EGA V: §6

(former EGA IV: §21)

Translation and Editing of his ‘prenotes’

by

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Section 21

Invertible sheaves and divisors relative to projective and multiprojective fibrations
linear systems of divisors

1) Invertible sheaves on a projective and multiprojective fibrations.
2) Representability of Div $\frac{L}{X/S}$: relative divisors on projective and multiprojective fibrations.
3) Linear systems of divisors and morphisms into projective fibrations.
4) Linear systems of divisors and invertible modules.

The results developed in this paragraph and in the following (ones) are already partly global in nature and they give, namely, complements desur (about) projective schemes using the global constructions of Chapter II and also have some result sof Chapter III and also purely local results of the present Chapter IV.

One of the aims of the present pragraph is to develop the language of "linear systems of divisors" connected on the one hand to the classificatin of morphisms into a projective fibration, on the other hand to the classification of invertiblel modules over a given prescheme. Let us note that the 'parameter schemes' really natural for the linear systems of divisors are the Brauer-Severi schemes which generalize projective fibrations and can be defined, for example, as fibrations that gecome isomorphic to a projective fibration after an étale surjective base extension.

Since their study uses descent theory developed in chapter V ([Tr] of the original design) and since also their classification is equivalent to the classification of group torsors of the projective group we postpone the study of such schemes and their connections with the notion of linear systems of divisors to Chapter VI of our scientific work. From a technical point of view, the main result of this section (paragraph) is Theorem 1.1, which determines the Picard group of projective fibration in terms of that of the base, and its first corollaries developed in Nos. 1 and 2.

Section 1

(Crossed out in the original until the point that we will indicate.)

Invertible sheaves and divisors on projective fibrations and [illegible] **projective** (tr: perhaps multiprojective) **linear systems of divisors.**

(1) Determination of invertible sheaves on projective fibrations. Application to the automorphisms of a projective fibration

Theorem 1.1. *Let S be a prescheme E a locally free module over S of finite rank ≥ 2 at every point $P = P(E)$ the projective fibration that it defines. Then for every invertible module L over P we may find a family $(S_n)n \in Z$ of open disjoint sets in S covering S indexed by Z and an invertible module M over S so thta the restriction of L to $f^{-1}(S_n)$ should be isomoprhic to that of $M \otimes O_P(1) = f^*(M)(n)$. Also, the family of the S_n is uniquely determined by these conditions, thus M (is determined) up to a unique isomorphism.*

Remark 1.2. If we make no assumption about the rank of E then S canonically decomposes into the sum prescheme of the opens S^0, S^1, S^2 such tht over them [???] sur ceux-ci the rank of E is respectively 0, 1 and ≥ 2. Then the determination of the invertible modules over P is reduced to the $f^{-1}(S^i)$ for $i = 0, 1, 2$. The case $i = 2$ is justifiable from 1.1, on the other hand $f^{-1}(S^1)$ is S^1 isomorphic to S^1 hence its Picard group is nothing else but $\mathrm{Pic}(S^1)$ finally $f^{-1}(S^0)$ is empty thus its Picard group is zero.

Corollary 1.3. *Under the assumptions of 1.1 let us assume that S is connected and non-empty. Then every invertible module L over P is isomorphic to a module of the form $f^*(M)(n)$ where $n \in Z$ and M is an invertible module over S. Also n is uniquely determined and M is determined up to a unique isomorphism by the giving of L.*

Another way to formulate this corollary is the following. Let us consider the natural homomorphisms $\mathrm{Pic}(S) \to \mathrm{Pic}(P)$ [illegible] $Z \to \mathrm{Pic}(P)$ the first one deduced from $f\colon P \to S$ the second one determined by the element $d(O_P(1))$ of $\mathrm{Pic}\, P$. We deduce a canonical homomorphism $\mathrm{Pic}(S) \times Z \to \mathrm{Pic}(P)$ defined anyway without any restrictive assumption on S or on E. This gives:

Corollary 1.4. *Under the conditions of 1.1 if $S \neq 0$ then the preceding homomorphism is injective and even bijective if S is connected.*

If we abandon the assumption about the rank being ≥ 2 it follows from 1.2 and 1.4 that the preceding homomorphism is again surjective if S is connected but not necessarily injective, the kernel being isomorphic to Z, resp. to $\mathrm{Pic}(S) \times Z$ if E is of rank 1, respectively of rank zero.

Let us prove 1.1 by starting from uniqueness. First of all if S is the spectrum of a field let us notice that $O(n)$ is not isomorphic to $O(m)$, i.e. $O(m-n)$ is not isomorphic to O_P que si [Fr] $n - m = 0$. This comes from the fact that $O(1)$ is ample and $\dim P \geq 1$ (from the assumption that rank $E \geq 2$): Indeed we may suppose $d = m - n \geq 0$ if we had $d > 0$ then $O(d)$ would be ample and could not be isomorphic to O_P except if P is quasi-affine thus finite (since it is proper ovr K). This alreadyp roves the uniqueness of the family $(S_n)n \in Z$ considered in 1.1. For the uniqueness of M up to unique isomorphism, we are reduced to the case $S = S_n$, I say that in this case we have as isomorphism (uniquely determined in terms of the isomorphism $L \to f^*(M)(n)$ $(**)M \to f_*(L(-n))$). Indeed the isomorhism $(**)L \to f^*(M)(n)$ defines an isomorphism $L(-n) \to f^*(M)$ and consequently an isomorphism of the second term in $(**)$ with $f_*(f^*(M))$ which by itself is isomorphic to M, (M being locally free) to $M \otimes f_*(O_P)$ since [or] [Fr] [ref] $f_*(O_P) \leftarrow O_S$, hence the isomorphism $(**)$.

Let us prove the existence of the (S_n), M. Due to the uniqueness already shown the question is local over S, i.e. we are reduced to proving the

Corollary 1.5. *Let E be locally free and of finite rank over S, $P = P(E)$, L an invertible module over P, $s \in S$, then there exists an open neighborhood U of s and an integer $n \in Z$ such that $L \mid f^{-1}(u)$ is isomorphic to $O_P(n) \mid f^{-1}(u)$.*

Of course since the rank of E at s is ≥ 2 the integer n is well defined.

In addition 1.5 is trivial since the rank of E at s is ≤ 1. Let us note that E, est dela frima [Fr], since the question being local we may suppose already$E = O_S^{r+1}$, hence $P = P_S^r$. By the brief procedure of Paragraph 8 [of EGA IV? Tr] we are reduced also to the case where S is noetherian. We proceed in two steps:
a) S is the spectrum of a field K. We see that L is [Nousavon que Fr] defined by a graded module of finite type L over the gradual ring $K[t_0, \dots, t_r]$. We also see that the restriction L' of L to the affine space epointe [Fr] punctured (?) $E_K^{r+1} - \{0\}$ is nothing else but the inverse image of L by the canonical projection morphism $q\colon E_K^{r+1} - (0) \to P_K^r$ and is therefore an invertible module. Let $i\colon E^{r+1} - \{0\} \to E^{r+1}$ be the canonical immersion it follows from the fact that the affine ring $K[t_0, \dots, t_r]$ of E^{r+1} is factorial, thus a fortiori its localization at the point 0 of E^{r+1} is factorial and from the fact that the latter ring is of dimension ≥ 2 that $i_*(L')$ is an invertible module thus corresponding to an invertible module [illegible letter] over $K[t_0, \dots, t_r]$. In addition $M = \lceil(E^{r+1} - 0, q^1(L))$ is graded in a natural fashion, finally the homomorphism $L \to M$ is evidently an isomorphism at all the points of E^{r+1} distinct from zero.

Thus after replacing L by M we are reduced to the case of an invertible L. But $K[t_0, \dots, t_r]$ being factorial, L is then free of rank one, initialy by ignoring (neglecting) its grading; but a standard lemma (which can be found in Bourbaki without a doubt) implies that it is given free of rank [illegible] as a graded module which implies that the associated module over P is isomorphic to an $O_P(n)$. From the editorial point of view it would be clumsy (maladroit [Fr]) to begin by considering an L of finite type.

Let us begin carvement by considering $L = \lceil(E^{r+1} - (0), q_*(L))$ defining the module $L = q_*(L)$) we see (Chapter II) that L is a graded module that determines precisely L if we prove that L is free of rank 1 as a graded module by the indicated reasoning.
b) General case. Is deduced from case a) due to III.4.6.5 by using the relation $H^1(P_K^r, O_P^{r_K}) = 0$ established in III.2. q.e.d.

A variant of 1.4. Let $Z(S)$ be the set of locally constant functions with integer values over S, we define an evident homomorphism: $Z(S) \to \mathrm{Pic}(P)$ the $n \in Z(S)$ correspond in fact to partitions $(S_n)n \in Z$ on disjoint open sets (among which some may be empty) and to such a partition one associates the invertible module $O_P(n)$ whose restrictions to $f^{-1}(S_n)$ is $O_P(n)$. We find thus a variant (* bis) $\mathrm{Pic}(S) \times Z(A) \to \mathrm{Pic}(P)$ and a statement more general and more satisfactory than 1.4 affirming that (under the conditions of 1.1) this is a bijection. (NB Since $S \neq 0$ then the canonical application $Z \to Z(S)$ associating to every $n \in Z$ the constant function of value n is injective, resp. surjective, and we recover 1.4 formally which should be expressed (or stated) in the most general form that I vien d'expliciter [Fr].

Let us remark also as a remark that in the language of the Picard scheme which will be introduced in Chapter V.1.1 in the preceding equivalent form s'enouce simplement [Fr] by saying that the canonical homomorphism $Z_S \to \mathrm{Pic}_{P/S}$ of constant group schemes Z over S into the Picard scheme, deduced from the section of the latter defined by $O_P(1)$ is an isomorhism.

Let P be a projective fibration over a field K. An invertible module L over P is said to be *of degree n* if L is isomorphic to $O_P(n)$; if dim $P \geq 1$ this determines n in terms of L but if dim $P \leq 0$ (i.e. P is empty or reduced to a point) then L is of degree n for every n. To say that L is of degree n means also, because of 1.1 and 1.2, that the class of L in $\mathrm{Pic}(P)$ is in the image of $\mathrm{Pic}\, S \times \{n\}$ for the homomorphism $(*)\mathrm{Pic}(S) \times Z \to \mathrm{Pic}(P)$ described above, i.e. that L is isomorphic to a module of the form $f^*(M)(n)$ where M is invertible over S. Also, if the fibrs of P are non-empty, i.e. E is everywhere of rank ≥ 1, then M is determined up to a unique isomorphism in terms of L as follows, again, from 1.1 and 1.2. (By the way A ce propos [Fr] I notice that it is proper to announce 1.1 also without any hypothesis about the rank of E: every invertible L over S can be taken in the form indicated in that statmenet; lousque [Fr] if the fibers of P are non-empty, i.e. the rank of E is ≥ 2 athen the partition of S is also determined uniquely by the choice of L. In this way th remark 1.2 is eliminated and passe [Fr] in the proof.)

Let $P^1 = P(E^1)$ be a second projective fibration, then the determination of $\mathrm{Pic}(P)$ allows in principle to determine the S-morphism $g\colon P \to P^1$, since these are defined by an invertible module $L(g^*O_P)$ over [illegible] and a homomorphism $f_*(L) \leftarrow E^1$ (such taht the associated homomorphism $f^*(E^1) \to L$ is surjective) modules an isomorphism of L. We say that $g\colon P \to P^1$ is of degree n if $L = g^*(O_P(1))$ is of degree n. It suffices evidently de savoir [Fr] determine the homomorphisms of degree n for every n given. Let us note tht if g is of degree n and P has fibers of dimension ≥ 1 we necessarily have $n \geq 0$ (since over a field K if dim $P \geq 1$ then $O_P(n)$ is generated by its sections only if $n \geq 0$.), of course we could restrict ourselves to the case where P has its fibers of dim ≥ 1 (by proceeding as in 1.2). Since we have $f_*(f^*(M)(n)) = M \otimes f_*(O_P(n)) = M \otimes \mathrm{Sym}^2(E)$ we see that the determination of the S-morphisms $g\colon P \to P^1$ is reduced to the determination of the couples (m, u) up to isomorphism where M is an invertible module over S and $u\colon E^1 \to M \otimes \mathrm{Sym}^n(E)$ is a homomoprhism such tat the corresponding homomorhism [illegible] $^*(E^1) \to f^*(M)(n)$ is an epimorphism. Then g determines a first invriant of a global nature over S, savoir [Fr] $(M) \in \mathrm{Pic}(S)$ and this invariant fixed by the chosen M, the corresponding g correspond to a certain subset of the quotient set $\mathrm{Hom}(E^1, M \otimes \mathrm{Sym}^n(E))/I(S, O_S)^*$, the passage to the quotient by the group $I(S, O_S)^*$ corresponds to "module isomorphism" in the description of the $g\colon P \to P^1$ via invertible modules (Nota bene: the endomorphisms, resp. automorphisms, of an invertible L over a projective fibration P correspond to sections, resp. invertible sections, of O_P over P or even to sections, resp. invertible sections, of $O_S \to f^*(O_P)$ over S.)

Special cases (particular cases)

1) $n = 0$ – then we must take the homomorphism $E^1 \to M$ that are surjective, i.e. everywhere non-zero modules isomorphism of M. We find exactly the morphisms $g\colon P \to P^1$ of the form h, where h is a section of P over S (savoir [Fr] those determined by the invertible quotient M of E^1). Thus the S-morphisms of degree 0 of P into P^1 are the constant morphisms relative to S.

[Here ends the crossed out part Translator]

2) $n = 1$ – we must take the homomorphisms $E^1 \to E \otimes M$ or $L \otimes M$ tht are *surjective* as one erifies immediately and the corresponding homomorphism $g\colon P \to P^1$ is nothing else but the composition $P(E) \to P(E \otimes M) \to P(E^1)$ where the first homomorphism is the canonical isomorphism described in Chapter II and the second is the canonical closed immersion deduced from the epimorphism $E^1 \to E \otimes M$. If we call linear the homomorphisms from P into P^1 which can be so described as such a composition we see that the morphisms $g\colon P \to P^1$ which are of degree 1 are exactly the *linear* ones.

To finish let us determine the isomorphisms of P with P^1.

Theorem 1.6. *Let S be a prescheme, $P = P(E)$ and $P^1 = P(E^1)$ two projective fibers over S defined by E and E^1 locally free of finite type. Then every S-isomorhism g [illegible] $: P \to P^1$ is definable as a composition $P(E) \to P(E \otimes M) \to P(E^1)$ where M is an invertible module over S, the first homomorphism is the canonical isomoprphism of Chapter II and the second is the isomorphism deduced from an isomorphism $u\colon E^1 \to E \otimes M$. Since the fibers of P are non-empty (resp.of* $\dim \geq 1$*) therefore M (respectively the couple (M, U)) is determined up to a unique isomorphism in terms of g.*

According to the above considerations we are reduced to proving that g is of degree one which reduces us to the cse where S is the spectrum of a field and also (bien sur [Fr]) we may suppose that $\dim P \geq 1$. But let us note that $O_P(1)$ is then intrinsically characterized (i.e. independently from the way that P is reduced as a projective fibration) as the generator of $\mathrm{Pic}(P)$ (between the two generators $O_P(1)$ and $O_P(-1)$) which is ample; consequently if $g\colon P \to P^1$ is an isomorphism then $g^*(O_P^1(1))$ is isomorphic to $O_P(1)$ and we (gagne) [Fr] – In local form less savante [Fr] we may announce:

Corollary 1.7. *Under the conditions of 1.6 every S-morphism $g\colon P \to P$ can be described in a neighborhood of each $s \in S$ usign an isomorphism $u\colon E \mid U \simeq E^1 \mid U$ the latter being well defined module multiplication by an element of $\lceil(U, O_U)^*$. In particular:*

Corollary 1.8. *Let S be a prescheme, $P = P(E)$ a projective fibration over S defined by E locally free of finite type, u an automorphism of P. Then u is determined in a neighborhood of every point $s \in S$ by an automorphism u of $E \mid U$ the latter being well defined by u module multiplication by an element of $\lceil(U, O_U)^*$.*

Remark 1.9. In 1.8 we could easily deduce that the group functor $\mathrm{Aut}_S(P)$ over S is representable by an affine prescheme of finite presentation over S, which can be also interpreted as the quotient group scheme of the linear group scheme $G \mid (E)$ by its center G_M. The group prescheme is called the prescheme of projective groups or simply projective group defined by E and is denoted $GP(E)$. If E is free $E \cong O_S^{r+1}$, thus $GP(E)$ is nothing else but the group prescheme $GP(r)_S$ deduced by base change $S \to \mathrm{Spec}\, Z$ of the analogous group scheme $GP(r)$ over $\mathrm{Spec}\, Z$ called the absolute projective group.

End of Appendix 1. Marginal remark next to Remark 1.9 partly illegible [illegible] $P(E^v \otimes E)$ defined by the non-vanishing of the "determinant".

Section 2
Relative divisors and invertible shaves on projective and multiprojective fibrations

2.1. Let, as in No. 1, $P = P(E)$ E locally free over S of rank ≥ 2 everywhere. We propose to determine the set $\mathrm{Div}(P/S)$ of relative divisors ≥ 0 over P p.r. (pr rapport ? with respect) to S. We see that to give such a divisor is the same as to give an invertible module L over P with a section ϕ of L transversally regular. But according to 1.1 (ignoring a possible partition of S if S is not connected) L is isomorphic to a $M \otimes O_P(n)$ where$M$ is an invertible module over S in addition determined up to a unique isomorphism in terms of L. In addition we see tht we have also the canonical isomorphisms

$$f_*(L) \simeq M \otimes f_*(O_P(n1)) \simeq M \otimes \mathrm{Sym}^n(E) \tag{*}$$

so that to give a section ϕ of L is the same as to give a section ψ of $m \otimes \mathrm{Sym}^n(E)$. Taking into account the fact that the fibrs of P/S are integral we see in addition tht ϕ is transverse regular (i.e. regular on each fiber) if and only if $\psi(s) \neq 0$ for every $s \in S$ or which is the same if and only if the homomorphism $\psi\colon M^v \to \mathrm{Sym}^n(E)$ deferred by ψ is "universally injective", i.e. locally an isomorhism onto a direct factor, or what is the same if its transpose $\psi\colon \mathrm{Sym}^n(E)^v \to M$ is surjective.

We say that a relative divisor D over P is *of degree* n if $O_x(D) = L$ is of degree n in the sense of the previous No. Since $D \geq 0$ this implies $n \geq 0$ since [illegible] (on avait ???) if we had (?) $n < 0$ every section of L over X were zero. By 1.1 if D is a relative divisor ≥ 0 over P then there exists a unique decomposition of D into the disjoint sum of open subsets $S_n (n \in N)$ such that for every $n \in N$, $L/P/S_n$ is fo degree n. This reduces the determination of the set of relative divisors ≥ 0 to the case of relative divisors ≥ 0 of given degree n.

This text replaces of course the 'abstraction faite' above (translated *ignoring*).

This being granted the above reflections give the

Proposition 2.2. *Under the above hypotheses we have a one-to-one correspondence between the set $\mathrm{Div}^n(P/S)$ of relative divisors ≥ 0 of degree n over P and of the set of invertible quotient modules M of $\mathrm{Sym}^n(E)^v$ (or, which is the same, of invertible submodules locally direct factors M^v of $\mathrm{Sym}^n(E)$.*

If D and M correspond to each other then $O_P(D)$ is canonically isomorhic to $M \otimes O_P(n)$ and the section S_D is identified by this isomorhism a ce q'on devin [Fr] taking into account $(*)$.

Let us notice that this description is compatible with leut [illegible] base change in S. Taking into account the interpretation of invertible quotient modules of $\mathrm{Sym}^n(E)^v$ as sections over S of $P(\mathrm{Sym}^n(E))$, we find here: (oui Fr)

Corollary 2.3. *The subfunctor $\mathrm{Div}^n_{P/S}$ of $\mathrm{Div}^+_{P/S}$ is representable canonically by $P(\mathrm{Sym}^n(E)^v)$. Taking into account the considerations of 2.1, it follows that:*

Corollary 2.5. *$\mathrm{Div}^+_{P/S}$ is representable canonically by the S-prescheme sum of the $P(\mathrm{Sym}^n_{P/S}(E)^v)$, $n \in N$.*

Corollary 2.5. *Let us now suppose that we are given a finite family $(E_i)_{i\in I}$ of locally free modules over S, hence (d'ou'des Fr) $P_i = P(E_i)$ and a $P =$ fibered product of the P_i also the multiprojective fibration over S defined by the (E_i). For simplification of notations we denote by $O_i(n)$ the inverse image sur (Fr) P of the invertible module $O_{P_i}(n)$ over P_i. For every system of integers $n = (Ni)_{i\in I} \in Z^I$ we put $O_P(n) = \otimes_i O_i(n_i) = \otimes_{I_{O_S}} O_{P_i}(n_i)$ ceci pose [Fr], 1.1 generalizes as follows:*

Proposition 2.6. *Let us assume that the E_i have (partout Fr) rank ≥ 2. Then for every invertible module L over the multiprojective fibration P there exists a decomposition of S into the disjoint sum of open sets S_n, $n \in Z^I$ and an invertible module M over S such that $L/P/S_n$ is isomorhic to $O_P(n)/P/S_n$. Also the S_n are determined uniquely and M is determined up to a unique isomorphism.*

The proof consists in an immediate reduction to 1.1. Under the conditions of 2.6, we may therefore associate to every invertible L over P a 'multidegree' $n = (Ni)_{i\in I} \in Z(S)^I$ which characterizes L up to a unique isomorhism provided $\mathrm{Pic}(S) = 0$. also we may interpret the Ni (called the "partial degree of L with respect to the factor P_i of index i") if we take for each i a section g_i of P_i over S (NB such sections exist in any case locally over S) and if we note that this system defines for each i an S-morphism $g_i\colon P_i \to P$; this granted (cui pose F4) we have $N_i = \mathrm{dig}\, h_i^*(L)$ we point out that in general the n_i are not integers but they are locally constant functions of S into Z.

Proceeding as in No. 1, we may deduce from 2.6 the determination of a morphism of one multiprojective fibration into another and in particular the determination of the automorphisms of multiprojective fibrations. More interesting for us because of par 25 about the resultant (?) and discriminant of forms will be the determination of relative divisors ≥ 0 on a multiprojective fibration.

Corollary 2.8. *If D is a relative divisor over P we define its multidegree as that of $O_P(D)$. As above, the determination of $\mathrm{Div}^+(P/S)$ is reduced to that of $\mathrm{Div}^n(P/S)$ for a given multidegree $n \in I$ which gives an isomorphism $L = O_x(D) \approx M \otimes_{O_S} O_P(n)$. But we have due to Chapter II, Par. 2 (**) $f_*(D) = M \otimes f_*(O_P(n)) = M \otimes_i \mathrm{Sym}_i^n(E_i)$. Proceeding now as in 2.2, we find the*

Propositioin 2.9. *With the preceding notations, to be recalled, we have one-to-one canonical correspondence between the set $\mathrm{Div}^n(P/S)$ of relative divisors of multidegree n over P and the set of invertible quotient modules M of $\otimes_i \mathrm{Sym}_i^n(E_i)^v$ (or, what is the same...) If D and M correspond to each other, then $O_P(D)$ is canonically isomorphic to $M \otimes O_P(n)$ and s_D is identified then a'ce qu'on devine [Fr], taking into account (**).*

Corollary 2.10. *The subfunctor $\mathrm{Div}^n_{P/S}$ of $\mathrm{Div}^+_{P/S}$ corresponding to relative divisors of multidegree n is canonically representable by the projective fibration $P(\otimes_i \mathrm{Sym}_i^n(E_i)^V)$ and $\mathrm{Div}^+_{P/S}$ is canonically representable by the sum prescheme of the latter for $n \in N^I$.*

Corollary 2.11. *The preceding very simple determination of $\mathrm{Div}_{P/S}$ is due to the very simple structure of Pic(P?S) (indeed to the "discrete" structure of the Picard prescheme $\mathrm{Pic}_{P/S} \ldots$) We may, abstracting from the reasoning just done, (abstraive le raisonment* *fait) [making the reasoning done abstract – Grothendieck's art of making things as general as possible – translator's remark] wich reduces essentially to establishing a relative representability (with respect to Pic).*

To do this let us take a morphism $f\colon X \to S$ proper and flat of finite presentation *and* an invertible module L over X. We propose to determine the subgroups $\mathrm{Div}^L(X/S)$ of $\mathrm{Div}^+(X/S)$ formed by relative positive divisors such that $O_X(D)$ is isomorphic to a module of the form $L \otimes_{O_X} M$ where M is an invertible module over S (depending on D). We assume that we have $f_*(X_X) \leftarrow O_S$ which implies that the above M (ci-dessus M [Fr]) is determined up to a unique isomorphism by D since $M = f_*(L^{-1} \otimes O_X(D))$. To give D corresponding to an L given is thus reduced to giving a transversally regular section ϕ of $L \otimes M$. But we have, because of Chapter III, Par. 7, that there exists a module Q of finite presentation over S whose formation commutes in addition with every base change and an isomorphism of functors in G (a quasi-coherent module variable over S) $f_*(L \otimes G)$ – $\mathrm{Hom}(Q, G)$ (To tell the truth in the loc. cit. we suppose that S is locally neotherian but if we get rid of this hypothesis in an evident way by a brief procedure taking into account the commuting of Q with base change). In particular, to give ϕ is equivalent to giving a homomorphism $\Psi\colon Q \to M$. A necessary condition for ϕ to be transversally regular and one which is sufficient if the fibers of X are integral is that ϕ should be $\neq \phi$ fiber by iber, which in terms of Ψ means simply that Ψ is surjective thus that Ψ corresponds to a section of the projective fibration $P(Q)$ over S. We obtain therefore the

Proposition 2.12. *In the above notations $\mathrm{Div}^L(X/S)$ is in a canonical bijective correspondence with the set of sections of $P(Q)$ over S corresponding to a quotient module M of Q such that the section ϕ of $L \otimes M$ defined by $\Psi\colon Q \to M$ should be transversally regular. Let us suppose now that the hypothesis $O_S \to f_*(O_X)$ is encove [Fr] even (???) true after every base change or what reduces to the same by III.7 that we have the condition $k(s) \to H^0(X_s, O_{X_s})$ for every $s \in S$. Thus 2.12 applies equall well to every X^1_s/X^1 by an arbitrary base change $S^1 \to S$. We obtain therefore the*

Theorem 2.13. *Let $f\colon X \to S$ be a morphism of finite presentation proper and that satisfying $(***)$ above L an invertible module over X, let us consider the subfunctor of $\mathrm{Div}^L_{X/S}$ of $\mathrm{Div}^+_{X/S}$ defined above in terms of L. There exists a module of finite presentation Q over S such that the preceding functor is representable by a sub-prescheme open retrocompact of the projective fibration $P(Q)$; since the fibers of X/S are geometrically integral thus $\mathrm{Div}^L_{S/S}$ is representable by the same fibration (by the fibration itself lui même [Fr]).*

The last assertion results immediately from the one that we have given before. For the general case we have already noted that we have a monomorphism $\mathrm{Div}^L_{S/S} \to P(Q)$ and we are reduced to proving that the latter is representable by an open quasi compact immersion, which reduces us to proving that if we take a section of $P(Q)$ over S, i.e. an invertible quotient module M of Q, d'ou [Fr] a section ϕ of $L \otimes M$ non-vanishing at any fiber then the subfunctor [illegible] of the final functor S over $(\mathrm{Sch})/S$ "consisting in making ϕ transversally regular" is representable by a retrocompact open subset of S. But the fact that it is representable by an open set gives the fact that f is proper and that the transverse regularity is an open condition (cf. par. 11...) the retrocompactness is seen immediately by reduction to the noetherian case.

Section 3
Linear systems of divisors and morphisms into projective fibrations

Let D be a family of divisors over X/S parametrized by T (we imply *in what follows positive*). A point $x \in X$ is called "fixed point" for this family of divisors if $\mathrm{pr}_1^{-1}(x) \subset D$ essentially so hat the set of non-fixed points is a complement of $\mathrm{pr}_1(X \underset{S}{\times} T - D)$, consequently if $T \to S$ is universally open (for example flat locally of finite presentation) then the set of fixed points is closed. We say that the family of divisors is without fixed points if the set Z of fixed points is empty. If Z is closed then $X - Z$ is the biggest oipen set such that the family of divisors of $X - Z$ parametrized by T induced by the given family in an evident sense is without fixed points. If the family D is "without fixed points" and if T is flatand locally of finite presentation over S with fibers (S_1) and geometrically irreducible then D is also a divisor relative to X (for $\mathrm{pr}_1 \colon X \underset{S}{\times} T \to X$): indeed D is defined locally at a point $z \in \operatorname{supp} D$ by one equation $\phi = 0$ and the equation induced on the fiber $T_{s\,k(x)}$ of the point $x \in X$ (over $s \in S$) is non-nilpotent at z (since otherwise $\mathrm{pr}_1^{-1}(x)D = V(\phi_x)$ would contain set-theoretically a neighborhood of Z in $T_{k(x)}$ therefore would contain $T_{s\,k(x)}$ i.e. x would be a fixed point which it is not) but since $T_{s\,k(x)}$ is irreducible and (S_1) it follows that ϕ_x is $O_{T\,k(x)}$-regular at Z. We obtain therefore a family of divisors of T/S parametered by X, i.e. a morphism

$$X \to \mathrm{Div}_{T/S}.$$

In the general case where the family of divisors of X may have fixed points, we obtain a family of divisors T/S parametered by $X - Z$, i.e. a morhism $x - Z \to \mathrm{Div}_{T/S}$ by replacing in the previous definition X by $X - T$.[1] Anyway the above proof shows that $X - Z$ is exactly the greatest open set U of X such that $d \mid U \underset{S}{\times} T$ is a divisor relative to U, i.e. such that its symmetric (image) ${}^tD$ is a family of divisors of T/S parametrized by U/S. We may remark that if X and T are both flat locally of finite presentation over S with fibers (S_1) and geometrically irreducible the symmetry $D \to {}^tD$ gives a one-to-one correspondence between families of divisors of X/S parametrized by T which are without fixed points and families of divisors of T/X parametrized by X which are without fixed points. If in this statement we wish to get rid of the specific assumption made about the fibers of X/S and T/S, it is convenient to replace the "fixed points" by "fixed points in a more general or extended sense" by understanding that by a fixed point in a general sense of D an $x \in X$ such that D is not a relative divisor to X at all the points of $\mathrm{pr}_1^{-1}(x)$. IF W is open, the points of $X \underset{S}{\times} T$ where D is a relative divisor with respect to X then the set of fixed points in the extended sense of D is equal to $\mathrm{pr}_1(x \times_S T - W)$ since $T \to S$ is proper it is therefore a closed subset Z' of X. In every case (any case) we obtain a familyof divisors of T/S parametrizeed by $X - Z'$. The assumption that the fibers of T/S are (S_1) and geometrically irreducible serves precisely to insure tht $Z = Z'$ (fixed points = fixed points in extended sense). Geometrically, let us suppose for simplicity that $S = \operatorname{Spec} k$, k

[1] I think $X - Z$ [Tr].

an algebraically closed field which is allowed for T/S flat and of finite presentation, by a base change, to say that $X \in X(k)$ is a fixed point (resp. fixed point in an extended sense) means that $x \in \operatorname{supp} D_t$ for every $t \in T(k)$ (respectively, that there exists a prime cycle T' associated to T such that $x \in D_t$ for every $t \in T'(k)$). An omission: The formation of the set of fixed points Z is compatible with base change in S; on the other hand $X - Z$ (assumed open, e.g. T/S flat locally of finite presentation) is universally schematically dense in X relative to S. This last fact results from Par. 11[2] andfrom the fact that for every $s \in S$ Z_s does not contain any point of X_s associated to O_{X_S} (indeed the support of divisor over X_s does not contain any such point). In the case where T is a projective fibration $Q = P(F)$ $D_{Q/S}$ is representable by the sum of $P(\mathrm{Sym}^n(F^v)) = P(n)$, we find a morphism $X - Z \to P(n)$. We say that the family of divisors D is *of degree* n if the preceding morphism factors by $P(n)$; if $X \not{-}\phi$ hence $X - Z \neq \phi$ the n in question is well defined by D. Besides, in addition, to define this notion we do not strictly need for rigor the result of representability announced above but only ot have defined the canonical monomorphisms

$$p(n) \to \mathrm{Div}_{p^v/S}$$

(N.B. we put$P = P(F^v)$ hence $Q = P(F) = P^v$ with the notations of the previous Nos.) We call *a linear system of divisors over X/S parametrized by the projective fibration $O = P^v$ every family of divisors over X/S parametrized by O which is of degree one, i.e. defining $f\colon X - Z \to P$.* Therefore to such a linear system of divisors and if the fibers of P^v are $\neq \phi$[3] is associated to a rational map of X into a projective fibration. Indeed, even better as a rational map [illegible] "pseudo-morphisme rel: S". By the very construction $D \mid (X - Z) \underset{S}{\mathrm{x}} P^v$ is nothing else but the inverse image by $(f x \mathrm{id}_{P^v})$ of the canonical divisor (the incidence divisor) H over $P \underset{S}{\mathrm{x}} P^v$. Hence the knowledge of $f\colon X - Z \to P$ allows us to reconstruct at least the family of divisors of $X - Z$ induced by D so that if the family is without fixed points it is determined by the associated morphism $f\colon X \to P$. Let us note that we obtain evidently a one-to-one correspondence between linear systems of divisors X/S parametrized by P^v and morphisms $f\colon X \to P$ such taht if $(f \underset{S}{\mathrm{x}} \mathrm{id})^{-1}(H)$ is a relative divisor over $X \underset{S}{\mathrm{x}} P^v$ with respect to P^v. This condition can be verified fiber by fiber and we obtain:

Proposition 3.1. *We have a one-to-one correspondence between the linear systems without fixed points of divisors over X/S parametrized by P^v and the morp;hisms $f\colon X \to P$ having the following property: for everys $\in S$, denoting by k an algebraic closure of $k(s)$ and for every associated prime cycle X' of X_k, $f(X') \subset P_{\bar{k}}$ is not contained in any hyperplane of $P_{\bar{k}}$. (If X has geometrically integral fibers this can be stated simply by saying that $f(X_{\bar{k}})$ is not contained in any hyperplane of $P_{\bar{k}}$).*

In general (i.e. if $Z \not{\phi}$) we can no longer affirm that the knowledge of f determines the family of divisors. The most trivial case of that where$P^v = S$ is of relative dimension zero. To give a linear system of divisors of X/S parametrized by S is equivalent to giving a relative Cartier divisor D over X relative to S, the associated morphism is the projection

[2] Tr: make reference more precise

[3] Illegible

$X - D \to S$ and we see that the knowledge of this morphism (which includes knowing its domain of definition) does not determine D. In this case also let us suppose for simplicity that X is reduced the domain of definition of f considered as a rational map of X into $P = S$ is not $X - D$ but X. In order to eliminate this type of unpleasant phenomena we limit ourselves to linear systems of divisors "*without fixed components*". In general if $S = \operatorname{Spec}(k)$ to give a family (not necessarily linear) of divisors of $X \mid S$ parametrized by T we call a "fixed component" of the family every irreducible component of codimension one of the set Z of fixed points of the family; we say that the family is "without fixed component" if it is without fixed component, i.e. if $\operatorname{codim}(Z, X) \geq 2$. this terminology can be extended immeditely to the case where S is arbitrary by considering fiber by fiber. The property of being without a fixed component is evidently stable under base change.

Proposition. *Let us suppose $X \to S$ is flat locally of finite presentation with fibers (S_2) and let D be a linear system of divisors without fixed components over X/S parametrized by P^v. Then D is uniquely determined by the knowledge of the corresponding morphism $f\colon X - Z \to P$ (Z = set of fixed points) and even by the knowledge of the class of f as a "pseudo-morphism relative to S", $X - Z$ is the domain of definition of the said class.*

For this notation and the *sorite* of "pseudo-morphism relative to S" see section [20.10] of EGA IV.[4] We must prove that if D' is another linear system of divisors without fixed component parametrized by P^v defining $f\colon X - Z' \to P$ and if f and f' coincide on an open set $U(X - Z) \cap (X - Z')$ scheme–theoretically dense relative to S then $D = D'$. Indeed since P is separated over S we may take $U = (X - Z) \cap (X - Z') = Z - Z''$ where $Z'' = Z \cup Z'$. Since Z'' is of codimension ≥ 2 over each fiber and since X has (S_2) fibers, it follows that for every $x \in Z''$ the fiberX_S is of depth ≥ 2 at x. We may certainly conclude (using the fact that X is flat locally of finite presentation over S) that every divisor over X (not necessarily transversal to the fibers) is known once we know its restrictions to $X - Z$, which gives the wanted conclusion.

Let J be the ideal which defines D, it evidently suffices to show that $J \to i_*(J \mid X - Z'')$ is an isomorphism (where $i\colon X - Z'' \to X$ denotes the canonical immersion), now the homomorphism $J \to \vartheta_x$ can be reconstructed in effect by applying the functor i_8 to $J \mid X - Z'' \to \vartheta_x \mid x - Z''$. But since J is invertible, it is flat over S and $X \in Z'' \Rightarrow \operatorname{prof}_x J_s \geq 2$. It is enough, therefore, to prove the:

Lemma 3.1. *Let $X \to S$ be of finite presentation, let F be a module over finite presentation over X, flat relative to S, T a closed subset of X.*

Let us assume that for every $x \in X$ over $x \in S$ we have $\operatorname{prof} xF_s \geq 1$ (resp. $\operatorname{prof} xF_s \geq 2$). The canonical homomorphism $F \to i_*(F \mid X - T)$ is injective (resp. bijective), where $i\colon X - T \to T$ ($i\colon X - T \to X$ should be Tr) is the canonical immersion.

bf Proof of the lemma: We may suppose S, X to be affine and by a brief procedure we suppose that S is noetherian.

Then the hypothesis implies by Par. 6 that we have $\operatorname{prof} xF \geq \operatorname{prof} xF_s$ for every $x \in X$ over $s \in S$, thus $\operatorname{prof} xF_s \geq 1$ (resp. ≥ 2) if $x \in T$. We conclude therefore by paragraph 5 of EGA 5. (NB: Pour bien faire this lemma ought to be in paragraph 11 under the heading:

[4] Only 20.1–20.6 exists in EGA IV (Tr)

elimination of noetherian hypothesis...) (EGA IV see e.g. 11.3 [Tr]). It finally remains to verify the last assertion of Prop. 2 [illegible] that $X - Z$ is exactly the domain of definition of the rational map relative to S defined by f. Let $U \supset X - Z$ be its domain, it follows therefore from Proposition 1 that $U \to p$ is associated to a linear system of divisors D' over U/S parametrized by p^v and we have $D' \mid (X-Z)x_s p^v = d \mid (X-Z)x_s p^v$. Applying the uniqueness result (already proven) to D' and $D \mid Ux_s p^v$, we see that the two latter divisors are equal, thus $D \mid Ux_s p^v$ does not have fixed points, i.e. $Z \cap U = \phi$ thus $U = X - Z$. q.e.d.

I regret (I repent) to have given the proposition in a messed up ([Tr]: the original is in much more picturesque off-color French.) form half way between the classical hypothesis and natural hypothesis and without giving the converse this I propose to announce:

Proposition 3.3. *Let $X \to S$ be flat locally of finite presentation $Q = P^v = P(E^v)$ a projective fibration over S defined by a locally free module of finite type $F = E^v$. Let us consider the set ϕ of linear module of finite type $F = E^v$. Let us consider the set ϕ of linear systems D of divisors over X parametrized by Q such that the set Z of fixed points of D satisfies the property: $z \in Z$ implies prof$zO_{X_s} \geq 2$ (where s is the image of z in S). Let $\not{U}$ be the set of pseudo-morphisms f relative to S of X into P such that the domain of definition $U = \ldots f$ satisfies the condition $z \in X - U \Rightarrow z$ (profO_{X_s}) ≥ 2 and $f_u = f \mid u \colon u \to p$ satisfies the non-degeneracy included in 3. Let us consider the natural map $D \to f_D$ of ϕ into U then:*

- a) *This map is injctive and for $D \in \phi$ the set of fixed points Z is nothign else but the complement U of the domain of definition of f_D.*
- b) *Let $f \in \not{U}$ and let U be the open set over which f is defined and such that $z \in X - U$ implies profO_{X_s}, $z \geq 2$ for example $U = U(f)$, the domain of definition of f). In order that f should give a $D \in \phi$ it is necessary and sufficient that putting $L_U = f_U(O_P(1))$ (where $f_U \colon U \to P$ is the morphism induced by f) the module $i_*(L_U)$ over X is invertible (where $i \colon U \to X$ is the canonical immersion). We remark that if the fibers of X over S satisfy (S_2) for example if they are normal, see geometrically [voire Fr] normal, the depth condition considered over a closed set Z of X in the proposition simply means that for every $s \in S$, Z_s is of codimension ≥ 2 in X_s; ϕ is therefore the system of linear systems of divisors for P which are without fixed components. On the other hand, if $S =$ Spec(k) and if X is normal then for every rational map $f \colon X \to P$ the set of definition $U(f)$ satisfies codim$(X - U(f), X) \geq 2$ (II.7) so that in this case U is formed by the set of all the rational maps of X into P.*

The proof of a) has already been given. In order to prove b) let us note that the formation of $i_*(L_U)$ commutes with every flat extension S' of S(ref) at least if $u \to x$ is quasi-compact the case to which we reduce without difficulty so that the condition to consider is invariant under base change faithfully flat quasi-compact (qu cp). We take $S' = P^v$ and we note that the hypothesis that $i'_*(L'_U)$ is invertible does not change if we replace L'_U by $L'_U \underset{S'}{\mathrm{x}} M'$ where M' is an invertible module over S' so that

$$i_*\left(L'_U \underset{S'}{\mathrm{x}} M'\right) \simeq i'_*\left(L'_U \underset{S'}{\mathrm{x}} M'\right)$$

We take $M' = O_P^v$ so that the mentioned condition means also that $i'_*(N')$ is an invertible module where

$$N' = \left(f_u \underset{S}{\times} \mathrm{id}_P\right)^* (O_P \times P^v(1,1)).$$

But $0(1,1)$ is precisely the invertible module defined by the *canonical divisor* [(word usage PB)] H of $P^v \underset{S}{\times} P$ such that N is nothing else but the invertible module defined by $D' = (f_u \times \mathrm{id}_P)(H)$. If f gives $D \in \phi$, D' is nothing else but $D \mid U \underset{S}{\times} P^v$ therefore $N' = N \mid U \times P^v$ where N is the invertible module over $X \underset{S}{\times} P$ defined by D and it follows form Lemma 3.2 above applied to $P \underset{S}{\times} P^v \to P$ that we have $i_*(N') \simeq N$ therefore $i'_*(N')$ is invertible. Conversely, if this condition is satisfied we prove that f gives a $D \in \phi$ or what evidently reduces to the same thing that the divisor D' can be extended to a relative divisor with respect to P^v over $X \underset{S}{\times} P$. It reduces to the same to say that it extends to a divisor D over $X \underset{S}{\times} P$ since D will automatically be a relative divisor with base P^v as results from the fact that U contains elements associated to O_{X_S}, $s \in S$, a condition that is stable under base change and in particular by $s' = P^v \to S$. But it follows fimmediately from Lemma 4.2 above that D' extends to a divisor D if and only if D' extends to an invertible module ou encore $i'_*(N')$ is invertible. It would be necessary to extend to a divisor D if and only if D' extends to an invertible module ou encore $i'_*(N')$ is invertible. It would be necessary to edit the end of the proof in terms of necessary and sufficient condition (without referring to it twice as I did) and first of all release the:

Corollary 3.4 of Lemma 3. *2. Let us suppose that $g\colon X \to S$ is flat and locally of finite presentation. Let T be a closed subset of X such that $x \in T$ implies proof $O_{X_s} \geq 2$ (where $s = g(x)$) let $U = X - T$ and let $i\colon U \to X$ be the canonical immersion. For every locally free module of finite type L over X let us consider its restriction $L' = L \mid U$. Then*

- *a) the functor $L \to L'$ is fully faithful and for everyL the canonical homomorphism $L \to i_*(L')$ is an isomorphism. In order that L should be of rank n it is necessary and sufficient that L' should be such.*
- *b) Let L' be a locally free module over X then L' is isomorphic to a restriction of a locally free module L if and only if $i_*(L')$ is locally free.*
- *c) Let us suppose that L' is an invertible module associated to a divisor D' over U. Then the condition mentioned in b) is also necessary and sufficient in order that D' should be a restriction of a divisor D over X which will therefore be unique (and is equal to the scheme theoretic closure of D' in X). For D' to be a divisor relative with respect to S it is necessary and sufficient that D should be such. We simply use the fact that every L satisfies the announced hypothesis for F in Lemma 3.2.*

Corollary 3.5. *Let us assume that the local rings of X are factorial (for example X regular). Thent he map $\phi \to \cancel{U}$ is bijective. In particular if X is a regular prescheme locally of finite type over a field k and P is a projective fibration over k there is a one-to-one correspondence between the set ϕ of linear systems of divisors with no fixed components over X parametrized by P^v and the set U of rational maps of X into P that over k do not factor thorugh any hyperplane of P_k. Indeed since the local rings of X are factorial it follows that every invertible module over U extends to an invertible module over X*

so that the condition mentioned in b) is automatically satisfied. On the other hand, by Auslander-Buchsbaum a regular local ring is factorial.

Section 4
Linear Systems of Divisors and Invertible Modules

Using the results of (Section 1) No. 1, we shall give a complete description of linear systems over X in terms of invertible sheaves over X. We may evidently suppose that $P^v \to S$ is surjective, then $X \underset{S}{\times} P^v \to P^v$ is also such. It is also (Fr 144) and according to the generalities of 20.3 (Reference hard to locate, ask AG for help)[5] *to give a divisor D over $X \underset{S}{\times} P^v$ reduces to giving an invertible module N over $X \underset{S}{\times} P^v$ and a regular section ϕ of the latter. The assumption that D is a linear system of divisors over X parametrized by P^v can be expressed therefore by the two conditions*

1) *the $\phi_t (t \in P^v)$ induced by ϕ on the fibers of $X \underset{S}{\times} P^v$ over P^v are regular (which entails that ϕ is regular) and*
2) *the $N_x (x \in X)$ induced by N on the fibers of $X \underset{S}{\times} P^v$ over X are of degree 1. However [(or)]*[6] *to give an N invertible over the projective fibration $X \underset{S}{\times} P^v$ over X satisfying the condition 2) above is equivalent due to No. 1 to giving an invertible module L over X, N being determined as a function of L by $N = Lx_{O_S} O_{P^v}(1)$ and L being furthermore determined in terms of N by $L \approx \mathrm{pr}_{1_*}(N(-1))$ where (-1) denotes the tensoring with $O_{P^v}(-1)$ over O_S.*

To give ϕ reduces to giving a section of $L \times O_P(1)$, i.e. a section of

$$\mathrm{pr}_{1_*}(L \otimes_{\mathcal{O}_x} \mathcal{O}_{Xxp^v}(1)) = L \otimes_{\mathcal{O}_s} \mathrm{pr}_{1_*}(\mathcal{O}_{Xxp^v}(1))$$

but because of III.2 we have $\mathrm{pr}_{1_*}(\mathcal{O}_{Xxp^v}(1)) = E_X^v$ so that to give ϕ is equivalent to giving a morphism $g^*(E) \to L$ or, what is the same, a morphism $u \colon E \to g_*(L)$ (N.B.$g \colon X \to S$ is the canonical projection). It remains to explain the condition 1) above in terms of u. Since the constructions tht we made commute with base change it suffices to express this condition fiber by fiber and take into account that the points of P with value sin an extension k of $k(s)$ correspond exactly to straight lines in $E(s)\ x_{k(s)}k$ this condition can be expressed simply by requiring that for every $t \in E(s)$ the corresponding section $u(s)$ of L_s over X_s should be regular and that the analogous condition should be verified after every extension of the base field. We see as usual tht it suffices to test this condition over an algebraic closure of k. To summarize:

Proposition 4.1. *Let $g \colon X \to S$ be a flat morphism locally of finite presentation. Let $P = P(E)$ be a projective fibration over X defined by E locally free of finite type, everywhere $\neq 0$, i.e., P has non-empty fibers, $P^v = P(E^v)$. Then there is a bijective correspondence between the set of linear systems of divisors over $X \mid S$ parametrized by P^v and the set of couples (up to isomorphism) (L, u) where L is an invertible module voer X and $u \colon E \to g_*(L)$ is a homomorphism such that for every $s \in S$ and for every point t of $E(s)x_{k(s)}k$ (for any extension k of $k(s)$ which we may suppose to be the algebraic closure of $k(s)$) the corresponding section $u(t)$ of $L_{s\,k}$ over $X_{s\,k}$ should be regular.*

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[5] Ask A.G., is it EGA 21.3 [Tr]?
[6] [Fr]

We note that if the fibers of X are geometrically integral this condition on u means simply that for every $s \in S$, $u(s): E(s) \to H^0(X_s, L_s)$ is injective a fact that we would also have to make explicit in 4.1 we would also have to recall (for convenience of reference) the construction of the divisor D in terms of (L, u) as the divisor of the evident section ϕ of $L \underset{O_S}{\mathrm{x}} O_{P^v}(1)$ defined by u.

Corollary 4.2. *Let us assume that $f: X \to S$ is proper flat and of finite presentation and with integral geometric fibers. Let L be an invertible module over X and $P = P(E)$ a projective fibration over S as in 5.1. There exists a module Q of finite presentation over S and an isomorphism of functors of the quasi-coherent O_S-module F: $Hom(Q, F) \to g_*(L \underset{O_S}{\mathrm{x}} F)$. Once this is assumed,*[7] *the linear systems of divisors on X parametrized by P^v and associated to L in the sense of 4.1 correspond bijectively to* surjective *homomorphism $Q \to E^v$ modulo multiplication by a section of O_S^*. the existence of Q is reduced by a brief procedure to the case of S noetherian and in this case it is nothing else but III.7.7.6 of EGA III [Tr] (the hypothesis about the fibers of X being anyway useless). Since E is locally free of finit etype, to give a homomorphism $E \to f(L)$ is equivalent to giving a section of $L \underset{O_S}{\mathrm{x}} E^v$ therefore to giving a homomorhism of $Q \to E^v$. It remains to express that the condition mentioned in 4.1 is really erified, which (due to the hypothesis made about the fibers of X/S) is reduced to verifying that fiber by fiber the corresponding homomorphism*

$$E(s) \to H^0(X_s, L_s) \simeq Hom_{k(s)}(Q(s), k(s))$$

is injective or again[8] *$Q(s) \to E^v(s)$ is surjective which by Nakayama means also that $Q \to E^v$ is surjective. The "modulo sections of O_S" (or O_{S^*} P.B.) becomes "modulo isomorhisms" in 4.1. We may interpret 4.1 in another way by using the fact that $P(Q)$ represents the subfunctor of $\mathrm{Div}_{X/S}$ define dby L by virtue of No. 2. Consequently a linear system of divisors parametrized byP^v and associated to L is interpreted as a morphism $P^v = P(E) \to P(Q)$ the linear character of the family of divisors defined by such a morphism can be interpreted therefore by the fact that this morphism should be "linear", i.e. precisely defined by a surjective morphism of $Q \to E^v$. We see also in this case that the morhism $P^v \to \mathrm{Div}_{S/S}$ is a monomorphism (since $P^v \to P(Q) \to \mathrm{Div}_{S/S}$ is such) a fact anyway more general cf. corollary below. Let us therefore agree to say that two linear families of divisors of $X \mid S$ parametrized by the projective fibers P^v, $P^{v'}$ are isomorphic if they are transformed one into another by an S-isomorphism $P^v \to P^{v'}$ (which will be anyway unique due to the fact that we have a monomorhisms (?) in (into) $\mathrm{Div}_{S/S}$). We may therefore express 4.2 by saying that the set of classes up to isomorhism of linear systems of divisors over X associated to L is in bijective canonical correspondence with the set Grass(Q) (S) and this correspondence is compatible with any base change. We see that the functor: $S' \to$ set of classes (mod isomorphism) of linear systems of divisors of $X_{S'} \mid_{S'}$ associated to $L_{S'}$ is representable by the S-prescheme Grass(Q).*

(Marginal Remarks Hard to Read, P.B.) [illegible ask AG]

[7] Ceci pose [Fr]
[8] ou encore

We should make explicit in 4.1 tht $L \mid X - Z$ is canonically isomorphic to $f^*(O_P(1))$ (with the notations of the previus No.) so tht in this case $D \in \phi$ mentione din 3.3, $L \mid X - Z$ is nothing else but the canonical and unique extension of $f^*(O_P(1))$ to an invertible sheaf over X.

Proposition 4.3. *Let D be a linear system of divisors over $X \mid S$ parametrized by P^v where $g: X \to S$ is a flat morphism of finite presentation.*

- *a) Let us suppose that g is of finite presentation and that for every $x \in S$ if we denote by k an algebraic closure of $k(s)$ then there eixsts a prime cycle T associated to X_k such that $k \to H^0(T, O_T)$ should be an isomorhism (a conditioni automatically satisfied if g is proper and surjective). Then the morphism $D: P^v \to Div_{S/S}$ is a monomorphism.*
- *b) Let us consider the map $u \to D \circ u$ of $Aut_S(P^v)$? into the set of families of divisors over X/S parametrized by P^v. Then if g is surjective the previous map is injective in particular $D = Du$ implies $u = id_{P^v}$ more generally the morphism of functors $Aut_S(P^v) \to$ Sys Lin $div_{X/S}$, P^v is a monomorphism.*

We note that under the hypothesis of a), b) is a trivial consequence of a); on the other hand, b) is valid under less restrictive assumptions than a). We point out that a) becomes false if we abandon the restrictive hypothesis that we have made: take for example $S = \operatorname{Spec} k$, X an open subset of P^1_k not containing two distinct points a, b of $P^1_k(k)$. Then the two points a and b define the same divisors of X (savoir the zero divisor [Fr]) without being identical.

Let is assume first of all that S is the spectrum of a field k which we can evidently (by a "descent") assume to be algebraically closed. Let T be as in a) and we given it the induced reduced structure, we have then a morphism ("induced divisor")

$$\mathrm{Div}_{X/k} \to \mathrm{Div}_{T/k}$$

and it suffices to show that the composition

$$P^v \to \mathrm{Div}_{T/k}$$

is a monomorphism. Since the latter is again a linear system of divisors, we are reduced to the case $X = T$, thus to the case where $H^0(X, \mathcal{O}_X) \xleftarrow{\sim} \mathcal{O}_S$. Thus for every S over k we have

$$g_{S*}(\mathcal{O}_{X_S}) \xleftarrow{\sim}$$

thus if L over X and $u: E \to$??? are as in 4.1(???) if two sections ϕ andψ of E_S everywhere non-zero are such that $u(\phi)$ and $u(\psi)$ are sections of L_S over X_S having the same divisor then they are deduced from each other by multiplication by an invertible section of $\mathcal{O}_{X_S}$,it follows that ψ is deduced from ϕ by multiplication by an invertible section of $\mathcal{O}_S$ thus ϕ and ψ define the same point of P^v with values in S. Since every point of P^v with values in S is defined locally over S by a section ϕ of E_S which does not vanish (cf. Chap I) a) follows. To prove b) we note the:

Lemma.

Let D be a linear system of divisors over X non-empty and locally of finite type over k algebraically closed parametrized by $P^v(E)$ and let us consider the corresponding morphism

$$f\colon X - Z \longrightarrow P$$

where Z is the base locus (set of fixed points in the original [Tr]). Then if $r = \mathrm{rank}_k E > 0$ there exist $r+1$ points $x_i 1 \leq i \leq r+1$ of $X(k) - Z(k)$ such that the $f(x_i)$ give a "projective base" of P, i.e. such that for every subset $\mathcal{J}$ of $[1, r+1]$ having r elements the $f(x_i)$ are not containe din any hyperplane of P.

We may evidently suppose that $Z = \phi$. Since by 4.1 (ref ??? [Tr]) $f(X)$ is not contained in any hyperplane of P we conclude from the beginning the existence of r points $(1 \leq i \leq r)$ such that the $f(x_i)$ are projectively independent in P, i.e. are defined by linearly independent forms over E. It remains to prove that there exists an $x_{r+1} = x$ in $X(k)$ such that $f(x)$ is not in any of the r hyperplanes H_i defined by the system of $(r-1)$ from among the $f(x_i)$. But in the contrary case taking into account the "sorites" [Fr] of Jacobson we would have

$$f(X) \subset \bigcup_i H_i$$

thus if X_0 is an irreducible component of X then $f(X_0)$ would be contained in one of the H_i which contradicts 3.1 or 4.1 (ref??? [Tr]). This being established, to prove b) we may evidently suppose $Z = \phi$ and using 3.1 or 4.1 [Tr] we are reduced to proving that an automorphism u of P^v is determined if we know the composition of its contragradient u^v in P with $f\colon X \to P$ and that the analogous assertion is true after every base change $S \to \mathrm{Spec}(k)$ by an automorphism u of P^v_S. But this results immediately from the previous lemma and from the determination of automorphisms $P^v(E) = P(E^v)$ done in section (or number [Tr]) one, which implies that the effect of an automorphism of a projective fibration over an S is known (relative to a module (Module [Fr]) locally free of finite type) if we know its effect on a projective basis in each fiber.

Let us now summarize the general case: S arbitrary. Of course, after a base change over S we are reduced in a) to proving that any two sections of P^v over S which define the same divisor over X are identical and in b) to proving that any two automorphisms of P^v which are such that $D \circ u = D \circ v$ are identical. We may suppose that S is affine, the case b) where we do not suppose expressly that g is of finite presentation but g is surjective we reduce ourselves immediately (due to the fact that g is open) to the case where X is also affine thus of finite presentation over S. By a brief procedure we reduce to the case of S being noetherian.

Now for a noetherian base scheme S and for a morphism of functors $F \xrightarrow{h} G$ over S (F and G are the functors $(\mathrm{Sch}/S) \to (\mathrm{Ens})$) we have very general criteria which will be made explicit in Ch. V which allow to affirm that if for every $s \in S$ the corresponding morphism $F_s \to G_s$ is a monomorphism then $F \to G$ is a monomorphism (NB we put $F_s = Fx_S S_{k(s)}$ and the same for G_s), making simple assumptions about F and G (verified for example if F and G are both representable by preschemes of finite type over S, but in the case in hand only the first functor is representable à priori). We will summarize the argument of Ch. V in the two particular cases which are of interest to us here. We have two sections u, v (of P^v respectively of the projective group $GP(E^v)$) of a prescheme of finite type F over S about which we want to prove that they are equal. To do this it is clearly sufficient to prove that they are equal after the base change

$$\mathrm{Spec}(\mathcal{O}_S, s/\mathfrak{M}^{n+1}) \to S,$$

which reduces us to the case of S artinian and local. $S = \mathrm{Spec}\, A$. We proceed by induction on the integer n such that $\mathfrak{M}^{n+1} = 0$ which allows us to suppose tht the two sections u v are equal modulo $\mathfrak{M}^n$. Then one is induced from the other by means of an element δ of

$$\mathrm{Hom}_k(u_0^*(\Omega^1_{F_0/k}), V)$$

where $k = A\mathfrak{M}$ is the residual field, $F_0 = F \otimes_A k$ is the reduced fiber $V = \mathfrak{M}^n = \mathfrak{M}^n/\mathfrak{M}^{n+1}$. It suffices to prove that $\delta = 0$ using the hypothesis $h(u) = h(v)$.

The general principle of verification is as follows: to start with we express that $h(u)$ and $h(v)$ coincide modulo $\mathfrak{M}^n$ we see that their "difference" can be written as an element δ' of $\mathrm{Hom}_k(w_0^*(\Omega^1_{G_0/k}, V)$ where $w_0 = h_0(u_0) = h_0(v_0)$ and where $G_0 = G \times_A k$, this element is nothing else but the one deduced from δ by composition with the natural homomorphism

$$h_0^*\colon w_0^*(\Omega^1_{G_0/k}) \longrightarrow u_0^*(\Omega^1_{F_0/k})$$

deduced from $h_0\colon F_0 \to G_0$. Since $h(u) = h(v)$ thus $\delta' = 0$ the composition of δ with the preceding homomorphism h_0^* is zero so that we see that h_0 is surjective it follows that $\delta = 0$ and we are done. Now the fact that $h_0\colon F_0 \to G_0$ is a monomorphism thus inducing a monomorphism for the set of points with value sin the dual numbers over k implies that indeed h_0^* is surjective (its transpose being injective). This reasoning is valid since G is representable which is however not the case in the case that we consider. We can however define a vector bundle $\mathcal{G}_{w_0}$ over k playing the role dual to $w_0^*(\Omega^1_{G_0/k})$ (illegible) tangent to G_0 at w_0 by expressing the "deviation" of two points of G which coincide modulo $\mathfrak{M}^n$ as an element of $\mathcal{G}_{w_0} \otimes_k V$. This is essentially straightforward and is contained in the systematic developmentsof par. 26 (? Infinitesimal extensions) which we review here. In the case a) G is the functor $\mathrm{Div}_{X/S}$ w_0 corresponds to a Cartier divisor D_0 over $X_0 = X \otimes_A k$ and we have to take $\Omega = H^0(D_0, n_{D_0/X_0})$ where $n$is the normal sheaf to D_0 in X_0, isomorphic also to the induced sheaf on D_0 by $\mathcal{O}_{X_0}(D_0)$ on D_0. In the case b) we may suppose that D has no fixed points and it is more convenient to interpret the situation in terms of morphisms into P (see 4.1) so that G becomes the functor

$$\mathrm{Hom}_S(X, P)$$

and G_{w_0} should be the space

$$\mathrm{Hom}_{\mathcal{O}_{X_0}}(f_0^*(\Omega^1_{P_0/k}, \mathcal{O}_{P_0})$$

In both cases we have a natural homomorphism

$$G_{u_0} \otimes_k V \to G_{w_0} \otimes_k V$$

(where G_{u_0} is the dual fo $u_0^*(\Omega^1_{F_0/k})$) expressing the passage from the deviation δ to the corresponding deviation δ' by the mapping h and the injectivity of this mapping results from the injectivity of $G_{u_0} \to G_{w_0}$ which 'elle provient du fair [Fr] that $h_0 \colon F_0 \to G_0$ is a monomorphism.

Practially it does not seem possible to write up that last part of the proof without referring to the small calculations of paragraph 25 (which it is out of the question to redo here in the particular case). We note that this does not give rise to a vicious circle since the par. 25 and the calculations that we have developed only depend on the rewrite of differential calculus from par. 16 and also 4.3 will not be used gain in Ch. IV except perhaps in the two following numbers or sections [Editors Note: Did Grothendieck intend this part as fragment of EGA IV, this seems very likely].

The interest of 4.3 a) is to prove that under the stated conditions the parametrizing projective fibration can be interpreted intrinsically the notion of class (up to an isomoprhism over the projective fibration parametrizing ???) of the linear system over X/S as being a subfunctor P^v of $\mathrm{Div}_{X/S}$ which satisfies certain properties (savoir [Fr] is representable by a projective fibration and the family of divisors defined by the canonical injection of the latter into $\mathrm{Div}_{S/S}$ is linear in the sense of No. 3), which is essentially the classical point of view (where a linear system of divisors is defined as a *set* of divisor ssatisfying certain conditions, compare 4.4). On the other hand 4.3 b) is equivalent to saying that if g is surjective then if two linear system sof divisors over X/S parametrized by two projective fibrations $P^v(P^v)'$ are isomorphic then there exists a unique isomoprhismf rom P^v to $(P^v)'$ (compatible with D and D') we may therefore say that a class (up to isomoprhism) of linear systems over S/X determines it parametrizing projective fibration *up to a unique isomoprhism.* Technically this result will allow us (once we have the descent theory of Chapter V [Editor - not yet written also numbering is of only historical interest]) to make *the faithfully flat descent for linear systems of divisors* – under the reservation always however to allow also as parametrizing fibrations "the twisted projective fibrations" which will be done in a future section.

Descending again to the earth, and even lower, to explain in vulgar terms the notion of a linear system we place ourselves for simplicity over a box field (although the statement will hold essentially as such over an affine base)

Proposition 4.4. *Let X be a prescheme of finite type over a field k, such that*

$$k \longrightarrow H^0(X, \mathcal{O}_X)$$

is an isomoprhism. To every linear system D of divisors over X parametrized by a projective fibrtion P^v over k we associate the set (!) Ens(D) of all the divisors over X of the form $D(t)$ where $t \in P^v(k)$

- *a) If d' is another linear system of divisors over X parametrized by a projective fibration $(P^v)'$ then D and D' are isomorphic if and only if Ens$(D) =$ Ens(D').*
- *b) Suppose k alg. closed or X geometrically integral.*
  In order that the set Δ of positive Cartier divisors over X should be of the form Ens(D) it is necessary and sufficient that there should exist a k-subspace of the vector space E of meromorphic functions on X such that for every $\phi \in E - (0)$ ϕ shuld be regular, i.e. div(ϕ) is defined and that Δ should be the set of div(ϕ) for $\phi \in E - (0)$

c) *Let E, E' be two k-vector subspaces of the meromorphic functions on X satisfying the assumption of* b) *then the sets of divisors Δ, Δ' defined by them are equal iff there exists a regular pseudo-function ϕ over X such that*

$$E' = \phi E.$$

If $E \neq (0)$, i.e. $\Delta \neq \phi$ such a ϕ is defined modulo multiplication by an element of k^8.

The proof is an easy exercise using 4.1 and I dispense with writing down the proof except if you protest this. In addition, it seems to me that 4.4 could profitably come before 4.3, being technically more trivial. Note also that if X is geometrically integral the condition on E stated in b) becomes *void*. The restriction made at the beginning of b) is attached to the fact that otherwise the condition announced for b) may not be true after passing to the alg. closure of k (it is easy to give examples) in every characteristic even if k is separably closed in char $p > 0$. For good measure we would have to announce b) without supplementary conditions on k or X but by announcing the condition over E and by passing to the algebraic closure of k (and noting that if X is geometrically integral this condition becomes void). By abuse of language, a *set* Δ of divisors of the form $\mathrm{Ens}(D)$ will be often called a linear system of divisors on X (slightly illegible)