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Parameterization of randomly measured points for least squares fitting of B-spline curves and surfaces

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The paper presents a simple technique to assign parameter values to randomly measured points for the least squares fitting of B-spline surfaces. The parameterization is realized by projecting the measured points to a base surface. The parameters of the projected points are then used as the parameters of the measured points. The base surface is in fact a first approximation of the final fitted surface, and it can usually be created from some approximate boundary information in the form of either points or curves. A similar technique can also be used for B-spline curve fitting.

Keywords: parameterization, interpolation, B-spline surfaces

Free-form curve and surface fitting or interpolation plays an important role in computer-aided geometric design (CAGD). In most of the cases, geometric designers are asked to create curves or surfaces from measured data points. These measured data often come from a digitizing device, e.g. a coordinate measuring machine (CMM) or a laser scanning device. The applications in mechanical engineering can often be found in the model, mould and die making industry for rapid prototyping and reverse engineering[1–3]. Typical examples are the construction of car and aircraft bodies, ship hulls, aeroplane wings, shoe insoles, telephone sets, turbine blades, and other household appliances such as fans, vases, tea pots and toys. In these fields of application, the designers often formalize their ideas by first constructing a wooden or clay model and then using advanced digitizing equipment and geometric modelling software to create a CAD model for further design or production. Applications can also be found in the computer-integrated manufacturing (CIM) environment, biomechanical engineering, electrical engineering, medical science and computer science for feature recognition and image processing.

This paper discusses the parameterization of digitized data, i.e. the determination of parameter values of digitized points in order to perform least squares (LSQ) fitting of B-spline curves and surfaces. Numerous papers have been published on the selection of such parameters. The commonly used ones are uniform parameters, cumulative chord length parameters, and centripetal parameterization parameters. All of these methods are based on the assumption that the points are arranged in a special pattern, i.e. chain points for curves, and grid points for surfaces (see *Figures 1a* and *b* for an illustration). It is becoming very difficult to use these parameterization methods if the measured points are irregularly spaced as shown in *Figure 1c*. With such a parameterization, the fitted surface is severely distorted (see *Figures 6b* and *c* for examples). These methods fail completely if the measured points are randomly distributed as shown in *Figure 1d*. A simple parameterization method is presented in this paper to tackle these problems. This method applies equally well to the least squares fitting of nonuniform rational B-spline (NURBS) curves and surfaces[4,5], and, as a matter of fact, it has been implemented for NURBS. For reasons of simplicity, the basic principles are explained using B-splines.

B-SPLINE CURVE AND SURFACE LSQ FITTING

B-spline curves and surfaces are defined by the following equations:

$$\mathbf{P}(u) = \sum_{i=1}^{n} B_i(u) \cdot \mathbf{v}_i \qquad (1)$$

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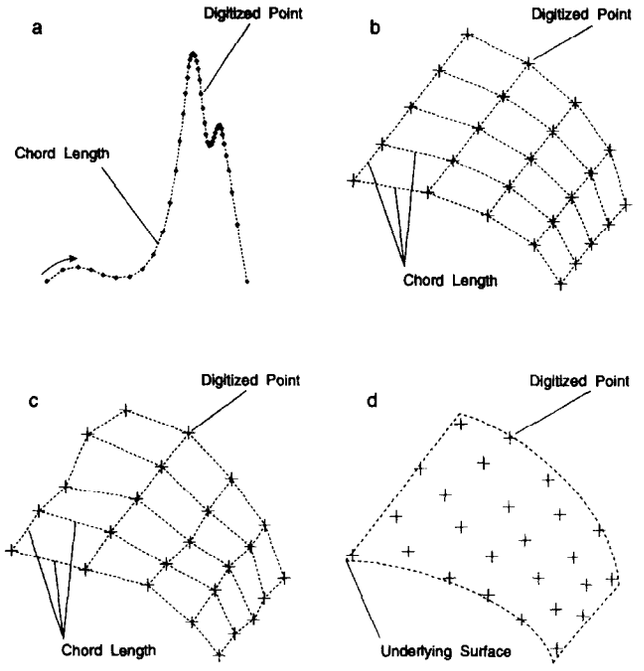

Figure 1 Distribution of measuring points; (a) chain selected curve points, (b) regular grid surface points, (c) irregular grid surface points, (d) randomly distributed surface points

$$\mathbf{P}(u,v) = \sum_{i=1}^{n_u} \sum_{j=1}^{n_v} B_{ui}(u) \cdot B_{vj}(v) \cdot \mathbf{v}_{ij} \tag{2}$$

where u for B-spline curves and u and v for B-spline surfaces are *location parameters* that uniquely locate a point on the curve or surface, and $B_i(u)$, $i = 1, 2, \ldots, n$, $B_{ui}(u)$, $i = 1, 2, \ldots, n_u$, and $B_{vj}(v)$, $j = 1, 2, \ldots, n_v$, are normalized B-splines that can be evaluated by the recursive Cox-de Boor algorithm[6,7]. A B-spline curve is completely defined by its order k ($k-1$ being the so-called degree), an ordered set of n control points $\mathbf{v}_i = [x_i, y_i, z_i]^T$, $i = 1, 2, \ldots, n$, and a knot sequence $\xi := \{\xi_i | i = 1, 2, \ldots, n+k\}$. Similarly, a B-spline surface can also be uniquely defined by its orders k_u and k_v, an ordered set of $n = n_u \times n_v$ control points $\mathbf{v}_{ij} = [x_{ij}, y_{ij}, z_{ij}]^T$, $i = 1, 2, \ldots, n_u$, $j = 1, 2, \ldots, n_v$, and two knot sequences $\xi := \{\xi_i | i = 1, 2, \ldots, n_u + k_u\}$ and $\zeta := \{\zeta_j | j = 1, 2, \ldots, n_v + k_v\}$.

In the context of B-spline curve fitting or interpolation, the aim is usually to find a set of control points $\{\mathbf{v}_i\}_1^n$ with a fixed number of control points n, order k, and a knot sequence ξ with which the B-spline curve fits or interpolates a set of curve points. Analogously for B-spline surfaces, a set of control points $\{\{\mathbf{v}_{ij}\}_{i=1}^{n_u}\}_{j=1}^{n_v}$ is to be found with a fixed number of control points n_u and n_v, orders k_u and k_v, and two knot sequences ξ and ζ to fit or interpolate the measured points.

Writing out Equations 1 and 2 in matrix form and switching the left and right sides yields a uniform representation

$$\begin{cases} \mathbf{b}^T(\cdot) \cdot \mathbf{X} = x(\cdot) \\ \mathbf{b}^T(\cdot) \cdot \mathbf{Y} = y(\cdot) \\ \mathbf{b}^T(\cdot) \cdot \mathbf{Z} = z(\cdot) \end{cases} \tag{3}$$

where $x(\cdot)$, $y(\cdot)$ and $z(\cdot)$ represent a point $\mathbf{P}$ on the curve or surface, $\mathbf{X}$, $\mathbf{Y}$ and $\mathbf{Z}$ represent the collection of x, y and z coordinates of the control points, and, for B-spline curves,

$$(\cdot) = (u) \tag{4}$$

$$\mathbf{X} = [x_1, x_2, \ldots, x_n]^T$$

$$\mathbf{Y} = [y_1, y_2, \ldots, y_n]^T$$

$$\mathbf{Z} = [z_1, z_2, \ldots, z_n]^T \tag{5}$$

$$\mathbf{b}(u) = [B_1(u), B_2(u), \ldots, B_n(u)]^T \tag{6}$$

and, for B-spline surfaces,

$$(\cdot) = (u, v), \tag{7}$$

$$\mathbf{X} = [x_1, x_2, \ldots, x_n]^T = [x_{11}, x_{12}, \ldots, x_{1n_v}, x_{21}, \ldots, x_{n_u n_v}]^T$$

$$\mathbf{Y} = [y_1, y_2, \ldots, y_n]^T = [y_{11}, y_{12}, \ldots, y_{1n_v}, y_{21}, \ldots, y_{n_u n_v}]^T$$

$$\mathbf{Z} = [z_1, z_2, \ldots, z_n]^T = [z_{11}, z_{12}, \ldots, z_{1n_v}, z_{21}, \ldots, z_{n_u n_v}]^T \tag{8}$$

$$\mathbf{b}(u,v) = \left[B_1(u,v), B_2(u,v), \ldots, B_n(u,v)\right]^T = \left[B_{u1}(u)B_{v1}(v), B_{u1}(u)B_{v2}(v), \ldots, B_{u1}(u)B_{vn_v}(v), B_{u2}(u)B_{v1}(v), \ldots, B_{un_u}(u)B_{vn_v}(v)\right]^T \tag{9}$$

where $n = n_u \cdot n_v$ is the total number of control points.

Let $\mathbf{m}$ be a set of m measured points (digitized either by a CMM or by other means)

$$\mathbf{m} := \left\{\bar{\mathbf{P}}_i = \left[\bar{x}_i, \bar{y}_i, \bar{z}_i\right]^T \mid i = 1, 2, \ldots, m\right\} \tag{10}$$

and $\mathbf{u} := \{u_i \mid i = 1, 2, \ldots, m\}$ for curve points and $\mathbf{u} := \{u_i \mid i = 1, 2, \ldots, m\}$ and $\mathbf{v} := \{v_i \mid i = 1, 2, \ldots, m\}$ for surface points be the *location parameters* allocated to $\mathbf{m}$ by some means that are examined in the sections below. By introducing the measured points together with their *location parameters* into Equations 3, we obtain the following observation equations:

$$\begin{aligned} &\mathbf{b}^T(\cdot_i) \cdot \mathbf{X} = \bar{x}_i \\ &\mathbf{b}^T(\cdot_i) \cdot \mathbf{Y} = \bar{y}_i \\ &\mathbf{b}^T(\cdot_i) \cdot \mathbf{Z} = \bar{z}_i \quad i = 1, 2, \ldots, m \end{aligned} \tag{11}$$

or, in compact matrix form,

$$\begin{aligned} &\mathbf{B} \cdot \mathbf{X} = \bar{\mathbf{X}} \\ &\mathbf{B} \cdot \mathbf{Y} = \bar{\mathbf{Y}} \\ &\mathbf{B} \cdot \mathbf{Z} = \bar{\mathbf{Z}} \end{aligned} \tag{12}$$

where

$$\bar{\mathbf{X}} = \left[\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_m\right]^T$$

$$\bar{\mathbf{Y}} = \left[\bar{y}_1, \bar{y}_2, \ldots, \bar{y}_m\right]^T$$

$$\bar{\mathbf{Z}} = \left[\bar{z}_1, \bar{z}_2, \ldots, \bar{z}_m\right]^T \tag{13}$$

represent the collection of x, y and z coordinates,

respectively, of the measured points, and

$$\mathbf{B} = \begin{bmatrix} B_1(\cdot_1) & B_2(\cdot_1) & B_3(\cdot_1) & \dots & B_n(\cdot_1) \\ B_1(\cdot_2) & B_2(\cdot_2) & B_3(\cdot_2) & \dots & B_n(\cdot_2) \\ \vdots & \vdots & \vdots & & \vdots \\ \vdots & \vdots & \vdots & & \vdots \\ \vdots & \vdots & \vdots & & \vdots \\ B_1(\cdot_m) & B_2(\cdot_m) & B_3(\cdot_m) & \dots & B_n(\cdot_m) \end{bmatrix}_{m \times n} \tag{14}$$

denotes the observation matrix. In these equations, $(\cdot_i) = (u_i)$ for B-spline curves and $(\cdot_i) = (u_i, v_i)$ for B-spline surfaces, $i = 1, 2, \dots, m$. Matrix **B** has the following properties:

- Matrix **B** is of full column rank if and only if the Schoenberg–Whitney conditions summarized in Appendix A are satisfied. For numerical computations, the condition number of $\mathbf{B}^T\mathbf{B}$ is also influenced by the actual positions of the available location parameters of a B-spline inside its definition domain. If, for example, all the parameters available to $B_i(u)$ are located at positions where $B_i(u)$ becomes almost zero, $\mathbf{B}^T\mathbf{B}$ becomes an ill conditioned matrix although the Schoenberg–Whitney conditions may be satisfied.
- If $n \gg k$ for curve fitting or $n_u \gg k_u$ and $n_v \gg k_v$ for surface fitting, **B** is a sparse matrix. After row permutation or reordering the points to be fitted, **B** becomes a banded matrix. Under the same conditions $\mathbf{B}^T\mathbf{B}$ is a banded, symmetric and positive defined matrix, i.e. $\forall\ \mathbf{X} \in \mathbb{R}^n$ and $\|\mathbf{X}\| \neq 0$, $\mathbf{X}^T(\mathbf{B}^T\mathbf{B})\mathbf{X} > 0$. The bandwidth of $\mathbf{B}^T\mathbf{B}$ is $2k-1$ for curve fitting, and $2(k_u - 1)n_v + 2k_v - 1$ for surface fitting.

From Equations 12, a solution interpolating through the digitized points can be achieved when $m = n$ and a least squares fitting solution can be achieved when $m > n$. To solve Equations 12 for an x coordinate for example, **X** can be found by minimizing

$$\min_{\mathbf{X}} S = (\mathbf{B}\cdot\mathbf{X} - \bar{\mathbf{X}})^T \cdot (\mathbf{B}\cdot\mathbf{X} - \bar{\mathbf{X}}) \tag{15}$$

All solutions of the fitting examples shown in this paper are realized by a stable algorithm using Householder transformations with stepwise refinement to achieve an accurate solution[8]. The numerical algorithm is available in the mathematical library[9] *NAG Fortran Library Mark* 15. Other stable algorithms that observe the banded structure of matrix **B** can also be found in References 10 and 11. The remaining problem is that of assigning parameter values **u** for curves and **u** and **v** for surfaces to the digitized points.

COMMONLY USED PARAMETERIZATION METHODS

For practical applications, there are three parameterization methods that are commonly used to assign the location parameters, i.e. uniform, cumulative chord length and centripetal model parameters. All of these methods can be applied to parameterize chain selected curve points or a grid of surface points. Uniform parameterization is the simplest way to assign parameters to digitized data. In this case,

$$u_i = \frac{i-1}{m-1} \quad 1 \le i \le m \tag{16}$$

for m curve points, and, analogously,

$$u_i = \frac{i-1}{m_u - 1} \quad 1 \le i \le m_u$$

$$v_j = \frac{j-1}{m_v - 1} \quad 1 \le j \le m_v \tag{17}$$

for $m = m_u \times m_v$ grid surface points. This method is sometimes unsatisfactory because the distribution of the measured points is not taken into account. When the measured points are unevenly spaced, a better choice has been suggested to be the cumulative chord length parameterization defined by

$$u_i = u_{i-1} + \frac{\|\bar{\mathbf{P}}_i - \bar{\mathbf{P}}_{i-1}\|}{\sum_{j=1}^{m-1} \|\bar{\mathbf{P}}_{j+1} - \bar{\mathbf{P}}_j\|} \quad 2 \le i \le m \tag{18}$$

with $u_1 = 0.0$ for curve points. A set of $m_u \times m_v$ grid points for surface fitting can also be parameterized analogously with the following formulae:

$$u_{ij} = \frac{\sum_{l=1}^{i-1} \|\bar{\mathbf{P}}_{l+1,j} - \bar{\mathbf{P}}_{l,j}\|}{\sum_{l=1}^{m_u-1} \|\bar{\mathbf{P}}_{l+1,j} - \bar{\mathbf{P}}_{l,j}\|}$$

$$2 \le i \le m_u;\ 1 \le j \le m_v \tag{19}$$

with $u_{1j} = 0.0$, $j = 1, 2, \dots, m_v$, and

$$v_{ij} = \frac{\sum_{l=1}^{j-1} \|\bar{\mathbf{P}}_{i,l+1} - \bar{\mathbf{P}}_{i,l}\|}{\sum_{l=1}^{m_v-1} \|\bar{\mathbf{P}}_{i,l+1} - \bar{\mathbf{P}}_{i,l}\|}$$

$$1 \le i \le m_u;\ 2 \le j \le m_v \tag{20}$$

with $v_{i1} = 0.0$, $i = 1, 2, \dots, m_u$. To avoid recomputations, all the chord lengths can be first computed and placed in a temporary working space. Chord length parameterization is in fact a scaled approximation of the arc length parameterization. In the case of surface fitting, the arc length refers to the arc length of the isometric curves on the surface. If the data points are more or less evenly spaced, Equation 18 approximates to Equation 16 for uniform parameterization, and similarly Equations 19 and 20 approximate to Equations 17.

A different approach, the centripetal model, is proposed by Lee in Reference 12:

$$u_i = u_{i-1} + \frac{\|\bar{\mathbf{P}}_i - \bar{\mathbf{P}}_{i-1}\|^{1/2}}{\sum_{j=1}^{m-1} \|\bar{\mathbf{P}}_{j+1} - \bar{\mathbf{P}}_j\|^{1/2}} \quad 2 \le i \le m \tag{21}$$

with $u_1 = 0.0$. This method observes the changing nature of the curvature for curve fitting[12]. A generalization of the above three methods, the exponential model*, is defined by Lee in the same paper[12]:

$$u_i = u_{i-1} + \frac{\|\bar{\mathbf{P}}_i - \bar{\mathbf{P}}_{i-1}\|^{e}}{\sum_{j=1}^{m-1} \|\bar{\mathbf{P}}_{j+1} - \bar{\mathbf{P}}_j\|^{e}} \quad 2 \le i \le m \tag{22}$$

with $u_1 = 0.0$ and $0.0 \le e \le 1.0$. When $e = 0.0$, Equation 22 reduces to the uniform parameterization, and, when $e = 1.0$, Equation 22 reduces to the cumulative chord length parameterization. As expected, decreasing the value of e causes the resulting shape of the fitted curve to tend to the shape coinciding with a uniform parameterization for fixed knots, while with increasing e the curve tends to that for cumulative chord length parameterization. The centripetal model is a method between the uniform and chord length parameterization with $e = 0.5$. Equations 21 and 22 for u_is can be extended to a grid of surface points for u_{ij}s similar to those in Equations 19 and 20.

In practice, some variants of the above parameterization methods exist, for example an averaging method for assigning location parameters to a set of grid points for surface fitting[5,13]. There are also some published methods that start from one of the above parameterizations† and try to optimize the parameters allocated to the digitized points, i.e. **u** for curves and **u** and **v** for surfaces, in such a way as to minimize the deviations between the digitized points and their corresponding fitted curve or surface points[14,15].

It should be noted that the parameterization of digitized points for fitting or interpolation and the parameterization of the knot parameters, i.e. the selection of $\xi := \{\xi_1, \xi_2, \ldots, \xi_{n+k}\}$ for B-spline curves and $\xi := [\xi_1, \xi_2, \ldots, \xi_{n_u+k_u}]$ and $\zeta := [\zeta_1, \zeta_2, \ldots, \zeta_{n_v+k_v}]$ for B-spline surfaces, are closely related. This case is different from that of polynomial curves and surfaces where only one curve segment or surface patch is considered. With a knot selection method consistent with the parameterization and the nature of the digitized points, one can use any kind of parameterization method to assign parameters to digitized points. *Figure 2* shows examples of B-spline curves fitted from the same digitized points using different parameterizations. A distinction is made between the parameterization of the digitized points themselves (there are five alternatives, with $e = 0.0$, 0.25, 0.5, 0.75 and 1.0 defined by Equation 22) and the parameterization of the knots (there are two alternatives, a uniform parameterization, and the average knot parameterization defined in Appendix B). With uniform knots, only some of the parameterization methods can achieve good results (see *Figures 2a–e*), while, with the average knots, all the parameterization methods can achieve reasonably good fitting results (see *Figures*

*In his original paper, this approach is called the *general exponent* method[12].

†Unless otherwise stated explicitly, parameterization in this paper usually refers to the parameterization of measured points, and not the knots.

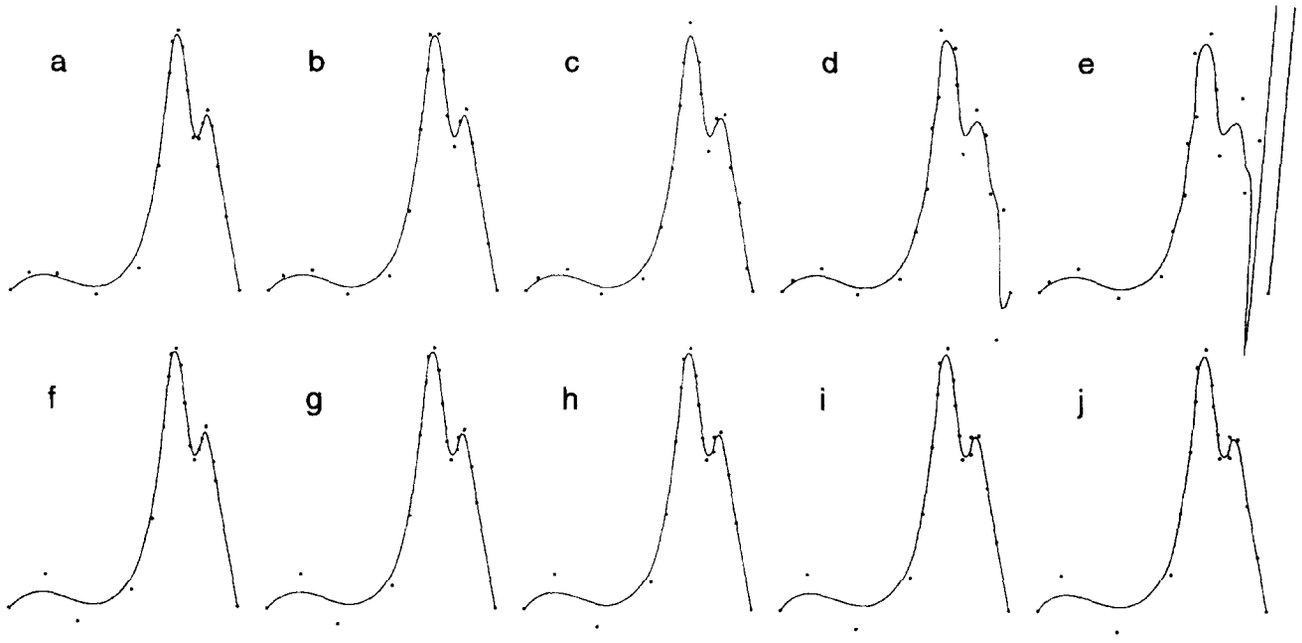

Figure 2 Parameterization for chain selected points
[With the general exponential parameterization model for the digitized points, the exponential values of (a)–(e) and (f)–(j) are 0.0 (uniform), 0.25, 0.5 (centripetal), 0.75, 1.0 (chord length). The knots of (a)–(e) are parameterized uniformly, and the knots of (f)–(j) are parameterized with the average method. In all the examples, the same order ($k = 4$) and number of control points ($n = 19$) are applied. The figures were obtained from 45 chain selected points which are shown in *Figure 1a* and *Figures 8a–c*. The dots indicate the locations of the control points.]

2f–j). This is mainly because of the following reasons:

- With this example, the measured points are taken more densely in areas where the curvature is sharper. With average knots, more knots are allocated to places where the curve changes rapidly.
- On the other hand, with average knots, it is almost sure that $\mathbf{B}^T\mathbf{B}$ will not become singular or ill conditioned. This is because, for all of the B-splines B_i, there are about equal numbers of $m \cdot k/n$ measured points allocated inside their definition domains $[\xi_i, \xi_{i+k}]$, and consequently the Schoenberg–Whitney conditions[10,16] are safely satisfied. For the examples of *Figures 2f–j*, this number is nine. This is, however, not the case for chord length parameterization of measured points in connection with uniform knots as illustrated in *Figure 2e*. If we count the number of measured points inside the definition domain of the individual B-splines, *Figure 2e* has the following unbalanced distribution {4, 7, 10, 11, 8, 6, 4, 5, 10, 12, 13, 19, 17, 15, 14, 7, 3, 2, 1}.

When parameter values of the measured points are fixed, there should, in fact, be an optimal set of knots for a fixed order and number of control points. On the other hand, if the knots are fixed first, there should also be a set of optimal location parameters $\{u_i\}_1^m$ for the measured points which will yield optimal fitting results. It is not the objective of this paper to produce such optimal parameterization, but to present an alternative method that has some advantages over other available methods. This alternative is called *base surface* parameterization for surface fitting and *base curve* parameterization for curve fitting.

BASE SURFACE PARAMETERIZATION

We shall use the concept of an *underlying surface*, by which we mean a surface that *best* fits the digitized points. In ideal cases, the underlying surface should be the original measured surface.

Parameterization procedure

The method presented here for the parameterization of measured points, i.e. the determination of location parameters for each of the measured points, uses a *base surface* (BS). A base surface is a first approximation of the final fitted surface. The basic strategy is that, by projecting digitized points onto the base surface, each of them will be associated with a projected point. We can then take the location parameters locating the projected points on the base surface as the location parameters of the digitized points for surface fitting. The parameterization procedure included the following steps:

- *Base surface creation:* A base surface can usually be defined from some characteristic curves approximating the underlying geometry. For most of the cases, it is sufficient to use four approximate boundaries. When dealing with potential complex surfaces, some other interior characteristic curves can also be used. These characteristic curves are often fitted from points either digitized specifically for this purpose, or picked up dynamically from the set of points for fitting the surface.
- *Parameter allocation via projection:* When a base surface is available, the parameterization is realized by projecting the measured points onto the base surface. This projection may be done either normally to the surface, or according to a given projection vector. The projection is realized by a minimization technique

$$d_i^2(u_i, v_i) = \min_{u,v} d_i^2(u, v) \tag{23}$$

where (u_i, v_i) are the location parameters of the projected point of $\bar{\mathbf{P}}_i = [\bar{x}_i, \bar{y}_i, \bar{z}_i]^T$, d_i is the distance from a surface point $\mathbf{P}(u, v) = [x(u,v), y(u,v), z(u,v)]^T$ to the digitized point $\bar{\mathbf{P}}_i$ in the case in which the projection vector is the base surface normal, and d_i is the distance from a surface point $\mathbf{P}(u,v)$ perpendicular to the projection line passing through the measured point $\bar{\mathbf{P}}_i$ in the case in which the direction of projection is fixed. In the second case, the projected point is in fact the intersection point of the projection line with the base surface. A numerical algorithm from a mathematical library[9] is used to perform the minimization process. After the projection, the parameter set $\{(u_i, v_i) \mid i = 1, 2, \ldots, m\}$ is used as the location parameter set of the measured points $\mathbf{m} := \{\bar{\mathbf{P}}_i = \{\bar{x}_i, \bar{y}_i, \bar{z}_i]^T \mid i = 1, 2, \ldots, m\}$ for LSQ fitting.

Definition of base surface

A base surface can be any parametric surface. It can be of any form, either truly freeform, or a particular simpler form, such as a planar, cylindrical or spherical surface. It should, however, satisfy following conditions:

- *Unique local mapping property:* There exists a projection vector (fixed or variable) that projects the underlying surface uniquely to the base surface within the digitized area. The so-called local mapping means that there exists a closed curve on the uv plane inside which all the projected points are located. The uniqueness implies that any two points on the underlying surface should have two different projected points on the uv plane within the above mentioned boundary, and a single point on the underlying surface can only have one projected point within the boundary. *Figure 3a* shows an ideal base surface for which all the measured points are uniquely projected to a unit square in the uv plane. In *Figure 3b*, the projection is unique, and the selected base surface is also suited. It should be noted, however, that there are cases in which *Figure 3b* will not work when the white area inside the unit square becomes too large, i.e. if there are insufficient measured points for some large area. In such a case, the Schoenberg–Whitney conditions may not be satisfied, and consequently the matrix $\mathbf{B}^T\mathbf{B}$ may be singular. Generally, there is no problem in the fitting process if the tinted area extends outside the unit square. It is also better to avoid such cases as the one in which the fitted surface

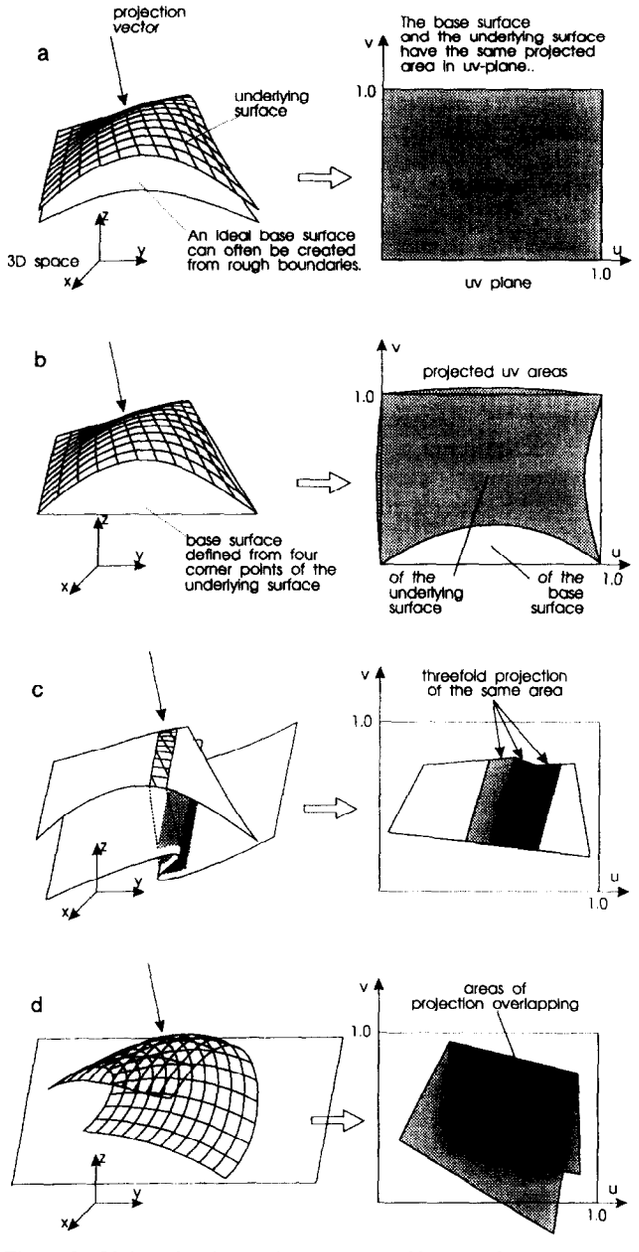

Figure 3 Unique local mapping property of base surface

covers a smaller area in 3D space than the underlying surface or the scattered points. The projection of *Figure 3c* is not unique, and the selected surface cannot be a base surface. The case shown in *Figure 3d* will not work either, as two different points can have the same projected point.

- *Smoothness and closeness of base surface*: The base surface should be as smooth and simple as possible, and yet still resemble the underlying surface. There should be a balance between these properties. For a stable and smooth fitting surface, the smoothness of the base surface is more important than the closeness of the base surface to the underlying surface. In terms of closeness, a base surface is usually satisfactory if the unique local mapping property is satisfied.
- *Parameterization of base surface*: The parameterization of a base surface directly affects the parameterization of the fitted surface. A good choice should be made by observing the underlying shape of the fitted surface. The parameterization of a base surface can usually be controlled by the parameterization of the characteristic curves. If, for example, a base surface is created from four boundaries, the parameterization of the base surface can then be controlled by the parameterization of the boundaries. Parameterization methods discussed in previous section can be applied for the parameterization of a base surface.

For the sake of popularity and a uniform database, it is better to use surfaces in B-spline form for B-spline surface fitting, or best to use surfaces in nonuniform rational B-spline form so that conical shapes can be used as base surfaces.

BASE SURFACE CONSTRUCTION

This section summarizes some commonly used methods for creating a base surface in B-spline form. Practical applications are, however, not limited to these cases and to the methods presented here.

Simple base surfaces

For some applications, four corner points $\mathbf{v} := \{\{\mathbf{v}_{ij}\}_{i=1}^{2}\}_{j=1}^{2}$ are sufficient to create a base surface (see, for example, *Figure 5*). In this case, the base surface can be defined as a bilinear B-spline surface over $\xi := \{0,0,1,1\}$ and $\zeta := \{0,0,1,1\}$ with $k_u = n_u = 2$ and $k_v = n_v = 2$, and with the four corner points $\mathbf{v}$ as the control points. For some other cases, two approximate boundary curves are neceessary (see, for example, *Figure 7*). Let us assume that the two boundaries are defined as B-spline curves using the same knot sequence $\xi := \{\xi_i\}_1^{n+k}$ with order k and number of control points n, and that the respective control points are $\{\mathbf{v}_{ij}\}_{i=1}^{n}$ for $j = 1, 2$. A B-spline surface can be defined that interpolates the two curves over knots $\xi := \{\xi_i\}_1^{n_u+k_u}$ and $\zeta := \{0,0,1,1\}$ with $k_u = k$ and $n_u = n$, $k_v = n_v = 2$, and the control points $\{\{\mathbf{v}_{ij}\}_{i=1}^{n_u}\}_{j=1}^{2}$. Other simple base surfaces in cylindrical or spherical form can also be created using NURBS[4,5].

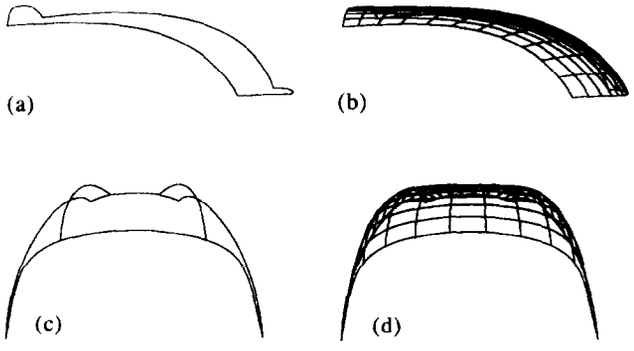

Figure 4 Two commonly used base surfaces; (a) characteristic curves, (b) base surface, (c) characteristic curves, (d) base surface [Base surface illustrated in (b) is defined using four boundaries illustrated in (a), and base surface in (d) is defined using four boundaries plus a set of section curves in (c). These base surfaces were used in Reference 2 for creating a CAD model of a scaled car body.]

BS from four boundaries

For most of the cases, a base surface can be defined from four approximate boundaries as shown in *Figures 4a* and *b*. This usually automatically guarantees an ideal base surface such as that in *Figure 3a*.

To create a base surface in B-spline form from its boundaries, one can use Farin's approach that defines a Coon's surface interpolating the four boundaries (see Reference 17, pp 354–356). Let us assume that we have four boundaries represented by B-spline curves. We further assume that the two nonadjacent boundaries are defined over the same set of knots with full multiple knots at the beginning and end of the knot sequence. An interpolation B-spline surface through the four boundaries can then be defined with the same two knot sequences of the boundaries and a set of control points bordered by the control points $\{\mathbf{v}_{1j}\}_{j=1}^{n_v}$, $\{\mathbf{v}_{n_u j}\}_{j=1}^{n_v}$, $\{\mathbf{v}_{i1}\}_{i=1}^{n_u}$ and $\{\mathbf{v}_{in_v}\}_{i=1}^{n_u}$. The control points $\mathbf{v}_{ij}$ of the surface are computed by bilinear blending:

$$\mathbf{v}_{ij} = [1-u_i \quad u_i]\begin{bmatrix}\mathbf{v}_{1j}\\ \mathbf{v}_{n_u j}\end{bmatrix} + [\mathbf{v}_{i1} \quad \mathbf{v}_{in_v}]\begin{bmatrix}1-v_j\\ v_j\end{bmatrix} - [1-u_i \quad u_i]\begin{bmatrix}\mathbf{v}_{11} & \mathbf{v}_{1n_v}\\ \mathbf{v}_{n_u 1} & \mathbf{v}_{n_u n_v}\end{bmatrix}\begin{bmatrix}1-v_j\\ v_j\end{bmatrix} \tag{24}$$

where u_i and v_j are defined as

$$u_i = (i-1)/(n_u-1) \quad i = 1, 2, \ldots, n_u$$
$$v_j = (j-1)/(n_v-1) \quad j = 1, 2, \ldots, n_v \tag{25}$$

To further increase the flexibility of the construction method, the control points at the four corners can be defined as the average of the two adjacent end control points of the two adjacent boundaries. This treatment allows one to use approximate boundary curves whose 'corner control points' do not precisely coincide, which is often the case for base surface creation.

BS from section curves

In the case of complex shapes, one may find it necessary to use more characteristic curves in creating a base surface. When a set of m approximate section curves are available, one can construct a base surface from these approximate section curves. Let us assume that all these section curves

$$\mathbf{P}_j(u) = \sum_{i=1}^{n} B_i(u)\cdot\mathbf{p}_{ij} \quad j = 1, 2, \ldots, m \tag{26}$$

are defined over the same knot sequence $\xi = \{\xi_i\}_1^{n+k}$ with the same order k, number of control points n, and parameter range $[\xi_k, \xi_{n+1}]$, where $\{\mathbf{p}_{ij}\}_{i=1}^{n}$, $j = 1, 2, \ldots, m$, are the $m \times n$ control points of the m curves. The B-spline surface interpolated through the m section curves is then defined as

$$\mathbf{P}(u,v) = \sum_{i=1}^{n_u}\sum_{j=1}^{n_v} B_{ui}(u)B_{vj}(v)\mathbf{v}_{ij} \tag{27}$$

where $n_u = n$, $k_u = k$, $n_v = m$, and $k_v = \min\{n_v, k_u\}$. The control points $\{\mathbf{v}_{ij}\}_{j=1}^{n_v}$, $i = 1, 2, \ldots, n_u$, are obtained by applying a curve interpolation scheme to the ith column of the control points

$$\{\mathbf{p}_{ij}\}_{j=1}^{n_v} \tag{28}$$

The location parameters for the above curve fitting can be defined as

$$v_j = (v_{0j} + v_{1j})/2 \quad j = 1, 2, \ldots, n_v \tag{29}$$

where $\{v_{0j}\}_{j=1}^{n_v}$ is the centripetal parameterization (or another) of $\{\mathbf{P}_j(0.0)\}_{j=1}^{n_v}$, and $\{v_{1j}\}_{j=1}^{n_v}$ is the parameterization (the same method for defining v_{0j}s) of $\{\mathbf{P}_j(1.0)\}_{j=1}^{n_v}$. A set of common knots $\zeta = \{\zeta_j\}_1^{n_v+k_v}$ can be defined according to the average method (see Appendix B). The ζ are then the v knots for the final fitted surface, and the $\xi = \{\xi_i\}_1^{n_u+k_u}$ are the u knots. When there are only two approximate section curves, a ruled surface is defined as a special case.

BS from boundaries plus interior section curves

This case is shown in *Figure 4c* and *d*, i.e. four boundaries plus some interior section curves. We will construct a fitting B-spline surface through all the curve nets. Let us assume that the family of section curves (including two of the boundaries)

$$\mathbf{P}_j(u) = \sum_{i=1}^{n_u} B_i(u)\cdot\tilde{\mathbf{p}}_{ij} \quad j = 1, 2, \ldots, m \tag{30}$$

are defined over the same knot sequence $\{\xi = \xi_i\}_1^{n_u+k_u}$ with the same order k_u, number of control points n_u, parameter range $[\xi_{k_u}, \xi_{n_u+1}]$, and full multiple knots at the beginning and end of the sequence, where $\{\tilde{\mathbf{p}}_{ij}\}_{i=1}^{n_u}$, $j = 1, 2, \ldots, m$, are the $m \times n_u$ control points. Let us further assume that the second family of two boundary curves are defined by

$$\mathbf{Q}_i(v) = \sum_{j=1}^{n} B_j(v)\cdot\tilde{\mathbf{q}}_{ij} \quad i = 1, 2 \tag{31}$$

over the same knot sequence $\tilde{\zeta} = \{\tilde{\zeta}_j\}_1^{n+k}$ with the same order k, number of control points n, parameter range $[\tilde{\zeta}_k, \tilde{\zeta}_{n+1}]$, and full multiple knots at the beginning and end of the sequence, where $\{\tilde{\mathbf{q}}_{ij}\}_{j=1}^{n}$, $i = 1, 2$, are the $2n$ control points. One of the above two boundaries is called the primary boundary, and the other the secondary. For the development of the algorithm, we shall assume that $\mathbf{Q}_1(v)$ is the primary boundary. The fitting process starts from the construction of a dense family of section curves and is realized as follows.

(1) Project $\{\mathbf{P}_j(0.0) = \tilde{\mathbf{p}}_{1j}\}_{j=1}^{m}$ to the primary curve and $\{\mathbf{P}_j(1.0) = \tilde{\mathbf{p}}_{n_u j}\}_{j=1}^{m}$ to the secondary curve

to obtain the following set of location parameters: $\{\tilde{v}_{0j}\}_{j=1}^{m}$ and $\{\tilde{v}_{1j}\}_{j=1}^{m}$.

(2) Define average sampling parameters $\{\bar{v}_j\}_{j=1}^{n}$ as

$$\bar{v}_j = \begin{cases} 0.0 & j = 1 \\ \left(\sum_{i=j}^{j+k} \tilde{\zeta}_i\right)/(k+1) & \text{otherwise} \\ 1.0 & j = n \end{cases} \quad (32)$$

Such selection guarantees that the primary curve can be reconstructed safely via a fitting process using the sampled points of $\{\bar{v}_j\}_{j=1}^{n}$.

(3) Insert all parameters of $\{\tilde{v}_{0j}\}_{j=1}^{m}$ into $\{\bar{v}_j\}_{j=1}^{n}$. For each of the $\tilde{v}_{0j}$s, if there exists an integer l such that

$$|\tilde{v}_{0j} - \bar{v}_l| = \min_i |\tilde{v}_{0j} - \bar{v}_i| \leq \frac{1.0}{2k(n-k+1)} \quad (33)$$

i.e. when $\bar{v}_l$ is very close to $\tilde{v}_{0j}$, $\bar{v}_l$ is replaced by $\tilde{v}_{0j}$ instead of insertion. When any insertion takes place, a new knot is also inserted at the centre of $[\tilde{\zeta}_I, \zeta_{I+1}]$ in $\tilde{\zeta} = \{\tilde{\zeta}_j\}_1^{n+k}$. The index I is chosen such that $\tilde{\zeta}_I \leq \tilde{v}_{0j} < \tilde{\zeta}_{I+1}$. Denote the combined set of parameters of $\{\tilde{v}_{0j}\}_{j=1}^{m}$ and $\{\bar{v}_j\}_{j=1}^{n}$ by $\{v_j\}_{j=1}^{n_v}$, and denote the new knot sequence after insertion by $\zeta = \{\zeta_j\}_1^{n_v+k_v}$. Here $k_v = k$ and $n \leq n_v < n+m$, depending on the times of insertion or replacement.

(4) Apply the following piecewise linear transformation from $\{v_j\}_{j=1}^{n_v}$ to $\{\bar{v}_j\}_{j=1}^{n_v}$ such that $\{\tilde{v}_{0j}\}_{j=1}^{m}$ is transformed to $\{\tilde{v}_{1j}\}_{j=1}^{m}$:

$$\bar{v}_j = \tilde{v}_{1I} + (v_j - \tilde{v}_{0I})\frac{\tilde{v}_{1,I+1} - \tilde{v}_{1I}}{\tilde{v}_{0,I+1} - \tilde{v}_{0I}}$$
$$\tilde{v}_{0I} \leq v_j < \tilde{v}_{0,I+1}; \quad j = 1, 2, \ldots, n_v \quad (34)$$

In the case in which $m = n_v$ and $\bar{v}_j = \tilde{v}_{1j}$, $j = 1, 2, \ldots, m$, the final fitted surface interpolates all the curve nets.

(5) Evaluate $\{\mathbf{Q}_1(v_j)\}_{j=1}^{n_v}$ and $\{\mathbf{Q}_2(\bar{v}_j)\}_{j=1}^{n_v}$. These are the starting and end control points of the section curves to be used to construct the fitting surface. Now we should have a stripe of rectangles bordered by some number of control points. As $\{\mathbf{P}_j(0.0) = \tilde{\mathbf{p}}_{1j}\}_{j=1}^{m} \subset \{\mathbf{Q}_1(v_j)\}_{j=1}^{n_v}$ and $\{\mathbf{P}_j(1.0) = \tilde{\mathbf{p}}_{n_u j}\}_{j=1}^{m} \subset \{\mathbf{Q}_2(\bar{v}_j)\}_{j=1}^{n_v}$, only some of them need to be evaluated.

(6) Compute the interior control points of these new section curves via Farin's approach as if a Coons' surface is being computed for each of the rectangles. Let us denote the complete set of control points for the n_v section curves as $\{\{\mathbf{p}_{ij}\}_{i=1}^{n_u}\}_{j=1}^{n_v}$. Please note that $\{\{\tilde{\mathbf{p}}_{ij}\}_{i=1}^{n_u}\}_{j=1}^{m} \subset \{\{\mathbf{p}_{ij}\}_{i=1}^{n_u}\}_{j=1}^{n_v}$.

With these section curves, we can then apply curve fitting schemes to each of the n_u columns of the control points $\{\mathbf{p}_{ij}\}_{j=1}^{n_v}$, $i = 1, \ldots, n_u$, whose location parameters are defined by $\{v_j\}_{j=1}^{n_v}$ with the common knots $\zeta = \{\zeta_j\}_1^{n_v+k_v}$. Let $\{\mathbf{v}_{ij}\}_{j=1}^{n_v}$, $i = 1, \ldots, n_u$, be the control points of the n_u interpolated curves. The fitting surface through the curve net is then defined by

$$\mathbf{P}(u, v) = \sum_{i=1}^{n_u} \sum_{j=1}^{n_v} B_{ui}(u) B_{vj}(v) \mathbf{v}_{ij} \quad (35)$$

over knots $\xi = \{\xi_i\}_1^{n_u+k_u}$ and $\zeta = \{\zeta_j\}_1^{n_v+k_v}$.

Compatibility of characteristic curves

When we create a base surface from approximate characteristic curves, we always assume that each family, either u or v, of the characteristic curves is defined over the same knots with the same starting and end parameters. When creating characteristic curves from points, these assumptions should already be observed. Otherwise, some or all of the curves under consideration can be recomputed. One can use degree elevation, knot insertion, corner cutting and linear reparameterization techniques (see References 17–21 and references therein) to do this without changing their original shape. This may result in a fitting surface with too many control points. As the accuracy of the base surface is not so important, we usually recompute the curves with a set of common knots via a fitting process. For each curve, a number of points are first sampled and reparameterized with, for example, the centripetal model (it is better to apply the same parameterization method to all the curves). A common knot sequence is then constructed. Average knot parameterization can also be applied here. One can simply collect all the location parameters of the sample points (of all the curves), reorganize the parameters in nondecreasing order, and then apply average knots. All the curves are further recomputed with their own location parameters and the common knots.

EXAMPLES AND DISCUSSIONS

For practical applications, parameterization with a base surface has the following properties:

- *Applicability*: Base surface parameterization applies to both regularly and randomly distributed points. In the case of randomly distributed points (for example, those of *Figure 7*), base surface parameterization produces reasonably good and stable results.
- *Closeness of fitting*: In the case of irregularly spaced points, parameterization with a base surface usually produces better results than other methods. Even in the case of regular points, base surface parameterization achieves slightly better fitting results than others for most of the cases. *Figure 5* gives examples of B-spline surface fitting with regular points, and *Figure 6* gives examples of fitting with irregular points. In these and other figures, *Max Devi*, *Min Devi*, *Avrg Devi* and *Std Devi* stand for maximum, minimum, average and standard deviations evaluated from the measured points perpendicular to the fitted B-spline curve or surface.

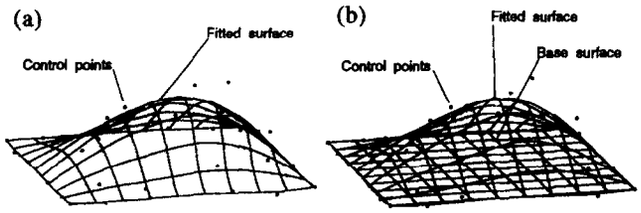

Figure 5 Surface fitting with regular grid points; (a) chord length parameterization, (b) parameterization with base surface created from four corner points of measured data
[The dots represent control points of the fitted surface. Average knots with orders $k_u = k_v = 4$ and a number of control points $n_u = n_v = 6$ are applied. The measured points are 225 simulated points with superimposed random errors (mean $\epsilon = 0.0170$, standard deviation $\sigma = 0.1014$). (a) *Max Devi* = 0.3620, *Min Devi* = −0.3469, *Avrg Devi* = 0.0028, *Std Devi* = 0.1204; (b) *Max Devi* = 0.3209, *Min Devi* = −0.2897, *Avrg Devi* = 0.0006, *Std Devi* = 0.1086.]

- *Parameterization*: The fitted surface using base surface parameterization behaves very well for any kind of points, regularly, irregularly or randomly distributed, while, with other existing parameterization methods, the fitted surface may be severely distorted if the measured points are not regularly distributed. See *Figure 6* for an example.

The example in *Figure 7* is a typical example of reverse engineering for CAD modelling from physical parts. The physical object in this example is a freeform spotlight as commonly used in shops, museums and other public or private places. The production of such freeform lamps starts by creating a mathematical or CAD model from a handmade physical workpiece produced by craftsmen. A coordinate measuring machine is first used to digitze the physical part. Least squares fitting techniques are then applied to identify all the geometric features from the digitized points of the CMM. The CAD model is further completed with ordinary techniques such as surface extension, intersection, trimming and filleting. Advanced design and production techniques can finally be applied to manufacture the spotlights[1,2]. *Figure 7* shows one surface of a spotlight.

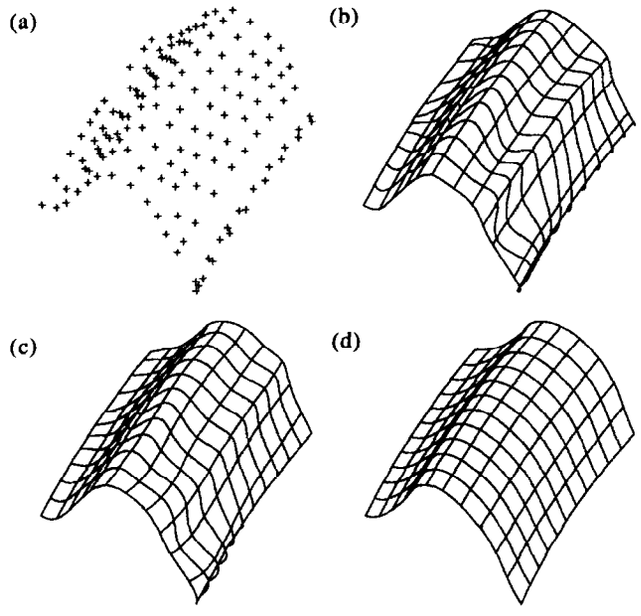

Figure 6 Surface fitting with irregular grid points; (a) 14×10 irregular grid points, (b) fitted surface with uniform parameterization, (c) fitted surface with centripetal parameterization, (d) fitted surface with base surface parameterization
[The base surface is created from four boundaries. For all the fitted surfaces, average knots with orders $k_u = k_v = 4$ and number of control points $n_u = 12$, $n_v = 8$ are applied. The measured surface is a surface of a profile cutter measured by a coordinate measuring machine. (a) 14×10 irregular grid points; (b) *Max Devi* = 0.0595, *Min Devi* = −0.0575, *Avrg Devi* = −0.0020, *Std Devi* = 0.0273; (c) *Max Devi* = 0.0514, *Min Devi* = −0.0514, *Avrg Devi* = −0.0014, *Std Devi* = 0.0185; (d) *Max Devi* = 0.0297, *Min Devi* = −0.0257, *Avrg Devi* = −0.0001, *Std Devi* = 0.0121.]

CASE FOR CURVE FITTING

For the case of B-spline curve fitting, a similar technique can be developed. Instead of using a base surface, a *base curve* is used for the parameterization of the measured points. The parameterization is realized by projecting the measured points to the base curve. The projection is also realized by a minimization algorithm. The parameters of the projected points are used as the parameters of the measured points. The base curve is created by a fitting process. Unlike the case of surfaces, the construction of the base curve uses all the measured points and not just some selected ones. This is simply because they are available, and using all the available information usually yields better results.

For chain selected points, one can directly apply curve fitting techniques and use the fitted curve as a base curve. For randomly ordered points, one can first create a 1st-step base curve and use this base curve to reorder the points. This 1st-step base curve can usually be created from a limited number of selected characteristic points of the measured data, including the two endpoints. The local mapping property should be observed for its creation; the smoothness and closeness can, however, be disregarded. By projecting all the points to this curve, each of the measured points is allocated a location parameter. Although such a set of parameters can directly be used as location parameters for fitting, and one can sometimes obtain good fitting results (see *Figure 8a*), these parameters are used at this stage mainly for reordering. After the points have been correctly ordered, one can do the same as if one is dealing with chain selected points.

The basic steps for curve fitting using base curves are shown in *Figures 8a–c*. In the case of randomly ordered points, a 1st-step base curve BC_1 is constructed from a limited number of points (in *Figure 8*, they are two endpoints). By projecting all the points to BC_1, each of the points is allocated a location parameter. All the points can then be reordered with these parameters. A new base curve BC_2 (see *Figure 8b*) is then created from the ordered set of points via a fitting process with existing parameterization techniques. Finally, curve C_2 (see *Figure 8c*) is fitted using BC_2 as a base curve for assigning the location parameters. In the case of chain selected points, one can go directly to the stage in *Figure 8b* to create the base curve BC_2. If we take the fitted curved C_2 as a new base curve and repeat the fitting process, an interesting question is that of whether the results could further be improved. The answer is 'yes' at the cost of extra computations. This also holds true for surface fitting with base surface parameterization. One can always improve the fitting results by taking the current fitted curve or surface as a new base curve or surface.

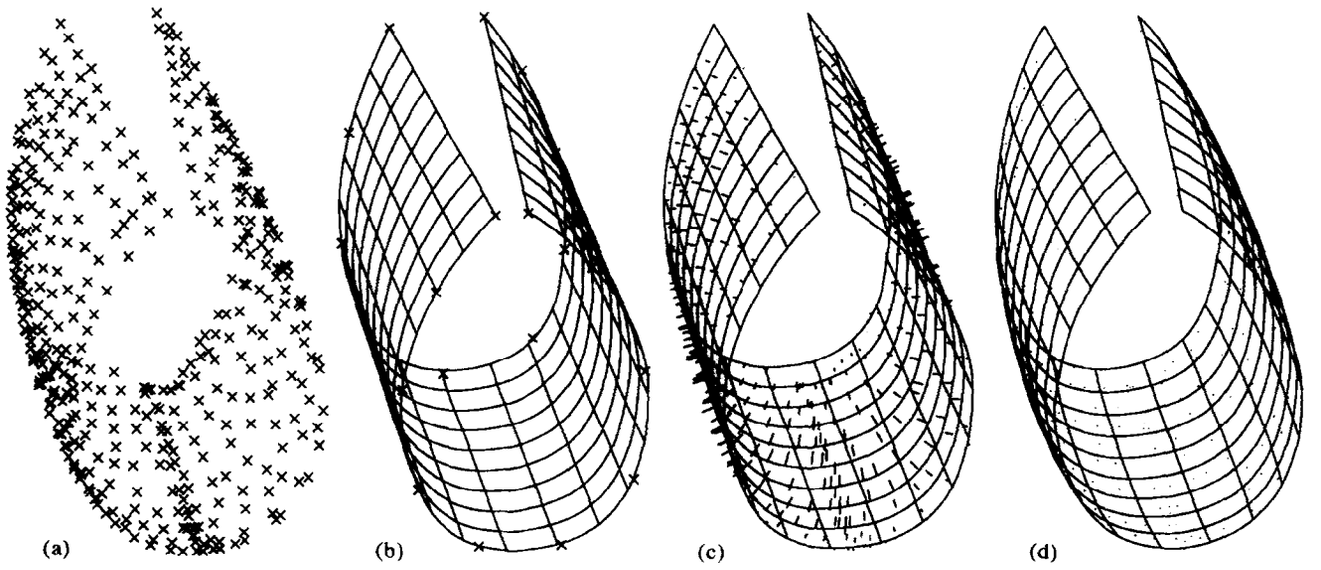

Figure 7 Surface fitting with randomly distributed points; (a) 408 randomly distributed points, (b) base surface fitted from 20 boundary points (uniform knots with $k_u = 3$, $n_u = 8$, $k_v = 2$, $n_v = 2$), (c) base surface with initial deviations from digitized points to the base surface, (d) final fitted surface with deviations (average knots with $k_u = 4$, $n_u = 17$, $k_v = 4$, $n_v = 5$)
[The underlying surface is a surface on a freeform spotlight. The points have been digitized with a coordinate measuring machine. (a) 408 randomly distributed points; (b) 20 boundary points and base surface; (c) *Max Devi* = 3.7871, *Min Devi* = −0.6674, *Avrg Devi* = 1.6094, *Std Devi* = 1.0907; (d) *Max Devi* = 0.1762, *Min Devi* = −0.2026, *Avrg Devi* = −0.0000, *Std Devi* = 0.0458.]

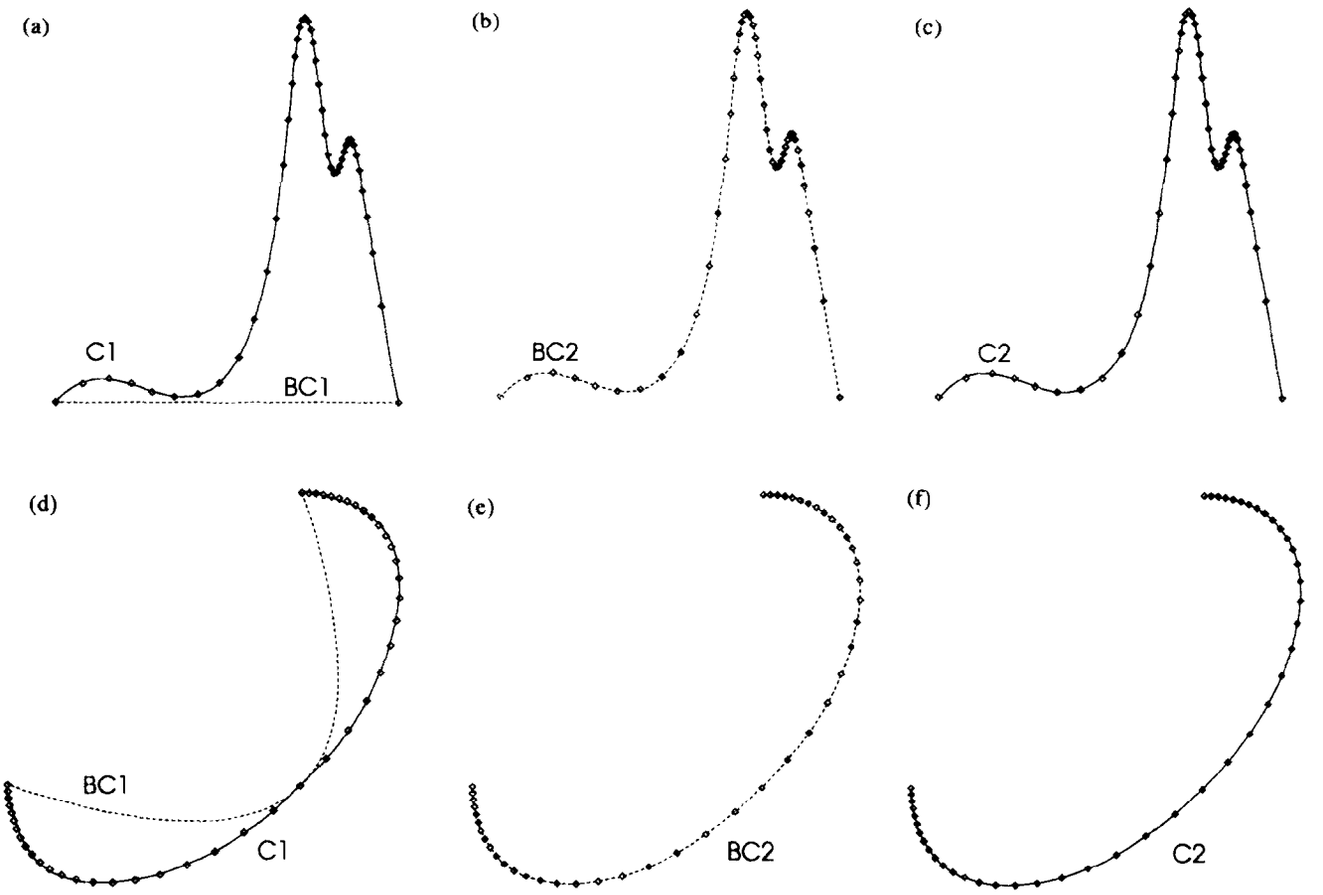

Figure 8 Curve fitting with various parameterizations; (a) C_1 fitted with BC_1 as a base curve which is created from two endpoints, (b) BC_2 fitted with centripetal model, (c) C_2 fitted with BC_2 as a base curve. (d), (e) and (f) show another example. In (d) BC_1 created from three selected characteristic points
[Average knots are used for all these examples. The order and number of control points of (a)–(c) are 4 and 19, and those of (d)–(f) are 4 and 9, except for BC_1 which uses 2, 2 for (a) and 3, 3 for (d). ◇: measured point, ---: base curves, ——: fitted curves. (a) *Max Devi* = 0.2740, *Min Devi* = 0.0010, *Avrg Devi* = 0.0454, *Std Devi* = 0.0618; (b) *Max Devi* = 0.3759, *Min Devi* = 0.0014, *Avrg Devi* = 0.0917, *Std Devi* = 0.0939; (c) *Max Devi* = 0.2973, *Min Devi* = 0.0003, *Avrg Devi* = 0.0675, *Std Devi* = 0.0707; (d) *Max Devi* = 0.2403, *Min Devi* = 0.0082, *Avrg Devi* = 0.1100, *Std Devi* = 0.0685; (e) *Max Devi* = 0.0796, *Min Devi* = 0.0010, *Avrg Devi* = 0.0261, *Std Devi* = 0.0192; (f) *Max Devi* = 0.0535, *Min Devi* = 0.0030, *Avrg Devi* = 0.0221, *Std Devi* = 0.0124.]

In *Figure 8a, curve* C_1 is a curve fitted directly from the original randomly ordered points using BC_1 as a base curve. For this example, using BC_1 produces better results than using BC_2 in terms of closeness of fit (see the deviation values in *Figure 8*). It is not, however, always true, and sometimes even BC_2 could be better than C_1 (see, for example, *Figures 8d–f*). This is mainly because of the fact that, for an ordered set of points, very good fitting results can be achieved with existing parameterization methods. In this sense, the application of the base curve parameterization to curve fitting is usually limited to the following cases. It can be used either to reorder randomly scattered curve points and then switch to existing solutions (from BC_1 to BC_2), or to improve fitting results by taking the fitted curve with existing parameterizations as a base curve (from BC_2 to C_2). Base curve parameterization can, however, provide very good insight into base surface parameterization.

CONCLUSIONS

This paper presents a simple method to parameterize randomly distributed points for curve and surface LSQ fitting. Parameterization is achieved by projecting measured points onto a base curve or base surface, and associating each measured point with its projected point. The location parameters of the projected points are then used as the location parameters of the measured points. Some commonly used methods to define a base surface from approximate characteristic curves of the underlying surface are presented. While existing parameterization methods do not work with randomly distributed points, the parameterization technique presented in this paper applies to regularly, irregularly and randomly distributed points.

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APPENDIX A

Invertibility of matrix $\mathbf{B}^T\mathbf{B}$

Schoenberg–Whitney conditions for curve fitting

Let $\{B_i\}_1^n$ be the n normalized B-splines defined over knots $\xi := \{\xi_i\}_1^{n+k}$, and $\mathbf{u} := \{u_i\}_1^m$ be the location parameters. The matrix $\mathbf{B}$ defined by Equation 14 for curve fitting is of full column rank if and only if there exists $\tilde{\mathbf{u}} = \{\tilde{u}_i\}_1^n \subset \mathbf{u}$ such that

$$B_i(\tilde{u}_i) \neq 0 \quad i = 1, 2, \ldots, n \tag{36}$$

or

$$\xi_i < \tilde{u}_i < \xi_{i+k} \quad i = 1, 2, \ldots, n \tag{37}$$

In the case of the presence of k-multiple knots over the definition domain of B_i, the ith condition for $\tilde{u}_i$ of Expression 37 can be relaxed to

$$\begin{aligned} &\xi_i \leq \tilde{u}_i < \xi_{i+k} \quad \xi_i = \ldots = \xi_{i+k-1} < \xi_{i+k} \\ &\xi_i < \tilde{u}_i \leq \xi_{i+k} \quad \xi_i < \xi_{i+1} = \ldots = \xi_{i+k} \end{aligned} \tag{38}$$

For more information, see Reference 10, p 200, for the case of interpolation, and Reference 16 for the case of fitting.

Schoenberg–Whitney conditions for surface fitting

Let $\{B_i\}_1^{n_u}$ and $\{B_j\}_1^{n_v}$ be the normalized B-splines defined over knots $\xi := \{\xi_i\}_1^{n_u+k_u}$ and $\zeta := \{\zeta_j\}_1^{n_v+k_v}$, and $\mathbf{u} := \{u_i \mid i = 1, 2, \ldots, m\}$ and $\mathbf{v} := \{v_i \mid i = 1, 2, \ldots, m\}$ be the location parameters. The matrix $\mathbf{B}$ defined by Equation 14 for surface fitting is of full column rank if and only if there exists $\tilde{\mathbf{u}} = \{\{\tilde{u}_{ij}\}_{i=1}^{n_u}\}_{j=1}^{n_v} \subset \mathbf{u}$ and there

exists $\tilde{\mathbf{v}} = \{\{\tilde{v}_{ij}\}_{i=1}^{n_u}\}_{j=1}^{n_v} \subset \mathbf{v}$ such that

$$B_i(\tilde{u}_{ij}) \cdot B_j(\tilde{v}_{ij}) \neq 0 \quad i = 1, 2, \ldots, n_u;\ j = 1, 2, \ldots, n_v \tag{39}$$

or

$$\begin{aligned} &\xi_i < \tilde{u}_{ij} < \xi_{i+k_u} \\ &\zeta_j < \tilde{v}_{ij} < \zeta_{j+k_v} \quad i = 1, 2, \ldots, n_u;\ j = 1, 2, \ldots, n_v \end{aligned} \tag{40}$$

In the case of the presence of k_u-multiple knots over the definition domain of B_i, the n_v conditions for $\tilde{u}_{ij}$ of Expressions 40 can be relaxed to

$$\begin{aligned} &\xi_i \leq \tilde{u}_{ij} < \xi_{i+k_u} \quad \xi_i = \ldots = \xi_{i+k_u-1} < \xi_{i+k_u} \\ &\xi_i < \tilde{u}_{ij} \leq \xi_{i+k_u} \quad \xi_i < \xi_{i+1} = \ldots = \xi_{i+k_u} \end{aligned} \tag{41}$$

where $j = 1, 2, \ldots, n_v$. In the case of the presence of k_v-multiple knots over the definition domain of B_j, the n_u conditions for $\tilde{v}_{ij}$ of Expressions 40 can be relaxed to

$$\begin{aligned} &\zeta_j \leq \tilde{v}_{ij} < \zeta_{j+k_v} \quad \zeta_j = \ldots = \zeta_{j+k_v-1} < \zeta_{j+k_v} \\ &\zeta_j < \tilde{v}_{ij} \leq \zeta_{j+k_v} \quad \zeta_j < \zeta_{j+1} = \ldots = \zeta_{j+k_v} \end{aligned} \tag{42}$$

where $i = 1, 2, \ldots, n_u$.

This theorem is a direct extension of the case of curve fitting.

APPENDIX B

Average knots

Let $\mathbf{u} := = \{u_i \mid i = 1, 2, \ldots, m\}$ be a set of nondecreasing location parameters allocated to $\mathbf{m}$, i.e.

$$u_i \leq u_{i+1} 1 \leq i \leq m - 1 \tag{43}$$

The average knots are defined as

$$\xi_i = \begin{cases} u_1 & 1 \leq i \leq k \\ \dfrac{\mu_i + \sum_{j=j_1+1}^{j_2} u_j}{j_2 - j_1 + 1} & k+1 \leq i \leq n \\ u_m & n+1 \leq i \leq n+k \end{cases} \tag{44}$$

where

$$\begin{aligned} \mu_i &= u_{j_c} + r \cdot (u_{j_c+1} - u_{j_c}) \\ j_1 &= j_c - h - (1-\delta) \cdot \text{int}\left(\frac{3(1-r)}{2}\right) \\ j_2 &= j_c + h + (1-\delta) \cdot \text{int}\left(\frac{3r}{2}\right) \end{aligned} \tag{45}$$

with

$$\begin{aligned} h &= \text{int}\left(\alpha \cdot \frac{k-1}{2} \cdot \frac{m}{n}\right) \\ r &= \text{rem}\left(\left(i - \frac{k+1}{2}\right) \cdot \frac{m}{n} + \frac{1}{2}\right) \\ j_c &= \text{int}\left(\left(i - \frac{k+1}{2}\right) \cdot \frac{m}{n} + \frac{1}{2}\right) \end{aligned} \tag{46}$$

and

$$\delta = \begin{cases} 1 & h = 0 \\ 0 & \text{otherwise} \end{cases} \tag{47}$$

In Equations 45 and 46, int(·) denotes the largest integer whose magnitude does not exceed that of (·), rem(·) is the difference between (·) and int(·). $\alpha \in [0.0, 1.0]$ is a constant that determines the number of location parameters to be taken into account in calculating an average. The larger the values of α, the more parameters are used for computing the average. When $\alpha = 1.0$, the ith knots are an average of $(k-1)m/n + 1$

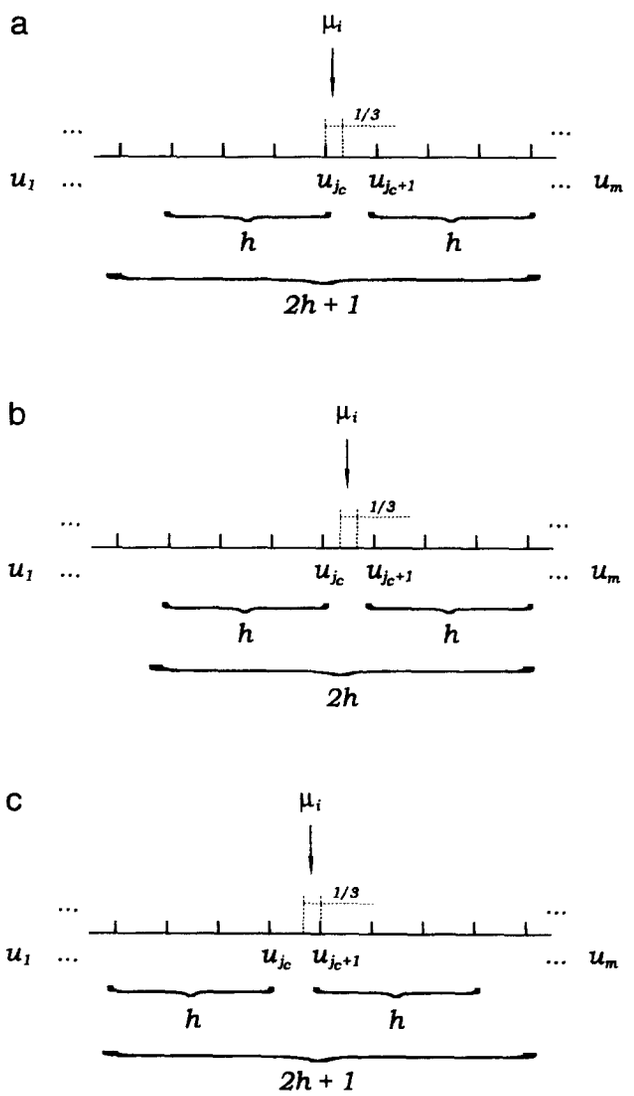

Figure 9 Number of parameters for computing average knots; (a) μ_i located on left side of $[u_{j_c}, u_{j_c+1}]$, (b) μ_i located in centre of $[u_{j_c}, u_{j_c+1}]$, (c) μ_i located on right side of $[u_{j_c}, u_{j_c+1}]$

parameters centred at the interval $[u_{j_c}, u_{j_c+1}]$. When $\alpha = 0.0$ or $h = 0.0$, Equations 44–47 simply return linear interpolation results as $\xi_i = \mu_i$, $i = k+1$, $k+2, \ldots, n$, defined by Equation 44. For randomly distributed points, one can set α to 1.0. See *Figure 9* for an illustration. Under the following conditions

- $m = n$, i.e. the case of interpolation,
- α is set to 1.0,
- the term μ_i is dropped and the denominator is replaced with $j_2 - j_1$ in Equation 44,

the above method produces the same results as those produced by the averaging method defined in References 5 and 13 for the case of interpolation.

In the case of random or surface points, all the parameters should first be arranged in nondecreasing order to satisfy Expressions 43 when computing average knots. For surface fitting, **u** and **v** parameters are treated independently. When computing u knots, for example, all the u-location parameters of the measured points are first reorganized in nondecreasing order. Any possible occurrence falling outside the parameter range $[-\epsilon, 1.0+\epsilon]$ in consequence is not taken into account. ϵ is a small and positive number, and usually $\epsilon = 1.0/(2k_u(n_u - k_u + 1))$. Let $\tilde{m}_u \leq m$ be the total number of parameter values falling inside $[-\epsilon, 1.0+\epsilon]$, k_u be the u order, n_u be the total number of control points in the u direction, and $\{\tilde{u}_i\}_1^{\tilde{m}_u}$ be the set of parameters. The u-average knots are then computed using Equations 44–47, except for the first k_u and last k_u knots, which are simply set to 0.0 and 1.0, respectively. Instead of m, k, n and $\{u_i\}_1^m$, we have, for this case, $\tilde{m}_u$, k_u, n_u and $\{\tilde{u}_i\}_1^{\tilde{m}_u}$. The average v knots are computed analogously by reordering the v parameters.

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