Mathematical Economics

Convex Analysis (I)

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Convex Sets

- A set $X \subseteq \mathbb{R}^n$ is said to be ***convex*** if

$$\forall x, y \in X, \forall \lambda \in [0, 1], \lambda x + (1-\lambda) y \in X.$$

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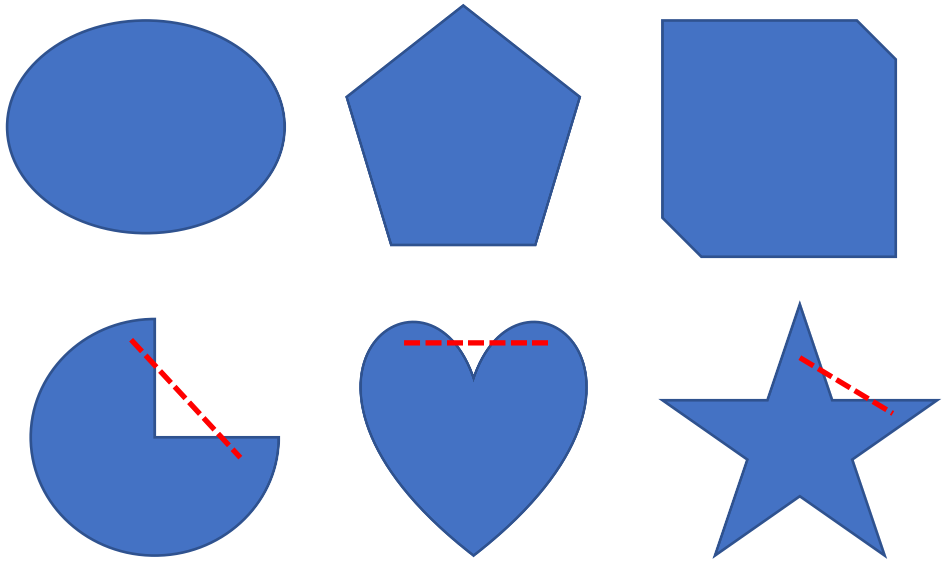
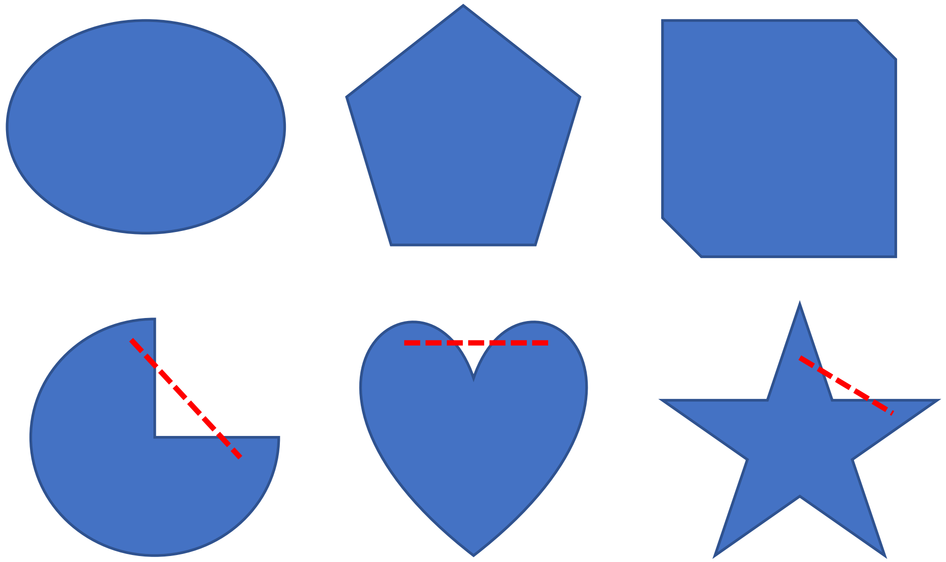


Convex Sets

- A set $X \subseteq \mathbb{R}^n$ is said to be ***convex*** if
$$\forall x, y \in X, \forall \lambda \in [0, 1], \lambda x + (1-\lambda)y \in X.$$
- For any $x_1, \cdots, x_m \in X$ and $\lambda_1, \cdots, \lambda_m \in [0, 1]$ with $\sum_{k=1}^{m} \lambda_k = 1$, the vector $\sum_{k=1}^{m} \lambda_k x_k$ is called a ***convex combination*** of the vectors $x_1, \cdots, x_m$.
- A set $X \subseteq \mathbb{R}^n$ is ***convex*** iff $\forall x_1, \cdots, x_m \in X$, $\forall\ \lambda_1, \cdots, \lambda_m \in [0, 1]$ with $\sum_{k=1}^{m} \lambda_k = 1$, $\sum_{k=1}^{m} \lambda_k x_k \in X$.



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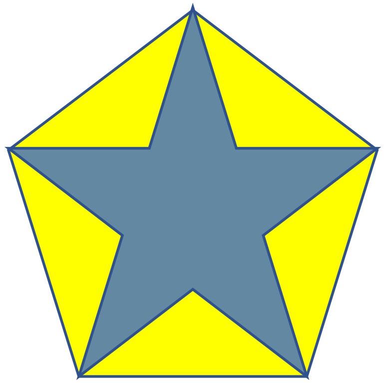
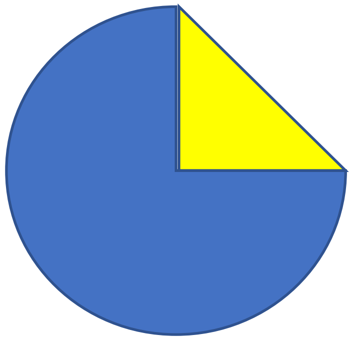
- A set $X \subseteq \mathbb{R}^n$ is ***convex*** iff $\forall x_1, \cdots, x_m \in X, \forall \lambda_1, \cdots, \lambda_m \in [0, 1]$ with $\sum_{k=1}^{m} \lambda_k = 1$, $\sum_{k=1}^{m} \lambda_k x_k \in X$.
- For any set $A \subseteq \mathbb{R}^n$, we define the ***convex hull*** of A, denoted by $\mathrm{Co}(A)$, as the set of all convex combinations of elements in A:

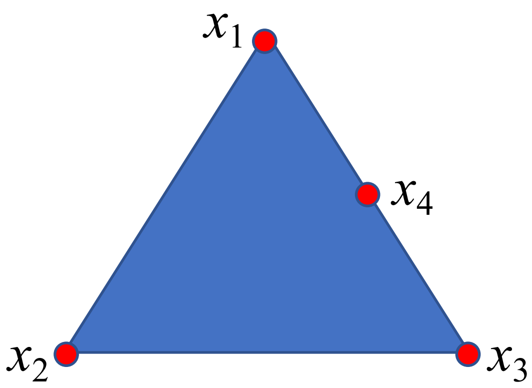
$$\mathrm{Co}(A) = \{a \in \mathbb{R}^n \mid \exists x_1, \cdots, x_m \in A, \exists \lambda_1, \cdots, \lambda_m \in [0, 1] \text{ with } \sum_{k=1}^{m} \lambda_k = 1, a = \sum_{k=1}^{m} \lambda_k x_k\}.$$

- A set A is convex iff $A = \mathrm{Co}(A)$. The convex hull of A is the *smallest* convex set containing A.





Extreme Points



- An ***extreme point*** of a convex set $X \subseteq \mathbb{R}^n$ is a point in X that cannot be expressed as a convex combination of distinct points, i.e., z is an ***extreme point*** of X if $z \in X$ and there are no x and y in X and λ in $(0, 1)$ s.t. $x \neq y$ and $z = \lambda x + (1-\lambda)y$.
- For any $a, b \in \mathbb{R}$ with $a < b$, $[a, b]$ has two extreme points a and b, (a, b) has no extreme points, and $(a, b]$ has one extreme point b.
- *An extreme point of a convex set must be a boundary point.* Thus $\forall x \in \mathbb{R}^n$, $\forall r \in \mathbb{R}_{++}$, an open ball $B_r(x)$ has no extreme points, and for the closed ball $\mathrm{Cl}(B_r(x))$ every boundary point is an extreme point. But not all boundary points of a convex set are extreme points.
- For any convex set $X \subseteq \mathbb{R}^n$, let

$$\mathrm{Extr}(X) = \{z \in X \mid z \text{ is an extreme point of } X\}.$$

The Carathéodory Theorem: Let $X \subseteq \mathbb{R}^n$. If $x \in \mathrm{Co}(X)$, then x can be expressed as a convex combination of no more than $n+1$ points in X.

The Minkowski-Krein-Milman Theorem: Every compact convex set in $\mathbb{R}^n$ is the convex hull of its extreme points.

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The Minkowski-Krein-Milman Theorem: Every compact convex set in $\mathbb{R}^n$ is the convex hull of its extreme points.

Corollary: Let $X \subseteq \mathbb{R}^n$ be compact and convex. Then any $x \in X$ can be expressed as a convex combination of no more than $n+1$ extreme points in X.

Connected Sets

- Two subsets A and B of $\mathbb{R}^n$ are said to be ***separated*** if
$$A \cap \mathrm{Cl}(B) = \mathrm{Cl}(A) \cap B = \emptyset.$$
- Two disjoint open sets must be separated.
- Separated sets are disjoint, but disjoint sets need not be separated: [0, 1] and (1, 2) are not separated, but (0, 1) and (1, 2) are separated.
- A set $X \subseteq \mathbb{R}^n$ is said to be ***connected*** if X is not a union of two nonempty separated sets.

Theorem 1: Let $X \subseteq \mathbb{R}$. Then X is connected iff $\forall x, y \in X$ with $x < y$, $\forall z \in (x, y)$, $z \in X$.

- See Exercise 1.
- In $\mathbb{R}$, a set is connected iff it is an interval (open, closed, or half-open; bounded or unbounded), iff it is convex.

- For any $A \subseteq \mathbb{R}^n$ and $r \in \mathbb{R}$, let
$$rA = \{x \in \mathbb{R}^n \mid \exists a \in A, x = ra\}.$$
- For any $A_1, A_2 \subseteq \mathbb{R}^n$, let
$$A_1 + A_2 = \{a_1 + a_2 \in \mathbb{R}^n \mid a_1 \in A_1, a_2 \in A_2\}.$$
- Example: Let $A = [1, 2]$, $B = [3, 4]$, and $r = 5$. Then
$$rA = [5, 10], rB = [15, 20], A + B = [4, 6].$$
- For any $A_1, \cdots, A_k \subseteq \mathbb{R}^n$, let
$$A_1 + \cdots + A_k = \{a_1 + \cdots + a_2 \in \mathbb{R}^n \mid a_1 \in A_1, \cdots, a_k \in A_k\}.$$

Theorem 2:

(1) $\forall A \subseteq \mathbb{R}^n$, $\mathrm{Co}(A) = \bigcap \{B \subseteq \mathbb{R}^n \mid A \subseteq B \text{ and } B \text{ is convex}\}$.

(2) If $\forall t \in T$, $A_t \subseteq \mathbb{R}^n$ is convex, then $\bigcap_{t\in T} A_t$ and $\times_{t\in T} A_t$ are convex.

(3) If $A \subseteq \mathbb{R}^n$ is convex, then $\forall r \in \mathbb{R}$, rA is convex.

(4) If $A \subseteq \mathbb{R}^n$ is open, then $\mathrm{Co}(A)$ is open.

(5) If $A \subseteq \mathbb{R}^n$ is compact, then $\mathrm{Co}(A)$ is compact.

(6) If $A \subseteq \mathbb{R}^n$ is convex, then $\mathrm{Int}(A)$ and $\mathrm{Cl}(A)$ are convex.

(7) If $A \subseteq \mathbb{R}^n$ is convex, then A is connected.

(8) If $A_1, \cdots, A_k$ are convex, then $A_1 + \cdots + A_k$ is convex.

(9) For any $A_1, \cdots, A_k \subseteq \mathbb{R}^n$,

$$\mathrm{Co}(A_1 + \cdots + A_k) = \mathrm{Co}(A_1) + \cdots + \mathrm{Co}(A_k).$$

(10) If none of the vectors $x_1, x_2, \cdots, x_m \in \mathbb{R}^n$ is a convex combination of the other vectors, then

$$\mathrm{Extr}(\mathrm{Co}\{x_1, \cdots, x_m\}) = \{x_1, \cdots, x_m\}.$$

Let $A = \{x \in \mathbb{R}^2 \mid |x_1| \geq 1;\ x_2 \geq 1/|x_1|\}$.
Then A is closed, but

$$\mathrm{Co}(A) = \{x \in \mathbb{R}^2 \mid x_2 > 0\}$$

is not closed.

- See Exercise 1.

The Intermediate Value Theorem: Let $X \subseteq \mathbb{R}^n$ be convex and $f: X \rightarrow \mathbb{R}$ continuous. If $\exists a, b \in X$, $f(a) < f(b)$, then $\forall z \in (f(a), f(b))$, $\exists \lambda \in (0, 1)$, $f(\lambda a + (1 - \lambda)b) = z$.

The Mean Value Theorem: Let $X \subseteq \mathbb{R}^n$ be open and convex, and $f: X \rightarrow \mathbb{R}$ differentiable. Then $\forall a, b \in X$, $\exists \lambda \in (0, 1)$,

$$f(a) - f(b) = Df(\lambda a + (1 - \lambda)b) \cdot (a - b).$$

Concave and Convex Functions

For any nonempty convex set $X \subseteq \mathbb{R}^n$, a function f: $X \to \mathbb{R}$ is

- ***concave*** if $\forall x, y \in X$, $\forall \lambda \in [0, 1]$, we have
$$f(\lambda x + (1-\lambda)\, y) \geq \lambda f(x) + (1-\lambda) f(y).$$
- ***convex*** if $\forall x, y \in X$, $\forall \lambda \in [0, 1]$, we have
$$f(\lambda x + (1-\lambda)\, y) \leq \lambda f(x) + (1-\lambda) f(y).$$
- ***strictly concave*** if $\forall x, y \in X$ with $x \neq y$, $\forall \lambda \in (0, 1)$, we have
$$f(\lambda x + (1-\lambda)\, y) > \lambda f(x) + (1-\lambda) f(y).$$
- ***strictly convex*** if $\forall x, y \in X$ with $x \neq y$, $\forall \lambda \in (0, 1)$, we have
$$f(\lambda x + (1-\lambda)\, y) < \lambda f(x) + (1-\lambda) f(y).$$
- Obviously, f is concave iff $-f$ is convex, and f is strictly concave iff $-f$ is strictly convex.

Theorem 3: For any nonempty convex set $X \subseteq \mathbb{R}^n$, if f: $X \to \mathbb{R}$ is both concave and convex, then for all x, $y \in X$ and $\lambda \in \mathbb{R}$ with $\lambda x + (1-\lambda)y \in X$, we have

$$f(\lambda x + (1-\lambda)y) = \lambda f(x) + (1-\lambda)f(y).$$

- That is, f is both concave and convex iff f is an ***affine*** function. Thus a ***linear*** function must be both concave and convex. (See Exercise 2.)

Theorem 3: For any nonempty convex set $X \subseteq \mathbb{R}^n$, if $f: X \rightarrow \mathbb{R}$ is both concave and convex, then for all $x, y \in X$ and $\lambda \in \mathbb{R}$ with $\lambda x + (1-\lambda)y \in X$, we have

$$f(\lambda x + (1-\lambda)y) = \lambda f(x) + (1-\lambda)f(y).$$

- Let $z = \lambda x + (1-\lambda)y$.

(1) If $\lambda \in [0, 1]$, then we are done.

(2) If $\lambda > 1$, then let $t = 1/\lambda \in (0, 1)$, and thus $x = tz + (1 - t)y$, which means that $f(x) = f(tz + (1-t)y) = tf(z) + (1-t)f(y)$. Hence we have

$$f(\lambda x + (1-\lambda)y) = f(z) = 1/tf(x) + (1-1/t)f(y) = \lambda f(x) + (1-\lambda)f(y).$$

(3) If $\lambda < 0$, then $y = (1/(1-\lambda))z + (-\lambda/(1-\lambda))x$, implying that

$$f(y) = f((1/(1-\lambda))z + (-\lambda/(1-\lambda))x) = (1/(1-\lambda))f(z) + (-\lambda/(1-\lambda))f(x).$$

Hence we have $f(\lambda x + (1-\lambda)y) = f(z) = \lambda f(x) + (1-\lambda)f(y)$.

Subgraph and Epigraph

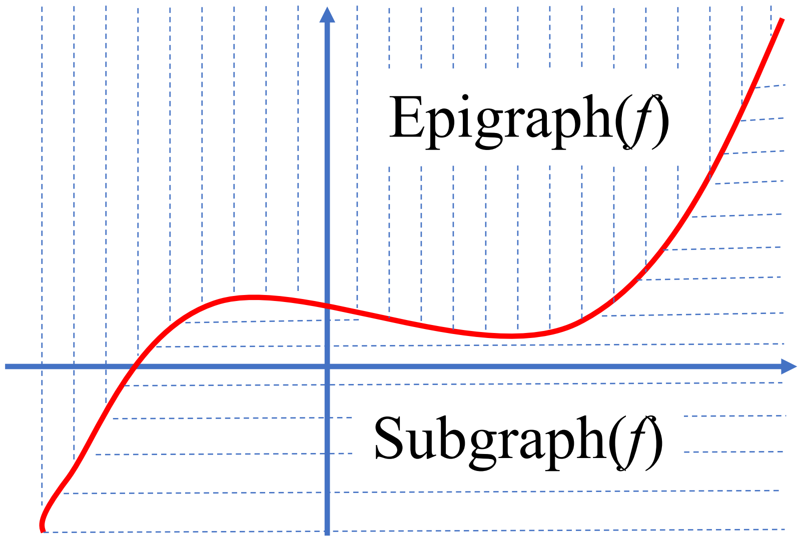
For any set $X \subseteq \mathbb{R}^n$ and any function $f: X \to \mathbb{R}$,

- the ***subgraph*** of f is defined to be

  $$\text{Subgraph}(f) = \{(x, y) \in X \times \mathbb{R}: f(x) \geq y\};$$

- the ***epigraph*** of f is defined to be

  $$\text{Epigraph}(f) = \{(x, y) \in X \times \mathbb{R}: f(x) \leq y\}.$$



Theorem 4: For any convex set $X \subseteq \mathbb{R}^n$,

- $f: X \to \mathbb{R}$ is concave iff its subgraph is convex;
- $f: X \to \mathbb{R}$ is convex iff its epigraph is convex.

Theorem 4: For any convex set $X \subseteq \mathbb{R}^n$,

- f: $X \to \mathbb{R}$ is concave iff its subgraph is convex;
- f: $X \to \mathbb{R}$ is convex iff its epigraph is convex.

- If f: $X \to \mathbb{R}$ is concave, then $\forall(x, y), (a, b) \in$ Subgraph(f), we have $f(x) \geq y$ and $f(a) \geq b$. Hence $\forall\lambda \in [0, 1]$, $f(\lambda x + (1-\lambda)a) \geq \lambda f(x) + (1-\lambda)f(a) \geq \lambda y + (1-\lambda)b$, which means that $(\lambda x+(1-\lambda)a, \lambda y+(1-\lambda)b) = \lambda(x, y)+(1-\lambda)(a, b) \in$ Subgraph(f). Therefore, Subgraph(f) must be convex.
- Let Subgraph(f) be convex. $\forall x, a \in X$, we have $(x, f(x)) \in$ Subgraph(f) and $(a, f(a)) \in$ Subgraph(f), and thus $\forall\lambda \in [0, 1]$, $\lambda(x, f(x))+(1-\lambda)(a, f(a)) = (\lambda x+(1-\lambda)a, \lambda f(x)+(1-\lambda)f(a)) \in$ Subgraph(f), which means that $f(\lambda x+(1-\lambda)a) \geq \lambda f(x) + (1-\lambda)f(a)$.

Theorem 5: Let $X \subseteq \mathbb{R}^n$ be open and convex and $f: X \to \mathbb{R}$ continuously differentiable. Then

(i) f is concave iff $Df(x) \cdot (y - x) \geq f(y) - f(x)$ for all $x, y \in X$;

(ii) f is convex iff $Df(x) \cdot (y - x) \leq f(y) - f(x)$ for all $x, y \in X$;

(iii) f is strictly concave iff $Df(x) \cdot (y - x) > f(y) - f(x)$ for all $x \neq y \in X$;

(iv) f is strictly convex iff $Df(x) \cdot (y - x) < f(y) - f(x)$ for all $x \neq y \in X$.

- See Exercise 1.

Theorem 6: Let $X \subseteq \mathbb{R}^n$ be open and convex and $f\colon X \to \mathbb{R}$ twice continuously differentiable. Then

(i) f is concave iff $D^2 f(x)$ is negative semidefinite for all $x \in X$;

(ii) if $D^2 f(x)$ is negative definite for all $x \in X$, then it is strictly concave.

(iii) f is convex iff $D^2 f(x)$ is positive semidefinite for all $x \in X$;

(iv) if $D^2 f(x)$ is positive definite for all $x \in X$, then it is strictly convex.

- See Exercise 1.

Example 1:

(1) $u(x, y) = x^2 + y^2$.

(2) Let u: $\mathbb{R}^2_{++} \to \mathbb{R}$ s.t. $u(x, y) = x^a y^b$, $a > 0$, $b > 0$.

(3) Let u: $\mathbb{R}^n \to \mathbb{R}$ s.t. $u(x) = \sum_{i=1}^{n} x_i^2$.

(4) $f(x) = x^4$, $g(x) = -x^4$.

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(4) $f(x) = x^4$, $g(x) = -x^4$.

(1) $H(x, y) = D^2u(x, y) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$, $|H_1| = 2 > 0$, $|H_2| = 4 > 0$.

Hence u is strictly convex.

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(4) $f(x) = x^4$, $g(x) = -x^4$.

(2) If $a + b < 1$, then u is strictly concave since

$$H(x) = D^2u(x, y) = \begin{pmatrix} a(a-1)x^{a-2}y^b & abx^{a-1}y^{b-1} \\ abx^{a-1}y^{b-1} & b(b-1)x^a y^{b-2} \end{pmatrix},$$

$$|H_1| = a(a-1)x^{a-2}y^b < 0,$$

$$|H_2| = ab(a-1)(b-1)x^{2a-2}y^{2b-2} - a^2b^2x^{2a-2}y^{2b-2}$$

$$= ab(1-a-b)x^{2a-2}y^{2b-2} > 0.$$

- If $a + b > 1$, then $|H_2| < 0$, and thus u is neither concave nor convex.

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(2) Let u: $\mathbb{R}^2_{++} \to \mathbb{R}$ s.t. $u(x, y) = x^a y^b$, $a > 0$, $b > 0$.

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(4) $f(x) = x^4$, $g(x) = -x^4$.

(2) If $a + b = 1$, then u is concave since

$$H(x) = \begin{pmatrix} a(a-1)x^{a-2}y^b & abx^{a-1}y^{b-1} \\ abx^{a-1}y^{b-1} & b(b-1)x^a y^{b-2} \end{pmatrix},$$

$$|H_1| = a(a-1)x^{a-2}y^b < 0,$$

$$|H_2| = ab(a-1)(b-1)x^{2a-2}y^{2b-2} - a^2b^2x^{2a-2}y^{2b-2}$$
$$= ab(1-a-b)x^{2a-2}y^{2b-2} \geq 0.$$

$$H^\pi(x) = \begin{pmatrix} b(b-1)x^a y^{b-2} & abx^{a-1}y^{b-1} \\ abx^{a-1}y^{b-1} & a(a-1)x^{a-2}y^b \end{pmatrix},$$

$$|H_1^\pi| = b(b-1)x^a y^{b-2} < 0,$$

$$|H_2^\pi| = ab(a-1)(b-1)x^{2a-2}y^{2b-2} - a^2b^2x^{2a-2}y^{2b-2}$$
$$= ab(1-a-b)x^{2a-2}y^{2b-2} \geq 0.$$

Example 1:
(1) $u(x, y) = x^2 + y^2$.
(2) Let u: $\mathbb{R}^2_{++} \to \mathbb{R}$ s.t. $u(x, y) = x^a y^b$, $a > 0$, $b > 0$.
(3) Let u: $\mathbb{R}^n \to \mathbb{R}$ s.t. $u(x) = \sum_{i=1}^{n} x_i^2$.
(4) $f(x) = x^4$, $g(x) = -x^4$.

(3) The function u must be strictly convex since

$$H(x) = \begin{pmatrix} 2 & 0 & \cdots & 0 \\ 0 & 2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 2 \end{pmatrix},$$

$$|H_1| = 2 > 0, \cdots, |H_n| = 2^n > 0.$$

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- f is strictly convex iff $Df(x) \cdot (y - x) < f(y) - f(x)$ for all $x \neq y \in X$.

(4) The function f must be strictly convex since

$$
\begin{aligned}
f'(x)(y-x) - y^4 + x^4 &= 4x^3(y-x) - y^4 + x^4 \\
&= (y-x)\,(4x^3 - (y^2 + x^2)(y+x)) \\
&= (y-x)\,(4x^3 - y^3 - xy^2 - x^2y - x^3) \\
&= (y-x)\,(3x^3 - y^3 - xy^2 - x^2y) \\
&= (y-x)\,(x^3 - y^3 + x^3 - xy^2 + x^3 - x^2y) \\
&= -(y-x)^2\,((x^2 + xy + y^2) + x(x+y) + x^2) \\
&= -(y-x)^2\,(3x^2 + 2xy + y^2) \\
&= -(y-x)^2\,((x + y)^2 + 2x^2) < 0 \ \forall x \neq y.
\end{aligned}
$$

Example 2:

(1) Let f: $[0, 10] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x, & \text{if } x \in [0, 5] \\ 10 - x, & \text{if } x \in (5, 10] \end{cases}.$$

(2) $f(x, y, z) = \max\{x, y, z\}$.

(3) $f(x, y, z) = \min\{x, y, z\}$.

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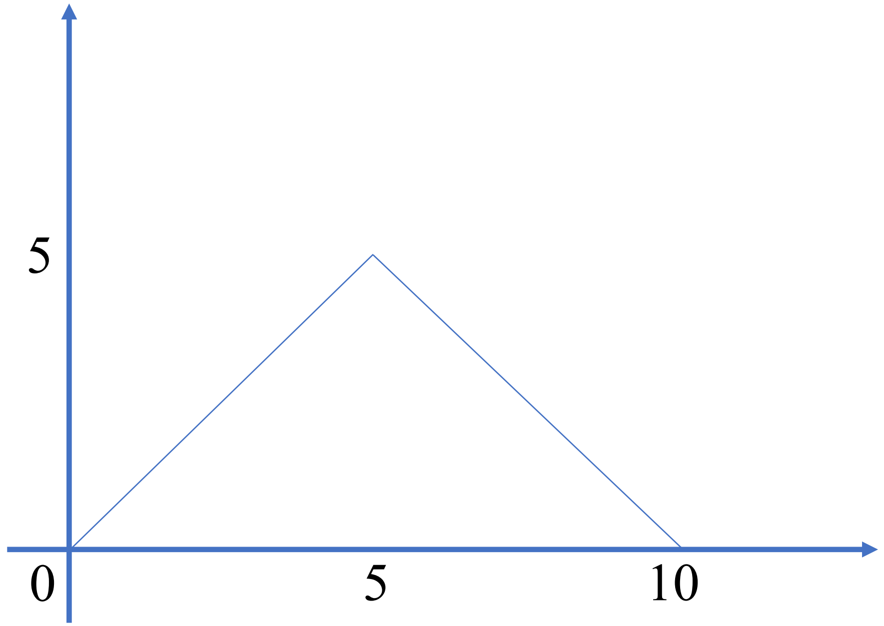
(1) f is concave since its subgraph is convex.



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(3) $f(x, y, z) = \min\{x, y, z\}$.

(2) f is convex since $\forall (x, y, z), (a, b, c) \in \mathbb{R}^3, \forall \lambda \in [0,1]$,

$$f(\lambda(x, y, z) + (1-\lambda)(a, b, c)) =$$

$$f(\lambda x + (1-\lambda)a, \lambda y + (1-\lambda)b, \lambda z + (1-\lambda)c) =$$

$$\max\{\lambda x + (1-\lambda)a, \lambda y + (1-\lambda)b, \lambda z + (1-\lambda)c\} \leq$$

$$\max\{\max\{\lambda x, \lambda y, \lambda z\} + (1-\lambda)a, \max\{\lambda x, \lambda y, \lambda z\} + (1-\lambda)b,$$

$$\max\{\lambda x, \lambda y, \lambda z\} + (1-\lambda)c\} =$$

$$\max\{\lambda x, \lambda y, \lambda z\} + \max\{(1-\lambda)a, (1-\lambda)b, (1-\lambda)c\} =$$

$$\lambda \max\{x, y, z\} + (1-\lambda)\max\{a, b, c\} =$$

$$\lambda f(x, y, z) + (1-\lambda)f(a, b, c).$$

Example 2:

(1) Let f: $[0, 10] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x, & \text{if } x \in [0,5] \\ 10 - x, & \text{if } x \in (5,10] \end{cases}.$$

(2) $f(x, y, z) = \max\{x, y, z\}$.

(3) $f(x, y, z) = \min\{x, y, z\}$.

(3) f is concave since $\forall (x, y, z), (a, b, c) \in \mathbb{R}^3, \forall \lambda \in [0, 1]$,

$$f(\lambda(x, y, z) + (1 - \lambda)(a, b, c)) =$$
$$f(\lambda x + (1 - \lambda)a, \lambda y + (1 - \lambda)b, \lambda z + (1 - \lambda)c) =$$
$$\min\{\lambda x + (1 - \lambda)a, \lambda y + (1 - \lambda)b, \lambda z + (1 - \lambda)c\} \geq$$
$$\min\{\min\{\lambda x, \lambda y, \lambda z\} + (1 - \lambda)a, \min\{\lambda x, \lambda y, \lambda z\} + (1 - \lambda)b,$$
$$\min\{\lambda x, \lambda y, \lambda z\} + (1 - \lambda)c\} =$$
$$\min\{\lambda x, \lambda y, \lambda z\} + \min\{(1 - \lambda)a, (1 - \lambda)b, (1 - \lambda)c\} =$$
$$\lambda \min\{x, y, z\} + (1 - \lambda)\min\{a, b, c\} =$$
$$\lambda f(x, y, z) + (1 - \lambda)f(a, b, c).$$

Exercise 1

(1) Prove Theorem 1.

(2) Prove Theorem 2.

(3) Show that a set $X \subseteq \mathbb{R}^n$ is connected iff X is not a union of two disjoint nonempty sets that are open in X, iff the only sets that are both open and closed in X are X and $\emptyset$.

(4) Determine the concavity and convexity of the function

$$u: \mathbb{R}_{++} \to \mathbb{R} \text{ s.t. } u(x) = x^a,\ a > 0.$$

(5) (Optional) Prove Theorem 5.

(6) (Optional) Prove Theorem 6.

(7) Let $X \subseteq \mathbb{R}^n$ be convex, $M = \{1, 2, \cdots, m\}$, and $\forall k \in M, f_k: X \to \mathbb{R}$ is concave. Show that the function $f: X \to \mathbb{R}$ defined by $f(x) = \min_{k \in M} \{f_k(x)\}$ for each $x \in X$ is concave.

(8) Let $X \subseteq \mathbb{R}^n$ be convex, $M = \{1, 2, \cdots, m\}$, and $\forall k \in M, f_k: X \to \mathbb{R}$ is convex. Show that the function $f: X \to \mathbb{R}$ defined by $f(x) = \max_{k \in M} \{f_k(x)\}$ for each $x \in X$ is convex.

Exercise 2 (Optional)

(1) For any nonempty set $X \subseteq \mathbb{R}^n$, $f\colon X \to \mathbb{R}$ is a ***linear*** function if for any $x_1, \cdots, x_k \in X$ and $r_1, \cdots, r_k \in \mathbb{R}$ with $r_1x_1 + \cdots + r_kx_k \in X$,

$$f(r_1x_1 + \cdots + r_kx_k) = r_1f(x_1) + \cdots + r_kf(x_k).$$

Show that f is linear iff $\exists a_1, \cdots, a_n \in \mathbb{R}$, $\forall x = (x_1, \cdots, x_n) \in X$,

$$f(x) = \sum_{i=1}^{n} a_ix_i.$$

(2) For any nonempty set $X \subseteq \mathbb{R}^n$, $f\colon X \to \mathbb{R}$ is a ***affine*** function if for any $x_1, \cdots, x_k \in X$ and $r_1, \cdots, r_k \in \mathbb{R}$ with $r_1 + \cdots + r_k = 1$ and $r_1x_1 + \cdots + r_kx_k \in X$,

$$f(r_1x_1 + \cdots + r_kx_k) = r_1f(x_1) + \cdots + r_kf(x_k).$$

Show that f is affine iff $\exists a_1, \cdots, a_n, b \in \mathbb{R}$, $\forall x = (x_1, \cdots, x_n) \in X$,

$$f(x) = \sum_{i=1}^{n} a_ix_i + b.$$

Thank you!