f_{(S')})_{(S'')} = f_{(S'')}$$

for each S-morphism $f : X \to Y$.

Corollary (3.3.10). — *If X and Y are S-preschemes, then there exists a canonical functorial isomorphism from the S'-prescheme $X_{(S')} \times_{S'} Y_{(S')}$ to the S'-prescheme $(X \times_S Y)_{(S')}$.*

Proof. We have, up to canonical isomorphism,

$$(X \times_S S') \times_{S'} (Y \times_S S') = X \times_S (Y \times_S S') = (X \times_S Y) \times_S S'$$

according to (3.3.9.1) and the associativity of products of S-preschemes. □

The functorial nature of the isomorphism defined in Corollary (3.3.10) can be expressed by the formula

(3.3.10.1) $$(u_{(S')}, v_{(S')})_{S'} = ((u, v)_S)_{(S')}$$

for each pair of S-morphisms $u : T \to X$, $v : T \to Y$.

In other words, the base change functor $X_{(S')}$ *commutes with products*; it also commutes with sums (3.2.8).

Corollary (3.3.11). — *Let Y be an S-prescheme, and $f : X \to Y$ a morphism which makes X a Y-prescheme (and, as a result, also an S-prescheme). The prescheme $X_{(S')}$ is then identified with the product $X \times_Y Y_{(S')}$, the projection $X \times_Y Y_{(S')} \to Y_{(S')}$ being identified with $f_{(S')}$.*

Proof. Let $\psi : Y \to S$ be the structure morphism of Y; we have the commutative diagram

$$\begin{tikzcd} S' \arrow[d] & Y_{(S')} \arrow[l] \arrow[d] & X_{(S')} \arrow[l, "f_{(S')}"'] \arrow[d] \\ S & Y \arrow[l, "\psi"'] & X. \arrow[l, "f"'] \end{tikzcd}$$

We have that $Y_{(S')}$ is identified with $S'_{(\psi)}$, and $X_{(S')}$ with $S'_{(\psi\circ f)}$; taking (3.3.9) and (3.3.4) into account, we thus deduce the corollary. □

(3.3.12). Let $f : X \to X'$ and $g : Y \to Y'$ be S-morphisms which are *monomorphisms* of preschemes (T, I, 1.1); then $f \times_S g$ is a *monomorphism*. Indeed, if p and q are the projections of $X \times_S Y$, p' and q' the projections of $X' \times_S Y'$, and u and v both S-morphisms $T \to X \times_S Y$, then the equation $(f \times_S g) \circ u = (f \times_S g) \circ v$ implies that $p' \circ (f \times_S g) \circ u = p' \circ (f \times_S g) \circ v$, or, in other words, that $f \circ p \circ u = f \circ p \circ v$, and since f is a monomorphism, $p \circ u = p \circ v$; using the fact that g is a monomorphism, we similarly obtain $q \circ u = q \circ v$, hence $u = v$. **I** | 111

It follows that, for each base change $S' \to S$,

$$f_{(S')} : X_{(S')} \longrightarrow Y_{(S')}$$

is a monomorphism.

(3.3.13). Let S and S' be affine schemes of rings A and A' (respectively); a morphism $S' \to S$ then corresponds to a ring homomorphism $A \to A'$. If X is an S-prescheme, we denote by $X_{(A')}$ or $X \otimes_A A'$ the S'-prescheme $X_{(S')}$; when X is also affine of ring B, $X_{(A')}$ is affine of ring $B_{(A')} = B \otimes_A A'$ obtained by extension of scalars from the A-algebra B to A'.

(3.3.14). With the notation of (3.3.6), for each S-*morphism* $f : S' \to X$, we have that $f' = (f, 1_{S'})_S$ is an S'-morphism $S' \to X' = X_{(S')}$ such that $p \circ f' = f$, $\pi' \circ f' = 1_{S'}$, or, in other words, an S'-*section of* X'; conversely, if f' is such an S'-section, then $f = p \circ f'$ is an S-morphism $S' \to X$. We thus define a canonical *bijective correspondence*

$$\operatorname{Hom}_S(S', X) \simeq \operatorname{Hom}_{S'}(S', X').$$

We say that f' is the *graph morphism* of f, and we denote it by Γ_f.

(3.3.15). Given a prescheme X, which we can always consider as a $\mathbf{Z}$-prescheme, it follows, in particular, from (3.3.14) that the X-*sections* of $X \otimes_{\mathbf{Z}} \mathbf{Z}[T]$ (where T is an indeterminate) correspond bijectively with *morphisms* $\mathbf{Z}[T] \to X$. We will show that these X-sections also correspond bijectively with *sections of the structure sheaf* $\mathscr{O}_X$ *over* X. Indeed, let (U_α) be a cover of X by affine open subsets; let $u : X \to X \otimes_{\mathbf{Z}} \mathbf{Z}[T]$ be an X-morphism, and let u_α be its restriction to U_α; if A_α is the ring of the affine scheme U_α, then $U_\alpha \otimes_{\mathbf{Z}} \mathbf{Z}[T]$ is an affine scheme of ring $A_\alpha[T]$ (3.2.2), and u_α canonically corresponds to an A_α-homomorphism $A_\alpha[T] \to A_\alpha$ (1.7.3). Now, since such a homomorphism is completely determined by the data of the image of T in A_α, let $s_\alpha \in A_\alpha = \Gamma(U_\alpha, \mathscr{O}_X)$, and if we suppose that the restrictions of u_α and u_β to an affine open subset $V \subset U_\alpha \cap U_\beta$ coincide, then we see immediately that s_α and s_β coincide on V; thus the family (s_α) consists of the restrictions to U_α of a section s of $\mathscr{O}_X$ over X; conversely, it is clear that such a section defines a family (u_α) of morphisms which are the restrictions to U_α of an X-morphism $X \to X \otimes_{\mathbf{Z}} \mathbf{Z}[T]$. This result is generalized in (**II**, 1.7.12).

3.4. Points of a prescheme with values in a prescheme; geometric points.

(3.4.1). Let X be a prescheme; for each prescheme T, we then denote by $X(T)$ the set $\operatorname{Hom}(T, X)$ of morphisms $T \to X$, and the elements of this set are called *the points of* X *with values in* T. If we associate to each morphism $f : T \to T'$ the map $u' \mapsto u' \circ f$ from $X(T')$ to $X(T)$, we see, for fixed X, that $X(T)$ is a *contravariant functor in* T, from the category of preschemes to that of sets. In addition, each morphism of preschemes $g : X \to Y$ defines a functorial homomorphism $X(T) \to Y(T)$, which sends $v \in X(T)$ to $g \circ v$.

(3.4.2). Given sets P, Q, and R, and maps $\varphi : P \to R$ and $\psi : Q \to R$, we define the *fibre product of* P *and* Q *over* R (relative to φ and ψ) as the subset of the product set $P \times Q$ consisting of the pairs (p, q) such that $\varphi(p) = \psi(q)$; **I** | 112
we denote it by $P \times_R Q$. Definition (3.2.1) of the product of S-preschemes can be interpreted, with the notation of (3.4.1), via the formula

(3.4.2.1) $$(X \times_S Y)(T) = X(T) \times_{S(T)} Y(T).$$

the maps $X(T) \to S(T)$ and $Y(T) \to S(T)$ corresponding to the structure morphisms $X \to S$ and $Y \to S$.

(3.4.3). If we are given a prescheme S and we consider only the S-preschemes and S-morphisms, then we will denote by $X(T)_S$ the set $\operatorname{Hom}_S(T, X)$ of S-morphisms $T \to X$, and suppress the subscript S when there is no chance of confusion; we say that the elements of $X(T)_S$ are the *points* (or S-*points*, when there is a possibility of confusion) *of the* S-*prescheme* X *with values in the* S-*prescheme* T. In particular, an S-*section* of X is none other than a *point of* X *with values in* S. The formula (3.4.2.1) can then be written as

(3.4.3.1) $$(X \times_S Y)(T)_S = X(T)_S \times Y(T)_S;$$

more generally, if Z is an S-prescheme, and X, Y, and T are Z-preschemes (thus *ipso facto* S-preschemes), then we have

(3.4.3.2) $$(X \times_Z Y)(T)_S = X(T)_S \times_{Z(T)_S} Y(T)_S.$$

We note that, to show that a triple (W, r, s) consisting of an S-prescheme W and S-morphisms $r : W \to X$ and $s : W \to Y$ is a product of X and Y (over Z), it suffices, by definition, to check that, for *each S-prescheme T*, the diagram

$$\begin{array}{ccc} W(T)_S & \xrightarrow{r'} & X(T)_S \\ {\scriptstyle s'}\downarrow & & \downarrow{\scriptstyle \varphi'} \\ Y(T)_S & \xrightarrow{\psi'} & Z(T)_S \end{array}$$

makes $W(T)_S$ the fibre product of $X(T)_S$ and $Y(T)_S$ over $Z(T)_S$, where r' and s' correspond to r and s, and φ' and ψ' to the structure morphisms $\varphi : X \to Z$ and $\psi : Y \to Z$.

(3.4.4). When T (resp. S) in the above is an affine scheme of ring B (resp. A), we replace T (resp. S) by B (resp. A) in the above notation, and we then call the elements of $X(B)$ the *points of X with values in the ring B*, and the elements of $X(B)_A$ the *points of the A-prescheme X with values in the A-algebra B*. We note that $X(B)$ and $X(B)_A$ are *covariant* functors in B. We similarly write $X(T)_A$ for the set of points of the A-prescheme X with values in the A-prescheme T.

(3.4.5). Consider, in particular, the case where T is of the form Spec(A), where A is a *local* ring; the elements of $X(A)$ then correspond bijectively to *local* homomorphisms $\mathcal{O}_x \to A$ for $x \in X$ (2.2.4); we say that the point x of the underlying space of X is the *location*[7] of the point of X with values in A to which it corresponds.

More specifically, we define the *geometric points* of a prescheme S to be the *points of X with values in a field K*: the data of such a point is equivalent to the data of its location x in the underlying subspace of X, and of an **I | 113**
extension K of $k(x)$; K will be called the *field of values* of the corresponding geometric point, and we say that this geometric point is *located at x*. We also define a map $X(K) \to X$, sending a geometric point with values in K to its location.

If $S' = \text{Spec}(K)$ is an S-prescheme (in other words, if K is considered as an extension of the residue field $k(s)$, where $s \in S$), and if X is an S-prescheme, then an element of $X(K)_S$, or, as we say, a *geometric point of X lying over s with values in K*, consists of the data of a $k(s)$-monomorphism from the residue field $k(x)$ to K, where x is a point of X *lying over s* (therefore $k(x)$ is an extension of $k(s)$).

In particular, if $S = \text{Spec}(K) = \{\xi\}$, *then the geometric points of X with values in K can be identified with the points $x \in X$ such that* $k(x) = K$; we say that these latter points are the *K-rational points of the K-prescheme X*; if K' is an extension of K, then the geometric points of X with values in K' bijectively correspond to the K'-rational points of $X' = X_{(K')}$ (3.3.14).

Lemma (3.4.6). — *Let X_i $(1 \leqslant i \leqslant n)$ be S-preschemes, s a point of S, and x_i $(1 \leqslant i \leqslant n)$ points of X_i lying over s. Then there exists an extension K of $k(s)$ and a geometric point of the product $Y = X_1 \times_S X_2 \times_S \cdots \times_S X_n$, with values in K, whose projections to the X_i are localized at the x_i.*

Proof. There exist $k(s)$-monomorphisms $k(x_i) \to K$, all in the same extension K of $k(s)$ (Bourbaki, *Alg.*, chap. V, §4, prop. 2). The compositions $k(s) \to k(x_i) \to K$ are all identical, and so the morphisms Spec$(K) \to X_i$ corresponding to the $k(x_i) \to K$ are all S-morphisms, and we thus conclude that they define a unique morphism Spec$(K) \to Y$. If y is the corresponding point of Y, it is clear that its projection in each of the X_i is x_i. □

Proposition (3.4.7). — *Let X_i $(1 \leqslant i \leqslant n)$ be S-preschemes, and, for each index i, let x_i be a point of X_i. For there to exist a point y of $Y = X_1 \times_S X_2 \times \ldots \times_S X_n$ whose image is x_i under the ith projection for each $1 \leqslant i \leqslant n$, it is necessary and sufficient that the x_i all lie above the same point s of S.*

Proof. The condition is evidently necessary; Lemma (3.4.6) proves that it is sufficient. □

In other words, if we denote by (X) the underlying set of X, we see that we have a canonical *surjective* function $(X \times_S Y) \to (X) \times_{(S)} (Y)$; we must point out that this function *is not injective* in general; in other words, *there can exist multiple distinct points z in $X \times_S Y$ that have the same projections $x \in X$ and $y \in Y$*; we have already seen this when S, X, and Y are prime spectra of fields k, K, and K' (respectively), since the tensor product $K \otimes_k K'$ has, in general, multiple distinct prime ideals (cf. (3.4.9)).

Corollary (3.4.8). — *Let $f : X \to Y$ be an S-morphism, and $f_{(S')} : X_{(S')} \to Y_{(S')}$ the S'-morphism induced by f by an extension $S' \to S$ of the base prescheme. Let p (resp. q) be the projection $X_{(S')} \to X$ (resp. $Y_{(S')} \to Y$); for every subset M of X, we have*

$$q^{-1}(f(M)) = f_{(S')}(p^{-1}(M)).$$

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[7] *[Trans.] We also say that* the geometric point lies over this x.

Proof. Indeed (3.3.11), $X_{(S')}$ can be identified with the product $X \times_Y Y_{(S')}$ thanks to the commutative diagram

$$\begin{tikzcd} X \arrow[d, "f"'] & X_{(S')} \arrow[l, "p"'] \arrow[d, "f_{(S')}"] \\ Y & Y_{(S')} \arrow[l, "q"'] \end{tikzcd}$$

By (3.4.7), the equation $q(y') = f(x)$ for $x \in M$ and $y' \in Y_{(S)}$ is equivalent to the existence of some $x' \in X_{(S')}$ such that $p(x') = x$ and $f_{(S')}(x') = y'$, whence the corollary. □

Lemma (3.4.6) can be made clearer in the following manner:

Proposition (3.4.9). — *Let X and Y be S-preschemes, x a point of X, and y a point of Y, with both x and y lying above the same point $s \in S$. The set of points of $X \times_S Y$ with projections x and y is in bijective correspondence with the set of types of extensions (?) composed of $k(x)$ and $k(y)$ considered as extensions of $k(s)$* (Bourbaki, *Alg.*, chap. VIII, §8, prop. 2).

Proof. Let p (resp. q) be the projection from $X \times_S Y$ to X (resp. Y), and E the subspace $p^{-1}(x) \cap q^{-1}(y)$ of the underlying space of $X \times_S Y$. First, note that the morphisms $\mathrm{Spec}(k(x)) \to S$ and $\mathrm{Spec}(k(y)) \to S$ factor as $\mathrm{Spec}(k(x)) \to \mathrm{Spec}(k(s)) \to S$ and $\mathrm{Spec}(k(y)) \to \mathrm{Spec}(k(s)) \to S$; since $\mathrm{Spec}(k(s)) \to S$ is a monomorphism (2.4.7), it follows from (3.2.4) that we have

$$P = \mathrm{Spec}(k(x)) \times_S \mathrm{Spec}(k(y)) = \mathrm{Spec}(k(x)) \times_{\mathrm{Spec}(k(s))} \mathrm{Spec}(k(y)) = \mathrm{Spec}(k(x) \otimes_{k(s)} k(y)).$$

We will define two maps, $\alpha : P_0 \to E$ and $\beta : E \to P_0$, inverse to one another (where P_0 denotes the underlying set of the prescheme P). If $i : \mathrm{Spec}(k(x)) \to X$ and $j : \mathrm{Spec}(k(y)) \to Y$ are the canonical morphisms (2.4.5), we take α to be the map of underlying spaces corresponding to the morphism $i \times_S j$. On the other hand, every $z \in E$ defines, by hypothesis, two $k(s)$-monomorphisms, $k(x) \to k(z)$ and $k(y) \to k(z)$, and thus a $k(s)$-monomorphism $k(x) \otimes_{k(s)} k(y) \to k(z)$, and thus a morphism $\mathrm{Spec}(k(z)) \to P$; $\beta(z)$ will be the image of z in P_0 under this morphism. The verification of the fact that $\alpha \circ \beta$ and $\beta \circ \alpha$ are the identity maps follows from (2.4.5) and the definition of the product (3.2.1). Finally, we know that P_0 is in bijective correspondence with the set of types of extensions (?) composed of $k(x)$ and $k(y)$ (Bourbaki, *Alg.*, chap. VIII, §8, prop. 1). □

3.5. Surjections and injections.

(3.5.1). In a general sense, consider a property **P** of morphisms of preschemes, and the following two propositions:

(i) If $f : X \to X'$ and $g : Y \to Y'$ are S-morphisms that have property **P**, then $f \times_S g$ also has property **P**.
(ii) If $f : X \to Y$ is an S-morphism that has property **P**, then every S'-morphism $f_{(S')} : X_{(S')} \to Y_{(S')}$, induced by f by an extension of the base prescheme, also has property **P**.

Since $f_{(S')} = f \times_S 1_{S'}$, we see that, if, for every prescheme X, the *identity* 1_X has property **P**, then (i) implies (ii); on the other hand, since $f \times_S g$ is the composite morphism

$$X \times_S Y \xrightarrow{f \times 1_Y} X' \times_S Y \xrightarrow{1_{X'} \times g} X' \times_S Y',$$

we see that, if the *composition* of two morphisms has property **P**, then so does the product $f \times_S g$, and so (ii) implies (i).

A first application of this remark is

Proposition (3.5.2). —

(i) *If $f : X \to X'$ and $g : Y \to Y'$ are surjective S-morphisms, then $f \times_S g$ is surjective.*
(ii) *If $f : X \to Y$ is a surjective S-morphism, then $f_{(S')}$ is surjective for every extension S' of the base prescheme.*

Proof. The composition of any two surjections being a surjection, it suffices to prove (ii); but this proposition follows immediately from (3.4.8) applied to $M = X$. □

Proposition (3.5.3). — *For a morphism $f : X \to Y$ to be surjective, it is necessary and sufficient that, for every field K and every morphism $\mathrm{Spec}(K) \to Y$, there exist an extension K' of K and a morphism $\mathrm{Spec}(K') \to X$ that make the following diagram commute:*

$$\begin{tikzcd} X \arrow[d, "f"'] & \mathrm{Spec}(K') \arrow[l] \arrow[d] \\ Y & \mathrm{Spec}(K). \arrow[l] \end{tikzcd}$$

Proof. The condition is sufficient because, for all $y \in Y$, it suffices to apply it to a morphism $\operatorname{Spec}(K) \to Y$ corresponding to a monomorphism $k(y) \to K$, with K being an extension of $k(y)$ (2.4.6). Conversely, suppose that f is surjective, and let $y \in Y$ be the image of the unique point of $\operatorname{Spec}(K)$; there exists some $x \in X$ such that $f(x) = y$; we will consider the corresponding monomorphism $k(y) \to k(x)$ (2.2.1); it then suffices to take K' to be the extension of $k(y)$ such that there exist $k(y)$-monomorphisms from $k(x)$ and K to K' (Bourbaki, *Alg.*, chap. V, §4, prop. 2); the morphism $\operatorname{Spec}(K') \to X$ corresponding to $k(x) \to K'$ is exactly that for which we are searching. □

With the language introduced in (3.4.5), we can say that *every geometric point of Y with values in K comes from a geometric point of X with values in an extension of K.*

Definition (3.5.4). — We say that a morphism $f : X \to Y$ of preschemes is *universally injective*, or a *radicial morphism*, if, for every field K, the corresponding map $X(K) \to Y(K)$ is injective.

It follows also from the definitions that every *monomorphism of preschemes* (T, 1.1) is radicial.

(3.5.5). For a morphism $f : X \to Y$ to be radicial, it suffices that the condition of Definition (3.5.4) hold for every *algebraically closed* field. In fact, if K is an arbitrary field, and K' an algebraically-closed extension of K, then the diagram

$$\begin{tikzcd} X(K) \arrow[r, "\alpha"] \arrow[d, "\varphi"'] & Y(K) \arrow[d, "\varphi'"] \\ X(K') \arrow[r, "\alpha'"] & Y(K') \end{tikzcd}$$

commutes, where φ and φ' come from the morphism $\operatorname{Spec}(K') \to \operatorname{Spec}(K)$, and α and α' corresponding to f. However, φ is injective, and so too is α', by hypothesis; hence α is necessarily injective.

Proposition (3.5.6). — *Let $f : X \to Y$ and $g : Y \to Z$ be two morphisms of preschemes.*

(i) *If f and g are radicial, then so is $g \circ f$.*
(ii) *Conversely, if $g \circ f$ is radicial, then so is f.*

Proof. Taking into account Definition (3.5.4), the proposition reduces to the corresponding claims for the maps $X(K) \to Y(K) \to Z(K)$, and these claims are evident. □

Proposition (3.5.7). —

(i) *If the S-morphisms $f : X \to X'$ and $g : X \to X'$ are radicial, then so is $f \times_S g$.*
(ii) *If the S-morphism $f : X \to Y$ is radicial, then so is $f_{(S')} : X_{(S')} \to Y_{(S')}$ for every extension $S' \to S$ of the base prescheme.*

Proof. Given (3.5.1), it suffices to prove (i). We have seen (3.4.2.1) that

$$(X \times_S Y)(K) = X(K) \times_{S(K)} Y(K),$$
$$(X' \times_S Y')(K) = X'(K) \times_{S(K)} Y'(K),$$

with the map $(X \times_S Y)(K) \to (X' \times_S Y')(K)$ corresponding to $f \times_S g$ thus being identified with $(u, v) \to (f \circ u, g \circ v)$, and the proposition then follows. □

Proposition (3.5.8). — *For a morphism $f = (\psi, \theta) : X \to Y$ to be radicial, it is necessary and sufficient for ψ to be injective and for the monomorphism $\theta^x : k(\psi(x)) \to k(x)$ to make $k(x)$ a radicial extension of $k(\psi(x))$ for every $x \in X$.*

Proof. We suppose that f is radicial and first show that the equation $\psi(x_1) = \psi(x_2) = y$ necessarily implies that $x_1 = x_2$. Indeed, there exists a field K, and an extension of $k(y)$, along with $k(y)$-monomorphisms $k(x_1) \to K$ and $k(x_2) \to K$ (Bourbaki, *Alg.*, chap. V, §4, prop. 2); the corresponding morphisms $u_1 : \operatorname{Spec}(K) \to X$ and $u_2 : \operatorname{Spec}(K) \to X$ are then such that $f \circ u_1 = f \circ u_2$, and so $u_1 = u_2$ by hypothesis, and this implies, in particular, that $x_1 = x_2$. We now consider $k(x)$ as the extension of $k(\psi(x))$ by means of θ^x: if $k(x)$ is not a radicial algebraically-closed extension, then there exist two distinct $k(\psi(x))$-monomorphisms from $k(x)$ to an algebraically-closed extension K of $k(\psi(x))$, and the two corresponding morphisms $\operatorname{Spec}(K) \to X$ would contradict the hypothesis. Conversely, taking (2.4.6) into account, it is immediate that the conditions stated are sufficient for f to be radicial. □

Corollary (3.5.9). — *If A is a ring, and S is a multiplicative set of A, then the canonical morphism* $\operatorname{Spec}(S^{-1}A) \to \operatorname{Spec}(A)$ *is radicial.*

Proof. Indeed, this morphism is a monomorphism (1.6.2). □

Corollary (3.5.10). — *Let $f : X \to Y$ be a radicial morphism, $g : Y' \to Y$ a morphism, and $X' = X_{(Y')} = X \times_Y Y'$. Then the radicial morphism $f_{(Y')}$ (3.5.7, ii) is a bijection from the underlying space of X to $g^{-1}(f(X))$; further, for every field K, the set $X'(K)$ can be identified with the subset of $Y'(K)$ given by the inverse image of the map $Y'(K) \to Y(K)$ (corresponding to g) from the subset $X(K)$ of $Y(K)$.*

Proof. The first claim follows from (3.5.8) and (3.4.8); the second, from the commutativity of the following diagram: **I | 117**

$$\begin{tikzcd} X'(K) \arrow[r] \arrow[d] & Y'(K) \arrow[d] \\ X(K) \arrow[r] & Y(K) \end{tikzcd}$$

□

Remark (3.5.11). — We say that a morphism $f = (\psi, \theta)$ of preschemes is *injective* if the map ψ is injective. For a morphism $f = (\psi, \theta) : X \to Y$ to be radicial, it is necessary and sufficient that, for every morphism $Y' \to Y$, the morphism $f_{(Y')} : X_{(Y')} \to Y'$ be injective (which justifies the terminology of a *universally injective* morphism). In fact, the condition is necessary by (3.5.7, ii) and (3.5.8). Conversely, the condition implies that ψ is injective; if, for some $x \in X$, the monomorphism $\theta^x : k(\psi(x)) \to k(x)$ were not radicial, then there would be an extension K of $k(\psi(x))$, and two distinct morphisms $\operatorname{Spec}(K) \to X$ corresponding to the same morphism $\operatorname{Spec}(K) \to Y$ (3.5.8). But then, setting $Y' = \operatorname{Spec}(K)$, there would be two distinct Y'-sections of $X_{(Y')}$ (3.3.14), which contradicts the hypothesis that $f_{(Y')}$ is injective.

3.6. Fibres.

Proposition (3.6.1). — *Let $f : X \to Y$ be a morphism, y a point of Y, and $\mathfrak{a}_y$ an ideal of definition for $\mathscr{O}_y$ for the $\mathfrak{m}_y$-preadic topology. Then the projection $p : X \times_Y \operatorname{Spec}(\mathscr{O}_y/\mathfrak{a}_y) \to X$ is a homeomorphism from the underlying space of the prescheme $X \times_Y \operatorname{Spec}(\mathscr{O}_y/\mathfrak{a}_y)$ to the fibre $f^{-1}(y)$ equipped with the topology induced from that of the underlying space of X.*

Proof. Since $\operatorname{Spec}(\mathscr{O}_y/\mathfrak{a}_y) \to Y$ is radicial ((3.5.4) and (2.4.7)), since $\operatorname{Spec}(\mathscr{O}_y/\mathfrak{a}_y)$ is a single point, and since the ideal $\mathfrak{m}_y/\mathfrak{a}_y$ is nilpotent by hypothesis (1.1.12), we already know ((3.5.10) and (3.3.4)) that p identifies, as *sets*, the underlying space of $X \times_Y \operatorname{Spec}(\mathscr{O}_y/\mathfrak{a}_y)$ with $f^{-1}(y)$; everything reduces to proving that p is a homeomorphism. By (3.2.7), the question is local on X and Y, and so we can suppose that $X = \operatorname{Spec}(B)$ and $Y = \operatorname{Spec}(A)$, with B being an A-algebra. The morphism p then corresponds to the homomorphism $1 \otimes \varphi : B \to B \otimes_A A'$, where $A' = A_y/\mathfrak{a}_y$ and φ is the canonical map from A to A'. Then every element of $B \otimes_A A'$ can be written as

$$\sum_i b_i \otimes \varphi(a_i)/\varphi(s) = \left(\sum_i (a_i b_i \otimes 1)\right)(1 \otimes \varphi(s))^{-1},$$

where $s \notin \mathfrak{j}_y$, and Proposition (1.2.4) applies. □

(3.6.2). Throughout the rest of this treatise, whenever we consider a fibre $f^{-1}(y)$ of a morphism as having the structure of a $k(y)$-prescheme, *it will always be the prescheme obtained by transporting the structure of $X \times_Y \operatorname{Spec}(k(y))$ by the projection to X.* We will also write this (latter) product as $X \times_Y k(y)$, or $X \otimes_{\mathscr{O}_Y} k(y)$; more generally, if B is an $\mathscr{O}_y$-algebra, we will denote by $X \times_Y B$ or $X \otimes_{\mathscr{O}_Y} B$ the product $X \times_Y \operatorname{Spec}(B)$.

With the preceding convention, it follows from (3.5.10) that the points of X with values in an extension K of $k(y)$ are identified with the *points of $f^{-1}(y)$ with values in K.*

(3.6.3). Let $f : X \to Y$ and $g : Y \to Z$ be two morphisms, and $h = g \circ f$ their composition; for all $z \in Z$, the fibre $h^{-1}(z)$ is a prescheme isomorphic to

$$X \times_Z \operatorname{Spec}(k(z)) = (X \times_Y Y) \times_Z \operatorname{Spec}(k(z)) = X \times_Y g^{-1}(z).$$

Inparticular, if U is an open subset of X, then the prescheme induced on $U \cap f^{-1}(y)$ by the prescheme $f^{-1}(y)$ is **I | 118**
isomorphic to $f_U^{-1}(y)$ (f_U being the restriction of f to U),

Proposition (3.6.4). — (Transitivity of fibres) *Let $f : X \to Y$ and $g : Y' \to Y$ be morphisms; let $X' = X \times_Y Y' = X_{(Y')}$ and $f' = f_{(Y')} : X' \to Y'$. For every $y' \in Y'$, if we let $y = g(y')$, then the prescheme $f'^{-1}(y')$ is isomorphic to $f^{-1}(y) \otimes_{k(y)} k(y')$.*

Proof. Indeed, it suffices to remark that the two preschemes $(X \otimes_Y k(y)) \otimes_{k(y)} k(y')$ and $(X \times_Y Y') \otimes_{Y'} k(y')$ are both canonically isomorphic to $X \times_Y \operatorname{Spec}(k(y'))$ by (3.3.9.1). □

In particular, if V is an open neighborhood of y in Y, and we denote by f_V the restriction of f to the induced prescheme on $f^{-1}(V)$, then the preschemes $f^{-1}(y)$ and $f_V^{-1}(y)$ are canonically identified.

Proposition (3.6.5). — *Let $f : X \to Y$ be a morphism, y a point of Y, Z the local prescheme* $\operatorname{Spec}(\mathscr{O}_y)$*, and $p = (\psi, \theta)$ the projection $X \times_Y Z \to X$; then p is a homeomorphism from the underlying space of $X \times_Y Z$ to the subspace $f^{-1}(Z)$ of X (*when the underlying space of Z is identified with a subspace of Y, cf. (2.4.2)*), and, for all $t \in X \times_Y Z$, letting $z = \psi(t)$, we have that $\theta_t^\sharp$ is an isomorphism from $\mathscr{O}_x$ to $\mathscr{O}_t$.*

Proof. Since Z (identified as a subspace of Y) is contained inside every affine open containing y (2.4.2), we can, as in (3.6.1), reduce to the case where $X = \operatorname{Spec}(A)$ and $Y = \operatorname{Spec}(B)$ are affine schemes, with A being a B-algebra. Then $X \times_Y Z$ is the prime spectrum of $A \otimes_B B_y$, and this ring is canonically identified with $S^{-1}A$, where S is the image of $B - \mathfrak{j}_y$ in A (**0**, 1.5.2); since p then corresponds to the canonical homomorphism $A \to S^{-1}A$, the proposition follows from (1.6.2). □

3.7. Application: reduction of a prescheme mod. $\mathfrak{J}$. *This section, which makes use of notions and results from Chapter I and Chapter II, will not be used in what follows in this treatise, and is only intended for readers familiar with classical algebraic geometry.*

(3.7.1). Let A be a ring, X an A-prescheme, and $\mathfrak{J}$ an ideal of A; then $X_0 = X \otimes_A (A/\mathfrak{J})$ is an $(A/\mathfrak{J})$-prescheme, which we sometimes say is induced from X by *reduction* mod. $\mathfrak{J}$.

. This terminology is used foremost when A is a *local ring* and $\mathfrak{J}$ its maximal ideal, in such a way that X_0 is a prescheme over the residue field $k = A/\mathfrak{J}$ of A.

When A is also integral, with field of fractions K, we can consider the K-prescheme $X' = X \otimes_A K$. By an abuse of language which we will not use, it has been said, up until now, that X_0 is *induced by* X' by reduction mod. $\mathfrak{J}$. In the case where this language was used, A was a local ring of dimension 1 (most often a discrete valuation ring) and it was implied (be it more or less explicitly) that the given K-prescheme X' was a closed subprescheme of a K-prescheme P' (in fact, a projective space of the form (?) $\mathbf{P}_K^r$, cf. (**II**, 4.1.1)), itself of the form $P' = P \otimes_A K$, where P is a given A-prescheme (in fact, the A-scheme $\mathbf{P}_A^r$, with the notation of (**II**, 4.1.1)). In our language, the definition of X_0 in terms of X' is formulated as follows:

We consider the affine scheme $Y = \operatorname{Spec}(A)$, formed of two points, the unique closed point $y = \mathfrak{J}$ and the generic point (0), the singleton set U of the generic point being thus an open $U = \operatorname{Spec}(K)$ in Y. If X is an A-prescheme (or, in other words, a Y-prescheme), then $X \otimes_A K = X'$ is exactly the prescheme induced by X on $\psi^{-1}(U)$, denoting by ψ the structure morphism $X \to Y$. In particular, if φ is the structure morphism $P \to Y$, then a closed subprescheme X' of $P' = \varphi^{-1}(U)$ is a (locally closed) subprescheme of P. If P is Noetherian (for example, if A is Noetherian and P is of finite type over A), then there exists a smaller closed subprescheme $X = \overline{X'}$ of G that through which X' factors (9.5.10), and X' is the prescheme induced by X on the open $\varphi^{-1}(U) \cap X$, and so is isomorphic to $X \otimes_A K$ (9.5.10). *The immersion of X' into $P' = P \otimes_A K$ thus lets us canonically consider X' as being of the form $X' = X \otimes_A K$, where X is an A-prescheme.* We can then consider the reduced mod. $\mathfrak{J}$ prescheme $X_0 = X \otimes_A k$, which is exactly the fibre $\psi^{-1}(y)$ of the closed point y. Up until now, lacking the adequate terminology, we had avoided explicitly introducing the A-prescheme X. One ought to, however, note that all the claims normally made about the "reduced mod. $\mathfrak{J}$" prescheme X_0 should be seen as consequences of more complicated claims about X itself, and cannot be satisfactorily formulated or understood except by interpreting them as such. It seems also that any hypotheses made on X_0 always reduce to hypotheses on X itself (independent of the prior data of an immersion of X' in $\mathbf{P}_K^r$), which lets us give more intrinsic statements. **I** | 119

(3.7.3). Lastly, we draw attention to a very particular fact, which has undoubtedly contributed to slowing the conceptual clarification of the situation envisaged here: if A is a discrete valuation ring, and if X is *proper* over A (which is indeed the case if X is a closed subprescheme of some $\mathbf{P}_A^r$, cf. (**II**, 5.5.4)), then the points of X with values in A and the points of X' with values in k are in bijective correspondence (**II**, 7.3.8). This is why we often believe that facts about X' have been proved, when in reality we have proved facts about X, and these remain valuable (in this form) whenever we no longer suppose that the base local ring is of dimension 1.

§4. Subpreschemes and immersion morphisms

4.1. Subpreschemes.

(4.1.1). As the notion of a quasi-coherent sheaf (**0**, 5.1.3) is local, a quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$ over a prescheme X can be defined by the following condition: for each affine open V of X, $\mathscr{F}|V$ is isomorphic to the sheaf associated to a $\Gamma(V, \mathscr{O}_X)$-module (1.4.1). It is clear that, over a prescheme X, the structure sheaf $\mathscr{O}_X$ is quasi-coherent, and that the kernels, cokernels, and images of homomorphisms of quasi-coherent $\mathscr{O}_X$-modules, as well as inductive limits and direct sums of quasi-coherent $\mathscr{O}_X$-modules, are also quasi-coherent (Theorem (1.3.7) and Corollary (1.3.9)).

Proposition (4.1.2). — *Let X be a prescheme, and $\mathscr{I}$ a quasi-coherent sheaf of ideals of $\mathscr{O}_X$. Then the support Y of the sheaf $\mathscr{O}_X/\mathscr{I}$ is closed, and if we denote by $\mathscr{O}_Y$ the restriction of $\mathscr{O}_X/\mathscr{I}$ to Y, then $(Y, \mathscr{O}_Y)$ is a prescheme.*

Proof. It evidently suffices (2.1.3) to consider the case where X is an affine scheme, and to show that, in this case, Y is closed in X, and is an *affine scheme.* Indeed, if $X = \operatorname{Spec}(A)$, then we have $\mathscr{O}_X = \widetilde{A}$ and $\mathscr{I} = \widetilde{\mathfrak{I}}$, where $\mathfrak{I}$ is an ideal of A (1.4.1); Y is then equal to the closed subset $V(\mathfrak{I})$ of X and can be identified with the prime spectrum of the ring $B = A/\mathfrak{I}$ (1.1.11); in addition, if φ is the canonical homomorphism $A \to B = A/\mathfrak{I}$, then the direct image ${}^a\varphi_*(\widetilde{B})$ is canonically identified with the sheaf $\widetilde{A}/\widetilde{\mathfrak{I}} = \mathscr{O}_X/\mathscr{I}$ (Proposition (1.6.3) and Corollary (1.3.9)), which finishes the proof. □ I | 120

We say that $(Y, \mathscr{O}_Y)$ is the *subprescheme* of $(X, \mathscr{O}_X)$ *defined by the sheaf of ideals* $\mathscr{I}$; this is a particular case of the more general notion of a *subprescheme*:

Definition (4.1.3). — We say that a ringed space $(Y, \mathscr{O}_Y)$ is a subprescheme of a prescheme $(X, \mathscr{O}_X)$ if:

1st. Y is a locally closed subspace of X;

2nd. if U denotes the largest open subset of X containing Y such that Y is closed in U (*equivalently*, the complement in X of the boundary of Y with respect to $\overline{Y}$), then $(Y, \mathscr{O}_Y)$ is a subprescheme of $(U, \mathscr{O}_X|U)$ defined by a quasi-coherent sheaf of ideals of $\mathscr{O}_X|U$.

We say that the subprescheme $(Y, \mathscr{O}_Y)$ of $(X, \mathscr{O}_X)$ is closed if Y is closed in X (in which case $U = X$).

It follows immediately from this definition and Proposition (4.1.2) that the closed subpreschemes of X are in canonical *bijective correspondence* with the quasi-coherent sheaves of ideals $\mathscr{J}$ of $\mathscr{O}_X$, since if two such sheaves $\mathscr{J}$ and $\mathscr{J}'$ have the same (closed) support Y, and if the restrictions of $\mathscr{O}_X/\mathscr{J}$ and $\mathscr{O}_X/\mathscr{J}'$ to Y are identical, then we have $\mathscr{J}' = \mathscr{J}$.

(4.1.4). Let $(Y, \mathscr{O}_Y)$ be a subprescheme of X, U the largest open subset of X such that Y is closed (and thus contained) in U, and V an open subset of X contained in U; then $V \cap Y$ is closed in V. In addition, if Y is defined by the quasi-coherent sheaf of ideals $\mathscr{J}$ of $\mathscr{O}_X|U$, then $\mathscr{J}|V$ is a quasi-coherent sheaf of ideals of $\mathscr{O}_X|V$, and it is immediate that the prescheme induced by Y on $Y \cap V$ is the closed subprescheme of V defined by the sheaf of ideals $\mathscr{J}|V$. Conversely:

Proposition (4.1.5). — *Let $(Y, \mathscr{O}_Y)$ be a ringed space such that Y is a subspace of X, and there exists a cover (V_α) of Y by open subsets of X such that, for each α, $Y \cap V_\alpha$ is closed in V_α, and the ringed space $(Y \cap V_\alpha, \mathscr{O}_Y|(Y \cap V_\alpha))$ is a closed subprescheme of the prescheme induced on V_α by X. Then $(Y, \mathscr{O}_Y)$ is a subprescheme of X.*

Proof. The hypotheses imply that Y is locally closed in X and that the largest open U in which Y is closed (and thus contained) contains all the V_α; we can thus reduce to the case where $U = X$ and Y is closed in X. We then define a quasi-coherent sheaf of ideals $\mathscr{J}$ of $\mathscr{O}_X$ by taking $\mathscr{J}|V_\alpha$ to be the sheaf of ideals of $\mathscr{O}_X|V_\alpha$ which define the closed subprescheme $(Y \cap V_\alpha, \mathscr{O}_Y|(Y \cap V_\alpha))$, and, for each open subset W of X not intersecting Y, $\mathscr{J}|W = \mathscr{O}_X|W$. We see immediately, by Definition (4.1.3) and (4.1.4), that there exists a unique sheaf of ideals $\mathscr{J}$ satisfying these conditions, and that it defines the closed subprescheme $(Y, \mathscr{O}_Y)$. □

In particular, the *induced* (by X) prescheme on an *open subset* of X is a *subprescheme* of X.

Proposition (4.1.6). — *A subprescheme (resp. a closed subprescheme) of a subprescheme (resp. closed subprescheme) of X is canonically identified with a subprescheme (resp. closed subprescheme) of X.* I | 121

Proof. Since a locally closed subset of a locally closed subspace of X is a locally closed subspace of X, it is clear (4.1.5) that the question is local and that we can thus suppose that X is affine; the proposition then follows from the canonical identification of $A/\mathfrak{J}'$ with $(A/\mathfrak{J})/(\mathfrak{J}'/\mathfrak{J})$, where A is some ring, and $\mathfrak{J}$ and $\mathfrak{J}'$ are ideals of A such that $\mathfrak{J} \subset \mathfrak{J}'$. □

In what follows, we will always make use of the above identification.

(4.1.7). Let Y be a subprescheme of a prescheme X, and denote by ψ the canonical injection $Y \to X$ of the *underlying subspaces*; we know that the inverse image $\psi^*(\mathscr{O}_X)$ is the restriction $\mathscr{O}_X|Y$ (**0**, 3.7.1). If, for each $y \in Y$, we denote by ω_y the canonical homomorphism $(\mathscr{O}_X)_y \to (\mathscr{O}_Y)_y$, then these homomorphisms are the restrictions to stalks of a *surjective* homomorphism ω of sheaves of rings $\mathscr{O}_X|Y \to \mathscr{O}_Y$: indeed, it suffices to check locally on Y, that is to say, we can suppose that X is affine and that the subprescheme Y is closed; if in this case $\mathscr{I}$ is the sheaf of ideals in $\mathscr{O}_X$ which defines Y, then the ω_y are none other than the restriction to stalks of the homomorphism

$\mathscr{O}_X|Y \to (\mathscr{O}_X/\mathscr{I})|Y$. We have thus defined a *monomorphism of ringed spaces* (**0**, 4.1.1) $j = (\psi, \omega^\flat)$ which is evidently a morphism $Y \to X$ of preschemes (2.2.1), and we call this the *canonical injection morphism.*

If $f : X \to Z$ is a morphism, we then say that the composite morphism $Y \xrightarrow{j} X \xrightarrow{f} Z$ is the *restriction* of f to the subprescheme Y.

(4.1.8). Conforming to the general definitions (T, I, 1.1), we will say that a morphism of preschemes $f : Z \to X$ is *majoré*[8] by the injection morphism $j : Y \to X$ of a subprescheme Y of X if f factors as $Z \xrightarrow{g} Y \xrightarrow{j} X$, where g is a morphism of preschemes; further, g is necessarily *unique* since j is a monomorphism.

Proposition (4.1.9). — *For a morphism $f : Z \to X$ to be factor through an injection morphism $j : Y \to X$, it is necessary and sufficient that $f(Z) \subset Y$ and, for all $z \in Z$, letting $y = f(z)$, that the homomorphism $(\mathscr{O}_X)_y \to \mathscr{O}_z$ corresponding to f factor as $(\mathscr{O}_Z)_y \to (\mathscr{O}_Y)_y \to \mathscr{O}_z$ (*or equivalently, *for the kernel of $(\mathscr{O}_X)_y \to \mathscr{O}_z$ to contain the kernel of $(\mathscr{O}_X)_y \to (\mathscr{O}_Y)_y$).*

Proof. The conditions are evidently necessary. To see that they are sufficient, we can reduce to the case where Y is a *closed* subprescheme of X, by replacing X by an open U such that Y is closed in U (4.1.3) if necessary; Y is then defined by a quasi-coherent sheaf of ideals $\mathscr{I}$ of $\mathscr{O}_X$. Let $f = (\psi, \theta)$, and let $\mathscr{J}$ be the sheaf of ideals of $\psi^*(\mathscr{O}_X)$, kernel of $\theta^\sharp : \psi^*(\mathscr{O}_X) \to \mathscr{O}_Z$; considering the properties of the functor ψ^* (**0**, 3.7.2), the hypotheses imply that, for each $z \in Z$, we have $(\psi^*(\mathscr{I}))_z \subset \mathscr{J}_z$, and, as a result, that $\psi^*(\mathscr{I}) \subset \mathscr{J}$. Thus $\theta^\sharp$ factors as

$$\psi^*(\mathscr{O}_X) \longrightarrow \psi^*(\mathscr{O}_X)/\psi^*(\mathscr{I}) = \psi^*(\mathscr{O}_X/\mathscr{I}) \xrightarrow{\omega} \mathscr{O}_Z,$$

the first arrow being the canonical homomorphism. Let ψ' be the continuous map $Z \to Y$ coinciding with ψ; it is clear that we have $\psi'^*(\mathscr{O}_Y) = \psi^*(\mathscr{O}_X/\mathscr{J})$; on the other hand, ω is evidently a local homomorphism, so $g = (\psi', \omega^\flat)$ is a morphism of preschemes $Z \to Y$ (2.2.1), which, according to the above, is such that $f = j \circ g$, **I** | 122
hence the proposition. □

Corollary (4.1.10). — *For an injection morphism $Z \to X$ to be factor through the injection morphism $Y \to X$, it is necessary and sufficient for Z to be a subprescheme of Y.*

We then write $Z \leqslant Y$, and this condition is evidently an *ordering* on the set of subpreschemes of X.

4.2. Immersion morphisms.

Definition (4.2.1). — We say that a morphism $f : Y \to X$ is an immersion (resp. a closed immersion, an open immersion) if it factors as $Y \xrightarrow{g} Z \xrightarrow{j} X$, where g is an isomorphism, Z is a subprescheme of X (resp. a closed subprescheme, a subprescheme induced by an open set), and j is the injection morphism.

The subprescheme Z and the isomorphism g are then determined in a *unique* way, since if Z' is a second subprescheme of X, j' the injection $Z' \to X$, and g' an isomorphism $Y \to Z'$ such that $j \circ g = j' \circ g'$, then we have $j' = j \circ g \circ g'^{-1}$, hence $Z' \leqslant Z$ (4.1.10), and we can similarly show that $Z \leqslant Z'$, hence $Z' = Z$, and, since j is a monomorphism of preschemes, $g' = g$.

We say that $f = j \circ g$ is the *canonical factorization* of the immersion f, and the subprescheme Z and the isomorphism g are those *associated to f*.

It is clear that an immersion is a *monomorphism* of preschemes (4.1.7) and *a fortiori* a *radicial* morphism (3.5.4).

Proposition (4.2.2). —

(a) *For a morphism $f = (\psi, \theta) : Y \to X$ to be an open immersion, it is necessary and sufficient for ψ to be a homeomorphism from Y to some open subset of X, and, for all $y \in Y$, that the homomorphism $\theta_y^\sharp : \mathscr{O}_{\psi(y)} \to \mathscr{O}_y$ be bijective.*

(b) *For a morphism $f = (\psi, \theta) : Y \to X$ to be an immersion (resp. a closed immersion), it is necessary and sufficient for ψ to be a homeomorphism from Y to some locally closed (resp. closed) subset of X, and, for all $y \in Y$, that the homomorphism $\theta_y^\sharp : \mathscr{O}_{\psi(y)} \to \mathscr{O}_y$ be surjective.*

Proof.

(a) The conditions are evidently necessary. Conversely, if they are satisfied, then it is clear that $\theta^\sharp$ is an isomorphism from $\mathscr{O}_Y$ to $\psi^*(\mathscr{O}_X)$, and $\psi^*(\mathscr{O}_X)$ is the sheaf induced by "transport of structure" via ψ^{-1} from $\mathscr{O}_X|\psi(Y)$; hence the conclusion.

[8] *[Trans.] There doesn't seem to be an English equivalent of this, except for 'bounded above', which doesn't make much sense in this context. We would normally just say that 'f factors through j', but to avoid having to entirely restructure the often-lengthy sentences in the original, we sometimes (but as little as we can) use 'majoré'.*

(b) The conditions are evidently necessary—we prove that they are sufficient. Consider first the particular case where we suppose that X is an affine scheme, and that $Z = \psi(Y)$ is *closed* in X. We then know (**0**, 3.4.6) that $\psi_*(\mathcal{O}_Y)$ has support equal to Z, and that, denoting its restriction to Z by $\mathcal{O}'_Z$, the ringed space $(Z, \mathcal{O}'_Z)$ is induced from $(Y, \mathcal{O}_Y)$ by transport of structure via the homeomorphism ψ considered as a map from Y to Z. Let us now show that $f_*(\mathcal{O}_Y) = \psi_*(\mathcal{O}_Y)$ is a *quasi-coherent* $\mathcal{O}_X$-module. Indeed, for all $x \notin Z$, $\psi_*(\mathcal{O}_Y)$ restricted to a suitable neighborhood of x is zero. On the contrary, if $z \in Z$, then we have $x = \psi(y)$ for some well-defined $y \in Y$; let V be an affine open neighborhood of y in Y; $\psi(V)$ is then open in Z, and so equal to the intersection of Z with an open subset U of X, and the restriction of U to $\psi_*(\mathcal{O}_Y)$ is identical to the restriction of U to the direct image $(\psi_V)_*(\mathcal{O}_Y|V)$, where ψ_V is the restriction of ψ to V. The restriction of the morphism (ψ, θ) to $(V, \mathcal{O}_Y|V)$ is a morphism from this aforementioned prescheme to $(X, \mathcal{O}_X)$, and, as a result, is of the form $({}^a\varphi, \widetilde{\varphi})$, where φ is the homomorphism from the ring $A = \Gamma(X, \mathcal{O}_X)$ to the ring $\Gamma(V, \mathcal{O}_Y)$ (1.7.3); we conclude that $(\psi_V)_*(\mathcal{O}_Y|V)$ is a quasi-coherent $\mathcal{O}_X$-module (1.6.3), which proves our assertion, due to the local nature of quasi-coherent sheaves. In addition, the hypothesis that ψ is a homeomorphism implies (**0**, 3.4.5) that, for all $y \in Y$, ψ_y is an isomorphism $(\psi_*(\mathcal{O}_Y))_{\psi(y)} \to \mathcal{O}_y$; since the diagram I | 123

$$\begin{tikzcd} \mathcal{O}_{\psi(y)} \arrow[r, "\theta_{\psi(y)}"] \arrow[d, "\psi_y \circ \rho_{\psi(y)}"'] & (\psi_*(\mathcal{O}_Y))_{\psi(y)} \arrow[d, "\psi_y"] \\ (\psi^*(\mathcal{O}_X))_y \arrow[r, "\theta^\sharp_y"] & \mathcal{O}_y \end{tikzcd}$$

is commutative, and the vertical arrows are the isomorphisms (**0**, 3.7.2), the hypothesis that $\theta^\sharp_y$ is surjective implies that $\theta_{\psi(y)}$ is surjective as well. Since the support of $\psi_*(\mathcal{O}_Y)$ is $Z = \psi(Y)$, θ is a *surjective* homomorphism from $\mathcal{O}_X = \widetilde{A}$ to the quasi-coherent $\mathcal{O}_X$-module $f_*(\mathcal{O}_Y)$. As a result, there exists a unique isomorphism ω from a sheaf quotient $\widetilde{A}/\widetilde{\mathfrak{J}}$ (with $\mathfrak{J}$ being an ideal of A) to $f_*(\mathcal{O}_Y)$ which, when composed with the canonical homomorphism $\widetilde{A} \to \widetilde{A}/\widetilde{\mathfrak{J}}$, gives θ (1.3.8); if $\mathcal{O}_Z$ denotes the restriction of $\widetilde{A}/\widetilde{\mathfrak{J}}$ to Z, then $(Z, \mathcal{O}_Z)$ is a subprescheme of $(X, \mathcal{O}_X)$, and f factors through the canonical injection of this subprescheme into X and the isomorphism (ψ_0, ω_0), where ψ_0 is ψ considered as a map from Y to Z, and ω_0 the restriction of ω to $\mathcal{O}_Z$.

We now pass to the general case. Let U be an affine open subset of X such that $U \cap \psi(Y)$ is closed in U and nonempty. By restricting f to the prescheme induced by Y on the open subset $\psi^{-1}(U)$, and by considering it as a morphism from this prescheme to the prescheme induced by X on U, we reduce to the first case; the restriction of f to $\psi^{-1}(U)$ is thus a closed immersion $\psi^{-1}(U) \to U$, canonically factoring as $j_U \circ g_U$, where g_U is an isomorphism from the prescheme $\psi^{-1}(U)$ to a subprescheme Z_U of U, and j_U is the canonical injection $Z_U \to U$. Let V be a second affine open subset of X such that $V \subset U$; since the restriction Z'_V of Z_U to V is a subprescheme of the prescheme V, the restriction of f to $\psi^{-1}(V)$ factors as $j'_V \circ g'_V$, where j'_V is the canonical injection $Z'_V \to V$ and g'_V is an isomorphism from $\psi^{-1}(V)$ to Z'_V. By the uniqueness of the canonical factorization of an immersion (4.2.1), we necessarily have that $Z'_V = Z_V$ and $g'_V = g_V$. We conclude (4.1.5) that there is a subprescheme Z of X whose underlying space is $\psi(Y)$, and whose restriction to each $U \cap \psi(Y)$ is Z_U; the g_U are then the restrictions to $\psi^{-1}(U)$ of an isomorphism $g : Y \to Z$ such that $f = j \circ g$, where j is the canonical injection $Z \to X$. □

Corollary (4.2.3). — *Let X be an affine scheme. For a morphism $f = (\psi, \theta) : Y \to X$ to be a closed immersion, it is necessary and sufficient for Y to be an affine scheme, and the homomorphism $\Gamma(\psi) : \Gamma(\mathcal{O}_X) \to \Gamma(\mathcal{O}_Y)$ to be surjective.*

Corollary (4.2.4). —

(a) *Let f be a morphism $Y \to X$, and (V_λ) a cover of $f(Y)$ by open subsets of X. For f to be an immersion (resp. an open immersion), it is necessary and sufficient for its restriction to each of the induced preschemes $f^{-1}(V_\lambda)$ to be an immersion (resp. an open immersion) into V_λ.* I | 124

(b) *Let f be a morphism $Y \to X$, and (V_λ) an open cover of X. For f to be a closed immersion, it is necessary and sufficient for its restriction to each of the induced preschemes $f^{-1}(V_\lambda)$ to be a closed immersion into V_λ.*

Proof. Let $f = (\psi, \theta)$; in the case (a), $\theta^\sharp_y$ is surjective (resp. bijective) for all $y \in Y$, and in the case (b), $\theta^\sharp_y$ is surjective for all $y \in Y$; it thus suffices to check, in case (a), that ψ is a homeomorphism from Y to a locally closed (resp. open) subset of X, and, in case (b), that ψ is a homeomorphism from Y to a closed subset of X. Now ψ is evidently injective, and sends each neighborhood of y in Y to a neighborhood of $\psi(y)$ is $\psi(Y)$ for all

$y \in Y$, by virtue of the hypothesis; in case (a), $\psi(Y) \cap V_\lambda$ is locally closed (resp. open) in V_λ, so $\psi(Y)$ is locally closed (resp. open) in the union of the V_λ, and *a fortiori* in X; in case (b), $\psi(Y) \cap V_\lambda$ is closed in V_λ, so $\psi(Y)$ is closed in X since $X = \bigcup_\lambda V_\lambda$. □

Proposition (4.2.5). — *The composition of any two immersions (resp. of two open immersions, of two closed immersions) is an immersion (resp. an open immersion, a closed immersion).*

Proof. This follows easily from (4.1.6). □

4.3. Products of immersions.

Proposition (4.3.1). — *Let $\alpha : X' \to X$, $\beta : Y' \to Y$ be two S-morphisms; if α and β are immersions (resp. open immersions, closed immersions), then $\alpha \times_S \beta$ is an immersion (resp. an open immersion, a closed immersion). In addition, if α (resp. β) identifies X' (resp. Y') with a subprescheme X'' (resp. Y'') of X (resp. Y), then $\alpha \times_S \beta$ identifies the underlying space of $X' \times_S Y'$ with the subspace $p^{-1}(X'') \cap q^{-1}(Y'')$ of the underlying space of $X \times_S Y$, where p and q denote the projections from $X \times_S Y$ to X and Y respectively.*

Proof. According to Definition (4.2.1), we can restrict to the case where X' and Y' are subpreschemes, and α and β the injection morphisms. The proposition has already been proven for the subpreschemes induced by open sets (3.2.7); since every subprescheme is a closed subprescheme of a prescheme induced by an open set (4.1.3), we can reduce to the case where X' and Y' are *closed* subpreschemes.

Let us first show that we can assume S to be *affine*. Let (S_λ) be a cover of S by affine open sets; if φ and ψ are the structure morphisms of X and Y, then let $X_\lambda = \varphi^{-1}(S_\lambda)$ and $Y_\lambda = \psi^{-1}(S_\lambda)$. The restriction X'_λ (resp. Y'_λ) of X' (resp. Y') to $X_\lambda \cap X'$ (resp. $Y_\lambda \cap Y'$) is a closed subprescheme of X_λ (resp. Y_λ), the preschemes X_λ, Y_λ, X'_λ, and Y'_λ can be considered as S_λ-preschemes, and the products $X_\lambda \times_S Y_\lambda$ and $X_\lambda \times_{S_\lambda} Y_\lambda$ (resp. $X'_\lambda \times_S Y'_\lambda$ and $X'_\lambda \times_{S_\lambda} Y'_\lambda$) are identical (3.2.5). If the proposition is true when S is affine, then the restriction of $\alpha \times_S \beta$ to each of the $X'_\lambda \times_S Y'_\lambda$ is thus an immersion (3.2.7). Since the product $X'_\lambda \times_S Y'_\mu$ (resp. $X_\lambda \times_S Y_\mu$) can be identified with $(X'_\lambda \cap X'_\mu) \times_S (Y'_\lambda \cap Y'_\mu)$ (resp. $(X_\lambda \cap X_\mu) \times_S (Y_\lambda \cap Y_\mu)$) (3.2.6.4), the restriction of $\alpha \times_S \beta$ to each of the $X'_\lambda \times_S Y'_\mu$ is **I** | 125
also an immersion; the same is true for $\alpha \times_S \beta$ by (4.2.4).

Next, we show that we can assume X and Y to be *affine*. Indeed, let (U_i) (resp. (V_j)) be a cover of X (resp. Y) by affine open sets, and let X'_i (resp. Y'_j) be the restriction of X' (resp. Y') to $X' \cap U_i$ (resp. $Y' \cap V_j$), which is a closed subprescheme of U_i (resp. V_j); $U_i \times_S V_j$ can be identified with the restriction of $X \times_S Y$ to $p^{-1}(U_i) \cap q^{-1}(V_j)$ (3.2.7); similarly, if p' and q' are the projections from $X' \times_S Y'$, then $X'_i \times_S Y'_j$ can be identified with the restriction of $X' \times_S Y'$ to $p'^{-1}(X'_i) \cap q'^{-1}(Y'_j)$. Set $\gamma = \alpha \times_S \beta$; we have, by definition, $p \circ \gamma = \alpha \circ p'$ and $q \circ \gamma = \beta \circ q'$; since $X'_i = \alpha^{-1}(U_i)$ and $Y'_j = \beta^{-1}(V_j)$, we also have $p'^{-1}(X'_i) = \gamma^{-1}(p^{-1}(U_i))$ and $q'^{-1}(Y'_j) = \gamma^{-1}(q^{-1}(V_j))$, hence

$$p'^{-1}(X'_i) \cap q'^{-1}(Y'_j) = \gamma^{-1}(p^{-1}(U_i) \cap q^{-1}(V_j)) = \gamma^{-1}(U_i \times_S V_j),$$

and we then conclude as in the previous part of the proof.

So suppose X, Y, and S are affine, and let B, C, and A be their respective rings. Then B and C are A-algebras, and X' and Y' are affine schemes whose rings are quotient algebras B' and C' of B and C respectively. In addition, we have $\alpha = ({}^a\rho, \widetilde{\rho})$ and $\beta = ({}^a\sigma, \widetilde{\sigma})$, where ρ and σ are (respectively) the canonical homomorphisms $B \to B'$ and $C \to C'$ (1.7.3). With that in mind, we know that $X \times_S Y$ (resp. $X' \times_S Y'$) is an affine scheme with ring $B \otimes_A C$ (resp. $B' \otimes_A C'$), and $\alpha \times_S \beta = ({}^a\tau, \widetilde{\tau})$, where τ is the homomorphism $\rho \otimes \sigma$ from $B \otimes_A C$ to $B' \otimes_A C'$ (Proposition (3.2.2) and Corollary (3.2.3)); since this homomorphism is surjective, $\alpha \times_S \beta$ is an immersion. In addition, if $\mathfrak{b}$ (resp. $\mathfrak{c}$) is the kernel of ρ (resp. σ), then the kernel of τ is $u(\mathfrak{b}) + v(\mathfrak{c})$, where u (resp. v) is the homomorphism $b \mapsto b \otimes 1$ (resp. $c \mapsto 1 \otimes c$). Since $p = ({}^a u, \widetilde{u})$ and $q = ({}^a v, \widetilde{v})$, this kernel corresponds, in the prime spectrum of $B \otimes_A C$, to the closed set $p^{-1}(X') \cap q^{-1}(Y')$ ((1.2.2.1) and Proposition (1.1.2, iii)), which finishes the proof. □

Corollary (4.3.2). — *If $f : X \to Y$ is an immersion (resp. an open immersion, a closed immersion) and an S-morphism, then $f_{(S')}$ is an immersion (resp. an open immersion, a closed immersion) for every extension $S' \to S$ of the base prescheme.*

4.4. Inverse images of a subprescheme.

Proposition (4.4.1). — *Let $f : X \to Y$ be a morphism, Y' a subprescheme (resp. a closed subprescheme, a prescheme induced by an open set) of Y, and $j : Y' \to Y$ the injection morphism. Then the projection $p : X \times_Y Y' \to X$ is an immersion (resp. a closed immersion, an open immersion); the underlying space of the subprescheme of X associated to p is $f^{-1}(Y')$; in addition, if j' is the injection morphism of this subprescheme into X, then for a morphism $h : Z \to X$ to be such that $f \circ h : Z \to Y$ factors through j, it is necessary and sufficient for h to factor through j'.*

Proof. Since $p = 1_X \times_Y j$ (3.3.4), the first claim follows from Proposition (4.3.1); the second is a particular case of Corollary (3.5.10) (after swapping the roles of X and Y'). Finally, if we have $f \circ h = j \circ h'$, where h' is a morphism $Z \to Y'$, then it follows from the definition of the product that we have $h = p \circ u$, where u is a morphism $Z \to X \times_Y Y'$, whence the last claim. □

We say that the subprescheme of X thus defined is the *inverse image* of the subprescheme Y' of Y under the morphism f, terminology which is consistent with that introduced more generally in (3.3.6). When we speak of $f^{-1}(Y')$ as a subprescheme of X, this will always be the subprescheme we mean. I | 126

When the preschemes $f^{-1}(Y')$ and X are identical, j' is the identity and each morphism $h : Z \to X$ thus factors through j', so *the morphism $f : X \to Y$ factors as $X \xrightarrow{g} Y' \xrightarrow{j} Y$.*

When y is a *closed* point of Y and $Y' = \operatorname{Spec}(k(y))$ is the smallest closed subprescheme of Y having $\{y\}$ as its underlying space (4.1.9), the closed subprescheme $f^{-1}(Y')$ is canonically isomorphic to the *fibre* $f^{-1}(y)$ defined in (3.6.2), and we will use this identification in all that follows.

Corollary (4.4.2). — *Let $f : X \to Y$ and $g : Y \to Z$ be morphisms, and $h = g \circ f$ their composition. For each subprescheme Z' of Z, the subpreschemes $f^{-1}(g^{-1}(Z'))$ and $h^{-1}(Z')$ of X are identical.*

Proof. This follows from the existence of the canonical isomorphism $X \times_Y (Y \times_Z Z') \simeq X \times_Z Z'$ (3.3.9.1). □

Corollary (4.4.3). — *Let X' and X'' be subpreschemes of X, and $j' : X' \to X$, and $j'' : X'' \to X$ their injection morphisms; then $j'^{-1}(X'')$ and $j''^{-1}(X')$ are both equal to the greatest lower bound $\inf(X', X'')$ of X' and X'' for the ordering $\leqslant$ on subpreschemes, and this is canonically isomorphic to $X' \times_X X''$.*

Proof. This follows immediately from Proposition (4.4.1) and Corollary (4.1.10). □

Corollary (4.4.4). — *Let $f : X \to Y$ be a morphism, and Y' and Y'' subpreschemes of Y; then we have $f^{-1}(\inf(Y', Y'')) = \inf(f^{-1}(Y'), f^{-1}(Y''))$.*

Proof. This follows from the existence of the canonical isomorphism between $(X \times_Y Y') \times_X (X \times_Y Y'')$ and $X \times_Y (Y' \times_Y Y'')$ (3.3.9.1). □

Proposition (4.4.5). — *Let $f : X \to Y$ be a morphism, and Y' a closed subprescheme of Y defined by a quasi-coherent sheaf of ideals $\mathscr{K}$ of $\mathcal{O}_Y$ (4.1.3); the closed subprescheme $f^{-1}(Y')$ of X is then defined by the quasi-coherent sheaf of ideals $f^*(\mathscr{K})\mathcal{O}_X$ of $\mathcal{O}_X$.*

Proof. The statement is evidently local on X and Y; it thus suffices to note that if A is a B-algebra and $\mathfrak{K}$ an ideal of B, then we have $A \otimes_B (B/\mathfrak{K}) = A/\mathfrak{K}A$, and to then apply (1.6.9). □ $\mathbf{Err}_{\mathrm{II}}$

Corollary (4.4.6). — *Let X' be a closed subprescheme of X defined by a quasi-coherent sheaf of ideals $\mathscr{J}$ of $\mathcal{O}_X$, and i the injection $X' \to X$; for the restriction $f \circ i$ of f to X' to factor through the injection $j : Y' \to Y$* (in other words, for it to factor as $j \circ g$, with g a morphism $X' \to Y'$), *it is necessary and sufficient that $f^*(\mathscr{K}) \subset \mathscr{J}$.*

Proof. It suffices to apply Proposition (4.4.1) to i, taking Proposition (4.4.5) into account. □

4.5. Local immersions and local isomorphisms.

Definition (4.5.1). — Let $f : X \to Y$ be a morphism of preschemes. We say that f is a local immersion at a point $x \in X$ if there exists an open neighborhood U of x in X and an open neighborhood V of $f(x)$ in Y such that the restriction of f to the induced prescheme U is a closed immersion of U into the induced prescheme V. We say that f is a local immersion if f is a local immersion at each point of X.

Definition (4.5.2). — We say that a morphism $f : X \to Y$ is a local isomorphism at a point $x \in X$ if there exists an open neighborhood U of x in X such that the restriction of f to the induced prescheme U is an open immersion of U into Y. We say that f is a local isomorphism if f is a local isomorphism at each point of X. I | 127

(4.5.3). An immersion (resp. a closed immersion) $f : X \to Y$ can be characterized as a local immersion such that f is a homeomorphism from the underlying space of X to a subset (resp. a closed subset) of Y. An open immersion f can be characterized as an *injective* local isomorphism.

Proposition (4.5.4). — *Let X be an irreducible prescheme, and $f : X \to Y$ a dominant injective morphism. If f is a local immersion, then f is an immersion, and $f(X)$ is open in Y.*

Proof. Let $x \in X$, and let U be an open neighborhood of x, and V an open neighborhood of $f(x)$ in Y such that the restriction of f to U is a closed immersion into V; since U is dense in X, $f(U)$ is dense in Y by hypothesis, so $f(U) = V$, and f is a homeomorphism from U to V; the hypothesis that f is injective implies that $f^{-1}(V) = U$, hence the proposition. □

Proposition (4.5.5). —

(i) *The composition of any two local immersions (resp. of two local isomorphisms) is a local immersion (resp. a local isomorphism).*

(ii) *Let $f : X \to X'$ and $g : Y \to Y'$ be two S-morphisms. If f and g are local immersions (resp. local isomorphisms), then so too is $f \times_S g$.*

(iii) *If an S-morphism f is a local immersion (resp. a local isomorphism), then so too is $f_{(S')}$ for every extension $S' \to S$ of the base prescheme.*

Proof. According to (3.5.1), it suffices to prove (i) and (ii).

(i) follows immediately from the transitivity of closed (resp. open) immersions (4.2.5) and from the fact that if f is a homeomorphism from X to a closed subset of Y, then for every open $U \subset X$, $f(U)$ is open in $f(X)$, so there exists an open subset V of Y such that $f(U) = V \cap f(X)$, and, as a result, $f(U)$ is closed in V.

To prove (ii), let p and q be the projections from $X \times_X Y$, and p' and q' the projections from $X' \times_S Y'$. There exists, by hypothesis, open neighborhoods U, U', V, and V' of $x = p(z)$, $x' = p'(z')$, $y = q(z)$, and $y' = q'(z')$ (respectively), such that the restrictions of f and g to U and V (respectively) are closed (resp. open) immersions into U' and V' (respectively). Since the underlying spaces of $U \times_S V$ and $U' \times_S V'$ can be identified with the open neighborhoods $p^{-1}(U) \cap q^{-1}(V)$ and $p'^{-1}(U') \cap q'^{-1}(V')$ of z and z' (respectively) (3.2.7), the proposition follows from Proposition (4.3.1). □

§5. Reduced preschemes; separation condition

5.1. Reduced preschemes.

Proposition (5.1.1). — *Let $(X, \mathcal{O}_X)$ be a prescheme, and $\mathcal{B}$ a quasi-coherent $\mathcal{O}_X$-algebra. Then there exists a unique quasi-coherent $\mathcal{O}_X$-module $\mathcal{N}$ whose stalk $\mathcal{N}_x$ at any $x \in X$ is the nilradical of the ring $\mathcal{B}_x$. When X is affine, and, consequently, $\mathcal{B} = \widetilde{B}$, where B is an algebra over $A(X)$, then we have $\mathcal{N} = \widetilde{\mathfrak{N}}$, where $\mathfrak{N}$ is the nilradical of B.*

Proof. The statement is local, so it suffices to show the latter claim. We know that $\widetilde{\mathfrak{N}}$ is a quasi-coherent $\mathcal{O}_X$-module (1.4.1), and that its stalk at a point $x \in X$ is the ideal $\mathfrak{N}_x$ of the ring of fractions B_x; it remains to prove that the nilradical of B_x is contained in $\mathfrak{N}_x$, the converse inclusion being evident. Let z/s be an element of the nilradical of B_x, with $z \in B$, and $s \notin \mathfrak{j}_x$; by hypothesis, there exists an integer k such that $(z/s)^k = 0$, which implies that there exists some $t \notin \mathfrak{j}_x$ such that $tz^k = 0$. We conclude that $(tz)^k = 0$, and, as a result, that $z/s = (tz)/(ts) \in \mathfrak{N}_x$. □ I | 128

We say that the quasi-coherent $\mathcal{O}_X$-module $\mathcal{N}$ thus defined is the *nilradical* of the $\mathcal{O}_X$-algebra $\mathcal{B}$; in particular, we denote by $\mathcal{N}_X$ the nilradical of $\mathcal{O}_X$.

Corollary (5.1.2). — *Let X be a prescheme; the closed subprescheme of X defined by the sheaf of ideals $\mathcal{N}_X$ is the only reduced subprescheme (**0**, 4.1.4) of X that has X as its underlying space; it is also the smallest subprescheme of X that has X as its underlying space.*

Proof. Since the structure sheaf of the closed subprescheme of Y defined by $\mathcal{N}_X$ is $\mathcal{O}_X/\mathcal{N}_X$, it is immediate that Y is reduced and has X as its underlying space, because $\mathcal{N}_x \neq \mathcal{O}_x$ for any $x \in X$. To show the other claims, note that a subprescheme Z of X that has X as its underlying space is defined by a sheaf of ideals $\mathcal{I}$ (4.1.3) such that $\mathcal{I}_x \neq \mathcal{O}_x$ for any $x \in X$. We can restrict to the case where X is affine, say $X = \operatorname{Spec}(A)$ and $\mathcal{I} = \widetilde{\mathfrak{I}}$, where $\mathfrak{I}$ is an ideal of A; then, for every $x \in X$, we have $\mathfrak{I}_x \subset \mathfrak{j}_x$, and so $\mathfrak{I}$ is contained in every prime ideal of A, and so also in their intersection $\mathfrak{N}$, the nilradical of A. This proves that Y is the small subprescheme of X that has X as its underlying space (4.1.9); furthermore, if Z is distinct from Y, we necessarily have $\mathcal{I}_x \neq \mathcal{N}_x$ for at least one $x \in X$, and so (5.1.1) Z is not reduced. □

Definition (5.1.3). — We define the reduced prescheme associated to a prescheme X, denoted by X_{red}, to be the unique reduced subprescheme of X that has X as its underlying space.

Saying that a prescheme X is reduced thus implies that $X = X_{\mathrm{red}}$.

Proposition (5.1.4). — *For the prime spectrum of a ring A to be a reduced (resp. integral) prescheme (2.1.7), it is necessary and sufficient for A to be a reduced (resp. integral) ring.*

Proof. Indeed, it follows immediately from (5.1.1) that the condition $\mathcal{N} = (0)$ is necessary and sufficient for $X = \operatorname{Spec}(A)$ to be reduced; the claim corresponding to integral rings is then a consequence of (1.1.13). □

Since every ring of fractions $\neq \{0\}$ of an integral ring is integral, it follows from (5.1.4) that, for every *locally integral* prescheme X, $\mathscr{O}_x$ is an *integral* ring for every $x \in X$. The converse is true whenever the underlying space of X is *locally Noetherian*: indeed, X is then reduced, and if U is an affine open subset of X, which is a Noetherian space, then U has only a finite number of irreducible components, and so its ring A has only a finite number of minimal prime ideals (1.1.14). If two of the irreducible components of U had a common point x, then $\mathscr{O}_x$ would have at least two distinct minimal prime ideals, and would thus not be integral; the components of U are thus open subsets that are pairwise disjoint, and each of them is thus integral.

(5.1.5). Let $f = (\psi, \theta) : X \to Y$ be a morphism of preschemes; the homomorphism $\theta_x^\sharp : \mathscr{O}_{\psi(x)} \to \mathscr{O}_x$ sends each nilpotent element of $\mathscr{O}_{\psi(x)}$ to a nilpotent element of $\mathscr{O}_x$; by passing to the quotients, $\theta^\sharp$ induces a homomorphism I | 129

$$\omega : \psi^*(\mathscr{O}_Y/\mathscr{N}_Y) \longrightarrow \mathscr{O}_X/\mathscr{N}_X.$$

It is clear that, for every $x \in X$, $\omega_x : \mathscr{O}_{\psi(x)}/\mathscr{N}_{\psi(x)} \to \mathscr{O}_x/\mathscr{N}_x$ is a local homomorphism, and so $(\psi, \omega^\flat)$ is a morphism of preschemes $X_{\text{red}} \to Y_{\text{red}}$, which we denote by f_{red}, and call the *reduced* morphism associated to f. It is immediate that, for morphisms $f : X \to Y$ and $g : Y \to Z$, we have $(g \circ f)_{\text{red}} = g_{\text{red}} \circ f_{\text{red}}$, and so we have defined X_{red} as a *functor*, *covariant* in X.

The preceding definition shows that the diagram

$$\begin{tikzcd} X_{\text{red}} \arrow[r, "f_{\text{red}}"] \arrow[d] & Y_{\text{red}} \arrow[d] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

is commutative, where the vertical arrows are the injection morphisms; in other words, $X_{\text{red}} \to X$ is a *functorial* morphism. We note in particular that, if X is reduced, then every morphism $f : X \to Y$ factors as $X \xrightarrow{f_{\text{red}}} Y_{\text{red}} \to Y$; in other words, f factors through the injection morphism $Y_{\text{red}} \to Y$.

Proposition (5.1.6). — *Let $f : X \to Y$ be a morphism; if f is surjective (resp. radicial, an immersion, a closed immersion, an open immersion, a local immersion, a local isomorphism), then so too is f_{red}. Conversely, if f_{red} is surjective (resp. radicial), then so too is f.*

Proof. The proposition is trivial if f is surjective; if f is radicial, then the proposition follows from the fact that, for every $x \in X$, the field $k(x)$ is the same for the preschemes X and X_{red} (3.5.8). Finally, if $f = (\psi, \theta)$ is an immersion, a closed immersion, or a local immersion (resp. an open immersion, or a local isomorphism), then the proposition follows from the fact that, if $\theta_x^\sharp$ is surjective (resp. bijective), then so too is the homomorphism obtained by passing to the quotients by the nilradicals $\mathscr{O}_{\psi(x)}$ and $\mathscr{O}_x$ ((5.1.2) and (4.2.2)) (cf. (5.5.12)). □

Proposition (5.1.7). — *If X and Y are S-preschemes, then the preschemes $X_{\text{red}} \times_{S_{\text{red}}} Y_{\text{red}}$ and $X_{\text{red}} \times_S Y_{\text{red}}$ are identical, and canonically identified with a subprescheme of $X \times_S Y$ that has the same underlying subspace as the two aforementioned products.*

Proof. The canonical identification of $X_{\text{red}} \times_S Y_{\text{red}}$ with a subprescheme of $X \times_S Y$ that has the same underlying space follows from (4.3.1). Furthermore, if φ and ψ are the structure morphisms $X_{\text{red}} \to S$ and $Y_{\text{red}} \to S$ (respectively), then they factor through S_{red} (5.1.5), and since $S_{\text{red}} \to S$ is a monomorphism, the first claim of the proposition follows from (3.2.4). □

Corollary (5.1.8). — *The preschemes $(X \times_S Y)_{\text{red}}$ and $(X_{\text{red}} \times_{S_{\text{red}}} Y_{\text{red}})_{\text{red}}$ are canonically identified with one another.*

Proof. This follows from (5.1.2) and (5.1.7). □

We note that, even if X and Y are reduced preschemes, $X \times_S Y$ might not be reduced, because the tensor product of two reduced algebras can have nilpotent elements.

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Proposition (5.1.9). — *Let X be a prescheme, and $\mathscr{I}$ a quasi-coherent sheaf of ideals of $\mathscr{O}_X$ such that $\mathscr{I}^n = 0$ for some integer $n > 0$. Let X_0 be the closed subprescheme $(X, \mathscr{O}_X/\mathscr{I})$ of X; for X to be an affine scheme, it is necessary and sufficient for X_0 to be an affine scheme.*

The condition is clearly necessary, so we will show that it is sufficient. If we set $X_k = (X, \mathscr{O}_X/\mathscr{I}^{k+1})$, it is enough to prove by induction on k that X_k is affine, and so we are led to consider the base case, where $\mathscr{I}^2 = 0$. We set

$$A = \Gamma(X, \mathscr{O}_X)$$
$$A_0 = \Gamma(X_0, \mathscr{O}_{X_0}) = \Gamma(X, \mathscr{O}_X/\mathscr{I}).$$

The canonical homomorphism $\mathcal{O}_X \to \mathcal{O}_X/\mathcal{I}$ induces a homomorphism of rings $\varphi : A \to A_0$. We will see below that φ is *surjective*, which implies that

$$0 \longrightarrow \Gamma(X,\mathcal{I}) \longrightarrow \Gamma(X,\mathcal{O}_X) \longrightarrow \Gamma(X,\mathcal{O}_X/\mathcal{I}) \longrightarrow 0 \tag{5.1.9.1}$$

is an *exact* sequence. We now prove, assuming that this is true, the proposition. Note that $\mathfrak{K} = \Gamma(X,\mathcal{I})$ is an ideal whose square is zero in A, and thus a module over $A_0 = A/\mathfrak{K}$. By hypothesis, we have $X_0 = \operatorname{Spec}(A)$, and, since the underlying spaces of X_0 and X are identical, $\mathfrak{K} = \Gamma(X_0,\mathcal{I})$; Additionally, since $\mathcal{I}^2 = 0$, $\mathcal{I}$ is a quasi-coherent $(\mathcal{O}_X/\mathcal{I})$-module, so we have $\mathcal{I} \cong \widetilde{\mathfrak{K}}$ and $\mathfrak{K}_x = \mathcal{I}_x$ for all $x \in X_0$ (1.4.1). With this in mind, let $X' = \operatorname{Spec}(A)$, and consider the morphism $f = (\psi,\theta) : X \to X'$ of preschemes that corresponds to the identity map $A \to \Gamma(X,\mathcal{O}_X)$ (2.2.4). For every affine open subset V of X, the diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & \Gamma(V,\mathcal{O}_X|V) \arrow[d] \\ A_0 = A/\mathfrak{K} \arrow[r] & \Gamma(V,\mathcal{O}_{X_0}|V) \end{tikzcd}$$

commutes, whence the diagram

$$\begin{tikzcd} X' & X \arrow[l, "f"'] \\ X'_0 \arrow[u, "j'"] & X_0 \arrow[l, "f_0"] \arrow[u, "j"'] \end{tikzcd}$$

also commutes (X'_0 being the closed subprescheme of X' defined by the quasi-coherent sheaf of ideals $\widetilde{R}$, and j and j' being the canonical injection morphisms). But since X_0 is affine, f_0 is an isomorphism, and since the underlying continuous maps of j and j' are identity maps, we see straight away that $\psi : X \to X'$ is a homeomorphism. Furthermore, the equation $\mathfrak{K}_x = \mathcal{I}_x$ shows that the restriction of $\theta^\sharp : \psi^*(\mathcal{O}_{X'}) \to \mathcal{O}_X$ is an *isomorphism* from $\psi^*(\widetilde{\mathfrak{K}})$ to $\mathcal{I}$; additionally, by passing to the quotients, $\theta^\sharp$ gives an *isomorphism* $\psi^*(\mathcal{O}_X/\widetilde{\mathfrak{K}}) \to \mathcal{O}_X/\mathcal{I}$, because f_0 is an isomorphism; we thus immediately conclude, by the 5 lemma (M, I, 1.1), that $\theta^\sharp$ is itself an isomorphism, and thus that f is an *isomorphism*, and thus that X is affine. So everything reduces to proving the exactitude of (5.1.9.1), which will follow from showing that $\mathrm{H}^1(X,\mathcal{I}) = 0$. But $\mathrm{H}^1(X,\mathcal{I}) = \mathrm{H}^1(X_0,\mathcal{I})$, and we have seen that **I** | 131
$\mathcal{I}$ is a quasi coherent $\mathcal{O}_{X_0}$-module. Our proof will thus follow from

Lemma (5.1.9.2). — *If Y is an affine scheme, and $\mathcal{F}$ a quasi-coherent $\mathcal{O}_Y$-module, then* $\mathrm{H}^1(Y,\mathcal{F}) = 0$.

Proof. This lemma will be proven in Chapter III, §1, as a consequence of the more general theorem that $\mathrm{H}^i(Y,\mathcal{F}) = 0$ for all $i > 0$. To give an independent proof, note that $\mathrm{H}^1(Y,\mathcal{F})$ can be identified with the module $\operatorname{Ext}^1_{\mathcal{O}_Y}(Y;\mathcal{O}_Y,\mathcal{F})$ of extensions classes of the $\mathcal{O}_Y$-module $\mathcal{O}_Y$ by the $\mathcal{O}_Y$-module $\mathcal{F}$ (T, 4.2.3); so everything reduces to proving that such an extension $\mathcal{G}$ is trivial. But, for all $y \in Y$, there is a neighbourhood V of y in Y such that $\mathcal{G}|V$ is isomorphic to $\mathcal{F}|Y \oplus \mathcal{O}_Y|V$ (**0**, 5.4.9); from this we conclude that $\mathcal{G}$ is a *quasi-coherent* $\mathcal{O}_Y$-module. If A is the ring of Y, then we have $\mathcal{F} = \widetilde{M}$ and $\mathcal{G} = \widetilde{N}$, where M and N are A-modules, and, by hypothesis, N is an extension of the A-module A by the A-module M (1.3.11). Since this extension is necessarily trivial, the lemma is proven, and thus so too is (5.1.9). □

Corollary (5.1.10). — *Let X be a prescheme such that $\mathcal{N}_X$ is nilpotent. For X to be an affine scheme, it is necessary and sufficient for X_{red} to be an affine scheme.*

5.2. Existence of a subprescheme with a given underlying space.

Proposition (5.2.1). — *For every locally closed subspace Y of the underlying space of a prescheme X, there exists exactly one reduced subprescheme of X that has Y as its underlying space.*

Proof. The uniqueness follows from (5.1.2), so it remains only to show the existence of the prescheme in question.

If X is affine, given by some ring A, and Y closed in X, then the proposition is immediate: $\mathfrak{j}(Y)$ is the largest ideal $\mathfrak{a} \subset A$ such that $V(\mathfrak{a}) = Y$, and it is equal to its radical (1.1.4, i), so $A/\mathfrak{j}(Y)$ is a reduced ring.

In the general case, for every affine open $U \subset X$ such that $U \cap Y$ is closed in U, consider the closed subprescheme Y_U of U defined by the sheaf of ideals associated to the ideal $\mathfrak{j}(U \cap Y)$ of $A(U)$, which is reduced. We can show that, if V is an affine open subset of X contained in U, then Y_V is *induced* by Y_U on $V \cap Y$; but this induced prescheme is a closed subprescheme (of V) which is reduced and has $V \cap Y$ as its underlying space; the uniqueness of Y_V thus implies our claim. □

Proposition (5.2.2). — *Let X be a reduced subprescheme of a prescheme Y; if Z is the closed reduced subprescheme of Y that has $\overline{X}$ as its underlying space, then X is a subprescheme induced on an open subset of Z.*

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Proof. There is indeed an open subset U of Y such that $X = U \cap \overline{X}$; since, by (5.2.2), X is a reduced subprescheme of Z, the subprescheme X is induced by Z on the open subspace X by uniqueness (5.2.1). □

Corollary (5.2.4). — *Let $f : X \to Y$ be a morphism, and X' (resp. Y') a closed subprescheme of X (resp. Y) defined by a quasi-coherent sheaf of ideals $\mathscr{J}$ (resp. $\mathscr{K}$) of $\mathscr{O}_X$ (resp. $\mathscr{O}_Y$). Suppose that X' is reduced, and that $f(X') \subset Y'$. Then $f^*(\mathscr{K})\mathscr{O}_X \subset \mathscr{J}$.*

Proof. Since, by (5.2.2), the restriction of f to X' factors as $X' \to Y' \to Y$, it suffices to apply (4.4.6). □

5.3. Diagonal; graph of a morphism.

(5.3.1). Let X be an S-prescheme; we define the *diagonal morphism* of X in $X \times_S X$, denoted by $\Delta_{X|S}$, or Δ_X, or even Δ if no confusion may arise, to be the S-morphism $(1_X, 1_X)_S$, or, in other words, the unique S-morphism Δ_X such that

(5.3.1.1) $$p_1 \circ \Delta_X = p_2 \circ \Delta_X = 1_X,$$

where p_1 and p_2 are the projections of $X \times_S X$ (Definition (3.2.1)). If $f : T \to X$ and $g : T \to Y$ are S-morphisms, we immediately have that

(5.3.1.2) $$(f, g)_S = (f \times_S g) \circ \Delta_{T|S}.$$

The reader will note that the preceding definition and the results stated in (5.3.1) to (5.3.8) are true in any category, *provided that the products used within exist in the category.*

Proposition (5.3.2). — *Let X and Y be S-preschemes; if we make the canonical identification between $(X \times Y) \times (X \times Y)$ and $(X \times X) \times (Y \times Y)$, then the morphism $\Delta_{X \times Y}$ is identified with $\Delta_X \times \Delta_Y$.*

Proof. Indeed, if $p_1 : X \times X \to X$ and $q_1 : Y \times Y \to Y$ are the projections onto the first component, then the projection onto the first component $(X \times Y) \times (X \times Y) \to X \times Y$ is identified with $p_1 \times q_1$, and we have

$$(p_1 \times q_1) \circ (\Delta_X \times \Delta_Y) = (p_1 \circ \Delta_X) \times (q_1 \circ \Delta_Y) = 1_{X \times Y}$$

and we can argue similarly for the projections onto the second component. □

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Corollary (5.3.4). — *For every extension $S' \to S$ of the base prescheme, $\Delta_{X_{S'}}$ is canonically identified with $(\Delta_X)_{(S')}$.*

Proof. It suffices to remark that $(X \times_S X)_{(S')}$ is canonically identified with $X_{(S')} \times_{S'} X_{(S')}$ (3.3.10). □

Proposition (5.3.5). — *Let X and Y be S-preschemes, and $\varphi : S \to T$ a morphism of preschemes, which lets us consider every S-prescheme as a T-prescheme. Let $f : X \to S$ and $g : Y \to S$ be the structure morphisms, p and q the projections of $X \times_S Y$, and $\pi = f \circ p = g \circ q$ the structure morphism $X \times_S Y \to S$. Then the diagram*

(5.3.5.1)
$$\begin{tikzcd} X \times_S Y \arrow[r, "(p,q)_T"] \arrow[d, "\pi"'] & X \times_T Y \arrow[d, "f \times_T g"] \\ S \arrow[r, "\Delta_{S|T}"] & S \times_T S \end{tikzcd}$$

commutes, and identifies $X \times_S Y$ with the product of the $(S \times_T S)$-preschemes S and $X \times_T Y$, and the projections with π and $(p, q)_T$. I | 133

Proof. By (3.4.3), we are led to proving the corresponding proposition *in the category of sets*, replacing X, Y, and S by $X(Z)_T$, $Y(Z)_T$, and $S(Z)_T$ (respectively), with Z being an arbitrary T-prescheme. But, in the category of sets, the proof is immediate and left to the reader. □

Corollary (5.3.6). — *The morphism $(p, q)_T$ can be identified (letting $P = S \times_T S$) with $1_{X \times_T Y} \times_P \Delta_S$.*

Proof. This follows from (5.3.5) and (3.3.4). □

Corollary (5.3.7). — *If $f : X \to Y$ is an S-morphism, then the diagram*

$$\begin{tikzcd} X \arrow[r, "(1_X,f)_S"] \arrow[d, "f"'] & X \times_S Y \arrow[d, "f \times_S 1_Y"] \\ Y \arrow[r, "\Delta_Y"] & Y \times_S Y \end{tikzcd}$$

commutes, and identifies X with the product of the $(Y \times_S Y)$-preschemes Y and $X \times_S Y$.

Proof. It suffices to apply (5.3.5), replacing S by Y, and T by S, and noting that $X \times_Y Y = X$ (3.3.3). □

Proposition (5.3.8). — *For $f : X \to Y$ to be a monomorphism of preschemes, it is necessary and sufficient for $\Delta_{X|Y}$ to be an isomorphism from X to $X \times_Y X$.*

Proof. Indeed, to say that f is a monomorphism implies that, for every Y-prescheme Z, the corresponding map $f' : X(Z)_Y \to Y(Z)_Y$ is an injection, and, since $Y(Z)_Y$ consists of a single element, this implies that $X(Z)_Y$ consists of a single element as well. But this can also be expressed by saying that $X(Z)_Y \times X(Z)_Y$ is canonically isomorphic to $X(Z)_Y$; the former is exactly the set $(X \times_Y X)(Z)_Y$ (3.4.3.1), which implies that $\Delta_{X|Y}$ is an isomorphism. □

Proposition (5.3.9). — *The diagonal morphism Δ_X is an immersion from X to $X \times_S X$.*

Proof. Indeed, since the continuous maps p_1 and Δ_X from the underlying spaces are such that $p_1 \circ \Delta_X$ is the identity, Δ_X is a homeomorphism from X to $\Delta_X(X)$. Similarly, the composite homomorphism $\mathcal{O}_x \to \mathcal{O}_{\Delta_X(x)} \to \mathcal{O}_x$ (composed of the homomorphisms corresponding to p_1 and Δ_X) is the identity, which means that the homomorphism corresponding to Δ_X is surjective; the proposition thus follows from (4.2.2). □

We say that the subprescheme of $X \times_S X$ associated to the immersion Δ_X (4.2.1) is *the diagonal* of $X \times_S X$.

Corollary (5.3.10). — *Under the hypotheses of (5.3.5), $(p,q)_T$ is an immersion.*

Proof. This follows from (5.3.6) and (4.3.1). □

We say (under the hypotheses of (5.3.5)) that $(p,q)_T$ is the *canonical immersion* of $X \times_S Y$ into $X \times_T Y$.

Corollary (5.3.11). — *Let X and Y be S-preschemes, and $f : X \to Y$ an S-morphism; then the graph morphism $\Gamma_f = (1_X, f)_S$ of f (3.3.14) is an immersion of X into $X \times_S Y$.*

Proof. This is a particular case of Corollary (5.3.10), where we replace S by Y, and T by S (cf. (5.3.7)). □ **I | 134**

The subprescheme of $X \times_S Y$ associated to the immersion Γ_f (4.2.1) is called *the graph* of the morphism f; the subpreschemes of $X \times_S Y$ that are graphs of morphisms $X \to Y$ are characterised by the property that the restriction to such a subprescheme G of the projection $p_1 : X \times_S Y \to X$ is an *isomorphism* g from G to X: G is the the graph of the morphism $p_2 \circ g^{-1}$, where p_2 is the projection $X \times_S Y \to Y$.

When we take, in particular, $X = S$, then the S-morphisms $S \to Y$ (which are exactly the *S-sections* of Y (2.5.5)) are equal to their graph morphisms; the subpreschemes of Y that are the graphs of S-sections (in other words, those that are isomorphic to S by the restriction of the structure morphism $Y \to S$) are then also called the *images* of these sections, or, by an abuse of language, the *S-sections* of Y.

Corollary (5.3.12). — *With the hypotheses and notation of (5.3.11), for every morphism $g : S' \to S$, let f' be the inverse image of f under g (3.3.7); then $\Gamma_{f'}$ is the inverse image of Γ_f under g.*

Proof. This is a particular case of (3.3.10.1). □

Corollary (5.3.13). — *Let $f : X \to Y$ and $g : Y \to Z$ be morphisms; if $g \circ f$ is an immersion (resp. a local immersion), then so too is f.*

Proof. Indeed, f factors as $X \xrightarrow{\Gamma_f} X \times_Z Y \xrightarrow{p_2} Y$. Furthermore, p_2 can be identified with $(g \circ f) \times_Z 1_Y$ (3.3.4); if $g \circ f$ is an immersion (resp. a local immersion), then so too is p_2 ((4.3.1) and (4.5.5)), and since Γ_f is an immersion (5.3.11), we are done, by (4.2.4) (resp. (4.5.5)). □

Corollary (5.3.14). — *Let $j : X \to Y$ and $g : Z \to Z$ be S-morphisms. If j is an immersion (resp. a local immersion), then so too is $(j, g)_S$.*

Proof. Indeed, if $p : Y \times_S Z \to Y$ is the projection onto the first component, then we have $j = p \circ (j, g)_S$, and it suffices to apply (5.3.13). □

Proposition (5.3.15). — *If $f : X \to Y$ is an S-morphism, then the diagram*

(5.3.15.1)
$$\begin{tikzcd} X \arrow[r, "\Delta_X"] \arrow[d, "f"'] & X \times_S X \arrow[d, "f \times_S f"] \\ Y \arrow[r, "\Delta_Y"] & Y \times_S Y \end{tikzcd}$$

commutes (in other words, Δ_X is a functorial morphism in the category of preschemes).

Proof. The proof is immediate and left to the reader. □

Corollary (5.3.16). — *If X is a subprescheme of Y, then the diagonal $\Delta_X(X)$ can be identified with a subprescheme of $\Delta_Y(Y)$, and the underlying space can be identified with*

$$\Delta_Y(Y) \cap p_1^{-1}(X) = \Delta_Y \cap p_2^{-1}(X)$$

(p_1 and p_2 being the projections of $Y \times_S Y$).

Proof. Applying (5.3.15) to the injection morphism $f : X \to Y$, we see that $f \times_S f$ is an immersion that identifies the underlying space of $X \times_S X$ with the subspace $p_1^{-1}(X) \cap p_2^{-1}(X)$ of $Y \times_S Y$ (4.3.1); further, if $z \in \Delta_Y(Y) \cap p_1^{-1}(X)$, then we have $z = \Delta_Y(y)$ and $y = p_1(z) \in X$, so $y = f(y)$, and $z = \Delta_Y(f(y))$ belongs to **I** | 135
$\Delta_X(X)$ by the commutativity of (5.3.15.1). □

Corollary (5.3.17). — *Let $f_1 : Y \to X$ and $f_2 : Y \to X$ be S-morphisms, and y a point of Y such that $f_1(y) = f_2(y) = x$, and such that the homomorphisms $k(x) \to k(y)$ corresponding to f_1 and f_2 are identical. Then, if $f = (f_1, f_2)_S$, the point $f(y)$ belongs to the diagonal $\Delta_{X|S}(X)$.*

Proof. The two homomorphisms $k(x) \to k(y)$ corresponding to f_i ($i = 1, 2$) define two S-morphisms $g_i : \mathrm{Spec}(k(y)) \to \mathrm{Spec}(k(x))$ such that the diagrams

$$\begin{tikzcd} \mathrm{Spec}(k(y)) \arrow[r, "g_i"] \arrow[d] & \mathrm{Spec}(k(x)) \arrow[d] \\ Y \arrow[r, "f_i"] & X \end{tikzcd}$$

commute. The diagram

$$\begin{tikzcd} \mathrm{Spec}(k(y)) \arrow[r, "(g_1, g_2)_S"] \arrow[d] & \mathrm{Spec}(k(x)) \times_S \mathrm{Spec}(k(x)) \arrow[d] \\ Y \arrow[r, "(f_1, f_2)_S"] & X \times_S X \end{tikzcd}$$

thus also commutes. But it follows from the equality $g_1 = g_2$ that the image under $(g_1, g_2)_S$ of the unique point of $\mathrm{Spec}(k(y))$ belongs to the diagonal of $\mathrm{Spec}(k(x)) \times_S \mathrm{Spec}(k(x))$; the conclusion then follows from (5.3.15). □

5.4. Separated morphisms and separated preschemes.

Definition (5.4.1). — We say that a morphism of preschemes $f : X \to Y$ is *separated* if the diagonal morphism $X \to X \times_Y X$ is a *closed* immersion; we then also say that X is a *separated prescheme over Y*, or a *Y-scheme*. We say that a prescheme X is separated if it is separated over $\mathrm{Spec}(\mathbf{Z})$; we then also say that X is a *scheme*[9] *(cf. (5.5.7))*.

By (5.3.9), for X to be separated over Y, it is necessary and sufficient for $\Delta_X(X)$ to be a *closed subspace* of the underlying space of $X \times_Y X$.

Proposition (5.4.2). — *Let $S \to T$ be a separated morphism. If X and Y are S-preschemes, then the canonical immersion $X \times_S Y \to X \times_T Y$ (5.3.10) is closed.*

Proof. Indeed, if we refer to the diagram in (5.3.5.1), we see that $(p, q)_T$ can be considered as being obtained from $\Delta_{S|T}$ by the extension $f \times_T g : X \times_T Y \to S \times_T S$ of the base prescheme $S \times_T S$; the proposition then follows from (4.3.2). □

Corollary (5.4.3). — *Let Y be an S-scheme, and $f : X \to Y$ an S-morphism. Then the graph morphism $\Gamma_f : X \to X \times_S Y$ (5.3.11) is a closed immersion.*

[9] *[Trans.] We repeat here the warning given at the very start of this translation: the early versions of the EGA use* prescheme *to mean is now usually called a scheme, and* scheme *for what is now usually called a separated scheme..* Grothendieck himself later said that the more modern terminology was preferable, but we have decided to keep this translation 'historically accurate' by using the older nomenclature.

Proof. This is a particular case of (5.4.2), where we replace S by Y, and T by S. □

Corollary (5.4.4). — *Let $f : X \to Y$ and $g : Y \to Z$ be morphisms, with g separated. If $g \circ f$ is a closed immersion, then so too is f.*

Proof. The proof using (5.4.3) is the same as that of (5.3.13) using (5.3.11). □

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Corollary (5.4.5). — *Let Z be an S-scheme, and $j : X \to Y$ and $g : X \to Z$ S-morphisms. If j is a closed immersion, then so too is $(j,g)_S : X \to Y \times_S Z$.*

Proof. The proof using (5.4.4) is the same as that of (5.3.14) using (5.3.13). □

Corollary (5.4.6). — *If X is an S-scheme, then every S-section of X (2.5.5) is a closed immersion.*

Proof. If $\varphi : X \to S$ is the structure morphism, and $\psi : S \to X$ an S-section of X, it suffices to apply (5.4.5) to $\varphi \circ \psi = 1_S$. □

Corollary (5.4.7). — *Let X be an integral prescheme with generic point s, and X an S-scheme. If two S-sections f and g are such that $f(s) = g(s)$, then $f = g$.*

Proof. Indeed, if $x = f(s) = g(s)$, then the homomorphisms $k(x) \to k(s)$ corresponding to f and g are necessarily identical. If $h = (f,g)_S$, we thus deduce (5.3.17) that $h(s)$ belongs to the diagonal $Z = \Delta_X(X)$; but since $S = \overline{\{s\}}$, and since Z is closed by hypothesis, we have $h(S) \subset Z$. It then follows from (5.2.2) that h factors as $S \to Z \to X \times_S X$, and we thus conclude that $f = g$, by definition of the diagonal. □

Remark (5.4.8). — If we suppose, conversely, that the conclusion of (5.4.3) is true when $f = 1_Y$, then we can conclude that Y is separated over S; similarly, if we suppose that the conclusion of (5.4.5) applies to the two morphisms $Y \xrightarrow{\Delta_Y} Y \times_Z Y \xrightarrow{p_1} Y$, then we can conclude that Δ_Y is a closed immersion, and thus that Y is separated over Z; finally, if we assume that the conclusion of (5.4.6) is true for the Y section Δ_Y of the Y-prescheme $Y \times_S Y \to Y$, then this implies that Y is separated over S.

5.5. Separation criteria.

Proposition (5.5.1). —

(i) *Every monomorphism of preschemes (and, in particular, every immersion) is a separated morphism.*
(ii) *The composition of any two separated morphisms is separated.*
(iii) *If $f : X \to X'$ and $g : Y \to Y'$ are separated S-morphisms, then $f \times_S g$ is separated.*
(iv) *If $f : X \to Y$ is a separated S-morphism, then the S'-morphism $f_{(S')}$ is separated for every extension $S' \to S$ of the base prescheme.*
(v) *If the composition $g \circ f$ is separated, then f is separated.*
(vi) *For a morphism f to be separated, it is necessary and sufficient for f_{red} (5.1.5) to be separated.*

Proof. Note that (i) is an immediate consequence of (5.3.8). If $f : X \to Y$ and $g : Y \to Z$ are morphisms, then the diagram

(5.5.1.1)
$$\begin{tikzcd} X \arrow[rr, "\Delta_{X|Z}"] \arrow[dr, "\Delta_{X|Y}"'] & & X \times_Z X \\ & X \times_Y X \arrow[ur, "j"'] & \end{tikzcd}$$

commutes (where j denotes the canonical immersion (5.3.10)), as can be immediately verified. If f and g are separated, then $\Delta_{X|Y}$ is a closed immersion by definition, and j is a closed immersion by (5.4.2), whence $\Delta_{X|Z}$ is a closed immersion by (4.2.4), which proves (ii). Given (i) and (ii), (iii) and (iv) are equivalent (3.5.1), so it I | 137
suffices to prove (iv). But $X_{(S')} \times_{Y_{(S')}} X_{(S')}$ is canonically identified with $(X \times_Y X) \times_Y Y_{(S')}$ by (3.3.11) and (3.3.9.1), and we immediately see that the diagonal morphism $\Delta_{X_{(S')}}$ can then be identified with $\Delta_X \times_Y 1_{Y_{(S')}}$; the proposition then follows from (4.3.1).

To prove (v), consider, as in (5.3.13), the factorisation $X \xrightarrow{\Gamma_f} X \times_Z Y \xrightarrow{p_2} Y$ of f, noting that $p_2 = (g \circ f) \times_Z 1_Y$; the hypothesis (that $g \circ f$ is separated) implies that g_2 is separated, by (iii) and (i), and, since Γ_f is an immersion, Γ_f is separated, by (i), whence f is separated, by (ii). Finally, to prove (vi), recall that the preschemes $X_{\text{red}} \times_{Y_{\text{red}}} X_{\text{red}}$

and $X_{\mathrm{red}} \times_Y X_{\mathrm{red}}$ are canonically identified with one another (5.1.7); if we denote by j the injection $X_{\mathrm{red}} \to X$, then the diagram

$$\begin{tikzcd} X_{\mathrm{red}} \arrow[r, "\Delta_{X_{\mathrm{red}}}"] \arrow[d, "j"'] & X_{\mathrm{red}} \times_Y X_{\mathrm{red}} \arrow[d, "j \times_Y j"] \\ X \arrow[r, "\Delta_X"] & X \times_Y X \end{tikzcd}$$

commutes (5.3.15), and the proposition follows from the fact that the vertical arrows are homeomorphisms of the underlying spaces (4.3.1). □

Corollary (5.5.2). — *If $f : X \to Y$ is separated, then the restriction of f to any subprescheme of X is separated.*

Proof. This follows from (5.5.1, i and ii). □

Corollary (5.5.3). — *If X and Y are S-preschemes such that Y is separated over S, then $X \times_S Y$ is separated over X.*

Proof. This is a particular case of (5.5.1, iv). □

Proposition (5.5.4). — *Let X be a prescheme, and assume that its underlying space is a* finite *union of closed subsets X_k ($1 \leqslant k \leqslant n$); for each k, consider the reduced subprescheme of X that has X_k as its underlying space* (5.2.1), *and denote this again by X_k. Let $f : X \to Y$ be a morphism, and, for each k, let Y_k be a closed subset of Y such that $f(X_k) \subset Y_k$; we again denote by Y_k the reduced subprescheme of Y that has Y_k as its underlying space, so that the restriction $X_k \to Y$ of f to X_k factors as $X_k \xrightarrow{f_k} Y_k \to Y$* (5.2.2). *For f to be separated, it is necessary and sufficient for all the f_k to be separated.*

Proof. The necessity follows from (5.5.1, i, ii, and v). Conversely, if the condition of the statement is satisfied, then each of the restrictions $X_k \to Y$ of f is separated (5.5.1, (i) and (ii)); if p_1 and p_2 are the projections of $X \times_Y X$, then the subspace $\Delta_{X_k}(X_k)$ can be identified with the subspace $\Delta_X(X) \cap p_1^{-1}(X_k)$ of the underlying space of $X \times_Y X$ (5.3.16); these subspaces are closed in $X \times_Y X$, and thus so too is their union $\Delta_X(X)$. □

Suppose, in particular, that the X_k are the *irreducible components* of X; then we can suppose that the Y_k are the irreducible components of Y (**0**, 2.1.5); Proposition (5.5.4) then, in this case, leads to the idea of separation in the case of *integral* preschemes (2.1.7). **I** | 138

Proposition (5.5.5). — *Let (Y_λ) be an open cover of a prescheme Y; for a morphism $f : X \to Y$ to be separated, it is necessary and sufficient for each of its restrictions $f^{-1}(Y_\lambda) \to Y_\lambda$ to be separated.*

Proof. If we set $X_\lambda = f^{-1}(Y_\lambda)$, everything reduces, by taking (4.2.4, b) and the identification of the products $X_\lambda \times_Y X_\lambda$ and $X_\lambda \times_{Y_\lambda} X_\lambda$ into account, to proving that the $X_\lambda \times_Y X_\lambda$ form a cover of $X \times_Y X$. But if we set $Y_{\lambda\mu} = Y_\lambda \cap Y_\mu$ and $X_{\lambda\mu} = X_\lambda \cap X_\mu = f^{-1}(Y_{\lambda\mu})$, then $X_\lambda \times_Y X_\mu$ can be identified with the product $X_{\lambda\mu} \times_{Y_{\lambda\mu}} X_{\lambda\mu}$ (3.2.6.4), and so also with $X_{\lambda\mu} \times_Y X_{\lambda\mu}$ (3.2.5), and finally with an open subset of $X_\lambda \times_Y X_\lambda$, which proves our claim (3.2.7). □

Proposition (5.5.4) allows us, by taking an affine open cover of Y, to restrict our study of separated morphisms to just those that take values in affine schemes.

Proposition (5.5.6). — *Let Y be an* affine *scheme, X a prescheme, and (U_α) a cover of X by affine open subsets. For a morphism $f : X \to Y$ to be separated, it is necessary and sufficient for $U_\alpha \cap U_\beta$ to be, for every pair of indices (α, β), an affine open subset,* and *for the ring $\Gamma(U_\alpha \cap U_\beta, \mathscr{O}_X)$ to be generated by the union of the canonical images of the rings $\Gamma(U_\alpha, \mathscr{O}_X)$ and $\Gamma(U_\beta, \mathscr{O}_X)$.*

Proof. The $U_\alpha \times_Y U_\beta$ form an open cover of $X \times_Y X$ (3.2.7); denoting the projections of $X \times_Y X$ by p and q, we have

$$\begin{aligned} \Delta_X^{-1}(U_\alpha \times_Y U_\beta) &= \Delta_X^{-1}(p^{-1}(U_\alpha) \cap q^{-1}(U_\beta)) \\ &= \Delta_X^{-1}(p^{-1}(U_\alpha)) \cap \Delta_X^{-1}(q^{-1}(U_\beta)) = U_\alpha \cap U_\beta; \end{aligned}$$

so everything reduces to proving that the restriction of Δ_X to $U_\alpha \cap U_\beta$ is a closed immersion into $U_\alpha \times_Y U_\beta$. But this restriction is exactly $(j_\alpha, j_\beta)_Y$, where j_α (resp. j_β) denotes the injection morphism from $U_\alpha \cap U_\beta$ to U_α (resp. U_β), as follows from the definitions. Since $U_\alpha \times_Y U_\beta$ is an affine scheme whose ring is canonically isomorphic to $\Gamma(U_\alpha, \mathscr{O}_X) \otimes_{\Gamma(Y, \mathscr{O}_Y)} \Gamma(U_\beta, \mathscr{O}_X)$ (3.2.2), we see that $U_\alpha \cap U_\beta$ must be an affine scheme, and that the map $h_\alpha \otimes h_\beta \mapsto h_\alpha h_\beta$ from the ring $A(U_\alpha \times_Y U_\beta)$ to $\Gamma(U_\alpha \cap U_\beta, \mathscr{O}_X)$ must be surjective (4.2.3), which finishes the proof. □

Corollary (5.5.7). — *An affine scheme is separated* (and is thus a *scheme*, which justifies the terminology of (5.4.1)).

Corollary (5.5.8). — *Let Y be an affine scheme; for $f : X \to Y$ to be a separated morphism, it is necessary and sufficient for X to be separated* (in other words, for X to be a *scheme*).

Proof. Indeed, we see that the criteria of (5.5.6) do not depend on f. □

Corollary (5.5.9). — *For a morphism $f : X \to Y$ to be separated, it is necessary and sufficient for the induced prescheme $f^{-1}(U)$ to be separated, for every open subset of U on which Y induces a separated prescheme, and it is sufficient for it to be the case for every affine open subset $U \subset Y$.*

Proof. The necessity of the condition follows from (5.5.4) and (5.5.1, ii); the sufficiency follows from (5.5.4) and (5.5.8), taking into account the existence of affine open covers of Y. □

In particular, if X and Y are affine schemes, then *every* morphism $X \to Y$ is separated.

Proposition (5.5.10). — *Let Y be a* scheme, *and $f : X \to Y$ a morphism. For every affine open subset U of X, and every affine open subset V of Y, $U \cap f^{-1}(V)$ is affine.*

Proof. Let p_1 and p_2 be the projections of $X \times_{\mathbf{Z}} Y$; the subspace $U \cap f^{-1}(V)$ is the image of $\Gamma_f(X) \cap p_1^{-1}(U) \cap p_2^{-1}(V)$ under p_1. But $p_1^{-1}(U) \cap p_2^{-1}(V)$ can be identified with the underlying space of the prescheme $U \times_{\mathbf{Z}} V$ **I | 139**
(3.2.7), and is thus an affine scheme (3.2.2); since $\Gamma_f(X)$ is closed in $X \times_{\mathbf{Z}} Y$ (5.4.3), $\Gamma_f(X) \cap p_1^{-1}(U) \cap p_2^{-1}(V)$ is closed in $U \times_{\mathbf{Z}} V$, and so the prescheme induced by the subprescheme of $X \times_{\mathbf{Z}} Y$ associated to Γ_f (4.2.1) on the open subset $\Gamma_f(X) \cap p_1^{-1}(U) \cap p_2^{-1}(V)$ of its underlying space is a closed subprescheme of an affine scheme, and thus an affine scheme (4.2.3). The proposition then follows from the fact that Γ_f is an immersion. □

Example (5.5.11). — The prescheme from Example (2.3.2) ("the projective line over a field K") is *separated*, because, for the cover (X_1, X_2) of X by affine open subsets, $X_1 \cap X_2 = U_{12}$ is affine, and $\Gamma(U_{12}, \mathscr{O}_X)$, the ring of rational fractions of the form $f(s)/s^m$ with $f \in K[s]$, is generated by $K[s]$ and $1/s$, so the conditions of (5.5.6) are satisfied.

With the same choice of X_1, X_2, U_{12}, and U_{21} as in Example (2.3.2), now take u_{12} to be the isomorphism which sends $f(s)$ to $f(t)$; we now obtain, by gluing, a *non-separated integral* prescheme X, because the first condition of (5.5.6) is satisfied, but not the second. It is immediate here that $\Gamma(X, \mathscr{O}_X) \to \Gamma(X_1, \mathscr{O}_X) = K[s]$ is an isomorphism; the inverse isomorphism defines a morphism $f : X \to \operatorname{Spec}(K[s])$ that is surjective, and for every $y \in \operatorname{Spec}(K[s])$ such that $\mathfrak{j}_y \neq (0)$, $f^{-1}(y)$ consists of a single point, but for $\mathfrak{j}_y = (0)$, $f^{-1}(y)$ consists of *two* distinct points (we say that X is the "affine line over K with the point 0 doubled").

We can also give examples where *neither* of the two conditions of (5.5.6) are satisfied. First note that, in the prime spectrum Y of the ring $A = K[s,t]$ of polynomials in two indeterminates over a field K, the open subset U given by the union of $D(s)$ and $D(t)$ is *not an affine open subset.* Indeed, if z is a section of $\mathscr{O}_Y$ over U, there exist two integers $m, n \geqslant 0$ such that $s^m z$ and $t^n z$ are the restrictions of polynomials in s and t to U (1.4.1), which is clearly possible only if the section z extends to a section over the whole of Y, identified with a polynomial in s and t. If U were an affine open subset, then the injection morphism $U \to Y$ would be an isomorphism (1.7.3), which is a contradiction, since $U \neq Y$.

With the above in mind, take two affine schemes Y_1 and Y_2, prime spectra of the rings $A_1 = K[s_1, t_2]$ and $A_2 = K[s_2, t_2]$ (respectively); take $U_{12} = D(s_1) \cup D(t_1)$ and $U_{21} = D(s_2) \cup D(t_2)$, and take u_{12} to be the restriction of an isomorphism $Y_2 \to Y_1$ to U_{21} corresponding to the isomorphism of rings that sends $f(s_1, t_1)$ to $f(s_2, t_2)$; we then have an example where the conditions of (5.5.6) are not satisfied (the integral prescheme thus obtained is called "the affine plane over K with the point 0 doubled").

Remark (5.5.12). — Given some property **P** of morphisms of preschemes, consider the following propositions.

(i) *Every closed immersion has property **P**.*
(ii) *The composition of any two morphisms that both have property **P** also has property **P**.*
(iii) *If $f : X \to X'$ and $g : Y \to Y'$ are S-morphisms that have property **P**, then $f \times_S g$ has property **P**.* **I | 140**
(iv) *If $f : X \to Y$ is an S-morphism that has property **P**, then every S'-morphism $f_{(S')}$ obtained by an extension $S' \to S$ of the base prescheme also has property **P**.*
(v) *If the composition $g \circ f$ of two morphisms $f : X \to Y$ and $g : Y \to Z$ has property **P**, and g is separated, then f has property **P**.*
(vi) *If a morphism $f : X \to Y$ has property **P**, then so too does f_{red}* (5.1.5).

If we suppose that (i) and (ii) are both true, then (iii) and (iv) are *equivalent*, and (v) and (vi) are *consequences* of (i), (ii), and (iii).

The first claim has already been shown (3.5.1). Consider the factorisation (5.3.13) $X \xrightarrow{\Gamma_f} X \times_Z Y \xrightarrow{p_2} Y$ of f; the equation $p_2 = (g \circ f) \times_Z 1_Y$ shows that, if $g \circ f$ has property **P**, then so too does p_2, by (iii); if g is separated, then Γ_f is a closed immersion (5.4.3), and so also has property **P**, by (i); finally, by (ii), f has property **P**.

Finally, consider the commutative diagram

$$\begin{tikzcd} X_{\mathrm{red}} \arrow[r, "f_{\mathrm{red}}"] \arrow[d] & Y_{\mathrm{red}} \arrow[d] \\ X \arrow[r, "f"] & Y, \end{tikzcd}$$

where the vertical arrows are the closed immersions (5.1.5), and thus have property **P**, by (i). The hypothesis that f has property **P** implies, by (ii), that $X_{\mathrm{red}} \xrightarrow{f_{\mathrm{red}}} Y_{\mathrm{red}} \to Y$ has property **P**; finally, since a closed immersion is separated (5.5.1, i), f_{red} has property **P**, by (v).

Note that, if we consider the propositions

(i') *Every immersion has property* ***P***;

(v') *If $g \circ f$ has property* ***P****, then so too does f*;

then the above arguments show that (v') is a consequence of (i'), (ii), and (iii).

(5.5.13). Note that (v) and (vi) are again consequences of (i), (iii), and

(ii') *If $j : X \to Y$ is a closed immersion, and $g : Y \to Z$ is a morphism that has property* ***P****, then $g \circ j$ has property* ***P***.

Similarly, (v') is a consequence of (i'), (iii), and

(ii") *If $j : X \to Y$ is an immersion, and $g : Y \to Z$ is a morphism that has property* ***P****, then $g \circ j$ has property* ***P***.

This follows immediately from the arguments of (5.5.12).

§6. Finiteness conditions

6.1. Noetherian and locally Noetherian preschemes.

Definition (6.1.1). — We say that a prescheme X is Noetherian (resp. locally Noetherian) if it is a finite union (resp. union) of affine open V_α in such a way that the ring of the of the induced scheme on each of the V_α is Noetherian.

It follows immediately from (1.5.2) that, if X is locally Noetherian, then the structure sheaf $\mathcal{O}_X$ is a *coherent sheaf of rings*, the question being a local one. Every *quasi-coherent $\mathcal{O}_X$-submodule* (resp. quasi-coherent quotient $\mathcal{O}_X$-module) of a *coherent $\mathcal{O}_X$*-module $\mathcal{F}$ is *coherent*, as the question is once again a local one, and it suffices to apply (1.5.1), (1.4.1), and (1.3.10), combined with the fact that a submodule (resp. quotient module) of a module of finite type over a Noetherian ring is of finite type. In particular, every *quasi-coherent sheaf of ideals* of $\mathcal{O}_X$ is *coherent*. **I** | 141

If a prescheme X is a finite union (resp. union) of open subsets W_λ in such a way that the preschemes induced on the W_λ are Noetherian (resp. locally Noetherian), it is clear that X is Noetherian (resp. locally Noetherian).

Proposition (6.1.2). — *For a prescheme X to be Noetherian, it is necessary and sufficient for it to be locally Noetherian and have a quasi-compact underlying space. The underlying space itself is then also Noetherian.*

Proof. The first claim follows immediately from the definitions and (1.1.10, ii). The second follows from (1.1.6) and the fact that every space that is a finite union of Noetherian subspaces is itself Noetherian (**0**, 2.2.3). ☐

Proposition (6.1.3). — *Let X be an affine scheme given by a ring A. The following conditions are equivalent:* (a) *X is Noetherian;* (b) *X is locally Noetherian;* (c) *A is Noetherian.*

Proof. The equivalence between (a) and (b) follows from (6.1.2) ant the fact that the underlying space of every affine scheme is quasi-compact (1.1.10); it is furthermore clear that (c) implies (a). To see that (a) implies (c), we remark that there is a finite cover (V_i) of X by affine open subsets such that the ring A_i of the prescheme induced on V_i is Noetherian. So let $(\mathfrak{a}_n)$ be an increasing sequence of ideals of A; by a canonical bijective correspondence, there is a corresponding sequence $(\widetilde{\mathfrak{a}}_n)$ of sheaves of ideals in $\widetilde{A} = \mathcal{O}_X$; to see that the sequence $(\mathfrak{a}_n)$ is stable (?), it suffices to prove that the sequence $(\widetilde{\mathfrak{a}}_n)$ is. But the restriction $\widetilde{\mathfrak{a}}_n|V_i$ is a quasi-coherent sheaf of ideals in $\mathcal{O}_X|V_i$, being the inverse image of $\widetilde{\mathfrak{a}}_n$ under the canonical injection $V_i \to X$ (**0**, 5.1.4); $\widetilde{\mathfrak{a}}_n|V_i$ is thus of the form $\widetilde{\mathfrak{a}}_{ni}$, where $\mathfrak{a}_{ni}$ is an ideal of A_i (1.3.7). Since A_i is Noetherian, the sequence $(\mathfrak{a}_{ni})$ is stable for all i, whence the proposition. ☐

We note that the above argument proves also that *if X is a Noetherian prescheme, then every increasing sequence of coherent sheaves of ideals of $\mathcal{O}_X$ is stable (?).*

Proposition (6.1.4). — *Every subprescheme of a Noetherian (resp. locally Noetherian) prescheme is Noetherian (resp. locally Noetherian).*

Proof. If suffices to give a proof for a Noetherian prescheme X; further, by definition (6.1.1), we can also restrict to the case where X is an affine scheme. Since every subprescheme of X is a closed subprescheme of a prescheme induced on an open subset (4.1.3), we can restrict to the case of a subprescheme Y, either closed or induced on an open subset of X. The proof in the case where Y is closed is immediate, since if A is the ring of X, we know that Y is an affine scheme given by the ring $A/\mathfrak{J}$, where $\mathfrak{J}$ is an ideal of A (4.2.3); since A is Noetherian (6.1.3), so too is $A/\mathfrak{J}$.

Now suppose that Y is open in X; the underlying space of Y is Noetherian (6.1.2), hence quasi-compact, and thus a finite union of open subsets $D(f_i)$ ($f_i \in A$); everything reduces to showing the proposition in the case where $Y = D(f)$ with $f \in A$. But then Y is an affine scheme whose ring is isomorphic to A_f (1.3.6); since A is Noetherian (6.1.3), so too is A_f. □

(6.1.5). We note that the *product* of two Noetherian S-preschemes is not necessarily Noetherian, even if the preschemes are affine, since the tensor product of two Noetherian algebras in not necessarily a Noetherian ring (cf. (6.3.8)).

Proposition (6.1.6). — *If X is a Noetherian prescheme, the nilradical $\mathscr{N}_X$ of $\mathscr{O}_X$ is nilpotent.*

Proof. We can in fact cover X with a finite number of affine open subsets U_i, and it suffices to prove that there exists whole numbers n_i such that $(\mathscr{N}_X|U_i)^{n_i} = 0$; if n is the largest of the n_i, then we will have $\mathscr{N}_X^n = 0$. We can thus restrict to the case where $X = \operatorname{Spec}(A)$ is affine, with A a Noetherian ring; by (5.1.1) and (1.3.13), it suffices to observe that the nilradical of A is nilpotent ([**Sam53b**, p. 127, cor. 4]). □

Corollary (6.1.7). — *Let X be a Noetherian prescheme; for X to be an affine scheme, it is necessary and sufficient that X_{red} be affine.*

Proof. This follows from (6.1.6) and (5.1.10). □

Lemma (6.1.8). — *Let X be a topological space, x a point of X, and U an open neighbourhood of x having only a finite number of irreducible components. Then there exists a neighbourhood V of x such that every open neighbourhood of x contained in V is connected.*

Proof. Let U_i ($1 \leqslant i \leqslant m$) be the irreducible components of U not containing x; the complement (in U) of the union of the U_i is an open neighbourhood V of X inside U, and thus so too in X; it is also, incidentally, the complement (in X) of the union of the irreducible components of X that do not contain x (**0**, 2.1.6). So let W be an open neighbourhood of X contained in V. The irreducible components of W are the intersections of W with the irreducible components of U (**0**, 2.1.6), so these components contain x; since they are connected, so too is W. □

Corollary (6.1.9). — *A locally Noetherian topological space is locally connected (which implies, amongst other things, that its connected components are open).*

Proposition (6.1.10). — *Let X be a locally Noetherian topological space. The following conditions are equivalent.*

(a) *The irreducible components of X are open.*
(b) *The irreducible components of X are exactly its connected components.*
(c) *The connected components of X are irreducible.*
(d) *Two distinct irreducible components of X have an empty intersection.*

Finally, if X is a prescheme, then these conditions are also equivalent to

(e) *For every $x \in X$, $\operatorname{Spec}(\mathscr{O}_x)$ is irreducible (or, in other words, the nilradical of $\mathscr{O}_x$ is prime).*

Proof. It is immediate that (a) implies (b), because an irreducible space is connected, and (a) implies that the irreducible components of X are the sets that are both open and closed. It is trivial that (b) implies (c); conversely, a closed set F containing a connected component C of X, with C distinct from F, cannot be irreducible, because not being connected means that F is the union of two disjoint nonempty sets that are both open and closed in F, and thus closed in X; as a result, (c) implies (b). We immediately conclude from this that (c) implies (d), since two distinct connected components have no points in common. **I** | 143

We have not yet used the fact that X is locally Noetherian. Suppose now that this is indeed the case, and we will show that (d) implies (a): by (**0**, 2.1.6), we can restrict ourselves to the case where the space X is Noetherian, and so has only a finite number of irreducible components. Since they are closed and pairwise disjoint, they are open.

Finally, the equivalence between (d) and (e) holds true even without the assumption that the underlying space of the prescheme X is locally Noetherian. We can in fact restrict ourselves to the case where $X = \operatorname{Spec}(A)$ is affine, by (**0**, 2.1.6); to say that x is contained in only one single irreducible component of X is to say that $\mathfrak{j}_x$ contains only one single minimal ideal of A (1.1.14), which is equivalent to saying that $\mathfrak{j}_x\mathscr{O}_x$ contains only one single minimal ideal of $\mathscr{O}_x$, whence the conclusion. □

Corollary (6.1.11). — *Let X be a locally Noetherian space. For X to be irreducible, it is necessary and sufficient that X be connected and nonempty, and that any two distinct irreducible components of X have an empty intersection. If X is a prescheme, this latter condition is equivalent to asking that* $\operatorname{Spec}(\mathscr{O}_x)$ *be irreducible for all $x \in X$.*

Proof. The second claim has already been shown in (6.1.10); the only thing thus remaining to show is that the conditions in the first claim are sufficient. But by (6.1.10), these conditions imply that the irreducible components of X are exactly its connected components, and since X is connected and nonempty, it is irreducible. □

Corollary (6.1.12). — *Let X be a locally Noetherian prescheme. For X to be integral, it is necessary and sufficient that X be connected and that $\mathscr{O}_x$ be integral for all $x \in X$.*

Proposition (6.1.13). — *Let X be a locally Noetherian prescheme, and let $x \in X$ be a point such that the nilradical $\mathscr{N}_x$ of $\mathscr{O}_x$ is prime (resp. such that $\mathscr{O}_x$ is reduced, resp. integral); then there exists an open neighbourhood U of x that is irreducible (resp. reduced, resp. integral).*

Proof. It suffices to consider two cases: where $\mathscr{N}_x$ is prime, and where $\mathscr{N}_x = 0$; the third hypotheses is a combination of the first two. If $\mathscr{N}_x$ is prime, then x belongs to only one single irreducible component Y of X (6.1.10); the union of the irreducible components of X that do not contain x is closed (the set of these components being locally finite), and the complement U of this union is thus open and contained in Y, and thus irreducible (**0**, 2.1.6) If $\mathscr{N}_x = 0$, we also have $\mathscr{N}_y = 0$ for any y in a neighbourhood of x, because $\mathscr{N}$ is quasi-coherent (5.1.1), and thus coherent, since X is locally Noetherian, and the conclusion then follows from (**0**, 5.2.2). □

6.2. Artinian preschemes.

Definition (6.2.1). — We say that a prescheme is *Artinian* if it is affine, and given by an Artinian ring.

Proposition (6.2.2). — *Given a prescheme X, the following conditions are equivalent:* **I** | 144

(a) *X is an Artinian scheme;*
(b) *X is Noetherian and its underlying space is discrete;*
(c) *X is Noetherian and the points of its underlying space are closed* (the T_1 condition).

When any of the above hold, the underlying space of X is finite, and the ring A of X is the direct sum of local (Artinian) rings of points of X.

Proof. We know that (a) implies the last claim ([**SZ60**, p. 205, th. 3]), so every prime ideal of A is thus maximal and is the inverse image of a maximal ideal of one of the local components of A, and so the space X is finite and discrete; (a) thus implies (b), and (b) clearly implies (c). To see that (c) implies (a), we first show that X is then finite; we can indeed restrict to the case where X is affine, and we know that a Noetherian ring whose prime ideals are all maximal is Artinian ([**SZ60**, p. 203]), whence our claim. The underlying space X is then discrete, the topological sum of a finite number of points x_i, and the local rings $\mathscr{O}_{x_i} = A_i$ are Artinian; it is clear that X is isomorphic to the prime spectrum affine scheme of the ring A (the direct sum of the A_i) (1.7.3). □

6.3. Morphisms of finite type.

Definition (6.3.1). — We say that a morphism $f : X \to Y$ is *of finite type* if Y is the union of a family (V_α) of affine open subsets having the following property:

(P) $f^{-1}(V_\alpha)$ is a finite union of affine open subsets $U_{\alpha i}$ that are such that each ring $A(U_{\alpha i})$ is an algebra of finite type over $A(V_\alpha)$.

We then say that X is a prescheme of finite type over Y, or a Y-prescheme of finite type.

Proposition (6.3.2). — *If $f : X \to Y$ is a morphism of finite type, then every affine open subset W of Y satisfies property* (P) *of* (6.3.1).

We first show

Lemma (6.3.2.1). — *If $T \subset Y$ is an affine open subset, satisfying property* (P)*, then, for every $g \in A(T)$, $D(g)$ also satisfies property* (P).

Proof. By hypothesis, $f^{-1}(T)$ is a finite union of affine open subsets Z_j, that are such that $A(Z_j)$ is an algebra of finite type over $A(T)$; let $\varphi_j : A(T) \to A(Z_j)$ be the homomorphism of rings corresponding to the restriction of f to Z_j (2.2.4), and set $g_j = \varphi_j(g)$; we then have $f^{-1}(D(g)) \cap Z_j = D(g_j)$ (1.2.2.2). But $A(D(g_j)) = A(Z_j)_{g_j} = A(Z_j)[1/g_j]$ is of finite type over $A(Z_j)$, and *a fortiori* over $A(T)$ by the hypothesis, and so also over $A(D(g)) = A(T)[1/g]$, which proves the lemma. □

Proof. With the above lemma, since W is quasi-compact (1.1.10), there exists a finite covering of W by sets of the form $D(g_i) \subset W$, where each g_i belongs to a ring $A(V_{\alpha(i)})$. Each $D(g_i)$, being quasi-compact, is a finite union of sets $D(h_{ik})$, where $h_{ik} \in A(W)$; if $\varphi_i : A(W) \to A(D(g_i))$ is the canonical map, then we have $D(h_{ik}) = D(\varphi_i(h_{ik}))$ by (1.2.2.2). By (6.3.2.1), each of the $f^{-1}(D(h_{ik}))$ admits a finite covering by affine open subsets U_{ijk}, that are such that the $A(U_{ijk})$ are algebras of finite type over $A(D(h_{ik})) = A(W)[1/h_{ik}]$, whence the proposition. □ **Err**$_{\mathrm{II}}$

We can thus say that the notion of a prescheme of finite type over Y is *local on* Y. I | 145

Proposition (6.3.3). — *Let X and Y be affine schemes; for X to be of finite type over Y, it is necessary and sufficient that $A(X)$ be an algebra of finite type over $A(Y)$.*

Proof. Since the condition clearly suffices, we show that it is necessary. Set $A = A(Y)$ and $B = A(X)$; by (6.3.2), there exists a finite affine open cover (V_i) of X such that each of the rings $A(V_i)$ is an A-algebra of finite type. Further, since the V_i are quasi-compact, we can cover each of them with a finite number of open subsets of the form $D(g_{ij}) \subset V_i$, where $g_{ij} \in B$; if φ_i is a homomorphism $B \to A(V_i)$ that corresponds to the canonical injection $V_i \to X$, then we have $B_{g_{ij}} = (A(V_i))_{\varphi_i(g_{ij})} = A(V_i)[1/\varphi_i(g_{ij})]$, so $B_{g_{ij}}$ is an A-algebra of finite type. We can thus restrict to the case where $V_i = D(g_i)$ with $g_i \in B$. By hypothesis, there exists a finite subset F_i of B and an integer $n_i \geqslant 0$ such that B_{g_i} is the algebra generated over A by the elements $b_i/g_i^{n_i}$, where the b_i run over all of F_i. Since there are only finitely many of the g_i, we can assume that all the n_i are equal to the same integer n. Further, since the $D(g_i)$ form a cover of X, the ideal generated in B by the g_i is equal to B, or, in other words, there exist $h_i \in B$ such that $\sum_i h_i g_i = 1$. So let F be the finite subset of B given by the union of the F_i, the set of the g_i, and the set of the h_i; we will show that the subring $B' = A[F]$ of B is equal to B. By hypothesis, for every $b \in B$ and every i, the canonical image of b in B_{g_i} is of the form $b_i'/g_i^{m_i}$, where $b_i' \in B'$; by multiplying the b_i' by suitable powers of the g_i, we can again assume that all the m_i are equal to the same integer m. By the definition of the ring of fractions, there is thus an integer N (dependant on b) such that $N \geqslant m$ and $g_i^N b \in B'$ for all i; but, *in the ring* B', the g_i^N generate the ideal B', because the g_i do (and the h_i belong to B'); there are thus $c_i \in B'$ such that $\sum_i c_i g_i^N = 1$, whence $b = \sum_i c_i g_i^N b \in B'$, Q.E.D. □

Proposition (6.3.4). —

(i) *Every closed immersion is of finite type.*
(ii) *The composition of any two morphisms of finite type is of finite type.*
(iii) *If $f : X \to X'$ and $g : Y \to Y'$ are S-morphisms of finite type, then $f \times_S g$ is of finite type.*
(iv) *If $f : X \to Y$ is an S-morphism of finite type, then $f_{(S')}$ is of finite type for any extension $g : S' \to S$ of the base prescheme.*
(v) *If the composition $g \circ f$ of two morphisms is of finite type, with g separated, then f is of finite type.*
(vi) *If a morphism f is of finite type, then f_{red} is of finite type.*

Proof. By (5.5.12), it suffices to prove (i), (ii), and (iv).

To show (i), we can restrict to the case of a canonical injection $X \to Y$, with X being a closed subprescheme of Y; further (6.3.2), we can assume that Y is affine, in which case X is also affine (4.2.3) and its ring is isomorphic to a quotient ring $A/\mathfrak{J}$, where A is the ring of Y and $\mathfrak{J}$ is an ideal of A; since $A/\mathfrak{J}$ is of finite type over A, the conclusion follows.

Now we show (ii). Let $f : X \to Y$ and $g : Y \to Z$ be two morphisms of finite type, and let U be an affine open subset of Z; g^{-1} admits a finite covering by affine open subsets V_i that are such that each $A(V_i)$ is an algebra of finite type over $A(U)$ (6.3.2); similarly. each of the f^{-1} admits a finite cover by affine open subsets W_{ij} that are such that each $A(W_{ij})$ is an algebra of finite type over $A(V_i)$, and so also an algebra of finite type over $A(U)$, whence the conclusion. I | 146

Finally, to show (iv), we can restrict to the case where $S = Y$; then $f_{(S')}$ is also equal to $f_{Y_{(S')}}$, where we consider f as a Y-morphism, and the base extension is $Y_{(S')} \to Y$ (3.3.9). So let p and q be the projections $X_{(S')} \to X$ and $X_{(S')} \to S'$. Let V be an affine open subset of S; $f^{-1}(V)$ is a finite union of affine open subsets W_i, each of which is such that $A(W_i)$ is an algebra of finite type over $A(V)$ (6.3.2). Let V' be an affine open subset of S' contained in $g^{-1}(V)$; since $f \circ p = g \circ q$, $q^{-1}(V')$ is contained in the union of the $p^{-1}(W_i)$; on the other hand, the intersection $p^{-1}(W_i) \cap q^{-1}(V')$ can be identified with the product $W_i \times VV'$ (3.2.7), which is an affine scheme whose ring is isomorphic to $A(W_i) \otimes_{A(V)} A(V')$ (3.2.2); this ring is, by hypothesis, an algebra of finite type over $A(V')$, which proves the proposition. □

Corollary (6.3.5). — *Let $f : X \to Y$ be an immersion morphism. If the underlying space of Y (resp. X) is locally Noetherian (resp. Noetherian), then f is of finite type.*

Proof. We can always assume that Y is affine (6.3.2); if the underlying space of Y is locally Noetherian, then we can further assume that it is Noetherian, and then the underlying space of X, which is a subspace, is also Noetherian. In other words, we can assume that Y is affine and that the underlying space of X is Noetherian; there then exists a covering of X by a finite number of affine open subsets $D(g_i) \subset Y$, where $g_i \in A(Y)$, that are such that the $X \cap D(g_i)$ are closed in $D(g_i)$ (and thus affine schemes (4.2.3)), because X is locally closed in Y (4.1.3). Then $A(X \cap D(g_i))$ is an algebra of finite type over $A(D(g_i))$, by (6.3.4, i) and (6.3.3), and $A(D(g_i)) = A(Y)_{g_i} = A(Y)[1/g_i]$ is of finite type over $A(Y)$, which finishes the proof. □

Corollary (6.3.6). — *Let $f : X \to Y$ and $g : Y \to Z$ be morphisms. If $g \circ f$ if of finite type, with either X Noetherian or $X \times_Z Y$ locally Noetherian, then f is of finite type.*

Proof. This follows immediately from the proof of (5.5.12) and from (6.3.5) applied to the immersion morphism Γ_f. □

Proposition (6.3.7). — *Let $f : X \to Y$ be a morphism of finite type; if Y is Noetherian (resp. locally Noetherian), then X is Noetherian (resp. locally Noetherian).*

Proof. We can restrict to proving the proposition for when Y is Noetherian. Then Y is a finite union of affine open subsets V_i that are such that the $A(V_i)$ are Noetherian rings. By (6.3.2), each of the $f^{-1}(V_i)$ is a finite union of affine open subsets W_{ij} that are such that the $A(W_{ij})$ are algebras of finite type over $A(V_i)$, and thus Noetherian rings; this proves that X is Noetherian. □

Corollary (6.3.8). — *Let X be a prescheme of finite type over S. For every base extension $S' \to S$ with S' Noetherian (resp. locally Noetherian), $X_{(S')}$ is Noetherian (resp. locally Noetherian).*

Proof. This follows from (6.3.7), since $X_{(S')}$ is of finite type over S' by (6.3.4, iv). □

We can also says that, for a product $X \times_S Y$ of S-preschemes, if *one* of the factors X or Y is *of finite type* over I | 147
S and *the other is Noetherian* (resp. *locally Noetherian*), then $X \times_S Y$ is *Noetherian* (resp. *locally Noetherian*).

Corollary (6.3.9). — *Let X be a prescheme of finite type over a locally Noetherian prescheme S. Then every S-morphism $f : X \to Y$ is of finite type.*

Proof. In fact, we can assume that S is Noetherian; if $\varphi : X \to S$ and $\psi : Y \to S$ are the structure morphisms, then we have $\varphi = \psi \circ f$, and X is Noetherian by (6.3.7); f is thus of finite type by (6.3.6). □

Proposition (6.3.10). — *Let $f : X \to Y$ be a morphism of finite type. For f to be surjective, it is necessary and sufficient that, for every* algebraically closed *field Ω, the map $X(\Omega) \to Y(\Omega)$ that corresponds to f (3.4.1) be surjective.*

Proof. The condition suffices, as we can see by considering, for all $y \in Y$, an algebraically closed extension Ω of $k(y)$, and the commutative diagram

$$\begin{tikzcd} X \arrow[dd, "f"'] & \\ & \mathrm{Spec}(\Omega) \arrow[ul] \arrow[dl] \\ Y & \end{tikzcd}$$

(cf. (3.5.3)). Conversely, suppose that f is surjective, and let $g : \{\xi\} = \mathrm{Spec}(\Omega) \to Y$ be a morphism, where Ω is an algebraically closed field. If we consider the diagram

$$\begin{tikzcd} X \arrow[d, "f"'] & X_{(\Omega)} \arrow[l] \arrow[d, "f_{(\Omega)}"] \\ Y & \mathrm{Spec}(\Omega), \arrow[l] \end{tikzcd}$$

then it suffices to show that there exists a *rational point over* Ω in $X_{(\Omega)}$ ((3.3.14), (3.4.3), and (3.4.4)). Since f is surjective, $X_{(\Omega)}$ is nonempty (3.5.10), and since f is of finite type, so too is $f_{(\Omega)}$ (6.4.3, iv); thus $X_{(\Omega)}$ contains a nonempty affine open subset Z such that $A(Z)$ is an non-null algebra of finite type over Ω. By Hilbert's Nullstellensatz [**Zar47**], there exists an Ω-homomorphism $A(Z) \to \Omega$, and thus a section of $X_{(\Omega)}$ over $\operatorname{Spec}(\Omega)$, which proves the proposition. □

6.4. Algebraic preschemes.

Definition (6.4.1). — Given a field K, we define an *algebraic K-prescheme* to be a prescheme X of finite type over K; K is called the base field of X. If in addition X is a scheme *(or if X is a K-*scheme*, which is equivalent* (5.5.8)*)*, we say that X is an *algebraic K-scheme.*

Every algebraic K-prescheme is *Noetherian* (6.3.7).

Proposition (6.4.2). — *Let X be an algebraic K-prescheme. For a point $x \in X$ to be closed, it is necessary and sufficient that $k(x)$ be an algebraic extension of K of finite degree.*

Proof. We can assume that X is affine, with the ring A of X being a K-algebra of finite type. Indeed, the affine open subsets U of X such that $A(U)$ is a K-algebra of finite type form a finite cover of X (6.3.1). The closed points of X are thus the points such that $\mathfrak{j}_x$ is a maximal ideal of A, or in other words, such that $A/\mathfrak{j}_x$ is a field **I | 148**
(necessarily equal to $k(x)$). Since $A/\mathfrak{j}_x$ is a K-algebra of finite type, we see that if x is closed, then $k(x)$ is a field that is an algebra of finite type over K, and so necessarily a K-algebra of *finite rank* [**Zar47**]. Conversely, if $k(x)$ is of finite rank over K, then so is $A/\mathfrak{j}_x \subset k(x)$, and since every integral ring that is also a K-algebra of finite rank is a field, we have that $A/\mathfrak{j}_x = k(x)$, and hence x is closed. □

Corollary (6.4.3). — *Let K be an algebraically-closed field, and X an algebraic K-prescheme; the closed points of X are then the rational points over K* (3.4.4) *and can be canonically identified with the points of X with values in K.*

Proposition (6.4.4). — *Let X be an algebraic prescheme over a field K. The following properties are equivalent.*

- (a) *X is Artinian.*
- (b) *The underlying space of X is discrete.*
- (c) *The underlying space of X has only a finite number of closed points.*
- (c') *The underlying space of X is finite.*
- (d) *The points of X are closed.*
- (e) *X is isomorphic to* $\operatorname{Spec}(A)$*, where A is a K-algebra of finite rank.*

Proof. Since X is Noetherian, it follows from (6.2.2) that the conditions (a), (b), and (d) are equivalent, and imply (c) and (c′); it is also clear that (e) implies (a). It remains to see that (c) implies (d) and (e); we can restrict to the case where X is affine. Then $A(X)$ is a K-algebra of finite type (6.3.3), and thus a Jacobson ring ([**CC**, p. 3-11 and 3-12]), in which there are, by hypothesis, only a finite number of maximal ideals. Since a finite intersection of prime ideals can only be a prime ideal if it is equal to one of the prime ideals being intersected, every prime ideal of $A(X)$ is thus maximal, whence (d). Further, we then know (6.2.2) that $A(X)$ is an Artinian K-algebra of finite type, and so necessarily of *finite rank* [**Zar47**]. □

(6.4.5). When the conditions of (6.4.4) are satisfied, we say that X is a scheme *finite over K* (cf. (**II**, 6.1.1)), or a *finite K-scheme*, of *rank* $[A:K]$, which we also denote by $\operatorname{rg}_K(X)$; if X and Y are finite K-schemes, we have

$$\operatorname{rg}_K(X \sqcup Y) = \operatorname{rg}_K(X) + \operatorname{rg}_K(Y), \tag{6.4.5.1}$$

$$\operatorname{rg}_K(X \times_K Y) = \operatorname{rg}_K(X)\operatorname{rg}_K(Y), \tag{6.4.5.2}$$

as a result of (3.2.2).

Corollary (6.4.6). — *Let X be a finite K-scheme. For every extension K' of K, $X \otimes_K K'$ as a finite K'-scheme, and its rank over K' is equal to the rank of X over K.*

Proof. If $A = A(X)$, then we have $[A \otimes_K K' : K'] = [A:K]$. □

Corollary (6.4.7). — *Let X be a scheme finite over a field K; we let $n = \sum_{x\in X}[k(X):K]_s$* (we recall that if K' is an extension of K, then $[K':K]_s$ is the *separable rank* of K' over k, the rank of the largest algebraic separable extension of K contained in K')*; then for every algebraically closed extension Ω of K, the underlying space of $X \otimes_K \Omega$ has* **I | 149**
exactly n points, which can be identified with the points of X with values in Ω.

Proof. We can clearly restrict to the case where the ring $A = A(X)$ is *local* (6.2.2); let $\mathfrak{m}$ be its maximal ideal, and $L = A/\mathfrak{m}$ its residue field, an algebraic extension of K. The points of X with values in Ω then correspond, bijectively, to the Ω-sections of $X \otimes_K \Omega$ ((3.4.1) and (3.3.14)), and also to the K-homomorphisms from L to Ω (1.7.3), whence the proposition (Bourbaki, *Alg.*, chap. V, §7, nº 5, prop. 8), taking (6.4.3) into account. □

(6.4.8). The number n defined in (6.4.7) is called the *separable rank* of A (or of X) over K, or also the *geometric number of points* of X; it is equal to the number of elements of $X(\Omega)_K$. It follows immediately from this definition that, for every extension K' of K, $X \otimes_K K'$ has the same geometric number of points as X. If we denote this number by $n(X)$, it is clear that, if X and Y are two schemes, finite over K, then

$$n(X \sqcup Y) = n(X) + n(Y). \tag{6.4.8.1}$$

Under the same hypotheses, we also have

$$n(X \times_K Y) = n(X)n(Y) \tag{6.4.8.2}$$

because of the interpretation of $n(X)$ as the number of elements of $X(\Omega)_K$ and Equation (3.4.3.1).

Proposition (6.4.9). — *Let K be a field, X and Y algebraic K-preschemes, $f : X \to Y$ a K-morphism, and Ω an algebraically closed extension of K of infinite transcendence degree over K. For f to be surjective, it is necessary and sufficient that the map $X(\Omega)_K \to Y(\Omega)_K$ that corresponds to f (3.4.1) be surjective.*

Proof. The necessity follows from (6.3.10), noting that f is necessarily of finite type (6.3.9). To see that the condition is sufficient, we argue as in (6.3.10), noting that, for every $y \in Y$, $k(y)$ is an extension of K of finite type, and so is K-isomorphic to a subfield of Ω. □

Remark (6.4.10). — We will see in chapter IV that the conclusion of (6.4.9) still holds without the hypothesis on the transcendence degree of Ω over K.

Proposition (6.4.11). — *If $f : X \to Y$ is a morphism of finite type, then, for every $y \in Y$, the fibre $f^{-1}(y)$ is an algebraic prescheme over the residue field $k(y)$, and for every $x \in f^{-1}(y)$, $k(x)$ is an extension of $k(x)$ of finite type.*

Proof. Since $f^{-1}(y) = X \otimes_Y k(y)$ (6.3.6), the proposition follows from (6.3.4, iv) and (6.3.3). □

Proposition (6.4.12). — *Let $f : X \to Y$ and $g : Y' \to Y$ be morphisms; set $X' = X \times_Y Y'$, and let $f' = f_{(Y')} : X' \to Y'$. Let $y' \in Y'$ and set $y = g(y')$; if the fibre $f^{-1}(y)$ is a finite algebraic scheme over $k(y)$, then the fibre $f'^{-1}(y)$ is a finite algebraic scheme over $k(y')$, and has the same rank and geometric number of points as $f^{-1}(y)$ does.*

Proof. Taking into account the transitivity of fibres (3.6.5), this follows immediately from (6.4.6) and (6.4.8). □

(6.4.13). Proposition (6.4.11) shows that the morphisms of finite type that correspond, intuitively, to the "algebraic families of algebraic varieties", with the points of Y playing the role of "parameters", which gives these morphisms **I** | 150
a "geometric" meaning. The morphisms which are not of finite type will show up in the following mostly in questions of "changing the base prescheme", by localisation or completion, for example.

6.5. Local determination of a morphism.

Proposition (6.5.1). — *Let X and Y be S-preschemes, with Y of finite type over S; let $x \in X$ and $y \in Y$ lie over the same point $s \in S$.*

(i) *If two S-morphisms $f = (\psi, \theta)$ and $f' = (\psi', \theta')$ from X to Y are such that $\psi(x) = \psi'(x) = y$, and the (local) $\mathcal{O}_s$-homomorphisms $\theta_x^\sharp$ and $\theta'^\sharp_x$ from $\mathcal{O}_y$ to $\mathcal{O}_x$ are identical, then f and f' agree on an open neighbourhood of x.*

(ii) *Suppose further that S is locally Noetherian. For every local $\mathcal{O}_s$-homomorphism $\varphi : \mathcal{O}_y \to \mathcal{O}_x$, there exists an open neighbourhood U of x in X, and an S-morphism $f = (\psi, \theta)$ from U to Y such that $\psi(x) = y$ and $\theta_x^\sharp = \varphi$.*

Proof.

(i) Since the question is local on S, X, and Y, we can assume that S, X, and Y are affine, given by rings A, B, and C (respectively), and with f and f' of the form $({}^a\varphi, \widetilde{\varphi})$ and $({}^a\varphi', \widetilde{\varphi}')$ (respectively), where φ and φ' are A-homomorphisms from C to B such that $\varphi^{-1}(\mathfrak{j}_x) = \varphi'^{-1}(\mathfrak{j}_x) = \mathfrak{j}_y$, and the homomorphisms φ_x and φ'_x from C_y to B_x, induced by φ and φ', are identical; we can further suppose that C is an A-algebra *of finite type*. Let c_i $(1 \leqslant i \leqslant n)$ be the generators of the A-algebra C, and set $b_i = \varphi(c_i)$ and $b'_i = \varphi'(c_i)$; by hypothesis, we have $b_i/1 = b'_i/1$ in the ring of fractions B_x $(1 \leqslant i \leqslant n)$. This implies that there exist elements $s_i \in B - \mathfrak{j}_x$ such that $s_i(b_i - b'_i) = 0$ for $1 \leqslant i \leqslant n$, and we can clearly assume that all the s_i are equal to a single element $g \in B - \mathfrak{j}_x$. From this, we conclude that we have $b_i/1 = b'_i/1$ for $1 \leqslant i \leqslant n$ in the ring of fractions B_g; if i_g is the canonical homomorphism $B \to B_g$, we then have $i_g \circ \varphi = i_g \circ \varphi'$; so the restrictions of f and f' to $D(g)$ are identical.

(ii) We can restrict to the situation as in (i), and further assume that the ring A is Noetherian. Let c_i ($1 \leqslant i \leqslant n$) be the generators of the A-algebra C, and let $\alpha : A[X_1, \ldots, X_n] \to C$ be the homomorphism of polynomial algebras that sends X_i to c_i for $1 \leqslant i \leqslant n$. Also let i_y be the canonical homomorphism $C \to C_y$, and consider the composite homomorphism

$$\beta : A[X_1, \ldots, X_n] \xrightarrow{\alpha} C \xrightarrow{i_y} C_y \xrightarrow{\varphi} B_x.$$

We denote by $\mathfrak{a}$ the kernel of β; since A is Noetherian, so too is $A[X_1, \ldots, X_n]$, and so $\mathfrak{a}$ admits a finite system of generators $Q_j(X_1, \ldots, X_n)$ ($1 \leqslant j \leqslant m$). Furthermore, each of the elements $\varphi(i_y(c_i))$ can be written in the form b_i/s_i, where $b_i \in B$ and $s_i \notin \mathfrak{j}_x$; we can further assume that all of the s_i are equal to a single element $g \in B - \mathfrak{j}_x$. With this, by hypothesis, we have $Q_j(b_1/g, \ldots, b_n/g) = 0$ in B_x; set **I | 151**

$$Q_j(X_1/T, \ldots, X_n/T) = P_j(X_1, \ldots, X_n, T)/T^{k_j}$$

where P_j is homogeneous of degree k_j. Then let $d_j = P_j(b_1, \ldots, b_n, g) \in B$. By hypothesis, we have $t_j d_j = 0$ for some $t_j \in B - \mathfrak{j}_x$ ($1 \leqslant j \leqslant m$), and we can clearly assume that all the t_j are equal to a single element $h \in B - \mathfrak{j}_x$; from this we conclude that $P_j(hb_1, \ldots, hb_n, hg) = 0$ for $1 \leqslant j \leqslant m$. With this, consider the homomorphism ρ from $A[X_1, \ldots, X_n]$ to the ring of fractions B_{hg} which sends X_i to hb_i/hg ($1 \leqslant i \leqslant n$); the image of $\mathfrak{a}$ under this homomorphism is 0, and is *a fortiori* the same as the image of the kernel $\alpha^{-1}(0)$ under ρ. So ρ factors as $A[X_1, \ldots, X_n] \xrightarrow{\alpha} C \xrightarrow{\gamma} B_{hg}$, with $\gamma(c_i) = hb_i/hg$, and it is clear that, if i_x is the canonical homomorphism $B_{hg} \to B_x$, then the diagram

(6.5.1.1)
$$\begin{tikzcd} C \arrow[r, "\gamma"] \arrow[d, "i_y"'] & B_{hg} \arrow[d, "i_x"] \\ C_y \arrow[r, "\varphi"] & B_x \end{tikzcd}$$

is commutative; we thus have $\varphi = \gamma_x$, and since φ is a local homomorphism, ${}^a\gamma(x) = y$; $f = ({}^a\gamma, \widetilde{\gamma})$ is thus an S-morphism from the neighbourhood $D(hg)$ of x to Y as claimed in the proposition.

□

Corollary (6.5.2). — *Under the hypotheses of* (6.5.1, ii)*, if, further, X is of finite type over S, then we can assume that the morphism f is of finite type.*

Proof. This follows from Corollary (6.3.6). □

Corollary (6.5.3). — *Suppose that the hypotheses of Proposition* (6.5.1, ii)*, and suppose further that Y is integral, and that φ is an injective homomorphism. Then we can assume that $f = ({}^a\gamma, \widetilde{\gamma})$, where γ is injective.*

Proof. Indeed, we can assume C to be integral (5.1.4), hence i_y injective; it then follows from the diagram (6.5.1.1) that γ is injective. □

Proposition (6.5.4). — *Let $f = (\psi, \theta) : X \to Y$ be a morphism of finite type, x a point of X, and $y = \psi(x)$.*

(i) *For f to be a local immersion at the point x* (4.5.1)*, it is necessary and sufficient that $\theta_x^\sharp : \mathcal{O}_y \to \mathcal{O}_x$ be surjective.*
(ii) *Assume further that Y is locally Noetherian. For f to be a local isomorphism at the point x* (4.5.2)*, it is necessary and sufficient that $\theta_x^\sharp$ be an isomorphism.*

Proof.

(ii) By (6.5.1), there exists an open neighbourhood V of Y and a morphism $g : V \to X$ such that $g \circ f$ (resp. $f \circ g$) is defined and agrees with the identity on a neighbourhood of x (resp. y), whence we can easily see that f is a local isomorphism.
(i) Since the question is local on X and Y, we can assume that X and Y are affine, given by rings A and B (respectively); we have $f = ({}^a\varphi, \widetilde{\varphi})$, where φ is a homomorphism of rings $B \to A$ that makes A a B-algebra of finite type; we have $\varphi^{-1}(\mathfrak{j}_x) = \mathfrak{j}_y$, and the homomorphism $\varphi_x : B_y \to A_x$ induced by φ is *surjective*. Let (t_i) ($1 \leqslant i \leqslant n$) be a system of generators of the B-algebra A; the hypothesis on φ_x implies that there exist $b_i \in B$ and some $c \in B - \mathfrak{j}_x$ such that, in the ring of fractions A_x, we have $t_i/1 = \varphi(b_i)/\varphi(c)$ for $1 \leqslant i \leqslant n$. Then (1.3.3) there exists some $a \in A - \mathfrak{j}_x$ such that, if we let $g = a\varphi(c)$, we also have $t_i/1 = a\varphi(b_i)/g$ *in the ring of fractions* A_g. With this, there exists, by hypothesis, a polynomial $Q(X_1, \ldots, X_n)$, with coefficients in the ring $\varphi(B)$, such that $a = Q(t_1, \ldots, t_n)$; let $Q(X_1/T, \ldots, X_n/T) = P(X_1, \ldots, X_n, T)/T^m$, where P is homogeneous of degree m. In the ring A_g, we **I | 152**

have

$$a/1 = a^m P(\varphi(b_1), \ldots, \varphi(b_n), \varphi(c))/g^m = a^m \varphi(d)/g^m$$

where $d \in B$. Since, in A_g, $g/1 = (a/1)(\varphi(c)/1)$ is invertible by definition, so too are $a/1$ and $\varphi(c)/1$, and we can thus write $a/1 = (\varphi(d)/1)(\varphi(c)/1)^{-m}$. From this we conclude that $\varphi(d)/1$ is also invertible in A_g. So let $h = cd$; since $\varphi(h)/1$ is invertible in A_g, the composite homomorphism $B \xrightarrow{\varphi} A \to A_g$ factors as $B \to B_h \xrightarrow{\gamma} A_g$ (**0**, 1.2.4). We will show that γ is *surjective*; it suffices to show that the image of B_h in A_g contains the $t_i/1$ and $(g/1)^{-1}$. But we have $(g/1)^{-1} = (\varphi(c)/1)^{m-1}(\varphi(d)/1)^{-1} = \gamma(c^m/h)$, and $a/1 = \gamma(d^{m+1}/h^m)$, so $(a\varphi(b_i))/1 = \gamma(b_i d^{m+1}/h^m)$, and since $t_i/1 = (a\varphi(b_i)/1)(g/1)^{-1}$, our claim is proved. The choice of h implies that $\psi(D(g)) \subset D(h)$, and we also know that the restriction of f to $D(g)$ is equal to $({}^a\gamma, \widetilde{\gamma})$; since γ is surjective, this restriction is a closed immersion of $D(g)$ into $D(h)$ (4.2.3).

□

Corollary (6.5.5). — *Let $f = (\psi, \theta) : X \to Y$ be a morphism of finite type. Assume that X is irreducible, and denote by x its generic point, and let $y = \psi(x)$.*

(i) *For f to be a local immersion at any point of X, it is necessary and sufficient that $\theta_x^\sharp : \mathcal{O}_y \to \mathcal{O}_x$ be surjective.*

(ii) *Assume further that Y is irreducible and locally Noetherian. For f to be a local isomorphism at any point of X, it is necessary and sufficient that y be the generic point of Y* (or, equivalently (**0**, 2.1.4), that f be a *dominant* morphism) *and that $\theta_x^\sharp$ be an isomorphism* (in other words, that f be *birational* (2.2.9)).

Proof. It is clear that (i) follows from (6.5.4, i), taking into account the fact that every nonempty open subset of X contains x; similarly, (ii) follows from (6.5.4, ii). □

6.6. Quasi-compact morphisms and morphisms locally of finite type.

Definition (6.6.1). — We say that a morphism $f : X \to Y$ is *quasi-compact* if the inverse image of any quasi-compact open subset of Y under f is quasi-compact.

Let $\mathfrak{B}$ be a base of the topology of Y consisting of quasi-compact open subsets (for example, affine open subsets); for f to be quasi-compact, it is necessary and sufficient that the inverse image of every set of $\mathfrak{B}$ under f be quasi-compact (or, equivalently, a *finite* union of affine open subsets), because every quasi-compact open subset of Y is a finite union of sets of $\mathfrak{B}$. For example, if X is *quasi-compact* and Y *affine*, then *every* morphism $f : X \to Y$ is quasi-compact: indeed, X is a finite union of affine open subsets U_i, and for every affine open subset V of Y, $U_i \cap f^{-1}(V)$ is affine (5.5.10), and so quasi-compact.

If $f : X \to Y$ is a quasi-compact morphism, it is clear that, for every open subset V of Y, the restriction of f to $f^{-1}(V)$ is a quasi-compact morphism $f^{-1}(V) \to V$. Conversely, if (U_α) is an open cover of Y, and $f : X \to Y$ a morphism such that the restrictions $f^{-1}(U_\alpha) \to U_\alpha$ are quasi-compact, then f is quasi-compact.

Definition (6.6.2). — We say that a morphism $f : X \to Y$ is *locally of finite type* if, for every $x \in X$, there exists an open neighbourhood U of x and an open neighbourhood $V \supset f(U)$ of y such that the restriction of f to U is a morphism of finite type from U to V. We then also say that X is a prescheme locally of finite type over Y, or a Y-prescheme locally of finite type. **I** | 153

It follows immediately from (6.3.2) that, if f is locally of finite type, then, for every open subset W of Y, the restriction of f to $f^{-1}(W)$ is a morphism $f^{-1}(W) \to W$ that is locally of finite type.

If Y is locally Noetherian and X locally of finite type over Y, then X is locally Noetherian thanks to (6.3.7).

Proposition (6.6.3). — *For a morphism $f : X \to Y$ to be of finite type, it is necessary and sufficient that it be quasi-compact and locally of finite type.*

Proof. The necessity of the conditions is immediate, given (6.3.1) and the remark following (6.6.1). Conversely, suppose that the conditions are satisfied, and let U be an affine open subset of Y, given by some ring A; for all $x \in f^{-1}(U)$, there is, by hypothesis, a neighbourhood $V(x) \subset f^{-1}(U)$ of x, and a neighbourhood $W(x) \subset U$ of $y = f(x)$ containing $f(V(x))$, and such that the restriction of f to $V(x)$ is a morphism $V(x) \to W(x)$ of finite type. Replacing $W(x)$ with a neighbourhood $W_1(x) \subset W(x)$ of x of the form $D(g)$ (with $g \in A$), and $V(x)$ with $V(x) \cap f^{-1}(W_1(x))$, we can assume that $W(x)$ is of the form $D(g)$, and thus of finite type over U (because its ring can be written as $A[1/g]$); so $V(x)$ is of finite type over U. Further, $f^{-1}(U)$ is quasi-compact by hypothesis, and so the finite union of open subsets $V(x_i)$, which finishes the proof. □

Proposition (6.6.4). —

(i) *An immersion $X \to Y$ is quasi-compact if it is closed, or if the underlying space of Y is locally Noetherian, or if the underlying space of X is Noetherian.*
(ii) *The composition of any two quasi-compact morphisms is quasi-compact.*
(iii) *If $f : X \to Y$ is a quasi-compact S-morphism, then so too is $f_{(S')} : X_{(S')} \to Y_{(S')}$ for any extension $g : S \to S'$ of the base prescheme.*
(iv) *If $f : X \to X'$ and $g : Y \to Y'$ are two quasi-compact S-morphisms, then $f \times_S g$ is quasi-compact.*
(v) *If the composition of any two morphisms $f : X \to Y$ and $g : Y \to Z$ is quasi-compact, and if either g is separated or the underlying space of X is locally Noetherian, then f is quasi-compact.*
(vi) *For a morphism f to be quasi-compact, it is necessary and sufficient that f_{red} be quasi-compact.*

Proof. We note that (vi) is evident because the property of being quasi-compact, for a morphism, depends only on the corresponding continuous map of underlying spaces. We will similarly prove the part of (v) corresponding to the case where the underlying space of X is locally Noetherian. Set $h = g \circ f$, and let U be a quasi-compact open subset of Y; $g(U)$ is quasi-compact (but not necessarily open) in Z, and so contained in a finite union of quasi-compact open subsets V_j (2.1.3), and $f^{-1}(U)$ is thus contained in the union of the $h^{-1}(V_j)$, which are quasi-compact subspaces of X, and thus Noetherian subspaces. We thus conclude (**0**, 2.2.3) that $f^{-1}(U)$ is a Noetherian space, and *a fortiori* quasi-compact.

To prove the other claims, it suffices to prove (i), (ii), and (iii) (5.5.12). But (ii) is evident, and (i) follows from (6.3.5) whenever the space Y is locally Noetherian or the space X is Noetherian, and is evident for a closed immersion. To show (iii), we can restrict to the case where $S = Y$ (3.3.11); let $f' = f_{(S')}$, and let U' **I** | 154
be a quasi-compact open subset of S'. For every $s' \in U'$, let T be an affine open neighbourhood of $g(s')$ in S, and let W be an affine open neighbourhood of s' contained in $U' \cap g^{-1}(T)$; it will suffice to show that $f'^{-1}(W)$ is quasi-compact; in other words, we can restrict to showing that, when S and S' are *affine*, the underlying space of $X \times_S S'$ is quasi-compact. But since X is then, by hypothesis, a finite union of affine open subsets V_j, $X \times_S S'$ is a union of the underlying spaces of the affine schemes $V_j \times_S S'$ ((3.2.2) and (3.2.7)), which proves the proposition. ☐

We note also that, if $X = X' \sqcup X''$ is the sum of two preschemes, a morphism $f : X \to Y$ is quasi-compact if and only if its restrictions to both X' and X'' are quasi-compact.

Proposition (6.6.5). — *Let $f : X \to Y$ be a quasi-compact morphism. For f to be dominant, it is necessary and sufficient that, for every generic point y of an irreducible component of Y, $f^{-1}(y)$ contain the generic point of an irreducible component of X.*

Proof. It is immediate that the condition is sufficient (even without assuming that f is quasi-compact). To see that it is necessary, consider an affine open neighbourhood U of y; $f^{-1}(U)$ is quasi-compact, and so a *finite* union of affine open subsets V_i, and the hypothesis that f be dominant implies that y belongs to the closure *in* U of one of the $f(V_i)$. We can clearly assume X and Y to be reduced; since the closure in X of an irreducible component of V_i is an irreducible component on X (**0**, 2.1.6), we can replace X by V_i, and Y by the closed reduced subprescheme of U that has $\overline{f(V_i)} \cap U$ as its underlying space (5.2.1), and we are thus led to proving the proposition when $X = \operatorname{Spec}(A)$ and $Y = \operatorname{Spec}(B)$ are affine and reduced. Since f is dominant, B is a subring of A (1.2.7), and the proposition then follows from the fact that every minimal prime ideal of B is the intersection of B with a minimal prime ideal of A (**0**, 1.5.8). ☐

Proposition (6.6.6). —

(i) *Every local immersion is locally of finite type.*
(ii) *If two morphisms $f : X \to Y$ and $g : Y \to Z$ are locally of finite type, then so too is $g \circ f$.*
(iii) *If $f : X \to Y$ is an S-morphism locally of finite type, then $f_{(S')} : X_{(S')} \to Y_{(S')}$ is locally of finite type for any extension $S' \to S$ of the base prescheme.*
(iv) *If $f : X \to X'$ and $g : Y \to Y'$ are S-morphisms locally of finite type, then $f \times_S g$ is locally of finite type.*
(v) *If the composition $g \circ f$ of two morphisms is locally of finite type, then f is locally of finite type.*
(vi) *If a morphism f is locally of finite type, then so too is f_{red}.*

Proof. By (5.5.12), it suffices to prove (i), (ii), and (iii). If $j : X \to Y$ is a local immersion then, for every $x \in X$, there is an open neighbourhood V of $j(x)$ in Y and an open neighbourhood U of x in X such that the restriction of j to U is a closed immersion $U \to V$ (4.5.1), and so this restriction is of finite type. To prove (ii), consider a point $x \in X$; by hypothesis, there is an open neighbourhood W of $g(f(x))$ and an open neighbourhood V of $f(x)$ such that $g(V) \subset W$ and such that V is of of finite type over W; furthermore, $f^{-1}(V)$ is locally of finite type over V (6.6.2), so there is an open neighbourhood U of x that is contained in $f^{-1}(V)$ and of is finite type over V; thus we have $g(f(U)) \subset W$, and that U is of finite type over W (6.3.4, ii). Finally, to prove (iii), we can restrict to the case where $Y = S$ (3.3.11); for every $x' \in X' = X_{(S')}$, let x be the image of x' in X, s the image of x in S, T an open neighbourhood of s, T' the inverse image of T in S', and U an open neighbourhood of x that is of finite type over T and whose image is contained in T; then $U \times_S T' = U \times_T T'$ is an open neighbourhood of x' (3.2.7) that is of finite type over T' (6.3.4, iv). ☐ **I | 155**

Corollary (6.6.7). — *Let X and Y be S-preschemes that are locally of finite type over S. If S is locally Noetherian, then $X \times_S Y$ is locally Noetherian.*

Proof. Indeed, X being locally of finite type over S means that it is locally Noetherian, and that $X \times_S Y$ is locally of finite type over X, and so $X \times_S Y$ is also locally Noetherian. ☐

Remark (6.6.8). — Proposition (6.3.10) and its proof extend immediately to the case where we suppose only that the morphism f is locally of finite type. Similarly, propositions (6.4.2) and (6.4.9) hold true when we suppose only that the preschemes X and Y in the claim are locally of finite type over the field K.

§7. Rational maps

7.1. Rational maps and rational functions.

(7.1.1). Let X and Y be preschemes, U and V dense open subsets of X, and f (resp. g) a morphism from U (resp. V) to Y; we say that f and g are *equivalent* if they agree on a dense open subset of $U \cap V$. Since a finite intersection of dense open subsets of X is a dense open subset of X, it is clear that this relation is an *equivalence relation.*

Definition (7.1.2). — Given preschemes X and Y, we define a rational map from X to Y to be an equivalence class of morphisms from a dense open subset of X to a dense open subset of Y, under the equivalence relation defined in (7.1.1). If X and Y are S-preschemes, we say that such a class is a rational S-map if there exists a representative of the class that is also an S-morphism. We define a rational S-section of X to be any rational S-map from S to X. We define a rational function on a prescheme X to be any rational X-section on the X-prescheme $X \otimes_{\mathbf{Z}} \mathbf{Z}[T]$ (where T is an indeterminate).

By an abuse of language, whenever we are discussing only S-preschemes, we will say “rational map” instead of “rational S-map” if no confusion may arise.

Let f be a rational map from X to Y, and U an open subset of X; if f_1 and f_2 are two morphisms belonging to the class of f, defined (respectively) on dense open subsets V and W of X, then the restrictions $f_1|(U \cap V)$ and $f_2|(U \cap W)$ agree on $U \cap V \cap W$, which is dense in U; the class f of morphisms thus defines a rational map from U to Y, called the *restriction of f to U*, and denoted by $f|U$.

If, to every S-morphism $f : X \to Y$, we take the corresponding rational S-map to which f belongs, we obtain a canonical map from $\operatorname{Hom}_S(X,Y)$ to the set of rational S-maps from X to Y. We denote by $\Gamma_{\mathrm{rat}}(X/Y)$ the set of rational Y-sections on X, and we thus have a canonical map $\Gamma(X/Y) \to \Gamma_{\mathrm{rat}}(X/Y)$. It is also clear that, if X and Y are S-preschemes, then the set of rational S-maps from X to Y is canonically identified with $\Gamma_{\mathrm{rat}}((X \times_S Y)/X)$ (3.3.14).

(7.1.3). It also follows from (7.1.2) and (3.3.14) that the rational functions on X are canonically identified with *equivalence classes of sections of the structure sheaf $\mathcal{O}_X$* over dense open subsets of X, where two such sections are equivalent if the agree on some dense open subset of X contained inside the intersection of the subsets on which they are defined. In particular, it follows that the rational functions on X form a *ring* $R(X)$.

(7.1.4). When X is an *irreducible* prescheme, every nonempty open subset of X is dense in X; so we can say that the nonempty open subsets of X are the *open neighbourhoods of the generic point* x of X. To say that two morphisms from nonempty open subsets of X to Y are equivalent thus means, in this case, that they have the *same germ* at the point x. In other words, the rational maps (resp. rational S-maps) $X \to Y$ are identified with the *germs of morphisms* (resp. *S-morphisms*) from nonempty open subsets of X to Y at the generic point x of X. In particular:

Proposition (7.1.5). — *If X is an irreducible prescheme, then the ring $R(X)$ of rational maps on X is canonically identified with the local ring $\mathcal{O}_x$ of the generic point x of X. It is a local ring of dimension 0, and thus a local Artinian ring when X is Noetherian; it is a field when X is integral, and, when X is further an affine scheme, it is identified with the field of fractions of $A(X)$.*

Proof. Given the above, and the identification of rational functions with sections of $\mathcal{O}_X$ over a dense open subset of X, the first claim is nothing but the definition of the fibre of a sheaf above a point. For the other claims, we can reduce to the case where X is affine, given by some ring A; then $\mathfrak{j}_x$ is the nilradical of A, and $\mathcal{O}_x$ is thus of dimension 0; if A is integral, then $\mathfrak{j}_x = (0)$, and $\mathcal{O}_x$ is thus the field of fractions of A. Finally, if A is Noetherian, we know ([**Sam53b**, p. 127, cor. 4]) that $\mathfrak{j}_x$ is nilpotent, and $\mathcal{O}_x = A_x$ Artinian. □

If X is *integral*, then the ring $\mathcal{O}_z$ is integral for all $z \in X$; every affine open subset U containing z also contains x, and $R(U)$, being equal to the field of fractions of $A(U)$, is identified with $R(X)$; we thus conclude that $R(X)$ can also be identified with the *field of fractions of* $\mathcal{O}_z$: the canonical identification of $\mathcal{O}_z$ to a subring of $R(X)$ consists of associating, to every germ of a section $s \in \mathcal{O}_z$, the unique rational function on X, class of a section of $\mathcal{O}_X$, (necessarily defined on a dense open subset of X) having s as its germ at the point z.

(7.1.6). Now suppose that X has a *finite* number of irreducible components X_i $(1 \leqslant i \leqslant n)$ (which will be the case whenever the underlying space of X is *Noetherian*); let X'_i be the open subset of X given by the complement of the $X_j \cap X_i$ for $j \neq i$ inside X_i; X'_i is irreducible, its generic point x_i is the generic point of X_i, and the X'_i are pairwise disjoint, with their union being dense in X (**0**, 2.1.6). For every dense open subset U of X, $U_i = U \cap X'_i$ is a nonempty dense open subset of X'_i, with the U_i being pairwise disjoint, and so $U' = \bigcup_i U'_i$ is dense in X. Giving a morphism from U' to Y consists of giving (arbitrarily) a morphism from each of the U_i to Y.

Thus:

Proposition (7.1.7). — *Let X and Y be two preschemes (resp. S-preschemes) such that X has a finite number of irreducible components X_i, with generic points x_i $(1 \leqslant i \leqslant n)$. If R_i is the set of germs of morphisms (resp. S-morphisms) from open subsets of X to Y at the point x_i, then the set of rational maps (resp. rational S-maps) from X to Y can be identified with the product of the R_i $(1 \leqslant i \leqslant n)$.*

Corollary (7.1.8). — *Let X be a Noetherian prescheme. The ring of rational functions on X is an Artinian ring, whose local components are the rings $\mathcal{O}_{x_i}$ of the generic points x_i of the irreducible components of X.*

Corollary (7.1.9). — *Let A be a Noetherian ring, and $X = \operatorname{Spec}(A)$. If Q is the complement of the union of the minimal prime ideals of A, then the ring of rational functions on X can be canonically identified with the ring of fractions $Q^{-1}A$.*

This will follow from the following lemma:

Lemma (7.1.9.1). — *For an element $f \in A$ to be such that $D(f)$ is dense in X, it is necessary and sufficient that $f \in Q$; every dense open subset of X contains an open subset of the form $D(f)$, where $f \in Q$.*

Proof. To show (7.1.9.1), we again denote by X_i $(1 \leqslant i \leqslant n)$ the irreducible components of X; if $D(f)$ is dense in X then $D(f) \cap X_i \neq \varnothing$ for $1 \leqslant i \leqslant n$, and vice-versa; but this means that $f \notin \mathfrak{p}_i$ for $1 \leqslant i \leqslant n$, where we set $\mathfrak{p}_i = \mathfrak{j}(X_i)$, and since the $\mathfrak{p}_i$ are the minimal prime ideals of A (1.1.14), the conditions $f \notin \mathfrak{p}_i$ $(1 \leqslant i \leqslant n)$ are equivalent to $f \in Q$, whence the first claim of the lemma. For the other claim, if U is a dense open subset of X, the complement of U is a set of the form $V(\mathfrak{a})$, where $\mathfrak{a}$ is an ideal which is not contained in any of the $\mathfrak{p}_i$; it is thus not contained in their union ([**Nor53**, p. 13]), and there thus exists some $f \in \mathfrak{a}$ belonging to Q; whence $D(f) \subset U$, which finishes the proof. □

(7.1.10). Suppose again that X is irreducible, with generic point x. Since every nonempty open subset U of X contains x, and thus also contains every $z \in X$ such that $x \in \overline{\{z\}}$, every morphism $U \to Y$ can be composed with the canonical morphism $\operatorname{Spec}(\mathcal{O}_x) \to X$ (2.4.1); and any two morphisms into Y from two nonempty open subsets of X which agree on a nonempty open subset of X give, by composition, the same morphism $\operatorname{Spec}(\mathcal{O}_x) \to Y$. In other words, to every rational map from X to Y there is a corresponding well-defined morphism $\operatorname{Spec}(\mathcal{O}_x) \to Y$.

Proposition (7.1.11). — *Let X and Y be two S-preschemes; suppose that X is irreducible with generic point x, and that Y is of finite type over S. Any two rational S-maps from X to Y that correspond to the same S-morphism $\operatorname{Spec}(\mathcal{O}_x) \to Y$ are then identical. If we further suppose S to be locally Noetherian, then every S-morphism from $\operatorname{Spec}(\mathcal{O}_x)$ to Y corresponds to exactly one rational S-map from X to Y.*

Proof. Taking into account that every nonempty subset of X is dense in X, this follows from (6.5.1). □

Corollary (7.1.12). — *Suppose that S is locally Noetherian, and that the other hypotheses of (7.1.11) are satisfied. The rational S-maps from X to Y can then be identified with points of the S-prescheme Y, with values in the S-prescheme* $\operatorname{Spec}(\mathcal{O}_x)$.

Proof. This is nothing but (7.1.11), with the terminology introduced in (3.4.1). □

Corollary (7.1.13). — *Suppose that the conditions of (7.1.12) are satisfied. Let s be the image of x in S. The data of a rational S-map from X to Y is equivalent to the data of a point y of Y over s along with a local $\mathcal{O}_s$-homomorphism $\mathcal{O}_y \to \mathcal{O}_x = R(X)$.*

Proof. This follows from (7.1.11) and (2.4.4). ☐

In particular:

Corollary (7.1.14). — *Under the conditions of (7.1.12), rational S-maps from X to Y depend only (for any given Y) on the S-prescheme* $\operatorname{Spec}(\mathcal{O}_x)$*, and, in particular, remain the same whenever X is replaced by* $\operatorname{Spec}(\mathcal{O}_z)$*, for any $z \in X$.*

Proof. Since $z \in \overline{\{x\}}$, x is the generic point of $Z = \operatorname{Spec}(\mathcal{O}_z)$, and $\mathcal{O}_{X,x} = \mathcal{O}_{Z,z}$. ☐

When X is integral, $R(X) = \mathcal{O}_x = k(x)$ is a field (7.1.5); the preceding corollaries then specialize to the following:

Corollary (7.1.15). — *Suppose that the conditions of (7.1.12) are satisfied, and further that X is integral. Let s be the image of x in S. Then rational S-maps from X to Y can be identified with the geometric points of $Y \otimes_S k(s)$ with values in the extension $R(X)$ of $k(s)$, or, in other words, every such map is equivalent to the data of a point $y \in Y$ above s along with a $k(s)$-monomorphism from $k(y)$ to $k(x) = R(X)$.*

Proof. The points of Y above s are identified with the points of $Y \otimes_S k(s)$ (3.6.3), and the local $\mathcal{O}_s$-homomorphisms $\mathcal{O}_y \to R(X)$ with the $k(s)$-monomorphisms $k(y) \to R(X)$. ☐

More precisely:

Corollary (7.1.16). — *Let k be a field, and X and Y two algebraic preschemes over k (6.4.1); suppose further that X is integral. Then the rational k-maps from X to Y can be identified with the geometric points of Y with values in the extension $R(X)$ of k (3.4.4).*

7.2. Domain of definition of a rational map.

(7.2.1). Let X and Y be preschemes, and f rational map from X to Y. We say that f is *defined at a point* $x \in X$ if there exists a dense open subset U of X that contains x, and a morphism $U \to Y$ belonging to the equivalence class of f. The set of points $x \in X$ where f is defined is called the *domain of definition* of f; it is clear that it is an open dense subset of X.

Proposition (7.2.2). — *Let X and Y be S-preschemes such that X is reduced and Y is separated over S. Let f be a rational S-map from X to Y, with domain of definition U_0. Then there exists exactly one S-morphism $U_0 \to Y$ belonging to the class of f.* **I** | 159

Since, for every morphism $U \to Y$ belonging to the class of f, we necessarily have $U \subset U_0$, it is clear that the proposition will be a consequence of the following:

Lemma (7.2.2.1). — *Under the hypotheses of (7.2.2), let U_1 and U_2 be two dense open subsets of X, and $f_i : U_i \to Y$ ($i = 1, 2$) two S-morphisms such that there exists an open subset $V \subset U_1 \cap U_2$, dense in X, and on which f_1 and f_2 agree. Then f_1 and f_2 agree on $U_1 \cap U_2$.*

Proof. We can clearly restrict to the case where $X = U_1 = U_2$. Since X (and thus V) is reduced, X is the smallest closed subprescheme of X containing V (5.2.2). Let $g = (f_1, f_2)_S : X \to Y \times_S Y$; since, by hypothesis, the diagonal $T = \Delta_Y(Y)$ is a closed subprescheme of $Y \times_S Y$, $Z = g^{-1}(T)$ is a closed subprescheme of X (4.4.1). If $h : V \to Y$ is the common restriction of f_1 and f_2 to V, then the restriction of g to V is $g' = (h, h)_S$, which factors as $g' = \Delta_Y \circ h$; since $\Delta_Y^{-1}(T) = Y$, we have that $g'^{-1}(T) = V$, and so Z is a closed subprescheme of X inducing V, thus containing V, which implies that $Z = X$. From the equation $g^{-1}(T) = X$, we deduce (4.4.1) that g factors as $\Delta_Y \circ f$, where f is a morphism $X \to Y$, which implies, by the definition of the diagonal morphism, that $f_1 = f_2 = f$. ☐

It is clear that the morphism $U_0 \to Y$ defined in (7.2.2) is the unique morphism of the class f that *cannot be extended* to a morphism from an open subset of X that strictly contains U_0. *Under the hypotheses of* (7.2.2), we can thus *identify* the rational maps from X to Y with the *non-extendible* (to strictly larger open subsets) morphisms from dense open subsets of X to Y. With this identification, Proposition (7.2.2) implies:

Corollary (7.2.3). — *With the hypotheses from (7.2.2) on X and Y, let U be a dense open subset of X. Then there exists a canonical bijective correspondence between S-morphisms from U to Y and rational S-maps from X to Y that are defined at all points of U.*

Proof. By (7.2.2), for every S-morphism f from U to Y, there exists exactly one rational S-map $\overline{f}$ from X to Y which extends f. ☐

Corollary (7.2.4). — *Let S be a scheme, X a reduced S-prescheme, Y an S-scheme, and $f : U \to Y$ an S-morphism from a dense open subset U of X to Y. If $\overline{f}$ is the rational* **Z***-map from X to Y that extends f, then $\overline{f}$ is an S-morphism (and thus the rational S-map from X to Y that extends f).*

Proof. Indeed, if $\varphi : X \to S$ and $\psi : Y \to S$ are the structure morphisms, U_0 the domain of definition of $\overline{f}$, and j the injection $U_0 \to X$, then it suffices to show that $\psi \circ \overline{f} = \varphi \circ j$, but this follows from (7.2.2.1), since f is an S-morphism. □

Corollary (7.2.5). — *Let X and Y be two S-preschemes; suppose that X is reduced, and that X and Y are separated over S. Let $p : Y \to X$ be an S-morphism (making Y an X-prescheme), U a dense open subset of X, and f a U-section of Y; then the rational map $\overline{f}$ from X to Y extending f is a rational X-section of Y.*

Proof. We have to show that $p \circ \overline{f}$ is the identity on the domain of definition of $\overline{f}$; since X is separated over S, this again follows from (7.2.2.1). □ **I | 160**

Corollary (7.2.6). — *Let X be a reduced prescheme, and U a dense open subset of X. Then there is a canonical bijective correspondence between sections of $\mathcal{O}_X$ over U and rational functions on X defined at every point of U.*

Proof. Taking (7.2.3), (7.1.2), and (7.1.3) into account, it suffices to note that the X-prescheme $X \otimes_{\mathbf{Z}} \mathbf{Z}[T]$ is separated over X (5.5.1, iv). □

Corollary (7.2.7). — *Let Y be a reduced prescheme, $f : X \to Y$ a separated morphism, U a dense open subset of Y, $g : U \to f^{-1}(U)$ a U-section of $f^{-1}(U)$, and Z the reduced subprescheme of X that has $\overline{g(U)}$ as its underlying space* (5.2.1). *For g to be the restriction of a Y-section of X* (in other words (7.2.5), for the rational map from Y to X extending g to be defined everywhere)*, it is necessary and sufficient for the restriction of f to Z to be an isomorphism from Z to Y.*

Proof. The restriction of f to $f^{-1}(U)$ is a separated morphism (5.5.1, i), so g is a closed immersion (5.4.6), and so $g(U) = Z \cap f^{-1}(U)$, and the subprescheme induced by Z on the open subset $g(U)$ of Z is identical to the closed subprescheme of $f^{-1}(U)$ associated to g (5.2.1). It is then clear that the stated condition is sufficient, because, if satisfied, and if $f_Z : Z \to Y$ is the restriction of f to Z, and $\overline{g} : Y \to Z$ is the inverse isomorphism, then $\overline{g}$ extends g. Conversely, if g is the restriction to U of a Y-section h of X, then h is a closed immersion (5.4.6), and so $h(Y)$ is closed, and, since it is contained in Z, is equal to Z, and it follows from (5.2.1) that h is necessarily an isomorphism from Y to the closed subprescheme Z of X. □

(7.2.8). Let X and Y be two S-preschemes, with X reduced, and Y separated over S. Let f be a rational S-map from X to Y, and let x be a point of X; we can compose f with the canonical S-morphism $\operatorname{Spec}(\mathcal{O}_x) \to X$ (2.4.1) provided that the intersection of $\operatorname{Spec}(\mathcal{O}_x)$ with the domain of definition of f is dense in $\operatorname{Spec}(\mathcal{O}_x)$ (identified with the set of $z \in X$ such that $x \in \overline{\{z\}}$ (2.4.2)). This will happen in the follow cases:

1st. X is *irreducible* (and thus *integral*), because then the generic point ξ of X is the generic point of $\operatorname{Spec}(\mathcal{O}_x)$; since the domain of definition U of f contains ξ, $U \cap \operatorname{Spec}(\mathcal{O}_x)$ contains ξ, and so is dense in $\operatorname{Spec}(\mathcal{O}_x)$.

2nd. X is *locally Noetherian*; our claim then follows from:

Lemma (7.2.8.1). — *Let X be a prescheme whose underlying space is locally Noetherian, and x a point of X. The irreducible components of* $\operatorname{Spec}(\mathcal{O}_x)$ *are the intersections of* $\operatorname{Spec}(\mathcal{O}_x)$ *with the irreducible components of X containing x. For an open subset $U \subset X$ to be such that $U \cap \operatorname{Spec}(\mathcal{O}_x)$ is dense* $\operatorname{Spec}(\mathcal{O}_x)$*, it is necessary and sufficient for it to have a nonempty intersection with the irreducible components of X that contain x* (which will be the case whenever U is *dense* in X).

Proof. It suffices to show just the first claim, since the second then follows. Since $\operatorname{Spec}(\mathcal{O}_x)$ is contained in every affine open subset U that contains x, and since the irreducible components of U that contain x are the intersections of U with the irreducible components of X containing x (**0**, 2.1.6), we can suppose that X is affine, given by some ring A. Since the prime ideals of A_x correspond bijectively to the prime ideals of A that are contained in $\mathfrak{j}_x$ (2.1.6), the minimal prime ideals of A_x correspond to the minimal prime ideals of A that are contained in $\mathfrak{j}_x$, hence the lemma. □ **I | 161**

With this in mind, suppose that we are in one of the two cases mentioned in (7.2.8). If U is the domain of definition of the rational S-map f, then we denote by f' the rational map from $\operatorname{Spec}(\mathcal{O}_x)$ to Y which agrees (taking (2.4.2) into account) with f on $U \cap \operatorname{Spec}(\mathcal{O}_x)$; we say that this rational map is *induced by f*.

Proposition (7.2.9). — *Let S be a locally Noetherian prescheme, X a reduced S-prescheme, and Y an S-scheme of finite type. Suppose further that X is either irreducible or locally Noetherian. Then let f be a rational S-map from X to Y, and x a point of X. For f to be defined at a point x, it is necessary and sufficient for the rational map f' from* $\mathrm{Spec}(\mathcal{O}_x)$ *to Y, induced by f (7.2.8), to be a morphism.*

Proof. The condition clearly being necessary (since $\mathrm{Spec}(\mathcal{O}_x)$ is contained in every open subset containing x), we show that it is sufficient. By (6.5.1), there exists an open neighbourhood U of x in X, and an S-morphism g from U to Y that induces f' on $\mathrm{Spec}(\mathcal{O}_x)$. If X is irreducible, then U is dense in X, and, by (7.2.3), we can suppose that g is a rational S-map. Further, the generic point of X belongs to both $\mathrm{Spec}(\mathcal{O}_x)$ and the domain of definition of f, and so s and g agree at this point, and thus on a nonempty open subset of X (6.5.1). But since f and g are rational S-maps, they are identical (7.2.3), and so f is defined at x.

If we now suppose that X is locally Noetherian, then we can suppose that U is Noetherian; then there are only a finite number of irreducible components X_i of X that contain x (7.2.8.1), and we can suppose that they are the only ones that have a nonempty intersection with U, by replacing, if needed, U with a smaller open subset (since there are only a finite number of irreducible components of X that have a nonempty intersection with U, because U is Noetherian). We then have, as above, that f and g agree on a nonempty open subset of each of the X_i. Taking into account the fact that each of the X_i is contained in $\overline{U}$, we consider the morphism f_1, defined on a dense open subset of $U \cup (X - \overline{U})$, equal to g on U, and to f on the intersection of $X - \overline{U}$ with the domain of definition of f. Since $U \cup (X - \overline{U})$ is dense in X, f_1 and f agree on a dense open subset of X, and since f is a rational map, f is an extension of f_1 (7.2.3), and is thus defined at the point x. □

7.3. Sheaf of rational functions.

(7.3.1). Let X be a prescheme. For every open subset $U \subset X$, we denote by $R(U)$ the ring of rational functions on U (7.1.3); this is a $\Gamma(U, \mathcal{O}_X)$-algebra. Further, if $V \subset U$ is a second open subset of X, then every section of $\mathcal{O}_X$ over a dense (in X) open subset of V gives, by restriction to V, a section over a dense (in X) open subset of V, and if two sections agree on a dense (in X) open subset of U, then their restrictions to V agree on a dense (in X) open subset of V. We can thus define a di-homomorphism of algebras $\rho_{V,U} : R(U) \to R(V)$, and it is clear that, if $U \supset V \supset W$ are open subsets of X, then we have $\rho_{W,U} = \rho_{W,V} \circ \rho_{V,U}$; the $R(U)$ thus define a *presheaf* of algebras on X.

Definition (7.3.2). — We define the sheaf of rational functions on a prescheme X, denoted by $\mathcal{R}(X)$, to be the $\mathcal{O}_X$-algebra associated to the presheaf defined by the $R(U)$. **I | 162**

For every prescheme X and open subset $U \subset X$, it is clear that the induced sheaf $\mathcal{R}(X)|U$ is exactly $\mathcal{R}(U)$.

Proposition (7.3.3). — *Let X be a prescheme such that the family (X_λ) of its irreducible components is locally finite (which is the case whenever the underlying space of X is locally Noetherian). Then the $\mathcal{O}_X$-module $\mathcal{R}(X)$ is quasi-coherent, and for every open subset U of X that has a nonempty intersection with only finitely many of the components X_λ, $R(U)$ is equal to $\Gamma(U, \mathcal{R}(X))$, and can be canonically identified with the direct sum of the local rings of the generic points of the X_λ such that $U \cap X_\lambda \neq \varnothing$.*

Proof. We can evidently restrict to the case where X has only a finite number of irreducible components X_i, with generic points x_i $(1 \leqslant i \leqslant n)$. The fact that $R(U)$ can be canonically identified with the direct sum of the $\mathcal{O}_{x_i} = R(X_i)$ such that $U \cap X_i \neq \varnothing$ then follows from (7.1.7). We will show that the presheaf $U \to R(U)$ satisfies the sheaf axioms, which will prove that $R(U) = \Gamma(U, \mathcal{R}(X))$. Indeed, it satisfies (F1) by what has already been discussed. To see that it satisfies (F2), consider a cover of an open subset U of X by open subsets $V_\alpha \subset U$; if the $s_\alpha \in R(V_\alpha)$ are such that the restrictions of s_α and s_β to $V_\alpha \cap V_\beta$ agree for every pair of indices, then we can conclude that, for every index i such that $U \cap X_i \neq \varnothing$, the components in $R(X_i)$ of all the s_α such that $V_\alpha \cap X_i \neq \varnothing$ are all the same; denoting this component by t_i, it is clear that the element of $R(U)$ that has the t_i as its components has s_α as its restriction to each V_α. Finally, to see that $\mathcal{R}(X)$ is quasi-coherent, we can restrict to the case where $X = \mathrm{Spec}(A)$ is affine; by taking U to be an affine open subset of the form $D(f)$, where $f \in A$, it follows from the above and from Definition (1.3.4) that we have $\mathcal{R}(X) = \widetilde{M}$, where M is the direct sum of the A-modules A_{x_i}. □

Corollary (7.3.4). — *Let X be a reduced prescheme that has only a finite number of irreducible components, and let X_i $(1 \leqslant i \leqslant n)$ be the closed reduced preschemes of X that have the irreducible components of X as their underlying spaces (5.2.1). If h_i is the canonical injection $X_i \to X$, then $\mathcal{R}(X)$ is the direct sum of the $\mathcal{O}_X$-algebras $(h_i)_*(\mathcal{R}(X_i))$.*

Corollary (7.3.5). — *If X is irreducible, then every quasi-coherent $\mathcal{R}(X)$-module $\mathcal{F}$ is a simple sheaf.*

Proof. It suffices to show that every $x \in X$ admits a neighbourhood U such that $\mathscr{F}|U$ is a simple sheaf (**0**, 3.6.2); in other words, we are led to considering the case where X is affine; we can further suppose that $\mathscr{F}$ is the cokernel of a homomorphism $(\mathscr{R}(X))^I \to (\mathscr{R}(X))^J$ (**0**, 5.1.3), and everything then follows from showing that $\mathscr{R}(X)$ is a simple sheaf; but this is evident, because $\Gamma(U, \mathscr{R}(X)) = R(X)$ for every nonempty open subset U, where U contains the generic point of X. □

Corollary (7.3.6). — *If X is irreducible, then, for every quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$, $\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{R}(X)$ is a simple sheaf; if, further, X is reduced (and thus integral), then $\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{R}(X)$ is isomorphic to a sheaf of the form $(\mathscr{R}(X))^{(I)}$.*

Proof. The second claim follows from the fact that $R(X)$ is a field. □

Proposition (7.3.7). — *Suppose that the prescheme X is locally integral or locally Noetherian. Then $\mathscr{R}(X)$ is a quasi-coherent $\mathscr{O}_X$-algebra; if, further, X is reduced (which will be the case whenever X is locally integral), then the canonical homomorphism $\mathscr{O}_X \to \mathscr{R}(X)$ is injective.* **I | 163**

Proof. The question being local, the first claim follows from (7.3.3); the second follows from (7.2.3). □

(7.3.8). [10] Let X and Y be two *integral* preschemes, which implies that $\mathscr{R}(X)$ (resp. $\mathscr{R}(Y)$) is a quasi-coherent $\mathscr{O}_X$-module (resp. $\mathscr{O}_Y$-module) (7.3.3). Let $f : X \to Y$ be a *dominant* morphism; then there exists a canonical homomorphism of $\mathscr{O}_X$-modules

(7.3.8.1) $$\tau : f^*(\mathscr{R}(Y)) \longrightarrow \mathscr{R}(X).$$

Proof. Suppose first that $X = \operatorname{Spec}(A)$ and $Y = \operatorname{Spec}(B)$ are affine, given by integral rings A and B, with f thus corresponding to an injective homomorphism $B \to A$ which extends to a monomorphism $L \to K$ from the field of fractions L of B to the field of fractions K of A. The homomorphism (7.3.8.1) then corresponds to the canonical homomorphism $L \otimes_B A \to K$ (1.6.5). In the general case, for each pair of nonempty affine open sets $U \subset X$ and $V \subset Y$ such that $f(U) \subset V$, we define, as above, a homomorphism $\tau_{U,V}$ and we immediately have that, if $U' \subset U$, $V' \subset V$, $f(U') \subset V'$, then $\tau_{U,V}$ extends $\tau_{U',V'}$, and hence our assertion. If x and y are the generic points of X and Y respectively, then we have $f(x) = y$,

$$(f^*(\mathscr{R}(Y)))_x = \mathscr{O}_y \otimes_{\mathscr{O}_y} \mathscr{O}_x = \mathscr{O}_x$$

(**0**, 4.3.1) and τ_x is thus an *isomorphism*. □

7.4. Torsion sheaves and torsion-free sheaves.

(7.4.1). Let X be an *integral* scheme. For every $\mathscr{O}_X$-module $\mathscr{F}$, the canonical homomorphism $\mathscr{O}_X \to \mathscr{R}(X)$ defines, by tensoring, a homomorphism (again said to be *canonical*) $\mathscr{F} \to \mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{R}(X)$, which, on each fibre, is exactly the homomorphism $z \to z \otimes 1$ from $\mathscr{F}_x$ to $\mathscr{F}_x \otimes_{\mathscr{O}_x} R(X)$. The *kernel* $\mathscr{T}$ of this homomorphism is a $\mathscr{O}_X$-submodule of $\mathscr{F}$, called the *torsion sheaf* of $\mathscr{F}$; it is quasi-coherent if $\mathscr{F}$ is quasi-coherent ((4.1.1) and (7.3.6)). We say that $\mathscr{F}$ is *torsion free* if $\mathscr{T} = 0$, and that $\mathscr{F}$ is a *torsion sheaf* if $\mathscr{T} = \mathscr{F}$. For every $\mathscr{O}_X$-module $\mathscr{F}$, $\mathscr{F}/\mathscr{T}$ is torsion free. We deduce from (7.3.5) that:

Proposition (7.4.2). — *If X is an integral prescheme, then every torsion-free quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$ is isomorphic to a subsheaf $\mathscr{G}$ of a simple sheaf of the form $(\mathscr{R}(X))^{(I)}$, generated (as a $\mathscr{R}(X)$-module) by $\mathscr{G}$.*

The cardinality of I is called the *rank* of $\mathscr{F}$; for every nonempty affine open subset U of X, the rank of $\mathscr{F}$ is equal to the rank of $\Gamma(U, \mathscr{F})$ as a $\Gamma(U, \mathscr{O}_X)$-module, as we see by considering the generic point of X, contained in U. In particular:

Corollary (7.4.3). — *On an integral prescheme X, every torsion-free quasi-coherent $\mathscr{O}_X$-module of rank 1 (in particular, every invertible $\mathscr{O}_X$-module) is isomorphic to a $\mathscr{O}_X$-submodule of $\mathscr{R}(X)$, and vice versa.*

Corollary (7.4.4). — *Let X be an integral prescheme, $\mathscr{L}$ and $\mathscr{L}'$ torsion-free $\mathscr{O}_X$-modules, and f (resp. f') a section of $\mathscr{L}$ (resp. $\mathscr{L}'$) over X. In order to have $f \otimes f' = 0$, it is necessary and sufficient for one of the sections f and f' to be zero.*

Proof. Let x be the generic point of X; we have, by hypothesis, that $(f \otimes f')_x = f_x \otimes f'_x = 0$. Since $\mathscr{L}_x$ and $\mathscr{L}'_x$ can be identified with $\mathscr{O}_X$-submodules of the field $\mathscr{O}_x$, the above equation leads to $f_x = 0$ or $f'_x = 0$, and thus $f = 0$ or $f' = 0$, since $\mathscr{L}$ and $\mathscr{L}'$ are torsion free (7.3.5). □

Proposition (7.4.5). — *Let X and Y be integral preschemes, and $f : X \to Y$ a dominant morphism. For every torsion-free quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$, $f_*(\mathscr{F})$ is a torsion-free $\mathscr{O}_Y$-module.*

Proof. Since f_* is left exact (**0**, 4.2.1), it suffices, by (7.4.2), to prove the proposition in the case where $\mathscr{F} = (\mathscr{R}(X))^{(I)}$. But every nonempty open subset U of Y contains the generic point of Y, so $f^{-1}(U)$ contains the generic point of X (**0**, 2.1.5), so we have that $\Gamma(U, f_*(\mathscr{F})) = \Gamma(f^{-1}(U), \mathscr{F}) = (R(X))^{(I)}$; in other words, $f_*(\mathscr{F})$ is the simple sheaf with fibre $(R(X))^{(I)}$, considered as a $\mathscr{R}(Y)$-module, and it is clearly torsion free. □ **I | 164**

Proposition (7.4.6). — *Let X be an integral prescheme, and x its generic point. For every quasi-coherent $\mathcal{O}_X$-module $\mathscr{F}$ of finite type, the following conditions are equivalent:* (a) *$\mathscr{F}$ is a torsion sheaf;* (b) $\mathscr{F}_x = 0$*;* (c) $\operatorname{Supp}(\mathscr{F}) \neq X$.

Proof. By (7.3.5) and (7.4.1), the equations $\mathscr{F}_x = 0$ and $\mathscr{F} \otimes_{\mathcal{O}_X} \mathscr{R}(X) = 0$ are equivalent, so (a) and (b) are equivalent; then $\operatorname{Supp}(\mathscr{F})$ is closed in X (**0**, 5.2.2), and since every nonempty open subset of X contains x, (b) and (c) are equivalent. □

(7.4.7). We generalise (by an abuse of language) the definitions of (7.4.1) to the case where X is a *reduced* prescheme having only a *finite* number of irreducible components; it then follows from (7.3.4) that the equivalence between *a)* and *c)* in (7.4.6) still holds true for such a prescheme.

§8. Chevalley schemes

8.1. Allied local rings. For each local ring A, we denote by $\mathfrak{m}(A)$ the maximal ideal of A.

Lemma (8.1.1). — *Let A and B be two local rings such that $A \subset B$; Then the following conditions are equivalent.*

(i) $\mathfrak{m}(B) \cap A = \mathfrak{m}(A)$.
(ii) $\mathfrak{m}(A) \subset \mathfrak{m}(B)$.
(iii) 1 *is not an element of the ideal of B generated by* $\mathfrak{m}(A)$.

Proof. It is evident that (i) implies (ii), and (ii) implies (iii); lastly, if (iii) is true, then $\mathfrak{m}(B) \cap A$ contains $\mathfrak{m}(A)$, and does not contain 1, and is thus equal to $\mathfrak{m}(A)$.

When the equivalent conditions of (8.1.1) are satisfied, we say that B *dominates* A; this is equivalent to saying that the injection $A \to B$ is a *local* homomorphism. It is clear that, in the set of local subrings of a ring R, the relation given by domination is an order. □

(8.1.2). Now consider a *field* R. For all subrings A of R, we denote by $L(A)$ the set of local rings $A_\mathfrak{p}$, where $\mathfrak{p}$ ranges over the prime spectrum of A; such local rings are identified with the subrings of R containing A. Since $\mathfrak{p} = (\mathfrak{p}A_\mathfrak{p}) \cap A$, the map $\mathfrak{p} \mapsto A_\mathfrak{p}$ from $\operatorname{Spec}(A)$ to $L(A)$ is bijective.

Lemma (8.1.3). — *Let R be a field, and A a subring of R. For a local subring M of R to dominate a ring $A_\mathfrak{p} \in L(A)$, it is necessary and sufficient that $A \subset M$; the local ring $A_\mathfrak{p}$ dominated by M is then unique, and corresponds to $\mathfrak{p} = \mathfrak{m}(M) \cap A$.*

Proof. If M dominates $A_\mathfrak{p}$, then $\mathfrak{m}(M) \cap A_\mathfrak{p} = \mathfrak{p}A_\mathfrak{p}$, by (8.1.1), whence the uniqueness of $\mathfrak{p}$; on the other hand, if $A \subset M$, then $\mathfrak{m}M \cap A = \mathfrak{p}$ is prime in A, and since $A - \mathfrak{p} \subset M$, we have that $A_\mathfrak{p} \subset M$ and $\mathfrak{p}A_\mathfrak{p} \subset \mathfrak{m}(M)$, so M dominates $A_\mathfrak{p}$. □

Lemma (8.1.4). — *Let R be a field, M and N local subrings of R, and P the subring of R generated by $M \cup N$. Then the following conditions are equivalent.* **I | 165**

(i) *There exists a prime ideal $\mathfrak{p}$ of P such that $\mathfrak{m}(M) = \mathfrak{p} \cap M$ and $\mathfrak{m}(N) = \mathfrak{p} \cap N$.*
(ii) *The ideal $\mathfrak{a}$ generated in P by $\mathfrak{m}(M) \cup \mathfrak{m}(N)$ is distinct from P.*
(iii) *There exists a local subring Q of R simultaneously dominating both M and N.*

Proof. It is clear that (i) implies (ii); conversely, if $\mathfrak{a} \neq P$, then $\mathfrak{a}$ is contained in a maximal ideal $\mathfrak{n}$ of P, and since $1 \notin \mathfrak{n}$, $\mathfrak{n} \cap M$ contains $\mathfrak{m}(M)$ and is distinct from M, so $\mathfrak{n} \cap M = \mathfrak{m}(M)$, and similarly $\mathfrak{n} \cap N = \mathfrak{m}(N)$. It is clear that, if Q dominates both M and N, then $P \subset Q$ and $\mathfrak{m}(M) = \mathfrak{m}(Q) \cap M = (\mathfrak{m}(Q) \cap P) \cap M$, and $\mathfrak{m}(N) = (\mathfrak{m}(Q) \cap P) \cap N$, so (iii) implies (i); the converse is evident when we take $Q = P_\mathfrak{p}$. □

When the conditions of (8.1.4) are satisfied, we say, with C. Chevalley, that the local rings M and N are *allied.*

Proposition (8.1.5). — *Let A and B be subrings of a field R, and C the subring of R generated by $A \cup B$. Then the following conditions are equivalent.*

(i) *For every local ring Q containing A and B, we have that $A_\mathfrak{p} = B_\mathfrak{q}$, where $\mathfrak{p} = \mathfrak{m}(Q) \cap A$ and $\mathfrak{q} = \mathfrak{m}(Q) \cap B$.*
(ii) *For all prime ideals $\mathfrak{r}$ of C, we have that $A_\mathfrak{p} = B_\mathfrak{q}$, where $\mathfrak{p} = \mathfrak{r} \cap A$ and $\mathfrak{q} = \mathfrak{r} \cap B$.*
(iii) *If $M \in L(A)$ and $N \in L(B)$ are allied, then they are identical.*
(iv) $L(A) \cap L(B) = L(C)$.

[10] *[Trans.] This paragraph was changed entirely in the Errata of EGA II.*

Proof. Lemmas (8.1.3) and (8.1.4) prove that (i) and (iii) are equivalent; it is clear that (i) implies (ii) by taking $Q = C_{\mathfrak{r}}$; conversely, (ii) implies (i), because if Q contains $A \cup B$ then it contains C, and if $\mathfrak{r} = \mathfrak{m}(Q) \cap C$, then $\mathfrak{p} = \mathfrak{r} \cap A$ and $\mathfrak{q} = \mathfrak{r} \cap B$, by (8.1.3). It is immediate that (iv) implies (i), because if Q contains $A \cup B$ then it dominates a local ring $C_{\mathfrak{r}} \in L(C)$ by (8.1.3); by hypothesis we have that $C_{\mathfrak{r}} \in L(A) \cap L(B)$, and (8.1.1) and (8.1.3) prove that $C_{\mathfrak{r}} = A_{\mathfrak{p}} = B_{\mathfrak{q}}$. We prove finally that (iii) implies (iv). Let $Q \in L(C)$; Q dominates some $M \in L(A)$ and some $N \in L(B)$ (8.1.3), so M and N, being allied, are identical by hypothesis. As we then have that $C \subset M$, we know that M dominates some $Q' \in L(C)$ (8.1.3), so Q dominates Q', whence necessarily (8.1.3) $Q = Q' = M$, so $Q \in L(A) \cap L(B)$. Conversely, if $Q \in L(A) \cap L(B)$, then $C \subset Q$, so (8.1.3) Q dominates some $Q'' \in L(C) \subset L(A) \cap L(B)$; Q and Q'', being allied, are identical, so $Q'' = Q \in L(C)$, which completes the proof. ☐

8.2. Local rings of an integral scheme.

(8.2.1). Let X be an *integral* prescheme, and R its field of rational functions, identical to the local ring of the generic point a of X; for all $x \in X$, we know that $\mathcal{O}_x$ can be canonically identified with a subring of R (7.1.5), and for every rational function $f \in R$, the domain of definition $\delta(f)$ of f is the open set of $x \in X$ such that $f \in \mathcal{O}_x$. It thus follows, from (7.2.6), that, for every open $U \subset X$, we have

$$\Gamma(U, \mathcal{O}_X) = \bigcap_{x \in U} \mathcal{O}_x. \tag{8.2.1.1}$$

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Proposition (8.2.2). — *Let X be an integral prescheme, and R its field of rational fractions. For X to be a scheme, it is necessary and sufficient for the relation "$\mathcal{O}_x$ and $\mathcal{O}_y$ are allied" (8.1.4), for points x and y of X, to imply that $x = y$.*

Proof. We suppose that this condition is satisfied, and aim to show that X is separated. Let U and V be two distinct affine open subsets of X, given by rings A and B (respectively), identified with subrings of R; U (resp. V) is thus identified (8.1.2) with $L(A)$ (resp. $L(B)$), and the hypotheses tell us (8.1.5) that C is the subring of R generated by $A \cup B$, and $W = U \cap V$ is identified with $L(A) \cap L(B) = L(C)$. Furthermore, we know ([**CC**], p. 5-03, 4 *bis*) that every subring E of R is equal to the intersection of the local rings belonging to $L(E)$; C is thus identified with the intersection of the rings $\mathcal{O}_z$ for $z \in W$, or, equivalently (8.2.1.1), with $\Gamma(W, \mathcal{O}_X)$. So consider the subprescheme induced by X on W; to the identity morphism $\varphi : C \to \Gamma(W, \mathcal{O}_X)$ there corresponds (2.2.4) a morphism $\Phi = (\psi, \theta) : W \to \operatorname{Spec}(C)$; we will see that Φ is an *isomorphism* of preschemes, whence W is an *affine* open subset. The identification of W with $L(C) = \operatorname{Spec}(C)$ shows that ψ is *bijective*. On the other hand, for all $x \in W$, $\theta_x^\sharp$ is the injection $C_{\mathfrak{r}} \to \mathcal{O}_x$, where $\mathfrak{r} = \mathfrak{m}_x \cap C$, and, by definition, $C_{\mathfrak{r}}$ is identified with $\mathcal{O}_x$, so $\theta_x^\sharp$ is bijective. It thus remains to show that ψ is a *homeomorphism*, or, in other words, that for every closed subset $F \subset W$, $\psi(F)$ is closed in $\operatorname{Spec}(C)$. But F is the intersection of W with a closed subspace of U of the form $V(\mathfrak{a})$, where $\mathfrak{a}$ is an ideal of A; we will show that $\psi(F) = V(\mathfrak{a}C)$, which proves our claim. In fact, the prime ideals of C containing $\mathfrak{a}C$ are the prime ideals of C containing $\mathfrak{a}$, and so are the ideals of the form $\psi(x) = \mathfrak{m}_x \cap C$, where $\mathfrak{a} \subset \mathfrak{m}_x$ and $x \in W$; since $\mathfrak{a} \subset \mathfrak{m}_x$ is equivalent to $x \in V(\mathfrak{a}) = W \cap F$ for $x \in U$, we do indeed have that $\psi(F) = V(\mathfrak{a}C)$.

It follows that X is separated, because $U \cap V$ is affine and its ring C is generated by the union $A \cup B$ of the rings of U and V (5.5.6).

Conversely, suppose that X is separated, and let x and y be points of X such that $\mathcal{O}_x$ and $\mathcal{O}_y$ are allied. Let U (resp. V) be an affine open subset containing x (resp. y), of ring A (resp. B); we then know that $U \cap V$ is affine and that its ring C is generated by $A \cup B$ (5.5.6). If $\mathfrak{p} = \mathfrak{m}_x \cap A$ and $\mathfrak{q} = \mathfrak{m}_y \cap B$, then $A_{\mathfrak{p}} = \mathcal{O}_x$ and $B_{\mathfrak{q}} = \mathcal{O}_y$, and since $A_{\mathfrak{p}}$ and $B_{\mathfrak{q}}$ are allied, there exists a prime ideal $\mathfrak{r}$ of C such that $\mathfrak{p} = \mathfrak{r} \cap A$ and $\mathfrak{q} = \mathfrak{r} \cap B$ (8.1.4). But then there exists a point $z \in U \cap V$ such that $\mathfrak{r} = \mathfrak{m}_z \cap C$, since $U \cap V$ is affine, and so evidently $x = z$ and $y = z$, whence $x = y$. ☐

Corollary (8.2.3). — *Let X be an integral scheme, and x and y points of X. In order for $x \in \overline{\{y\}}$, it is necessary and sufficient for $\mathcal{O}_x \subset \mathcal{O}_y$, or, equivalently, for every rational function defined at x to also be defined at y.*

Proof. The condition is evidently necessary because the domain of definition $\delta(f)$ of a rational function $f \in R$ is open; we now show that it is sufficient. If $\mathcal{O}_x \subset \mathcal{O}_y$, then there exists a prime ideal $\mathfrak{p}$ of $\mathcal{O}_x$ such that $\mathcal{O}_y$ dominates $(\mathcal{O}_x)_{\mathfrak{p}}$ (8.1.3); but (2.4.2) there exists some $z \in X$ such that $x \in \overline{\{z\}}$ and $\mathcal{O}_z = (\mathcal{O}_x)_{\mathfrak{p}}$; since $\mathcal{O}_z$ and $\mathcal{O}_y$ are allied, we have that $z = y$ by (8.2.2), whence the corollary. ☐

Corollary (8.2.4). — *If X is an integral scheme then the map $x \to \mathcal{O}_x$ is injective; equivalently, if x and y are two distinct points of X, then there exists a rational function defined at one of these points but not the other.*

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Proof. This follows from (8.2.3) and the axiom (T_0) (2.1.4). ☐

Corollary (8.2.5). — *Let X be an integral scheme whose underlying space is Noetherian; letting f range over the field R of rational functions on X, the sets $\delta(f)$ generate the topology of X.*

In fact, every closed subset of X is thus a finite union of irreducible closed subsets, or, in other words, of the form $\overline{\{y\}}$ (2.1.5). But, if $x \notin \overline{\{y\}}$, then there exists a rational function f defined at x but not at y (8.2.3), or, equivalently, we have that $x \in \delta(f)$ and that $\delta(f)$ is not contained in $\overline{\{y\}}$. The complement of $\overline{\{y\}}$ is thus a union of sets of the form $\delta(f)$, and, by virtue of the first remark, every open subset of X is the union of finite intersections of open sets of the form $\delta(f)$.

(8.2.6). Corollary (8.2.5) shows that the topology of X is entirely characterised by the data of the local rings $(\mathcal{O}_x)_{x\in X}$ that have R as their field of fractions. It is equivalent to say that the closed subsets of X are defined in the following manner: given a finite subset $\{x_1,\ldots,x_n\}$ of X, consider the set of $y \in X$ such that $\mathcal{O}_y \subset \mathcal{O}_{x_i}$ for at least one index i, and these sets (over all choices of $\{x_1,\ldots,x_n\}$) are the closed subsets of X. Further, once the topology on X is known, the structure sheaf $\mathcal{O}_X$ is also determined by the family of the $\mathcal{O}_x$, since $\Gamma(U,\mathcal{O}_X) = \bigcap_{x\in U}\mathcal{O}_x$, by (8.2.1.1). The family $(\mathcal{O}_X)_{x\in X}$ thus completely determines the prescheme X when X is an integral scheme whose underlying space is Noetherian.

Proposition (8.2.7). — *Let X and Y be integral schemes, $f : X \to Y$ a dominant morphism* (2.2.6)*, and K (resp. L) the field of rational functions on X (resp. Y). Then L can be identified with a subfield of K, and, for all $x \in X$, $\mathcal{O}_{f(x)}$ is the unique local ring of Y dominated by $\mathcal{O}_x$.*

Proof. If $f = (\psi,\theta)$ and a is the generic point of X, then $\psi(a)$ is the generic point of Y (**0**, 2.1.5); $\theta_a^\sharp$ is then a monomorphism of fields, from $L = \mathcal{O}_{\psi(a)}$ to $K = \mathcal{O}_a$. Since every nonempty affine open subset U of Y contains $\psi(a)$, it follows from (2.2.4) that the homomorphism $\Gamma(U,\mathcal{O}_Y) \to \Gamma(\psi^{-1}(U),\mathcal{O}_X)$ corresponding to f is the restriction of $\theta_a^\sharp$ to $\Gamma(U,\mathcal{O}_Y)$. So, for every $x \in X$, $\theta_x^\sharp$ is the restriction to $\mathcal{O}_{\psi(a)}$ of $\theta_a^\sharp$, and is thus a monomorphism. We also know that $\theta_x^\sharp$ is a local homomorphism, so, if we identify L with a subfield of K by $\theta_a^\sharp$, $\mathcal{O}_{\psi(x)}$ is dominated by $\mathcal{O}_x$ (8.1.1); it is also the only local ring of Y dominated by $\mathcal{O}_x$, since two local rings of Y that are allied are identical (8.2.2). □

Proposition (8.2.8). — *Let X be an* irreducible *prescheme, $f : X \to Y$ a local immersion (resp. local isomorphism), and suppose further that f is separated. Then f is an immersion (resp. an open immersion).*

Proof. Let $f = (\psi,\theta)$; it suffices, in both cases, to prove that ψ is a *homeomorphism* from X to $\psi(X)$ (4.5.3). Replacing f by f_{red} ((5.1.6) and (5.5.1, vi)), we can assume that X and Y are *reduced.* If Y' is the closed reduced subprescheme of Y that has $\overline{\psi(X)}$ as its underlying space, then f factors as $X \xrightarrow{f'} Y' \xrightarrow{j} Y$, where j is the canonical injection (5.2.2). It follows from (5.5.1, v) that f' is again a separated morphism; further, f' is again a local **I | 168**
immersion (resp. a local isomorphism), because, since the condition is local on X and Y, we can restrict to the case where f is a closed immersion (resp. open immersion), and our claim then follows immediately from (4.2.2).

We can thus suppose that f is a *dominant* morphism, which leads to the fact that Y is, itself, irreducible (**0**, 2.1.5), and so X and Y are both *integral.* Further, the condition being local on Y, we can suppose that Y is an affine scheme; since f is separated, X is a scheme (5.5.1, ii), and we are finally at the hypotheses of Proposition (8.2.7). Then, for all $x \in X$, $\theta_x^\sharp$ is injective; but the hypothesis that f is a local immersion implies that $\theta_x^\sharp$ is surjective (4.2.2), so $\theta_x^\sharp$ is bijective, or, equivalently (with the identification of Proposition (8.2.7)) we have that $\mathcal{O}_{\psi(x)} = \mathcal{O}_x$. This implies, by Corollary (8.2.4), that ψ is an *injective* map, which already proves the proposition when f is a local isomorphism (4.5.3). When we suppose that f is only a local immersion, for all $x \in X$ there exists an open neighborhood U of x in X and an open neighborhood V of $\psi(x)$ in Y such that the restriction of ψ to U is a homeomorphism from U to a *closed* subset of V. But U is dense in X, so $\psi(U)$ is dense in Y and *a fortiori* in V, which proves that $\psi(U) = V$; since ψ is injective, $\psi^{-1}(V) = U$ and this proves that ψ is a homeomorphism from X to $\psi(X)$. □

8.3. Chevalley schemes.

(8.3.1). Let X be a *Noetherian* integral scheme, and R its field of rational functions; we denote by X' the set of local subrings $\mathcal{O}_x \subset R$, where x ranges over all points of X. The set X' satisfies the following three conditions.

- (Sch. 1) For all $M \in X'$, R is the field of fractions of M.
- (Sch. 2) There exists a finite set of Noetherian subrings A_i of R such that $X' = \bigcup_i L(A_i)$, and, for all pairs of indices i, j, the subring A_{ij} of R generated by $A_i \cup A_j$ is an algebra of finite type over A_i.
- (Sch. 3) Any two elements M and N of X' that are allied are identical.

We have seen in (8.2.1) that (Sch. 1) is satisfied, and (Sch. 3) follows from (8.2.2). To show (Sch. 2), it suffices to cover X by a finite number of affine open subsets U_i whose rings are Noetherian, and to take $A_i = \Gamma(U_i,\mathcal{O}_X)$; the hypothesis that X is a scheme implies that $U_i \cap U_j$ is affine, and also that $\Gamma(U_i \cap U_j,\mathcal{O}_X) = A_{ij}$ (5.5.6); further, since the space U_i is Noetherian, the immersion $U_i \cap U_j \to U_i$ is of finite type (6.3.5), so A_{ij} is an A_i-algebra of finite type (6.3.3).

(8.3.2). The structures whose axioms are (Sch. 1), (Sch. 2), and (Sch. 3) generalise "schemes", in the sense of C. Chevalley, who additionally supposes that R is an extension of finite type of a field K, and that the A_i are K-algebras of finite type (which renders a part of (Sch. 2) useless) [**CC**]. Conversely, if we have such a structure on a set X', then we can associate to it an integral scheme X by using the remarks from (8.2.6): the underlying space of X is equal to X' endowed with the topology defined in (8.2.6), and with the sheaf $\mathcal{O}_X$ such that $\Gamma(U, \mathcal{O}_X) = \bigcap_{x \in U} \mathcal{O}_x$ for all open $U \subset X$, with the evident definition of restriction homomorphisms. We leave to the reader the task of verifying that we thus obtain an integral scheme, whose local rings are the elements of X'; we will not use this result in what follows.

§9. Supplement on quasi-coherent sheaves

9.1. Tensor product of quasi-coherent sheaves.

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Proposition (9.1.1). — *Let X be a prescheme (resp. a locally Noetherian prescheme). Let $\mathcal{F}$ and $\mathcal{G}$ be quasi-coherent (resp. coherent) $\mathcal{O}_X$-modules; then $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ is quasi-coherent (resp. coherent); it is further of finite type if both $\mathcal{F}$ and $\mathcal{G}$ are of finite type. If $\mathcal{F}$ admits a finite presentation and if $\mathcal{G}$ is quasi-coherent (resp. coherent), then $\mathcal{H}om(\mathcal{F}, \mathcal{G})$ is quasi-coherent (resp. coherent).*

Proof. Being a local proposition, we can suppose that X is affine (resp. Noetherian affine); further, if $\mathcal{F}$ is coherent, then we can assume that it is the cokernel of a homomorphism $\mathcal{O}_X^m \to \mathcal{O}_X^n$. The claims pertaining to quasi-coherent sheaves then follow from Corollaries (1.3.12) and (1.3.9); the claims pertaining to coherent sheaves follow from Theorem (1.5.1) and from the fact that if M and N are modules of finite type over a Noetherian ring A then $M \otimes_A N$ and $\mathrm{Hom}_A(M, N)$ are both A-modules of finite type. □

Definition (9.1.2). — Let X and Y be S-preschemes, p and q the projections of $X \times_S Y$, and $\mathcal{F}$ (resp.$\mathcal{G}$) a quasi-coherent $\mathcal{O}_X$-module (resp. quasi-coherent $\mathcal{O}_Y$-module). We define the tensor product of $\mathcal{F}$ and $\mathcal{G}$ over $\mathcal{O}_S$ (*or* over S), denoted by $\mathcal{F} \otimes_{\mathcal{O}_S} \mathcal{G}$ (*or* $\mathcal{F} \otimes_S \mathcal{G}$) to be the tensor product $p^*(\mathcal{F}) \otimes_{\mathcal{O}_{X \times_S Y}} q^*(\mathcal{G})$ over the prescheme $X \times_S Y$.

If X_i $(1 \leqslant i \leqslant n)$ are S-preschemes, and $\mathcal{F}_i$ $(1 \leqslant i \leqslant n)$ are quasi-coherent $\mathcal{O}_{X_i}$-modules, then we similarly define the tensor product $\mathcal{F}_1 \otimes_S \mathcal{F}_2 \otimes_S \cdots \otimes_S \mathcal{F}_n$ over the prescheme $Z = X_1 \times_S X_2 \times_S \cdots \times_S X_n$; it is a *quasi-coherent* $\mathcal{O}_Z$-module by virtue of (9.1.1) and (**0**, 5.1.4); it is further *coherent* if all the $\mathcal{F}_i$ are coherent *and* Z is *locally Noetherian*, by virtue of (9.1.1), (**0**, 5.3.11), and (6.1.1).

Note that, if we take $X = Y = S$, then Definition (9.1.2) gives us back the tensor product of $\mathcal{O}_S$-modules. Furthermore, since $q^*(\mathcal{O}_Y) = \mathcal{O}_{X \times_S Y}$ (**0**, 4.3.4), the product $\mathcal{F} \otimes_S \mathcal{O}_Y$ is canonically identified with $p^*(\mathcal{F})$, and, in the same way, $\mathcal{O}_X \otimes_S \mathcal{G}$ is canonically identified with $q^*(\mathcal{G})$. In particular, if we take $Y = S$ and denote by f the structure morphism $X \to Y$, then we have that $\mathcal{O}_X \otimes_Y \mathcal{G} = f^*(\mathcal{G})$: the ordinary tensor product and the inverse image thus appear as particular cases of the general tensor product.

Definition (9.1.2) leads immediately to the fact that, for fixed X and Y, $\mathcal{F} \otimes_S \mathcal{G}$ is a *right-exact additive covariant bifunctor* in $\mathcal{F}$ and $\mathcal{G}$.

Proposition (9.1.3). — *Let S, X, and Y be affine schemes of rings A, B, and C (respectively), with B and C being A-algebras. Let M (resp. N) be a B-module (resp. C-module), and $\mathcal{F} = \widetilde{M}$ (resp. $\mathcal{G} = \widetilde{N}$) the associated quasi-coherent sheaf; then $\mathcal{F} \otimes_S \mathcal{G}$ is canonically isomorphic to the sheaf associated to the $(B \otimes_A C)$-module $M \otimes_A N$.*

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Proof. According to Proposition (1.6.5), $\mathcal{F} \otimes_S \mathcal{G}$ is canonically isomorphic to the sheaf associated to the $(B \otimes_A C)$-module

$$\big(M \otimes_B (B \otimes_A C)\big) \otimes_{B \otimes_A C} \big((B \otimes_A C) \otimes_C N\big)$$

and, by the canonical isomorphisms between tensor products, this latter module is isomorphic to

$$M \otimes_B (B \otimes_A C) \otimes_C N = (M \otimes_B B) \otimes_A (C \otimes_C N) = M \otimes_A N.$$

□

Proposition (9.1.4). — *Let $f : T \to X$ and $g : T \to Y$ be S-morphisms, and $\mathcal{F}$ (resp. $\mathcal{G}$) a quasi-coherent $\mathcal{O}_X$-module (resp. quasi-coherent $\mathcal{O}_Y$-module). Then*

$$(f, g)_S^*(\mathcal{F} \otimes_S \mathcal{G}) = f^*(\mathcal{F}) \otimes_{\mathcal{O}_T} g^*(\mathcal{G}).$$

Proof. If p, q are the projections of $X \times_S Y$, then the formula follows from the equalities $(f,g)_S^* \circ p^* = f^*$ and $(f,g)_S^* \circ q^* = g^*$ (**0**, 3.5.5), and the fact that the inverse image of a tensor product of algebraic sheaves is the tensor product of their inverse images (**0**, 4.3.3). □

Corollary (9.1.5). — *Let $f : X \to X'$ and $g : Y \to Y'$ be S-morphisms, and $\mathscr{F}'$ (resp. $\mathscr{G}'$) a quasi-coherent $\mathcal{O}_{X'}$-module (resp. quasi-coherent $\mathcal{O}_{Y'}$-module). Then*

$$(f,g)_S^*(\mathscr{F}' \otimes_S \mathscr{G}') = f^*(\mathscr{F}') \otimes_S g^*(\mathscr{G}')$$

Proof. This follows from (9.1.4) and the fact that $f \times_S g = (f \circ p, g \circ q)_S$, where p and q are the projections of $X \times_S Y$. □

Corollary (9.1.6). — *Let X, Y, and Z be S-preschemes, and $\mathscr{F}$ (resp. $\mathscr{G}$, $\mathscr{H}$) a quasi-coherent $\mathcal{O}_X$-module (resp. quasi-coherent $\mathcal{O}_Y$-module, quasi-coherent $\mathcal{O}_Z$-module); then the sheaf $\mathscr{F} \otimes_S \mathscr{G} \otimes_S \mathscr{H}$ is the inverse image of $(\mathscr{F} \otimes_S \mathscr{G}) \otimes_S \mathscr{H}$ by the canonical isomorphism from $X \times_S Y \times_S Z$ to $(X \times_S Y) \times_S Z$.*

Proof. This isomorphism is given by $(p_1, p_2)_S \times_S p_3$, where p_1, p_2, and p_3 are the projections of $X \times_S Y \times_S Z$. Similarly, the inverse image of $\mathscr{G} \otimes_S \mathscr{F}$ under the canonical isomorphism from $X \times_S Y$ to $Y \times_S X$ is $\mathscr{F} \otimes_S \mathscr{G}$. □

Corollary (9.1.7). — *If X is an S-prescheme, then every quasi-coherent $\mathcal{O}_X$-module $\mathscr{F}$ is the inverse image of $\mathscr{F} \otimes_S \mathcal{O}_S$ by the canonical isomorphism from X to $X \times_S S$ (3.3.3).*

Proof. This isomorphism is $(1_X, \varphi)_S$, where φ is the structure morphism $X \to S$, and the corollary follows from (9.1.4) and the fact that $\varphi^*(\mathcal{O}_S) = \mathcal{O}_X$. □

(9.1.8). Let X be an S-prescheme, $\mathscr{F}$ a quasi-coherent $\mathcal{O}_X$-module, and $\varphi : S' \to S$ a morphism; we denote by $\mathscr{F}_{(\varphi)}$ or $\mathscr{F}_{(S')}$ the quasi-coherent sheaf $\mathscr{F} \otimes_S \mathcal{O}_{S'}$ over $X \times_S S' = X_{(\varphi)} = X_{(S')}$; so $\mathscr{F}_{(S')} = p^*(\mathscr{F})$, where p is the projection $X_{(S')} \to X$.

Proposition (9.1.9). — *Let $\varphi'' : S'' \to S'$ be a morphism. For every quasi-coherent $\mathcal{O}_X$-module $\mathscr{F}$ on the S-prescheme X, $(\mathscr{F}_{(\varphi)})_{(\varphi')}$ is the inverse image of $\mathscr{F}_{(\varphi \circ \varphi')}$ by the canonical isomorphism $(X_{(\varphi)})_{(\varphi')} \simeq X_{(\varphi \circ \varphi')}$ (3.3.9).*

Proof. This follows immediately from the definitions and from (3.3.9), and is written

$$(\mathscr{F} \otimes_S \mathcal{O}_{S'}) \otimes_{S'} \mathcal{O}_{S''} = \mathscr{F} \otimes_S \mathcal{O}_{S''}. \tag{9.1.9.1}$$

□

Proposition (9.1.10). — *Let Y be an S-prescheme, and $f : X \to Y$ an S-morphism. For every quasi-coherent $\mathcal{O}_Y$-module $\mathscr{G}$ and every morphism $S' \to S$, we have that $(f_{(S')})^*(\mathscr{G}_{(S')}) = (f^*(\mathscr{G}))_{(S')}$.*

Proof. This follows immediately from the commutativity of the diagram **I | 171**

$$\begin{tikzcd} X_{(S')} \arrow[r, "f_{(S')}"] \arrow[d] & Y_{(S')} \arrow[d] \\ X \arrow[r, "f"] & Y. \end{tikzcd}$$

□

Corollary (9.1.11). — *Let X and Y be S-preschemes, and $\mathscr{F}$ (resp. $\mathscr{G}$) a quasi-coherent $\mathcal{O}_X$-module (resp. quasi-coherent $\mathcal{O}_Y$-module). Then the inverse image of the sheaf $(\mathscr{F}_{(S')}) \otimes_{(S')} (\mathscr{G}_{(S')})$ by the canonical isomorphism $(X \times_S Y)_{(S')} \simeq (X_{(S')}) \times_{S'} (Y_{(S')})$ (3.3.10) is equal to $(\mathscr{F} \otimes_S \mathscr{G})_{(S')}$.*

Proof. If p and q are the projections of $X \times_S Y$, then the isomorphism in question is nothing but $(p_{(S')}, q_{(S')})_{S'}$; the corollary then follows from Propositions (9.1.4) and (9.1.10). □

Proposition (9.1.12). — *With the notation from Definition (9.1.2), let z be a point of $X \times_S Y$, and let $x = p(z)$, and $y = q(z)$; the stalk $(\mathscr{F} \otimes_S \mathscr{G})_z$ is isomorphic to $(\mathscr{F}_x \otimes_{\mathcal{O}_x} \mathcal{O}_z) \otimes_{\mathcal{O}_z} (\mathscr{G}_y \otimes_{\mathcal{O}_y} \mathcal{O}_z) = \mathscr{F}_x \otimes_{\mathcal{O}_x} \mathcal{O}_z \otimes_{\mathcal{O}_y} \otimes \mathscr{G}_y$.*

Proof. Since we can reduce to the affine case, the proposition follows from Equation (1.6.5.1). □

Corollary (9.1.13). — *If $\mathscr{F}$ and $\mathscr{G}$ are of finite type, then*

$$\operatorname{Supp}(\mathscr{F} \otimes_S \mathscr{G}) = p^{-1}(\operatorname{Supp}(\mathscr{F})) \cap q^{-1}(\operatorname{Supp}(\mathscr{G})).$$

Proof. Since $p^*(\mathscr{F})$ and $q^*(\mathscr{G})$ are both of finite type over $\mathscr{O}_{X\times_S Y}$, we reduce, by Proposition (9.1.12) and by (**0**, 1.7.5), to the case where $\mathscr{G} = \mathscr{O}_Y$, that is, it remains to prove the following equation:

$$\mathrm{Supp}(p^*(\mathscr{F})) = p^{-1}(\mathrm{Supp}(\mathscr{F})). \tag{9.1.13.1}$$

The same reasoning as in (**0**, 1.7.5) leads us to prove that, for all $z \in X \times_S Y$, we have $\mathscr{O}_z/\mathfrak{m}_x\mathscr{O}_z \neq 0$ (with $x = p(z)$), which follows from the fact that the homomorphism $\mathscr{O}_x \to \mathscr{O}_z$ is *local*, by hypothesis. □

We leave it to the reader to extend the results in this section to the more general case of an arbitrary (but finite) number of factors, instead of just two.

9.2. Direct image of a quasi-coherent sheaf.

Proposition (9.2.1). — *Let $f : X \to Y$ be a morphism of preschemes. We suppose that there exists a cover (Y_α) of Y by affine opens having the following property: every $f^{-1}(Y_\alpha)$ admits a* finite *cover $(X_{\alpha i})$ by affine opens that are contained in $f^{-1}(Y_\alpha)$ and that are such that every intersection $X_{\alpha i} \cap X_{\alpha j}$ is itself a* finite *union of affine opens. With these hypotheses, for every quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$, $f_*(\mathscr{F})$ is a quasi-coherent $\mathscr{O}_Y$-module.*

Proof. Since this is a local condition on Y, we can assume that Y is equal to one of the Y_α, and thus omit the indices α.

(a) First, suppose that the $X_i \cap X_j$ are themselves *affine* opens. We set $\mathscr{F}_i = \mathscr{F}|X_i$ and $\mathscr{F}_{ij} = \mathscr{F}|(X_i \cap X_j)$, and let $\mathscr{F}'_i$ and $\mathscr{F}'_{ij}$ be the images of $\mathscr{F}_i$ and $\mathscr{F}_{ij}$ (respectively) by the restriction of f to X_i and to $X_i \cap X_j$ (respectively); we know that the $\mathscr{F}'_i$ and $\mathscr{F}'_{ij}$ are quasi-coherent (1.6.3). Set $\mathscr{G} = \bigoplus_i \mathscr{F}'_i$ and $\mathscr{H} = \bigoplus_{i,j} \mathscr{F}'_{ij}$; $\mathscr{G}$ and $\mathscr{H}$ are quasi-coherent $\mathscr{O}_Y$-modules; we will define a homomorphism $u : \mathscr{G} \to \mathscr{H}$ such that $f_*(\mathscr{F})$ is the *kernel* of u; it will follow from this that $f_*(\mathscr{F})$ is quasi-coherent (1.3.9). It suffices to define u as a homomorphism of presheaves; taking into account the definitions of $\mathscr{G}$ and $\mathscr{H}$, it thus suffices, for every open subset $W \subset Y$, to define a homomorphism **I | 172**

$$u_W : \bigoplus_i \Gamma(f^{-1}(W) \cap X_i, \mathscr{F}) \longrightarrow \bigoplus_{i,j} \Gamma(f^{-1}(W) \cap X_i \cap X_j, \mathscr{F})$$

that satisfies the usual compatibility conditions when we let W vary. If, for every section $s_i \in \Gamma(f^{-1}(W) \cap X_i, \mathscr{F})$, we denote by $s_{i|j}$ its restriction to $f^{-1}(W) \cap X_i \cap X_j$, then we set

$$u_W\big((s_i)\big) = (s_{i|j} - s_{j|i})$$

and the compatibility conditions are clearly satisfied. To prove that the kernel $\mathscr{R}$ of u is $f_*(\mathscr{F})$, we define a homomorphism from $f_*(\mathscr{F})$ to $\mathscr{R}$ by sending each section $s \in \Gamma(f^{-1}(W), \mathscr{F})$ to the family (s_i), where s_i is the restriction of s to $f^{-1}(W) \cap X_i$; axioms (F1) and (F2) of sheaves (G, II, 1.1) tell us that this homomorphism is *bijective*, which finishes the proof in this case.

(b) In the general case, the same reasoning applies once we have established that the $\mathscr{F}_{ij}$ are quasi-coherent. But, by hypothesis, $X_i \cap X_j$ is a finite union of affine opens X_{ijk}; and since the X_{ijk} are affine opens *in a scheme*, the intersection of any two of them is again an affine open (5.5.6). We are thus led to the first case, and so we have proved Proposition (9.2.1).

□

Corollary (9.2.2). — *The conclusion of Proposition* (9.2.1) *holds true in each of the following cases:*

(a) *f is separated and quasi-compact;*
(b) *f is separated and of finite type;*
(c) *f is quasi-compact, and the underlying space of X is locally Noetherian.*

Proof. In case (a), the $X_{\alpha i} \cap X_{\alpha j}$ are affine (5.5.6). Case (b) is a particular example of case (a) (6.6.3). Finally, in case (c), we can reduce to the case where Y is affine and the underlying space of X is Noetherian; then X admits a finite cover of affine opens (X_i), and the $X_i \cap X_j$, being quasi-compact, are finite unions of affine opens (2.1.3). □

9.3. Extension of sections of quasi-coherent sheaves.

Theorem (9.3.1). — *Let X be a prescheme whose underlying space is Noetherian, or a scheme whose underlying space is quasi-compact. Let $\mathscr{L}$ be an invertible $\mathscr{O}_X$-module* (**0**, 5.4.1)*, f a section of $\mathscr{L}$ over X, X_f the open set of $x \in X$ such that $f(x) \neq 0$* (**0**, 5.5.1)*, and $\mathscr{F}$ a quasi-coherent $\mathscr{O}_X$-module.*

(i) *If $s \in \Gamma(X, \mathscr{F})$ is such that $s|X_f = 0$, then there exists an integer $n > 0$ such that $s \otimes f^{\otimes n} = 0$.*
(ii) *For every section $s \in \Gamma(X_f, \mathscr{F})$, there exists an integer $n > 0$ such that $s \otimes f^{\otimes n}$ extends to a section of $\mathscr{F} \otimes \mathscr{L}^{\otimes n}$ over X.*

Proof.

(i) Since the underlying space of X is quasi-compact, and thus the union of finitely-many affine opens U_i with $\mathscr{L}|U_i$ isomorphic to $\mathscr{O}_X|U_i$, we can reduce to the case where X is affine and $\mathscr{L}=\mathscr{O}_X$. In this case, f can be identified with an element of $A(X)$, and we have that $X_f=D(f)$; s can be identified with an element of an $A(X)$-module M, and $s|X_f$ to the corresponding element of M_f, and the result is then trivial, recalling the definition of a module of fractions. **I** | 173

(ii) Again, X is a finite union of affine opens U_i $(1\leqslant i\leqslant r)$ such that $\mathscr{L}|U_i\cong\mathscr{O}_X|U_i$, and, for every i, $(s\otimes f^{\otimes n})|(U_i\cap X_f)$ can be identified (by the aforementioned isomorphism) with $(f|(U_i\cap X_f))^n(s|(U_i\cap X_f))$. We then know (1.4.1) that there exists an integer $n>0$ such that, for all i, $(s\otimes f^{\otimes n})|(U_i\cap X_f)$ extends to a section s_i of $\mathscr{F}\otimes\mathscr{L}^{\otimes n}$ over U_i. Let $s_{i|j}$ be the restriction of s_i to $U_i\cap U_j$; by definition we have that $s_{i|j}-s_{j|i}=0$ on $X_f\cap U_i\cap U_j$. But, if X is a Noetherian space, then $U_i\cap U_j$ is quasi-compact; if X is a scheme, then $U_i\cap U_j$ is an affine open (5.5.6), and so again quasi-compact. By virtue of (i), there thus exists an integer m (independent of i and j) such that $(s_{i|j}-s_{j|i})\otimes f^{\otimes m}=0$. It immediately follows that there exists a section s' of $\mathscr{F}\otimes\mathscr{L}^{\otimes(n+m)}$ over X that restricts to $s_i\otimes f^{\otimes m}$ over each U_i, and restricts to $s\otimes f^{\otimes(n+m)}$ over X_f.

□

The following corollaries give an interpretation of Theorem (9.3.1) in a more algebraic language:

Corollary (9.3.2). — *With the hypotheses of (9.3.1), consider the graded ring $A_*=\Gamma_*(\mathscr{L})$ and the graded A_*-module $M_*=\Gamma_*(\mathscr{L},\mathscr{F})$ (**0**, 5.4.6). If $f\in A_n$, where $n\in\mathbf{Z}$, then there is a canonical isomorphism $\Gamma(X_f,\mathscr{F})\simeq((M_*)_f)_0$ (*the subgroup of the module of fractions $(M_*)_f$ consisting of elements of degree 0*).*

Corollary (9.3.3). — *Suppose that the hypotheses of (9.3.1) are satisfied, and suppose further that $\mathscr{L}=\mathscr{O}_X$. Then, setting $A=\Gamma(X,\mathscr{O}_X)$ and $M=\Gamma(X,\mathscr{F})$, the A_f-module $\Gamma(X_f,\mathscr{F})$ is canonically isomorphic to M_f.*

Proposition (9.3.4). — *Let X be a Noetherian prescheme, $\mathscr{F}$ a coherent $\mathscr{O}_X$-module, and $\mathscr{J}$ a coherent sheaf of ideals in $\mathscr{O}_X$, such that the support of $\mathscr{F}$ is contained in that of $\mathscr{O}_X/\mathscr{J}$. Then there exists an integer $n>0$ such that $\mathscr{J}^n\mathscr{F}=0$.*

Proof. Since X is a union of finitely-many affine opens whose rings are Noetherian, we can suppose that X is affine, given by some Noetherian ring A; then $\mathscr{F}=\widetilde{M}$, where $M=\Gamma(X,\mathscr{F})$ is an A-module of finite type, and $\mathscr{J}=\widetilde{\mathfrak{J}}$, where $\mathfrak{J}=\Gamma(X,\mathscr{J})$ is an ideal of A ((1.4.1) and (1.5.1)). Since A is Noetherian, $\mathfrak{J}$ admits a finite system of generators f_i $(1\leqslant i\leqslant m)$. By hypothesis, every section of $\mathscr{F}$ over X is zero on each of the $D(f_i)$; if s_j $(1\leqslant j\leqslant q)$ are sections of $\mathscr{F}$ that generate M, then there exists an integer h, independent of i and j, such that $f_i^h s_j=0$ (1.4.1), whence $f_i^h s=0$ for all $s\in M$. We thus conclude that, if $n=mh$, then $\mathfrak{J}^nM=0$, and so the corresponding $\mathscr{O}_X$-module $\mathscr{J}^n\mathscr{F}=\widetilde{\mathfrak{J}^nM}$ (1.3.13) is zero. □

Corollary (9.3.5). — *With the hypotheses of (9.3.4), there exists a closed subprescheme Y of X, whose underlying space is the support of $\mathscr{O}_X/\mathscr{J}$, such that, if $j:Y\to X$ is the canonical injection, then $\mathscr{F}=j_*(j^*(\mathscr{F}))$.*

Proof. First, note that the supports of $\mathscr{O}_X/\mathscr{J}$ and $\mathscr{O}_X/\mathscr{J}^n$ are the same, since, if $\mathscr{J}_x=\mathscr{O}_x$, then $\mathscr{J}_x^n=\mathscr{O}_x$, and we also have that $\mathscr{J}_x^n\subset\mathscr{J}_x$ for all $x\in X$. We can, thanks to (9.3.4), thus suppose that $\mathscr{J}\mathscr{F}=0$; we can then take Y to be the closed subprescheme of X defined by $\mathscr{J}$, and since $\mathscr{F}$ is then an $(\mathscr{O}_X/\mathscr{J})$-module, the conclusion follows immediately. □

9.4. Extension of quasi-coherent sheaves.

(9.4.1). Let X be a topological space, $\mathscr{F}$ a sheaf of sets (resp. of groups, of rings) on X, U an open subset of X, $\psi:U\to X$ the canonical injection, and $\mathscr{G}$ a subsheaf of $\mathscr{F}|U=\psi^*(\mathscr{F})$. Since ψ_* is left exact, $\psi_*(\mathscr{G})$ is a subsheaf of $\psi_*(\psi^*(\mathscr{F}))$; we denote by ρ the canonical homomorphism $\mathscr{F}\to\psi_*(\psi^*(\mathscr{F}))$ (**0**, 3.5.3), and we denote by $\overline{\mathscr{G}}$ the subsheaf $\rho^{-1}(\psi_*(\mathscr{G}))$ of $\mathscr{F}$. It follows immediately from the definitions that, for every open subset V of X, $\Gamma(V,\overline{\mathscr{G}})$ consists of sections $s\in\Gamma(V,\mathscr{F})$ whose restriction to $V\cap U$ is a section of $\mathscr{G}$ over $V\cap U$. We thus have that $\overline{\mathscr{G}}|U=\psi^*(\overline{\mathscr{G}})=\mathscr{G}$, and that $\overline{\mathscr{G}}$ is the *largest* subsheaf of $\mathscr{F}$ that restricts to $\mathscr{G}$ over U; we say that $\overline{\mathscr{G}}$ is the *canonical extension* of the subsheaf $\mathscr{G}$ of $\mathscr{F}|U$ to a subsheaf of $\mathscr{F}$. **I** | 174

Proposition (9.4.2). — *Let X be a prescheme, and U an open subset of X such that the canonical injection $j:U\to X$ is a quasi-compact morphism* (which will be the case for *all* U if the underlying space of X is *locally Noetherian* (6.6.4, i)). *Then:*

(i) *for every quasi-coherent $(\mathscr{O}_X|U)$-module $\mathscr{G}$, $j_*(\mathscr{G})$ is a quasi-coherent $\mathscr{O}_X$-module, and $j_*(\mathscr{G})|U=j^*(j_*(\mathscr{G}))=\mathscr{G}$;*

(ii) *for every quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$ and every quasi-coherent $(\mathscr{O}_X|U)$-submodule $\mathscr{G}$, the canonical extension $\overline{\mathscr{G}}$ of $\mathscr{G}$ (9.4.1) is a quasi-coherent $\mathscr{O}_X$-submodule of $\mathscr{F}$.*

Proof. If $j = (\psi, \theta)$ (ψ being the injection $U \to X$ of underlying spaces), then, by definition, we have that $j_*(\mathscr{G}) = \psi_*(\mathscr{G})$ for every $(\mathscr{O}_X|U)$-module $\mathscr{G}$, and, further, that $j^*(\mathscr{H}) = \psi^*(\mathscr{H}) = \mathscr{H}|U$ for every $\mathscr{O}_X$-module $\mathscr{H}$, by definition of the prescheme induced over an open subset. So (i) is thus a particular case of ((9.2.2, a)); for the same reason, $j_*(j^*(\mathscr{F}))$ is quasi-coherent, and since $\overline{\mathscr{G}}$ is the inverse image of $j_*(\mathscr{G})$ by the homomorphism $\rho : \mathscr{F} \to j_*(j^*(\mathscr{F}))$, (ii) follows from (4.1.1). ☐

Note that the hypothesis that the morphism $j : U \to X$ is quasi-compact holds whenever the open subset U is *quasi-compact* and X is a *scheme*: indeed, U is then a union of finitely-many affine opens U_i, and, for every affine open V of X, $V \cap U_i$ is an affine open (5.5.6), and thus quasi-compact.

Corollary (9.4.3). — *Let X be a prescheme, and U a quasi-compact open subset of X such that the injection morphism $j : U \to X$ is quasi-compact. Suppose as well that every quasi-coherent $\mathscr{O}_X$-module is the inductive limit of its quasi-coherent $\mathscr{O}_X$-submodules of finite type* (which will be the case if X is an *affine scheme*). *Then let $\mathscr{F}$ be a quasi-coherent $\mathscr{O}_X$-module, and $\mathscr{G}$ a quasi-coherent $(\mathscr{O}_X|U)$-submodule of $\mathscr{F}|U$* of finite type. *Then there exists a quasi-coherent $\mathscr{O}_X$-submodule $\mathscr{G}'$ of $\mathscr{F}$* of finite type *such that $\mathscr{G}'|U = \mathscr{G}$.*

Proof. We have $\mathscr{G} = \overline{\mathscr{G}}|U$, and $\overline{\mathscr{G}}$ is quasi-coherent, from (9.4.2), so the inductive limit of its quasi-coherent $\mathscr{O}_X$-submodules $\mathscr{H}_\lambda$ of finite type. It follows that $\mathscr{G}$ is the inductive limit of the $\mathscr{H}_\lambda|U$, and thus equal to one of the $\mathscr{H}_\lambda|U$, since it is of finite type (**0**, 5.2.3). ☐

Remark (9.4.4). — Suppose that for *every* affine open $U \subset X$, the injection morphism $U \to X$ is quasi-compact. Then, if the conclusion of (9.4.3) holds for every affine open U and for every quasi-coherent $(\mathscr{O}_X|U)$-submodule $\mathscr{G}$ of $\mathscr{F}|U$ of finite type, it follows that $\mathscr{F}$ is the inductive limit of its quasi-coherent $\mathscr{O}_X$-submodules of finite **I** | 175
type. Indeed, for every affine open $U \subset X$, we have that $\mathscr{F}|U = \widetilde{M}$, where M is an $A(U)$-module, and since the latter is the inductive limit of its quasi-coherent submodules of finite type, $\mathscr{F}|U$ is the inductive limit of its $(\mathscr{O}_X|U)$-submodules of finite type (1.3.9). But, by hypothesis, each of these submodules is induced on U by a quasi-coherent $\mathscr{O}_X$-submodule $\mathscr{G}_{\lambda,U}$ of $\mathscr{F}$ of finite type. The finite sums of the $\mathscr{G}_{\lambda,U}$ are again quasi-coherent $\mathscr{O}_X$-modules of finite type, because this is a local property, and the case where X is affine was covered in (1.3.10); it is clear then that $\mathscr{F}$ is the inductive limit of these finite sums, whence our claim.

Corollary (9.4.5). — *Under the hypotheses of Corollary (9.4.3), for every quasi-coherent $(\mathscr{O}_X|U)$-module $\mathscr{G}$ of finite type, there exists a quasi-coherent $\mathscr{O}_X$-module $\mathscr{G}'$ of finite type such that $\mathscr{G}'|U = \mathscr{G}$.*

Proof. Since $\mathscr{F} = j_*(\mathscr{G})$ is quasi-coherent (9.4.2) and $\mathscr{F}|U = \mathscr{G}$, it suffices to apply Corollary (9.4.3) to $\mathscr{F}$. ☐

Lemma (9.4.6). — *Let X be a prescheme, L a well-ordered set, $(V_\lambda)_{\lambda \in L}$ a cover of X by affine opens, and U an open subset of X; for all $\lambda \in L$, we set $W_\lambda = \bigcup_{\mu<\lambda} V_\mu$. Suppose that: (1) for every $\lambda \in L$, $V_\lambda \cap W_\lambda$ is quasi-compact; and (2) the immersion morphism $U \to X$ is quasi-compact. Then, for every quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$ and every quasi-coherent $(\mathscr{O}_X|U)$-submodule $\mathscr{G}$ of $\mathscr{F}|U$* of finite type*, there exists a quasi-coherent $\mathscr{O}_X$-submodule $\mathscr{G}'$ of $\mathscr{F}$* of finite type *such that $\mathscr{G}'|U = \mathscr{G}$.*

Proof. Let $U_\lambda = U \cup W_\lambda$; we will define a family $(\mathscr{G}'_\lambda)$ by induction, where $\mathscr{G}'_\lambda$ is a quasi-coherent $(\mathscr{O}_X|U_\lambda)$-submodule of $\mathscr{F}|U_\lambda$ of finite type, such that $\mathscr{G}'_\lambda|U_\mu = \mathscr{G}'_\mu$ for $\mu < \lambda$ and $\mathscr{G}'_\lambda|U = \mathscr{G}$. The unique $\mathscr{O}_X$-submodule $\mathscr{G}'$ of $\mathscr{F}$ such that $\mathscr{G}'|U_\lambda = \mathscr{G}'$ for all $\lambda \in L$ (**0**, 3.3.1) will then give us what we want. So suppose that the $\mathscr{G}'_\mu$ are defined and have the preceding properties for $\mu < \lambda$; if λ does not have a predecessor then we take $\mathscr{G}'_\lambda$ to be the unique $(\mathscr{O}_X|U_\lambda)$-submodule of $\mathscr{F}|U_\lambda$ such that $\mathscr{G}'_\lambda|U_\mu = \mathscr{G}'_\mu$ for all $\mu < \lambda$, which is allowed since the U_μ with $\mu < \lambda$ then form a cover of U_λ. If, conversely, $\lambda = \mu + 1$, then $U_\lambda = U_\mu \cup V_\mu$, and it suffices to define a quasi-coherent $(\mathscr{O}_X|V_\mu)$-submodule $\mathscr{G}''_\mu$ of $\mathscr{F}|V_\mu$ of finite type such that

$$\mathscr{G}''_\mu|(U_\mu \cap V_\mu) = \mathscr{G}'_\mu|(U_\mu \cap V_\mu);$$

and then to take $\mathscr{G}'_\lambda$ to be the $(\mathscr{O}_X|U_\lambda)$-submodule of $\mathscr{F}|U_\lambda$ such that $\mathscr{G}'_\lambda|U_\mu = \mathscr{G}'_\mu$ and $\mathscr{G}'_\lambda|V_\mu = \mathscr{G}''_\mu$ (**0**, 3.3.1). But, since V_μ is affine, the existence of $\mathscr{G}''_\mu$ is guaranteed by (9.4.3) as soon as we show that $U_\mu \cap V_\mu$ is quasi-compact; but $U_\mu \cap V_\mu$ is the union of $U \cap V_\mu$ and $W_\mu \cap V_\mu$, which are both quasi-compact by virtue of the hypotheses. ☐

Theorem (9.4.7). — *Let X be a prescheme, and U an open subset of X. Suppose that one of the following conditions is verified:*

(a) *the underlying space of X is locally Noetherian;*
(b) *X is a quasi-compact scheme and U is a quasi-compact open.*

Then, for every quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$ and every quasi-coherent $(\mathscr{O}_X|U)$-submodule $\mathscr{G}$ of $\mathscr{F}|U$ of finite type*, there exists a quasi-coherent $\mathscr{O}_X$-submodule $\mathscr{G}'$ of $\mathscr{F}$* of finite type *such that $\mathscr{G}'|U = \mathscr{G}$.*

Proof. Let $(V_\lambda)_{\lambda\in L}$ be a cover of X by affine opens, with L assumed to be finite in case (b); since L is equipped with the structure of a well-ordered set, it suffices to check that the conditions of (9.4.6) are satisfied. It is clear in the case of (a), since the spaces V_λ are Noetherian. For case (b), the $V_\lambda \cap \lambda_\mu$ are affine (5.5.6), and thus quasi-compact, and, since L is finite, $V_\lambda \cap W_\lambda$ is quasi-compact. Whence the theorem. □ **I | 176**

Corollary (9.4.8). — *Under the hypotheses of (9.4.7), for every quasi-coherent $(\mathscr{O}_X|U)$-module $\mathscr{G}$ of finite type, there exists a quasi-coherent $\mathscr{O}_X$-module $\mathscr{G}'$ of finite type such that $\mathscr{G}'|U = \mathscr{G}$.*

Proof. It suffices to apply (9.4.7) to $\mathscr{F} = j_*(\mathscr{G})$, which is quasi-coherent (9.4.2) and such that $\mathscr{F}|U = \mathscr{G}$. □

Corollary (9.4.9). — *Let X be a prescheme whose underlying space is locally Noetherian, or a quasi-compact scheme. Then every quasi-coherent $\mathscr{O}_X$-module is the inductive limit of its quasi-coherent $\mathscr{O}_X$-submodules of finite type.*

Proof. This follows from Theorem (9.4.7) and Remark (9.4.4). □

Corollary (9.4.10). — *Under the hypotheses of (9.4.9), if a quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$ is such that every quasi-coherent $\mathscr{O}_X$-submodule of finite type of $\mathscr{F}$ is generated by its sections over X, then $\mathscr{F}$ is generated by its sections over X.*

Proof. Let U be an affine open neighborhood of a point $x \in X$, and let s be a section of $\mathscr{F}$ over U; the $\mathscr{O}_X$-submodule $\mathscr{G}$ of $\mathscr{F}|U$ generated by s is quasi-coherent and of finite type, so there exists a quasi-coherent $\mathscr{O}_X$-submodule $\mathscr{G}'$ of $\mathscr{F}$ of finite type such that $\mathscr{G}'|U = \mathscr{G}$ (9.4.7). By hypothesis, there thus exists a finite number of sections t_i of $\mathscr{G}'$ over X and of sections a_i of $\mathscr{O}_X$ over a neighborhood $V \subset U$ of x such that $s|V = \sum_i a_i(t_i|V)$, which proves the corollary. □

9.5. Closed image of a prescheme; closure of a subprescheme.

Proposition (9.5.1). — *Let $f : X \to Y$ be a morphism of preschemes such that $f_*(\mathscr{O}_X)$ is a quasi-coherent $\mathscr{O}_Y$-module (which will be the case if f is quasi-compact* and, *in addition, either f is separated or X is locally Noetherian (9.2.2)). Then there exists a smaller subprescheme Y' of Y such that f factors through the canonical injection $j : Y' \to Y$ (*or, equivalently (4.4.1), such that the subprescheme $f^{-1}(Y')$ of X is *identical* to X*).*

More precisely:

Corollary (9.5.2). — *Under the conditions of (9.5.1), let $f = (\psi,\theta)$, and let $\mathscr{J}$ be the (quasi-coherent) kernel of the homomorphism $\theta : \mathscr{O}_Y \to f_*(\mathscr{O}_X)$. Then the closed subprescheme Y' of Y defined by $\mathscr{J}$ satisfies the conditions of (9.5.1).*

Proof. Since the functor ψ^* is exact, the canonical factorization $\theta : \mathscr{O}_Y \to \mathscr{O}_Y/\mathscr{J} \xrightarrow{\theta'} \psi_*(\mathscr{O}_X)$ gives (**0**, 3.5.4.3) a factorization $\theta^\sharp : \psi^*(\mathscr{O}_Y) \to \psi^*(\mathscr{O}_Y)/\psi^*(\mathscr{J}) \xrightarrow{\theta'^\sharp} \mathscr{O}_X$; since $\theta^\sharp_x$ is a local homomorphism for every $x \in X$, the same is true of ${\theta'}^\sharp_x$; if we denote by ψ_0 the continuous map ψ considered as a map from X to Y', and by θ_0 **Err$_{\mathrm{II}}$** the restriction $\theta'|X' : (\mathscr{O}_Y/\mathscr{J})|Y' \to \psi_*(\mathscr{O}_X)|Y' = (\psi_0)_*(\mathscr{O}_X)$, then we see that $f_0 = (\psi_0,\theta_0)$ is a morphism of preschemes $X \to Y'$ (2.2.1) such that $f = j \circ f_0$. Now, if Y'' is a second closed subprescheme of Y, defined by a **I | 177** quasi-coherent sheaf of ideals $\mathscr{J}'$ of $\mathscr{O}_Y$, such that f factors through the injection $j' : Y'' \to Y$, then we should immediately have that $\psi(X) \subset Y''$, and thus that $Y' \subset Y''$, since Y'' is closed. Furthermore, for all $y \in Y''$, θ should factor as $\mathscr{O}_y \to \mathscr{O}_y/\mathscr{J}'_y \to (\psi_*(\mathscr{O}_X))_y$, which, by definition, leads to $\mathscr{J}'_y \subset \mathscr{J}_y$, and thus X' is a closed subprescheme of Y'' (4.1.10). □

Definition (9.5.3). — Whenever there exists a smaller subprescheme Y' of Y such that f factors through the canonical injection $j : Y' \to Y$, we say that Y' is the *closed image* prescheme of X under the morphism f.

Proposition (9.5.4). — *If $f_*(\mathscr{O}_X)$ is a quasi-coherent $\mathscr{O}_Y$-module, then the underlying space of the closed image of X under f is the closure $\overline{f(X)}$ in Y.*

Proof. As the support of $f_*(\mathscr{O}_X)$ is contained in $\overline{f(X)}$, we have (with the notation of (9.5.2)) $\mathscr{J}_y = \mathscr{O}_y$ for $y \notin \overline{f(X)}$, thus the support of $\mathscr{O}_Y/\mathscr{J}$ is contained in $\overline{f(X)}$. In addition, this support is closed and contains $f(X)$: indeed, if $y \in f(X)$, the unit element of the ring $(\psi_*(\mathscr{O}_X))_y$ is not zero, being the germ at y of the section

$$1 \in \Gamma(X,\mathscr{O}_X) = \Gamma(Y,\psi_*(\mathscr{O}_X));$$

since it is the image under θ of the unit element of $\mathscr{O}_y$, the latter does not belong to $\mathscr{J}_y$, hence $\mathscr{O}_y/\mathscr{J}_y \neq 0$; this finishes the proof. □

Proposition (9.5.5). — (Transitivity of closed images). *Let $f : X \to Y$ and $g : Y \to Z$ be two morphisms of preschemes; we suppose that the closed image Y' of X under f exists, and that, if g' is the restriction of g to Y', then the closed image Z' of Y' under g' exists. Then the closed image of X under $g \circ f$ exists and is equal to Z'.*

Proof. It suffices (9.5.1) to show that Z' is the smallest closed subprescheme Z_1 of Z such that the closed subprescheme $(g \circ f)^{-1}(Z_1)$ of X (equal to $f^{-1}(g^{-1}(Z_1))$ by Corollary (4.4.2)) is equal to X; it is equivalent to say that Z' is the smallest closed subprescheme of Z such that f factors through the injection $g^{-1}(Z_1) \to Y$ (4.4.1). By virtue of the existence of the closed image Y', every Z_1 with this property is such that $g^{-1}(Z_1)$ factors through Y', which is equivalent to saying that $j^{-1}(g^{-1}(Z_1)) = g'^{-1}(Z_1) = Y'$, denoting by j the injection $Y' \to Y$. By the definition of Z', we indeed conclude that Z' is the smallest closed subprescheme of Z satisfying the preceding condition. □

Corollary (9.5.6). — *Let $f : X \to Y$ be an S-morphism such that Y is the closed image of X under f. Let Z be an S-scheme; if two S-morphisms g_1, g_2 from Y to Z are such that $g_1 \circ f = g_2 \circ f$, then $g_1 = g_2$.*

Proof. Let $h = (g_1, g_2)_S : Y \to Z \times_S Z$; since the diagonal $T = \Delta_Z(Z)$ is a closed subprescheme of $Z \times_S Z$, $Y' = h^{-1}(T)$ is a closed subprescheme of Y (4.4.1). Let $u = g_1 \circ f = g_2 \circ f$; we then have, by definition of the product, $h' = h \circ f = (u, u)_S$, so $h \circ f = \Delta_Z \circ u$; since $\Delta_Z^{-1}(T) = Z$, we have $h'^{-1}(T) = u^{-1}(Z) = X$, so $f^{-1}(Y') = X$. From this, we conclude (4.4.1) that the f factors through the canonical injection $Y' \to Y$, so $Y' = Y$ by hypothesis; it then follows (4.4.1) that h factors as $\Delta_Z \circ v$, where v is a morphism $Y \to Z$, which implies that $g_1 = g_2 = v$. □

Remark (9.5.7). — If X and Y are S-schemes, Proposition (9.5.6) implies that, when Y is the closed image of X under f, f is an *epimorphism* in the category of *S-schemes* (T, 1.1). We will show in Chapter V that, conversely, if the closed image Y' of X under f exists and if f is an epimorphism of S-schemes, then we necessarily have that $Y' = Y$. **I** | 178

Proposition (9.5.8). — *Suppose that the hypotheses of (9.5.1) are satisfied, and let Y' be the closed image of X under f. For every open V of Y, let $f_V : f^{-1}(V) \to V$ be the restriction of f; then the closed image of $f^{-1}(V)$ under f_V in V exists and is equal to the prescheme induced by Y' on the open $V \cap Y'$ of Y'* (in other words, to the subprescheme $\inf(V, Y)$ of Y (4.4.3)).

Proof. Let $X' = f^{-1}(V)$; since the direct image of $\mathcal{O}_{X'}$ by f_V is exactly the restriction of $f_*(\mathcal{O}_X)$ to V, it is clear that the kernel $\mathcal{J}'$ of the homomorphism $\mathcal{O}_V \to (f_V)_*(\mathcal{O}_{X'})$ is the restriction of $\mathcal{J}$ to V, from which the proposition quickly follows. □

We will see that this result can be understood as saying that taking the closed image commutes with an extension $Y_1 \to Y$ of the base prescheme, which is an *open immersion*. We will see in Chapter IV that it is the same for an extension $Y_1 \to Y$ which is a *flat* morphism, provided that f is separated and quasi-compact.

Proposition (9.5.9). — *Let $f : X \to Y$ be a morphism such that the closed image Y' of X under f exists.*

(i) *If X is reduced, then so too is Y'.*
(ii) *If the hypotheses of Proposition (9.5.1) are satisfied and X is irreducible (resp. integral), then so too is Y'.*

Proof. By hypothesis, the morphism f factors as $X \xrightarrow{g} Y' \xrightarrow{j} Y$, where j is the canonical injection. Since X is reduced, g factors as $X \xrightarrow{h} {Y'}_{\text{red}} \xrightarrow{j'} Y'$, where j' is the canonical injection (5.2.2), and it then follows from the definition of Y' that $Y'_{\text{red}} = Y'$. If, moreover, the conditions of Proposition (9.5.1) are satisfied, then it follows from (9.5.4) that $f(X)$ is dense in Y'; if X is irreducible, then so is Y' (**0**, 2.1.5). The claim about integral preschemes follows from the conjunction of the two others. □

Proposition (9.5.10). — *Let Y be a subprescheme of a prescheme X, such that the canonical injection $i : Y \to X$ is a quasi-compact morphism. Then there exists a smaller closed subprescheme $\overline{Y}$ of X containing Y; its underlying space is the closure of that of Y; the latter is open in its closure, and the prescheme Y is induced on this open by $\overline{Y}$.*

Proof. It suffices to apply Proposition (9.5.1) to the injection j, which is separated (5.5.1) and quasi-compact by hypothesis; (9.5.1) thus proves the existence of $\overline{Y}$, and (9.5.4) shows that its underlying space is the closure of Y in X; since Y is locally closed in X, it is open in $\overline{Y}$, and the last claim comes from (9.5.8) applied to an open subset V of X such that Y is closed in V. □

With the above notation, if the injection $V \to X$ is quasi-compact, and if $\mathcal{J}$ is the quasi-coherent sheaf of ideals of $\mathcal{O}_X|V$ defining the closed subprescheme Y of V, it follows from Proposition (9.5.1) that the quasi-coherent sheaf of ideals of $\mathcal{O}_X$ defining $\overline{Y}$ is the canonical extension (9.4.1) $\overline{\mathcal{J}}$ of $\mathcal{J}$, because it is clearly the largest quasi-coherent subsheaf of ideals of $\mathcal{O}_X$ inducing $\mathcal{J}$ on V.

Corollary (9.5.11). — *Under the hypotheses of Proposition (9.5.10), every section of $\mathscr{O}_{\overline{Y}}$ over an open V of $\overline{Y}$ that is zero on $V \cap Y$ is zero.* **I | 179**

Proof. By Proposition (9.5.8), we can reduce to the case where $V = \overline{Y}$. If we take into account that the sections of $\mathscr{O}_{\overline{Y}}$ over $\overline{Y}$ canonically correspond to the $\overline{Y}$-sections of $\overline{Y} \otimes_Z Z[T]$ (3.3.15) and that the latter is separated over $\overline{Y}$, then the corollary appears as a specific case of (9.5.6). □

When there exists a smaller closed subprescheme Y' of X containing a subprescheme Y of X, we say that Y' is the *closure* of Y in X, when there is little cause for confusion.

9.6. Quasi-coherent sheaves of algebras; change of structure sheaf.

Proposition (9.6.1). — *Let X be a prescheme, and $\mathscr{B}$ a quasi-coherent $\mathscr{O}_X$-algebra (**0**, 5.1.3). For a $\mathscr{B}$-module $\mathscr{F}$ to be quasi-coherent (on the ringed space $(X, \mathscr{B})$) it is necessary and sufficient that $\mathscr{F}$ be a quasi-coherent $\mathscr{O}_X$-module.*

Proof. Since the question is local, we can assume X to be affine, given by some ring A, and thus $\mathscr{B} = \widetilde{B}$, where B is an A-algebra (1.4.3). If $\mathscr{F}$ is quasi-coherent on the ringed space $(X, \mathscr{B})$ then we can also assume that $\mathscr{F}$ is the cokernel of a $\mathscr{B}$-homomorphism $\mathscr{B}^{(I)} \to \mathscr{B}^{(J)}$; since this homomorphism is also an $\mathscr{O}_X$-homomorphism of $\mathscr{O}_X$-modules, and $\mathscr{B}^{(I)}$ and $\mathscr{B}^{(J)}$ are quasi-coherent $\mathscr{O}_X$-modules (1.3.9, ii), $\mathscr{F}$ is also a quasi-coherent $\mathscr{O}_X$-module (1.3.9, i).

Conversely, if $\mathscr{F}$ is a quasi-coherent $\mathscr{O}_X$-module, then $\mathscr{F} = \widetilde{M}$, where M is a B-module (1.4.3); M is isomorphic to the cokernel of a B-homomorphism $B^{(I)} \to B^{(J)}$, so $\mathscr{F}$ is a $\mathscr{B}$-module isomorphic to the cokernel of the corresponding homomorphism $\mathscr{B}^{(I)} \to \mathscr{B}^{(J)}$ (1.3.13), which finishes the proof. □

In particular, if $\mathscr{F}$ and $\mathscr{G}$ are two quasi-coherent $\mathscr{B}$-modules, then $\mathscr{F} \otimes_{\mathscr{B}} \mathscr{G}$ is a quasi-coherent $\mathscr{B}$-module; similarly for $\mathscr{H}om(\mathscr{F}, \mathscr{G})$ whenever we further suppose that $\mathscr{F}$ admits a finite presentation (1.3.13).

(9.6.2). Given a prescheme X, we say that a quasi-coherent $\mathscr{O}_X$-algebra $\mathscr{B}$ is of *finite type* if, for all $x \in X$, there exists an open *affine* neighborhood U of x such that $\Gamma(U, \mathscr{B}) = B$ is an algebra of finite type over $\Gamma(U, \mathscr{O}_X) = A$. We then have that $\mathscr{B}|U = \widetilde{B}$ and, for all $f \in A$, the induced $(\mathscr{O}_X|D(f))$-algebra $\mathscr{B}|D(f)$ is of finite type, because it is isomorphic to $(B_f)^\sim$, and $B_f = B \otimes_A A_f$ is clearly an algebra of finite type over A_f. Since the $D(f)$ form a basis for the topology of U, we thus conclude that if $\mathscr{B}$ is a quasi-coherent $\mathscr{O}_X$-algebra of finite type then, for every open V of X, $\mathscr{B}|V$ is a quasi-coherent $(\mathscr{O}_X|V)$-algebra of finite type.

Proposition (9.6.3). — *Let X be a locally Noetherian prescheme. Then every quasi-coherent $\mathscr{O}_X$-algebra $\mathscr{B}$ of finite type is a coherent sheaf of rings (**0**, 5.3.7).*

Proof. We can once again restrict to the case where X is an affine scheme given by a Noetherian ring A, and where $\mathscr{B} = \widetilde{B}$, with B being an A-algebra of finite type; B is then a Noetherian ring. With this, it remains to prove that the kernel $\mathscr{N}$ of a $\mathscr{B}$-homomorphism $\mathscr{B}^m \to \mathscr{B}$ is a $\mathscr{B}$-module of finite type; but it is isomorphic (as **I | 180**
a $\mathscr{B}$-module) to $\widetilde{N}$, where N is the kernel of the corresponding homomorphism of B-modules $B^m \to B$ (1.3.13). Since B is Noetherian, the submodule N of B^m is a B-module of finite type, so there exists a homomorphism $B^p \to B^m$ with image N; since the sequence $B^p \to B^m \to B$ is exact, so too is the corresponding sequence $\mathscr{B}^p \to \mathscr{B}^m \to \mathscr{B}$ (1.3.5), and since $\mathscr{N}$ is the image of $\mathscr{B}^p \to \mathscr{B}^m$ (1.3.9, i), this proves the proposition. □

Corollary (9.6.4). — *Under the hypotheses of (9.6.3), for a $\mathscr{B}$-module $\mathscr{F}$ to be coherent, it is necessary and sufficient that it be a quasi-coherent $\mathscr{O}_X$-module and a $\mathscr{B}$-module of finite type. If this is the case, and if $\mathscr{G}$ is a $\mathscr{B}$-submodule or a quotient module of $\mathscr{F}$, then in order for $\mathscr{G}$ to be a coherent $\mathscr{B}$-module, it is necessary and sufficient that it be a quasi-coherent $\mathscr{O}_X$-module.*

Proof. Taking (9.6.1) into account, the conditions on $\mathscr{F}$ are clearly necessary; we will show that they are sufficient. We can restrict to the case where X is affine, given by some Noetherian ring A, where $\mathscr{B} = \widetilde{B}$, with B an A-algebra of finite type, where $\mathscr{F} = \widetilde{M}$, with M a B-module, and where there exists a surjective $\mathscr{B}$-homomorphism $\mathscr{B}^m \to \mathscr{F} \to 0$. We then have the corresponding exact sequence $B^m \to M \to 0$, so M is a B-module of finite type; further, the kernel P of the homomorphism $B^m \to M$ is then a B-module of finite type, since B is Noetherian. We thus conclude (1.3.13) that $\mathscr{F}$ is the cokernel of a $\mathscr{B}$-homomorphism $\mathscr{B}^m \to \mathscr{B}^n$, and is thus coherent, since $\mathscr{B}$ is a coherent sheaf of rings (**0**, 5.3.4). The same reasoning shows that a quasi-coherent $\mathscr{B}$-submodule (resp. a quotient $\mathscr{B}$-module) of $\mathscr{F}$ is of finite type, from whence the second part of the corollary. □

Proposition (9.6.5). — *Let X be a quasi-compact scheme, or a prescheme whose underlying space is Noetherian. For all quasi-compact $\mathscr{O}_X$-algebras $\mathscr{B}$ of finite type, there exists a quasi-coherent $\mathscr{O}_X$-submodule $\mathscr{E}$ of $\mathscr{B}$ of finite type such that $\mathscr{E}$ generates (**0**, 4.1.3) the $\mathscr{O}_X$-algebra $\mathscr{B}$.*

Proof. In fact, by hypothesis, there exists a finite cover (U_i) of X consisting of affine opens such that $\Gamma(U_i, \mathscr{B}) = B_i$ is an algebra of finite type over $\Gamma(U_i, \mathscr{O}_X) = A_i$; let E_i be a A_i-submodule of B_i of finite type that generates the A_i-algebra B_i; thanks to (9.4.7), there exists a $\mathscr{O}_X$-submodule $\mathscr{E}_i$ of $\mathscr{B}$, quasi-coherent and of finite type, such that $\mathscr{E}_i|U_i = \widetilde{E}_i$. It is clear that the sum $\mathscr{E}$ of the $\mathscr{E}_i$ is the desired object. □

Proposition (9.6.6). — *Let X be a prescheme whose underlying space is locally Noetherian, or a quasi-compact scheme. Then every quasi-coherent $\mathscr{O}_X$-algebra $\mathscr{B}$ is the inductive limit of its quasi-coherent $\mathscr{O}_X$-subalgebras of finite type.*

Proof. In fact, it follows from (9.4.9) that $\mathscr{B}$ is the inductive limit (as an $\mathscr{O}_X$-module) of its quasi-coherent $\mathscr{O}_X$-submodules of finite type; the latter generating quasi-coherent $\mathscr{O}_X$*-subalgebras* of $\mathscr{B}$ of finite type (1.3.14), and so $\mathscr{B}$ is *a fortiori* their inductive limit. □

§10. Formal schemes

10.1. Formal affine schemes.

(10.1.1). Let A be an *admissible* topological ring (**0**, 7.1.2); for each ideal of definition $\mathfrak{J}$ of A, $\operatorname{Spec}(A/\mathfrak{J})$ can be identified with the closed subspace $V(\mathfrak{J})$ of $\operatorname{Spec}(A)$ (1.1.11), the set of *open* prime ideals of A; this topological space does not depend on the ideal of definition $\mathfrak{J}$ considered; we denote this topological space by $\mathfrak{X}$. Let $(\mathfrak{J}_\lambda)$ be a fundamental system of neighborhoods of 0 in A, consisting of ideals of definition, and for each λ, let $\mathscr{O}_\lambda$ be the structure sheaf of $\operatorname{Spec}(A/\mathfrak{J}_\lambda)$; this sheaf is induced on $\mathfrak{X}$ by $\widetilde{A}/\widetilde{\mathfrak{J}}_\lambda$ (which is zero outside of $\mathfrak{X}$). For $\mathfrak{J}_\mu \subset \mathfrak{J}_\lambda$, the canonical homomorphism $A/\mathfrak{J}_\mu \to A/\mathfrak{J}_\lambda$ thus defines a homomorphism $u_{\lambda\mu} : \mathscr{O}_\mu \to \mathscr{O}_\lambda$ of sheaves of rings (1.6.1), and $(\mathscr{O}_\lambda)$ is a *projective system of sheaves of rings* for these homomorphisms. Since the topology of $\mathfrak{X}$ admits a basis consisting of quasi-compact open subsets, we can associate to each $\mathscr{O}_\lambda$ a *pseudo-discrete sheaf of topological rings* (**0**, 3.8.1) which have $\mathscr{O}_\lambda$ as the underlying sheaf of rings (without topologies), and that we denote also by $\mathscr{O}_\lambda$; and the $\mathscr{O}_\lambda$ give again a *projective system of sheaves of topological rings* (**0**, 3.8.2). We denote by $\mathscr{O}_\mathfrak{X}$ the *sheaf of topological rings* on $\mathfrak{X}$, the projective limit of the system $(\mathscr{O}_\lambda)$; for each *quasi-compact* open subset U of $\mathfrak{X}$, $\Gamma(U, \mathscr{O}_\mathfrak{X})$ is a topological ring, the projective limit of the system of *discrete* rings $\Gamma(U, \mathscr{O}_\lambda)$ (**0**, 3.2.6). **I** | 181

Definition (10.1.2). — Given an admissible topological ring A, we define the formal spectrum of A, denoted by $\operatorname{Spf}(A)$, to be the closed subspace $\mathfrak{X}$ of $\operatorname{Spec}(A)$ consisting of the open prime ideals of A. We say that a topologically ringed space is a formal affine scheme if it is isomorphic to a formal spectrum $\operatorname{Spf}(A) = \mathfrak{X}$ equipped with a sheaf of topological rings $\mathscr{O}_\mathfrak{X}$ which is the projective limit of sheaves of pseudo-discrete topological rings $(\widetilde{A}/\widetilde{\mathfrak{J}}_\lambda)|\mathfrak{X}$, where $\mathfrak{J}_\lambda$ varies over the filtered set of ideals of definition of A.

When we speak of a *formal spectrum* $\mathfrak{X} = \operatorname{Spf}(A)$ as a formal affine scheme, it will always be as the topologically ringed space $(\mathfrak{X}, \mathscr{O}_\mathfrak{X})$ where $\mathscr{O}_\mathfrak{X}$ is defined as above.

We note that every *affine scheme* $X = \operatorname{Spec}(A)$ can be considered as a formal affine scheme in only one way, by considering A as a discrete topological ring: the topological rings $\Gamma(U, \mathscr{O}_X)$ are then discrete whenever U is quasi-compact (but not, in general, when U is an arbitrary open subset of X).

Proposition (10.1.3). — *If $\mathfrak{X} = \operatorname{Spf}(A)$, where A is an admissible ring, then $\Gamma(\mathfrak{X}, \mathscr{O}_X)$ is topologically isomorphic to A.*

Proof. Indeed, since $\mathfrak{X}$ is closed in $\operatorname{Spec}(A)$, it is quasi-compact, and so $\Gamma(\mathfrak{X}, \mathscr{O}_\mathfrak{X})$ is topologically isomorphic to the projective limit of the discrete rings $\Gamma(\mathfrak{X}, \mathscr{O}_\lambda)$; but $\Gamma(\mathfrak{X}, \mathscr{O}_\lambda)$ is isomorphic to $A/\mathfrak{J}_\lambda$ (1.3.7); since A is separated and complete, it is topologically isomorphic to $\varprojlim A/\mathfrak{J}_\lambda$ (**0**, 7.2.1), whence the proposition. □

Proposition (10.1.4). — *Let A be an admissible ring, $\mathfrak{X} = \operatorname{Spf}(A)$, and, for every $f \in A$, let $\mathfrak{D}(f) = D(f) \cap \mathfrak{X}$; then the topologically ringed space $(\mathfrak{D}(f), \mathscr{O}_\mathfrak{X}|\mathfrak{D}(f))$ is isomorphic to the formal affine spectrum $\operatorname{Spf}(A_{\{f\}})$* (**0**, 7.6.15).

Proof. For every ideal of definition $\mathfrak{J}$ of A, the discrete ring $S_f^{-1}A/S_f^{-1}\mathfrak{J}$ is canonically identified with $A_{\{f\}}/\mathfrak{J}_{\{f\}}$ (**0**, 7.6.9), so, by (1.2.5) and (1.2.6), the topological space $\operatorname{Spf}(A_{\{f\}})$ is canonically identified with $\mathfrak{D}(f)$. Further, for every quasi-compact open subset U of $\mathfrak{X}$ contained in $\mathfrak{D}(f)$, $\Gamma(U, \mathscr{O}_\lambda)$ can be identified with the module of sections of the structure sheaf of $\operatorname{Spec}(S_f^{-1}A/S_f^{-1}\mathfrak{J}_\lambda)$ over U (1.3.6), so, setting $\mathfrak{Y} = \operatorname{Spf}(A_{\{f\}})$, $\Gamma(U, \mathscr{O}_\mathfrak{X})$ can be identified with the module of sections $\Gamma(U, \mathscr{O}_\mathfrak{Y})$, which proves the proposition. □

(10.1.5). As a sheaf of rings *without topology*, the structure sheaf $\mathscr{O}_\mathfrak{X}$ of $\operatorname{Spf}(A)$ admits, for every $x \in \mathfrak{X}$, a fibre which, by (10.1.4), can be identified with the inductive limit $\varinjlim A_{\{f\}}$ for the $f \notin \mathfrak{j}_x$. Then, by (**0**, 7.6.17) and (**0**, 7.6.18):

Proposition (10.1.6). — *For every $x \in \mathfrak{X} = \operatorname{Spf}(A)$, the fibre $\mathscr{O}_x$ is a local ring whose residue field is isomorphic to $k(x) = A_x/\mathfrak{j}_x A_x$. If, further, A is adic and Noetherian, then $\mathscr{O}_x$ is a Noetherian ring.*

Since $k(x)$ is not reduced at 0, we conclude from this that the *support* of the ring of sheaves $\mathscr{O}_\mathfrak{X}$ is *equal to* $\mathfrak{X}$.

10.2. Morphisms of formal affine schemes.

(10.2.1). Let A, B be two admissible rings, and let $\varphi : B \to A$ be a *continuous* morphism. The continuous map ${}^a\varphi : \mathrm{Spec}(A) \to \mathrm{Spec}(B)$ (1.2.1) then maps $\mathfrak{X} = \mathrm{Spf}(A)$ to $\mathfrak{Y} = \mathrm{Spf}(B)$, since the inverse image under φ of an open prime ideal of A is an open prime ideal of B. Conversely, for all $g \in B$, φ defines a continuous homomorphism $\Gamma(\mathfrak{D}(g), \mathcal{O}_{\mathfrak{Y}}) \to \Gamma(\mathfrak{D}(\varphi(g)), \mathcal{O}_{\mathfrak{X}})$ according to (10.1.4), (10.1.3), and (**0**, 7.6.7); since these homomorphisms satisfy the compatibility conditions for the restrictions corresponding to the change from g to a multiple of g, and since $\mathfrak{D}(\varphi(g)) = {}^a\varphi^{-1}(\mathfrak{D}(g))$, they define a *continuous* homomorphism of sheaves of topological rings $\mathcal{O}_{\mathfrak{Y}} \to {}^a\varphi_*(\mathcal{O}_{\mathfrak{X}})$ (**0**, 3.2.5) that we denote by $\widetilde{\varphi}$; we have thus defined a morphism $\Phi = ({}^a\varphi, \widetilde{\varphi})$ of topologically ringed spaces $\mathfrak{X} \to \mathfrak{Y}$. We note that, as a homomorphism of sheaves without topology, $\widetilde{\varphi}$ defines a homomorphism $\widetilde{\varphi}_x^\sharp : \mathcal{O}_{{}^a\varphi(x)} \to \mathcal{O}_x$ on the stalks, for all $x \in \mathfrak{X}$.

Proposition (10.2.2). — *Let A and B be admissible topological rings, and let $\mathfrak{X} = \mathrm{Spf}(A)$ and $\mathfrak{Y} = \mathrm{Spf}(B)$. For a morphism $u = (\psi, \theta) : \mathfrak{X} \to \mathfrak{Y}$ of topologically ringed spaces to be of the form $({}^a\varphi, \widetilde{\varphi})$, where φ is a continuous ring homomorphism $B \to A$, it is necessary and sufficient that $\theta_x^\sharp$ be a local homomorphism $\mathcal{O}_{\varphi(x)} \to \mathcal{O}_x$ for all $x \in \mathfrak{X}$.*

Proof. The condition is necessary: let $\mathfrak{p} = \mathfrak{j}_x \in \mathrm{Spf}(A)$, and let $\mathfrak{q} = \varphi^{-1}(\mathfrak{j}_x)$; if $g \notin \mathfrak{q}$, then we have $\varphi(g) \notin \mathfrak{p}$, and it is immediate that the homomorphism $B_{\{g\}} \to A_{\{\varphi(g)\}}$ induced by φ (**0**, 7.6.7) sends $\mathfrak{q}_{\{g\}}$ to a subset of $\mathfrak{p}_{\{\varphi(g)\}}$; by passing to the inductive limit, we see (taking (10.1.5) and (**0**, 7.6.17) into account) that $\widetilde{\varphi}_x^\sharp$ is a local homomorphism.

Conversely, let (ψ, θ) be a morphism satisfying the condition of the proposition; by (10.1.3), θ defines a continuous ring homomorphism

$$\varphi = \Gamma(\theta) : B = \Gamma(\mathfrak{Y}, \mathcal{O}_{\mathfrak{Y}}) \longrightarrow \Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}) = A.$$

By virtue of the hypothesis on θ, for the section $\varphi(g)$ of $\mathcal{O}_{\mathfrak{X}}$ over $\mathfrak{X}$ to be an invertible germ at the point x, it is necessary and sufficient that g be an invertible germ at the point $\psi(x)$. But, by (**0**, 7.6.17), the sections of $\mathcal{O}_{\mathfrak{X}}$ (resp. $\mathcal{O}_{\mathfrak{Y}}$) over $\mathfrak{X}$ (resp. $\mathfrak{Y}$) that have a non-invertible germ at the point x (resp. $\psi(x)$) are exactly the elements of $\mathfrak{j}_x$ (resp. $\mathfrak{j}_{\psi(x)}$); the above remark thus shows that ${}^a\varphi = \psi$. Finally, for all $g \in B$ the diagram **I** | 183

$$\begin{tikzcd}
B = \Gamma(\mathfrak{Y}, \mathcal{O}_{\mathfrak{Y}}) \arrow[r, "\varphi"] \arrow[d] & \Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}) = A \arrow[d] \\
B_{\{g\}} = \Gamma(\mathfrak{D}(g), \mathcal{O}_{\mathfrak{Y}}) \arrow[r, "\Gamma(\theta_{\mathfrak{D}(g)})"] & \Gamma(\mathfrak{D}(\varphi(g)), \mathcal{O}_{\mathfrak{X}}) = A_{\{\varphi(g)\}}
\end{tikzcd}$$

is commutative; by the universal property of completed rings of fractions (**0**, 7.6.6), $\theta_{\mathfrak{D}(g)}$ is equal to $\widetilde{\varphi}_{\mathfrak{D}(g)}$ for all $g \in B$, and so (**0**, 3.2.5) we have $\theta = \widetilde{\varphi}$. □

We say that a morphism (ψ, θ) of topologically ringed spaces satisfying the condition of Proposition (10.2.2) is a *morphism of formal affine schemes*. We can say that the functors $\mathrm{Spf}(A)$ in A and $\Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ in $\mathfrak{X}$ define an *equivalence* between the category of admissible rings and the opposite category of formal affine schemes (T, I, 1.2).

(10.2.3). As a particular case of (10.2.2), note that, for $f \in A$, the canonical injection of the formal affine scheme induced by $\mathfrak{X}$ on $\mathfrak{D}(f)$ corresponds to the continuous canonical homomorphism $A \to A_{\{f\}}$. Under the hypotheses of Proposition (10.2.2), let h be an element of B, and g an element of A that is a multiple of $\varphi(h)$; we then have $\psi(\mathfrak{D}(g)) \subset \mathfrak{D}(h)$; the restriction of u to $\mathfrak{D}(g)$, considered as a morphism from $\mathfrak{D}(g)$ to $\mathfrak{D}(h)$, is the unique morphism v making the diagram

$$\begin{tikzcd}
\mathfrak{D}(g) \arrow[r, "v"] \arrow[d] & \mathfrak{D}(h) \arrow[d] \\
\mathfrak{X} \arrow[r, "u"] & \mathfrak{Y}
\end{tikzcd}$$

commutate.

This morphism corresponds to the unique continuous homomorphism $\varphi' : B_{\{h\}} \to A_{\{g\}}$ (**0**, 7.6.7) making the diagram

$$\begin{tikzcd}
A \arrow[d] & B \arrow[l, "\varphi"'] \arrow[d] \\
A_{\{g\}} & B_{\{h\}} \arrow[l, "\varphi'"]
\end{tikzcd}$$

commutate.

10.3. Ideals of definition for a formal affine scheme.

(10.3.1). Let A be an admissible ring, $\mathfrak{J}$ an open ideal of A, and $\mathfrak{X}$ the formal affine scheme $\operatorname{Spf}(A)$. Let $(\mathfrak{J}_\lambda)$ be the set of those ideals of definition for A that are contained in $\mathfrak{J}$; then $\widetilde{\mathfrak{J}}/\widetilde{\mathfrak{J}}_\lambda$ is a sheaf of ideals of $\widetilde{A}/\widetilde{\mathfrak{J}}_\lambda$. Denote by $\mathfrak{J}^\Delta$ the projective limit of the sheaves on $\mathfrak{X}$ induced by $\widetilde{\mathfrak{J}}/\widetilde{\mathfrak{J}}_\lambda$, which is identified with a *sheaf of ideals* of $\mathscr{O}_\mathfrak{X}$ (**0**, 3.2.6). For every $f \in A$, $\Gamma(\mathfrak{D}(f), \mathfrak{J}^\Delta)$ is the projective limit of the $S_f^{-1}\mathfrak{J}/S_f^{-1}\mathfrak{J}_\lambda$, or, in other words, can be identified with the open ideal $\mathfrak{J}_{\{f\}}$ of the ring $A_{\{f\}}$ (**0**, 7.6.9), and, in particular, $\Gamma(\mathfrak{X}, \mathfrak{J}^\Delta) = \mathfrak{J}$; we conclude (the $\mathfrak{D}(f)$ forming a basis for the topology of $\mathfrak{X}$) that

$$\mathfrak{J}^\Delta|\mathfrak{D}(f) = (\mathfrak{J}_{\{f\}})^\Delta. \tag{10.3.1.1}$$

(10.3.2). With the notation of (10.3.1), for all $f \in A$, the canonical map from $A_{\{f\}} = \Gamma(\mathfrak{D}(f), \mathscr{O}_\mathfrak{X})$ to $\Gamma(\mathfrak{D}(f), (\widetilde{A}/\widetilde{\mathfrak{J}})|\mathfrak{X}) =$ I | 184
$S_f^{-1}A/S_f^{-1}\mathfrak{J}$ is *surjective* and has $\Gamma(\mathfrak{D}(f), \mathfrak{J}^\Delta) = \mathfrak{J}_{\{f\}}$ as its kernel (**0**, 7.6.9); these maps thus define a *surjective* continuous homomorphism, said to be *canonical*, from the sheaf of topological rings $\mathscr{O}_\mathfrak{X}$ to the sheaf of discrete rings $(\widetilde{A}/\widetilde{\mathfrak{J}})|\mathfrak{X}$, whose kernel is $\mathfrak{J}^\Delta$; this homomorphism is none other than $\widetilde{\varphi}$ (10.2.1), where φ is the continuous homomorphism $A \to A/\mathfrak{J}$; the morphism $({}^a\varphi, \widetilde{\varphi}) : \operatorname{Spec}(A/\mathfrak{J}) \to \mathfrak{X}$ of formal affine schemes (where ${}^a\varphi$ is the identity homeomorphism from $\mathfrak{X}$ to itself) is also said to be *canonical*. We thus have, according to the above, a *canonical isomorphism*

$$\mathscr{O}_\mathfrak{X}/\mathfrak{J}^\Delta \simeq (\widetilde{A}/\widetilde{\mathfrak{J}})|\mathfrak{X}. \tag{10.3.2.1}$$

It is clear (since $\Gamma(\mathfrak{X}, \mathfrak{J}^\Delta) = \mathfrak{J}$) that the map $\mathfrak{J} \to \mathfrak{J}^\Delta$ is *strictly increasing*; according to the above, for $\mathfrak{J} \subset \mathfrak{J}'$, the sheaf $\mathfrak{J}'^\Delta/\mathfrak{J}^\Delta$ is canonically isomorphic to $\widetilde{\mathfrak{J}'}/\widetilde{\mathfrak{J}} = (\mathfrak{J}'/\mathfrak{J})^\sim$.

(10.3.3). The hypotheses and notation being the same as in (10.3.1), we say that a sheaf of ideals $\mathscr{J}$ of $\mathscr{O}_\mathfrak{X}$ is a *sheaf of ideals of definition* for $\mathfrak{X}$ (or an *ideal sheaf of definition* for $\mathfrak{X}$) if, for all $x \in \mathfrak{X}$, there exists an open neighborhood of x of the form $\mathfrak{D}(f)$, where $f \in A$, such that $\mathscr{J}|\mathfrak{D}(f)$ is of the form $\mathfrak{H}^\Delta$, where $\mathfrak{H}$ is an ideal of definition for $A_{\{f\}}$.

Proposition (10.3.4). — *For all $f \in A$, each sheaf of ideals of definition for $\mathfrak{X}$ induces a sheaf of ideals of definition for $\mathfrak{D}(f)$.*

Proof. This follows from (10.3.1.1). ☐

Proposition (10.3.5). — *If A is an admissible ring, then every sheaf of ideals of definition for $\mathfrak{X} = \operatorname{Spf}(A)$ is of the form $\mathfrak{J}^\Delta$, where $\mathfrak{J}$ is a uniquely determined ideal of definition for A.*

Proof. Let $\mathscr{J}$ be a sheaf of ideals of definition of $\mathfrak{X}$; by hypothesis, and since $\mathfrak{X}$ is quasi-compact, there are finitely-many elements $f_i \in A$ such that the $\mathfrak{D}(f_i)$ cover $\mathfrak{X}$ and such that $\mathscr{J}|\mathfrak{D}(f_i) = \mathfrak{H}_i^\Delta$, where $\mathfrak{H}_i$ is an ideal of definition for $A_{\{f_i\}}$. For each i, there exists an open ideal $\mathfrak{K}_i$ of A such that $(\mathfrak{K}_i)_{\{f_i\}} = \mathfrak{H}_i$ (**0**, 7.6.9); let $\mathfrak{K}$ be an ideal of definition for A containing all the $\mathfrak{K}_i$. The canonical image of $\mathscr{J}/\mathfrak{K}^\Delta$ in the structure sheaf $(A/\mathfrak{K})^\sim$ of $\operatorname{Spec}(A/\mathfrak{K})$ (10.3.2) is thus such that its restriction to $\mathfrak{D}(f_i)$ is equal to its restriction to $(\mathfrak{K}_i/\mathfrak{K})^\sim$; we conclude that this canonical image is a *quasi-coherent* sheaf on $\operatorname{Spec}(A/\mathfrak{K})$, and so is of the form $(\mathfrak{J}/\mathfrak{K})^\sim$, where $\mathfrak{J}$ is an ideal of definition for A containing $\mathfrak{K}$ (1.4.1), whence $\mathscr{J} = \mathfrak{J}^\Delta$ (10.3.2); in addition, since for each i there exists an integer n_i such that $\mathfrak{H}_i^{n_i} \subset \mathfrak{K}_{\{f_i\}}$, we will have, by setting n to be the largest of the n_i, that $(\mathscr{J}/\mathfrak{K}^\Delta)^n = 0$, and, as a result (10.3.2), that $((\mathfrak{J}/\mathfrak{K})^\sim)^n = 0$, whence finally that $(\mathfrak{J}/\mathfrak{K})^n = 0$ (1.3.13), which proves that $\mathfrak{J}$ is an ideal of definition for A (**0**, 7.1.4). ☐

Proposition (10.3.6). — *Let A be an adic ring, and $\mathfrak{J}$ an ideal of definition for A such that $\mathfrak{J}/\mathfrak{J}^2$ is an $(A/\mathfrak{J})$-module of finite type. For any integer $n > 0$, we then have $(\mathfrak{J}^\Delta)^n = (\mathfrak{J}^n)^\Delta$.*

Proof. For all $f \in A$, we have (since $\mathfrak{J}^n$ is an open ideal)

$$(\Gamma(\mathfrak{D}(f), \mathfrak{J}^\Delta))^n = (\mathfrak{J}_{\{f\}})^n = (\mathfrak{J}^n)_{\{f\}} = \Gamma(\mathfrak{D}(f^n), (\mathfrak{J}^n)^\Delta)$$

by (10.3.1.1) and (**0**, 7.6.12). The result then follows from the fact that $(\mathfrak{J}^\Delta)^n$ is associated to the presheaf I | 185
$U \mapsto (\Gamma(U, \mathfrak{J}^\Delta))^n$ (**0**, 4.1.6), since the $\mathfrak{D}(f)$ form a basis for the topology of $\mathfrak{X}$. ☐

(10.3.7). We say that a family $(\mathscr{J}_\lambda)$ of sheaves of ideals of definition for $\mathfrak{X}$ is a *fundamental system of sheaves of ideals of definition* if each sheaf of ideals of definition for $\mathfrak{X}$ contains one of the $\mathscr{J}_\lambda$; since $\mathscr{J}_\lambda = \mathfrak{J}_\lambda^\Delta$, it is equivalent to say that the $\mathfrak{J}_\lambda$ form a *fundamental system of neighborhoods of* 0 in A. Let (f_α) be a family of elements of A such that the $\mathfrak{D}(f_\alpha)$ cover $\mathfrak{X}$. If $(\mathscr{J}_\lambda)$ is a filtered decreasing family of sheaves of ideals of $\mathscr{O}_\mathfrak{X}$ such that, for each

α, the family $(\mathscr{J}_\lambda|\mathfrak{D}(f_\alpha))$ is a fundamental system of sheaves of ideals of definition for $\mathfrak{D}(f_\alpha)$, then $(\mathscr{J}_\lambda)$ is a fundamental system of sheaves of ideals of definition for $\mathfrak{X}$. Indeed, for each sheaf of ideals of definition $\mathscr{J}$ for $\mathfrak{X}$, there is a finite cover of $\mathfrak{X}$ by $\mathfrak{D}(f_i)$ such that, for each i, $\mathscr{J}_{\lambda_i}|\mathfrak{D}(f_i)$ is a sheaf of ideals of definition for $\mathfrak{D}(f_i)$ that is contained in $\mathscr{J}|\mathfrak{D}(f_i)$. If μ is an index such that $\mathscr{J}_\mu \subset \mathscr{J}_{\lambda_i}$ for all i, then it follows from (10.3.3) that $\mathscr{J}_\mu$ is a sheaf of ideals of definition for $\mathfrak{X}$, evidently contained in $\mathscr{J}$, whence our claim.

10.4. Formal preschemes and morphisms of formal preschemes.

(10.4.1). Given a topologically ringed space $\mathfrak{X}$, we say that an open $U \subset \mathfrak{X}$ is a *formal affine open* (resp. a *formal adic affine open*, resp. a *formal Noetherian affine open*) if the topologically ringed space induced on U by $\mathfrak{X}$ is a formal affine scheme (resp. a scheme whose ring is adic, resp. adic and Noetherian).

Definition (10.4.2). — A *formal prescheme* is a topologically ringed space $\mathfrak{X}$ which admits a formal affine open neighborhood for each point. We say that the formal prescheme $\mathfrak{X}$ is adic (resp. locally Noetherian) if each point of $\mathfrak{X}$ admits a formal adic (resp. Noetherian) open neighborhood. We say that $\mathfrak{X}$ is Noetherian if it is locally Noetherian and its underlying space is quasi-compact (and hence Noetherian).

Proposition (10.4.3). — *If $\mathfrak{X}$ is a formal prescheme (resp. a locally Noetherian formal prescheme), then the formal affine (resp. Noetherian affine) open sets form a basis for the topology of $\mathfrak{X}$.*

Proof. This follows from Definition (10.4.2) and Proposition (10.1.4) by taking into account that, if A is an adic Noetherian ring, then so too is $A_{\{f\}}$ for all $f \in A$ (**0**, 7.6.11). □

Corollary (10.4.4). — *If $\mathfrak{X}$ is a formal prescheme (resp. a locally Noetherian formal prescheme, resp. a Noetherian formal prescheme), then the topologically ringed space induced on each open set of $\mathfrak{X}$ is also a formal prescheme (resp. a locally Noetherian formal prescheme, resp. a Noetherian formal prescheme).*

Definition (10.4.5). — Given two formal preschemes $\mathfrak{X}$ and $\mathfrak{Y}$, a morphism (of formal preschemes) from $\mathfrak{X}$ to $\mathfrak{Y}$ is a morphism (ψ, θ) of topologically ringed spaces such that, for all $x \in \mathfrak{X}$, $\theta_x^\sharp$ is a local homomorphism $\mathscr{O}_{\psi(x)} \to \mathscr{O}_x$.

It is immediate that the composition of any two morphisms of formal preschemes is again a morphism of formal preschemes; the formal preschemes thus form a *category*, and we denote by $\operatorname{Hom}(\mathfrak{X}, \mathfrak{Y})$ the set of morphisms from a formal prescheme $\mathfrak{X}$ to a formal prescheme $\mathfrak{Y}$. **I** | 186

If U is an open subset of $\mathfrak{X}$, then the canonical injection into $\mathfrak{X}$ of the formal prescheme induced on U by $\mathfrak{X}$ is a morphism of formal preschemes (and in fact a *monomorphism* of topologically ringed spaces (**0**, 4.1.1)).

Proposition (10.4.6). — *Let $\mathfrak{X}$ be a formal prescheme, and $\mathfrak{S} = \operatorname{Spf}(A)$ a formal affine scheme. There exists a canonical bijective equivalence between the morphisms from the formal prescheme $\mathfrak{X}$ to the formal prescheme $\mathfrak{S}$ and the continuous homomorphisms from the ring A to the topological ring $\Gamma(\mathfrak{X}, \mathscr{O}_\mathfrak{X})$.*

Proof. The proof is similar to that of (2.2.4), by replacing "homomorphism" with "continuous homomorphism", "affine open" with "formal affine open", and by using Proposition (10.2.2) instead of Theorem (1.7.3); we leave the details to the reader. □

(10.4.7). Given a formal prescheme $\mathfrak{S}$, we say that the data of a formal prescheme $\mathfrak{X}$ and a morphism $\varphi : \mathfrak{X} \to \mathfrak{S}$ defines a formal prescheme $\mathfrak{X}$ *over* $\mathfrak{S}$ or an *formal $\mathfrak{S}$-prescheme*, φ being called the *structure morphism* of the $\mathfrak{S}$-prescheme $\mathfrak{X}$. If $\mathfrak{S} = \operatorname{Spf}(A)$, where A is an admissible ring, then we also say that the formal $\mathfrak{S}$-prescheme $\mathfrak{X}$ is a *formal A-prescheme* or a formal prescheme *over A*. An arbitrary formal prescheme can be considered as a formal prescheme over **Z** (equipped with the discrete topology).

If $\mathfrak{X}$ and $\mathfrak{Y}$ are formal $\mathfrak{S}$-preschemes, then we say that a morphism $u : \mathfrak{X} \to \mathfrak{Y}$ is a *$\mathfrak{S}$-morphism* if the diagram

$$\begin{tikzcd} \mathfrak{X} \arrow[rr, "u"] \arrow[dr] & & \mathfrak{Y} \arrow[dl] \\ & \mathfrak{S} & \end{tikzcd}$$

(where the downwards arrows are the structure morphisms) is commutative. With this definition, the formal $\mathfrak{S}$-preschemes (for some fixed $\mathfrak{S}$) form a *category*. We denote by $\operatorname{Hom}_\mathfrak{S}(\mathfrak{X}, \mathfrak{Y})$ the set of $\mathfrak{S}$-morphisms from a formal $\mathfrak{S}$-prescheme $\mathfrak{X}$ to a formal $\mathfrak{S}$-prescheme $\mathfrak{Y}$. When $\mathfrak{S} = \operatorname{Spf}(A)$, we sometimes say *$A$-morphism* instead of *$\mathfrak{S}$-morphism*.

(10.4.8). Since every affine scheme can be considered as a formal affine scheme (10.1.2), every (usual) prescheme can be considered as a formal prescheme. In addition, it follows from Definition (10.4.5) that, for the *usual* preschemes, the morphisms (resp. S-morphisms) of *formal* preschemes coincide with the morphisms (resp. S-morphisms) defined in §2.

10.5. Sheaves of ideals of definition for formal preschemes.

(10.5.1). Let $\mathfrak{X}$ be a formal prescheme; we say that an $\mathcal{O}_{\mathfrak{X}}$-ideal $\mathscr{J}$ is a *sheaf of ideals of definition* (or an *ideal sheaf of definition*) for $\mathfrak{X}$ if every $x \in \mathfrak{X}$ has a formal affine open neighborhood U such that $\mathscr{J}|U$ is a sheaf of ideals of definition for the formal affine scheme induced on U by $\mathfrak{X}$ (10.3.3); by (10.3.1.1) and Proposition (10.4.3), for each open $V \subset \mathfrak{X}$, $\mathscr{J}|V$ is then a sheaf of ideals of definition for the formal prescheme induced on V by $\mathfrak{X}$.

We say that a family $(\mathscr{J}_\lambda)$ of sheaves of ideals of definition for $\mathfrak{X}$ is a *fundamental system of sheaves of ideals of definition* if there exists a cover (U_α) of $\mathfrak{X}$ by formal affine open sets such that, for each α, the family of the $\mathscr{J}_\lambda|U_\alpha$ is a fundamental system of sheaves of ideals of definition (10.3.6) for the formal affine scheme induced on U_α by $\mathfrak{X}$. It follows from the last remark of (10.3.7) that, when $\mathfrak{X}$ is a formal affine scheme, this definition coincides with the definition given in (10.3.7). For an open subset V of $\mathfrak{X}$, the restrictions $\mathscr{J}_\lambda|V$ then form a fundamental system of sheaves of ideals of definition for the formal prescheme induced on V, according to (10.3.1.1). If $\mathfrak{X}$ is a *locally Noetherian* formal prescheme, and $\mathscr{J}$ is a sheaf of ideals of definition for $\mathfrak{X}$, then it follows from Proposition (10.3.6) that the powers $\mathscr{J}^n$ form a fundamental system of sheaves of ideals of definition for $\mathfrak{X}$. **I** | 187

(10.5.2). Let $\mathfrak{X}$ be a formal prescheme, and $\mathscr{J}$ a sheaf of ideals of definition for $\mathfrak{X}$. Then the ringed space $(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}/\mathscr{J})$ is a (usual) *prescheme*, which is affine (resp. locally Noetherian, resp. Noetherian) when $\mathfrak{X}$ is a formal affine scheme (resp. a locally Noetherian formal scheme, resp. a Noetherian formal scheme); we can reduce to the affine case, and then the proposition has already been proved in (10.3.2). In addition, if $\theta : \mathcal{O}_{\mathfrak{X}} \to \mathcal{O}_{\mathfrak{X}}/\mathscr{J}$ is the canonical homomorphism, then $u = (1_{\mathfrak{X}}, \theta)$ is a *morphism* (said to be *canonical*) of formal preschemes $(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}/\mathscr{J}) \to (\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$, because, again, this was proved in the affine case (10.3.2), to which it is immediately reduced.

Proposition (10.5.3). — *Let $\mathfrak{X}$ be a formal prescheme, and $(\mathscr{J}_\lambda)$ a fundamental system of sheaves of ideals of definition for $\mathfrak{X}$. Then the sheaf of topological rings $\mathcal{O}_{\mathfrak{X}}$ is the projective limit of the pseudo-discrete sheaves of rings* (**0**, 3.8.1) $\mathcal{O}_{\mathfrak{X}}/\mathscr{J}_\lambda$.

Proof. Since the topology of $\mathfrak{X}$ admits a basis of formal quasi-compact affine open sets (10.4.3), we reduce to the affine case, where the proposition is a consequence of Proposition (10.3.5), (10.3.2), and Definition (10.1.1). □

It is not true that any formal prescheme admits a sheaf of ideals of definition. However:

Proposition (10.5.4). — *Let $\mathfrak{X}$ be a locally Noetherian formal prescheme. There exists a largest sheaf of ideals of definition $\mathscr{T}$ for $\mathfrak{X}$; this is the unique sheaf of ideals of definition $\mathscr{J}$ such that the prescheme $(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}/\mathscr{J})$ is reduced. If $\mathscr{J}$ is a sheaf of ideals of definition for $\mathfrak{X}$, then $\mathscr{T}$ is the inverse image under $\mathcal{O}_{\mathfrak{X}} \to \mathcal{O}_{\mathfrak{X}}/\mathscr{J}$ of the nilradical of $\mathcal{O}_{\mathfrak{X}}/\mathscr{J}$.*

Proof. Suppose first that $\mathfrak{X} = \operatorname{Spf}(A)$, where A is an adic Noetherian ring. The existence and the properties of $\mathscr{T}$ follow immediately from Propositions (10.3.5) and (5.1.1), taking into account the existence and the properties of the largest ideal of definition for A ((**0**, 7.1.6) and (**0**, 7.1.7)). $\mathbf{Err}_{\mathrm{II}}$

To prove the existence and the properties of $\mathscr{T}$ in the general case, it suffices to show that, if $U \supset V$ are two Noetherian formal affine open subsets of X, then the largest sheaf of ideals of definition $\mathscr{T}_U$ for U induces the largest sheaf of ideals of definition $\mathscr{T}_V$ for V; but as $(V, (\mathcal{O}_{\mathfrak{X}}|V)/(\mathscr{T}_U|V))$ is reduced, this follows from the above. □

We denote by $\mathfrak{X}_{\mathrm{red}}$ the (usual) reduced prescheme $(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}/\mathscr{T})$.

Corollary (10.5.5). — *Let $\mathfrak{X}$ be a locally Noetherian formal prescheme, and $\mathscr{T}$ the largest sheaf of ideals of definition for $\mathfrak{X}$; for each open subset V of $\mathfrak{X}$, $\mathscr{T}|V$ is the largest sheaf of ideals of definition for the formal prescheme induced on V by $\mathfrak{X}$.*

Proposition (10.5.6). — *Let $\mathfrak{X}$ and $\mathfrak{Y}$ be formal preschemes, $\mathscr{J}$ (resp. $\mathscr{K}$) be a sheaf of ideals of definition for $\mathfrak{X}$ (resp. $\mathfrak{Y}$), and $f : \mathfrak{X} \to \mathfrak{Y}$ a morphism of formal preschemes.*

(i) *If $f^*(\mathscr{K})\mathcal{O}_{\mathfrak{X}} \subset \mathscr{J}$, then there exists a unique morphism $f' : (\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}/\mathscr{J}) \to (\mathfrak{Y}, \mathcal{O}_{\mathfrak{Y}}/\mathscr{K})$ of usual preschemes making the diagram*

(10.5.6.1)
$$\begin{tikzcd} (\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}) \arrow[r, "f"] & (\mathfrak{Y}, \mathcal{O}_{\mathfrak{Y}}) \\ (\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}/\mathscr{J}) \arrow[u] \arrow[r, "f'"] & (\mathfrak{Y}, \mathcal{O}_{\mathfrak{Y}}/\mathscr{K}) \arrow[u] \end{tikzcd}$$

commutate, where the vertical arrows are the canonical morphisms.

(ii) *Suppose that $\mathfrak{X} = \mathrm{Spf}(A)$ and $\mathfrak{Y} = \mathrm{Spf}(B)$ are formal affine schemes, $\mathscr{J} = \mathfrak{J}^\Delta$ and $\mathscr{K} = \mathfrak{K}^\Delta$, where $\mathfrak{J}$ (resp. $\mathfrak{K}$) is an ideal of definition for A (resp. B), and $f = ({}^a\varphi, \widetilde{\varphi})$, where $\varphi : B \to A$ is a continuous homomorphism; for $f^*(\mathscr{K})\mathscr{O}_\mathfrak{X} \subset \mathscr{J}$ to hold, it is necessary and sufficient that $\varphi(\mathfrak{K}) \subset \mathfrak{J}$, and, in this case, f' is then the morphism $({}^a\varphi', \widetilde{\varphi'})$, where $\varphi' : B/\mathfrak{K} \to A/\mathfrak{J}$ is the homomorphism induced from φ by passing to quotients.*

Proof.

(i) If $f = (\psi, \theta)$, then the hypotheses imply that the image under $\theta^\sharp : \psi^*(\mathscr{O}_\mathfrak{Y}) \to \mathscr{O}_\mathfrak{X}$ of the sheaf of ideals $\psi^*(\mathscr{K})$ of $\psi^*(\mathscr{O}_\mathfrak{Y})$ is contained in $\mathscr{J}$ (**0**, 4.3.5). By passing to quotients, we thus obtain from $\theta^\sharp$ a homomorphism of sheaves of rings

$$\omega : \psi^*(\mathscr{O}_\mathfrak{Y}/\mathscr{K}) = \psi^*(\mathscr{O}_\mathfrak{Y})/\psi^*(\mathscr{K}) \longrightarrow \mathscr{O}_\mathfrak{X}/\mathscr{J};$$

furthermore, since, for all $x \in \mathfrak{X}$, $\theta_x^\sharp$ is a *local* homomorphism, so too is ω_x. The morphism of ringed spaces $(\psi, \omega^\flat)$ is thus (2.2.1) the unique morphism f' of ringed spaces whose existence was claimed.

(ii) The canonical functorial correspondence between morphisms of formal affine schemes and continuous homomorphisms of rings (10.2.2) shows that, in the case considered, the relation $f^*(\mathscr{K})\mathscr{O}_\mathfrak{X} \subset \mathfrak{J}$ implies that we have $f' = ({}^a\varphi', \widetilde{\varphi'})$, where $\varphi' : B/\mathfrak{K} \to A/\mathfrak{J}$ is the unique homomorphism making the diagram

(10.5.6.2)
$$\begin{tikzcd} B \arrow[r, "\varphi"] \arrow[d] & A \arrow[d] \\ B/\mathfrak{K} \arrow[r, "\varphi'"] & A/\mathfrak{J} \end{tikzcd}$$

commutate. The existence of φ' thus implies that $\varphi(\mathfrak{K}) \subset \mathfrak{J}$. Conversely, if this condition is satisfied, then, denoting by φ' the unique homomorphism making the diagram (10.5.6.2) commutate and setting $f' = ({}^a\varphi', \widetilde{\varphi'})$, it is clear that the diagram (10.5.6.1) is commutative; considering the homomorphisms ${}^a\varphi^*(\mathscr{O}_\mathfrak{Y}) \to \mathscr{O}_\mathfrak{X}$ and ${}^a\varphi'^*(\mathscr{O}_\mathfrak{Y}/\mathscr{K}) \to \mathscr{O}_\mathfrak{X}/\mathscr{J}$ corresponding to f and f' respectively then leads to the fact that this implies the relation $f^*(\mathscr{K})\mathscr{O}_\mathfrak{X} \subset \mathscr{J}$.

□

It is clear that the correspondence $f \mapsto f'$ defined above is *functorial.*

10.6. Formal preschemes as inductive limits of preschemes.

(10.6.1). Let $\mathfrak{X}$ be a formal prescheme, and $(\mathscr{J}_\lambda)$ a fundamental system of sheaves of ideals of definition for $\mathfrak{X}$; for each λ, let f_λ be the canonical morphism $(\mathfrak{X}, \mathscr{O}_\mathfrak{X}/\mathscr{J}_\lambda) \to \mathfrak{X}$ (10.5.2); for $\mathscr{J}_\mu \subset \mathscr{J}_\lambda$, the canonical morphism $\mathscr{O}_\mathfrak{X}/\mathscr{J}_\mu \to \mathscr{O}_\mathfrak{X}/\mathscr{J}_\lambda$ defines a canonical morphism $f_{\mu\lambda} : (\mathfrak{X}, \mathscr{O}_\mathfrak{X}/\mathscr{J}_\lambda) \to (\mathfrak{X}, \mathscr{O}_\mathfrak{X}/\mathscr{J}_\mu)$ of (usual) preschemes such that **I** | 189 $f_\lambda = f_\mu \circ f_{\mu\lambda}$. The preschemes $X_\lambda = (\mathfrak{X}, \mathscr{O}_\mathfrak{X}/\mathscr{J}_\lambda)$ and the morphisms $f_{\mu\lambda}$ thus form (by (10.4.8)) an *inductive system* in the category of formal preschemes.

Proposition (10.6.2). — *With the notation of (10.6.1), the formal prescheme $\mathfrak{X}$ and the morphisms f_λ form an inductive limit (T, I, 1.8) of the system $(X_\lambda, f_{\mu\lambda})$ in the category of formal preschemes.*

Proof. Let $\mathfrak{Y}$ be a formal prescheme, and, for each index λ, let

$$g_\lambda = (\psi_\lambda, \theta_\lambda) : X_\lambda \longrightarrow \mathfrak{Y}$$

be a morphism such that $g_\lambda = g_\mu \circ f_{\mu\lambda}$ for $\mathscr{J}_\mu \subset \mathscr{J}_\lambda$. This latter condition and the definition of the X_λ imply first of all that the ψ_λ are identical to a single continuous map $\psi : \mathfrak{X} \to \mathfrak{Y}$ of the underlying spaces; in addition, the homomorphisms $\theta_\lambda^\sharp : \psi^*(\mathscr{O}_\mathfrak{Y}) \to \mathscr{O}_{X_i} = \mathscr{O}_\mathfrak{X}/\mathscr{J}_\lambda$ form a *projective system* of homomorphisms of sheaves of rings. By passing to the projective limit, there is an induced homomorphism $\omega : \psi^*(\mathscr{O}_\mathfrak{Y}) \to \varprojlim \mathscr{O}_\mathfrak{X}/\mathscr{J}_\lambda = \mathscr{O}_\mathfrak{X}$, and it is clear that the morphism $g = (\psi, \omega^\flat)$ of *ringed spaces* is the *unique* morphism making the diagrams

(10.6.2.1)
$$\begin{tikzcd} X_\lambda \arrow[rr, "g_\lambda"] \arrow[dr, "f_\lambda"'] & & \mathfrak{Y} \\ & \mathfrak{X} \arrow[ur, "g"'] & \end{tikzcd}$$

commutative. It remains to prove that g is a morphism of *formal preschemes*; the question is local on $\mathfrak{X}$ and $\mathfrak{Y}$, so we can assume that $\mathfrak{X} = \mathrm{Spf}(A)$ and $\mathfrak{Y} = \mathrm{Spf}(B)$, with A and B admissible rings, and with $\mathscr{J}_\lambda = \mathfrak{J}_\lambda^\Delta$, where $(\mathfrak{J}_\lambda)$ is a fundamental system of ideals of definition for A (10.3.5); since $A = \varprojlim A/\mathfrak{J}_\lambda$, the existence of a morphism g of formal affine schemes making the diagrams (10.6.2.1) commutate then follows from the bijective correspondence

(10.2.2) between morphisms of formal affine schemes and continuous ring homomorphisms, and from the definition of the projective limit. But the uniqueness of g as a morphism of ringed spaces shows that it coincides with the morphism in the beginning of the proof. ☐

The following proposition establishes, under certain additional conditions, the existence of the inductive limit of a given inductive system of (usual) preschemes in the category of formal preschemes:

Proposition (10.6.3). — *Let $\mathfrak{X}$ be a topological space, and $(\mathcal{O}_i, u_{ji})$ a projective system of sheaves of rings on $\mathfrak{X}$, with* $\mathbf{N}$ *for its set of indices. Let $\mathscr{J}_i$ be the kernel of $u_{0i} : \mathcal{O}_i \to \mathcal{O}_0$. Suppose that:*

(a) *the ringed space $(\mathfrak{X}, \mathcal{O}_i)$ is a prescheme X_i;*
(b) *for all $x \in \mathfrak{X}$ and all i, there exists an open neighborhood U_i of x in $\mathfrak{X}$ such that the restriction $\mathscr{J}_i|U_i$ is nilpotent; and* $\mathbf{Err}_{\mathrm{II}}$
(c) *the homomorphisms u_{ji} are surjective.* **I** | 190

Let $\mathcal{O}_{\mathfrak{X}}$ be the sheaf of topological rings given by the projective limit of the pseudo-discrete sheaves of rings $\mathcal{O}_i$, and let $u_i : \mathcal{O}_{\mathfrak{X}} \to \mathcal{O}_i$ be the canonical homomorphism. Then the topologically ringed space $(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ is a formal prescheme; the homomorphisms u_i are surjective; their kernels $\mathscr{J}^{(i)}$ form a fundamental system of sheaves of ideals of definition for $\mathfrak{X}$, and $\mathscr{J}^{(0)}$ is the projective limit of the sheaves of ideals $\mathscr{J}_i$.

Proof. We first note that, on each stalk, u_{ji} is a surjective homomorphism and *a fortiori* a local homomorphism; thus $v_{ij} = (1_{\mathfrak{X}}, u_{ji})$ is a morphism of preschemes $X_j \to X_i$ $(i \geqslant j)$ (2.2.1). Suppose first that each X_i is an affine scheme of ring A_i. There exists a *ring* homomorphism $\varphi_{ji} : A_i \to A_j$ such that $u_{ji} = \widetilde{\varphi_{ji}}$ (1.7.3); as a result (1.6.3), the sheaf $\mathcal{O}_j$ is a quasi-coherent $\mathcal{O}_i$-module over X_i (for the external law defined by u_{ji}), associated to A_j considered as an A_i-module by means of φ_{ji}. For all $f \in A_i$, let $f' = \varphi_{ji}(f)$; by hypothesis, the open sets $D(f)$ and $D(f')$ are identical in $\mathfrak{X}$, and the homomorphism from $\Gamma(D(f), \mathcal{O}_i) = (A_i)_f$ to $\Gamma(D(f), \mathcal{O}_j) = (A_j)_{f'}$ corresponding to u_{ji} is exactly $(\varphi_{ji})_f$ (1.6.1). But when we consider A_j as an A_i-module, $(A_j)_{f'}$ is the $(A_i)_f$-module $(A_j)_f$, so we also have $u_{ji} = \widetilde{\varphi_{ji}}$, where φ_{ji} is now considered as a homomorphism of A_i*-modules*. Then, since u_{ji} is surjective, we conclude that φ_{ji} is also surjective (1.3.9), and if $\mathfrak{I}_{ji}$ is the kernel of φ_{ji}, then the kernel of u_{ji} is a quasi-coherent $\mathcal{O}_i$-module equal to $\widetilde{\mathfrak{I}_{ji}}$. In particular, we have $\mathscr{J}_i = \widetilde{\mathfrak{I}_i}$, where $\mathfrak{I}_i$ is the kernel of $\varphi_{0i} : A_i \to A_0$. Hypothesis (b) implies that $\mathscr{J}_i$ is *nilpotent*: indeed, since $\mathfrak{X}$ is quasi-compact, we can cover $\mathfrak{X}$ by a finite number of open sets U_k such that $(\mathscr{J}_i|U_k)^{n_k} = 0$, and, by setting n to be the largest of the n_k, we have $\mathscr{J}_i^n = 0$. We thus conclude that $\mathfrak{I}_i$ is nilpotent (1.3.13). Then the ring $A = \varprojlim A_i$ is admissible (**0**, 7.2.2), the canonical homomorphism $\varphi_i : A \to A_i$ is surjective, and its kernel $\mathfrak{I}^{(i)}$ is equal to the projective limit of the $\mathfrak{I}_{ik}$ for $k \geqslant i$; the $\mathfrak{I}^{(i)}$ form a fundamental system of neighborhoods of 0 in A. The claims of Proposition (10.6.3) follow in this case from (10.1.1) and (10.3.2), with $(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ being $\operatorname{Spf}(A)$.

Again, in this particular case, we note that, if $f = (f_i)$ is an element of the projective limit $A = \varprojlim A_i$, then all the open sets $D(f_i)$ (which are affine open sets in X_i) can be identified with the open subset $\mathfrak{D}(f)$ of $\mathfrak{X}$, and the prescheme induced on $\mathfrak{D}(f)$ by X_i thus being identified with the affine scheme $\operatorname{Spec}((A_i)_{f_i})$.

In the general case, we remark first that, for every quasi-compact open subset U of $\mathfrak{X}$, each of the $\mathscr{J}_i|U$ is nilpotent, as shown by the above reasoning. We will show that, for every $x \in \mathfrak{X}$, there exists an open neighborhood U of x in $\mathfrak{X}$ which is an *affine open set* for *all* the X_i. Indeed, we take U to be an affine open set for X_0, and observe that $\mathcal{O}_{X_0} = \mathcal{O}_{X_i}/\mathscr{J}_i$. Since, by the above, $\mathscr{J}_i|U$ is nilpotent, U is also an affine open set for each X_i, by Proposition (5.1.9). This being so, for each U satisfying the preceding conditions, the study of the affine case as above shows that $(U, \mathcal{O}_X|U)$ is a formal prescheme whose $\mathscr{J}^{(i)}|U$ form a fundamental system of sheaves of ideals of definition, and $\mathscr{J}^{(0)}|U$ is the projective limit of the $\mathscr{J}_i|U$; whence the conclusion. ☐

Corollary (10.6.4). — *Suppose that, for $i \geqslant j$, the kernel of u_{ji} is $\mathscr{J}_i^{j+1}$ and that $\mathscr{J}_1/\mathscr{J}_1^2$ is of finite type over $\mathcal{O}_0 = \mathcal{O}_1/\mathscr{J}_1$. Then $\mathfrak{X}$ is an adic formal prescheme, and if $\mathscr{J}^{(n)}$ is the kernel of $\mathcal{O}_{\mathfrak{X}} \to \mathcal{O}_n$, then $\mathscr{J}^{(n)} = \mathscr{J}^{n+1}$ and $\mathscr{J}/\mathscr{J}^2$ is isomorphic to $\mathscr{J}_1$. If, in addition, X_0 is locally Noetherian (resp. Noetherian), then $\mathfrak{X}$ is locally Noetherian (resp. Noetherian).* **I** | 191

Proof. Since the underlying spaces of $\mathfrak{X}$ and X_0 are the same, the question is local, and we can suppose that all the X_i are affine; taking into account the fact that $\mathscr{J}_{ij} = \widetilde{\mathfrak{I}_{ji}}$ (with the notation of Proposition (10.3.6)), we can immediately reduce the problem to the corresponding claims of Proposition (**0**, 7.2.7) and Corollary (**0**, 7.2.8), by noting that $\mathfrak{I}_1/\mathfrak{I}_1^2$ is then an A_0-module of finite type (1.3.9). ☐

In particular, *every locally Noetherian formal prescheme* $\mathfrak{X}$ is the inductive limit of a sequence (X_n) of locally Noetherian (usual) preschemes satisfying the conditions of Proposition (10.3.6) and Corollary (10.6.4): it suffices to consider a sheaf of ideals of definition $\mathscr{J}$ for $\mathfrak{X}$ (10.5.4) and to take $X_n = (\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}/\mathscr{J}^{n+1})$ ((10.5.1) and Proposition (10.6.2)).

Corollary (10.6.5). — *Let A be an admissible ring. For the formal affine scheme $\mathfrak{X} = \mathrm{Spf}(A)$ to be Noetherian, it is necessary and sufficient for A to be adic and Noetherian.*

Proof. The condition is evidently sufficient. Conversely, suppose that $\mathfrak{X}$ is Noetherian, and let $\mathfrak{J}$ be an ideal of definition for A, and $\mathscr{J} = \mathfrak{J}^\Delta$ the corresponding sheaf of ideals of definition for $\mathfrak{X}$. The (usual) preschemes $X_n = (\mathfrak{X}, \mathscr{O}_\mathfrak{X}/\mathscr{J}^{n+1})$ are then affine and Noetherian, so the rings $A_n = A/\mathfrak{J}^{n+1}$ are Noetherian (6.1.3), whence we conclude that $\mathfrak{J}/\mathfrak{J}^2$ is an $A/\mathfrak{J}$-module of finite type. Since the $\mathscr{J}^n$ form a fundamental system of sheaves of ideals of definition for $\mathfrak{X}$ (10.5.1), we have $\mathscr{O}_\mathfrak{X} = \varprojlim \mathscr{O}_\mathfrak{X}/\mathscr{J}^n$ (10.5.3); we thus conclude (10.1.3) that A is topologically isomorphic to $\varprojlim A/\mathfrak{J}^n$, which is adic and Noetherian (**0**, 7.2.8). ☐

Remark (10.6.6). — With the notation of Proposition (10.6.3), let $\mathscr{F}_i$ be an $\mathscr{O}_i$-module, and suppose we are given, for $i \geqslant i$, a v_{ij}-morphism $\theta_{ji} : \mathscr{F}_i \to \mathscr{F}_j$ such that $\theta_{kj} \circ \theta_{ji} = \theta_{ki}$ for $k \leqslant j \leqslant i$. Since the underlying continuous map of v_{ij} is the identity, θ_{ji} is a homomorphism of sheaves of abelian groups to the space $\mathfrak{X}$; in addition, if $\mathscr{F}$ is the projective limit of the projective system $(\mathscr{F}_i)$ of sheaves of abelian groups, then the fact that the θ_{ji} are v_{ij}-morphisms lets us define an $\mathscr{O}_\mathfrak{X}$-module structure on $\mathscr{F}$ by passing to the projective limit; when equipped with this structure, we say that $\mathscr{F}$ is the *projective limit* (with respect to the θ_{ji}) of the system of $\mathscr{O}_i$-modules $(\mathscr{F}_i)$. In the particular case where $v_{ij}^*(\mathscr{F}_i) = \mathscr{F}_j$ and θ_{ji} is the *identity*, we say that $\mathscr{F}$ is the projective limit of a system $(\mathscr{F}_i)$ such that $v_{ij}^*(\mathscr{F}_i) = \mathscr{F}_j$ for $j \leqslant i$ (without mentioning the θ_{ji}).

(10.6.7). Let $\mathfrak{X}$ and $\mathfrak{Y}$ be formal preschemes, $\mathscr{J}$ (resp. $\mathscr{K}$) a sheaf of ideals of definition for $\mathfrak{X}$ (resp. $\mathfrak{Y}$), and $f : \mathfrak{X} \to \mathfrak{Y}$ a morphism such that $f^*(\mathscr{K})\mathscr{O}_\mathfrak{X} \subset \mathscr{J}$. We then have, for every integer $n > 0$, that $f^*(\mathscr{K}^n)\mathscr{O}_\mathfrak{X} = (f^*(\mathscr{K})\mathscr{O}_\mathfrak{X})^n \subset \mathscr{J}^n$; f thus induces (10.5.6) a morphism of (usual) preschemes $f_n : X_n \to Y_n$ by setting $X_n = (\mathfrak{X}, \mathscr{O}_\mathfrak{X}/\mathscr{J}^{n+1})$ and $Y_n = (\mathscr{Y}, \mathscr{O}_\mathfrak{Y}/\mathscr{K}^{n+1})$, and it immediately follows from the definitions that the diagrams

$$
\begin{tikzcd}
X_m \arrow[r, "f_m"] \arrow[d] & Y_m \arrow[d] \\
X_n \arrow[r, "f_n"] & Y_n
\end{tikzcd}
\tag{10.6.7.1}
$$

commute for $m \leqslant n$; in other words, (f_n) is an *inductive system* of morphisms. **I** | 192

(10.6.8). Conversely, let (X_n) (resp. (Y_n)) be an inductive system of (usual) preschemes satisfying conditions (b) and (c) of Proposition (10.6.3), and let $\mathfrak{X}$ (resp. $\mathfrak{Y}$) be its inductive limit. By definition of the inductive limit, each sequence (f_n) of morphisms $X_n \to Y_n$ forms an inductive system that admits an *inductive limit* $f : \mathfrak{X} \to \mathfrak{Y}$, which is the unique morphism of formal preschemes that makes the diagrams

$$
\begin{tikzcd}
X_n \arrow[r, "f_n"] \arrow[d] & Y_n \arrow[d] \\
\mathfrak{X} \arrow[r, "f"] & \mathfrak{Y}
\end{tikzcd}
$$

commutate.

Proposition (10.6.9). — *Let $\mathfrak{X}$ and $\mathfrak{Y}$ be locally Noetherian formal preschemes, and $\mathscr{J}$ (resp. $\mathscr{K}$) be a sheaf of ideals of definition for $\mathfrak{X}$ (resp. $\mathfrak{Y}$); the map $f \mapsto (f_n)$ defined in (10.6.7) is a bijection from the set of morphisms $f : \mathfrak{X} \to \mathfrak{Y}$ such that $f^*(\mathscr{K})\mathscr{O}_\mathfrak{X} \subset \mathscr{J}$ to the set of sequences (f_n) of morphisms that make the diagrams (10.6.7.1) commutate.*

Proof. If f is the inductive limit of this sequence, then it remains to show that $f^*(\mathscr{K})\mathscr{O}_\mathfrak{X} \subset \mathscr{J}$. The statement, being local on $\mathfrak{X}$ and $\mathfrak{Y}$, can be reduced to the case where $\mathfrak{X} = \mathrm{Spf}(A)$ and $\mathfrak{Y} = \mathrm{Spf}(B)$ are affine, with A and B adic Noetherian rings, and with $\mathscr{J} = \mathfrak{J}^\Delta$ and $\mathscr{K} = \mathfrak{K}^\Delta$, where $\mathfrak{J}$ (resp. $\mathfrak{K}$) is an ideal of definition for A (resp. B). We then have that $X_n = \mathrm{Spec}(A_n)$ and $Y_n = \mathrm{Spec}(B_n)$, with $A_n = A/\mathfrak{J}^{n+1}$ and $B_n = B/\mathfrak{K}^{n+1}$, by Proposition (10.3.6) and (10.3.2); $f_n = ({}^a\varphi_n, \widetilde{\varphi_n})$, where the homomorphisms $\varphi_n : B_n \to A_n$ forms a projective system, thus $f = ({}^a\varphi, \widetilde{\varphi})$, and so $f = ({}^a\varphi, \widetilde{\varphi})$, where $\varphi = \varprojlim \varphi_n$. The commutativity of the diagram (10.6.7.1) for $m = 0$ then gives the condition $\varphi_n(\mathfrak{K}/\mathfrak{K}^{n+1}) \subset \mathfrak{J}/\mathfrak{J}^{n+1}$ for all n, so, by passing to the projective limit, we have $\varphi(\mathfrak{K}) \subset \mathfrak{J}$, which implies that $f^*(\mathscr{K})\mathscr{O}_\mathfrak{X} \subset \mathscr{J}$ (10.5.6, ii). ☐

Corollary (10.6.10). — *Let $\mathfrak{X}$ and $\mathfrak{Y}$ be locally Noetherian formal preschemes, and $\mathscr{T}$ the largest sheaf of ideals of definition for $\mathfrak{X}$ (10.5.4).*

(i) *For every sheaf of ideals of definition $\mathscr{K}$ for $\mathfrak{Y}$ and every morphism $f : \mathfrak{X} \to \mathfrak{Y}$, we have $f^*(\mathscr{K})\mathscr{O}_\mathfrak{X} \subset \mathscr{T}$.*

(ii) *There is a canonical bijective correspondence between* $\mathrm{Hom}(\mathfrak{X}, \mathfrak{Y})$ *and the set of sequences (f_n) of morphisms making the diagrams (10.6.7.1) commute, where $X_n = (\mathfrak{X}, \mathscr{O}_\mathfrak{X}/\mathscr{T}^{n+1})$ and $Y_n = (\mathfrak{Y}, \mathscr{O}_\mathfrak{Y}/\mathscr{K}^{n+1})$.*

Proof. (ii) follows immediately from (i) and Proposition (10.6.9). To prove (i), we can reduce to the case where $\mathfrak{X} = \mathrm{Spf}(A)$ and $\mathfrak{Y} = \mathrm{Spf}(B)$, with A and B Noetherian, and with $\mathscr{T} = \mathfrak{T}^\Delta$ and $\mathscr{K} = \mathfrak{K}^\Delta$, where $\mathfrak{T}$ is the largest ideal of definition for A and $\mathfrak{K}$ is an ideal of definition for B. Let $f = ({}^a\varphi, \widetilde{\varphi})$, where $\varphi : B \to A$ is a continuous homomorphism; since the elements of $\mathfrak{K}$ are topologically nilpotent (**0**, 7.1.4, ii), so too are those of $\varphi(\mathfrak{K})$, and so $\varphi(\mathfrak{K}) \subset \mathfrak{T}$, since $\mathfrak{T}$ is the set of topologically nilpotent elements of A (**0**, 7.1.6); hence, by Proposition (10.5.6, ii), we are done. □

Corollary (10.6.11). — *Let $\mathfrak{S}$, $\mathfrak{X}$, $\mathfrak{Y}$ be locally Noetherian formal preschemes, and $f : \mathfrak{X} \to \mathfrak{S}$ and $g : \mathfrak{Y} \to \mathfrak{S}$ the morphisms that make $\mathfrak{X}$ and $\mathfrak{Y}$ formal $\mathfrak{S}$-preschemes. Let $\mathscr{J}$ (resp. $\mathscr{K}$, $\mathscr{L}$) be a sheaf of ideals of definition for $\mathfrak{S}$ (resp. $\mathfrak{X}$, $\mathfrak{Y}$), and suppose that $f^*(\mathscr{J})\mathscr{O}_{\mathfrak{X}} \subset \mathscr{K}$ and $g^*(\mathscr{J})\mathscr{O}_{\mathfrak{Y}} = \mathscr{L}$; set $S_n = (\mathfrak{S}, \mathscr{O}_{\mathfrak{S}}/\mathscr{J}^{n+1})$, $X_n = (\mathfrak{X}, \mathscr{O}_{\mathfrak{X}}/\mathscr{K}^{n+1})$, and $Y_n = (\mathfrak{Y}, \mathscr{O}_{\mathfrak{Y}}/\mathscr{L}^{n+1})$. Then there exists a canonical bijective correspondence between $\mathrm{Hom}_{\mathfrak{S}}(\mathfrak{X},\mathfrak{Y})$ and the set of sequences* **I** | 193
(u_n) of S_n-morphisms $u_n : X_n \to Y_n$ making the diagrams (10.6.7.1) commute.

Proof. For each $\mathfrak{S}$-morphism $u : \mathfrak{X} \to \mathfrak{Y}$, we have by definition that $f = g \circ u$, whence

$$u^*(\mathscr{L})\mathscr{O}_{\mathfrak{X}} = u^*(g^*(\mathscr{J}\mathscr{O}_{\mathfrak{Y}})\mathscr{O}_{\mathfrak{X}} = f^*(\mathscr{J})\mathscr{O}_{\mathfrak{X}} \subset \mathscr{K},$$

and the corollary then follows from Proposition (10.6.9). □

We note that, for $m \leqslant n$, the data of a morphism $f_n : X_n \to Y_n$ determines a unique morphism $f_m : X_m \to Y_m$ making the diagram (10.6.7.1) commutate, since we immediately see that we can reduce to the affine case; we have thus defined a map $\varphi_{mn} : \mathrm{Hom}_{S_n}(X_n, Y_n) \to \mathrm{Hom}_{S_m}(X_m, Y_m)$, and the $\mathrm{Hom}_{S_n}(X_n, Y_n)$ form, with the φ_{mn}, a *projective system of sets*; Corollary (10.6.11) then says that there is a canonical bijection

$$\mathrm{Hom}_{\mathfrak{S}}(\mathfrak{X},\mathfrak{Y}) \simeq \varprojlim_n \mathrm{Hom}_{S_n}(X_n, Y_n).$$

10.7. Products of formal preschemes.

(10.7.1). Let $\mathfrak{S}$ be a formal prescheme; formal $\mathfrak{S}$-preschemes form a category, and we can define a notion of a *product* of formal $\mathfrak{S}$-preschemes.

Proposition (10.7.2). — *Let $\mathfrak{X} = \mathrm{Spf}(B)$ and $\mathfrak{Y} = \mathrm{Spf}(C)$ be formal affine schemes over a formal affine scheme $\mathfrak{S} = \mathrm{Spf}(A)$. Let $\mathfrak{Z} = \mathrm{Spf}(B\widehat{\otimes}_A C)$, and let p_1 and p_2 be the $\mathfrak{S}$-morphisms corresponding (10.2.2) to the canonical (continuous) A-homomorphisms $\rho : B \to B\widehat{\otimes}_A C$ and $\sigma : C \to B\widehat{\otimes}_A C$; then $(\mathfrak{Z}, p_1, p_2)$ is a product of the formal affine $\mathfrak{S}$-schemes $\mathfrak{X}$ and $\mathfrak{Y}$.*

Proof. By Proposition (10.4.6), it suffices to check that, if we associate, to each continuous A-homomorphism $\varphi : B\widehat{\otimes}_A C \to D$ (where D is an admissible ring which is a topological A-algebra), the pair $(\varphi \circ \rho, \varphi \circ \sigma)$, then this defines a bijection

$$\mathrm{Hom}_A(B\widehat{\otimes}_A C, D) \simeq \mathrm{Hom}_A(B, D) \times \mathrm{Hom}_A(C, D),$$

which is exactly the universal property of the completed tensor product (**0**, 7.7.6). □

Proposition (10.7.3). — *Given formal $\mathfrak{S}$-preschemes $\mathfrak{X}$ and $\mathfrak{Y}$, their product $\mathfrak{X} \times_{\mathfrak{S}} \mathfrak{Y}$ exists.*

Proof. The proof is similar to that of Theorem (3.2.6), replacing affine schemes (resp. affine open sets) by formal affine schemes (resp. formal affine open sets), and replacing Proposition (3.2.2) by Proposition (10.7.2). □

All the formal properties of the product of preschemes ((3.2.7) and (3.2.8), (3.3.1) and (3.3.12)) hold true without modification for the product of formal preschemes.

(10.7.4). Let $\mathfrak{S}$, $\mathfrak{X}$, and $\mathfrak{Y}$ be formal preschemes, and let $f : \mathfrak{X} \to \mathfrak{S}$ and $g : \mathfrak{Y} \to \mathfrak{S}$ be morphisms. Suppose that there exist, in $\mathfrak{S}$, $\mathfrak{X}$, and $\mathfrak{Y}$ respectively, three fundamental systems of sheaves of ideals of definitions $(\mathscr{J}_\lambda)$, $(\mathscr{K}_\lambda)$, and $(\mathscr{L}_\lambda)$, all having the same set I of indices, and such that $f^*(\mathscr{J}_\lambda)\mathscr{O}_{\mathfrak{X}} \subset \mathscr{K}_\lambda$ and $g^*(\mathscr{J}_\lambda)\mathscr{O}_{\mathfrak{Y}} \subset \mathscr{L}_\lambda$ for all λ. Set $S_\lambda = (\mathfrak{S}, \mathscr{O}_{\mathfrak{S}}/\mathscr{J}_\lambda)$, $X_\lambda = (\mathfrak{X}, \mathscr{O}_{\mathfrak{X}}/\mathscr{K}_\lambda)$, and $Y_\lambda = (\mathfrak{Y}, \mathscr{O}_{\mathfrak{Y}}/\mathscr{L}_\lambda)$; for $\mathscr{J}_\mu \subset \mathscr{J}_\lambda$, $\mathscr{K}_\mu \subset \mathscr{K}_\lambda$, and $\mathscr{L}_\mu \subset \mathscr{L}_\lambda$, note that S_λ (resp. X_λ, Y_λ) is a closed subprescheme of S_μ (resp. X_μ, Y_μ) that has the *same* underlying space (10.6.1). Since **I** | 194
$S_\lambda \to S_\mu$ is a monomorphism of preschemes, we see that the products $X_\lambda \times_{S_\lambda} Y_\lambda$ and $X_\lambda \times_{S_\mu} Y_\lambda$ are identical (3.2.4), since $X_\lambda \times_{S_\mu} Y_\lambda$ can be identified with a closed subprescheme of $X_\mu \times_{S_\mu} Y_\mu$ that has the *same* underlying space (4.3.1). With this in mind, the product $\mathfrak{X} \times_{\mathfrak{S}} \mathfrak{Y}$ is the *inductive limit* of the usual preschemes $X_\lambda \times_{S_\lambda} Y_\lambda$: indeed, as we see in Proposition (10.6.2), we can reduce to the case where $\mathfrak{S}$, $\mathfrak{X}$, and $\mathfrak{Y}$ are formal affine schemes. Taking into account both Proposition (10.5.6, ii) and the hypotheses on the fundamental systems of sheaves of ideals of definition for $\mathfrak{S}$, $\mathfrak{X}$, and $\mathfrak{Y}$, we immediately see that our claim follows from the definition of the completed tensor product of two algebras (**0**, 7.7.1).

Furthermore, let $\mathfrak{Z}$ be a formal $\mathfrak{S}$-prescheme, $(\mathscr{M}_\lambda)$ a fundamental system of ideals of definition for $\mathfrak{Z}$ having I for its set of indices, and let $u : \mathfrak{Z} \to \mathfrak{X}$ and $v : \mathfrak{Z} \to \mathfrak{Y}$ be $\mathfrak{S}$-morphisms such that $u^*(\mathscr{K}_\lambda)\mathscr{O}_\mathfrak{Z} \subset \mathscr{M}_\lambda$ and $v^*(\mathscr{L}_\lambda)\mathscr{O}_\mathfrak{Z} \subset \mathscr{M}_\lambda$ for all λ. If we set $Z_\lambda = (\mathfrak{Z}, \mathscr{O}_\mathfrak{Z}/\mathscr{M}_\lambda)$, and if $u_\lambda : Z_\lambda \to X_\lambda$ and $v_\lambda : Z_\lambda \to Y_\lambda$ are the S_λ-morphisms corresponding to u and v (10.5.6), then we immediately have that $(u, v)_\mathfrak{S}$ is the inductive limit of the S_λ-morphisms $(u_\lambda, v_\lambda)_{S_\lambda}$.

The ideas of this section apply, in particular, to the case where $\mathfrak{S}$, $\mathfrak{X}$, and $\mathfrak{Y}$ are locally Noetherian, taking the systems consisting of the powers of a sheaf of ideals of definition (10.5.1) as the fundamental systems of sheaves of ideals of definition . However, we note that $\mathfrak{X} \times_\mathfrak{S} \mathfrak{Y}$ is not necessarily locally Noetherian (see however (10.13.5)).

10.8. Formal completion of a prescheme along a closed subset.

(10.8.1). Let X be a *locally Noetherian* (usual) prescheme, and X' a closed subset of the underlying space of X; we denote by Φ the set of *coherent* sheaves of ideals $\mathscr{J}$ of $\mathscr{O}_X$ such that the support of $\mathscr{O}_X/\mathscr{J}$ is X'. The set Φ is nonempty ((5.2.1), (4.1.4), (6.1.1)); we order it by the relation $\supset$.

Lemma (10.8.2). — *The ordered set Φ is filtered; if X is Noetherian, then, for all $\mathscr{J}_0 \in \Phi$, the set of powers $\mathscr{J}_0^n$ $(n > 0)$ is cofinal in Φ.*

Proof. If $\mathscr{J}_1$ and $\mathscr{J}_2$ are in Φ, and if we set $\mathscr{J} = \mathscr{J}_1 \cap \mathscr{J}_2$, then $\mathscr{J}$ is coherent since $\mathscr{O}_X$ is coherent ((6.1.1) and (**0**, 5.3.4)), and we have $\mathscr{J}_x = (\mathscr{J}_1)_x \cap (\mathscr{J}_2)_x$ for all $x \in X$, whence $\mathscr{J}_x = \mathscr{O}_x$ for $x \notin X'$, and $\mathscr{J}_x \neq \mathscr{O}_x$ for $x \in X'$, which proves that $\mathscr{J} \in \Phi$. On the other hand, if X is Noetherian and if $\mathscr{J}_0$ and $\mathscr{J}$ are in Φ, then there exists an integer $n > 0$ such that $\mathscr{J}_0^n(\mathscr{O}_X/\mathscr{J}) = 0$ (9.3.4), which implies that $\mathscr{J}_0^n \subset \mathscr{J}$. □

(10.8.3). Now let $\mathscr{F}$ be a *coherent* $\mathscr{O}_X$-module; for all $\mathscr{J} \in \Phi$, we have that $\mathscr{F} \otimes_{\mathscr{O}_X} (\mathscr{O}_X/\mathscr{J})$ is a coherent $\mathscr{O}_X$-module (9.1.1) with support contained in X', and we will usually identify it with its restriction to X'. When $\mathscr{J}$ varies over Φ, these sheaves form a *projective system* of sheaves of abelian groups.

Definition (10.8.4). — Given a closed subset X' of a locally Noetherian prescheme X and a coherent $\mathscr{O}_X$-module $\mathscr{F}$, we define the *completion of $\mathscr{F}$ along X'*, denoted by $\mathscr{F}_{/X'}$ (or $\widehat{\mathscr{F}}$ when there is little chance of confusion), to be the restriction to X' of the sheaf $\varprojlim_\Phi (\mathscr{F} \otimes_{\mathscr{O}_X} (\mathscr{O}_X/\mathscr{J}))$; we say that its sections over X' are the *formal sections of $\mathscr{F}$ along X'*. **I** | 195

It is immediate that, for every open $U \subset X$, we have $(\mathscr{F}|U)_{/(U\cap X')} = (\mathscr{F}_{/X'})|(U \cap X')$.

By passing to the projective limit, it is clear that $(\mathscr{O}_X)_{/X'}$ is a sheaf of rings, and that $\mathscr{F}_{/X'}$ can be considered as an $(\mathscr{O}_X)_{/X'}$-module. In addition, since there exists a basis for the topology of X' consisting of quasi-compact open sets, we can consider $(\mathscr{O}_X)_{/X'}$ (resp. $\mathscr{F}_{/X'}$) as a *sheaf of topological rings* (resp. of *topological groups*), the projective limit of the *pseudo-discrete* sheaves of rings (resp. groups) $\mathscr{O}_X/\mathscr{F}$ (resp. $\mathscr{F} \otimes_{\mathscr{O}_X} (\mathscr{O}_X/\mathscr{F}) = \mathscr{F}/\mathscr{J}\mathscr{F}$), and, by passing to the projective limit, $\mathscr{F}_{/X'}$ then becomes a *topological $(\mathscr{O}_X)_{/X'}$-module* ((**0**, 3.8.1) and (**0**, 8.2)); recall that, for every *quasi-compact* open $U \subset X$, $\Gamma(U \cap X', (\mathscr{O}_X)_{/X'})$ (resp. $\Gamma(U \cap X', \mathscr{F}_{/X'})$) is then the projective limit of the discrete rings (resp. groups) $\Gamma(U, \mathscr{O}_X/\mathscr{J})$ (resp. $\Gamma(U, \mathscr{F}/\mathscr{J}\mathscr{F})$).

Now, if $u : \mathscr{F} \to \mathscr{G}$ is a homomorphism of $\mathscr{O}_X$-modules, then there are canonically induced homomorphisms $u_\mathscr{J} : \mathscr{F} \otimes_{\mathscr{O}_X} (\mathscr{O}_X/\mathscr{J}) \to \mathscr{G} \otimes_{\mathscr{O}_X} (\mathscr{O}_X/\mathscr{J})$ for all $\mathscr{J} \in \Phi$, and these homomorphisms form a projective system. By passing to the projective limit and restricting to X', these give a continuous $(\mathscr{O}_X)_{/X'}$-homomorphism $\mathscr{F}_{/X'} \to \mathscr{G}_{/X'}$, denoted $u_{/X'}$ or $\widehat{u}$, and called the *completion* of the homomorphism u along X'. It is clear that, if $v : \mathscr{G} \to \mathscr{H}$ is a second homomorphism of $\mathscr{O}_X$-modules, then we have $(v \circ u)_{/X'} = (v_{/X'}) \circ (u_{/X'})$, hence $\mathscr{F}_{/X'}$ is a *covariant additive functor* in $\mathscr{F}$ from the category of coherent $\mathscr{O}_X$-modules to the category of topological $(\mathscr{O}_X)_{/X'}$-modules.

Proposition (10.8.5). — *The support of $(\mathscr{O}_X)_{/X'}$ is X'; the topologically ringed space $(X', (\mathscr{O}_X)_{/X'})$ is a locally Noetherian formal prescheme, and, if $\mathscr{J} \in \Phi$, then $\mathscr{J}_{/X'}$ is a sheaf of ideals of definition for this formal prescheme. If $X = \operatorname{Spec}(A)$ is an affine scheme with Noetherian ring A, $\mathscr{J} = \widetilde{\mathfrak{J}}$ for some ideal $\mathfrak{J}$ of A, and $X' = V(\mathfrak{J})$, then $(X', (\mathscr{O}_X)_{/X'})$ is canonically identified with $\operatorname{Spf}(\widehat{A})$, where $\widehat{A}$ is the separated completion of A with respect to the $\mathfrak{J}$-preadic topology.*

Proof. We can evidently reduce to proving the latter claim. We know (**0**, 7.3.3) that the separated completion $\widehat{\mathfrak{J}}$ of $\mathfrak{J}$ with respect to the $\mathfrak{J}$-preadic topology can be identified with the ideal $\mathfrak{J}\widehat{A}$ of $\widehat{A}$, where $\widehat{A}$ is the Noetherian $\widehat{\mathfrak{J}}$-adic ring such that $\widehat{A}/\widehat{\mathfrak{J}}^n = A/\mathfrak{J}^n$ (**0**, 7.2.6). This latter equation shows that the open prime ideals of $\widehat{A}$ are the ideals $\widehat{\mathfrak{p}} = \mathfrak{p}\widehat{\mathfrak{J}}$, where $\mathfrak{p}$ is a prime ideal of A containing $\mathfrak{J}$, and that we have $\widehat{\mathfrak{p}} \cap A = \mathfrak{p}$, and hence $\operatorname{Spf}(\widehat{A}) = X'$. Since $\mathscr{O}_X/\mathscr{J}^n = (A/\mathfrak{J}^n)^\sim$, the proposition follows immediately from the definitions. □

We say that the formal prescheme defined above is the *completion of X along X'*, and we denote it by $X_{/X'}$ or $\widehat{X}$ when there is little chance of confusion. When we take $X' = X$, we can set $\mathscr{J} = 0$, and we thus have $X_{/X} = X$.

It is clear that, if U is a subprescheme induced on an open subset of X, then $U_{/(U\cap X')}$ is canonically identified with the formal subprescheme induced on $X_{/X'}$ by the open subset $U\cap X'$ of X'.

Corollary (10.8.6). — *The (usual) prescheme $\widehat{X}_{\mathrm{red}}$ is the unique reduced subprescheme of X having X' as its underlying space (5.2.1). For $\widehat{X}$ to be Noetherian, it is necessary and sufficient for $\widehat{X}_{\mathrm{red}}$ tob e Noetherian, and it suffices that X be Noetherian.*

Proof. Since $\widehat{X}_{\mathrm{red}}$ is determined locally (10.5.4), we can assume that X is an affine scheme of some Noetherian ring A; with the notation of Proposition (10.8.5), the ideal $\mathfrak{T}$ of topologically nilpotent elements of $\widehat{A}$ is the inverse image under the canonical map $\widehat{A}\to\widehat{A}/\widehat{\mathfrak{J}}=A/\mathfrak{J}$ of the nilradical of $A/\mathfrak{J}$ (**0**, 7.1.3), so $\widehat{A}/\mathfrak{T}$ is isomorphic to the quotient of $A/\mathfrak{J}$ by its nilradical. The first claim then follows from Propositions (10.5.4) and (5.1.1). If $\widehat{X}_{\mathrm{red}}$ is Noetherian, then so too is its underlying space X', and so the $X'_n=\operatorname{Spec}(\mathscr{O}_X/\mathscr{J}^n)$ are Noetherian (6.1.2), and thus so too is $\widehat{X}$ (10.6.4); the converse is immediate, by Proposition (6.1.2). □

(10.8.7). The canonical homomorphisms $\mathscr{O}_X\to\mathscr{O}_X/\mathscr{J}$ (for $\mathscr{J}\in\Phi$) form a projective system, and give, by passing to the projective limit, a homomorphism of sheaves of rings $\theta:\mathscr{O}_X\to\psi_*((\mathscr{O}_X)_{/X'})=\varprojlim_{\Phi}(\mathscr{O}_X/\mathscr{J})$, where ψ denotes the canonical injection $X'\to X$ of the underlying spaces. We denote by i (or i_X) the morphism (said to be *canonical*)

$$(\psi,\theta):X_{/X'}\longrightarrow X$$

of ringed spaces.

By taking tensor products, for every coherent $\mathscr{O}_X$-module $\mathscr{F}$, the canonical homomorphisms $\mathscr{O}_X\to\mathscr{O}_X/\mathscr{J}$ give homomorphisms $\mathscr{F}\to\mathscr{F}\otimes_{\mathscr{O}_X}(\mathscr{O}_X/\mathscr{J})$ of $\mathscr{O}_X$-modules which form a projective system, and thus give, by passing to the projective limit, a canonical functorial homomorphism $\gamma:\mathscr{F}\to\psi_*(\mathscr{F}_{/X'})$ of $\mathscr{O}_X$-modules.

Proposition (10.8.8). —

(i) *The functor $\mathscr{F}_{/X'}$ (in $\mathscr{F}$) is exact.*
(ii) *The functorial homomorphism $\gamma^\sharp:i^*(\mathscr{F})\to\mathscr{F}_{/X'}$ of $(\mathscr{O}_X)_{/X'}$-modules is an isomorphism.*

Proof.

(i) It suffices to prove that, if $0\to\mathscr{F}'\to\mathscr{F}\to\mathscr{F}''\to 0$ is an exact sequence of coherent $\mathscr{O}_X$-modules, and if U is an affine open subset of X with Noetherian ring A, then the sequence

$$0\longrightarrow\Gamma(U\cap X',\mathscr{F}'_{/X'})\longrightarrow\Gamma(U\cap X',\mathscr{F}_{/X'})\longrightarrow\Gamma(U\cap X',\mathscr{F}''_{/X'})\longrightarrow 0$$

is exact. We have that $\mathscr{F}|U=\widetilde{M}$, $\mathscr{F}'|U=\widetilde{M'}$, and $\mathscr{F}''|U=\widetilde{M''}$, where M, M', and M'' are three A-modules of finite type such that the sequence $0\to M'\to M\to M''\to 0$ is exact ((1.5.1) and (1.3.11)); let $\mathscr{J}\in\Phi$, and let $\mathfrak{J}$ be an ideal of A such that $\mathscr{J}|U=\widetilde{\mathfrak{J}}$. We then have

$$\Gamma(U\cap X',\mathscr{F}\otimes_{\mathscr{O}_X}\mathscr{O}_X/\mathscr{J}^n)=M\otimes_A(A/\mathfrak{J}^n)$$

(3.12); so, by definition of the projective limit, we have

$$\Gamma(U\cap X',\mathscr{F}_{/X'})=\varprojlim_n(M\otimes_A(A/\mathfrak{J}^n))=\widehat{M},$$

the separated completion of M with respect to the $\mathfrak{J}$-preadic topology, and similarly

$$\Gamma(U\cap X',\mathscr{F}'_{/X'})=\widehat{M'},\ \Gamma(U\cap X',\mathscr{F}''_{/X'})=\widehat{M''};$$

our claim then follows, since A is Noetherian, and since the functor $\widehat{M}$ in M is exact on the category of A-modules of finite type (**0**, 7.3.3). **I** | 197

(ii) The question is local, so we can assume that we have an exact sequence $\mathscr{O}_X^m\to\mathscr{O}_X^n\to\mathscr{F}\to 0$ (**0**, 5.3.2); since $\gamma^\sharp$ is functorial, and the functors $i^*(\mathscr{F})$ and $\mathscr{F}_{/X'}$ are right exact (by (i) and (**0**, 4.3.1)), we have the commutative diagram

(10.8.8.1)
$$\begin{tikzcd} i^*(\mathscr{O}_X^m) \arrow[r] \arrow[d, "\gamma^\sharp"'] & i^*(\mathscr{O}_X^n) \arrow[r] \arrow[d, "\gamma^\sharp"'] & i^*(\mathscr{F}) \arrow[r] \arrow[d, "\gamma^\sharp"'] & 0 \\ (\mathscr{O}_X^m)_{/X'} \arrow[r] & (\mathscr{O}_X^n)_{/X'} \arrow[r] & \mathscr{F}_{/X'} \arrow[r] & 0 \end{tikzcd}$$

whose rows are exact. Furthermore, the functors $i^*(\mathscr{F})$ and $\mathscr{F}_{/X'}$ commute with finite direct sums ((**0**, 3.2.6) and (**0**, 4.3.2)), and we thus reduce to proving our claim for $\mathscr{F}=\mathscr{O}_X$. We have $i^*(\mathscr{O}_X)=(\mathscr{O}_X)_{/X'}=\mathscr{O}_{\widehat{X}}$ (**0**, 4.3.4), and that $\gamma^\sharp$ is a homomorphism of $\mathscr{O}_{\widehat{X}}$*-modules*; so it suffices to check that $\gamma^\sharp$

sends the unit section of $\mathscr{O}_{\widehat{X}}$ over an open subset of X' to itself, which is immediate, and shows, in this case, that $\gamma^\sharp$ is the identity.

□

Corollary (10.8.9). — *The morphism $i : X_{/X'} \to X$ of ringed spaces is flat.*

Proof. This follows from (**0**, 6.7.3) and Proposition (10.8.8, i). □

Corollary (10.8.10). — *If $\mathscr{F}$ and $\mathscr{G}$ are coherent $\mathscr{O}_X$-modules, then there exist canonical functorial (in $\mathscr{F}$ and $\mathscr{G}$) isomorphisms*

$$(\mathscr{F}_{/X'}) \otimes_{(\mathscr{O}_X)_{/X'}} (\mathscr{G}_{/X'}) \simeq (\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G})_{/X'}, \tag{10.8.10.1}$$

$$(\mathscr{H}om_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G}))_{/X'} \simeq \mathscr{H}om_{(\mathscr{O}_X)_{/X'}}(\mathscr{F}_{/X'}, \mathscr{G}_{/X'}). \tag{10.8.10.2}$$

Proof. This follows from the canonical identification of $i^*(\mathscr{F})$ with $\mathscr{F}_{/X'}$; the existence of the first isomorphism is then a result which holds for all morphisms of ringed spaces (**0**, 4.3.3.1), and the second is a result which holds for all flat morphisms (**0**, 6.7.6), by Corollary (10.8.9). □

Proposition (10.8.11). — *For every coherent $\mathscr{O}_X$-module $\mathscr{F}$, the canonical homomorphism $\Gamma(X, \mathscr{F}) \to \Gamma(X', \mathscr{F}_{/X'})$ induced by $\mathscr{F} \to \mathscr{F}_{/X'}$ has kernel consisting of the zero sections in some neighborhood of X'.*

Proof. It follows from the definition of $\mathscr{F}_{/X'}$ that the canonical image of such a section is zero. Conversely, if $s \in \Gamma(X, \mathscr{F})$ has a zero image in $\Gamma(X', \mathscr{F}_{/X'})$, then it suffices to see that every $x \in X'$ admits a neighborhood in X in which s is zero, and we can thus reduce to the case where $X = \operatorname{Spec}(A)$ is affine, A Noetherian, $X' = V(\mathfrak{J})$ for some ideal $\mathfrak{J}$ of A, and $\mathscr{F} = \widetilde{M}$ for some A-module M of finite type. Then $\Gamma(X', \mathscr{F}_{/X'})$ is the separated completion $\widehat{M}$ of M for the $\mathfrak{J}$-preadic topology, and the homomorphism $\Gamma(X, \mathscr{F}) \to \Gamma(X', \mathscr{F}_{/X'})$ is the canonical homomorphism $M \to \widehat{M}$. We know (**0**, 7.3.7) that the kernel of this homomorphism is the set of the $z \in M$ killed by an element of $1 + \mathfrak{J}$. So we have $(1 + f)s = 0$ for some $f \in \mathfrak{J}$; for every $x \in X'$ we have $(1_x + f_x)s_x = 0$, and, since $1_x + f_x$ is invertible in $\mathscr{O}_x$ ($\mathfrak{J}_x\mathscr{O}_x$ being contained in the maximal ideal of $\mathscr{O}_x$), we have $s_x = 0$, which proves the proposition. □

Corollary (10.8.12). — *The support of $\mathscr{F}_{/X'}$ is equal to* $\operatorname{Supp}(\mathscr{F}) \cap X'$.

Proof. It is clear that $\mathscr{F}_{/X'}$ is an $(\mathscr{O}_X)_{/X'}$-module of finite type ((10.8.8, ii) and (**0**, 5.2.4)), so its support is closed (**0**, 5.2.2) and evidently contained in $\operatorname{Supp}(\mathscr{F}) \cap X'$. To show that it is equal to the latter set, we immediately reduce to proving that the equation $\Gamma(X', \mathscr{F}_{/X'}) = 0$ implies that $\operatorname{Supp}(\mathscr{F}) \cap X' = \varnothing$; this follows from Proposition (10.8.11) and Theorem (1.4.1). □ **I | 198**

Corollary (10.8.13). — *Let $u : \mathscr{F} \to \mathscr{G}$ be a homomorphism of coherent $\mathscr{O}_X$-modules. For $u_{/X'} : \mathscr{F}_{/X'} \to \mathscr{G}_{/X'}$ to be zero, it is necessary and sufficient for u to be zero on a neighborhood of X'.*

Proof. By Proposition (10.8.8, ii), $u_{/X'}$ can be identified with $i^*(u)$, so, if we consider u as a section over X of the sheaf $\mathscr{H} = \mathscr{H}om_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G})$, then $u_{/X'}$ is the section over X' of $i^*(\mathscr{H}) = \mathscr{H}_{/X'}$ to which it canonically corresponds ((10.8.10.2) and (**0**, 4.4.6)). It thus suffices to apply Proposition (10.8.11) to the coherent $\mathscr{O}_X$-module $\mathscr{H}$. □

Corollary (10.8.14). — *Let $u : \mathscr{F} \to \mathscr{G}$ be a homomorphism of coherent $\mathscr{O}_X$-modules. For $u_{/X'}$ to be a monomorphism (resp. an epimorphism), it is necessary and sufficient for u to be a monomorphism (resp. an epimorphism) on a neighborhood of X'.*

Proof. Let $\mathscr{P}$ and $\mathscr{N}$ be the cokernel and kernel (respectively) of u, so that we have the exact sequence $0 \to \mathscr{N} \xrightarrow{v} \mathscr{F} \xrightarrow{u} \mathscr{G} \xrightarrow{w} \mathscr{P} \to 0$, hence (10.8.8, i) the exact sequence

$$0 \longrightarrow \mathscr{N}_{/X'} \xrightarrow{v_{/X'}} \mathscr{F}_{/X'} \xrightarrow{u_{/X'}} \mathscr{G}_{/X'} \xrightarrow{w_{/X'}} \mathscr{P}_{/X'} \longrightarrow 0.$$

If $u_{/X'}$ is a monomorphism (resp. an epimorphism), then we have $v_{/X'} = 0$ (resp. $w_{/X'} = 0$), so there exists a neighborhood of X' on which $v = 0$ (resp. $w = 0$) by Corollary (10.8.13). □

10.9. Extension of morphisms to completions.

(10.9.1). Let X and Y be locally Noetherian (usual) preschemes, $f : X \to Y$ a morphism, and X' (resp. Y') a closed subset of the underlying space X (resp. Y) such that $f(X') \subset Y'$. Let $\mathscr{J}$ (resp. $\mathscr{K}$) be a sheaf of ideals of $\mathscr{O}_X$ (resp. $\mathscr{O}_Y$) such that the support of $\mathscr{O}_X/\mathscr{J}$ (resp. $\mathscr{O}_Y/\mathscr{K}$) is X' (resp. Y') and $f^*(\mathscr{K})\mathscr{O}_X \subset \mathscr{J}$; we note that there always exist such sheaves of ideals, since, for example, we can take $\mathscr{J}$ to be the largest sheaf of ideals of $\mathscr{O}_X$ defining a subprescheme of X with underlying space X' (5.2.1), and the hypothesis $f(X') \subset Y'$ implies that $f^*(\mathscr{K})\mathscr{O}_X \subset \mathscr{J}$ (5.2.4). For every integer $n > 0$ we have $f^*(\mathscr{K}^n)\mathscr{O}_X \subset \mathscr{J}^n$ (**0**, 4.3.5); as a result (4.4.6), if we set $X'_n = (X', \mathscr{O}_X/\mathscr{J}^{n+1})$ and $Y'_n = (Y', \mathscr{O}_Y/\mathscr{K}^{n+1})$, then f induces a morphism $f_n : X'_n \to Y'_n$, and it is immediate that the f_n form an inductive system. We denote its inductive limit (10.6.8) by $\widehat{f} : X_{/X'} \to Y_{/Y'}$, and we say (by abuse of language) that $\widehat{f}$ is the *extension of f to the completions of X and Y along X' and Y'*. It can be checked immediately that this morphism does not depend on the choice of sheaves of ideals $\mathscr{J}$ and $\mathscr{K}$ satisfying the above conditions. It suffices to consider the case where X and Y are Noetherian affine schemes with rings A and B (respectively); then $\mathscr{J} = \widetilde{\mathfrak{J}}$ and $\mathscr{K} = \widetilde{\mathfrak{K}}$, where $\mathfrak{J}$ (resp. $\mathfrak{K}$) is an ideal of A (resp. B), f corresponds to a ring homomorphism $\varphi : B \to A$ such that $\varphi(\mathfrak{K}) \subset \mathfrak{J}$ ((4.4.6) and (1.7.4)); $\widehat{f}$ *is then the morphism corresponding* (10.2.2) *to the continuous homomorphism* $\widehat{\varphi} : \widehat{B} \to \widehat{A}$, where $\widehat{A}$ (resp. $\widehat{B}$) is the separated completion of A (resp. B) with respect to the $\mathfrak{J}$-preadic (resp. $\mathfrak{K}$-preadic) topology (10.6.8); we know that, if we replace $\mathscr{J}$ by another sheaf of ideals **I** | 199
$\mathscr{J}' = \widetilde{\mathfrak{J}}'$ such that the support of $\mathscr{O}_X/\mathscr{J}'$ is X', then the $\mathfrak{J}$-preadic and $\mathfrak{J}'$-preadic topologies on A are the same (10.82).

We note that, by definition, the continuous map $X' \to Y'$ of the underlying spaces of $X_{/X'}$ and $Y_{/Y'}$ corresponding to $\widehat{f}$ is exactly the restriction to X' of f.

(10.9.2). It follows immediately from the above definition that the diagram of morphisms of ringed spaces

$$\begin{tikzcd} \widehat{X} \arrow[r, "\widehat{f}"] \arrow[d, "i_X"'] & \widehat{Y} \arrow[d, "i_Y"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

commutes, with the vertical arrows being the canonical morphisms (10.8.7).

(10.9.3). Let Z be a third prescheme, $g : Y \to Z$ a morphism, and Z' a closed subset of Z such that $g(Y') \subset Z'$. If $\widehat{g}$ denotes the completion of the morphism g along Y' and Z', then it immediately follows from (10.9.1) that we have $(g \circ f)^\wedge = \widehat{g} \circ \widehat{f}$.

Proposition (10.9.4). — *Let X and Y be locally Noetherian S-preschemes, with Y of finite type over S. Let f and g be S-morphisms from X to Y such that $f(X') \subset Y'$ and $g(X') \subset Y'$. For $\widehat{f} = \widehat{g}$ to hold, it is necessary and sufficient for f and g to coincide on a neighborhood of X'.*

Proof. The condition is evidently sufficient (even without the finiteness hypothesis on Y). To see that it is necessary, we remark first that the hypothesis $\widehat{f} = \widehat{g}$ implies that $f(x) = g(x)$ for all $x \in X'$. Also, the question being local, we can assume that X and Y are affine open neighborhoods of x and $y = f(x) = g(x)$ respectively (with Noetherian rings), that S is affine, and that $\Gamma(Y, \mathscr{O}_Y)$ is a $\Gamma(S, \mathscr{O}_S)$-algebra of finite type (6.3.3). Then f and g correspond to $\Gamma(S, \mathscr{O}_S)$-homomorphisms ρ and σ (respectively) from $\Gamma(Y, \mathscr{O}_Y)$ to $\Gamma(X, \mathscr{O}_X)$ (1.7.3), and, by hypothesis, the extensions by continuity of these homomorphisms to the separated completion of $\Gamma(Y, \mathscr{O}_Y)$ are the same. We conclude from Proposition (10.8.11) that, for every section $s \in \Gamma(Y, \mathscr{O}_Y)$, the sections $\rho(s)$ and $\sigma(s)$ coincide on a neighborhood (depending on s) of X'; since $\Gamma(Y, \mathscr{O}_Y)$ is an algebra of finite type over $\Gamma(S, \mathscr{O}_S)$, we have that there exists a neighborhood V of X' such that $\rho(s)$ and $\sigma(s)$ coincide on V for *every* section $s \in \Gamma(Y, \mathscr{O}_Y)$. If $h \in \Gamma(X, \mathscr{O}_X)$ is such that $D(h)$ is a neighborhood of x contained in V, then we conclude from the above and from Theorem (1.4.1, d) that f and g coincide on $D(h)$. ☐

Proposition (10.9.5). — *Under the hypotheses of* (10.9.1), *for every coherent $\mathscr{O}_Y$-module $\mathscr{G}$, there exists a canonical functorial isomorphism of $(\mathscr{O}_X)_{/X'}$-modules*

$$(f^*(\mathscr{G}))_{/X'} \simeq \widehat{f}^*(\mathscr{G}_{/Y'}).$$

Proof. If we canonically identify $(f^*(\mathscr{G}))_{/X'}$ with $i_X^*(f^*(\mathscr{G}))$, and $\widehat{f}^*(\mathscr{G}_{/Y'})$ with $\widehat{f}^*(i_Y^*(\mathscr{G}))$ (10.8.8), then the proposition follows immediately from the commutativity of the diagram in (10.9.2). ☐

(10.9.6). Now let $\mathscr{F}$ be a coherent $\mathscr{O}_X$-module, and let $\mathscr{G}$ be a coherent $\mathscr{O}_Y$-module. If $u : \mathscr{G} \to \mathscr{F}$ is an f-morphism, then it corresponds to an $\mathscr{O}_X$-homomorphism $u^\sharp : f^*(\mathscr{G}) \to \mathscr{F}$, thus, by completion, to a continuous $(\mathscr{O}_X)_{/X'}$-homomorphism $(u^\sharp)_{/X'} : (f^*(\mathscr{G}))_{/X'} \to \mathscr{F}_{/X'}$, and, by Proposition (10.9.5), there exists a unique $\widehat{f}$-morphism $v : \mathscr{G}_{/Y'} \to \mathscr{F}_{/X'}$ such that $v^\sharp = (u^\sharp)_{/X'}$. If we consider the triples $(\mathscr{F}, X, X')$ ($\mathscr{F}$ being a coherent $\mathscr{O}_X$-module, and I | 200
X' a closed subset of X) as a *category*, with the morphisms $(\mathscr{F}, X, X') \to (\mathscr{G}, Y, Y')$ consisting of a morphism of preschemes $f : X \to Y$ such that $f(X') \subset Y'$ and an f-morphism $u : \mathscr{G} \to \mathscr{F}$, then we can say that $(X_{/X'}, \mathscr{F}_{/X'})$ is a *functor* in $(\mathscr{F}, X, X')$ with values in the category of pairs $(\mathfrak{Z}, \mathscr{H})$ consisting of a locally Noetherian formal prescheme $\mathfrak{Z}$ and an $\mathscr{O}_{\mathfrak{Z}}$-module $\mathscr{H}$, with the morphisms of the latter category being the pairs consisting of a morphism g of formal preschemes and a g-morphism.

Proposition (10.9.7). — *Let S, X, and Y be locally Noetherian preschemes, $g : X \to S$ and $h : Y \to S$ morphisms, S' a closed subset of S, and X' (resp. Y') a closed subset of X (resp. Y) such that $g(X') \subset S'$ (resp. $h(Y') \subset S'$); let $Z = X \times_S Y$; suppose that Z is locally Noetherian, and let $Z' = p^{-1}(X') \cap q^{-1}(Y')$, where p and q are the projections of $X \times_S Y$. With these conditions, the completion $Z_{/Z'}$ can be identified with the product $(X_{/X'}) \times_{S_{/S'}} (Y_{/Y'})$ of formal $S_{/S'}$-preschemes, where the structure morphisms are identified with $\widehat{g}$ and $\widehat{h}$, and the projections with $\widehat{p}$ and $\widehat{q}$.*

Proof. It is immediate that the question is local for S, X, and Y, and we thus reduce to the case where $S = \operatorname{Spec}(A)$, $X = \operatorname{Spec}(B)$, $Y = \operatorname{Spec}(C)$, $S' = V(\mathfrak{J})$, $X' = V(\mathfrak{K})$, and $Y' = V(\mathfrak{L})$, with $\mathfrak{J}$, $\mathfrak{K}$, and $\mathfrak{L}$ ideals such that $\varphi(\mathfrak{J}) \subset \mathfrak{K}$ and $\psi(\mathfrak{J}) \subset \mathfrak{L}$, where we denote by φ and ψ the homomorphisms $A \to B$ and $A \to C$ which correspond to g and h (respectively). We know that $Z = \operatorname{Spec}(B \otimes_A C)$ and that $Z' = V(\mathfrak{M})$, where $\mathfrak{M}$ is the ideal $\operatorname{Im}(\mathfrak{K} \otimes_A C) + \operatorname{Im}(B \otimes_A \mathfrak{L})$. The conclusion follows (10.7.2) from the fact that the completed tensor product $(\widehat{B} \otimes_{\widehat{A}} \widehat{C})^\wedge$ (where $\widehat{A}$, $\widehat{B}$, and $\widehat{C}$ are, respectively, the separated completions of A, B, and C with respect to the $\mathfrak{J}$-, $\mathfrak{K}$-, and $\mathfrak{L}$-preadic topologies) is the separated completion of the tensor product $B \otimes_A C$ with respect to the $\mathfrak{M}$-preadic topology (**0**, 7.7.2). □

In addition, we note that, if T is a locally Noetherian S-prescheme, $u : T \to X$ and $v : T \to Y$ both S-morphisms, and T' a closed subset of T such that $u(T') \subset X'$ and $v(T') \subset Y'$, then the extension to the completion $((u, v)_S)^\wedge$ can be identified with $(\widehat{u}, \widehat{v})_{S_{/S'}}$.

Corollary (10.9.8). — *Let X and Y be locally Noetherian S-preschemes such that $X \times_S Y$ is locally Noetherian; let S' be a closed subset of S, and X' (resp. Y') a closed subset of X (resp. Y) whose image in S is contained in S'. For every S-morphism $f : X \to Y$ such that $f(X') \subset Y'$, the graph morphism $\Gamma_{\widehat{f}}$ can be identified with the extension $(\Gamma_f)^\wedge$ of the graph morphism of f.*

Corollary (10.9.9). — *Let X and Y be locally Noetherian preschemes, $f : X \to Y$ a morphism, Y' a closed subset of Y, and $X' = f^{-1}(Y')$. Then the prescheme $X_{/X'}$ can be identified, by the commutative diagram*

$$\begin{tikzcd} X \arrow[d, "f"'] & X_{/X'} \arrow[l] \arrow[d, "\widehat{f}"] \\ Y & Y_{/Y'} \arrow[l] \end{tikzcd}$$

with the product $X \times_Y (Y_{/Y'})$ of formal preschemes.

Proof. It suffices to apply Proposition (10.9.7), replacing S and S' by Y, and X and X' by X. □

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Remark (10.9.10). — If S is the sum $X_1 \sqcup X_2$ (3.1), X' the union $X_1' \cup X_2'$, where X_i' is a closed subset of X_i $(i = 1, 2)$, then we have $X_{/X'} = X_{1/X_1'} \sqcup X_{2/X_2'}$.

10.10. Application to coherent sheaves on formal affine schemes.

(10.10.1). In this paragraph, A denotes an *adic Noetherian ring*, and $\mathfrak{J}$ an ideal of definition for A. Let $X = \operatorname{Spec}(A)$, and $\mathfrak{X} = \operatorname{Spf}(A)$, which can be identified with the closed subset $V(\mathfrak{J})$ of X (10.1.2). In addition, Definitions (10.1.2) and (10.8.4) show that the *formal affine scheme* $\mathfrak{X}$ is identical the completion $X_{/\mathfrak{X}}$ of the affine scheme X along the closed subset $\mathfrak{X}$ of its underling space. To every coherent $\mathscr{O}_X$-module $\mathscr{F}$, corresponds an $\mathscr{O}_{\mathfrak{X}}$-module of finite type $\mathscr{F}_{/\mathfrak{X}}$, which is a sheaf of topological modules over the sheaf of topological rings $\mathscr{O}_{\mathfrak{X}}$. Every coherent $\mathscr{O}_X$-module $\mathscr{F}$ is of the form $\widetilde{M}$, where M is an A-module of finite type (1.5.1); we set $(\widetilde{M})_{/X} = M^\Delta$. In addition, if $u : M \to N$ is an A-homomorphism of A-modules of finite type, then it corresponds to a homomorphism $\widetilde{u} : \widetilde{M} \to \widetilde{N}$, and, as a result, to a continuous homomorphism $\widetilde{u}_{/X'} : (\widetilde{M})_{/X'} \to (\widetilde{N})_{/X'}$, which we denote by u^Δ. It is immediate that $(v \circ u)^\Delta = v^\Delta \circ u^\Delta$; we have thus defined a *covariant additive functor* M^Δ from the category of A-modules of finite type to the category of $\mathscr{O}_{\mathfrak{X}}$-modules of finite type. When A is a *discrete* ring, we have $M^\Delta = \widetilde{M}$.

Proposition (10.10.2). —

(i) *M^{Δ} is an exact functor in M, and there exists a canonical functorial isomorphism of A-modules $\Gamma(\mathfrak{X}, M^{\Delta}) \simeq M$.*
(ii) *If M and N are A-modules of finite type, then there exist canonical functorial isomorphisms*

(10.10.2.1)
$$(M \otimes_A N)^{\Delta} \simeq M^{\Delta} \otimes_{\mathscr{O}_{\mathfrak{X}}} N^{\Delta},$$

(10.10.2.2)
$$(\mathrm{Hom}_A(M,N))^{\Delta} \simeq \mathscr{H}om_{\mathscr{O}_{\mathfrak{X}}}(M^{\Delta}, N^{\Delta}).$$

(iii) *The map $u \mapsto u^{\Delta}$ is a functorial isomorphism*

(10.10.2.3)
$$\mathrm{Hom}_A(M,N) \simeq \mathrm{Hom}_{\mathscr{O}_X}(M^{\Delta}, N^{\Delta}).$$

Proof. The exactness of M^{Δ} follows from the exactness of the functors $\widetilde{M}$ (1.3.5) and $\mathscr{F}_{/X'}$ (10.8.8). By definition, $\Gamma(X, M^{\Delta})$ is the separated completion of the A-module $\Gamma(X, \widetilde{M}) = M$ with respect to the $\mathfrak{J}$-preadic topology; but, since A is complete and M is of finite type, we know (**0**, 7.3.6) that M is separated and complete, which proves (i). The isomorphism (10.10.2.1) (resp. (10.10.2.2)) comes from the composition of the isomorphisms (1.3.12, i) and (10.8.10.1) (resp. (1.3.12, ii) and (10.8.10.2)). Finally, since $\mathrm{Hom}_A(M,N)$ is an A-module of finite type, we can apply (i), which identifies $\Gamma(\mathfrak{X}, (\mathrm{Hom}_A(M,N))^{\Delta})$ with $\mathrm{Hom}_A(M,N)$, and we can use (10.10.2.2), which proves that the homomorphism (10.10.2.3) is an isomorphism. □

We deduce from Proposition (10.10.2) a series of results analogous to those of Theorem (1.3.7) and Corollary (1.3.12), whose formulation we leave to the reader. **I** | 202

We note that the exactness property of M^{Δ}, applied to the exact sequence $0 \to \mathfrak{J} \to A \to A/\mathfrak{J} \to 0$, shows that the sheaf of ideals of $\mathscr{O}_{\mathfrak{X}}$ denoted here by $\mathfrak{J}^{\Delta}$ coincides with the one denoted also by $\mathfrak{J}^{\Delta}$ in (10.3.1), by (10.3.2).

Proposition (10.10.3). — *Under the hypotheses of* (10.10.1), *$\mathscr{O}_{\mathfrak{X}}$ is a coherent sheaf of rings.*

Proof. If $f \in A$, then we know that $A_{\{f\}}$ is an adic Noetherian ring (**0**, 7.6.11), and, since the question is local, we reduce (10.1.4) to proving that the kernel of the homomorphism $v : \mathscr{O}_{\mathfrak{X}}^n \to \mathscr{O}_{\mathfrak{X}}$ is an $\mathscr{O}_{\mathfrak{X}}$-module of finite type. We then have $v = u^{\Delta}$, where u is an A-homomorphism $A^n \to A$ (10.10.2); since A is Noetherian, the kernel of u is of finite type, or, equivalently, we have a homomorphism $A^m \xrightarrow{w} A^n$ such that the sequence $A^m \xrightarrow{w} A^n \xrightarrow{u} A$ is exact. We conclude (10.10.2) that the sequence $\mathscr{O}_{\mathfrak{X}}^m \xrightarrow{w^{\Delta}} \mathscr{O}_{\mathfrak{X}}^n \xrightarrow{v} \mathscr{O}_{\mathfrak{X}}$ is exact, which proves that the kernel of v is of finite type. □

(10.10.4). With the above notation, set $A_n = A/\mathfrak{J}^{n+1}$, and let S_n be the affine scheme $\mathrm{Spec}(A_n) = (\mathfrak{X}, \mathscr{O}_{\mathfrak{X}}/\mathscr{J}^{n+1})$, with $\mathscr{J} = \mathfrak{J}^{\Delta}$ the sheaf of ideals of definition for $\mathscr{O}_{\mathfrak{X}}$ corresponding to the ideal $\mathfrak{J}$. Let u_{mn} be the morphism of preschemes $X_m \to X_n$ corresponding to the canonical homomorphism $A_n \to A_m$ for $m \leqslant n$; the formal scheme $\mathfrak{X}$ is the inductive limit of the X_n with respect to the u_{mn} (10.6.3).

Proposition (10.10.5). — *Under the hypothesis of* (10.10.1), *let $\mathscr{F}$ be an $\mathscr{O}_{\mathfrak{X}}$-module. The following conditions are equivalent:*

(a) *$\mathscr{F}$ is a coherent $\mathscr{O}_{\mathfrak{X}}$-module;*
(b) *$\mathscr{F}$ is isomorphic to the projective limit* (10.6.6) *of a sequence $(\mathscr{F}_n)$ of coherent $\mathscr{O}_{X_n}$-modules such that $u_{mn}^*(\mathscr{F}_n) = \mathscr{F}_m$; and*
(c) *there exists an A-module M of finite type (determined up to canonical isomorphism by Proposition* (10.10.2, i)*) such that $\mathscr{F}$ is isomorphic to M^{Δ}.*

Proof. We first show that (b) implies (c). We have $\mathscr{F}_n = \widetilde{M_n}$, where M_n is an A_n-module of finite type, and the hypotheses imply that $M_m = M_n \otimes_{A_n} A_m$ for $m \leqslant n$ (1.6.5); the M_n thus form a projective system for the canonical di-homomorphisms $M_n \to M_m$ $(m \leqslant n)$, and it follows immediately from the definition of the A_n that this projective system satisfies the conditions of (**0**, 7.2.9); as a result, its projective limit M is an A-module of finite type such that $M_n = M \otimes_A A_n$ for all n. We deduce that $\mathscr{F}_n$ is induced over X_n by $\widetilde{M} \otimes_{\mathscr{O}_X} (\mathscr{O}_X/\widetilde{\mathfrak{J}}^{n+1})$, and so $\mathscr{F} = M^{\Delta}$ by Definition (10.8.4).

Conversely, (c) implies (b); indeed, if u_n is the immersion morphism $X_n \to X$, then $u_n^*(\widetilde{M}) = (M \otimes_A A_n)^{\sim}$ is induced over X_n by $\widetilde{M} \otimes_{\mathscr{O}_X} (\mathscr{O}_X/\widetilde{\mathfrak{J}}^{n+1})$, and $M^{\Delta} = \varprojlim u_n^*(\widetilde{M})$ by Definition (10.8.4); since $u_m = u_n \circ u_{mn}$ for $m \leqslant n$, the $\mathscr{F}_n = u_n^*(\widetilde{M})$ satisfy the conditions of (b), whence our claim.

We now show that (c) implies (a): indeed, we have, by definition, that $\mathscr{O}_{\mathfrak{X}} = A^{\Delta}$; since M is the cokernel of a homomorphism $A^m \to A^n$, it follows from Proposition (10.10.2) that M^{Δ} is the cokernel of a homomorphism $\mathscr{O}_{\mathfrak{X}}^m \to \mathscr{O}_{\mathfrak{X}}^n$, and, since the sheaf of rings $\mathscr{O}_{\mathfrak{X}}$ is coherent (10.10.3), so too is M^{Δ} (**0**, 5.3.4). **I** | 203

Finally, (a) implies (b). Considered as an $\mathcal{O}_{\mathfrak{X}}$-module, we have that $\mathcal{O}_{X_n} = \mathcal{O}_{\mathfrak{X}}/\mathcal{J}^{n+1} = A_n^\Delta$; but $\mathcal{F}_n = \mathcal{F} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{O}_{X_n}$ is a coherent $\mathcal{O}_{\mathfrak{X}}$-module (**0**, 5.3.5), and, since it is also an $\mathcal{O}_{X_n}$-module, and $\mathcal{J}^{n+1}$ is coherent, we conclude that $\mathcal{F}_n$ is a coherent $\mathcal{O}_{X_n}$-module (**0**, 5.3.10), and it is immediate that $u_{mn}^*(\mathcal{F}_n) = \mathcal{F}_m$ for $m \leqslant n$ (recalling that the continuous map $X_m \to X_n$ of the underlying spaces is the identity on $\mathfrak{X}$). The sheaf $\mathcal{G} = \varprojlim \mathcal{F}_n$ is thus a coherent $\mathcal{O}_{\mathfrak{X}}$-module, since we have seen that (b) implies (a). The canonical homomorphisms $\mathcal{F} \to \mathcal{F}_n$ form a projective system, which, by passing to the limit, gives a canonical homomorphism $w : \mathcal{F} \to \mathcal{G}$, and it remains only to prove that w is bijective. The question is now *local*, so we can reduce to the case where $\mathcal{F}$ is the cokernel of a homomorphism $\mathcal{O}_{\mathfrak{X}}^p \to \mathcal{O}_{\mathfrak{X}}^q$; since this homomorphism is of the form v^Δ, where v is a homomorphism $A^m \to A^n$ (10.10.2), $\mathcal{F}$ is isomorphic to M^Δ, where $M = \operatorname{Coker} v$ (10.10.2). We then have, by Proposition (10.10.2), that $\mathcal{F}_n = M^\Delta \otimes_{\mathcal{O}_{\mathfrak{X}}} A_n^\Delta = (M \otimes_A A_n)^\Delta$, and, since the $\mathfrak{J}$-adic topology on $M \otimes_A A_n$ is discrete, we have $(M \otimes_A A_n)^\Delta = (M \otimes_A A_n)^\sim$ (as an $\mathcal{O}_{X_n}$-module); we have seen above that $M^\Delta = \varprojlim \mathcal{F}_n$, and w is thus the identity in this case. □

Corollary (10.10.6). — *If $\mathcal{F}$ satisfies condition* (b) *of Proposition* (10.10.5), *then the projective system $(\mathcal{F}_n)$ is isomorphic to the system of the $\mathcal{F} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{O}_{X_n}$.*

(10.10.7). Now let A and B be adic Noetherian rings, and $\varphi : B \to A$ a continuous homomorphism; we denote by $\mathfrak{J}$ (resp. $\mathfrak{K}$) an ideal of definition for A (resp. B) such that $\varphi(\mathfrak{K}) \subset \mathfrak{J}$, and we set $X = \operatorname{Spec}(A)$, $Y = \operatorname{Spec}(B)$, $\mathfrak{X} = \operatorname{Spf}(A)$, and $\mathfrak{Y} = \operatorname{Spf}(B)$. Let $f : X \to Y$ be the morphism of preschemes corresponding to φ (1.6.1), and $\widehat{f} : \mathfrak{X} \to \mathfrak{Y}$ its extension to the completions (10.9.1), which is also a morphism of formal preschemes that corresponds to φ (10.2.2).

Proposition (10.10.8). — *For every B-module N of finite type, there exists a canonical functorial isomorphism of $\mathcal{O}_{\mathfrak{X}}$-modules*

$$\widehat{f}^*(N^\Delta) \simeq (N \otimes_B A)^\Delta.$$

Proof. Denoting by $i_X : \mathfrak{X} \to X$ and $i_Y : \mathfrak{Y} \to Y$ the canonical morphisms, we have (10.8.8), up to canonical functorial isomorphisms, $N^\Delta = i_Y^*(\widetilde{N})$ and

$$(N \otimes_B A)^\Delta = i_X^*((N \otimes_B A)^\sim) = i_X^*(f^*(\widetilde{N}))$$

(1.6.5); the proposition then follows from the commutativity of the diagram in (10.9.2). □

Corollary. — *For every ideal $\mathfrak{b}$ of B, we have $\widehat{f}^*(\mathfrak{b}^\Delta)\mathcal{O}_{\mathfrak{X}} = (\mathfrak{b}A)^\Delta$.*

Proof. Let j be the canonical injection $\mathfrak{b} \to B$, to which corresponds the canonical injection $j^\Delta : \mathfrak{b}^\Delta \to \mathcal{O}_{\mathfrak{Y}}$ of sheaves of $\mathcal{O}_{\mathfrak{Y}}$-modules; by definition, $\widehat{f}^*(\mathfrak{b}^\Delta)\mathcal{O}_{\mathfrak{X}}$ is the image of the homomorphism $\widehat{f}^*(j^\Delta) : \widehat{f}^*(\mathfrak{b}^\Delta) \to \mathcal{O}_{\mathfrak{X}} = \widehat{f}^*(\mathcal{O}_{\mathfrak{Y}})$; but this homomorphism can be identified with $(j \otimes 1)^\Delta : (\mathfrak{b} \otimes_B A)^\Delta \to \mathcal{O}_{\mathfrak{X}} = (B \otimes_B A)^\Delta$ by Proposition (10.10.8). Since the image of $j \otimes 1$ is the ideal $\mathfrak{b}A$ of A, the image of $(j \otimes 1)^\Delta$ is thus $(\mathfrak{b}A)^\Delta$, by Proposition (10.10.2), whence the conclusion. □

10.11. Coherent sheaves on formal preschemes.

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Proposition (10.11.1). — *If $\mathfrak{X}$ is a locally Noetherian formal prescheme, then the sheaf of rings $\mathcal{O}_{\mathfrak{X}}$ is coherent, and every sheaf of ideals of definition for $\mathfrak{X}$ is coherent.*

Proof. The question is local, so we can reduce to the case of a Noetherian affine formal scheme, and the proposition then follows from Propositions (10.10.3) and (10.10.5). □

(10.11.2). Let $\mathfrak{X}$ be a locally Noetherian formal prescheme, $\mathcal{J}$ a sheaf of ideals of definition for $\mathfrak{X}$, and X_n the locally Noetherian (usual) prescheme $(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}/\mathcal{J}^{n+1})$, so that $\mathfrak{X}$ is the *inductive limit* of the sequence (X_n) with respect to the canonical morphisms $u_{mn} : X_m \to X_n$ (10.6.3). With this notation:

Theorem (10.11.3). — *For an $\mathcal{O}_{\mathfrak{X}}$-module $\mathcal{F}$ to be coherent, it is necessary and sufficient for it to be isomorphic to a projective limit of a sequence $(\mathcal{F}_n)$, where the $\mathcal{F}_n$ are coherent $\mathcal{O}_{X_n}$-modules such that $u_{mn}^*(\mathcal{F}_n) = \mathcal{F}_m$ for $m \leqslant n$* (10.6.6). *The projective system $(\mathcal{F}_n)$ is then isomorphic to the system of the $u_n^*(\mathcal{F}) = \mathcal{F} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{O}_{X_n}$, where u_n is the canonical morphism $X_n \to \mathfrak{X}$.*

Proof. The question is local, so we can reduce to the case where $\mathfrak{X}$ is a Noetherian affine formal scheme, and the theorem then is a consequence of Proposition (10.10.5) and Corollary (10.10.6). □

We can thus say that *the data of a coherent $\mathcal{O}_{\mathfrak{X}}$-module is equivalent to the data of a projective system $(\mathcal{F}_n)$ of coherent $\mathcal{O}_{X_n}$-modules such that $u_{mn}(\mathcal{F}_n) = \mathcal{F}_m$ for $m \leqslant n$.*

Corollary (10.11.4). — *If $\mathscr{F}$ and $\mathscr{G}$ are coherent $\mathscr{O}_{\mathfrak{X}}$-modules, then we can (with the notation of Theorem (10.11.3)) define a canonical functorial isomorphism*

$$\mathrm{Hom}_{\mathscr{O}_{\mathfrak{X}}}(\mathscr{F},\mathscr{G}) \simeq \varprojlim_n \mathrm{Hom}_{\mathscr{O}_{X_n}}(\mathscr{F}_n,\mathscr{G}_n). \tag{10.11.4.1}$$

Proof. The projective limit on the right-hand side is understood to be taken with respect to the maps $\theta_n \mapsto u^*_{mn}(\theta_n)$ ($m \leqslant n$) from $\mathrm{Hom}_{\mathscr{O}_{X_n}}(\mathscr{F}_n,\mathscr{G}_n)$ to $\mathrm{Hom}_{\mathscr{O}_{X_m}}(\mathscr{F}_m,\mathscr{F}_m)$. The homomorphism (10.11.4.1) sends an element $\theta \in \mathrm{Hom}_{\mathscr{O}_{\mathfrak{X}}}(\mathscr{F},\mathscr{G})$ to the sequence $(u^*_n(\theta))$; we see that we can define an inverse homomorphism of the above by sending a projective system $(\theta_n) \in \varprojlim_n \mathrm{Hom}_{\mathscr{O}_{X_n}}(\mathscr{F}_n,\mathscr{G}_n)$ to its projective limit in $\mathrm{Hom}_{\mathscr{O}_{\mathfrak{X}}}(\mathscr{F},\mathscr{G})$, taking into account Theorem (10.11.3). □

Corollary (10.11.5). — *For a homomorphism $\theta : \mathscr{F} \to \mathscr{G}$ to be surjective, it is necessary and sufficient for the corresponding homomorphism $\theta_0 = u^*_0(\theta) : \mathscr{F}_0 \to \mathscr{G}_0$ to be surjective.*

Proof. The question is local, so we reduce to the case where $\mathfrak{X} = \mathrm{Spf}(A)$ with A an adic Noetherian ring, $\mathscr{F} = M^\Delta$, $\mathscr{G} = N^\Delta$, and $\theta = u^\Delta$, where M and N are A-modules of finite type, and u is a homomorphism $M \to N$; we then have that $\theta_0 = \widetilde{u_0}$, where u_0 is the homomorphism $u \otimes 1 : M \otimes_A A/\mathfrak{J} \to N \otimes_A A/\mathfrak{J}$; the conclusion follows from the fact θ and u (resp. θ_0 and u_0) are simultaneously surjective ((1.3.9) and (10.10.2)) and that u and u_0 are simultaneously surjective (**0**, 7.1.14). □

(10.11.6). Theorem (10.11.3) shows that we can consider every coherent $\mathscr{O}_{\mathfrak{X}}$-module $\mathscr{F}$ as a *topological $\mathscr{O}_{\mathfrak{X}}$-module*, considering it as a projective limit of *pseudo-discrete* sheaves of groups $\mathscr{F}_n$ (**0**, 3.8.1). It then follows from Corollary (10.11.4) that every homomorphism $u : \mathscr{F} \to \mathscr{G}$ of coherent $\mathscr{O}_{\mathfrak{X}}$-modules is automatically *continuous* (**0**, 3.8.2). **I** | 205
Furthermore, if $\mathscr{H}$ is a coherent $\mathscr{O}_{\mathfrak{X}}$-submodule of a coherent $\mathscr{O}_{\mathfrak{X}}$-module $\mathscr{F}$, then, for every open $U \subset \mathfrak{X}$, $\Gamma(U,\mathscr{H})$ is a *closed* subgroup of the topological group $\Gamma(U,\mathscr{F})$, since the functor Γ is left exact, and $\Gamma(U,\mathscr{H})$ is the kernel of the homomorphism $\Gamma(U,\mathscr{F}) \to \Gamma(U,\mathscr{F}/\mathscr{H})$, which is *continuous* by the above, since $\mathscr{F}/\mathscr{G}$ is coherent (**0**, 5.3.4); our claim follows from the fact that $\Gamma(U,\mathscr{F}/\mathscr{H})$ is a separated topological group.

Proposition (10.11.7). — *Let $\mathscr{F}$ and $\mathscr{G}$ be coherent $\mathscr{O}_{\mathfrak{X}}$-modules. We can define (with the notation of Theorem (10.11.3)) canonical functorial isomorphisms of topological $\mathscr{O}_{\mathfrak{X}}$-modules (10.11.6)*

$$\mathscr{F} \otimes_{\mathscr{O}_{\mathfrak{X}}} \mathscr{G} \simeq \varprojlim_n (\mathscr{F}_n \otimes_{\mathscr{O}_{X_n}} \mathscr{G}_n), \tag{10.11.7.1}$$

$$\mathscr{H}\!om_{\mathscr{O}_{\mathfrak{X}}}(\mathscr{F},\mathscr{G}) \simeq \varprojlim_n \mathscr{H}\!om_{\mathscr{O}_{X_n}}(\mathscr{F}_n,\mathscr{G}_n). \tag{10.11.7.2}$$

Proof. The existence of the isomorphism (10.11.7.1) follows from the formula

$$\mathscr{F}_n \otimes_{\mathscr{O}_{X_n}} \mathscr{G}_n = (\mathscr{F} \otimes_{\mathscr{O}_{\mathfrak{X}}} \mathscr{O}_{X_n}) \otimes_{\mathscr{O}_{X_n}} (\mathscr{G} \otimes_{\mathscr{O}_{\mathfrak{X}}} \mathscr{O}_{X_n}) = (\mathscr{F} \otimes_{\mathscr{O}_{\mathfrak{X}}} \mathscr{G}) \otimes_{\mathscr{O}_{\mathfrak{X}}} \mathscr{O}_{X_n}$$

and from Theorem (10.11.3). The isomorphism (10.11.7.2), where both sides are considered as sheaves of modules without topology, follows from the definition of the sections of $\mathscr{H}\!om_{\mathscr{O}_{\mathfrak{X}}}(\mathscr{F},\mathscr{G})$ and $\mathscr{H}\!om_{\mathscr{O}_{X_n}}(\mathscr{F}_n,\mathscr{G}_n)$, and from the existence of the isomorphism (10.11.4.1), mapping a prescheme induced on an arbitrary Noetherian formal affine open set to $\mathfrak{X}$. It remains to prove that the isomorphism (10.11.7.2) is bicontinuous over a quasi-compact set, and we can thus reduce to the case where $\mathfrak{X} = \mathrm{Spf}(A)$ with A an adic Noetherian ring, and hence (10.10.5) to the case where $\mathscr{F} = M^\Delta$ and $\mathscr{G} = N^\Delta$, with M and N both A-modules of finite type; taking (10.10.2.1), (10.10.2.3), and Corollary (1.3.12, ii) into account, we reduce to showing that the canonical isomorphism $\mathrm{Hom}_A(M,N) \simeq \varprojlim_n \mathrm{Hom}_{A_n}(M_n,N_n)$ (with $M_n = M \otimes_A A_n$ and $N_n = N \otimes_A A_n$) is continuous, which has already been proved in (**0**, 7.8.2). □

(10.11.8). Since $\mathrm{Hom}_{\mathscr{O}_{\mathfrak{X}}}(\mathscr{F},\mathscr{G})$ is the group of sections of the sheaf of topological groups $\mathscr{H}\!om_{\mathscr{O}_{\mathfrak{X}}}(\mathscr{F},\mathscr{G})$, it is equipped with a group topology. If $\mathfrak{X}$ is *Noetherian*, then it follows from (10.11.7.2) that the subgroups $\mathrm{Hom}_{\mathscr{O}_{\mathfrak{X}}}(\mathscr{F},\mathscr{J}^n\mathscr{G})$ (for arbitrary n) form a fundamental system of neighborhoods of 0 in this group.

Proposition (10.11.9). — *Let $\mathfrak{X}$ be a Noetherian formal prescheme, and $\mathscr{F}$ and $\mathscr{G}$ coherent $\mathscr{O}_{\mathfrak{X}}$-modules. In the topological group $\mathrm{Hom}_{\mathscr{O}_{\mathfrak{X}}}(\mathscr{F},\mathscr{G})$, the surjective (resp. injective, bijective) homomorphisms form an open set.*

Proof. By Corollary (10.11.5), the set of surjective homomorphisms in $\mathrm{Hom}_{\mathscr{O}_{\mathfrak{X}}}(\mathscr{F},\mathscr{G})$ is the inverse image under the continuous map $\mathrm{Hom}_{\mathscr{O}_{\mathfrak{X}}}(\mathscr{F},\mathscr{G}) \to \mathrm{Hom}_{\mathscr{O}_{X_0}}(\mathscr{F}_0,\mathscr{G}_0)$ of a subset of the discrete group $\mathrm{Hom}_{\mathscr{O}_{X_0}}(\mathscr{F}_0,\mathscr{G}_0)$, whence the first claim. To show the second, we cover $\mathfrak{X}$ by a finite number of Noetherian formal affine subsets U_i. For $\theta \in \mathrm{Hom}_{\mathscr{O}_{\mathfrak{X}}}(\mathscr{F},\mathscr{G})$ to be injective, it is necessary and sufficient for all of the images under the (continuous) restriction maps $\mathrm{Hom}_{\mathscr{O}_{\mathfrak{X}}}(\mathscr{F},\mathscr{G}) \to \mathrm{Hom}_{\mathscr{O}_{\mathfrak{X}|U_i}}(\mathscr{F}|U_i,\mathscr{G}|U_i)$ to be injective; we can thus reduce to the affine case, and then this has already been proved in (**0**, 7.8.3). □

10.12. Adic morphisms of formal preschemes. I | 206

(10.12.1). Let $\mathfrak{X}$ and $\mathfrak{S}$ be *locally Noetherian* formal preschemes; we say that a morphism $f : \mathfrak{X} \to \mathfrak{S}$ is *adic* if there exists an ideal of definition $\mathscr{J}$ of $\mathfrak{S}$ such that $\mathscr{K} = f^*(\mathscr{J})\mathscr{O}_{\mathfrak{X}}$ is an ideal of definition of $\mathfrak{X}$; we then also say that $\mathfrak{X}$ is an *adic $\mathfrak{S}$-prescheme* (for f). Whenever this is the case, for *every* ideal of definition $\mathscr{J}_1$ of $\mathfrak{S}$, $\mathscr{K}_1 = f^*(\mathscr{J}_1)\mathscr{O}_{\mathfrak{X}}$ is an ideal of definition of $\mathfrak{X}$. Indeed, the question being local, we can assume that $\mathfrak{X}$ and $\mathfrak{S}$ are Noetherian and affine; there then exists a whole number n such that $\mathscr{J}^n \subset \mathscr{J}_1$ and $\mathscr{J}_1^n \subset \mathscr{J}$ ((10.3.6) and (**0**, 7.1.4)), whence $\mathscr{K}^n \subset \mathscr{K}_1$ and $\mathscr{K}_1^n \subset \mathscr{K}$. The first of these relations shows that $\mathscr{K}_1 = \mathfrak{K}_1^{\Delta}$, where $\mathfrak{K}_1$ is an open ideal of $A = \Gamma(\mathfrak{X}, \mathscr{O}_{\mathfrak{X}})$, and the second shows that $\mathfrak{K}_1$ is an ideal of definition of A (**0**, 7.1.4), whence our claim.

It follows immediately from the above that, if $\mathfrak{X}$ and $\mathfrak{Y}$ are adic $\mathfrak{S}$-preschemes, then *every $\mathfrak{S}$-morphism* $u : \mathfrak{X} \to \mathfrak{Y}$ *is adic*: indeed, if $f : \mathfrak{X} \to \mathfrak{S}$ and $g : \mathfrak{Y} \to \mathfrak{S}$ are the structure morphisms, and $\mathscr{J}$ is an ideal of definition of $\mathfrak{S}$, then we have $f = g \circ u$, and so $u^*(g^*(\mathscr{J})\mathscr{O}_{\mathfrak{Y}})\mathscr{O}_{\mathfrak{X}} = f^*(\mathscr{J})\mathscr{O}_{\mathfrak{X}}$ is an ideal of definition of $\mathfrak{X}$, and, by hypothesis, $g^*(\mathscr{J})\mathscr{O}_{\mathfrak{Y}}$ is an ideal of definition of $\mathfrak{Y}$.

(10.12.2). In what follows, we suppose that we have some fixed locally Noetherian formal prescheme $\mathfrak{S}$, and some ideal of definition $\mathscr{J}$ of $\mathfrak{S}$; we set $S_n = (\mathfrak{S}, \mathscr{O}_{\mathfrak{S}}/\mathscr{J}^{n+1})$. The (locally Noetherian) adic $\mathfrak{S}$-preschemes clearly form a *category*. We say that an inductive system (X_n) of locally Noetherian (usual) S_n-preschemes is an *adic inductive (S_n)-system* if the structure morphisms $f_n : X_n \to S_n$ are such that, for $m \leqslant n$, the diagrams

$$\begin{tikzcd} X_n \arrow[d, "f_n"'] & X_m \arrow[l] \arrow[d, "f_m"] \\ S_n & S_m \arrow[l] \end{tikzcd}$$

commute and *identify X_m with the product* $X_n \times_{S_n} S_m = (X_n)_{(S_m)}$. The adic inductive systems form a *category*: it suffices in fact to define a morphism $(X_n) \to (Y_n)$ of such systems to be an *inductive system of S_n-morphisms* $u_n : X_n \to Y_n$ such that u_m is identified with $(u_n)_{(S_m)}$ for $m \leqslant n$. With this in mind:

Theorem (10.12.3). — *There is a canonical equivalence between the category of adic $\mathfrak{S}$-preschemes and the category of adic inductive (S_n)-systems.*

The equivalence in question is obtained in the following way: if $\mathfrak{X}$ is an adic $\mathfrak{S}$-prescheme, and $f : \mathfrak{X} \to \mathfrak{S}$ is the structure morphism, then $\mathscr{K} = f^*(\mathscr{J})\mathscr{O}_{\mathfrak{X}}$ is an ideal of definition of $\mathfrak{X}$, and we associate to $\mathfrak{X}$ the inductive system of the $X_n = (\mathfrak{X}, \mathscr{O}_{\mathfrak{X}}/\mathscr{K}^{n+1})$, with the structure morphism $f_n : X_n \to S_n$ corresponding to f (10.5.6). We first show that (X_n) is an *adic inductive system*: if $f = (\psi, \theta)$, then $\psi^*(\mathscr{J})\mathscr{O}_{\mathfrak{X}} = \mathscr{K}$, so $\psi^*(\mathscr{J}^n)\mathscr{O}_{\mathfrak{X}} = \mathscr{K}^n$ for all n, and (by exactness of the functor ψ^*) $\mathscr{K}^{m+1}/\mathscr{K}^{n+1} = \psi^*(\mathscr{J}^{m+1}/\mathscr{J}^{n+1})(\mathscr{O}_{\mathfrak{X}}/\mathscr{K}^{n+1})$ for $m \leqslant n$; our conclusion thus follows from (4.4.5). Furthermore, it can be immediately verified that a $\mathfrak{S}$-morphism $u : \mathfrak{X} \to \mathfrak{Y}$ of adic $\mathfrak{S}$-preschemes corresponds (with the obvious notation) to an inductive system of S_n-morphisms $u_n : X_n \to Y_n$ I | 207
such that u_m is identified with $(u_n)_{(S_m)}$ for $m \leqslant n$.

The fact that this equivalence is well defined will follow from the more-precise following proposition.

Proposition (10.12.3.1). — *Let (X_n) be an inductive system of S_n-preschemes; suppose that the structure morphisms $f_n : X_n \to S_n$ are such that the diagrams in* (10.12.2.1) *commute and identify X_m with $X_n \times_{S_n} S_m$ for $m \leqslant n$. Then the inductive system (X_n) satisfies conditions* (b) *and* (c) *of* (10.6.3)*; let $\mathfrak{X}$ be the inductive limit, and $f : \mathfrak{X} \to \mathfrak{S}$ the morphism given by the inductive limit of the inductive system (f_n). Then, if X_0 is locally Noetherian, $\mathfrak{X}$ is locally Noetherian, and f is an adic morphism.*

Proof. Since the sheaf of ideals of $\mathscr{O}_{S_n}$ that defines the subprescheme S_m of S_n is nilpotent, by (4.4.5), so too is the sheaf of ideals of $\mathscr{O}_{X_n}$ that defines the subprescheme X_m of X_n, and so the conditions of (10.6.3) are satisfied. The question being local on $\mathfrak{X}$ and $\mathfrak{S}$, we can assume that $\mathfrak{S} = \operatorname{Spf}(A)$, $\mathscr{J} = \mathfrak{J}^{\Delta}$ (with A a Noetherian $\mathfrak{J}$-adic ring), and $X_n = \operatorname{Spec}(B_n)$; if $A_n = A/\mathfrak{J}^{n+1}$, then the hypothesis implies that B_0 is Noetherian, and if we set $\mathfrak{J}_n = \mathfrak{J}/\mathfrak{J}^{n+1}$, then $B_m = B_n/\mathfrak{J}_n^{m+1}B_n$. The kernel of $B_n \to B_0$ is thus $\mathfrak{K}_n = \mathfrak{J}_n B_n$, and the kernel of $B_n \to B_m$ is $\mathfrak{K}_n^{m+1}$ for $m \leqslant n$; further, since A_1 is Noetherian, $\mathfrak{J}_1$ is of finite type over A_1, and so $\mathfrak{K}_1 = \mathfrak{K}_1/\mathfrak{K}_1^2$ is of finite type over B_1, and *a fortiori* of finite type over $B_0 = B_1/\mathfrak{K}_1$; the fact that $\mathfrak{X}$ is Noetherian then follows from (10.6.4); if $B = \varprojlim B_n$, then we have $\mathfrak{X} = \operatorname{Spf}(B)$, and, if $\mathfrak{K}$ is the kernel of $B \to B_0$, then $B_n = B/\mathfrak{K}^{n+1}$. If $\rho_n : A/\mathfrak{J}^{n+1} \to B/\mathfrak{K}^{n+1}$ is the homomorphism corresponding to f_n, then we have that

$$\mathfrak{K}/\mathfrak{K}^{n+1} = (B/\mathfrak{K}^{n+1})\rho_n(\mathfrak{J}/\mathfrak{J}^{n+1})$$

since the homomorphism $\rho : A \to B$ corresponding to f is equal to $\varprojlim \rho_n$, and that the ideal $\mathfrak{J}B$ of B is dense in $\mathfrak{K}$, and, since every ideal of B is closed (**0**, 7.3.5), we also have that $\mathfrak{K} = \mathfrak{J}B$. If $\mathscr{K} = \mathfrak{K}^{\Delta}$, the equality $f^*(\mathscr{J})\mathscr{O}_{\mathfrak{X}} = \mathscr{K}$ then follows from (10.10.9), and finishes the proof. □

(10.12.3.2). The above equivalence gives, for adic $\mathfrak{S}$-preschemes $\mathfrak{X}$ and $\mathfrak{Y}$, a *canonical bijection*

$$\mathrm{Hom}_{\mathfrak{S}}(\mathfrak{X},\mathfrak{Y}) \simeq \varprojlim_n \mathrm{Hom}_{S_n}(X_n, Y_n)$$

where the projective limit is relative to the maps $u_n \to (u_n)_{(S_m)}$ for $m \leqslant n$.

10.13. Morphisms of finite type.

Proposition (10.13.1). — *Let $\mathfrak{Y}$ be a locally Noetherian formal prescheme, $\mathscr{K}$ an ideal of definition of $\mathfrak{Y}$, and $f : \mathfrak{X} \to \mathfrak{Y}$ a morphism of formal preschemes. Then the following conditions are equivalent.*

(a) *X is locally Noetherian, f is an adic morphism (10.12.1), and, if we set $\mathscr{J} = f^*(\mathscr{K})\mathscr{O}_{\mathfrak{X}}$, then the morphism $f_0 : (\mathfrak{X}, \mathscr{O}_{\mathfrak{X}}/\mathscr{J}) \to (\mathfrak{Y}, \mathscr{O}_{\mathfrak{Y}}/\mathscr{K})$ induced by f is of finite type.*

(b) *$\mathfrak{X}$ is locally Noetherian, and is the inductive limit of an adic inductive (Y_n)-system (X_n) such that the morphism $X_0 \to Y_0$ is of finite type.* I | 208

(1) Every point of $\mathfrak{Y}$ has a Noetherian formal affine open neighbourhood V which satisfies the following property:

*(Q) $f^{-1}(V)$ is a finite union of Noetherian formal affine open subsets U_i such that the Noetherian adic ring $\Gamma(U_i, \mathscr{O}_{\mathfrak{X}})$ is topologically isomorphic to the quotient of a formal series algebra, restricted (**0**, 7.5.1) to $\Gamma(V, \mathscr{O}_{\mathfrak{Y}})$, by an ideal (which is necessarily closed).*

Proof. It is immediate that (a) implies (b), by (10.12.3). To show that (b) implies (c), we can, since the question is local on $\mathfrak{Y}$, assume that $\mathfrak{Y} = \mathrm{Spf}(B)$, where B is Noetherian and adic; let $\mathscr{K} = \mathfrak{K}^\Delta$, with $\mathfrak{K}$ an ideal of definition of B. Since, by hypothesis, X_0 is of finite type over Y_0, X_0 is a finite union of affine open subsets U_i such that the ring A_{i0} of the affine scheme induced by X_0 on U_i is an algebra of finite type over the ring $B/\mathfrak{K}$ of Y_0 (6.3.2). By (5.1.9), U_i is also an affine open subset in each of the Noetherian preschemes X_n, and, if A_{in} is the ring of the affine scheme induced by X_n on U_i, then hypothesis (b) implies, for $m \leqslant n$, that A_{im} is isomorphic to $A_{in}/\mathfrak{K}^{m+1}A_{in}$. Consequently, the formal prescheme induced by $\mathfrak{X}$ on U_i is isomorphic to $\mathrm{Spf}(A_i)$, where $A_i = \varprojlim_n A_{in}$ (10.6.4); A_i is a $\mathfrak{K}A_i$-adic ring, and $A_i/\mathfrak{K}A_i$, being isomorphic to A_{i0}, is an algebra of finite type over $B/\mathfrak{K}$. We thus conclude (**0**, 7.5.5) that A_i is topologically isomorphic to a quotient of a formal series algebra restricted to B (by a necessarily closed ideal, because such an algebra is Noetherian (**0**, 7.5.4)).

To show that (c) implies (a), we can restrict to the case where $\mathfrak{X} = \mathrm{Spf}(A)$ is also affine, with A a Noetherian adic ring isomorphic to a quotient of a formal series algebra, restricted to B, by a closed ideal. Then (**0**, 7.5.5) $A/\mathfrak{K}A$ is an algebra of finite type over $B/\mathfrak{K}$, and $\mathfrak{K}A = \mathfrak{J}$ is an ideal of definition of A, and so, by (10.10.9), the conditions of (a) are satisfied. □

We note that, if the conditions of Proposition (10.13.1) are satisfied, then property (a) holds true for *any* ideal of definition $\mathscr{K}$ of $\mathfrak{Y}$ (by *(c)*), and so, in property (b), *all* the f_n are morphisms of finite type.

Corollary (10.13.2). — *If the conditions of (10.13.1) are satisfied, then every Noetherian formal affine open subset V of $\mathfrak{Y}$ has property (Q), and, if $\mathfrak{Y}$ is Noetherian, then so too is $\mathfrak{X}$.*

Proof. This follows immediately from (10.13.1) and (6.3.2). □

Definition (10.13.3). — When the equivalent properties (a), (b), and (c) of (10.13.1) are satisfied, we say that the morphism f is of finite type, or that $\mathfrak{X}$ is a formal $\mathfrak{Y}$-prescheme of finite type, or a formal prescheme of finite type over $\mathfrak{Y}$.

Corollary (10.13.4). — *Let $\mathfrak{X} = \mathrm{Spf}(A)$ and $\mathfrak{Y} = \mathrm{Spf}(B)$ be Noetherian formal affine schemes; for $\mathfrak{X}$ to be of finite type over $\mathfrak{Y}$, it is necessary and sufficient for the Noetherian adic ring A to be isomorphic to the quotient of a formal series algebra, restricted to B, by some closed ideal.*

Proof. With the notation of (10.13.1), if $\mathfrak{X}$ is of finite type over $\mathfrak{Y}$, then $A/\mathfrak{K}A$ is a $(B/\mathfrak{K})$-algebra of finite type by (6.3.3), and $\mathfrak{K}A$ is an ideal of definition of A (10.10.9). We are then done, by (**0**, 7.5.5). □

Proposition (10.13.5). —

(i) *The composition of any two morphisms (of formal preschemes) of finite type is again of finite type.* I | 209

(ii) *Let $\mathfrak{X}$, $\mathfrak{S}$, and $\mathfrak{S}'$ be locally Noetherian (resp. Noetherian) formal preschemes, and $f : \mathfrak{X} \to \mathfrak{S}$ and $\mathfrak{X} \to \mathfrak{S}'$ morphisms. If f is of finite type, then $\mathfrak{X} \times_{\mathfrak{S}} \mathfrak{S}'$ is locally Noetherian (resp. Noetherian) and of finite type over $\mathfrak{S}'$.*

(iii) *Let $\mathfrak{S}$ be a locally Noetherian formal prescheme, and $\mathfrak{X}'$ and $\mathfrak{Y}'$ formal $\mathfrak{S}$-preschemes such that $\mathfrak{X}' \times_{\mathfrak{S}} \mathfrak{Y}'$ is locally Noetherian. If $\mathfrak{X}$ and $\mathfrak{Y}$ are locally Noetherian formal $\mathfrak{S}$-preschemes, and $f : \mathfrak{X} \to \mathfrak{X}'$ and $g : \mathfrak{Y} \to \mathfrak{Y}'$ are $\mathfrak{S}$-morphisms of finite type, then $\mathfrak{X} \times_{\mathfrak{S}} \mathfrak{Y}$ is locally Noetherian, and $f \times_{\mathfrak{S}} g$ is a $\mathfrak{S}$-morphism of finite type.*

Proof. By the formal argument of (3.5.1), (iii) follows from (i) and (ii), so it suffices to prove (i) and (ii).

Let $\mathfrak{X}$, $\mathfrak{Y}$, and $\mathfrak{Z}$ be locally Noetherian formal preschemes, and $f : \mathfrak{X} \to \mathfrak{Y}$ and $g : \mathfrak{Y} \to \mathfrak{Z}$ morphisms of finite type. If $\mathscr{L}$ is an ideal of definition of $\mathfrak{Z}$, then $\mathscr{K} = g^*(\mathscr{L})\mathcal{O}_{\mathfrak{Y}}$ is an ideal of definition of $\mathfrak{Y}$, and $\mathscr{J} = f^*(g^*(\mathscr{L}))\mathcal{O}_{\mathfrak{X}}$ is an ideal of definition for $\mathfrak{X}$. Let $X_0 = (\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}/\mathscr{J})$, $Y_0 = (\mathfrak{Y}, \mathcal{O}_{\mathfrak{Y}}/\mathscr{K})$, and $Z_0 = (\mathfrak{Z}, \mathcal{O}_{\mathfrak{Z}}/\mathscr{L})$, and let $f_0 : X_0 \to Y_0$ and $g_0 : Y_0 \to Z_0$ be the morphisms corresponding to f and g (respectively). Since, by hypothesis, f_0 and g_0 are of finite type, so too is $g_0 \circ f_0$ (6.3.4), which corresponds to $g \circ f$; thus $g \circ f$ is of finite type, by (10.13.1).

Under the conditions of (ii), $\mathfrak{S}$ (resp. $\mathfrak{X}$, $\mathfrak{S}'$) is the inductive limit of a sequence (S_n) (resp. (X_n), (S'_n)) of locally Noetherian preschemes, and we can assume (10.13.1) that $X_m = X_n \times_{S_n} S_m$ for $m \leqslant n$. The formal prescheme $\mathfrak{X} \times_{\mathfrak{S}} \mathfrak{S}'$ is then the inductive limit of the preschemes $X_n \times_{S_n} S'_n$ (10.7.4), and we have that

$$X_m \times_{S_m} S'_m = (X_n \times_{S_n} S_m) \times_{S_m} S'_m = (X_n \times_{S_n} S'_n) \times_{S'_n} S'_m.$$

Furthermore, $X_0 \times_{S_0} S'_0$ is locally Noetherian, since X_0 is of finite type over S_0 (6.3.8). We thus conclude (10.12.3.1), first of all, that $\mathfrak{X} \times_{\mathfrak{S}} \mathfrak{S}'$ is locally Noetherian; then, since $X_0 \times_{S_0} S'_0$ is of finite type over S'_0 (6.3.8), it follows from (10.12.3.1) and (10.13.1) that $\mathfrak{X} \times_{\mathfrak{S}} \mathfrak{S}'$ is of finite type over $\mathfrak{S}'$, which proves (ii) (the claim about Noetherian preschemes being an immediate consequence of (6.3.8)). □

Corollary (10.13.6). — *Under the hypotheses of (10.9.9), if f is a morphism of finite type, then so too is its extension $\widehat{f}$ to the completions.*

10.14. Closed subpreschemes of formal preschemes.

Proposition (10.14.1). — *Let $\mathfrak{X}$ be a locally Noetherian formal prescheme, and $\mathscr{A}$ a coherent sheaf of ideals of $\mathcal{O}_{\mathfrak{X}}$. If $\mathfrak{Y}$ if the (closed) support of $\mathcal{O}_{\mathfrak{X}}/\mathscr{A}$, then the topologically ringed space $(\mathfrak{Y}, (\mathcal{O}_{\mathfrak{X}}/\mathscr{A})|\mathfrak{Y})$ is a locally Noetherian formal prescheme that is Noetherian if $\mathfrak{X}$ is.*

Proof. Note that $\mathcal{O}_{\mathfrak{X}}/\mathfrak{A}$ is coherent by (10.10.3) and (**0**, 5.3.4), so its support $\mathfrak{Y}$ is closed (**0**, 5.2.2). Let $\mathscr{J}$ be an ideal of definition of $\mathfrak{X}$, and let $X_n = (\mathfrak{X}/\mathcal{O}_{\mathfrak{X}}/\mathscr{J}^{n+1})$; the sheaf of rings $\mathcal{O}_{\mathfrak{X}}/\mathscr{A}$ is the projective limit of the sheaves $\mathcal{O}_{\mathfrak{X}}/(\mathscr{A} + \mathscr{J}^{n+1}) = (\mathcal{O}_{\mathfrak{X}}/\mathscr{A}) \otimes_{\mathcal{O}_{\mathfrak{X}}} (\mathcal{O}_{\mathfrak{X}}/\mathscr{J}^{n+1})$ (10.11.3), all of which have support $\mathfrak{Y}$. The sheaf $(\mathscr{A} + \mathscr{J}^{n+1})/\mathscr{J}^{n+1}$ is a coherent $\mathcal{O}_{\mathfrak{X}}$-module, since $\mathscr{J}^{n+1}$ is coherent, and so $(\mathscr{A} + \mathscr{J}^{n+1})/\mathscr{J}^{n+1}$ is also a coherent $(\mathcal{O}_{\mathfrak{X}}/\mathscr{J}^{n+1})$-module (**0**, 5.3.10); if Y_n is the closed subprescheme of X_n defined by this sheaf of ideals, it is immediate that $(\mathfrak{Y}, (\mathcal{O}_{\mathfrak{X}}/\mathscr{A})|\mathfrak{Y})$ is the formal prescheme given by the inductive limit of the Y_n, and, since the **I** | 210
conditions of (10.6.4) are satisfied, this proves that this formal prescheme is locally Noetherian, and further Noetherian if $\mathfrak{X}$ is (since then Y_0 is, by (6.1.4)). □

Definition (10.14.2). — We define a closed subprescheme of a formal prescheme $\mathfrak{X}$ to be any formal prescheme of the form $(\mathfrak{Y}, (\mathcal{O}_{\mathfrak{X}}/\mathscr{A})|\mathfrak{Y})$ with $\mathscr{A}$ a coherent ideal of $\mathcal{O}_{\mathfrak{X}}$; we say that this prescheme is the subprescheme **Err**$_{\text{II}}$
defined by $\mathscr{A}$.

It is clear that the correspondence thus defined between coherent ideals of $\mathcal{O}_{\mathfrak{X}}$ and closed subpreschemes of **Err**$_{\text{II}}$
$\mathfrak{X}$ is bijective.

The morphism of topologically ringed spaces $j = (\psi, \theta) : \mathfrak{Y} \to \mathfrak{X}$, where ψ is the injection $\mathfrak{Y} \to \mathfrak{X}$ and $\theta^\sharp$ the canonical homomorphism $\mathcal{O}_{\mathfrak{X}} \to \mathcal{O}_{\mathfrak{X}}/\mathscr{A}$, is evidently (10.4.5) a morphism of formal preschemes, and we call it the *canonical injection* from $\mathfrak{Y}$ to $\mathfrak{X}$. Note that, if $\mathfrak{X} = \operatorname{Spf}(A)$, or if A is Noetherian and adic, then $\mathscr{A} = \mathfrak{a}^\Delta$, where $\mathfrak{a}$ is an ideal of A (10.10.5), and it then follows immediately from the above that $\mathfrak{Y} = \operatorname{Spf}(A/\mathfrak{a})$, up to isomorphism, and that j corresponds (10.2.2) to the canonical homomorphism $A \to A/\mathfrak{a}$.

We say that a morphism $f : \mathfrak{Z} \to \mathfrak{X}$ of locally Noetherian formal preschemes is a *closed immersion* if it factors as $\mathfrak{Z} \xrightarrow{g} \mathfrak{Y} \xrightarrow{j} \mathfrak{X}$, where g is an isomorphism from $\mathfrak{Z}$ to a closed subprescheme $\mathfrak{Y}$ of $\mathfrak{X}$, and j is the canonical injection. Since j is a monomorphism of ringed spaces, g and $\mathfrak{Y}$ are necessarily *unique*.

Proposition (10.14.3). — *A closed immersion is a morphism of finite type.*

Proof. We can immediately restrict to the case where $\mathfrak{X}$ is a formal affine scheme $\operatorname{Spf}(A)$, and $\mathfrak{Y} = \operatorname{Spf}(A/\mathfrak{a})$; the proposition then follows from Proposition (10.13.1, *c*). □

Lemma (10.14.4). — *Let $f : \mathfrak{Y} \to \mathfrak{X}$ be a morphism of locally Noetherian formal preschemes, and let (U_α) be a cover of $f(\mathfrak{Y})$ by Noetherian formal affine open subsets of $\mathfrak{X}$ such that the $f^{-1}(U_\alpha)$ are Noetherian formal affine open subsets of $\mathfrak{Y}$. For f to be a closed immersion, it is necessary and sufficient for $f(\mathfrak{Y})$ to be a closed subset of $\mathfrak{X}$ and, for all α, for the restriction of f to $f^{-1}(U_\alpha)$ to correspond (10.4.6) to a surjective homomorphism $\Gamma(U_\alpha, \mathcal{O}_{\mathfrak{X}}) \to \Gamma(f^{-1}(U_\alpha), \mathcal{O}_{\mathfrak{Y}})$.*

Proof. The conditions are clearly necessary. Conversely, if the conditions are satisfied, and if we denote by $\mathfrak{a}_\alpha$ the kernel of $\Gamma(U_\alpha, \mathcal{O}_{\mathfrak{X}}) \to \Gamma(f^{-1}(U_\alpha), \mathcal{O}_{\mathfrak{Y}})$, then we can define a coherent sheaf of ideals $\mathscr{A}$ of $\mathcal{O}_{\mathfrak{X}}$ by setting $\mathscr{A}|U_\alpha = \mathfrak{a}_\alpha^\Delta$ and taking $\mathscr{A}$ to be zero on the complement of the union of the U_α. Since $f(\mathfrak{Y})$ is closed, and since the support of $\mathfrak{a}_\alpha^\Delta$ is $U_\alpha \cap f(\mathfrak{Y})$, everything reduces to proving that $\mathfrak{a}_\alpha^\Delta$ and $\mathfrak{a}_\beta^\Delta$ induce the same sheaf on any Noetherian formal affine open subset $V \subset U_\alpha \cap U_\beta$. But the restriction to $f^{-1}(U_\alpha)$ of f is a closed immersion of this formal prescheme into U_α, $f^{-1}(V)$ is a Noetherian formal affine open subsets of $f^{-1}(U_\alpha)$, and the restriction of f to $f^{-1}(V)$ is a closed immersion; if $\mathfrak{b}$ is the kernel of the surjective homomorphism $\Gamma(V, \mathscr{O}_{\mathfrak{X}}) \to \Gamma(f^{-1}(V), \mathscr{O}_{\mathfrak{Y}})$ corresponding to this restriction, then it is immediate (10.10.2) that $\mathfrak{a}_\alpha^\Delta$ induces $\mathfrak{b}^\Delta$ on V. The sheaf of ideals $\mathscr{A}$ being thus defined, it is then clear that $f = g \circ j$, where $j : \mathfrak{Z} \to \mathfrak{X}$ is the canonical injection of the closed subprescheme $\mathfrak{Z}$ of $\mathfrak{X}$ defined by $\mathscr{A}$, and that g is an isomorphism from $\mathfrak{Y}$ to $\mathfrak{Z}$. ☐

Proposition (10.14.5). —

(i) *If $f : \mathfrak{Z} \to \mathfrak{Y}$ and $g : \mathfrak{Y} \to \mathfrak{X}$ are closed immersions of locally Noetherian formal preschemes, then $g \circ f$ is a closed immersion.* I | 211

(ii) *Let $\mathfrak{X}$, $\mathfrak{Y}$, and $\mathfrak{S}$ be locally Noetherian formal preschemes, $f : \mathfrak{X} \to \mathfrak{S}$ a closed immersion, and $g : \mathfrak{Y} \to \mathfrak{S}$ a morphism. Then the morphism $\mathfrak{X} \times_{\mathfrak{S}} \mathfrak{Y} \to \mathfrak{Y}$ is a closed immersion.*

(iii) *Let $\mathfrak{S}$ be a locally Noetherian formal prescheme, and $\mathfrak{X}'$ and $\mathfrak{Y}'$ locally Noetherian formal $\mathfrak{S}$-preschemes such that $\mathfrak{X}' \times_{\mathfrak{S}} \mathfrak{Y}'$ is locally Noetherian. If $\mathfrak{X}$ and $\mathfrak{Y}$ are locally Noetherian $\mathfrak{S}$-preschemes, and $f : \mathfrak{X} \to \mathfrak{X}'$ and $g : \mathfrak{Y} \to \mathfrak{Y}'$ are $\mathfrak{S}$-morphisms that are closed immersions, then $f \times_{\mathfrak{S}} g$ is a closed immersion.*

Proof. By (3.5.1), it again suffices to prove (i) and (ii).

To prove (i), we can assume that $\mathfrak{Y}$ (resp. $\mathfrak{Z}$) is a closed subprescheme of $\mathfrak{X}$ (resp. $\mathfrak{Y}$) defined by a coherent sheaf $\mathscr{J}$ (resp. $\mathscr{K}$) of ideals of $\mathscr{O}_{\mathfrak{X}}$ (resp. $\mathscr{O}_{\mathfrak{Y}}$); if ψ is the injection $\mathfrak{Y} \to \mathfrak{X}$ of underlying spaces, then $\psi_*(\mathscr{K})$ is a coherent sheaf of ideals of $\psi_*(\mathscr{O}_{\mathfrak{Y}}) = \mathscr{O}_{\mathfrak{X}}/\mathscr{J}$ (**0**, 5.3.12), and thus also a coherent $\mathscr{O}_{\mathfrak{X}}$-module (**0**, 5.3.10); the kernel $\mathscr{K}_1$ of $\mathscr{O}_{\mathfrak{X}} \to (\mathscr{O}_{\mathfrak{X}}/\mathscr{J})/\psi_*(\mathscr{K})$ is thus a coherent sheaf of ideals of $\mathscr{O}_{\mathfrak{X}}$ (**0**, 5.3.4), and $\mathscr{O}_{\mathfrak{X}}/\mathscr{K}_1$ is isomorphic to $\psi_*(\mathscr{O}_{\mathfrak{Y}}/\mathscr{K})$, which proves that $\mathfrak{Z}$ is an isomorphism to a closed subprescheme of $\mathfrak{X}$.

To prove (ii), we can immediately restrict to the case where $\mathfrak{S} = \mathrm{Spf}(A)$, $\mathfrak{X} = \mathrm{Spf}(B)$, and $\mathfrak{Y} = \mathrm{Spf}(C)$, with A a Noetherian $\mathfrak{J}$-adic ring, $B = A/\mathfrak{a}$ (where $\mathfrak{a}$ is an ideal of A), and C a Noetherian topological adic A-algebra. Everything then reduces to proving that the homomorphism $C \to C\widehat{\otimes}_A(A/\mathfrak{a})$ is *surjective*: but $A/\mathfrak{a}$ is an A-module of finite type, and its topology is the $\mathfrak{J}$-adic topology; it then follows from (**0**, 7.7.8) that $C\widehat{\otimes}_A(A/\mathfrak{a})$ can be identified with $C \otimes_A (A/\mathfrak{a}) = C/\mathfrak{a}C$, whence our claim. ☐

Corollary (10.14.6). — *Under the hypotheses of (10.14.5, ii), let $p : \mathfrak{X} \times_{\mathfrak{S}} \mathfrak{Y} \to \mathfrak{X}$ and $q : \mathfrak{X} \times_{\mathfrak{S}} \mathfrak{Y} \to \mathfrak{Y}$ be the projections, so that the diagram*

$$\begin{tikzcd} \mathfrak{X} \arrow[d, "f"'] & \mathfrak{X} \times_{\mathfrak{S}} \mathfrak{Y} \arrow[l, "p"'] \arrow[d, "q"] \\ \mathfrak{S} & \mathfrak{Y} \arrow[l, "g"'] \end{tikzcd}$$

commutes. For every coherent $\mathscr{O}_{\mathfrak{X}}$ module $\mathscr{F}$, we then have a canonical isomorphism of $\mathscr{O}_{\mathfrak{Y}}$-modules

(10.14.6.1) $$u : g^* f_*(\mathscr{F}) \simeq q_* p^*(\mathscr{F}).$$

Proof. We know that defining a homomorphism $g^* f_*(\mathscr{F}) \to q_* p^*(\mathscr{F})$ is equivalent to defining a homomorphism $f_*(\mathscr{F}) \to g_* q_* p^*(\mathscr{F}) = f_* p_* p^*(\mathscr{F})$ (**0**, 4.4.3): we take $u = f_*(\rho)$, where ρ is the canonical homomorphism $\mathscr{F} \to p_* p^*(\mathscr{F})$ (**0**, 4.4.3). To see that u is an isomorphism, we can immediately restrict to the case where $\mathfrak{S}$, $\mathfrak{X}$, and $\mathfrak{Y}$ are formal spectra of Noetherian adic rings A, B, and C (respectively), satisfying the conditions in (10.14.5, ii) above; we then have $\mathscr{F} = M^\Delta$, where M is an $(A/\mathfrak{a})$-module of finite type (10.10.5), and the two sides of (10.14.6.1) can then be identified, respectively, by (10.10.8), with $(C \otimes_A M)^\Delta$ and $((C/\mathfrak{a}C) \otimes_{A/\mathfrak{a}} M)^\Delta$, whence the corollary, since $(C/\mathfrak{a}C) \otimes_{A/\mathfrak{a}} M = (C \otimes_A (A/\mathfrak{a})) \otimes_{A/\mathfrak{a}} M$ is canonically identified with $C \otimes_A M$. ☐

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Corollary (10.14.7). — *Let X be a locally Noetherian usual prescheme, Y a closed subprescheme of X, j the canonical injection $Y \to X$, X' a closed subset of X, and $Y' = Y \cap X'$; then $\widehat{j} : Y_{/Y'} \to X_{/X'}$ is a closed immersion, and, for every coherent $\mathscr{O}_Y$-module $\mathscr{F}$, we have*

$$\widehat{j}_*(\mathscr{F}_{/Y'}) = (j_*(\mathscr{F}))_{/X'}.$$

Proof. Since $Y' = j^{-1}(X')$, it suffices to use (10.9.9) and to apply (10.14.5) and (10.14.6). ☐

10.15. Separated formal preschemes.

Definition (10.15.1). — Let $\mathfrak{S}$ be a formal prescheme, $\mathfrak{X}$ a formal $\mathfrak{S}$-prescheme, and $f : \mathfrak{X} \to \mathfrak{S}$ the structure morphism. We define the diagonal morphism $\Delta_{\mathfrak{X}|\mathfrak{S}} : \mathfrak{X} \to \mathfrak{X} \times_{\mathfrak{S}} \mathfrak{X}$ (also denoted by $\Delta_{\mathfrak{X}}$) to be the morphism $(1_{\mathfrak{X}}, 1_{\mathfrak{X}})_{\mathfrak{S}}$. We say that $\mathfrak{X}$ is separated over $\mathfrak{S}$, or is a formal $\mathfrak{S}$-scheme, or that f is a separated morphism, if the image of the underlying space of $\mathfrak{X}$ under $\Delta_{\mathfrak{X}}$ is a closed subset of the underlying space of $\mathfrak{X} \times_{\mathfrak{S}} \mathfrak{X}$. We say that a formal prescheme $\mathfrak{X}$ is separated, or is a formal scheme, if it is separated over **Z**.

Proposition (10.15.2). — *Suppose that the formal preschemes $\mathfrak{S}$ and $\mathfrak{X}$ are inductive limits of sequences (S_n) and (X_n) (respectively) of usual preschemes, and that the morphism $f : \mathfrak{X} \to \mathfrak{S}$ is the inductive limit of a sequence of morphisms $f_n : X_n \to S_n$. For f to be separated, it is necessary and sufficient for the morphism $f_0 : X_0 \to S_0$ to be separated.*

Proof. Indeed, $\Delta_{\mathfrak{X}|\mathfrak{S}}$ is then the inductive limit of the sequence of morphisms $\Delta_{X_n|S_n}$ (10.7.4), and the image of the underlying space of $\mathfrak{X}$ (resp. of $\mathfrak{X} \times_{\mathfrak{S}} \mathfrak{X}$ under $\Delta_{\mathfrak{X}|\mathfrak{S}}$) is identical to the image of the underlying space of X_0 (resp. of $X_0 \times_{S_0} X_0$) under $\Delta_{X_0|S_0}$; whence the conclusion. □

Proposition (10.15.3). — *Suppose that all the formal preschemes (resp. morphisms of formal preschemes) in what follows are inductive limits of sequences of usual preschemes (resp. of morphisms of usual preschemes).*

(i) *The composition of any two separated morphisms is separated.*
(ii) *If $f : \mathfrak{X} \to \mathfrak{X}'$ and $g : \mathfrak{Y} \to \mathfrak{Y}'$ are separated $\mathfrak{S}$-morphisms, then $f \times_{\mathfrak{S}} g$ is separated.*
(iii) *If $f : \mathfrak{X} \to \mathfrak{Y}$ is a separated $\mathfrak{S}$-morphism, then the $\mathfrak{S}'$-morphism $f_{(\mathfrak{S}')}$ is separated for every extension $\mathfrak{S}' \to \mathfrak{S}$ of the base formal prescheme.*
(iv) *If the composition $g \circ f$ of two morphisms is separated, then f is separated.*

(In the above, it is implicit that if the same formal prescheme $\mathfrak{Z}$ is mentioned more than once in the same proposition, we consider it as the inductive limit of the *same* sequence (Z_n) of usual preschemes wherever it is mentioned, and the morphisms from $\mathfrak{Z}$ to another formal prescheme (resp. from a formal prescheme to $\mathfrak{Z}$) as inductive limits of morphisms from Z_n to some usual preschemes (resp. from some usual preschemes to Z_n)).

Proof. With the notation of (10.15.2), we have, in fact, that $(g \circ f)_0 = g_0 \circ f_0$ and $(f \times_{\mathfrak{S}} g)_0 = f_0 \times_{S_0} g_0$; the claims of (10.15.3) are then immediate consequences of (10.15.2) and the corresponding claims in (5.5.1) for usual preschemes. □

We leave it to the reader to state, for the same type of formal preschemes and morphisms as in (10.15.3), the propositions corresponding to (5.5.5), (5.5.9), and (5.5.10) (by replacing "affine open subset" by "formal affine open subset satisfying condition (b) of (10.6.3)"). **I** | 213

A similar argument also shows that every *Noetherian* formal affine scheme is separated, which justifies the terminology.

Proposition (10.15.4). — *Let $\mathfrak{S}$ be a locally Noetherian formal prescheme, and $\mathfrak{X}$ and $\mathfrak{Y}$ locally Noetherian formal $\mathfrak{S}$-preschemes such that $\mathfrak{X}$ or $\mathfrak{Y}$ is of finite type over $\mathfrak{S}$ (so that $\mathfrak{X} \times_{\mathfrak{S}} \mathfrak{Y}$ is locally Noetherian) and such that $\mathfrak{Y}$ is separated over $\mathfrak{S}$. Let $f : \mathfrak{X} \to \mathfrak{Y}$ be an $\mathfrak{S}$-morphism; then the graph morphism $\Gamma_f(1_{\mathfrak{X}}, f)_{\mathfrak{S}} : \mathfrak{X} \to \mathfrak{X} \times_{\mathfrak{S}} \mathfrak{Y}$ is a closed immersion.*

Proof. We can assume that $\mathfrak{S}$ is the inductive limit of a sequence (S_n) of locally Noetherian preschemes, $\mathfrak{X}$ (resp. $\mathfrak{Y}$) the inductive limit of a sequence (X_n) (resp. (Y_n)) of S_n-preschemes, and f the inductive limit of a sequence $(f_n : X_n \to Y_n)$ of S_n-morphisms; then $\mathfrak{X} \times_{\mathfrak{S}} \mathfrak{Y}$ is the inductive limit of the sequence $(X_n \times_{S_n} Y_n)$, and Γ_f the inductive limit of the sequence (Γ_{f_n}) (10.7.4); by hypothesis, Y_0 is separated over S_0 (10.15.2), so the space $\Gamma_{f_0}(X_0)$ is a closed subspace of $X_0 \times_{S_0} Y_0$; since the underlying spaces of $\mathfrak{X} \times_{\mathfrak{S}} \mathfrak{Y}$ (resp. $\Gamma_f(\mathfrak{X})$) and $X_0 \times_{S_0} Y_0$ (resp. $\Gamma_{f_0}(X_0)$) are identical, we already see that $\Gamma_f(\mathfrak{X})$ is a *closed* subspace of $\mathfrak{X} \times_{\mathfrak{S}} \mathfrak{Y}$. Now note that, when (U, V) runs over the set of pairs consisting of a Noetherian formal affine open subset U (resp. V) of $\mathfrak{X}$ (resp $\mathfrak{Y}$) such that $f(U) \subset V$, the open subsets $U \times_S V$ form a cover of $\Gamma_f(\mathfrak{X})$ in $\mathfrak{X} \times_{\mathfrak{S}} \mathfrak{Y}$, and, if $f_U : U \to V$ is the restriction of f to U, then $\Gamma_{f_U} : U \to U \times_{\mathfrak{S}} V$ is the restriction of Γ_f to U. If we show that Γ_{f_U} is a closed immersion, then Γ_f will be a closed immersion (10.14.4), or, in other words, we are led to consider the case where $\mathfrak{S} = \operatorname{Spf}(A)$, $\mathfrak{X} = \operatorname{Spf}(B)$, and $\mathfrak{Y} = \operatorname{Spf}(C)$ are affine (with A, B, and C Noetherian adics), with f corresponding to a continuous A-homomorphism $\varphi : C \to B$; then Γ_f corresponds to the unique continuous homomorphism $\omega : B \widehat{\otimes}_A C \to B$ which, when composed with the canonical homomorphisms $B \to B \widehat{\otimes}_A C$ and $C \to B \widehat{\otimes}_A C$, gives the identity and φ (respectively). But it is clear that ω is *surjective*, whence our claim. □

Corollary (10.15.5). — *Let $\mathfrak{S}$ be a locally Noetherian formal prescheme, and $\mathfrak{X}$ an $\mathfrak{S}$-prescheme of finite type; for $\mathfrak{X}$ to be separated over $\mathfrak{S}$, it is necessary and sufficient for the diagonal morphism $\mathfrak{X} \to \mathfrak{X} \times_{\mathfrak{S}} \mathfrak{X}$ to be a closed immersion.*

Proposition (10.15.6). — *A closed immersion $j : \mathfrak{Y} \to \mathfrak{X}$ of locally Noetherian formal preschemes is a separated morphism.*

Proof. With the notation of (10.14.2), $j_0 : Y_0 \to X_0$ is a closed immersion, thus a separated morphism, and so it suffices to apply (10.15.2). □

Proposition (10.15.7). — *Let X be a locally Noetherian (usual) prescheme, X' a closed subset of X, and $\widehat{X} = X_{/X'}$. For $\widehat{X}$ to be separated, it is necessary and sufficient that $\widehat{X}_{\text{red}}$ be separated, and it is sufficient that X be separated.*

Proof. With the notation of (10.8.5), for $\widehat{X}$ to be separated, it is necessary and sufficient for X'_0 to be separated (10.15.2), and since $\widehat{X}_{\text{red}} = (X'_0)_{\text{red}}$, it is equivalent to ask for $\widehat{X}_{\text{red}}$ to be separated (5.5.1, vi). □

CHAPTER II

Elementary global study of some classes of morphisms (EGA II)

Summary

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The various classes of morphisms studied in this chapter are used extensively in cohomological methods; further study, using these methods, will be done in Chapter III, where we use especially §§2, 4, and 5 of Chapter II. Section §8 can be omitted on a first reading: it gives some supplements to the formalism developed in §§1 and 3, reducing to easy applications of this formalism, and we will use it less consistently than the other results of this chapter.

§1. Affine morphisms

1.1. S-preschemes and $\mathscr{O}_S$-algebras.

(1.1.1). Let S be a prescheme, X an S-prescheme, and $f : X \to S$ its structure morphism. We know (**0**, 4.2.4) that the direct image $f_*(\mathscr{O}_X)$ is an $\mathscr{O}_S$-algebra, which we denote $\mathscr{A}(X)$ when there is little chance of confusion; if **II** | 6
U is an open subset of S, then we have

$$\mathscr{A}(f^{-1}(U)) = \mathscr{A}(X)|U.$$

Similarly, for every $\mathscr{O}_X$-module $\mathscr{F}$ (resp. every $\mathscr{O}_X$-algebra $\mathscr{B}$), we write $\mathscr{A}(\mathscr{F})$ (resp. $\mathscr{A}(\mathscr{B})$) for the direct image $f_*(\mathscr{F})$ (resp. $f_*(\mathscr{B})$) which is an $\mathscr{A}(X)$-module (resp. an $\mathscr{A}(X)$-algebra) and not only an $\mathscr{O}_S$-module (resp. an $\mathscr{O}_S$-algebra).

(1.1.2). Let Y be a second S-prescheme, $g : Y \to S$ its structure morphism, and $h : X \to Y$ an S-morphism; we then have the commutative diagram

(1.1.2.1)
$$\begin{tikzcd} X \arrow[rr, "h"] \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\ & S. & \end{tikzcd}$$

We have by definition $h = (\psi, \theta)$, where $\theta : \mathscr{O}_Y \to h_*(\mathscr{O}_X) = \psi_*(\mathscr{O}_X)$ is a homomorphism of sheaves of rings; we induce (**0**, 4.2.2) a homomorphism of $\mathscr{O}_S$-algebras $g_*(\theta) : g_*(\mathscr{O}_Y) \to g_*(h_*(\mathscr{O}_X)) = f_*(\mathscr{O}_X)$, in other words, a homomorphism of $\mathscr{O}_S$-algebras $\mathscr{A}(Y) \to \mathscr{A}(X)$, which we denote by $\mathscr{A}(h)$. If $h' : Y \to Z$ is a second S-morphism, then it is immediate that $\mathscr{A}(h' \circ h) = \mathscr{A}(h) \circ \mathscr{A}(h')$. We havve thus define a *contravariant functor* $\mathscr{A}(X)$ from the category of S-preschemes to the category of $\mathscr{O}_S$-algebras.

Now let $\mathscr{F}$ be an $\mathscr{O}_X$-module, $\mathscr{G}$ an $\mathscr{O}_Y$-module, and $u : \mathscr{G} \to \mathscr{F}$ an h-morphism, that is (**0**, 4.4.1) a homomorphism of $\mathscr{O}_Y$-modules $\mathscr{G} \to h_*(\mathscr{F})$. Then $g_*(u) : g_*(\mathscr{G}) \to g_*(h_*(\mathscr{F})) = f_*(\mathscr{F})$ is a homomorphism $\mathscr{A}(\mathscr{G}) \to \mathscr{A}(\mathscr{F})$ of $\mathscr{O}_S$-modules, which we denote by $\mathscr{A}(u)$; in addition, the pair $(\mathscr{A}(h), \mathscr{A}(u))$ form a *di-homomorphism* from the $\mathscr{A}(Y)$-module $\mathscr{A}(\mathscr{G})$ to the $\mathscr{A}(X)$-module $\mathscr{A}(\mathscr{F})$.

(1.1.3). If we fix the prescheme S, then we can consider the pairs $(X, \mathscr{F})$, where X is an S-prescheme and $\mathscr{F}$ is an $\mathscr{O}_X$-module, as forming a *category*, by defining a *morphism* $(X, \mathscr{F}) \to (Y, \mathscr{G})$ as a pair (h, u), where $h : X \to Y$ is an S-morphism and $u : \mathscr{G} \to \mathscr{F}$ is an h-morphism. We can then say that $(\mathscr{A}(X), \mathscr{A}(\mathscr{F}))$ is a *contravariant functor* with values in the category whose objects are pairs consisting of an $\mathscr{O}_S$-algebra and a module over that algebra, and the morphisms are the di-homomorphisms.

1.2. Affine preschemes over a prescheme.

Definition (1.2.1). — Let X be an S-prescheme, $f : X \to S$ its structure morphism. We say that X is *affine over* S if there exists a cover (S_α) of S by affine open sets such that for all α, the induced prescheme on X by the open set $f^{-1}(S_\alpha)$ is affine.

Example (1.2.2). — Every closed subprescheme of S is an affine S-prescheme over S ((**I**, 4.2.3) and (**I**, 4.2.4)).

Remark (1.2.3). — An affine prescheme X over S is not necessarily an affine scheme, as the example $X = S$ shows (1.2.2). On the other hand, if an affine scheme X is an S-prescheme, then X is not necessarily affine over S (see Example (1.3.3)). However, remember that if S is a *scheme*, then every S-prescheme which is an affine scheme is affine over S (**I**, 5.5.10). **II** | 7

Proposition (1.2.4). — *Every S-prescheme which is affine over S is separated over S (in other words, it is an S-scheme).*

Proof. This follows immediately from (**I**, 5.5.5) and (**I**, 5.5.8). ☐

Proposition (1.2.5). — *Let X be an S-scheme affine over S, $f : X \to S$ its structure morphism. For every open $U \subset S$, $f^{-1}(U)$ is affine over U.*

Proof. By Definition (1.2.1), we can reduce to the case where $S = \operatorname{Spec}(A)$ and $X = \operatorname{Spec}(B)$ are affine; then $f = ({}^a\varphi, \widetilde{\varphi})$, where $\varphi : A \to B$ is a homomorphism. As the $D(g)$ for $g \in A$ form a basis for S, we reduce to the case where $U = D(g)$; but we then know that $f^{-1}(U) = D(\varphi(g))$ (**I**, 1.2.2.2), hence the proposition. ☐

Proposition (1.2.6). — *Let X be an S-scheme affine over S, $f : X \to S$ its structure morphism. For every quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$, $f_*(\mathscr{F})$ is a quasi-coherent $\mathscr{O}_S$-module.*

Proof. Taking into account Proposition (1.2.4), this follows from (**I**, 9.2.2, a). ☐

In particular, the $\mathscr{O}_S$-algebra $\mathscr{A}(X) = f_*(\mathscr{O}_X)$ is *quasi-coherent.*

Proposition (1.2.7). — *Let X be an S-scheme affine over S. For every S-prescheme Y, the map $h \mapsto \mathscr{A}(h)$ from the set* $\operatorname{Hom}_S(Y, X)$ *to the set* $\operatorname{Hom}(\mathscr{A}(X), \mathscr{A}(Y))$ (1.1.2) *is bijective.*

Proof. Let $f : X \to S$ and $g : Y \to S$ be the structure morphisms. First, suppose that $S = \operatorname{Spec}(A)$ and $X = \operatorname{Spec}(B)$ are affine; we must prove that for every homomorphism $\omega : f_*(\mathscr{O}_X) \to g_*(\mathscr{O}_Y)$ of $\mathscr{O}_S$-algebras, there exists a unique S-morphism $h : Y \to X$ such that $\mathscr{A}(h) = \omega$. By definition, for every open $U \subset S$, ω defines a homomorphism $\omega_U = \Gamma(U, \omega) : \Gamma(f^{-1}(U), \mathscr{O}_X) \to \Gamma(g^{-1}(U), \mathscr{O}_Y)$ of $\Gamma(U, \mathscr{O}_S)$-algebras. In particular, for $U = S$, this gives a homomorphism $\varphi : \Gamma(X, \mathscr{O}_X) \to \Gamma(Y, \mathscr{O}_Y)$ of $\Gamma(S, \mathscr{O}_S)$-algebras, to which corresponds a well-defined S-morphism $h : Y \to X$, since X is affine (**I**, 2.2.4). It remains to prove that $\mathscr{A}(h) = \omega$, in other words, for every open set U of a basis for S, ω_U coincides with the homomorphism of algebras φ_U corresponding to the S-morphism $g^{-1}(U) \to f^{-1}(U)$, a restiction of h. We can reduce to the case where $U = D(\lambda)$, with $\lambda \in S$; then, if $f = ({}^a\rho, \widetilde{\rho})$, where $\rho : A \to B$ is a ring homomorphism, we have $f^{-1}(U) = D(\mu)$, where $\mu = \rho(\lambda)$, and $\Gamma(f^{-1}(U), \mathscr{O}_X)$ is the ring of fractions B_μ; the diagram

$$\begin{tikzcd} B \arrow[r, "\varphi"] \arrow[d] & \Gamma(Y, \mathscr{O}_Y) \arrow[d] \\ B_\mu \arrow[r, "\varphi_U"] & \Gamma(g^{-1}(U), \mathscr{O}_Y) \end{tikzcd}$$

is commutative, and so is the analogous diagram where φ_U is replaced by ω_U; the equality $\varphi_U = \omega_U$ then follows from the universal property of rings of fractions (**0**, 1.2.4).

We now pass to the general case; let (S_α) be a cover of S by affine open sets such that the $f^{-1}(S_\alpha)$ are affine. **II** | 8
Then every homomorphism $\omega : \mathscr{A}(X) \to \mathscr{A}(Y)$ of $\mathscr{O}_S$-algebras gives by restriction a family of homomorphisms

$$\omega_\alpha : \mathscr{A}(f^{-1}(S_\alpha)) \longrightarrow \mathscr{A}(g^{-1}(S_\alpha))$$

of $\mathscr{O}_{S_\alpha}$-algebras, hence a family of S_α-morphisms $h_\alpha : g^{-1}(S_\alpha) \to f^{-1}(S_\alpha)$ by the above. It remains to see that for every affine open set of a basis for $S_\alpha \cap S_\beta$, the restriction of h_α and h_β to $g^{-1}(U)$ coincide, which is evident since by the above, these restrictions both correspond to the homomorphism $\mathscr{A}(X)|U \to \mathscr{A}(Y)|U$, a restriction of ω. □

Corollary (1.2.8). — *Let X and Y be two S-schemes which are affine over S. For an S-morphism $h : Y \to X$ to be an isomorphism, it is necessary and sufficient for $\mathscr{A}(h) : \mathscr{A}(X) \to \mathscr{A}(Y)$ to be an isomorphism.*

Proof. This follows immediately from Proposition (1.2.7) and from the functorial nature of $\mathscr{A}(X)$. □

1.3. Affine preschemes over S associated to an $\mathscr{O}_S$-algebra.

Proposition (1.3.1). — *Let S be a prescheme. For every quasi-coherent $\mathscr{O}_S$-algebra $\mathscr{B}$, there exists a prescheme X affine over S, defined up to unique S-isomorphism, such that $\mathscr{A}(X) = \mathscr{B}$.*

Proof. The uniqueness follows from Corollary (1.2.8); we prove the existence of X. For every affine open $U \subset S$, let X_U be the prescheme $\operatorname{Spec}(\Gamma(U, \mathscr{B}))$; as $\Gamma(U, \mathscr{B})$ is a $\Gamma(U, \mathscr{O}_S)$-algebras, X_U is an S-prescheme (**I**, 1.6.1). In addition, as $\mathscr{B}$ is quasi-coherent, the $\mathscr{O}_S$-algebra $\mathscr{A}(X_U)$ canonically identifies with $\mathscr{B}|U$ ((**I**, 1.3.7), (**I**, 1.3.13), (**I**, 1.6.3)). Let V be a second affine open subset of S, and let $X_{U,V}$ be the prescheme induced by X_U on $f_U^{-1}(U \cap V)$, where f_U denotes the structure morphism $X_U \to S$; $X_{U,V}$ and $X_{V,U}$ are affine over $U \cap V$ (1.2.5), and by definition $\mathscr{A}(X_{U,V})$ and $\mathscr{A}(X_{V,U})$ canonically identify with $\mathscr{B}|(U \cap V)$. Hence there is (1.2.8) a canonical S-isomorphism $\theta_{U,V} : X_{U,V} \to X_{U,V}$; in addition, if W is a third affine open subset of S, and if $\theta'_{U,V}$, $\theta'_{V,W}$, and $\theta'_{U,W}$ are the restrictions of $\theta_{U,V}$, $\theta_{V,W}$, and $\theta_{U,W}$ to the inverse images of $U \cap V \cap W$ in X_V, X_W, and X_W respectively under the structure morphisms, then we have $\theta'_{U,V} \circ \theta'_{V,W} = \theta'_{U,W}$. As a result, there exists a prescheme X, a cover (T_U) of X by affine open sets, and for every U an isomorphism $\varphi_U : X_U \to T_U$, such that φ_U maps $f_U^{-1}(U \cap V)$ to $T_U \cap T_V$, and we have $\theta_{U,V} = \varphi_U^{-1} \circ \varphi_V$ (**I**, 2.3.1). The morphism $g_U = f_U \circ \varphi_U^{-1}$ makes T_U an S-prescheme, and the morphisms g_U and g_V coincide on $T_U \cap T_V$, hence X is an S-prescheme. In addition, it is clear by definition that X is affine over S and that $\mathscr{A}(T_U) = \mathscr{B}|U$, hence $\mathscr{A}(X) = \mathscr{B}$. □

We say that the S-scheme X defined in this way is *associated to the $\mathscr{O}_S$-algebra $\mathscr{B}$*, or is the *spectrum of $\mathscr{B}$*, and we denote it by $\operatorname{Spec}(\mathscr{B})$.

Corollary (1.3.2). — *Let X be a prescheme affine over S, $f : X \to S$ the structure morphism. For every affine open $U \subset S$, the induced prescheme on $f^{-1}(U)$ is the affine scheme with ring $\Gamma(U, \mathscr{A}(X))$.* **II** | 9

Proof. As we can suppose that X is associated to an $\mathscr{O}_S$-algebra by Propositions (1.2.6) and (1.3.1), the corollary follows from the construction of X described in Proposition (1.3.1). □

Example (1.3.3). — Let S be the affine plane over a field K, where the point 0 has been doubled (**I**, 5.5.11); with the notation of (**I**, 5.5.11), S is the union of two affine open sets Y_1 and Y_2; if f is the open immersion $Y_1 \to S$, then $f^{-1}(Y_2) = Y_1 \cap Y_2$ is not an affine open set in Y_1 (*loc. cit.*), hence we have an example of an affine scheme which is not affine over S.

Corollary (1.3.4). — *Let S be an affine scheme; for an S-prescheme X to be affine over S, it is necessary and sufficient for X to be an affine scheme.*

Corollary (1.3.5). — *Let X be a prescheme affine over a prescheme S, and let Y be an X-prescheme. For Y to be affine over S, it is necessary and sufficient for Y to be affine over X.*

Proof. We immediately reduce to the case where S is an affine scheme, and then we can reduce to the case where X is an affine scheme (1.3.4); the two conditions of the statement then give that Y is an affine scheme (1.3.4). □

(1.3.6). Let X be a prescheme affine over S. To define a prescheme Y affine *over* X, it is equivalent, by Corollary (1.3.5), to give a prescheme Y affine *over* S, and an S-morphism $g : Y \to X$; in other words (Proposition (1.3.1) and (1.2.7)), it is equivalent to give a quasi-coherent $\mathscr{O}_S$-algebra $\mathscr{B}$ and a homomorphism $\mathscr{A}(X) \to \mathscr{B}$ of $\mathscr{O}_S$-algebras (which can be considered as defining on $\mathscr{B}$ an $\mathscr{A}(X)$-algebra structure). If $f : X \to S$ is the structure morphism, then we have $\mathscr{B} = f_*(g_*(\mathscr{O}_Y))$.

Corollary (1.3.7). — *Let X be a prescheme affine over S; for X to be of finite type over S, it is necessary and sufficient for the quasi-coherent $\mathscr{O}_S$-algebra $\mathscr{A}(X)$ to be of finite type* (**I**, 9.6.2).

Proof. By definition (**I**, 9.6.2), we can reduce to the case where S is affine; then X is an affine scheme (1.3.4), and if $S = \operatorname{Spec}(A)$, $X = \operatorname{Spec}(B)$, then $\mathscr{A}(X)$ is the $\mathscr{O}_S$-algebra $\widetilde{B}$; as $\Gamma(U, \widetilde{B}) = B$, the corollary follows from (**I**, 9.6.2) and (**I**, 6.3.3). □

Corollary (1.3.8). — *Let X be a prescheme affine over S; for X to be reduced, it is necessary and sufficient for the quasi-coherent $\mathscr{O}_X$-algebra $\mathscr{A}(X)$ to be reduced* (**0**, 4.1.4).

Proof. The question is local on S; by Corollary (1.3.2), the corollary follows from (**I**, 5.1.1) and (**I**, 5.1.4). □

1.4. Quasi-coherent sheaves over a prescheme affine over S.

Proposition (1.4.1). — *Let X be a prescheme affine over S, Y an S-prescheme, and $\mathscr{F}$ (resp. $\mathscr{G}$) a quasi-coherent $\mathscr{O}_X$-module (resp. an $\mathscr{O}_Y$-module). Then the map $(h,u) \mapsto (\mathscr{A}(h), \mathscr{A}(u))$ from the set of morphism $(Y,\mathscr{G}) \to (X,\mathscr{F})$ to the set of di-homomorphisms $(\mathscr{A}(X),\mathscr{A}(\mathscr{F})) \to (\mathscr{A}(Y),\mathscr{A}(\mathscr{G}))$ ((1.1.2) and (1.1.3)) is bijective.*

Proof. The proof follows exactly as that of Proposition (1.2.7) by using (**I**, 2.2.5) and (**I**, 2.2.4), and the details are left to the reader. □

Corollary (1.4.2). — *If, in addition to the hypotheses of Proposition (1.4.1), we suppose that Y is affine over S, then for (h,u) to be an isomorphism, it is necessary and sufficient for $(\mathscr{A}(h),\mathscr{A}(u))$ to be a di-isomorphism.*

Proposition (1.4.3). — *For every pair $(\mathscr{B},\mathscr{M})$ consisting of a quasi-coherent $\mathscr{O}_S$-algebra $\mathscr{B}$ and a quasi-coherent $\mathscr{B}$-module $\mathscr{M}$* (considered as an $\mathscr{O}_S$-module or as a $\mathscr{B}$-module, which are equivalent (**I**, 9.6.1)), *there exists a pair $(X,\mathscr{F})$ consisting of a prescheme X affine over S and of a quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$, such that $\mathscr{A}(X) = \mathscr{B}$ and $\mathscr{A}(\mathscr{F}) = \mathscr{M}$; in addition, this couple is determined up to unique isomorphism.* II | 10

Proof. The uniqueness follows from Proposition (1.4.1) and Corollary (1.4.2); the existence is proved as in Proposition (1.3.1), and we leave the details to the reader. □

We denote by $\widetilde{\mathscr{M}}$ the $\mathscr{O}_X$-module $\mathscr{F}$, and we say that it is *associated* to the quasi-coherent $\mathscr{B}$-module $\mathscr{M}$; for every affine open $U \subset S$, $\mathscr{M}|p^{-1}(U)$ (where p is the structure morphism $X \to S$) is canonically isomorphic to $(\Gamma(U,\mathscr{M}))^{\sim}$.

Corollary (1.4.4). — *On the category of quasi-coherent $\mathscr{B}$-modules, $\widetilde{\mathscr{M}}$ is an additive covariant exact functor in $\mathscr{M}$, which commutes with inductive limitd and direct sums.*

Proof. We immediately reduce to the case where S is affine, and the corollary then follows from (**I**, 1.3.5), (**I**, 1.3.9), and (**I**, 1.3.11). □

Corollary (1.4.5). — *Under the hypotheses of Proposition (1.4.3), for $\widetilde{\mathscr{M}}$ to be an $\mathscr{O}_X$-module of finite type, it is necessary and sufficient for $\mathscr{M}$ to be a $\mathscr{B}$-module of finite type.*

Proof. We immediately reduce to the case where $S = \operatorname{Spec}(A)$ is an affine scheme. Then $\mathscr{B} = \widetilde{B}$, where B is an A-algebra of finite type (**I**, 9.6.2), and $\mathscr{M} = \widetilde{M}$, where M is a B-module (**I**, 1.3.13); *over the prescheme X*, $\mathscr{O}_X$ is associated to the ring B and $\widetilde{\mathscr{M}}$ to the B-module M; for $\widetilde{\mathscr{M}}$ to be of finite type, it is therefore necessary and sufficient for M to be of finite type (**I**, 1.3.13), hence our assertion. □

Proposition (1.4.6). — *Let Y be a prescheme affine over S, X and X' two preschemes affine over Y* (hence also over S (1.3.5))*. Let $\mathscr{B} = \mathscr{A}(Y)$, $\mathscr{A} = \mathscr{A}(X)$, and $\mathscr{A}' = \mathscr{A}(X')$. Then $X \times_Y X'$ is affine over Y* (thus also over S)*, and $\mathscr{A}(X \times_Y X')$ canonically identifies with $\mathscr{A} \otimes_{\mathscr{B}} \mathscr{A}'$.*

Proof. By (**I**, 9.6.1), $\mathscr{A} \otimes_{\mathscr{B}} \mathscr{A}'$ is a quasi-coherent $\mathscr{B}$-algebra, thus also a quasi-coherent $\mathscr{O}_S$-algebra (**I**, 9.6.1); let Z be the spectrum of $\mathscr{A} \otimes_{\mathscr{B}} \mathscr{A}'$ (1.3.1). The canonical $\mathscr{B}$-homomorphisms $\mathscr{A} \to \mathscr{A} \otimes_{\mathscr{B}} \mathscr{A}'$ and $\mathscr{A}' \to \mathscr{A} \otimes_{\mathscr{B}} \mathscr{A}'$ correspond (1.2.7) to Y-morphisms $Z \to X$ and $p' : Z \to X'$. To see that the triple (Z,p,p') is a product $X \times_Y X'$, we can reduce to the case where S is an affine scheme with ring C (**I**, 3.2.6.4). But then Y, X, and X' are affine schemes (1.3.4) whose rings B, A, and A' are C-algebras such that $\mathscr{B} = \widetilde{B}$, $\mathscr{A} = \widetilde{A}$, and $\mathscr{A}' = \widetilde{A'}$. We then know (**I**, 1.3.13) that $\mathscr{A} \otimes_{\mathscr{B}} \mathscr{A}'$ canonically identifies with the $\mathscr{O}_S$-algebra $(A \otimes_B A')^{\sim}$, hence the ring $A(Z)$ identifies with $A \otimes_B A'$ and the morphisms p and p' correspond to the canonical homomorphisms $A \to A \otimes_B A'$ and $A' \to A \otimes_B A'$. The proposition then follows from (**I**, 3.2.2). □

Corollary (1.4.7). — *Let $\mathscr{F}$ (resp. $\mathscr{F}'$) be a quasi-coherent $\mathscr{O}_X$-module (resp. $\mathscr{O}_{X'}$-module); then $\mathscr{A}(\mathscr{F} \otimes_Y \mathscr{F}')$ canonically identifies with $\mathscr{A}(\mathscr{F}) \otimes_{\mathscr{A}(Y)} \mathscr{A}(\mathscr{F}')$.*

Proof. We know that $\mathscr{F} \otimes_Y \mathscr{F}'$ is quasi-coherent over $X \times_Y X'$ (**I**, 9.1.2). Let $g : Y \to S$, $f : X \to Y$, and $f' : X' \to Y$ be the structure morphisms, such that the structure morphism $h : Z \to S$ is equal to $g \circ f \circ p$ and to $g \circ f' \circ p'$. We define a canonical homomorphism II | 11

$$\mathscr{A}(\mathscr{F}) \otimes_{\mathscr{A}(Y)} \mathscr{A}(\mathscr{F}') \longrightarrow \mathscr{A}(\mathscr{F} \otimes_Y \mathscr{F}')$$

in the following way: for every open $U \subset S$, we have canonical homomorphisms $\Gamma(f^{-1}(g^{-1}(U)),\mathscr{F}) \to \Gamma(h^{-1}(U),p^*(\mathscr{F}))$ and $\Gamma(f'^{-1}(g^{-1}(U)),\mathscr{F}') \to \Gamma(h^{-1}(U),p'^*(\mathscr{F}'))$ (**0**, 4.4.3), thus we obtain a canonical homomorphism

$$\Gamma(f^{-1}(g^{-1}(U)),\mathscr{F}) \otimes_{\Gamma(g^{-1}(U),\mathscr{O}_Y)} \Gamma(f'^{-1}(g^{-1}(U)),\mathscr{F}') \longrightarrow \Gamma(h^{-1}(U),p^*(\mathscr{F})) \otimes_{\Gamma(h^{-1}(U),\mathscr{O}_Z)} \Gamma(h^{-1}(U),p'^*(\mathscr{F}')).$$

To see that we have defined an isomorphism of $\mathscr{A}(Z)$-modules, we can reduce to the case where S (and as a result X, X', Y, and $X\times_Y X'$) are affine scheme, and (with the notation of Proposition (1.4.6)), $\mathscr{F} = \widetilde{M}$, $\mathscr{F}' = \widetilde{M'}$, where M (resp. M') is an A-module (resp. an A'-module). Then $\mathscr{F}\otimes_Y \mathscr{F}'$ identifies with the sheaf on $X\times_Y X'$ associated to the $(A\otimes_B A')$-module $M\otimes_B M'$ (**I**, 9.1.3), and the corollary follows from the canonical identification of the $\mathscr{O}_S$-modules $(M\otimes_B M')^\sim$ and $\widetilde{M}\otimes_{\widetilde{B}} \widetilde{M'}$ (where M, M', and B are considered as C-modules) ((**I**, 1.3.12) and (**I**, 1.6.3)). ☐

If we apply Corollary (1.4.7) in particular to the case where $X = Y$ and $\mathscr{F}' = \mathscr{O}_{X'}$, then we see that the $\mathscr{A}'$-module $\mathscr{A}(f'^*(\mathscr{F}))$ identifies with $\mathscr{A}(\mathscr{F})\otimes_{\mathscr{B}}\mathscr{A}'$.

(1.4.8). In particular, when $X = X' = Y$ (X being affine over S), we see that if $\mathscr{F}$ and $\mathscr{G}$ are two quasi-coherent $\mathscr{O}_X$-modules, then we have

$$\mathscr{A}(\mathscr{F}\otimes_{\mathscr{O}_X}\mathscr{G}) = \mathscr{A}(\mathscr{F})\otimes_{\mathscr{A}(X)}\mathscr{A}(\mathscr{G}) \tag{1.4.8.1}$$

up to canonical functorial isomorphism. If in addition $\mathscr{F}$ admits a finite presentation, then it follows from (**I**, 1.6.3) and (**I**, 1.3.12) that

$$\mathscr{A}(\mathscr{H}om_X(\mathscr{F},\mathscr{G})) = \mathscr{H}om_{\mathscr{A}(X)}(\mathscr{A}(\mathscr{F}),\mathscr{A}(\mathscr{G})) \tag{1.4.8.2}$$

up to canonical isomorphism.

Remark (1.4.9). — If X and X' are two preschemes affine over S, then the sum $X\sqcup X'$ is also affine over S, as the sum of two affine schemes is an affine scheme.

Proposition (1.4.10). — *Let S be a prescheme, $\mathscr{B}$ a quasi-coherent $\mathscr{O}_S$-algebra, and $X = \operatorname{Spec}(\mathscr{B})$. For a quasi-coherent sheaf of ideals $\mathscr{J}$ of $\mathscr{B}$, $\widetilde{\mathscr{J}}$ is quasi-coherent sheaf of ideals of $\mathscr{O}_X$, and the closed subprescheme Y of X defined by $\widetilde{\mathscr{J}}$ is canonically isomorphic to* $\operatorname{Spec}(\mathscr{B}/\mathscr{J})$.

Proof. It follows immediately from (**I**, 4.2.3) that Y is affine over S; by Proposition (1.3.1), we reduce to the case where S is affine, and the proposition then follows immediately from (**I**, 4.1.2). ☐

We can also express the result of Proposition (1.4.10) by saying that if $h : \mathscr{B}\to\mathscr{B}'$ is a *surjective* homomorphism of quasi-coherent $\mathscr{O}_S$-algebras, $\mathscr{A}(h) : \operatorname{Spec}(\mathscr{B}')\to\operatorname{Spec}(\mathscr{B})$ is a *closed immersion.*

Proposition (1.4.11). — *Let S be a prescheme, $\mathscr{B}$ a quasi-coherent $\mathscr{O}_S$-algebra, and $X = \operatorname{Spec}(\mathscr{B})$. For every quasi-coherent sheaf of ideals $\mathscr{K}$ of $\mathscr{O}_S$, we have (denoting by f the structure morphism $X\to S$) $f^*(\mathscr{K})\mathscr{O}_X = (\mathscr{K}\mathscr{B})^\sim$ up to canonical isomorphism.* **II** | 12

Proof. The question being local on S, we can reduce to the case where $S = \operatorname{Spec}(A)$ is affine, and in this case the proposition is none other than (**I**, 1.6.9). ☐

1.5. Change of base prescheme.

Proposition (1.5.1). — *Let X be a prescheme affine over S. For every extension $g : S'\to S$ of the base prescheme, $X' = X_{(S')} = X\times_S S'$ is affine over S'.*

Proof. If f' is the projection $X'\to S'$, then it suffices to prove that $f'^{-1}(U')$ is an affine open set for every affine open subset U' of S' such that $g(U')$ is contained in an affine open subset U of S (1.2.1); we can thus reduce to the case where S and S' are affine, and it suffices to prove that X' is then an affine scheme (1.3.4). But then (1.3.4) X is an affine scheme, and if A, A', and B are the rings of S, S', and X respectively, then we know that X' is the affine scheme with ring $A'\otimes_A B$ (**I**, 3.2.2). ☐

Corollary (1.5.2). — *Under the hypotheses of Proposition (1.5.1), let $f : X\to S$ be the structure morphism, $f' : X'\to S'$ and $g' : X'\to X$ the projections, such that the diagram*

$$\begin{tikzcd} X \arrow[d, "f"'] & X' \arrow[l, "g'"'] \arrow[d, "f'"] \\ S & S' \arrow[l, "g"] \end{tikzcd}$$

is commutative. For every quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$, there exists a canonical isomorphism of $\mathscr{O}_{S'}$-modules

(1.5.2.1) $$u : g^*(f_*(\mathscr{F})) \simeq f'_*(g'^*(\mathscr{F})).$$

In particular, there exists a canonical isomorphism from $\mathscr{A}(X')$ to $g^*(\mathscr{A}(X))$.

Proof. To define u, it suffices to define a homomorphism

$$v : f_*(\mathscr{F}) \longrightarrow g_*(f'_*(g'^*(\mathscr{F}))) = f_*(g'_*(g'^*(\mathscr{F})))$$

and to set $u = v^\sharp$ (**0**, 4.4.3). We take $v = f_*(\rho)$, where ρ is the canonical homomorphism $\mathscr{F} \to g'_*(g'^*(\mathscr{F}))$ (**0**, 4.4.3). To prove that u is an isomorphism, we can reduce to the case where S and S', hence X and X', are affine; with the notation of Proposition (1.5.1), we then have $\mathscr{F} = \widetilde{M}$, where M is a B-module. We then note immediately that $g^*(f_*(\mathscr{F}))$ and $f'_*(g'^*(\mathscr{F}))$ are both equal to the $\mathscr{O}_{S'}$-module associated to the A'-module $A' \otimes_A M$ (where M is considered as an A-module), and that u is the homomorphism associated to the identity ((**I**, 1.6.3), (**I**, 1.6.5), (**I**, 1.6.7)). □

Remark (1.5.3). — We do not have that Corollary (1.5.2) remains true when X is not assumed affine over S, even when $S' = \operatorname{Spec}(k(s))$ ($s \in S$) and $S' \to S$ is the canonical morphism (**I**, 2.4.5)—in which case X' is none other than the *fibre* $f^{-1}(s)$ (**I**, 3.6.2). In other words, when X is not affine over S, the operation "direct image of **II** | 13
quasi-coherent sheaves" does not commute with the operation of "passing to fibres". However, we will see in Chapter III (**III**, 4.2.4) a result in this sense, of an "asymptotic" nature, valid for *coherent* sheaves on X when f is proper (5.4) and S is Noetherian.

Corollary (1.5.4). — *For every prescheme X affine over S and every $s \in S$, the fibre $f^{-1}(s)$ (where f denoted the structure morphism $X \to S$) is an affine scheme.*

Proof. It suffices to apply Proposition (1.5.1) with $S' = \operatorname{Spec}(k(s))$ and to use Corollary (1.3.4). □

Corollary (1.5.5). — *Let X be an S-prescheme, S' a prescheme affine over S; then $X' = X_{(S')}$ is a prescheme affine over X. In addition, if $f : X \to S$ is the structure morphism, then there is a canonical isomorphism of $\mathscr{O}_X$-algebras $\mathscr{A}(X') \simeq f^*(\mathscr{A}(S'))$, and for every quasi-coherent $\mathscr{A}(S')$-module $\mathscr{M}$, a canonical di-isomorphism $f^*(\mathscr{M}) \simeq \mathscr{A}(f'^*(\widetilde{\mathscr{M}}))$, denoting by $f' = f_{(S')}$ the structure morphism $X' \to S'$.*

Proof. It suffices to swap the roles of X and S' in (1.5.1) and (1.5.2). □

(1.5.6). Now let S, S' be two preschemes, $q : S' \to S$ a morphism, $\mathscr{B}$ (resp. $\mathscr{B}'$) a quasi-coherent $\mathscr{O}_S$-algebra (resp. $\mathscr{O}_{S'}$-algebra), $u : \mathscr{B} \to \mathscr{B}'$ a q-morphism (that is, a homomorphism $\mathscr{B} \to q_*(\mathscr{B}')$ of $\mathscr{O}_S$-algebras). If $X = \operatorname{Spec}(\mathscr{B})$, $X' = \operatorname{Spec}(\mathscr{B}')$, then we canonically obtain a morphism

$$v = \operatorname{Spec}(u) : X' \longrightarrow X$$

such that the diagram

(1.5.6.1) $$\begin{tikzcd} X' \arrow[r, "v'"] \arrow[d] & X \arrow[d] \\ S' \arrow[r, "q"] & S \end{tikzcd}$$

is commutative (the vertical arrows being the structure morphisms). Indeed, the data of u is equivalent to that of a homomorphism of quasi-coheren $\mathscr{O}_{S'}$-algebras $u^\sharp : q^*(\mathscr{B}) \to \mathscr{B}'$ (**0**, 4.4.3); this thus canonically defines an S'-morphism

$$w : \operatorname{Spec}(\mathscr{B}') \longrightarrow \operatorname{Spec}(q^*(\mathscr{B}))$$

such that $\mathscr{A}(w) = u^\sharp$ (1.2.7). On the other hand, it follows from (1.5.2) that $\operatorname{Spec}(q^*(\mathscr{B}))$ canonically identifies with $X \times_S S'$; the morphism v is the composition $X' \xrightarrow{w} X \times_S S' \xrightarrow{p_1} X$ of w with the first projection, and the commutativity of (1.5.6.1) follows from the definitions. Let U (resp. U') be an affine open of S (resp. S') such that $q(U') \subset U$, $A = \Gamma(U, \mathscr{O}_S)$, $A' = \Gamma(U', \mathscr{O}_{S'})$ their rings, $B = \Gamma(U, \mathscr{B})$, $B' = \Gamma(U', \mathscr{B}')$; the restriction of u to a $(q|U')$-morphism: $\mathscr{B}|U \to \mathscr{B}'|U'$ corresponds to a di-homomorphism of algebras $B \to B'$; if V, V' are the inverse images of U, U' in X, X' respectively, under the structure morphisms, then the morphism $V' \to V$, the restriction of v, corresponds (**I**, 1.7.3) to the above di-homomorphism.

(1.5.7). Under the same hypotheses as in (1.5.6), let $\mathscr{M}$ be a quasi-coherent $\mathscr{B}$-module; there is then a canonical isomorphism of $\mathscr{O}_{X'}$-modules

(1.5.7.1) $$v^*(\widetilde{\mathscr{M}}) \simeq (q^*(\mathscr{M}) \otimes_{q^*(\mathscr{B})} \mathscr{B}')^\sim.$$ **II** | 14

Indeed, the canonical isomorphism (1.5.2.1) gives a canonical isomorphism from $p_1^*(\widetilde{\mathscr{M}})$ to the sheaf on $\operatorname{Spec}(q^*(\mathscr{B}))$ associated to the $q^*(\mathscr{B})$-module $q^*(\mathscr{M})$, and it then suffices to apply (1.4.7).

1.6. Affine morphisms.

(1.6.1). We say that a morphism $f: X \to Y$ of preschemes is *affine* if it defines X as a prescheme affine over Y. The properties of preschemes affine over another translates as follows in this language:

Proposition (1.6.2). —

(i) *A closed immersion is affine.*
(ii) *The composition of two affine morphisms is affine.*
(iii) *If $f: X \to Y$ is an affine S-morphism, then $f_{(S')}: X_{(S')} \to Y_{(Y')}$ is affine for every base change $S' \to S$.*
(iv) *If $f: X \to Y$ and $f': X' \to Y'$ are two affine S-morphisms, then*

$$f \times_S f': X \times_S X' \longrightarrow Y \times_S Y'$$

is affine.
(v) *If $f: X \to Y$ and $g: Y \to Z$ are two morphisms such that $g \circ f$ is affine and g is separated, then f is affine.*
(vi) *If f is affine, then so if f_{red}.*

Proof. By (**I**, 5.5.12), it suffices to prove (i), (ii), and (iii). But (i) is none other than Example (1.2.2), and (ii) is none other than Corollary (1.3.5); finally, (iii) follows from Proposition (1.5.1), since $X_{(S')}$ identifies with the product $X \times_Y Y_{(S')}$ (**I**, 3.3.11). □

Corollary (1.6.3). — *If X is an affine scheme and Y is a scheme, then every morphism $f: X \to Y$ is affine.*

Proposition (1.6.4). — *Let Y be a locally Noetherian prescheme, $f: X \to Y$ a morphism of finite type. For f to be affine, it is necessary and sufficient for f_{red} to be.*

Proof. By (1.6.2, vi), we see only need to prove the sufficiency of the condition. It suffices to prove that if Y is affine and Noetherian, then X is affine; but Y_{red} is then affine, so the same is true for X_{red} by hypothesis. Now X is Noetherian, so the conclusion follows from (**I**, 6.1.7). □

1.7. Vector bundle associated to a sheaf of modules.

(1.7.1). Let A be a ring, E an A-module. Recall that we call the *symmetric algebra* on E and denote by $\mathbf{S}(E)$ (or $\mathbf{S}_A(E)$) the quotient algebra of the tensor algebra $\mathbf{T}(E)$ by the two-sided ideal generated by the elements $x \otimes y - y \otimes x$, where x and y vary over E. The algebra $\mathbf{S}(E)$ is characterized by the following universal property: if σ is the canonical map $E \to \mathbf{S}(E)$ (obtained by composing $E \to \mathbf{T}(E)$ with the canonical map $\mathbf{T}(E) \to \mathbf{S}(E)$), then every A-linear map $E \to B$, where B is a *commutative* A-algebra, factors uniquely as $E \xrightarrow{\sigma} \mathbf{S}(E) \xrightarrow{g} B$, where g is an A-homomorphism *of algebras*. We immediately deduce from this characterization that for two A-modules E and F, we have

$$\mathbf{S}(E \oplus F) = \mathbf{S}(E) \oplus \mathbf{S}(F)$$

up to canonical isomorphism; in addition, $\mathbf{S}(E)$ is a covariant functor in E, from the category of A-modules to that of commutative A-algebras; finally, the above characterization also shows that if $E = \varinjlim E_\lambda$, then we have $\mathbf{S}(E) = \varinjlim \mathbf{S}(E_\lambda)$ up to canonical isomorphism. By abuse of language, a product $\sigma(x_1)\sigma(x_2)\cdots\sigma(x_n)$, where $x_i \in E$, is often denoted by $x_1 x_2 \cdots x_n$ if no confusion follows. The algebra $\mathbf{S}(E)$ is *graded*, $\mathbf{S}_n(E)$ being the set of linear combinations of n elements of E $(n \geqslant 0)$; the algebra $\mathbf{S}(A)$ is canonically isomorphic to the polynomial algebra $A[T]$ is an indeterminate, and the algebra $\mathbf{S}(A^n)$ with the polynomial algebra in n indeterminates $A[T_1, \ldots, T_n]$. **II** | 15

(1.7.2). Let φ be a ring homomorphism $A \to B$. If F is a B-module, then the canonical map $F \to \mathbf{S}(F)$ gives a canonical map $F_{[\varphi]} \to \mathbf{S}(F)_{[\varphi]}$, which thus factors as $F_{[\varphi]} \to \mathbf{S}(F_{[\varphi]}) \to \mathbf{S}(F)_{[\varphi]}$; the canonical homomorphism $\mathbf{S}(F_{[\varphi]}) \to \mathbf{S}(F)_{[\varphi]}$ is surjective, but not necessarily bijective. If E is an A-module, then every di-homomorphism $E \to F$ (that is to say, every A-homomorphism $E \to F_{[\varphi]}$) thus canonically gives an A-homomorphism of algebras $\mathbf{S}(E) \to \mathbf{S}(F_{[\varphi]}) \to \mathbf{S}(F)_{[\varphi]}$, that is to say a di-homomorphism of algebras $\mathbf{S}(E) \to \mathbf{S}(F)$.

With the same notations, for every A-module E, $\mathbf{S}(E \otimes_A B)$ canonically identifies with the algebra $\mathbf{S}(E) \otimes_A B$; this follows immediately from the universal property of $\mathbf{S}(E)$ (1.7.1).

(1.7.3). Let R be a multiplicative subset of the ring A; apply (1.7.2) to the ring $B = R^{-1}A$, and remembering that $R^{-1}E = E \otimes_A R^{-1}A$, we see that we have $\mathbf{S}(R^{-1}E) = R^{-1}\mathbf{S}(E)$ up to canonical isomorphism. In addition, if $R' \supset R$ is a second multiplicative subset of A, then the diagram

$$\begin{tikzcd} R^{-1}E \arrow[r] \arrow[d] & R'^{-1}E \arrow[d] \\ \mathbf{S}(R^{-1}E) \arrow[r] & \mathbf{S}(R'^{-1}E) \end{tikzcd}$$

is commutative.

(1.7.4). Now let $(S, \mathscr{A})$ be a ringed space, and let $\mathscr{E}$ be a $\mathscr{A}$-module over S. If to any open $U \subset S$ we associate the $\Gamma(U, \mathscr{A})$-module $\mathbf{S}(\Gamma(U, \mathscr{E}))$, then we define (see the functorial nature of $\mathbf{S}(E)$ (1.7.2)) a presheaf of algebras; we say that the associated sheaf, which we denote by $\mathbf{S}(\mathscr{E})$ or $\mathbf{S}_{\mathscr{A}}(\mathscr{E})$ is the *symmetric $\mathscr{A}$-algebra* on the $\mathscr{A}$-module $\mathscr{E}$. It follows immediately from (1.7.1) that $\mathbf{S}(\mathscr{E})$ is a solution to a universal problem: every homomorphism of $\mathscr{A}$-modules $\mathscr{E} \to \mathscr{B}$, where $\mathscr{B}$ is an $\mathscr{A}$-algebra, factors uniquely as $\mathscr{E} \to \mathbf{S}(\mathscr{E}) \to \mathscr{B}$, the second arrow being a homomorphism of $\mathscr{A}$-algebras. There is thus a bijective correspondence between homomorphisms $\mathscr{E} \to \mathscr{B}$ of $\mathscr{A}$-modules and homomorphisms $\mathbf{S}(\mathscr{E}) \to \mathscr{B}$ of $\mathscr{A}$-algebras. In particular, every homomorphism $u : \mathscr{E} \to \mathscr{F}$ of $\mathscr{A}$-modules defines a homomorphism $\mathbf{S}(u) : \mathbf{S}(\mathscr{E}) \to \mathbf{S}(\mathscr{F})$ of $\mathscr{A}$-algebras, and $\mathbf{S}(\mathscr{E})$ is thus a covariant functor in $\mathscr{E}$. **II** | 16

By (1.7.2) and the commutativity of $\mathbf{S}$ with inductive limts, we have $(\mathbf{S}(\mathscr{E}))_x = \mathbf{S}(\mathscr{E}_x)$ for every point $x \in S$. If $\mathscr{E}, \mathscr{F}$ are two $\mathscr{A}$-modules, then $\mathbf{S}(\mathscr{E} \oplus \mathscr{F})$ canonically identifies with $\mathbf{S}(\mathscr{E}) \otimes_A \mathbf{S}(\mathscr{F})$, as we see for the corresponding presheaves.

We also note that $\mathbf{S}(\mathscr{E})$ is a graded $\mathscr{A}$-algebra, the infinite direct sum of the $\mathbf{S}_n(\mathscr{E})$, where the $\mathscr{A}$-module $\mathbf{S}_n(\mathscr{E})$ is the sheaf associated to the presheaf $U \mapsto \mathbf{S}_n(\Gamma(U, \mathscr{E}))$. If we take in particular $\mathscr{E} = \mathscr{A}$, then we see that $\mathbf{S}_{\mathscr{A}}(\mathscr{A})$ identifies with $\mathscr{A}[T] = \mathscr{A} \otimes_{\mathbf{Z}} \mathbf{Z}[T]$ (T indeterminate, $\mathbf{Z}$ being considered as a simple sheaf).

(1.7.5). Let $(T, \mathscr{B})$ be a second ringed space, f a morphism $(S, \mathscr{A}) \to (T, \mathscr{B})$. If $\mathscr{F}$ is a $\mathscr{B}$-module, then $\mathbf{S}(f^*(\mathscr{F}))$ canonically identifies with $f^*(\mathbf{S}(\mathscr{F}))$; indeed, if $f = (\psi, \theta)$, then by definition (**0**, 4.3.1),

$$\mathbf{S}(f^*(\mathscr{F})) = \mathbf{S}(\psi^*(\mathscr{F}) \otimes_{\psi^*(\mathscr{B})} \mathscr{A}) = \mathbf{S}(\psi^*(\mathscr{F})) \otimes_{\psi^*(\mathscr{B})} \mathscr{A}$$

(1.7.2); for every open U of S and every section h of $\mathbf{S}(\psi^*(\mathscr{F}))$ over U, h coincides, in a neighborhood V of every point $s \in U$, with an element of $\mathbf{S}(\Gamma(V, \psi^*(\mathscr{F})))$; if we refer to the definition of $\psi^*(\mathscr{F})$ (**0**, 3.7.1) and take into account that every element of $\mathbf{S}(E)$ for a module E is a linear combination of a finite number of products of elements of E, then we see that there is a neighborhood W of $\psi(s)$ in T, a section h' of $\mathbf{S}(\mathscr{F})$ over W, and a neighborhood $V' \subset V \cap \psi^{-1}(W)$ of s such that h coincides with $t \mapsto h'(\psi(t))$ over V'; hence out assertion.

Proposition (1.7.6). — *Let A be a ring, $S = \operatorname{Spec}(A)$ its prime spectrum, $\mathscr{E} = \widetilde{M}$ the $\mathscr{O}_S$-module associated to an A-module M; then the $\mathscr{O}_S$-algebra $\mathbf{S}(\mathscr{E})$ is associated to the A-algebra $\mathbf{S}(M)$.*

Proof. For every $f \in A$, $\mathbf{S}(M_f) = (\mathbf{S}(M))_f$ (1.7.3), and the proposition thus folloes from the definition (**I**, 1.3.4). ☐

Corollary (1.7.7). — *If S is a prescheme, $\mathscr{E}$ a quasi-coherent $\mathscr{O}_S$-module, then the $\mathscr{O}_S$-algebra $\mathbf{S}(\mathscr{E})$ is quasi-coherent. If in addition $\mathscr{E}$ is of finite type, then each of the $\mathscr{O}_S$-modules $\mathbf{S}_n(\mathscr{E})$ is of finite type.*

Proof. The first assertion is an immediate consequence of (1.7.6) and of (**I**, 1.4.1); the second follows from the fact that if E is an A-module of finite type, then $\mathbf{S}_n(E)$ is an A-module of finite type; we then apply (**I**, 1.3.13). ☐

Definition (1.7.8). — Let $\mathscr{E}$ be a quasi-coherent $\mathscr{O}_S$-module. We call the *vector bundle over S defined by $\mathscr{E}$* and denote by $\mathbf{V}(\mathscr{E})$ the spectrum (1.3.1) of the quasi-coherent $\mathscr{O}_S$-algebra $\mathbf{S}(\mathscr{E})$.

By (1.2.7), for every S-prescheme X, there is a canonical bijective correspondence between the S-morphisms $X \to \mathbf{V}(\mathscr{E})$ and the homomorphisms of *$\mathscr{O}_S$-algebras* $\mathbf{S}(\mathscr{E}) \to \mathscr{A}(X)$, thus also between these S-morphisms and the homomorphisms of *$\mathscr{O}_S$-modules* $\mathscr{E} \to \mathscr{A}(X) = f_*(\mathscr{O}_X)$ (where f is the structure morphism $X \to S$). In particular:

(1.7.9). Take for X a subprescheme induced by S on an *open* $U \subset S$. Then the S-morphisms $U \to \mathbf{V}(\mathscr{E})$ are none other that the U-sections (**I**, 2.5.5) of the U-prescheme induces by $\mathbf{V}(\mathscr{E})$ on the open $p^{-1}(U)$ (where p is the structure morphism $\mathbf{V}(\mathscr{E}) \to S$). From what we have just seen, these U-sections bijectively correspond to homomorphisms of $\mathscr{O}_S$-modules $\mathscr{E} \to j_*(\mathscr{O}_S|U)$ (where j is the canonical injection $U \to S$), or equivalently **II** | 17
(**0**, 4.4.3) with the $(\mathscr{O}_S|U)$-homomorphisms $j^*(\mathscr{E}) = \mathscr{E}|U \to \mathscr{O}_S|U$. In addition, it is immediate that the restriction

to an open $U' \subset U$ of an S-morphism $U \to \mathbf{V}(\mathscr{E})$ corresponds to the restriction to U' of the corresponding homomorphism $\mathscr{E}|U \to \mathscr{O}_S|U$. We conclude that *the sheaf of germs of S-sections* of $\mathbf{V}(\mathscr{E})$ canonically identifies with the *dual* $\mathscr{E}^\vee$ of $\mathscr{E}$.

In particular, if we set $X = U = S$, then the *zero* homomorphism $\mathscr{E} \to \mathscr{O}_S$ corrresponds to a canonical S-section of $\mathbf{V}(\mathscr{E})$, called the *zero S-section* (cf. (8.3.3)).

(1.7.10). Now take X to be the spectrum $\{\xi\}$ of a field K; the structure morphism $f : X \to S$ then corresponfs to a monomorphism $k(s) \to K$, where $s = f(\xi)$ (**I**, 2.4.6); the S-morphisms $\{\xi\} \to \mathbf{V}(\mathscr{E})$ are none other than the *geometric points of* $\mathbf{V}(\mathscr{E})$ *with values in the extension K of* $k(s)$ (**I**, 3.4.5), points which are localized at the points of $p^{-1}(s)$. The set of these points, which we can call *the rational geometric fibre over K* of $\mathbf{V}(\mathscr{E})$ *over the point s*, is identified by (1.7.8) with the set of homomorphisms of $\mathscr{O}_S$-modules $\mathscr{E} \to f_*(\mathscr{O}_X)$, or, equivalently (**0**, 4.4.3) with the set of homomorphisms of $\mathscr{O}_X$-modules $f^*(\mathscr{E}) \to \mathscr{O}_X = K$. But we have by definition (**0**, 4.3.1) $f^*(\mathscr{E}) = \mathscr{E}_s \otimes_{\mathscr{O}_s} K = \mathscr{E}^s \otimes_{k(s)} K$, setting $\mathscr{E}^s = \mathscr{E}_s/\mathfrak{m}_s\mathscr{E}_s$; the geometric fibre of $\mathbf{V}(\mathscr{E})$ rational over K over s thus identifies with the *dual* of the *K-vector space* $\mathscr{E}^s \otimes_{k(s)} K$; if $\mathscr{E}^s$ or K is of finite dimension over $k(s)$, then this dual also identifies with $(\mathscr{E}^s)^\vee \otimes_{k(s)} K$, denoting by $(\mathscr{E}^s)^\vee$ the dual of the $k(s)$-vector space $\mathscr{E}^s$.

Proposition (1.7.11). —

(i) $\mathbf{V}(\mathscr{E})$ *is a contravariant functor in $\mathscr{E}$ from the category of quasi-coherent $\mathscr{O}_S$-modules to the category of affine S-schemes.*
(ii) *If $\mathscr{E}$ is an $\mathscr{O}_S$-module of finite type, then* $\mathbf{V}(\mathscr{E})$ *is of finite type over S.*
(iii) *If $\mathscr{E}$ and $\mathscr{F}$ are two quasi-coherent $\mathscr{O}_S$-modules, then* $\mathbf{V}(\mathscr{E} \oplus \mathscr{F})$ *canonically identifies with* $\mathbf{V}(\mathscr{E}) \times_S \mathbf{V}(\mathscr{F})$.
(iv) *Let $g : S' \to S$ be a morphism; for every quasi-coherent $\mathscr{O}_S$-module $\mathscr{E}$,* $\mathbf{V}(g^*(\mathscr{E}))$ *canonically identifies with* $\mathbf{V}(\mathscr{E})_{(S')} = \mathbf{V}(\mathscr{E}) \times_S S'$.
(v) *A surjective homomorphism $\mathscr{E} \to \mathscr{F}$ of quasi-coherent $\mathscr{O}_S$-modules corresponds to a closed immersion* $\mathbf{V}(\mathscr{F}) \to \mathbf{V}(\mathscr{E})$.

Proof. (i) is an immediate consequence of (1.2.7), taking into account that every homomorphism of $\mathscr{O}_S$-modules $\mathscr{E} \to \mathscr{F}$ canonically defines a homomorphism of $\mathscr{O}_S$-algebras $\mathbf{S}(\mathscr{E}) \to \mathbf{S}(\mathscr{F})$. (ii) follows immediately from the definition (**I**, 6.3.1) and the fact that if E is an A-module of finite type, then $\mathbf{S}(E)$ is an A-algebra of finite type. To prove (iii), it suffices to start with the canonical isomorphism $\mathbf{S}(\mathscr{E} \oplus \mathscr{F}) \simeq \mathbf{S}(\mathscr{E}) \otimes_{\mathscr{O}_S} \mathbf{S}(\mathscr{F})$ (1.7.4) and to apply (1.4.6). Similarly, to prove (iv), it suffices to start with the canonical isomorphism $\mathbf{S}(g^*(\mathscr{E})) \simeq g^*(\mathbf{S}(\mathscr{E}))$ (1.7.5) and to apply (1.5.2). Finally, to establish (v), it suffices to remark that if the homomorphism $\mathscr{E} \to \mathscr{F}$ is surjective, then so is the corresponding homomorphism $\mathbf{S}(\mathscr{E}) \to \mathbf{S}(\mathscr{F})$ of $\mathscr{O}_S$-algebras, and the conclusion follows from (1.4.10). □

(1.7.12). Take in particular $\mathscr{E} = \mathscr{O}_S$; the prescheme $\mathbf{V}(\mathscr{O}_S)$ is the affine S-scheme, spectrum of the $\mathscr{O}_S$-algebra $\mathbf{S}(\mathscr{O}_S)$ which identifies with the $\mathscr{O}_S$-algebra $\mathscr{O}_S[T] = \mathscr{O}_S \otimes_{\mathbf{Z}} \mathbf{Z}[T]$ (T indeterminate); this is evident when $S = \operatorname{Spec}(\mathbf{Z})$, by virtue of (1.7.6), and we pass from there to the general case by considering the structure morphism $S \to \operatorname{Spec}(\mathbf{Z})$ and using (1.7.11, iv). Because of this result, we set $\mathbf{V}(\mathscr{O}_S) = S[T]$, and we thus have the formula **II** | 18

$$S[T] = S \otimes_{\mathbf{Z}} \mathbf{Z}[T]. \tag{1.7.12.1}$$

The identification of the sheaf of germs of S-sections of $S[T]$ with $\mathscr{O}_S$, already seen in (**I**, 3.3.15), here in a more general context, as a special case of (1.7.9).

(1.7.13). For every S-prescheme X, we have seen (1.7.8) that $\operatorname{Hom}_S(X, S[T])$ canonically identifies with $\operatorname{Hom}_{\mathscr{O}_S}(\mathscr{O}_S, \mathscr{A}(X))$, which is canonically isomorphic to $\Gamma(S, \mathscr{A}(X))$, and as a result is equipped with the structure of a ring; in addition, to every S-morphism $h : X \to Y$ there corresponds a morphism $\Gamma(\mathscr{A}(h)) : \Gamma(S, \mathscr{A}(Y)) \to \Gamma(S, \mathscr{A}(X))$ for the ring structures (1.1.2). When we equip $\operatorname{Hom}_S(X, S[T])$ with a ring structure as defined, then we can see that $\operatorname{Hom}(X, S[T])$ can be considered as a *contravariant functor* in X, from the category of S-preschemes to that of rings. On the other hand, $\operatorname{Hom}_S(X, \mathbf{V}(\mathscr{E}))$ is likewise identified with $\operatorname{Hom}_{\mathscr{O}_S}(\mathscr{E}, \mathscr{A}(X))$ (where $\mathscr{A}(X)$ is considered as an $\mathscr{O}_S$*-module*); as a result, we can canonically give a *module* structure on the ring $\operatorname{Hom}_S(X, S[T])$, and we see as above that the pair

$$(\operatorname{Hom}_S(X, S[T]), \operatorname{Hom}(X, \mathbf{V}(\mathscr{E})))$$

is a contravariant functor in X, with values in the category whose elements are the pairs (A, M) consisting of a ring A and an A-module M, the morphisms being di-homomorphisms.

We will interpret these facts by saying that $S[T]$ is an *S-scheme of rings* and that $\mathbf{V}(\mathscr{E})$ is an *S-scheme of modules* on the S-scheme of rings $S[T]$ (cf. Chapter 0, §8).

(1.7.14). We will see that the structure of an S-scheme of modules defined on the S-scheme $\mathbf{V}(\mathscr{E})$ allows us to reconstruct the $\mathscr{O}_S$-module $\mathscr{E}$ up to unique isomorphism: for this, we will show that $\mathscr{E}$ is canonically isomorphic to a $\mathscr{O}_S$-submodule of $\mathbf{S}(\mathscr{E}) = \mathscr{A}(\mathbf{V}(\mathscr{E}))$, defined by means of this structure. Indeed (1.7.4) the set $\operatorname{Hom}_{\mathscr{O}_S}(\mathbf{S}(\mathscr{E}), \mathscr{A}(X))$ of homomorphisms of *$\mathscr{O}_S$-algebras* canonical identifies with $\operatorname{Hom}_{\mathscr{O}_S}(\mathscr{E}, \mathscr{A}(X))$, the set of homomorphisms of *$\mathscr{O}_S$-modules*: if h and h' are two elements of this latter set, s_i $(1 \leqslant i \leqslant n)$ sections of $\mathscr{E}$ over an open $U \subset S$, t a section of $\mathscr{A}(X)$ over U, then we have by definition

$$(h + h')(s_1 s_2 \cdots s_n) = \prod_{i=1}^{n} (h(s_i) + h'(s_i))$$

and

$$(t \cdot h)(s_1 s_2 \cdots s_n) = t^n \prod_{i=1}^{n} h(s_i).$$

This being so, if z is a section of $\mathbf{S}(\mathscr{E})$ over U, then $h \mapsto h(z)$ is a map from $\operatorname{Hom}_S(X, \mathbf{V}(\mathscr{E})) = \operatorname{Hom}_{\mathscr{O}_S}(\mathbf{S}(\mathscr{E}), \mathscr{A}(X))$ to $\Gamma(U, \mathscr{A}(X))$. We will show that $\mathscr{E}$ identifies with a submodule of $\mathbf{S}(\mathscr{E})$ such that, *for every open $U \subset S$, every* **II | 19**
section z of this $\mathscr{O}_S$-submodule of U, and every S-prescheme X, the map $h \mapsto h(z)$ from $\operatorname{Hom}_{\mathscr{O}_S}(\mathbf{S}(\mathscr{E})|U, \mathscr{A}(X)|U)$ *to* $\Gamma(U, \mathscr{A}(X))$ *is a homomorphism of* $\Gamma(U, \mathscr{A}(X))$*-modules.*

It is immediate that $\mathscr{E}$ has this property; to show the converse, we can reduce to proving that when $S = \operatorname{Spec}(A)$, $\mathscr{E} = \widetilde{M}$, a section z of $\mathbf{S}(\mathscr{E})$ over S that (for $U = S$) has the property stated above is necessarily a section of $\mathscr{E}$; we then have $z = \sum_{n=0}^{\infty} z_n$, where $z_n \in \mathbf{S}_n(M)$, and it is a question of proving that $z_n = 0$ for $n \neq 1$. Set $B = \mathbf{S}(M)$ and take for X the prescheme $\operatorname{Spec}(B[T])$, where T is an indeterminate. The set $\operatorname{Hom}_{\mathscr{O}_S}(\mathbf{S}(\mathscr{E}), \mathscr{A}(X))$ identifies with the set of ring homomorphisms $h : B \to B[T]$ (**I**, 1.3.13), and from what we saw above, we have $(T \cdot h)(z) = \sum_{n=0}^{\infty} T^n h(z_n)$: the hypothesis on z implies that we have $\sum_{n=0}^{\infty} T^n h(z_n) = T \cdot \sum_{n=0}^{\infty} h(z_n)$ for every homomorphism h. In in particular we take for h the canonical injection, then $\sum_{n=0}^{\infty} T^n z_n = T \cdot \sum_{n=0}^{\infty} z_n$, which implies the conclusion $z_n = 0$ for $n \neq 1$.

Proposition (1.7.15). — *Let Y be a prescheme whose underlying space is Noetherian, or a quasi-compact scheme. Every affine Y-scheme X of finite type over Y is Y-isomorphic to a closed Y-subscheme of the form* $\mathbf{V}(\mathscr{E})$*, where $\mathscr{E}$ is a quasi-coherent $\mathscr{O}_Y$-module of finite type.*

Proof. The quasi-coherent $\mathscr{O}_Y$-algebra $\mathscr{A}(X)$ is of finite type (1.3.7). The hypotheses imply that $\mathscr{A}(X)$ is generated by a quasi-coherent $\mathscr{O}_Y$-submodule of finite type $\mathscr{E}$ (**I**, 9.6.5); by definition, this implies that the canonical homomorphism $\mathbf{S}(\mathscr{E}) \to \mathscr{A}(X)$ canonically extending the injection $\mathscr{E} \to \mathscr{A}(X)$ is *surjective*; the conclusion then follows from (1.4.10). □

§2. Homogeneous prime spectra

2.1. Generalities on graded rings and modules.

Notation (2.1.1). — Given a ring S *graded* in positive degrees, we denote by S_n the subset of S consisting of homogeneous elements of degree n $(n \geqslant 0)$, by S_+ the (direct) sum of the S_n for $n > 0$; we have $1 \in S_0$, S_0 is a subring of S, S_+ is a graded ideal of S, and S is the direct sum of S_0 and S_+. If M is a *graded* module over S (with positive or negative degrees), we similarly denote by M_n the S_0-module consisting of homogeneous elements of M of degree n (with $n \in \mathbf{Z}$).

For every integer $d > 0$, we denote by $S^{(d)}$ the direct sum of the S_{nd}; by considering the elements of S_{nd} as homogeneous of degree n, the S_{nd} define on $S^{(d)}$ a graded ring structure.

For every integer k such that $0 \leqslant k \leqslant d-1$, we denote by $M^{(d,k)}$ the direct sum of the M_{nd+k} $(n \in \mathbf{Z})$; this is **II | 20**
a graded $S^{(d)}$-module when we consider the elements of M_{nd+k} as homogeneous of degree n. We write $M^{(d)}$ in place of $M^{(d,0)}$.

With the above notation, for every integer n (positive or negative), we denote by $M(n)$ the graded S-module defined by $(M(n))_k = M_{n+k}$ for every $k \in \mathbf{Z}$. In particular, $S(n)$ will be a graded S-module such that $(S(n))_k = S_{n+k}$, by agreeing to set $S_n = 0$ for $n < 0$. We say that a graded S-module M is *free* if it is isomorphic, considered as a *graded* module, to a direct sum of modules of the form $S(n)$; as $S(n)$ is a monogeneous S-module, generated by the element 1 of S considered as an element of degree $-n$, it is equivalent to say that M admits a *basis* over S consisting of *homogeneous* elements.

We say that a graded S-module M *admits a finite presentation* if there exists an exact sequence $P \to Q \to M \to 0$, where P and Q are finite direct sums of modules of the form $S(n)$ and the homomorphisms are of degree 0 (cf. (2.1.2)).

(2.1.2). Let M and N be two graded S-modules; we define on $M \otimes_S N$ a *graded* S-module structure in the following way. On the tensor product $M \otimes_{\mathbf{Z}} N$, we can define a graded $\mathbf{Z}$-module structure (where $\mathbf{Z}$ is graded by $\mathbf{Z}_0 = \mathbf{Z}$, $\mathbf{Z}_n = 0$ for $n \neq 0$) by setting $(M \otimes_{\mathbf{Z}} N)_q = \bigoplus_{m+n=q} M_m \otimes_{\mathbf{Z}} N_n$ (as M and N are respectively direct sums of the M_m and the N_n, we know that we can canonically identify $M \otimes_{\mathbf{Z}} N$ with the direct sum of all the $M_m \otimes_{\mathbf{Z}} N_n$). This being so, we have $M \otimes_S N = (M \otimes_{\mathbf{Z}} N)/P$, where P is the $\mathbf{Z}$-submodule of $M \otimes_{\mathbf{Z}} N$ generated by the elements $(xs) \otimes y - x \otimes (sy)$ for $x \in M$, $y \in N$, $s \in S$; it is clear that P is a *graded* $\mathbf{Z}$-submodule of $M \otimes_{\mathbf{Z}} N$, and we see immediately that we obtain a graded S-module structure on $M \otimes_S N$ by passing to the quotient.

For two graded S-modules M and N, recall that a homomorphism $u : M \to N$ of S-modules is said to be *of degree* k if $u(M_j) \subset N_{j+k}$ for all $j \in \mathbf{Z}$. If H_n denotes the set of all the homomorphisms of degree n from M to N, then we denote by $\operatorname{Hom}_S(M,N)$ the (direct) *sum* of the H_n ($n \in \mathbf{Z}$) in the S-module H of all the homomorphisms (of S-modules) from M to N; in general, $\operatorname{Hom}_S(M,N)$ is not equal to the later. However, we have $H = \operatorname{Hom}_S(M,N)$ when M is *of finite type*; indeed, we can then suppose that M is generated by a finite number of homogeneous elements x_i ($1 \leqslant i \leqslant n$), and every homomorphism $u \in H$ can be written in a unique way as $\sum_{k \in \mathbf{Z}} u_k$, where for each k, $u_k(x_i)$ is equal to the homogeneous component of degree $k + \deg(x_i)$ of $u(x_i)$ ($1 \leqslant i \leqslant n$), which implies that $u_k = 0$ except for a finite number of indices; we have by definition that $u_k \in H_k$, hence the conclusion.

We say that the elements of degree 0 of $\operatorname{Hom}_S(M,N)$ are the *homomorphisms of graded S-modules.* It is clear that $S_m H_n \subset H_{m+n}$, so the H_n define on $\operatorname{Hom}_S(M,N)$ a graded S-module structure.

It follows immediately from these definitions that we have

(2.1.2.1) $$M(m) \otimes_S N(n) = (M \otimes_S N)(m+n),$$

(2.1.2.2) $$\operatorname{Hom}_S(M(m), N(n)) = (\operatorname{Hom}_S(M,N))(n-m),$$

for two graded S-modules M and N. **II** | 21

Let S and S' be two graded rings; a homomorphism of *graded rings* $\varphi : S \to S'$ is a homomorphism of rings such that $\varphi(S_n) \subset S'_n$ for all $n \in \mathbf{Z}$ (in other words, φ must be a homomorphism *of degree* 0 of graded $\mathbf{Z}$-modules). The data of such a homomorphism defines on S' a *graded* S'-module structure; equipped with this structure and its graded ring structure, we say that S' is a *graded S'-algebra.*

If M is also a graded S-module, then the tensor product $M \otimes_S S'$ of *graded* S-modules is equipped in a natural way with a *graded* S'-module structure, the grading being defined as above.

Lemma (2.1.3). — *Let S be a ring graded in positive degrees. For a subset E of S_+ consisting of homogeneous elements to generate S_+ as an S-module, it is necessary and sufficient for E to generate S an an S_0-algebra.*

Proof. The condition is evidently sufficient; we show that it is necessary. Let E_n (resp. E^n) be the set of elements of E equal to n (resp. $\leqslant n$); it suffices to show, by induction on $n > 0$, that S_n is the S_0-module generated by the elements of degree n which are products of elements of E^n. This is evident for $n = 1$ by virtue of the hypothesis; the latter also shows that $S_n = \sum_{p=0}^{n-1} S_p E_{n-p}$, and the induction argument is then immediate. □

Corollary (2.1.4). — *For S_+ to be an ideal of finite type, it is necessary and sufficient for S to be an S_0-algebra of finite type.*

Proof. We can always assume that a finite system of generators of the S_0-algebra S (resp. of the S-ideal S_+) consists of homogeneous elements, by replacing each of the generators considered by its homogeneous components. □

Corollary (2.1.5). — *For S to be Noetherian, it is necessary and sufficient for S_0 to be Noetherian and for S to be an S_0-algebra of finite type.*

Proof. The condition is evidently sufficient; it is necessary, since S_0 is isomorphic to S/S_+ and S_+ must be an ideal of finite type (2.1.4). □

Lemma (2.1.6). — *Let S be a ring graded in positive degrees, which is an S_0-algebra of finite type. Let M be a graded S-module of finite type. Then:*

(i) *The M_n are S_0-modules of finite type, and there exists an integer n_0 such that $M_n = 0$ for $n \leqslant n_0$.*
(ii) *There exists an integer n_1 and an integer $h > 0$ such that, for every integer $n \geqslant n_1$, we have $M_{n+h} = S_h M_n$.*
(iii) *For every pair of integers (d,k) such that $d > 0$, $0 \leqslant k \leqslant d-1$, $M^{(d,k)}$ is an $S^{(d)}$-module of finite type.*
(iv) *For every integer $d > 0$, $S^{(d)}$ is an S_0-algebra of finite type.*
(v) *There exists an integer $h > 0$ such that $S_{mh} = (S_h)^m$ for all $m > 0$.*
(vi) *For every integer $n > 0$, there exists an integer m_0 such that $S_m \subset S_+^n$ for all $m \geqslant m_0$.*

Proof. We can assume that S is generated (as an S_0-algebra) by homogeneous elements f_i, of degrees h_i $(1 \leqslant i \leqslant r)$, and M is generated (as an S-module) by homogeneous elements x_j of degrees k_j $(1 \leqslant j \leqslant s)$. It is clear that M_n is formed by linear combinations, with coefficients in S_0, of elements $f_1^{\alpha_1} \cdots f_r^{\alpha_r} x_j$ such that the α_i **II** | 22
are integers $\geqslant 0$ satisfying $k_j + \sum_i \alpha_i h_i = n$; for each j, there are only finitly many systems (α_i) satisfying this equation, since the h_i are > 0, hence the first assertion of (i); the second is evident. On the other hand, let h be the l.c.m. of the h_i and set $g_i = f_i^{h/h_i}$ $(1 \leqslant i \leqslant r)$ such that all the g_i are of degree h; let z_μ be the elements of M of the form $f_1^{\alpha_1} \cdots f_r^{\alpha_r} x_j$ with $0 \leqslant \alpha_i < h/h_i$ for $1 \leqslant i \leqslant r$; there are finitely many of these elements, so let n_1 be the largest of their degrees. It is clear that for $n \geqslant n_1$, every element of M_{n+h} is a linear combination of the z_μ whose cofficients are monomials of degree > 0 with respect to the g_i, so we have $M_{n+h} = S_h M_n$, which establishes (ii). In a similar way, we see (for all $d > 0$) that an element of $M^{(d,k)}$ is a linear combinations, with coeffients in S_0, of elements of the form $g^d f_1^{\alpha_1} \cdots f_r^{\alpha_r} x_j$ with $0 \leqslant \alpha_i < d$, g being a homogeneous element of S; hence (iii); (iv) then follows from (iii) and from Lemma (2.1.3), by taking $M = S_+$, since $(S_+)^{(d)} = (S^{(d)})_+$. The assertion of (v) is deduced from (ii) by taking $M = S$. Finally, for a given n, there are finitely many systems (α_i) such that $\alpha_i \geqslant 0$ and $\sum_i \alpha_i < n$, so if m_0 is the largest value of the sum $\sum_i \alpha_i h_i$ of these systems, then we have $S_m \subset S_+^n$ for $m > m_0$, which proves (vi). □

Corollary (2.1.7). — *If S is Noetherian, then so is $S^{(d)}$ for every integer $d > 0$.*

Proof. This follows from (2.1.5) and (2.1.6, iv). □

(2.1.8). Let $\mathfrak{p}$ be a *graded* prime ideal of the graded ring S; $\mathfrak{p}$ is thus a direct sum of the subgroups $\mathfrak{p}_n = \mathfrak{p} \cap S_n$. Suppose that $\mathfrak{p}$ *does not contain* S_+. Then if $f \in S_+$ is not in $\mathfrak{p}$, the relation $f^n x \in \mathfrak{p}$ is equivalent to $x \in \mathfrak{p}$; in particular, if $f \in S_d$ $(d > 0)$, for all $x \in S_{m-nd}$, then the relation $f^n x \in \mathfrak{p}_m$ is equivalent to $x \in \mathfrak{p}_{m-nd}$.

Proposition (2.1.9). — *Let n_0 be an integer > 0; for all $n \geqslant n_0$, let $\mathfrak{p}_n$ be a subgroup of S_n. For there to exist a graded prime ideal $\mathfrak{p}$ of S not containing S_+ and such that $\mathfrak{p} \cap S_n = \mathfrak{p}_n$ for all $n \geqslant n_0$, it is necessary and sufficient for the following coniditions to be satisfied:*

(1st) *$S_m \mathfrak{p}_n \subset \mathfrak{p}_{m+n}$ for all $m \geqslant 0$ and all $n \geqslant n_0$.*
(2nd) *For $m \geqslant n_0$, $n \geqslant n_0$, $f \in S_m$, $g \in S_n$, the relation $fg \in \mathfrak{p}_{m+n}$ implies $f \in \mathfrak{p}_m$ or $g \in \mathfrak{p}_n$.*
(3rd) *$\mathfrak{p}_n \neq S_n$ for at least one $n \geqslant n_0$.*

In addition, the graded prime ideal $\mathfrak{p}$ is then unique.

Proof. It is evident that the conditions (1st) and (2nd) are necessary. In addition, if $\mathfrak{p} \not\supset S_+$, then there exists at least one $k > 0$ such that $\mathfrak{p} \cap S_k \neq S_k$; if $f \in S_k$ is not in $\mathfrak{p}$, the relation $\mathfrak{p} \cap S_n = S_n$ implies $\mathfrak{p} \cap S_{n-mk} = S_{n-mk}$ according to (2.1.8); therefore, if $\mathfrak{p} \cap S_n = S_n$ for a certain value of n, we would have $\mathfrak{p} \supset S_+$ contrary to the hypothesis, which proves that (3rd) is necessary. Conversely, suppose that the conditions (1st), (2nd), and (3rd) are satisfied. Note that if for an integer $d \geqslant n_0$, $f \in S_d$ is not in $\mathfrak{p}_d$, then, if $\mathfrak{p}$ exists, $\mathfrak{p}_m$, for $m < n_0$, is necessarily equal to the set of the $x \in S_m$ such that $f^r x \in \mathfrak{p}_{m+rd}$, except for a finite number of values of r. This already proves that if $\mathfrak{p}$ exists, then it is unique. It remains to show that if we define the $\mathfrak{p}_m$ for $m < n_0$ by the previous condition, then $\mathfrak{p} = \sum_{n=0}^{\infty} \mathfrak{p}_n$ is a prime ideal. First, note that by virtue of (2nd), for $m \geqslant n_0$, $\mathfrak{p}_m$ is also defined as the set of the $x \in S_m$ such that $f^r x \in \mathfrak{p}_{m+rd}$ except for a finite number of values of r. This being so, if $g \in S_m$, **II** | 23
$x \in \mathfrak{p}_n$, then we have $f^r g x \in \mathfrak{p}_{m+n+rd}$ except for a finite number of values of r, so $gx \in \mathfrak{p}_{m+n}$, which proves that $\mathfrak{p}$ is an ideal of S. To establish that this ideal is prime, in other words that the ring $S/\mathfrak{p}$, graded by the subgroups $S_n/\mathfrak{p}_n$, is an integral domain, it suffices (by considering the components of higher degree of two elements of $S/\mathfrak{p}$) to prove that if $x \in S_m$ and $y \in S_n$ are such that $x \notin \mathfrak{p}_m$ and $y \notin \mathfrak{p}_n$, then $xy \notin \mathfrak{p}_{m+n}$. If not, for r large enough, we would have $f^{2r} xy \in \mathfrak{p}_{m+n+2rd}$; but we have $f^r y \notin \mathfrak{p}_{n+rd}$ for all $r > 0$; it then follows from (2nd) that, except for a finite number of values of r, we have $f^r x \in \mathfrak{p}_{m+rd}$, and we conclude that $x \in \mathfrak{p}_m$ contrary to the hypothesis. □

(2.1.10). We say that a subset $\mathfrak{J}$ of S_+ is an *ideal of* S_+ if it is an ideal of S, and $\mathfrak{J}$ is a *graded prime ideal of* S_+ if it is the intersection of S_+ and a graded prime ideal of S *not containing* S_+ (this prime ideal is also unique according to Proposition (2.1.9)). If $\mathfrak{J}$ is an ideal of S_+, the *radical of* $\mathfrak{J}$ *in* S_+ is the set of elements of S_+ which have a power in $\mathfrak{J}$, in other words the set $\mathfrak{r}_+(\mathfrak{J}) = \mathfrak{r}(\mathfrak{J}) \cap S_+$; in particular, the radical of 0 in S_+ is then called the *nilradical* of S_+ and denoted by $\mathfrak{N}_+$: this is the set of nilpotent elements of S_+. If $\mathfrak{J}$ is an *graded* ideal of S_+, then its radical $\mathfrak{r}_+(\mathfrak{J})$ is a *graded* ideal: by passing to the quotient ring $S/\mathfrak{J}$, we can reduce to the case $\mathfrak{J} = 0$, and it remains to see that if $x = x_h + x_{h+1} + \cdots + x_k$ is nilpotent, then so are the $x_i \in S_i$ $(1 \leqslant h \leqslant i \leqslant k)$; we can assume $x_k \neq 0$ and the component of highest degree of x^n is then x_k^n, hence x_k is nilpotent, and we then argue by induction on k. We say that the graded ring S is *essentially reduced* if $\mathfrak{N}_+ = 0$, in other words, if S_+ does not contain nilpotent elements $\neq 0$.

(2.1.11). We note that if, in the graded ring S, an element x is a zero-divisor, then so is its component of highest degree. We say that a ring S is *essentially integral* if the ring S_+ (*without the unit element*) does not contain a zero-divisor and is $\neq 0$; it suffices that a homogeneous element $\neq 0$ in S_+ is not a zero-divisor in this ring. It is clear that if $\mathfrak{p}$ is a graded prime ideal of S_+, then $S/\mathfrak{p}$ is essentially integral.

Let S be an essentially integral graded ring, and let $x_0 \in S_0$: if there then exists *a* homogeneous element $f \neq 0$ of S_+ such that $x_0 f = 0$, then we have $x_0 S_+ = 0$, since we have $(x_0 g)f = (x_0 f)g = 0$ for all $g \in S_+$, and the hypothesis thus implies $x_0 g = 0$. For S to be integral, it is necessary and sufficient for S_0 to be integral and the annihilator of S_+ in S_0 to be 0.

2.2. Rings of fractions of a graded ring.

(2.2.1). Let S be a graded ring, in positive degrees, f a *homogeneous* element of S, of degree $d > 0$; then the ring of fractions $S' = S_f$ is graded, taking for S'_n the set of the x/f^k, where $x \in S_{n+kd}$ with $k \geqslant 0$ (we observe here that n can take arbitrary negative values); we denote the subring $S'_0 = (S_f)_0$ of S' consisting of elements *of degree* 0 by the notation $S_{(f)}$.

If $f \in S_d$, then the monomials $(f/1)^h$ in S_f (h a positive or negative integer) form a *free system* over the ring $S_{(f)}$, and the set of their linear combinations is none other than the ring $(S^{(d)})_f$, which is thus *isomorphic to* $S_{(f)}[T,T^{-1}] = S_{(f)} \otimes_{\mathbf{Z}} \mathbf{Z}[T,T^{-1}]$ (where T is an indeterminate). Indeed, if we have a relation $\sum_{h=-a}^{b} z_h (f/1)^h = 0$ with $z_h = x_h/f^m$, where the x_h are in S_{md}, then this relation is equivalent by definition to the existence of a $k > -a$ such that $\sum_{h=-a}^{b} f^{h+k} x_h = 0$, and as the degrees of the terms of this sum are distinct, we have $f^{h+k}x_h = 0$ for all h, hence $z_h = 0$ for all h. **II | 24**

If M is a graded S-module, then $M' = M_f$ is a graded S_f-module, M'_n being the set of the z/f^k with $z \in M_{n+kd}$ ($k \geqslant 0$); we denote by $M_{(f)}$ the set of the homomogenous elements of degree 0 of M'; it is immediate that $M_{(f)}$ is an $S_{(f)}$-module and that we have $(M^{(d)})_f = M_{(f)} \otimes_{S_{(f)}} (S^{(d)})_f$.

Lemma (2.2.2). — *Let d and e be integers > 0, $f \in S_d$, $g \in S_e$. There exists a canonical ring isomorphism*

$$S_{(fg)} \simeq (S_{(f)})_{g^d/f^e};$$

if we canonically identify these two rings, then there exists a canonical module isomorphism

$$M_{(fg)} \simeq (M_{(f)})_{g^d/f^e}.$$

Proof. Indeed, fg divides $f^e g^d$, and this latter element divides $(fg)^{de}$, so the graded rings S_{fg} and $S_{f^e g^d}$ are canonically identified; on the other hand, $S_{f^e g^d}$ also identifies with $(S_{f^e})_{g^d/1}$ (**0**, 1.4.6), and as $f^e/1$ is invertible in S_{f^e}, $S_{f^e g^d}$ also identifies with $(S_{f^e})_{g^d/f^e}$. The element g^d/f^e is of degree 0 in S_{f^e}; we immediately conclude that the subring of $(S_{f^e})_{g^d/f^e}$ consisting of elements of degree 0 is $(S_{(f^e)})_{g^d/f^e}$, and as we evidently have $S_{(f^e)} = S_{(f)}$, this proves the first part of the proposition; the second is established in a similar way. □

(2.2.3). Under the hypotheses of (2.2.2), it is clear that the canonical homomorphism $S_f \to S_{fg}$ (**0**, 1.4.1), which sends x/f^k to $g^k x/(fg)^k$, is of degree 0, thus gives by restriction a *canonical homomorphism* $S_{(f)} \to S_{(fg)}$, such that the diagram

$$\begin{tikzcd} & S_{(f)} \arrow[dl] \arrow[dr] & \\ S_{(fg)} \arrow[rr, "\sim"] & & (S_{(f)})_{g^d/f^e} \end{tikzcd}$$

is commutative. We similarly define a canonical homomorphism $M_{(f)} \to M_{(fg)}$.

Lemma (2.2.4). — *If f and g are two homogeneous elements of S_+, then the ring $S_{(fg)}$ is generated by the union of the canonical images of $S_{(f)}$ and $S_{(g)}$.*

Proof. By virtue of Lemma (2.2.2), it suffices to see that $1/(g^d/f^e) = f^{d+e}/(fg)^d$ belongs to the canonical image of $S_{(g)}$ in $S_{(fg)}$, which is evident by definition. □

Proposition (2.2.5). — *Let d be an integer > 0 and let $f \in S_d$. Then there exists a canonical ring isomorphisms $S_{(f)} \simeq S^{(d)}/(f-1)S^{(d)}$; if we identify these two rings by this isomorphism, then there exists a canonical module isomorphism $M_{(f)} \simeq M^{(d)}/(f-1)M^{(d)}$.*

Proof. The first of these isomorphisms is defined by sending x/f^n, where $x \in S_{nd}$, to the element $\overline{x}$, the class of x mod. $(f-1)S^{(d)}$; this map is well-defined, because we have the congruence $f^h x \equiv x\,(\mathrm{mod.}\,(f-1)S^{(d)})$ for all $x \in S^{(d)}$, so if $f^h x = 0$ for an $h > 0$, then we have $\overline{x} = 0$. On the other hand, if $x \in S_{nd}$ is such that $x = (f-1)y$ **II | 25** with $y = y_{hd} + y_{(h+1)d} + \cdots + y_{kd}$ with $y_{jd} \in S_{jd}$ and $y_{hd} \neq 0$, then we necessarily have $h = n$ and $x = -y_{hd}$, as well as the relations $y_{(j+1)d} = f y_{jd}$ for $h \leqslant j \leqslant k-1$, $f y_{kd} = 0$, which ultimately gives $f^{k-n}x = 0$; we send every class $\overline{x}$ mod. $(f-1)S^{(d)}$ of an element $x \in S_{nd}$ to the element x/f^n of $S_{(f)}$, since the preceding remark shows that this map is well-defined. It is immediate that these two maps thus defined are ring homomorphisms, each the reciprocal of the other. We proceed exactly the same way for M. □

Corollary (2.2.6). — *If S is Noetherian, then so is $S_{(f)}$ for f homogeneous of degree* > 0.

Proof. This follows immediately from Corollary (2.1.7) and Proposition (2.2.5). □

(2.2.7). Let T be a multiplicative subset of S_+ consisting of *homogeneous* elements; $T_0 = T \cup \{1\}$ is then a multiplicative subset of S; as the elements of T_0 are homogeneous, the ring $T_0^{-1}S$ is still graded in the evident way; we denote by $S_{(T)}$ the subring of $T_0^{-1}S$ consisting of elements of order 0, that is to say, the elements of the form x/h, where $h \in T$ and x is homogeneous of degree equal to that of h. We know (**0**, 1.4.5) that $T_0^{-1}S$ is canonically identified with the inductive limit of the rings S_f, where f varies over T (with respect to the canonical homomorphisms $S_f \to S_{fg}$); as this identification respects the degrees, it identifies $S_{(T)}$ with the *inductive limit* of the $S_{(f)}$ for $f \in T$. For every graded S-module M, we similarly define the module $M_{(T)}$ (over the ring $S_{(T)}$) consisting of elements of degree 0 of $T_0^{-1}M$, and we see that this module is the inductive limit of the $M_{(f)}$ for $f \in T$.

If $\mathfrak{p}$ is a graded prime ideal of S_+, then we denote by $S_{(\mathfrak{p})}$ and $M_{(\mathfrak{p})}$ the ring $S_{(T)}$ and the module $M_{(T)}$ respectively, where T is the set of *homogeneous* elements of S_+ which do not belong to $\mathfrak{p}$.

2.3. Homogeneous prime spectrum of a graded ring.

(2.3.1). Given a graded ring S, in positive degrees, we call the *homogeneous prime spectrum* of S and denote it by $\mathrm{Proj}(S)$ the set of graded prime ideals of S_+ (2.1.10), or equivalently the set of graded prime ideals of S *not containing* S_+; we will define a *scheme* structure having $\mathrm{Proj}(S)$ as the underlying set.

(2.3.2). For every subset E of S, let $V_+(E)$ be the set of graded prime ideals of S containing S and not containing S_+; this is thus the subset $V(E) \cap \mathrm{Proj}(S)$ of $\mathrm{Spec}(S)$. From (**I**, 1.1.2) we deduce:

(2.3.2.1) $$V_+(0) = \mathrm{Proj}(S),\ V_+(S) = V_+(S_+) = \varnothing,$$

(2.3.2.2) $$V_+\left(\bigcup_\lambda E_\lambda\right) = \bigcap_\lambda V_+(E_\lambda),$$

(2.3.2.3) $$V_+(EE') = V_+(E) \cup V_+(E').$$

We do not change $V_+(E)$ by replacing E with the graded ideal generated by E; in addition, if $\mathfrak{J}$ is a graded ideal of S, then we have

(2.3.2.4) $$V_+(\mathfrak{J}) = V_+\left(\bigcup_{q \geqslant n}(\mathfrak{J} \cap S_q)\right)$$

for all $n > 0$: indeed, if $\mathfrak{p} \in \mathrm{Proj}(S)$ contains the homogeneous elements of $\mathfrak{J}$ of degree $\geqslant n$, then as by hypothesis **II** | 26
there exists a homogeneous element $f \in S_d$ not contained in $\mathfrak{p}$, for every $m \geqslant 0$ and every $x \in S_m \cap \mathfrak{J}$, we have $f^r x \in \mathfrak{J} \cap S_{m+rd}$ for all but finitely many values of r, so $f^r x \in \mathfrak{p} \cap S_{m+rd}$, which implies that $x \in \mathfrak{p} \cap S_m$ (2.1.9).

Finally, we have, for every graded ideal $\mathfrak{J}$ of S,

(2.3.2.5) $$V_+(\mathfrak{J}) = V_+(\mathfrak{r}_+(\mathfrak{J})).$$

§3. Homogeneous spectrum of a sheaf of graded algebras

3.1. Homogeneous spectrum of a quasi-coherent graded $\mathscr{O}_Y$-algebra.

§4. Projective bundles; ample sheaves

4.1. Definition of projective bundles.

§5. Quasi-affine morphisms; quasi-projective morphisms; proper morphisms; projective morphisms

5.1. Quasi-affine morphisms.

Definition (5.1.1). — We define a quasi-affine scheme to be a scheme isomorphic to some subscheme induced on some quasi-compact open subset of an affine scheme. We say that a morphism $f : X \to Y$ is quasi-affine, or that X (considered as a Y-prescheme via f) is a quasi-affine Y-scheme, if there exists a cover (U_α) of Y by affine open subsets such that the $f^{-1}(U_\alpha)$ are quasi-affine schemes.

It is clear that a quasi-affine morphism is *separated* ((**I**, 5.5.5) and (**I**, 5.5.8)) and *quasi-compact* (**I**, 6.6.1); every affine morphisms is evidently quasi-affine.

Recall that, for any prescheme X, setting $A = \Gamma(X, \mathscr{O}_X)$, the identity homomorphism $A \to A = \Gamma(X, \mathscr{O}_X)$ defines a morphism $X \to \operatorname{Spec}(A)$, said to be *canonical* (**I**, 2.2.4); this is nothing but the canonical morphism defined in (4.5.1) for the specific case where $\mathscr{L} = \mathscr{O}_X$, if we remember that $\operatorname{Proj}(A[T])$ is canonically identified with $\operatorname{Spec}(A)$ (3.1.7).

Proposition (5.1.2). — *Let X be a quasi-compact scheme or a prescheme whose underlying space is Noetherian, and A the ring $\Gamma(X, \mathscr{O}_X)$. The following conditions are equivalent.*

(a) *X is a quasi-affine scheme.*
(b) *The canonical morphism $u : X \to \operatorname{Spec}(A)$ is an open immersion.*
(b') *The canonical morphism $u : X \to \operatorname{Spec}(A)$ is a homeomorphism from X to some subspace of the underlying space of* $\operatorname{Spec}(A)$.
(c) *The $\mathscr{O}_X$-module $\mathscr{O}_X$ is very ample relative to u* (4.4.2).
(c') *The $\mathscr{O}_X$-module $\mathscr{O}_X$ is ample* (4.5.1).
(d) *When f ranges over A, the X_f form a basis for the topology of X.*
(d') *When f varies over A, the X_f that are affine form a cover of X.* **II** | 95
(e) *Every quasi-coherent $\mathscr{O}_X$-module is generated by its sections over X.*
(e') *Every quasi-coherent sheaf of ideals of $\mathscr{O}_X$ of finite type is generated by its sections over X.*

Proof. It is clear that (b) implies (a), and (a) implies (c) by (4.4.4, b) applied to the identity morphism (taking into account the remark preceding this proposition); Furthermore, (c) implies (c′) (4.5.10, i), and (c′), (b), and (b′) are all equivalent by (4.5.2, b) and (4.5.2, b′). Finally, (c′) is the same as each of (d), (d′), (e), and (e′) by (4.5.2, a), (4.5.2, a′), (4.5.2, c), and (4.5.5, d″). □

We further observe that, with the previous notation, the X_f that are affine form a *basis* for the topology of X, and that the canonical morphism u is *dominant* (4.5.2).

Corollary (5.1.3). — *Let X be a quasi-compact prescheme. If there exists a morphism $v : X \to Y$ from X to some affine scheme Y (which would be a homeomorphism from X to some open subspace of Y), then X is quasi-affine.*

Proof. There exists a family (g_α) of sections of $\mathscr{O}_Y$ over Y such that the $D(g_\alpha)$ form a basis for the topology of $v(X)$; if $v = (\psi, \theta)$ and we set $f_\alpha = \theta(g_\alpha)$, then we have $X_{f_\alpha} = \psi^{-1}(D(g_\alpha))$ (**I**, 2.2.4.1), so the X_{f_α} form a basis for the topology of X, and the criterion (5.1.2, d) is satisfied. □

Corollary (5.1.4). — *If X is a quasi-affine scheme, then* every *invertible $\mathscr{O}_X$-module is very ample (relative to the canonical morphism), and* a fortiori *ample.*

Proof. Such a module $\mathscr{L}$ is generated by its sections over X (5.1.2, e), so $\mathscr{L} \otimes \mathscr{O}_X = \mathscr{L}$ is very ample (4.4.8). □

Corollary (5.1.5). — *Let X be a quasi-compact prescheme. If there exists an invertible $\mathscr{O}_X$-module $\mathscr{L}$ such that $\mathscr{L}$ and $\mathscr{L}^{-1}$ are ample, then X is a quasi-affine scheme.*

Proof. Indeed, $\mathscr{O}_X = \mathscr{L} \otimes \mathscr{L}^{-1}$ is then ample (4.5.7). □

Proposition (5.1.6). — *Let $f : X \to Y$ be a quasi-compact morphism. Then the following conditions are equivalent.*

(a) *The morphism f is quasi-affine.*
(b) *The $\mathscr{O}_Y$-algebra $f_*(\mathscr{O}_X) = \mathscr{A}(X)$ is quasi-coherent, and the canonical morphism $X \to \operatorname{Spec}(\mathscr{A}(X))$ corresponding to the identity morphism $\mathscr{A}(X) \to \mathscr{A}(X)$* (1.2.7) *is an open immersion.*
(b') *The $\mathscr{O}_Y$-algebra $\mathscr{A}(X)$ is quasi-coherent, and the canonical morphism $X \to \operatorname{Spec}(\mathscr{A}(X))$ is a homeomorphism from X to some subspace of* $\operatorname{Spec}(\mathscr{A}(X))$.
(c) *The $\mathscr{O}_X$-module $\mathscr{O}_X$ is very ample for f.*

(c') *The $\mathscr{O}_X$-module $\mathscr{O}_X$ is ample for f.*
(d) *The morphism f is separated, and, for every quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$, the canonical homomorphism $\sigma : f^*(f_*(\mathscr{F})) \to \mathscr{F}$ (**0**, 4.4.3) is surjective.*

Furthermore, whenever f is quasi-affine, every invertible $\mathscr{O}_X$-module $\mathscr{L}$ is very ample relative to f.

Proof. The equivalence between (a) and (c′) follows from the local (on Y) character of the f-ampleness (4.6.4), Definition (5.1.1), and the criterion (5.1.2, c′). The other properties are local on Y and thus follow immediately from (5.1.2) and (5.1.4), taking into account the fact that $f_*(\mathscr{F})$ is quasi-coherent whenever f is separated (**I**, 9.2.2, a). □ **II** | 96

Corollary (5.1.7). — *Let $f : X \to Y$ be a quasi-affine morphism. For every open subset U of Y, the restriction $f^{-1}(U) \to U$ of f is quasi-affine.*

Corollary (5.1.8). — *Let Y be an affine scheme, and $f : X \to Y$ a quasi-compact morphism. For f to be quasi-affine, it is necessary and sufficient for X to be a quasi-affine scheme.*

Proof. This is an immediate consequence of (5.1.6) and (4.6.6). □

Corollary (5.1.9). — *Let Y be a quasi-compact scheme or a prescheme whose underlying space is Noetherian, and $f : X \to Y$ a morphism of* finite type. *If f is quasi-affine, then there exists a quasi-coherent $\mathscr{O}_Y$-subalgebra $\mathscr{B}$ of $\mathscr{A}(X) = f_*(\mathscr{O}_X)$ of* finite type (**I**, 9.6.2) *such that the morphism $X \to \operatorname{Spec}(\mathscr{B})$ corresponding to the canonical injection $\mathscr{B} \to \mathscr{A}(X)$ is an immersion. Further, every quasi-coherent $\mathscr{O}_Y$-subalgebra $\mathscr{B}'$ of finite type over $\mathscr{A}(X)$ containing $\mathscr{B}$ has the same property.*

Proof. Indeed, $\mathscr{A}(X)$ is the inductive limit of its quasi-coherent $\mathscr{O}_Y$-subalgebras of finite type (**I**, 9.6.5); the result is then a particular case of (3.8.4), taking into account the identification of $\operatorname{Spec}(\mathscr{A}(X))$ with $\operatorname{Proj}(\mathscr{A}(X)[T])$ (3.1.7). □

Proposition (5.1.10). —

(i) *A quasi-compact morphism $X \to Y$ that is a homeomorphism from the underlying space of X to some subspace of the underlying space of Y (so, in particular, any closed immersion) is quasi-affine.*
(ii) *The composition of any two quasi-affine morphisms is quasi-affine.*
(iii) *If $f : X \to Y$ is a quasi-affine S-morphism, then $f_{(S')} : X_{(S')} \to Y_{(S')}$ is a quasi-affine morphism for any extension $S' \to S$ of the base prescheme.*
(iv) *If $f : X \to Y$ and $g : X' \to Y'$ are quasi-affine S-morphisms, then $f \times_S g$ is quasi-affine.*
(v) *If $f : X \to Y$ and $g : Y \to Z$ are morphisms such that $g \circ f$ is quasi-affine, and if g is separated or the underlying space of X is locally Noetherian, then f is quasi-affine.*
(vi) *If f is a quasi-affine morphism, then so is f_{red}.*

Proof. Taking into account the criterion (5.1.6, c′), all of (i), (iii), (iv), (v), and (vi) follow immediately from (4.6.13, i *bis*), (4.6.13, iii), (4.6.13, iv), (4.6.13, v), and (4.6.13, vi) (respectively). To prove (ii), we can restrict to the case where Z is affine, and then the claim follows directly from applying (4.6.13, ii) to $\mathscr{L} = \mathscr{O}_X$ and $\mathscr{K} = \mathscr{O}_Y$. □

Remark (5.1.11). — Let $f : X \to Y$ and $g : Y \to Z$ be morphisms such that $X \times_Z Y$ is locally Noetherian. Then the graph immersion $\Gamma_f : X \to X \times_Z Y$ is quasi-affine, since it is quasi-compact (**I**, 6.3.5), and since (**I**, 5.5.12) shows that, in (v), the conclusion still holds true if we remove the hypothesis that g is separated.

Proposition (5.1.12). — *Let $f : X \to Y$ be a quasi-compact morphism, and $g : X' \to X$ a quasi-affine morphism. If $\mathscr{L}$ is an ample (for f) $\mathscr{O}_X$-module, then $g^*(\mathscr{L})$ is an ample (for $f \circ g$) $\mathscr{O}_{X'}$-module.*

Proof. Since $\mathscr{O}_{X'}$ is very ample for g, and the question is local on Y (4.6.4), it follows from (4.6.13, ii) that there exists (for Y affine) an integer n such that

$$g^*(\mathscr{L}^{\otimes n}) = (g^*(\mathscr{L}))^{\otimes n}$$

is ample for $f \circ g$, and so $g^*(\mathscr{L})$ is ample for $f \circ g$ (4.6.9) □

5.2. Serre's criterion.

Theorem (5.2.1). — (Serre's criterion). *Let X be a quasi-compact scheme or a prescheme whose underlying space is Noetherian. The following conditions are equivalent.*

(a) *X is an affine scheme.*

(b) *There exists a family of elements $f_\alpha \in A = \Gamma(X, \mathcal{O}_X)$ such that the X_{f_α} are affine, and such that the ideal generated by the f_α in A is equal to A itself.*

(c) *The functor $\Gamma(X, \mathscr{F})$ is exact in $\mathscr{F}$ on the category of quasi-coherent $\mathcal{O}_X$-modules, or, in other words, if*

$$0 \longrightarrow \mathscr{F}' \longrightarrow \mathscr{F} \longrightarrow \mathscr{F}'' \longrightarrow 0 \tag{*}$$

is an exact sequence of quasi-coherent $\mathcal{O}_X$-modules, then the sequence

$$0 \longrightarrow \Gamma(X, \mathscr{F}') \longrightarrow \Gamma(X, \mathscr{F}) \longrightarrow \Gamma(X, \mathscr{F}'') \longrightarrow 0$$

is also exact.

(c') *Condition* (c) *holds for every exact sequence $(*)$ of quasi-coherent $\mathcal{O}_X$-modules such that $\mathscr{F}$ is isomorphic to a $\mathcal{O}_X$-submodule of $\mathcal{O}_X^n$ for some finite n.*

(d) $\mathrm{H}^1(X, \mathscr{F}) = 0$ *for every quasi-coherent $\mathcal{O}_X$-module $\mathscr{F}$.*

(d′) $\mathrm{H}^1(X, \mathscr{J}) = 0$ *for every quasi-coherent sheaf of ideals $\mathscr{J}$ of $\mathcal{O}_X$.*

Proof. It is evident that (a) implies (b); furthermore, (b) implies that the X_{f_α} cover X, because, by hypothesis, the section 1 is a linear combination of the f_α, and the $D(f_\alpha)$ thus cover $\mathrm{Spec}(A)$. The final claim of (4.5.2) thus implies that $X \to \mathrm{Spec}(A)$ is an isomorphism.

We know tha (a) implies (c) (**I**, 1.3.11), and (c) trivially implies (c′). We now prove that (c′) implies (b). First of all, (c′) implies that, for every *closed* point $x \in X$ and every open neighbourhood U of x, there exists some $f \in A$ such that $x \in X_f \subset X - U$. Let $\mathscr{J}$ (resp. $\mathscr{J}'$) be the quasi-coherent sheaf of ideals of $\mathcal{O}_X$ defining the reduced closed subprescheme of X that has $X - U$ (resp. $(X - U) \cup \{x\}$) as its underlying space (**I**, 5.2.1); it is clear that we have $\mathscr{J}' \subset \mathscr{J}$, and that $\mathscr{J}'' = \mathscr{J} / \mathscr{J}'$ is a quasi-coherent $\mathcal{O}_X$ module that has support equal to $\{x\}$, and such that $\mathscr{J}''_x = k(x)$. Hypothesis (c′) applied to the exact sequence $0 \to \mathscr{J}' \to \mathscr{J} \to \mathscr{J}'' \to 0$ shows that $\Gamma(X, \mathscr{J}) \to \Gamma(X, \mathscr{J}'')$ is surjective. The section of $\mathscr{J}''$ whose germ at x is 1_x is thus the image of some section $f \in \Gamma(X, \mathscr{J}) \subset \Gamma(X, \mathcal{O}_X)$, and we have, by definition, that $f(x) = 1_x$ and $f(y) = 0$ in $X - U$, which establishes our claim. Now, if U is affine, then so is X_f (**I**, 1.3.6), so the union of the X_f that are affine ($f \in A$) is an open set Z that contains *all the closed points* of X; since X is a quasi-compact Kolmogoroff space, we necessarily have $Z = X$ (**0**, 2.1.3). Because X is quasi-compact, there are a *finite* number of elements $f_i \in A$ ($1 \leqslant i \leqslant n$) such that the X_{f_i} are affine and cover X. So consider the homomorphism $\mathcal{O}_X^n \to \mathcal{O}_X$ defined by the sections f_i (**0**, 5.1.1); since, for all $x \in X$, at least one of the $(f_i)_x$ is invertible, this homomorphism is *surjective*, and we thus have an exact sequence $0 \to \mathscr{R} \to \mathcal{O}_X^n \to \mathcal{O}_X \to 0$, where $\mathscr{R}$ is a quasi-coherent $\mathcal{O}_X$-submodule of $\mathcal{O}_X$. It then follows from **II** | 98
(c′) that the corresponding homomorphism $\Gamma(X, \mathcal{O}_X^n) \to \Gamma(X, \mathcal{O}_X)$ is surjective, which proves (b).

Finally, (a) implies (d) (**I**, 5.1.9.2), and (d) trivially implies (d′). It remains to show that (d′) implies (c′). But if $\mathscr{F}'$ is a quasi-coherent $\mathcal{O}_X$-submodule of $\mathcal{O}_X^n$, then the filtration $0 \subset \mathcal{O}_X \subset \mathcal{O}_X^2 \subset \ldots \subset \mathcal{O}_X^n$ defines a filtration of $\mathscr{F}'$ given by the $\mathscr{F}'_k = \mathscr{F} \cap \mathcal{O}_X^k$ ($0 \leqslant k \leqslant n$), which are quasi-coherent $\mathcal{O}_X$-modules (**I**, 4.1.1), and $\mathscr{F}'_{k+1} / \mathscr{F}'_k$ is isomorphic to a quasi-coherent $\mathcal{O}_X$-submodule of $\mathcal{O}_X^{k+1} / \mathcal{O}_X^k = \mathcal{O}_X$, which is to say, a quasi-coherent sheaf of ideals of $\mathcal{O}_X$. Hypothesis (d′) thus implies that $\mathrm{H}^1(X, \mathscr{F}'_{k+1} / \mathscr{F}'_k) = 0$; the exact cohomology sequence $\mathrm{H}^1(X, \mathscr{F}'_k) \to \mathrm{H}^1(X, \mathscr{F}'_{k+1}) \to \mathrm{H}^1(X, \mathscr{F}'_{k+1} / \mathscr{F}'_k) = 0$ then lets us prove by induction on k that $H^1(X, \mathscr{F}'_k) = 0$ for all k. □

Remark (5.2.1.1). — When X is a *Noetherian* prescheme, we can replace "quasi-coherent" by "coherent" in the statements of (c′) and (d′). Indeed, in the proof of the fact that (c′) implies (b), $\mathscr{J}$ and $\mathscr{J}'$ are then *coherent* sheaves of ideals, and, furthermore, every quasi-coherent submodule of a coherent module is coherent (**I**, 6.1.1); whence the conclusion.

Corollary (5.2.2). — *Let $f : X \to Y$ be a separated quasi-compact morphism. The following conditions are equivalent.*

(a) *The morphism f is an affine morphism.*

(b) *The functor f_* is exact on the category of quasi-coherent $\mathcal{O}_X$-modules.*

(c) *For every quasi-coherent $\mathcal{O}_X$-module $\mathscr{F}$, we have* $\mathrm{R}^1 f_*(\mathscr{F}) = 0$.

(c') *for every quasi-coherent sheaf of ideals $\mathscr{J}$ of $\mathcal{O}_X$, we have* $\mathrm{R}^1 f_*(\mathscr{J}) = 0$.

Proof. All these conditions are local on Y, by definition of the functor $\mathrm{R}^1 f_*$ (T, 3.7.3), and so we can assume that Y is affine. If f is affine, then X is affine, and property (b) is nothing more than (**I**, 1.6.4). Conversely, we now show that (b) implies (a): for every quasi-coherent $\mathcal{O}_X$-module $\mathscr{F}$, we have that $f_*(\mathscr{F})$ is a quasi-coherent

$\mathscr{O}_Y$-module (**I**, 9.2.2, a). By hypothesis, the functor $f_*(\mathscr{F})$ is exact in $\mathscr{F}$, and the functor $\Gamma(Y, \mathscr{G})$ is exact in $\mathscr{G}$ (in the category of quasi-coherent $\mathscr{O}_Y$-modules) because Y is affine (**I**, 1.3.11); so $\Gamma(Y, f_*(\mathscr{F})) = \Gamma(X, \mathscr{F})$ is exact in $\mathscr{F}$, which proves our claim, by (5.2.1, c).

If f is affine, then $f^{-1}(U)$ is affine for every affine open subset U of Y (1.3.2), and so $\mathrm{H}^1(f^{-1}(U), \mathscr{F}) = 0$ (5.2.1, d), which, by definition, implies that $\mathrm{R}^1 f_*(\mathscr{F}) = 0$. Finally, suppose that condition (c′) is satisfied; the exact sequence of terms of low degree in the Leray spectral sequence (G, II, 4.17.1 and I, 4.5.1) give, in particular, the exact sequence

$$0 \longrightarrow \mathrm{H}^1(Y, f_*(\mathscr{J})) \longrightarrow \mathrm{H}^1(X, \mathscr{J}) \longrightarrow \mathrm{H}^0(Y, \mathrm{R}^1 f_*(\mathscr{J})).$$

Since Y is affine, and $f_*(\mathscr{J})$ quasi-coherent (**I**, 9.2.2, a), we have that $\mathrm{H}^1(Y, f_*(\mathscr{J})) = 0$ (5.2.1); hypothesis (c′) thus implies that $\mathrm{H}^1(X, \mathscr{J}) = 0$, and we conclude, by (5.2.1), that X is an affine scheme. □

Corollary (5.2.3). — *If $f : X \to Y$ is an affine morphism, then, for every quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$, the canonical homomorphism $\mathrm{H}^1(Y, f_*(\mathscr{F})) \to \mathrm{H}^1(X, \mathscr{F})$ is bijective.*

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Proof. We have the exact sequence

$$0 \longrightarrow \mathrm{H}^1(Y, f_*(\mathscr{F})) \longrightarrow \mathrm{H}^1(X, \mathscr{F}) \longrightarrow \mathrm{H}^0(Y, \mathrm{R}^1 f_*(\mathscr{F}))$$

of terms of low degree in the Leray spectral sequence, and the conclusion follows from (5.2.2). □

Remark (5.2.4). — In Chapter III, §1, we prove that, if X is affine, then we have $\mathrm{H}^i(X, \mathscr{F}) = 0$ for all $i > 0$ and all quasi-coherent $\mathscr{O}_X$-modules $\mathscr{F}$.

5.3. Quasi-projective morphisms.

Definition (5.3.1). — We say that a morphism $f : X \to Y$ is *quasi-projective*, or that X (considered as a Y-prescheme via f) is *quasi-projective over Y*, or that X is a *quasi-projective Y-scheme*, if f is of finite type and there exists an invertible f-ample $\mathscr{O}_X$-module.

We note that this notion *is not local on Y*: the counterexamples of Nagata [**Nag58b**] and Hironaka show that, even if X and Y are non-singular algebraic schemes over an algebraically closed field, every point of Y can have an affine neighbourhood U such that $f^{-1}(U)$ is quasi-projective over U, without f being quasi-projective.

We note that a quasi-projective morphism is necessarily *separated* (4.6.1). When Y is quasi-compact, it is equivalent to say either that f is quasi-projective, or that f is of finite type and there exists a *very ample* (relative to f) $\mathscr{O}_X$-module ((4.6.2) and (4.6.11)). Further:

Proposition (5.3.2). — *Let Y be a quasi-compact scheme or a prescheme whose underlying space is Noetherian, and let X be a Y-prescheme. The following conditions are equivalent.*

(a) *X is a quasi-projective Y-scheme.*
(b) *X is of finite type over Y, and there exists some quasi-coherent $\mathscr{O}_Y$-module $\mathscr{E}$ of finite type such that X is Y-isomorphic to a subprescheme of* $\mathbf{P}(\mathscr{E})$.
(c) *X is of finite type over Y, and there exists some quasi-coherent graded $\mathscr{O}_Y$-algebra $\mathscr{S}$ such that $\mathscr{S}_1$ is of finite type and generates $\mathscr{S}$, and such that X is Y-isomorphic to a induced subprescheme on some everywhere-dense open subset of* $\mathrm{Proj}(\mathscr{S})$.

Proof. This follows immediately from the previous remark and from (4.4.3), (4.4.6), and (4.4.7). □

We note that, whenever Y is a *Noetherian* prescheme, we can, in conditions (b) and (c) of (5.3.2), remove the hypothesis that X is of finite type over Y, since this automatically satisfied (**I**, 6.3.5).

Corollary (5.3.3). — *Let Y be a quasi-compact scheme such that there exists an ample $\mathscr{O}_Y$-module $\mathscr{L}$* (4.5.3). *For a Y-scheme X to be quasi-projective, it is necessary and sufficient for it to be of finite type over Y and also isomorphic to a Y-subscheme of a projective bundle of the form $\mathbf{P}^r_Y$.*

Proof. If $\mathscr{E}$ is a quasi-coherent $\mathscr{O}_Y$-module of finite type, then $\mathscr{E}$ is isomorphic to a quotient of an $\mathscr{O}_Y$-module $\mathscr{L}^{\otimes(-n)} \otimes_{\mathscr{O}_Y} \mathscr{O}_Y^k$ (4.5.5), and so $\mathbf{P}(\mathscr{E})$ is isomorphic to a closed subscheme of $\mathbf{P}_Y^{k-1}$ ((4.1.2) and (4.1.4)). □

Proposition (5.3.4). —

(i) *A quasi-affine morphism of finite type (and, in particular, a quasi-compact immersion, or an affine morphism of finite type) is quasi-projective.*
(ii) *If $f : X \to Y$ and $g : Y \to Z$ are quasi-projective, and if Z is quasi-compact, then $g \circ f$ is quasi-projective.* **II** | 100
(iii) *If $f : X \to Y$ is a quasi-projective S-morphism, then $f_{(S')} : X_{(S')} \to Y_{(S')}$ is quasi-projective for every extension $S' \to S$ of the base prescheme.*

(iv) *If $f : X \to Y$ and $g : X' \to Y'$ are quasi-projective S-morphisms, then $f \times_S g$ is quasi-projective.*
(v) *If $f : X \to Y$ and $g : Y \to Z$ are morphisms such that $g \circ f$ is quasi-projective, and if g is separated or X locally Noetherian, then f is quasi-projective.*
(vi) *If f is a quasi-projective morphism, then so is f_{red}.*

Proof. (i) follows from (5.1.6) and (5.1.10, i). The other claims are immediate consequences of Definition (5.3.1), of the properties of morphisms of finite type (**I**, 6.3.4), and of (4.6.13). ☐

Remark (5.3.5). — We note that we can have f_{red} being quasi-projective without f being quasi-projective, even if we assume that Y is the spectrum of an algebra of finite rank over $\mathbf{C}$ and that f is proper.

Corollary (5.3.6). — *If X and X' are quasi-projective Y-schemes, then $X \sqcup X'$ is a quasi-projective Y-scheme.*

Proof. This follows from (4.6.18). ☐

5.4. Proper morphisms and universally closed morphisms.

Definition (5.4.1). — We say that a morphism of preschemes $f : X \to Y$ is *proper* if it satisfies the following two conditions:

(a) f is separated and of finite type; and
(b) for every prescheme Y' and every morphism $Y' \to Y$, the projection $f_{(Y')} : X \times_Y Y' \to Y'$ is a closed morphism (**I**, 2.2.6).

When this is the case, we also say that X (considered as a Y-prescheme with structure morphism f) is proper over Y.

It is immediate that conditions (a) and (b) are *local* on Y. To show that the image of a closed subset Z of $X \times_Y Y'$ under the projection $q : X \times_Y Y' \to Y'$ is closed in Y, it suffices to see that $q(Z) \cap U'$ is closed in U' for every affine open subset U' of Y'; since $q(Z) \cap U' = q(Z \cap q^{-1}(U'))$, and since $q^{-1}(U')$ can be identified with $X \times_Y U'$ (**I**, 4.4.1), we see that to satisfy condition (b) of Definition (5.4.1), we can restrict to the case where Y is an *affine* scheme. We further see (5.3.6) that, if Y is locally Noetherian, then we can even restrict to proving (b) in the case where Y' is of finite type over Y.

It is clear that every proper morphism is *closed*.

Proposition (5.4.2). —

(i) *A closed immersion is a proper morphism.*
(ii) *The composition of two proper morphisms is proper.*
(iii) *If X and Y are S-preschemes, and $f : X \to Y$ a proper S-morphism, then $f_{(S')} : X_{(S')} \to Y_{(S')}$ is proper for every extension $S' \to S$ of the base prescheme.*
(iv) *If $f : X \to Y$ and $g : X' \to Y'$ are proper S-morphisms, then $f \times_S g : X \times_S Y \to X' \times_S Y'$ is a proper S-morphism.* **II** | 101

Proof. It suffices to prove (i), (ii), and (iii) (**I**, 3.5.1). In each of the three cases, verifying condition (a) of (5.4.1) follows from previous results ((**I**, 5.5.1) and (6.4.3)); it remains to verify condition (b). It is immediate in case (i), because if $X \to Y$ is a closed immersion, then so is $X \times_Y Y' \to Y \times_Y Y' = Y'$ ((**I**, 4.3.2) and (3.3.3)). To prove (ii), consider two proper morphisms $X \to Y$ and $Y \to Z$, and a morphism $Z' \to Z$. We can write $X \times_Z Z' = X \times_Y (Y \times_Z Z')$ (**I**, 3.3.9.1), and so the projection $X \times_Z Z' \to Z'$ factors as $X \times_Y (Y \times_Z Z') \to Y \times_Z Z' \to Z'$. Taking the initial remark into account, (ii) follows from the fact that the composition of two closed morphisms is closed. Finally, for every morphism $S' \to S$, we can identify $X_{(S')}$ with $X \times_Y Y_{(S')}$ (**I**, 3.3.11); for every morphism $Z \to Y_{(S')}$, we can write

$$X_{(S')} \times_{Y_{(S')}} Z = (X \times_Y Y_{(S')}) \times_{Y_{(S')}} Z = X \times_Y Z;$$

since by hypothesis $X \times_Y Z \to Z$ is closed, this proves (iii). ☐

Corollary (5.4.3). — *Let $f : X \to Y$ and $g : Y \to Z$ be morphisms such that $g \circ f$ is proper.*

(i) *If g is separated, then f is proper.*
(ii) *If g is separated and of finite type, and if f is surjective, then g is proper.*

Proof. (i) follows from (5.4.2) by the general procedure (**I**, 5.5.12). To prove (ii), we need only verify that condition (b) of Definition (5.4.1) is satisfied. For every morphism $Z' \to Z$, the diagram

$$\begin{tikzcd} X \times_Z Z' \arrow[r, "f \times 1_{Z'}"] \arrow[dr, "p"'] & Y \times_Z Z' \arrow[d, "p'"] \\ & Z' \end{tikzcd}$$

(where p and p' are the projections) commutes (**I**, 3.2.1); furthermore, $f \times 1_{Z'}$ is surjective because f is surjective (**I**, 3.5.2), and p is a closed morphism by hypothesis. Every closed subset F of $Y \times_Z Z'$ is thus the image under $f \times 1_{Z'}$ of some closed subset E of $X \times_Z Z'$, so $p'(F) = p(E)$ is closed in Z' by hypothesis, whence the corollary. □

Corollary (5.4.4). — *If X is a proper prescheme over Y, and $\mathscr{S}$ a quasi-coherent $\mathscr{O}_Y$-algebra, then every Y-morphism $f : X \to \mathrm{Proj}(\mathscr{S})$ is proper (and* a fortiori *closed).*

Proof. The structure morphism $p : \mathrm{Proj}(\mathscr{S}) \to Y$ is separated, and $p \circ f$ is proper by hypothesis. □

Corollary (5.4.5). — *Let $f : X \to Y$ be a separated morphism of finite type. Let $(X_i)_{1\leqslant i\leqslant n}$ (resp. $(Y_i)_{1\leqslant i\leqslant n}$) be a finite family of closed subpreschemes of X (resp. Y), and j_i (resp. h_i) the canonical injection $X_i \to X$ (resp. $Y_i \to Y$). Suppose that the underlying space of X is the union of the X_i, and that, for all i, there is a morphism $f_i : X_i \to Y_i$, such that the diagram*

$$\begin{tikzcd} X_i \arrow[r, "f_i"] \arrow[d, "j_i"'] & Y_i \arrow[d, "h_i"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

commutes. Then, for f to be proper, it is necessary and sufficient for all of the f_i to be proper.

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Proof. If f is proper, then so is $f \circ j_i$, because j_i is a closed immersion (5.4.2); since h_i is a closed immersion, and thus a separated morphism, f_i is proper, by (5.4.3). Conversely, suppose that all of the f_i are proper, and consider the prescheme Z given by the *sum* of the X_i; let u be the morphism $Z \to X$ which reduces to j_i on each X_i. The restriction of $f \circ u$ to each X_i is equal to $f \circ j_i = h_i \circ f_i$, and is thus proper, because both the h_i and the f_i are (5.4.2); it then follows immediately from Definition (5.4.1) that u is proper. But since by hypothesis u is surjective, we conclude that f is proper by (5.4.3). □

Corollary (5.4.6). — *Let $f : X \to Y$ be a separated morphism of finite type; for f to be proper, it is necessary and sufficient for $f_{\mathrm{red}} : X_{\mathrm{red}} \to Y_{\mathrm{red}}$ to be proper.*

Proof. This is a particular case of (5.4.5), with $n = 1$, $X_1 = X_{\mathrm{red}}$, and $Y_1 = Y_{\mathrm{red}}$ (**I**, 5.1.5). □

(5.4.7). If X and Y are Noetherian preschemes, and $f : X \to Y$ a separated morphism of finite type, then we can, to show that f is proper, restrict to the the case of *dominant* morphisms and *integral* preschemes. Indeed, let X_i $(1 \leqslant i \leqslant n)$ be the (finitely many) irreducible components of X, and consider, for each i, the unique reduced closed subprescheme of X that has X_i as its underlying space, which we again denote by X_i (**I**, 5.2.1). Let Y_i be the unique reduced closed subprescheme of Y that has $\overline{f(X_i)}$ as its underlying space. If g_i (resp. h_i) is the injection morphism $X_i \to X$ (resp. $Y_i \to Y$), then we conclude that $f \circ g_i = h_i \circ f_i$, where f_i is a dominant morphism $X_i \to Y_i$ (**I**, 5.2.2); we are then under the right conditions to apply (5.4.5), and for f to be proper, it is necessary and sufficient for all the f_i to be proper.

Corollary (5.4.8). — *Let X and Y be separated S-preschemes of finite type over S, and $f : X \to Y$ an S-morphism. For f to be proper, it is necessary and sufficient that, for every S-prescheme S', the morphism $f \times_S 1_{S'} : X \times_S S' \to Y \times_S S'$ be closed.*

Proof. First note that, if $g : X \to S$ and $h : Y \to S$ are the structure morphisms, then we have, by definition, $g = h \circ f$, and so f is separated and of finite type ((**I**, 5.5.1) and (6.3.4)). If f is proper, then so is $f \times_S 1_{S'}$ (5.4.2); *a fortiori*, $f \times_S 1_{S'}$ is closed. Conversely, suppose that the conditions of the statement are satisfied, and let Y' be a Y-prescheme; Y' can also be considered as an S-prescheme, and since $Y \to S$ is separated, $X \times_Y Y'$ can be identified with a closed subprescheme of $X \times_S Y'$ (**I**, 5.4.2). In the commutative diagram

$$\begin{tikzcd} X \times_Y Y' \arrow[r, "f \times 1_{Y'}"] \arrow[d] & Y \times_Y Y' = Y' \arrow[d] \\ X \times_S Y' \arrow[r, "f \times 1_{S'}"] & Y \times_S Y', \end{tikzcd}$$

the vertical arrows are closed immersions; it thus immediately follows that if $f \times 1_{S'}$ is a closed morphism, then so is $f \times 1_{Y'}$ □

Remark (5.4.9). — We say that a morphism $f : X \to Y$ is *universally closed* if it satisfies condition (b) of Definition (5.4.1). The reader will observe that, in (5.4.2) to (5.4.8), we can replace every occurrence of "proper" with "universally closed" without changing the validity of the results (and in the hypotheses of (5.4.3), (5.4.5), (5.4.6), and (5.4.8), we can omit the finiteness conditions). **II | 103**

(5.4.10). Let $f : X \to Y$ be a morphism of finite type. We say that a closed subset Z of X is *proper on* Y (or Y*-proper*, or *proper for* f) if the restriction of f to a closed subprescheme of X, with underlying space Z (**1**, 5.2.1), is *proper*. Since this restriction is then separated, it follows from (5.4.6) and (**I**, 5.5.1, vi) that the preceding property *does not depend* on the closed subprescheme of X that has Z as its underlying space. If $g : X' \to X$ is a *proper* morphism, then $g^{-1}(Z)$ is a *proper* subset of X': if T is a subprescheme of X that has Z as its underlying space, it suffices to note that the restriction of g to the closed subprescheme $g^{-1}(T)$ of X' is a proper morphism $g^{-1}(T) \to T$, by (5.4.2, iii), and to then apply (5.4.2, ii). Further, if X'' is a Y*-scheme* of finite type, and $u : X \to X''$ a Y-morphism, then $u(Z)$ is a *proper* subset of X''; indeed, let us take T to be the reduced closed subprescheme of X having Z as its underlying space; then the restriction of f to T is proper, and thus so is the restriction of u to T (5.4.3, i), thus $u(Z)$ is closed in X''; let T'' be a closed subprescheme of X'' having $u(Z)$ as its underlying space (**I**, 5.2.1), such that $u|T$ factors as $T \xrightarrow{v} T'' \xrightarrow{j} X''$, where j is the canonical injection (**I**, 5.2.2), and v is thus proper and surjective (5.4.5); if g is the restriction to T'' of the structure morphism $X'' \to Y$, then g is separated and of finite type, and we have that $f|T = g \circ v$; it thus follows from (5.4.3, ii) that g is proper, whence our assertion.

It follows, in particular, from these remarks that, if Z is a Y-proper subset of X, then

(1) for every closed subprescheme X' of X, $Z \cap X'$ is a Y-proper subset of X'; and
(2) if X is a subprescheme of a Y-scheme of finite type X'', then Z is also a Y-proper subset of X'' (and so, in particular, is *closed in* X'').

5.5. Projective morphisms.

Proposition (5.5.1). — *Let X be a Y-prescheme. The following conditions are equivalent.*

(a) *X is Y-isomorphic to a* closed *subprescheme of a projective bundle* $\mathbf{P}(\mathscr{E})$*, where $\mathscr{E}$ is a quasi-coherent $\mathscr{O}_Y$-module of finite type.*
(b) *There exists a quasi-coherent graded $\mathscr{O}_Y$-algebra $\mathscr{S}$ such that $\mathscr{S}_1$ is of finite type and generates $\mathscr{S}$, and such that X is Y-isomorphic to* $\operatorname{Proj}(\mathscr{S})$.

Proof. Condition (a) implies (b), by (3.6.2, ii): if $\mathscr{J}$ is a quasi-coherent graded sheaf of ideals of $\mathbf{S}(\mathscr{E})$, then the quasi-coherent graded $\mathscr{O}_Y$-algebra $\mathscr{S} = \mathbf{S}(\mathscr{E})/\mathscr{J}$ is generated by $\mathscr{S}_1$, and $\mathscr{S}_1$, the canonical image of $\mathscr{E}$, is an $\mathscr{O}_Y$-module of finite type. Condition (b) implies (a) by (3.6.2) applied to the case where $\mathscr{M} \to \mathscr{S}_1$ is the identity map. ☐

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Definition (5.5.2). — We say that a Y-prescheme X is *projective* on Y, or is a projective Y-scheme, if it satisfies either of the (equivalent) conditions (a) and (b) of (5.5.1). We say that a morphism $f : X \to Y$ is projective if it makes X a projective Y-scheme.

It is clear that if $f : X \to Y$ is projective, then there exists a *very ample* (relative to f) $\mathscr{O}_X$-module (4.4.2).

Theorem (5.5.3). —

(i) *Every projective morphism is quasi-projective and proper.*
(ii) *Conversely, let Y be a quasi-compact scheme or a prescheme whose underlying space is Noetherian; then every morphism $f : X \to Y$ that is quasi-projective and proper is projective.*

Proof.

(i) It is clear that if $f : X \to Y$ is projective, then it is of finite type and quasi-projective (thus, in particular, separated); furthermore, it follows immediately from (5.5.1, b) and (3.5.3) that if f is projective, then so is $f \times_Y 1_{Y'} : X \times_Y Y' \to Y'$ for every morphism $Y' \to Y$. To show that f is universally closed, it is thus enough to show that a projective morphism f is *closed*. Since the question is local on Y, we can suppose that $Y = \operatorname{Spec}(A)$, thus (5.5.1) $X = \operatorname{Proj}(S)$, where S is a graded A-algebra generated by a finite number of elements of S_1. For all $y \in Y$, the fibre $f^{-1}(y)$ can be identified with $\operatorname{Proj}(S) \times_Y \operatorname{Spec}(k(y))$ (**I**, 3.6.1), and so also with $\operatorname{Proj}(S \otimes_A k(y))$ (2.8.10); so $f^{-1}(y)$ is empty if and only if $S \otimes_A k(y)$ satisfies condition (TN) (2.7.4), or, in other words, if $S_n \otimes_A k(y) = 0$ for sufficiently large n. But since $(S_n)_y$ is an $\mathscr{O}_y$-module of finite type, the preceding condition implies that $(S_n)_y = 0$ for sufficiently large N, by

Nakayama's lemma. If $\mathfrak{a}_n$ is the annihilator in A of the A-module S_n, then the preceding condition also implies that $\mathfrak{a}_n \subset \mathfrak{j}_n$ for sufficiently large n (**0**, 1.7.4). But since $S_n S_1 = S_{n+1}$, by hypothesis, we have that $\mathfrak{a}_n \subset \mathfrak{a}_{n+1}$, and if $\mathfrak{a}$ is the union of the $\mathfrak{a}_n$, then we see that $f(X) = V(\mathfrak{a})$, which proves that $f(X)$ is closed in Y. If now X' is an arbitrary closed subset of X, then there exists a closed subprescheme of X that has X' as its underlying space (**I**, 5.2.1), and it is clear (5.5.1, a) that the morphism $X' \to X \xrightarrow{f} Y$ is projective, and so $f(X')$ is closed in Y.

(ii) The hypothesis on Y and the fact that f is quasi-projective implies the existence of a quasi-coherent $\mathscr{O}_Y$-module $\mathscr{E}$ of finite type, as well as a Y-immersion $j : X \to \mathbf{P}(\mathscr{E})$ (5.3.2). But since f is proper, j is *closed*, by (5.4.4), and so f is projective.

□

Remark (5.5.4). —

(i) Let $f : X \to Y$ be a morphism such that f is proper, such that there exists a *very ample* (relative to f) $\mathscr{O}_X$-module $\mathscr{L}$, and such that the quasi-coherent $\mathscr{O}_Y$-module $\mathscr{E} = f_*(\mathscr{L})$ is *of finite type*. Then f is a *projective* morphism: indeed (4.4.4), there is then a Y-immersion $r : X \to \mathbf{P}(\mathscr{E})$, and, since f is proper, r is a *closed* immersion (5.4.4). We will see in Chapter III, §3, that when Y is *locally Noetherian*, the third condition above ($\mathscr{E}$ being of finite type) is a consequence of the first two, and so the first two conditions *characterise*, in this case, the projective morphisms, and if Y is quasi-compact, then we can replace the second condition (the existence of a very ample (relative to f) $\mathscr{O}_X$-module $\mathscr{L}$) by the hypothesis that there exists an *ample* (relative to f) $\mathscr{O}_X$-module (4.6.11).

(ii) Let Y be a quasi-compact scheme such that there exists an ample $\mathscr{O}_Y$-module. For a Y-scheme X to be *projective*, it is necessary and sufficient for it to be Y-isomorphic to a *closed* Y-subscheme of a projective bundle of the form $\mathbf{P}_Y^r$. The condition is clearly sufficient. Conversely, if X is projective over Y, then **II** | 105
it is quasi-projective, and so there exists a Y-immersion j of X into some $\mathbf{P}_Y^r$ (5.3.3) that is *closed*, by (5.4.4) and (5.5.3).

(iii) The argument of (5.5.3) shows that, for every prescheme Y and every integer $r \geqslant 0$, the structure morphism $\mathbf{P}_Y^r \to Y$ is *surjective*, because if we set $\mathscr{S} = \mathbf{S}_{\mathscr{O}_Y}(\mathscr{O}_Y^{r+1})$, then we evidently have $\mathscr{S}_y = \mathbf{S}_{k(y)}(k(y)^{r+1})$ (1.7.3), and so $(\mathscr{S}_n)_y \neq 0$ for any $y \in Y$ or any $n \geqslant 0$.

(iv) It follows from the examples of Nagata [**Nag58b**] that there exist proper morphisms that are not quasi-projective.

Proposition (5.5.5). —

(i) *A closed immersion is a projective morphism.*

(ii) *If $f : X \to Y$ and $g : Y \to Z$ are projective morphisms, and if Z is a quasi-compact scheme or a prescheme whose underlying space is Noetherian, then $g \circ f$ is projective.*

(iii) *If $f : X \to Y$ is a projective S-morphism, then $f_{(S')} : X_{(S')} \to Y_{(S')}$ is projective for every extension $S' \to S$ of the base prescheme.*

(iv) *If $f : X \to Y$ and $g : X' \to Y'$ are projective S-morphisms, then so is $f \times_S g$.*

(v) *If $g \circ f$ is a projective morphism, and if g is separated, then f is projective.*

(vi) *If f is projective, then so is f_{red}.*

Proof. (i) follows immediately from (3.1.7). We have to show (iii) and (iv) separately, because of the restriction introduced on Z in (ii) (cf. (**I**, 3.5.1)). To show (iii), we restrict to the case where $S = Y$ (**I**, 3.3.11), and the claim then immediately follows from (5.5.1, b) and (3.5.3). To show (iv), we are immediately led to the case where $X = \mathbf{P}(\mathscr{E})$ and $X = \mathbf{P}(\mathscr{E}')$, where $\mathscr{E}$ (resp. $\mathscr{E}'$) is a quasi-coherent $\mathscr{O}_Y$-module (resp. quasi-coherent $\mathscr{O}_{Y'}$-module) of finite type. Let p and p' be the canonical projections of $T = Y \times_S Y'$ to Y and Y' (respectively); by (4.1.3.1), we have $\mathbf{P}(p^*(\mathscr{E})) = \mathbf{P}(\mathscr{E}) \times_Y T$ and $\mathbf{P}(p'^*(\mathscr{E}')) = \mathbf{P}(\mathscr{E}') \times_{Y'} T$; whence

$$\begin{aligned}\mathbf{P}(p^*(\mathscr{E})) \times_T \mathbf{P}(p'^*(\mathscr{E}')) &= (\mathbf{P}(\mathscr{E}) \times_Y T) \times_T (T \times_{Y'} \mathbf{P}(\mathscr{E}')) \\ &= \mathbf{P}(\mathscr{E}) \times_Y (T \times_{Y'} \mathbf{P}(\mathscr{E}')) = \mathbf{P}(\mathscr{E}) \times_S \mathbf{P}(\mathscr{E}')\end{aligned}$$

by replacing T with $Y \times_S Y'$, and using (**I**, 3.3.9.1). But $p^*(\mathscr{E})$ and $p'^*(\mathscr{E}')$ are of finite type over T (**0**, 5.2.4), and thus so is $p^*(\mathscr{E}) \otimes_{\mathscr{O}_T} p'^*(\mathscr{E}')$; since $\mathbf{P}(p^*(\mathscr{E})) \times_T \mathbf{P}(p'^*(\mathscr{E}'))$ can be identified with a closed subprescheme of $p^*(\mathscr{E}) \otimes_{\mathscr{O}_T} p'^*(\mathscr{E}')$ (4.3.3), this proves (iv). To show (v) and (vi), we can apply (**I**, 5.5.13), because every closed subprescheme of a projective Y-scheme is a projective Y-scheme, by (5.5.1, a).

It remains to prove (ii); by the hypothesis on Z, this follows from (5.5.3), (5.3.4, ii), and (5.4.2, ii). □

Proposition (5.5.6). — *If X and X' are projective Y-schemes, then $X \sqcup X'$ is a projective Y-scheme.*

Proof. This is an evident consequence of (5.5.2) and (4.3.6). ☐

Proposition (5.5.7). — *Let X be a projective Y-scheme, and $\mathscr{L}$ a Y-ample $\mathscr{O}_X$-module; then, for every section f of $\mathscr{L}$ over X, X_f is affine over Y.*

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Proof. Since the question is local on Y, we can assume that $Y = \operatorname{Spec}(A)$; furthermore, $X_{f^{\otimes n}} = X_f$, so by replacing $\mathscr{L}$ with some suitable $\mathscr{L}^{\otimes n}$, we can assume that $\mathscr{L}$ is very ample relative to the structure morphism $q : X \to Y$ (4.6.11). The canonical homomorphism $\sigma : q^*(q_*(\mathscr{L})) \to \mathscr{L}$ is thus surjective, and the corresponding morphism

$$r = r_{\mathscr{L},\sigma} : X \longrightarrow P = \mathbf{P}(q_*(\mathscr{L}))$$

is an immersion such that $\mathscr{L} = r^*(\mathscr{O}_P(1))$ (4.4.4); furthermore, since X is proper over Y, the immersion r is closed (5.4.4). But by definition, $f \in \Gamma(Y, q_*(\mathscr{L}))$, and $\sigma^\flat$ is the identity of $q_*(\mathscr{L})$; it then follows from Equation (3.7.3.1) that we have $X_f = r^{-1}(D_+(f))$; so X_f is a closed subprescheme of the affine scheme $D_+(f)$, and is thus also an affine scheme. ☐

In the particular case where $Y = X$, we obtain (taking (4.6.13, i) into account) the following corollary, whose direct proof is immediate anyway:

Corollary (5.5.8). — *Let X be a prescheme, and $\mathscr{L}$ an invertible $\mathscr{O}_X$-module. For every section f of $\mathscr{L}$ over X, X_f is affine over X (and thus also an affine scheme whenever X is an affine scheme).*

5.6. Chow's lemma.

Theorem (5.6.1). — (Chow's lemma). *Let S be a prescheme, and X an S-scheme of finite type. Suppose that the following conditions are satisfied:*

(a) *S is Noetherian;*
(b) *S is a quasi-compact scheme, and X has a finite number of irreducible components.*

Under these hypotheses,

(i) *there exists a* quasi-projective *S-scheme X', and an S-morphism $f : X' \to X$ that is both*projective *and* surjective*;*
(ii) *we can take X' and f to be such that there exists an open subset $U \subset X$ for which $U' = f^{-1}(U)$ is dense in X', and for which the restriction of f to U' is an isomorphism $U' \simeq U$; and*
(iii) *if X is reduced (resp. irreducible, integral), then we can assume that X' is reduced (resp. irreducible, integral).*

Proof. The proof proceeds in multiple steps.

(A) We can first restrict to the case where X is *irreducible.* Indeed, in hypothesis (a), X is Noetherian, and so, in the two hypotheses, the irreducible components X_i of X are finite in number. If the theorem is shown to be true for each of the reduced closed preschemes of X having the X_i as their underlying spaces, and if X'_i and $f_i : X'_i \to X_i$ are the prescheme and the morphism corresponding to X_i (respectively), then the prescheme X' given by the *sum* of the X'_i, and the morphism $f : X' \to X$ whose restriction to each X'_i is $j_i \circ f_i$ (where j_i is the canonical injection $X_i \to X$) satisfy the conclusion of the theorem. It is immediate that X' is reduced if all of the X'_i are; furthermore, we can satisfy (ii) by taking U to be the union of the sets $U_i \cap \complement\left(\bigcup_{j\neq i} X_j\right)$. Finally, since the X'_i are quasi-projective over S, so is X' (5.3.6); **II | 107**
similarly, the morphisms $X'_i \to X$ are projective by (5.5.5, i) and (5.5.5, ii), and so f is projective (5.5.6), and is clearly surjective, by definition.

(B) Now suppose that X is *irreducible.* Since the structure morphism $r : X \to S$ is of finite type, there exists a finite cover (S_i) of S by affine open subsets, and for each i there is a finite cover (T_{ij}) of $r^{-1}(S_i)$ by affine open subsets, and the morphisms $T_{ij} \to S_i$ are of finite type, and so quasi-projective (5.3.4, i); since in both hypotheses (a) and (b) the immersion $S_i \to S$ is quasi-compact, it is also quasi-projective (5.3.4, i), and so the restriction of r to T_{ij} is a quasi-projective morphism (5.3.4, ii). Denote the T_{ij} by U_k $(1 \leqslant k \leqslant n)$. There exists, for each index k, an open immersion $\varphi_k : U_k \to P_k$, where P_k is projective over S ((5.3.2) and (5.5.2)). Let $U = \bigcap_k U_k$; since X is irreducible, and the U_k nonempty, U is nonempty, and thus dense in X; the restrictions of the φ_k to U define a morphism

$$\varphi : U \longrightarrow P = P_1 \times_S P_2 \times_S \cdots \times_S P_n$$

such that the diagrams

(5.6.1.1)
$$\begin{tikzcd} U \arrow[r, "\varphi"] \arrow[d, "j_k"'] & P \arrow[d, "p_k"] \\ U_k \arrow[r, "\varphi_k"] & P_k \end{tikzcd}$$

commute, where j_k is the canonical injection $U \to U_k$, and p_k the canonical projection $P \to P_k$. If j is the canonical injection $U \to X$, then the morphism $\psi = (j, \varphi)_S : U \to X \times_S P$ is an *immersion* (**I**, 5.3.14). In hypothesis (a), $X \times_S P$ is locally Noetherian ((3.4.1), (**I**, 6.3.7), and (**I**, 6.3.8)); in hypothesis (b), $X \times_S P$ is a quasi-compact scheme ((**I**, 5.5.1) and (**I**, 6.6.4)); in both cases, the *closure* X' in $X \times_S P$ of the subprescheme Z associated to ψ (and so with underlying space $\psi(U)$) exists, and ψ factors as

(5.6.1.2)
$$\psi : U \xrightarrow{\psi'} X' \xrightarrow{h} X \times_S P$$

where ψ' is an *open immersion* and h a *closed immersion* (**I**, 9.5.10). Let $q_1 : X \times_S P \to X$ and $q_2 : X \times_S P \to P$ be the canonical projections; we set

(5.6.1.3)
$$f : X' \xrightarrow{h} X \times_S P \xrightarrow{q_1} X,$$

(5.6.1.4)
$$g : X' \xrightarrow{h} X \times_S P \xrightarrow{q_2} P.$$

We will see that X' and f satisfy the conclusion of the theorem.

(C) First we show that f is *projective* and *surjective*, and that the restriction of f to $U' = f^{-1}(U)$ is an *isomorphism* from U' to U. Since the P_k are projective over S, so is P (5.5.5, iv), and so $X \times_S P$ is projective over X (5.5.5, iii), and thus so is X', which is a closed subprescheme of $X \times_S P$. Furthermore, we have $f \circ \psi' = q_1 \circ (h \circ \psi') = q_1 \circ \psi = j$, so $f(X')$ contains the open everywhere-dense subset U of X; but f is a *closed* morphism (5.5.3), so $f(X') = X$. Now note that $q_1^{-1}(U) = U \times_S P$ is induced on an open subset of $X \times_S P$, and, by definition, the prescheme $U' = h^{-1}(U \times_S P)$ is induced by X' on the open subset U'; it is thus the closure *relative to* $U \times_S P$ of the prescheme Z (**I**, 9.5.8). But the immersion **II** | 108
ψ factors as $U \xrightarrow{\Gamma_\varphi} U \times_S P \xrightarrow{j \times 1} X \times_S P$, and since P is separated over S, the graph morphism Γ_φ is a closed immersion (**I**, 5.4.3), and so Z is a *closed* subprescheme of $U \times_S P$, whence $U' = Z$. Since ψ is an immersion, the restriction of f to U' is an isomorphism onto U, and the inverse of ψ'; finally, by the definition of X', U' is dense in X'.

(D) We now show that g is an *immersion*, which will imply that X' is *quasi-projective* over S, because P is projective over S. Set

$$\begin{aligned} V_k &= \varphi_k(U_k) \quad \text{(open subset of } P_k) \\ W_k &= p_k^{-1}(V_k) \quad \text{(open subset of } P) \\ U'_k &= f^{-1}(U_k) \quad \text{(open subset of } X') \\ U''_k &= g^{-1}(W_k) \quad \text{(open subset of } X'). \end{aligned}$$

It is clear that the U'_k form an open cover of X'; we will first see that the U''_k also form an open cover of X', by showing that $U'_k \subset U''_k$. For this, it will suffice to show that the diagram

(5.6.1.5)
$$\begin{tikzcd} U'_k \arrow[r, "g|U'_k"] \arrow[d, "f|U'_k"'] & P \arrow[d, "p_k"] \\ U_k \arrow[r, "\varphi_k"] & P_k \end{tikzcd}$$

commutes. But the prescheme $U'_k = h^{-1}(U_k \times_S P)$ is induced by X' on the open subset U'_k, and is thus the closure of $Z = U' \subset U'_k$ relative to U'_k (**I**, 9.5.8). To show the commutativity of (5.6.1.5), it thus suffices (since P_k is an S-scheme) to show that composing the diagram with the canonical injection $U' \to U'_k$ (or, equivalently, thanks to the isomorphism from U' to U, with ψ) gives us a commutative diagram (**I**, 9.5.6). But, by definition, the diagram thus obtains is exactly (5.6.1.1), whence our claim.

The W_k thus form an open cover of $g(X')$; to show that g is an immersion, it suffices to show that each of the restrictions $g|U''_k$ is an immersion into W_k (**I**, 4.2.4). For this, consider the morphism $u_k : W_k \xrightarrow{p_k} V_k \xrightarrow{\varphi_k^{-1}} U_k \to X$; since X is separated over S, the graph morphism $\Gamma_{u_k} : W_k \to X \times_S W_k$ is a

closed immersion (**I**, 5.4.3), and so the graph $T_k = \Gamma_{u_k}(W_k)$ is a closed subprescheme of $X \times_S W$; if we show that $U' \to X \times_S W_k$ factors through this subprescheme, then the map from the subprescheme induced by X' on the open subset X''_k of X' to $X \times_S W_k$ will also factor through this graph, by (**I**, 9.5.8). Since the restriction of q_2 to T_k is an isomorphism onto W_k, the restriction of g to X''_k will be an immersion into W_k, and our claim will be proven. Let v_k be the canonical injection $U' \to X \times_S W_k$; we have to show that there exists a morphism $w_k : U' \to W_k$ such that $v_k = \Gamma_{u_k} \circ w_k$. By the definition of the product, it suffices to prove that $q_1 \circ v_k = u_k \circ q_2 \circ v_k$ (**I**, 3.2.1), or, by composing on the right with **II | 109**
the isomorphism $\psi' : U \to U'$, that $q_1 \circ \psi = u_k \circ q_2 \circ \psi$. But since $q_1 \circ \psi = j$ and $q_2 \circ \psi = \varphi$, our claim follows from the commutativity of (5.6.1.1), taking into account the definition of u_k.

(E) It is clear that since U, and thus U', is irreducible, so is the X' from the preceding construction, and the morphism f is thus *birational* (**I**, 2.2.9). If in addition X is reduced, then so is U', and hence X' is also reduced (**I**, 9.5.9). This finishes the proof.

□

Corollary (5.6.2). — *Suppose that one of the hypotheses,* (a) *and* (b), *of* (5.6.1) *is satisfied. For X to be proper over S, it is necessary and sufficient for there to exist a projective scheme X' over S, and a surjective S—morphism $f : X' \to X$ (which is thus projective, by* (5.5.5, v)*). Whenever this is the case, we can further choose f to be such that there exists a dense open subset U of X for which the restriction of f to $f^{-1}(U)$ is an isomorphism $f^{-1}(U) \simeq U$, and for which $f^{-1}(U)$ is dense in X'. If in addition X is irreducible (resp. reduced), then we can assume that X' is also irreducible (resp. reduced); when X and X' are irreducible, f is a birational morphism.*

Proof. The condition is sufficient, by (5.5.3) and (5.4.3, ii). It is necessary because, with the notation of (5.6.1), if X is proper over S, then X' is proper over S, because it is projective over X, and thus proper over X (5.5.3), and our claim follows from (5.4.2, ii); furthermore, since X' is quasi-projective over S, it is projective over S, by (5.5.3). □

Corollary (5.6.3). — *Let S be a locally Noetherian prescheme, and X an S-scheme of finite type over S, with structure morphism $f_0 : X \to S$. For X to be proper over S, it is necessary and sufficient that, for every morphism* of finite type *$S' \to S$, $(f_0)_{(S')} : X_{(S')} \to S'$ be a closed morphism. It even suffices for this condition to be verified only for every S-prescheme of the form $S' = S \otimes_{\mathbf{Z}} \mathbf{Z}[T_1, \ldots, T_n]$ (where the T_i are indeterminates).*

Proof. The condition being clearly necessary, we now show that it is sufficient. Since the question is local on S and S' (5.4.1), we can suppose that S and S' are affine and Noetherian. By Chow's lemma, there exists a projective S-scheme P, an immersion $j : X' \to P$, and a surjective projective morphism $f : X' \to X$, such that the diagram

$$\begin{tikzcd} X \arrow[d, "f_0"'] & X' \arrow[l, "f"'] \arrow[d, "j"] \\ S & P \arrow[l, "r"] \end{tikzcd}$$

commutes. Since P is of finite type over S, the first hypothesis implies that the projection $q_2 : X \times_S P \to P$ is a *closed* morphism. But the immersion j is the composition of q_2 and the morphism $f \times 1$ from $X' \times_S P$ to $X \times_S P$; but f, being projective, is proper (5.5.3), and so $f \times 1$ is closed. We thus conclude that j is a closed immersion, and thus proper (5.4.2, i). Furthermore, the structure morphism $r : P \to S$ is projective, and thus proper (5.5.3), so $f_0 \circ f = r \circ j$ is proper (5.4.2, ii); finally, since f is surjective, f_0 is proper, by (5.4.3).

To prove the proposition using only the second, weaker hypothesis (where S' is of the form $S \otimes_{\mathbf{Z}} \mathbf{Z}[T_1, \ldots, T_n]$), it suffices to show that it implies the first. But, if S' is affine and of finite type over $S = \operatorname{Spec}(A)$, then we have **II | 110**
$S' = \operatorname{Spec}(A[c_1, \ldots, c_n])$ (**I**, 6.3.3), and S' is thus isomorphic to a closed subprescheme of $S'' = \operatorname{Spec}(A[T_1, \ldots, T_n])$ (where the T_i are indeterminates). In the commutative diagram

$$\begin{tikzcd} X \times_S S' \arrow[r, "1_X \times j"] \arrow[d, "(f_0)_{(S')}"'] & X \times_S S'' \arrow[d, "(f_0)_{(S'')}"] \\ S' \arrow[r, "j"] & S'' \end{tikzcd}$$

both j and $1_X \times j$ are closed immersions (**I**, 4.3.1), and $(f_0)_{(S')}$ is closed by hypothesis; thus $(f_0)_{(S'')}$ is also closed. □

§6. Integral morphisms and finite morphisms

6.1. Preschemes integral over another prescheme.

§7. Valuative criteria

7.1. Reminder on valuation rings.

§8. Blowup schemes; based cones; projective closure

8.1. Blowup preschemes.

(8.1.1). Let Y be a prescheme, and, for every integer $n \geqslant 0$, let $\mathscr{I}_n$ be a quasi-coherent sheaf of ideals of $\mathcal{O}_Y$; suppose that the following conditions are satisfied:

$$\mathscr{I}_0 = \mathcal{O}_Y, \ \mathscr{I}_n \subset \mathscr{I}_m \text{ for } m \leqslant n, \tag{8.1.1.1}$$

$$\mathscr{I}_m \mathscr{I}_n \subset \mathscr{I}_{m+n} \text{ for any } m, n. \tag{8.1.1.2}$$ II | 153

We note that these hypotheses imply

$$\mathscr{I}_1^n \subset \mathscr{I}_n. \tag{8.1.1.3}$$

Set

$$\mathscr{S} = \bigoplus_{n \geqslant 0} \mathscr{I}_n. \tag{8.1.1.4}$$

It follows from (8.1.1.1) and (8.1.1.2) that $\mathscr{S}$ is a quasi-coherent graded $\mathcal{O}_Y$-algebra, and thus defines a Y-scheme $X = \operatorname{Proj}(\mathscr{S})$. If $\mathscr{J}$ is an *invertible* sheaf of ideals of $\mathcal{O}_Y$, then $\mathscr{I}_n \otimes_{\mathcal{O}_Y} \mathscr{J}^{\otimes n}$ is canonically identified with $\mathscr{I}_n \mathscr{J}^n$. If we then replace the $\mathscr{I}_n$ by the $\mathscr{I}_n \mathscr{J}^n$, and, in doing so, replace $\mathscr{S}$ by a quasi-coherent $\mathcal{O}_Y$-algebra $\mathscr{S}_{(\mathscr{J})}$, then $X_{(\mathscr{J})} = \operatorname{Proj}(\mathscr{S}_{(\mathscr{J})})$ is canonically isomorphic to X (3.1.8).

(8.1.2). Suppose that Y is *locally integral*, so that the sheaf $\mathscr{R}(Y)$ of rational functions is a quasi-coherent $\mathcal{O}_Y$-algebra (**1**, 7.3.7). We say that a $\mathcal{O}_Y$-submodule $\mathscr{I}$ of $\mathscr{R}(Y)$ is a *fractional ideal* of $\mathscr{R}(Y)$ if it is of *finite type* (**0**, 5.2.1). Suppose we have, for all $n \geqslant 0$, a quasi-coherent fractional ideal $\mathscr{I}_n$ of $\mathscr{R}(Y)$, such that $\mathscr{I}_0 = \mathcal{O}_Y$, and such that condition (8.1.1.2) (but not necessarily the second condition (8.1.1.1)) is satisfied; we can then again define a quasi-coherent graded $\mathcal{O}_Y$-algebra by Equation (8.1.1.4), and the corresponding Y-scheme $X = \operatorname{Proj}(\mathscr{S})$; we will again have a canonical isomorphism from X to $X_{\mathscr{J}}$ for every *invertible* fractional ideal $\mathscr{J}$ of $\mathscr{R}(Y)$.

Definition (8.1.3). — Let Y be a prescheme (resp. a locally integral prescheme), and $\mathscr{I}$ a quasi-coherent ideal of $\mathcal{O}_Y$ (resp. a quasi-coherent fractional ideal of $\mathscr{R}(Y)$). We say that the Y-scheme $X = \operatorname{Proj}(\bigoplus_{n \geqslant 0} \mathscr{I}^n)$ is obtained by blowing up the ideal $\mathscr{I}$, or is the blow-up prescheme of Y relative to $\mathscr{I}$. When $\mathscr{I}$ is a quasi-coherent ideal of $\mathcal{O}_Y$, and Y' is the closed subprescheme of Y defined by $\mathscr{I}$, we also say that X is the Y-scheme obtained by blowing up Y'.

By definition, $\mathscr{S} = \bigoplus_{n \geqslant 0} \mathscr{I}^n$ is then generated by $\mathscr{S}_1 = \mathscr{I}$; if $\mathscr{I}$ is an $\mathcal{O}_Y$-module of *finite type*, then X is *projective* over Y (5.5.2). Without any hypotheses on $\mathscr{I}$, the $\mathcal{O}_X$-module $\mathcal{O}_X(1)$ is *invertible* (3.2.5) and *very ample*, by (4.4.3) applied to the structure morphism $X \to Y$.

We note that, if $j : X \to Y$ is the structure morphism, then the restriction of f to $f^{-1}(Y - Y')$ is an *isomorphism* to $Y - Y'$ whenever $\mathscr{I}$ is an *ideal of* $\mathcal{O}_Y$ and Y' is the closed subprescheme that it defines: indeed, the question being local on Y, it suffices to assume that $\mathscr{I} = \mathcal{O}_Y$, and our claim then follows from (3.1.7).

If we replace $\mathscr{I}$ by $\mathscr{I}^d$ $(d > 0)$, then the blow-up Y-scheme X is replaced by a canonically isomorphic Y-scheme X' (8.1.1); similarly, for every *invertible* ideal (resp. *invertible* fractional ideal) $\mathscr{J}$, the blow-up prescheme $X_{(\mathscr{J})}$ relative to the ideal $\mathscr{I}\mathscr{J}$ is canonically isomorphic to X (8.1.1).

In particular, whenever $\mathscr{I}$ is an *invertible* ideal (resp. *invertible* fractional ideal), the Y-scheme obtained by blowing up $\mathscr{I}$ is *isomorphic to* Y (3.1.7).

Proposition (8.1.3). — *Let Y be an integral prescheme.*

(i) *For every sequence $(\mathscr{I}_n)$ of quasi-coherent fractional ideals of $\mathscr{R}(Y)$ that satisfies (8.1.1.2) and such that $\mathscr{I}_0 = \mathcal{O}_Y$, the Y-scheme $X = \operatorname{Proj}(\bigoplus_{n \geqslant 0} \mathscr{I}^n)$ is integral, and the structure morphism $f : X \to Y$ is dominant.* II | 154
(ii) *Let $\mathscr{I}$ be a quasi-coherent fractional ideal of $\mathscr{R}(Y)$, and let X be the Y-scheme given by the blow up of Y relative to $\mathscr{I}$. If $\mathscr{I} \neq 0$, then the structure morphism $f : X \to Y$ is then birational and surjective.*

Proof.

(i) This follows from the fact that $\mathscr{S} = \bigoplus_{n\geqslant 0} \mathscr{I}_n$ is an *integral* $\mathscr{O}_Y$-algebra ((3.1.12) and (3.1.14)), since, for all $y \in Y$, $\mathscr{O}_y$ is an integral ring (**I**, 5.1.4).

(ii) By (i), X is integral; if, furthermore, x and y are the generic points of X and Y (respectively), then we have $f(x) = y$, and it remains to show that $k(x)$ is of rank 1 over $k(y)$. But x is also the generic point of the fibre $f^{-1}(y)$; if ψ is the canonical morphism $Z \to Y$, where $Z = \operatorname{Spec}(k(y))$, then the prescheme $f^{-1}(y)$ can be identified with $\operatorname{Proj}(\mathscr{S}')$, where $\mathscr{S}' = \psi^*(\mathscr{S})$ (3.5.3). But it is clear that $\mathscr{S}' = \bigoplus_{n\geqslant 0} (\mathscr{I}_y)^n$, and, since $\mathscr{I}$ is a quasi-coherent fractional ideal of $\mathscr{R}(Y)$ that is not zero, $\mathscr{I}_y \neq 0$ (**I**, 7.3.6), whence $\mathscr{I}_y = k(y)$; then $\operatorname{Proj}(\mathscr{S}')$ can be identified with $\operatorname{Spec}(k(y))$ (3.1.7), whence the conclusion.

□

We show a *converse* of (8.1.4) in (**III**, 2.3.8).

(8.1.5). We return to the setting and notation of (8.1.1). By definition, the injection homomorphisms $\mathscr{I}_{n+1} \to \mathscr{I}_n$ (8.1.1.1) define, for every $k \in \mathbf{Z}$, an injective homomorphism of degree zero of graded $\mathscr{S}$-modules

$$u_k : \mathscr{S}_+(k+1) \longrightarrow \mathscr{S}(k); \tag{8.1.5.1}$$

since $\mathscr{S}_+(k+1)$ and $\mathscr{S}(k+1)$ are canonically **(TN)**-isomorphic, they give a canonical correspondence between u_k and an injective homomorphism of $\mathscr{O}_X$-modules (3.4.2):

$$\widetilde{u}_k : \mathscr{O}_X(k+1) \longrightarrow \mathscr{O}_X(k). \tag{8.1.5.2}$$

Recall as well (3.2.6) that we have defined canonical homomorphisms

$$\lambda : \mathscr{O}_X(h) \otimes_{\mathscr{O}_X} \mathscr{O}_X(k) \longrightarrow \mathscr{O}_X(h+k) \tag{8.1.5.3}$$

and, since the diagram

$$\begin{tikzcd}
\mathscr{S}(h) \otimes_{\mathscr{S}} \mathscr{S}(k) \otimes_{\mathscr{S}} \mathscr{S}(l) \arrow[r] \arrow[d] & \mathscr{S}(h+k) \otimes_{\mathscr{S}} \mathscr{S}(l) \arrow[d] \\
\mathscr{S}(h) \otimes_{\mathscr{S}} \mathscr{S}(k+l) \arrow[r] & \mathscr{S}(h+k+l)
\end{tikzcd}$$

commutes, it follows from the functoriality of the λ (3.2.6) that the homomorphisms (8.1.5.3) define the structure of a *quasi-coherent graded $\mathscr{O}_X$-algebra* on

$$\mathscr{S}_X = \bigoplus_{n\in\mathbf{Z}} \mathscr{O}_X(n). \tag{8.1.5.4}$$

Furthermore, the diagram

$$\begin{tikzcd}
\mathscr{S}(h) \otimes_{\mathscr{S}} \mathscr{S}(k+1) \arrow[r] \arrow[d, "1\otimes u_k"'] & \mathscr{S}(h+k+1) \arrow[d, "u_{k+h}"] \\
\mathscr{S}(h) \otimes_{\mathscr{S}} \mathscr{S}(k) \arrow[r] & \mathscr{S}(h+k)
\end{tikzcd}$$

commutes; the functoriality of the λ then implies that we have a commutative diagram

$$\begin{tikzcd}
\mathscr{O}_X(h) \otimes_{\mathscr{O}_X} \mathscr{O}_X(k+1) \arrow[r, "\lambda"] \arrow[d, "1\otimes\widetilde{u}_k"'] & \mathscr{O}_X(h+k+1) \arrow[d, "\widetilde{u}_{k+h}"] \\
\mathscr{O}_X(h) \otimes_{\mathscr{O}_X} \mathscr{O}_X(k) \arrow[r, "\lambda"] & \mathscr{O}_X(h+k)
\end{tikzcd}$$

where the horizontal arrows are the canonical homomorphisms. We can thus say that the $\widetilde{u}_k$ define an *injective homomorphism* (of degree zero) *of graded $\mathscr{S}_X$-modules*

$$\widetilde{u} : \mathscr{S}_X(1) \longrightarrow \mathscr{S}_X. \tag{8.1.5.6}$$

(8.1.6). Keeping the notation from (8.1.5), we now note that, for $n \geqslant 0$, the composite homomorphism $\widetilde{v}_n = \widetilde{u}_{n-1} \circ \widetilde{u}_{n-2} \circ \ldots \circ \widetilde{u}_0$ is an *injective* homomorphism $\mathscr{O}_X(n) \to \mathscr{O}_X$; we denote by $\mathscr{I}_{n,X}$ its image, which is thus a quasi-coherent ideal of $\mathscr{O}_X$, *isomorphic* to $\mathscr{O}_X(n)$. Furthermore, the diagram

$$\begin{tikzcd}
\mathscr{O}_X(m) \otimes_{\mathscr{O}_X} \mathscr{O}_X(n) \arrow[r, "\lambda"] \arrow[d, "\widetilde{v}_m\otimes\widetilde{v}_n"'] & \mathscr{O}_X(m+n) \arrow[d, "\widetilde{v}_{m+n}"] \\
\mathscr{O}_X \arrow[r, "\mathrm{id}"'] & \mathscr{O}_X
\end{tikzcd}$$

commutes for $m \geqslant 0$, $n \geqslant 0$. We thus deduce the following inclusions:

(8.1.6.1) $$\mathscr{I}_{0,X} = \mathscr{O}_X, \quad \mathscr{I}_{n,X} \subset \mathscr{I}_{m,X} \qquad \text{for } 0 \leqslant m \leqslant n;$$

(8.1.6.2) $$\mathscr{I}_{m,X}\mathscr{I}_{n,X} \subset \mathscr{I}_{m+n,X} \qquad \text{for } m \geqslant 0,\ n \geqslant 0.$$

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Proposition (8.1.7). — *Let Y be a prescheme, $\mathscr{I}$ a quasi-coherent ideal of $\mathscr{O}_Y$, and $X = \operatorname{Proj}(\bigoplus_{n\geqslant 0}\mathscr{I}^n)$ the Y-scheme given by blowing up $\mathscr{I}$. We then have, for all $n > 0$, a canonical isomorphism*

(8.1.7.1) $$\mathscr{O}_X(n) \xrightarrow{\sim} \mathscr{I}^n\mathscr{O}_X = \mathscr{I}_{n,X}$$

(cf. (0, 4.3.5)), and thus that $\mathscr{I}^n\mathscr{O}_X$ is a very-ample invertible $\mathscr{O}_X$-module if $n > 0$.

Proof. The last claim is immediate, since $\mathscr{O}_X(1)$ is invertible (3.2.5) and very ample for Y by definition ((4.4.3) and (4.4.9)). Also by definition, the image of v_n is exactly $\mathscr{I}^n\mathscr{S}$, and (8.1.7.1) then follows from the exactness of the functor $\widetilde{\mathscr{M}}$ (3.2.4) and from Equation (3.2.4.1). □

Corollary (8.1.8). — *Under the hypotheses of (8.1.7), if $f : X \to Y$ is the structure morphism, and Y' the closed subprescheme of Y defined by $\mathscr{I}$, then the closed subprescheme $X' = f^{-1}(Y')$ of X is defined by $\mathscr{I}\mathscr{O}_X$ (which is canonically isomorphic to $\mathscr{O}_X(1)$), from which we obtain a canonical short exact sequence*

(8.1.8.1) $$0 \longrightarrow \mathscr{O}_X(1) \longrightarrow \mathscr{O}_X \longrightarrow \mathscr{O}_{X'} \longrightarrow 0.$$

Proof. This follows from (8.1.7.1) and from (**I**, 4.4.5). □

(8.1.9). Under the hypotheses of (8.1.7), we can be more precise about the structure of the $\mathscr{I}_{n,X}$. Note that the homomorphism

$$\widetilde{u}_{-1} : \mathscr{O}_X \longrightarrow \mathscr{O}_X(-1)$$

canonically corresponds to a section s of $\mathscr{O}_X(-1)$ over X, which we call the *canonical section* (relative to $\mathscr{I}$) (**0**, 5.1.1). In the diagram in (8.1.5.5), the horizontal arrows are isomorphisms (3.2.7); by replacing h with k, and k with -1 in this diagram, we obtain that $\widetilde{u}_k = 1_k \otimes \widetilde{u}_{-1}$ (where 1_k denotes the identity on $\mathscr{O}_X(h)$), or, equivalently, that the homomorphism $\widetilde{u}_k$ is given exactly by *tensoring with the canonical section s* (for all $k \in \mathbf{Z}$). The homomorphism $\widetilde{u}$ (8.1.5.6) can then be understood in the same way.

Thus, for all $n \geqslant 0$, the homomorphism $\widetilde{v}_n : \mathscr{O}_X(n) \to \mathscr{O}_X$ is given exactly by tensoring with $s^{\otimes n}$; we thus deduce:

Corollary (8.1.10). — *With the notation of (8.1.8), the underlying space of X' is the set of $x \in X$ such that $s(x) = 0$, where s denotes the canonical section of $\mathscr{O}_X(-1)$.*

Proof. Indeed, if c_x is a generator of the fibre $(\mathscr{O}_X(1))_x$ at a point x, then $s_x \otimes c_x$ is canonically identified with a generator of the fibre of $\mathscr{I}_{1,X}$ at the point x, and is thus invertible if and only if $s_x \notin \mathfrak{m}_x(\mathscr{O}_x(-1))_x$, or, equivalently, if and only if $s(x) \neq 0$. □

Proposition. — *Let Y be an integral prescheme, $\mathscr{I}$ a quasi-coherent fractional ideal of $\mathscr{R}(Y)$, and X the Y-scheme given by blowing up $\mathscr{I}$. Then $\mathscr{I}\mathscr{O}_X$ is an invertible $\mathscr{O}_X$-module that is very ample for Y.*

Proof. The question being local on Y (4.4.5), we can reduce to the case where $Y = \operatorname{Spec}(A)$, with A some integral ring of ring of fractions K, and $\mathscr{I} = \widetilde{\mathfrak{J}}$, with $\mathfrak{J}$ some fractional ideal of K; there then exists an element $a \neq 0$ of A such that $a\mathfrak{J} \subset A$. Let $S = \bigoplus_{n\geqslant 0}\mathfrak{J}^n$; the map $x \mapsto ax$ is an A-isomorphism from $\mathfrak{J}^{n+1} = (S(1))_n$ to $a\mathfrak{J}^{n+1} = a\mathfrak{J}S_n \subset \mathfrak{J}^n = S_n$, and thus defines a (TN)-isomorphism of degree zero of graded S-modules $S_+(1) \to a\mathfrak{J}S$. **II | 157**
On the other hand, $x \mapsto a^{-1}x$ is an isomorphism of degree zero of graded S-modules $a\mathfrak{J}S \xrightarrow{\sim} \mathfrak{J}S$. We thus obtain, by composition (3.2.4), an isomorphism of $\mathscr{O}_X$-modules $\mathscr{O}_X(1) \xrightarrow{\sim} \mathscr{I}\mathscr{O}_X$, and, since S is generated by $S_1 = \mathfrak{J}$, $\mathscr{O}_X(1)$ is invertible (3.2.5) and very ample ((4.4.3) and (4.4.9)), whence our claim. □

8.2. Preliminary results on the localisation of graded rings.

(8.2.1). Let S be a graded ring, but not assumed (for the moment) to be only in positive degree. We define

(8.2.1.1) $$S^{\geqslant} = \bigoplus_{n\geqslant 0} S_n, \qquad S^{\leqslant} = \bigoplus_{n\leqslant 0} S_n$$

which are both graded subrings of S, in only positive and negative degrees (respectively). If f is a homogeneous elements of degree d (positive or negative) of S, then the ring of fraction $S_f = S'$ is again endowed with the structure of a graded ring, by taking S'_n ($n \in \mathbf{Z}$) to be the set of the x/f^k for $x \in S_{n+kd}$ ($k \geqslant 0$); we define $S_{(f)} = S'_0$, and will write $S_f^{\geqslant}$ and $S_f^{\leqslant}$ for $S'^{\geqslant}$ and $S'^{\leqslant}$ (respectively). If $d > 0$, then

$$(S^{\geqslant})_f = S_f \tag{8.2.1.2}$$

since, if $x \in S_{n+kd}$ with $n+kd < 0$, then we can write $x/f^k = xf^h/f^{h+k}$, and we also have that $n+(h+k)d > 0$ for h sufficiently large and > 0. We thus conclude, by definition, that

$$(S^{\geqslant})_{(f)} = (S_f^{\geqslant})_0 = S_{(f)}. \tag{8.2.1.3}$$

If M is a graded S-module, then we similarly define

$$M^{\geqslant} = \bigoplus_{n\geqslant 0} M_n, \qquad M^{\leqslant} = \bigoplus_{n\leqslant 0} M_n \tag{8.2.1.4}$$

which are (respectively) a graded $S^{\geqslant}$-module and a graded $S^{\leqslant}$-module, and their intersection is the S_0 module M_0. If $f \in S_d$, then we define M_f to be the graded S_f-module whose elements of degree n are the z/f^k for $z \in M_{n+kd}$ ($k \geqslant 0$); we denote by $M_{(f)}$ the set of elements of degree zero of M_f, and this is an $S_{(f)}$-module, and we will write $M_f^{\geqslant}$ and $M_f^{\leqslant}$ to mean $(M_f)^{\geqslant}$ and $(M_f)^{\leqslant}$ (respectively). If $d > 0$, then we see, as above, that

$$(M^{\geqslant})_f = M_f \tag{8.2.1.5}$$

and

$$(M^{\geqslant})_{(f)} = (M_f^{\geqslant})_0 = M_{(f)}. \tag{8.2.1.6}$$

(8.2.2). Let $\mathbf{z}$ be an indeterminate, we we will call the *homogenisation variable.* If S is a graded ring (in positive or negative degrees), then the polynomial algebra[1]

$$\widehat{S} = S[\mathbf{z}] \tag{8.2.2.1}$$

is a graded S-algebra, where we define the degree of $f\mathbf{z}^n$ ($n \geqslant 0$), with f homogeneous, as **II** | 158

$$\deg(f\mathbf{z}^n) = n + \deg f. \tag{8.2.2.2}$$

Lemma (8.2.3). — (i) *There are canonical isomorphisms of (non-graded) rings*

$$\widehat{S}_{(\mathbf{z})} \xrightarrow{\sim} \widehat{S}/(\mathbf{z}-1)\widehat{S} \xrightarrow{\sim} S. \tag{8.2.3.1}$$

(ii) *There is a canonical isomorphism of (non-graded) rings*

$$\widehat{S}_{(f)} \xrightarrow{\sim} S_f^{\leqslant} \tag{8.2.3.2}$$

for all $f \in S_d$ with $d > 0$.

Proof. The first of the isomorphisms in (8.2.3.1) was defined in (2.2.5), and the second is trivial; the isomorphism $\widehat{S}_{(\mathbf{z})} \xrightarrow{\sim} S$ thus defined thus gives a correspondence between $x\mathbf{z}^n/\mathbf{z}^{n+k}$ (where $\deg(x) = k$ for $k \geqslant -n$) and the element x. The homomorphism (8.2.3.2) gives a correspondence between $x\mathbf{z}^n/f^k$ (where $\deg(x) = kd - n$) and the element x/f^k of degree $-n$ in $S_f^{\leqslant}$, and it is again clear that this does indeed give an isomorphism. □

(8.2.4). Let M be a graded S-module. It is clear that the S-module

$$\widehat{M} = M \otimes_S \widehat{S} = M \otimes_S S[\mathbf{z}] \tag{8.2.4.1}$$

is the direct sum of the S-modules $M \otimes S\mathbf{z}^n$, and thus of the abelian groups $M_k \otimes S\mathbf{z}^n$ ($k \in \mathbf{Z}$, $n \geqslant 0$); we define on $\widehat{M}$ the structure of a graded $\widehat{S}$-module by setting

$$\deg(x \otimes \mathbf{z}^n) = n + \deg x \tag{8.2.4.2}$$

for all homogeneous x in M. We leave it to the reader to prove the analogue of (8.2.3):

Lemma (8.2.5). — (i) *There is a canonical di-isomorphism of (non-graded) modules*

$$\widehat{M}_{(\mathbf{z})} \xrightarrow{\sim} M. \tag{8.2.5.1}$$

(ii) *For all $f \in S_d$ ($d > 0$), there is a di-isomorphism of (non-graded) modules*

$$\widehat{M}_{(f)} \xrightarrow{\sim} M_f^{\leqslant}. \tag{8.2.5.2}$$

[1] This should not be confused with the use of the notation $\widehat{S}$ to denote the completed separation of a ring.

(8.2.6). Let S be a *positively*-graded ring, and consider the decreasing sequence of graded ideals of S

(8.2.6.1) $$S_{[n]} = \bigoplus_{m\geqslant n} S_m \qquad (n \geqslant 0)$$

(so, in particular, we have $S_{[0]} = S$ and $S_{[1]} = S_+$). Since it is evident that $S_{[m]}S_{[n]} \subset S_{[m+n]}$, we can define a *graded ring* $S^\natural$ by setting

(8.2.6.2) $$S^\natural = \bigoplus_{n\geqslant 0} S^\natural_n \quad \text{with} \quad S^\natural_n = S_{[n]}.$$

$S^\natural_0$ is then the ring S considered as a *non-graded* ring, and $S^\natural$ is thus an $S^\natural_0$-algebra. For every homogeneous element $f \in S_d$ $(d > 0)$, we denote by $f^\natural$ the element f considered as belonging to $S_{[d]} = S^\natural_d$. With this notation: **II | 159**

Lemma (8.2.7). — *Let S be a positively-graded ring, and f a homogeneous element of S_d $(d > 0)$. There are canonical ring isomorphisms*

(8.2.7.1) $$S_f \xrightarrow{\sim} \bigoplus_{n\in\mathbf{Z}} S(n)_{(f)}$$

(8.2.7.2) $$(S^{\geqslant}_f)_{f/1} \xrightarrow{\sim} S_f$$

(8.2.7.3) $$S^\natural_{(f^\natural)} \xrightarrow{\sim} S^{\geqslant}_f$$

where the first two are isomorphisms of graded rings.

Proof. It is immediate, by definition, that we have $(S_f)_n = (S(n)_f)_0$, whence the isomorphism in (8.2.7.1), which is exactly the identity. Next, since $f/1$ is invertible in S_f, there is a canonical isomorphism $S_f \xrightarrow{\sim} (S^{\geqslant}_f)_{f/1} = (S_f)_{f/1}$, by (8.2.1.2) applied to S_f; the inverse isomorphism is, by definition, the isomorphism in (8.2.7.2). Finally, if $x = \sum_{m\geqslant n} y_m$ is an element of $S_{[n]}$ with $n = kd$, then the element $x/(f^\natural)^k$ corresponds to the element $\sum_m y_m/f^k$ of $S^{\geqslant}_f$, and we can quickly verify that this defines an isomorphism (8.2.7.3). □

(8.2.8). If M is a graded S-module, then we similarly define, for all $n \in \mathbf{Z}$,

(8.2.8.1) $$M_{[n]} = \bigoplus_{m\geqslant n} M_m$$

and, since $S_{[m]}M_{[n]} \subset M_{[m+n]}$ $(m \geqslant 0)$, we can define a graded $S^\natural$-module $M^\natural$ by setting

(8.2.8.2) $$M^\natural = \bigoplus_{n\in\mathbf{Z}} \quad \text{with} \quad M^\natural_n = M_{[n]}.$$

We leave to the reader the proof of:

Lemma (8.2.9). — *With the notation of (8.2.7) and (8.2.8), there are canonical di-isomorphisms of modules*

(8.2.9.1) $$M_f \xrightarrow{\sim} \bigoplus_{n\in\mathbf{Z}} M(n)_{(f)}$$

(8.2.9.2) $$(M^{\geqslant}_f)_{f/1} \xrightarrow{\sim} M_f$$

(8.2.9.3) $$M^\natural_{(f^\natural)} \xrightarrow{\sim} M^{\geqslant}_f$$

where the first two are di-isomorphisms of graded modules.

Lemma (8.2.10). — *Let S be a positively-graded ring.*

(i) *For $S^\natural$ to be an $S^\natural_0$-algebra of finite type (resp. a Noetherian $S^\natural_0$-algebra), it is necessary and sufficient for S to be an $S^\natural_0$-algebra of finite type (resp. a Noetherian $S^\natural_0$-algebra).*
(ii) *For $S^\natural_{n+1} = S^\natural_1 S^\natural_n$ $(n \geqslant n_0)$, it is necessary and sufficient for $S_{n+1} = S_1 S_n$ $(n \geqslant n_0)$.*
(iii) *For $S^\natural_n = S^\natural_1$ $(n \geqslant n_0)$, it is necessary and sufficient for $S_n = S^n_1$ $(n \geqslant n_0)$.*
(iv) *If (f_α) is a set of homogeneous elements of S_+ such that S_+ is the radical in S_+ of the ideal of S_+ generated by the f_α, then $S^\natural_+$ is the radical in $S^\natural_+$ of the ideal of $S^\natural_+$ generated by the $f^\natural_\alpha$.*

Proof. (i) If $S^\natural$ is an $S_0^\natural$-algebra of finite type, then $S_+ = S_1^\natural$ is a module of finite type over $S = S_0^\natural$, by (2.1.6, i), and so S is an S_0-algebra of finite type (2.1.4); if $S^\natural$ is a Noetherian ring, then so too is $S_0^\natural = S$ (2.1.5). Conversely, if S is an S_0-algebra of finite type, then we know (2.1.6, ii) that there exist $h > 0$ **II | 160**
and $m_0 > 0$ such that $S_{n+h} = S_h S_n$ for $n \geqslant m_0$; we can clearly assume that $m_0 \geqslant h$. Furthermore, the S_m are S_0-modules of finite type (2.1.6, i). So, if $n \geqslant m_0 + h$, then $S_n^\natural = S_h S_{n-h}^\natural = S_h^\natural S_{n-h}^\natural$; and if $m < m_0 + h$ then, letting $E = S_{m_0} + \ldots + S_{m_0+h-1}$, we have that

$$S_m^\natural = S_m + \ldots + S_{m_0+h-1} + S_h E + S_h^2 E + \ldots.$$

For $1 \leqslant m \leqslant m_0$, let G_m be the union of the finite systems of generators of the S_0-modules S_i for $m \leqslant i \leqslant m_0 + h - 1$, thought of as a subset of $S_{[m]}$. For $m_0 + 1 \leqslant m \leqslant m_0 + h - 1$, let G_m be the union of the finite system of generators of the S_0-modules S_i for $m \leqslant i \leqslant m_0 + h - 1$ and of $S_h E$, thought of as a subset of $S_{[m]}$. It is clear that $S_m^\natural = S_0^\natural G_m$ for $1 \leqslant m \leqslant m_0 + h - 1$, and thus the union G of the G_m for $1 \leqslant m \leqslant m_0 + h - 1$ is a system of generators of the $S_0^\natural$-algebra $S^\natural$. We thus conclude that, if $S = S_0^\natural$ is a Noetherian ring, then so too is $S^\natural$.

(ii) It is clear that, if $S_{n+1} = S_1 S_n$ for $n \geqslant n_0$, then $S_{n+1}^\natural = S_1 S_n^\natural$, and *a fortiori* $S_{n+1}^\natural = S_1^\natural S_n^\natural$ for $n \geqslant n_0$. Conversely, this last equality can be written as

$$S_{n+1} + S_{n+2} + \ldots = (S_1 + S_2 + \ldots)(S_n + S_{n+1} + \ldots)$$

and comparing terms of degree $n+1$ (in S) on both sides gives that $S_{n+1} = S_1 S_n$.

(iii) If $S_n = S_1^n$ for $n \geqslant n_0$, then $S_n^\natural = S_1^n + S_1^{n+1} + \ldots$; since $S_1^\natural$ contains $S_1 + S_1^2 + \ldots$, we have that $S_n^\natural \subset S_1^{\natural n}$, and thus $S_n^\natural = S_1^{\natural n}$ for $n \geqslant n_0$. Conversely, the only terms of $S_1^{\natural n} = (S_1 + S_2 + \ldots)^n$ that are of degree n in S are those of S_1^n; the equality $S_n^\natural = S_1^{\natural n}$ thus implies that $S_n = S_1^n$.

(iv) It suffices to show that, if an element $g \in S_{k+h}$ is considered as an element of $S_k^\natural$ ($k > 0$, $h \geqslant 0$), then there exists an integer $n > 0$ such that g^n is a linear combination (in $S_{kn}^\natural$) of the $f_\alpha^\natural$ with coefficients in $S^\natural$. By hypothesis, there exists an integer m_0 such that, for $m \geqslant m_0$, we have, *in* S, that $g^m = \sum_\alpha c_{\alpha m} f_\alpha$, where the indices α here are *independent of* m; furthermore, we can clearly assume that the $c_{\alpha m}$ are homogeneous, with

$$\deg(c_{\alpha m}) = m(k+h) - \deg f_\alpha$$

in S. So take m_0 sufficiently large enough to ensure that $km_0 > \deg f_\alpha$ for all the f_α that appear in g^{m_0}; for all α, let $c'_{\alpha m}$ be the element $c_{\alpha m}$ considered as having degree $km - \deg f_\alpha$ *in* $S^\natural$; we then have, in $S^\natural$, that $g^m = \sum_\alpha c'_{\alpha m} f_\alpha^\natural$, which finishes the proof.

□

(8.2.11). Consider the graded S_0-algebra

(8.2.11.1) $$S^\natural \otimes_S S_0 = S^\natural / S_+ S^\natural = \bigoplus_{n \geqslant 0} S_{[n]} / S_+ S_{[n]}.$$

Since S_n is a quotient S_0-module of $S_{[n]}/S_+ S_{[n]}$, there is a canonical homomorphism of graded S_0-algebras

(8.2.11.2) $$S^\natural \otimes_S S_0 \longrightarrow S$$

which is clearly *surjective*, and thus corresponds (2.9.2) to a canonical *closed immersion*

(8.2.11.3) $$\operatorname{Proj}(S) \longrightarrow \operatorname{Proj}(S^\natural \otimes_S S_0).$$

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Proposition (8.2.12). — *The canonical morphism (8.2.11.3) is bijective. For the homomorphism (8.2.11.2) to be (TN)-bijective, it is necessary and sufficient for there to exist some n_0 such that $S_{n+1} = S_1 S_n$ for $n \geqslant n_0$. If this latter condition is satisfied, then (8.2.11.3) is an isomorphism; the converse is true whenever S is Noetherian.*

Proof. To prove the first claim, it suffices (2.8.3) to show that the kernel $\mathfrak{I}$ of the homomorphism (8.2.11.2) consists of *nilpotent* elements. But if $f \in S_{[n]}$ is an element whose class modulo $S_+ S_{[n]}$ belongs to this kernel, then this implies that $f \in S_{[n+1]}$; then f^{n+1}, considered as an element of $S_{[n(n+1)]}$, is also an element of $S_+ S_{[n(n+1)]}$, since it can be written as $f \cdot f^n$; so the class of f^{n+1} modulo $S_+ S_{[n(n+1)]}$ is zero, which proves our claim. Since the hypothesis that $S_{n+1} = S_1 S_n$ for $n \geqslant n_0$ is equivalent to $S_{n+1}^\natural = S_1^\natural S_n^\natural$ for $n \geqslant n_0$ (8.2.10, ii), this hypothesis is equivalent, by definition, to the fact that (8.2.11.2) is (TN)-injective, and thus (TN)-bijective, and so (8.2.11.3) is an isomorphism, by (2.9.1). Conversely, if (8.2.11.3) is an isomorphism, then the sheaf $\widetilde{\mathfrak{I}}$ on $\operatorname{Proj}(S^\natural \otimes_S S_0)$ is zero (2.9.2, i); since $S^\natural \otimes_S S_0$ is Noetherian, as a quotient of $S^\natural$ (8.2.10, i), we conclude from (2.7.3) that $\mathfrak{I}$ satisfies condition (TN), and so $S_{n+1}^\natural = S_1^\natural S_n^\natural$ for $n \geqslant n_0$, and this finishes the proof, by (8.2.10, ii). □

(8.2.13). Consider now the canonical injections $(S_+)^n \to S_{[n]}$, which define an injective homomorphism of degree zero of graded rings

(8.2.13.1) $$\bigoplus_{n\geqslant 0}(S_+)^n \longrightarrow S^\natural.$$

Proposition (8.2.14). — *For the homomorphism (8.2.13.1) to be a (TN)-isomorphism, it is necessary and sufficient for there to exist some n_0 such that $S_n = S_1^n$ for all $n \geqslant n_0$. Whenever this is the case, the morphisms corresponding to (8.2.13.1) is everywhere defined and also an isomorphism*

$$\operatorname{Proj}(S^\natural) \xrightarrow{\sim} \operatorname{Proj}(\bigoplus_{n\geqslant 0}(S_+)^n);$$

the converse is true whenever S is Noetherian.

Proof. The first two claims are evident, given (8.2.10, iii) and (2.9.1). The third will follow from (8.2.10, i and iii) and the following lemma:

Lemma (8.2.14.1). — *Let T be a positively-graded ring that is also a T_0-algebra of finite type. If the morphism corresponding to the injective homomorphism $\bigoplus_{n\geqslant 0} T_1^n \to T$ is everywhere defined and also an isomorphism* $\operatorname{Proj}(T) \to \operatorname{Proj}(\bigoplus_{n\geqslant 0} T_1^n)$, *then there exists some n_0 such that $T_n = T_1^n$ for $n \geqslant n_0$.*

Let g_i $(1 \leqslant i \leqslant r)$ be generators of the T_0-module T_1. The hypothesis implies first of all that the $D_+(g_i)$ cover $\operatorname{Proj}(T)$ (2.8.1). Let $(h_j)_{1\leqslant j\leqslant s}$ be a system of homogeneous elements of T_+, with $\deg(h_j) = n_j$, that form, with the g_i, a system of generators of the ideal T_+, or, equivalently (2.1.3), a system of generators of T as a T_0-algebra; if we set $T' = \bigoplus_{n\geqslant 0} T_1^n$, then the element $h_j/g_i^{n_j}$ of the ring $T_{(g_i)}$ must, by hypothesis, belong to the subring $T'_{(g_i)}$, and so there exists some integer k such that $T_1^k h_j \subset T_1^{k+n_j}$ for all j. We thus conclude, by induction on r, that $T_1^k h_j^r \subset T'$ for all $r \geqslant 1$, and, by definition of the h_j, we thus have that $T_1^k T \subset T'$. Also, there exists, for all j, an integer m_j such that $h_j^{m_j}$ belongs to the ideal of T generated by the g_i (2.3.14), so $h_j^{m_j} \in T_1 T$, and $h_j^{m_j k} \in T_1^k T \subset T'$. There is thus an integer $m_0 \geqslant k$ such that $h_j^m \in T_1^{mn}$ for $m \geqslant m_0$. So, if q is the largest of the integers n_j, then $n_0 = qsm_0 + k$ is the required number. Indeed, an element of S_n, for $n \geqslant n_0$, is the sum of monomials belonging to $T_1^\alpha u$, where u is a product of powers of the h_j; if $\alpha \geqslant k$, then it follows from the above that $T_1^\alpha u \subset T_1^n$; in the other case, one of the exponents of the h_j is $\geqslant m_0$, so $u \in T_1^\beta v$, where $\beta \geqslant k$ and v is again a product of powers of the h_j; we can then reduce to the previous case, and so we conclude that $T_1^\alpha u \subset T_1^n$ in all cases. ☐ **II | 162**

Remark (8.2.15). — The condition $S_n = S_1^n$ for $n \geqslant n_0$ clearly implies that $S_{n+1} = S_1 S_n$ for $n \geqslant n_0$, but the converse is not necessarily true, even if we assume that S is Noetherian. For example, let K be a field, $A = K[\mathbf{x}]$, and $B = K[\mathbf{y}]/\mathbf{y}^2 K[\mathbf{y}]$, where $\mathbf{x}$ and $\mathbf{y}$ are indeterminates, with $\mathbf{x}$ taken to have degree 1 and $\mathbf{y}$ to have degree 2, and let $S = A \otimes_K B$, so that S is a graded algebra over K that has a basis given by the elements 1, $\mathbf{x}^n$ $(n \geqslant 1)$, and $\mathbf{x}^n\mathbf{y}$ $(n \geqslant 0)$. It is immediate that $S_{n+1} = S_1 S_n$ for $n \geqslant 2$, but $S_1^n = K\mathbf{x}^n$ while $S_n = K\mathbf{x}^n + K\mathbf{x}^n\mathbf{y}$ for $n \geqslant 2$.

8.3. Based cones.

(8.3.1). Let Y be a prescheme; in all of this section, we will consider only *Y-preschemes* and *Y-morphisms*. Let $\mathscr{S}$ be a quasi-coherent *positively*-graded $\mathscr{O}_Y$-algebra; *we further assume that $\mathscr{S}_0 = \mathscr{O}_Y$*. Following the notation introduced in (8.2.2), we let

(8.3.1.1) $$\widehat{\mathscr{S}} = \mathscr{S}[\mathbf{z}] = \mathscr{S} \otimes_{\mathscr{O}_Y} \mathscr{O}_Y[\mathbf{z}]$$

which we consider as a positively-graded $\mathscr{O}_Y$-algebra by defining the degrees as in (8.2.2.2), so that, for every affine open subset U of Y, we have

$$\Gamma(U, \widehat{\mathscr{S}}) = (\Gamma(U, \mathscr{S}))[\mathbf{z}].$$

In what follows, we write

(8.3.1.2) $$X = \operatorname{Proj}(\mathscr{S}), \quad C = \operatorname{Spec}(\mathscr{S}), \quad \widehat{C} = \operatorname{Proj}(\widehat{\mathscr{S}})$$

(where, in the definition of C, we consider $\mathscr{S}$ as a non-graded $\mathscr{O}_Y$-algebra), and we say that C (resp. $\widehat{C}$) is the *affine cone* (resp. *projective cone*) defined by $\mathscr{S}$; we will sometimes say "cone" instead of "affine cone". By an abuse of language, we also say that C (resp. $\widehat{C}$) is the *affine cone based at X (?)* (resp. the *projective cone based at X (?)*)[2], with the implicit understanding that the prescheme X is given in the form $\operatorname{Proj}(\mathscr{S})$; finally, we say that $\widehat{C}$ is the *projective closure* of C (with the data of $\mathscr{S}$ being implicit in the structure of C).

[2] *[Trans.] A more literal translation of the French (*cône projetant (affine/projectif)*) would be the* projecting (affine/projective) cone, *but it seems that this terminology already exists to mean something else.*

Proposition (8.3.2). — *There exist canonical Y-morphisms*

$$Y \xrightarrow{\varepsilon} C \xrightarrow{i} \widehat{C} \tag{8.3.2.1}$$

$$X \xrightarrow{j} \widehat{C} \tag{8.3.2.2}$$

such that ε and j are closed immersions, and i is an affine morphism, which is a dominant open immersion, for which

$$i(C) = \widehat{C} \setminus j(X); \tag{8.3.2.3}$$

furthermore, $\widehat{C}$ is the smallest closed subprescheme of $\widehat{C}$ containing $i(C)$.

Proof. To define i, consider the open subset of $\widehat{C}$ given by

$$\widehat{C}_{\mathbf{z}} = \operatorname{Spec}(\widehat{\mathscr{S}}/(\mathbf{z}-1)\widehat{\mathscr{S}}) \tag{8.3.2.4}$$

(3.1.4), where $\mathbf{z}$ is canonically identified with a section of $\mathscr{S}$ over Y. The isomorphism $i : C \xrightarrow{\sim} \widehat{C}_{\mathbf{z}}$ then corresponds to the canonical isomorphism (8.2.3.1)

$$\widehat{\mathscr{S}}/(\mathbf{z}-1)\widehat{\mathscr{S}} \xrightarrow{\sim} \mathscr{S}.$$

The morphism ε corresponds to the augmentation homomorphism $\mathscr{S} \to \mathscr{S}_0 = \mathscr{O}_Y$, which has kernel $\mathscr{S}_+$ (1.2.7), and, since the latter is surjective, ε is a closed immersion (1.4.10). Finally, j corresponds (3.5.1) to the surjective homomorphism of degree zero $\widehat{\mathscr{S}} \to \mathscr{S}$, which restricts to the identity on $\mathscr{S}$ and is zero on $\mathbf{z}\widehat{\mathscr{S}}$, which is its kernel; j is everywhere defined, and is a closed immersion, by (3.6.2).

To prove the other claims of (8.3.2), we can clearly restrict to the case where $Y = \operatorname{Spec}(A)$ is affine, and $\mathscr{S} = \widetilde{S}$, with S a graded A-algebra, whence $\widehat{\mathscr{S}} = (\widehat{S})^{\sim}$; the homogeneous elements f of S_+ can then be identified with sections of $\widehat{\mathscr{S}}$ over Y, and the open subset of $\widehat{C}$, denoted $D_+(f)$ in (2.3.3), can then be written as $\widehat{C}_f$ (3.1.4); similarly, the open subset of C denoted $D(f)$ in (**I**, 1.1.1) can be written as C_f (**0**, 5.5.2). With this in mind, it follows from (2.3.14) and from the definition of $\widehat{S}$ that, in this case, the open subsets $\widehat{C}_{\mathbf{z}} = i(C)$ and $\widehat{C}_f$ (with f homogeneous in S_+) form a *cover* of $\widehat{C}$. Furthermore, with this notation,

$$i^{-1}(\widehat{C}_f) = C_f; \tag{8.3.2.5}$$

indeed, $\widehat{C}_f \cap i(C) = \widehat{C}_f \cap \widehat{C}_{\mathbf{z}} = \widehat{C}_{f\mathbf{z}} = \operatorname{Spec}(\widehat{S}_{(f\mathbf{z})})$. But, if $d = \deg(f)$, then $\widehat{S}_{(f\mathbf{z})}$ is canonically isomorphic to $(\widehat{S}_{(\mathbf{z})})_{f/\mathbf{z}^d}$ (2.2.2), and it follows from the definition of the isomorphism in (8.2.3.1) that the image of $(\widehat{S}_{(\mathbf{z})})_{f/\mathbf{z}^d}$ under the corresponding isomorphism of rings of fractions is exactly S_f. Since $C_f = \operatorname{Spec}(S_f)$, this proves (8.3.2.5) and shows, at the same time, that the morphism i is affine; furthermore, the restriction of i to C_f, thought of as a morphism to $\widehat{C}_f$, corresponds (**I**, 1.7.3) to the canonical homomorphism $\widehat{S}_{(f)} \to \widehat{S}_{(f\mathbf{z})}$, and, by the above and (8.2.3.2), we can claim the following result:

(8.3.2.6). If $Y = \operatorname{Spec}(A)$ is affine, and $\mathscr{S} = \widetilde{S}$, then, for every homogeneous f in S_+, $\widehat{C}_f$ is canonically identified with $\operatorname{Spec}(S_f^{\leqslant})$, and the morphism $C_f \to \widehat{C}_f$ given by restricting i then corresponds to the canonical injection $S_f^{\leqslant} \to S_f$.

Now note that (for Y affine) the complement of $\widehat{C}_{\mathbf{z}}$ in $\widehat{C} = \operatorname{Proj}(\widehat{S})$ is, by definition, the set of graded prime ideals of $\widehat{S}$ containing $\mathbf{z}$, which is exactly $j(X)$, by definition of j, which proves (8.3.2.3). **II** | 164

Finally, to prove the last claim of (8.3.2), we can assume that Y is affine. With the above notation, note that, in the ring $\widehat{S}$, $\mathbf{z}$ is not a zero divisor; since $i(C) = \widehat{C}$, it suffices to prove the following lemma:

Lemma (8.3.2.7). — *Let T be a positively-graded ring, $Z = \operatorname{Proj}(T)$, and g a homogeneous element of T of degree $d > 0$. If g is not a zero divisor in T, then Z is the smallest closed subprescheme of Z that contains $Z_g = D_+(g)$.*

By (**I**, 4.1.9), the question is local on Z; for every homogeneous element $h \in T_e$ ($e > 0$), it thus suffices to prove that Z_h is the smallest closed subprescheme of Z_h that contains Z_{gh}; it follows from the definitions and from (**I**, 4.3.2) that this condition is equivalent to asking for the canonical homomorphism $T_{(h)} \to T_{(gh)}$ to be *injective*. But this homomorphism can be identified with the canonical homomorphism $T_{(h)} \to (T_{(h)})_{g^e/h^d}$ (2.2.3). But since g^e is not a zero divisor in T, g^e/h^d is not a zero divisor in T_h (nor *a fortiori* in $T_{(h)}$), since the fact that $(g^e/h^d)(t/h^m) = 0$ (for $t \in T$ and $m > 0$) implies the existence of some $n > 0$ such that $h^n g^e t = 0$, whence $h^n t = 0$, and thus $t/h^m = 0$ in T_h. This thus finishes the proof (**0**, 1.2.2). ☐

(8.3.3). We will often identify the affine cone C with the subprescheme induced by the projective cone $\widehat{C}$ on the open subset $i(C)$ by means of the open immersion i. The closed subprescheme of C associated to the closed immersion ε is called the *vertex prescheme* (?) of C; we also say that ε, which is a Y-section of C, is the *vertex section* (?), or the *null section*, or C; we can identify Y with the vertex prescheme (?) of C by means of ε. Also, $i \circ \varepsilon$ is a Y-section of $\widehat{C}$, and thus also a closed immersion (**I**, 5.4.6), corresponding to the canonical surjective homomorphism of degree zero $\widehat{\mathscr{S}} = \mathscr{S}[\mathbf{z}] \to \mathscr{O}_Y[\mathbf{z}]$ (3.1.7), whose kernel is $\mathscr{S}_+[\mathbf{z}] = \mathscr{S}_+\widehat{\mathscr{S}}$; the subprescheme of $\widehat{C}$ associated to this closed immersion is also called the *vertex prescheme* (?) of $\widehat{C}$, and $i \circ \varepsilon$ the *vertex section* (?) of $\widehat{C}$; it can be identified with Y by means of $i \circ \varepsilon$. Finally, the closed subprescheme of $\widehat{C}$ associated to j is called the *part at infinity* of $\widehat{C}$, and can be identified with X by means of j.

(8.3.4). The subpreschemes of C (resp. $\widehat{C}$) induced on the *open* subsets

$$E = C - \varepsilon(Y), \qquad \widehat{E} = \widehat{C} - i(\varepsilon(Y)) \tag{8.3.4.1}$$

are called (by an abuse of language) the *pointed affine cone* and the *pointed projective cone* (respectively) defined by $\mathscr{S}$; we note that, despite this nomenclature, E *is not necessarily affine* over Y, nor $\widehat{E}$ projective over Y (8.4.3). When we identify C with $i(C)$, we thus have the underlying spaces

$$C \cup \widehat{E} = \widehat{C}, \qquad C \cap \widehat{E} = E \tag{8.3.4.2}$$

so that $\widehat{C}$ can be considered as being obtained by *gluing* the open subpreschemes C and $\widehat{E}$; furthermore, by (8.3.2.3),

$$E = \widehat{E} - j(X). \tag{8.3.4.3}$$

If $Y = \operatorname{Spec}(A)$ is affine, then, with the notation of (8.3.2),

$$E = \bigcup C_f, \qquad \widehat{E} = \bigcup \widehat{C}_f, \qquad C_f = C \cap \widehat{C}_f \tag{8.3.4.4}$$

where f runs over the set of homogeneous elements of S_+ (or only a subset M of this set, with M generating an ideal of S_+ whose radical in S_+ is S_+ itself, or, equivalently, such that the X_f for $f \in M$ cover X (2.3.14)). The gluing of C and $\widehat{C}_f$ along C_f is thus determined by the injection morphisms $C_f \to C$ and $C_f \to \widehat{C}_f$, which, as we have seen (8.3.2.6), correspond (respectively) to the canonical homomorphisms $S \to S_f$ and $S_f^{\leqslant} \to S_f$.

Proposition (8.3.5). — *With the notation of* (8.3.1) *and* (8.3.4), *the morphism associated* (3.5.1) *to the canonical injection* $\varphi : \mathscr{S} \to \widehat{\mathscr{S}} = \mathscr{S}[\mathbf{z}]$ *is a surjective affine morphism (called the canonical retraction)*

$$p : \widehat{E} \longrightarrow X \tag{8.3.5.1}$$

such that

$$p \circ j = 1_X. \tag{8.3.5.2}$$

Proof. To prove the proposition, we can restrict to the case where Y is affine. Taking into account the expression in (8.3.4.4) for $\widehat{E}$, the fact that the domain of definition $G(\varphi)$ of p is equal to $\widehat{E}$ will follow from the first of the following claims:

(8.3.5.3). If $Y = \operatorname{Spec}(A)$ is affine, and $\mathscr{S} = \widetilde{S}$, then, for all homogeneous $f \in S_+$,

$$p^{-1}(X_f) = \widehat{C}_f \tag{8.3.5.4}$$

and the restriction of p to $\widehat{C}_f = \operatorname{Spec}(S_f^{\leqslant})$, thought of as a morphism from $\widehat{C}_f$ to X_f, corresponds to the canonical injection $S_{(f)} \to S_f^{\leqslant}$. If, further, $f \in S_1$, then $\widehat{C}_f$ is isomorphic to $X_f \otimes_{\mathbf{Z}} \mathbf{Z}[T]$ (where T is an indeterminate).

Indeed, Equation (8.3.5.4) is exactly a particular case of (2.8.1.1), and the second claim is exactly the definition of $\operatorname{Proj}(\varphi)$ whenever Y is affine (2.8.1). Then Equation (8.3.5.2) and the fact that p is surjective show that the composition $\mathscr{S} \to \widehat{\mathscr{S}} \to \mathscr{S}$ of the canonical homomorphisms is the identity on $\mathscr{S}$. Finally, the last claim of (8.3.5.3) follows from the fact that $S_f^{\leqslant}$ is isomorphic to $S_{(f)}[T]$ whenever $f \in S_1$ (2.2.1). □

Corollary (8.3.6). — *The restriction*

$$\pi : E \longrightarrow X \tag{8.3.6.1}$$

of p *to* E *is a surjective affine morphism. If* Y *is affine and* f *homogeneous in* S_+, *then*

$$\pi^{-1}(X_f) = C_f \tag{8.3.6.2}$$

and the restriction of π *to* C_f *corresponds to the canonical injection* $S_{(f)} \to S_f$. *If, further,* $f \in S_1$, *then* C_f *is isomorphic to* $X_f \otimes_{\mathbf{Z}} \mathbf{Z}[T, T^{-1}]$ *(where* T *is an indeterminate).*

Proof. Equation (8.3.6.2) follows immediately from (8.3.5.3) and (8.3.2.5), and shows the surjectivity of π; we have already seen that the immersion i, restricted to C_f, corresponds to the injection $S_f^{\leqslant} \to S_f$ (8.3.2). Finally, the last claim is a consequence of the fact that, for $f \in S_1$, S_f is isomorphic to $S_{(f)}[T, T^{-1}]$ (2.2.1). □ **II | 166**

Remark (8.3.7). — Whenever Y is affine, the elements of the underlying space of E are the (not-necessarily-graded) prime ideals $\mathfrak{p}$ of S not containing S_+, by definition of the immersion ε (8.3.2). For such an ideal $\mathfrak{p}$, the $\mathfrak{p} \cap S_n$ clearly satisfy the conditions of (2.1.9), and so there exists exactly one *graded* prime ideal $\mathfrak{q}$ of S such that $\mathfrak{q} \cap S_n = \mathfrak{p} \cap S_n$ for all n; the map $\pi : E \to X$ of underlying spaces can then be understood via the equation

$$\pi(\mathfrak{p}) = \mathfrak{q}. \tag{8.3.7.1}$$

Indeed, to prove this equation, it suffices to consider some homogeneous f in S_+ such that $\mathfrak{p} \in D(f)$, and to note that $\mathfrak{q}_{(f)}$ is the inverse image of $\mathfrak{p}_f$ under the injection $S_{(f)} \to S_f$.

Corollary (8.3.8). — *If $\mathscr{S}$ is generated by $\mathscr{S}_1$, then the morphisms p and π are of finite type; for all $x \in X$, the fibre $p^{-1}(x)$ is isomorphic to* $\operatorname{Spec}(k(x)[T])$, *and the fibre π^{-1} isomorphic to* $\operatorname{Spec}(k(x)[T, T^{-1}])$

Proof. This follows immediately from (8.3.5) and (8.3.6) by noting that, whenever Y is affine and S is generated by S_1, the X_f, for $f \in S_1$, form a cover of X (2.3.14). □

Remark (8.3.9). — The pointed affine cone corresponding to the graded $\mathscr{O}_Y$-algebra $\mathscr{O}_Y[T]$ (where T is an indeterminate) can be identified with $G_m = \operatorname{Spec}(\mathscr{O}_Y[T, T^{-1}])$, since it is exactly C_T, as we have seen in (8.3.2) (see (8.4.4) for a more general result). This prescheme is canonical endowed with the structure of a "*Y-scheme in commutative groups*". This idea will be explained in detail later on, but, for now, can be quickly summarised as follows. A Y-scheme in groups is a Y-scheme G endowed with two Y-morphisms, $p : G \times_Y G \to G$ and $s : G \to G$, that satisfy conditions formally analogous to the axioms of the composition law and the symmetry law of a group: the diagram

$$\begin{tikzcd} G \times G \times G \arrow[r, "p\times 1"] \arrow[d, "1\times p"'] & G \times G \arrow[d, "p"] \\ G \times G \arrow[r, "p"'] & G \end{tikzcd}$$

should commute ("associativity"), and there should be a condition which corresponds to the fact that, for groups, the maps

$$(x, y) \longmapsto (x, x^{-1}, y) \longmapsto (x, x^{-1}y) \longmapsto x(x^{-1}y)$$

and

$$(x, y) \longmapsto (x, x^{-1}, y) \longmapsto (x, yx^{-1}) \longmapsto (yx^{-1})x$$

should both reduce to $(x, y) \mapsto y$; the sequence of morphisms corresponding, for example, to the first composite map is

$$G \times G \xrightarrow{(1,s)\times 1} G \times G \times G \xrightarrow{1\times p} G \times G \xrightarrow{p} G$$

and the reader should write down the second sequence. **II | 167**

It is immediate (**I**, 3.4.3) that the data of a structure of a Y-scheme in groups on a Y-scheme G is equivalent to the data, for *every* Y-prescheme Z, of a *group* structure on the set $\operatorname{Hom}_Y(Z, G)$, where these structures should be such that, for every Y-morphism $Z \to Z'$, the corresponding map $\operatorname{Hom}_Y(Z', G) \to \operatorname{Hom}_Y(Z, G)$ is a group homomorphism. In the particular case of G_m that we consider here, $\operatorname{Hom}_Y(Z, G)$ can be identified with the set of Z-sections of $Z \times_Y G_m$ (**I**, 3.3.14), and thus with the set of Z-sections of $\operatorname{Spec}(\mathscr{O}_Z[T, T^{-1}])$; finally, the same reasoning as in (**I**, 3.3.15) shows that this set is canonically identified with the set of *invertible* elements of the ring $\Gamma(Z, \mathscr{O}_Z)$, and the group structure on this set is the structure coming from the multiplication in the ring $\Gamma(Z, \mathscr{O}_Z)$. The reader can verify that the morphisms p and s from above are obtained in the following way: they correspond, by (1.2.7) and (1.4.6), to the homomorphisms of $\mathscr{O}_Y$-algebras

$$\pi : \mathscr{O}_Y[T, T^{-1}] \longrightarrow \mathscr{O}_Y[T, T^{-1}, T', T'^{-1}]$$
$$\sigma : \mathscr{O}_Y[T, T^{-1}] \longrightarrow \mathscr{O}_Y[T, T^{-1}]$$

and are entirely defined by the data of $\pi(T) = TT'$ and $\sigma(T) = T^{-1}$.

With this in mind, G_m can be considered as a "*universal domain of operators*" for every *affine cone* $C = \operatorname{Spec}(\mathscr{S})$, where $\mathscr{S}$ is a quasi-coherent positively-graded $\mathscr{O}_Y$-algebra. This means that we can canonically define a Y-morphism $G_m \times_Y C \to C$ which has the formal properties of an external law of a set endowed with a group of operators; or, again, as above for schemes in groups, we can give, for every Y-prescheme Z, an external law on $\operatorname{Hom}_Y(Z, C)$, having the group $\operatorname{Hom}_Y(Z, G_m)$ as its set of operators, with the usual axioms of sets endowed with a group of operators, and a compatibility condition with respect to the Y-morphisms $Z \to Z'$. In the current case, the morphism $G_m \times_Y C \to C$ is defined by the data of a homomorphism of $\mathcal{O}_Y$-algebras $\mathcal{S} \to \mathcal{S} \otimes_{\mathcal{O}_Y} \mathcal{O}_Y[T, T^{-1}] = \mathcal{S}[T, T^{-1}]$, which associates, to each section $s_n \in \Gamma(U, \mathcal{S}_n)$ (where U is an open subset of Y), the section $s_n T^n \in \Gamma(U, \mathcal{S} \otimes_{\mathcal{O}_Y} \mathcal{O}_Y[T, T^{-1}])$.

Conversely, suppose that we are given a quasi-coherent, *a priori non-graded*, $\mathcal{O}_Y$-algebra, and, on $C = \operatorname{Spec}(\mathcal{S})$, a structure of a "*$Y$-scheme in sets endowed with a group of operators*" that has the Y-scheme in groups G_m as its domain of operators; then we canonically obtain a *grading* of $\mathcal{O}_Y$-algebras on $\mathcal{S}$. Indeed, the data of a Y-morphism $G_m \times_Y C \to C$ is equivalent to that of a homomorphism of $\mathcal{O}_Y$-algebras $\psi : \mathcal{S} \to \mathcal{S}[T, T^{-1}]$, which can be written as $\psi = \sum_{n \in \mathbf{Z}} \psi_n T^n$, where the $\psi_n : \mathcal{S} \to \mathcal{S}$ are homomorphisms of $\mathcal{O}_Y$-modules (with $\psi_n(s) = 0$ except for finitely many n for every section $s \in \Gamma(U, \mathcal{S})$, for any open subset U of Y). We can then prove that the axioms of sets endowed with a group of operators imply that the $\psi_n(\mathcal{S}) = \mathcal{S}_n$ define a grading (in positive or negative degree) of $\mathcal{O}_Y$-algebras on $\mathcal{S}$, with the ψ_n being the corresponding projectors. We also have the notation of a structure of an "*affine cone*" on every affine Y-scheme, defined in a "geometric" way without any reference to any prior grading. We will not further develop this point of view here, and we leave the work of **II** | 168
precisely formulating the definitions and results corresponding to the information given above to the reader.

8.4. Projective closure of a vector bundle.

(8.4.1). Let Y be a prescheme, and $\mathcal{E}$ a quasi-coherent $\mathcal{O}_Y$-module. If we take $\mathcal{S}$ to be the graded $\mathcal{O}_Y$-algebra $\mathbf{S}_{\mathcal{O}_Y}(\mathcal{E})$, then Definition (8.3.1.1) shows that $\widehat{\mathcal{S}}$ can be identified with $\mathbf{S}_{\mathcal{O}_Y}(\mathcal{E} \oplus \mathcal{O}_Y)$. With the affine cone $\operatorname{Spec}(\mathcal{S})$ defined by $\mathcal{S}$ being, by definition, $\mathbf{V}(\mathcal{E})$, and $\operatorname{Proj}(\mathcal{S})$ being, by definition, $\mathbf{P}(\mathcal{E})$, we see that:

Proposition (8.4.2). — *The projective closure of a vector bundle* $\mathbf{V}(\mathcal{E})$ *on* Y *is canonically isomorphic to* $\mathbf{P}(\mathcal{E} \oplus \mathcal{O}_Y)$*, and the part at infinity of the latter is canonically isomorphic to* $\mathbf{P}(\mathcal{E})$*.*

Remark (8.4.3). — Take, for example, $\mathcal{E} = \mathcal{O}_Y^r$ with $r \geqslant 2$; then the pointed cones E and $\widehat{E}$ defined by $\mathcal{S}$ are nether affine nor projective on Y if $Y \neq \varnothing$. The second claim is immediate, because $\widehat{C} = \mathbf{P}(\mathcal{O}_Y^{r+1})$ is projective on Y, and the underlying spaces of E and $\widehat{E}$ are non-closed open subsets of $\widehat{C}$, and so the canonical immersions $E \to \widehat{C}$ and $\widehat{E} \to \widehat{C}$ are not projective (5.5.3), and we conclude by appealing to (5.5.5, v). Now, supposing, for example, that $Y = \operatorname{Spec}(A)$ is affine, and $r = 2$, then $C = \operatorname{Spec}(A[T_1, T_2])$, and E is then the prescheme induced by C on the open subset $D(T_1) \cup D(T_2)$; but we have already seen that the latter is not affine (**I**, 5.5.11); *a fortiori* $\widehat{E}$ cannot be affine, since E is the open subset where the section $\mathbf{z}$ over $\widehat{E}$ does not vanish (8.3.2).

However:

Proposition (8.4.4). — *If* $\mathcal{L}$ *is an invertible* $\mathcal{O}_Y$*-module, then there are canonical isomorphisms for both the pointed cones* E *and* $\widehat{E}$ *corresponding to* $C = \mathbf{V}(\mathcal{L})$*:*

$$\operatorname{Spec}\left(\bigoplus_{n \in \mathbf{Z}} \mathcal{L}^{\otimes n}\right) \xrightarrow{\sim} E \tag{8.4.4.1}$$

$$\mathbf{V}(\mathcal{L}^{-1}) \xrightarrow{\sim} \widehat{E}. \tag{8.4.4.2}$$

Furthermore, there exists a canonical isomorphism from the projective closure of $\mathbf{V}(\mathcal{L})$ *to the projective closure of* $\mathbf{V}(\mathcal{L}^{-1})$ *that sends the null section (resp. the part at infinity) of the former to the part at infinity (resp. the null section) of the second.*

Proof. We have here that $\mathcal{S} = \bigoplus_{n \geqslant 0} \mathcal{L}^{\otimes n}$; the canonical injection

$$\mathcal{S} \longrightarrow \bigoplus_{n \in \mathbf{Z}} \mathcal{L}^{\otimes n}$$

defines a canonical dominant morphism

$$\operatorname{Spec}\left(\bigoplus_{n \in \mathbf{Z}} \mathcal{L}^{\otimes n}\right) \longrightarrow \mathbf{V}(\mathcal{L}) = \operatorname{Spec}\left(\bigoplus_{n \geqslant 0} \mathcal{L}^{\otimes n}\right) \tag{8.4.4.3}$$

and it suffices to prove that this morphism is an isomorphism from the scheme $\operatorname{Spec}(\bigoplus_{n \in \mathbf{Z}} \mathcal{L}^{\otimes n})$ to E. The question being local on Y, we can assume that $Y = \operatorname{Spec}(A)$ is affine and that $\mathcal{L} = \mathcal{O}_Y$, and so $\mathcal{S} = (A[T])^\sim$ **II** | 169
and $\bigoplus_{n \in \mathbf{Z}} \mathcal{L}^{\times n} = (A[T, T^{-1}])^\sim$. But $A[T, T^{-1}]$ is the ring of fractions $A[T]_T$ of $A[T]$, and thus (8.4.4.3) identifies $\bigoplus_{n \in \mathbf{Z}} \mathcal{L}^{\otimes n}$ (?) with the prescheme induced by $C = \mathbf{V}(\mathcal{L})$ on the open subset $D(T)$; the complement $V(T)$ of this open subset in C is the underlying space of the closed subprescheme of C defined by the ideal $TA[T]$, which is exactly the null section of C, and so $E = D(T)$.

The isomorphism in (8.4.4.2) will be a consequence of the last claim, since $\mathbf{V}(\mathscr{L}^{-1})$ is the complement of the part at infinity of its projective closure, and $\widehat{E}$ is the complement of the null section of the projective closure $C = \mathbf{V}(\mathscr{L})$. But these projective closures are $\mathbf{P}(\mathscr{L}^{-1} \oplus \mathscr{O}_Y)$ and $\mathbf{P}(\mathscr{L} \oplus \mathscr{O}_Y)$ (respectively); but we can write $\mathscr{L} \oplus \mathscr{O}_Y = \mathscr{L} \otimes (\mathscr{L}^{-1} \oplus \mathscr{O}_Y)$. The existence of the desired canonical isomorphism then follows from (4.1.4), and everything reduces to showing that this isomorphism swaps the null sections and the parts at infinity. For this, we can reduce to the case where $Y = \operatorname{Spec}(A)$ is affine, $L = Ac$, and $L^{-1} = Ac'$, with the canonical isomorphism $L \otimes L^{-1} \to A$ sending $c \otimes c'$ to the element 1 of A. Then $\mathbf{S}(L \oplus A)$ is the tensor product of $A[\mathbf{z}]$ with $\bigoplus_{n\geqslant 0} Ac^{\otimes n}$, and $\mathbf{S}(L^{-1} \oplus A)$ is the tensor product of $A[\mathbf{z}]$ with $\bigoplus_{n\geqslant 0} Ac'^{\otimes n}$, and the isomorphism defined in (4.1.4) sends $\mathbf{z}^h \otimes c'^{\otimes(n-h)}$ to the element $\mathbf{z}^{n-h} \otimes c^{\otimes h}$. But, in $\mathbf{P}(\mathscr{L}^{-1} \oplus \mathscr{O}_Y)$, the part at infinity is the set of points where the section $\mathbf{z}$ vanishes, and the null section is the section of points where the section c' vanishes; since we have analogous definitions for $\mathbf{P}(\mathscr{L} \oplus \mathscr{O}_Y)$, the conclusion follows immediately from the above explanation. □

8.5. Functorial behaviour.

(8.5.1). Let Y and Y' be prescheme, $q : Y' \to Y$ a morphism, and $\mathscr{S}$ (resp. $\mathscr{S}'$) a *positively*-graded quasi-coherent $\mathscr{O}_Y$-algebra (resp. *positively*-graded quasi-coherent $\mathscr{O}_{Y'}$-algebra). Consider a q-morphism of graded algebras

(8.5.1.1)
$$\varphi : \mathscr{S} \longrightarrow \mathscr{S}'.$$

We know (1.5.6) that this corresponds, canonically, to a morphism

$$\Phi = \operatorname{Spec}(\varphi) : \operatorname{Spec}(\mathscr{S}') \longrightarrow \operatorname{Spec}(\mathscr{S})$$

such that the diagram

(8.5.1.2)
$$\begin{tikzcd} C' \arrow[r, "\Phi"] \arrow[d] & C \arrow[d] \\ Y' \arrow[r, "q"'] & Y \end{tikzcd}$$

commutes, where we write $C = \operatorname{Spec}(\mathscr{S})$ and $C' = \operatorname{Spec}(\mathscr{S}')$. *Suppose, further, that* $\mathscr{S}_0 = \mathscr{O}_Y$ *and* $\mathscr{S}'_0 = \mathscr{O}_{Y'}$; let $\varepsilon : Y \to C$ and $\varepsilon : Y' \to C'$ be the canonical immersions (8.3.2); we then have a commutative diagram

(8.5.1.3)
$$\begin{tikzcd} Y' \arrow[r, "q"] \arrow[d, "\varepsilon'"'] & Y \arrow[d, "\varepsilon"] \\ C' \arrow[r, "\Phi"'] & C \end{tikzcd}$$

which corresponds to the diagram **II** | 170

$$\begin{tikzcd} \mathscr{S} \arrow[r, "\varphi"] \arrow[d] & \mathscr{S}' \arrow[d] \\ \mathscr{O}_Y \arrow[r] & \mathscr{O}_{Y'} \end{tikzcd}$$

where the vertical arrows are the augmentation homomorphisms, and so the commutativity follows from the hypothesis that φ is assumed to be a homomorphism of *graded* algebras.

Proposition (8.5.2). — *If E (resp. E') is the pointed affine cone defined by $\mathscr{S}$ (resp. $\mathscr{S}'$), then $\Phi^{-1}(E) \subset E'$; if, further,* $\operatorname{Proj}(\varphi) : G(\varphi) \to \operatorname{Proj}(\mathscr{S})$ *is everywhere defined (or, equivalently, if $G(\varphi) = \operatorname{Proj}(\mathscr{S}')$), then $\Phi^{-1}(E) = E'$, and conversely.*

Proof. The first claim follows from the commutativity of (8.5.1.3). To prove the second, we can restrict to the case where $Y = \operatorname{Spec}(A)$ and $Y' = \operatorname{Spec}(A')$ are affine, and $\mathscr{S} = \widetilde{S}$ and $\mathscr{S}' = \widetilde{S}'$. For every homogeneous f in S_+, writing $f' = \varphi(f)$, we have that $\Phi^{-1}(C_f) = C'_{f'}$ (**I**, 2.2.4.1); saying that $G(\varphi) = \operatorname{Proj}(S')$ implies that the radical (in S'_+) of the ideal generated by the $f' = \varphi(f)$ is S'_+ itself ((2.8.1) and (2.3.14)), and this is equivalent to saying that the $C'_{f'}$ cover E' (8.3.4.4). □

(8.5.3). The q-morphism φ canonically extends to a q-morphism of graded algebras

(8.5.3.1)
$$\widehat{\varphi} : \widehat{\mathscr{S}} \longrightarrow \widehat{\mathscr{S}'}$$

by letting $\widehat{\varphi}(\mathbf{z}) = \mathbf{z}$. This induces a morphism

$$\widehat{\Phi} = \operatorname{Proj}(\widehat{\varphi}) : G(\widehat{\varphi}) \longrightarrow \widehat{C} = \operatorname{Proj}(\widehat{\mathscr{S}})$$

such that the diagram

$$\begin{tikzcd} G(\widehat{\varphi}) \arrow[r, "\widehat{\Phi}"] \arrow[d] & \widehat{C} \arrow[d] \\ Y' \arrow[r, "q"'] & Y \end{tikzcd}$$

commutes (3.5.6). It follows immediately from the definitions that, if we write $i : C \to \widehat{C}$ and $i' : C' \to \widehat{C'}$ to mean the canonical open immersions (8.3.2), then $i'(C') \subset G(\widehat{\varphi})$, and the diagram

(8.5.3.2)
$$\begin{tikzcd} C' \arrow[r, "\Phi"] \arrow[d, "i"'] & C \arrow[d, "i'"] \\ G(\widehat{\varphi}) \arrow[r, "\widehat{\Phi}"'] & \widehat{C} \end{tikzcd}$$

commutes. Finally, if we let $X = \operatorname{Proj}(\mathscr{S})$ and $X' = \operatorname{Proj}(\mathscr{S}')$, and if $j : X \to \widehat{C}$ and $j' : X' \to \widehat{C'}$ are the canonical closed immersions (8.3.2), then it follows from the definition of these immersions that $j'(G(\varphi)) \subset G(\widehat{\varphi})$, and that the diagram **II** | 171

(8.5.3.3)
$$\begin{tikzcd} G(\varphi) \arrow[r, "\operatorname{Proj}(\varphi)"] \arrow[d, "j'"'] & X \arrow[d, "j"] \\ G(\widehat{\varphi}) \arrow[r, "\widehat{\Phi}"'] & \widehat{C} \end{tikzcd}$$

commutes.

Proposition (8.5.4). — *If $\widehat{E}$ (resp. $\widehat{E'}$) is the pointed projective cone defined by $\mathscr{S}$ (resp. by $\mathscr{S}'$), then $\widehat{\Phi}^{-1}(\widehat{E}) \subset \widehat{E'}$; furthermore, if $p : \widehat{E} \to X$ and $p' : \widehat{E'} \to X'$ are the canonical retractions, then $p'(\widehat{\Phi}^{-1}(\widehat{E})) \subset G(\widehat{\varphi})$, and the diagram*

(8.5.4.1)
$$\begin{tikzcd} \widehat{\Phi}^{-1}(\widehat{E}) \arrow[r, "\widehat{\Phi}"] \arrow[d, "p'"'] & \widehat{E} \arrow[d, "p"] \\ G(\varphi) \arrow[r, "\operatorname{Proj}(\varphi)"'] & X \end{tikzcd}$$

commutes. If $\operatorname{Proj}(\varphi)$ *is everywhere defined, then so too is $\widehat{\Phi}$, and we have that $\widehat{\Phi}^{-1}(\widehat{E}) = \widehat{E'}$*

Proof. The first claim follows from the commutativity of Diagrams (8.5.1.3) and (8.5.3.2), and the two following claims from the definition of the canonical retractions (8.3.5) and the definition of $\widehat{\varphi}$. To see that $\widehat{\Phi}$ is everywhere defined whenever $\operatorname{Proj}(\varphi)$ is, we can restrict to the case where $Y = \operatorname{Spec}(A)$ and $Y' = \operatorname{Spec}(A')$ are affine, and where $\mathscr{S} = \widetilde{S}$ and $\mathscr{S}' = \widetilde{S'}$; the hypothesis is that, when f runs over the set of homogeneous elements of S_+, the radical in S'_+ of the ideal generated in S'_+ by the $\varphi(f)$ is S'_+ itself; we thus immediately conclude that the radical in $(S'[\mathbf{z}])_+$ of the ideal generated by $\mathbf{z}$ and the $\varphi(f)$ is $(S'[\mathbf{z}])_+$ itself, whence our claim; this also shows that $\widehat{E'}$ is the union of the $\widehat{C'}_{\varphi(f)}$, and hence equal to $\widehat{\Phi}^{-1}(\widehat{E})$. □

Corollary (8.5.5). — *Whenever* $\operatorname{Proj}(\varphi)$ *is everywhere defined, the inverse image under $\widehat{\Phi}$ of the underlying space of the part at infinity (resp. of the vertex prescheme) of $\widehat{C'}$ is the underlying space of the part at infinity (resp. of the vertex prescheme) of $\widehat{C}$.*

Proof. This follows immediately from (8.5.4) and (8.5.2), taking into account the equalities (8.3.4.1) and (8.3.4.2). □

8.6. A canonical isomorphism for pointed cones.

(8.6.1). Let Y be a prescheme, $\mathscr{S}$ a quasi-coherent positively-graded $\mathscr{O}_Y$-algebra *such that* $\mathscr{S}_0 = \mathscr{O}_Y$, and let X be the Y-scheme $\operatorname{Proj}(\mathscr{S})$. We are going to apply the results of (8.5) to the case where $Y' = X$, and $q : X \to Y$ is the structure morphism; let

(8.6.1.1)
$$\mathscr{S}_X = \bigoplus_{n\in\mathbf{Z}} \mathscr{O}_X(n)$$

which is a quasi-coherent graded $\mathscr{O}_X$-algebra, with multiplication defined by means of the canonical homomor- **II** | 172 phisms (3.2.6.1)

$$\mathcal{O}_X(m)\otimes_{\mathcal{O}_X}\mathcal{O}_X(n)\longrightarrow\mathcal{O}_X(m+n)$$

whose associativity is ensured by the commutative diagram in (2.5.11.4). Let $\mathcal{S}'$ be the quasi-coherent positively-graded $\mathcal{O}_X$-subalgebra $\mathcal{S}_X^{\geqslant}=\bigoplus_{n\geqslant 0}\mathcal{O}_X(n)$ of $\mathcal{S}_X$.

Finally, consider the canonical q-morphism

$$\alpha:\mathcal{S}\longrightarrow\mathcal{S}_X^{\geqslant} \tag{8.6.1.2}$$

defined in (3.3.2.3) as a homomorphism $\mathcal{S}\to q_*(\mathcal{S}_X)$, but which clearly sends $\mathcal{S}$ to $q_*(\mathcal{S}_X^{\geqslant})$. Write

$$C_X=\operatorname{Spec}(\mathcal{S}_X^{\geqslant}),\quad \widehat{C}_X=\operatorname{Proj}(\mathcal{S}_X^{\geqslant}[\mathbf{z}]),\quad X'=\operatorname{Proj}(\mathcal{S}_X^{\geqslant}) \tag{8.6.1.3}$$

and denote by E_X and $\widehat{E}_X$ the corresponding pointed affine and pointed projective cones (respectively); denote the canonical morphisms defined in (8.3) by $\varepsilon_X:X\to C_X$, $i_X:C\to\widehat{C}_X$, $j_X:X'\to\widehat{C}_X$, $p_X:\widehat{E}_X\to X'$, and $\pi_X:E_X\to X'$.

Proposition (8.6.2). — *The structure morphism $u:X'\to X$ is an* isomorphism, *and the morphism* $\operatorname{Proj}(\alpha)$ *is everywhere defined and identical to u. The morphism* $\operatorname{Proj}(\widehat{\alpha}):\widehat{C}_X\to\widehat{C}$ *is everywhere defined, and its restrictions to $\widehat{E}_X$ and E_X are* isomorphisms *to $\widehat{E}$ and E (respectively). Finally, if we identify X' with X via u, then the morphisms p_X and π_X are identified with the structure morphisms of the X-preschemes $\widehat{E}_X$ and E_X.*

Proof. We can clearly restrict to the case where $Y=\operatorname{Spec}(A)$ is affine, and $\mathcal{S}=\widetilde{S}$; then X is the union of affine open subsets X_f, where f runs over the set of homogeneous elements of S_+, with the ring of each X_f being $S_{(f)}$. It follows from (8.2.7.1) that

$$\Gamma(X_f,\mathcal{S}_X^{\geqslant})=S_f^{\geqslant}. \tag{8.6.2.1}$$

So $u^{-1}(X_f)=\operatorname{Proj}(S_f^{\geqslant})$. But if $f\in S_d$ $(d>0)$, then $\operatorname{Proj}(S_f^{\geqslant})$ is canonically isomorphic to $\operatorname{Proj}((S_f^{\geqslant})^{(d)})$ (2.4.7), and we also know that $(S_f^{\geqslant})^{(d)}=(S^{(d)})_f^{\geqslant}$ can be identified with $S_{(f)}[T]$ (2.2.1) by the map $T\mapsto f/1$; we thus conclude (3.1.7) that the structure morphism $u^{-1}(X_f)\to X_f$ is an isomorphism, whence the first claim. To prove the second, note that the restriction $u^{-1}(X_f)\cap G(\alpha)\to X=\operatorname{Proj}(S)$ of $\operatorname{Proj}(\alpha)$ corresponds to the canonical map $x\mapsto x/1$ from S to $S_f^{\geqslant}$ (2.6.2); we thus deduce, first of all, that $G(\alpha)=X'$, and then, taking into account the fact that $u^{-1}(X_f)=(u^{-1}(X_f))_{f/1}$, that it follows from (2.8.1.1) that the image of $u^{-1}(X_f)$ under $\operatorname{Proj}(\alpha)$ is contained in X_f, and the restriction of $\operatorname{Proj}(\alpha)$ to $u^{-1}(X_f)$, thought of as a morphism to $X_f=\operatorname{Spec}(S_{(f)})$, is indeed identical to that of u. Finally, applying (8.3.5.4) to p_X instead of p, we see that $p_X^{-1}(u^{-1}(X_f))=\operatorname{Spec}((S_f^{\geqslant})_{f/1}^{\leqslant})$, and this open subset is, by (8.5.4.1), the inverse image under $\operatorname{Proj}(\widehat{\alpha})$ of $p^{-1}(X_f)=\operatorname{Spec}(S_f^{\leqslant})$ (8.3.5.3). Taking (8.2.3.2) into account, the restriction of $\operatorname{Proj}(\widehat{\alpha})$ to $p_X^{-1}(u^{-1}(X_f))$ corresponds to the isomorphism inverse to (8.2.7.2), restricted to $S_f^{\leqslant}$, whence the third claim; the last claim is evident by definition. II | 173

We note also that it follows from the commutative diagram in (8.5.3.2) that *the restriction to C_X of* $\operatorname{Proj}(\widehat{\alpha})$ *is exactly the morphism* $\operatorname{Spec}(\alpha)$. ☐

Corollary (8.6.3). — *Considered as X-schemes, $\widehat{E}_X$ is canonically isomorphic to* $\operatorname{Spec}(\mathcal{S}_X^{\leqslant})$, *and E_X to* $\operatorname{Spec}(\mathcal{S}_X)$.

Proof. Since we know that the morphisms p_X and π_X are affine ((8.3.5) and (8.3.6)), it suffices (given (1.3.1)) to prove the corollary in the case where $Y=\operatorname{Spec}(A)$ is affine and $\mathcal{S}=\widetilde{S}$. The first claim follows from the existence of the canonical isomorphisms (8.2.7.2) $(S_f^{\geqslant})_{f/1}^{\leqslant}\xrightarrow{\sim}S_f^{\leqslant}$ and from the fact that these isomorphisms are compatible with the map sending f to fg (where f and g are homogeneous in S_+). Similarly, applying (8.3.6.2) to π_X instead of π, we see that $\pi_X^{-1}(u^{-1}(X_f))=\operatorname{Spec}((S_f^{\geqslant})_{f/1})$ for f homogeneous in S_+, and the second claim then follows from the existence of the canonical isomorphisms (8.2.7.2) $(S_f^{\geqslant})_{f/1}\xrightarrow{\sim}S_f$.

We can then say that $\widehat{C}_X$, thought of as an X-scheme, is given by *gluing* the affine X-schemes $C_X=\operatorname{Spec}(\mathcal{S}_X^{\geqslant})$ and $\widehat{E}_X=\operatorname{Spec}(\mathcal{S}_X^{\leqslant})$ over X, where the intersection of the two affine X-schemes is the open subset $E_X=\operatorname{Spec}(\mathcal{S}_X)$. ☐

Corollary (8.6.4). — *Assume that $\mathcal{O}_X(1)$ is an invertible $\mathcal{O}_X$-module, and that $\mathcal{S}_X$ is isomorphic to $\bigoplus_{n\in\mathbf{Z}}(\mathcal{O}_X(1))^{\otimes n}$ (which will be the case, in particular, whenever $\mathcal{S}$ is generated by $\mathcal{S}_1$* ((3.2.5) *and* (3.2.7)*)). Then the pointed projective cone $\widehat{E}$ can be identified with the rank-1 vector bundle* $\mathbf{V}(\mathcal{O}_X(-1))$ *on X, and the pointed affine cone E with the subprescheme of this vector bundle induced on the complement of the null section. With this identification, the canonical retraction $\widehat{E}\to X$ is identified with the structure morphism of the X-scheme* $\mathbf{V}(\mathcal{O}_X(-1))$*. Finally, there exists a canonical Y-morphism* $\mathbf{V}(\mathcal{O}_X(1))\to C$*, whose restriction to the complement of the null section of* $\mathbf{V}(\mathcal{O}_X(1))$ *is an isomorphism from this complement to the pointed affine cone E.*

Proof. If we write $\mathscr{L} = \mathscr{O}_X(1)$, then $\mathscr{S}_X^{\geqslant}$ is identical to $\mathbf{S}_{\mathscr{O}_X}(\mathscr{L})$, and so $\widehat{E}_X$ is canonically identified with $\mathbf{V}(\mathscr{L}^{-1})$, by (8.6.3), and C_X with $\mathbf{V}(\mathscr{L})$. The morphism $\mathbf{V}(\mathscr{L}) \to C$ is the restriction of $\operatorname{Proj}(\widehat{\alpha})$, and the claims of the corollary are then particular cases of (8.6.2). □

We note that the inverse image under the morphism $\mathbf{V}(\mathscr{O}_X(1)) \to C$ of the underlying space of the vertex prescheme of C is the underlying space of the null section of $\mathbf{V}(\mathscr{O}_X(1))$ (8.5.5); but, in general, the corresponding subpreschemes of C and of $\mathbf{V}(\mathscr{O}_X(1))$ are not isomorphic. This problem will be studied below.

8.7. Blowing up based cones.

(8.7.1). Under the conditions of (8.6.1), we have, writing $r = \operatorname{Proj}(\widehat{\alpha})$, a commutative diagram

(8.7.1.1)
$$\begin{tikzcd} X \arrow[r, "i_X \circ \varepsilon_X"] \arrow[d, "q"'] & \widehat{C}_X \arrow[d, "r"] \\ Y \arrow[r, "i \circ \varepsilon"] & \widehat{C} \end{tikzcd}$$

by (8.5.1.3) and (8.5.3.2); furthermore, the restriction of r to the complement $\widehat{C}_X - i_X(\varepsilon_X(X))$ of the null section is an *isomorphism* to the complement $\widehat{C} - i(\varepsilon(Y))$ of the null section, by (8.6.2). If we suppose, to simplify things, that Y is affine, that $\mathscr{S}$ is of finite type and generated by $\mathscr{S}_1$, and that X is projective over Y and $\widehat{C}_X$ projective over X (5.5.1), then $\widehat{C}_X$ is projective over Y (5.5.5, ii), and *a fortiori* over $\widehat{C}$ (5.5.5, v). We then have a projective Y-morphism $r : \widehat{C}_X \to \widehat{C}$ (whose restriction to C_X is a projective Y-morphism $C_X \to C$) that *contracts X to Y* (?) and that induces an *isomorphism* when we restrict to the *complements of X and Y*. We thus have a connection between C_X and C, analogous to that which exists between a blow-up prescheme and the original prescheme (8.1.3). We will effectively show that C_X can be identified with the homogeneous spectrum of a graded $\mathscr{O}_C$-algebra. II | 174

(8.7.2). Keeping the notation of (8.6.1), consider, for all $n \geqslant 0$, the quasi-coherent ideal

(8.7.2.1)
$$\mathscr{S}_{[n]} = \bigoplus_{m \geqslant n} \mathscr{S}_m$$

of the graded $\mathscr{O}_Y$-algebra $\mathscr{S}$. It is clear that

(8.7.2.2)
$$\mathscr{S}_{[0]} = \mathscr{S}, \qquad \mathscr{S}_{[n]} \subset \mathscr{S}_{[m]} \qquad \text{for } m \leqslant n$$

(8.7.2.3)
$$\mathscr{S}_n \mathscr{S}_{[m]} \subset \mathscr{S}_{[m+n]}.$$

Consider the $\mathscr{O}_C$-module associated to $\mathscr{S}_{[n]}$, which is a quasi-coherent ideal of $\mathscr{O}_C = \widetilde{\mathscr{S}}$ (1.4.4)

(8.7.2.4)
$$\mathscr{I}_n = (\mathscr{S}_{[n]})^{\sim}.$$

We thus deduce, from (8.7.2.2) and (8.7.2.3), using (1.4.4) and (1.4.8.1), the analogous formulas

(8.7.2.5)
$$\mathscr{I}_{[0]} = \mathscr{O}_C, \qquad \mathscr{I}_{[n]} \subset \mathscr{I}_{[m]} \qquad \text{for } m \leqslant n$$

(8.7.2.6)
$$\mathscr{I}_n \mathscr{I}_{[m]} \subset \mathscr{I}_{[m+n]}.$$

We are thus in the setting of (8.1.1), which leads us to introduce the quasi-coherent graded $\mathscr{O}_C$-algebra

(8.7.2.7)
$$\mathscr{S}^{\natural} = \bigoplus_{n \geqslant 0} \mathscr{I}_n = \left(\bigoplus_{n \geqslant 0} \mathscr{S}_{[n]} \right)^{\sim}.$$

Proposition (8.7.3). — *There is a canonical C-isomorphism*

(8.7.3.1)
$$h : C_X \xrightarrow{\sim} \operatorname{Proj}(\mathscr{S}^{\natural}).$$

Proof. Suppose first of all that $Y = \operatorname{Spec}(A)$ is affine, so that $\mathscr{S} = \widetilde{S}$, with S a positively-graded A-algebra, and $C = \operatorname{Spec}(S)$. Definition (8.2.7.4) then shows, with the notation of (8.2.6), that $\mathscr{S}^{\natural} = (S^{\natural})^{\sim}$. To define (8.7.3.1), consider a homogeneous element $f \in S_d$ $(d > 0)$ and the corresponding element $f^{\natural} \in S^{\natural}$ (8.2.6); the S-isomorphism in (8.2.7.3) then defines a C-isomorphism

(8.7.3.2)
$$\operatorname{Spec}(S_f^{\geqslant}) \xrightarrow{\sim} \operatorname{Spec}(S^{\natural}_{(f^{\natural})}).$$

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But with the notation of (8.6.2), if $v : C_X \to X$ is the structure morphism, then it follows from (8.6.2.1) that $v^{-1}(X_f) = \operatorname{Spec}(S_f^{\geqslant})$. We also have that $\operatorname{Spec}(S^{\natural}_{(f^{\natural})}) = D_+(f^{\natural})$, which means that (8.7.3.2) defines an isomorphism $v^{-1}(X_f) \to D_+(f^{\natural})$. Furthermore, if $g \in S_e$ $(e > 0)$, then the diagram

$$\begin{tikzcd} v^{-1}(X_{fg}) \arrow[r, "\sim"] \arrow[d] & D_+(f^{\natural}g^{\natural}) \arrow[d] \\ v^{-1}(X_f) \arrow[r, "\sim"] & D_+(f^{\natural}) \end{tikzcd}$$

commutes, by definition of the isomorphism in (8.2.7.3). Finally, by definition, S_+ is generated by the homogeneous f, and so it follows from (8.2.10, iv) and from (2.3.14) that the $D_+(f^{\natural})$ form a cover of $\operatorname{Proj}(S^{\natural})$, and that the $v^{-1}(X_f)$ form a cover of C_X, since the X_f form a cover of X; in this case, we have thus defined the isomorphism (8.7.3.1).

To prove (8.7.3) in the general case, it suffices to show that, if U and U' are affine open subsets of Y, given by rings A and A' (respectively), and such that $U' \subset U$, then, setting $\mathscr{S}|U = \widetilde{S}$ and $\mathscr{S}|U' = \widetilde{S}'$, the diagram

(8.7.3.3)
$$\begin{tikzcd} C_{U'} \arrow[r] \arrow[d] & \operatorname{Proj}(S'^{\natural}) \arrow[d] \\ C_U \arrow[r] & \operatorname{Proj}(S^{\natural}) \end{tikzcd}$$

commutes. But S is canonically identified with $S \otimes_A A'$, and so $S'^{\natural}$ is canonically identified with

$$S^{\natural} \otimes_S S' = S^{\natural} \otimes_A A';$$

thus $\operatorname{Proj}(S'^{\natural}) = \operatorname{Proj}(S^{\natural}) \times_U U'$ (2.8.10); similarly, if $X = \operatorname{Proj}(S)$ and $X' = \operatorname{Proj}(S')$, then $X' = X \times_U U'$ and $\mathscr{S}_{X'} = \mathscr{S}_X \otimes_{\mathscr{O}_U} U'$ (3.5.4), or, equivalently, $\mathscr{S}_{X'} = j^*(\mathscr{S}_X)$, where j is the projection $X' \to X$. We then (1.5.2) have that $C_{U'} = C_U \times_X X' = C_U \times_U U'$, and the commutativity of (8.7.3.3) is then immediate. □

Remark (8.7.4). — (i) The end of the proof of (8.7.3) can be immediately generalised in the following way. Let $g : Y' \to Y$ be a morphism, $\mathscr{S}' = g^*(\mathscr{S})$, and $X' = \operatorname{Proj}(\mathscr{S}')$; then we have a commutative diagram

(8.7.4.1)
$$\begin{tikzcd} C_{X'} \arrow[r] \arrow[d] & \operatorname{Proj}(\mathscr{S}'^{\natural}) \arrow[d] \\ C_X \arrow[r] & \operatorname{Proj}(\mathscr{S}^{\natural}) \end{tikzcd}$$

Now let $\varphi : \mathscr{S}'' \to \mathscr{S}$ be a homomorphism of graded $\mathscr{O}_Y$-algebras such that, if we write $X'' = \operatorname{Proj}(\mathscr{S}'')$, then $u = \operatorname{Proj}(\varphi) : X \to X''$ is everywhere defined; we also have a Y-morphism $v : C \to C''$ **II | 176**
(with $C'' = \operatorname{Spec}(\mathscr{S}'')$) such that $\mathscr{A}(v) = \varphi$, and, since φ is a homomorphism of graded algebras, φ induces a v-morphism of graded algebras $\psi : \mathscr{S}''^{\natural} \to \mathscr{S}^{\natural}$ (1.4.1). Furthermore, it follows from (8.2.10, iv) and from the hypothesis on φ that $\operatorname{Proj}(\psi)$ is everywhere defined. Finally, taking (3.5.6.1) into account, there is a canonical u-morphism $\mathscr{S}_{X''} \to \mathscr{S}_X$, whence (1.5.6) a morphism $w : C_{X''} \to C_X$. With this in mind, the diagram

(8.7.4.2)
$$\begin{tikzcd} C_{X''} \arrow[r, "\sim"] \arrow[d, "w"'] & \operatorname{Proj}(\mathscr{S}''^{\natural}) \arrow[d, "\operatorname{Proj}(\psi)"] \\ C_X \arrow[r, "\sim"] & \operatorname{Proj}(\mathscr{S}^{\natural}) \end{tikzcd}$$

is commutative, as we can immediately verify by restricting to the case where Y is affine.

(ii) Note that, by (8.7.2.5) and (8.7.2.6), we have $\mathscr{I}_1^m \subset \mathscr{I}_m \subset \mathscr{I}_1$ for all $m > 0$. But, by definition, $\mathscr{I}_1 = (\mathscr{S}_+)^{\sim}$, and so $\mathscr{I}_1$ defines the closed subprescheme $\varepsilon(Y)$ in C ((1.4.10) and (8.3.2)); we thus conclude that, for all $m > 0$, *the support of $\mathscr{O}_C/\mathscr{I}_m$ is contained in the underlying space of the vertex prescheme* $\varepsilon(Y)$; on the inverse image of the pointed affine cone E, the structure morphism $\operatorname{Proj}(\mathscr{S}^{\natural}) \to C$ thus restricts to an *isomorphism* (by (8.7.3) and (8.7.1)). Furthermore, by canonically identifying C with an open subset of $\widehat{C}$ (8.3.3), we can clearly extend the ideals $\mathscr{I}_m$ of $\mathscr{O}_C$ to ideals $\mathscr{J}_m$ of $\mathscr{O}_{\widehat{C}}$, by asking for it to agree with $\mathcal{O}_{\widehat{C}}$ on the open subset $\widehat{E}$ of $\widehat{C}$. If we define $\mathscr{T} = \bigoplus_{n\geqslant 0} \mathscr{J}_m$, which is a quasi-coherent graded $\mathcal{O}_{\widehat{C}}$-algebra, we can extend the isomorphism (8.7.3.1) to a $\widehat{C}$-isomorphism

$$\widehat{C}_X \xrightarrow{\sim} \operatorname{Proj}(\mathscr{T}). \tag{8.7.4.3}$$

Indeed, over $\widehat{E}$, it follows from the above that $\operatorname{Proj}(\mathscr{T})$ is canonically identified with $\widehat{E}$, and we thus define the isomorphism (8.7.4.3) over $\widehat{E}$ by asking for it to agree with the canonical isomorphism $\widehat{E}_X \to \widehat{E}$ (8.6.2); it is clear that this isomorphism and (8.7.3.1) then agree over $\widehat{E}$.

Corollary (8.7.5). — *Suppose that there exists some $n_0 > 0$ such that*

$$\mathscr{S}_{n+1} = \mathscr{S}_1\mathscr{S}_n \qquad \textit{for } n \geqslant n_0. \tag{8.7.5.1}$$

Then the vertex subprescheme (?) of C_X (isomorphic to X) is the inverse image under the canonical morphism $r : C_X \to C$ of the vertex subprescheme of C (isomorphic to Y). Conversely, if this property is true, and if we further assume that Y is Noetherian and that $\mathscr{S}$ is of finite type, then there exists some $n_0 > 0$ such that (8.7.5.1) holds true.

Proof. The first claim being local on Y, we can assume that $Y = \operatorname{Spec}(A)$ is affine, so that $\mathscr{S} = \widetilde{S}$, with S a positively-graded A-algebra. The claim then follows from (8.2.12), since $\operatorname{Proj}(S^\natural \otimes_S S_0) = C_X \times_C \varepsilon(Y)$ (by the identification in (8.7.3.1)), or, in other words, since this prescheme is the inverse image of $\varepsilon(Y)$ in C_X (**I**, 4.4.1). The converse also follows from (8.2.12) whenever Y is Noetherian affine and S is of finite type. If Y is Noetherian (but not necessarily affine) and $\mathscr{S}$ is of finite type, then there exists a finite cover of Y by Noetherian affine open subsets U_i, and we then deduce from the above that, for all i, there exists an integer n_i such that $\mathscr{S}_{n+1}|U_i = (\mathscr{S}_1|U_i)(\mathscr{S}_n|U_i)$ for $n \geqslant n_i$; the largest of the n_i then ensures that (8.7.5.1) holds true. □ **II** | 177

(8.7.6). Now consider the C-prescheme Z given by *blowing up* the *vertex subprescheme* $\varepsilon(Y)$ in the affine cone C; by Definition (8.1.3), it is exactly the prescheme $\operatorname{Proj}(\bigoplus_{n\geqslant 0} \mathscr{S}_+^n)$; the canonical injection

$$\iota : \bigoplus_{n\geqslant 0} \mathscr{S}_+^n \longrightarrow \mathscr{S}^\natural \tag{8.7.6.1}$$

defines (by the identification in (8.7.3)) a canonical dominant C-morphism

$$G(\iota) \longrightarrow Z \tag{8.7.6.2}$$

where $G(\iota)$ is an open subset of C_X (3.5.1); note that it could be the case that $G(\iota) \neq C_X$, as shown by the example where $Y = \operatorname{Spec}(K)$, with K a field, and $\mathscr{S} = \widetilde{S}$, with $S = K[\mathbf{y}]$, where $\mathbf{y}$ is an indeterminate *of degree 2*; if R_n denotes the set $(S_+)^n$, thought of as a subset of $S_{[n]} = S_n^\natural$, then $S_+^\natural$ is not the radical in $S_+^\natural$ of the ideal generated by the union of the R_n (cf. (2.3.14)).

Corollary (8.7.7). — *Assume that there exists some $n_0 > 0$ such that*

$$\mathscr{S}_n = \mathscr{S}_1^n \quad \textit{for } n \geqslant n_0. \tag{8.7.7.1}$$

Then the canonical morphism (8.7.6.2) is everywhere defined, and is an isomorphism $C_X \xrightarrow{\sim} Z$. Conversely, if this property is true, and if we further assume that Y is Noetherian and that $\mathscr{S}$ is of finite type, then there exists some n_0 such that (8.7.7.1) holds true.

Proof. The first claim is local on Y, and thus follows from (8.2.14); the converse follows similarly, arguing as in (8.7.5). □

Remark (8.7.8). — Since condition (8.7.7.1) implies (8.7.5.1), we see that, whenever it holds true, not only can C_X be identified with the prescheme given by blowing up the vertex (identified with Y) of the affine cone C, but also the vertex (identified with X) of C_X can be identified with the closed subprescheme given by the inverse image of the vertex Y of C. Furthermore, hypothesis (8.7.7.1) implies that, on $X = \operatorname{Proj}(\mathscr{S})$, the $\mathcal{O}_X$-modules $\mathcal{O}_X(n)$ are invertible ((3.2.5) and (3.2.9)), and that $\mathcal{O}_X(n) = \mathscr{L}^{\otimes n}$ with $\mathscr{L} = \mathcal{O}_X(1)$ ((3.2.7) and (3.2.9)); by Definition (8.6.1.1), C_X is thus the *vector bundle* $\mathbf{V}(\mathscr{L})$ on X, and its vertex is the *null section* of this vector bundle.

8.8. Ample sheaves and contractions.

(8.8.1). Let Y be a prescheme, $f : X \to Y$ a *separated* and *quasi-compact* morphism, and $\mathscr{L}$ an invertible $\mathcal{O}_X$-module that is *ample relative to f*. Consider the positively-graded $\mathcal{O}_Y$-algebra

$$\mathscr{S} = \mathcal{O}_Y \oplus \bigoplus_{n\geqslant 1} f_*(\mathscr{L}^{\otimes n}) \tag{8.8.1.1}$$

which is quasi-coherent (**I**, 9.2.2, a). There is a canonical homomorphisms of graded $\mathcal{O}_X$-algebras **II** | 178

(8.8.1.2)
$$\tau : f^*(\mathscr{S}) \longrightarrow \bigoplus_{n\geqslant 0} \mathscr{L}^{\otimes n}$$

which, in degrees $\geqslant 1$, agrees with the canonical homomorphism $\sigma : f^*(f_*(\mathscr{L}^{\otimes n})) \to \mathscr{L}^{\otimes n}$ (**0**, 4.4.3), and is the identity in degree 0. The hypothesis that $\mathscr{L}$ is f-ample then implies ((4.6.3) and (3.6.1)) that the corresponding Y-morphism

(8.8.1.3)
$$r = r_{\mathscr{L},\tau} : X \longrightarrow P = \operatorname{Proj}(\mathscr{S})$$

is everywhere defined and is a *dominant open immersion*, and that

(8.8.1.4)
$$r^*(\mathscr{O}_P(n)) = \mathscr{L}^{\otimes n} \qquad \text{for all } n \in \mathbf{Z}.$$

Proposition (8.8.2). — *Let $C = \operatorname{Spec}(\mathscr{S})$ be the affine cone defined by $\mathscr{S}$; if $\mathscr{L}$ is f-ample, then there exists a canonical Y-morphism*

(8.8.2.1)
$$g : V = \mathbf{V}(\mathscr{L}) \longrightarrow C$$

such that the diagram

(8.8.2.2)
$$\begin{tikzcd} X \arrow[r, "j"] \arrow[d, "f"'] & \mathbf{V}(\mathscr{L}) \arrow[r, "\pi"] \arrow[d, "g"] & X \arrow[d, "f"] \\ Y \arrow[r, "\varepsilon"] & C \arrow[r, "\psi"] & Y \end{tikzcd}$$

commutes, where ψ and π are the structure morphisms, and j and ε the canonical immersions sending X and Y (respectively) to the null section of $\mathbf{V}(\mathscr{L})$ and the vertex prescheme of C (respectively). Furthermore, the restriction of g to $\mathbf{V}(\mathscr{L}) - j(X)$ is an open immersion

(8.8.2.3)
$$\mathbf{V}(\mathscr{L}) - j(X) \longrightarrow E = C - \varepsilon(Y)$$

into the pointed affine cone E corresponding to $\mathscr{S}$.

Proof. With the notation of (8.8.1), let $\mathscr{S}_P^{\geqslant} = \bigoplus_{n\geqslant 0} \mathscr{O}_P(n)$ and $C_p = \operatorname{Spec}(\mathscr{S}_P^{\geqslant})$. We know (8.6.2) that there is a canonical morphism $h = \operatorname{Spec}(\alpha) : C_p \to C$ such that the diagram

(8.8.2.4)
$$\begin{tikzcd} C_P \arrow[r] \arrow[d, "h"'] & P \arrow[d, "p"] \\ C \arrow[r, "\psi"] & Y \end{tikzcd}$$

commutes; furthermore, if $\varepsilon_P : P \to C_P$ is the canonical immersion, then the diagram

(8.8.2.5)
$$\begin{tikzcd} P \arrow[r, "p"] \arrow[d, "\varepsilon_P"'] & C_P \arrow[d, "h"] \\ Y \arrow[r, "\varepsilon"] & C \end{tikzcd}$$

commutes (8.7.1.1), and, finally, the restriction of H to the pointed affine cone E_P is an *isomorphism* $E_P \xrightarrow{\sim} E$ (8.6.2). It follows from (8.8.1.4) that

$$r^*(\mathscr{S}_P^{\geqslant}) = \mathbf{S}_{\mathscr{O}_X}(\mathscr{L})$$

and so we have a canonical P-morphism $q : \mathbf{V}(\mathscr{L}) \to C_P$, with the commutative diagram **II** | 179

(8.8.2.6)
$$\begin{tikzcd} \mathbf{V}(\mathscr{L}) \arrow[r, "\pi"] \arrow[d, "q"'] & X \arrow[d, "r"] \\ C_P \arrow[r] & P \end{tikzcd}$$

identifying $\mathbf{V}(\mathscr{L})$ with the product $C_P \times_P X$ (1.5.2); since r is an open immersion, so too is q (**I**, 4.3.2). Furthermore, the restriction of q to $\mathbf{V}(\mathscr{L}) - j(X)$ sends this prescheme to E_p, by (8.5.2), and the diagram

(8.8.2.7)
$$\begin{tikzcd} X \arrow[r, "j"] \arrow[d, "r"'] & \mathbf{V}(\mathscr{L}) \arrow[d, "q"] \\ P \arrow[r, "\varepsilon_P"] & C_P \end{tikzcd}$$

is commutative (since it is a particular case of (8.5.1.3)). The claims of (8.8.2) immediately follow from these facts, by taking g to be the composite morphism $h \circ q$. □

Remark (8.8.3). — Assume further that Y is a *Noetherian* prescheme, and that f is a *proper* morphism. Since r is then *proper*(5.4.4), and thus closed, and since it is also a dominant open immersion, r is necessarily an *isomorphism* $X \xrightarrow{\sim} P$. Furthermore, we will see, in Chapter III (**III**, 2.3.5.1), that $\mathscr{S}$ is then necessarily an $\mathscr{O}_Y$-algebra *of finite type*. It then follows that $\mathscr{S}^\natural$ is an $\mathscr{S}_0^\natural$-algebra *of finite type* ((8.2.10, i) and (8.7.2.7)); since C_P is C-isomorphic to $\operatorname{Proj}(\mathscr{S}^\natural)$ (8.7.3), we see that the morphism $h : C_P \to C$ is *projective*; since the morphism r is an isomorphism, so too is $q : \mathbf{V}(\mathscr{L}) \to C_P$, and we thus conclude that the morphism $g : \mathbf{V}(\mathscr{L}) \to C$ is *projective*. Furthermore, since the restriction of h to E_P is an isomorphism to E, and since q is an isomorphism, the restriction (8.8.2.3) of g is an isomorphism $\mathbf{V}(\mathscr{L}) - j(X) \xrightarrow{\sim} E$.

If we further assume that L is *very ample* for f, then, as we will also see in Chapter III (**III**, 2.3.5.1), there exists some integer $n_0 > 0$ such that $\mathscr{S}_n = \mathscr{S}_1^n$ for $n \geqslant n_0$. We then conclude, by (8.7.7), that $\mathbf{V}(\mathscr{L})$ can be identified with the prescheme Z given by *blowing up the vertex prescheme* (identified with Y) *in the affine cone* C, and that the *null section* of $\mathbf{V}(\mathscr{L})$ (identified with Y) is the *inverse image* of the vertex subprescheme Y of C.

Some of the above results can in fact be proven even without the Noetherian hypothesis:

Corollary (8.8.4). — *Let Y be a prescheme (resp. a quasi-compact scheme), $f : X \to Y$ a proper morphism, and $\mathscr{L}$ an invertible $\mathscr{O}_X$-module that is ample relative to f. Then the morphism in* (8.8.2.1) *is proper (resp. projective), and its restriction* (8.8.2.3) *is an isomorphism.*

Proof. To prove that g is proper, we can restrict to the case where Y is affine, and it then suffices to consider the case where Y is a quasi-compact scheme. The same arguments as in (8.8.3) first of all show that r is an *isomorphism* $X \xrightarrow{\sim} P$; then q is also an isomorphism, and, since the restriction of h to E_P is an isomorphism $E_P \xrightarrow{\sim} E$, we have already seen that (8.8.2.3) is an isomorphism. It remains only to prove that g is *projective*.

Since f is of finite type, by hypothesis, we can apply (3.8.5) to the homomorphism τ from (8.8.1.2): there is an integer $d > 0$ and a quasi-coherent $\mathscr{O}_Y$-submodule $\mathscr{E}$ of finite type of $\mathscr{S}_d$ such that, if $\mathscr{S}'$ is the $\mathscr{O}_Y$-subalgebra of $\mathscr{S}$ generated by $\mathscr{E}$, and $\tau' = \tau \circ q^*(\varphi)$ (where φ is the canonical injection $\mathscr{S}' \to \mathscr{S}$), then $r' = r_{\mathscr{L},\tau'}$ is an immersion **II | 180**

$$X \longrightarrow P' = \operatorname{Proj}(\mathscr{S}').$$

Furthermore, since φ is injective, r' is also a *dominant immersion* (3.7.6); the same argument as for r then shows that r' is a *surjective closed immersion*; since r' factors as $X \xrightarrow{r} \operatorname{Proj}(\mathscr{S}) \xrightarrow{\Phi} \operatorname{Proj}(\mathscr{S}')$, where $\Phi = \operatorname{Proj}(\varphi)$, we thus conclude that Φ is also a *surjective closed immersion*. But this implies that Φ is an *isomorphism*; we can restrict to the case where $Y = \operatorname{Spec}(A)$ is affine, and $\mathscr{S} = \widetilde{S}$ and $\mathscr{S}' = \widetilde{S}'$, with S a graded A-algebra and S' a graded subalgebra of S. For every homogeneous element $t \in S'$, we have that $S'_{(t)}$ is a subring of $S_{(t)}$; if we return to the definition of $\operatorname{Proj}(\varphi)$ (2.8.1), we see that it suffices to prove that, if B' is a subring of a ring B, and if the morphism $\operatorname{Spec}(B) \to \operatorname{Spec}(B')$ corresponding to the canonical injection $B' \to B$ is a closed immersion, then this morphism is necessarily an *isomorphism*; but this follows from (**I**, 4.2.3). Furthermore, $\Phi^*(\mathscr{O}_{P'}(n)) = \mathscr{O}_P(n)$ ((3.5.2, ii) and (3.5.4)), and so $r'^*(\mathscr{O}_{P'}(n))$ is isomorphic to $\mathscr{L}^{\otimes n}$ (4.6.3). Let $\mathscr{S}'' = \mathscr{S}'^{(d)}$, so that (3.1.8, i) X is canonically identified with $P'' = \operatorname{Proj}(\mathscr{S}'')$, and $\mathscr{L}'' = \mathscr{L}^{\otimes d}$ with $\mathscr{O}_{P''}(1)$ (3.2.9, ii).

Now, if $C'' = \operatorname{Spec}(\mathscr{S}'')$, then $\mathscr{S}_{P''}^{\geqslant} = \bigoplus_{n \geqslant 0} \mathscr{O}_{P''}(n)$ can be identified with $\bigoplus_{n \geqslant 0} \mathscr{L}''^{\otimes n}$, and thus $C_{P''} = \operatorname{Spec}(\mathscr{S}_{P''}^{\geqslant})$ with $\mathbf{V}(\mathscr{L}'')$; we also know (8.7.3) that $C_{P''}$ is C''-isomorphic to $\operatorname{Proj}(\mathscr{S}''^\natural)$; by the definition of $\mathscr{S}''$, we know that $\mathscr{S}''^\natural$ is generated by $\mathscr{S}_1''^\natural$, and that $\mathscr{S}_1''^\natural$ is of finite type over $\mathscr{S}_0''^\natural = \mathscr{S}''$ ((8.2.10, i and iii)), and so $\operatorname{Proj}(\mathscr{S}''^\natural)$ is *projective* over C'' (5.5.1). Consider the diagram

(8.8.4.1)
$$\begin{tikzcd} \mathbf{V}(\mathscr{L}) \arrow[r, "g"] \arrow[d, "u"'] & \operatorname{Spec}(\mathscr{S}) = C \arrow[d, "v"] \\ \mathbf{V}(\mathscr{L}'') \arrow[r, "g''"] & \operatorname{Spec}(\mathscr{S}'') = C'' \end{tikzcd}$$

where g and g'' correspond, by (1.5.6), to the canonical j-morphisms

$$\mathscr{S} \longrightarrow \bigoplus_{n \geqslant 0} \mathscr{L}^{\otimes n} \quad \text{and} \quad \mathscr{S}'' \longrightarrow \bigoplus_{n \geqslant 0} \mathscr{L}''^{\otimes n}$$

(3.3.2.3) (see (8.8.5) below), and v and u to the inclusion morphisms $\mathscr{S}'' \to \mathscr{S}$ and $\bigoplus_{n \geqslant 0} \mathscr{L}^{\otimes nd} \to \bigoplus_{n \geqslant 0} \mathscr{L}^{\otimes n}$ (respectively); it is immediate (3.3.2) that this diagram is commutative. We have just seen that g'' is a projective morphism; we also know that u is a *finite* morphism. Since the question is local on X, we can assume that X is affine of ring A, and that $\mathscr{L} = \mathscr{O}_X$; everything then reduces to noting that the ring $A[T]$ is a module of finite type over its subring $A[T^d]$ (with T an indeterminate). Since Y is a quasi-compact scheme, and since C'' is affine over Y, we know that C'' is also a quasi-compact scheme, and so $g'' \circ u$ is a projective morphism (5.5.5, ii); by commutativity of (8.8.4.1), $v \circ g$ is also projective, and, since v is affine, thus separated, we finally conclude that g is projective (5.5.5, v). □ II | 181

(8.8.5). Consider again the situation in (8.8.1). We will see that the morphism $g : \mathbf{V}(\mathscr{L}) \to C$ can be also be defined in a way that works for any invertible (but not necessarily ample) $\mathscr{O}_X$-module $\mathscr{L}$. For this, consider the f-morphism

(8.8.5.1)
$$\tau^\flat : \mathscr{S} \longrightarrow \bigoplus_{n\geqslant 0} \mathscr{L}^{\otimes n}$$

corresponding to the morphism τ from (8.8.1.2). This induces (1.5.6) a morphism $g' : V \to C$ such that, if $\pi : V \to X$ and $\psi : C \to Y$ are the structure morphisms, the diagrams

(8.8.5.2)
$$\begin{tikzcd} X \arrow[d, "f"'] & V \arrow[l, "\pi"'] \arrow[d, "g'"] \\ Y & C \arrow[l, "\psi"] \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "j"] \arrow[d, "f"'] & V \arrow[d, "g'"] \\ Y \arrow[r, "\varepsilon"] & C \end{tikzcd}$$

commute ((8.5.1.2) and (8.5.1.3)). We will show that (if we assume that $\mathscr{L}$ is f-ample) *the morphisms g and g' are identical.*

The question being local on Y, we can assume that $Y = \operatorname{Spec}(A)$ is affine, and (by (8.8.1.3)) identify X with an open subset of $P = \operatorname{Proj}(S)$, where $S = A \oplus \bigoplus_{n\geqslant 0} \Gamma(X, \mathscr{L}^{\otimes n})$; we then deduce, by (8.8.1.4), that $\Gamma(X, \mathscr{O}_P(n)) = \Gamma(X, \mathscr{L}^{\otimes n})$ for all $n \in \mathbf{Z}$. Taking into account the definition of $h = \operatorname{Spec}(\alpha)$, where α is the canonical p-morphism $\widetilde{S} \to \mathscr{S}_P^{\geqslant}$ (8.6.1.2), we have to show that the restriction to X of $\alpha^\sharp : p^*(\widetilde{S}) \to \mathscr{S}_P^{\geqslant}$ is identical to τ. Taking (0, 4.4.3) into account, it suffices to show that, if we compose the canonical homomorphism $\alpha_n : S_n \to \Gamma(P, \mathscr{O}_P(n))$ with the restriction homomorphism $\Gamma(P, \mathscr{O}_P(n)) \to \Gamma(X, \mathscr{O}_P(n)) = \Gamma(X, \mathscr{L}^{\otimes n})$, then we obtain the identity, for all $n > 0$; but this follows immediately from the definition of the algebra S and of α_n (2.6.2).

Proposition (8.8.6). — *Assume (with the notation of* (8.8.5)*) that, if we write $f = (f_0, \lambda)$, then the homomorphism $\lambda : \mathscr{O}_Y \to j_*(\mathscr{O}_X)$ is bijective; then:*

(i) *if we write $g = (g_0, \mu)$, then $\mu : \mathscr{O}_C \to g_*(\mathscr{O}_V)$ is an isomorphism; and*
(ii) *if X is integral (resp. locally integral and normal), then C is integral (resp. normal).*

Proof. Indeed, the f-morphism $\tau^\flat$ is then an *isomorphism*

$$\tau^\flat : \mathscr{S} = \psi_*(\mathscr{O}_C) \longrightarrow f_*(\pi_*(\mathscr{O}_V)) = \psi_*(g_*(\mathscr{O}_V))$$

and the Y-morphism g can be considered as that for which the homomorphism $\mathscr{A}(g)$ (1.1.2) is equal to $\tau^\flat$. To see that μ is an isomorphism of $\mathscr{O}_C$-modules, it suffices (1.4.2) to see that $\mathscr{A}(\mu) : \psi_*(\mathscr{O}_C) \to \psi_*(g_*(\mathscr{O}_V))$ is an isomorphism. But, by Definition (1.1.2), we have that $\mathscr{A}(\mu) = \mathscr{A}(g)$, whence the conclusion of (i).

To prove (ii), we can restrict to the case where Y is affine, and so $\mathscr{S} = \widetilde{S}$, with $S = \bigoplus_{n\geqslant 0} \Gamma(X, \mathscr{L}^{\otimes n})$; the II | 182
hypothesis that X is integral implies that the ring S is integral (**I**, 7.4.4), and thus so too is C (**I**, 5.1.4). To show that C is normal, we will use the following lemma:

Lemma (8.8.6.1). — *Let Z be a normal integral prescheme. Then the ring $\Gamma(Z, \mathscr{O}_Z)$ is integral and integrally closed.*

Proof. It follows from (**I**, 8.2.1.1) that $\Gamma(Z, \mathscr{O}_Z)$ is the intersection, in the field of rational functions $R(Z)$, of the integrally closed rings $\mathscr{O}_z$ over all $z \in Z$. □

With this in mind, we first show that V is *locally integral* and *normal*; for this, we can restrict to the case where $X = \operatorname{Spec}(A)$ is affine, with ring A integral and integrally closed (6.3.8), and where $\mathscr{L} = \mathscr{O}_X$. Since then $V = \operatorname{Spec}(A[T])$, and $A[T]$ is integral and integrally closed [**Jaf60**, p. 99], this proves our claim. For every affine open subset U of C, $g^{-1}(U)$ is quasi-compact, since the morphism g is quasi-compact; since V is locally integral, the connected components of $g^{-1}(U)$ are open integral preschemes in $g^{-1}(U)$, and thus finite in number, and, since V is normal, these preschemes are also normal (6.3.8). Then $\Gamma(U, \mathscr{O}_C)$, which is equal to $\Gamma(g^{-1}(U), \mathscr{O}_V)$, by (i), is the direct sum (?) of finitely-many integral and integrally closed rings (8.8.6.1), which proves that C is normal (6.3.4). □

8.9. Grauert's ampleness criterion: statement. We intend to show that the properties proven in (8.8.2) *characterise* f-ample $\mathscr{O}_X$-modules, and, more precisely, to prove the following criterion:

Theorem (8.9.1). — (Grauert's criterion). *Let Y be a prescheme, $p : X \to Y$ a separated and quasi-compact morphism, and $\mathscr{L}$ an invertible $\mathscr{O}_X$-module. For $\mathscr{L}$ to be ample relative to p, it is necessary and sufficient for there to exist a Y-prescheme C, a Y-section $\varepsilon : Y \to C$ of C, and a Y-morphism $q : \mathbf{V}(\mathscr{L}) \to C$, satisfying the following properties:*

(i) *the diagram*

(8.9.1.1)
$$\begin{tikzcd} X \arrow[r, "j"] \arrow[d, "p"'] & \mathbf{V}(\mathscr{L}) \arrow[d, "q"] \\ Y \arrow[r, "\varepsilon"] & C \end{tikzcd}$$

commutes, where j is the null section of the vector bundle $\mathbf{V}(\mathscr{L})$; and

(ii) *the restriction of q to $\mathbf{V}(\mathscr{L}) - j(X)$ is a quasi-compact open immersion*
$$\mathbf{V}(\mathscr{L}) - j(X) \longrightarrow X$$
whose image does not intersect $\varepsilon(Y)$.

Note that, if C is *separated* over Y, we can, in condition (ii), remove the hypothesis that the open immersion is quasi-compact; to see that this property (of quasi-compactness) is in fact a consequence of the other conditions, we can restrict to the case where Y is affine, and the claim then follows from (**I**, 5.5.1)i and (**I**, 5.5.10). We can also remove the same hypothesis if we assume that X is Noetherian, since then V is also Noetherian, and the claim follows from (**I**, 6.3.5). **II** | 183

Corollary (8.9.2). — *If the morphism $p : X \to Y$ is proper, then we can, in the statement of Theorem* (8.9.1)*, assume that q is proper, and replace "open immersion" by "isomorphism".*

In a more suggestive manner, we can say (whenever $p : X \to Y$ is proper) that $\mathscr{L}$ *is ample relative to p if and only if we can* "contract" *the null section of the vector bundle* $\mathbf{V}(\mathscr{L})$ *to the base prescheme Y*. An important particular case is that where Y is the spectrum of a field, and where the operation of "contraction" consists of contract the null section $\mathbf{V}(\mathscr{L})$ *to a single point.*

(8.9.3). The necessity of the conditions in Theorem (8.9.1) and Corollary (8.9.2) follow immediately from (8.8.2) and (8.8.4).

To show that the conditions of (8.9.1) suffices, consider a slightly more general situation. For this, let (with the notation of (8.8.2))
$$\mathscr{S}' = \bigoplus_{n\geqslant 0} \mathscr{L}^{\otimes n}$$
and
$$V = \mathbf{V}(\mathscr{L}) = \operatorname{Spec}(\mathscr{S}').$$

The closed subprescheme $j(X)$, null section of $\mathbf{V}(\mathscr{L})$, is defined by the quasi-coherent sheaf of ideals $\mathscr{J} = (\mathscr{S}'_+)^\sim$ of $\mathscr{O}_V$ (1.4.10). This $\mathscr{O}_V$-module is *invertible*, since this property is local on X, and this reduces to remarking that the ideal $TA[T]$ in a ring of polynomials $A[T]$ is a free cyclic $A[T]$-module. Furthermore, it is immediate (again, because the question is local on X) that
$$\mathscr{L} = j^*(\mathscr{J})$$
and
$$j_*(\mathscr{L}) = \mathscr{J}/\mathscr{J}^2.$$

Now, if
$$\pi : \mathbf{V}(\mathscr{L}) \longrightarrow X$$
is the structure morphism, then $\pi_*(\mathscr{J}) = \mathscr{S}'_+$ and $\pi_*(\mathscr{J}/\mathscr{J}^2) = \mathscr{L}$; there are thus canonical homomorphisms $\mathscr{L} \to \pi_*(\mathscr{J}) \to \mathscr{L}$, the first being the canonical injection $\mathscr{L} \to \mathscr{S}'_+$, and the second the canonical projection from $\mathscr{S}'_+$ to $\mathscr{S}'_1 = \mathscr{L}$, and their composition being the identity. We can also canonically embed $\pi_*(\mathscr{J}) = \mathscr{S}'_+ = \bigoplus_{n\geqslant 1} \mathscr{L}^{\otimes n}$ into the *product* $\prod_{n\geqslant 1} \mathscr{L}^{\otimes n} = \varprojlim_n \pi_*(\mathscr{J}/\mathscr{J}^{n+1})$ (since $\pi_*(\mathscr{J}/\mathscr{J}^{n+1}) = \mathscr{L} \oplus \mathscr{L}^{\otimes 2} \oplus \ldots \oplus \mathscr{L}^{\otimes n}$), and we thus have canonical homomorphisms

(8.9.3.1)
$$\mathscr{L} \longrightarrow \varprojlim \pi_*(\mathscr{J}/\mathscr{J}^{n+1}) \longrightarrow \mathscr{L}$$

whose composition is the identity.

With this in mind, the generalisation of (8.9.1) that we are going to prove is the following:

Proposition (8.9.4). — *Let Y be a prescheme, V a Y-prescheme, and X a closed subprescheme of V defined by an ideal $\mathscr{J}$ of $\mathscr{O}_V$, which is an* invertible *$\mathscr{O}_V$-module; if $j : X \to V$ is the canonical injection, then let $\mathscr{L} = j^*(\mathscr{J}) = \mathscr{J} \otimes_{\mathscr{O}_V} \mathscr{O}_X$,* **II** | 184 *so that $j_*(\mathscr{L}) = \mathscr{J}/\mathscr{J}^2$. Assume that the structure morphism $p : X \to Y$ is separated and quasi-compact, and that the following conditions are satisfied:*

(i) *there exists a Y-morphism $\pi : V \to X$ of finite type such that $\pi \circ j = 1_X$, and so $\pi_*(\mathscr{J}/\mathscr{J}^2) = \mathscr{L}$;*
(ii) *there exists a homomorphism of $\mathscr{O}_X$-modules $\varphi : \mathscr{L} \to \varprojlim \pi_*(\mathscr{J}/\mathscr{J}^{n+1})$ such that the composition*

$$\mathscr{L} \xrightarrow{\varphi} \varprojlim \pi_*(\mathscr{J}/\mathscr{J}^{n+1}) \xrightarrow{\alpha} \pi_*(\mathscr{J}/\mathscr{J}^2) = \mathscr{L}$$

(where α is the canonical homomorphism) is the identity;
(iii) *there exists a Y-prescheme C, a Y-section ε of C, and a Y-morphism $q : V \to C$ such that the diagram*

(8.9.4.1)
$$\begin{tikzcd} X \arrow[r, "j"] \arrow[d, "p"'] & V \arrow[d, "q"] \\ Y \arrow[r, "\varepsilon"'] & C \end{tikzcd}$$

commutes; and
(iv) *the restriction of q to $W = V - j(X)$ is a quasi-compact open immersion into C, whose image does not intersect $\varepsilon(Y)$.*

Then $\mathscr{L}$ is ample relative to p.

8.10. Grauert's ampleness criterion: proof.

Lemma (8.10.1). —

8.11. Uniqueness of contractions.

8.12. Quasi-coherent sheaves on based cones.

(8.12.1). Let us take the hypotheses and notation of (8.3.1). Let $\mathscr{M}$ be a *quasi-coherent graded $\mathscr{S}$-module*; to avoid any confusion, we denote by $\widetilde{\mathscr{M}}$ the quasi-coherent $\mathscr{O}_C$-module associated to $\mathscr{M}$ (1.4.3) when $\mathscr{M}$ is considered as **II** | 192 a *nongraded* $\mathscr{S}$-module, and by $\mathscr{P}roj_0(\mathscr{M})$ the quasi-coherent $\mathscr{O}_X$-module associated to $\mathscr{M}$, $\mathscr{M}$ being considered this time as a graded $\mathscr{S}$-module (in other words, the $\mathscr{O}_X$-module denoted by $\widetilde{\mathscr{M}}$ in (3.2.2)). In addition, we set

(8.12.1.1)
$$\mathscr{M}_X = \mathscr{P}roj_0(\mathscr{M}) = \bigoplus_{n\in\mathbf{Z}} \mathscr{P}roj_0(\mathscr{M}(n));$$

the quasi-coherent graded $\mathscr{O}_X$-algebra $\mathscr{S}_X$ being defined by (8.6.1.1), $\mathscr{P}roj(\mathscr{M})$ is equipped with a structure of a *(quasi-coherent) graded $\mathscr{S}_X$-module*, by means of the canonical homomorphisms (3.2.6.1)

(8.12.1.2)
$$\mathscr{O}_X(m) \otimes_{\mathscr{O}_X} \mathscr{P}roj_0(\mathscr{M}(n)) \longrightarrow \mathscr{P}roj_0(\mathscr{S}(m) \otimes_{\mathscr{S}} \mathscr{M}(n)) \longrightarrow \mathscr{P}roj_0(\mathscr{M}(m+n)),$$

the verification of the axioms of sheaves of modules being done using the commutative diagram in (2.5.11.4).

If $Y = \operatorname{Spec}(A)$ is affine, $\mathscr{S} = \widetilde{S}$, and $\mathscr{M} = \widetilde{M}$, where S is a graded A-algebra and M is a graded S-module, then, for every homogeneous element $f \in S_+$, we have

(8.12.1.3)
$$\Gamma(X_f, \mathscr{P}roj(\widetilde{M})) = M_f$$

by the definitions and (8.2.9.1).

Now consider the quasi-coherent graded $\widehat{\mathscr{S}}$-module

(8.12.1.4)
$$\widehat{\mathscr{M}} = \mathscr{M} \otimes_{\mathscr{S}} \widehat{\mathscr{S}}$$

($\widehat{\mathscr{S}}$ defined by (8.3.1.1)); we deduce a quasi-coherent graded $\mathscr{O}_{\widehat{C}}$-module $\mathscr{P}roj_0(\widehat{\mathscr{M}})$, which we will also denote by

(8.12.1.5)
$$\mathscr{M}^\square = \mathscr{P}roj_0(\widehat{\mathscr{M}}).$$

It is clear (3.2.4) that $\mathscr{M}^\square$ is an additive functor which is *exact* in $\mathscr{M}$, commuting with direct sums and with inductive limits.

CHAPTER III

Cohomological study of coherent sheaves (EGA III)

build hack [CC]

Summary

This chapter gives the fundamental theorems concerning the cohomology of coherent algebraic sheaves, with the exception of theorems explaining the theory of residues (duality theorems), which will be the subject of a later chapter. Amongst all those included here, there are essentially six fundamental theorems, and each one is the subject of one of the first six chapters. These results will prove to be essential tools in all that follows, even in questions which are not truly cohomological in their nature; the reader will see the first such examples starting from §4. §7 gives some more technical results, but ones which are constantly used in applications. Finally, in §§8–11, we will develop certain results, related to the duality of coherent sheaves, that are particularly important for applications, and which can be explained even before the introduction of the full general theory of residues.

The content of §§1 and 2 is due to J.-P. Serre, and the reader will observe that we have had only to follow (FAC). §8 and 9 are equally inspired by (FAC) (the changes necessitated by the different contexts, however, being less evident). Finally, as we said in the Introduction, §4 should be considered as the formalisation, in modern language, of the fundamental "invariance theorem" of Zariski's "theory of holomorphic functions".

We draw attention to the fact that the results of n°3.4 (and the preliminary propositions of (**0**, 13.4 to 13.7)) will not be used in what follows Chapter III, and can thus be skipped in a first reading.

§1. Cohomology of affine schemes

1.1. Review of the exterior algebra complex.

(1.1.1). Let A be a ring, $\mathbf{f} = (f_i)_{1\leqslant i\leqslant r}$ a system of r elements of A. The *exterior algebra complex* $K_\bullet(\mathbf{f})$ corresponding to $\mathbf{f}$ is a chain complex (G, I, 2.2) defined in the following way: the graded A-module $K_\bullet(\mathbf{f})$ is equal to the *exterior algebra* $\wedge(A^r)$, graded in the usual way, and the boundary map is the *interior multiplication* $i_\mathbf{f}$ by $\mathbf{f}$ considered as an element of the dual $(A^r)^\vee$; we recall that $i_\mathbf{f}$ is an *antiderivation* of degree -1 of $\wedge(A^r)$, and if $(\mathbf{e}_i)_{1\leqslant i\leqslant r}$ is the canonical basis of A^r, then we have $i_\mathbf{f}(\mathbf{e}_i) = f_i$; the verification of the condition $i_\mathbf{f} \circ i_\mathbf{f} = 0$ is immediate.

An equivalent definition is the following: for each i, we consider a chain complex $K_\bullet(f_i)$ defined as follows: $K_0(f_i) = K_1(f_i) = A$, $K_n(f_i) = 0$ for $n \neq 0, 1$: the boundary map is defined by the condition that $d_1 : A \to A$ is *multiplication by* f_i. We then take $K_\bullet(\mathbf{f})$ to be the *tensor product* $K_\bullet(f_1) \otimes K_\bullet(f_2) \otimes \cdots \otimes K_\bullet(f_r)$ (G, I, 2.7) with its total degree; the verification of the isomorphism from this complex to the complex defined above is immediate.

[1] EGA IV does not depend on §§8–11, and will probably be published before these chapters. *[Trans.] These last four chapters were never published.*

(1.1.2). For every A-module M, we define the *chain complex*

$$K_\bullet(\mathbf{f}, M) = K_\bullet(\mathbf{f}) \otimes_A M \tag{1.1.2.1}$$

and the *cochain complex* (G, I, 2.2)

$$K^\bullet(\mathbf{f}, M) = \mathrm{Hom}_A(K_\bullet(\mathbf{f}, M)). \tag{1.1.2.2}$$

If g is a k-cochain of this latter complex, and if we set

$$g(i_1, \ldots, i_k) = g(\mathbf{e}_{i_1} \wedge \cdots \wedge \mathbf{e}_{i_k}),$$

then g identifies with an *alternating* map from $[1, r]^k$ to M, and it follows from the above definitions that we have

$$d^k g(i_1, i_2, \ldots, i_{k+1}) = \sum_{h=1}^{k+1} (-1)^{h-1} f_{i_h} g(i_1, \ldots, \widehat{i_h}, \ldots, i_{k+1}). \tag{1.1.2.3}$$

(1.1.3). From the above complexes, we deduce as usual the *homology and cohomology A-modules* (G, I, 2.2) III | 83

$$\mathrm{H}_\bullet(\mathbf{f}, M) = \mathrm{H}_\bullet(K_\bullet(\mathbf{f}, M)), \tag{1.1.3.1}$$

$$\mathrm{H}^\bullet(\mathbf{f}, M) = \mathrm{H}^\bullet(K^\bullet(\mathbf{f}, M)). \tag{1.1.3.2}$$

We define an *A-isomorphism* $K_\bullet(\mathbf{f}, M) \simeq K^\bullet(\mathbf{f}, M)$ by sending each chain $z = \sum(\mathbf{e}_{i_1} \wedge \cdots \wedge \mathbf{e}_{i_k}) \otimes z_{i_1, \ldots, i_k}$ to the cochain g_z such that $g_z(j_1, \ldots, j_{r-k}) = \varepsilon z_{i_1, \ldots, i_k}$, where $(j_h)_{1 \leqslant h \leqslant r-k}$ is the strictly increasing sequence complementary to the strictly increasing sequence $(i_h)_{1 \leqslant h \leqslant k}$ in $[1, r]$ and $\varepsilon = (-1)^\nu$, where ν is the number of inversions of the permutation $i_1, \ldots, i_k, j_1, \ldots, j_{r-k}$ of $[1, r]$. We verify that $g_{dz} = d(g_z)$, which gives an isomorphism

$$\mathrm{H}^i(\mathbf{f}, M) \simeq \mathrm{H}_{r-i}(\mathbf{f}, M) \text{ for } 0 \leqslant i \leqslant r. \tag{1.1.3.3}$$

In this chapter, we will especially consider the cohomology modules $\mathrm{H}^\bullet(\mathbf{f}, M)$.

For a given $\mathbf{f}$, it is immediate (G, I, 2.1) that $M \mapsto \mathrm{H}^\bullet(\mathbf{f}, M)$ is a *cohomological functor* (T, II, 2.1) from the category of A-modules to the category of graded A-modules, zero in degrees < 0 and $> r$. In addition, we have

$$\mathrm{H}^0(\mathbf{f}, M) = \mathrm{Hom}_A(A/(\mathbf{f}), M), \tag{1.1.3.4}$$

denoting by $(\mathbf{f})$ the ideal of A generated by $f_1, \ldots, f_r$; this follows immediately from (1.1.2.3), and it is clear that $\mathrm{H}^0(\mathbf{f}, M)$ identifies with the submodule of M *killed by* $(\mathbf{f})$. Similarly, we have by (1.1.2.3) that

$$\mathrm{H}^r(\mathbf{f}, M) = M\Big/\left(\sum_{i=1}^r f_i M\right) = (A/(\mathbf{f})) \otimes_A M. \tag{1.1.3.5}$$

We will use the following known result, which we will recall a proof of to be complete:

Proposition (1.1.4). — *Let A be a ring, $\mathbf{f} = (f_i)_{1 \leqslant i \leqslant r}$ a finite family of elements of A, and M an A-module. If, for $1 \leqslant i \leqslant r$, the scaling $z \mapsto f_i \cdot z$ on $M_{i-1} = M/(f_1 M + \cdots + f_{i-1} M)$ is injective, then we have $\mathrm{H}^i(\mathbf{f}, M) = 0$ for $i \neq r$.*

It suffices to prove that $\mathrm{H}_i(\mathbf{f}, M) = 0$ for all $i > 0$ according to (1.1.3.3). We argue by induction on r, the case $r = 0$ being trivial. Set $\mathbf{f}' = (f_i)_{1 \leqslant i \leqslant r-1}$; this family satisfies the conditions in the statement, so if we set $L_\bullet = K_\bullet(\mathbf{f}', M)$, then we have $\mathrm{H}_i(L_\bullet) = 0$ for $i > 0$ by hypothesis, and $\mathrm{H}_0(L_\bullet) = M_{r-1}$ by virtue of (1.1.3.3) and (1.1.3.5). To abbreviate, set $K_\bullet = K_\bullet(f_r) = K_0 \oplus K_1$, with $K_0 = K_1 = A$, $d_1 : K_1 \to K_0$ multiplication by f_r; we have by definition (1.1.1) that $K_\bullet(\mathbf{f}, M) = K_\bullet \otimes_A L_\bullet$. We have the following lemma:

Lemma (1.1.4.1). — *Let $K_\bullet$ be a chain complex of free A-modules, zero except in dimensions 0 and 1. For every chain complex $L_\bullet$ of A-modules, we have an exact sequence*

$$0 \longrightarrow \mathrm{H}_0(K_\bullet \otimes \mathrm{H}_p(L_\bullet)) \longrightarrow \mathrm{H}_p(K_\bullet \otimes L_\bullet) \longrightarrow \mathrm{H}_1(K_\bullet \otimes \mathrm{H}_{p-1}(L_\bullet)) \longrightarrow 0$$

for every index p. III | 84

This is a particular case of an exact seqence of low-order terms of the Künneth spectral sequence (M, XVII, 5.2 (a) and G, I, 5.5.2); it can be proved directly as follows. Consider K_0 and K_1 as chain complexes (zero in dimensions $\neq 0$ and $\neq 1$ respectively); we then have an exact sequence of complexes

$$0 \longrightarrow K_0 \otimes L_\bullet \longrightarrow K_\bullet \otimes L_\bullet \longrightarrow K_1 \otimes L_\bullet \longrightarrow 0,$$

to which we can apply the exact sequence in homology

$$\cdots \longrightarrow \mathrm{H}_{p+1}(K_1 \otimes L_\bullet) \xrightarrow{\partial} \mathrm{H}_p(K_0 \otimes L_\bullet) \longrightarrow \mathrm{H}_p(K_\bullet \otimes L_\bullet) \longrightarrow \mathrm{H}_p(K_1 \otimes L_\bullet) \xrightarrow{\partial} \mathrm{H}_{p-1}(K_0 \otimes L_\bullet) \longrightarrow \cdots.$$

But it is evident that $\mathrm{H}_p(K_0 \otimes L_\bullet) = K_0 \otimes \mathrm{H}_p(L_\bullet)$ and $\mathrm{H}_p(K_1 \otimes L_\bullet) = K_1 \otimes \mathrm{H}_{p-1}(L_\bullet)$ for all p; in addition, we verify immediately that the operator $\partial : K_1 \otimes \mathrm{H}_p(L_\bullet) \to K_0 \otimes \mathrm{H}_p(L_\bullet)$ is none other than $d_1 \otimes 1$; the lemma thus follows from the above exact sequence and the definition of $\mathrm{H}_0(K_\bullet \otimes \mathrm{H}_p(L_\bullet))$ and $\mathrm{H}_1(K_\bullet \otimes \mathrm{H}_{p-1}(L_\bullet))$.

The lemma having been established, the end of the proof of Proposition (1.1.4) is immediate: the induction hypothesis of Lemma (1.1.4.1) gives $\mathrm{H}_p(K_\bullet \otimes L_\bullet) = 0$ for $p \geqslant 2$; in addition if we show that $\mathrm{H}_1(K_\bullet, \mathrm{H}_0(L_\bullet)) = 0$, then we also deduce from Lemma (1.1.4.1) that $\mathrm{H}_1(K_\bullet \otimes L_\bullet) = 0$; but by definition, $\mathrm{H}_1(K_\bullet, \mathrm{H}_0(L_\bullet))$ is none other than the kernel of the scaling $z \mapsto f_r \cdot z$ on M_{r-1}, and as by hypothesis this kernel is zero, this finishes the proof.

(1.1.5). Let $\mathbf{g} = (g_i)_{1 \leqslant i \leqslant r}$ be a second sequence of r elements of A, and set $\mathbf{fg} = (f_i g_i)_{1 \leqslant i \leqslant r}$. We can define a canonical homomorphism of complexes

(1.1.5.1)
$$\varphi_{\mathbf{g}} : K_\bullet(\mathbf{fg}) \longrightarrow K_\bullet(\mathbf{f})$$

as the canonical extension to the exterior algebra $\wedge(A^r)$ of the A-linear map $(x_1, \ldots, x_r) \mapsto (g_1 x_1, \ldots, g_r x_r)$ from A^r to itself. To see that we have a homomorphism of complexes, it suffices to note, in general, that if $u : E \to F$ is an A-linear map, and if $\mathbf{x} \in F^\vee$ and $\mathbf{y} = {}^t u(\mathbf{x}) \in E^\vee$, then we have the formula

(1.1.5.2)
$$(\wedge u) \circ i_{\mathbf{y}} = i_{\mathbf{x}} \circ (\wedge u);$$

indeed, the two elements are antiderivations of $\wedge F$, and it suffies to check that they coincide on F, which follows immediately from the definitions.

When we identify $K_\bullet(\mathbf{f})$ with the tensor product of the $K_\bullet(f_i)$ (1.1.1), $\varphi_{\mathbf{g}}$ is the tensor product of the φ_{g_i}, where φ_{g_i} is the identity in degree 0 and multiplication by g_i in degree 1.

(1.1.6). In particular, for every pair of integers m and n such that $0 \leqslant n \leqslant m$, we have homomorphisms of complexes

(1.1.6.1)
$$\varphi_{\mathbf{f}^{m-n}} : K_\bullet(\mathbf{f}^m) \longrightarrow K_\bullet(\mathbf{f}^n)$$

and as a result, homomorphisms

(1.1.6.2)
$$\varphi_{\mathbf{f}^{m-n}} : K^\bullet(\mathbf{f}^n, M) \longrightarrow K^\bullet(\mathbf{f}^m, M),$$

(1.1.6.3)
$$\varphi_{\mathbf{f}^{m-n}} : \mathrm{H}^\bullet(\mathbf{f}^n, M) \longrightarrow \mathrm{H}^\bullet(\mathbf{f}^m, M).$$

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The latter homomorphisms evidently satisfy the transitivity condition $\varphi_{\mathbf{f}^{m-p}} = \varphi_{\mathbf{f}^{m-n}} \circ \varphi_{\mathbf{f}^{n-p}}$ for $p \leqslant n \leqslant m$; they therefore define two *inductive systems* of A-modules; we set

(1.1.6.4)
$$C^\bullet((\mathbf{f}), M) = \varinjlim_n K^\bullet(\mathbf{f}^n, M),$$

(1.1.6.5)
$$\mathrm{H}^\bullet((\mathbf{f}), M) = \mathrm{H}^\bullet(C^\bullet((\mathbf{f}), M)) = \varinjlim_n \mathrm{H}^\bullet(\mathbf{f}^n, M),$$

the last equality following from the fact that passing to the inductive limit commutes with the functor $\mathrm{H}^\bullet$ (G, I, 2.1). We will later see (1.4.3) that $\mathrm{H}^\bullet((\mathbf{f}), M)$ does not depend on the *ideal* $(\mathbf{f})$ of A (and similarly on the $(\mathbf{f})$-pre-adic topology on A), which justifies the notations.

It is clear that $M \mapsto C^\bullet((\mathbf{f}), M)$ is an exact A-linear functor, and $M \mapsto \mathrm{H}^\bullet((\mathbf{f}), M)$ is a cohomological functor.

(1.1.7). Set $\mathbf{f} = (f_i) \in A^r$ and $\mathbf{g} = (g_i) \in A^r$; denote by $e_{\mathbf{g}}$ the left multiplication by the vector $\mathbf{g} \in A^r$ on the exterior algebra $\wedge(A^r)$; we know that we have the *homotopy formula*

(1.1.7.1)
$$i_{\mathbf{f}} e_{\mathbf{g}} + e_{\mathbf{g}} i_{\mathbf{f}} = \langle \mathbf{g}, \mathbf{f} \rangle 1$$

in the A-module A^r (1 denotes the identity automorphism of A^r); this relation also implies that *in the complex* $K_\bullet(\mathbf{f})$ we have

(1.1.7.2)
$$d e_{\mathbf{g}} + e_{\mathbf{g}} d = \langle \mathbf{g}, \mathbf{f} \rangle 1.$$

If the ideal $(\mathbf{f})$ is equal to A, then there exists a $\mathbf{g} \in A^r$ such that $\langle \mathbf{g}, \mathbf{f} \rangle = \sum_{i=1}^r g_i f_i = 1$. As a result (G, I, 2.4):

Proposition (1.1.8). — *Suppose that the ideal* $(\mathbf{f})$ *generated by the* f_i *is equal to* A. *Then the complex* $K_\bullet(\mathbf{f})$ *is homotopically trivial, and so are the complexes* $K_\bullet(\mathbf{f}, M)$ *and* $K^\bullet(\mathbf{f}, M)$ *for every* A*-module* M.

Corollary (1.1.9). — *If* $(\mathbf{f}) = A$, *then we have* $\mathrm{H}^\bullet(\mathbf{f}, M) = 0$ *and* $\mathrm{H}^\bullet((\mathbf{f}), M) = 0$ *for every* A*-module* M.

Proof. Indeed, we then have $(\mathbf{f}^n) = A$ for all n. ☐

Remark (1.1.10). — With the same notations as above, set $X = \mathrm{Spec}(A)$ and Y the closed subprescheme of X defined by the ideal $(\mathbf{f})$. We will prove in §9 that $\mathrm{H}^\bullet((\mathbf{f}), M)$ is isomorphic to the cohomology $\mathrm{H}^\bullet_Y(X, \widetilde{M})$ corresponding to the antifilter Φ of closed subsets of Y (T, 3.2). We will also show that Proposition (1.2.3) applied to X and to $\mathscr{F} = \widetilde{M}$ is a particular case of an exact sequence in cohomology

$$\cdots \longrightarrow \mathrm{H}^p_Y(X, \mathscr{F}) \longrightarrow \mathrm{H}^p(X, \mathscr{F}) \longrightarrow \mathrm{H}^p(X - Y, \mathscr{F}) \longrightarrow \mathrm{H}^{p+1}_Y(X, \mathscr{F}) \longrightarrow \cdots.$$

1.2. Čech cohomology of an open cover.

Notation (1.2.1). — In this section, we denote:

(1) X a prescheme;
(2) $\mathscr{F}$ a quasi-coherent $\mathscr{O}_X$-module; **III** | 86
(3) $A = \Gamma(X, \mathscr{O}_X)$, $M = \Gamma(X, \mathscr{F})$;
(4) $\mathbf{f} = (f_i)_{1 \leqslant i \leqslant r}$ a finite system of elements of A;
(5) $U_i = X_{f_i}$, the open set (**0**, 5.5.2) of the $x \in X$ such that $f_i(x) \neq 0$;
(6) $U = \bigcup_{i=1}^r U_i$;
(7) $\mathfrak{U}$ the cover $(U_i)_{1 \leqslant i \leqslant r}$ of U.

(1.2.2). Suppose that X is either a prescheme whose underlying space is *Noetherian* or a *scheme* whose underlying space is *quasi-compact*. We then know (**I**, 9.3.3) that we have $\Gamma(U_i, \mathscr{F}) = M_{f_i}$. We set

$$U_{i_0 i_1 \cdots i_p} = \bigcap_{k=0}^p U_{i_k} = X_{f_{i_0} f_{i_1} \cdots f_{i_p}}$$

(**0**, 5.5.3); so we also have

$$\Gamma(U_{i_0 i_1 \cdots i_p}, \mathscr{F}) = M_{f_{i_0} f_{i_1} \cdots f_{i_p}}. \tag{1.2.2.1}$$

We have (**0**, 1.6.1) that $M_{f_{i_0} f_{i_1} \cdots f_{i_p}}$ identifies with the inductive limit $\varinjlim_n M^{(n)}_{i_0 i_1 \cdots i_p}$, where the inductive system is formed by the $M^{(n)}_{i_0 i_1 \cdots i_p} = M$, the homomorphisms $\varphi_{nm} : M^{(m)}_{i_0 i_1 \cdots i_p} \to M^{(n)}_{i_0 i_1 \cdots i_p}$ being multiplication by $(f_{i_0} f_{i_1} \cdots f_{i_p})^{n-m}$ for $m \leqslant n$. We denote by $C^p_n(M)$ the set of *alternating* maps from $[1, r]^{p+1}$ to M (for all n); these A-modules also form an inductive system with respect to the φ_{nm}. If $C^p(\mathfrak{U}, \mathscr{F})$ is the group of *alternating* Čech p-cochains relative to the cover $\mathfrak{U}$, with coefficients in $\mathscr{F}$ (G, II, 5.1), then it follows from the above that we can write

$$C^p(\mathfrak{U}, \mathscr{F}) = \varinjlim_n C^p_n(M). \tag{1.2.2.2}$$

With the notations of (1.1.2), $C^p_n(M)$ identifies with $K^{p+1}(\mathbf{f}^n, M)$, and the map φ_{nm} identifies with the map $\varphi_{\mathbf{f}^{n-m}}$ defined in (1.1.6). We thus have, for every $p \geqslant 0$, a canonical functorial isomorphism

$$C^p(\mathfrak{U}, \mathscr{F}) \simeq C^{p+1}((\mathbf{f}), M). \tag{1.2.2.3}$$

In addition, the formula (1.1.2.3) and the definition of the cohomology of a cover (G, II, 5.1) shows that the isomorphisms (1.2.2.3) are compatible with the coboundary maps.

Proposition (1.2.3). — *If X is a prescheme whose underlying space in Noetherian or a scheme whose underlying space is quasi-compact, then there exists a canonical functorial isomorphism in $\mathscr{F}$*

$$\mathrm{H}^p(\mathfrak{U}, \mathscr{F}) \simeq \mathrm{H}^{p+1}((\mathbf{f}), M) \textit{ for } p \geqslant 1. \tag{1.2.3.1}$$

In addition, we have a functorial exact sequence in $\mathscr{F}$

$$0 \longrightarrow \mathrm{H}^0((\mathbf{f}), M) \longrightarrow M \longrightarrow \mathrm{H}^0(\mathfrak{U}, \mathscr{F}) \longrightarrow \mathrm{H}^1((\mathbf{f}), M) \longrightarrow 0. \tag{1.2.3.2}$$

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Proof. The isomorphisms (1.2.3.1) are immediate consequences of what we saw in (1.2.2). On the other hand, we have $C^0(\mathfrak{U}, \mathscr{F}) = C^1((\mathbf{f}), M)$; as a result, $\mathrm{H}^0(\mathfrak{U}, \mathscr{F})$ identifies with the subgroup of 1-cocycles of $C^1((\mathbf{f}), M)$; as $M = C^0((\mathbf{f}), M)$, the exact sequence (1.2.3.2) is none other than the one given by the definition of the cohomology groups $\mathrm{H}^0((\mathbf{f}), M)$ and $\mathrm{H}^1((\mathbf{f}), M)$. ☐

Corollary (1.2.4). — *Suppose that the X_{f_i} are quasi-compact and that there exists $g_i \in \Gamma(U, \mathscr{F})$ such that $\sum_i g_i(f_i|U) = 1|U$. Then for every quasi-coherent $(\mathscr{O}_X|U)$-module $\mathscr{G}$, we have $\mathrm{H}^p(\mathfrak{U}, \mathscr{G}) = 0$ for $p > 0$; if in addition $U = X$, then the canonical homomorphism (1.2.3.2) $M \to \mathrm{H}^0(\mathfrak{U}, \mathscr{F})$ is bijective.*

Proof. As by hypothesis the $U_i = X_{f_i}$ are quasi-compact, so is U, and we can reduce to the case where $U = X$; the hypothesis then implies that $\mathrm{H}^p((\mathbf{f}), M) = 0$ for all $p \geqslant 0$ (1.1.9). The corollary then follows immediately from (1.2.3.1) and (1.2.3.2). ☐

We note that since $\mathrm{H}^0(\mathfrak{U},\mathscr{F})=\mathrm{H}^0(U,\mathscr{F})$ (G, II, 5.2.2), we have again proved (I, 1.3.7) as a special case.

Remark (1.2.5). — Suppose that X is an *affine scheme*; then the $U_i=X_{f_i}=D(f_i)$ are affine open sets, as well as the $U_{i_0 i_1\cdots i_p}$ (but U is not necessarily affine). In this case, the functors $\Gamma(X,\mathscr{F})$ and $\Gamma(U_{i_0 i_1\cdots i_p},\mathscr{F})$ are exact in $\mathscr{F}$ (I, 1.3.11). If we have an exact sequence $0\to\mathscr{F}'\to\mathscr{F}\to\mathscr{F}''\to 0$ of quasi-coherent $\mathscr{O}_X$-modules, then the sequence of complexes

$$0\longrightarrow C^\bullet(\mathfrak{U},\mathscr{F}')\longrightarrow C^\bullet(\mathfrak{U},\mathscr{F})\longrightarrow C^\bullet(\mathfrak{U},\mathscr{F}'')\longrightarrow 0$$

is exact, and thus gives an exact sequence in cohomology

$$\cdots\longrightarrow \mathrm{H}^p(\mathfrak{U},\mathscr{F})\longrightarrow \mathrm{H}^p(\mathfrak{U},\mathscr{F})\longrightarrow \mathrm{H}^p(\mathfrak{U},\mathscr{F}'')\xrightarrow{\partial}\mathrm{H}^{p+1}(\mathfrak{U},\mathscr{F}')\longrightarrow\cdots.$$

On the other hand, if we set $M'=\Gamma(X,\mathscr{F}')$ and $M''=\Gamma(X,\mathscr{F}'')$, then the sequence $0\to M'\to M\to M''\to 0$ is exact; as $C^\bullet((\mathbf{f}),M)$ is an exact functor in M, we also have the exact sequence in cohomology

$$\cdots\longrightarrow \mathrm{H}^p((\mathbf{f}),M')\longrightarrow \mathrm{H}^p((\mathbf{f}),M)\longrightarrow \mathrm{H}^p((\mathbf{f}),M'')\xrightarrow{\partial}\mathrm{H}^{p+1}((\mathbf{f}),M')\longrightarrow\cdots.$$

This being so, as the diagram

$$\begin{tikzcd}
0 \arrow[r] & C^\bullet(\mathfrak{U},\mathscr{F}') \arrow[r]\arrow[d] & C^\bullet(\mathfrak{U},\mathscr{F}) \arrow[r]\arrow[d] & C^\bullet(\mathfrak{U},\mathscr{F}'') \arrow[r]\arrow[d] & 0 \\
0 \arrow[r] & C^\bullet((\mathbf{f}),M') \arrow[r] & C^\bullet((\mathbf{f}),M) \arrow[r] & C^\bullet((\mathbf{f}),M'') \arrow[r] & 0
\end{tikzcd}$$

is commutative, we conclude that the diagrams III | 88

(1.2.5.1)
$$\begin{tikzcd}
\mathrm{H}^p(\mathfrak{U},\mathscr{F}'') \arrow[r, "\partial"]\arrow[d] & \mathrm{H}^{p+1}(\mathfrak{U},\mathscr{F}') \arrow[d] \\
\mathrm{H}^{p+1}((\mathbf{f}),M'') \arrow[r, "\partial"] & \mathrm{H}^{p+2}((\mathbf{f}),M')
\end{tikzcd}$$

are commutative for all p (G, I, 2.1.1).

1.3. Cohomology of an affine scheme.

Theorem (1.3.1). — *Let X be an affine scheme. For every quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$, we have $\mathrm{H}^p(X,\mathscr{F})=0$ for all $p>0$.*

Proof. Let $\mathfrak{U}$ be a finite cover of X by the affine open sets $X_{f_i}=D(f_i)$ $(1\leqslant i\leqslant r)$; we then know that the ideal of $A=\Gamma(X,\mathscr{O}_X)$ generated by the f_i is equal to A. We thus conclude from Corollary (1.2.4) that we have $\mathrm{H}^p(\mathfrak{U},\mathscr{F})=0$ for $p>0$. As there are finite covers of X by affine open sets which are arbitrarily fine (I, 1.1.10), the definition of Čech cohomology (G, II, 5.8) shows that we also have $\check{\mathrm{H}}^p(X,\mathscr{F})=0$ for $p>0$. But this also applies to every prescheme X_f for $f\in A$ (I, 1.3.6), hence $\check{\mathrm{H}}^p(X_f,\mathscr{F})=0$ for $p>0$. As we have $X_f\cap X_g=X_{fg}$, we deduce that we also have $\mathrm{H}^p(X,\mathscr{F})=0$ for all $p>0$, by virtue of (G, II, 5.9.2). ☐

Corollary (1.3.2). — *Let Y be a prescheme, $f:X\to Y$ an affine morphism* (II, 1.6.1). *For every quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$, we have $\mathrm{R}^q f_*(\mathscr{F})=0$ for $q>0$.*

Proof. By definition $\mathrm{R}^q f_*(\mathscr{F})$ is the $\mathscr{O}_Y$-module associated to the presheaf $U\mapsto\mathrm{H}^q(f^{-1}(U),\mathscr{F})$, where U varies over the open subsets of Y. But the affine open sets form a basis for Y, and for such an open set U, $f^{-1}(U)$ is affine (II, 1.3.2), hence $\mathrm{H}^q(f^{-1}(U),\mathscr{F})=0$ by Theorem (1.3.1), which proves the corollary. ☐

Corollary (1.3.3). — *Let Y be a prescheme, $f:X\to Y$ an affine morphism. For every quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$, the canonical homomorphism $\mathrm{H}^p(Y,f_*(\mathscr{F}))\to\mathrm{H}^p(X,\mathscr{F})$ (0, 12.1.3.1) is bijective for all p.*

Proof. It suffices (by (**0**, 12.1.7)) to show that the edge homomorphisms $''E_2^{p0}=\mathrm{H}^p(Y,f_*(\mathscr{F}))\to\mathrm{H}^p(X,\mathscr{F})$ of the second spectral sequence of the composite functor Γf_* are bijective. But the E_2-term of this spectral sequence is given by $''E_2^{pq}=\mathrm{H}^p(Y,\mathrm{R}^q f_*(\mathscr{F}))$ (G, II, 4.17.1), so it follows from Corollary (1.3.2) that $''E_2^{pq}=0$ for $q>0$, and the spectral sequence degenerates; hence our assertion (**0**, 11.1.6). ☐

Corollary (1.3.4). — *Let $f:X\to Y$ be an affine morphism, $g:Y\to Z$ a morphism. For every quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$, the canonical homomorphism $\mathrm{R}^p g_*(f_*(\mathscr{F}))\to\mathrm{R}^p(g\circ f)_*(\mathscr{F})$ (0, 12.2.5.1) is bijective for all p.* III | 89

Proof. It suffices to note that, according to Corollary (1.3.3), for every affine open subset W of Z, the canonical homomorphism $\mathrm{H}^p(g^{-1}(W),f_*(\mathscr{F}))\to\mathrm{H}^p(f^{-1}(g^{-1}(W)),\mathscr{F})$ is bijective; this proves that the homomorphism of presheaves defining the canonical homomorphism $\mathrm{R}^p g_*(f_*(\mathscr{F}))\to\mathrm{R}^p(g\circ f)_*(\mathscr{F})$ is bijective (**0**, 12.2.5). ☐

1.4. Application to the cohomology of arbitrary preschemes.

Proposition (1.4.1). — *Let X be a scheme, $\mathfrak{U} = (U_\alpha)$ be a cover of X by affine open sets. For every quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$, the cohomology modules $\mathrm{H}^\bullet(X, \mathcal{F})$ and $\mathrm{H}^\bullet(\mathfrak{U}, \mathcal{F})$ (over $\Gamma(X, \mathcal{O}_X)$) are canonically isomorphic.*

Proof. As X is a scheme, every finite intersection V of open sets in the cover $\mathfrak{U}$ is affine (**I**, 5.5.6), so $\mathrm{H}^q(V, \mathcal{F}) = 0$ for $g \geqslant 1$ by Theorem (1.3.1). The proposition then follows from a theorem of Leray (G, II, 5.4.1). □

Remark (1.4.2). — We note that the result of Proposition (1.4.1) is still true when the finite intersections of the sets U_α are affine, even when we do not necessarily assume that X is a scheme.

Corollary (1.4.3). — *Let X be a scheme with quasi-compact underlying space, $A = \Gamma(X, \mathcal{O}_X)$, and $\mathbf{f} = (f_i)_{1 \leqslant i \leqslant r}$ a finite sequence of elements of A such that the X_{f_i} (notation of (1.2.1)) are affine. Then (with the notations of (1.2.1)), for every quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$, we have a canonical isomorphism which is functorial in $\mathcal{F}$*

$$\mathrm{H}^q(U, \mathcal{F}) \simeq \mathrm{H}^{q+1}((\mathbf{f}), M) \textit{ for } q \geqslant 1, \tag{1.4.3.1}$$

and an exact sequence which is functorial in $\mathcal{F}$

$$0 \longrightarrow \mathrm{H}^0((\mathbf{f}), M) \longrightarrow M \longrightarrow \mathrm{H}^0(U, \mathcal{F}) \longrightarrow \mathrm{H}^1((\mathbf{f}), M) \longrightarrow 0. \tag{1.4.3.2}$$

Proof. This follows immediately from Propositions (1.4.1) and (1.2.3). □

(1.4.4). If X is an *affine scheme*, then it follows from Remark (1.2.5) and Proposition (1.4.1) that for all $q \geqslant 0$, the diagrams

$$\begin{tikzcd} \mathrm{H}^q(U, \mathcal{F}'') \arrow[r, "\partial"] \arrow[d] & \mathrm{H}^{q+1}(U, \mathcal{F}') \arrow[d] \\ \mathrm{H}^{q+1}((\mathbf{f}), M'') \arrow[r, "\partial"] & \mathrm{H}^{q+2}((\mathbf{f}), M') \end{tikzcd} \tag{1.4.4.1}$$

corresponding to an exact sequence $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ of quasi-coherent $\mathcal{O}_X$-modules (with the notations of Remark (1.2.5)) are commutative. **III** | 90

Proposition (1.4.5). — *Let X be a quasi-compact scheme, $\mathcal{L}$ an invertible $\mathcal{O}_X$-module, and consider the graded ring $A_* = \Gamma_*(\mathcal{L})$ (**0**, 5.4.6); then $\mathrm{H}^\bullet(\mathcal{F}, \mathcal{L}) = \bigoplus_{n \in \mathbf{Z}} \mathrm{H}^\bullet(X, \mathcal{F} \otimes \mathcal{L}^{\otimes n})$ is a graded A_*-module, and for all $f \in A_n$, we have a canonical isomorphism*

$$\mathrm{H}^\bullet(X_f, \mathcal{F}) \simeq (\mathrm{H}^\bullet(\mathcal{F}, \mathcal{L}))_{(f)} \tag{1.4.5.1}$$

of $(A_*)_{(f)}$-modules.

Proof. As X is a quasi-compact scheme, we can calculate the cohomology of all the $\mathcal{O}_X$-modules $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$ using the same finite cover $\mathfrak{U} = (U_i)$ consisting of the affine open sets such that the restriction $\mathcal{L}|U_i$ is isomorphic to $\mathcal{O}_X|U_i$ for each i (1.4.1). It is then immediate that the $U_i \cap X_f$ are affine open sets (**I**, 1.3.6), and we can thus calculate the cohomology $\mathrm{H}^\bullet(X_f, \mathcal{F} \otimes \mathcal{L}^{\otimes n})$ using the cover $\mathfrak{U}|X_f = (U_i \cap X_f)$ (1.4.1). It is immediate that for all $f \in A_n$, multiplication by f defines a homomorphism $C^\bullet(\mathfrak{U}, \mathcal{F} \otimes \mathcal{L}^m) \to C^\bullet(\mathfrak{U}, \mathcal{F} \otimes \mathcal{L}^{\otimes(m+n)})$, hence a homomorphism $\mathrm{H}^\bullet(\mathfrak{U}, \mathcal{F} \otimes \mathcal{L}^{\otimes m}) \to \mathrm{H}^\bullet(\mathfrak{U}, \mathcal{F} \otimes \mathcal{L}^{\otimes(m+n)})$, which establishes the first assertion. On the other hand, for a given $f \in A_n$, it follows from (**I**, 9.3.2) that we have an isomorphism of complexes of $(A_*)_{(f)}$-modules

$$C^\bullet(\mathfrak{U}|X_f, \mathcal{F}) \simeq \left(C^\bullet\left(\mathfrak{U}, \bigoplus_{n \in \mathbf{Z}} \mathcal{F} \otimes \mathcal{L}^{\otimes n}\right)\right)_{(f)},$$

taking into account (**I**, 1.3.9, ii). Passing to the cohomology of these complexes, we induce the isomorphism (1.4.5.1), recalling that the functor $M \mapsto M_{(f)}$ is exact on the category of graded A_*-modules. □

Corollary (1.4.6). — *Suppose that the hypotheses of Proposition (1.4.5) are satisfied, and in addition suppose that $\mathcal{L} = \mathcal{O}_X$. If we set $A = \Gamma(X, \mathcal{O}_X)$, then for all $f \in A$, we have a canonical isomorphism $\mathrm{H}^\bullet(X_f, \mathcal{F}) \simeq (\mathrm{H}^\bullet(X, \mathcal{F}))_f$ of A_f-modules.*

Corollary (1.4.7). — *Let X be a quasi-compact scheme, f an element of $\Gamma(X, \mathcal{O}_X)$.*

(i) *Suppose that the open set X_f is affine. Then for every quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$, every $i > 0$, and every $\xi \in \mathrm{H}^i(X, \mathcal{F})$, there exists an integer $n > 0$ such that $f^n \xi = 0$.*

(ii) *Conversely, suppose that X_f is quasi-compact and that for every quasi-coherent sheaf of ideals $\mathcal{J}$ of $\mathcal{O}_X$ and every $\zeta \in \mathrm{H}^1(X, \mathcal{J})$, there exists an $n > 0$ such that $f^n \zeta = 0$. Then X_f is affine.*

Proof.

(i) If X_f is affine, then we have $\mathrm{H}^i(X_f,\mathscr{F})=0$ for all $i>0$ (1.3.1), so the assertion follows directly from Corollary (1.4.6).

(ii) By virtue of Serre's criterion (**II**, 5.2.1), it suffices to prove that for every quasi-coherent sheaf of ideals $\mathscr{K}$ of $\mathscr{O}_X|X_f$, we have $\mathrm{H}^1(X_f,\mathscr{K})=0$. As X_f is a quasi-compact open set in a quasi-compact scheme X, there exists a quasi-coherent sheaf of ideals $\mathscr{J}$ of $\mathscr{O}_X$ such that $\mathscr{K}=\mathscr{J}|X_f$ (**I**, 9.4.2). According to Corollary (1.4.6), we have $\mathrm{H}^1(X_f,\mathscr{K})=(\mathrm{H}^1(X,\mathscr{J}))_f$, and the hypothesis implies that the right hand side is zero, hence the assertion.

□

Remark (1.4.8). — We note that Corollary (1.4.7, i) gives a simpler proof of the relation (**II**, 4.5.13.2).

Lemma (1.4.9). — *Let X be a quasi-comact scheme, $\mathfrak{U}=(U_i)_{1\leqslant i\leqslant n}$ a finite cover of X by affine open sets, and $\mathscr{F}$ a quasi-coherent $\mathscr{O}_X$-module. The complex of sheaves $\mathscr{C}^\bullet(\mathfrak{U},\mathscr{F})$ defined by the cover $\mathfrak{U}$ (G, II, 5.2) is then a quasi-coherent $\mathscr{O}_X$-module.* III | 91

Proof. It follows from the definitions (G, II, 5.2) that $\mathscr{C}^p(\mathfrak{U},\mathscr{F})$ is the direct sum of the direct image sheaves of the $\mathscr{F}|U_{i_0\cdots i_p}$ under the canonical injection $U_{i_0\cdots i_p}\to X$. The hypothesis that X is a scheme implies that these injections are affine morphisms (**I**, 5.5.6), hence the $\mathscr{C}^p(\mathfrak{U},\mathscr{F})$ are quasi-coherent (**II**, 1.2.6). □

Proposition (1.4.10). — *Let $u:X\to Y$ be a separated and quasi-compact morphism. For every quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$, the $\mathrm{R}^q u_*(\mathscr{F})$ are quasi-coherent $\mathscr{O}_Y$-modules.*

Proof. The question is local on Y, so we can suppose that Y is affine. Then X is a finite union of affine open sets U_i $(1\leqslant i\leqslant n)$; let $\mathfrak{U}$ be the cover (U_i). In addition, as Y is a scheme, it follows from (**I**, 5.5.10) that for every affine open $V\subset Y$, the canonical injection $u^{-1}(V)\to X$ is an affine morphism; we conclude (Proposition (1.4.1) and (G, II, 5.2)) that we have a canonical isomorphism

(1.4.10.1) $$\mathrm{H}^\bullet(u^{-1}(V),\mathscr{F})\simeq\mathrm{H}^\bullet(\Gamma(V,\mathscr{K}^\bullet)),$$

where we set $\mathscr{K}^\bullet=u_*(\mathscr{C}^\bullet(\mathfrak{U},\mathscr{F}))$. According to Lemma (1.4.1) and (**I**, 9.2.2), $\mathscr{K}^\bullet$ is a quasi-coherent $\mathscr{O}_Y$-module; moreover, it constitutes a *complex of sheaves* since so is $\mathscr{C}^\bullet(\mathfrak{U},\mathscr{F})$. It then follows from the definition of the cohomology $\mathscr{H}^\bullet(\mathscr{K}^\bullet)$ (G, II, 4.1) that the latter consists of quasi-coherent $\mathscr{O}_Y$-modules (**I**, 4.1.1). As (for V affine in Y) the functor $\Gamma(V,\mathscr{G})$ is exact in $\mathscr{G}$ on the category of quasi-coherent $\mathscr{O}_Y$-modules, we have (G, II, 4.1)

(1.4.10.2) $$\mathrm{H}^\bullet(\Gamma(V,\mathscr{K}^\bullet))=\Gamma(V,\mathscr{H}^\bullet(\mathscr{K}^\bullet)).$$

Finally, we note that it follows from the definition of the canonical homomorphism

$$\mathrm{H}^\bullet(\mathfrak{U},\mathscr{F})\longrightarrow\mathrm{H}^\bullet(X,\mathscr{F}),$$

given in (G, II, 5.2), that if $V'\subset V$ is a second affine open subset of Y, then the diagram

$$\begin{tikzcd}
\mathrm{H}^\bullet(u^{-1}(V),\mathscr{F}) \arrow[r, "\sim"] \arrow[d] & \mathrm{H}^\bullet(\Gamma(V,\mathscr{K}^\bullet)) \arrow[d] \\
\mathrm{H}^\bullet(u^{-1}(V'),\mathscr{F}) \arrow[r, "\sim"] & \mathrm{H}^\bullet(\Gamma(V',\mathscr{K}^\bullet))
\end{tikzcd}$$

is commutative. We thus conclude from the above that the isomorphisms (1.4.10.1) define an isomorphism of $\mathscr{O}_Y$-modules

(1.4.10.3) $$\mathrm{R}^\bullet u_*(\mathscr{F})\simeq\mathscr{H}^\bullet(\mathscr{K}^\bullet),$$

and as a result, $\mathrm{R}^\bullet u_*(\mathscr{F})$ is quasi-coherent. □ III | 92

In addition, it follows from (1.4.10.3), (1.4.10.2), and (1.4.10.1) that:

Corollary (1.4.11). — *Under the hypotheses of Proposition (1.4.10), for every affine open set V of Y, the canonical homomorphism*

(1.4.11.1) $$\mathrm{H}^q(u^{-1}(V),\mathscr{F})\longrightarrow\Gamma(V,\mathrm{R}^1u_*(\mathscr{F}))$$

is an isomorphism for all $q\geqslant 0$.

Corollary (1.4.12). — *Suppose that the hypotheses of Proposition (1.4.10) are satisfied, and in addition suppose that Y is quasi-compact. Then there exists an integer $r>0$ such that for every quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$ and every integer $q>r$, we have $\mathrm{R}^q u_*(\mathscr{F})=0$. If Y is affine, then we can take for r an integer such that there exists a cover of X consisting of r open affine sets.*

Proof. As we can cover Y by a finite number of affine open sets, we can reduce to proving the second assertion, by virtue of Corollary (1.4.11). If $\mathfrak{U}$ is a cover of X by r affine open sets, then we have $\mathrm{H}^q(\mathfrak{U},\mathscr{F})=0$ for $q>r$, since the cochains of $C^q(\mathfrak{U},\mathscr{F})$ are alternating; the assertion thus follows from Proposition (1.4.1). □

Corollary (1.4.13). — *Suppose that the hypotheses of Proposition (1.4.10) are satsified, and in addition suppose that $Y=\operatorname{Spec}(A)$ is affine. Then for every quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$ and every $f\in A$, we have*

$$\Gamma(Y_f,\mathrm{R}^q u_*(\mathscr{F}))=(\Gamma(Y,\mathrm{R}^q u_*(\mathscr{F}))_f$$

up to canonical isomorphism.

Proof. This follows from the fact that $\mathrm{R}^q u_*(\mathscr{F})$ is a quasi-coherent $\mathscr{O}_Y$-module (**I**, 1.3.7). □

Proposition (1.4.14). — *Let $f:X\to Y$ be a separated and quasi-compact morphism, $g:Y\to Z$ an affine morphism. For every quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$, the canonical homomorphism $\mathrm{R}^p(g\circ f)_*(\mathscr{F})\to g_*(\mathrm{R}^p f_*(\mathscr{F}))$ (**0**, 12.2.5.2) is bijective for all p.*

Proof. For every affine open subset W of Z, $g^{-1}(W)$ is an affine open subset of Y. The homomorphism of presheaves defining the canonical homomorphism

$$\mathrm{R}^p(g\circ f)_*(\mathscr{F})\longrightarrow g_*(\mathrm{R}^p f_*(\mathscr{F}))$$

(**0**, 12.2.5) is thus bijective by Corollary (1.4.11). □

Proposition (1.4.15). — *Let $u:X\to Y$ be a separated morphism of finite type, $v:Y'\to Y$ a flat morphism of preschemes (**0**, 6.7.1); let $u'=u_{(Y')}$, such that we have the commutative diagram*

(1.4.15.1)
$$\begin{tikzcd} X \arrow[d, "u"'] & X'=X_{(Y')} \arrow[l, "v'"'] \arrow[d, "u'"] \\ Y & Y'. \arrow[l, "v"'] \end{tikzcd}$$

Then for every quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$, $\mathrm{R}^q u'_*(\mathscr{F}')$ is canonically isomorphic to $\mathrm{R}^q u_*(\mathscr{F})\otimes_{\mathscr{O}_Y}\mathscr{O}_{Y'}=v^*(\mathrm{R}^q u_*(\mathscr{F}))$ for all $q\geqslant 0$, where $\mathscr{F}'=v'^*(\mathscr{F})=\mathscr{F}\otimes_{\mathscr{O}_Y}\mathscr{O}_{Y'}$.

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Proof. The canonical homomorphism $\rho:\mathscr{F}\to v'_*(v'^*(\mathscr{F}))$ (**0**, 4.4.3.2) defines by functoriality a homomorphism

(1.4.15.2)
$$\mathrm{R}^q u_*(\mathscr{F})\longrightarrow \mathrm{R}^q u_*(v'_*(\mathscr{F}')).$$

On the other hand, we have, by setting $w=u\circ v'=v\circ u'$, the canonical homomorphisms (**0** ,12.2.5.1 and 12.2.5.2)

(1.4.15.3)
$$\mathrm{R}^q u_*(v'_*(\mathscr{F}'))\longrightarrow \mathrm{R}^q w_*(\mathscr{F}')\longrightarrow v_*(\mathrm{R}^q u'_*(\mathscr{F}')).$$

Composing (1.4.15.3) and (1.4.15.2), we have a homomorphism

$$\psi:\mathrm{R}^q u_*(\mathscr{F})\longrightarrow v_*(\mathrm{R}^q u'_*(\mathscr{F}')),$$

and finally we obtain a canonical homomorphism (whose definition does not make *any assumptions* on v)

(1.4.15.4)
$$\psi^\sharp: v^*(\mathrm{R}^q u_*(\mathscr{F}))\longrightarrow \mathrm{R}^q u'_*(\mathscr{F}'),$$

and it is necessary to prove that it is an isomorphism when v is *flat*. It is clear that the question is local on Y and Y', and we can therefore suppose that $Y=\operatorname{Spec}(A)$ and $Y'=\operatorname{Spec}(B)$; we will also use the following lemma:

Lemma (1.4.15.5). — *Let $\varphi:A\to B$ be a ring homomorphism, $Y=\operatorname{Spec}(A)$, $X=\operatorname{Spec}(B)$, $f:X\to Y$ the morphism corresponding to φ, and M a B-module. For the $\mathscr{O}_X$-module $\widetilde{M}$ to be f-flat (**0**, 6.7.1), it is necessary and sufficient for M to be a flat A-module. In particular, for the morphism f to be flat, it is necessary and sufficient for B to be a flat A-module.*

This follows from the definition (**0**, 6.7.1) and from (**0**, 6.3.3), taking into account (**I**, 1.3.4).

This being so, it follows from (1.4.11.1) and the definitions of the homomorphisms (1.4.15.3) (cf. (**0**, 12.2.5)) that ψ then corresponds to the composite morphism

$$\mathrm{H}^q(X,\mathscr{F})\xrightarrow{\rho_q}\mathrm{H}^q(X,v'_*(v'^*(\mathscr{F})))\xrightarrow{\theta_q}\mathrm{H}^q(X',v'^*(v'_*(v'^*(\mathscr{F}))))\xrightarrow{\sigma_q}\mathrm{H}^q(X',v'^*(\mathscr{F})),$$

where ρ_q and σ_q are the homomorphisms in cohomology corresponding to the canonical morphisms ρ and $\sigma : v'^*(v'_*(\mathscr{G}')) \to \mathscr{G}'$, and θ_q is the φ-morphism (**0**, 12.1.3.1) relative to the $\mathscr{O}_X$-module $v'_*(v'^*(\mathscr{F}))$. But by the functoriality of θ_q, we have the commutative diagram

$$\begin{tikzcd} \mathrm{H}^q(X, \mathscr{F}) \arrow[r, "\rho_q"] \arrow[d, "\theta_q"'] & \mathrm{H}^q(X, v'_*(v'^*(\mathscr{F}))) \arrow[d, "\theta_q"] \\ \mathrm{H}^q(X', v'^*(\mathscr{F})) \arrow[r, "v'^*(\rho_q)"] & \mathrm{H}^q(X', v'^*(v'_*(v'^*(\mathscr{F})))), \end{tikzcd}$$

and as by definition (**0**, 4.4.3) $v'^*(\rho)$ is the inverse of σ, we see that the composite morphism considered above is finally none other than θ_q; as a result, $\psi^\sharp$ is the associated B-homomorphism $\mathrm{H}^q(X, \mathscr{F}) \otimes_A B \to \mathrm{H}^q(X', \mathscr{F}')$. As u is of finite type, X is a finite union of affine open sets U_i $(1 \leqslant i \leqslant r)$; let $\mathfrak{U}$ be the cover (U_i). As v is an affine morphism, so is v' (**II**, 1.6.5, iii), and as a result the $U'_i = v'^{-1}(U_i)$ form an affine open cover $\mathfrak{U}'$ of X'. We then know (**0**, 12.1.4.2) that the diagram **III | 94**

$$\begin{tikzcd} \mathrm{H}^q(\mathfrak{U}, \mathscr{F}) \arrow[r, "\theta_q"] \arrow[d] & \mathrm{H}^q(\mathfrak{U}', \mathscr{F}') \arrow[d] \\ \mathrm{H}^q(X, \mathscr{F}) \arrow[r, "\theta_q"] & \mathrm{H}^q(X', \mathscr{F}') \end{tikzcd}$$

is commutative, and the vertical arrows are isomorphisms since X and X' are schemes (**I**, 1.4.1). As a result, it suffices to prove that the canonical φ-morphism $\theta_q : \mathrm{H}^q(\mathfrak{U}, \mathscr{F}) \to \mathrm{H}^q(\mathfrak{U}', \mathscr{F}')$ is such that the associated B-homomorphism

$$\mathrm{H}^q(\mathfrak{U}, \mathscr{F}) \otimes_A B \longrightarrow \mathrm{H}^q(\mathfrak{U}', \mathscr{F}')$$

is an isomorphism. For every sequence $\mathbf{s} = (i_k)_{0 \leqslant k \leqslant p}$ of $p+1$ indices of $[1, r]$, set $U_{\mathbf{s}} = \bigcap_{k=0}^p U_{i_k}$, $U'_{\mathbf{s}} = \bigcap_{k=0}^p U'_{i_k} = v'^{-1}(U_{\mathbf{s}})$, $M_{\mathbf{s}} = \Gamma(U_{\mathbf{s}}, \mathscr{F})$, and $M'_{\mathbf{s}} = \Gamma(U'_{\mathbf{s}}, \mathscr{F}')$. The canonical map $M_{\mathbf{s}} \otimes_A B \to M'_{\mathbf{s}}$ is an isomorphism (**I**, 1.6.5), hence the canonical map $C^p(\mathfrak{U}, \mathscr{F}) \otimes_A B \to C^p(\mathfrak{U}', \mathscr{F}')$ is an isomorphism, by which $d \otimes 1$ identifies with the coboundary map $C^p(\mathfrak{U}', \mathscr{F}') \to C^{p+1}(\mathfrak{U}', \mathscr{F}')$. As B is a *flat* A-module, it follows from the definition of the cohomology modules that the canonical map $\mathrm{H}^q(\mathfrak{U}, \mathscr{F}) \otimes_A B \to \mathrm{H}^q(\mathfrak{U}', \mathscr{F}')$ is an isomorhism (**0**, 6.1.1). This result will later be generalized in §6. ☐

Corollary (1.4.16). — *Let A be a ring, X an A-scheme of finite type, and B an A-algebra which is faithfully flat over A. For X to be affine, it is necessary and sufficient for $X \otimes_A B$ to be.*

Proof. The condition is evidently necessary (**I**, 3.2.2); we show that it is sufficient. As X is separated over A and the morphism $\operatorname{Spec}(B) \to \operatorname{Spec}(A)$ is flat, it follows from Proposition (1.4.1) that we have

(1.4.16.1) $$\mathrm{H}^i(X \otimes_A B, \mathscr{F} \otimes_A B) = \mathrm{H}^i(X, \mathscr{F}) \otimes_A B$$

for every $i \geqslant 0$ and every quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$. If $X \otimes_A B$ is affine, the left hand side of (1.4.16.1) is zero for $i = 1$, hence so is $\mathrm{H}^1(X, \mathscr{F})$ since B is a faithfully flat A-module. As X is a quasi-compact scheme, we finish the proof by Serre's criterion (**II**, 5.2.1). ☐

Proposition (1.4.17). — *Let X be a prescheme, $0 \to \mathscr{F} \xrightarrow{u} \mathscr{G} \xrightarrow{v} \mathscr{H} \to 0$ an exact sequence of $\mathscr{O}_X$-modules. If $\mathscr{F}$ and $\mathscr{H}$ are quasi-coherent, then so is $\mathscr{G}$.* **III | 95**

Proof. The question is local on X, so we can suppose that $X = \operatorname{Spec}(A)$ is affine, and it then suffices to prove that $\mathscr{G}$ satisfies the conditions (d1) and (d2) of (**I**, 1.4.1) (with $V = X$). The verification of (d2) is immediate, because if $t \in \Gamma(X, \mathscr{G})$ is zero when restricted to $D(f)$, then so is its image $v(t) \in \Gamma(X, \mathscr{H})$; therefore there exists an $m > 0$ such that $f^m v(t) = v(f^m t) = 0$ (**I**, 1.4.1), and as Γ is left exact, $f^m t = u(s)$, where $s \in \Gamma(X, \mathscr{F})$; as u is injective, the restriction of s to $D(f)$ is zero, hence (**I**, 1.4.1) there exists an integer $n > 0$ such that $f^n s = 0$; we finally deduce that $f^{m+n} t = u(f^n s) = 0$.

We now check (d1); let $t' \in \Gamma(D(f), \mathscr{G})$; as $\mathscr{H}$ is quasi-coherent, there exists an integer m such that $f^m v(t') = v(f^m t')$ extends to a section $z \in \Gamma(X, \mathscr{H})$ (**I**, 1.4.1). But in virtue of Theorem (1.3.1) (or (**I**, 5.1.9.2)) applied to the quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$, the sequence $\Gamma(X, \mathscr{G}) \to \Gamma(X, \mathscr{H}) \to 0$ is exact, so there exists $t \in \Gamma(X, \mathscr{G})$ such that $z = v(t)$; we thus see that $v(f^m t' - t'') = 0$, denoting by t'' the restriction of t to $D(f)$; thus we have $f^m t' - t'' = u(s')$, where $s' \in \Gamma(D(f), \mathscr{F})$. But as $\mathscr{F}$ is quasi-coherent, there exists an integer $n > 0$

such that $f^n s'$ extends to a section $s \in \Gamma(X, \mathscr{F})$; as $f^{m+n}t' - f^n t'' = u(f^n s')$, we see that $f^{m+n}t'$ is the restriction to $D(f)$ of a section $f^n t + u(f^n s) \in \Gamma(X, \mathscr{G})$, which finishes the proof. □

§2. Cohomological study of projective morphisms

2.1. Explicit calculations of certain cohomology groups.

§3. Finiteness theorem for proper morphisms

3.1. The dévissage lemma.

Definition (3.1.1). — Let C be an abelian category. We say that a subset C' of the set of objects of C is *exact* if $0 \in \mathsf{C}'$ and if, for every exact sequence $0 \to A' \to A \to A'' \to 0$ in C such that two of the objects A, A', A'' are in C', then the third is also in C'.

Theorem (3.1.2). — *Let X be a Noetherian prescheme; we denote by C the abelian category of coherent $\mathscr{O}_X$-modules. Let C' be an exact subset of C, X' a closed subset of the underlying space of X. Suppose that for every closed irreducible subset Y of X', with generic point y, there exists an $\mathscr{O}_X$-module $\mathscr{G} \in \mathsf{C}'$ such that $\mathscr{G}_y$ is a $k(y)$-vector space of dimension 1. Then every coherent $\mathscr{O}_X$-module with support contained in X' is in C' (and in particular, if $X' = X$, then we have $\mathsf{C}' = \mathsf{C}$).*

Proof. Consider the following property $\mathbf{P}(Y)$ of a closed subset Y of X': every coherent $\mathscr{O}_X$-module with support contained in Y is in C'. By virtue of the principle of Noetherian induction (**0**, 2.2.2), we see that we can reduce to showing that *if Y is a closed subset of X' such that the property $\boldsymbol{P}(Y')$ is true for every closed subset Y' of Y, distinct from Y, then $\boldsymbol{P}(Y)$ is true.*

Therefore, let $\mathscr{F} \in \mathsf{C}$ have support contained in Y, and we show that $\mathscr{F} \in \mathsf{C}'$. Denote also by Y the reduced closed subprescheme of X having Y for its underlying space (**I**, 5.2.1); it is defined by a coherent sheaf of ideals $\mathscr{J}$ of $\mathscr{O}_X$. We know (**I**, 9.3.4) that there exists an integer $n > 0$ such that $\mathscr{J}^n \mathscr{F} = 0$; for $1 \leqslant k \leqslant n$, we thus have an exact sequence

$$0 \longrightarrow \mathscr{J}^{k-1}\mathscr{F}/\mathscr{J}^k\mathscr{F} \longrightarrow \mathscr{F}/\mathscr{J}^k\mathscr{F} \longrightarrow \mathscr{F}/\mathscr{J}^{k-1}\mathscr{F} \longrightarrow 0$$

of coherent $\mathscr{O}_X$-modules ((**I**, 5.3.6) and (**I**, 5.3.3)); as C' is exact, we see, by induction on k, that it suffices to show that each of the $\mathscr{F}_k = \mathscr{J}^{k-1}\mathscr{F}/\mathscr{J}^k\mathscr{F}$ is in C'. We thus reduce to proving that $\mathscr{F} \in \mathsf{C}'$ under the additional hypothesis that $\mathscr{J}\mathscr{F} = 0$; it is equivalent to say that $\mathscr{F} = j_*(j^*(\mathscr{F}))$, where j is the canonical injection $Y \to X$. Let us now consider two cases:

(a) Y is *reducible.* Let $Y = Y' \cap Y''$, where Y' and Y'' are closed subsets of Y, distinct from Y; denote also by Y' and Y'' the reduced closed subpreschemes of X having Y and Y'' for their respective underlying spaces, which are defined respectively by sheaves of ideals $\mathscr{J}'$ and $\mathscr{J}''$ of $\mathscr{O}_X$. Set $\mathscr{F}' = \mathscr{F} \otimes_{\mathscr{O}_X} (\mathscr{O}_X/\mathscr{J}')$ and $\mathscr{F}'' = \mathscr{F} \otimes_{\mathscr{O}_X} (\mathscr{O}_X/\mathscr{J}'')$. The canonical homomorphisms $\mathscr{F} \to \mathscr{F}'$ and $\mathscr{F} \to \mathscr{F}''$ thus define a homomorphism $u : \mathscr{F} \to \mathscr{F}' \oplus \mathscr{F}''$. We show that for every $z \notin Y' \cap Y''$, the homomorphism $u_z : \mathscr{F}_z \to \mathscr{F}'_z \oplus \mathscr{F}''_z$ is *bijective.* Indeed, we have $\mathscr{J}' \cap \mathscr{J}'' = \mathscr{J}$, since the question is local and the above equality follows from ((**I**, 5.2.1) and (**I**, 1.1.5)); if $z \notin Y''$, then we have $\mathscr{J}'_z = \mathscr{J}_z$, hence $\mathscr{F}'_z = \mathscr{F}_z$ and $\mathscr{F}''_z = 0$, which establishes our assertion in this case; we reason similarly for $z \notin Y'$. As a result, the kernel and cokernel of u, which are in C (**0**, 5.3.4), have their support in $Y' \cap Y''$, and thus is in C' by hypothesis; for the same reason, $\mathscr{F}'$ and $\mathscr{F}''$ are in C', hence also $\mathscr{F}' \oplus \mathscr{F}''$, as C' is exact. The conclusion then follows from the consideration of the two exact sequences **III | 116**

$$0 \longrightarrow \operatorname{Im} u \longrightarrow \mathscr{F}' \oplus \mathscr{F}'' \longrightarrow \operatorname{Coker} u \longrightarrow 0,$$

$$0 \longrightarrow \operatorname{Ker} u \longrightarrow \mathscr{F} \longrightarrow \operatorname{Im} u \longrightarrow 0,$$

and the hypothesis that C' is exact.

(b) Y is irreducible, and as a result, the subprescheme Y of X is *integral.* If y is its generic point, then we have $(\mathscr{O}_Y)_y = k(y)$, and as $j^*(\mathscr{F})$ is a coherent $\mathscr{O}_Y$-module, $\mathscr{F}_y = (j^*(\mathscr{F}))_y$ is a $k(y)$-vector space of finite dimension m. By hypothesis, there is a coherent $\mathscr{O}_X$-module $\mathscr{G} \in \mathsf{C}'$ (necessarily of support Y) such that $\mathscr{G}_y$ is a $k(y)$-vector space of dimension 1. As a result, there is a $k(y)$-isomorphism $(\mathscr{G}_y)^m \simeq \mathscr{F}_y$, which is also an $\mathscr{O}_Y$-isomorphism, and as $\mathscr{G}^m$ and $\mathscr{F}$ are coherent, there exists an open neighborhood W of y in X and an isomorphism $\mathscr{G}^m|W \simeq \mathscr{F}|W$ (**0**, 5.2.7). Let $\mathscr{H}$ be the graph of this isomorphism, which is a coherent $(\mathscr{O}_X|W)$-submodule of $(\mathscr{G}^m \oplus \mathscr{F})|W$, canonically isomorphic to $\mathscr{G}^m|W$ and to $\mathscr{F}|W$; there thus exists a coherent $\mathscr{O}_X$-submodule $\mathscr{H}_0$ of $\mathscr{G}^m \oplus \mathscr{F}$, inducing $\mathscr{H}$ on W and 0 on $X - Y$, since $\mathscr{G}^m$ and $\mathscr{F}$ have Y for their support (**I**, 9.4.7). The restrictions $v : \mathscr{H}_0 \to \mathscr{G}^m$ and $w : \mathscr{H}_0 \to \mathscr{F}$ of the canonical projections of $\mathscr{G}^m \oplus \mathscr{F}$ are then homomorphisms of coherent $\mathscr{O}_X$-modules, which, on W and on $X - Y$, reduce to isomorphisms; in other words, the kernels and cokernels of v

and w have their suppoer in the closed set $Y-(Y\cap W)$, distinct from Y. They are in C'; on the other hand, we have $\mathscr{G}^m\in\mathsf{C}'$ since $\mathscr{G}\in\mathsf{C}'$ and since C' is exact. We conclude successively, by the exactness of C', that $\mathscr{H}_0\in\mathsf{C}'$, then $\mathscr{F}\in\mathsf{C}'$. Q.E.D.

□

Corollary (3.1.3). — *Suppose that the exact subset C' of C has in addition the property that any coherent direct factor of a coherent $\mathscr{O}_X$-module $\mathscr{M}\in\mathsf{C}'$ is also in C'. In this case, the conclusion of Theorem (3.1.2) is still valid when the condition "$\mathscr{G}_y$ is a $k(y)$-vector space of dimension 1" is replaced by $\mathscr{G}_y\neq 0$ (this is equivalent to $\operatorname{Supp}(\mathscr{G})=Y$).*

Proof. The reasoning of Theorem (3.1.2) must be modified only in the case (b); now $\mathscr{G}_y$ is a $k(y)$-vector space of dimension $q>0$, and as a result, we have an $\mathscr{O}_Y$-isomorphism $(\mathscr{G}_y)^m\simeq(\mathscr{F}_y)^q$; the end of the reasoning in Theorem (3.1.2) then proves that $\mathscr{F}^q\in\mathsf{C}'$, and the additional hypothesis on C' implies that $\mathscr{F}\in\mathsf{C}'$. □

3.2. The finiteness theorem: the case of usual schemes.

Theorem (3.2.1). — *Let Y be a locally Noetherian prescheme, $f:X\to Y$ a proper morphism. For every coherent $\mathscr{O}_X$-module $\mathscr{F}$, the $\mathscr{O}_Y$-modules $\mathrm{R}^q f_*(\mathscr{F})$ are coherent for $q\geqslant 0$.*

Proof. The question being local on Y, we can suppose Y Noetherian, thus X Noetherian (**I**, 6.3.7). The coherent $\mathscr{O}_X$-modules $\mathscr{F}$ for which the conclusion of Theorem (3.2.1) is true forms an *exact* subset C' of the category C of coherent $\mathscr{O}_X$-modules. Indeed, let $0\to\mathscr{F}'\to\mathscr{F}\to\mathscr{F}''\to 0$ is an exact sequence of coherent **III | 117**
$\mathscr{O}_X$-modules; suppose for example that $\mathscr{F}'$ and $\mathscr{F}''$ belong to C'; we have the long exact sequence in cohomology

$$\mathrm{R}^{q-1}f_*(\mathscr{F}'')\xrightarrow{\partial}\mathrm{R}^q f_*(\mathscr{F}')\longrightarrow\mathrm{R}^q f_*(\mathscr{F})\longrightarrow\mathrm{R}^q f_*(\mathscr{F}'')\xrightarrow{\partial}\mathrm{R}^{q+1}f_*(\mathscr{F}'),$$

in which by hypothesis the outer four terms are coherent; it is the same for the middle term $\mathrm{R}^q f_*(\mathscr{F})$ by ((**0**, 5.3.4) and (**0**, 5.3.3)). We show in the same way that when $\mathscr{F}$ and $\mathscr{F}'$ (resp. $\mathscr{F}$ and $\mathscr{F}''$) are in C', then so is $\mathscr{F}''$ (resp. $\mathscr{F}'$). In addition, every coherent *direct factor* $\mathscr{F}'$ of an $\mathscr{O}_X$-module $\mathscr{F}\in\mathsf{C}'$ belongs to C': indeed, $\mathrm{R}^q f_*(\mathscr{F}')$ is then a direct factor of $\mathrm{R}^q f_*(\mathscr{F})$ (G, II, 4.4.4), therefore it is of finite type, and as it is quasi-coherent (1.4.10), it is coherent, as Y is Noetherian. By virtue of Corollary (3.1.3), we reduce to proving that when X is *irreducible* with generic point x, there exists *one* coherent $\mathscr{O}_X$-module $\mathscr{F}$ belonging to C', such that $\mathscr{F}_x\neq 0$: indeed, if this point is established, then it can be applied to any irreducible closed subprescheme Y of X, since if $j:Y\to X$ is the canonical injection, then $f\circ j$ is proper (**II**, 5.4.2), and if $\mathscr{G}$ is a coherent $\mathscr{O}_Y$-module with support Y, then $j_*(\mathscr{G})$ is a coherent $\mathscr{O}_X$-module such that $\mathrm{R}^q(f\circ j)_*(\mathscr{G})=\mathrm{R}^q f_*(j_*(\mathscr{G}))$ (G, II, 4.9.1), therefore we can apply COrollary (3.1.3).

By virtue of Chow's lemma (**II**, 5.6.2), there exists an irreducible prescheme X' an a *projective* and surjective morphism $g:X'\to X$ such that $f\circ g:X'\to Y$ is *projective*. There exists an ample $\mathscr{O}_X$-module $\mathscr{L}$ for g (**II**, 5.3.1); we apply the fundamental theorem of projective morphisms (2.2.1) to $g:X'\to X$ and with $\mathscr{L}$: there thus exixts an integer n such that $\mathscr{F}=g_*(\mathscr{O}_{X'}(n))$ is a coherent $\mathscr{O}_X$-module and $\mathrm{R}^q g_*(\mathscr{O}_{X'}(n))=0$ for all $q>0$; in addition, as $g^*(g_*(\mathscr{O}_{X'}(n)))\to\mathscr{O}_{X'}(n)$ is surjective for n large enough (2.2.1), we see that we can suppose, at the generic point x of X, that we have $\mathscr{F}_x\neq 0$ (**II**, 3.4.7). On the other hand, as $f\circ g$ is projective as Y is Noetherian, the $\mathrm{R}^q(f\circ g)_*(\mathscr{O}_{X'}(n))$ are *coherent* (2.2.1). This being so, $\mathrm{R}^\bullet(f\circ g)_*(\mathscr{O}_{X'}(n))$ is the abutment of a Leray spectral sequence, whose E_2-term is given by $E_2^{pq}=\mathrm{R}^p f_*(\mathrm{R}^q g_*(\mathscr{O}_{X'}(n)))$; the above shows that this spectral sequence degenerates, and we then know (**0**, 11.1.6) that $E_2^{p0}=\mathrm{R}^p f_*(\mathscr{F})$ is isomorphic to $\mathrm{R}^p(f\circ g)_*(\mathscr{O}_{X'}(n))$, which finishes the proof. □

Corollary (3.2.2). — *Let Y be a locally Noetherian prescheme. For every proper morphism $f:X\to Y$, the direct image under f of any coherent $\mathscr{O}_X$-module is a coherent $\mathscr{O}_Y$-module.*

Corollary (3.2.3). — *Let A be a Noetherian ring, X a proper scheme over A; for every coherent $\mathscr{O}_X$-module $\mathscr{F}$, the $\mathrm{H}^p(X,\mathscr{F})$ are A-modules of finite type, and there exists an integer $r>0$ such that for every coherent $\mathscr{O}_X$-module $\mathscr{F}$ and all $p>r$, $\mathrm{H}^p(X,\mathscr{F})=0$.*

Proof. The second assertion has already been proved (1.4.12); the first follows from the finiteness theorem (3.2.1), taking into account Corollary (1.4.11). □

In particular, if X is a *proper algebraic scheme* over a field k, then, for every coherent $\mathscr{O}_X$-module $\mathscr{F}$, the $\mathrm{H}^p(X,\mathscr{F})$ are *finite-dimensional* k-vector spaces.

Corollary (3.2.4). — *Let Y be a locally Noetherian prescheme, $f:X\to Y$ a morphism of finite type. For every coherent $\mathscr{O}_X$-module $\mathscr{F}$ whose support in proper over Y (**II**, 5.4.10), the $\mathscr{O}_Y$-modules $\mathrm{R}^q f_*(\mathscr{F})$ are coherent.*

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Proof. The question being local on Y, we can suppose Y Noetherian, and it is the same for X (**I**, 6.3.7). By hypothesis, every closed subprescheme Z of X whose underlying space is $\mathrm{Supp}(\mathscr{F})$ is proper over Y, in other words, if $j : Z \to X$ is the canonical injection, then $f \circ j : Z \to Y$ is proper. We can suppose that Z is such that $\mathscr{F} = j_*(\mathscr{G})$, where $\mathscr{G} = j^*(\mathscr{F})$ is a coherent $\mathscr{O}_Z$-module (**I**, 9.3.5); as we have $\mathrm{R}^q f_*(\mathscr{F}) = \mathrm{R}^q(f \circ j)_*(\mathscr{G})$ by Corollary (1.3.4), the conclusion follows immediately from Theorem (3.2.1). □

3.3. Generalization of the finiteness theorem (usual schemes).

Proposition (3.3.1). — *Let Y be a Noetherian prescheme, $\mathscr{S}$ a quasi-coherent $\mathscr{O}_Y$-algebra of finite type, graded in positive degrees, $Y' = \mathrm{Proj}(\mathscr{S})$, and $g : Y' \to Y$ the structure morphism. Let $f : X \to Y$ be a proper morphism, $\mathscr{S}' = f^*(\mathscr{S})$, $\mathscr{M} = \bigoplus_{k \in \mathbf{Z}} \mathscr{M}_k$ a quasi-coherent graded $\mathscr{S}'$-module of finite type. Then the $\mathrm{R}^p f_*(\mathscr{M}) = \bigoplus_{k\in\mathbf{Z}} \mathrm{R}^p f_*(\mathscr{M}_k)$ are graded $\mathscr{S}$-modules of finite type for all p. Suppose in addition that the $\mathscr{S}$ are generated by $\mathscr{S}_1$; then, for every $p \in \mathbf{Z}$, there exists an integer k_p such that for all $k \geqslant k_p$ and all $r > 0$, we have*

$$\mathrm{R}^p f_*(\mathscr{M}_{k+r}) = \mathscr{S}_r \mathrm{R}^p f_*(\mathscr{M}_k). \tag{3.3.1.1}$$

Proof. The first assertion is identical to the statement of Theorem (2.4.1, i), where we have simply replaced “projective morphism” by “proper morphism”. In the proof of Theorem (2.4.1, i), the hypothesis on f was only used to show (with the notation of this proof) that $\mathrm{R}^p f'_*(\widetilde{\mathscr{M}})$ is a coherent $\mathscr{O}_{Y'}$-module. With the hypothesis of Proposition (3.3.1), f' is proper (**II**, 5.4.2, iii), so we can resume without change in the proof of Theorem (2.4.1, i), thanks to the finiteness theorem (3.2.1).

As for the second assertion, it suffices to remark that there is a finite affine open cover (U_i) of Y such that the restrictions to the U_i of the two sides of (3.3.1.1) are equal for all $k \geqslant k_{p,i}$ (**II**, 2.1.6, ii); it suffices to take for k_p the largest of the $k_{p,i}$. □

Corollary (3.3.2). — *Let A be a Notherian ring, $\mathfrak{m}$ an ideal of A, X a proper A-scheme, and $\mathscr{F}$ a coherent $\mathscr{O}_X$-module. Then, for all $p \geqslant 0$, the direct sum $\bigoplus_{k\geqslant 0} \mathrm{H}^p(X, \mathfrak{m}^k\mathscr{F})$ is a module of finite type over the ring $S = \bigoplus_{k\geqslant 0} \mathfrak{m}^k$; in particular, there exists an integer $k_p \geqslant 0$ such that for all $k \geqslant k_p$ and all $r > 0$, we have*

$$\mathrm{H}^p(X, \mathfrak{m}^{k+r}\mathscr{F}) = \mathfrak{m}^r \mathrm{H}^p(X, \mathfrak{m}^k\mathscr{F}). \tag{3.3.2.1}$$

Proof. It suffices to apply Proposition (3.3.1) with $Y = \mathrm{Spec}(A)$, $\mathscr{S} = \widetilde{S}$, $\mathscr{M}_k = \mathfrak{m}^k\mathscr{F}$, taking into account Corollary (1.4.11). □

It should be remembered that the S-module structure on $\bigoplus_{k\geqslant 0} \mathrm{H}^p(X, \mathfrak{m}^k\mathscr{F})$ is obtained by considering, for every $a \in \mathfrak{m}^r$, the map $\mathrm{H}^p(X, \mathfrak{m}^k\mathscr{F}) \to \mathrm{H}^p(X, \mathfrak{m}^{k+r}\mathscr{F})$, which comes from the passage to cohomology of the multiplication map $\mathfrak{m}^r\mathscr{F} \to \mathfrak{m}^{k+r}\mathscr{F}$ defined by a (2.4.1).

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3.4. Finiteness theorem: the case of formal schemes. The results of this section (except the definition (3.4.1)) will not be used in the rest of this chapter.

(3.4.1). Let $\mathfrak{X}$ and $\mathfrak{S}$ be two locally Noetherian formal preschemes (**I**, 10.4.2), $f : \mathfrak{X} \to \mathfrak{S}$ a morphism of formal preschemes. We say that f is a *proper* morphism if it satisfies the following conditions:

1st. f *is a morphism of finite type* (**I**, 10.13.3).

2nd. *If $\mathscr{K}$ is a sheaf of ideals of definition for $\mathfrak{S}$ and if we set $\mathscr{J} = f^*(\mathscr{K})\mathscr{O}_{\mathfrak{X}}$, $X_0 = (\mathfrak{X}, \mathscr{O}_{\mathfrak{X}}/\mathscr{J})$, $S_0 = (\mathfrak{S}, \mathscr{O}_{\mathfrak{S}}/\mathscr{K})$, then the morphism $f_0 : X_0 \to S_0$ induced by f* (**I**, 10.5.6) *is proper.*

It is immediate that this definition does not depend on the sheaf of ideals of definition $\mathscr{K}$ for $\mathscr{S}$ considered; indeed, if $\mathscr{K}'$ is a second sheaf of ideals of definition such that $\mathscr{K}' \subset \mathscr{K}$, and if we set $\mathscr{J}' = f^*(\mathscr{K}')\mathscr{O}_{\mathfrak{X}}$, $X'_0 = (\mathfrak{X}, \mathscr{O}_{\mathfrak{X}}/\mathscr{J}')$, $S'_0 = (\mathscr{S}, \mathscr{O}_{\mathfrak{S}}/\mathscr{K}')$, then the morphism $f'_0 : X'_0 \to S'_0$ induced by f is such that the diagram

$$\begin{tikzcd} X_0 \arrow[r, "f_0"] \arrow[d, "i"'] & S_0 \arrow[d, "j"] \\ X'_0 \arrow[r, "f'_0"] & S'_0 \end{tikzcd}$$

is commutative, i and j being surjective immersions; it is equivalent to say that f_0 or f'_0 is proper, by virtue of (**II**, 5.4.5).

We note that, for all $n \geqslant 0$, if we set $X_n = (\mathfrak{X}, \mathscr{O}_{\mathfrak{X}}/\mathscr{J}^{n+1})$, $S_n = (\mathfrak{S}, \mathscr{O}_{\mathfrak{S}}/\mathscr{K}^{n+1})$, then the morphism $f_n : X_n \to S_n$ induced by f (**I**, 10.5.6) is proper for all n whenever it is for $n = 0$ (**II**, 5.4.6).

If $g : Y \to Z$ is a proper morphism of locally Noetherian usual preschemes, Z' a closed subset of Z, Y' a closed subset of Y such that $g(Y') \subset Z'$, then the extension $\widehat{g} : Y_{/Y'} \to Z_{/Z'}$ of g to the completions (**I**, 10.9.1) is a proper morphism of formal preschemes, as it follows from the definition and from (**II**, 5.4.5).

Let $\mathfrak{X}$ and $\mathfrak{S}$ be two locally Noetherian formal preschemes, $f : \mathfrak{X} \to \mathfrak{S}$ a morphism *of finite type* (**I**, 10.13.3); the notation being the same as above, we say that a subset Z of the underlying space of $\mathfrak{X}$ is *proper* over $\mathfrak{S}$ (or proper for f) if, considered as a subset of X_0, Z is *proper over* S_0 (**II**, 5.4.10). All the properties of proper subsets of usual preschemes stated in (**II**, 5.4.10) are still true for the proper subsets of formal preschemes, as it follows immediately from the definitions.

Theorem (3.4.2). — *Let $\mathfrak{X}$ and $\mathfrak{Y}$ be locally Noetherian formal preschemes, $f : \mathfrak{X} \to \mathfrak{Y}$ a proper morphism. For every coherent $\mathscr{O}_{\mathfrak{X}}$-module $\mathscr{F}$, the $\mathscr{O}_{\mathfrak{Y}}$-modules $\mathrm{R}^q f_*(\mathscr{F})$ are coherent for all $q \geqslant 0$.*

Let $\mathscr{J}$ be a sheaf of ideals of definition for $\mathfrak{Y}$, $\mathscr{K} = f^*(\mathscr{J})\mathscr{O}_{\mathfrak{X}}$, and consider the $\mathscr{O}_{\mathfrak{X}}$-modules

(3.4.2.1) $$\mathscr{F}_k = \mathscr{F} \otimes_{\mathscr{O}_{\mathfrak{Y}}} (\mathscr{O}_{\mathfrak{Y}}/\mathscr{J}^{k+1}) = \mathscr{F}/\mathscr{K}^{k+1}\mathscr{F} \quad (k \geqslant 0)$$

which evidently form a *projective system* of topological $\mathscr{O}_{\mathfrak{X}}$-modules, such that $\mathscr{F} = \varprojlim_k \mathscr{F}_k$ (**I**, 10.11.3). On the other hand, it follows from Theorem (3.4.2) that each of the $\mathrm{R}^q f_*(\mathscr{F})$, being coherent, is naturally equipped with a topological $\mathscr{O}_{\mathfrak{Y}}$-module structure (**I**, 10.11.6), and so are the $\mathrm{R}^q f_*(\mathscr{F}_k)$. The canonical homomorphisms $\mathscr{F} \to \mathscr{F}_k = \mathscr{F}/\mathscr{K}^{k+1}\mathscr{F}$ canonically correspond to homomorphisms **III | 120**

$$\mathrm{R}^q f_*(\mathscr{F}) \longrightarrow \mathrm{R}^q f_*(\mathscr{F}_k),$$

which are necessarily continuous for the topological $\mathscr{O}_{\mathfrak{Y}}$-module structures above (**I**, 10.11.6), and form a projective system, giving the limit a canonical functorial homomorphism

(3.4.2.2) $$\mathrm{R}^q f_*(\mathscr{F}) \longrightarrow \varprojlim_k \mathrm{R}^q f_*(\mathscr{F}_k),$$

which will be a continuous homomorphism of topological $\mathscr{O}_{\mathfrak{Y}}$-modules. We will prove along with Theorem (3.4.2) the

Corollary (3.4.3). — *Each of the homomorphisms (3.4.2.2) is a topological isomorphism. In addition, if $\mathfrak{Y}$ is Noetherian, then the projective system $(\mathrm{R}^q f_*(\mathscr{F}/\mathscr{K}^{k+1}\mathscr{F}))_{k\geqslant 0}$ satisfies the* (ML)*-condition* (**0**, 13.1.1).

We will begin by establishing Theorem (3.4.2) and Corollary (3.4.3) when Y is a Noetherian formal affine scheme (**I**, 10.4.1):

Corollary (3.4.4). — *Under the hypotheses of Theorem (3.4.2), suppose in addition that $\mathfrak{Y} = \mathrm{Spf}(A)$, where A is an adic Noetherian ring. Let $\mathfrak{J}$ be an ideal of definition for A, and set $\mathscr{F}_k = \mathscr{F}/\mathfrak{J}^{k+1}\mathscr{F}$ for $k \geqslant 0$. Then the $\mathrm{H}^n(\mathfrak{X}, \mathscr{F})$ are A-modules of finite type; the projective system $(\mathrm{H}^n(\mathfrak{X}, \mathscr{F}_k))_{k\geqslant 0}$ satisfies the* (ML)*-condition for all n; if we set*

(3.4.4.1) $$N_{n,k} = \mathrm{Ker}\big(\mathrm{H}^n(\mathfrak{X}, \mathscr{F}) \longrightarrow \mathrm{H}^n(\mathfrak{X}, \mathscr{F}_k)\big)$$

(also equal to $\mathrm{Im}(\mathrm{H}^n(\mathfrak{X}, \mathfrak{J}^{k+1}\mathscr{F}) \to \mathrm{H}^n(\mathfrak{X}, \mathscr{F}))$ by the exact sequence in cohomology), then the $N_{n,k}$ define on $\mathrm{H}^n(\mathfrak{X}, \mathscr{F})$ a $\mathfrak{J}$-good filtration (**0**, 13.7.7)*; finally, the canonical homomorphism*

(3.4.4.2) $$\mathrm{H}^n(\mathfrak{X}, \mathscr{F}) \longrightarrow \varprojlim_k \mathrm{H}^n(\mathfrak{X}, \mathscr{F}_k)$$

is a topological isomorphism for all n (the left hand side being equipped with the $\mathfrak{J}$-adic topology, the $\mathrm{H}^n(\mathfrak{X}, \mathscr{F}_k)$ with the discrete topology).

Set

$$S = \mathrm{gr}(A) = \bigoplus_{k\geqslant 0} \mathfrak{J}^k/\mathfrak{J}^{k+1}, \quad \mathscr{M} = \mathrm{gr}(\mathscr{F}) = \bigoplus_{k\geqslant 0} \mathfrak{J}^k\mathscr{F}/\mathfrak{J}^{k+1}\mathscr{F}.$$

We know that $\mathfrak{J}^\Delta$ is a sheaf of ideals of definition for $\mathfrak{Y}$ (**I**, 10.3.1); let $\mathscr{K} = f^*(\mathfrak{J}^\Delta)\mathscr{O}_{\mathfrak{X}}$, $X_0 = (\mathfrak{X}, \mathscr{O}_{\mathfrak{X}}/\mathscr{K})$, $Y_0 = (\mathfrak{Y}, \mathscr{O}_{\mathfrak{Y}}/\mathfrak{J}^\Delta) = \mathrm{Spec}(A_0)$, with $A_0 = A/\mathfrak{J}$. It is clear that the $\mathscr{M}_k = \mathfrak{J}^k\mathscr{F}/\mathfrak{J}^{k+1}\mathscr{F}$ are coherent $\mathscr{O}_{X_0}$-modules (**I**, 10.11.3). Consider on the other hand the quasi-coherent graded $\mathscr{O}_{X_0}$-algebra

(3.4.4.4) $$\mathscr{S} = \mathscr{O}_{X_0} \otimes_{A_0} S = \mathrm{gr}(\mathscr{O}_{\mathfrak{X}}) = \bigoplus_{k\geqslant 0} \mathscr{K}^k/\mathscr{K}^{k+1}.$$

The hypothesis that $\mathscr{F}$ is a $\mathscr{O}_{\mathfrak{X}}$-module of finite type implies first that $\mathscr{M}$ is a graded $\mathscr{S}$-module *of finite type.* **III | 121**
Indeed, the question is local on $\mathfrak{X}$, and we can thus suppose that $\mathfrak{X} = \mathrm{Spf}(B)$, where B is an adic Noetherian ring, and $\mathscr{F} = N^\Delta$, where N is a B-module of finite type (**I**, 10.10.5); we have in addition $X_0 = \mathrm{Spec}(B_0)$, where $B_0 = B/\mathfrak{J}B$, and the quasi-coherent $\mathscr{O}_{X_0}$-modules $\mathscr{S}$ and $\mathscr{M}$ are respectively equal to $\widetilde{S'}$ and $\widetilde{M'}$, where $S' = \bigoplus_{k\geqslant 0}((\mathfrak{J}^k/\mathfrak{J}^{k+1}) \otimes_{A_0} B_0)$ and $M' = \bigoplus_{k\geqslant 0}((\mathfrak{J}^k/\mathfrak{J}^{k+1}) \otimes_{A_0} N_0)$, with $N_0 = N/\mathfrak{J}N$; we then evidently have $M' = S' \otimes_{B_0} N_0$, and as N_0 is a B_0-module of finite type, M' is a S'-module of finite type, hence our assertion (**I**, 1.3.13).
As the morphism $f_0 : X_0 \to Y_0$ is *proper* by hypothesis, we can apply Corollary (3.3.2) to $\mathscr{S}$, $\mathscr{M}$, and the morphism f_0: taking into account Corollary (1.4.11), we conclude that for *all* $n \geqslant 0$, $\bigoplus_{k\geqslant 0} \mathrm{H}^n(X_0, \mathscr{M}_k)$ is a

graded S-module *of finite type*. This proves that the condition (F_n) of (**0**, 13.7.7) is satisfied for *all* $n \geqslant 0$, when we consider the strictly projective system $(\mathscr{F}/\mathfrak{J}^k\mathscr{F})_{k\geqslant 0}$ of sheaves of abelian groups on X_0, each equipped with its natural "filtered A-module" structure. We can thus apply (**0**, 13.7.7), which proves that:

1st. The projective system $(\mathrm{H}^n(\mathfrak{X},\mathscr{F}_k))_{k\geqslant 0}$ satisfies the (ML)-condition.

2nd. If $\mathrm{H}'^n = \varprojlim_k \mathrm{H}^n(\mathfrak{X},\mathscr{F}_k)$, then H'^n is an A-module of finite type.

3rd. The filtration defined on H'^n by the kernels of the canonical homomorphisms $\mathrm{H}'^n \to \mathrm{H}^n(\mathfrak{X},\mathscr{F}_k)$ is $\mathfrak{J}$-good.

Note that on the other hand, if we set $X_k = (\mathfrak{X}, \mathscr{O}_{\mathfrak{X}}/\mathscr{K}^{k+1})$, then $\mathscr{F}_k$ is a coherent $\mathscr{O}_{X_k}$-module (**I**, 10.11.3), and if U is an affine open set in X_0, then U is also an affine open set in each of the X_k (**I**, 5.1.9), so $\mathrm{H}^n(U,\mathscr{F}_k) = 0$ for all $n > 0$ and all k (1.3.1) and $\mathrm{H}^0(U,\mathscr{F}_k) \to \mathrm{H}^0(U,\mathscr{F}_h)$ is surjective for $h \leqslant k$ (**I**, 1.3.9). We are thus in the conditions of (**0**, 13.3.2) and applying (**0**, 13.3.1) proves that H'^n canonically identifies with $\mathrm{H}^n(\mathfrak{X}, \varprojlim_k \mathscr{F}_k) = \mathrm{H}^n(\mathfrak{X},\mathscr{F})$; this finishes the proof of Corollary (3.4.4).

(3.4.5). We return to the proof of (3.4.2) and (3.4.3). We first prove the propositions for the case $\mathfrak{Y} = \operatorname{Spf}(A)$ envissaged in (3.4.4); for this, for all $g \in A$, apply (3.4.4) to the Noetherian affine formal scheme induced on the open set $\mathfrak{Y}_g = \mathfrak{D}(g)$ of $\mathfrak{Y}$, which is equal to $\operatorname{Spf}(A_{\{g\}})$, and to the formal prescheme induced by $\mathfrak{X}$ on $f^{-1}(\mathfrak{Y}_g)$; note that $\mathfrak{Y}_g$ is also an affine open set in the prescheme $Y_k = (\mathfrak{Y}, \mathscr{O}_{\mathfrak{Y}}/(\mathfrak{J}^\Delta)^{k+1})$, and as $\mathscr{F}_k$ is a coherent $\mathscr{O}_{X_k}$-module, we have

$$\mathrm{H}^n(f^{-1}(\mathfrak{Y}_g),\mathscr{F}_k) = \Gamma(\mathfrak{Y}_g, \mathrm{R}^n f_*(\mathscr{F}_k))$$

for all $k \geqslant 0$ by virtue of Corollary (1.4.11). The canonical homomorphism

$$\mathrm{H}^n(f^{-1}(\mathfrak{Y}_g),\mathscr{F}) \longrightarrow \varprojlim_k \Gamma(\mathfrak{Y}_g, \mathrm{R}^n f_*(\mathscr{F}_k))$$

is an isomorphism; but we have (**0**, 3.2.6)

$$\varprojlim_k \Gamma(\mathfrak{Y}_g, \mathrm{R}^n f_*(\mathscr{F}_k)) = \Gamma(\mathfrak{Y}_g, \varprojlim_k \mathrm{R}^n f_*(\mathscr{F}_k)),$$

and as the sheaf $\mathrm{R}^n f_*(\mathscr{F})$ is the sheaf associated to the presheaf $\mathfrak{Y}_g \mapsto \mathrm{H}^n(f^{-1}(\mathfrak{Y}_g),\mathscr{F})$ on the $\mathfrak{Y}_g$ (**0**, 3.2.1), **III** | 122
we have shown that the homormorphism (3.4.2.2) is *bijective*. Let us now prove that $\mathrm{R}^n f_*(\mathscr{F})$ is a coherent $\mathscr{O}_{\mathfrak{Y}}$-module, and more precisely that we have

$$\mathrm{R}^n f_*(\mathscr{F}) = \left(\mathrm{H}^n(\mathfrak{X},\mathscr{F})\right)^\Delta. \tag{3.4.5.1}$$

With the above notation, we have, since $\mathscr{F}_k$ is a coherent $\mathscr{O}_{X_k}$-module (1.4.13),

$$\Gamma(\mathfrak{Y}_g, \mathrm{R}^n f_*(\mathscr{F}_k)) = (\Gamma(\mathfrak{Y}, \mathrm{R}^n f_*(\mathscr{F}_k)))_g = (\mathrm{H}^n(\mathfrak{X},\mathscr{F}_k))_g.$$

Now the $\mathrm{H}^n(\mathfrak{X},\mathscr{F}_k)$ form a projective system satisfying (ML), and their projective limit $\mathrm{H}^n(\mathfrak{X},\mathscr{F})$ is an A-module of finite type. We conclude (**0**, 13.7.8) that we have

$$\varprojlim_k \left((\mathrm{H}^n(\mathfrak{X},\mathscr{F}_k))_g\right) = \mathrm{H}^n(\mathfrak{X},\mathscr{F}) \otimes_A A_{\{g\}} = \Gamma(\mathfrak{Y}_g, (\mathrm{H}^n(\mathfrak{X},\mathscr{F}))^\Delta),$$

taking into account (**I**, 10.10.8) applied to A and $A_{\{g\}}$; this proves (3.4.5.1) since $\Gamma(\mathfrak{Y}_g, \mathrm{R}^n f_*(\mathscr{F})) = \varprojlim_k \Gamma(\mathfrak{Y}_g, \mathrm{R}^n f_*(\mathscr{F}_k))$.

As (3.4.2.2) is then an isomorphism of coherent $\mathscr{O}_{\mathfrak{Y}}$-modules, it is necessarily a *topological* isomorphism (**I**, 10.11.6). Finally, it follows from the relations $\mathrm{R}^n f_*(\mathscr{F}_k) = (\mathrm{H}^n(\mathfrak{X},\mathscr{F}_k))^\Delta$ that the projective system $(\mathrm{R}^n f_*(\mathscr{F}_k))_{k\geqslant 0}$ satisfies (ML) (**I**, 10.10.2).

Once (3.4.2) and (3.4.3) are proved in the case where the formal prescheme $\mathfrak{Y}$ is affine Noetherian, it is immediate to pass to the general case for (3.4.2) and the first assertion of (3.4.3), which are local on $\mathfrak{Y}$. As for the second assertion of (3.4.3), it suffices, $\mathfrak{Y}$ being Noetherian, to cover it by a finite number of Noetherian affine open sets U_i and to note that the restricyions of the projective system $(\mathrm{R}^q f_*(\mathscr{F}_k))$ to each of the U_i satisfies (ML).

Along the way, we have in addition proved:

Corollary (3.4.6). — *Under the hypotheses of Corollary (3.4.4), the canonical homomorphism*

$$\mathrm{H}^q(\mathfrak{X},\mathscr{F}) \longrightarrow \Gamma(\mathfrak{Y}, \mathrm{R}^q f_*(\mathscr{F})) \tag{3.4.6.1}$$

is bijective.

§4. The fundamental theorem of proper morphisms. Applications

4.1. The fundamental theorem.

§5. An existence theorem for coherent algebraic sheaves

5.1. Statement of the theorem.

§6. Local and global Tor functors; Künneth formula

6.1. Introduction.

§7. Base change for homological functors of sheaves of modules

7.1. Functors of *A*-modules.

CHAPTER IV

Local study of schemes and their morphisms (EGA IV)

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Summary

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The subjects discussed in the chapter call for the following remarks.

(a) The common property of all the subjects discussed is that they all related to *local* properties of preschemes or morphisms, i.e. considered at a point, or the points of a fibre, or on a (non-specified) neighbourhood of a point or of a fibre. These properties are generally of a *topological*, *differential*, or *dimensional* nature (i.e. bringing the ideas of *dimension* and *depth* into play), and are linked to the properties of the *local rings* at the points considered. One type of problem is the relating, for a given morphisms $f : X \to Y$ and point $x \in X$, of the properties of X at x with those of Y at $y = f(x)$ and those of the fibre $X_y = f^{-1}(y)$ at x. Another is the determining of the topological nature (for example, the constructibility, or the fact of being open or closed) of the set of points $x \in X$ at which X has a certain property, or for which the fibre $X_{f(x)}$ passing through x has a certain property at x. Similarly, we are interested in the topological nature of the set of points $y \in Y$ such that X has a certain property at all the points of the fibre X_y, or those such that this fibre itself has a certain property.

(b) The most important idea for the following chapters is that of *flat morphisms of finite presentation*, as well as the particular cases of *smooth morphisms* and *étale morphisms*. Their detailed study (as well as that of connected questions) really starts in §11.

(c) Sections §§1–10 can be considered as being preliminary in nature, and as developing three types of techniques, used, not only in the other sections of the chapter, but also, of course, in the follow chapters:

[1] The order and content of §§11–21 are given only as an indication of what the titles will be, and will possibly be modified before their publication. *[Trans.] This was indeed the case: many of §§11–21 ended up having entirely different titles.*

(c1) Sections §§1–4 are envisaged as treating the diverse aspects of the idea of *change of base*, above all in relation with the conditions of *finiteness* or *flatness*; we there initiate the technique of *descent*, with its most elementary aspects (the questions of "effectiveness" linked to this technique will be studied in Chapter V).

(c2) Sections §§5–7 are focused on what we may call *Noetherian* techniques, since the preschemes considered are always locally Noetherian, whereas, on the contrary, there is generally no finiteness condition imposed on the *morphisms*; this is essentially due to the fact that the ideas of dimension and depth are hardly manageable except in the case of Noetherian local rings. Recall that §7 constitutes a "delicate (?)" theory of Noetherian local rings, not much used in what follows in the chapter.

(c3) Sections §§8–10 describe, amongst other things, the means of *eliminating the Noetherian hypotheses* on the preschemes considered, by substituting such hypotheses for suitable ones of *finiteness* ("finite presentation") on the *morphisms* considered: the advantage of this substitution is that the latter such hypotheses (those of finiteness on the morphisms) are *stable under base change*, which is not the case for the Noetherian hypotheses on the preschemes. The technique permitting this substitution relies, in some part, on the use of the idea of the *projective limit* of preschemes, thanks to which we can reduce a question to the same question with *Noetherian* hypotheses; on the other hand, it relies on the systematic use of *constructible sets*, which have the double interest of being preserved under taking inverse images (of arbitrary morphisms) and by direct images **IV-1** | 224 (of morphisms of finite presentation), and having manageable topological properties in locally Noetherian preschemes. The same techniques often even allow to restrict to the case of more specific Noetherian rings, for example the **Z**-*algebras of finite type*, and it is here that the properties of "excellent" rings (studied in §7) intervene in a decisive manner. Independently of the question of elimination of Noetherian hypotheses, the techniques of §§8–10, elementary in nature, find constant use in nearly all applications.

§1. Relative finiteness conditions. Constructible sets of preschemes

In this section. we will resume the exposé of "finiteness conditions" for a morphism of preschemes $f : X \to Y$ given in (**I**, 6.3 and 6.6). There are essentially two notions of "finiteness" of a *global* nature on X, that of *quasi-compact* morphism (defined in (**I**, 6.6.1)) and that of a *quasi-separated* morphism; on the other hand, there are two notions of "finiteness" of a *local* nature on X, that of a morphism *locally of finite type* (defined in (**I**, 6.6.2)) and that of a morphism *locally of finite presentation.* By combining these local notions with the preceding global notions, we obtain the notion of a morphism *of finite type* (defined in (**I**, 6.3.1)) and of a morphism *of finite presentation.* For the convenience of the reader, we will give again in this section the properties stated in (**I**, 6.3 and 6.6), referring to their labels in Chapter I for their proofs.

In n^{os}1.8 and 1.9, we complete, in the context of preschemes, and making use of the previous notions of finiteness, the results on constructible sets given in ($\mathbf{0}_{\text{III}}$, §9).

1.1. Quasi-compact morphisms.

Definition (1.1.1). — We say that a morphism of preschemes $f : X \to Y$ is *quasi-compact* if the continous map f from the topological space X to the topological space Y is quasi-compact (**0**, 9.1.1), in other words, if the inverse image $f^{-1}(U)$ of every quasi-compact open subset U of Y is quasi-compact (cf. (**I**, 6.6.1)).

If $\mathfrak{B}$ is a basis for the topology of Y consisting of affine open sets, then for f to be quasi-compact, it is necessary and sufficient that for all $V \in \mathfrak{B}$, $f^{-1}(V)$ is a *finite union of affine open sets.* For example, if Y is affine and X is quasi-compact, *every* morphism $f : X \to Y$ is quasi-compact (**I**, 6.6.1).

If $f : X \to Y$ is a quasi-compact morphism, then it is clear that for every open subset V of Y, the restriction of f to $f^{-1}(V)$ is a quasi-compact morphism $f^{-1}(V) \to V$. Conversely, if (U_α) is an open cover of Y and $f : X \to Y$ is a morphism such that the restrictions $f^{-1}(U_\alpha) \to U_\alpha$ are quasi-compact, then f is quasi-compact. As a result, if $f : X \to Y$ is an S-morphism of S-preschemes, and if there exists an open cover (S_λ) of S such that **IV-1** | 225 the restrictions $g^{-1}(S_\lambda) \to h^{-1}(S_\lambda)$ of f (where g and h are the structure morphisms) are quasi-compact, then f is quasi-compact.

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