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D-modules and Microlocal Calculus

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代数解析概論

DAISU KAISEKI GAIRON
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by Masaki Kashiwara

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Contents

Preface

This book provides an introduction to the general theory of D-modules, whose framework was built up by my teacher Mikio Sato, Takahiro Kawai, and me. As an application of D-module theory, I planned to write how the geometry of flag manifolds and the representation theory of semisimple Lie algebras are related to each other through the theory. Since an excellent Japanese book ([HT]) by Ryoshi Hotta and Toshiyuki Tanisaki covering this topic has been already published, I have decided to write, instead, a book on the algorithm for b-functions using microlocal analysis. For this reason, the title of this book has been changed from the one announced.* I apologize for any confusion this may have caused.

It was around 1969 when I started to study D-modules and microlocal analysis with Sato and Kawai. Among the results I have obtained through over 30 years' study, the microlocal algorithm for b-functions is one of the most beautiful ones (its only defect is a limited range of applications).

I would like to thank Kiyoshi Takeuchi, Toshiyuki Tanisaki, and Kenji Iohara for pointing out many mistakes in preliminary versions of the book, and the staff from Iwanami Shoten for their cooperation.

Masaki Kashiwara
Paris, March 2000

*The original Japanese book had been announced to be published under the title of "Algebraic Analysis and Representation Theory."

Preface to the English Edition

This book was originally written in Japanese, and was published by Iwanami Shoten on March 28, 2000. The Japanese title is "Foundation of Algebraic Analysis". In this edition, I changed the title in order to express the contents more exactly. The phrase "microlocal calculus" in the present title is not a popular terminology. As explained in the introduction, I intend by this to send the message that the microlocal point of view helps concrete calculations, as powerfully as the Cauchy integral formula provides the values of many definite integrals.

Finally, but not the least, special thanks go to the translator, Mutsumi Saito.

Masaki Kashiwara
Kyoto, August 2002

Introduction

The study of D-modules was launched when Mikio Sato gave a colloquium talk on them at the Department of Mathematics, University of Tokyo, in 1960. Let us briefly explain his motivation for studying D-modules.

The general form of a system of linear partial differential equations with unknown functions $u_1, \ldots, u_p$ in $x = (x_1, \ldots, x_n)$ is

$$\sum_{j=1}^{p} P_{ij}(x, \partial) u_j = 0 \quad (i = 1, \ldots, q), \tag{0.1}$$

where $P_{ij}(x, \partial)$ $(i = 1, \ldots, q,\ j = 1, \ldots, p)$ are linear partial differential operators, and each of them is written as

$$\sum_{\alpha \in \mathbb{Z}^n_{\geq 0}} a_\alpha(x) \partial^\alpha, \quad \partial^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}, \quad \alpha = (\alpha_1, \ldots, \alpha_n).$$

The unknown functions $u_1, \ldots, u_p$ are not intrinsic in the system of equations; they are just dummies for the purpose of writing the system in an explicit form. This point of view is a starting point for the introduction of D-modules. As a simple example, let us consider the equation

$$\left(x \frac{d}{dx} - \lambda\right) u = 0, \tag{0.2}$$

where λ is a complex number. Let $v(x) = xu(x)$; then this is transformed into the equation

$$\left(x \frac{d}{dx} - \lambda - 1\right) v(x) = 0, \tag{0.3}$$

because

$$\left(x \frac{d}{dx} - \lambda - 1\right) x = x \left(x \frac{d}{dx} - \lambda\right).$$

Conversely, if v is a solution to equation (0.3), then, assuming that $\lambda \neq -1$, we see that $u = \frac{1}{\lambda+1}\frac{d}{dx}v$ satisfies equation (0.2), since

$$\left(x\frac{d}{dx} - \lambda\right)\frac{d}{dx} = \frac{d}{dx}\left(x\frac{d}{dx} - \lambda - 1\right).$$

Moreover, by letting $v = xu(x)$ for a solution u to equation (0.2), we obtain $u = \frac{1}{\lambda+1}\frac{d}{dx}v$, since

$$\frac{1}{\lambda+1}\frac{d}{dx}v - u = \frac{1}{\lambda+1}\left(\frac{d}{dx}x - \lambda - 1\right)u = 0.$$

Conversely, by letting $u = \frac{1}{\lambda+1}\frac{d}{dx}v$ for a solution v to equation (0.3), we obtain $v = xu$, since

$$xu - v = \frac{1}{\lambda+1}\left(x\frac{d}{dx} - \lambda - 1\right)v = 0.$$

By the transformations $v = xu$ and $u = \frac{1}{\lambda+1}\frac{d}{dx}v$, we thus see that equations (0.2) and (0.3) are equivalent to each other. Hence they mean the same, although they look different. In other words, taking u or v as an unknown function is quite artificial. Wc can formulate this idea by using D-modules.

Let us consider equation (0.1). Let D be the (noncommutative) ring of linear partial differential operators, and $P : D^{\oplus q} \to D^{\oplus p}$ the map given by

$$(Q_1, \ldots, Q_q) \mapsto \left(\sum_{i=1}^{q} Q_i P_{i1}, \ldots, \sum_{i=1}^{q} Q_i P_{ip}\right).$$

This map P is left D-linear, and its cokernel M is the D-module corresponding to equation (0.1). We can derive the solutions to (0.1) from M as follows: Let F be a space of functions in which we want to find the solutions. Various spaces can be taken as F, such as the space of C^∞-functions, the space of distributions, or the space of holomorphic functions, depending on the problem being considered. Since we can apply differential operators to the functions belonging to F, we regard F as a left D-module. Then

$$\mathrm{Hom}_D(D^{\oplus p}, F) = F^{\oplus p} = \{(u_1, \ldots, u_p); u_1, \ldots, u_p \in F\}.$$

By a well known property (left exactness) of Hom_D,

$$\begin{aligned} \mathrm{Hom}_D(M, F) &= \mathrm{Ker}(\mathrm{Hom}_D(D^{\oplus p}, F) \xrightarrow{P} \mathrm{Hom}_D(D^{\oplus q}, F)) \\ &= \{(u_1, \ldots, u_p) \in F^{\oplus p}; u_1, \ldots, u_p \,\text{satisfy (0.1)}\}. \end{aligned}$$

Hence the space of solutions to (0.1) in F equals $\mathrm{Hom}_D(M, F)$, which depends only on M. Conversely, given a D-module M, for each isomorphism

$$M \simeq \mathrm{Coker}(D^{\oplus q} \xrightarrow{P} D^{\oplus p}), \tag{0.4}$$

we obtain equation (0.1). Then (0.1) is considered an explicit presentation of M corresponding to the isomorphism (0.4). There are many such isomorphisms for a given D-module. Each isomorphism gives a different explicit presentation (0.1). In the previous example, one D-module has two distinct isomorphisms; it is isomorphic to two cokernels:

$$\mathrm{Coker}(D \xrightarrow{x\frac{d}{dx}-\lambda} D) \simeq \mathrm{Coker}(D \xrightarrow{x\frac{d}{dx}-\lambda-1} D),$$

which lead to (0.2) and (0.3) respectively.

From this point of view, D-modules were introduced, and the theory has grown into a rich one with many subtheories such as characteristic varieties, microlocal analysis, holonomic systems (maximally overdetermined systems), etc. We give an outline of the theory in this book.

Today, besides the theory of linear partial differential equations, which is a root of D-module theory, the theory is also applied to representation theory, conformal field theory, etc. For applications to representation theory, [HT] has a detailed exposition. In this book, we describe applications of the theory to b-functions; in particular, we give an algorithm to compute b-functions using microlocal analysis. Sato seems to have expected that the D-module theory (in particular the theory of holonomic systems) would be useful for explicit computations. This is analogous to Cauchy's integration theorem, which has not only theoretical beauty and importance, but also an application to explicit computations of definite integrals through Cauchy's residue formula. The theory of b-functions described in this book is a good example of applicability of microlocal analysis of holonomic systems to explicit computations.

Conventions and Notation. In this book, a manifold means a nonsingular complex analytic manifold. A morphism of manifolds $f : X \to Y$ is said to be smooth if the corresponding maps of tangent spaces $T_xX \to T_{f(x)}Y$ are surjective at all $x \in X$.

We abbreviate the dimension $\dim X$ of a manifold X to d_X, and write $d_{X/Y} = d_X - d_Y$.

We simply call a sheaf of rings a ring, and a sheaf of modules a module.

$\rightarrowtail$ indicates injectivity, and $\twoheadrightarrow$ surjectivity.

For an analytic set Z, we say that a property (P) holds at a generic point p on Z if there exists a nowhere dense closed analytic subset Y such that (P) holds at all $p \in Z \setminus Y$.

CHAPTER 1

Basic Properties of D-modules

1.1. D-modules

Let X be an n-dimensional complex manifold, and $\mathcal{O}_X$ the sheaf of holomorphic functions on X. A (holomorphic) *differential operator* on X is a sheaf morphism P from $\mathcal{O}_X$ to $\mathcal{O}_X$, locally written with a coordinate system $(x_1, \ldots, x_n)$ and with holomorphic functions $a_\alpha(x)$ as

$$(Pu)(x) = \sum_{\alpha \in \mathbb{Z}^n_{\geq 0}} a_\alpha(x) \partial_x^\alpha u(x) \quad (u \in \mathcal{O}_X).$$

Here for $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n_{\geq 0}$, we set

$$\partial_x^\alpha u = \frac{\partial^{\alpha_1 + \cdots + \alpha_n} u}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}},$$

and assume $a_\alpha(x) = 0$ except for finitely many α. We call P a *differential operator of order at most m* if $a_\alpha(x) = 0$ for all α with $|\alpha| := \alpha_1 + \cdots + \alpha_n > m$. We denote by $\mathcal{D}_X$ the sheaf of differential operators on X. The sheaf $\mathcal{D}_X$ has a ring structure with the composition of operators as multiplication, and $\mathcal{O}_X$ is a subring of $\mathcal{D}_X$. Let Θ_X denote the sheaf of vector fields on X. Then $\mathcal{D}_X$ contains Θ_X. Note that Θ_X is a left $\mathcal{O}_X$-submodule of $\mathcal{D}_X$, but not a right $\mathcal{O}_X$-submodule.

Let $v \in \Theta_X$ and $a \in \mathcal{O}_X$. Then

$$[v, a] := va - av = v(a),$$

where we denote by $v(a)$ the holomorphic function obtained by operating by v on a. This is an immediate consequence of the equation $v(ab) = v(a)b + av(b)$, and shows that $\mathcal{D}_X$ is a non-commutative ring. Accordingly the study of its structure is more complicated than algebraic geometry (the study of commutative rings).

The simplest $\mathcal{D}_X$-module is $\mathcal{O}_X$. It is generated by 1 as a $\mathcal{D}_X$-module, and the kernel of the map $\mathcal{D}_X \to \mathcal{O}_X$ defined by the generator

1 is $\mathcal{D}_X\Theta_X \subset \mathcal{D}_X$. Hence we have an exact sequence of $\mathcal{D}_X$-modules

$$\mathcal{D}_X \otimes_{\mathcal{O}_X} \Theta_X \xrightarrow{\delta_1} \mathcal{D}_X \longrightarrow \mathcal{O}_X \longrightarrow 0, \tag{1.1}$$

where

$$\delta_1(P \otimes v) = Pv, \quad P \in \mathcal{D}_X,\ v \in \Theta_X.$$

For a $\mathcal{D}_X$-module $\mathcal{F}$, we thus have

$$\mathcal{H}om_{\mathcal{D}_X}(\mathcal{O}_X, \mathcal{F}) = \left\{ u \in \mathcal{F};\ \frac{\partial}{\partial x_i}u = 0\ (i = 1, \cdots, n) \right\}$$

by using a coordinate system $x = (x_1, \cdots, x_n)$. In the correspondence seen in the Introduction, $\mathcal{O}_X$ corresponds to the equations

$$\frac{\partial}{\partial x_1}u = \cdots = \frac{\partial}{\partial x_n}u = 0.$$

In Proposition 1.6, we will extend (1.1) to the left to obtain an exact sequence

$$\begin{aligned} 0 \to \mathcal{D}_X \otimes_{\mathcal{O}_X} \bigwedge^n \Theta_X &\xrightarrow{\delta_n} \mathcal{D}_X \otimes_{\mathcal{O}_X} \bigwedge^{n-1} \Theta_X \longrightarrow \cdots \\ &\xrightarrow{\delta_2} \mathcal{D}_X \otimes_{\mathcal{O}_X} \Theta_X \xrightarrow{\delta_1} \mathcal{D}_X \longrightarrow \mathcal{O}_X \longrightarrow 0. \end{aligned}$$

To see a relation between this and the de Rham complex

$$\Omega_X^\bullet : \Omega_X^0 \xrightarrow{d} \Omega_X^1 \xrightarrow{d} \cdots \xrightarrow{d} \Omega_X^n,$$

we shall consider differential homomorphisms in the next section.

1.2. Differential Homomorphisms of $\mathcal{O}_X$-modules

Let X be a complex manifold.

DEFINITION 1.1. Let $\mathcal{F}$ and $\mathcal{G}$ be $\mathcal{O}_X$-modules. A $\mathbb{C}$-linear sheaf homomorphism $f : \mathcal{F} \to \mathcal{G}$ is called a *differential homomorphism* if for every $s \in \mathcal{F}$ there exist finitely many $P_j \in \mathcal{D}_X$ and $v_j \in \mathcal{G}$ such that

$$f(as) = \sum_j P_j(a)v_j$$

for all $a \in \mathcal{O}_X$.

Let $\mathcal{D}iff(\mathcal{F}, \mathcal{G})$ denote the sheaf of differential homomorphisms from $\mathcal{F}$ to $\mathcal{G}$. Then $\mathcal{D}_X = \mathcal{D}iff(\mathcal{O}_X, \mathcal{O}_X)$.

LEMMA 1.2. *Let $\mathcal{F}$, $\mathcal{G}$, and $\mathcal{H}$ be $\mathcal{O}_X$-modules, and $f : \mathcal{F} \to \mathcal{G}$ and $g : \mathcal{G} \to \mathcal{H}$ differential homomorphisms. Then $g \circ f : \mathcal{F} \to \mathcal{H}$ is also a differential homomorphism.*

Let us consider the right $\mathcal{D}_X$-module $\mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{D}_X$. The right $\mathcal{D}_X$-module structure gives $\mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{D}_X$ an $\mathcal{O}_X$-module structure. By tensoring $\mathcal{G}$ with the left $\mathcal{O}_X$-linear homomorphism

$$\mathcal{D}_X \to \mathcal{O}_X \ (\mathcal{D}_X \ni P \to P(1) \in \mathcal{O}_X),$$

we obtain

$$P_{\mathcal{G}} : \mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{D}_X \to \mathcal{G}.$$

Note that this is not $\mathcal{O}_X$-linear. The morphism $P_{\mathcal{G}}$ induces

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{D}_X) \to \mathcal{H}om_{\mathbb{C}}(\mathcal{F}, \mathcal{G}).$$

This is the map sending $\varphi \in \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{D}_X)$ to $P_{\mathcal{G}} \circ \varphi$. If $\varphi(s) = \sum v_j \otimes P_j$ for $s \in \mathcal{F}$, then $P_{\mathcal{G}} \circ \varphi(as) = \sum P_j(a) v_j$. Hence $P_{\mathcal{G}} \circ \varphi$ is a differential homomorphism.

PROPOSITION 1.3. *$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{D}_X) \to \mathcal{D}\!\mathit{iff}(\mathcal{F}, \mathcal{G})$ is an isomorphism.*

PROOF. First we prove the proposition when $\mathcal{F} = \mathcal{O}_X$.

(1) (injectivity) Taking a coordinate system $x = (x_1, \dots, x_n)$, we have $\mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{D}_X \cong \bigoplus_\alpha \mathcal{G} \otimes \partial_x^\alpha$, and thus

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{D}_X) \cong \bigoplus_\alpha \mathcal{G} \otimes \partial_x^\alpha.$$

Let K be the kernel of the homomorphism in question, and put

$$F_m(K) := \{\textstyle\sum v_\alpha \otimes \partial_x^\alpha \in K; v_\alpha = 0 \text{ for } |\alpha| > m\}.$$

Then $K = \bigcup_m F_m(K)$. We shall inductively prove that $F_m(K) = 0$. Let $\sum v_\alpha \otimes \partial_x^\alpha$ belong to $F_m(K)$. Then we have $\sum(\partial_x^\alpha a) v_\alpha = 0$ for all $a \in \mathcal{O}_X$. We have to show that $v_\alpha = 0$ for all α. First we see that $F_0(K) = 0$ by putting $a = 1$. Next suppose that $m \neq 0$ and $F_{m-1}(K) = 0$. For all i

$$\begin{aligned} 0 &= \sum \partial_x^\alpha (x_i a) v_\alpha - \sum x_i (\partial_x^\alpha a) v_\alpha \\ &= \sum_{\alpha_i > 0} \alpha_i (\partial_x^{\alpha - \delta_i} a) v_\alpha, \end{aligned}$$

where δ_i is the i-th unit vector $(0, \dots, \overset{i}{\check{1}}, 0, \dots, 0)$. Hence the induction hypothesis leads to $v_\alpha = 0$ $(\alpha_i \neq 0)$. Finally we also see that $v_0 = 0$, since $v_0 \otimes \partial^0 \in F_0(K) = 0$.

(2) (surjectivity) Let $f \in \mathcal{D}\!\mathit{iff}(\mathcal{O}_X, \mathcal{G})$ and $f(a) = \sum_i P_i(a)v_i$. Then $f = P_{\mathcal{G}} \circ \varphi$ for $\varphi \in \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{G} \otimes \mathcal{D}_X)$ with $\varphi(1) = \sum v_i \otimes P_i$.

We have thus proved the proposition in the case when $\mathcal{F} = \mathcal{O}_X$. Next we prove it in the general case.

(1) (surjectivity) Let $f \in \mathcal{D}\!\mathit{iff}(\mathcal{F}, \mathcal{G})$. Since the map $\mathcal{O}_X \ni a \mapsto f(as)$ belongs to $\mathcal{D}\!\mathit{iff}(\mathcal{O}_X, \mathcal{G})$ for every $s \in \mathcal{F}$, there exists a unique $\varphi(s) \in \mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{D}_X$ satisfying

$$P_{\mathcal{G}}(\varphi(s)a) = f(as) \qquad (a \in \mathcal{O}_X).$$

It is immediate that $\varphi \in \mathcal{H}om(\mathcal{F}, \mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{D}_X)$ is right $\mathcal{O}_X$-linear.

(2) (injectivity) If $\psi \in \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G} \otimes \mathcal{D}_X)$ satisfies $P_{\mathcal{G}} \circ \psi = 0$, then for every $s \in \mathcal{F}$ we have $P_{\mathcal{G}}\psi(as) = 0$ $(a \in \mathcal{O}_X)$, and thus $\psi(s) = 0$. □

COROLLARY 1.4.

$$\mathcal{H}om_{\mathcal{D}_X^{\mathrm{op}}}(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{D}_X, \mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{D}_X) \xrightarrow{\sim} \mathcal{D}\!\mathit{iff}(\mathcal{F}, \mathcal{G}),$$

where $\mathcal{D}_X^{\mathrm{op}}$ denotes the opposite ring of $\mathcal{D}_X$ (cf. § 1.4).

It is enough to note that

$$\mathcal{H}om_{\mathcal{D}_X^{\mathrm{op}}}(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{D}_X, \mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{D}_X) \simeq \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{D}_X).$$

PROPOSITION 1.5. *Let $\mathcal{F}, \mathcal{G}$, and $\mathcal{H}$ be $\mathcal{O}_X$-modules. Then the diagram*

$$\begin{array}{ccc}
\begin{array}{c}\mathcal{H}om_{\mathcal{D}_X^{\mathrm{op}}}(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{D}_X, \mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{D}_X) \\ \otimes_{\mathbb{C}} \mathcal{H}om_{\mathcal{D}_X^{\mathrm{op}}}(\mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{D}_X, \mathcal{H} \otimes_{\mathcal{O}_X} \mathcal{D}_X)\end{array} & \longrightarrow & \begin{array}{c}\mathcal{D}\!\mathit{iff}(\mathcal{F}, \mathcal{G}) \\ \otimes_{\mathbb{C}} \mathcal{D}\!\mathit{iff}(\mathcal{G}, \mathcal{H})\end{array} \\
\downarrow & & \downarrow \\
\mathcal{H}om_{\mathcal{D}_X^{\mathrm{op}}}(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{D}_X, \mathcal{H} \otimes_{\mathcal{O}_X} \mathcal{D}_X) & \longrightarrow & \mathcal{D}\!\mathit{iff}(\mathcal{F}, \mathcal{H})
\end{array}$$

is commutative, where the vertical arrows are the homomorphisms obtained by composition.

Let us apply the above argument to the de Rham complex

$$\cdots \to 0 \to \Omega_X^0 \xrightarrow{d} \Omega_X^1 \xrightarrow{d} \cdots \xrightarrow{d} \Omega_X^n \to 0 \to \cdots,$$

where d's are the exterior derivatives.

Take a coordinate system $x = (x_1, \dots, x_n)$. Then

$$d(a\omega) = da \wedge \omega + ad\omega = \sum \frac{\partial a}{\partial x_i} dx_i \wedge \omega + ad\omega. \tag{1.2}$$

Hence every d is a differential homomorphism, and Corollary 1.4 gives a complex of right $\mathcal{D}_X$-modules

$$\cdots \to 0 \to \Omega_X^0 \otimes_{\mathcal{O}_X} \mathcal{D}_X \xrightarrow{d} \cdots \xrightarrow{d} \Omega_X^n \otimes_{\mathcal{O}_X} \mathcal{D}_X \to 0 \to \cdots. \tag{1.3}$$

Equation (1.2) leads to

$$d(\omega \otimes P) = \sum_{i=1}^{n} dx_i \wedge \omega \otimes \partial_i P + d\omega \otimes P \quad (\omega \in \Omega_X^\bullet,\ P \in \mathcal{D}_X). \tag{1.4}$$

Hence

$$dP = \sum_{i=1}^{n} dx_i \otimes \partial_i P, \tag{1.5}$$

and

$$d(\omega \wedge \theta) = d\omega \wedge \theta + (-1)^p \omega \wedge d\theta \quad (\omega \in \Omega_X^p,\ \theta \in \Omega_X^q \otimes_{\mathcal{O}_X} \mathcal{D}_X). \tag{1.6}$$

Conversely, these two formulas characterize the differential d of $\Omega_X^\bullet \otimes \mathcal{D}_X$. For every left $\mathcal{D}_X$-module $\mathcal{M}$, we obtain a complex

$$\cdots \to 0 \to \Omega_X^0 \otimes_{\mathcal{O}_X} \mathcal{M} \xrightarrow{d}$$
$$\cdots \xrightarrow{d} \Omega_X^{n-1} \otimes_{\mathcal{O}_X} \mathcal{M} \xrightarrow{d} \Omega_X^n \otimes_{\mathcal{O}_X} \mathcal{M}_X \to 0 \to \cdots$$

by applying the functor $\bullet \otimes_{\mathcal{D}_X} \mathcal{M}$ to (1.3). Here

$$du = \sum_i dx_i \otimes \partial_i u \quad (u \in \mathcal{M}), \tag{1.7}$$

$$d(\omega \wedge \theta) = d\omega \wedge \theta + (-1)^p \omega \wedge d\theta \quad (\omega \in \Omega_X^p,\ \theta \in \Omega_X^q \otimes \mathcal{M}). \tag{1.8}$$

We call this complex the *de Rham complex* of $\mathcal{M}$, and denote it by $DR_X(\mathcal{M})$. For example, (1.3) is nothing but $DR_X(\mathcal{D}_X)$.

By applying the functor

$$\mathcal{H}om_{\mathcal{D}_X^{\mathrm{op}}}(\bullet, \mathcal{D}_X) : \mathrm{Mod}(\mathcal{D}_X^{\mathrm{op}}) \to \mathrm{Mod}(\mathcal{D}_X)^{\mathrm{op}}$$

to (1.3), we obtain a complex of left $\mathcal{D}$-modules

$$\cdots \leftarrow 0 \leftarrow \mathcal{D}_X \otimes \mathcal{O}_X \xleftarrow{\delta} \mathcal{D}_X \otimes \Theta_X \leftarrow \mathcal{D}_X \otimes \bigwedge^2 \Theta_X \leftarrow$$
$$\cdots \leftarrow \mathcal{D}_X \otimes \bigwedge^n \Theta_X \leftarrow 0 \leftarrow \cdots,$$

since we have

$$\mathcal{H}om_{\mathcal{D}_X^{\mathrm{op}}}(\Omega_X^p \otimes_{\mathcal{O}_X} \mathcal{D}_X, \mathcal{D}_X) \simeq \mathcal{H}om_{\mathcal{O}_X}(\Omega_X^p, \mathcal{D}_X) \simeq \mathcal{D}_X \otimes_{\mathcal{O}_X} \bigwedge^p \Theta_X.$$

Explicitly, δ is written as

$$\begin{aligned}
&\delta(P \otimes v_1 \wedge \cdots \wedge v_p) \\
&\quad = \sum_i (-1)^{i-1} P v_i \otimes v_1 \wedge \cdots \wedge \hat{v}_i \wedge \cdots \wedge v_p \\
&\qquad + \sum_{i<j} (-1)^{i+j} P \otimes [v_i, v_j] \wedge v_1 \wedge \cdots \wedge \hat{v}_i \wedge \cdots \wedge \hat{v}_j \wedge \cdots \wedge v_p,
\end{aligned}$$

where $\hat{}$ indicates deleting the term underneath. We thus obtain the sequence stated in the last paragraph of § 1.1.

PROPOSITION 1.6. *The complex*

$$0 \leftarrow \mathcal{O}_X \leftarrow \mathcal{D} \leftarrow \mathcal{D} \otimes_{\mathcal{O}_X} \Theta_X \leftarrow \mathcal{D} \otimes_{\mathcal{O}_X} \bigwedge^2 \Theta_X \leftarrow$$
$$\cdots \leftarrow \mathcal{D} \otimes_{\mathcal{O}_X} \bigwedge^n \Theta_X \leftarrow 0$$

is exact.

Later (Proposition 4.2), we will give a proof of this in a more general setting.

1.3. A System of Generators of D

Let Θ_X denote the sheaf of vector fields on a manifold X. Then Θ_X is a subsheaf of $\mathcal{D}_X$, and $\Theta_X \to \mathcal{D}_X$ is (left) $\mathcal{O}_X$-linear. The ring $\mathcal{D}_X$ is generated by $\mathcal{O}_X$ and Θ_X, and their fundamental relations are the following:

(1.9) $\quad \mathcal{O}_X \to \mathcal{D}_X$ is a ring homomorphism,

(1.10) $\quad \Theta_X \to \mathcal{D}_X$ is left $\mathcal{O}_X$-linear,

(1.11) $\quad \Theta_X \to \mathcal{D}_X$ is a Lie algebra homomorphism,

(1.12) $\quad [v, a] = v(a)$ for all $v \in \Theta_X$, $a \in \mathcal{O}_X$,

where we denote by $v(a)$ the element of $\mathcal{O}_X$ obtained by differentiating a with respect to v. To be more precise, we formulate the above as follows:

LEMMA 1.7. *Let R be a sheaf of rings on X, and $\iota : \mathcal{O}_X \to R$ and $\varphi : \Theta_X \to R$ sheaf morphisms such that*

(1) $\iota : \mathcal{O}_X \to R$ *is a ring homomorphism,*
(2) $\varphi : \Theta_X \to R$ *is left* $\mathcal{O}_X$*-linear, where a left* $\mathcal{O}_X$*-module structure of* R *is given through* ι,
(3) $\varphi : \Theta_X \to R$ *is a Lie algebra homomorphism, where Lie brackets in* R *are commutators, and*
(4) $[\varphi(v), \iota(a)] = \iota(v(a))$ *for all* $v \in \Theta_X$, $a \in \mathcal{O}_X$.

Then there exists a unique ring homomorphism $\Phi : \mathcal{D}_X \to R$ *such that* $\mathcal{O}_X \longrightarrow \mathcal{D}_X \xrightarrow{\Phi} R$ *coincides with* ι, *and* $\Theta_X \longrightarrow \mathcal{D}_X \xrightarrow{\Phi} R$ *with* φ.

PROOF. This is a local problem. Using a local coordinate system, we have

$$\Phi(\sum a_\alpha(x)\partial^\alpha) = \sum_\alpha \iota(a_\alpha(x))\varphi(\partial_1)^{\alpha_1} \cdots \varphi(\partial_n)^{\alpha_n}.$$

Conversely, when we define Φ as above, Φ is a ring homomorphism (details are omitted). □

In particular, if M is an $\mathcal{O}_X$-module and if a sheaf morphism $\varphi : \Theta_X \times M \to M$ satisfies

$$\varphi(v, s) \text{ is } \mathcal{O}_X\text{-linear in } v \in \Theta_X \text{ and } \mathbb{C}\text{-linear in } s \in M, \tag{1.13}$$

$$\varphi(v, as) = a\varphi(v, s) + v(a)s \quad (\forall v \in \Theta_X,\ a \in \mathcal{O}_X,\ s \in M), \tag{1.14}$$

$$\varphi([v_1, v_2], s) = \varphi(v_1, \varphi(v_2, s)) - \varphi(v_2, \varphi(v_1, s)), \tag{1.15}$$

then there exists a unique $\mathcal{D}_X$-module structure on M such that $a \cdot s = as$, $v \cdot s = \varphi(v, s)$ $(a \in \mathcal{O}_X,\ v \in \Theta_X,\ s \in M)$. In fact, since the morphisms $\mathcal{O}_X \to \mathcal{E}nd_{\mathbb{C}}(M)$ and $\Theta_X \xrightarrow{\varphi} \mathcal{E}nd_{\mathbb{C}}(M)$ satisfy the conditions in Lemma 1.7, they can be uniquely extended to a ring homomorphism $\mathcal{D}_X \to \mathcal{E}nd_{\mathbb{C}}(M)$.

1.4. Left $\mathcal{D}_X$-modules and Right $\mathcal{D}_X$-modules

Since $\mathcal{D}_X$ is a non-commutative ring, a left $\mathcal{D}_X$-module and a right $\mathcal{D}_X$-module are different notions. We mainly treat left $\mathcal{D}_X$-modules in this book, since the category of left $\mathcal{D}_X$-modules and that of right $\mathcal{D}_X$-modules turn out to be equivalent. In this equivalence of categories, the left $\mathcal{D}_X$-module $\mathcal{O}_X$ corresponds to a right $\mathcal{D}_X$-module $\Omega_X := \Omega_X^{\dim X}$. We shall consider this correspondence in this section.

In what follows, we denote by $\mathcal{A}^{\mathrm{op}}$ the opposite ring of a ring $\mathcal{A}$. Namely,

$$\mathcal{A}^{\mathrm{op}} = \{a^{\mathrm{op}}\,;\, a \in \mathcal{A}\},$$

and its multiplication is given by $a^{\mathrm{op}}b^{\mathrm{op}} = (ba)^{\mathrm{op}}$. Hence a left $\mathcal{A}$-module and a right $\mathcal{A}^{\mathrm{op}}$-module, and a right $\mathcal{A}$-module and a left $\mathcal{A}^{\mathrm{op}}$-module, are respectively the same notions. Form now on, we call a left $\mathcal{A}$-module simply an $\mathcal{A}$-module.

(a) Lie Derivatives and Exterior Derivatives. On the sheaf of exterior differential forms $\Omega_X^\bullet = \bigoplus_i \Omega_X^i$, the sheaf Θ_X acts in two ways. Recall that for $v \in \Theta_X$ its *inner derivative* $i_v \in \mathcal{E}nd(\Omega_X^\bullet)$ is characterized by the following properties:

$$i_{av} = ai_v = i_v a \quad (a \in \mathcal{O}_X,\ v \in \Theta_X)\,, \tag{1.16}$$

$$i_v(\omega_1 \wedge \omega_2) = (i_v\omega_1) \wedge \omega_2 + (-1)^p \omega_1 \wedge (i_v\omega_2) \qquad (\omega_1 \in \Omega_X^p,\ \omega_2 \in \Omega_X^\bullet), \tag{1.17}$$

$$i_v(\mathcal{O}_X) = 0\,, \tag{1.18}$$

$$i_v(\omega) = \langle v, \omega\rangle \in \mathcal{O}_X \quad (\omega \in \Omega_X^1)\,, \tag{1.19}$$

where $\langle\ ,\ \rangle$ is the coupling of Θ_X and Ω_X^1. Hence i_v is an operator (of degree -1) from Ω_X^p to Ω_X^{p-1}. In contrast, the *Lie derivative* $L_v \in \mathcal{E}nd(\Omega_X^\bullet)$ is characterized by the following properties:

$$L_v(\omega_1 \wedge \omega_2) = L_v(\omega_1) \wedge \omega_2 + \omega_1 \wedge L_v(\omega_2), \tag{1.20}$$

$$L_v(a) = v(a) \quad (a \in \mathcal{O}_X), \tag{1.21}$$

$$dL_v = L_v d\,. \tag{1.22}$$

Hence L_v is an operator (of degree 0) from Ω_X^p to itself. The Lie derivatives also satisfy

$$[L_{v_1}, L_{v_2}] = L_{[v_1, v_2]} \tag{1.23}$$

for all $v_1, v_2 \in \Theta_X$. The exterior derivative d and these two derivatives are related by

$$L_v = di_v + i_v d. \tag{1.24}$$

LEMMA 1.8. *For every $a \in \mathcal{O}_X$ and every $v \in \Theta_X$, the equality*

$$L_{av} = L_v a \tag{1.25}$$

holds as operators on $\Omega_X := \Omega_X^{\dim X}$.

PROOF. Let $\omega \in \Omega_X^\bullet$. Then

$$\begin{aligned} L_{av}(\omega) &= d(ai_v\omega) + ai_v d\omega = da \wedge i_v\omega + ai_v d\omega + ai_v d\omega \\ &= da \wedge i_v\omega + aL_v(\omega)\,, \end{aligned}$$

and

$$\begin{aligned} i_v(da \wedge \omega) &= i_v(da) \wedge \omega - da \wedge i_v \omega \\ &= v(a)\omega - da \wedge i_v \omega \,. \end{aligned}$$

Hence

$$L_{av}(\omega) = aL_v(\omega) + v(a)\omega - i_v(da \wedge \omega)\,. \tag{1.26}$$

This gives the desired result, since $da \wedge \omega = 0$ for $\omega \in \Omega_X$. □

Define

$$\varphi : \Theta_X \to \mathcal{E}nd_{\mathbb{C}}(\Omega_X)^{\mathrm{op}}$$

by $v \mapsto -L_v$, and $\mathcal{O}_X \to \mathcal{E}nd_{\mathbb{C}}(\Omega_X)^{\mathrm{op}}$ by multiplication. Then equation (1.25) means that φ is left $\mathcal{O}_X$-linear. In addition, the other conditions in Lemma 1.7 are satisfied, and hence the above two morphisms uniquely extend to a ring homomorphism $\mathcal{D}_X \to \mathcal{E}nd_{\mathbb{C}}(\Omega_X)^{\mathrm{op}}$. This gives a right $\mathcal{D}_X$-module structure on Ω_X. By definition, the equality

$$\omega v = -L_v \omega \tag{1.27}$$

holds for every $v \in \Theta_X$ and every $\omega \in \Omega_X$.

This action is related to integration by parts. Namely, $a \in \mathcal{O}_X$, $\omega \in \Omega_X$, and $P \in \mathcal{D}_X$ formally satisfy

$$\int (\omega P)a = \int \omega P(a),$$

that is, there exists a differential form η of degree $(n-1)$ such that

$$(\omega P)a - \omega P(a) = d\eta.$$

As an example, say $P = v \in \Theta_X$. Then it is easy to see that

$$(\omega v)a - \omega v(a) = -(L_v \omega)a - \omega v(a) = -d(a i_v \omega).$$

(b) Left $\mathcal{D}_X$-modules Versus Right $\mathcal{D}_X$-modules. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module (i.e., an $\mathcal{O}_X$-module locally isomorphic to $\mathcal{O}_X$). Then $\mathcal{L}^{\otimes -1} = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X)$ is also an invertible $\mathcal{O}_X$-module, and $\mathcal{L}^{\otimes -1} \otimes_{\mathcal{O}_X} \mathcal{L} \cong \mathcal{O}_X$. For $t \in \mathcal{L}^{\otimes -1}$ and $s \in \mathcal{L}$, we denote by $\langle t, s\rangle \in \mathcal{O}_X$ the image of $t \otimes s$ under this isomorphism. Let $\mathcal{A}$ be an $\mathcal{O}_X$-ring. Namely, $\mathcal{A}$ is a ring with a given ring homomorphism $\mathcal{O}_X \to \mathcal{A}$. There exists a natural ring structure on $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes -1}$. Its multiplication is given by

$$(s_1 \otimes a_1 \otimes t_1) \circ (s_2 \otimes a_2 \otimes t_2) = s_1 \otimes a_1 \langle t_1, s_2\rangle a_2 \otimes t_2$$

for $s_i \in \mathcal{L}$, $t_i \in \mathcal{L}^{\otimes -1}$, $a_i \in \mathcal{A}$ $(i = 1, 2)$. Let $\mathcal{M}$ be an $\mathcal{A}$-module. Then $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{M}$ is a left $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes -1}$-module; the action is given by $(s \otimes a \otimes t) \circ (s' \otimes u) = s \otimes a\langle t, s'\rangle u$. We thus obtain the following proposition.

PROPOSITION 1.9. *Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module, and $\mathcal{A}$ an $\mathcal{O}_X$-ring. Then the category* $\mathrm{Mod}(\mathcal{A})$ *of left $\mathcal{A}$-modules and the category* $\mathrm{Mod}(\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes -1})$ *of left $(\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes -1})$-modules are equivalent to each other by*

$$\mathrm{Mod}(\mathcal{A}) \ni \mathcal{M} \mapsto \mathcal{L} \otimes_{\mathcal{O}} \mathcal{M} \in \mathrm{Mod}(\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes -1}).$$

Note that $(\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes -1})^{\mathrm{op}} \simeq \mathcal{L}^{\otimes -1} \otimes_{\mathcal{O}_X} \mathcal{A}^{\mathrm{op}} \otimes_{\mathcal{O}_X} \mathcal{L}$.

PROPOSITION 1.10.

$$\mathcal{D}_X^{\mathrm{op}} \cong \Omega_X \otimes_{\mathcal{O}_X} \mathcal{D}_X \otimes_{\mathcal{O}_X} \Omega_X^{\otimes -1}. \tag{1.28}$$

PROOF. The right $\mathcal{D}_X$-module structure of Ω_X gives a morphism $\mathcal{D}_X^{\mathrm{op}} \to \mathcal{E}nd_{\mathbb{C}}(\Omega_X)$, and its image is contained in $\mathcal{D}iff(\Omega_X, \Omega_X) = \Omega_X \otimes_{\mathcal{O}_X} \mathcal{D}_X \otimes_{\mathcal{O}_X} \Omega_X^{\otimes -1}$. Hence we obtain $\varphi : \mathcal{D}_X^{\mathrm{op}} \to \Omega_X \otimes_{\mathcal{O}_X} \mathcal{D}_X \otimes_{\mathcal{O}_X} \Omega_X^{\otimes -1}$. Since $\Omega_X^{\otimes -1} \otimes_{\mathcal{O}_X} \mathcal{D}_X^{\mathrm{op}} \otimes_{\mathcal{O}_X} \Omega_X = (\Omega_X \otimes_{\mathcal{O}_X} \mathcal{D}_X \otimes_{\mathcal{O}_X} \Omega_X^{\otimes -1})^{\mathrm{op}}$, we also obtain $\psi := (\Omega_X^{\otimes -1} \otimes \varphi \otimes \Omega_X)^{\mathrm{op}} : \Omega_X \otimes_{\mathcal{O}_X} \mathcal{D}_X \otimes_{\mathcal{O}_X} \Omega_X^{\otimes -1} \to \mathcal{D}_X^{\mathrm{op}}$. It is easy to check that ψ and φ are inverse to each other (cf. Remark 1.12). $\square$

COROLLARY 1.11. *The category* $\mathrm{Mod}(\mathcal{D}_X^{\mathrm{op}})$ *of right $\mathcal{D}_X$-modules and the category* $\mathrm{Mod}(\mathcal{D}_X)$ *of left $\mathcal{D}_X$-modules are equivalent to each other by*

$$\mathrm{Mod}(\mathcal{D}_X) \ni \mathcal{M} \mapsto \Omega_X \otimes_{\mathcal{O}_X} \mathcal{M} \in \mathrm{Mod}(\mathcal{D}_X^{\mathrm{op}}).$$

REMARK 1.12. Let $(x_1, \cdots, x_n)$ be a coordinate system, and let dx denote $dx_1 \cdots dx_n$. Then the isomorphism (1.28) is explicitly given by

$$(\mathcal{D}_X)^{\mathrm{op}} \ni P \mapsto dx \otimes P^* \otimes dx^{\otimes -1} \in \Omega_X \otimes_{\mathcal{O}_X} \mathcal{D}_X \otimes_{\mathcal{O}_X} \Omega_X^{\otimes -1},$$

where

$$P^* = \sum (-1)^{|\alpha|} \partial^\alpha a_\alpha(x)$$

is the *formal adjoint* of $P = \sum a_\alpha(x) \partial^\alpha$. We have $(PQ)^* = Q^* P^*$ and $P^{**} = P$. The latter corresponds to the claim that φ and ψ are inverse to each other in the proof of Proposition 1.10.

CHAPTER 2

Characteristic Varieties

2.1. Cotangent Bundles

The ring $\mathcal{D}_X$ is non-commutative. Hence the study of its structure is more complicated than algebraic geometry (the study of commutative rings). However, we can derive objects in the theory of commutative algebra from $\mathcal{D}_X$ as follows.

Take a local coordinate system $(x_1, \ldots, x_n)$. For a differential operator $P = \sum_{\alpha \in \mathbb{Z}^n_{\geq 0}} a_\alpha(x)\partial^\alpha$, we call

$$p(x, \xi) := \sum a_\alpha(x)\xi^\alpha$$

the *total symbol* of P, where $\xi^\alpha := \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n}$ for $\alpha = (\alpha_1, \ldots, \alpha_n)$. This is a function in $(x, \xi) = (x_1, \ldots, x_n, \xi_1, \ldots, \xi_n)$, highly depending on the choice of a coordinate system (cf. (7.7)). Let $Q = \sum b_\alpha(x)\partial_x^\alpha$ be another differential operator, and $q(x, \xi) = \sum b_\alpha(x)\xi^\alpha$ its total symbol. Then the total symbol $r(x, \xi)$ of $R = PQ$ is given by the following formula:

$$r(x, \xi) = \sum_{\alpha \in \mathbb{Z}^n_{\geq 0}} \frac{1}{\alpha!}(\partial_\xi^\alpha p(x, \xi))(\partial_x^\alpha q(x, \xi)) .$$

where $\alpha! = \prod_{i=1}^n \alpha_i!$. Since $p(x, \xi)$ is a polynomial in ξ, we have $\partial_\xi^\alpha p(x, \xi) = 0$ except for finitely many α, and thus the right hand side of the above formula is a finite sum in α. The formula is easily deduced from Leibniz's rule

$$\partial_x^\alpha(a(x)b(x)) = \sum_{\beta \in \mathbb{Z}^n_{\geq 0}} \binom{\alpha}{\beta}(\partial_x^\beta a)(x)(\partial_x^{\alpha-\beta} b)(x),$$

where $\binom{\alpha}{\beta} = \prod_{i=1}^{n} \binom{\alpha_i}{\beta_i}$ for $\alpha = (\alpha_1, \dots, \alpha_n)$ and $\beta = (\beta_1, \dots, \beta_n)$. Set

$$F_m(\mathcal{D}_X) = \{ P \in \mathcal{D}_X \,;\, P = \sum_{\substack{|\alpha| \le m \\ \alpha \in \mathbb{Z}^n_{\ge 0}}} a_\alpha(x) \partial_x^\alpha \},$$

where $|\alpha| = \alpha_1 + \cdots + \alpha_n$. Although this definition uses a coordinate system, the following proposition shows that it does not depend on the choice of a coordinate system.

PROPOSITION 2.1. (1) *$F_m(\mathcal{D}_X) = 0$ for $m < 0$.*

(2) *For $m \ge 0$,*

$$F_m(\mathcal{D}_X) = \{ P \in \mathcal{D}_X \,;\, [P, \mathcal{O}_X] \subset F_{m-1}(\mathcal{D}_X) \}.$$

(3) $F_0(\mathcal{D}_X) = \mathcal{O}_X$, $F_1(\mathcal{D}_X) = \mathcal{O}_X \oplus \Theta_X$.

Hence we can inductively define $F_m(\mathcal{D}_X)$ by (1) and (2).

Let p, q be the total symbols of $P \in F_{m_1}(\mathcal{D}_X)$, $Q \in F_{m_2}(\mathcal{D}_X)$ respectively. Put $p_j = \sum_{|\alpha|=j} p_\alpha(x)\xi^\alpha$ and $q_j = \sum_{|\alpha|=j} q_\alpha(x)\xi^\alpha$. Let r be the total symbol of PQ. Then

$$\begin{aligned} (2.1) \quad r \;=\;& p_{m_1}(x,\xi) q_{m_2}(x,\xi) \\ &+ \left(p_{m_1-1}(x,\xi) q_{m_2}(x,\xi) + p_{m_1}(x,\xi) q_{m_2-1}(x,\xi) \right. \\ &\quad \left. + \sum_i \frac{\partial p_{m_1}}{\partial \xi_i} \frac{\partial q_{m_2}}{\partial x_i} \right) \\ &+ (\text{terms of degree less than } m_1 + m_2 - 1 \text{ in } \xi). \end{aligned}$$

This immediately gives the following proposition.

PROPOSITION 2.2. (1) $F_{m_1}(\mathcal{D}_X) F_{m_2}(\mathcal{D}_X) \subset F_{m_1+m_2}(\mathcal{D}_X)$.

(2) $[F_{m_1}(\mathcal{D}_X), F_{m_2}(\mathcal{D}_X)] \subset F_{m_1+m_2-1}(\mathcal{D}_X)$.

Set

$$\mathrm{Gr}^F_n(\mathcal{D}_X) = F_n(\mathcal{D}_X)/F_{n-1}(\mathcal{D}_X), \quad \mathrm{Gr}^F(\mathcal{D}_X) = \bigoplus_{n \in \mathbb{Z}} \mathrm{Gr}^F_n(\mathcal{D}_X).$$

The multiplication

$$F_n(\mathcal{D}_X) \times F_m(\mathcal{D}_X) \to F_{n+m}(\mathcal{D}_X)$$

induces

$$\mathrm{Gr}^F_n(\mathcal{D}_X) \times \mathrm{Gr}^F_m(\mathcal{D}_X) \to \mathrm{Gr}^F_{n+m}(\mathcal{D}_X).$$

This gives $\mathrm{Gr}^F(\mathcal{D}_X)$ a ring structure. By Proposition 2.2 (2), the ring $\mathrm{Gr}^F(\mathcal{D}_X)$ is commutative. Since

$$\mathrm{Gr}_0^F(\mathcal{D}_X) = \mathcal{O}_X, \quad \mathrm{Gr}_1^F(\mathcal{D}_X) = \Theta_X,$$

$\Theta_X \to \mathrm{Gr}_1^F(\mathcal{D}_X) \to \mathrm{Gr}^F(\mathcal{D}_X)$ can be extended to an $\mathcal{O}_X$-algebra homomorphism

$$S_{\mathcal{O}_X}(\Theta_X) \to \mathrm{Gr}^F(\mathcal{D}_X), \tag{2.2}$$

where $S_{\mathcal{O}_X}(\Theta_X)$ is the symmetric algebra of Θ_X over $\mathcal{O}_X$. By using local coordinates, $\Theta_X = \bigoplus_{i=1}^n \mathcal{O}_X \frac{\partial}{\partial x_i}$. Hence, by corresponding ξ_i to $\partial/\partial x_i$, we have

$$S_{\mathcal{O}_X}(\Theta_X) = \mathcal{O}_X \otimes \mathbb{C}[\xi_1, \dots, \xi_n].$$

Since

$$\mathrm{Gr}^F(\mathcal{D}_X) \quad = \bigoplus_{\alpha \in \mathbb{Z}^n_{\geq 0}} \mathcal{O}_X \partial_x^\alpha ,$$

the homomorphism (2.2), which is given by $\xi^\alpha \mapsto \partial_x^\alpha$, is an isomorphism.

We have thus obtained the following theorem.

THEOREM 2.3. *There is an isomorphism of graded rings*

$$S_{\mathcal{O}_X}(\Theta_X) \xrightarrow{\sim} \mathrm{Gr}^F(\mathcal{D}_X).$$

We denote by σ_m the homomorphism defined by $F_m(\mathcal{D}_X) \to \mathrm{Gr}_m^F(\mathcal{D}_X) \subset \mathrm{Gr}^F(\mathcal{D}_X) \simeq S_{\mathcal{O}_X}(\Theta_X)$. By using local coordinates, we have

$$\sigma_m(P) = \sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha \in S^m_{\mathcal{O}_X}(\Theta_X)$$

for $P = \sum_{|\alpha|\leq m} a_\alpha(x)\partial_x^\alpha$. This is called the *principal symbol* (of order m) of P. We have thus succeeded to derive an element of the commutative algebra $S_{\mathcal{O}_X}(\Theta_X)$, namely the principal symbol, from a differential operator. Although the principal symbol is only a part of a differential operator (indeed, it is the part of the highest degree of the total symbol), it carries a great deal of information on $\mathcal{D}_X$, as seen in the following.

Let $\pi_X : T^*X \to X$ denote the cotangent bundle. Then

$$S_{\mathcal{O}_X}(\Theta_X) \subset \pi_{X*}\mathcal{O}_{T^*X}.$$

Hence a section of $S_{\mathcal{O}_X}(\Theta_X)$ can be regarded as a function on T^*X. Let $P \in F_{m_1}(\mathcal{D}_X)$ and $Q \in F_{m_2}(\mathcal{D}_X)$. Then $[P, Q] \in F_{m_1+m_2-1}(\mathcal{D}_X)$. By Proposition 2.2 (2), the commutator $[\ ,\ ]$ induces a map

$$\mathrm{Gr}^F_{m_1}(\mathcal{D}_X) \times \mathrm{Gr}^F_{m_2}(\mathcal{D}_X) \to \mathrm{Gr}^F_{m_1+m_2-1}(\mathcal{D}_X).$$

This can be explicitly calculated by equation (2.1) in local coordinates:

$$\begin{aligned}&\sigma_{m_1+m_2-1}([P, Q])\\ &= \sum_i \left(\frac{\partial \sigma_{m_1}(P)}{\partial \xi_i} \frac{\partial \sigma_{m_2}(Q)}{\partial x_i} - \frac{\partial \sigma_{m_2}(Q)}{\partial \xi_i} \frac{\partial \sigma_{m_1}(P)}{\partial x_i} \right).\end{aligned}$$

Given functions $f(x, \xi)$ and $g(x, \xi)$ in x and ξ, set

$$\{f, g\} = \sum_i \left(\frac{\partial f}{\partial \xi_i} \frac{\partial g}{\partial x_i} - \frac{\partial g}{\partial \xi_i} \frac{\partial f}{\partial x_i} \right). \tag{2.3}$$

Then

$$\sigma_{m_1+m_2-1}([P, Q]) = \{\sigma_{m_1}(P), \sigma_{m_2}(Q)\}. \tag{2.4}$$

Considering f and g as functions on T^*X, we see that $\{\ ,\ \}$ is independent of the choice of a local coordinate system as below. We have a canonical 1-form ω_X on T^*X. For every point $p \in T^*X$, a 1-form ω_p at a point $\pi_X(p)$ of X is determined by the definition of T^*X. The canonical 1-form ω_X is defined by $\omega_X(p) = \pi_X^* \omega_p$. In local coordinates,

$$\omega_X = \sum_i \xi_i dx_i .$$

At every point p, the 2-form $\theta_X = d\omega_X$ gives an anti-symmetric bilinear form on $T_p(T^*X)$. This is nondegenerate and gives $H : T_p^*(T^*X) \xrightarrow{\sim} T_p(T^*X)$ by

$$\langle \theta_X, v \wedge H(\eta) \rangle = \langle \eta, v \rangle \quad (\eta \in T_p^*(T^*X),\ v \in T_p(T^*X)).$$

Explicitly in local coordinates,

$$H : \begin{aligned} d\xi_i &\mapsto \partial/\partial x_i , \\ dx_i &\mapsto -\partial/\partial \xi_i . \end{aligned} \tag{2.5}$$

In particular, $H_f = H(df)$ is a vector field on T^*X for a function f on T^*X. This is called a *Hamiltonian* of f.

DEFINITION 2.4. For functions f and g on T^*X, set $\{f, g\} = H_f(g)$ and call it the *Poisson bracket* of f and g.

By (2.5), the Poisson bracket $\{\ ,\ \}$ is expressed as (2.3) in local coordinates. This shows that $\{\ ,\ \}$ is determined by (T^*X, θ_X). A pair $(\mathcal{X}, \theta)$ of a manifold $\mathcal{X}$ and a closed 2-form θ on $\mathcal{X}$ is called a *symplectic manifold* if θ gives a nondegenerate anti-symmetric bilinear form on $T_p\mathcal{X}$ for every p. On such a manifold, we can define a Poisson bracket in the same way as above. Hence the Poisson bracket is a notion depending only on a symplectic structure of $\mathcal{X}$. By tracing back the above argument, we can determine a 2-form from the Poisson bracket.

In the formula (2.4), the commutator $[\ ,\ ]$ expresses the noncommutativity of $\mathcal{D}_X$. Hence, symbolically speaking, the noncommutativity of $\mathcal{D}_X$ determines a symplectic structure of T^*X.

2.2. Characteristic Varieties

In the previous section, we have obtained from $\mathcal{D}_X$, first a commutative object $S_{\mathcal{O}_X}(\Theta_X)$ by taking Gr^F, next the cotangent bundle T^*X as its geometric object, and finally the symplectic structure on T^*X as a reflection of the noncommutativity of $\mathcal{D}_X$.

In this section, we consider $\mathcal{D}_X$-modules and derive a commutative object from each of them. Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module (see the Appendix, § A.1, for coherent sheaves). Then $\mathcal{M}$ is locally generated by a finite number of sections $u_1, \dots, u_N$. Consider

$$F_m(\mathcal{M}) = \sum_{\nu=1}^{N} F_m(\mathcal{D}_X)u_\nu.$$

Then this satisfies the following:

$$\mathcal{M} = \bigcup_m F_m(\mathcal{M}). \tag{2.6}$$

$$F_m(\mathcal{D}_X)F_l(\mathcal{M}) \subset F_{m+l}(\mathcal{M}). \tag{2.7}$$

$$\bigoplus_m F_m(\mathcal{M}) \text{ is locally finitely generated as a } \bigoplus_m F_m(\mathcal{D}_X)\text{-module.} \tag{2.8}$$

DEFINITION 2.5. A sequence $\{F_m(\mathcal{M})\}_{m\in\mathbb{Z}}$ of subsheaves of $\mathcal{M}$ is called a *filtration* as a D-module if it satisfies (2.6) and (2.7), and a *coherent filtration* if it satisfies (2.8) in addition (see the Appendix, § A.1).

As seen in the above, a coherent $\mathcal{D}_X$-module locally has a coherent filtration. This is however not unique. In this section, while using a coherent filtration, we aim at extracting properties of a coherent $\mathcal{D}_X$-module independent of the choice of a coherent filtration. Let $\{F_m(\mathcal{M})\}$ be a coherent filtration of $\mathcal{M}$. Set

$$\text{(2.9)}\quad \mathrm{Gr}_m^F(\mathcal{M}) = F_m(\mathcal{M})/F_{m-1}(\mathcal{M}), \quad \mathrm{Gr}^F(\mathcal{M}) = \bigoplus_m \mathrm{Gr}_m^F(\mathcal{M}).$$

Then $\mathrm{Gr}^F(\mathcal{M})$ has a structure of $\mathrm{Gr}^F(\mathcal{D}_X)$-module, and it is a coherent $\mathrm{Gr}^F(\mathcal{D}_X)$-module. Since $\mathrm{Gr}^F(\mathcal{D}_X) \subset \pi_{X*}\mathcal{O}_{T^*X}$, we have $\pi_X^{-1}\mathrm{Gr}^F(\mathcal{D}_X) \subset \mathcal{O}_{T^*X}$.

For a module M over $S_{\mathcal{O}_X}(\Theta_X) \cong \mathrm{Gr}^F(\mathcal{D}_X)$, set

$$M^\sim = \mathcal{O}_{T^*X} \otimes_{\pi_X^{-1}\mathrm{Gr}^F(\mathcal{D}_X)} \pi_X^{-1}M.$$

It is known that $\mathcal{O}_{T^*X}$ is faithfully flat over $\pi_X^{-1}\mathrm{Gr}^F(\mathcal{D}_X)$. In particular, $M \mapsto M^\sim$ is an exact functor from $\mathrm{Mod}(\mathrm{Gr}^F\mathcal{D}_X)$ to $\mathrm{Mod}(\mathcal{O}_{T^*X})$, and $(\mathrm{Gr}^F\mathcal{M})^\sim$ is a coherent $\mathcal{O}_{T^*X}$-module.

THEOREM 2.6. $\mathrm{Supp}((\mathrm{Gr}^F\mathcal{M})^\sim) \subset T^*X$ *is independent of the choice of a coherent filtration* $F(\mathcal{M})$ *of* $\mathcal{M}$.

PROOF. Let $F'(\mathcal{M})$ be another coherent filtration of $\mathcal{M}$. We need to show that

$$\text{(2.10)}\qquad \mathrm{Supp}((\mathrm{Gr}^{F'}\mathcal{M})^\sim) = \mathrm{Supp}((\mathrm{Gr}^F\mathcal{M})^\sim).$$

By the assumption (2.8), locally for N large enough we have $F'_m(\mathcal{M}) \subset F_{m+N}(\mathcal{M})$ $(\forall m \in \mathbb{Z})$. Then $F''_m(\mathcal{M}) := F_{m+N}(\mathcal{M})$ is also a coherent filtration and satisfies $\mathrm{Gr}^{F''}(\mathcal{M}) = \mathrm{Gr}^F(\mathcal{M})$. Hence, replacing F by F'', we can assume $F'_m(\mathcal{M}) \subset F_m(\mathcal{M})$ $(\forall m \in \mathbb{Z})$ without loss of generality. For N large enough, we have

$$\text{(2.11)}\qquad F'_m(\mathcal{M}) \subset F_m(\mathcal{M}) \subset F'_{m+N}(\mathcal{M}) \quad (\forall m \in \mathbb{Z}).$$

We shall prove (2.10) by induction on $N \geq 0$.

When $N = 0$, it is trivial.

When $N = 1$, we have the following exact sequences:

$$\text{(2.12)}\quad 0 \to \bigoplus_m \frac{F'_m(\mathcal{M})}{F_{m-1}(\mathcal{M})} \to \bigoplus_m \frac{F_m(\mathcal{M})}{F_{m-1}(\mathcal{M})} \to \bigoplus_m \frac{F_m(\mathcal{M})}{F'_m(\mathcal{M})} \to 0,$$

$$0 \to \bigoplus_m \frac{F_{m-1}(\mathcal{M})}{F'_{m-1}(\mathcal{M})} \to \bigoplus_m \frac{F'_m(\mathcal{M})}{F'_{m-1}(\mathcal{M})} \to \bigoplus_m \frac{F'_m(\mathcal{M})}{F_{m-1}(\mathcal{M})} \to 0.$$

Put

$$L' = \bigoplus_m \frac{F'_m(\mathcal{M})}{F_{m-1}(\mathcal{M})}, \quad L'' = \bigoplus_m \frac{F_m(\mathcal{M})}{F'_m(\mathcal{M})}.$$

Then they are coherent $\mathrm{Gr}^F(\mathcal{D}_X)$-modules. By (2.12) we have two exact sequences of $\mathrm{Gr}^F(\mathcal{D}_X)$-modules:

$$0 \to L' \to \mathrm{Gr}^F(\mathcal{M}) \to L'' \to 0,$$
$$0 \to L'' \to \mathrm{Gr}^{F'}(\mathcal{M}) \to L' \to 0.$$

Since $\sim$ is an exact functor, we obtain two exact sequences of $\mathcal{O}_{T^*X}$-modules:

$$0 \to \tilde{L}' \to (\mathrm{Gr}^F\mathcal{M})^\sim \to \tilde{L}'' \to 0,$$
$$0 \to \tilde{L}'' \to (\mathrm{Gr}^{F'}\mathcal{M})^\sim \to \tilde{L}' \to 0.$$

Hence we have

$$\mathrm{Supp}((\mathrm{Gr}^F\mathcal{M})^\sim) = \mathrm{Supp}\tilde{L}' \cup \mathrm{Supp}\tilde{L}'' = \mathrm{Supp}((\mathrm{Gr}^{F'}\mathcal{M})^\sim). \tag{2.13}$$

We have thus proved (2.10) in the case $N = 1$.

Next we prove (2.10) when $N > 1$. Put $F''_m(\mathcal{M}) = F_{m-1}(\mathcal{M}) + F'_m(\mathcal{M})$. Then this is also a coherent filtration, and satisfies

$$F''_m(\mathcal{M}) \subset F_m(\mathcal{M}) \subset F''_{m+1}(\mathcal{M}).$$

Hence, as above, we have

$$\mathrm{Supp}((\mathrm{Gr}^F\mathcal{M})^\sim) = \mathrm{Supp}((\mathrm{Gr}^{F''}\mathcal{M})^\sim). \tag{2.14}$$

We also have

$$F'_m(\mathcal{M}) \subset F''_m(\mathcal{M}) \subset F'_{m+N-1}(\mathcal{M}).$$

Hence, by the induction hypothesis, we obtain

$$\mathrm{Supp}((\mathrm{Gr}^{F'}\mathcal{M})^\sim) = \mathrm{Supp}((\mathrm{Gr}^{F''}\mathcal{M})^\sim). \tag{2.15}$$

Combining (2.14) and (2.15), we obtain

$$\mathrm{Supp}((\mathrm{Gr}^F\mathcal{M})^\sim) = \mathrm{Supp}((\mathrm{Gr}^{F'}\mathcal{M})^\sim).$$

□

By this theorem, $\mathrm{Supp}((\mathrm{Gr}^F\mathcal{M})^\sim)$ does not depend on the choice of F, and thus globally defines a closed analytic subset of T^*X. This is called the *characteristic variety* of $\mathcal{M}$ and denoted by $\mathrm{Ch}(\mathcal{M})$. The following is its precise definition.

DEFINITION 2.7. The characteristic variety $\mathrm{Ch}(\mathcal{M})$ of a coherent $\mathcal{D}_X$-module $\mathcal{M}$ is the unique closed subset of T^*X satisfying the following property: For any open subset U of X and any coherent filtration F of $\mathcal{M}|_U$, we have

$$\mathrm{Ch}(\mathcal{M}) \cap \pi_X^{-1}(U) = \mathrm{Supp}(\mathcal{O}_{T^*U} \otimes_{\pi_X^{-1}\mathrm{Gr}^F(\mathcal{D}_U)} \pi_X^{-1}\mathrm{Gr}^F(\mathcal{M}|_U)).$$

We give some properties of characteristic varieties of D-modules in what follows.

PROPOSITION 2.8.

$$\mathrm{Supp}(\mathcal{M}) = \pi_X(\mathrm{Ch}(\mathcal{M}))\,.$$

PROPOSITION 2.9. $\mathrm{Ch}(\mathcal{M})$ *is a homogeneous (i.e., invariant under the action of* $\mathbb{C}^*$ *on* T^*X*) closed analytic subset.*

These are obvious.

PROPOSITION 2.10. *If* $0 \to \mathcal{M}' \xrightarrow{\varphi} \mathcal{M} \xrightarrow{\psi} \mathcal{M}'' \to 0$ *is an exact sequence of coherent* $\mathcal{D}_X$*-modules, then*

$$\mathrm{Ch}(\mathcal{M}) = \mathrm{Ch}(\mathcal{M}') \cup \mathrm{Ch}(\mathcal{M}'')\,.$$

PROOF. Since this is local on X, we may assume that $\mathcal{M}$ has a coherent filtration F by Lemma A.26. Put $F_m(\mathcal{M}'') = \psi(F_m(\mathcal{M}))$ and $F_m(\mathcal{M}') = \varphi^{-1}(F_m(\mathcal{M}))$. Then $F(\mathcal{M}')$ and $F(\mathcal{M}'')$ are respectively coherent filtrations of $\mathcal{M}'$ and $\mathcal{M}''$ by Lemma A.28. Moreover, the sequence

$$0 \to \mathrm{Gr}^F(\mathcal{M}') \to \mathrm{Gr}^F(\mathcal{M}) \to \mathrm{Gr}^F(\mathcal{M}'') \to 0$$

is exact, and accordingly

$$0 \to (\mathrm{Gr}^F\mathcal{M}')^{\sim} \to (\mathrm{Gr}^F\mathcal{M})^{\sim} \to (\mathrm{Gr}^F\mathcal{M}'')^{\sim} \to 0$$

is also exact. Hence we obtain

$$\begin{aligned}\mathrm{Ch}(\mathcal{M}) &= \mathrm{Supp}(\mathrm{Gr}^F\mathcal{M})^{\sim} = \mathrm{Supp}(\mathrm{Gr}^F\mathcal{M}')^{\sim} \cup \mathrm{Supp}(\mathrm{Gr}^F\mathcal{M}'')^{\sim}\\ &= \mathrm{Ch}(\mathcal{M}') \cup \mathrm{Ch}(\mathcal{M}'').\end{aligned}$$

□

Suppose that a $\mathcal{D}_X$-module $\mathcal{M}$ is generated by a single element u. Put $\mathcal{I} = \{P \in \mathcal{D}_X; Pu = 0\}$. Then $\mathcal{M} \cong \mathcal{D}_X/\mathcal{I}$. Introduce a filtration of $\mathcal{M}$ by $F_m(\mathcal{M}) = F_m(\mathcal{D}_X)u$. Then, by definition, $\mathrm{Ch}(\mathcal{M})$ equals $\mathrm{Supp}((\mathrm{Gr}^F\mathcal{M})^{\sim})$. Put $F_m(\mathcal{I}) = \mathcal{I} \cap F_m(\mathcal{D}_X)$. Since $\mathrm{Gr}^F(\mathcal{M}) = \mathrm{Gr}^F(\mathcal{D}_X)/\mathrm{Gr}^F(\mathcal{I})$, we have

$$\mathrm{Ch}(\mathcal{M}) = \{\, p \in T^*X\,;\ \sigma_m(P)(p) = 0 \text{ for all } m \text{ and all } P \in F_m(\mathcal{I})\,\}.$$

Hence, for $\mathcal{I} = \sum \mathcal{D}_X P_j$ with $P_j \in F_{m_j}(\mathcal{D}_X)$, we have

$$\mathrm{Ch}(\mathcal{M}) \subset \bigcap_j \sigma_{m_j}(P_j)^{-1}(0).$$

Here the equality does not however hold in general, for we do not have

$$\mathrm{Gr}^F(\mathcal{I}) = \sum(\mathrm{Gr}^F\mathcal{D}_X)\sigma_{m_j}(P_j) \tag{2.16}$$

in general. We call $\{P_j\}$ an *involutive system of generators* of $\mathcal{I}$ if equation (2.16) holds. Generally speaking, it is not easy to check whether a given system of generators of a coherent ideal of $\mathcal{D}_X$ is involutive or not.

The condition of involutivity (2.16) can be restated as follows:

LEMMA 2.11. *A set of operators $\{P_j\}$, with $P_j \in F_{m_j}(\mathcal{D}_X)$, is an involutive system of $\sum \mathcal{D}_X P_j$ if and only if for $l \in \mathbb{Z}_{\geq 0}$ and $a_j \in S^{l-m_j}_{\mathcal{O}_X}(\Theta_X) = \mathrm{Gr}^F_{l-m_j}(\mathcal{D}_X)$ satisfying $\sum a_j\sigma_{m_j}(P_j) = 0$ there locally exist $A_j \in F_{l-m_j}(\mathcal{D}_X)$ with $a_j = \sigma_{l-m_j}(A_j)$ such that $\sum A_jP_j = 0$.*

PROOF. Assume that $\{P_j\}$ is an involutive system of generators of $\mathcal{I} := \sum \mathcal{D}_X P_j$. We shall first show that

$$F_l(\mathcal{I}) \subset \sum F_{l-m_j}(\mathcal{D}_X)P_j \tag{2.17}$$

by induction on l. By (2.16) we have

$$F_l(\mathcal{I}) \subset \sum F_{l-m_j}(\mathcal{D}_X)P_j + F_{l-1}(\mathcal{I}),$$

and the induction hypothesis implies

$$F_{l-1}(\mathcal{I}) \subset \sum F_{l-1-m_j}(\mathcal{D}_X)P_j.$$

Hence we have shown (2.17). Now assume that $a_j \in S^{l-m_j}(\Theta_X) = \mathrm{Gr}^F_{l-m_j}(\mathcal{D}_X)$ satisfy $\sum a_j\sigma_{m_j}(P_j) = 0$. Take $A'_j \in F_{l-m_j}(\mathcal{D}_X)$ such that $\sigma_{l-m_j}(A'_j) = a_j$. Then $B := \sum A'_jP_j$ belongs to $F_{l-1}(\mathcal{I})$, and hence (2.17) implies that there exist $A''_j \in F_{l-1-m_j}(\mathcal{D}_X)$ such that $B = \sum A''_jP_j$. Then $A_j := A'_j - A''_j$ satisfy $\sigma_{l-m_j}(A_j) = a_j$ and $\sum A_jP_j = 0$.

Next we shall prove the converse. We have

$$F_{l-1}(\mathcal{D}_X) \cap (\sum F_{l-m_j}(\mathcal{D}_X)P_j) \subset \sum F_{l-1-m_j}(\mathcal{D}_X)P_j. \tag{2.18}$$

Indeed, suppose $B_j \in F_{l-m_j}(\mathcal{D}_X)$ and $C := \sum B_jP_j \in F_{l-1}(\mathcal{D}_X)$. Then $a_j := \sigma_{l-m_j}(B_j)$ satisfy $\sum a_j\sigma_{m_j}(P_j) = 0$. Hence we can find $A_j \in F_{l-m_j}(\mathcal{D}_X)$ with $\sigma_{l-m_j}(A_j) = a_j$ such that $\sum A_jP_j = 0$. Since $C_j := B_j - A_j$ belong to $F_{l-1-m_j}(\mathcal{D}_X)$, and $C = \sum C_jP_j$ belongs to

$\sum F_{l-1-m_j}(\mathcal{D}_X)P_j$, we have (2.18). By induction on l', (2.18) leads to

$$F_l(\mathcal{D}_X)\cap\left(\sum F_{l'-m_j}(\mathcal{D}_X)P_j\right)\subset\sum F_{l-m_j}(\mathcal{D}_X)P_j$$

for $l'>l$. Therefore we obtain (2.17) and then (2.16). □

From results in algebraic geometry, we have the following sufficient condition.

PROPOSITION 2.12. *If $P_j\in F_{m_j}(\mathcal{D}_X)$ $(j=1,\dots,N)$ satisfy the following conditions, then $\{P_j\}$ is an involutive system of $\mathcal{I}=\sum\mathcal{D}_XP_j$.*

(1) *$\bigcap_{j=1}^N\sigma_{m_j}(P_j)^{-1}(0)$ is of codimension N.*

(2)

$$[P_j,P_k]=\sum_l Q_{jkl}P_l$$

for some $Q_{jkl}\in F_{m_j+m_k-m_l-1}(\mathcal{D}_X)$.

PROOF. Let $p_j=\sigma_{m_j}(P_j)$.

As is well known in algebraic geometry, the condition (1) implies the following:

If $a_j\in\mathrm{Gr}^F_{l-m_j}(\mathcal{D}_X)$ satisfy $\sum a_jp_j=0$, then there exist $r_{ij}\in\mathrm{Gr}^F_{l-m_i-m_j}(\mathcal{D}_X)$ such that

$$a_i=\sum_j r_{ij}p_j,\quad r_{ij}=-r_{ji}\,.$$

Take $R_{ij}\in F_{l-m_i-m_j}(\mathcal{D}_X)$ with $\sigma_{l-m_i-m_j}(R_{ij})=r_{ij}$ and $R_{ij}=-R_{ji}$. Then $A_i=\sum_j R_{ij}P_j$ satisfy $\sigma_{l-m_i}(A_i)=a_i$ and

$$\begin{aligned}\sum_i A_iP_i&=\sum_{i,j}R_{ij}P_jP_i\\&=\sum_{i<j}R_{ij}[P_j,P_i]\\&=\sum_{i<j}\sum_k R_{ij}Q_{jik}P_k\,.\end{aligned}$$

Put $S_k=\sum_{i<j}R_{ij}Q_{jik}$. Then $S_k\in F_{l-m_k-1}(\mathcal{D}_X)$, $\sum(A_i-S_i)P_i=0$, and $\sigma_{l-m_i}(A_i-S_i)=a_i$. □

From the proof of Theorem 2.6, we see the following. Given a closed analytic subset V of T^*X, let $\mathrm{Mod}_V(\mathcal{O}_{T^*X})$ denote the abelian category of coherent $\mathcal{O}_{T^*X}$-modules with support on V, and $K(V)$

its Grothendieck group. Namely, $K(V)$ is the $\mathbb{Z}$-module spanned by $\{[F]; F \in \mathrm{Mod}_V(\mathcal{O}_{T^*X})\}$ with the defining relations $[F'] + [F''] = [F]$ for all exact sequences $0 \to F' \to F \to F'' \to 0$. Then, as an element of $K(\mathrm{Ch}(\mathcal{M}))$, $[\mathrm{Gr}^F(\mathcal{M})^\sim]$ does not depend on the choice of a coherent filtration F. In particular, for an irreducible closed analytic subset V of T^*X, the multiplicity of $\mathrm{Gr}^F(\mathcal{M})^\sim$ on V (see for example [T], §3.5) does not depend on the choice of a coherent filtration F, when V is an irreducible component of $V \cup \mathrm{Ch}(\mathcal{M})$. We call this the *multiplicity* of $\mathcal{M}$ on V and denote it by $\mathrm{mult}_V(\mathcal{M})$. Indeed, mult_V gives a homomorphism

$$\mathrm{mult}_V : K(\mathrm{Ch}(\mathcal{M})) \to \mathbb{Z}.$$

Hence we have the following proposition.

PROPOSITION 2.13. *Let $0 \to \mathcal{M}' \to \mathcal{M} \to \mathcal{M}'' \to 0$ be an exact sequence of coherent $\mathcal{D}_X$-modules, and V an irreducible component of $V \cup \mathrm{Ch}(\mathcal{M})$. Then*

$$\mathrm{mult}_V(\mathcal{M}) = \mathrm{mult}_V(\mathcal{M}') + \mathrm{mult}_V(\mathcal{M}'').$$

PROPOSITION 2.14. *Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module, and $f : \mathcal{M} \to \mathcal{M}$ a monomorphism of $\mathcal{D}_X$-modules. Then $\mathrm{Ch}(\mathcal{M}/f(\mathcal{M}))$ is a nowhere dense subset of $\mathrm{Ch}(\mathcal{M})$.*

PROOF. Let F be a coherent filtration of $\mathcal{N} := \mathcal{M}/f(\mathcal{M})$. By the exact sequence

$$0 \to \mathcal{M} \to \mathcal{M} \to \mathcal{N} \to 0,$$

we have

$$\mathrm{mult}_V(\mathcal{N}) = \mathrm{mult}_V(\mathcal{M}) - \mathrm{mult}_V(\mathcal{M}) = 0$$

for any irreducible component V of $\mathrm{Ch}(\mathcal{M})$. Hence the multiplicity of $(\mathrm{Gr}^F\mathcal{N})^\sim$ on V is 0. Therefore the support of $(\mathrm{Gr}^F\mathcal{N})^\sim$ cannot contain V. □

2.3. Involutivity

As explained in the sequel, the characteristic varieties of coherent $\mathcal{D}_X$-modules cannot be arbitrary and have an 'involutivity' property. Let $\mathcal{M} = \mathcal{D}_X/\mathcal{I}$ ($\mathcal{I}$ is a coherent left ideal of $\mathcal{D}_X$). Then $\mathrm{Ch}(\mathcal{M}) = \{p \in T^*X\,;\, a(p) = 0 \text{ for all } a \in \mathrm{Gr}^F(\mathcal{I})\}$.

Conversely, suppose that $\mathrm{Gr}^F(\mathcal{I}) = \{a \in \mathrm{Gr}^F(\mathcal{D}_X)\,;\, a|_{\mathrm{Ch}(\mathcal{M})} = 0\}$. For a and b with $a|_{\mathrm{Ch}(\mathcal{M})} = b|_{\mathrm{Ch}(\mathcal{M})} = 0$, take A and B in $\mathcal{I}$ with

$a = \sigma(A)$ and $b = \sigma(B)$. Then

$$\{a, b\} = \sigma([A, B])$$

vanishes on $\mathrm{Ch}(\mathcal{M})$, since $[A, B] \in \mathcal{I}$. In general, a closed analytic subset V of T^*X is said to be *involutive* if its defining ideal

$$I_V := \{ a \in \mathcal{O}_{T^*X} \, ; \, a|_V = 0 \}$$

satisfies $\{I_V, I_V\} \subset I_V$ (cf. § A.4). We have the following theorem.

THEOREM 2.15. *Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module. Then* $\mathrm{Ch}(\mathcal{M})$ *is involutive.*

This is not easy to prove. If there exists a coherent filtration F of $\mathcal{M}$ such that $a\,\mathrm{Gr}^F(\mathcal{M}) = 0$ for all $a \in \mathrm{Gr}^F(\mathcal{D}_X)$ satisfying $a|_{\mathrm{Ch}(\mathcal{M})} = 0$, then it is not hard to prove that $\mathrm{Ch}(\mathcal{M})$ is involutive by the same argument as above. However we do not have such a filtration F in general. We shall give a proof of this theorem in § 7.6.

Let us admit this theorem for the time being and proceed with our arguments. Since the dimension of any involutive closed analytic subset is greater than or equal to $\dim X$, we obtain the following proposition.

PROPOSITION 2.16. *For every coherent $\mathcal{D}_X$-module $\mathcal{M}$, the dimension of* $\mathrm{Ch}(\mathcal{M})$ *at every point is greater than or equal to* $\dim X$.

Since $\dim T^*X = 2 \dim X$, the dimension of $\mathrm{Ch}(\mathcal{M})$ is greater than or equal to half of that of T^*X. Roughly speaking, we should have $\dim X - \mathrm{codim}\mathrm{Ch}(\mathcal{M}) \geq 0$, because solutions to $\mathcal{M}$ are expressed as functions in $(\dim X - \mathrm{codim}\mathrm{Ch}(\mathcal{M}))$-many independent variables.

2.4. Codimension Filtration

In algebraic geometry, it is known that, for a coherent $\mathcal{O}_X$-module $\mathcal{F}$ and a closed analytic subset Z of a manifold X, $\Gamma_Z(\mathcal{F}) = \{ u \in \mathcal{F} \, ; \, \mathrm{supp}\, u \subset Z \}$ is coherent. We consider an analogous problem: for a coherent $\mathcal{D}_X$-module $\mathcal{M}$ and a closed analytic subset V of T^*X, is $\{ u \in \mathcal{M} \, ; \, \mathrm{Ch}(\mathcal{D}_X u) \subset V \}$ a coherent $\mathcal{D}_X$-module? The answer is not affirmative in general, as in the following example.

EXAMPLE 2.17. Let $X = \mathbb{C}$, $V = T^*_X X = \{ (x, \xi) \, ; \, \xi = 0 \} \subset T^*X$, and $\lambda \in \mathbb{C} \setminus \mathbb{Z}$. Put $\mathcal{M} = \mathcal{D}_X / \mathcal{D}_X (x\partial_x - \lambda)$ and $\mathcal{N} = \{ u \in \mathcal{M} \, ; \, \mathrm{Ch}(\mathcal{D}_X u) \subset V \}$. Then

$$\mathcal{N}|_{X - \{0\}} = \mathcal{M}|_{X - \{0\}},$$

since

$$\mathrm{Ch}(\mathcal{M}) = T^*_X X \cup T^*_{\{0\}} X.$$

Furthermore $\mathcal{N}_0 = 0$, since $\mathcal{M}_0$ is an irreducible $(\mathcal{D}_X)_0$-module. Thus $\mathcal{N} = \mathcal{M}_{X\setminus\{0\}}$, and hence $\mathcal{N}$ is not coherent.

However, we have the following theorem.

THEOREM 2.18. *For any coherent $\mathcal{D}_X$-module $\mathcal{M}$ and any $k \in \mathbb{Z}$,*

$$\{\, u \in \mathcal{M}\,;\ \mathrm{codim}(\mathrm{Ch}(\mathcal{D}_X u)) \geq k \,\}$$

is a coherent $\mathcal{D}_X$-submodule of $\mathcal{M}$.

To prove this theorem, we first prove the following theorem.

THEOREM 2.19. *Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module. Then:*

(1) $\mathcal{E}xt^k_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_X) = 0$ *for* $k < \mathrm{codim}(\mathrm{Ch}(\mathcal{M}))$.

(2) $\mathrm{codim}(\mathrm{Ch}(\mathcal{E}xt^k_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_X))) \geq k$.

(3) $\mathrm{Ch}(\mathcal{E}xt^k_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_X)) \subset \mathrm{Ch}(\mathcal{M})$.

Here $\mathcal{E}xt^k_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_X)$ is a right coherent $\mathcal{D}_X$-module, and

$$\mathrm{Ch}(\mathcal{E}xt^k_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_X))$$

denotes the characteristic variety of the corresponding left $\mathcal{D}_X$-module $\mathcal{E}xt^k_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_X) \otimes_{\mathcal{O}_X} \Omega_X^{\otimes -1}$.

The following counterpart in commutative algebra to this theorem is well known (cf. [Hotta2], §8. 6).

PROPOSITION 2.20. *Let Z be a nonsingular variety, and $\mathcal{F}$ a coherent $\mathcal{O}_Z$-module. Then:*

(1) $\mathcal{E}xt^k_{\mathcal{O}_Z}(\mathcal{F}, \mathcal{O}_Z) = 0$ *for* $k < \mathrm{codim}(\mathrm{Supp}\,\mathcal{F})$.

(2) $\mathrm{codim}(\mathrm{Supp}(\mathcal{E}xt^k_{\mathcal{O}_Z}(\mathcal{F}, \mathcal{O}_Z))) \geq k$.

We shall prove Theorem 2.19 by reducing it to Proposition 2.20.

Locally we have an exact sequence

$$0 \leftarrow \mathcal{M} \leftarrow \mathcal{L}_0 \xleftarrow{d} \mathcal{L}_1 \xleftarrow{d} \cdots, \tag{2.19}$$

where $\mathcal{L}_j = \bigoplus_k \mathcal{D}_X u_{jk}$ are finitely generated free $\mathcal{D}_X$-modules. Take integers a_{jk} and define a coherent filtration $F(\mathcal{L}_j)$ of $\mathcal{L}_j$ by

$$F_m(\mathcal{L}_j) = \bigoplus_k F_{m-a_{jk}}(\mathcal{D}_X) u_{jk}.$$

Let $F(\mathcal{M})$ be the filtration defined as the image of the coherent filtration $F(\mathcal{L}_0)$ of $\mathcal{L}_0$. Then this is also coherent. Moreover, we can take an exact sequence (2.19) so that d preserves the filtrations F, and the sequences

$$0 \leftarrow \mathrm{Gr}_m^F(\mathcal{M}) \leftarrow \mathrm{Gr}_m^F(\mathcal{L}_0) \leftarrow \mathrm{Gr}_m^F(\mathcal{L}_1) \leftarrow \cdots \tag{2.20}$$

are exact for all m.

Put $\mathcal{L}_j^* = \mathcal{H}om_{\mathcal{D}_X}(\mathcal{L}_j, \mathcal{D}_X)$. Then this is a right $\mathcal{D}_X$-module, and we have a complex of right $\mathcal{D}_X$-modules

$$\mathcal{L}_\bullet^* : 0 \to \mathcal{L}_0^* \to \mathcal{L}_1^* \to \cdots .$$

By the definition of $\mathcal{E}xt$,

$$\mathcal{E}xt^k_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_X) = \mathcal{H}^k(\mathcal{L}_\bullet^*), \tag{2.21}$$

and

$$\begin{aligned} \mathcal{E}xt^k_{\mathrm{Gr}^F(\mathcal{D}_X)}&(\mathrm{Gr}^F(\mathcal{M}), \mathrm{Gr}^F(\mathcal{D}_X)) \\ &= \mathcal{H}^k(\mathcal{H}om_{\mathrm{Gr}^F(\mathcal{D}_X)}(\mathrm{Gr}^F(\mathcal{L}_\bullet), \mathrm{Gr}^F(\mathcal{D}_X)). \end{aligned} \tag{2.22}$$

Define a filtration $F(\mathcal{L}_j^*)$ of the right $\mathcal{D}_X$-module

$$\mathcal{L}_j^* = \mathcal{H}om_{\mathcal{D}_X}(\mathcal{L}_j, \mathcal{D}_X) = \bigoplus_k v_{jk}\mathcal{D}_X$$

(v_{jk} is the basis dual to u_{jk}) by

$$F_m(\mathcal{L}_j^*) = \{\varphi \in \mathcal{H}om_{\mathcal{D}_X}(\mathcal{L}_j, \mathcal{D}_X); \varphi(F_k(\mathcal{L}_j)) \subset F_{k+m}(\mathcal{D}_X)\ (\forall k)\}.$$

Then $F_m(\mathcal{L}_j^*) = \bigoplus_k v_{jk} F_{m+a_{jk}}(\mathcal{D}_X)$, and it is a coherent filtration of $\mathcal{L}_j^*$. Furthermore we have

$$\mathcal{H}om_{\mathrm{Gr}^F(\mathcal{D}_X)}(\mathrm{Gr}^F(\mathcal{L}_j), \mathrm{Gr}^F(\mathcal{D}_X)) = \mathrm{Gr}^F(\mathcal{L}_j^*),$$

and thus by (2.22)

$$\mathcal{E}xt^k_{\mathrm{Gr}^F(\mathcal{D}_X)}(\mathrm{Gr}^F(\mathcal{M}), \mathrm{Gr}^F(\mathcal{D}_X)) = \mathcal{H}^k(\mathrm{Gr}^F(\mathcal{L}_\bullet^*)).$$

Put $\mathcal{Z}^k = \mathrm{Ker}(\mathcal{L}_k^* \to \mathcal{L}_{k+1}^*)$ and $\mathcal{J}^k = \mathrm{Im}(\mathcal{L}_{k-1}^* \to \mathcal{L}_k^*)$. Then by definition $\mathcal{H}^k(\mathcal{L}_\bullet^*) = \mathcal{Z}^k/\mathcal{J}^k$. The filtration $F_m(\mathcal{Z}^k) := F_m(\mathcal{L}_k^*) \cap \mathcal{Z}^k$ of $\mathcal{Z}^k$ is coherent. Define a coherent filtration $F(\mathcal{H}^k(\mathcal{L}_\bullet^*))$ of $\mathcal{H}^k(\mathcal{L}_\bullet^*)$ as the image of $F(\mathcal{Z}^k)$. Then we have the following lemma.

LEMMA 2.21. *The graded module* $\mathrm{Gr}^F(\mathcal{H}^k(\mathcal{L}_\bullet^*))$ *is isomorphic to a subquotient of* $\mathcal{H}^k(\mathrm{Gr}^F(\mathcal{L}_\bullet^*))$.

PROOF. Let d^k denote the differential $\mathcal{L}_k^* \to \mathcal{L}_{k+1}^*$. Then

$$\begin{aligned} \mathcal{H}^k(\mathrm{Gr}_m^F(\mathcal{L}_\bullet^*)) &= \frac{F_m(\mathcal{L}_k^*) \cap (d^k)^{-1}(F_{m-1}(\mathcal{L}_{k+1}^*))}{F_{m-1}(\mathcal{L}_k^*) + d^{k-1}(F_m(\mathcal{L}_{k-1}^*))} \\ &\supset \frac{F_m(\mathcal{Z}^k)}{F_{m-1}(\mathcal{Z}^k) + d^{k-1}F_m(\mathcal{L}_{k-1}^*)}. \end{aligned}$$

On the other hand, we have

$$\mathrm{Gr}_m^F(\mathcal{H}^k(\mathcal{L}_\bullet^*)) = \frac{F_m(\mathcal{Z}^k)}{F_{m-1}(\mathcal{Z}^k) + \mathcal{J}^k \cap F_m(\mathcal{Z}^k)}.$$

Since $F_{m-1}(\mathcal{Z}^k) + d^{k-1}F_m(\mathcal{L}_{k-1}^*) \subset F_{m-1}(\mathcal{Z}^k) + \mathcal{J}^k \cap F_m(\mathcal{Z}^k)$, we obtain the assertion. □

Consequently $\mathrm{Gr}^F(\mathcal{E}xt_{\mathcal{D}_X}^k(\mathcal{M}, \mathcal{D}_X))$ is isomorphic to a subquotient of $\mathcal{E}xt_{\mathrm{Gr}^F(\mathcal{D}_X)}^k(\mathrm{Gr}^F(\mathcal{M}), \mathrm{Gr}^F(\mathcal{D}_X))$, and thus

$$\mathrm{Ch}(\mathcal{E}xt_{\mathcal{D}_X}^k(\mathcal{M}, \mathcal{D}_X)) \subset \mathrm{Supp}(\mathcal{E}xt_{\mathcal{O}_{T^*X}}^k((\mathrm{Gr}^F\mathcal{M})^{\sim}, \mathcal{O}_{T^*X})). \tag{2.23}$$

We are now ready to prove Theorem 2.19 (1). By Proposition 2.20 (1)

$$\mathcal{E}xt_{\mathcal{O}_{T^*X}}^k(\mathrm{Gr}^F(\mathcal{M})^{\sim}, \mathcal{O}_{T^*X}) = 0, \quad k < \mathrm{codim}(\mathrm{Supp}(\mathrm{Gr}^F\mathcal{M})^{\sim}),$$

and by definition $\mathrm{Ch}(\mathcal{M}) = \mathrm{Supp}((\mathrm{Gr}^F\mathcal{M}))^{\sim}$.

Hence $\mathrm{Ch}(\mathcal{E}xt_{\mathcal{D}_X}^k(\mathcal{M}, \mathcal{D}_X)) = \emptyset$, when $k < \mathrm{codim}\mathrm{Ch}(\mathcal{M})$. We have thus proved (1).

We immediately obtain Theorem 2.19 (2), since the codimension of $\mathrm{Supp}(\mathcal{E}xt_{\mathcal{O}_{T^*X}}^k(\mathrm{Gr}^F(\mathcal{M})^{\sim}, \mathcal{O}_{T^*X}))$ is greater than or equal to k by Proposition 2.20 (2).

(3) is deduced from the inclusion

$$\mathrm{Supp}(\mathcal{E}xt_{\mathcal{O}_{T^*X}}^k(\mathrm{Gr}^F(\mathcal{M})^{\sim}, \mathcal{O}_{T^*X})) \subset \mathrm{Supp}\ \mathrm{Gr}^F(\mathcal{M})^{\sim}.$$

We have thus completed the proof of Theorem 2.19.

To prove Theorem 2.18, we introduce the following notation:

$$T_p(\mathcal{M}) := \mathcal{E}xt_{\mathcal{D}_X}^0(\tau^{\geq p}\mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_X), \mathcal{D}_X).$$

Here we have used notation from § A.2 of the Appendix. For example, $\tau^{\geq p}$ is given in (A.5). $T_p(\mathcal{M})$ is a coherent $\mathcal{D}_X$-module.

We obtain a distinguished triangle

$$\begin{aligned}&\mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\tau^{\geq p+1}\mathbb{R}\mathcal{H}om(\mathcal{M},\mathcal{D}_X),\mathcal{D}_X)\\ &\qquad\longrightarrow \mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\tau^{\geq p}\mathbb{R}\mathcal{H}om(\mathcal{M},\mathcal{D}_X),\mathcal{D}_X)\\ &\qquad\qquad\longrightarrow \mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{E}xt^p(\mathcal{M},\mathcal{D}_X)[-p],\mathcal{D}_X)\xrightarrow{+1}\end{aligned}$$

by applying $\mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\bullet,\mathcal{D}_X)$ to the distinguished triangle

$$\begin{aligned}\mathcal{E}xt^p(\mathcal{M},\mathcal{D}_X)[-p] &\longrightarrow \tau^{\geq p}\mathbb{R}\mathcal{H}om(\mathcal{M},\mathcal{D}_X)\\ &\longrightarrow \tau^{\geq p+1}\mathbb{R}\mathcal{H}om(\mathcal{M},\mathcal{D}_X) \xrightarrow{+1}\end{aligned}$$

given in (A.6). Take cohomology modules to obtain an exact sequence

$$\begin{aligned}&\mathcal{E}xt^{p-1}(\mathcal{E}xt^p(\mathcal{M},\mathcal{D}_X),\mathcal{D}_X)\\ &\qquad\to T_{p+1}(\mathcal{M})\to T_p(\mathcal{M})\to \mathcal{E}xt^p(\mathcal{E}xt^p(\mathcal{M},\mathcal{D}_X),\mathcal{D}_X).\end{aligned}$$

By Theorem 2.19, we have $\operatorname{codim}\operatorname{Ch}(\mathcal{E}xt^p(\mathcal{M},\mathcal{D}_X))\geq p$ and, in turn, $\mathcal{E}xt^{p-1}(\mathcal{E}xt^p(\mathcal{M},\mathcal{D}_X),\mathcal{D}_X)=0$. Hence we obtain an exact sequence

$$0\to T_{p+1}(\mathcal{M})\to T_p(\mathcal{M})\to \mathcal{E}xt^p(\mathcal{E}xt^p(\mathcal{M},\mathcal{D}_X),\mathcal{D}_X). \tag{2.24}$$

Since $\operatorname{codim}\operatorname{Ch}(\mathcal{E}xt^p(\mathcal{E}xt^p(\mathcal{M},\mathcal{D}_X),\mathcal{D}_X))\geq p$, we obtain

$$\operatorname{codim}\operatorname{Ch}(T_p(\mathcal{M}))\geq p \tag{2.25}$$

by descending induction on p. (Note that Theorem 3.7 leads to $T_p(\mathcal{M})=0$ $(p>\dim X)$.) Furthermore, by

$$\tau^{\geq 0}\mathbb{R}\mathcal{H}om(\mathcal{M},\mathcal{D}_X)\cong \mathbb{R}\mathcal{H}om(\mathcal{M},\mathcal{D}_X)$$

and

$$\mathcal{M}\cong \mathbb{R}\mathcal{H}om(\mathbb{R}\mathcal{H}om(\mathcal{M},\mathcal{D}_X),\mathcal{D}_X)$$

we have $T_0(\mathcal{M})=\mathcal{M}$ (cf. § A.3(b)). We have thus obtained a decreasing sequence of coherent $\mathcal{D}_X$-submodules of $\mathcal{M}$

$$T_{\dim X}(\mathcal{M})\subset T_{\dim X-1}(\mathcal{M})\subset\cdots\subset T_1(\mathcal{M})\subset T_0(\mathcal{M})=\mathcal{M}.$$

LEMMA 2.22.

$$\operatorname{codim}\ \operatorname{Ch}(\mathcal{M})\geq p \quad \textit{implies} \quad T_p(\mathcal{M})=\mathcal{M}.$$

PROOF. Since $\mathcal{E}xt^k(\mathcal{M},\mathcal{D}_X)=0$ $(k<p)$, we have

$$\mathbb{R}\mathcal{H}om(\mathcal{M},\mathcal{D}_X)\cong \tau^{\geq p}\mathbb{R}\mathcal{H}om(\mathcal{M},\mathcal{D}_X).$$

Hence $T_p(\mathcal{M})\cong\mathcal{M}$. □

Theorem 2.18 is deduced from the following theorem.

THEOREM 2.23.

$$T_p(\mathcal{M}) = \{ u \in \mathcal{M} ; \operatorname{codim} \operatorname{Ch}(\mathcal{D}_X u) \geq p \}.$$

PROOF. Since we have proved $\operatorname{codim}\operatorname{Ch}(T_p(\mathcal{M})) \geq p$, it remains to show that any $u \in \mathcal{M}$ satisfying $\operatorname{codim}\operatorname{Ch}(\mathcal{D}_X u) \geq p$ belongs to $T_p(\mathcal{M})$. Let $\mathcal{N} = \mathcal{D}_X u$. Then $\mathcal{N} = T_p(\mathcal{N})$ by Lemma 2.22. Hence we obtain $\mathcal{N} \subset T_p(\mathcal{M})$ by the commutative diagram

$$\begin{tikzcd} T_p(\mathcal{N}) \arrow[r, "\sim"] \arrow[d] & \mathcal{N} \arrow[d] \\ T_p(\mathcal{M}) \arrow[r, hook] & \mathcal{M}. \end{tikzcd}$$

□

Since $T_p(\mathcal{M})$ is coherent, we have obtained Theorem 2.18. Moreover, we have the following theorem.

THEOREM 2.24. *The codimension of every irreducible component of* $\operatorname{Ch}(T_p(\mathcal{M})/T_{p+1}(\mathcal{M}))$ *equals* p.

PROOF. Put $\mathcal{N} = T_p(\mathcal{M})/T_{p+1}(\mathcal{M})$. Then $T_p(\mathcal{N}) = \mathcal{N}$, and $T_{p+1}(\mathcal{N}) = 0$. Indeed, the inverse image of $T_{p+1}(\mathcal{N})$ under the canonical homomorphism $T_p(\mathcal{M}) \to \mathcal{N}$ is contained in $T_{p+1}(\mathcal{M})$, since the codimension of its characteristic variety is greater than or equal to $p+1$.

Hence it suffices to show that $\operatorname{Ch}(\mathcal{M})$ is purely p-codimensional when $\mathcal{M}$ satisfies $T_p(\mathcal{M}) = \mathcal{M}$ and $T_{p+1}(\mathcal{M}) = 0$. Let $\operatorname{Ch}(\mathcal{M}) = V_0 \cup V_1$, where V_0 is purely p-codimensional, and V_1 is of codimension $\geq p+1$. By (2.24) we have

$$\mathcal{M} \subset \mathcal{E}xt^p(\mathcal{E}xt^p(\mathcal{M}, \mathcal{D}_X), \mathcal{D}_X).$$

Let $\mathcal{N} = \mathcal{E}xt^p(\mathcal{M}, \mathcal{D}_X)$. Then

$$\operatorname{Ch}(\mathcal{M}) \subset \operatorname{Ch}(\mathcal{E}xt^p(\mathcal{N}, \mathcal{D}_X)) \subset \operatorname{Ch}(\mathcal{N}) \subset \operatorname{Ch}(\mathcal{M}),$$

and thus $\operatorname{Ch}(\mathcal{N}) = \operatorname{Ch}(\mathcal{M})$. For a coherent filtration F of $\mathcal{M}$, there exists a coherent filtration F of $\mathcal{N}$ such that $\operatorname{Gr}^F(\mathcal{N})$ is a subquotient of $\mathcal{E}xt^p(\operatorname{Gr}^F(\mathcal{M}), \operatorname{Gr}^F(\mathcal{D}))$. Proposition 2.20 (1) implies that $\mathcal{E}xt^p(\operatorname{Gr}^F(\mathcal{M}), \operatorname{Gr}^F(\mathcal{D}))^\sim$ vanishes outside V_0, and thus

$$\operatorname{Supp}(\mathcal{E}xt^p(\operatorname{Gr}^F(\mathcal{M}), \operatorname{Gr}^F(\mathcal{D}))^\sim) \subset V_0.$$

Therefore

$$\operatorname{Ch}(\mathcal{N}) = \operatorname{Supp}((\operatorname{Gr}^F \mathcal{N})^\sim) \subset V_0,$$

and $\mathrm{Ch}(\mathcal{M}) = \mathrm{Ch}(\mathcal{N}) = V_0$. $\square$

CHAPTER 3

Construction of D-modules

3.1. Tensor Products

Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be left $\mathcal{D}_X$-modules. Then there exists a natural left $\mathcal{D}_X$-module structure in $\mathcal{M}_1 \otimes_{\mathcal{O}_X} \mathcal{M}_2$. For $v \in \Theta_X$, $s_1 \in \mathcal{M}_1$, and $s_2 \in \mathcal{M}_2$, similarly to Leibniz's rule, set

only for Θ_X

$$v(s_1 \otimes s_2) = vs_1 \otimes s_2 + s_1 \otimes vs_2.$$

Then this is well-defined, satisfies the conditions in Lemma 1.7, and hence gives a $\mathcal{D}_X$-module structure to $\mathcal{M}_1 \otimes_{\mathcal{O}_X} \mathcal{M}_2$. We can express this by using a coproduct à la Hopf algebra as follows. Regard $\mathcal{D}_X$ as a left $\mathcal{O}_X$-module, and consider $\mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{D}_X$. This is clearly a right $\mathcal{D}_X \otimes_{\mathbb{C}} \mathcal{D}_X$-module. Moreover $\mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{D}_X$ is generated by $1 \otimes 1$ as a right $\mathcal{D}_X \otimes_{\mathbb{C}} \mathcal{D}_X$-module, and its right annihilator ideal $\mathcal{K}$ is generated by $\{ a \otimes 1 - 1 \otimes a \,;\, a \in \mathcal{O}_X \}$. Therefore

$$\mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{D}_X \simeq (\mathcal{D}_X \otimes_{\mathbb{C}} \mathcal{D}_X)/\mathcal{K},$$

and

$$\begin{aligned} &\mathcal{E}nd_{\mathcal{D}_X \otimes_{\mathbb{C}} \mathcal{D}_X}(\mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{D}_X) \\ &\xrightarrow{\sim} \{ P \in \mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{D}_X \,;\, P(a \otimes 1 - 1 \otimes a) = 0 \text{ for all } a \in \mathcal{O}_X \}. \end{aligned}$$

This isomorphism is given by $\mathcal{E}nd_{\mathcal{D}_X \otimes_{\mathbb{C}} \mathcal{D}_X}(\mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{D}_X) \ni \varphi \mapsto \varphi(1 \otimes 1) \in \mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{D}_X$. We define a ring homomorphism

$$\Delta : \mathcal{D}_X \to \mathcal{E}nd_{\mathcal{D}_X \otimes_{\mathbb{C}} \mathcal{D}_X}(\mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{D}_X)$$

as follows:

(3.1) $$\Delta(a) = a \otimes_{\mathcal{O}_X} 1 = 1 \otimes_{\mathcal{O}_X} a \in \mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{D}_X \text{ for } a \in \mathcal{O}_X;$$

$\Delta(a)$ belongs to $\mathcal{E}nd_{\mathcal{D}_X \otimes_{\mathbb{C}} \mathcal{D}_X}(\mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{D}_X)$, since $(a \otimes_{\mathcal{O}_X} 1)(b \otimes 1 - 1 \otimes b) = 0$. Also

(3.2) $$\Delta(v) = v \otimes 1 + 1 \otimes v \text{ for } v \in \Theta_X;$$

$\Delta(v) \in \mathcal{E}nd_{\mathcal{D}_X \otimes_{\mathbb{C}} \mathcal{D}_X}(\mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{D}_X)$, since

$$\begin{aligned}(v \otimes_{\mathcal{O}_X} 1 + 1 \otimes_{\mathcal{O}_X} v)&(a \otimes 1 - 1 \otimes a)\\ &= v(a) \otimes_{\mathcal{O}_X} 1 - 1 \otimes_{\mathcal{O}_X} v(a) = 0\end{aligned}$$

for all $a \in \mathcal{O}_X$.

Furthermore, it is easy to check that Δ satisfies the conditions in Lemma 1.7, and hence Δ can be extended to a ring homomorphism

$$\Delta : \mathcal{D}_X \to \mathcal{E}nd_{\mathcal{D}_X \otimes_{\mathbb{C}} \mathcal{D}_X}(\mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{D}_X).$$

Thus $\mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{D}_X$ is a two-sided $(\mathcal{D}_X, \mathcal{D}_X \otimes_{\mathbb{C}} \mathcal{D}_X)$-module. Hence, for left $\mathcal{D}_X$-modules $\mathcal{M}_1$ and $\mathcal{M}_2$,

$$\mathcal{M}_1 \otimes_{\mathcal{O}_X} \mathcal{M}_2 = (\mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{D}_X) \otimes_{\mathcal{D}_X \otimes_{\mathbb{C}} \mathcal{D}_X} (\mathcal{M}_1 \otimes_{\mathbb{C}} \mathcal{M}_2)$$

is a left $\mathcal{D}_X$-module. An explicit expression of the action is as follows. Let $P \in \mathcal{D}_X$ and

$$\Delta(P) = \sum_i P_i^1 \otimes_{\mathcal{O}_X} P_i^2 \in \mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{D}_X.$$

Then

$$P(s_1 \otimes s_2) = \sum_i P_i^1 s_1 \otimes P_i^2 s_2 \tag{3.3}$$

for $s_1 \in \mathcal{M}_1$ and $s_2 \in \mathcal{M}_2$. In what follows, we denote by $\mathcal{M}_1 \overset{D}{\otimes} \mathcal{M}_2$ the $\mathcal{D}_X$-module $\mathcal{M}_1 \otimes_{\mathcal{O}_X} \mathcal{M}_2$.

EXERCISE 1. Prove that $\Delta : \mathcal{D}_X \to \mathcal{E}nd_{\mathcal{D}_X \otimes_{\mathbb{C}} \mathcal{D}_X}(\mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{D}_X)$ is an isomorphism.

PROPOSITION 3.1. *Given a $\mathcal{D}_X$-module $\mathcal{M}$ and an $\mathcal{O}_X$-module F, there exists a canonical isomorphism of $\mathcal{D}_X$-modules*

$$\mathcal{D}_X \otimes_{\mathcal{O}_X} (F \otimes_{\mathcal{O}_X} \mathcal{M}) \xrightarrow{\sim} (\mathcal{D}_X \otimes_{\mathcal{O}_X} F) \overset{D}{\otimes} \mathcal{M}.$$

Here $\mathcal{D}_X \otimes_{\mathcal{O}_X} F$ is the tensor product of a right $\mathcal{O}_X$-module $\mathcal{D}_X$ and a (left) $\mathcal{O}_X$-module F, and thus it is a left $\mathcal{D}_X$-module. Similarly, $\mathcal{D}_X \otimes_{\mathcal{O}_X} (F \otimes_{\mathcal{O}_X} \mathcal{M})$ is the tensor product of a right $\mathcal{O}_X$-module $\mathcal{D}_X$ and a left $\mathcal{O}_X$-module $F \otimes_{\mathcal{O}_X} \mathcal{M}$.

PROOF. An $\mathcal{O}_X$-module homomorphism

$$F \otimes_{\mathcal{O}_X} \mathcal{M} \to (\mathcal{D}_X \otimes_{\mathcal{O}_X} F) \otimes_{\mathcal{O}_X} \mathcal{M} \quad (s \otimes u \mapsto (1 \otimes s) \otimes u)$$

can be extended to a $\mathcal{D}_X$-module homomorphism

$$\varphi : \mathcal{D}_X \otimes_{\mathcal{O}_X} (F \otimes_{\mathcal{O}_X} \mathcal{M}) \to (\mathcal{D}_X \otimes_{\mathcal{O}_X} F) \overset{D}{\otimes} \mathcal{M}.$$

It is easy to see that the image of $F_m(\mathcal{D}_X \otimes_{\mathcal{O}_X} (F \otimes_{\mathcal{O}_X} \mathcal{M})) := F_m(\mathcal{D}_X) \otimes_{\mathcal{O}_X} (F \otimes_{\mathcal{O}_X} \mathcal{M})$ by φ is contained in

$$\begin{aligned} F_m((\mathcal{D}_X \otimes_{\mathcal{O}_X} F) \overset{D}{\otimes} \mathcal{M}) &:= (F_m(\mathcal{D}_X) \otimes_{\mathcal{O}_X} F) \otimes_{\mathcal{O}_X} \mathcal{M} \\ &\subset (\mathcal{D}_X \otimes_{\mathcal{O}_X} F) \overset{D}{\otimes} \mathcal{M}. \end{aligned}$$

Hence φ induces a homomorphism from $\mathrm{Gr}^F_m(\mathcal{D}_X \otimes_{\mathcal{O}_X} (F \otimes_{\mathcal{O}_X} \mathcal{M})) = (\mathrm{Gr}^F_m \mathcal{D}_X) \otimes_{\mathcal{O}_X} F \otimes_{\mathcal{O}_X} \mathcal{M}$ to $\mathrm{Gr}^F_m((\mathcal{D}_X \otimes_{\mathcal{O}_X} F) \otimes_{\mathcal{O}_X} \mathcal{M}) = (\mathrm{Gr}^F_m \mathcal{D}_X) \otimes_{\mathcal{O}_X} F \otimes_{\mathcal{O}_X} \mathcal{M}$. It is easy to check that this is the identity. By Proposition A.17, φ is an isomorphism. □

EXERCISE 2. Using a coordinate system, show that the inverse ψ of $\varphi : \mathcal{D}_X \otimes_{\mathcal{O}_X} (F \otimes \mathcal{M}) \xrightarrow{\sim} (\mathcal{D}_X \otimes F) \overset{D}{\otimes} \mathcal{M}$ is given as follows:

$$\psi((\partial^\alpha \otimes s) \otimes u) = \sum (-1)^{|\beta|} \binom{\alpha}{\beta} \partial^{\alpha-\beta} \otimes (s \otimes \partial^\beta u).$$

REMARK 3.2. By setting $F = \mathcal{O}_X$ in Proposition 3.1, we see that

$$\mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{M} \cong \mathcal{D}_X \overset{D}{\otimes} \mathcal{M} \tag{3.4}$$

for any $\mathcal{D}_X$-module $\mathcal{M}$. Note that the left $\mathcal{D}_X$-module structure of the left hand side depends only on the $\mathcal{O}_X$-module structure of $\mathcal{M}$.

LEMMA 3.3. *Given a left $\mathcal{D}_X$-module $\mathcal{M}$ and a right $\mathcal{D}_X$-module $\mathcal{N}$, the tensor product $\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{N}$ has a right $\mathcal{D}_X$-module structure given by*

$$(m \otimes n)v = -(vm) \otimes n + m \otimes (nv)$$

for $m \in \mathcal{M},\ n \in \mathcal{N},\ v \in \Theta_X$.

We leave its direct proof to the readers. Since $\mathcal{N} \otimes_{\mathcal{O}_X} \Omega_X^{\otimes -1}$ is a left $\mathcal{D}_X$-module, $\mathcal{M} \overset{D}{\otimes} (\mathcal{N} \otimes_{\mathcal{O}_X} \Omega_X^{\otimes -1})$ has a left $\mathcal{D}_X$-module structure. Its corresponding right $\mathcal{D}_X$-module is $\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{N}$

REMARK 3.4. Each side of (3.4) has a right $\mathcal{D}_X$-module structure. The left hand side has the right $\mathcal{D}_X$-module structure determined by Lemma 3.3 as the tensor product of a right $\mathcal{D}_X$-module $\mathcal{D}_X$ and a left $\mathcal{D}_X$-module $\mathcal{M}$. The right hand side has the right $\mathcal{D}_X$-module structure induced from the right $\mathcal{D}_X$-module structure of $\mathcal{D}_X$. Then (3.4) is an isomorphism as $(\mathcal{D}_X, \mathcal{D}_X)$-bimodules.

PROPOSITION 3.5. *Let $\mathcal{N}$ be a right $\mathcal{D}_X$-module, and $\mathcal{M}_1$ and $\mathcal{M}_2$ left $\mathcal{D}_X$-modules. Then*

$$\mathcal{N} \otimes_{\mathcal{D}_X} (\mathcal{M}_1 \overset{D}{\otimes} \mathcal{M}_2) \simeq (\mathcal{N} \otimes_{\mathcal{O}_X} \mathcal{M}_1) \otimes_{\mathcal{D}_X} \mathcal{M}_2 .$$

PROOF. Each side is a quotient of $\mathcal{N} \otimes_{\mathcal{O}_X} \mathcal{M}_1 \otimes_{\mathcal{O}_X} \mathcal{M}_2$. The left hand side is the one divided by the submodule generated by

$$\begin{aligned} sv \otimes (t_1 \otimes t_2) &- s \otimes v(t_1 \otimes t_2) \\ &= sv \otimes t_1 \otimes t_2 - s \otimes vt_1 \otimes t_2 - s \otimes t_1 \otimes vt_2 \end{aligned}$$

for $s \in \mathcal{N}$, $t_i \in \mathcal{M}_i$ $(i = 1, 2)$, $v \in \Theta_X$.

The right hand side is the one divided by the submodule generated by

$$\begin{aligned} (s \otimes t_1)v \otimes t_2 &- (s \otimes t_1) \otimes vt_2 \\ &= sv \otimes t_1 \otimes t_2 - s \otimes vt_1 \otimes t_2 - s \otimes t_1 \otimes vt_2. \end{aligned}$$

Hence they are isomorphic. $\square$

Similarly to $\bullet \overset{D}{\otimes} \bullet$, $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N})$ has a left $\mathcal{D}_X$-module structure for left $\mathcal{D}_X$-modules $\mathcal{M}$ and $\mathcal{N}$. On $\varphi \in \mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N})$, $v \in \Theta_X$ acts by

$$(v\varphi)(u) = v(\varphi(u)) - \varphi(v \cdot u) \quad (u \in \mathcal{M}).$$

This can be extended to an action of $\mathcal{D}_X$ by Lemma 1.7. By the definition, we have a canonical homomorphism of $\mathcal{D}_X$-modules

$$\mathcal{M}_1 \overset{D}{\otimes} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}_1, \mathcal{M}_2) \to \mathcal{M}_2$$

and also an isomorphism

$$\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}_1 \overset{D}{\otimes} \mathcal{M}_2, \mathcal{M}_3) \simeq \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}_1, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}_2, \mathcal{M}_3)) \tag{3.5}$$

for $\mathcal{D}_X$-modules $\mathcal{M}_k$ $(k = 1, 2, 3)$.

3.2. Homological Properties of D-modules

Since we develop the theory by using derived categories in this book, the homological finiteness of $\mathcal{D}_X$ is indispensable. In this section, we prove the following two theorems:

THEOREM 3.6. *The global dimension of $\mathcal{D}_X$ is less than or equal to $1 + 2\dim X$ for any manifold X.*

Namely, we have $\mathrm{Ext}^k_{\mathcal{D}_X}(\mathcal{M}, \mathcal{N}) = 0$ for any $k > 1 + 2\dim X$ and any $\mathcal{D}_X$-modules $\mathcal{M}, \mathcal{N}$.

THEOREM 3.7. *Let X be an n-dimensional manifold. Then, for any $x \in X$, the global dimension* $\mathrm{gl.dim}\,\mathcal{D}_{X,x}$ *of $\mathcal{D}_{X,x}$ equals n.*

We shall prove Theorem 3.6 by reducing it to the following theorem, due to Golovin.

THEOREM 3.8 ([Gol]). *Let X be a manifold with* $\dim X > 0$. *Then the global dimension of $\mathcal{O}_X$ equals* $\dim X + 1$.

In the previous section, we defined a functor

$$\mathcal{H}om_{\mathcal{O}_X}(\bullet,\bullet) : \mathrm{Mod}(\mathcal{D}_X)^{\mathrm{op}} \times \mathrm{Mod}(\mathcal{D}_X) \to \mathrm{Mod}(\mathcal{D}_X).$$

Denote its derived functor by

$$\mathbb{R}\mathcal{H}om_{\mathcal{O}_X}(\bullet,\bullet) : D^-(\mathcal{D}_X)^{\mathrm{op}} \times D^+(\mathcal{D}_X) \to D^+(\mathcal{D}_X).$$

Since injective $\mathcal{D}_X$-modules are injective $\mathcal{O}_X$-modules, the diagram

$$\begin{tikzcd}
D^-(\mathcal{D}_X)^{\mathrm{op}} \times D^+(\mathcal{D}_X) \arrow[r, "\mathbb{R}\mathcal{H}om_{\mathcal{O}_X}"] \arrow[d] & D^+(\mathcal{D}_X) \arrow[d] \\
D^-(\mathcal{O}_X)^{\mathrm{op}} \times D^+(\mathcal{O}_X) \arrow[r, "\mathbb{R}\mathcal{H}om_{\mathcal{O}_X}"'] & D^+(\mathcal{O}_X)
\end{tikzcd}$$

is commutative. By Golovin's theorem,

$$H^i\mathbb{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M},\mathcal{N}) = 0 \quad (i > 1 + \dim X) \tag{3.6}$$

for any $\mathcal{D}_X$-modules $\mathcal{M}$ and $\mathcal{N}$. Recall from (3.5) that

$$\mathcal{H}om_{\mathcal{D}_X}(\mathcal{O}_X, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{M},\mathcal{N})) \simeq \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\mathcal{N}).$$

LEMMA 3.9.

$$\mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{O}_X, \mathbb{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M},\mathcal{N})) \simeq \mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\mathcal{N}).$$

PROOF. It suffices to show that

$$H^i\mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{O}_X, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{M},\mathcal{N})) = 0 \quad (i \neq 0) \tag{3.7}$$

for any injective $\mathcal{D}_X$-module $\mathcal{N}$. Take the locally free resolution $\mathcal{D}_X \otimes_{\mathcal{O}_X} \bigwedge^{\bullet} \Theta_X \to \mathcal{O}_X$ of $\mathcal{O}_X$ (Proposition 1.6). Then

$$\begin{aligned}
&\mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{O}_X, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{M},\mathcal{N})) \\
&\cong \mathcal{H}om_{\mathcal{D}_X}(\mathcal{D}_X \otimes_{\mathcal{O}_X} \overset{\bullet}{\bigwedge} \Theta_X, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{M},\mathcal{N})).
\end{aligned}$$

By the isomorphism (3.5),

$$\begin{aligned}
&\mathcal{H}om_{\mathcal{D}_X}(\mathcal{D}_X \otimes_{\mathcal{O}_X} \overset{\bullet}{\bigwedge} \Theta_X, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{M},\mathcal{N})) \\
&\simeq \mathcal{H}om_{\mathcal{D}_X}((\mathcal{D}_X \otimes_{\mathcal{O}_X} \overset{\bullet}{\bigwedge} \Theta_X) \overset{D}{\otimes} \mathcal{M}, \mathcal{N}).
\end{aligned}$$

Since $(\mathcal{D}_X \otimes_{\mathcal{O}_X} \bigwedge^{\bullet} \Theta_X) \overset{D}{\otimes} \mathcal{M} \to \mathcal{O}_X \overset{D}{\otimes} \mathcal{M}$ is a quasi-isomorphism,

$$\mathcal{H}om_{\mathcal{D}_X}(\mathcal{D}_X \otimes_{\mathcal{O}_X} \bigwedge^{\bullet} \Theta_X, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{M},\mathcal{N})) \simeq \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\mathcal{N}).$$

Hence we obtain (3.7). □

Let $\mathcal{M}$, $\mathcal{N} \in \mathrm{Mod}(\mathcal{D}_X)$. Since $H^i \mathbb{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M},\mathcal{N}) = 0$ for $i > \dim X + 1$, $\mathbb{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M},\mathcal{N})$ can be represented by a complex K with $K^i = 0$ for $i > \dim X + 1$ or $i < 0$. Hence we have isomorphisms

$$\begin{aligned}\mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\mathcal{N}) &\simeq \mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{O}_X, K) \\ &\simeq \mathcal{H}om_{\mathcal{D}_X}(\mathcal{D}_X \otimes_{\mathcal{O}_X} \bigwedge^{\bullet} \Theta_X, K).\end{aligned}$$

The latter leads to $H^i \mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{O}_X, K) = 0$ $(i > 1 + 2\dim X)$. We thus obtain $\mathrm{gl.dim}\, \mathcal{D}_X \le 1 + 2\dim X$.

REMARK 3.10. Furthermore, we have $\mathrm{gl.dim}\, \mathcal{D}_X = 1 + 2\dim X$ when $\dim X > 0$. Indeed, $\mathrm{Ext}^{2n+1}_{\mathcal{D}_{\mathbb{C}^n}}(\mathcal{O}_{\mathbb{C}^n,\{0\}}, \mathcal{D}^{\oplus I}_{\mathbb{C}^n})$ does not vanish, whenever $\#I = \infty$. We give a sketch of its proof. Let $X = \mathbb{C}^n$. Since $\mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{O}_X, \mathcal{D}_X^{\oplus I}) \simeq {\mathcal{O}_X}^{\oplus I}[-n]$, we have $\mathrm{Ext}^{2n+1}_{\mathcal{D}_X}(\mathcal{O}_{X,\{0\}}, \mathcal{D}_X^{\oplus I}) \simeq H^{n+1}_{\{0\}}(X; {\mathcal{O}_X}^{\oplus I})$. We prove that this does not vanish by induction on n. Suppose $n > 1$, and let $f : X \to Y = \mathbb{C}^{n-1}$ be the projection. Then we have an exact sequence

$$0 \longrightarrow f^{-1}\mathcal{O}_Y^{\oplus I} \longrightarrow {\mathcal{O}_X}^{\oplus I} \xrightarrow{\partial_n} {\mathcal{O}_X}^{\oplus I} \longrightarrow 0$$

and accordingly an exact sequence

$$H^{n+1}_{\{0\}}(X; {\mathcal{O}_X}^{\oplus I}) \to H^{n+2}_{\{0\}}(X; f^{-1}\mathcal{O}_Y^{\oplus I}) \to H^{n+2}_{\{0\}}(X; {\mathcal{O}_X}^{\oplus I}) = 0.$$

Note that

$$H^k_{\{0\}}(X; f^{-1}\mathcal{F}) = H^{k-2}_{\{0\}}(Y; \mathcal{F})$$

for any sheaf $\mathcal{F}$ on Y. We thus have

$$H^{n+2}_{\{0\}}(X; f^{-1}{\mathcal{O}_Y}^{\oplus I}) = H^n_{\{0\}}(Y; {\mathcal{O}_Y}^{\oplus I}) \neq 0$$

and $H^{n+1}_{\{0\}}(X; {\mathcal{O}_X}^{\oplus I}) \neq 0$.

Now suppose $n = 1$, and let $X = \mathbb{P}^1_{\mathbb{C}}$. By the exact sequence

$$\begin{aligned} 0 = H^1(X; {\mathcal{O}_X}^{\oplus I}) &\to H^1(X \backslash \{0\}; {\mathcal{O}_X}^{\oplus I}) \\ &\to H^2_{\{0\}}(X; {\mathcal{O}_X}^{\oplus I}) \to H^2(X; {\mathcal{O}_X}^{\oplus I}) = 0, \end{aligned}$$

we have $H^2_{\{0\}}(X;\mathcal{O}_X{}^{\oplus I}) = H^1(X\setminus\{0\};\mathcal{O}_X{}^{\oplus I})$. Hence it is enough to show that $H^1(\mathbb{C};\mathcal{O}_{\mathbb{C}}^{\oplus I}) \neq 0$. We may assume $I = \mathbb{Z}_{>0}$. Let $\mathcal{D}b_{\mathbb{C}}$ denote the sheaf of distributions. From the soft resolution of $\mathcal{O}_{\mathbb{C}}^{\oplus I}$

$$0 \longrightarrow \mathcal{O}_{\mathbb{C}}^{\oplus I} \longrightarrow \mathcal{D}b_{\mathbb{C}}^{\oplus I} \xrightarrow{\partial/\partial\bar{z}} \mathcal{D}b_{\mathbb{C}}^{\oplus I} \longrightarrow 0,$$

we have

$$H^1(\mathbb{C};\mathcal{O}_{\mathbb{C}}^{\oplus I}) = \Gamma(\mathbb{C};\mathcal{D}b_{\mathbb{C}}^{\oplus I})/\partial_{\bar{z}}\Gamma(\mathbb{C};\mathcal{D}b_{\mathbb{C}}^{\oplus I}).$$

Let

$$u_n = \partial_{\bar{z}}(z-n)^{-1} \in \Gamma(\mathbb{C};\mathcal{D}b_{\mathbb{C}}).$$

Since $\operatorname{supp}(u_n) = \{n\}$, we have $u = \bigoplus u_n \in \Gamma(\mathbb{C};\mathcal{D}b_{\mathbb{C}}^{\oplus I})$. Suppose $\partial_{\bar{z}}v_n = u_n$; then $\operatorname{supp}(v_n) = \mathbb{C}$. Hence $u \notin \partial_{\bar{z}}\Gamma(\mathbb{C};\mathcal{D}b_{\mathbb{C}}^{\oplus I})$.

We shall next prove Theorem 3.7. For this, we use the involutivity of characteristic varieties. By Theorem 3.6, we already know $\operatorname{gl.dim}\mathcal{D}_{X,x} < \infty$. First we reduce the proof to the vanishing of $\operatorname{Ext}^i_{\mathcal{D}_{X,x}}(M,\mathcal{D}_{X,x})$ by the following lemma.

LEMMA 3.11. *Suppose that the projective dimension of a finitely generated module M over a Noetherian ring A is finite, and that $\operatorname{Ext}^j_A(M,A) = 0$ $(j > r)$ for some $r \in \mathbb{Z}_{\geq 0}$. Then the projective dimension of M is less than or equal to r.*

PROOF. By the assumption, M has a projective resolution of length m

$$0 \leftarrow M \leftarrow P_0 \leftarrow \cdots \xleftarrow{\varphi} P_m \leftarrow 0,$$

where $P_0,\ldots,P_m$ are finitely generated projective modules. It is then enough to show that M has a projective resolution of length $m-1$ if $\operatorname{Ext}^m_A(M,A) = 0$. Since $\operatorname{Ext}^m_A(M,A)$ is the cokernel of $\operatorname{Hom}_A(P_{m-1},A) \xrightarrow{\varphi^*} \operatorname{Hom}_A(P_m,A)$, the homomorphism φ^* is surjective. Hence $\operatorname{Hom}_A(P_{m-1},A)\otimes_A P_m \to \operatorname{Hom}_A(P_m,A)\otimes_A P_m$ is also surjective. Since P_m is a finitely generated projective module, $\operatorname{Hom}(N,A)\otimes_A P_m \xrightarrow{\sim} \operatorname{Hom}_A(N,P_m)$ for any A-module N. Thus

$$\operatorname{Hom}_A(P_{m-1},P_m) \to \operatorname{Hom}_A(P_m,P_m)$$

is surjective. In particular, there exists $\psi \in \operatorname{Hom}_A(P_{m-1},P_m)$ such that $\psi\circ\varphi = \mathrm{id}_{P_m}$. Hence $\operatorname{Coker}(\varphi)$ is a direct summand of P_{m-1}, and thus projective. We have thus obtained a projective resolution of M of length $m-1$:

$$0 \leftarrow M \leftarrow P_0 \leftarrow \cdots \leftarrow P_{m-2} \leftarrow \operatorname{Coker}(\varphi) \leftarrow 0.$$

□

By this lemma, it suffices to show that $\mathcal{E}xt^k_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_X) = 0$ $(k > n)$ for any coherent $\mathcal{D}_X$-module $\mathcal{M}$. By Theorem 2.19,

$$\operatorname{codim}\bigl(\operatorname{Ch}(\mathcal{E}xt^k_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_X))\bigr) \geq k > n.$$

Hence $\mathcal{E}xt^k_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_X) = 0$ by Proposition 2.16. Thus $\operatorname{gl.dim} \mathcal{D}_{X,x} \leq n$. Since $\operatorname{Ext}^n_{\mathcal{D}_{X,x}}(\mathcal{O}_{X,x}, \mathcal{D}_{X,x}) = \Omega_{X,x}$, we conclude that

$$\operatorname{gl.dim} \mathcal{D}_{X,x} = n.$$

3.3. Dual Module

Since $\mathcal{D}_X \otimes_{\mathcal{O}_X} \Omega_X^{\otimes -1}$ is a $\mathcal{D}_X \otimes_{\mathbb{C}} \mathcal{D}_X$-module, $\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_X \otimes_{\mathcal{O}_X} \Omega_X^{\otimes -1})$ is a $\mathcal{D}_X$-module for any $\mathcal{D}_X$-module $\mathcal{M}$. However, if $\operatorname{Ch}(\mathcal{M})$ is not equal to T^*X, then $\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_X \otimes_{\mathcal{O}_X} \Omega_X^{\otimes -1})$ vanishes. Hence, for the duality, we need to discuss the higher cohomologies as well. For this purpose, we use the language of derived categories, explained in the Appendix, § A.2. Let $D(\mathcal{D}_X)$ denote the derived category of the category $\operatorname{Mod}(\mathcal{D}_X)$ of $\mathcal{D}_X$-modules. Let $D^+(\mathcal{D}_X)$, $D^-(\mathcal{D}_X)$, and $D^b(\mathcal{D}_X)$ denote the full subcategories of $D(\mathcal{D}_X)$ consisting of complexes bounded above, below, and above and below respectively. Let $D^b_{\mathrm{coh}}(\mathcal{D}_X)$ denote the full subcategory of $D^b(\mathcal{D}_X)$ consisting of $\mathcal{M} \in D^b(\mathcal{D}_X)$ such that all $H^i(\mathcal{M})$ are coherent $\mathcal{D}_X$-modules. $D^b_{\mathrm{coh}}(\mathcal{D}_X)$ is a triangulated category.

As seen in the Appendix, § A.2, $\mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\bullet, \mathcal{D}_X \otimes_{\mathcal{O}_X} \Omega_X^{\otimes -1})$ is a functor from $D^-(\mathcal{D}_X)$ to $D^+(\mathcal{D}_X)$. Define a contravariant functor

$$\mathbb{D}_X : D^-(\mathcal{D}_X) \to D^+(\mathcal{D}_X) \tag{3.8}$$

by

$$\mathbb{D}_X(\mathcal{M}) = \mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_X \otimes_{\mathcal{O}_X} \Omega_X^{\otimes -1})[d_X]$$

$(d_X := \dim X)$. The shift $[d_X]$ has been adjusted so that $\mathbb{D}_X$ sends $\mathcal{O}_X$ to itself. Since the cohomological dimension of $\mathcal{D}_X$ is finite, $\mathbb{D}_X$ preserves $D^b(\mathcal{D}_X)$. Moreover, it preserves $D^b_{\mathrm{coh}}(\mathcal{D}_X)$ as well. Hence by § A.3(b)

$$\mathrm{id}_{D^b_{\mathrm{coh}}(\mathcal{D}_X)} \xrightarrow{\sim} \mathbb{D}_X \circ \mathbb{D}_X. \tag{3.9}$$

The left derived functor of

$$\bullet \overset{D}{\otimes} \bullet : \operatorname{Mod}(\mathcal{D}_X) \times \operatorname{Mod}(\mathcal{D}_X) \to \operatorname{Mod}(\mathcal{D}_X)$$

sends a pair of bounded complexes in $D^b(\mathcal{D}_X) \times D^b(\mathcal{D}_X)$ to a bounded complex in $D^b(\mathcal{D}_X)$. We denote this by

$$\bullet \overset{\mathbb{D}}{\otimes} \bullet : D^b(\mathcal{D}_X) \times D^b(\mathcal{D}_X) \to D^b(\mathcal{D}_X).$$

The next proposition immediately follows from the corresponding statements on $\bullet \overset{D}{\otimes} \bullet$.

PROPOSITION 3.12. *Let $\mathcal{M}_k \in D^b(\mathcal{D}_X)$ $(k = 1, 2, 3)$. Then*

(1) $\mathcal{M}_1 \overset{\mathbb{D}}{\otimes} \mathcal{M}_2 \simeq \mathcal{M}_2 \overset{\mathbb{D}}{\otimes} \mathcal{M}_1$,

(2) $(\mathcal{M}_1 \overset{\mathbb{D}}{\otimes} \mathcal{M}_2) \overset{\mathbb{D}}{\otimes} \mathcal{M}_3 \simeq \mathcal{M}_1 \overset{\mathbb{D}}{\otimes} (\mathcal{M}_2 \overset{\mathbb{D}}{\otimes} \mathcal{M}_3)$,

(3) $\mathcal{O}_X \overset{\mathbb{D}}{\otimes} \mathcal{M}_1 \simeq \mathcal{M}_1$.

For $\mathcal{M} \in D^-(\mathcal{D}_X)$ and $\mathcal{N} \in D^b(\mathcal{D}_X)$, we have a morphism

$$\mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_X) \overset{\mathbb{L}}{\otimes}_{\mathcal{D}_X} \mathcal{N} \to \mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{N}). \tag{3.10}$$

This is an isomorphism, if $\mathcal{M} \in D^b_{\text{coh}}(\mathcal{D}_X)$ or $\mathcal{N} \in D^b_{\text{coh}}(\mathcal{D}_X)$.

If $\mathcal{N}_1, \mathcal{N}_2 \in D^b(\mathcal{D}_X)$, then by Proposition 3.5

$$\Omega_X \overset{\mathbb{L}}{\otimes}_{\mathcal{D}_X} (\mathcal{N}_1 \overset{\mathbb{D}}{\otimes} \mathcal{N}_2) \simeq (\Omega_X \otimes_{\mathcal{O}_X} \mathcal{N}_1) \overset{\mathbb{L}}{\otimes}_{\mathcal{D}_X} \mathcal{N}_2. \tag{3.11}$$

Hence, by using $\Omega_X = \mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{O}_X, \mathcal{D}_X)[d_X]$ and (3.10), we see that for $\mathcal{N}_1, \mathcal{N}_2 \in D^b(\mathcal{D}_X)$

$$\begin{aligned} &\mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{O}_X, \mathcal{N}_1 \overset{\mathbb{D}}{\otimes} \mathcal{N}_2) \\ &\simeq \mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{O}_X, \mathcal{D}_X) \overset{\mathbb{L}}{\otimes}_{\mathcal{D}_X} (\mathcal{N}_1 \overset{\mathbb{D}}{\otimes} \mathcal{N}_2) \\ &\simeq (\Omega_X \otimes_{\mathcal{O}_X} \mathcal{N}_1) \overset{\mathbb{L}}{\otimes}_{\mathcal{D}_X} \mathcal{N}_2[-d_X]. \end{aligned} \tag{3.12}$$

Hence for $\mathcal{M}, \mathcal{N} \in D^b(\mathcal{D}_X)$ we have by (3.12)

$$\begin{aligned} &\mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{O}_X, (\mathbb{D}_X\mathcal{M}) \overset{\mathbb{D}}{\otimes} \mathcal{N}) \\ &\simeq (\Omega_X \otimes_{\mathcal{O}_X} \mathbb{D}_X\mathcal{M}) \overset{\mathbb{L}}{\otimes}_{\mathcal{D}_X} \mathcal{N}[-d_X] \\ &\simeq \mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_X) \overset{\mathbb{L}}{\otimes}_{\mathcal{D}_X} \mathcal{N}. \end{aligned}$$

We have thus obtained a morphism

$$\mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{O}_X, (\mathbb{D}_X\mathcal{M}) \overset{\mathbb{D}}{\otimes} \mathcal{N}) \to \mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{N}) \tag{3.13}$$

from (3.10). If $\mathcal{M} \in D^b_{\text{coh}}(\mathcal{D}_X)$ or $\mathcal{N} \in D^b_{\text{coh}}(\mathcal{D}_X)$, then (3.13) is an isomorphism. Under the conditions $\mathcal{M}, \mathcal{N} \in D^b_{\text{coh}}(\mathcal{D}_X)$, we have

$$\mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{N}) \simeq \mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathbb{D}_X\mathcal{N}, \mathbb{D}_X\mathcal{M}). \tag{3.14}$$

LEMMA 3.13. *Suppose that a $\mathcal{D}_X$-module $\mathcal{M}$ is coherent over $\mathcal{O}_X$ (cf. Proposition 4.43). Then*

$$\mathbb{D}_X\mathcal{M} \simeq \mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathcal{O}_X). \tag{3.15}$$

PROOF. By tensoring $\mathcal{M}$ and the flat resolution $\mathcal{D}_X \otimes_{\mathcal{O}_X} \bigwedge^\bullet \Theta_X$ of $\mathcal{O}_X$ in Proposition 1.6, we obtain

$$0 \leftarrow \mathcal{M} \leftarrow \mathcal{D}_X \overset{D}{\otimes} \mathcal{M} \leftarrow \cdots \leftarrow (\mathcal{D}_X \otimes_{\mathcal{O}_X} \bigwedge^{d_X} \Theta_X) \overset{D}{\otimes} \mathcal{M} \leftarrow 0.$$

By Proposition 3.1, we have

$$(\mathcal{D}_X \otimes_{\mathcal{O}_X} \bigwedge^{\bullet} \Theta_X) \overset{D}{\otimes} \mathcal{M} \simeq \mathcal{D}_X \otimes_{\mathcal{O}_X} (\bigwedge^{\bullet} \Theta_X \otimes_{\mathcal{O}_X} \mathcal{M}).$$

Hence we obtain the resolution
(3.16)

$$0 \leftarrow \mathcal{M} \leftarrow \mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{M} \leftarrow \cdots \leftarrow \mathcal{D}_X \otimes_{\mathcal{O}_X} (\bigwedge^{d_X} \Theta_X \otimes_{\mathcal{O}_X} \mathcal{M}) \leftarrow 0.$$

Hence $\mathbb{D}_X \mathcal{M}$ is represented by the complex

$$\begin{aligned}
&\mathcal{H}om_{\mathcal{D}_X}(\mathcal{D}_X \otimes_{\mathcal{O}_X} (\bigwedge^{\bullet} \Theta_X \otimes_{\mathcal{O}_X} \mathcal{M}), \mathcal{D}_X \otimes_{\mathcal{O}_X} \Omega_X^{\otimes -1})[d_X] \\
&\quad \cong \mathcal{H}om_{\mathcal{O}_X}(\bigwedge^{\bullet} \Theta_X \otimes_{\mathcal{O}_X} \mathcal{M}, \mathcal{D}_X \otimes_{\mathcal{O}_X} \Omega_X^{\otimes -1})[d_X] \\
&\quad \cong \mathcal{H}om_{\mathcal{O}_X}(\Omega_X \otimes_{\mathcal{O}_X} \bigwedge^{\bullet} \Theta_X \otimes_{\mathcal{O}_X} \mathcal{M}, \Omega_X \otimes_{\mathcal{O}_X} \mathcal{D}_X \otimes_{\mathcal{O}_X} \Omega_X^{\otimes -1})[d_X].
\end{aligned}$$

Since $\Omega_X \otimes_{\mathcal{O}_X} \bigwedge^\bullet \Theta_X \simeq \Omega_X^\bullet$, it is isomorphic to

$$\begin{aligned}
\mathcal{H}om_{\mathcal{O}_X}(\Omega_X^\bullet \otimes_{\mathcal{O}_X} \mathcal{M}, \mathcal{O}_X) &\otimes_{\mathcal{O}_X} (\Omega_X \otimes_{\mathcal{O}_X} \mathcal{D}_X \otimes_{\mathcal{O}_X} \Omega_X^{\otimes -1})[d_X] \\
&\simeq (\bigwedge^{\bullet} \Theta_X \otimes_{\mathcal{O}_X} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathcal{O}_X)) \\
&\qquad \otimes_{\mathcal{O}_X} (\Omega_X \otimes_{\mathcal{O}_X} \mathcal{D}_X \otimes_{\mathcal{O}_X} \Omega_X^{\otimes -1})[d_X].
\end{aligned}$$

In Proposition 1.10, we proved $\Omega_X \otimes_{\mathcal{O}_X} \mathcal{D}_X \otimes_{\mathcal{O}_X} \Omega_X^{\otimes -1} \simeq \mathcal{D}_X^{\mathrm{op}}$, and hence

$$\begin{aligned}
(\bigwedge^{\bullet} \Theta_X \otimes_{\mathcal{O}_X} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathcal{O}_X)) &\otimes_{\mathcal{O}_X} (\Omega_X \otimes_{\mathcal{O}_X} \mathcal{D}_X \otimes_{\mathcal{O}_X} \Omega_X^{\otimes -1})[d_X] \\
&\simeq \mathcal{D}_X \otimes (\bigwedge^{\bullet} \Theta_X \otimes_{\mathcal{O}_X} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathcal{O}_X)).
\end{aligned}$$

This is quasi-isomorphic to $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathcal{O}_X)$ by (3.16). □

For $\mathcal{M} \in D^b(\mathcal{D}_X)$, set

$$\operatorname{Supp}(\mathcal{M}) := \bigcup_i \operatorname{Supp}(H^i(\mathcal{M})),$$

and, for $\mathcal{M} \in D^b_{\mathrm{coh}}(\mathcal{D}_X)$,

$$\mathrm{Ch}(\mathcal{M}) := \bigcup_i \mathrm{Ch}(H^i(\mathcal{M})).$$

PROPOSITION 3.14. *For $\mathcal{M} \in D^b_{\mathrm{coh}}(\mathcal{D}_X)$, we have*

$$\mathrm{Ch}(\mathbb{D}_X\mathcal{M}) = \mathrm{Ch}(\mathcal{M}).$$

PROOF. The proof of the inclusion $\mathrm{Ch}(\mathbb{D}_X\mathcal{M}) \subset \mathrm{Ch}(\mathcal{M})$ is reduced to the case $\mathcal{M} \in \mathrm{Mod}(\mathcal{D}_X)$, which is nothing but Theorem 2.19 (3). Then, by applying this to $\mathbb{D}_X\mathcal{M}$, we obtain the inverse inclusion

$$\mathrm{Ch}(\mathcal{M}) = \mathrm{Ch}(\mathbb{D}_X\mathbb{D}_X\mathcal{M}) \subset \mathrm{Ch}(\mathbb{D}_X\mathcal{M}).$$

□

3.4. Algebraic Relative Cohomology

In this section, we define (algebraic) relative cohomology as a method of obtaining new $\mathcal{D}$-modules from a $\mathcal{D}$-module. For a closed analytic subset Z of a complex manifold X, consider the functor Γ_Z sending a sheaf $\mathcal{F}$ on X to $\Gamma_Z(\mathcal{F}) := \{ s \in \mathcal{F}; \mathrm{supp}(s) \subset Z \}$. Its derived functor sends $\mathcal{D}_X$-modules to $\mathcal{D}_X$-modules, but the images of coherent modules by $H^i\mathbb{R}\Gamma_Z$ are far from being coherent modules. If $\mathcal{F}$ is a coherent module, then by Hilbert's Nullstellensatz we have $\Gamma_Z(\mathcal{F}) = \{ s ; I_Z^m s = 0,\ m \gg 0 \}$. Here I_Z is the coherent ideal of $\mathcal{O}_X$ consisting of the holomorphic functions vanishing on Z. Accordingly, for any $\mathcal{O}_X$-module $\mathcal{F}$, set

$$\begin{aligned} \Gamma_{[Z]}(\mathcal{F}) &:= \{ s \in \mathcal{F} ; I_Z^m s = 0,\ m \gg 0 \} \\ &= \varinjlim_m \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X/I_Z^m, \mathcal{F}), \end{aligned}$$

and

$$\Gamma_{[X\setminus Z]}(\mathcal{F}) := \varinjlim_m \mathcal{H}om_{\mathcal{O}_X}(I_Z^m, \mathcal{F}).$$

If I is a coherent ideal of $\mathcal{O}_X$ with $\mathrm{Supp}(\mathcal{O}_X/I) = Z$, then locally $I_Z^k \subset I \subset I_Z$ for $k > 0$ large enough. Hence

$$\begin{aligned} \Gamma_{[Z]}(\mathcal{F}) &= \varinjlim_m \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X/I^m, \mathcal{F}), \\ \Gamma_{[X\setminus Z]}(\mathcal{F}) &= \varinjlim_m \mathcal{H}om_{\mathcal{O}_X}(I^m, \mathcal{F}). \end{aligned}$$

Obviously

$$\mathrm{Supp}\,\Gamma_{[Z]}(\mathcal{F}) \subset Z, \quad (\Gamma_{[X\setminus Z]}(\mathcal{F}))|_{X\setminus Z} \simeq \mathcal{F}|_{X\setminus Z}.$$

Since $Z_1 \supset Z_2$ induces $I_{Z_1} \subset I_{Z_2}$, there exist homomorphisms $I_{Z_1}^m \to I_{Z_2}^m$ and $\mathcal{O}_X/I_{Z_1}^m \to \mathcal{O}_X/I_{Z_2}^m$. By taking the inductive limit on m after applying $\mathcal{H}om_{\mathcal{O}_X}(\bullet, \mathcal{F})$ to them, we obtain the following lemma.

LEMMA 3.15. *Suppose $Z_1 \supset Z_2$. Then there exist canonical homomorphisms*

$$\Gamma_{[Z_2]}(\mathcal{F}) \rightarrowtail \Gamma_{[Z_1]}(\mathcal{F}) \rightarrowtail \mathcal{F}, \tag{3.17}$$

$$\mathcal{F} \to \Gamma_{[X\setminus Z_2]}(\mathcal{F}) \to \Gamma_{[X\setminus Z_1]}(\mathcal{F}). \tag{3.18}$$

By applying the left exact functor $\mathcal{H}om_{\mathcal{O}_X}(\bullet, \mathcal{F})$ to the exact sequence $0 \to I_Z^m \to \mathcal{O}_X \to \mathcal{O}_X/I_Z^m \to 0$, we obtain an exact sequence

$$0 \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X/I_Z^m, \mathcal{F}) \to \mathcal{F} \to \mathcal{H}om_{\mathcal{O}_X}(I_Z^m, \mathcal{F}).$$

Since $\varinjlim_m$ is an exact functor, we have an exact sequence

$$0 \to \Gamma_{[Z]}(\mathcal{F}) \to \mathcal{F} \to \Gamma_{[X\setminus Z]}(\mathcal{F}). \tag{3.19}$$

Note the similarity to the exact sequence for an open inclusion $j : X\setminus Z \hookrightarrow X$

$$0 \to \Gamma_Z(\mathcal{F}) \to \mathcal{F} \to j_*j^{-1}\mathcal{F}.$$

Suppose that Z is a hypersurface $f(x) = 0$ for a holomorphic function $f(x)$, and let $I = \mathcal{O}_X f$. Then, since $I^m = \mathcal{O}_X f^m$ is a free $\mathcal{O}_X$-module, we have

$$\begin{aligned}\mathcal{H}om_{\mathcal{O}_X}(I^m, \mathcal{F}) &= \mathcal{H}om_{\mathcal{O}_X}(I^m, \mathcal{O}_X) \otimes_{\mathcal{O}_X} \mathcal{F} \\ &= \mathcal{O}_X f^{-m} \otimes_{\mathcal{O}_X} \mathcal{F}.\end{aligned}$$

Note that $\varinjlim_m \mathcal{O}_X f^{-m} = \mathcal{O}_X[f^{-1}]$, and that taking tensor product and taking inductive limit commute, to obtain the following lemma.

LEMMA 3.16. *If $Z = f^{-1}(0)$, then*

$$\Gamma_{[X\setminus Z]}(\mathcal{F}) = \mathcal{O}_X[f^{-1}] \otimes_{\mathcal{O}_X} \mathcal{F}.$$

Let us consider some properties of $\Gamma_{[Z]}$ and $\Gamma_{[X\setminus Z]}$. We use the Artin–Rees lemma for the proof of the following proposition.

PROPOSITION 3.17. *Let $\mathcal{G}$ be a coherent $\mathcal{O}_X$-module. Then for any $\mathcal{O}_X$-module $\mathcal{F}$ we have*

(1)

$$\begin{aligned}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{G}, \Gamma_{[X\setminus Z]}(\mathcal{F})) &\simeq \Gamma_{[X\setminus Z]}(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{G}, \mathcal{F})) \\ &\simeq \varinjlim_m \mathcal{H}om_{\mathcal{O}_X}(I_Z^m \mathcal{G}, \mathcal{F}),\end{aligned}$$

(2)

$$\begin{aligned}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{G}, \Gamma_{[Z]}(\mathcal{F})) &\simeq \Gamma_{[Z]}(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{G}, \mathcal{F})) \\ &\simeq \varinjlim_m \mathcal{H}om_{\mathcal{O}_X}(\mathcal{G}/I_Z^m \mathcal{G}, \mathcal{F}).\end{aligned}$$

PROOF. Since (2) is easy to prove, we prove only (1). By definition,

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{G}, \Gamma_{[X\setminus Z]}(\mathcal{F})) = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{G}, \varinjlim_m \mathcal{H}om_{\mathcal{O}_X}(I_Z^m, \mathcal{F})). \tag{3.20}$$

By the coherence of $\mathcal{G}$, $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{G}, \bullet)$ commutes with $\varinjlim_m$. Hence (3.20) is computed as follows:

$$\begin{aligned}&\varinjlim_m \mathcal{H}om_{\mathcal{O}_X}(\mathcal{G}, \mathcal{H}om_{\mathcal{O}_X}(I_Z^m, \mathcal{F})) \\ &\quad \simeq \varinjlim_m \mathcal{H}om_{\mathcal{O}_X}(I_Z^m \otimes_{\mathcal{O}_X} \mathcal{G}, \mathcal{F}).\end{aligned} \tag{3.21}$$

Similarly we have

$$\begin{aligned}\Gamma_{[X\setminus Z]}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{G}, \mathcal{F}) &\simeq \varinjlim_m \mathcal{H}om_{\mathcal{O}_X}(I_Z^m, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{G}, \mathcal{F})) \\ &\simeq \varinjlim_m \mathcal{H}om_{\mathcal{O}_X}(I_Z^m \otimes_{\mathcal{O}_X} \mathcal{G}, \mathcal{F}).\end{aligned}$$

By Proposition A.37, locally there exists m_0 large enough so that, for every $m \geq 0$, $I_Z^{m+m_0} \otimes_{\mathcal{O}_X} \mathcal{G} \to I_Z^m \otimes_{\mathcal{O}_X} \mathcal{G}$ is uniquely factored as $I_Z^{m+m_0} \otimes_{\mathcal{O}_X} \mathcal{G} \twoheadrightarrow I_Z^{m+m_0}\mathcal{G} \to I_Z^m \otimes_{\mathcal{O}_X} \mathcal{G}$. Hence

$$\varinjlim_m \mathcal{H}om_{\mathcal{O}_X}(I_Z^m \otimes_{\mathcal{O}_X} \mathcal{G}, \mathcal{F}) \simeq \varinjlim_m \mathcal{H}om_{\mathcal{O}_X}(I_Z^m \mathcal{G}, \mathcal{F}).$$

□

PROPOSITION 3.18.

(1)

$$\begin{aligned}\Gamma_{[X\setminus Z_1]}\Gamma_{[X\setminus Z_2]}(\mathcal{F}) &\simeq \Gamma_{[X\setminus (Z_1 \cup Z_2)]}(\mathcal{F}) \\ &\simeq \varinjlim_m \mathcal{H}om_{\mathcal{O}_X}(I_{Z_1}^m I_{Z_2}^m, \mathcal{F}) \\ &\simeq \varinjlim_m \mathcal{H}om_{\mathcal{O}_X}(I_{Z_1}^m \cap I_{Z_2}^m, \mathcal{F}).\end{aligned}$$

(2) $\Gamma_{[Z_1]}\Gamma_{[Z_2]}(\mathcal{F}) \simeq \Gamma_{[Z_1 \cap Z_2]}(\mathcal{F})$.

PROOF. (1) By Proposition 3.17,

$$\mathcal{H}om(I_{Z_1}^{m_1}, \Gamma_{[X\setminus Z_2]}(\mathcal{F})) \simeq \varinjlim_{m_2} \mathcal{H}om_{\mathcal{O}_X}(I_{Z_1}^{m_1} I_{Z_2}^{m_2}, \mathcal{F}) .$$

Hence

$$\begin{aligned}\Gamma_{[X\setminus Z_1]}\Gamma_{[X\setminus Z_2]}(\mathcal{F}) &\simeq \varinjlim_{m_1}\varinjlim_{m_2} \mathcal{H}om_{\mathcal{O}_X}(I_{Z_1}^{m_1} I_{Z_2}^{m_2}, \mathcal{F}) \\ &\simeq \varinjlim_{m} \mathcal{H}om_{\mathcal{O}_X}(I_{Z_1}^{m} I_{Z_2}^{m}, \mathcal{F}) .\end{aligned}$$

The first isomorphism of (1) follows from $I_{Z_1}^m I_{Z_2}^m = (I_{Z_1} I_{Z_2})^m$ and $\mathrm{Supp}(\mathcal{O}_X/I_{Z_1}I_{Z_2}) = Z_1 \cup Z_2$. By Corollary A.36, locally for every m_1 there exists m_0 such that $I_{Z_1}^{m_1} \cap I_{Z_2}^{m+m_0} = I_{Z_2}^{m}(I_{Z_1}^{m_1} \cap I_{Z_2}^{m_0})$ for all $m > 0$. Hence

$$I_{Z_1}^{m_1} \cap I_{Z_2}^{m+m_0} \subset I_{Z_1}^{m_1} I_{Z_2}^{m} \subset I_{Z_1}^{m_1} \cap I_{Z_2}^{m} ,$$

and thus

$$\varinjlim_{m_2} \mathcal{H}om_{\mathcal{O}_X}(I_{Z_1}^{m_1} I_{Z_2}^{m_2}, \mathcal{F}) \simeq \varinjlim_{m_2} \mathcal{H}om_{\mathcal{O}_X}(I_{Z_1}^{m_1} \cap I_{Z_2}^{m_2}, \mathcal{F}) .$$

Therefore

$$\begin{aligned}\Gamma_{[X\setminus Z_1]}\Gamma_{[X\setminus Z_2]}(\mathcal{F}) &\simeq \varinjlim_{m_1}\varinjlim_{m_2} \mathcal{H}om_{\mathcal{O}_X}(I_{Z_1}^{m_1} \cap I_{Z_2}^{m_2}, \mathcal{F}) \\ &\simeq \varinjlim_{m} \mathcal{H}om_{\mathcal{O}_X}(I_{Z_1}^{m} \cap I_{Z_2}^{m}, \mathcal{F}).\end{aligned}$$

(2) It follows from Proposition 3.17 (2) that

$$\Gamma_{[Z_1]}\Gamma_{[Z_2]}(\mathcal{F}) \simeq \varinjlim_{m_1}\varinjlim_{m_2} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X/(I_{Z_1}^{m_1} + I_{Z_2}^{m_2}), \mathcal{F}) .$$

They are isomorphic to $\varinjlim_{m} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X/(I_{Z_1} + I_{Z_2})^m, \mathcal{F})$, since

$$(I_{Z_1} + I_{Z_2})^{2m} \subset I_{Z_1}^m + I_{Z_2}^m \subset (I_{Z_1} + I_{Z_2})^m .$$

Hence (2) follows from $\mathrm{Supp}(\mathcal{O}_X/(I_{Z_1} + I_{Z_2})) = Z_1 \cap Z_2$. □

LEMMA 3.19. *We have the following canonical exact sequences:*

(1)

$$0 \to \Gamma_{[X\setminus(Z_1\cap Z_2)]}(\mathcal{F}) \to \Gamma_{[X\setminus Z_1]}(\mathcal{F}) \oplus \Gamma_{[X\setminus Z_2]}(\mathcal{F}) \to \Gamma_{[X\setminus(Z_1\cup Z_2)]}(\mathcal{F}).$$

(2) $0 \to \Gamma_{[Z_1\cap Z_2]}(\mathcal{F}) \to \Gamma_{[Z_1]}(\mathcal{F}) \oplus \Gamma_{[Z_2]}(\mathcal{F}) \to \Gamma_{[Z_1\cup Z_2]}(\mathcal{F}).$

Here in (1) *the homomorphisms* $\Gamma_{[X\setminus(Z_1\cap Z_2)]}(\mathcal{F}) \to \Gamma_{[X\setminus Z_\nu]}(\mathcal{F})$ $(\nu = 1, 2)$ *and* $\Gamma_{[X\setminus Z_1]}(\mathcal{F}) \to \Gamma_{[X\setminus(Z_1\cup Z_2)]}(\mathcal{F})$ *are the ones in* (3.18), *and the homomorphism* $\Gamma_{[X\setminus Z_2]}(\mathcal{F}) \to \Gamma_{[X\setminus(Z_1\cup Z_2)]}(\mathcal{F})$ *is the one in* (3.18) *with the sign changed. Likewise in* (2).

PROOF. The sequences

$$0 \to I^m_{Z_1} \cap I^m_{Z_2} \to I^m_{Z_1} \oplus I^m_{Z_2} \to I^m_{Z_1} + I^m_{Z_2} \to 0$$

are exact for all m. By applying the left exact functor $\mathcal{H}om_{\mathcal{O}_X}(\bullet, \mathcal{F})$ to these and taking the inductive limit with respect to m, we obtain (1) from Proposition 3.18. Similarly for (2). □

LEMMA 3.20.

$$\begin{aligned} \Gamma_{[Z]}(\mathcal{F})_x &\xrightarrow{\sim} \Gamma_{[Z]}(\mathcal{F}_x), \\ \Gamma_{[X\setminus Z]}(\mathcal{F})_x &\xrightarrow{\sim} \Gamma_{[X\setminus Z]}(\mathcal{F}_x). \end{aligned}$$

PROOF. By the coherence of I^n_Z,

$$\mathcal{H}om_{\mathcal{O}_X}(I^n_Z, \mathcal{F})_x \simeq \mathcal{H}om_{\mathcal{O}_{X,x}}(I^n_{Z,x}, \mathcal{F}_x) \simeq \mathcal{H}om_{\mathcal{O}_X}(I^n_Z, \mathcal{F}_x).$$

By taking their inductive limits, we obtain

$$\Gamma_{[X\setminus Z]}(\mathcal{F})_x \xrightarrow{\sim} \Gamma_{[X\setminus Z]}(\mathcal{F}_x).$$

The proof of the first isomorphism is similar. □

LEMMA 3.21.

$$\begin{aligned} \Gamma_{[X\setminus Z_2]}\Gamma_{[Z_1]}(\mathcal{F}) &\simeq \Gamma_{[Z_1]}\Gamma_{[X\setminus Z_2]}(\mathcal{F}) \\ &\simeq \varinjlim_{m,n} \mathcal{H}om_{\mathcal{O}_X}(I^n_{Z_2}/(I^m_{Z_1} I^n_{Z_2}), \mathcal{F}). \end{aligned}$$

PROOF. Similarly to the proof of (3.20), we have

$$\begin{aligned} \Gamma_{[X\setminus Z_2]}\Gamma_{[Z_1]}(\mathcal{F}) &\simeq \varinjlim_{n,m} \mathcal{H}om_{\mathcal{O}_X}(I^n_{Z_2} \otimes (\mathcal{O}_X/I^m_{Z_1}), \mathcal{F}) \\ &\simeq \varinjlim_{n,m} \mathcal{H}om_{\mathcal{O}_X}(I^n_{Z_2}/(I^m_{Z_1} I^n_{Z_2}), \mathcal{F}). \end{aligned}$$

$\Gamma_{[Z_1]}\Gamma_{[X\setminus Z_2]}(\mathcal{F})$ is isomorphic to the same by a similar computation. □

We will use this lemma for showing that $\Gamma_{[Z_1]}\Gamma_{[X\setminus Z_2]}(\mathcal{F})$ depends only on $Z_1\setminus Z_2$.

LEMMA 3.22. *If* $Z_2 \supset Z_1$, *then* $\Gamma_{[X\setminus Z_2]}\Gamma_{[Z_1]}(\mathcal{F}) = 0$.

PROOF. We have

$$\Gamma_{[X\setminus Z_2]}\Gamma_{[Z_1]}(\mathcal{F}) = \varinjlim_{n,m} \mathcal{H}om_{\mathcal{O}_X}(I_{Z_2}^n/(I_{Z_1}^m I_{Z_2}^n), \mathcal{F}).$$

This vanishes, since $I_{Z_2}^{m+n} \subset I_{Z_1}^m I_{Z_2}^n$. □

PROPOSITION 3.23. *If* $Z_1\setminus Z_2 = Z_1'\setminus Z_2'$, *then*

$$\Gamma_{[Z_1]}\Gamma_{[X\setminus Z_2]}(\mathcal{F}) \simeq \Gamma_{[Z_1']}\Gamma_{[X\setminus Z_2']}(\mathcal{F}).$$

PROOF. First we show that

$$\Gamma_{[Z_1][X\setminus Z_2]}(\mathcal{F}) \simeq \Gamma_{[Z_1]}\Gamma_{[X\setminus (Z_1\cap Z_2)]}(\mathcal{F}). \tag{3.22}$$

Applying the left exact functor $\Gamma_{[Z_1]}$ to the exact sequence in Lemma 3.19

$$0 \to \Gamma_{[X\setminus (Z_1\cap Z_2)]}(\mathcal{F}) \to \Gamma_{[X\setminus Z_1]}(\mathcal{F}) \oplus \Gamma_{[X\setminus Z_2]}(\mathcal{F}) \to \Gamma_{[X\setminus (Z_1\cup Z_2)]}(\mathcal{F}).$$

and using $\Gamma_{[Z_1]}\Gamma_{[X\setminus Z_1]}(\mathcal{F}) = 0$ and $\Gamma_{[Z_1]}\Gamma_{[X\setminus (Z_1\cup Z_2)]}(\mathcal{F}) = 0$, which are the result of Lemma 3.22, we obtain (3.22).

Put $Z_0 = Z_1 \cap Z_1'$. Then $Z_0\setminus Z_2 = Z_0\setminus Z_2'$. In the exact sequence (3.19) for $\mathcal{G} = \Gamma_{[Z_1]}\Gamma_{[X\setminus Z_2]}(\mathcal{F})$

$$0 \to \Gamma_{[Z_0]}(\mathcal{G}) \to \mathcal{G} \to \Gamma_{[X\setminus Z_0]}(\mathcal{G}),$$

we obtain

$$\Gamma_{[Z_0]}(\mathcal{G}) = \Gamma_{[Z_0\cap Z_1]}\Gamma_{[X\setminus Z_2]}(\mathcal{F}) = \Gamma_{[Z_0]}\Gamma_{[X\setminus Z_2]}(\mathcal{F})$$

from Proposition 3.18(2). By Proposition 3.18(1),

$$\Gamma_{[X\setminus Z_0]}(\mathcal{G}) = \Gamma_{[Z_1]}\Gamma_{[X\setminus (Z_0\cup Z_2)]}(\mathcal{F}),$$

and this vanishes by Lemma 3.22. Hence

$$\mathcal{G} = \Gamma_{[Z_0]}\Gamma_{[X\setminus Z_2]}(\mathcal{F}) = \Gamma_{[Z_0]}\Gamma_{[X\setminus (Z_0\cap Z_2)]}(\mathcal{F}).$$

The proposition follows from this, since $Z_0 \cap Z_2 = Z_0 \cap Z_2'$. □

DEFINITION 3.24. Suppose that $S \subset X$ can be written as $S = Z_1\setminus Z_2$ for some closed analytic subsets Z_1 and Z_2. Then we set

$$\Gamma_{[S]}(\mathcal{F}) = \Gamma_{[Z_1]}\Gamma_{[X\setminus Z_2]}(\mathcal{F}),$$

which is independent of the choice of Z_1 and Z_2 by Proposition 3.23.

The following proposition is obvious.

PROPOSITION 3.25. *Let* S, S_1, *and* S_2 *be differences of closed analytic subsets of* X. *Then*

(1) $\Gamma_{[S]}$ *is a left exact functor from* $\mathrm{Mod}(\mathcal{O}_X)$ *to* $\mathrm{Mod}(\mathcal{O}_X)$.

(2) $\Gamma_{[S_1]} \circ \Gamma_{[S_2]} = \Gamma_{[S_1 \cap S_2]}$.

Let $\mathcal{M}$ be a $\mathcal{D}_X$-module, and Z a closed analytic set. We show that $\Gamma_{[Z]}(\mathcal{M})$ and $\Gamma_{[X\setminus Z]}(\mathcal{M})$ have $\mathcal{D}_X$-module structures. Since $\mathcal{D}_X$ is flat as a right $\mathcal{O}_X$-module, we have $\mathcal{D}_X \otimes_{\mathcal{O}_X} I_Z^n \simeq \mathcal{D}_X I_Z^n$. Hence

$$\Gamma_{[X\setminus Z]}(\mathcal{M}) = \varinjlim_n \mathcal{H}om_{\mathcal{D}_X}(\mathcal{D}_X I_Z^n, \mathcal{M}).$$

LEMMA 3.26. *For all ideals I of $\mathcal{O}_X$,*

$$I^{n+m} F_m(\mathcal{D}_X) \subset F_m(\mathcal{D}_X) I^n .$$

PROOF. We prove the inclusion by induction on m.

It is trivial when $m = 0$.

When $m = 1$, we obtain $I^{n+1} v \subset F_1(\mathcal{D}_X) I^n$ for $v \in \Theta_X$ from the equation

$$a_0 \cdots a_n v = v \cdot a_0 \cdots a_n - \sum_{i=0}^{n} [v, a_i] a_0 \cdots \hat{a}_i \cdots a_n \in F_1(\mathcal{D}_X) I^n$$

for $a_0, \ldots, a_n \in I$.

Suppose $m > 1$. Then, since $F_m(\mathcal{D}_X) = F_1(\mathcal{D}_X) F_{m-1}(\mathcal{D}_X)$, we have

$$\begin{aligned} I^{n+m} F_m(\mathcal{D}_X) &= I^{n+m} F_1(\mathcal{D}_X) F_{m-1}(\mathcal{D}_X) \\ &\subset F_1(\mathcal{D}_X) I^{n+m-1} F_{m-1}(\mathcal{D}_X) \\ &\subset F_1(\mathcal{D}_X) F_{m-1}(\mathcal{D}_X) I^n . \end{aligned}$$

□

Hence for $P \in F_m(\mathcal{D}_X)$ we obtain a homomorphism of left $\mathcal{D}_X$-modules

$$\mathcal{D}_X I_Z^{n+m} \to \mathcal{D}_X I_Z^n$$

by multiplying P from the right. Applying $\mathcal{H}om_{\mathcal{D}_X}(\bullet, \mathcal{M})$ to this, we obtain

$$\mathcal{H}om_{\mathcal{D}_X}(\mathcal{D}_X I_Z^n, \mathcal{M}) \to \mathcal{H}om_{\mathcal{D}_X}(\mathcal{D}_X I_Z^{n+m}, \mathcal{M}).$$

Further, applying $\varinjlim_n$ to this, we obtain

$$\Gamma_{[X\setminus Z]}(\mathcal{M}) \to \Gamma_{[X\setminus Z]}(\mathcal{M}).$$

Regarding this as the action of $P \in \mathcal{D}_X$, we provide $\Gamma_{[X\setminus Z]}(\mathcal{M})$ with a $\mathcal{D}_X$-module structure. Accordingly $\Gamma_{[Z]}(\mathcal{M}) = \mathrm{Ker}(\mathcal{M} \to \Gamma_{[X\setminus Z]}(\mathcal{M}))$ also has a $\mathcal{D}_X$-module structure. We have thus defined

$\mathcal{D}_X$-module structures in $\Gamma_{[Z]}(\mathcal{M})$ and $\Gamma_{[X\setminus Z]}(\mathcal{M})$. More generally, for a difference S of closed analytic subsets, we have a functor

$$\Gamma_{[S]} : \mathrm{Mod}(\mathcal{D}_X) \to \mathrm{Mod}(\mathcal{D}_X).$$

Since $\Gamma_{[S]}$ is left exact, we can define its right derived functors:

$$\mathbb{R}\Gamma_{[S]} \begin{array}{l} : \quad D^b(\mathcal{D}_X) \to D^b(\mathcal{D}_X)\,, \\ : \quad D^b(\mathcal{O}_X) \to D^b(\mathcal{O}_X)\,. \end{array}$$

We denote $H^k\mathbb{R}\Gamma_{[S]}(\mathcal{M})$ by $H^k_{[S]}(\mathcal{M})$ and call it the *k-th algebraic relative cohomology module* of $\mathcal{M}$. Put

$$\mathcal{J} = \left\{ \mathcal{M} \in \mathrm{Mod}(\mathcal{D}_X)\,; \quad \begin{array}{l} \mathcal{M}_x \text{ is an injective } \mathcal{O}_{X,x}\text{-module} \\ \text{for every } x \in X \end{array} \right\}.$$

PROPOSITION 3.27. *Let Z be a closed analytic set, and S a difference of closed analytic sets.*

(0) *For every $\mathcal{M} \in \mathrm{Mod}(\mathcal{D}_X)$, there exists $\mathcal{N} \in \mathcal{J}$ with a monomorphism $\mathcal{M} \rightarrowtail \mathcal{N}$.*

(1) *If $0 \to \mathcal{M}' \to \mathcal{M} \to \mathcal{M}'' \to 0$ is exact, and if $\mathcal{M}', \mathcal{M} \in \mathcal{J}$, then $\mathcal{M}'' \in \mathcal{J}$.*

(2) *If $0 \to \mathcal{M}' \to \mathcal{M} \to \mathcal{M}'' \to 0$ is exact, and if $\mathcal{M}' \in \mathcal{J}$, then*

$$0 \to \Gamma_{[S]}(\mathcal{M}') \to \Gamma_{[S]}(\mathcal{M}) \to \Gamma_{[S]}(\mathcal{M}'') \to 0$$

is exact.

(3) *If $\mathcal{M} \in \mathcal{J}$, then $\Gamma_{[S]}(\mathcal{M}) \in \mathcal{J}$.*

(4) *If $\mathcal{M} \in \mathcal{J}$, then*

$$0 \to \Gamma_{[Z\cap S]}(\mathcal{M}) \to \Gamma_{[S]}(\mathcal{M}) \to \Gamma_{[S\setminus Z]}(\mathcal{M}) \to 0$$

is exact.

PROOF. (0) and (1) are trivial.

Let $S = Z_1 \setminus Z_2$. Then

$$(\Gamma_{[S]}\mathcal{M})_x = \varinjlim_{n,m} \mathrm{Hom}_{\mathcal{O}_{X,x}}((I^n_{Z_2}/I^m_{Z_1}I^n_{Z_2})_x, \mathcal{M}_x),$$

which leads to (2).

Suppose $\mathcal{M} \in \mathcal{J}$. Then, for every $x \in X$, applying the exact functor $\mathrm{Hom}_{(\mathcal{O}_X)_x}(\bullet, \mathcal{M}_x)$ to the exact sequence

$$0 \to (I^m_Z)_x \to (\mathcal{O}_X)_x \to (\mathcal{O}_X/I^m_Z)_x \to 0\,,$$

we obtain an exact sequence

$$0 \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X/I^m_Z, \mathcal{M})_x \to \mathcal{M}_x \to \mathcal{H}om_{\mathcal{O}_X}(I^m_Z, \mathcal{M})_x \to 0.$$

Hence

$$0 \to \Gamma_{[Z]}(\mathcal{M}) \to \mathcal{M} \to \Gamma_{[X\backslash Z]}(\mathcal{M}) \to 0 \tag{3.23}$$

is exact.

We show (3). We need to show it when $S = X\backslash Z$ and when $S = Z$ for a closed analytic subset Z. Thanks to (3.23), we need to do it only for $\Gamma_{[Z]}(\mathcal{M})$. To do this, we show that

$$\mathcal{H}om_{\mathcal{O}_X}(G, \Gamma_{[Z]}\mathcal{M}) \to \mathcal{H}om_{\mathcal{O}_X}(G', \Gamma_{[Z]}\mathcal{M}) \to 0 \tag{3.24}$$

is exact for an exact sequence $0 \to G' \to G$ of coherent $\mathcal{O}_X$-modules. Recall that

$$\mathcal{H}om_{\mathcal{O}_X}(G, \Gamma_{[Z]}\mathcal{M}) \simeq \varinjlim_m \mathcal{H}om_{\mathcal{O}_X}(G/I_Z^m G, \mathcal{M}) \tag{3.25}$$

by Proposition 3.17. Since $G' \cap I_Z^{m+r} G \subset I_Z^m G'$ for $r \gg 0$ by Corollary A.36, it follows that

$$\mathcal{H}om_{\mathcal{O}_X}(G', \Gamma_{[Z]}\mathcal{M}) \simeq \varinjlim_m \mathcal{H}om_{\mathcal{O}_X}(G'/(G' \cap I_Z^m G), \mathcal{M}).$$

Hence we obtain (3.24) from the exact sequence

$$\mathcal{H}om_{\mathcal{O}_X}(G/I_Z^m G, \mathcal{M}) \to \mathcal{H}om_{\mathcal{O}_X}(G'/(G' \cap I_Z^m G), \mathcal{M}) \to 0$$

induced from $G'/(G' \cap I_Z^m G) \subset G/I_Z^m G$. We obtain (4) by applying (3.23) to $\Gamma_{[S]}\mathcal{M}$. □

The similar result holds for $\mathcal{O}_X$-modules. Hence we have the following corollary.

COROLLARY 3.28. *The following diagram commutes:*

$$\begin{array}{ccc} D^b(\mathcal{D}_X) & \xrightarrow{\mathbb{R}\Gamma_{[S]}} & D^b(\mathcal{D}_X) \\ \downarrow & & \downarrow \\ D^b(\mathcal{O}_X) & \xrightarrow{\mathbb{R}\Gamma_{[S]}} & D^b(\mathcal{O}_X). \end{array}$$

Hence we can compute $\mathbb{R}\Gamma_{[S]}(\mathcal{M})$ as follows: Let $\mathcal{K}$ be a complex composed of objects in $\mathcal{J}$, and $\mathcal{M} \to \mathcal{K}$ a quasi-isomorphism. Then

$$\mathbb{R}\Gamma_{[S]}(\mathcal{M}) \simeq \Gamma_{[S]}(\mathcal{K}).$$

In particular, we have the following.

THEOREM 3.29. (1) *Let S_1 and S_2 be differences of closed analytic sets. Then*

$$\mathbb{R}\Gamma_{[S_1]}\mathbb{R}\Gamma_{[S_2]}(\mathcal{M}) \cong \mathbb{R}\Gamma_{[S_1\cap S_2]}(\mathcal{M})$$

for any $\mathcal{M} \in D^b(\mathcal{D}_X)$.

(2) *Let Z be a closed analytic set. Then for any $\mathcal{M} \in D^b(\mathcal{D}_X)$ there exists a distinguished triangle*

$$\mathbb{R}\Gamma_{[Z]}(\mathcal{M}) \longrightarrow \mathcal{M} \longrightarrow \mathbb{R}\Gamma_{[X\setminus Z]}(\mathcal{M}) \xrightarrow{+1} .$$

PROOF. (1) is trivial.

Let us show (2). Take a complex $\mathcal{K}$ composed of objects in $\mathcal{J}$ and a quasi-isomorphism $\mathcal{M} \to \mathcal{K}$. Then from Proposition 3.27(4) we obtain an exact sequence of $\mathcal{D}_X$-modules

$$0 \to \Gamma_{[Z]}(\mathcal{K}^n) \to \mathcal{K}^n \to \Gamma_{[X\setminus Z]}(\mathcal{K}^n) \to 0.$$

(2) follows from this and Lemma A.40. □

PROPOSITION 3.30. *Let $F \in \mathrm{Mod}(\mathcal{D}_X)$. Then*

$$\mathcal{H}^k\mathbb{R}\Gamma_{[Z]}(F) \simeq \varinjlim_m \mathcal{E}xt^k_{\mathcal{O}_X}(\mathcal{O}_X/I_Z^m, F),$$

$$\mathcal{H}^k\mathbb{R}\Gamma_{[X\setminus Z]}(F) \simeq \varinjlim_m \mathcal{E}xt^k_{\mathcal{O}_X}(I_Z^m, F).$$

In particular, $H^k_{[Z]}(F) := H^k\mathbb{R}\Gamma_{[Z]}(F) = 0$ for $k > \dim X$.

PROOF. Take an injective resolution $F \to \mathcal{I}^\bullet$. Then

$$\mathbb{R}\Gamma_{[Z]}(F) \simeq \Gamma_{[Z]}(\mathcal{I}^\bullet) \simeq \varinjlim_m \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X/I_Z^m, \mathcal{I}^\bullet).$$

Since $\varinjlim_m$ is exact,

$$\begin{aligned} H^k\mathbb{R}\Gamma_{[Z]}(F) &\simeq \varinjlim_m H^k(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X/I_Z^m, \mathcal{I}^\bullet)) \\ &\simeq \varinjlim_m \mathcal{E}xt^k_{\mathcal{O}_X}(\mathcal{O}_X/I_Z^m, F). \end{aligned}$$

Since the injective dimension of $\mathcal{O}_{X,x}$ equals $\dim X$,

$$\mathcal{E}xt^k_{\mathcal{O}_X}(\mathcal{O}_X/I_Z^m, F)_x = \mathrm{Ext}^k_{\mathcal{O}_{X,x}}((\mathcal{O}_X/I_Z^m)_x, F_x) = 0$$

for $k > \dim X$. Hence $H^k\mathbb{R}\Gamma_{[Z]}(F) = 0$ for $k > \dim X$. The assertion for $\mathbb{R}\Gamma_{[X\setminus Z]}$ can be proved similarly. □

Using the same argument as above and Proposition 3.17, we obtain the following proposition.

PROPOSITION 3.31. *Let $\mathcal{G}$ be a coherent $\mathcal{O}_X$-module, and let $\mathcal{F} \in D^b(\mathcal{O}_X)$. Then*

$$\begin{aligned} \mathbb{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{G}, \mathbb{R}\Gamma_{[Z]}\mathcal{F}) &\simeq \mathbb{R}\Gamma_{[Z]}\mathbb{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{G}, \mathcal{F}), \\ H^k(\mathbb{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{G}, \mathbb{R}\Gamma_{[Z]}\mathcal{F})) &\simeq \varinjlim_m \mathcal{E}xt^k_{\mathcal{O}_X}(\mathcal{G}/I_Z^m\mathcal{G}, \mathcal{F}). \end{aligned}$$

When Z is a hypersurface $f^{-1}(0)$ ($f \in \mathcal{O}_X$), we have

$$\Gamma_{[X\setminus Z]}(\mathcal{M}) = \mathcal{O}_X[1/f] \overset{D}{\otimes} \mathcal{M} = \mathcal{M}_f,$$

where $\mathcal{M}_f = \varinjlim(\mathcal{M} \xrightarrow{f} \mathcal{M} \xrightarrow{f} \cdots)$. Hence this functor is exact with respect to $\mathcal{M}$, and thus

$$\mathbb{R}\Gamma_{[X\setminus f^{-1}(0)]}(\mathcal{M}) = \mathcal{O}_X[1/f] \overset{D}{\otimes} \mathcal{M}.$$

PROPOSITION 3.32. *Let Z be a closed analytic subset of X, and let $\mathcal{M} \in \mathrm{Mod}(\mathcal{D}_X)$. If $\Gamma_{[Z]}(\mathcal{M}) \xrightarrow{\sim} \mathcal{M}$, then $\mathbb{R}\Gamma_{[Z]}(\mathcal{M}) \xrightarrow{\sim} \mathcal{M}$.*

PROOF. Locally $Z = \{f_1 = \cdots = f_N = 0\}$ ($f_i \in \mathcal{O}_X$). We prove the proposition by induction on N.

(1) Suppose $N = 1$ and $Z = f^{-1}(0)$ ($f \in \mathcal{O}_X$). Then

$$\mathbb{R}\Gamma_{[Z]}(\mathcal{M}) = (\mathcal{M} \to \mathcal{M}_f).$$

Since for any $u \in \mathcal{M}$ there exists n such that $f^n u = 0$, we have $\mathcal{M}_f = 0$.

(2) Suppose $N > 1$, $Z_1 = \{f_2 = \cdots = f_N = 0\}$, and $Z_2 = \{f_1 = 0\}$. Then $\Gamma_{[Z_1]}(\mathcal{M}) = \Gamma_{[Z_2]}(\mathcal{M}) = \mathcal{M}$. By the induction hypothesis, we have $\mathbb{R}\Gamma_{[Z_1]}(\mathcal{M}) = \mathcal{M}$ and $\mathbb{R}\Gamma_{[Z_2]}(\mathcal{M}) = \mathcal{M}$. Hence

$$\mathbb{R}\Gamma_{[Z]}(\mathcal{M}) = \mathbb{R}\Gamma_{[Z_1]}\mathbb{R}\Gamma_{[Z_2]}(\mathcal{M}) = \mathbb{R}\Gamma_{[Z_1]}(\mathcal{M}) = \mathcal{M}.$$

□

COROLLARY 3.33. *Let $\mathcal{M} \in D^b(\mathcal{D}_X)$. If $\Gamma_{[Z]}\mathcal{H}^i(\mathcal{M}) = \mathcal{H}^i(\mathcal{M})$, then*

$$\mathbb{R}\Gamma_{[Z]}(\mathcal{M}) = \mathcal{M}.$$

PROPOSITION 3.34. *Let $\mathcal{M} \in D^b(\mathcal{D}_X)$. If $\mathbb{R}\Gamma_{[Z]}(\mathcal{M}) \xrightarrow{\sim} \mathcal{M}$, then*

$$\mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{N}) \xleftarrow{\sim} \mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathbb{R}\Gamma_{[Z]}\mathcal{N})$$

for any $\mathcal{N} \in D^b(\mathcal{D}_X)$.

PROOF. From the distinguished triangle

$$\mathbb{R}\Gamma_{[Z]}(\mathcal{N}) \longrightarrow \mathcal{N} \longrightarrow \mathbb{R}\Gamma_{[X\setminus Z]}(\mathcal{N}) \xrightarrow{+1},$$

we obtain

$$\begin{aligned}\mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathbb{R}\Gamma_{[Z]}\mathcal{N}) &\longrightarrow \mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{N}) \\ &\longrightarrow \mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathbb{R}\Gamma_{[X\setminus Z]}\mathcal{N}) \xrightarrow{+1} .\end{aligned}$$

Hence we want $\mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathbb{R}\Gamma_{[X\setminus Z]}\mathcal{N}) = 0$. Since we have $\mathbb{R}\Gamma_{[Z]}\mathbb{R}\Gamma_{[X\setminus Z]}\mathcal{N} = 0$, the proposition is reduced to

$$\mathbb{R}\Gamma_{[Z]}(\mathcal{N}) = 0 \implies \mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{N}) = 0. \tag{3.26}$$

To prove this, it is enough to show that

$$H^0(U; \mathbb{R}\operatorname{Hom}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{N}[i])) = 0$$

for all open sets U. Replacing U by X, and $\mathcal{N}[i]$ by $\mathcal{N}$, we show that

$$\operatorname{Hom}_{D^b(\mathcal{D}_X)}(\mathcal{M}, \mathcal{N}) = H^0(X; \mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{N}))$$

vanishes. Let $\varphi \in \operatorname{Hom}_{D^b(\mathcal{D}_X)}(\mathcal{M}, \mathcal{N})$. Then by the commutative diagram

$$\begin{tikzcd} \mathbb{R}\Gamma_{[Z]}(\mathcal{M}) \arrow[r, "\mathbb{R}\Gamma_{[Z]}\varphi"] \arrow[d, "\wr"'] & \mathbb{R}\Gamma_{[Z]}(\mathcal{N}) = 0 \arrow[d] \\ \mathcal{M} \arrow[r, "\varphi"] & \mathcal{N}, \end{tikzcd}$$

we see that $\varphi = 0$. ☐

PROPOSITION 3.35.

$$\mathbb{R}\Gamma_{[Z]}(\mathcal{M}) = \mathbb{R}\Gamma_{[Z]}(\mathcal{O}_X) \overset{\mathbb{D}}{\otimes} \mathcal{M} .$$

PROOF. We shall prove the more general statement

$$\mathbb{R}\Gamma_{[Z]}(\mathcal{N}) \overset{\mathbb{D}}{\otimes} \mathcal{M} \xrightarrow{\sim} \mathbb{R}\Gamma_{[Z]}(\mathcal{N} \overset{\mathbb{D}}{\otimes} \mathcal{M}). \tag{3.27}$$

In order to prove this, we shall first prove that

$$\mathbb{R}\Gamma_{[Z]}(\mathcal{F}) \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_X} \mathcal{G} \simeq \mathbb{R}\Gamma_{[Z]}(\mathcal{F} \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_X} \mathcal{G}) \tag{3.28}$$

for $\mathcal{F}, \mathcal{G} \in D^b(\mathcal{O}_X)$. We may assume that all cohomology modules of $\mathcal{G}$ are coherent. Then, setting $\mathcal{G}^* = \mathbb{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{G}, \mathcal{O}_X)$, we have

$$\mathbb{R}\Gamma_{[Z]}(\mathcal{F}) \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_X} \mathcal{G} \simeq \mathbb{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{G}^*, \mathbb{R}\Gamma_{[Z]}(\mathcal{F}))$$

and

$$\mathbb{R}\Gamma_{[Z]}(\mathcal{F} \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_X} \mathcal{G}) = \mathbb{R}\Gamma_{[Z]}\mathbb{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{G}^*, \mathcal{F}).$$

Then (3.28) follows from Proposition 3.31.

Now we shall show (3.27). Since

$$\mathbb{R}\Gamma_{[Z]}(\mathbb{R}\Gamma_{[Z]}(\mathcal{N})\overset{\mathbb{D}}{\otimes}\mathcal{M}) \xrightarrow{\sim} \mathbb{R}\Gamma_{[Z]}(\mathcal{N})\overset{\mathbb{D}}{\otimes}\mathcal{M}$$

by (3.28), the morphism $\mathbb{R}\Gamma_{[Z]}(\mathcal{N})\overset{\mathbb{D}}{\otimes}\mathcal{M} \to \mathcal{N}\overset{\mathbb{D}}{\otimes}\mathcal{M}$ is uniquely factored as

$$\mathbb{R}\Gamma_{[Z]}(\mathcal{N})\overset{\mathbb{D}}{\otimes}\mathcal{M} \to \mathbb{R}\Gamma_{[Z]}(\mathcal{N}\overset{\mathbb{D}}{\otimes}\mathcal{M}) \to \mathcal{N}\overset{\mathbb{D}}{\otimes}\mathcal{M}$$

by Proposition 3.34. The first arrow is an isomorphism by (3.28). □

Let Z_ν be analytic subsets ($\nu = 1, 2$). Then

$$\begin{aligned}\mathbb{R}\Gamma_{[Z_1]}(\mathcal{O}_X)\overset{\mathbb{D}}{\otimes}\mathbb{R}\Gamma_{[Z_2]}(\mathcal{O}_X) &\simeq \mathbb{R}\Gamma_{[Z_1]}\mathbb{R}\Gamma_{[Z_2]}(\mathcal{O}_X)\\ &\simeq \mathbb{R}\Gamma_{[Z_1\cap Z_2]}(\mathcal{O}_X).\end{aligned} \tag{3.29}$$

The next proposition immediately follows from Proposition 2.20.

PROPOSITION 3.36. *Let Z be a closed analytic subset of X of codimension $\geq l$. Then*

$$\mathcal{H}^k_{[Z]}(\mathcal{O}_X) = 0 \quad (k < l).$$

PROOF. By Proposition 2.20, $\mathcal{E}xt^k_{\mathcal{O}_X}(\mathcal{O}_X/I_Z^m, \mathcal{O}_X) = 0$ for $k < l$. □

For a purely l-codimensional closed analytic set Z, we define a $\mathcal{D}_X$-module $\mathcal{B}_{Z|X}$ by

$$\mathcal{B}_{Z|X} := \mathcal{H}^l_{[Z]}(\mathcal{O}_X)\,. \tag{3.30}$$

If Z is the hypersurface defined by $f(x) = 0$, then

$$\mathcal{H}^k_{[Z]}(\mathcal{O}_X) = \begin{cases} \mathcal{O}_X[f^{-1}]/\mathcal{O}_X & (k = 1)\,,\\ 0 & (k \neq 1)\,.\end{cases} \tag{3.31}$$

A closed analytic subset of X of codimension l is said to be locally of *complete intersection* if it locally equals the zero set of l holomorphic functions. If $Z = \{\, x\,;\, f_1(x) = \cdots = f_l(x) = 0\,\}$ is of complete intersection of codimension l, then

$$\mathbb{R}\Gamma_{[Z]}(\mathcal{O}_X) \simeq \mathcal{B}_{f_1^{-1}(0)|X}[-1]\overset{\mathbb{D}}{\otimes}\mathcal{B}_{f_2^{-1}(0)|X}[-1]\overset{\mathbb{D}}{\otimes}\cdots\overset{\mathbb{D}}{\otimes}\mathcal{B}_{f_l^{-1}(0)|X}[-1]$$

by (3.29). The cohomology modules of the left hand side vanish at degree less than l, and those of the right hand side vanish at degree greater than l. Hence we have the following proposition.

PROPOSITION 3.37. *Suppose that* $Z = \{x\,;\, f_1(x) = \cdots = f_l(x) = 0\}$ *is of complete intersection of codimension* l. *Then* $\mathcal{H}^k_{[Z]}(\mathcal{O}_X) = 0$ $(k \neq l)$, *and*

$$\mathcal{B}_{Z|X} := \mathcal{H}^l_{[Z]}(\mathcal{O}_X) \simeq \frac{\mathcal{O}_X[1/(f_1\cdots f_l)]}{\sum_{k=1}^{l}\mathcal{O}_X[1/(f_1\cdots \hat{f}_k\cdots f_l)]}\,.$$

Let Z_ν be an analytic subset of complete intersection of codimension l_ν $(\nu = 1, 2)$, and $Z = Z_1 \cap Z_2$ of codimension $(l_1 + l_2)$. Hence Z is also an analytic subset of complete intersection. Then

$$\mathcal{B}_{Z_1|X} \overset{D}{\otimes} \mathcal{B}_{Z_2|X} \xrightarrow{\sim} \mathcal{B}_{Z|X}\,. \tag{3.32}$$

Note that, for $\alpha(u \otimes v) = (-1)^{l_1 l_2} v \otimes u$,

$$\begin{tikzcd} \mathcal{B}_{Z_1|X} \overset{D}{\otimes} \mathcal{B}_{Z_2|X} \arrow[r] \arrow[d, "\alpha"] & \mathcal{B}_{Z|X} \\ \mathcal{B}_{Z_2|X} \overset{D}{\otimes} \mathcal{B}_{Z_1|X} \arrow[ur] & \end{tikzcd} \tag{3.33}$$

is commutative. This is deduced from the sign rule (A.9) and the fact that (3.29) is symmetric in Z_1 and Z_2.

Given $f \in \mathcal{O}_X$, we denote by $\delta(f)$ the section of $\mathcal{B}_{f^{-1}(0)|X} = \mathcal{O}_X[f^{-1}]/\mathcal{O}_X$ corresponding to $1/f$. For an analytic subset of complete intersection $Z = \{x\,;\, f_1(x) = \cdots = f_l(x) = 0\}$, we denote by $\delta(f_1)\cdot\cdots\cdot\delta(f_l) \in \mathcal{B}_{Z|X}$ the image of $\delta(f_1) \otimes \cdots \otimes \delta(f_l)$ by the composite in (3.29)

$$\mathcal{B}_{f_1^{-1}(0)|X} \overset{D}{\otimes} \cdots \overset{D}{\otimes} \mathcal{B}_{f_l^{-1}(0)|X} \to \mathcal{B}_{Z|X}\,.$$

It follows from (3.33) that $\delta(f_1)\delta(f_2) = -\delta(f_2)\delta(f_1)$.

LEMMA 3.38. *Suppose that* $Z = \{f_1 = \cdots = f_l = 0\}$ *is of codimension* l. *Put*

$$Z' = \{f_1 = \cdots = f_{l-1} = 0\}, \quad I = \sum_{i=1}^{l-1} \mathcal{O}_X f_i\,.$$

If $a \in \mathcal{O}_X$ *satisfies* $a|_{Z'} = 0$ (*i.e.*, $a^m \in I$, $m \gg 0$), *then*

$$\delta(f_1)\cdots\delta(f_{l-1})\delta(f_l + a) = \sum (-1)^k a^k \delta(f_1)\cdots\delta(f_{l-1})\delta(f_l^{1+k})\,.$$

Note that $a^k\delta(f_1)\cdots\delta(f_{l-1}) = 0$ for $k \gg 0$.

PROOF. We have an exact sequence

$$0 \to \mathcal{B}_{Z'|X} \to \mathcal{B}_{Z'|X}[1/f_l] \to \mathcal{B}_{Z|X} \to 0\,.$$

Let $j : X\backslash Z \hookrightarrow X$. Then

$$\mathcal{B}_{Z'|X}[1/f_l] \quad \hookrightarrow \quad j_* j^{-1}\mathcal{B}_{Z'|X}\,, \tag{3.34}$$

and $\mathcal{B}_{Z'|X}[1/f_l] = \mathcal{B}_{Z'|X}[1/(f_l + a)]$. Indeed, the kernel of (3.34) equals $\Gamma_{[Z]}\mathcal{B}_{Z'|X} = H^{l-1}\mathbb{R}\Gamma_{[Z]}(\mathcal{O}_X) = 0$.

Put $u = 1/(f_1 \cdots f_{l-1}) \in \mathcal{B}_{Z'|X}$. Then, outside Z,

$$\begin{aligned} u\frac{1}{f_l + a} \;=\; u\frac{1}{f_l(1 + a/f_l)} \;&=\; u\Big(\frac{(-1)^m a^m}{(f_l + a)f_l^m} + \sum_{k=0}^{m-1}(-1)^k\frac{a^k}{f_l^{k+1}}\Big) \\ &=\; u\left(\sum_{k=0}^{m-1}(-1)^k\frac{a^k}{f_l^{k+1}}\right). \end{aligned} \tag{3.35}$$

Hence (3.35) holds in $\mathcal{B}_{Z'|X}[1/f_l] = \mathcal{B}_{Z'|X}[1/(f_l+a)]$ as well, and this finishes the proof. $\square$

LEMMA 3.39. *Let Z be a submanifold of complete intersection of codimension l. For $f_1, \cdots, f_l \in \mathcal{O}_X$ with $I_Z = \mathcal{O}_X f_1 + \cdots + \mathcal{O}_X f_l$,*

$$\mathcal{B}_{Z|X} \otimes_{\mathcal{O}_X} \Omega_X^l \ni \frac{df_1}{f_1} \wedge \cdots \wedge \frac{df_l}{f_l} := \frac{1}{f_1 \cdots f_l} \otimes df_1 \wedge \cdots \wedge df_l$$

is independent of the choice of $f_1, \cdots, f_l$.

PROOF. Suppose $I_Z = \mathcal{O}_X f_1' + \cdots + \mathcal{O}_X f_l'$. Then, in a neighborhood of each point of Z, there exists a matrix $A \in GL_l(\mathcal{O}_X)$ such that

$$(f_1', \cdots, f_l') = (f_1, \cdots, f_l)A.$$

Since $GL_l(\mathcal{O}_X)$ is generated by

$$\begin{bmatrix} a & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & & & & & \\ & \ddots & & & & \\ & & 0 & 1 & & \\ & & 1 & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{bmatrix},$$

$$\begin{bmatrix} 1 & b & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix} \qquad (a \in \mathcal{O}_X^\times,\ b \in \mathcal{O}_X),$$

it is enough to show that the following three equations hold in $\mathcal{B}_{f_1^{-1}(0)\cap f_2^{-1}(0)|X} \otimes_{\mathcal{O}_X} \Omega_X^2$:

$$\begin{cases} \text{(a)} & \frac{1}{(af_1)} \cdot \frac{1}{f_2} d(af_1) \wedge df_2 = \frac{1}{f_1} \cdot \frac{1}{f_2} df_1 \wedge df_2\,, \\ \text{(b)} & \frac{1}{f_1} \cdot \frac{1}{f_2} df_1 \wedge df_2 = \frac{1}{f_2} \cdot \frac{1}{f_1} df_2 \wedge df_1\,, \\ \text{(c)} & \frac{1}{f_1} \cdot \frac{1}{af_1+f_2} df_1 \wedge d(af_1+f_2) = \frac{1}{f_1} \cdot \frac{1}{f_2} df_1 \wedge df_2\,. \end{cases}$$

(a) follows from $d(af_1) \wedge df_2 = f_1 da \wedge df_2 + a df_1 \wedge df_2$ and $f_1 \left(\frac{1}{f_1} \cdot \frac{1}{f_2}\right) = 0$.

(b) follows from $\frac{1}{f_1} \cdot \frac{1}{f_2} = -\frac{1}{f_2} \cdot \frac{1}{f_1}$.

(c) follows from $df_1 \wedge d(af_1 + f_2) = df_1 \wedge df_2 + f_1 da \wedge df_2$, $\frac{1}{f_1} \cdot \frac{1}{(af_1+f_2)} = \frac{1}{f_1} \cdot \frac{1}{f_2}$ and $f_1 \left(\frac{1}{f_1} \cdot \frac{1}{f_2}\right) = 0$. □

When $X = \mathbb{C}^n$, and $Z = \{x_1 = \cdots = x_l = 0\}$,

$$\mathcal{B}_{Z|X} = \frac{\mathcal{O}_X[1/x_1 \cdots x_l]}{\sum_{i=1}^{l} \mathcal{O}_X[1/x_1 \cdots \hat{x}_i \cdots x_l]}\,.$$

Any element of $\mathcal{O}_X[1/x_1 \cdots x_l]$ is written as $\sum_{\alpha\in\mathbb{Z}^l} a_\alpha(x'')x'^\alpha$ with $x' = (x_1, \ldots, x_l)$ and $x'' = (x_{l+1}, \ldots, x_n)$. For any k, if $\alpha_k \geq 0$, then $x'^\alpha \in \mathcal{O}_X[1/x_1 \cdots \hat{x}_k \cdots x_l]$. Hence any element of $\mathcal{B}_{Z|X}$ can be uniquely written as a finite sum

$$\sum_{\alpha_1,\ldots,\alpha_l>0} a_\alpha(x'')\frac{1}{x'^\alpha},$$

or $\sum_{\alpha\in\mathbb{Z}^l_{\geq 0}} c_\alpha(x'')\partial_{x_1}^{\alpha_1} \cdots \partial_{x_l}^{\alpha_l} \delta(x_1) \cdots \delta(x_l)$. Hence, regarding $\mathcal{O}_Z$ as a subsheaf of $\mathcal{O}_X$, we have

$$\mathcal{B}_{Z|X} = \bigoplus_{\alpha\in(\mathbb{Z}_{>0})^l} \mathcal{O}_Z \frac{1}{x'^\alpha}\,.$$

Put $\delta(x') = \delta(x_1) \cdots \delta(x_l) = \frac{1}{x_1 \cdots x_l} \in \mathcal{B}_{Z|X}$. Then

$$x_k \delta(x') = 0 \qquad (1 \leq k \leq l)\,,$$
$$\partial_k \delta(x') = 0 \qquad (l < k \leq n)\,.$$

Hence

$$\mathcal{B}_{Z|X} = \mathcal{D}_X / \left(\sum_{k=1}^{l} \mathcal{D}_X x_k + \sum_{k=l+1}^{n} \mathcal{D}_X \partial_k \right).$$

We have thus obtained the following theorem.

THEOREM 3.40. *Suppose that $Z \subset X$ is a submanifold of codimension l. Then*

$$\mathrm{Ch}(\mathcal{B}_{Z|X}) = T_Z^* X .$$

Even if $\mathcal{M}$ is a coherent $\mathcal{D}_X$-module, $H^i_{[Z]}(\mathcal{M})$ may not be a coherent $\mathcal{D}_X$-module (but it is a pseudo-coherent $\mathcal{D}_X$-module in the sense of Definition A.5). For example, $H^l_{[Z]}(\mathcal{D}_X) = \mathcal{B}_{Z|X} \overset{D}{\otimes} \mathcal{D}_X$ is not a coherent $\mathcal{D}_X$-module for a submanifold Z of X of codimension $l > 0$.

Let X_1, X_2 be copies of X, and $X \hookrightarrow X_1 \times X_2$ the diagonal embedding. Then $\mathcal{B}_{X|X_1 \times X_2}$ is a $\mathcal{D}_{X_1 \times X_2}$-module. Since $\mathcal{D}_{X_1} \otimes_{\mathbb{C}} \mathcal{D}_{X_2} \subset \mathcal{D}_{X_1 \times X_2}|_X$, we may regard $\mathcal{B}_{X|X_1 \times X_2}$ as a $\mathcal{D}_{X_1} \otimes_{\mathbb{C}} \mathcal{D}_{X_2}$-module. Thus $\mathcal{B}_{X|X_1 \times X_2} \otimes_{\mathcal{O}} \Omega_{X_2}$ is a $(\mathcal{D}_{X_1}, \mathcal{D}_{X_2})$-module. Then

$$\mathcal{D}_X \simeq \mathcal{B}_{X|X_1 \times X_2} \otimes_{\mathcal{O}} \Omega_{X_2} \tag{3.36}$$

as $(\mathcal{D}_{X_1}, \mathcal{D}_{X_2})$-modules. Indeed, $\mathcal{B}_{X|X_1 \times X_2} \otimes_{\mathcal{O}} \Omega_{X_2}$ has a canonical element by Lemma 3.39, which corresponds to $1 \in \mathcal{D}_X$.

For the isomorphism $a : X_1 \times X_2 \to X_1 \times X_2$ defined by $(x_1, x_2) \mapsto (x_2, x_1)$,

$$\begin{aligned} a_*(\mathcal{B}_{X|X_1 \times X_2} \otimes_{\mathcal{O}} \Omega_{X_2}) &= \mathcal{B}_{X|X_1 \times X_2} \otimes_{\mathcal{O}} \Omega_{X_1} \\ &= \Omega_{X_1} \otimes_{\mathcal{O}} (\mathcal{B}_{X|X_1 \times X_2} \otimes_{\mathcal{O}} \Omega_{X_2}) \otimes_{\mathcal{O}} \Omega_{X_2}^{\otimes -1}. \end{aligned}$$

The corresponding isomorphism,

$$\mathcal{D}_X \simeq \Omega_X \otimes_{\mathcal{O}} \mathcal{D}_X \otimes_{\mathcal{O}} \Omega_X^{\otimes -1},$$

is the one defined in Proposition 1.10.

CHAPTER 4

Functorial Properties of D-modules

4.1. Pull-backs of D-modules

Let $f : X \to Y$ be a morphism of complex manifolds. Then we can construct a $\mathcal{D}_X$-module by lifting a $\mathcal{D}_Y$-module.

Set $\mathcal{D}_{X\to Y} = \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{D}_Y$. This sheaf on X is a right $f^{-1}\mathcal{D}_Y$-module by the multiplication of $f^{-1}\mathcal{D}_Y$. We show that this also has a left $\mathcal{D}_X$-module structure. Note first that $\mathcal{D}_{X\to Y}$ has a natural left $\mathcal{O}_X$-module structure. Let $\sum a_i \otimes w_i$ ($a_i \in \mathcal{O}_X$, $w_i \in \Theta_Y$) be the image of $v \in \Theta_X$ by the natural morphism $\Theta_X \to \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\Theta_Y$. Then

$$v(a \otimes P) = v(a) \otimes P + \sum_i aa_i \otimes w_i P \quad (a \in \mathcal{O}_X,\ P \in \mathcal{D}_Y)$$

is independent of the choice of $\sum a_i \otimes w_i$. Indeed, since we can define a morphism $(\mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\Theta_Y) \otimes_{\mathbb{C}} f^{-1}\mathcal{D}_Y \to \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{D}_Y$ by $(a\otimes w)\otimes P \mapsto a\otimes wP$, we obtain a morphism $\Theta_X \otimes_{\mathbb{C}} (\mathcal{O}_X \otimes_{\mathbb{C}} f^{-1}\mathcal{D}_Y) \to \mathcal{D}_{X\to Y}$. Moreover we can easily check $v(af^*(b) \otimes P) = v(a \otimes bP)$ for $v \in \Theta_X$, $a \in \mathcal{O}_X$, $b \in \mathcal{O}_Y$, and $P \in \mathcal{D}_Y$. Thus we obtain a morphism $\Theta_X \otimes_{\mathbb{C}} \mathcal{D}_{X\to Y} \to \mathcal{D}_{X\to Y}$.

Since this action satisfies the conditions in Lemma 1.7, it can be extended to an action of $\mathcal{D}_X$. As this action of $\mathcal{D}_X$ and that of $f^{-1}\mathcal{D}_Y$ from the right commute, $\mathcal{D}_{X\to Y}$ is a $(\mathcal{D}_X, f^{-1}\mathcal{D}_Y)$-module. Let $1_{X\to Y}$ denote the canonical element $1 \otimes 1 \in \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{D}_Y$ of $\mathcal{D}_{X\to Y}$. Then

$$\begin{aligned} v 1_{X\to Y} &= \sum a_i 1_{X\to Y} w_i\,, \\ 1_{X\to Y} b &= f^*(b) 1_{X\to Y} \quad (b \in \mathcal{O}_Y). \end{aligned}$$

Thus, if $\mathcal{N}$ is a left $\mathcal{D}_Y$-module, then $\mathcal{D}_{X\to Y} \otimes_{f^{-1}\mathcal{D}_Y} f^{-1}\mathcal{N}$ is a left $\mathcal{D}_X$-module. We denote this by $Df^*\mathcal{N}$. As $\mathcal{O}_X$-modules,

$$Df^*\mathcal{N} \;\cong\; \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{D}_Y \otimes_{f^{-1}\mathcal{D}_Y} f^{-1}\mathcal{N}$$

$$\cong \quad \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{N}.$$

In coordinate systems $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_m)$ of X and Y respectively, for $a \in \mathcal{O}_X$ and $u \in \mathcal{N}$,

$$\frac{\partial}{\partial x_i}(a \otimes u) = \frac{\partial a}{\partial x_i} \otimes u + \sum_{j=1}^{m} a \frac{\partial f_j(x)}{\partial x_i} \otimes \frac{\partial}{\partial y_j} u,$$

where $f(x) = (f_1(x), \ldots, f_m(x))$.

Even if $\mathcal{N}$ is a coherent $\mathcal{D}_Y$-module, $Df^*\mathcal{N}$ may not be a coherent $\mathcal{D}_X$-module.

EXAMPLE 4.1. Let $X = \mathbb{C}$ and $Y = \mathbb{C}$ with coordinates x and y respectively. Let $f(x) = x^2$. Then $\mathcal{D}_Y = \bigoplus \mathcal{O}_Y \partial_y^n$, $Df^*(\mathcal{D}_Y) = \bigoplus_{n \geq 0} \mathcal{O}_X \otimes_{\mathbb{C}} \mathbb{C}\partial_y^n$, and

$$\partial_x(a \otimes \partial_y^n) = \frac{\partial a}{\partial x} \otimes \partial_y^n + 2ax \otimes \partial_y^{n+1}.$$

While $Df^*(\mathcal{D}_Y)$ is isomorphic to $\mathcal{D}_X$ in $X \backslash \{0\}$, it is not coherent over $\mathcal{D}_X$ on a neighborhood of $x = 0$; $Df^*(\mathcal{D}_Y) = \sum_{n=0}^{\infty} \mathcal{O}_X (x^{-1}\partial_x)^n \subset \mathcal{D}_X[x^{-1}]$.

Let us study the structure of $\mathcal{D}_{X \to Y}$ for a smooth morphism $f : X \to Y$. Locally we can write $f(x, y) = y$ in appropriate coordinate systems $y = (y_1, \ldots, y_m)$ of Y and $(x, y) = (x_1, \ldots, x_n, y_1, \ldots, y_m)$ of X. It follows from $1_{X \to Y}\partial_{y_i} = \partial_{y_i} 1_{X \to Y}$ that $\mathcal{D}_{X \to Y}$ is generated by $1_{X \to Y}$ as a $\mathcal{D}_X$-module. Since $f_*(\partial_{x_i}) = 0$,

$$\frac{\partial}{\partial x_i} 1_{X \to Y} = 0.$$

Let $\Omega^\bullet_{X/Y}$ denote the sheaf of differential forms relative to f, that is,

$$\Omega^1_{X/Y} = \operatorname{Coker}(f^*\Omega^1_Y \to \Omega^1_X), \tag{4.1}$$

$$\Omega^p_{X/Y} = \bigwedge^p \Omega^1_{X/Y} = \Omega^p_X / \left(\operatorname{Im}(f^*\Omega^1_Y \to \Omega^1_X) \wedge \Omega^{p-1}_X\right). \tag{4.2}$$

Then we have the relative de Rham complex

$$\Omega^\bullet_{X/Y} : \Omega^0_{X/Y} \xrightarrow{d^0_{X/Y}} \Omega^1_{X/Y} \xrightarrow{d^1_{X/Y}} \cdots \xrightarrow{d^{n-1}_{X/Y}} \Omega^n_{X/Y} \longrightarrow 0 \longrightarrow \cdots$$

(n is the dimension of fibers of f). Since the differentials $d^\bullet_{X/Y}$ are differential homomorphisms of order 1, we obtain a complex of right $\mathcal{D}_X$-modules

$$\Omega^\bullet_{X/Y} \otimes_{\mathcal{O}_X} \mathcal{D}_X : \Omega^0_{X/Y} \otimes_{\mathcal{O}_X} \mathcal{D}_X \to \cdots \to \Omega^n_{X/Y} \otimes_{\mathcal{O}_X} \mathcal{D}_X \tag{4.3}$$

by Corollary 1.4. Applying $\mathcal{H}om_{\mathcal{D}_X}(\bullet, \mathcal{D}_X)$ to this, we obtain

$$\mathcal{D}_X \otimes_{\mathcal{O}_X} \bigwedge^{\bullet} \Theta_{X/Y} : \mathcal{D}_X \leftarrow \mathcal{D}_X \otimes_{\mathcal{O}_X} \Theta_{X/Y} \leftarrow \cdots \leftarrow \mathcal{D}_X \otimes_{\mathcal{O}_X} \bigwedge^{n} \Theta_{X/Y},$$

where

$$\Theta_{X/Y} = \mathcal{H}om_{\mathcal{O}_X}(\Omega^1_{X/Y}, \mathcal{O}_X) = \operatorname{Ker}(\Theta_X \to f^*\Theta_Y). \tag{4.4}$$

PROPOSITION 4.2. *Let $f : X \to Y$ be a smooth morphism with fibers of dimension n. Then*

$$\begin{aligned} 0 \leftarrow \mathcal{D}_{X\to Y} \leftarrow \mathcal{D}_X \leftarrow \mathcal{D}_X \otimes_{\mathcal{O}_X} \Theta_{X/Y} \leftarrow& \\ \cdots \leftarrow \mathcal{D}_X \otimes_{\mathcal{O}_X} \bigwedge^{n} \Theta_{X/Y} \leftarrow 0& \end{aligned} \tag{4.5}$$

is exact.

PROOF. Define a coherent filtration of $\mathcal{D}_X \otimes_{\mathcal{O}_X} \bigwedge^k \Theta_{X/Y}$ by

$$F_m(\mathcal{D}_X \otimes_{\mathcal{O}_X} \bigwedge^{k} \Theta_{X/Y}) = F_{m-k}(\mathcal{D}_X) \otimes_{\mathcal{O}_X} \bigwedge^{k} \Theta_{X/Y},$$

and equip $\mathcal{D}_{X\to Y}$ with the quotient filtration from $\mathcal{D}_X$. Then (4.5) is a complex of filtered modules. Taking Gr^F, we obtain a complex of $\operatorname{Gr}^F(\mathcal{D}_X)$-modules

$$0 \leftarrow \operatorname{Gr}^F \mathcal{D}_{X\to Y} \leftarrow \operatorname{Gr}^F \mathcal{D}_X \leftarrow \operatorname{Gr}^F \mathcal{D}_X \otimes_{\mathcal{O}} \Theta_{X/Y} \leftarrow \cdots . \tag{4.6}$$

Locally write $\Theta_{X/Y} = \bigoplus_{i=1}^n \mathcal{O}_X v_i$. Then (4.6) is the Koszul complex with respect to $\{v_1, \ldots, v_n\}$ (cf. [Hotta2], §8.4). This is exact, since the codimension of the set of zeros of $\{v_1, \ldots, v_n\}$ in T^*X equals n. By Proposition A.17, (4.5) is exact. □

Letting Y be a set of one point $\{\mathrm{pt}\}$ in this proposition, we obtain Proposition 1.6.

4.2. Derived Functors of Inverse Images

The language of derived categories is most suitable for the systematic study of inverse images Df^*. In what follows, let us use the terminology in the Appendix. For a manifold X, let $\operatorname{Mod}(\mathcal{D}_X)$ denote the abelian category of $\mathcal{D}_X$-modules, $D(\mathcal{D}_X)$ the derived category of $\operatorname{Mod}(\mathcal{D}_X)$, and $D^b(\mathcal{D}_X)$ the subcategory of $D(\mathcal{D}_X)$ consisting of all bounded complexes. Let $f : X \to Y$ be a morphism of manifolds. Then

$$Df^* : \operatorname{Mod}(\mathcal{D}_Y) \to \operatorname{Mod}(\mathcal{D}_X)$$

is a right exact functor. Let us consider its left derived functor $\mathbb{L}Df^*$:

$$\mathbb{L}Df^* : D^-(\mathcal{D}_Y) \to D^-(\mathcal{D}_X).$$

This sends $D^b(\mathcal{D}_Y)$ to $D^b(\mathcal{D}_X)$, since the flat dimension of $\mathcal{O}_Y$ is finite. We denote this functor by $\mathbb{D}f^*$;

$$\mathbb{D}f^* : D^b(\mathcal{D}_Y) \to D^b(\mathcal{D}_X).$$

For a bounded complex $\mathcal{N}^\bullet$ of $\mathcal{D}_Y$-modules, by taking a bounded complex $\mathcal{L}^\bullet$ of $\mathcal{D}_Y$-modules flat over $\mathcal{O}_Y$ with a quasi-isomorphism $\mathcal{L}^\bullet \to \mathcal{N}^\bullet$, we can express $\mathbb{D}f^*\mathcal{N}^\bullet$ as $Df^*\mathcal{L}^\bullet$. The following propositions immediately follow from the corresponding propositions for Df^*.

PROPOSITION 4.3. *Let $f : X \to Y$ be a morphism of manifolds, and let $\mathcal{N}_k \in D^b(\mathcal{D}_Y)$ $(k = 1, 2)$. Then*

$$\mathbb{D}f^*(\mathcal{N}_1 \overset{\mathbb{D}}{\otimes} \mathcal{N}_2) \simeq \mathbb{D}f^*\mathcal{N}_1 \overset{\mathbb{D}}{\otimes} \mathbb{D}f^*\mathcal{N}_2.$$

PROPOSITION 4.4. *Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of manifolds. Then*

$$\mathbb{D}f^* \circ \mathbb{D}g^* \simeq \mathbb{D}(g \circ f)^*.$$

4.3. External Tensor Product

Let X, Y be two manifolds, and $q_1 : X \times Y \to X$, $q_2 : X \times Y \to Y$ the projections. For an $\mathcal{O}_X$-module $\mathcal{F}$ and an $\mathcal{O}_Y$-module $\mathcal{G}$, we define an $\mathcal{O}_{X\times Y}$-module $\mathcal{F} \boxtimes \mathcal{G}$ by

$$\begin{aligned} \mathcal{F} \boxtimes \mathcal{G} &:= (\mathcal{O}_{X\times Y} \otimes_{q_1^{-1}\mathcal{O}_X} \mathcal{F}) \otimes_{q_2^{-1}\mathcal{O}_Y} q_2^{-1}\mathcal{G} \\ &= \mathcal{O}_{X\times Y} \otimes_{q_1^{-1}\mathcal{O}_X \otimes_{\mathbb{C}} q_2^{-1}\mathcal{O}_Y} (q_1^{-1}\mathcal{F} \otimes_{\mathbb{C}} q_2^{-1}\mathcal{G}). \end{aligned}$$

It is well known that this functor is exact with respect to $\mathcal{F}$ and $\mathcal{G}$. Furthermore,

$$\mathrm{Supp}(\mathcal{F} \boxtimes \mathcal{G}) = (\mathrm{Supp}\,\mathcal{F}) \times (\mathrm{Supp}\,\mathcal{G}).$$

Similarly, $\mathcal{D}_{X\times Y}$ contains $q_1^{-1}\mathcal{D}_X \otimes_{\mathbb{C}} q_2^{-1}\mathcal{D}_Y$ as a subring. For a $\mathcal{D}_X$-module $\mathcal{M}$ and a $\mathcal{D}_Y$-module $\mathcal{N}$, set

$$\mathcal{M} \overset{D}{\boxtimes} \mathcal{N} := \mathcal{D}_{X\times Y} \otimes_{q_1^{-1}\mathcal{D}_X \otimes_{\mathbb{C}} q_2^{-1}\mathcal{D}_Y} (q_1^{-1}\mathcal{M} \otimes_{\mathbb{C}} q_2^{-1}\mathcal{N}).$$

We call this object the *external tensor product* of $\mathcal{M}$ and $\mathcal{N}$. We have

$$\mathcal{M} \overset{D}{\boxtimes} \mathcal{N} = Dq_1^*\mathcal{M} \overset{\mathbb{D}}{\otimes} Dq_2^*\mathcal{N}.$$

As $\mathcal{O}_{X\times Y}$-modules, $\mathcal{M}\overset{D}{\boxtimes}\mathcal{N} \simeq \mathcal{M}\boxtimes\mathcal{N}$. Hence $\mathcal{M}\overset{D}{\boxtimes}\mathcal{N}$ is an exact functor with respect to $\mathcal{M}$ and $\mathcal{N}$. Hence it gives a functor in the derived categories

$$\bullet \overset{\mathbb{D}}{\boxtimes} \bullet : D^b(\mathcal{D}_X) \times D^b(\mathcal{D}_Y) \to D^b(\mathcal{D}_{X\times Y}).$$

Since $\mathcal{D}_X \overset{D}{\boxtimes} \mathcal{D}_Y = \mathcal{D}_{X\times Y}$, $\mathcal{M}\overset{D}{\boxtimes}\mathcal{N}$ is coherent if $\mathcal{M}$ and $\mathcal{N}$ are coherent. The characteristic variety of $\mathcal{M}\overset{D}{\boxtimes}\mathcal{N}$ is given in the following proposition. We identify $T^*(X\times Y)$ with $T^*X \times T^*Y$.

PROPOSITION 4.5. *For a coherent $\mathcal{D}_X$-module $\mathcal{M}$ and a coherent $\mathcal{D}_Y$-module $\mathcal{N}$,*

$$\mathrm{Ch}(\mathcal{M}\overset{D}{\boxtimes}\mathcal{N}) = \mathrm{Ch}(\mathcal{M}) \times \mathrm{Ch}(\mathcal{N}).$$

PROOF. Let $F(\mathcal{M})$ and $F(\mathcal{N})$ be coherent filtrations of $\mathcal{M}$ and $\mathcal{N}$ respectively. Define a coherent filtration of $\mathcal{M}\overset{D}{\boxtimes}\mathcal{N}$ by

$$F_k(\mathcal{M}\overset{D}{\boxtimes}\mathcal{N}) = \sum_{k=i+j} F_i(\mathcal{M})\boxtimes F_j(\mathcal{N}) \subset \mathcal{M}\overset{D}{\boxtimes}\mathcal{N}.$$

Then $\mathrm{Gr}^F(\mathcal{M}\overset{D}{\boxtimes}\mathcal{N}) = \mathrm{Gr}^F(\mathcal{M})\boxtimes \mathrm{Gr}^F(\mathcal{N})$. Hence

$$\mathrm{Gr}^F(\mathcal{M}\overset{D}{\boxtimes}\mathcal{N})^\sim = (\mathrm{Gr}^F\mathcal{M})^\sim \boxtimes (\mathrm{Gr}^F\mathcal{N})^\sim,$$

and accordingly

$$\mathrm{Supp}(\mathrm{Gr}^F(\mathcal{M}\overset{D}{\boxtimes}\mathcal{N})^\sim) = \mathrm{Supp}(\mathrm{Gr}^F\mathcal{M})^\sim \times \mathrm{Supp}(\mathrm{Gr}^F\mathcal{N})^\sim.$$

□

4.4. Coherence of Inverse Images

As in Example 4.1, the functor Df^* does not preserve coherence of $\mathcal{D}$-modules in general. We can however give a sufficient condition for an inverse image to be coherent in terms of characteristic variety, as we describe it below.

Let $f : X \to Y$ be a morphism of manifolds. Then we have a canonical morphism f_d

$$f_d : X \underset{Y}{\times} T^*Y \to T^*X.$$

This is the morphism gathering all homomorphisms $T^*_{f(x)}Y \to T^*_x X$ ($x \in X$) dual to $T_x X \to T_{f(x)}Y$. Let

$$f_\pi : X \underset{Y}{\times} T^*Y \to T^*Y$$

denote the canonical projection. Note that we have

$$f_\pi^* \omega_Y = f_d^* \omega_X,$$

where ω_X and ω_Y denote the canonical 1-forms on T^*X and T^*Y respectively. We have

$$\begin{array}{ccc} T^*X & \xleftarrow{\ f_d\ } & X \underset{Y}{\times} T^*Y \\ & & \Big\downarrow{\scriptstyle f_\pi} \\ & & T^*Y \end{array}$$

Let $T^*_X X$ denote the zero section of the vector bundle $\pi_X : T^*X \to X$, and set

$$\overset{\circ}{T}{}^*X := T^*X \setminus T^*_X X.$$

In a coordinate system (x, ξ),

$$T^*_X X = \{(x, \xi); \xi = 0\}, \quad \overset{\circ}{T}{}^*X = \{(x, \xi); \xi \neq 0\}.$$

Set

$$T^*_X Y := f_d^{-1}(T^*_X X) \subset X \underset{Y}{\times} T^*Y.$$

When f is an embedding, $T^*_X Y$ is called the *conormal bundle* to X.

DEFINITION 4.6. Let $f : X \to Y$ be a morphism of manifolds, and $\mathcal{N}$ a coherent $\mathcal{D}_Y$-module. We say that X is *non-characteristic* for $\mathcal{N}$ if

$$(4.7) \qquad T^*_X Y \cap f_\pi^{-1} \operatorname{Ch} \mathcal{N} \subset X \underset{Y}{\times} T^*_Y Y.$$

THEOREM 4.7. *Let $f : X \to Y$ be a morphism of manifolds, and $\mathcal{N}$ a coherent $\mathcal{D}_Y$-module. Suppose that X is non-characteristic for $\mathcal{N}$. Then*

(1) $H^k(\mathbb{D}f^*\mathcal{N}) = 0$ $(k \neq 0)$, *or* $\mathcal{T}or_k^{f^{-1}\mathcal{O}_Y}(\mathcal{O}_X, f^{-1}\mathcal{N}) = 0$ $(k \neq 0)$ (*for the notation, see* §A. 3 (a)),

(2) $H^0(\mathbb{D}f^*\mathcal{N}) = Df^*\mathcal{N}$ *is a coherent $\mathcal{D}_X$-module, and*

(3) $\operatorname{Ch}(Df^*\mathcal{N}) = f_d f_\pi^{-1} \operatorname{Ch}(\mathcal{N})$.

REMARK 4.8. By the assumption of being non-characteristic (4.7),

$$f_d : f_\pi^{-1}\operatorname{Ch}(\mathcal{N}) \to T^*X$$

is finite (i.e., it is closed and all fibers are finite). Hence $f_d f_\pi^{-1}\operatorname{Ch}(\mathcal{N})$ is a closed analytic subset of T^*X.

PROOF. We divide the proof of this theorem into four steps:

(Step 1) the case when X is a hypersurface of Y,
(Step 2) the case when f is an embedding,
(Step 3) the case when f is smooth,
(Step 4) the general case.

Among the four steps, the first one is most essential.

(Step 1) The case when X is a hypersurface of Y.

Among the statements (1), (2), and (3) in Theorem 4.7, the inclusion $\operatorname{Ch}(Df^*\mathcal{N}) \subset f_d f_\pi^{-1}\operatorname{Ch}(\mathcal{N})$ can be proved rather elementarily. However, its equality in (3) is not easy to prove. While a proof using multiplicities is given in [KF], we give a proof using Gabber's theorem, Theorem 7.33.

The statements are local. Take a coordinate system $(t, x) = (t, x_1, x_2, \dots, x_n)$ of Y so that $X = \{(t, x) \in Y; t = 0\}$. Take a coordinate system $(t, x; \tau, \xi)$ of T^*Y and a coordinate system (x, ξ) of T^*X so that $\omega_Y = \tau dt + \langle \xi, dx\rangle$ and $\omega_X = \langle \xi, dx\rangle$. Then $f_\pi : X \underset{Y}{\times} T^*Y \to T^*Y$ is the embedding given by $t = 0$, and $f_d : X \underset{Y}{\times} T^*Y \to T^*X$ is the map given by $(x, \tau, \xi) \mapsto (x, \xi)$. Put $V = \operatorname{Ch}\mathcal{N}$. In what follows, we work in a neighborhood of $(t, x) = (0, 0)$. By the assumption of being non-characteristic, $(t, x; \tau, \xi) = (0, 0; 1, 0)$ does not belong to V. Hence there exists a holomorphic function $f(t, x; \tau, \xi)$ such that it is a homogeneous polynomial in (τ, ξ) and vanishes on V, and such that $f(0, 0; 1, 0) \neq 0$. Hence $(\partial/\partial\tau)^m f$ never vanishes at $(t, x) = 0$ for some nonnegative integer m.

LEMMA 4.9. *There exists no non-empty involutive analytic subset in* $V \cap X \underset{Y}{\times} T^*Y$.

PROOF. Let W be an involutive analytic subset in $V \cap X \underset{Y}{\times} T^*Y$. Since f and t vanish on W, so does $H_t^m f = (-1)^m (\partial/\partial\tau)^m f$, which does not have zeros. Hence W is empty. $\square$

We now use Corollary 7.38. By this, there exists a coherent filtration F of $\mathcal{N}$ such that

$$\Gamma_{V \cap X \underset{Y}{\times} T^*Y}((\mathrm{Gr}^F \mathcal{N})^\sim) = 0.$$

Since $\mathrm{Supp}((\mathrm{Gr}^F \mathcal{N})^\sim) = V$, we have $\Gamma_{X \underset{Y}{\times} T^*Y}((\mathrm{Gr}^F \mathcal{N})^\sim) = 0$. Thus $t : (\mathrm{Gr}^F \mathcal{N})^\sim \to (\mathrm{Gr}^F \mathcal{N})^\sim$ is injective, and, in turn,

$$t : \mathrm{Gr}^F(\mathcal{N}) \to \mathrm{Gr}^F(\mathcal{N}) \text{ is injective.} \tag{4.8}$$

It follows from Proposition A.17 that $t : \mathcal{N} \to \mathcal{N}$ is also injective.

From the resolution $0 \to \mathcal{O}_Y \xrightarrow{t} \mathcal{O}_Y \to \mathcal{O}_X \to 0$ of $\mathcal{O}_X$ by free $\mathcal{O}_Y$-modules, we obtain

$$H^k(\mathbb{D}f^*\mathcal{N}) = \begin{cases} \mathcal{N}/t\mathcal{N} & (k=0), \\ \mathrm{Ker}(t : \mathcal{N} \to \mathcal{N}) & (k=-1), \\ 0 & (k \neq 0, -1). \end{cases}$$

We have thus obtained (1).

Put $\mathcal{M} = Df^*\mathcal{N} = \mathcal{N}/t\mathcal{N}$, and define a filtration $F(\mathcal{M})$ of $\mathcal{M}$ by

$$F_k(\mathcal{M}) = \mathrm{Im}(F_k(\mathcal{N}) \to \mathcal{M}).$$

Then by (4.8) the sequence

$$0 \to \mathrm{Gr}^F(\mathcal{N}) \xrightarrow{t} \mathrm{Gr}^F(\mathcal{N}) \to \mathrm{Gr}^F(\mathcal{M}) \to 0$$

is exact. Since $f_d : f_\pi^{-1}\,\mathrm{Supp}((\mathrm{Gr}^F \mathcal{N})^\sim) \to T^*X$ is finite, we see that

$$(\mathrm{Gr}^F \mathcal{M})^\sim = f_{d*} f_\pi^* (\mathrm{Gr}^F \mathcal{N})^\sim,$$

that $(\mathrm{Gr}^F \mathcal{M})^\sim$ is a coherent $\mathcal{O}_{T^*X}$-module, and that

$$\mathrm{Supp}((\mathrm{Gr}^F \mathcal{M})^\sim) = f_d f_\pi^{-1}\,\mathrm{Supp}((\mathrm{Gr}^F \mathcal{N})^\sim). \tag{4.9}$$

Hence $\mathrm{Gr}^F(\mathcal{M})$ is a coherent $\mathrm{Gr}^F(\mathcal{D}_X)$-module, and by Proposition A.33 $\mathcal{M}$ is a coherent $\mathcal{D}_X$-module. Moreover from (4.9) we obtain (3).

(Step 2) The case when $f : X \to Y$ is an embedding.

We use induction on the codimension d of X in Y. The theorem is trivial when $d = 0$ and has been proved in Step 1 when $d = 1$. Suppose $d > 1$. Since the statements are local on X, we may assume that $f : X \to Y$ factors as $X \xrightarrow{f_1} Z \xrightarrow{f_2} Y$, where Z is a hypersurface of Y. By the induction hypothesis, the theorem holds for f_1 and f_2. Hence the theorem is reduced to the following lemma.

LEMMA 4.10. *Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of manifolds, and suppose that Theorem 4.7 (locally) holds for f and g.*

(1) *If $g \circ f$ is non-characteristic for a coherent $\mathcal{D}_Z$-module $\mathcal{N}$, then g is non-characteristic for $\mathcal{N}$ in a neighborhood of $f(X)$, and f is non-characteristic for $Dg^*\mathcal{N}$.*

(2) *Theorem 4.7 holds for $g \circ f$.*

PROOF. We have the following commutative diagram:

$$\begin{array}{ccccc} T^*X & \xleftarrow{f_d} & X \underset{Y}{\times} T^*Y & \xleftarrow{\varphi} & X \underset{Z}{\times} T^*Z \\ & & {\scriptstyle f_\pi}\downarrow & & \downarrow{\scriptstyle \psi} \\ & & T^*Y & \xleftarrow{g_d} & Y \underset{Z}{\times} T^*Z \\ & & & & \downarrow{\scriptstyle g_\pi} \\ & & & & T^*Z. \end{array}$$

We have $(g \circ f)_d = f_d \circ \varphi$, $(g \circ f)_\pi = g_\pi \circ \psi$, and $\psi^{-1}T^*_Y Z \subset T^*_X Z$. Put $V = \mathrm{Ch}(\mathcal{N}) \subset T^*Z$.

First we prove (1). By the assumption,

$$\psi^{-1}(g_\pi^{-1}V \cap T^*_Y Z) \subset (g \circ f)_\pi^{-1} V \cap T^*_X Z \subset X \underset{Z}{\times} T^*_Z Z.$$

This implies that g is non-characteristic for $\mathcal{N}$ in a neighborhood of $f(X)$. Hence $\mathrm{Ch}(Dg^*\mathcal{N}) = g_d g_\pi^{-1} V$. Since $\varphi^{-1}T^*_X Y = T^*_X Z$,

$$\begin{aligned} (f_\pi^{-1} g_d g_\pi^{-1} V) \cap T^*_X Y &= \varphi\psi^{-1} g_\pi^{-1} V \cap T^*_X Y \\ &= \varphi((g \circ f)_\pi^{-1} V \cap T^*_X Z) \\ &\subset \varphi(X \underset{Z}{\times} T^*_Z Z) = X \underset{Y}{\times} T^*_Y Y. \end{aligned}$$

We have thus proved (1). Hence

$$\mathbb{D}(g \circ f)^*\mathcal{N} = \mathbb{D}f^* \circ \mathbb{D}g^*\mathcal{N} = \mathbb{D}f^*(Dg^*\mathcal{N}) = Df^*Dg^*\mathcal{N},$$

which is coherent.

Moreover, we obtain

$$\begin{aligned} \mathrm{Ch}(Df^*Dg^*\mathcal{N}) &= f_d f_\pi^{-1} \mathrm{Ch}(Dg^*\mathcal{N}) \\ &= f_d f_\pi^{-1} g_d g_\pi^{-1} \mathrm{Ch}(\mathcal{N}) \\ &= f_d \varphi \psi^{-1} g_\pi^{-1} \mathrm{Ch}(\mathcal{N}) \\ &= (g \circ f)_d (g \circ f)_\pi^{-1} \mathrm{Ch}(\mathcal{N}). \end{aligned}$$

□

(Step 3) The case when f is smooth.

In this case, we may assume that $X = Y \times Z$, and that f is the projection of X onto Y. Then

$$\mathbb{D}f^*\mathcal{N} = \mathcal{N} \overset{\mathbb{D}}{\boxtimes} \mathcal{O}_Z.$$

Hence the theorem immediately follows from results in § 4.3.

(Step 4) The general case.

Let $Z = X \times Y$. Then $f : X \to Y$ factors as $X \xrightarrow{j} Z \xrightarrow{p} Y$, where $j(x) = (x, f(x))$ and $p(x, y) = y$. The theorem holds for j and p, since j is an embedding and p is smooth. Hence by Lemma 4.10 it holds for f as well.

We have thus completed the proof of Theorem 4.7. □

As a corollary of Theorem 4.7, we have the next proposition about the characteristic variety of a tensor product of D-modules. Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be coherent $\mathcal{D}_X$-modules. We say that $\mathcal{M}_1$ and $\mathcal{M}_2$ are *non-characteristic* if

$$\mathrm{Ch}(\mathcal{M}_1) \cap \mathrm{Ch}(\mathcal{M}_2) \subset T^*_X X.$$

For subsets V_1 and V_2 of T^*X, set

$$V_1 \underset{X}{+} V_2 = p_0(V_1 \underset{X}{\times} V_2),$$

where $p_0 : T^*X \underset{X}{\times} T^*X \to T^*X$ is the addition map $((x, \xi_1), (x, \xi_2)) \mapsto (x, \xi_1 + \xi_2)$. Hence

$$V_1 \underset{X}{+} V_2 = \{\, (x, \xi_1 + \xi_2)\, ;\, (x, \xi_1) \in V_1,\ (x, \xi_2) \in V_2 \,\}.$$

If V_1 and V_2 are homogeneous closed analytic subsets of T^*X satisfying $V_1 \cap V_2 \subset T^*_X X$, then $V_1 \underset{X}{\times} V_2 \to T^*X$ is finite, and $p_0(V_1 \underset{X}{\times} V_2)$ is also a homogeneous closed analytic subset of T^*X.

PROPOSITION 4.11. *Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be non-characteristic coherent $\mathcal{D}_X$-modules. Then*

(1) $H^k(\mathcal{M}_1 \overset{\mathbb{D}}{\otimes} \mathcal{M}_2) = 0$ $(k \neq 0)$,

(2) $H^0(\mathcal{M}_1 \overset{\mathbb{D}}{\otimes} \mathcal{M}_2) = \mathcal{M}_1 \overset{D}{\otimes} \mathcal{M}_2$ *is a coherent $\mathcal{D}_X$-module, and*

(3) $\mathrm{Ch}(\mathcal{M}_1 \overset{\mathbb{D}}{\otimes} \mathcal{M}_2) = \mathrm{Ch}(\mathcal{M}_1) \underset{X}{+} \mathrm{Ch}(\mathcal{M}_2)$.

PROOF. Let $i : X \to X \times X$ be the diagonal embedding ($x \mapsto (x, x)$), and $q_\nu : X \times X \to X$ ($\nu = 1, 2$) the ν-th projection. Then

$$\begin{aligned} \mathbb{D}i^*(\mathcal{M}_1 \overset{D}{\boxtimes} \mathcal{M}_2) &= \mathbb{D}i^*(\mathbb{D}q_1^*\mathcal{M}_1 \overset{\mathbb{D}}{\otimes} \mathbb{D}q_2^*\mathcal{M}_2) \\ &= \mathbb{D}i^*\mathbb{D}q_1^*\mathcal{M}_1 \overset{\mathbb{D}}{\otimes} \mathbb{D}i^*\mathbb{D}q_2^*\mathcal{M}_2 = \mathcal{M}_1 \overset{\mathbb{D}}{\otimes} \mathcal{M}_2. \end{aligned}$$

As seen in § 4.3, $\mathcal{N} := \mathcal{M}_1 \overset{D}{\boxtimes} \mathcal{M}_2$ is a coherent $\mathcal{D}_{X\times X}$-module, and

$$\mathrm{Ch}(\mathcal{N}) = \mathrm{Ch}(\mathcal{M}_1) \times \mathrm{Ch}(\mathcal{M}_2).$$

Since $\mathcal{M}_1$ and $\mathcal{M}_2$ are non-characteristic, i is non-characteristic for $\mathcal{N}$. Hence from Theorem 4.7 we obtain (1), (2), and the equality $\mathrm{Ch}(Di^*\mathcal{N}) = i_d i_\pi^{-1}\mathrm{Ch}(\mathcal{N})$. (3) follows from this, $i_d = p_0$, and $i_\pi^{-1}\mathrm{Ch}(\mathcal{N}) = \mathrm{Ch}(\mathcal{M}_1) \underset{X}{\times} \mathrm{Ch}(\mathcal{M}_2)$. □

As in the following, taking the dual and taking the inverse image commute under the assumption of being non-characteristic.

THEOREM 4.12. *Let $f : X \to Y$ be a morphism of manifolds, and $\mathcal{N}$ a coherent $\mathcal{D}_Y$-module. Suppose that f is non-characteristic for $\mathcal{N}$. Then*

$$\mathbb{D}_X\mathbb{D}f^*\mathcal{N} \simeq \mathbb{D}f^*\mathbb{D}_Y\mathcal{N}.$$

PROOF. By (3.13),

$$\mathrm{Hom}_{D^b(\mathcal{D}_Y)}(\mathcal{N}, \mathcal{N}) \simeq \mathrm{Hom}_{D^b(\mathcal{D}_Y)}(\mathcal{O}_Y, \mathbb{D}_Y\mathcal{N} \overset{\mathbb{D}}{\otimes} \mathcal{N}).$$

Hence we have a canonical map $\mathcal{O}_Y \to \mathbb{D}_Y\mathcal{N} \overset{\mathbb{D}}{\otimes} \mathcal{N}$. Applying the functor $\mathbb{D}f^*$ to this, we obtain

$$\mathcal{O}_X \simeq \mathbb{D}f^*\mathcal{O}_Y \to \mathbb{D}f^*(\mathbb{D}_Y\mathcal{N} \overset{\mathbb{D}}{\otimes} \mathcal{N}) \simeq (\mathbb{D}f^*\mathbb{D}_Y\mathcal{N}) \overset{\mathbb{D}}{\otimes} \mathbb{D}f^*\mathcal{N}.$$

Since $\mathbb{D}f^*\mathcal{N} \in D^b_{\mathrm{coh}}(\mathcal{D}_X)$,

$$\begin{aligned} &\mathrm{Hom}_{D^b(\mathcal{D}_X)}(\mathbb{D}_X\mathbb{D}f^*\mathcal{N}, \mathbb{D}f^*\mathbb{D}_Y\mathcal{N}) \\ &\simeq \mathrm{Hom}_{D^b(\mathcal{D}_X)}(\mathcal{O}_X, \mathbb{D}f^*\mathbb{D}_Y\mathcal{N} \overset{\mathbb{D}}{\otimes} \mathbb{D}_X^2\mathbb{D}f^*\mathcal{N}) \\ &\simeq \mathrm{Hom}_{D^b(\mathcal{D}_X)}(\mathcal{O}_X, \mathbb{D}f^*\mathbb{D}_Y\mathcal{N} \overset{\mathbb{D}}{\otimes} \mathbb{D}f^*\mathcal{N}). \end{aligned}$$

We have thus constructed a morphism

$$\mathbb{D}_X\mathbb{D}f^*\mathcal{N} \to \mathbb{D}f^*\mathbb{D}_Y\mathcal{N}.$$

We shall show that this is an isomorphism. As in the proof of Theorem 4.7, it suffices to prove it in the case when f is an embedding and in the case when f is smooth.

(1) The case when f is an embedding.

We can reduce this case to the case when X is a hypersurface of Y. We may assume that Y is an open subset of $\mathbb{C}\times\mathbb{C}^n$, and that $X = \{(t,x) \in Y \subset \mathbb{C}\times\mathbb{C}^n;\, t=0\}$. Taking suitable $u = (u_1,\dots,u_N)^T$ and $A(t,x,\partial_x) \in M_N(\mathcal{D}_Y)$, we may also assume $\mathcal{N} = \sum \mathcal{D}_Y u_i$ with the fundamental relation $\partial_t u = Au$. Then

$$\mathbb{D}f^*\mathcal{N} = \bigoplus_{i=1}^{N} \mathcal{D}_X u_i \simeq \mathcal{D}_X^{\oplus N},$$

$$\mathbb{D}_X\mathbb{D}f^*\mathcal{N} \simeq \mathcal{D}_X^{\oplus N}[n].$$

Furthermore, $\mathbb{D}_Y\mathcal{N} = \tilde{\mathcal{N}}[n]$. Here $\tilde{\mathcal{N}} = \sum_{i=1}^N \mathcal{D}_Y v_i$ with the fundamental relation $\partial_t v = A^* v$ for $v = (v_1,\dots,v_N)^T$, where $(A^*)_{ij}$ denotes the adjoint operator (Remark 1.12) of the (i,j)-component $A_{ij} \in \mathcal{D}_Y$ of A. We have $\mathbb{D}f^*\tilde{\mathcal{N}}[n] \simeq \mathcal{D}_X^{\oplus N}[n]$, and

$$\mathbb{D}_X\mathbb{D}f^*\mathcal{N} \to \mathbb{D}f^*\mathbb{D}_Y\mathcal{N}$$

is an isomorphism.

(2) The case when f is smooth.

We can reduce this case to the case when $\mathcal{N} = \mathcal{D}_Y$. Then $\mathbb{D}f^*\mathcal{N} = \mathcal{D}_{X\to Y}$, and

$$\begin{aligned}
\mathbb{D}_X\mathbb{D}f^*\mathcal{N} &= \mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{D}_{X\to Y}, \mathcal{D}_X \otimes_{\mathcal{O}_X} \Omega_X^{\otimes -1})[d_X] \\
&\simeq \mathcal{D}_{Y\leftarrow X} \otimes_{\mathcal{O}_X} \Omega_X^{\otimes -1}[-d_{X/Y}][d_X] \\
&\simeq \mathcal{D}_{X\to Y} \otimes_{\mathcal{O}_Y} \Omega_Y^{\otimes -1}[d_Y] \\
&\simeq \mathbb{D}f^*(\mathcal{D}_Y \otimes_{\mathcal{O}_Y} \Omega_Y^{\otimes -1})[d_Y] \\
&\simeq \mathbb{D}f^*\mathbb{D}_Y\mathcal{N}.
\end{aligned}$$

(See §4.6 for the notation, and note that we have used Proposition 4.19.) □

4.5. Cauchy–Kovalevskaya Theorem

A special case of the following theorem is known as the Cauchy–Kovalevskaya Theorem.

THEOREM 4.13. *Let $f : X \to Y$ be a morphism of complex manifolds, and $\mathcal{N}$ a coherent $\mathcal{D}_Y$-module. If f is non-characteristic for*

$\mathcal{N}$, *then*

$$f^{-1}\mathbb{R}\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{N},\mathcal{O}_Y) \to \mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathbb{D}f^*\mathcal{N},\mathcal{O}_X)$$

is an isomorphism, where this morphism is obtained from

$$f^{-1}\mathbb{R}\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{N},\mathcal{O}_Y) \to \mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathbb{D}f^*\mathcal{N},\mathbb{D}f^*\mathcal{O}_Y)$$

and $\mathbb{D}f^*\mathcal{O}_Y \cong \mathcal{O}_X$.

To see the meaning of this theorem, let us consider the following particular case (indeed, this case is essential).

Let Y be an open subset of $\mathbb{C}^n$, let $X = \{x = (x_1,\cdots,x_n) \in Y; x_1 = 0\}$, and let f be the standard embedding. In addition, let $\mathcal{N} = \mathcal{D}_Y/\mathcal{D}_Y P$ for $P(x,\partial_x) \in F_m(\mathcal{D}_Y)(Y)$ with $\sigma_m(P) \not\equiv 0$. Then $\mathrm{Ch}(\mathcal{N}) = \{\,(x,\xi)\,;\,\sigma_m(P)(x,\xi)=0\,\}$. Hence f is non-characteristic if and only if

$$\sigma_m(P)(0,x_2,\ldots,x_n;\ 1,0,\ldots,0) \neq 0,$$

which is equivalent to $a_{(m,0,\cdots,0)}(0,x_2,\ldots,x_n) \neq 0$ for

$$P(x,\partial_x) = \sum_{|\alpha|\le m} a_\alpha(x)\partial_x^\alpha.$$

This condition is usually referred as "the initial hyperplane $\{x_1 = 0\}$ is non-characteristic for the equation $Pu = 0$".

Let u be the generator $1 \mod \mathcal{D}_Y P$ of $\mathcal{N}$. Then $Pu = 0$, and

$$\mathbb{D}f^*\mathcal{N} \cong \bigoplus_{j=0}^{m-1} \mathcal{D}_X u_j \quad (u_j = 1_{X\to Y}\otimes \partial_1^j u).$$

Thus $\mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathbb{D}f^*\mathcal{N},\mathcal{O}_X) = \mathcal{O}_X^{\oplus m}$. Therefore the theorem is equivalent to saying

$$H^j(f^{-1}\mathbb{R}\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{N},\mathcal{O}_Y)) = 0 \quad (j\neq 0), \tag{4.10}$$

$$f^{-1}\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{N},\mathcal{O}_Y) \xrightarrow{\sim} \mathcal{O}_X^{\oplus m}. \tag{4.11}$$

Since $0 \longrightarrow \mathcal{D}_Y \xrightarrow{P} \mathcal{D}_Y \longrightarrow \mathcal{N} \longrightarrow 0$ is exact,

$$\begin{aligned}\mathbb{R}\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{N},\mathcal{O}_Y) &= (\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{D}_Y,\mathcal{O}_Y) \xrightarrow{P} \mathcal{H}om_{\mathcal{D}_Y}(\mathcal{D}_Y,\mathcal{O}_Y))\\ &= (\mathcal{O}_Y \xrightarrow{P} \mathcal{O}_Y).\end{aligned}$$

Hence

$$H^j(\mathbb{R}\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{N},\mathcal{O}_Y)) = \begin{cases} \mathrm{Ker}(\mathcal{O}_Y \xrightarrow{P} \mathcal{O}_Y) & (j=0), \\ \mathrm{Coker}(\mathcal{O}_Y \xrightarrow{P} \mathcal{O}_Y) & (j=1), \\ 0 & (j\neq 0,1). \end{cases}$$

The condition (4.10) means the surjectivity of $\mathcal{O}_Y \xrightarrow{P} \mathcal{O}_Y$, and the condition (4.11) means the isomorphism

$$f^{-1}\operatorname{Ker}(\mathcal{O}_Y \xrightarrow{P} \mathcal{O}_Y) \xrightarrow{\sim} \mathcal{O}_X^{\oplus m}. \tag{4.12}$$

The homomorphism (4.12) is given as follows: $g \in \operatorname{Ker}(\mathcal{O}_Y \xrightarrow{P} \mathcal{O}_Y)$ corresponds to $\varphi : \mathcal{M} \to \mathcal{O}_Y$ satisfying $\varphi(u) = g$. Then $(Df^*\varphi)(u_j) = f^*(\varphi(\partial_1^j u)) = \partial_1^j g|_X$. Namely, (4.12) is the homomorphism corresponding g with $Pg = 0$ to $(g|_X, \partial_1 g|_X, \dots, \partial_1^{m-1} g|_X) \in \mathcal{O}_X^{\oplus m}$. Thus the isomorphism (4.12) is translated into the statement that, for given holomorphic functions v_j $(0 \le j < m)$ on X, we can uniquely solve the system of equations

$$\begin{cases} Pg = 0, \\ \partial_1^j g|_X = v_j \quad (0 \le j < m). \end{cases}$$

This is the famous Cauchy-Kovalevskaya Theorem.

PROOF OF THE THEOREM. (Step 1) The case when X is a hyperplane of Y.

We can write $\mathcal{N} = \sum_j \mathcal{D}_Y u_j$ locally on X. Set $\overset{\circ}{T}{}^*_X Y = T^*_X Y \cap \overset{\circ}{T}{}^* Y$. Since $\operatorname{Ch}(\mathcal{D}_Y u_j) \cap \overset{\circ}{T}{}^*_X Y = \emptyset$, there exist $P_j \in \mathcal{D}_Y$ such that $\sigma(P_j)^{-1}(0) \cap \overset{\circ}{T}{}^*_X Y = \emptyset$, and $P_j u_j = 0$. Put $\mathcal{L}_0 = \bigoplus_j (\mathcal{D}_Y/\mathcal{D}_Y P_j)$. Then we have an exact sequence of coherent $\mathcal{D}_Y$-modules

$$0 \to \mathcal{N}' \to \mathcal{L}_0 \to \mathcal{N} \to 0.$$

Since $\operatorname{Ch}(\mathcal{N}') \subset \operatorname{Ch}(\mathcal{L}_0)$, f is non-characteristic for $\mathcal{N}'$. Repeating this process, we obtain an exact sequence

$$\cdots \to \mathcal{L}_1 \to \mathcal{L}_0 \to \mathcal{N} \to 0,$$

where each $\mathcal{L}_k$ is of the form $\bigoplus_j (\mathcal{D}_Y/\mathcal{D}_Y P_{kj})$ with $\sigma(P_{kj})^{-1}(0) \cap \overset{\circ}{T}{}^*_X Y = \emptyset$. Hence $\mathcal{L}_\bullet$ and $\mathcal{N}$ are quasi-isomorphic, where $\mathcal{L}_\bullet$ denotes the complex

$$\to \mathcal{L}_1 \to \cdots \to \mathcal{L}_0 \to 0 \to \cdots.$$

Thus $Df^*\mathcal{L}_\bullet \simeq \mathbb{D}f^*\mathcal{N}$.

By the classical Cauchy-Kovalevskaya Theorem, each

$$f^{-1}\mathbb{R}\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{L}_k, \mathcal{O}_Y) \to \mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(Df^*\mathcal{L}_k, \mathcal{O}_X)$$

is an isomorphism, and hence

$$f^{-1}\mathbb{R}\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{L}_\bullet, \mathcal{O}_Y) \to \mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(Df^*\mathcal{L}_\bullet, \mathcal{O}_X)$$

is also an isomorphism. We obtain the desired result, since $\mathcal{L}_\bullet$ and $\mathcal{N}$ are isomorphic in $D^-(\mathcal{D}_Y)$, and $Df^*\mathcal{L}_\bullet$ and $\mathbb{D}f^*\mathcal{N}$ in $D^-(\mathcal{D}_X)$.

(Step 2) The case when f is an embedding.

Since the assertion is local on X, we factor $f : X \hookrightarrow Y$ into $X \hookrightarrow X' \hookrightarrow Y$, where X' is a hyperplane of Y. Let f' be $X \hookrightarrow X'$, and let f'' be $X' \hookrightarrow Y$. By Lemma 4.10, f'' is non-characteristic for $\mathcal{N}$, and f' for $\mathbb{D}f''^*\mathcal{N}$ (in a neighborhood of X). Hence

$$\begin{aligned} &f^{-1}\mathbb{R}\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{N}, \mathcal{O}_Y) \simeq f'^{-1}f''^{-1}\mathbb{R}\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{N}, \mathcal{O}_Y) \\ &\simeq f'^{-1}\mathbb{R}\mathcal{H}om_{\mathcal{D}'_X}(\mathbb{D}f''^*\mathcal{N}, \mathcal{O}_{X'}) \simeq \mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathbb{D}f'^*\mathbb{D}f''^*\mathcal{N}, \mathcal{O}_X). \end{aligned}$$

We obtain the desired result from this, since $\mathbb{D}f^*\mathcal{N} \cong \mathbb{D}f'^*\mathbb{D}f''^*\mathcal{N}$.

(Step 3) The case when f is smooth.

In this case, $\mathcal{D}_{X\to Y}$ is a coherent $\mathcal{D}_X$-module and flat over $f^{-1}\mathcal{D}_Y$. First we prove the following lemma, the special case $\mathcal{M} = \mathcal{D}_Y$ of the theorem.

LEMMA 4.14. *$f^{-1}\mathcal{O}_Y \xrightarrow{\sim} \mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{D}_{X\to Y}, \mathcal{O}_X)$ in $D^b(f^{-1}\mathcal{D}_Y)$.*

PROOF. By Proposition 4.2, we have an exact sequence

$$0 \leftarrow \mathcal{D}_{X\to Y} \leftarrow \mathcal{D}_X \leftarrow \mathcal{D}_X \otimes_{\mathcal{O}_X} \Theta_{X/Y} \leftarrow \mathcal{D}_X \otimes_{\mathcal{O}_X} \bigwedge^2 \Theta_{X/Y} \leftarrow \cdots .$$

Hence $\mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{D}_{X\to Y}, \mathcal{O}_X)$ can be represented by the complex $\mathcal{H}om_{\mathcal{D}_X}(\mathcal{D}_X \otimes_{\mathcal{O}_X} \bigwedge^\bullet \Theta_{X/Y}, \mathcal{O}_X) \simeq \Omega^\bullet_{X/Y}$.

It is well known that the cohomologies of the relative de Rham complex $\Omega^\bullet_{X/Y}$ are given by

$$\mathcal{H}^j(\Omega^\bullet_{X/Y}) = \begin{cases} f^{-1}\mathcal{O}_Y & (j = 0), \\ 0 & (j \neq 0). \end{cases}$$

□

Using this lemma, we prove the theorem when f is smooth. We have

$$\begin{aligned} &\mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathbb{D}f^*\mathcal{N}, \mathcal{O}_X) \\ &\quad = \mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{D}_{X\to Y} \overset{\mathbb{L}}{\otimes}_{f^{-1}\mathcal{D}_Y} f^{-1}\mathcal{N},\ \mathcal{O}_X) \\ &\quad \simeq \mathbb{R}\mathcal{H}om_{f^{-1}\mathcal{D}_Y}(f^{-1}\mathcal{N},\ \mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{D}_{X\to Y}, \mathcal{O}_X)) \\ &\quad \simeq \mathbb{R}\mathcal{H}om_{f^{-1}\mathcal{D}_Y}(f^{-1}\mathcal{N}, f^{-1}\mathcal{O}_Y) \\ &\quad \simeq f^{-1}\mathbb{R}\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{N}, \mathcal{O}_Y). \end{aligned}$$

In the last isomorphism, we have used the coherence of $\mathcal{N}$.

(Step 4) The general case.

Let $j : X \to Z = X \times Y$ $(x \mapsto (x, f(x)))$ be the graph morphism of f, and $p : Z \to Y$ the projection. Then $f = p \circ j$. By Lemma 4.10, j is non-characteristic for $\mathbb{D}p^*\mathcal{N}$. Hence, by the results in Step 2 and Step 3,

$$\begin{aligned}
&\mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathbb{D}f^*\mathcal{N}, \mathcal{O}_X)\\
&\quad \simeq \mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathbb{D}j^*\mathbb{D}p^*\mathcal{N}, \mathcal{O}_X)\\
&\quad \simeq j^{-1}\mathbb{R}\mathcal{H}om_{\mathcal{D}_Z}(\mathbb{D}p^*\mathcal{N}, \mathcal{O}_Z) \simeq j^{-1}p^{-1}\mathbb{R}\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{N}, \mathcal{O}_Y)\,.
\end{aligned}$$

We have thus completed the proof of Theorem 4.13. □

As a corollary of Theorem 4.13, we deduce the following more general statement: Theorem 4.13 is the case $\mathcal{N}_2 = \mathcal{O}_Y$.

COROLLARY 4.15. *Let $f : X \to Y$ be a morphism of complex manifolds, and let $\mathcal{N}_1$ and $\mathcal{N}_2$ be coherent $\mathcal{D}_Y$-modules. We assume the following:*

(4.13) *$\mathcal{N}_1$ and $\mathcal{N}_2$ are non-characteristic, i.e.,*

$$\mathrm{Ch}(\mathcal{N}_1) \cap \mathrm{Ch}(\mathcal{N}_2) \subset T^*_Y Y,$$

(4.14) *f is non-characteristic for $(\mathrm{Ch}\,\mathcal{N}_1)\underset{Y}{+}(\mathrm{Ch}\,\mathcal{N}_2)$, i.e.,*

$$T^*_X Y \cap f_\pi^{-1}(\mathrm{Ch}\,\mathcal{N}_1 \underset{Y}{+} \mathrm{Ch}\,\mathcal{N}_2) \subset X \underset{Y}{\times} T^*_Y Y\,.$$

Then

$$f^{-1}\mathbb{R}\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{N}_1, \mathcal{N}_2) \xrightarrow{\sim} \mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathbb{D}f^*\mathcal{N}_1,\, \mathbb{D}f^*\mathcal{N}_2)\,.$$

PROOF. By (3.13),

$$\mathbb{R}\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{N}_1, \mathcal{N}_2) \simeq \mathbb{R}\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{O}_Y,\, \mathbb{D}_Y\mathcal{N}_1 \overset{\mathbb{D}}{\otimes} \mathcal{N}_2)\,.$$

Proposition 3.14 says $\mathrm{Ch}(\mathbb{D}_Y\mathcal{N}_1) = \mathrm{Ch}(\mathcal{N}_1)$. Hence, by Proposition 4.11, $(\mathbb{D}_Y\mathcal{N}_1) \overset{\mathbb{D}}{\otimes} \mathcal{N}_2$ belongs to $D^b_{\mathrm{coh}}(\mathcal{D}_Y)$. Thanks to 3.14, $\mathcal{N} = (\mathbb{D}_Y\mathcal{N}_1) \overset{\mathbb{D}}{\otimes} \mathcal{N}_2 \in D^b_{\mathrm{coh}}(\mathcal{D}_Y)$ satisfies

$$f^{-1}\mathbb{R}\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{O}_Y, \mathcal{N}) \simeq f^{-1}\mathbb{R}\mathcal{H}om_{\mathcal{D}_Y}(\mathbb{D}_Y\mathcal{N}, \mathcal{O}_Y)\,.$$

Since $\mathrm{Ch}(\mathbb{D}_Y\mathcal{N}) = \mathrm{Ch}(\mathcal{N}) = \mathrm{Ch}(\mathcal{N}_1)\underset{Y}{+}\mathrm{Ch}(\mathcal{N}_2)$ by Propositions 3.14 and 4.11, f is non-characteristic for $\mathbb{D}_Y\mathcal{N}$. Hence Theorem 4.13 says

$$f^{-1}\mathbb{R}\mathcal{H}om_{\mathcal{D}_Y}(\mathbb{D}_Y\mathcal{N}, \mathcal{O}_Y) \simeq \mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathbb{D}f^*\mathbb{D}_Y\mathcal{N}, \mathcal{O}_X)\,.$$

By Theorem 4.12,

$$\mathbb{D}f^*\mathbb{D}_Y\mathcal{N} = \mathbb{D}_X\mathbb{D}f^*\mathcal{N}\,.$$

Hence

$$
\begin{aligned}
&\mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathbb{D}f^*\mathbb{D}_Y\mathcal{N},\, \mathcal{O}_X)\\
&\quad\simeq \mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathbb{D}_X\mathbb{D}f^*\mathcal{N},\, \mathcal{O}_X)\\
&\quad\simeq \mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{O}_X,\, \mathbb{D}f^*\mathcal{N})\\
&\quad\simeq \mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{O}_X,\, \mathbb{D}f^*\mathbb{D}_Y\mathcal{N}_1 \overset{\mathbb{D}}{\otimes} \mathbb{D}f^*\mathcal{N}_2)\\
&\quad\simeq \mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{O}_X,\, \mathbb{D}_X\mathbb{D}f^*\mathcal{N}_1 \overset{\mathbb{D}}{\otimes} \mathbb{D}f^*\mathcal{N}_2)\\
&\quad\simeq \mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathbb{D}f^*\mathcal{N}_1,\, \mathbb{D}f^*\mathcal{N}_2)\,.
\end{aligned}
$$

□

REMARK 4.16. If $f : X \hookrightarrow Y$ is an inclusion, then

$$\mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathbb{D}f^*\mathcal{N}, \mathcal{O}_X) \simeq \mathbb{R}\mathcal{H}om_{f^{-1}\mathcal{D}_Y}(f^{-1}\mathcal{N}, \widehat{\mathcal{O}}_{X/Y})$$

for any $\mathcal{D}_Y$-module $\mathcal{N}$, where $\widehat{\mathcal{O}}_{X/Y}$ is the completion of $\mathcal{O}_Y$ along X defined by

$$\widehat{\mathcal{O}}_{X/Y} := \varprojlim_k \mathcal{O}_Y / I_X^k\,.$$

Here $I_X \subset \mathcal{O}_Y$ is the defining ideal of X. The completion $\widehat{\mathcal{O}}_{X/Y}$ is an $f^{-1}\mathcal{D}_Y$-module. If Y is an open subset of $\mathbb{C}^{n+m} = \{(x,y); x \in \mathbb{C}^n,\ y \in \mathbb{C}^m\}$, and if $X = Y \cap \mathbb{C}^n \times \{0\}$, then for $U \subset X$

$$\Gamma(U, \widehat{\mathcal{O}}_{X/Y}) = \{\sum a_\alpha(x) y^\alpha; a_\alpha(x) \in \mathcal{O}_X(U)\},$$

the set of formal power series in y with coefficients in $\mathcal{O}_X(U)$.

In this case, $\mathcal{D}_{X\hookrightarrow Y}$ equals $\bigoplus \mathcal{D}_X \partial_y^\alpha$ as a $\mathcal{D}_X$-module, and thus

$$\mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{D}_{X\hookrightarrow Y}, \mathcal{O}_X) = \prod_\alpha \mathcal{H}om_{\mathcal{D}_X}(\mathcal{D}_X \partial_y^\alpha, \mathcal{O}_X).$$

We have an isomorphism

$$\mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{D}_{X\hookrightarrow Y}, \mathcal{O}_X) \simeq \widehat{\mathcal{O}}_{X/Y}$$

given by

$$\prod_\alpha \mathcal{H}om_{\mathcal{D}_X}(\mathcal{D}_X \partial_y^\alpha, \mathcal{O}_X) \ni (\varphi_\alpha)_\alpha \mapsto \sum_\alpha \varphi_\alpha(\partial_y^\alpha)\frac{y^\alpha}{\alpha!} \in \widehat{\mathcal{O}}_{X/Y}.$$

Hence

$$\begin{aligned}
&\mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathbb{D}f^*\mathcal{N}, \mathcal{O}_X)\\
&\simeq \mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{D}_{X\hookrightarrow Y}\overset{\mathbb{L}}{\otimes}_{f^{-1}\mathcal{D}_Y} f^{-1}\mathcal{N}, \mathcal{O}_X)\\
&\simeq \mathbb{R}\mathcal{H}om_{f^{-1}\mathcal{D}_Y}(f^{-1}\mathcal{N}, \mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{D}_{X\hookrightarrow Y}, \mathcal{O}_X))\\
&\simeq \mathbb{R}\mathcal{H}om_{f^{-1}\mathcal{D}_Y}(f^{-1}\mathcal{N}, \widehat{\mathcal{O}}_{X/Y}).
\end{aligned}$$

4.6. Integrals of D-modules

Supposing that $v(y) = \int u(x,y)dx$ makes sense for a function $u(x,y)$, let us consider how to derive differential equations for $v(y)$ from those for $u(x,y)$. Supposing, in addition, that Stokes's Theorem $\int \frac{\partial}{\partial x_i} u(x,y)dx = 0$ holds, we obtain $\int \frac{\partial}{\partial x_i} S(x,y,\partial_x,\partial_y)u(x,y)dx = 0$ for all differential operators $S(x,y,\partial_x,\partial_y)$. Hence, for

$$Q(y,\partial_y) = \sum_i \partial_{x_i} S_i(x,y,\partial_x,\partial_y) + P(x,y,\partial_x,\partial_y),$$

we have

$$Q(y,\partial_y)v(y) = \int P(x,y,\partial_x,\partial_y)u(x,y)dx.$$

Furthermore, if $P(x,y,\partial_x,\partial_y)u = 0$, then $Q(y,\partial_y)v(y) = 0$. Let us describe the above consideration in the language of D-modules. Let X be a manifold with a coordinate system (x,y), and Y a submanifold with a coordinate system y. Then the above consideration means that we associate a $\mathcal{D}_X$-module $\mathcal{M}$ to a $\mathcal{D}_Y$-module

$$\mathcal{M}\Big/\Big(\sum_i \frac{\partial}{\partial x_i}\mathcal{M}\Big) = \Big(\mathcal{D}_X\Big/\sum \frac{\partial}{\partial x_i}\mathcal{D}_X\Big) \otimes_{\mathcal{D}_X} \mathcal{M}.$$

We shall formulate this below.

Let $f : X \to Y$ be a morphism of manifolds. Set $d_X := \dim X$, $d_Y := \dim Y$, and $d_{X/Y} = d_X - d_Y$. The right $\mathcal{D}_Y$-module structure of $\mathcal{D}_Y$ gives a left $\mathcal{D}_Y$-module structure of $\mathcal{D}_Y \otimes_{\mathcal{O}_Y} \Omega_Y^{\otimes -1}$. Its inverse image by f is a left $\mathcal{D}_X$-module $f^{-1}(\mathcal{D}_Y \otimes_{\mathcal{O}_Y} \Omega_Y^{\otimes -1}) \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X$. Switching this to a right $\mathcal{D}_X$-module, we obtain a right $\mathcal{D}_X$-module $f^{-1}(\mathcal{D}_Y \otimes_{\mathcal{O}_Y} \Omega_Y^{\otimes -1}) \otimes_{f^{-1}\mathcal{O}_Y} \Omega_X$. Letting $\Omega_{X/Y} := \Omega_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\Omega_Y^{\otimes -1}$, we have $f^{-1}(\mathcal{D}_Y \otimes_{\mathcal{O}_Y} \Omega_Y^{\otimes -1}) \otimes_{f^{-1}\mathcal{O}_Y} \Omega_X \simeq f^{-1}\mathcal{D}_Y \otimes_{f^{-1}\mathcal{O}_Y} \Omega_{X/Y}$. Since the right action and the left action of $\mathcal{D}_Y$ on itself commute, $f^{-1}\mathcal{D}_Y \otimes_{f^{-1}\mathcal{O}_Y} \Omega_{X/Y}$ has an $(f^{-1}\mathcal{D}_Y, \mathcal{D}_X)$-module structure.

NOTATION 4.17. We denote by $\mathcal{D}_{Y \leftarrow X}$ the $(f^{-1}\mathcal{D}_Y, \mathcal{D}_X)$-module $f^{-1}\mathcal{D}_Y \otimes_{f^{-1}\mathcal{O}_Y} \Omega_{X/Y}$.

REMARK 4.18. In § 4.1, we have defined $\mathcal{D}_{X \to Y}$, a $(\mathcal{D}_X, f^{-1}\mathcal{D}_Y)$-module. Since the category of left D-modules and that of right D-modules are equivalent, we can switch $\mathcal{D}_{X \to Y}$ to a right $\mathcal{D}_X$- and left $f^{-1}\mathcal{D}_Y$-module, that is, an $(f^{-1}\mathcal{D}_Y, \mathcal{D}_X)$-module. The result is $\Omega_X \otimes_{\mathcal{O}_X} \mathcal{D}_{X \to Y} \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\Omega_Y^{\otimes -1}$. Using $\Omega_Y \otimes_{\mathcal{O}_Y} \mathcal{D}_Y \otimes_{\mathcal{O}_Y} \Omega_Y^{\otimes -1} \simeq \mathcal{D}_Y^{\mathrm{op}}$, we obtain $\Omega_X \otimes_{\mathcal{O}_X} \mathcal{D}_{X \to Y} \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\Omega_Y^{\otimes -1} \simeq \mathcal{D}_{Y \leftarrow X}$.

Let us explicitly describe the action of $\mathcal{D}_X$ on $\mathcal{D}_{Y \leftarrow X}$. The action of $\mathcal{O}_X \subset \mathcal{D}_X$ is obtained from the $\mathcal{O}_X$-module structure of $\Omega_{X/Y}$. The action of $\Theta_X \subset \mathcal{D}_X$ is given as follows: Let $\sum a_i \otimes w_i$ be the image of $v \in \Theta_X$ by $\Theta_X \to \mathcal{O}_X \otimes_{\mathcal{O}_Y} \Theta_Y$. Then, for $\omega \in \Omega_X$, a generator θ of Ω_Y, and $P \in \mathcal{D}_Y$,

$$(P \otimes (\omega/\theta))v = \sum_i P w_i \otimes a_i \omega/\theta - P \otimes L_v \omega/\theta + \sum_i P \otimes a_i \left(\frac{L_{w_i}\theta}{\theta} \right) \omega/\theta .$$

Taking a coordinate system $x = (x_1, \dots, x_n)$ of X and a coordinate system $y = (y_1, \dots, y_m)$ of Y, we have

$$(P \otimes a(x) dx/dy)\partial_{x_i} = \sum_j P \partial_{y_j} \otimes \frac{\partial f_j(x)}{\partial x_i} a(x) dx/dy - P \otimes \frac{\partial a(x)}{\partial x_i} dx/dy,$$

where $f(x) = (f_1(x), \dots, f_m(x))$, $dx = dx_1 \wedge \cdots \wedge dx_n$, and $dy = dy_1 \wedge \cdots \wedge dy_m$.

If $f : X \to Y$ is smooth, then the right $\mathcal{D}_X$-module structure of $\mathcal{D}_{Y \leftarrow X}$ is rather easy. Recall the complex (4.3) defined in § 4.1. Similarly to Proposition 4.2, we obtain the following proposition.

PROPOSITION 4.19. *Let $f : X \to Y$ be a smooth morphism of fiber dimension n. Then:*

(1) *$\mathcal{D}_{Y \leftarrow X}$ is a coherent right $\mathcal{D}_X$-module.*

(2) *The sequence*

$$0 \to \Omega^0_{X/Y} \otimes_{\mathcal{O}_X} \mathcal{D}_X \to \cdots \to \Omega^n_{X/Y} \otimes_{\mathcal{O}_X} \mathcal{D}_X \to \mathcal{D}_{Y \leftarrow X} \to 0$$

is exact.

(3) $\mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{D}_{X \to Y}, \mathcal{D}_X) \simeq \mathcal{D}_{Y \leftarrow X}[-n]$ *in* $D^b(f^{-1}\mathcal{D}_Y \otimes \mathcal{D}_X^{\mathrm{op}})$,
$\mathbb{R}\mathcal{H}om_{\mathcal{D}_X^{\mathrm{op}}}(\mathcal{D}_{Y \leftarrow X}, \mathcal{D}_X) \simeq \mathcal{D}_{X \to Y}[-n]$ *in* $D^b(\mathcal{D}_X \otimes f^{-1}\mathcal{D}_Y^{\mathrm{op}})$.

We have the following proposition as well; we omit its proof.

PROPOSITION 4.20. *Let $f : X \to Y$ be an embedding. Then:*

(1) $\mathcal{D}_{Y\leftarrow X}$ *and* $\mathcal{D}_{X\to Y}$ *are coherent over* $f^{-1}\mathcal{D}_Y$.

(2) $\mathbb{R}\mathcal{H}om_{\mathcal{D}_Y^{\mathrm{op}}}(\mathcal{D}_{X\to Y}, \mathcal{D}_Y) \simeq \mathcal{D}_{Y\leftarrow X}[-\operatorname{codim}_Y X]$ *in* $D^b(f^{-1}\mathcal{D}_Y \otimes \mathcal{D}_X^{\mathrm{op}})$,

$\mathbb{R}\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{D}_{Y\leftarrow X}, \mathcal{D}_Y) \simeq \mathcal{D}_{X\to Y}[-\operatorname{codim}_Y X]$ *in* $D^b(\mathcal{D}_X \otimes f^{-1}\mathcal{D}_Y^{\mathrm{op}})$.

Let $\mathcal{M}$ be a $\mathcal{D}_X$-module. If we defined its integral by a $\mathcal{D}_Y$-module $f_*(\mathcal{D}_{Y\leftarrow X} \otimes_{\mathcal{D}_X} \mathcal{M})$, it would equal

$$f_*(\operatorname{Coker}(\Omega_{X/Y}^{d_{X/Y}-1} \otimes_{\mathcal{O}_X} \mathcal{M} \to \Omega_{X/Y}^{d_{X/Y}} \otimes_{\mathcal{O}_X} \mathcal{M}))$$

for f smooth. However we should carry out its systematic study in the derived categories. Let $\mathcal{M} \in D^b(\mathcal{D}_X)$. Since $\mathcal{D}_{Y\leftarrow X}$ is an $(f^{-1}\mathcal{D}_Y, \mathcal{D}_X)$-module, and since the flat dimension of $\mathcal{D}_X$ is finite, $\mathcal{D}_{Y\leftarrow X} \overset{\mathbb{L}}{\otimes}_{\mathcal{D}_X} \mathcal{M}$ belongs to $D^b(f^{-1}\mathcal{D}_Y)$. Hence $\mathbb{R}f_*(\mathcal{D}_{Y\leftarrow X} \overset{\mathbb{L}}{\otimes}_{\mathcal{D}_X} \mathcal{M})$ belongs to $D^b(\mathcal{D}_Y)$.

DEFINITION 4.21. We define the *integral* $\mathbb{D}f_*(\mathcal{M})$ *of* $\mathcal{M}$ *along fibers of* f by

$$\mathbb{D}f_*(\mathcal{M}) = \mathbb{R}f_*(\mathcal{D}_{Y\leftarrow X} \overset{\mathbb{L}}{\otimes}_{\mathcal{D}_X} \mathcal{M}).$$

Thus $\mathbb{D}f_*$ is the functor obtained as the composite of the following two functors:

$$D^b(\mathcal{D}_X) \xrightarrow{\mathcal{D}_{Y\leftarrow X} \overset{\mathbb{L}}{\otimes}_{\mathcal{D}_X} \bullet} D^b(f^{-1}\mathcal{D}_Y) \xrightarrow{\mathbb{R}f_*} D^b(\mathcal{D}_Y).$$

Similarly we define $\mathbb{D}f_! : D^b(\mathcal{D}_X) \to D^b(\mathcal{D}_Y)$ by the composite

$$D^b(\mathcal{D}_X) \xrightarrow{\mathcal{D}_{Y\leftarrow X} \overset{\mathbb{L}}{\otimes}_{\mathcal{D}_X} \bullet} D^b(f^{-1}\mathcal{D}_Y) \xrightarrow{\mathbb{R}f_!} D^b(\mathcal{D}_Y).$$

4.7. Coherence of Integrals

Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module, and $f : X \to Y$ a morphism of manifolds. In this section, we give a sufficient condition for $\mathbb{D}f_*\mathcal{M}$ to be a coherent $\mathcal{D}_Y$-module. To state a condition for $\mathcal{M}$, we introduce the following notion.

DEFINITION 4.22. We call an $\mathcal{O}_X$-module F a *good* $\mathcal{O}_X$*-module* if it satisfies the following condition: For every relatively compact open subset U of X, there exists a directed family $\{G_i\}_i$ of coherent $\mathcal{O}_U$-submodules of $F|_U$ such that

$$F|_U = \sum_i G_i, \tag{4.15}$$

where $\{G_i\}_i$ being a *directed family* means that for any i and i' there exists i'' such that $G_i + G_{i'} \subset G_{i''}$.

All good $\mathcal{O}_X$-modules are pseudo-coherent $\mathcal{O}_X$-modules (for the definition of pseudo-coherence, see Definition A.5).

PROPOSITION 4.23. (1) *If F is a good $\mathcal{O}_X$-module, then the intersection $F_1 \cap F_2$ of its coherent submodules F_1 and F_2 is also coherent.*

(2) *If $\varphi : F \to F'$ is a homomorphism of good $\mathcal{O}_X$-modules, then $\operatorname{Ker}\varphi$ and $\operatorname{Coker}\varphi$ are also good $\mathcal{O}_X$-modules.*

(3) *If $0 \to F' \to F \to F'' \to 0$ is an exact sequence of $\mathcal{O}_X$-modules, and if F' and F'' are good $\mathcal{O}_X$-modules, then F is also a good $\mathcal{O}_X$-module.*

PROOF. (1) follows from the pseudo-coherence of F. Indeed, $F_1 \cap F_2 \simeq \operatorname{Ker}(F_1 \oplus F_2 \to F)$.

(2) To show that $\operatorname{Ker}\varphi$ is a good $\mathcal{O}_X$-module, it is enough to show that, for any coherent $\mathcal{O}_X$-submodule G of F, $G \cap \operatorname{Ker}\varphi$ is a coherent $\mathcal{O}_X$-module. This follows from $G \cap \operatorname{Ker}\varphi = \operatorname{Ker}(G \to F')$.

To show that $F'' := \operatorname{Coker}\varphi$ is a good $\mathcal{O}_X$-module, it is enough to show that, for any coherent submodule G' of F', the image of G' by $p : F' \to F''$ is coherent. We may assume that F is the union of a directed family $\{G_i\}_i$ of coherent $\mathcal{O}_X$-modules. Since $\varphi(G_i) \cap G'$ is a directed family of coherent modules, $\varphi(F) \cap G' = \bigcup_i (\varphi(G_i) \cap G')$ is coherent by the Noetherian property of $\mathcal{O}_X$ (Definition A.7). Hence $p(G') = G'/(\varphi(F) \cap G')$ is also coherent.

(3) Let φ be the homomorphism $F \to F''$. We may assume that F' and F'' are the unions of directed families of coherent $\mathcal{O}_X$-submodules. Let $F' = \bigcup F'_i$ and $F'' = \bigcup F''_j$. Let U be a relatively compact open subset of X. Fix one j. For every $x \in \bar{U}$, there exists a finitely generated submodule $G \subset F$ defined on a neighborhood of x such that $\varphi(G) = F''_j$. Since F is pseudo-coherent, G is coherent. Hence there exist a finite open covering $\bar{U} \subset \bigcup V_\nu$ and a coherent $\mathcal{O}_{V_\nu}$-submodule G_ν of $F|_{V_\nu}$ such that $\varphi(G_\nu) = F''_j|_{V_\nu}$. Take $V'_\nu \Subset V_\nu$ so that $\bar{U} \subset \bigcup V'_\nu$. Then, for any μ and ν, there exists i such that

$(F' \cap (G_\mu + G_\nu))|_{\bar{V}'_\mu \cap \bar{V}'_\nu} \subset F'_i$. Since μ and ν belong to a finite set, we can take i independent of μ and ν. Since $(F' + G_\mu)|_{V_\mu \cap V_\nu} = (F' + G_\nu)|_{V_\mu \cap V_\nu}$,

$$G_\mu|_{V'_\mu \cap V'_\nu} \subset (F' \cap (G_\mu + G_\nu) + G_\nu)|_{V'_\mu \cap V'_\nu} \subset (F'_i + G_\nu)|_{V'_\mu \cap V'_\nu}.$$

Hence

$$(F'_i + G_\mu)|_{V'_\mu \cap V'_\nu} = (F'_i + G_\nu)|_{V'_\mu \cap V'_\nu}.$$

By patching them together, we have an $\mathcal{O}_U$-submodule F_j of $F|_U$ such that $F_j|_{V'_\nu \cap U} = (F'_i + G_\nu)|_{V'_\nu \cap U}$. For any i', $F'_{i'} + F_j$ is a coherent $\mathcal{O}_U$-submodule, since the sequence

$$0 \to F'_{i'} + F'_i \to F'_{i'} + F_j \to F''_j \to 0$$

is exact. It is clear that $\{F'_{i'} + F_j\}$ satisfies the condition (4.15). $\square$

DEFINITION 4.24. A coherent $\mathcal{D}_X$-module $\mathcal{M}$ is called a *good* $\mathcal{D}_X$*-module* if it is a good $\mathcal{O}_X$-module.

Let $D^b_{good}(\mathcal{D}_X)$ denote the full subcategory of $D^b(\mathcal{D}_X)$ consisting of $\mathcal{M}$ such that all $H^i(\mathcal{M})$ are good $\mathcal{D}_X$-modules. By Proposition 4.23, $D^b_{good}(\mathcal{D}_X)$ is a triangulated subcategory of $D^b(\mathcal{D}_X)$. For $\mathcal{M} \in D^b_{good}(\mathcal{D}_X)$, set

$$\mathrm{Ch}(\mathcal{M}) = \bigcup_i \mathrm{Ch}\,\mathcal{H}^i(\mathcal{M}), \quad \mathrm{Supp}\,\mathcal{M} = \bigcup_i \mathrm{Supp}\,\mathcal{H}^i(\mathcal{M})$$

THEOREM 4.25. *Let $f : X \to Y$ be a morphism of manifolds, and let $\mathcal{M} \in D^b_{good}(\mathcal{D}_X)$. Suppose that the morphism* $\mathrm{Supp}\,\mathcal{M} \to Y$ *is proper. Then $\mathbb{D}f_*(\mathcal{M}) \in D^b_{good}(\mathcal{D}_Y)$.*

PROOF. We can reduce the proof to the case when $\mathcal{M}$ is a good $\mathcal{D}_X$-module. We shall prove that $H^i\,\mathbb{D}f_*(\mathcal{M})$ are good $\mathcal{D}_Y$-modules by descending induction on i. (Note that, for example, if $i > 2\dim X$, then $H^i\,\mathbb{D}f_*(\mathcal{M}) = 0$.) We may assume that $\mathcal{M}$ is generated by a coherent $\mathcal{O}_X$-submodule $\mathcal{F}$, by shrinking Y if necessary. Consider an exact sequence of good $\mathcal{D}_X$-modules

$$0 \to \mathcal{M}' \to \mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{M} \to 0.$$

From this, we obtain an exact sequence

$$\begin{aligned}(4.16)\qquad & H^i\,\mathbb{D}f_*(\mathcal{M}') \to H^i\,\mathbb{D}f_*(\mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{F}) \to H^i\,\mathbb{D}f_*(\mathcal{M}) \\ \to\ & H^{i+1}\,\mathbb{D}f_*(\mathcal{M}') \to H^{i+1}\,\mathbb{D}f_*(\mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{F}).\end{aligned}$$

LEMMA 4.26. *For any $\mathcal{O}_X$-module $\mathcal{F}$,*

$$\mathbb{D}f_!(\mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{F}) = \mathcal{D}_Y \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_Y} \mathbb{R}f_!(\Omega_{X/Y} \otimes_{\mathcal{O}_X} \mathcal{F}).$$

PROOF. Indeed,

$$\begin{aligned}
\mathbb{D}f_!(\mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{F}) &\simeq \mathbb{R}f_!(\mathcal{D}_{Y\leftarrow X} \overset{\mathbb{L}}{\otimes}_{\mathcal{D}_X} \mathcal{D}_X \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_X} \mathcal{F}) \\
&\simeq \mathbb{R}f_!(\mathcal{D}_{Y\leftarrow X} \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_X} \mathcal{F}) \\
&\simeq \mathbb{R}f_!(f^{-1}\mathcal{D}_Y \overset{\mathbb{L}}{\otimes}_{f^{-1}\mathcal{O}_Y} \Omega_{X/Y} \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_X} \mathcal{F}) \\
&\simeq \mathcal{D}_Y \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_Y} \mathbb{R}f_!(\Omega_{X/Y} \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_X} \mathcal{F}).
\end{aligned}$$

□

It follows from this lemma and the fact that $R^k f_!(\Omega_{X/Y} \otimes_{\mathcal{O}_X} \mathcal{F})$ is a coherent $\mathcal{O}_X$-module (Grauert's theorem) that

$$H^k(\mathbb{D}f_!(\mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{F})) \simeq \mathcal{D}_Y \otimes_{\mathcal{O}_Y} R^k f_!(\Omega_{X/Y} \otimes_{\mathcal{O}_X} \mathcal{F})$$

is a good $\mathcal{D}_Y$-module. By the induction hypothesis, $H^{i+1}\,\mathbb{D}f_*\mathcal{M}'$ is also a good $\mathcal{D}_Y$-module. Hence

$$\mathcal{N} := \operatorname{Ker}(H^{i+1}\,\mathbb{D}f_*\mathcal{M}' \to H^{i+1}\,\mathbb{D}f_*(\mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{F}))$$

is also a good $\mathcal{D}_Y$-module. From (4.16), we have an exact sequence

$$H^i\,\mathbb{D}f_*(\mathcal{D}_X \otimes \mathcal{F}) \to H^i(\mathbb{D}f_*\mathcal{M}) \to \mathcal{N} \to 0.$$

Since $\mathcal{N}$ and $H^i\,\mathbb{D}f_*(\mathcal{D}_X \otimes \mathcal{F})$ are locally finitely generated, $H^i\mathbb{D}f_*\mathcal{M}$ is also a locally finitely generated $\mathcal{D}_Y$-module. Applying this to $\mathcal{M}'$, we see that $H^i\mathbb{D}f_*\mathcal{M}'$ is a locally finitely generated $\mathcal{D}_Y$-module. Since $H^i(\mathbb{D}f_*(\mathcal{D}_X \otimes \mathcal{F}))$ is a good $\mathcal{D}_X$-module,

$$\mathcal{N}' := \operatorname{Coker}(H^i\mathbb{D}f_*\mathcal{M}' \to H^i(\mathbb{D}f_*(\mathcal{D}_X \otimes \mathcal{F}))$$

is also a good $\mathcal{D}_Y$-module. It follows from an exact sequence $0 \to \mathcal{N}' \to H^i\mathbb{D}f_*\mathcal{M} \to \mathcal{N} \to 0$ and Proposition 4.23(3) that $H^i\mathbb{D}f_*\mathcal{M}$ is also a good $\mathcal{D}_Y$-module. We have thus proved Theorem 4.25. □

We can estimate the characteristic variety of $\mathbb{D}f_*\mathcal{M}$ from above by the following theorem. Note the similarity to Theorem 4.7. Indeed, similarly to Theorem 4.7, the following theorem with the equality instead of the inclusion holds on condition that $f_\pi : f_d^{-1}\operatorname{Ch}(\mathcal{M}) \to T^*Y$ is finite, although we do not prove it.

THEOREM 4.27. *Let $f : X \to Y$ be a morphism of manifolds, and let $\mathcal{M} \in D^b_{good}(\mathcal{D}_X)$. Suppose that the morphism* $\operatorname{Supp}\mathcal{M} \to Y$ *is proper. Then*

$$\operatorname{Ch}(\mathbb{D}f_*\mathcal{M}) \subset f_\pi f_d^{-1}\operatorname{Ch}(\mathcal{M}).$$

Moreover, if f is an embedding, then the equality holds.

PROOF. We may assume that $\mathcal{M}$ is a good $\mathcal{D}_X$-module, and that $\mathcal{M}$ globally has a coherent filtration $F(\mathcal{M})$. By factoring $X \to Y$ into $X \hookrightarrow X \times Y \to Y$ and by using the same argument as in Lemma 4.10, we can reduce the proof to the case when f is an embedding and the case when $X = Y \times Z \to Y$.

When $f : X \to Y$ is an embedding, we may assume locally that $Y = X \times \mathbb{C}^n$ and $f(x) = (x, 0)$. Then

$$\mathbb{D}f_*\mathcal{M} = \mathcal{M} \overset{D}{\boxtimes} \mathcal{B}_{\{0\}|\mathbb{C}^n}.$$

Hence the desired result follows from $\operatorname{Ch}(\mathcal{B}_{\{0\}|\mathbb{C}^n}) = T^*_{\{0\}}\mathbb{C}^n$ and Proposition 4.5.

Next we consider the case when $X = Y \times Z$ and $f : X \to Y$ is given by $(y, z) \mapsto y$. Then

$$\mathbb{D}f_!(\mathcal{M}) = \mathbb{D}f_!(\Omega_Z \overset{\mathbb{L}}{\otimes}_{\mathcal{D}_Z} \mathcal{M}).$$

Since Ω_Z is quasi-isomorphic to the complex of right $\mathcal{D}_Z$-modules $\Omega_Z^\bullet \otimes_{\mathcal{O}_Z} \mathcal{D}_Z$ associated with the de Rham complex $\Omega_Z^\bullet$, we have

$$\Omega_Z \overset{\mathbb{L}}{\otimes}_{\mathcal{D}_Z} \mathcal{M} \simeq \Omega_Z^\bullet \otimes_{\mathcal{O}_Z} \mathcal{M},$$

where $\Omega_Z^\bullet \otimes_{\mathcal{O}_Z} \mathcal{M}$ is the de Rham complex of $\mathcal{M}$ (with respect to Z). Define a filtration $F(\Omega_Z^\bullet \otimes_{\mathcal{O}_Z} \mathcal{M})$ of $\Omega_Z^\bullet \otimes_{\mathcal{O}_Z} \mathcal{M}$ by

$$F_k(\Omega_Z^i \otimes_{\mathcal{O}_Z} \mathcal{M}) = \Omega_Z^i \otimes_{\mathcal{O}_Z} F_{k+i}(\mathcal{M}).$$

Then $F(\Omega_Z^\bullet \otimes \mathcal{M})$ is a complex of filtered $f^{-1}\mathcal{D}_Y$-modules. Consider the morphism

$$\mathbb{R}f_*(F_k(\Omega_Z^\bullet \otimes \mathcal{M})) \to \mathbb{D}f_*\mathcal{M}$$

in $D(\mathcal{O}_Y)$. Then

$$R^i f_*(F_k(\Omega_Z^\bullet \otimes \mathcal{M})) := H^i\mathbb{R}f_*(F_k(\Omega_Z^\bullet \otimes \mathcal{M}))$$

is a coherent $\mathcal{O}_Y$-module. Let $F_k(H^i\mathbb{D}f_*\mathcal{M})$ be the image of

$$R^i f_*(F_k(\Omega_Z^\bullet \otimes \mathcal{M})) \to H^i\mathbb{D}f_*\mathcal{M}.$$

Then $F_k(H^i\mathbb{D}f_*\mathcal{M})$ are also coherent $\mathcal{O}_Y$-modules and satisfy

$$F_l(\mathcal{D}_Y)F_k(H^i\mathbb{D}f_*\mathcal{M}) \subset F_{k+l}(H^i\mathbb{D}f_*\mathcal{M}).$$

Put

$$\begin{aligned} N_k^i &= \operatorname{Coker}(R^i f_*(F_{k-1}(\Omega_Z^\bullet \otimes \mathcal{M})) \to R^i f_* F_k(\Omega_Z^\bullet \otimes \mathcal{M}))\,, \\ N^i &= \bigoplus_k N_k^i. \end{aligned}$$

Then we have an epimorphism

$$N^i \twoheadrightarrow \operatorname{Gr}^F H^i(\mathbb{D} f_* \mathcal{M}).$$

Furthermore, by the exact sequence

$$0 \to F_{k-1}(\Omega_Z^\bullet \otimes \mathcal{M}) \to F_k(\Omega_Z^\bullet \otimes \mathcal{M}) \to \operatorname{Gr}_k^F(\Omega_Z^\bullet \otimes \mathcal{M}) \to 0,$$

we see that

$$N_k^i \hookrightarrow R^i f_*(\operatorname{Gr}_k^F(\Omega_Z^\bullet \otimes \mathcal{M})).$$

We have thus obtained a monomorphism

$$N^i \hookrightarrow R^i f_*(\operatorname{Gr}^F(\Omega_Z^\bullet \otimes \mathcal{M}))$$

of $\operatorname{Gr}^F(\mathcal{D}_Y)$-modules.

We have

$$\operatorname{Gr}_k^F(\Omega_Z^i \otimes \mathcal{M}) = \Omega_Z^i \otimes \operatorname{Gr}_{k+i}^F(\mathcal{M}),$$

and the homomorphism

$$\operatorname{Gr}^F(\Omega_Z^i \otimes \mathcal{M}) \to \operatorname{Gr}^F(\Omega_Z^{i+1} \otimes \mathcal{M})$$

is given by

$$\Omega_Z^i \otimes \operatorname{Gr}^F(\mathcal{M}) \ni \omega \otimes u \mapsto \sum \eta_\nu \wedge \omega \otimes v_\nu u,$$

where $v_\nu \in \Theta_Z$ is a basis of Θ_Z, and $\eta_\nu \in \Omega_Z^1$ is its dual basis. Hence we have

$$\operatorname{Gr}^F(\Omega_Z^\bullet \otimes \mathcal{M}) \simeq (\Omega_Z^\bullet \otimes \operatorname{Gr}^F \mathcal{D}_Z) \otimes_{\operatorname{Gr}^F \mathcal{D}_Z} \operatorname{Gr}^F \mathcal{M},$$

the complex obtained by applying the functor $\bullet \otimes_{\operatorname{Gr}^F \mathcal{D}_Z} \operatorname{Gr}^F \mathcal{M}$ to the Koszul complex $\Omega_Z^\bullet \otimes_{\mathcal{O}_Z} \operatorname{Gr}^F \mathcal{D}_Z$. Since $\Omega_Z^\bullet \otimes_{\mathcal{O}_Z} \operatorname{Gr}^F \mathcal{D}_Z$ is quasi-isomorphic to the $\operatorname{Gr}^F(\mathcal{D}_Z)$-module Ω_Z, we have

$$\operatorname{Gr}^F(\Omega_Z^\bullet \otimes \mathcal{M}) \simeq \Omega_Z \overset{\mathbb{L}}{\otimes}_{\operatorname{Gr}^F(\mathcal{D}_Z)} \operatorname{Gr}^F(\mathcal{M})$$

in $D^b(\mathrm{Gr}^F \mathcal{D}_X)$. Thus we see that $\mathrm{Gr}^F H^i(\mathbb{D}f_*\mathcal{M})$ is a subquotient of $R^i f_*(\Omega_Z \overset{\mathbb{L}}{\otimes}_{\mathrm{Gr}^F(\mathcal{D}_Z)} \mathrm{Gr}^F \mathcal{M})$. Accordingly we obtain

$$\begin{aligned}(4.17)\qquad &\mathrm{Ch}\, H^i(\mathbb{D}f_*\mathcal{M}) \\ &= \mathrm{Supp}(\mathrm{Gr}^F H^i(\mathbb{D}f_*\mathcal{M}))^\sim \\ &\subset \mathrm{Supp}(R^i f_*(\Omega_Z \overset{\mathbb{L}}{\otimes}_{\mathrm{Gr}^F(\mathcal{D}_Z)} \mathrm{Gr}^F \mathcal{M}))^\sim.\end{aligned}$$

Since $f_d : T^*Y \times Z \hookrightarrow T^*X$ is an embedding, and since

$$(\Omega_Z \overset{\mathbb{L}}{\otimes}_{\mathrm{Gr}^F(\mathcal{D}_Z)} \mathrm{Gr}^F \mathcal{M})^\sim \simeq \mathbb{L}f_d^*(\Omega_Z \otimes_{\mathcal{O}_Z} \mathrm{Gr}^F \mathcal{M}^\sim),$$

we have

$$\mathbb{R}f_*(\Omega_Z \overset{\mathbb{L}}{\otimes}_{\mathrm{Gr}^F(\mathcal{D}_Z)} \mathrm{Gr}^F \mathcal{M})^\sim = \mathbb{R}f_{\pi *}\mathbb{L}f_d^*(\Omega_Z \otimes_{\mathcal{O}_Z} \mathrm{Gr}^F \mathcal{M}^\sim).$$

Therefore

$$\begin{aligned}(4.18)\qquad &\mathrm{Supp}(R^i f_*(\Omega_Z \overset{\mathbb{L}}{\otimes}_{\mathrm{Gr}^F(\mathcal{D}_Z)} \mathrm{Gr}^F \mathcal{M}))^\sim \\ &\subset f_\pi f_d^{-1} \mathrm{Supp}(\mathrm{Gr}^F \mathcal{M})^\sim \\ &= f_\pi f_d^{-1} \mathrm{Ch}(\mathcal{M}).\end{aligned}$$

From this and (4.17), we obtain the desired result. □

We have the following formula for a relation between integrals and inverse images of D-modules.

THEOREM 4.28 (Projection Formula). *For any $\mathcal{M} \in D^b(\mathcal{D}_X)$ and $\mathcal{N} \in D^b(\mathcal{D}_Y)$,*

$$\mathbb{D}f_!(\mathbb{D}f^*\mathcal{N} \overset{\mathbb{D}}{\otimes} \mathcal{M}) \simeq \mathcal{N} \overset{\mathbb{D}}{\otimes} \mathbb{D}f_!\mathcal{M}.$$

PROOF. By definition,

$$\begin{aligned}\mathbb{D}f^*\mathcal{N} \overset{\mathbb{D}}{\otimes} \mathcal{M} \quad &= \quad (\mathcal{D}_{X\to Y} \overset{\mathbb{L}}{\otimes}_{f^{-1}\mathcal{D}_Y} f^{-1}\mathcal{N}) \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_X} \mathcal{M} \\ &\cong \quad (\mathcal{D}_{X\to Y} \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_X} \mathcal{M}) \overset{\mathbb{L}}{\otimes}_{f^{-1}\mathcal{D}_Y} f^{-1}\mathcal{N}.\end{aligned}$$

By the projection formula for sheaves,

$$\begin{aligned}&\mathbb{D}f_!(\mathbb{D}f^*\mathcal{N} \overset{\mathbb{D}}{\otimes} \mathcal{M}) \\ &\simeq \mathbb{R}f_!(\mathcal{D}_{Y\leftarrow X} \overset{\mathbb{L}}{\otimes}_{\mathcal{D}_X} (\mathcal{D}_{X\to Y} \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_X} \mathcal{M}) \overset{\mathbb{L}}{\otimes}_{f^{-1}\mathcal{D}_Y} f^{-1}\mathcal{N}) \\ &\simeq \mathbb{R}f_!(\mathcal{D}_{Y\leftarrow X} \overset{\mathbb{L}}{\otimes}_{\mathcal{D}_X} (\mathcal{D}_{X\to Y} \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_X} \mathcal{M})) \overset{\mathbb{L}}{\otimes}_{\mathcal{D}_Y} \mathcal{N},\end{aligned}$$

where $\mathbb{R}f_!(\mathcal{D}_{Y\leftarrow X}\overset{\mathbb{L}}{\otimes}_{\mathcal{D}_X}(\mathcal{D}_{X\to Y}\overset{\mathbb{L}}{\otimes}_{\mathcal{O}_X}\mathcal{M}))$ is an object of $D^b(\mathcal{D}_Y\otimes_{\mathbb{C}}\mathcal{D}_Y^{\mathrm{op}})$. Since

$$\mathbb{D}f_!\mathcal{M}\overset{\mathbb{D}}{\otimes}\mathcal{N}=(\mathbb{D}f_!\mathcal{M}\overset{\mathbb{D}}{\otimes}\mathcal{D}_Y)\overset{\mathbb{L}}{\otimes}_{\mathcal{D}_Y}\mathcal{N},$$

it suffices to prove that
(4.19)

$$\mathbb{R}f_!(\mathcal{D}_{Y\leftarrow X}\overset{\mathbb{L}}{\otimes}_{\mathcal{D}_X}(\mathcal{D}_{X\to Y}\overset{\mathbb{L}}{\otimes}_{\mathcal{O}_X}\mathcal{M}))\simeq\mathbb{R}f_!(\mathcal{D}_{Y\leftarrow X}\overset{\mathbb{L}}{\otimes}_{\mathcal{D}_X}\mathcal{M})\overset{\mathbb{D}}{\otimes}_{\mathcal{O}_Y}\mathcal{D}_Y$$

in $D^b(\mathcal{D}_Y\otimes_{\mathbb{C}}\mathcal{D}_Y^{\mathrm{op}})$. Since the right hand side of (4.19) is isomorphic to

$$\mathbb{R}f_!((\mathcal{D}_{Y\leftarrow X}\overset{\mathbb{L}}{\otimes}_{\mathcal{D}_X}\mathcal{M})\overset{\mathbb{L}}{\otimes}_{f^{-1}\mathcal{O}_Y}f^{-1}\mathcal{D}_Y),$$

(4.19) is reduced to an isomorphism
(4.20)

$$\mathcal{D}_{Y\leftarrow X}\overset{\mathbb{L}}{\otimes}_{\mathcal{D}_X}(\mathcal{D}_{X\to Y}\overset{\mathbb{L}}{\otimes}_{\mathcal{O}_X}\mathcal{M})\simeq(\mathcal{D}_{Y\leftarrow X}\overset{\mathbb{L}}{\otimes}_{\mathcal{D}_X}\mathcal{M})\overset{\mathbb{L}}{\otimes}_{f^{-1}\mathcal{O}_Y}f^{-1}\mathcal{D}_Y$$

in $D^b(f^{-1}\mathcal{D}_Y\otimes_{\mathbb{C}}f^{-1}\mathcal{D}_Y^{\mathrm{op}})$. The left hand side of (4.20) is isomorphic to

$$(\mathcal{D}_{Y\leftarrow X}\overset{\mathbb{L}}{\otimes}_{\mathcal{D}_X}(\mathcal{D}_{X\to Y}\overset{\mathbb{L}}{\otimes}_{\mathcal{O}_X}\mathcal{D}_X))\overset{\mathbb{L}}{\otimes}_{\mathcal{D}_X}\mathcal{M},$$

and the right hand side of (4.20) is isomorphic to

$$(\mathcal{D}_{Y\leftarrow X}\overset{\mathbb{L}}{\otimes}_{f^{-1}\mathcal{O}_Y}f^{-1}\mathcal{D}_Y)\overset{\mathbb{L}}{\otimes}_{\mathcal{D}_X}\mathcal{M}.$$

Hence (4.20) has been reduced to an isomorphism

$$\mathcal{D}_{Y\leftarrow X}\overset{\mathbb{L}}{\otimes}_{\mathcal{D}_X}(\mathcal{D}_{X\to Y}\overset{\mathbb{L}}{\otimes}_{\mathcal{O}_X}\mathcal{D}_X)\simeq\mathcal{D}_{Y\leftarrow X}\overset{\mathbb{L}}{\otimes}_{f^{-1}\mathcal{O}_Y}f^{-1}\mathcal{D}_Y \tag{4.21}$$

in $D^b(f^{-1}\mathcal{D}_Y\otimes_{\mathbb{C}}f^{-1}\mathcal{D}_Y^{\mathrm{op}}\otimes_{\mathbb{C}}\mathcal{D}_X^{\mathrm{op}})$.

By Proposition 3.1, $\mathcal{D}_{X\to Y}\overset{\mathbb{D}}{\otimes}\mathcal{D}_X$ is isomorphic to $\mathcal{D}_X\otimes_{\mathcal{O}_X}\mathcal{D}_{X\to Y}\simeq\mathcal{D}_X\otimes_{\mathcal{O}_Y}\mathcal{D}_Y$ as a $\mathcal{D}_X$-module, and hence it is flat over $\mathcal{D}_X$. By Proposition 3.5,

$$\begin{aligned}\mathcal{D}_{Y\leftarrow X}\overset{\mathbb{L}}{\otimes}_{\mathcal{D}_X}(\mathcal{D}_{X\to Y}\otimes_{\mathcal{O}_X}\mathcal{D}_X) &\simeq \mathcal{D}_{Y\leftarrow X}\otimes_{\mathcal{D}_X}(\mathcal{D}_{X\to Y}\otimes_{\mathcal{O}_X}\mathcal{D}_X)\\ &\simeq (\mathcal{D}_{Y\leftarrow X}\otimes_{\mathcal{O}_X}\mathcal{D}_{X\to Y})\otimes_{\mathcal{D}_X}\mathcal{D}_X\\ &\simeq \mathcal{D}_{Y\leftarrow X}\otimes_{\mathcal{O}_X}\mathcal{D}_{X\to Y}\end{aligned}$$

as $\mathcal{D}_Y\otimes_{\mathbb{C}}\mathcal{D}_Y^{\mathrm{op}}\otimes_{\mathbb{C}}\mathcal{D}_X^{\mathrm{op}}$-modules; their $\mathcal{D}_X^{\mathrm{op}}$-module structure is given by the tensor product of the $\mathcal{D}_X^{\mathrm{op}}$-module $\mathcal{D}_{Y\leftarrow X}$ and the $\mathcal{D}_X$-module $\mathcal{D}_{X\to Y}$.

We have thus reduced (4.21) to the following lemma. □

LEMMA 4.29. *As $f^{-1}\mathcal{D}_Y \otimes_{\mathbb{C}} f^{-1}\mathcal{D}_Y^{\mathrm{op}} \otimes_{\mathbb{C}} \mathcal{D}_X^{\mathrm{op}}$-modules,*

$$\mathcal{D}_{Y\leftarrow X} \otimes_{\mathcal{O}_X} \mathcal{D}_{X\to Y} \simeq \mathcal{D}_{Y\leftarrow X} \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{D}_Y \,,$$

where $\otimes_{f^{-1}\mathcal{O}_Y}$ on the right is the tensor product of the left $f^{-1}\mathcal{O}_Y$-module $\mathcal{D}_{Y\leftarrow X}$ and the left $f^{-1}\mathcal{O}_Y$-module $f^{-1}\mathcal{D}_Y$.

PROOF. Let $\Delta_f \subset X \times Y$ be the graph of f, and $\Delta_Y \subset Y \times Y$ the diagonal set of Y. Rewrite everything in terms of left modules. Let Y_1 and Y_2 be copies of Y. Then the lemma is equivalent to an isomorphism

$$\mathcal{B}_{\Delta_f|X\times Y_1} \otimes_{\mathcal{O}_X} \mathcal{B}_{\Delta_f|X\times Y_2} \simeq \mathcal{B}_{\Delta_f|X\times Y_1} \otimes_{\mathcal{O}_{Y_1}} \mathcal{B}_{\Delta_Y|Y_1\times Y_2}$$

as $\mathcal{D}_X \otimes_{\mathbb{C}} \mathcal{D}_{Y_1} \otimes_{\mathbb{C}} \mathcal{D}_{Y_2}$-modules. Both sides are isomorphic to $\mathcal{B}_{\Delta|X\times Y_1\times Y_2}$, where

$$\Delta = \{\, (x, y_1, y_2) \in X \times Y_1 \times Y_2 \,;\, f(x) = y_1 = y_2 \,\}\,.$$

$\square$

4.8. D-modules Supported by a Submanifold

In this section, we show that the category of coherent D-modules supported by a closed submanifold is equivalent to that of coherent D-modules on the closed submanifold.

THEOREM 4.30. *Let Z be a closed submanifold of X, and $j : Z \hookrightarrow X$ the inclusion map. Suppose that $\mathcal{M}$ is a $\mathcal{D}_X$-module satisfying $\Gamma_{[Z]}(\mathcal{M}) \simeq \mathcal{M}$.*

(1) *For $\mathcal{N} := j^{-1}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{D}_{X\hookleftarrow Z}, \mathcal{M})$, we have*

$$\mathbb{D}j_*\mathcal{N} \xrightarrow{\sim} \mathcal{M}\,.$$

If $\mathcal{M}$ is a coherent $\mathcal{D}_X$-module, then $\mathcal{N}$ is a coherent $\mathcal{D}_Z$-module.

(2) $\mathcal{E}xt^j_{\mathcal{D}_X}(\mathcal{D}_{X\hookleftarrow Z}, \mathcal{M}) = 0$ $(j \neq 0)$.

(3) *For any $\mathcal{D}_Z$-module $\mathcal{N}$, we have*

$$\mathcal{N} \simeq j^{-1}\mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{D}_{X\hookleftarrow Z}, \mathbb{D}j_*\mathcal{N})\,.$$

In particular, the category of coherent $\mathcal{D}_X$-modules supported by Z is equivalent to the category of coherent $\mathcal{D}_Z$-modules.

REMARK 4.31. Since $\mathcal{D}_{X\hookleftarrow Z}$ is flat over $\mathcal{D}_Z^{\mathrm{op}}$, the cohomologies of $\mathbb{D}j_*\mathcal{N} = \mathcal{D}_{X\hookleftarrow Z}\overset{\mathbb{L}}{\otimes}_{\mathcal{D}_Z}\mathcal{N}$ are concentrated at degree 0, namely $\mathbb{D}j_*\mathcal{N} = \mathcal{D}_{X\hookleftarrow Z}\otimes_{\mathcal{D}_Z}\mathcal{N}$. The morphism $\mathbb{D}j_*\mathcal{N}\to\mathcal{M}$ in (1) is the one given by

$$\mathcal{D}_{X\hookleftarrow Z}\otimes_{\mathcal{D}_Z}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{D}_{X\hookleftarrow Z},\mathcal{M})\to\mathcal{M}\quad(a\otimes\varphi\mapsto\varphi(a)).$$

If $\mathcal{M}$ is coherent, then $\Gamma_{[Z]}\mathcal{M}\simeq\mathcal{M}$ is equivalent to $\operatorname{Supp}\mathcal{M}\subset Z$.

PROOF. Note that the theorem is local on X. If $Z\subset Z'\subset X$, then

$$\mathcal{D}_{X\hookleftarrow Z}\simeq\mathcal{D}_{X\hookleftarrow Z'}\overset{\mathbb{L}}{\otimes}_{\mathcal{D}_{Z'}}\mathcal{D}_{Z'\hookleftarrow Z},$$

and hence

$$\begin{aligned}&\mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{D}_{X\hookleftarrow Z},\mathcal{M})\\&\simeq\mathbb{R}\mathcal{H}om_{\mathcal{D}_{Z'}}(\mathcal{D}_{Z'\hookleftarrow Z},\mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{D}_{X\hookleftarrow Z'},\mathcal{M})).\end{aligned}$$

By induction on $\operatorname{codim}Z$, it is enough to prove the theorem in the case $\operatorname{codim}Z=1$. Take a coordinate system $x=(x_1,\dots,x_n)$ of X so that $Z=\{x_1=0\}$.

We prove (1) first. Since $\mathcal{D}_{X\hookleftarrow Z}=\mathcal{D}_X/\mathcal{D}_Xx_1$,

$$\mathcal{N}=\{\,u\in\mathcal{M}\,;\,x_1u=0\,\}\subset\mathcal{M}.$$

From $\mathcal{D}_{X\hookleftarrow Z}\simeq\bigoplus\partial_1^m\mathcal{D}_Z$, we obtain an isomorphism of $\mathcal{D}_Z$-modules $\mathbb{D}j_*\mathcal{N}=\mathcal{D}_{X\hookleftarrow Z}\overset{\mathbb{L}}{\otimes}_{\mathcal{D}_Z}\mathcal{N}\simeq\bigoplus_{m\geq 0}(\mathbb{C}\partial_1^m\otimes_{\mathbb{C}}\mathcal{N})$. The homomorphism

$$\mathbb{D}j_*\mathcal{N}\simeq\bigoplus_{m\geq 0}(\mathbb{C}\partial_1^m\otimes_{\mathbb{C}}\mathcal{N})\to\mathcal{M}\tag{4.22}$$

is given by the multiplication. First we show the surjectivity of (4.22). Put $\mathcal{L}:=\sum_{m\geq 0}\partial_1^m\mathcal{N}$, the image. Let $u\in\mathcal{M}$. Then $x_1^mu=0$ for m large enough, by the assumption. By induction on $m\geq 1$, we show that $u\in\mathcal{L}$. When $m=1$, it is trivial. Suppose $m>1$. By

$$x_1^{m-1}(x_1\partial_1+m)=\partial_1x_1^m,$$

we have $x_1^{m-1}(x_1\partial_1+m)u=0$, and thus by the induction hypothesis

$$(x_1\partial_1+m)u\in\mathcal{L}\,.$$

In the meantime, again by the induction hypothesis, $x_1u\in\mathcal{L}$, and hence $\partial_1x_1u=(x_1\partial_1+1)u\in\mathcal{L}$. Therefore $(m-1)u\in\mathcal{L}$, and we have thus proved that $\mathcal{L}=\mathcal{M}$.

Next we show the injectivity of (4.22). It suffices to show that, if $\sum_{m=0}^{a} \partial_1^m u_m = 0$ with $u_m \in \mathcal{N}$ $(0 \leq m \leq a)$, then $u_a = 0$. We use induction on a. When $a = 0$, it is trivial. Suppose $a > 0$. Then

$$0 = x_1 \sum_{m=0}^{a} \partial_1^m u_m = \sum_{m=0}^{a} (\partial_1^m x_1 - m\partial_1^{m-1}) u_m = -\sum_{m=1}^{a} m\partial_1^{m-1} u_m.$$

Hence, from the induction hypothesis, we obtain $u_a = 0$.

Next we show the coherence of $\mathcal{N}$. By the surjectivity of (4.22), $\mathcal{M}$ is generated by $\mathcal{N}$ as a $\mathcal{D}_X$-module and hence locally generated by some $u_1, \dots, u_N \in \mathcal{N}$. Applying $\mathbb{D}j_*$ to the $\mathcal{D}_Z$-module homomorphism $\mathcal{N}_0 = \mathcal{D}_Z^{\oplus N} \to \mathcal{N}$ defined by the generators above, we obtain a $\mathcal{D}_X$-module morphism

$$\mathcal{M}_0 := \mathbb{D}j_*\mathcal{N}_0 \to \mathbb{D}j_*\mathcal{N} = \mathcal{M},$$

which is surjective. Since $\mathbb{D}j_*$ is faithfully exact, $\mathcal{N}_0 \to \mathcal{N}$ is surjective. In particular, $\mathcal{N}$ is locally finitely generated. Since $\mathcal{M}_1 := \operatorname{Ker}(\mathcal{M}_0 \to \mathcal{M})$ is a coherent $\mathcal{D}_X$-module supported by Z, $\mathcal{N}_1 := j^{-1}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{D}_{X \leftarrow Z}, \mathcal{M}_1)$ is also locally finitely generated. Since

$$0 \to \mathbb{D}j_*\mathcal{N}_1 \to \mathbb{D}j_*\mathcal{N}_0 \to \mathbb{D}j_*\mathcal{N} \to 0$$

is exact, $0 \to \mathcal{N}_1 \to \mathcal{N}_0 \to \mathcal{N} \to 0$ is exact, and hence we obtain the coherence of $\mathcal{N}$.

We prove (2) next. Since $\mathcal{D}_{X \leftarrow Z} = \mathcal{D}_X / \mathcal{D}_X x_1$, we have

$$\mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{D}_{X \leftarrow Z}, \mathcal{M}) = (\mathcal{M} \xrightarrow{x_1} \mathcal{M}),$$

and hence

$$\mathcal{E}xt^j_{\mathcal{D}_X}(\mathcal{D}_{X \leftarrow Z}, \mathcal{M}) = \begin{cases} 0 & (j \geq 2), \\ \mathcal{M}/(x_1\mathcal{M}) & (j = 1). \end{cases}$$

Thus it suffices to show that all $u \in \mathcal{M}$ belong to $x_1\mathcal{M}$. We may assume $u = \partial_1^m v$ for $m \geq 0$ and v with $x_1 v = 0$. Then

$$\begin{aligned} x_1 \partial_1^{m+1} v &= (\partial_1^{m+1} x_1 - (m+1)\partial_1^m) v \\ &= -(m+1)\partial_1^m v = -(m+1)u\,. \end{aligned}$$

Hence we obtain $u \in x_1\mathcal{M}$.

We finally prove (3). It follows from $\mathbb{D}j_*\mathcal{N} = \bigoplus \mathbb{C}\partial_1^m \otimes_{\mathbb{C}} \mathcal{N}$ and $x_1\partial_1^m = (\partial_1^m x_1 - m\partial_1^{m-1})$ that

$$x_1(\partial_1^m \otimes u) = -m\partial_1^{m-1} \otimes u \quad (u \in \mathcal{N}).$$

Hence

$$x_1 : \mathbb{D}j_*\mathcal{N} \to \mathbb{D}j_*\mathcal{N}$$

is surjective, and its kernel equals $\mathcal{N}$. Thus

$$\mathbb{R}\mathcal{H}om(\mathcal{D}_{X\hookleftarrow Z}, \mathbb{D}j_*\mathcal{N}) = (\mathbb{D}j_*\mathcal{N} \xrightarrow{x_1} \mathbb{D}j_*\mathcal{N}) = \mathcal{N}$$

in $D^b(\mathcal{D}_Z)$. □

For a coherent $\mathcal{D}_X$-module $\mathcal{M}$ supported by Z,

$$\mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{D}_{X\hookleftarrow Z}, \mathcal{M}) = \mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{D}_{X\hookleftarrow Z}, \mathcal{D}_X)\overset{\mathbb{L}}{\otimes}_{\mathcal{D}_X}\mathcal{M}\,.$$

By Proposition 4.20,

$$\mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{D}_{X\hookleftarrow Z}, \mathcal{D}_X) = \mathcal{D}_{Z\hookrightarrow X}[-\operatorname{codim} Z].$$

Hence we can express $\mathcal{N}$, by using $\mathbb{D}j^*$, as

$$\mathcal{N} = j^{-1}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{D}_{X\hookleftarrow Z}, \mathcal{M}) = \mathbb{D}j^*\mathcal{M}[-\operatorname{codim} Z].$$

In particular, when $\mathcal{M} = \mathcal{D}_{X\hookleftarrow Z} = \mathbb{D}j_*\mathcal{D}_Z$, by Theorem 4.30(3),

$$\mathcal{D}_Z \cong \mathcal{D}_{Z\hookrightarrow X}\overset{\mathbb{L}}{\otimes}_{\mathcal{D}_X}\mathcal{D}_{X\hookleftarrow Z}[-\operatorname{codim} Z]\,.$$

Hence we obtain the following proposition.

PROPOSITION 4.32. *For any $\mathcal{N} \in D^b(\mathcal{D}_Z)$, we have*

$$\mathbb{D}j^*\mathbb{D}j_*\mathcal{N} \cong \mathcal{N}[\operatorname{codim} Z]\,.$$

4.9. Adjunction Formula

The goal of this section is the following formula. Set $d_X := \dim X$, $d_Y := \dim Y$, and $d_{X/Y} := d_X - d_Y$.

THEOREM 4.33 (Adjunction Formula). *Let $f : X \to Y$ be a morphism. Let $\mathcal{M} \in D^b(\mathcal{D}_X)$ and $\mathcal{N} \in D^b(\mathcal{D}_Y)$.*

(1) *We have a canonical morphism*

$$\mathbb{R}f_*\mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathbb{D}f^*\mathcal{N})[d_X] \to \mathbb{R}\mathcal{H}om_{\mathcal{D}_Y}(\mathbb{D}f_!\mathcal{M}, \mathcal{N})[d_Y].$$

(2) *If $\mathcal{M} \in D^b_{good}(\mathcal{D}_X)$, and if $\operatorname{Supp}(\mathcal{M})$ is proper over Y, then the morphism in (1) is an isomorphism.*

The strategy for the proof is reducing the proof to the corresponding proposition for $\mathcal{O}$-modules. First let us construct the morphism in (1). For this purpose, we construct the following *trace morphism*.

PROPOSITION 4.34. *In $D^b(\mathcal{D}_Y)$, we have a canonical morphism*

$$\operatorname{Tr}_f : \mathbb{D}f_!\mathcal{O}_X[d_X] \to \mathcal{O}_Y[d_Y].$$

REMARK 4.35. (1) We have an $(f^{-1}\mathcal{O}_Y, \mathcal{D}_X)$-module homomorphism $\Omega_{X/Y} \otimes_{\mathcal{O}_X} \mathcal{D}_X \to \mathcal{D}_{Y\leftarrow X}$, which leads to a morphism

$$\Omega_{X/Y} = \Omega_{X/Y} \otimes_{\mathcal{O}_X} \mathcal{D}_X \overset{\mathbb{L}}{\otimes}_{\mathcal{D}_X} \mathcal{O}_X \to \mathcal{D}_{Y\leftarrow X} \overset{\mathbb{L}}{\otimes}_{\mathcal{D}_X} \mathcal{O}_X$$

in $D^b(f^{-1}\mathcal{O}_Y)$. Hence in $D^b(\mathcal{O}_Y)$ we obtain

$$\mathbb{R}f_!(\Omega_{X/Y})[d_X] \longrightarrow \mathbb{D}f_!(\mathcal{O}_X)[d_X] \xrightarrow{\mathrm{Tr}_f} \mathcal{O}_Y[d_Y],$$

the trace morphism of $\mathcal{O}$-modules.

(2) If f is an isomorphism, then Tr_f is also an isomorphism.

(3) Given $f : X \to Y$ and $g : Y \to Z$, the composite

$$\mathbb{D}(g \circ f)_! \mathcal{O}_X[d_X] \simeq \mathbb{D}g_! \, \mathbb{D}f_! \mathcal{O}_X[d_X] \xrightarrow{\mathrm{Tr}_f} \mathbb{D}g_! \mathcal{O}_Y[d_Y] \xrightarrow{\mathrm{Tr}_g} \mathcal{O}_Z[d_Z]$$

coincides with $\mathrm{Tr}_{g\circ f}$.

We construct Tr_f by using explicit complexes. First we express $\mathcal{D}_{Y\leftarrow X} \overset{\mathbb{L}}{\otimes}_{\mathcal{D}_X} \bullet$ as an explicit complex by using a flat resolution of $\mathcal{D}_{Y\leftarrow X}$. Recall that

$$\mathcal{D}_{Y\leftarrow X} \simeq f^{-1}(\mathcal{D}_Y \otimes_{\mathcal{O}_Y} \Omega_Y^{\otimes -1}) \otimes_{f^{-1}\mathcal{O}_Y} \Omega_X.$$

Since $\mathcal{O}_X \underset{\text{qis}}{\simeq} (\mathcal{D}_X \otimes_{\mathcal{O}_X} \bigwedge^{\bullet} \Theta_X)$ (qis indicates quasi-isomorphism. see §A. 2 (c)), we obtain a resolution of $\mathcal{D}_{Y\leftarrow X}$ as an $f^{-1}\mathcal{D}_Y \otimes_{\mathbb{C}} \mathcal{D}_X^{\mathrm{op}}$-module

$$\begin{aligned}
\mathcal{D}_{Y\leftarrow X} &\underset{\text{qis}}{\overset{\sim}{\leftarrow}} (f^{-1}(\mathcal{D}_Y \otimes_{\mathcal{O}_Y} \Omega_Y^{\otimes -1}) \otimes_{f^{-1}\mathcal{O}_Y} \Omega_X) \otimes_{\mathcal{O}_X} (\mathcal{D}_X \otimes_{\mathcal{O}_X} \overset{\bullet}{\bigwedge} \Theta_X) \\
&\simeq f^{-1}(\mathcal{D}_Y \otimes_{\mathcal{O}_Y} \Omega_Y^{\otimes -1}) \otimes_{f^{-1}\mathcal{O}_Y} \Omega_X \otimes_{\mathcal{O}_X} \overset{\bullet}{\bigwedge} \Theta_X \otimes_{\mathcal{O}_X} \mathcal{D}_X \\
&\simeq f^{-1}(\mathcal{D}_Y \otimes_{\mathcal{O}_Y} \Omega_Y^{\otimes -1}) \otimes_{f^{-1}\mathcal{O}_Y} \Omega_X^{\bullet}[d_X] \otimes_{\mathcal{O}_X} \mathcal{D}_X.
\end{aligned}$$

Since the right hand side is a complex of flat $\mathcal{D}_X^{\mathrm{op}}$-modules, we have

$$\begin{aligned}
(4.23)\quad \mathcal{D}_{Y\leftarrow X} &\overset{\mathbb{L}}{\otimes}_{\mathcal{D}_X} \mathcal{M} \\
&\simeq (f^{-1}(\mathcal{D}_Y \otimes \Omega_Y^{\otimes -1}) \otimes_{f^{-1}\mathcal{O}_Y} \Omega_X^{\bullet}[d_X] \otimes_{\mathcal{O}_X} \mathcal{D}_X) \otimes_{\mathcal{D}_X} \mathcal{M} \\
&\simeq f^{-1}(\mathcal{D}_Y \otimes_{\mathcal{O}_Y} \Omega_Y^{\otimes -1}) \otimes_{f^{-1}\mathcal{O}_Y} \Omega_X^{\bullet}[d_X] \otimes_{\mathcal{O}_X} \mathcal{M}
\end{aligned}$$

for any $\mathcal{D}_X$-module $\mathcal{M}$.

REMARK 4.36. The complex $\mathcal{D}_Y \otimes_{\mathcal{O}} \Omega_Y^{\otimes -1} \otimes_{\mathcal{O}} \Omega_X^{\bullet} \otimes_{\mathcal{O}} \mathcal{M}$ is also given as follows: In § 1.2, we have constructed a complex $DR_X(\mathcal{M}) = \Omega_X^{\bullet} \otimes_{\mathcal{O}_X} \mathcal{M}$. Since its differentials are differential homomorphisms over $\mathcal{O}_X$, they can be regarded as differential homomorphisms of $f^{-1}\mathcal{O}_Y$-modules as well. Hence, by the result in § 1.2, we obtain a complex of right $f^{-1}\mathcal{D}_Y$-modules $DR_X(\mathcal{M}) \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{D}_Y$. Transforming this into the category of left $\mathcal{D}_Y$-modules, we obtain the right hand side of (4.23). Similarly we can obtain (4.24) and (4.25), below.

To compute the image of (4.23) by $\mathbb{R}f_!$, let us take, as a soft resolution of a $\mathcal{D}_X$-module $\mathcal{O}_X$, the Dolbeault resolution

$$\mathcal{D}b_X^{(0,\bullet)} : \mathcal{D}b_X^{(0,0)} \xrightarrow{\bar\partial} \mathcal{D}b_X^{(0,1)} \longrightarrow \cdots \longrightarrow \mathcal{D}b_X^{(0,d_X)},$$

where $\mathcal{D}b_X^{(p,q)}$ denotes the sheaf of (p,q)-forms with distributions as coefficients. Note that $\bar\partial$ commutes with the action of $\mathcal{D}_X$. Hence in $D^b(f^{-1}\mathcal{D}_Y)$ we have

$$\begin{aligned}
(4.24)\qquad & \mathcal{D}_{Y\leftarrow X} \overset{\mathbb{L}}{\otimes}_{\mathcal{D}_X} \mathcal{O}_X \\
& \simeq f^{-1}(\mathcal{D}_Y \otimes_{\mathcal{O}_Y} \Omega_Y^{\otimes -1}) \otimes_{f^{-1}\mathcal{O}_Y} \Omega_X^{\bullet}[d_X] \otimes_{\mathcal{O}_X} \mathcal{D}b_X^{(0,\bullet)} \\
& \simeq f^{-1}(\mathcal{D}_Y \otimes_{\mathcal{O}_Y} \Omega_Y^{\otimes -1}) \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{D}b_X^{(\bullet,\bullet)}[d_X].
\end{aligned}$$

Since $\mathcal{D}b_X^{(p,q)}$ is soft, $f^{-1}(\mathcal{D}_Y \otimes \Omega_Y^{\otimes -1}) \otimes \mathcal{D}b_X^{(\bullet,\bullet)}[d_X]$ is a complex of soft sheaves. Hence in $D^b(\mathcal{D}_Y)$

$$\begin{aligned}
(4.25)\quad \mathbb{D}f_!\mathcal{O}_X & \simeq f_!(f^{-1}(\mathcal{D}_Y \otimes_{\mathcal{O}_Y} \Omega_Y^{\otimes -1}) \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{D}b_X^{(\bullet,\bullet)})[d_X] \\
& \simeq \mathcal{D}_Y \otimes_{\mathcal{O}_Y} \Omega_Y^{\otimes -1} \otimes_{\mathcal{O}_Y} f_!(\mathcal{D}b_X^{(\bullet,\bullet)})[d_X].
\end{aligned}$$

LEMMA 4.37. *We have a homomorphism*

$$(4.26)\qquad f_!(\mathcal{D}b_X^{(p,q)}) \to \mathcal{D}b_Y^{(p-d_{X/Y},q-d_{X/Y})}$$

that commutes with ∂ and $\bar\partial$, where $d_{X/Y} := \dim X - \dim Y$.

PROOF. Note that, for any open subset $V \subset Y$, $\Gamma(V; \mathcal{D}b_Y^{(p,q)})$ is the (topologically) dual space of $\Gamma_c(V; \mathcal{C}_Y^{\infty(d_Y-p,d_Y-q)})$, where $\mathcal{C}_Y^{\infty(p,q)}$ denotes the sheaf of (p,q)-forms on Y with C^∞-functions as coefficients. Let $\varphi \in \Gamma(V; f_!\mathcal{D}b_X^{(p,q)})$. Take a C^∞-function $a \in C^\infty(f^{-1}V)$ so that it equals 1 on a neighborhood of $\mathrm{supp}(\varphi)$, and $\mathrm{supp}(a)$ is proper over V. Then

$$\Gamma_c(V;\mathcal{C}_Y^{\infty(d_X-p,d_X-q)}) \underset{af^*}{\longrightarrow} \Gamma_c(f^{-1}V;\mathcal{C}_X^{\infty(d_X-p,d_X-q)}) \underset{\varphi}{\longrightarrow} \mathbb{C}$$

is continuous and gives an element of $\Gamma(V;\mathcal{D}b_Y^{(p-d_{X/Y},q-d_{X/Y})})$. This does not depend on the choice of a. We have thus constructed the homomorphism (4.26). The commutativity with ∂ and $\bar\partial$ immediately follows from that of $f^*: f^{-1}\mathcal{C}_Y^{\infty(\bullet,\bullet)} \to \mathcal{C}_X^{\infty(\bullet,\bullet)}$ with ∂ and $\bar\partial$. □

From this lemma, we have obtained

$$f_!(\mathcal{D}b_X^{(\bullet,\bullet)}) \to \mathcal{D}b_Y^{(\bullet,\bullet)}[-2d_{X/Y}].$$

Composing this with (4.25), we obtain

$$\mathbb{D}f_!\mathcal{O}_X \to \mathcal{D}_Y \otimes_{\mathcal{O}_Y} \Omega_Y^{\otimes -1} \otimes_{\mathcal{O}_Y} \mathcal{D}b_Y^{(\bullet,\bullet)}[d_X - 2d_{X/Y}].$$

Putting $X = Y$ in (4.25), we have

$$\mathcal{O}_Y \simeq \mathcal{D}_Y \otimes_{\mathcal{O}_Y} \Omega_Y^{\otimes -1} \otimes_{\mathcal{O}_Y} \mathcal{D}b_Y^{(\bullet,\bullet)}[d_Y].$$

Hence we have obtained a morphism in $D^b(\mathcal{D}_Y)$,

$$\mathbb{D}f_!\mathcal{O}_X \to \mathcal{O}_Y[d_X - 2d_{X/Y} - d_Y].$$

We have thus proved Proposition 4.34.

REMARK 4.38. On many occasions, we take, as a trace morphism, the one multiplied by $(2\pi\sqrt{-1})^{-d_{X/Y}}$. As below, this adjustment is related to the Cauchy integration formula.

We can construct $\mathcal{D}_X$-module theory on an algebraic variety defined over a subfield k of $\mathbb{C}$. Associated to a proper morphism of algebraic varieties $f: X \to Y$, we can define

$$\mathrm{Tr}_f : \mathbb{D}f_*\mathcal{O}_X[d_X] \to \mathcal{O}_Y[d_Y].$$

Let X^{an} denote the complex manifold corresponding to an algebraic variety X. Then we have a functor $F \mapsto F^{an}$ from the category of $\mathcal{D}_X$-modules to that of $\mathcal{D}_{X^{an}}$-modules. The morphism defined by

$$\begin{tikzcd}
(\mathbb{D}f_*\mathcal{O}_X)^{an}[d_X] \arrow[r, "(\mathrm{Tr}_f)^{an}"] \arrow[d, equal] & (\mathcal{O}_Y)^{an}[d_Y] \arrow[d, equal] \\
\mathbb{D}f_{an*}\mathcal{O}_{X^{an}}[d_X] \arrow[r] & \mathcal{O}_{Y^{an}}[d_Y]
\end{tikzcd}$$

coincides with the one obtained by multiplication of $(2\pi\sqrt{-1})^{-d_{X/Y}}$ to the one constructed by use of distributions in Proposition 4.34. Note that $(2\pi\sqrt{-1})^{-d_{X/Y}} \notin k$ in general.

For example, let $X = \mathbb{P}^1(\mathbb{C})$ and $Y = \{\mathrm{pt}\}$. Then

$$H^0(\mathbb{D}f_*\mathcal{O}_X[1]) = H^1(X;\Omega^1_X) \cong \mathbb{C}.$$

Let $U_0 = X\backslash\{\infty\}$, $U_\infty = X\backslash\{0\}$, and $\Gamma(U_0 \cap U_\infty;\Omega^1_X) \ni u = z^{-1}dz$. Then the image $[u]$ of u by $\Gamma(U_0\cap U_\infty;\Omega^1_X) \to H^1(X;\Omega^1_X)$ is a basis of $H^1(X;\Omega^1_X)$. Its image by

$$\mathrm{Tr}_f : H^0(\mathbb{D}f_*\mathcal{O}_X[1]) \to H^0(\mathcal{O}_Y) = \mathbb{C}$$

obtained in Proposition 4.34 is given as follows: Take sections $u_\nu \in \Gamma(U_\nu;\mathcal{D}b^{(1,0)})$ $(\nu = 0,\infty)$ so that $u = u_0 - u_\infty$ (on $U_0 \cap U_\infty$). Since $\bar\partial u_0 = \bar\partial u_\infty$ on $U_0 \cap U_\infty$, there exists $v \in \Gamma(X;\mathcal{D}b^{(1,1)})$ such that $v|_{U_\nu} = \bar\partial u_\nu$ $(\nu = 0,\infty)$. Then

$$\mathrm{Tr}_f[u] = \int_X v.$$

Put $D = \{z; |z| < 1\}$. Taking u_0 and u_∞ so that they are C^∞ on a neighborhood of ∂D, we have

$$\begin{aligned}\int_X v &= \int_D v + \int_{X\backslash D} v = \int_D \bar\partial u_0 + \int_{X\backslash D} \bar\partial u_\infty \\ &= \int_{\partial D} u_0 - \int_{\partial D} u_\infty = \int_{\partial D}\frac{dz}{z} = 2\pi\sqrt{-1}.\end{aligned}$$

The last equation is the Cauchy integration formula.

Now let us prove Theorem 4.33 (1). Let $\mathcal{N} \in D^b(\mathcal{D}_Y)$. By the isomorphism

$$\mathbb{D}f_!(\mathbb{D}f^*\mathcal{N}) \simeq \mathbb{D}f_!(\mathbb{D}f^*\mathcal{N} \overset{\mathbb{D}}{\otimes} \mathcal{O}_X) \simeq \mathcal{N} \overset{\mathbb{D}}{\otimes} \mathbb{D}f_!\mathcal{O}_X$$

obtained from the projection formula (Theorem 4.28) and by $\mathrm{Tr}_f : \mathbb{D}f_!\mathcal{O}_X[d_X] \to \mathcal{O}_Y[d_Y]$, we have

$$\mathrm{Tr} : \mathbb{D}f_!\,\mathbb{D}f^*\mathcal{N}[d_X] \to \mathcal{N}[d_Y].$$

Hence we obtain

$$\begin{aligned}&\mathbb{R}f_*\mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\,\mathbb{D}f^*\mathcal{N})[d_X] \\ &\quad\longrightarrow \mathbb{R}\mathcal{H}om_{\mathcal{D}_Y}(\mathbb{D}f_!\mathcal{M},\,\mathbb{D}f_!\,\mathbb{D}f^*\mathcal{N}[d_X]) \\ &\quad\xrightarrow{\mathrm{Tr}} \mathbb{R}\mathcal{H}om_{\mathcal{D}_Y}(\mathbb{D}f_!\mathcal{M},\,\mathcal{N}[d_Y]).\end{aligned}$$

We have thus obtained (1).

Next let us prove Theorem 4.33 (2). We prove it in the case $\mathcal{N} = \mathcal{D}_Y$, first. In this case, the morphism is

$$(4.27)\quad \mathbb{R}f_*\mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_{X\to Y})[d_X] \to \mathbb{R}\mathcal{H}om_{\mathcal{D}_Y}(\mathbb{D}f_!\mathcal{M}, \mathcal{D}_Y)[d_Y].$$

Similarly to the above, we can prove that this is a morphism in $D^b(\mathcal{D}_Y^{\mathrm{op}})$. Tensoring $\Omega_Y^{\otimes -1}$ to this, we carry this to the derived category of $\mathcal{D}_Y$-modules. Since $\mathcal{M} \in D^b_{good}(\mathcal{D}_X)$, we have

$$\begin{aligned}
&\mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_{X\to Y}) \otimes \Omega_Y^{\otimes -1} \\
&\quad\simeq (\Omega_X \otimes_{\mathcal{O}} \mathcal{D}_{X\to Y} \otimes_{\mathcal{O}} \Omega_Y^{\otimes -1}) \overset{\mathbb{L}}{\otimes}_{\mathcal{D}_X} \mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_X \otimes_{\mathcal{O}} \Omega_X^{\otimes -1}) \\
&\quad\simeq \mathcal{D}_{Y\leftarrow X} \overset{\mathbb{L}}{\otimes}_{\mathcal{D}_X} \mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_X \otimes_{\mathcal{O}} \Omega_X^{\otimes -1}).
\end{aligned}$$

Hence the result of carrying (4.27) to $D^b(\mathcal{D}_Y)$ is

$$\begin{aligned}
\mathbb{R}f_*(\mathcal{D}_{Y\leftarrow X} \overset{\mathbb{L}}{\otimes}_{\mathcal{D}_X} \mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_X \otimes_{\mathcal{O}} \Omega_X^{\otimes -1}))[d_X] \\
\to \mathbb{R}\mathcal{H}om_{\mathcal{D}_Y}(\mathbb{D}f_!\mathcal{M}, \mathcal{D}_Y \otimes_{\mathcal{O}} \Omega_Y^{\otimes -1})[d_Y].
\end{aligned}$$

Hence the assertion that (4.27) is an isomorphism is equivalent to the following proposition.

PROPOSITION 4.39. *Let $\mathcal{M} \in D^b_{good}(\mathcal{D}_X)$. Suppose that* $\operatorname{Supp}\mathcal{M}$ *is proper over Y. Then*

$$\mathbb{D}f_*\mathbb{D}_X(\mathcal{M}) \xrightarrow{\sim} \mathbb{D}_Y\mathbb{D}f_!\mathcal{M}.$$

PROOF. We may assume that $\mathcal{M}$ is a good $\mathcal{D}_X$-module. Locally on Y, $\mathcal{M}$ has a resolution

$$0 \to \mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{F}_N \to \cdots \to \mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{F}_0 \to \mathcal{M} \to 0,$$

where $\mathcal{F}_i$ are coherent $\mathcal{O}_X$-modules, and their supports are proper over X. Hence it suffices to prove that, for any coherent $\mathcal{O}_X$-module $\mathcal{F}$ with support proper over Y, (4.27) is an isomorphism in case $\mathcal{M} = \mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{F}$. Then in $D^b(f^{-1}\mathcal{D}_Y^{\mathrm{op}})$

$$\begin{aligned}
\mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_{X\to Y}) &= \mathbb{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{D}_{X\to Y}) \\
&= \mathbb{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{D}_Y) \\
&= \mathbb{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X) \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{D}_Y.
\end{aligned}$$

Hence

$$\mathbb{R}f_*\mathbb{R}\mathcal{H}om(\mathcal{M}, \mathcal{D}_{X\to Y})[d_X] \cong \mathbb{R}f_*\mathbb{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X) \otimes_{\mathcal{O}_Y} \mathcal{D}_Y[d_X].$$

On the other hand,

$$\begin{aligned} \mathbb{D}f_!\mathcal{M} &= \mathbb{R}f_!(\mathcal{D}_{Y\leftarrow X}\otimes_{\mathcal{D}_X}\mathcal{M}) \\ &\cong \mathbb{R}f_!(f^{-1}\mathcal{D}_Y\otimes_{f^{-1}\mathcal{O}_Y}\Omega_{X/Y}\otimes_{\mathcal{O}_X}\mathcal{F}) \\ &\cong \mathcal{D}_Y\otimes_{\mathcal{O}_Y}\mathbb{R}f_!(\Omega_{X/Y}\otimes_{\mathcal{O}_X}\mathcal{F}). \end{aligned}$$

Hence

$$\begin{aligned} &\mathbb{R}\mathcal{H}om_{\mathcal{D}_Y}(\mathbb{D}f_*\mathcal{M},\mathcal{D}_Y)[d_Y] \\ &\quad\cong \mathbb{R}\mathcal{H}om_{\mathcal{O}_Y}(\mathbb{R}f_*(\Omega_{X/Y}\otimes_{\mathcal{O}_X}\mathcal{F}),\mathcal{D}_Y)[d_Y] \\ &\quad\cong \mathbb{R}\mathcal{H}om_{\mathcal{O}_Y}(\mathbb{R}f_*(\Omega_{X/Y}\otimes_{\mathcal{O}_X}\mathcal{F}),\mathcal{O}_Y)\otimes_{\mathcal{O}_Y}\mathcal{D}_Y[d_Y]. \end{aligned}$$

We have thus reduced the proposition to

$$\begin{aligned} &\mathbb{R}f_*\mathbb{R}\mathcal{H}om_{\mathcal{O}_X}(\Omega_{X/Y}\otimes_{\mathcal{O}_X}\mathcal{F},\Omega_{X/Y})[d_X] \\ &\xrightarrow{\sim} \mathbb{R}\mathcal{H}om(\mathbb{R}f_*(\Omega_{X/Y}\otimes_{\mathcal{O}_X}\mathcal{F}),\mathcal{O}_Y)[d_Y] \\ &\simeq \mathbb{R}f_*\mathbb{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{O}_X)[d_X]. \end{aligned}$$

The last isomorphism is known as the *duality theorem* for complex manifolds (a generalization of the Serre duality). □

Let us derive Theorem 4.33 (2) from the result that (4.27) is an isomorphism. Since

$$\begin{aligned} &\mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\,\mathbb{D}f^*\mathcal{N}) \\ &\quad\cong \mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\mathcal{D}_{X\to Y}\overset{\mathbb{L}}{\otimes}_{f^{-1}\mathcal{D}_Y}f^{-1}\mathcal{N}) \\ &\quad\cong \mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\mathcal{D}_{X\to Y})\overset{\mathbb{L}}{\otimes}_{f^{-1}\mathcal{D}_Y}f^{-1}\mathcal{N}, \end{aligned}$$

we have

$$\mathbb{R}f_!\mathbb{R}\mathcal{H}om(\mathcal{M},\,\mathbb{D}f^*\mathcal{N})\simeq\mathbb{R}f_!\mathbb{R}\mathcal{H}om(\mathcal{M},\mathcal{D}_{X\to Y})\overset{\mathbb{L}}{\otimes}_{\mathcal{D}_Y}\mathcal{N}.$$

On the other hand,

$$\mathbb{R}\mathcal{H}om_{\mathcal{D}_Y}(\mathbb{D}f_!\mathcal{M},\mathcal{N})\cong\mathbb{R}\mathcal{H}om_{\mathcal{D}_Y}(\mathbb{D}f_!\mathcal{M},\mathcal{D}_Y)\overset{\mathbb{L}}{\otimes}_{\mathcal{D}_Y}\mathcal{N}.$$

Since (4.27) is an isomorphism, we obtain (2). We have thus completed the proof of Theorem 4.33.

4.10. Adjunction Formula (Inverse Form)

In this section, we prove the adjunction formula inverse to Theorem 4.33. Its proof is much more elementary than that of Theorem 4.33. Note that the condition on $\mathcal{M}$ is weaker.

THEOREM 4.40. *Let $f : X \to Y$ be a morphism. Let $\mathcal{M} \in D^b(\mathcal{D}_X)$, and $\mathcal{N} \in D^b_{\mathrm{coh}}(\mathcal{D}_Y)$. Suppose that f is smooth. Then*

$$(4.28)\quad \mathbb{R}f_*\mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathbb{D}f^*\mathcal{N},\mathcal{M})[d_X] \xrightarrow{\sim} \mathbb{R}\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{N},\mathbb{D}f_*\mathcal{M})[d_Y]\,.$$

PROOF. Let us transform the left hand side. We have

$$\begin{aligned}&\mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathbb{D}f^*\mathcal{N},\mathcal{M})\\ &\quad= \mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{D}_{X\to Y}\otimes_{f^{-1}\mathcal{D}_Y} f^{-1}\mathcal{N},\mathcal{M})\\ &\quad\simeq \mathbb{R}\mathcal{H}om_{f^{-1}\mathcal{D}_Y}(f^{-1}\mathcal{N},\mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{D}_{X\to Y},\mathcal{M}))\,.\end{aligned}$$

Since f is smooth, $\mathcal{D}_{X\to Y}$ is a coherent $\mathcal{D}_X$-module. Hence

$$\mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{D}_{X\to Y},\mathcal{M}) \cong \mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{D}_{X\to Y},\mathcal{D}_X)\overset{\mathbb{L}}{\otimes}_{\mathcal{D}_X}\mathcal{M}$$

in $D^b(f^{-1}\mathcal{D}_Y)$. By Proposition 4.19,

$$\mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{D}_{X\to Y},\mathcal{D}_X) \cong \mathcal{D}_{Y\leftarrow X}[-d_{X/Y}]\,.$$

Hence the left hand side of (4.28) is isomorphic to

$$\mathbb{R}f_*(\mathbb{R}\mathcal{H}om_{f^{-1}\mathcal{D}_Y}(f^{-1}\mathcal{N},\mathcal{D}_{Y\leftarrow X}\overset{\mathbb{L}}{\otimes}_{\mathcal{D}_X}\mathcal{M}))[d_Y]$$

and in turn to

$$\mathbb{R}\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{N},\mathbb{R}f_*(\mathcal{D}_{Y\leftarrow X}\overset{\mathbb{L}}{\otimes}_{\mathcal{D}_X}\mathcal{M}))[d_Y].$$

□

REMARK 4.41. We have assumed that f is smooth in Theorem 4.40. Instead of this assumption, the theorem also holds under the assumptions that $\mathcal{M} \in D^b_{good}(\mathcal{D}_X)$, that $\mathcal{N}$ is a coherent $\mathcal{D}_Y$-module, and that f is non-characteristic for $\mathcal{N}$.

4.11. Holonomic Systems

For any coherent $\mathcal{D}_X$-module $\mathcal{M}$ on an n-dimensional manifold X, we have $\dim \mathrm{Ch}(\mathcal{M}) \geq n$. If $\dim \mathrm{Ch}(\mathcal{M}) = n$, i.e., $\mathrm{Ch}(\mathcal{M})$ is Lagrangian (§A. 4), $\mathcal{M}$ is called a *holonomic $\mathcal{D}_X$-module*. For instance, $\mathcal{O}_X$ and $\mathcal{B}_{Z|X}$ (Z is a closed submanifold of X) are holonomic $\mathcal{D}_X$-modules. As below, for any holonomic $\mathcal{D}_X$-module $\mathcal{M}$, the stalk of $\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\mathcal{O}_X)$ at any point is finite-dimensional, or equivalently the dimension of holomorphic solutions is finite.

Since $\mathcal{E}xt^k_{\mathcal{D}_X}(\mathcal{M},\mathcal{D}_X) = 0$ when $k > n$ or $k < \mathrm{codim}(\mathrm{Ch}(\mathcal{M}))$, it follows that $\mathcal{E}xt^k_{\mathcal{D}_X}(\mathcal{M},\mathcal{D}_X) = 0$ ($k \neq n$) for any holonomic $\mathcal{D}_X$-module $\mathcal{M}$. Hence

$$\mathbb{D}_X\mathcal{M} = \mathcal{E}xt^n_{\mathcal{D}_X}(\mathcal{M},\mathcal{D}_X\otimes_{\mathcal{O}_X}\Omega_X^{\otimes -1})\,.$$

Consequently $\mathbb{D}_X\mathcal{M}$ is again a holonomic $\mathcal{D}_X$-module. Let $Hol(X)$ denote the abelian category of holonomic $\mathcal{D}_X$-modules. Then

$$\mathbb{D}_X : Hol(X)^{\mathrm{op}} \to Hol(X)$$

is an equivalence of abelian categories.

In particular, descending chains starting from a holonomic $\mathcal{D}_X$-module $\mathcal{M}$ and ascending chains starting from $\mathbb{D}_X\mathcal{M}$ have a one-to-one correspondence. Therefore we obtain the following proposition.

PROPOSITION 4.42. *For any holonomic $\mathcal{D}_X$-module $\mathcal{M}$ and any $x \in X$, $\mathcal{M}_x$ is a $(\mathcal{D}_X)_x$-module of finite length.*

Although the following proposition can be elementarily proved, we prove it by using Theorem 4.13.

PROPOSITION 4.43. *For a coherent $\mathcal{D}_X$-module $\mathcal{M}$, the following three conditions are equivalent:*

(1) $\mathrm{Ch}(\mathcal{M}) \subset T^*_X X$,
(2) *$\mathcal{M}$ is coherent over $\mathcal{O}_X$,*
(3) *$\mathcal{M}$ is locally isomorphic to $\mathcal{O}_X^{\oplus m}$ as a $\mathcal{D}_X$-module.*

PROOF. Clearly (3) $\Longrightarrow$ (2) $\Longrightarrow$ (1).

(1) $\Longrightarrow$ (2). Let $F(\mathcal{M})$ be a coherent filtration of $\mathcal{M}$. Then, by the assumption,

$$\mathrm{Gr}^F_m(\mathcal{D}_X)\,\mathrm{Gr}^F(\mathcal{M}) = 0$$

for m large enough. Since $\mathrm{Gr}^F \mathcal{M}$ is finitely generated over $\mathrm{Gr}^F \mathcal{D}_X$, $\mathrm{Gr}^F_m \mathcal{M} = 0$ $(m \gg 0)$. Hence $\mathcal{M} = F_m(\mathcal{M})$ $(m \gg 0)$.

(1)+(2) $\Longrightarrow$ (3). Let $x \in X$, and let $j : \{x\} \hookrightarrow X$ be the embedding. Thanks to (1), we may apply Corollary 4.15 to obtain

$$V := \mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{O}_X, \mathcal{M})_x \simeq \mathbb{R}\mathcal{H}om_{\mathbb{C}}(\mathbb{C}, \mathbb{D}j^*\mathcal{M}).$$

By Theorem 4.7, $H^j(V) = 0$ $(j \neq 0)$, and V is finite-dimensional. Furthermore, on a neighborhood of x, we obtain a $\mathcal{D}_X$-module homomorphism $\varphi : V \otimes \mathcal{O}_X \to \mathcal{M}$. Since $\mathbb{D}j^*(V \otimes \mathcal{O}_X) \to \mathbb{D}j^*\mathcal{M}$ is an isomorphism, φ is surjective by Nakayama's lemma. Let $\mathcal{N} = \mathrm{Ker}\,\varphi$. Then it follows from $\mathrm{Ch}(\mathcal{N}) \subset T^*_X X$ that $\mathcal{N}$ is coherent over $\mathcal{O}_X$, and $\mathbb{D}j^*\mathcal{N} = 0$. Again by Nakayama's lemma, we get $\mathcal{N} = 0$. □

As to finiteness of solution spaces, we have the following theorem.

THEOREM 4.44. *Let $\mathcal{M}$ and $\mathcal{N}$ be holonomic $\mathcal{D}_X$-modules. Then $\mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{N})$ belongs to $D^b_c(\mathbb{C}_X)$ (see § 5.3).*

We do not prove this theorem; we only prove the following theorem, which will be used later.

THEOREM 4.45. *Let $\mathcal{M}$ and $\mathcal{N}$ be holonomic $\mathcal{D}_X$-modules. Then $\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\mathcal{N})_x$ is finite-dimensional at any point x in X.*

PROOF. We use induction on the dimension of $Z := \mathrm{Supp}(\mathcal{N})$.

The holonomicity of $\mathbb{D}_X\mathcal{M}$ leads to that of $\Gamma_Z(\mathbb{D}_X\mathcal{M})$ and, in turn, that of $\mathcal{M}' := \mathbb{D}_X(\Gamma_Z(\mathbb{D}_X\mathcal{M}))$. Applying the exact contravariant functor $\mathbb{D}_X$ to the monomorphism $\Gamma_Z(\mathbb{D}_X\mathcal{M}) \rightarrowtail \mathbb{D}_X\mathcal{M}$, we obtain an epimorphism $\mathcal{M} \twoheadrightarrow \mathbb{D}_X(\Gamma_Z(\mathbb{D}_X\mathcal{M})) = \mathcal{M}'$. We show that the monomorphism

$$\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}',\mathcal{N}) \rightarrowtail \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\mathcal{N}) \tag{4.29}$$

is an isomorphism. Since $\mathrm{Supp}(\mathbb{D}_X\mathcal{N}) \subset Z$,

$$\mathcal{H}om_{\mathcal{D}_X}(\mathbb{D}_X\mathcal{N},\Gamma_Z(\mathbb{D}_X\mathcal{M})) \to \mathcal{H}om_{\mathcal{D}_X}(\mathbb{D}_X\mathcal{N},\mathbb{D}_X\mathcal{M})$$

is an isomorphism. Using (3.14), we see that this is isomorphic to the morphism (4.29). We have thus proved that (4.29) is an isomorphism. Hence, replacing $\mathcal{M}$ by $\mathcal{M}'$, we may assume $\mathrm{Supp}(\mathcal{M}) \subset Z$ from the beginning.

Let $\Lambda = \mathrm{Ch}(\mathcal{M}) \cup \mathrm{Ch}(\mathcal{N})$. Then Λ is Lagrangian, $\pi(\Lambda) = Z$, and hence $\Lambda \supset T^*_Z X$. Since $\Lambda_0 = \overline{\Lambda \backslash T^*_Z X}$ is also Lagrangian, $S := \pi_X(\Lambda_0) \cup Z_{\mathrm{sing}}$ is a nowhere dense closed submanifold of Z (see Proposition A.53), where Z_{sing} denotes the set of singular points of Z. Let $X_0 = X \backslash S$ and $Z_0 = Z \backslash S$. Then

$$\mathrm{Ch}(\mathcal{M}|_{X_0}) \cup \mathrm{Ch}(\mathcal{N}|_{X_0}) \subset T^*_{Z_0} X_0. \tag{4.30}$$

Let $i : Z_0 \hookrightarrow X_0$ be the closed embedding. By Theorem 4.30, there exist coherent $\mathcal{D}_{Z_0}$-modules $\mathcal{M}_0$ and $\mathcal{N}_0$ such that $\mathcal{M}|_{X_0} = \mathbb{D}i_*\mathcal{M}_0$ and $\mathcal{N}|_{X_0} = \mathbb{D}i_*\mathcal{N}_0$. Since $\mathrm{Ch}(\mathcal{N}_0) \cup \mathrm{Ch}(\mathcal{M}_0) \subset T^*_{Z_0} Z_0$ by (4.30), each of $\mathcal{M}_0$ and $\mathcal{N}_0$ is locally isomorphic to the direct sum of a finite number of $\mathcal{O}_{Z_0}$. Hence $\mathcal{H}om_{\mathcal{D}_{Z_0}}(\mathcal{M}_0,\mathcal{N}_0)$ is locally isomorphic to the direct sum of a finite number of $\mathbb{C}_{Z_0}$. Accordingly, so is $j^{-1}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\mathcal{N}) = i_*\mathcal{H}om_{\mathcal{D}_{Z_0}}(\mathcal{M}_0,\mathcal{N}_0)$ on X_0, where $j : X_0 \hookrightarrow X$ is the open embedding. Thus $j_*j^{-1}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\mathcal{N})$ has a finite-dimensional stalk at every point. In the exact sequence

$$\begin{aligned} 0 \to \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\Gamma_S(\mathcal{N}))_x \\ \to \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\mathcal{N})_x \to j_*j^{-1}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\mathcal{N})_x, \end{aligned}$$

$\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \Gamma_S(\mathcal{N}))_x$ is finite-dimensional by the induction hypothesis on $\dim(\operatorname{Supp}\mathcal{N})$. Therefore $\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{N})_x$ is finite-dimensional.

$\square$

CHAPTER 5

Regular Holonomic Systems

In this chapter, we provide some facts about regular holonomic systems without proofs. For details, see for example [HT] or [KK].

5.1. Ordinary Differential Equations with Regular Singularities

Let X be an open subset of $\mathbb{C}$ and $P(x,\partial)$ an ordinary differential operator of order m. Put $\mathcal{M} = \mathcal{D}/\mathcal{D}P$, and write

$$P(x,\partial) = \sum_{k=0}^{m} a_k(x)\partial^k.$$

In a neighborhood of x_0 with $a_m(x_0) \neq 0$, $\mathcal{M} \simeq \mathcal{O}_X^{\oplus m}$ as $\mathcal{D}$-modules. Hence $Pu = 0$ has m linearly independent holomorphic solutions. In a neighborhood of x_0 with $a_m(x_0) = 0$, it is not easy in general to know their behavior. For example, while $L = \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)$ is a locally constant sheaf of rank m on $X \setminus \{x_0\}$, a general algorithm for its monodromy is not known. An algorithm is well known, however, in the case of differential equations with regular singularities, defined below.

THEOREM 5.1. *The following two conditions are equivalent:*

(1) *Let* $r = \operatorname{ord}_{x=x_0} a_m(x)$. *Then* $\operatorname{ord}_{x=x_0} a_j(x) \geq r-(m-j)$ *for all* j, *where* $\operatorname{ord}_{x=x_0} f(x)$ *denotes the order of zero of* $f(x)$ *at* $x = x_0$. (*We agree that* $\operatorname{ord}_{x=x_0} 0 = \infty$.)
(2) $P(x,\partial)u = 0$ *has* m *linearly independent solutions of the form*

$$(x-x_0)^{\lambda}\sum_{j=0}^{s} u_j(x)(\log(x-x_0))^j$$

for some $s \in \mathbb{Z}_{\geq 0}$ *and* $\lambda \in \mathbb{C}$, *where* $u_j(x)$ *are holomorphic on a neighborhood of* $x = x_0$.

If these two equivalent conditions are satisfied, x_0 is called a *regular singularity* of $Pu = 0$. A zero of $a_m(x)$ not satisfying the above conditions is called an *irregular singularity*.

Suppose that $x_0 = 0$ is a regular singularity. Replacing P by $x^l P$ or $\partial^l P$ for some l, we may assume $m = r$. Then P is written as $\sum_{j=0}^{m} c_j(x)x^j\partial^j$ for some holomorphic functions $c_0, \ldots, c_m$. Furthermore, $c_m(0) \neq 0$. Hence, multiplying by $c_m(x)^{-1}$, we may assume $P = (x\partial)^m + \sum_{j=0}^{m-1} c_j(x)(x\partial)^j$. Here we have used the equality $x^j\partial^j = (x\partial)(x\partial - 1)\cdots(x\partial - j + 1)$.

We define a coherent filtration $F(\mathcal{M})$ of

$$\mathcal{M} = \mathcal{D}/\mathcal{D}P = \mathcal{D}u \qquad (u = 1 \mod \mathcal{D}P),$$

by

$$F_k(\mathcal{M}) = \sum_{j=0}^{m-1} F_k(\mathcal{D})(x\partial)^j u.$$

Then

$$(x\partial)F_k(\mathcal{M}) \subset F_k(\mathcal{M}),$$

or

$$x\xi \operatorname{Gr}^F(\mathcal{M}) = 0.$$

Conversely we can show that $x = 0$ is a regular singularity of $\mathcal{M}$ if $\mathcal{M}$ admits a coherent filtration satisfying the above condition. Since

$$\operatorname{Ch}(\mathcal{M}) \subset \{ (x, \xi) ; x\xi = 0 \},$$

for any coherent filtration F of $\mathcal{M}$, there exists $N > 0$ such that

$$(x\xi)^N \operatorname{Gr}^F(\mathcal{M}) = 0,$$

even if $x = 0$ is an irregular singularity. The mentioned characterization of a regular singularity means that we can take 1 as the above N.

5.2. Holonomic Modules with Regular Singularities

The notion of regular singularities of linear ordinary differential equations in the previous section can be generalized to that of holonomic $\mathcal{D}_X$-modules as follows:

Let X be a manifold and $\mathcal{M}$ a holonomic $\mathcal{D}_X$-module. Let Λ be the characteristic variety of $\mathcal{M}$ and I_Λ the defining ideal of Λ in $\operatorname{Gr}^F(\mathcal{D}_X)$. Locally on X, for any coherent filtration $F(\mathcal{M})$ of any $\mathcal{M}$, we have $I_\Lambda^N \operatorname{Gr}^F(\mathcal{M}) = 0$ for $N > 0$ large enough.

DEFINITION 5.2. We say that $\mathcal{M}$ is *regular holonomic* if $\mathcal{M}$ locally admits a coherent filtration $F(\mathcal{M})$ satisfying $I_\Lambda \operatorname{Gr}^F(\mathcal{M}) = 0$.

We will study microlocal properties of regular holonomic modules in § 8.7.

We denote by $RH(X)$ the category of regular holonomic $\mathcal{D}_X$-modules, a full subcategory of $\mathrm{Mod}(\mathcal{D}_X)$. It is known that $RH(X)$ satisfies the following properties. (The proof of (1) is easy.)

PROPOSITION 5.3. (1) *Suppose that $\mathcal{M}' \to \mathcal{M} \to \mathcal{M}''$ is an exact sequence of coherent $\mathcal{D}_X$-modules. If $\mathcal{M}'$ and $\mathcal{M}''$ belong to $RH(X)$, then so does $\mathcal{M}$.*

(2) *If $\mathcal{M}$ is regular holonomic, then there exists a coherent filtration $F(\mathcal{M})$ satisfying the condition in Definition 5.2 globally on X.*

Let $D^b_{rh}(\mathcal{D}_X)$ denote the full subcategory of $D^b(\mathcal{D}_X)$ consisting of $\mathcal{M} \in D^b(\mathcal{D}_X)$ all of whose cohomology modules $H^i(\mathcal{M})$ belong to $RH(X)$. Then $D^b_{rh}(\mathcal{D}_X)$ is a triangulated category. The category $D^b_{rh}(\mathcal{D}_X)$ is preserved under many functors of $\mathcal{D}_X$-modules, as follows.

THEOREM 5.4.

(1) *If $\mathcal{M} \in D^b_{rh}(\mathcal{D}_X)$, then $\mathbb{D}_X\mathcal{M} \in D^b_{rh}(\mathcal{D}_X)$.*

(2) *If $\mathcal{M}, \mathcal{M}' \in D^b_{rh}(\mathcal{D}_X)$, then $\mathcal{M} \overset{\mathbb{D}}{\otimes} \mathcal{M}' \in D^b_{rh}(\mathcal{D}_X)$.*

(3) *Let $f : X \to Y$ be a morphism of manifolds. Then*

$$\mathbb{D}f^* : D^b(\mathcal{D}_Y) \to D^b(\mathcal{D}_X)$$

sends $D^b_{rh}(\mathcal{D}_Y)$ into $D^b_{rh}(\mathcal{D}_X)$.

(4) *Let $f : X \to Y$ and $\mathcal{M} \in D^b_{rh}(\mathcal{D}_X)$. If $\operatorname{Supp}(\mathcal{M})$ is proper over Y, then $\mathbb{D}f_!\mathcal{M} \in D^b_{rh}(\mathcal{D}_Y)$.*

(5) *Let S be a difference of closed analytic subsets of X. Then*

$$\mathbb{R}\Gamma_{[S]} : D^b(\mathcal{D}_X) \to D^b(\mathcal{D}_X)$$

preserves $D^b_{rh}(\mathcal{D}_X)$.

The proofs of these properties are rather difficult. One reason for this is that the above definition of regularity depends on a microlocal property (i.e., a property on T^*X), and hence the definition does not get along with non-microlocal functors. The following proposition is an analytic characterization of regular holonomic modules.

PROPOSITION 5.5. *Let $\mathcal{M} \in D^b(\mathcal{D}_X)$. Suppose that all cohomology modules of $\mathcal{M}$ are holonomic. Then the following three conditions are equivalent:*

(1) $\mathcal{M} \in D^b_{rh}(\mathcal{D}_X)$.

(2) *For any $x \in X$,*

$$\mathbb{R}\operatorname{Hom}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_{X,x}) \xrightarrow{\sim} \mathbb{R}\operatorname{Hom}_{\mathcal{D}_X}(\mathcal{M}, \widehat{\mathcal{O}}_{X,x}),$$

where $\widehat{\mathcal{O}}_{X,x}$ is the $\mathcal{D}_{X,x}$-module of formal power series at x.

(3) *For any $x \in X$,*

$$\mathbb{R}\operatorname{Hom}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_{\{x\}|X}) \xrightarrow{\sim} \mathbb{R}\operatorname{Hom}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}^{\infty}_{\{x\}|X}),$$

where $\mathcal{B}^{\infty}_{\{x\}|X} = H^n_{\{x\}}(X; \mathcal{O}_X)$.

Roughly speaking, (2) means that all formal solutions of $\mathcal{M}$ converge. (3) is dual to (2). Indeed,

$$\mathbb{R}\operatorname{Hom}_{\mathbb{C}}(\mathbb{R}\operatorname{Hom}_{\mathcal{D}_X}(\mathcal{M}, \widehat{\mathcal{O}}_{X,x}), \mathbb{C}) \xrightarrow{\sim} \mathbb{R}\operatorname{Hom}_{\mathcal{D}_X}(\mathbb{D}_X\mathcal{M}, \mathcal{B}_{\{x\}|X})$$

and

$$\mathbb{R}\operatorname{Hom}_{\mathbb{C}}(\mathbb{R}\operatorname{Hom}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_{X,x}), \mathbb{C}) \xrightarrow{\sim} \mathbb{R}\operatorname{Hom}_{\mathcal{D}_X}(\mathbb{D}_X\mathcal{M}, \mathcal{B}^{\infty}_{\{x\}|X})$$

hold for all $\mathcal{M} \in D^b_{rh}(\mathcal{D}_X)$.

5.3. Riemann–Hilbert Correspondence

For a holonomic $\mathcal{D}_X$-module $\mathcal{M}$, set

$$\begin{aligned} \mathrm{DR}_X(\mathcal{M}) &:= \mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{O}_X, \mathcal{M}), \\ \dot{\mathrm{Sol}}_X(\mathcal{M}) &:= \mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X) = \mathrm{DR}_X(\mathbb{D}_X\mathcal{M}) \\ &= \mathbb{R}\mathcal{H}om_{\mathbb{C}}(\mathrm{DR}_X(\mathcal{M}), \mathbb{C}_X). \end{aligned}$$

(The last equation is not trivial.) They are objects of the derived category $D^b(\mathbb{C}_X)$ of $\mathrm{Mod}(\mathbb{C}_X)$, where $\mathrm{Mod}(\mathbb{C}_X)$ is the abelian category of sheaves of $\mathbb{C}$-vector spaces on X. Furthermore, their cohomology modules are constructible.

DEFINITION 5.6. A sheaf F of vector spaces on X is said to be *constructible* if it satisfies the following conditions:

(1) For every $x \in X$, F_x is finite-dimensional.

(2) There exists a finite increasing sequence of closed analytic subsets of X

$$\emptyset = X_0 \subset X_1 \subset \cdots \subset X_r = X$$

such that $F|_{X_k \setminus X_{k-1}}$ are locally constant sheaves for all k.

Let $D_c^b(\mathbb{C}_X)$ denote the full subcategory of $D^b(\mathbb{C}_X)$ consisting of objects in $D^b(\mathbb{C}_X)$ all of whose cohomology modules are constructible. If Z is an analytic subset of codimension $\geq d$, then

$$\mathcal{H}^j \mathbb{R}\Gamma_Z(\mathcal{O}_X) = 0 \quad (j < d),$$

and hence

$$\mathbb{R}\Gamma_Z(\mathrm{DR}_X(\mathcal{M})) = \mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathbb{D}_X\mathcal{M}, \mathbb{R}\Gamma_Z(\mathcal{O}_X))$$

satisfies

$$H^j \mathbb{R}\Gamma_Z(\mathrm{DR}_X(\mathcal{M})) = 0 \quad (j < d).$$

In general, $F \in D_c^b(\mathbb{C}_X)$ is called a *perverse sheaf* if

$$\operatorname{codim}(\operatorname{Supp} H^j(F)) \geq j\,, \tag{5.1}$$

$$\operatorname{codim}(\operatorname{Supp} H^j(\mathbb{R}\mathcal{H}om_{\mathbb{C}}(F, \mathbb{C}_X))) \geq j. \tag{5.2}$$

The condition (5.2) is equivalent to the condition that $\mathcal{H}_Z^j(F) = 0$ $(j < \operatorname{codim} Z)$ for any closed analytic subset Z. Thus $\mathrm{DR}_X(\mathcal{M})$ is a perverse sheaf. Let $\mathrm{Perv}(\mathbb{C}_X)$ denote the full subcategory of $D_c^b(\mathbb{C}_X)$ consisting of perverse sheaves. The following theorem generalizes Hilbert's 21st problem, which asks for a linear ordinary differential equation with a given monodromy.

THEOREM 5.7. (1) $\mathrm{DR}_X : D_{rh}^b(\mathcal{D}_X) \to D_c^b(\mathbb{C}_X)$ *is an equivalence of triangulated categories.*

(2) *Let* $\mathrm{Reg}(\mathcal{D}_X)$ *denote the abelian category of regular holonomic* $\mathcal{D}_X$*-modules. Then*

$$\mathrm{DR}_X : \mathrm{Reg}(\mathcal{D}_X) \to \mathrm{Perv}(\mathbb{C}_X)$$

is an equivalence of additive categories.

Since $\mathrm{Reg}(\mathcal{D}_X)$ is an abelian category, so is $\mathrm{Perv}(\mathbb{C}_X)$. While $\mathrm{Reg}(\mathcal{D}_X)$ is algebraic, $\mathrm{Perv}(\mathbb{C}_X)$ is topological. The point of the theorem is that these two are the same. For its applications to representation theory, see [HT].

CHAPTER 6

b-functions

6.1. Motivation for b-functions

Let $f(x)$ be a nonzero holomorphic function. Then there exist a nonzero polynomial $b(s)$ and a differential operator $P(s) \in \mathcal{D}_X \otimes \mathbb{C}[s]$ such that

$$P(s)f(x)^{s+1} = b(s)f(x)^s. \tag{6.1}$$

(The proof of its existence is one of the aims of this chapter.) The monic polynomial of the smallest degree among such $b(s)$ is called the *b-function* of $f(x)$. This has been introduced by Mikio Sato from the following viewpoint: Let $f(x)$ be a real analytic function on $\mathbb{R}^n$. Then

$$f(x)_+^s := \begin{cases} f(x)^s & (f(x) > 0)\,, \\ 0 & (f(x) \leq 0) \end{cases}$$

is continuous in x and holomorphic in s when $\operatorname{Re} s > 0$. Let us continue this analytically to the whole space $s \in \mathbb{C}$ by using the b-function. Since $P(s)f(x)_+^{s+1} = b(s)f(x)_+^s$ for $\operatorname{Re} s \gg 0$, it also holds for $\operatorname{Re} s > 0$ by the holomorphy in s. For $\operatorname{Re} s > -1$, define

$$f(x)_+^s = \frac{1}{b(s)}P(s)f(x)_+^{s+1}.$$

Then this is a distribution that coincides with the original function on $\operatorname{Re} s > 0$ and a meromorphic function in s with possible poles at zeros of $b(s)$. For general s, repeat this process, and define

$$f(x)_+^s = \frac{1}{b(s)b(s+1)\cdots b(s+N-1)}P(s)\cdots P(s+N-1)f(x)_+^{s+N}.$$

Then this is defined for $\operatorname{Re} s > -N$. If $b(s) = \prod_{j=1}^m (s - \lambda_j)$, then $f(x)_+^s$ is a meromorphic function in s with poles possibly at $\lambda_j, \lambda_j - 1, \cdots, \lambda_j - N + 1$. By taking N larger and larger, we see that $f(x)_+^s$ is meromorphically continued to the whole space $s \in \mathbb{C}$, and that its poles are contained in $\{\, \lambda_j - k\,;\, j = 1, \ldots, m,\ \ k = 0, 1, 2, \ldots \}$.

EXAMPLE 6.1. Let $X = \mathbb{R}$, and $f(x) = x$. Then

$$\partial_x x^{s+1} = (s+1)x^s.$$

Hence $b(s) = s + 1$ is the b-function of f.

EXAMPLE 6.2. Let $X = \mathbb{R}^n$, and $f(x) = x_1^2 + \cdots + x_n^2$. Then

$$\begin{aligned} \partial_i f(x)^{s+1} &= 2(s+1)x_i f(x)^s, \\ \partial_i^2 f(x)^{s+1} &= 4s(s+1)x_i^2 f(x)^{s-1} + 2(s+1)f(x)^s. \end{aligned}$$

Hence $\Delta = \sum_{i=1}^n \partial_i^2$ satisfies

$$\begin{aligned} \Delta f(x)^{s+1} &= 4s(s+1)f(x)f(x)^{s-1} + 2n(s+1)f(x)^s \\ &= 4(s+1)(s+n/2)f(x)^s. \end{aligned}$$

Thus $b(s) = (s+1)(s+n/2)$ is the b-function.

In Example 6.1, x_+^s has a pole of order 1 at $s = -1$, and its residue $(s+1)x_+^s|_{s=-1}$ equals $\delta(x)$, because $(s+1)x_+^s = \partial_x x_+^{s+1}$ leads to

$$(s+1)x_+^s|_{s=-1} = \partial_x x_+^0 = \delta(x).$$

Similarly x_+^s has poles of order 1 at $s = -1, -2, \cdots$, and its residue at $s = -1-n$ equals $\frac{(-1)^n}{n!}\delta^{(n)}(x)$. Thus $b(s)$ describes the position of poles of the analytic continuation of $f(x)^s$ in s. This is one of the motivations for b-functions. Moreover, a relation between $b(s)$ and the eigenvalues of local monodromies of $f(x)$ has been discovered by Malgrange [Ma]; we do not go into details about this subject.

6.2. $\mathcal{D}$-module Generated by $f(x)^s$

Let X be a complex manifold of dimension n, and $f(x)$ a nonzero holomorphic function on X. Then $f^{-1}(0)$ is a hypersurface of X. We consider $f(x)^s$ as a symbol and define a $\mathcal{D}_X$-module $\mathcal{D}_X[s]f(x)^{s+\mathbb{Z}}$. This is $(\mathbb{C}[s] \otimes \mathcal{O}_X[f^{-1}])f(x)^s = \bigcup_m \mathbb{C}[s] \otimes \mathcal{O}_X f(x)^{s-m}$ as an $\mathcal{O}_X$-module, and the action of $v \in \Theta_X$ is defined by

$$v(a(s)\otimes\varphi)f(x)^s = (a(s)\otimes v(\varphi))f(x)^s + (sa(s)\otimes\varphi f^{-1}v(f))f(x)^s$$

$$(a(s) \in \mathbb{C}[s],\ \varphi \in \mathcal{O}_X[f^{-1}]).$$

Then this is a $\mathcal{D}_X$-module by Lemma 1.7. We intuitively understand this by regarding $f(x)^s$ as a multivalued function and letting $\mathcal{D}_X$ act on it. By the definition,

$$\Gamma_{f^{-1}(0)}(\mathcal{D}_X[s]f(x)^{s+\mathbb{Z}}) = 0.$$

Since the multiplication of s onto $\mathcal{D}_X[s]f(x)^{s+\mathbb{Z}}$ commutes with the action of $\mathcal{D}_X$, the ring $\mathcal{D}_X[s] := \mathbb{C}[s]\otimes\mathcal{D}_X$ also acts on $\mathcal{D}_X[s]f(x)^{s+\mathbb{Z}}$. Define an operator t on $\mathcal{D}_X[s]f(x)^{s+\mathbb{Z}}$ by

$$(6.2)\quad t(a(s)f(x)^s) = (a(s+1)f(x))f(x)^s \quad (a(s) \in \mathbb{C}[s]\otimes\mathcal{O}_X[f^{-1}]).$$

This is an operator shifting s by 1. This clearly commutes with the action of $\mathcal{D}_X$, but not with s, and satisfies

$$(6.3)\qquad ts - st = t.$$

More generally, for all polynomials $a(s)$,

$$ta(s) = a(s+1)t.$$

Let $\mathcal{D}_X[s]f(x)^s$ denote the $\mathcal{D}_X[s]$-submodule of $\mathcal{D}_X[s]f(x)^{s+\mathbb{Z}}$ generated by $f(x)^s$. Set

$$I_f := \{\, P(s) \in \mathcal{D}[s]\,;\, P(s)f(x)^s = 0\,\}.$$

Then

$$I_f = \{\, P(s) \in \mathcal{D}[s]\,;\, P(m)f(x)^m = 0 \text{ for all } m \in \mathbb{Z}_{\geq 0}\,\}.$$

(The proof is left to the reader.) Clearly $\mathcal{D}_X[s]f(x)^s \cong \mathcal{D}_X[s]/I_f$. The operator t preserves $\mathcal{D}_X[s]f(x)^s$ and the $\mathcal{D}_X$-submodule $\mathcal{D}_X f(x)^s$ generated by $f(x)^s$, for $tf(x)^s = f(x)\cdot f(x)^s \in \mathcal{D}_X f(x)^s$, and t commutes with the action of $\mathcal{D}_X$.

Set

$$\begin{aligned}\mathcal{N}_f &= \mathcal{D}_X[s]f(x)^s,\\ \mathcal{N}_f' &= \mathcal{D}_X f(x)^s.\end{aligned}$$

Then $\mathcal{N}_f'$ is a coherent $\mathcal{D}_X$-module, $\mathcal{N}_f = \sum_{k=0}^{\infty} s^k\mathcal{D}_X f(x)^s$, and $\sum_{k=0}^{m} s^k\mathcal{D}_X f(x)^s$ is a coherent $\mathcal{D}_X$-module for any m. Indeed, since $I_f \cap (\sum_{k=0}^{m} s^k F_l(\mathcal{D}_X))$ are coherent $\mathcal{O}_X$-modules for all l, it follows from Theorem A.29 that $I_f\cap(\sum_{k=0}^{m} s^k\mathcal{D}_X)$ is a coherent $\mathcal{D}_X$-module.

Hence $\mathcal{N}_f$ is a union of coherent $\mathcal{D}_X$-submodules, but its coherence is not obvious. (We will prove its coherency later.) Let $\mathbb{C}[s,t]$ denote the ring generated by s and t with the defining relation (6.3). Then $\mathbb{C}[s,t] = \bigoplus_{i,j\in\mathbb{Z}_{\geq 0}}\mathbb{C}s^it^j = \bigoplus_{i,j\in\mathbb{Z}_{\geq 0}}\mathbb{C}t^js^i$. Let $\mathcal{D}_X[s,t]$ denote the ring $\mathcal{D}_X\otimes\mathbb{C}[s,t]$. Then $\mathcal{N}_f$ is a $\mathcal{D}_X[s,t]$-module, and $\mathcal{N}_f/t\mathcal{N}_f$ is a $\mathcal{D}_X[s]$-module generated by $f(x)^s \bmod t\mathcal{N}_f$. Thus the b-function $b_f(s)$ of f is understood to be the minimal polynomial of $s \in \operatorname{End}_{\mathcal{D}_X}(\mathcal{N}_f/t\mathcal{N}_f)$.

Let us consider the structure of $\mathcal{N}_f$. A coherent $\mathcal{D}_X$-module $\mathcal{M}$ is called a *subholonomic* $\mathcal{D}_X$-module if $\dim \operatorname{Ch}(\mathcal{M}) \leq \dim X + 1$.

LEMMA 6.3. *Let $df(x_0) \neq 0$. Then, in a neighborhood of x_0, $\mathcal{N}'_f = \mathcal{N}_f$, $\mathcal{N}_f$ is subholonomic, and*

(1) $\mathcal{N}_f \cong \mathcal{D}_X \xrightarrow{f} \mathbb{C}$,

(2) $\mathcal{E}xt^k_{\mathcal{D}_X}(\mathcal{N}_f, \mathcal{D}_X) = 0$ *for* $k \neq n-1$,

(3) $\mathrm{Ch}(\mathcal{N}_f) = \{\, s\,df(x)\,;\ s \in \mathbb{C},\ x \in X \,\}$.

PROOF. Take a local coordinate system $(x_1, \cdots, x_n)$ so that $f = x_1$. The equality $\mathcal{N}_f = \mathcal{N}'_f$ follows from $x_1\partial_1 f^s = sf^s$. It is not difficult to see that $\mathcal{D}_X \cap I_f = \sum_{j=2}^{n} \mathcal{D}_X \partial_j$. Hence

$$\mathcal{D}_X \xrightarrow{f} \mathbb{C} \simeq \mathcal{D}_X/(\mathcal{D}_X \cap I_f).$$

Thus we obtain (1), from which (2) and (3) immediately follow. $\square$

Thus, when df does not vanish, the structure of $\mathcal{N}_f$ is rather easy. The following lemma is well known.

LEMMA 6.4. *There exists a neighborhood U of $f^{-1}(0)$ such that $x \in U \backslash f^{-1}(0)$ implies $df(x) \neq 0$.*

Since the b-function of $f(x)$ equals 1 outside $f^{-1}(0)$, we need to study the b-function only in a neighborhood of $f^{-1}(0)$. Hence, from now on, we assume the following:

(6.4) For any $x \in X \backslash f^{-1}(0)$, $df(x) \neq 0$.

PROPOSITION 6.5. *Under the assumption (6.4), $\mathcal{N}'_f$ is a subholonomic $\mathcal{D}_X$-module.*

PROOF. By the results in § 2.4,

$$\tilde{\mathcal{N}} := \{\, u \in \mathcal{N}'_f\,;\ \operatorname{codim}\mathrm{Ch}(\mathcal{D}_X u) \geq n-1 \,\}$$

is coherent and subholonomic. Since $\mathcal{N}_f|_{X\backslash f^{-1}(0)}$ is subholonomic, $\tilde{\mathcal{N}}|_{X\backslash f^{-1}(0)} = \mathcal{N}_f|_{X\backslash f^{-1}(0)}$. Hence the support of $\mathcal{N}'_f/\tilde{\mathcal{N}}$ is contained in $f^{-1}(0)$. Let u be the section of $\mathcal{N}'_f/\tilde{\mathcal{N}}$ corresponding to $f^s \in \mathcal{N}'_f$. By Hilbert's Nullstellensatz, we locally have $f^N u = 0$ for some N. Thus $f^N f^s \in \tilde{\mathcal{N}}$. Since $t^N \in \mathrm{End}_{\mathcal{D}_X}(\mathcal{N}'_f)$ is a monomorphism sending f^s to f^{s+N}, the endomorphism t^N embeds $\mathcal{N}'_f$ into $\tilde{\mathcal{N}}$. Hence $\mathrm{Ch}(\mathcal{N}'_f) \subset \mathrm{Ch}(\tilde{\mathcal{N}})$, and we conclude that $\mathcal{N}'_f$ is subholonomic. $\square$

LEMMA 6.6. *If there exists a vector field v such that $vf = f$, then f (locally) has a b-function.*

PROOF. Since $vf^s = sf^s$, we have $\mathcal{N}_f = \mathcal{N}'_f$. Hence $\mathcal{N}_f$ is coherent and subholonomic. By Proposition 2.14, $\mathcal{N}_f/t\mathcal{N}_f$ is a holonomic $\mathcal{D}_X$-module. By Theorem 4.45, $\mathcal{E}nd_{\mathcal{D}_X}(\mathcal{N}_f/t\mathcal{N}_f)_{x_0}$ is finite-dimensional for any $x_0 \in X$. Let $\tilde{s}$ be the element of $\mathcal{E}nd_{\mathcal{D}_X}(\mathcal{N}_f/t\mathcal{N}_f)_{x_0}$ induced from s. Then there exists a nonzero polynomial b such that $b(\tilde{s}) = 0$, or equivalently $b(s)f^s \in t\mathcal{N}_f$. Thus we conclude the existence of the b-function of f. $\square$

THEOREM 6.7. *Every $f(x)$ (locally) has a b-function.*

PROOF. Define a holomorphic function $g(w,x)$ on $Z = \mathbb{C} \times X$ by $wf(x)$. Since $w\frac{\partial}{\partial w}g = g$, by Lemma 6.6, there exist $P \in \mathcal{D}_Z[s]$ and a nonzero $b(s) \in \mathbb{C}[s]$ such that

$$P(s, w, x, \partial_w, \partial_x)g^{s+1} = b(s)g^s.$$

In a neighborhood of $w = 1$, since we can write

$$P = \tilde{P}(s, w, x, \partial_x) + Q(s, w, x, \partial_w, \partial_x) \cdot (\partial_w - (s+1)w^{-1}),$$

we have

$$\tilde{P}(s, w, x, \partial_x)g^{s+1} = b(s)g^s$$

because $\frac{\partial}{\partial w}g^{s+1} = (s+1)w^{-1}g^{s+1}$. By putting $w = 1$, we obtain

$$\tilde{P}(s, 1, x, \partial_x)f(x)^{s+1} = b(s)f(x)^s.$$

$\square$

6.3. Rationality of b-functions

In this section, we make a detailed study of properties of b-functions and $\mathcal{N}_f = \mathcal{D}_X[s]f(x)^s$ by using Hironaka's desingularization theorem. We assume (6.4) in the sequel.

Set

$$W_f := \{(x, s\,d\log f(x));\, f(x) \neq 0,\ s \in \mathbb{C}\}^-,$$

where $^-$ indicates the closure. Since $\{\,(x, s\,d\log f(x))\,;\, f(x) \neq 0\,\}$ is an $(n+1)$-dimensional involutive submanifold, W_f is an $(n+1)$-dimensional involutive closed analytic subset of T^*X. We prove the following two theorems in this section:

THEOREM 6.8. *The $\mathcal{D}_X$-module $\mathcal{N}_f$ is coherent, and* $\mathrm{Ch}(\mathcal{N}_f) = W_f$.

THEOREM 6.9. *For every holomorphic function $f(x)$, the roots of its b-function are negative and rational.*

Clearly $\mathrm{Ch}(\mathcal{N}_f') \supset W_f$, for this holds on $X \setminus f^{-1}(0)$ by Lemma 6.3.

We use Hironaka's desingularization theorem to reduce the proofs of these theorems to a case where $f(x)$ has a very simple form. First we consider the simple case.

LEMMA 6.10. *The b-function of a monomial in n variables*

$$f(x) = x_1^{a_1} \cdots x_r^{a_r} \quad (1 \le r \le n,\ a_1, \ldots, a_r \ge 1)$$

can be divided by $\prod_{j=1}^{r} \prod_{\nu=1}^{a_j} \left(s + \frac{\nu}{a_j}\right)$. *(In fact, they coincide, although we do not need this fact.)*

PROOF. The assertion follows from the equality

$$\left(\prod_{j=1}^{r} \partial_{x_j}^{a_j}\right) f(x)^{s+1} = \prod_{j=1}^{r} \prod_{k=0}^{a_j - 1} (a_j(s+1) - k) f(x)^s. \tag{6.5}$$

□

LEMMA 6.11. *Let $f(x)$ be the one given in Lemma 6.10. Then:*

(1) $\mathcal{N}_f = \mathcal{N}_f'$.

(2) $I_f \cap \mathcal{D}_X = \sum_{j,k} \mathcal{D}_X \left(\frac{1}{a_j} x_j \partial_j - \frac{1}{a_k} x_k \partial_k\right) + \sum_{l=r+1}^{n} \mathcal{D}_X \partial_l$.

(3)

$$\mathrm{Ch}(\mathcal{N}_f) = W_f = \left\{ (x, \xi);\ \begin{array}{l} \frac{1}{a_1} x_1 \xi_1 = \cdots = \frac{1}{a_r} x_r \xi_r, \\ \xi_{r+1} = \cdots = \xi_n = 0 \end{array} \right\}.$$

PROOF. (1) follows from $sf^s = a_1^{-1} x_1 \partial_1 f^s$. The following facts are easy to prove; we omit their proofs:

$$W_f = \left\{ (x, \xi);\ \begin{array}{l} \frac{1}{a_1} x_1 \xi_1 = \cdots = \frac{1}{a_r} x_r \xi_r, \\ \xi_{r+1} = \cdots = \xi_n = 0 \end{array} \right\}, \tag{6.6}$$

$$\begin{aligned} &\{ a(x,\xi) \in \mathcal{O}_X[\xi_1, \cdots, \xi_n];\ a|_{W_f} = 0 \} \\ &= \left\{ a(x,\xi) \in \mathcal{O}_X[\xi_1, \cdots, \xi_n];\ a\left(x, s\frac{\partial f}{\partial x_1}, \cdots, s\frac{\partial f}{\partial x_n}\right) = 0 \right\} \\ &= \sum_{1 \le j,k \le r} \mathcal{O}_X[\xi_1, \cdots, \xi_n] \left(\frac{1}{a_j} x_j \xi_j - \frac{1}{a_k} x_k \xi_k\right) \\ &\qquad + \sum_{l=r+1}^{n} \mathcal{O}_X[\xi_1, \cdots, \xi_n] \xi_l. \end{aligned} \tag{6.7}$$

Let I' be the right hand side of (2).

Clearly $I_f \cap \mathcal{D}_X \supset I'$. We show that $I_f \cap \mathcal{D}_X \subset I'$. Let P be a differential operator of order at most m in I_f. We prove $P \in I'$ by induction on m. The principal symbol $\sigma_m(P)(x, \xi)$ of P satisfies $\sigma_m(P)(x, sdf) = 0$. Hence by (6.7) we can write

$$\sigma_m(P)(x,\xi) = \sum_{1\le i,j\le r} \varphi_{ij}(x,\xi)\left(\frac{1}{a_i}x_i\xi_i - \frac{1}{a_j}x_j\xi_j\right) + \sum_{l=r+1}^{n} \psi_l(x,\xi)\xi_l.$$

Thus

$$Q := P - \sum_{1\le i,j\le r} \varphi_{ij}(x,\partial)\left(\frac{1}{a_i}x_i\partial_i - \frac{1}{a_j}x_j\partial_j\right) - \sum_{l=r+1}^{n} \psi_l(x,\partial)\partial_l$$

is of order at most $(m-1)$ and belongs to I_f. By the induction hypothesis, $Q \in I'$, and in turn $P \in I'$. We have thus proved (2). The assertion (3) immediately follows from (2). □

We have thus proved Theorems 6.8 and 6.9 when $f(x)$ is a monomial. For a general $f(x)$, we prove it by using the following *desingularization theorem* due to Hironaka.

THEOREM 6.12. *Let $f(x)$ be a nonzero holomorphic function on X. Then there exists a projective holomorphic map $p : X' \to X$ satisfying the following conditions:*

(1) *X' is a nonsingular manifold.*
(2) *$X' \backslash p^{-1}f^{-1}(0) \to X \backslash f^{-1}(0)$ is an isomorphism.*
(3) *For every $q \in p^{-1}f^{-1}(0)$, there exists a local coordinate system $(x_1, \cdots, x_n)$ of X' with q as the origin such that $f \circ p = x_1^{a_1} \cdots x_r^{a_r}$ $(a_j > 0)$.*

Take $p : X' \to X$ as above, and let $f' = f \circ p$. The roots of the b-function $b_{f'}(s)$ of f' are negative and rational by Lemma 6.10. We estimate $b_f(s)$ by using $b_{f'}(s)$. The $\mathcal{D}_{X'}$-module $\mathcal{N}_{f'} := \mathcal{D}_{X'}[s]f'(x)^s$ is a subholonomic $\mathcal{D}_{X'}$-module. Hence, by Theorem 4.25, all cohomology modules of $\tilde{\mathcal{N}} = \mathbb{D}p_*\mathcal{N}_{f'}$ are coherent. Furthermore, they are $\mathcal{D}_X[s,t]$-modules.

LEMMA 6.13.

$$\operatorname{Ch} \mathcal{H}^j(\tilde{\mathcal{N}}) \subset W_f \cup \Lambda,$$

where Λ is an isotropic analytic subset contained in $\pi_X^{-1}f^{-1}(0)$.

PROOF. By Theorem 4.27,

$$\mathrm{Ch}\,\mathcal{H}^j(\tilde{\mathcal{N}}) \subset p_\pi p_d^{-1} W_{f'}.$$

Since p is an isomorphism outside $f^{-1}(0)$, we have

$$p_\pi p_d^{-1} W_{f'} \subset W_f \cup p_\pi p_d^{-1}(W_{f'} \cap \pi_{X'}^{-1} f'^{-1}(0)).$$

By Lemma 6.11, $W_{f'} \cap \pi_{X'}^{-1} f'^{-1}(0)$ is an isotropic analytic subset. Hence, by Proposition A.54, $p_\pi p_d^{-1}(W_{f'} \cap \pi_{X'}^{-1} f'^{-1}(0))$ is also an isotropic subset of T^*X. □

Note that $\Omega_X \ni \omega \mapsto p^*\omega \in \Omega_{X'}$ induces an $\mathcal{O}_{X'}$-module homomorphism $p^*\Omega_X \to \Omega_{X'}$. By tensoring $p^*\Omega_X^{\otimes -1}$ to both sides, we obtain a canonical homomorphism $\mathcal{O}_{X'} \to \Omega_{X'/X} := \Omega_{X'} \otimes \Omega_X^{\otimes -1}$. This is, of course, an isomorphism outside $f'^{-1}(0)$.

By definition, we have

$$\begin{aligned} \tilde{\mathcal{N}} &= \mathbb{R}p_*(\mathcal{D}_{X\leftarrow X'} \overset{\mathbb{L}}{\otimes}_{\mathcal{D}_{X'}} \mathcal{N}_{f'}), \\ \mathcal{D}_{X\leftarrow X'} &= p^{-1}\mathcal{D}_X \otimes_{p^{-1}\mathcal{O}_X} \Omega_{X'/X}. \end{aligned}$$

From $1 \in \mathcal{D}_X$ and $\mathcal{O}_{X'} \to \Omega_{X'/X}$ we obtain a $(p^{-1}\mathcal{O}_X, \mathcal{O}_{X'})$-module homomorphism

$$\mathcal{O}_{X'} \to p^{-1}\mathcal{D}_X \otimes_{p^{-1}\mathcal{O}_X} \Omega_{X'/X} = \mathcal{D}_{X\leftarrow X'},$$

while from $f'^s \in \mathcal{N}_{f'}$ we obtain an $\mathcal{O}_{X'}$-module homomorphism $\mathcal{O}_{X'} \to \mathcal{N}_{f'}$. By taking the tensor product of the above two, we obtain a morphism in $D^b(p^{-1}\mathcal{O}_X)$,

$$\mathcal{O}_{X'} \to \mathcal{D}_{X\leftarrow X'} \overset{\mathbb{L}}{\otimes}_{\mathcal{D}_{X'}} \mathcal{N}_{f'}. \tag{6.8}$$

We have thus obtained a morphism in $D^b(\mathcal{O}_X)$,

$$\mathcal{O}_X \to \mathbb{R}p_*(\mathcal{O}_{X'}) \to \mathbb{R}p_*(\mathcal{D}_{X\leftarrow X'} \overset{\mathbb{L}}{\otimes}_{\mathcal{D}_{X'}} \mathcal{N}_{f'}) = \tilde{\mathcal{N}}.$$

This yields a section u of $\mathcal{H}^0(\tilde{\mathcal{N}})$. Clearly $\tilde{\mathcal{N}}$ and $\mathcal{N}_f$ are isomorphic outside $f^{-1}(0)$, and u corresponds to $f(x)^s$ by this isomorphism. Furthermore, $tu = f(x)u$. Indeed, denoting by η the morphism (6.8), we see that

$$\begin{array}{ccc} \mathcal{O}_{X'} & \xrightarrow{\ \eta\ } & \mathcal{D}_{X\leftarrow X'} \overset{\mathbb{L}}{\otimes}_{\mathcal{D}_{X'}} \mathcal{N}_{f'} \\ {\scriptstyle p^*f}\downarrow & & \downarrow{\scriptstyle t} \\ \mathcal{O}_{X'} & \xrightarrow{\ \eta\ } & \mathcal{D}_{X\leftarrow X'} \overset{\mathbb{L}}{\otimes}_{\mathcal{D}_{X'}} \mathcal{N}_{f'} \end{array}$$

is commutative.

Put $J = \{ P \in \mathcal{D}_X[s] \, ; \, Pu = 0 \}$. Then, since J and I_f coincide outside $f^{-1}(0)$, J is contained in $I_f = \{ P \in \mathcal{D}_X[s] \, ; \, Pf^s = 0 \}$.

We have thus obtained the following diagram of $\mathcal{D}_X[s,t]$-modules:

$$\begin{tikzcd} \mathcal{D}_X[s]u = \mathcal{D}_X[s]/J \arrow[r, hook] \arrow[d, "\beta"', two heads] & \mathcal{H}^0(\tilde{\mathcal{N}}) \\ \mathcal{N}_f . & \end{tikzcd}$$

Outside $f^{-1}(0)$, the three $\mathcal{D}_X$-modules above are isomorphic. Since $\mathcal{H}^0(\tilde{\mathcal{N}})$ is a coherent $\mathcal{D}_X$-module, $\mathcal{D}_X[s]/J$ and $\mathcal{N}_f$ are also coherent. Moreover,

$$\mathrm{Ch}(\mathcal{N}_f) \subset \mathrm{Ch}(\mathcal{D}_X[s]/J) \subset \mathrm{Ch}\, \mathcal{H}^0(\tilde{\mathcal{N}}) .$$

By Lemma 6.13, there exists an isotropic set Λ such that

$$\mathrm{Ch}\, \mathcal{H}^0(\tilde{\mathcal{N}}) \subset W_f \cup \Lambda.$$

Hence $\mathrm{Ch}(\mathcal{N}_f) \subset W_f \cup \Lambda$.

Since we have $\mathrm{Supp}(T_n(\mathcal{N}_f)) \subset f^{-1}(0)$ and $\Gamma_{f^{-1}(0)}(\mathcal{N}_f) = 0$, we obtain $T_n(\mathcal{N}_f) = 0$. Hence, by Theorem 2.24, $\mathrm{Ch}(\mathcal{N}_f)$ is purely $(n+1)$-dimensional. From this, $\mathrm{Ch}(\mathcal{N}_f) \subset W_f$ follows. We have thus proved Theorem 6.8.

Next we prove Theorem 6.9. Since $b_{f'}(s)\mathcal{N}_{f'} \subset t\mathcal{N}_{f'}$, there exists a $\mathcal{D}_{X'}$-module homomorphism $\varphi : \mathcal{N}_{f'} \to \mathcal{N}_{f'}$ such that

$$b_{f'}(s) = t\varphi. \tag{6.9}$$

Applying the functor $\mathcal{H}^0\mathbb{D}p_*$ to this, we obtain a commutative diagram

$$\begin{tikzcd} \mathcal{H}^0(\tilde{\mathcal{N}}) \arrow[rr, "b_{f'}(s)"] \arrow[dr, "\tilde{\varphi}"'] & & \mathcal{H}^0(\tilde{\mathcal{N}}), \\ & \mathcal{H}^0(\tilde{\mathcal{N}}) \arrow[ur, "t"'] & \end{tikzcd}$$

where $\tilde{\varphi} = \mathcal{H}^0(\mathbb{D}p_*(\varphi))$.

Let $\mathcal{M} = \mathcal{H}^0(\tilde{\mathcal{N}})/\mathcal{D}_X[s]u$. Then $\mathcal{M}$ is a coherent $\mathcal{D}_X$-module and has a structure of $\mathcal{D}_X[s,t]$-module. Since its support is contained in $f^{-1}(0)$, we have

$$\mathrm{Ch}(\mathcal{M}) \subset (W_f \cap f^{-1}(0)) \cup \Lambda.$$

Hence $\mathcal{M}$ is a holonomic $\mathcal{D}_X$-module.

LEMMA 6.14. *Let $\mathcal{M}$ be a holonomic $\mathcal{D}_X$-module and a $\mathcal{D}_X[s,t]$-module. Then the action of t on $\mathcal{M}$ is nilpotent, i.e., $t^m = 0$ for m large enough.*

PROOF. For $x \in X$, $V := \mathcal{E}nd_{\mathcal{D}_X}(\mathcal{M})_x$ is a finite-dimensional vector space. Since $t, s \in \mathrm{End}(V)$ satisfy $(s+1)t = ts$, the action of t on V is nilpotent. □

By this lemma, $t^m \mathcal{H}^0(\tilde{\mathcal{N}}) \in \mathcal{D}_X[s]u$ for m large enough.

We have $b_{f'}(s)b_{f'}(s+1)u = b_{f'}(s+1)t\tilde{\varphi}(u) = tb_{f'}(s)\tilde{\varphi}(u) = t^2\tilde{\varphi}^2(u)$. Repeat this to obtain

$$b_{f'}(s)b_{f'}(s+1)\cdots b_{f'}(s+m)u = t^{m+1}\tilde{\varphi}^{m+1}(u).$$

Note that $t^m\tilde{\varphi}^{m+1}(u) \in \mathcal{D}_X[s]u$, and apply $\beta : \mathcal{D}_X[s]u \to \mathcal{N}_f$ to obtain

$$b_{f'}(s)\cdots b_{f'}(s+m)f(x)^s \in t\mathcal{N}_f.$$

Hence $b_f(s)$ divides $b_{f'}(s)\cdots b_{f'}(s+m)$. Thus Theorem 6.9 follows from the result that the roots of $b_{f'}(s)\cdots b_{f'}(s+m)$ are negative and rational. We have thus completed the proof of Theorems 6.8 and 6.9.

6.4. b-functions in the Case of Quasi-homogeneous Isolated Singularities

We call $x_0 \in f^{-1}(0)$ an *isolated singularity* if there exists a neighborhood U of x_0 such that $df(x_0) = 0$ and $df(x) \neq 0$ $(x \in U \setminus \{x_0\})$, and a *quasi-homogeneous isolated singularity* if in addition there exists a vector field v such that $vf = f$. As to quasi-homogeneous isolated singularities, the following is known.

THEOREM 6.15 (Saito [SaitoK]). *Let x_0 be a quasi-homogeneous isolated singularity of $f(x)$. Then there exists a local coordinate system with x_0 as the origin such that*

$$\sum a_i x_i \frac{\partial}{\partial x_i} f = f \quad (a_i \in \mathbb{Q}_{>0}).$$

Hence we may assume $v = \sum a_i x_i \frac{\partial}{\partial x_i}$. Then $f(x)$ is a linear combination of monomials $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ satisfying $\sum a_i \alpha_i = 1$. (There are only finitely many such α.)

Let us calculate the b-function of such an f. Let $b(s)$ be the b-function of $f(x)$. Then there exists $P \in \mathcal{D}_X$ such that $Pf^{s+1} = b(s)f^s$. Put $s = -1$ to obtain

$$fP(1) = b(-1). \tag{6.10}$$

Evaluate both sides at $x = 0$ to see that $b(-1) = 0$. Hence we have a polynomial b' with $b(s) = (s+1)b'(s)$. In turn, we have $P(1) = 0$ by (6.10), and hence $P \in \sum_{i=1}^n \mathcal{D}_X \partial_i$. Let $P = \sum_{i=1}^n Q_i \partial_i$. Then, since

$$\partial_i f^{s+1} = (s+1) f_i f^s \qquad (f_i := \partial f / \partial x_i),$$

we have

$$\sum_i Q_i f_i f^s = b'(s) f^s.$$

Conversely, if the above equality holds, then we have $Pf^{s+1} = (s+1)b'(s)f^s$ for $P = \sum Q_i \partial_i$.

Hence $b'(s)$ is minimal among the polynomials satisfying $b'(s)f^s \in \sum_i \mathcal{D}_X f_i f^s$. Let

$$\mathcal{M} = \mathcal{D}_X f^s / (\sum_i \mathcal{D}_X f_i f^s).$$

Since $v(f_i) = (1 - a_i) f_i$, we have

$$(v - (1 - a_i) - s) f_i f^s = 0.$$

Thus $\sum \mathcal{D}_X f_i f^s$ is a $\mathcal{D}_X[s]$-submodule of $\mathcal{N}_f = \mathcal{D}_X f^s$, and $\mathcal{M}$ is a $\mathcal{D}_X[s]$-module. The condition $b'(s)f^s \in \sum \mathcal{D}_X f_i f^s$ is equivalent to the condition $b'(s)\mathcal{M} = 0$. By the assumption on f, the support of $\mathcal{M}$ is the origin. Hence, by Theorem 4.30, $\mathcal{M} = \mathcal{B}_{\{0\}|X} \otimes V$ for a finite-dimensional vector space V. Thus $V = \mathrm{Hom}_{\mathcal{D}_X}(\mathcal{B}_{\{0\}|X}, \mathcal{M})$ has a $\mathbb{C}[s]$-module structure. The condition for $b'(s)$ is hence rewritten as $b'(s)|_V = 0$. Since $\Omega_X \otimes_{\mathcal{D}_X} \mathcal{B}_{\{0\}|X} = \mathbb{C}$, we have $V \cong \Omega_X \otimes_{\mathcal{D}_X} \mathcal{M}$.

PROPOSITION 6.16.

$$\mathcal{D}_X \cap I_f := \{ P \in \mathcal{D}_X \, ; \, Pf^s = 0 \} = \sum_{i,j} \mathcal{D}_X (f_i \partial_j - f_j \partial_i) \, .$$

PROOF. For $P \in F_m(\mathcal{D}_X) \cap I_f$, we have $\sigma_m(P)(x, df) = 0$. The following lemma holds.

LEMMA 6.17. *Let $\varphi_1, \ldots, \varphi_r \in \mathcal{O}_X$. Suppose that the codimension of $\varphi_1^{-1}(0) \cap \cdots \cap \varphi_r^{-1}(0)$ equals r. Then*

$$\begin{aligned}\{ \psi(x,\xi) \in \mathcal{O}_X[\xi_1, \ldots, \xi_r] \, ; \, \psi(x, s\varphi_1, \ldots, s\varphi_r) = 0 \} \\ = \sum_{1 \le i,j \le r} \mathcal{O}_X[\xi_1, \ldots, \xi_r](\varphi_i \xi_j - \varphi_j \xi_i).\end{aligned}$$

By this lemma, we can write

$$\sigma_m(P) = \sum_{1 \le i,j \le r} a_{ij}(x,\xi)(f_i \xi_j - f_j \xi_i),$$

where $a_{ij}(x,\xi)$ is homogeneous of degree $(m-1)$ in ξ. Hence $R = P - \sum a_{ij}(x,\partial)(f_i\partial_j - f_j\partial_i) \in F_{m-1}(\mathcal{D}_X)$, and $Rf^s = 0$. By induction on m, we have $R \in \sum \mathcal{D}_X(f_i\partial_j - f_j\partial_i)$. Hence we conclude that $P \in \sum \mathcal{D}_X(f_i\partial_j - f_j\partial_i)$. □

Since $f_i\partial_j - f_j\partial_i = \partial_j f_i - \partial_i f_j$,

$$\mathcal{M} \cong \frac{\mathcal{D}_X}{\sum \mathcal{D}_X(\partial_j f_i - \partial_i f_j) + \sum \mathcal{D}_X f_i} = \frac{\mathcal{D}_X}{\sum \mathcal{D}_X f_i}.$$

In this expression, s acts by $1 \mapsto v$.

Hence

$$V \cong \Omega_X \otimes_{\mathcal{D}_X} \mathcal{M} = \Omega_X/(\sum \Omega_X f_i).$$

Here s acts by $\omega \mapsto \omega v = -L_v\omega$. Letting $dx = dx_1 \wedge \cdots \wedge dx_n$, we have an isomorphism

$$\mathcal{O}_X/(\sum \mathcal{O}_X f_i) \xrightarrow{\sim} V \quad (u \mapsto udx).$$

Let us consider how s acts on $\mathcal{O}_X/(\sum \mathcal{O}_X f_i)$ under this correspondence. We have $-L_v(udx) = -v(u)dx - uL_v(dx)$ for $u \in \mathcal{O}_X$, and

$$L_v(dx) = \sum_{i=1}^{n} dx_1 \cdots dx_{i-1} dv(x_i) dx_{i+1} \cdots dx_n = \left(\sum_{i=1}^{n} a_i\right) dx.$$

Hence s acts on $\mathcal{O}_X/(\sum \mathcal{O}_X f_i)$ as $-(v + \sum_{i=1}^{n} a_i)$. Since the vector space $\mathcal{O}_X/(\sum \mathcal{O}_X f_i)$ is spanned by monomials x^β, and since each x^β is an eigenvector of s, s acts on $\mathcal{O}_X/(\sum \mathcal{O}_X f_i)$ diagonally. Let λ_j be the distinct eigenvalues of s. Then the minimal polynomial of s is the product of distinct $s - \lambda_j$. Summarizing this, we have the following theorem.

THEOREM 6.18. *Let $f(x)$ be a polynomial with a quasi-homogeneous isolated singularity; $\sum a_i x_i \partial_i f = f$. Let λ_j be the eigenvalues of $v = \sum a_i x_i \partial_i$ on the finite-dimensional vector space*

$$\mathcal{O}_X/(\sum_{i=1}^{n} \mathcal{O}_X(\partial f/\partial x_i)).$$

Then the b-function b_f of $f(x)$ is given by

$$b_f(s) = (s+1)\prod_j{}' \left(s + \lambda_j + \sum_{i=1}^{n} a_i\right),$$

where $\prod'$ indicates the multiplicity-free product.

EXAMPLE 6.19. Let

$$f(x) = \sum_{i=1}^{n} x_i^{m_i} \quad (m_i \geq 2)\,.$$

Then $\sum \frac{1}{m_i} x_i \partial_i f = f$. The vector space $\mathcal{O}_X/(\sum_{i=1}^n \mathcal{O}_X f_i)$ has a basis

$$x^\beta \quad (\beta = (\beta_1, \dots, \beta_n),\ 0 \leq \beta_i < m_i - 1).$$

Hence the eigenvalues of $v = \sum \dfrac{1}{m_i} x_i \partial_i$ are $\sum \dfrac{\beta_i}{m_i}$. Since $\sum \dfrac{\beta_i}{m_i} + \sum \dfrac{1}{m_i} = \sum (\beta_i + 1)/m_i$,

$$b_f(s) = (s+1) {\prod}' \left(s + \sum_{i=1}^{n} \frac{\nu_i}{m_i} \right),$$

where $\prod'$ indicates the multiplicity-free product for $\nu = (\nu_1, \dots, \nu_n)$ $(1 \leq \nu_i < m_i)$.

For example, we recover $b(s) = (s+1)(s+n/2)$ for $f(x) = \sum_{i=1}^n x_i^2$.

Let $n = 2$ and $f(x) = x_1^2 + x_2^3$. Then

$$\begin{aligned} b(s) &= (s+1)\left(s + \frac{1}{2} + \frac{1}{3}\right)\left(s + \frac{1}{2} + \frac{2}{3}\right) \\ &= (s+1)\left(s + \frac{5}{6}\right)\left(s + \frac{7}{6}\right). \end{aligned}$$

REMARK 6.20. $\dim \mathcal{O}/(\sum \mathcal{O} f_i)$ is an important invariant for f, called the Milnor number of f.

6.5. b-function and Irreducibility

Let f be a holomorphic function on X, and $b_f(s)$ its b-function. By definition, there exists a differential operator $P(s) \in \mathcal{D}_X[s]$ such that $P(s) f^{s+1} = b_f(s) f^s$. Given $\lambda \in \mathbb{C}$, set

$$\mathcal{N}_\lambda := \mathcal{D}_X / \mathcal{J}_\lambda,$$

where

$$\begin{aligned} I_f &= \{\, G(s) \in \mathcal{D}_X[s]\,;\, G(s) f(x)^s = 0 \,\}, \\ \mathcal{J}_\lambda &= \{\, G(\lambda)\,;\, G(s) \in I_f \,\} \subset \mathcal{D}_X\,. \end{aligned}$$

We denote by f^λ the section $1 \mod \mathcal{J}_\lambda$ of $\mathcal{N}_\lambda$. If $Pf^\lambda = 0$ for $P \in \mathcal{D}_X$, then there exists $Q(s) \in \mathcal{D}_X[s]$ such that

$$Pf^s = (s-\lambda)Q(s)f^s \tag{6.11}$$

in $\mathcal{N}_f$. For, by the definition, there exists $G(s) \in I_f$ such that $P = G(\lambda)$, and hence we obtain (6.11) by letting $G(s) = G(\lambda) - (s-\lambda)Q(s)$.

In this section, we investigate $\mathcal{N}_\lambda$ by using the b-function of f. From now on, we simply write $\mathcal{N}$ for $\mathcal{N}_f$.

LEMMA 6.21. *The following four conditions are equivalent:*

(1) $b_f(\lambda) \neq 0$.
(2) $\varphi_\lambda : \mathcal{N}_{\lambda+1} \to \mathcal{N}_\lambda$ $(\varphi_\lambda(f^{\lambda+1}) = f \cdot f^\lambda)$ *is injective.*
(3) $\varphi_\lambda : \mathcal{N}_{\lambda+1} \to \mathcal{N}_\lambda$ *is surjective.*
(4) $\varphi_\lambda : \mathcal{N}_{\lambda+1} \to \mathcal{N}_\lambda$ *is an isomorphism.*

PROOF. Consider the following commutative diagram:

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
 & 0 \arrow[d] & 0 \arrow[d] & K_0 \arrow[d] & \\
0 \arrow[r] & \mathcal{N} \arrow[r, "t"] \arrow[d, "s-\lambda-1"'] & \mathcal{N} \arrow[r] \arrow[d, "s-\lambda"] & \mathcal{N}/t\mathcal{N} \arrow[r] \arrow[d, "s-\lambda"] & 0 \\
0 \arrow[r] & \mathcal{N} \arrow[r, "t"] \arrow[d] & \mathcal{N} \arrow[r] \arrow[d] & \mathcal{N}/t\mathcal{N} \arrow[r] \arrow[d] & 0 \\
 & \mathcal{N}_{\lambda+1} \arrow[r, "\varphi_\lambda"] \arrow[d] & \mathcal{N}_\lambda \arrow[r] \arrow[d] & K_1 \arrow[d] & \\
 & 0 & 0 & 0, &
\end{tikzcd}$$

where each column is exact, and the first and second rows are also exact. Hence, from the snake lemma (see, for example, [T], p. 35), we obtain an exact sequence

$$0 \longrightarrow K_0 \longrightarrow \mathcal{N}_{\lambda+1} \xrightarrow{\varphi_\lambda} \mathcal{N}_\lambda \longrightarrow K_1 \longrightarrow 0. \tag{6.12}$$

Since $\mathcal{N}/t\mathcal{N}$ is holonomic, it is of finite length (Proposition 4.42). Hence the three properties of $s - \lambda : \mathcal{N}/t\mathcal{N} \to \mathcal{N}/t\mathcal{N}$: isomorphism, surjectivity, and injectivity are equivalent. We thus obtain

$$b_f(\lambda) \neq 0 \iff K_0 = K_1 = 0 \iff K_0 = 0 \iff K_1 = 0.$$

Translating this by the exact sequence (6.12), we obtain the equivalence of (1) through (4). $\square$

PROPOSITION 6.22. *If none of $\lambda - 1, \lambda - 2, \lambda - 3, \ldots$ is a root of $b(s)$, then*

$$\mathcal{N}_\lambda \to \mathcal{O}_X[f^{-1}] \overset{D}{\otimes} \mathcal{N}_\lambda$$

is an isomorphism, or equivalently $\mathbb{R}\Gamma_{[f^{-1}(0)]}(\mathcal{N}_\lambda) = 0$.

PROOF. By the assumption, $\mathcal{N}_\lambda \xrightarrow{\varphi_{\lambda-1}} \mathcal{N}_{\lambda-1} \longrightarrow \cdots$ are isomorphisms. Hence $\mathcal{O}_X[f^{-1}] \otimes_{\mathcal{O}_X} \mathcal{O}_X f^\lambda \subset \mathcal{O}_X[f^{-1}] \overset{D}{\otimes} \mathcal{N}_\lambda$ is contained in the image of $\mathcal{N}_\lambda$, and thus so is $\mathcal{D}_X(\mathcal{O}_X[f^{-1}] \otimes_{\mathcal{O}_X} \mathcal{O}_X f^\lambda) = \mathcal{O}_X[f^{-1}] \otimes_{\mathcal{O}_X} \mathcal{D}_X f^\lambda$. We have thus proved the surjectivity of $\mathcal{N}_\lambda \to \mathcal{O}_X[f^{-1}] \overset{D}{\otimes} \mathcal{N}_\lambda$.

Next we prove the injectivity of $\mathcal{N}_\lambda \to \mathcal{O}_X[f^{-1}] \overset{D}{\otimes} \mathcal{N}_\lambda$. For this, it suffices to show that $f : \mathcal{N}_\lambda \to \mathcal{N}_\lambda$ is injective. Let $fQf^\lambda = 0$. Then there exists $R(s) \in \mathcal{D}[s]$ such that $fQf^s = (s - \lambda)R(s)f^s$. For $m > 0$ large enough, there exists $T(s) \in \mathcal{D}_X[s]$ such that

$$R(s)f^m = fT(s).$$

Hence $fQf^s = (s-\lambda)fT(s)f^{s-m}$, or $(Qf^m)\cdot f^{s-m} = (s-\lambda)T(s)f^{s-m}$. In $\mathcal{N}_{\lambda-m}$, we thus have $(Qf^m)\cdot f^{\lambda-m} = 0$. Since $\mathcal{N}_\lambda \to \mathcal{N}_{\lambda-m}$ ($f^\lambda \mapsto f^m \cdot f^{\lambda-m}$) is an isomorphism, we obtain $Qf^\lambda = 0$. $\square$

Note that $\mathcal{O}_X[f^{-1}] \overset{D}{\otimes} \mathcal{N}_\lambda$ is the $\mathcal{D}_X$-module consisting of (multi-valued) functions of the form af^λ ($a \in \mathcal{O}_X[f^{-1}]$).

Let us prove a proposition dual to Proposition 6.22. We can rephrase Proposition 6.22 as follows:

(a) $\mathcal{N}_\lambda$ has no nonzero coherent submodule with support on $f^{-1}(0)$.

(b) If $\mathcal{M}$ is a holonomic module that has no nonzero coherent submodule with support on $f^{-1}(0)$, and if we are given a monomorphism $\mathcal{N}_\lambda \rightarrowtail \mathcal{M}$ that is an isomorphism on $X \setminus f^{-1}(0)$, then $\mathcal{N}_\lambda \to \mathcal{M}$ is an isomorphism.

PROPOSITION 6.23. *If none of* $\lambda, \lambda+1, \lambda+2, \dots$ *is a root of* $b(s)$, *then*

(1) $\mathcal{N}_\lambda$ *has no nonzero coherent quotient with support* $f^{-1}(0)$, *and*

(2) *if* $\mathcal{M}$ *is a holonomic module that has no nonzero coherent quotient with support on* $f^{-1}(0)$, *and if we are given an epimorphism* $\mathcal{M} \twoheadrightarrow \mathcal{N}_\lambda$ *that is an isomorphism on* $X \backslash f^{-1}(0)$, *then* $\mathcal{M} \to \mathcal{N}_\lambda$ *is an isomorphism.*

Namely, $\mathbb{R}\Gamma_{[f^{-1}(0)]}(\mathbb{D}_X \mathcal{N}_\lambda) = 0$, *or* $\mathcal{N}_\lambda \xleftarrow{\sim} \mathbb{D}_X(\mathcal{O}_X[f^{-1}] \overset{D}{\otimes} \mathbb{D}_X \mathcal{N}_\lambda)$.

PROOF. By the assumption,

$$\cdots \to \mathcal{N}_{\lambda+2} \to \mathcal{N}_{\lambda+1} \to \mathcal{N}_\lambda$$

are isomorphisms.

(1) Let $0 \longrightarrow \mathcal{L} \longrightarrow \mathcal{N}_\lambda \xrightarrow{\varphi} \mathcal{M} \longrightarrow 0$, and suppose $\mathrm{Supp}(\mathcal{M}) \subset f^{-1}(0)$. Then $f^m \varphi(f^\lambda) = 0$ for $m \gg 0$. Hence $f^m \cdot f^\lambda$ belongs to $\mathcal{L}$. Since $\mathcal{L}$ contains the image of $\mathcal{N}_{\lambda+m} \to \mathcal{N}_\lambda$, we have $\mathcal{L} = \mathcal{N}_\lambda$. We have thus proved that $\mathcal{M} = 0$.

(2) Suppose that $\psi : \mathcal{M} \twoheadrightarrow \mathcal{N}_\lambda$ satisfies the condition in (2). Take $u \in \mathcal{M}$ such that $\psi(u) = f^\lambda \in \mathcal{N}_\lambda$.

Fix any $G(s) \in \mathcal{D}_X[s]$. Then there exist $H(s) \in \mathcal{D}[s]$ and $a \in \mathbb{Z}_{\geq 0}$ such that

$$G(\lambda+n) f^n = f^{n-a} H(\lambda+n) \quad (n \geq a);$$

for example, for $G = \partial_i$, we can take $f^a\left(\partial_i + (s-\lambda) f^{-1} \frac{\partial f}{\partial x_i}\right)$ as $H(s)$. If we assume $G(s) \in I_f$ in addition, then $H(\lambda+n) f^\lambda = 0$ on $X \backslash f^{-1}(0)$. Hence $\mathrm{supp}(H(\lambda+n)u) \subset f^{-1}(0)$. Expand $H(\lambda+s) = \sum s^k H_k$ in s. Then $f^N H_k u = 0$ for $N > 0$ large enough, since $\mathrm{supp}(H_k u) \subset f^{-1}(0)$. Hence, for all $n \geq a$, $f^N H(\lambda+n) u = 0$. Thus, for $n \geq a+N$, we obtain

$$G(\lambda+n)(f^n u) = f^{n-a} H(\lambda+n) u = 0.$$

We have thus proved that, for any $G(s) \in I_f$, there exists N such that

$$G(\lambda+n)(f^n u) = 0 \quad (n \geq N).$$

Since I_f is locally finitely generated over $\mathcal{D}_X[s]$, we can take N independent of $G(s)$. Hence we can define a homomorphism ξ from $\mathcal{N}_{\lambda+N} \cong \mathcal{D}_X / \{G(\lambda+N); G(s) \in I_f\}$ to $\mathcal{M}$ by

$$\xi : f^{\lambda+N} \mapsto f^N u.$$

Furthermore, define $\tilde{\xi} : \mathcal{N}_\lambda \to \mathcal{M}$ by $\mathcal{N}_\lambda \xleftarrow{\sim} \mathcal{N}_{\lambda+N} \xrightarrow{\xi} \mathcal{M}$. Then, by $\mathcal{N}_{\lambda+N} \xrightarrow{\sim} \mathcal{N}_\lambda$, we have $\psi \circ \tilde{\xi} = \mathrm{id}_{\mathcal{N}_\lambda}$. Hence $\mathrm{Ker}(\psi)$ is a quotient of $\mathcal{M}$. Since the support of $\mathrm{Ker}(\psi)$ is contained in $f^{-1}(0)$, we obtain $\mathrm{Ker}(\psi) = 0$ by the assumption on $\mathcal{M}$. $\square$

COROLLARY 6.24. *For any* $\lambda \in \mathbb{C}$,

$$\mathcal{O}_X[f^{-1}] \overset{D}{\otimes} \mathcal{D}_X f^\lambda \simeq \mathcal{N}_{\lambda-n} \quad (n \in \mathbb{Z},\ n \gg 0),$$

$$\mathbb{D}_X(\mathcal{O}_X[f^{-1}] \overset{D}{\otimes} \mathbb{D}_X(\mathcal{D}_X f^\lambda)) \simeq \mathcal{N}_{\lambda+n} \quad (n \in \mathbb{Z},\ n \gg 0).$$

COROLLARY 6.25. *Assume that none of* $\lambda, \lambda+1, \lambda+2, \dots$ *is a root of* $b_f(s)$. *Let* $\mathcal{M}_\lambda := \mathcal{D}_X f^\lambda \subset \mathcal{O}_X[f^{-1}] \otimes \mathcal{N}_\lambda$. *Then, for every* $p \in X$, $(\mathcal{M}_\lambda)_p$ *is an irreducible* $\mathcal{D}_{X,p}$*-module.*

PROOF. Let $\mathcal{L}$ be a nonzero $\mathcal{D}_X$-submodule of $\mathcal{M}_\lambda$. Then the support of $\mathcal{M}_\lambda/\mathcal{L}$ is contained in $f^{-1}(0)$. By Proposition 6.23, we see that $\mathcal{M}_\lambda/\mathcal{L} = 0$. $\square$

CHAPTER 7

Ring of Formal Microdifferential Operators

7.1. Microlocalization

As we have seen in § 2.2, the characteristic variety of $\mathcal{D}/\mathcal{D}P$ for a differential operator P is the zero set of $\sigma_m(P)$. Hence, similarly to localization in algebraic geometry, it is natural to consider an extension ring $\mathcal{E}$ of $\pi^{-1}\mathcal{D}$ in which P^{-1} exists at points of the cotangent bundle where $\sigma_m(P)$ does not vanish. If we consider such an extension ring $\mathcal{E}$, then $\mathcal{E} \otimes_{\pi^{-1}\mathcal{D}} \pi^{-1}(\mathcal{D}/\mathcal{D}P)$ clearly vanishes outside the zero set of $\sigma_m(P)$. In this section, we construct such an extension ring $\mathcal{E}$ of $\pi^{-1}\mathcal{D}$. In this book, we use total symbols to construct $\mathcal{E}$. For a more natural construction, see [SKK].

Let us take a local coordinate system x of a manifold X. For a differential operator $P = \sum a_\alpha(x)\partial_x^\alpha$, set

$$p_k(x,\xi) = \sum_{|\alpha|=k} a_\alpha(x)\xi^\alpha$$

and call $\{p_k(x,\xi)\}_k$ the *total symbol* of P. This is the decomposition of the total symbol defined in §2. 1 into its homogeneous components.

We have seen in § 2.1 that the total symbol $\{r_k(x,\xi)\}_k$ of the product R of two differential operators P and Q with total symbols $\{p_k(x,\xi)\}_k$ and $\{q_k(x,\xi)\}_k$ is given by

$$r_k(x,\xi) = \sum_{k=i+j-|\alpha|} \frac{1}{\alpha!}(\partial_\xi^\alpha p_i(x,\xi))(\partial_x^\alpha q_j(x,\xi)). \tag{7.1}$$

Since P and Q are differential operators, $p_k(x,\xi)$ and $q_k(x,\xi)$ are polynomials in ξ. Equation (7.1) itself however makes sense when $p_k(x,\xi)$ and $q_k(x,\xi)$ are only holomorphic in (x,ξ). Thus we shall extend the ring $\mathcal{D}_X$ by imposing only the condition that $p_k(x,\xi)$ is homogeneous of degree k.

From now on, for a manifold X, let $\mathcal{O}_{T^*X}(m)$ denote the sheaf of holomorphic functions homogeneous of degree m along the fibers of T^*X, i.e., the sheaf of holomorphic functions f satisfying

$$\sum_{i=1}^{n} \xi_i \frac{\partial}{\partial \xi_i} f(x,\xi) = mf(x,\xi).$$

DEFINITION 7.1. Let $\widehat{\mathcal{E}}_{\mathbb{C}^n}(m)$ denote the sheaf on $T^*\mathbb{C}^n = \mathbb{C}^n \times \mathbb{C}^n$ defined as follows: For an open set Ω in $T^*\mathbb{C}^n$,

$\widehat{\mathcal{E}}_{\mathbb{C}^n}(m)(\Omega)$

$= \{\{p_k(x,\xi)\}_{k\in\mathbb{Z}}; p_k(x,\xi)$ satisfies the following (1) and (2)$\}$:

(1) $p_k(x,\xi) \in \Gamma(\Omega;\ \mathcal{O}_{T^*\mathbb{C}^n}(k))$,

(2) $p_k(x,\xi) = 0 \quad (k > m)$.

Clearly, $\widehat{\mathcal{E}}_{\mathbb{C}^n}(m)$ is a sheaf of $\mathbb{C}$-vector spaces, and $\widehat{\mathcal{E}}_{\mathbb{C}^n}(m-1) \subset \widehat{\mathcal{E}}_{\mathbb{C}^n}(m)$. Let $\sigma_m : \widehat{\mathcal{E}}_{\mathbb{C}^n}(m) \to \mathcal{O}_{T^*\mathbb{C}^n}(m)$ be the homomorphism that maps $\{p_k(x,\xi)\}_k$ to $p_m(x,\xi)$. Then we have the following exact sequence:

$$0 \longrightarrow \widehat{\mathcal{E}}_{\mathbb{C}^n}(m-1) \longrightarrow \widehat{\mathcal{E}}_{\mathbb{C}^n}(m) \xrightarrow{\sigma_m} \mathcal{O}_{T^*\mathbb{C}^n}(m) \longrightarrow 0. \tag{7.2}$$

Set $\widehat{\mathcal{E}}_{\mathbb{C}^n} = \bigcup_m \widehat{\mathcal{E}}_{\mathbb{C}^n}(m)$.

DEFINITION 7.2. A section of $\widehat{\mathcal{E}}_{\mathbb{C}^n}$ is called a *formal microdifferential operator*.

Next define the multiplication in $\widehat{\mathcal{E}}_{\mathbb{C}^n}$ by equation (7.1). Since $p_i(x,\xi) = 0$ $(i \gg 0)$ and $q_j(x,\xi) = 0$ $(j \gg 0)$, the right hand side of equation (7.1) is a finite sum for every k.

LEMMA 7.3. (1) *$\widehat{\mathcal{E}}_{\mathbb{C}^n}$ is a sheaf of rings.*

(2) $\widehat{\mathcal{E}}_{\mathbb{C}^n}(m_1) \cdot \widehat{\mathcal{E}}_{\mathbb{C}^n}(m_2) \subset \widehat{\mathcal{E}}_{\mathbb{C}^n}(m_1+m_2)$.

(3) *If $P \in \widehat{\mathcal{E}}_{\mathbb{C}^n}(m_1)$ and $Q \in \widehat{\mathcal{E}}_{\mathbb{C}^n}(m_2)$, then*

$$[P,Q] := PQ - QP \in \widehat{\mathcal{E}}_{\mathbb{C}^n}(m_1+m_2-1)$$

and

$$\sigma_{m_1+m_2-1}([P,Q]) = \{\sigma_{m_1}(P), \sigma_{m_2}(Q)\}.$$

Among them, only the associativity of the multiplication is not obvious, but this can be reduced to the case of $\mathcal{D}_{\mathbb{C}^n}$ by Lemma 7.5. Although a direct proof is not hard, we omit it and leave it to the reader.

7.2. Formal Microdifferential Operators on Manifolds

In the previous section, we defined formal microdifferential operators by using a coordinate system. For a manifold X, we patch them together to define a sheaf on T^*X. First we show the following for a differential operator $P(x, \partial) = \{p_k(x, \xi)\}_k$.

LEMMA 7.4. *Let $f(x)$ be a holomorphic function defined on an open set in $\mathbb{C}^n$, and $\varphi(t)$ a holomorphic function in a variable t. Then*

$$P(x, \partial)\varphi(f(x)) = \sum_k A_k(x)\varphi^{(k)}(f(x)),$$

where $\varphi^{(k)}(t) = \left(\frac{d}{dt}\right)^k \varphi(t)$, $\operatorname{grad} f = \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}\right)$, and

$$A_k(x) = \sum_{\substack{s \in \mathbb{Z}_{\geq 0} \\ \alpha_1, \cdots, \alpha_s \in \mathbb{Z}^n_{\geq 0} \\ |\alpha_j| \geq 2 \ (\forall j = 1, \cdots, s) \\ l = k + \sum_{j=1}^s (|\alpha_j| - 1)}} \frac{\prod_{j=1}^s (\partial_x^{\alpha_j} f(x))}{s!\, \alpha_1! \cdots \alpha_s!} \partial_\xi^{\alpha_1 + \cdots + \alpha_s} p_l(x, \operatorname{grad} f). \tag{7.3}$$

PROOF. We may assume that $P(x, \partial)$ does not contain x. Thus it suffices to prove the equation formally in t for $P(\partial) = e^{t\langle y, \partial \rangle}$. Then

$$\begin{aligned} & P(\partial)\varphi(f(x)) \\ &= \varphi(f(x + ty)) \\ &= \varphi\Big(f(x) + t\sum_{j=1}^n y_j \frac{\partial f}{\partial x_j} + \sum_{|\alpha| \geq 2} \frac{(ty)^\alpha}{\alpha!} \partial_x^\alpha f(x)\Big) \\ &= \sum_k \frac{1}{k!}\Big(t\sum_{j=1}^n y_j \frac{\partial f}{\partial x_j} + \sum_{|\alpha| \geq 2} \frac{(ty)^\alpha}{\alpha!} \partial_x^\alpha f(x)\Big)^k \varphi^{(k)}(f(x)). \end{aligned} \tag{7.4}$$

Hence

$$A_k = \frac{1}{k!}\Big(t\sum y_j \frac{\partial f}{\partial x_j} + \sum_{|\alpha| \geq 2} \frac{(ty)^\alpha}{\alpha!} \partial_x^\alpha f(x)\Big)^k.$$

Since $\partial_\xi^\alpha P(\xi) = (ty)^\alpha e^{t\langle y, \xi \rangle}$,

$$\partial_\xi^\alpha p_{l+|\alpha|}(\xi) = \frac{1}{l!}(ty)^\alpha (t\langle y, \xi \rangle)^l. \tag{7.5}$$

Thus we have

$$
\begin{aligned}
&A_k \\
&= \sum_s \frac{1}{(k-s)!\,s!} \langle ty, \operatorname{grad} f\rangle^{k-s} \left(\sum_{|\alpha|\geq 2} \frac{(ty)^\alpha}{\alpha!} \partial_x^\alpha f(x) \right)^s \\
&= \sum_s \frac{1}{(k-s)!\,s!} \langle ty, \operatorname{grad} f\rangle^{k-s} \sum_{|\alpha_1|,\cdots,|\alpha_s|\geq 2} \frac{(ty)^{\alpha_1+\cdots+\alpha_s}}{\alpha_1!\cdots\alpha_s!} \prod_{j=1}^s \partial_x^{\alpha_j} f(x) \\
&= \sum \frac{1}{s!} \partial_\xi^{\alpha_1+\cdots+\alpha_s} p_{k-s+|\alpha_1|+\cdots+|\alpha_s|}(\operatorname{grad} f) \frac{1}{\alpha_1!\cdots\alpha_s!} \prod_{j=1}^s \partial_x^{\alpha_j} f(x).
\end{aligned}
$$

□

Calculating the first three terms, we have

$$
\begin{aligned}
(7.6) \qquad A_m(x) \;=\; & p_m(x, \operatorname{grad} f) \\
&+ \frac{1}{2} \sum_{i,j} \frac{\partial^2 f(x)}{\partial x_i \partial x_j} \frac{\partial^2}{\partial \xi_i \partial \xi_j} p_{m+1}(x, \operatorname{grad} f) \\
&+ \frac{1}{6} \sum_{i,j,k} \frac{\partial^3 f}{\partial x_i \partial x_j \partial x_k} \frac{\partial^3}{\partial \xi_i \partial \xi_j \partial \xi_k} p_{m+2} \\
&+ \frac{1}{8} \sum_{i,j,k,l} \frac{\partial^2 f}{\partial x_i \partial x_j} \frac{\partial^2 f}{\partial x_k \partial x_l} \frac{\partial^4 p_{m+2}}{\partial \xi_i \partial \xi_j \partial \xi_k \partial \xi_l} \\
&+ (\text{terms with } p_{m+3}, p_{m+4}, \cdots).
\end{aligned}
$$

From this formula, we deduce a formula for coordinate change of differential operators. Let $x = (x_1, \cdots, x_n)$ and $\tilde{x} = (\tilde{x}_1, \cdots, \tilde{x}_n)$ be two coordinate systems of a manifold X, and (x, ξ) and $(\tilde{x}, \tilde{\xi})$ the corresponding coordinate systems of T^*X respectively. Since $\omega_X = \sum \xi_i dx_i = \sum \tilde{\xi}_j d\tilde{x}_j$, we have

$$\xi_i = \sum_j \tilde{\xi}_j \frac{\partial \tilde{x}_j}{\partial x_i}.$$

Express one differential operator on X as $P(x, \partial_x) = \sum p_k(x, \partial_x)$ in the x-coordinates and as $\tilde{P}(\tilde{x}, \partial_{\tilde{x}}) = \sum \tilde{p}_k(\tilde{x}, \partial_{\tilde{x}})$ in the $\tilde{x}$-coordinates. Then

$$\tilde{P}(\tilde{x}, \partial_{\tilde{x}}) e^{\langle \tilde{x}, \tilde{\xi}\rangle} = \sum \tilde{p}_k(\tilde{x}, \tilde{\xi}) e^{\langle \tilde{x}, \tilde{\xi}\rangle}.$$

Applying equation (7.3) to $\varphi(t) = e^t$ and $f(x) = \langle \tilde{x}, \tilde{\xi}\rangle$, we obtain

$$(7.7) \qquad \tilde{p}_k(\tilde{x}, \tilde{\xi}) = \sum \frac{\prod_{j=1}^s (\partial_x^{\alpha_j} \langle \tilde{x}, \tilde{\xi}\rangle)}{s!\,\alpha_1! \cdots \alpha_l!} (\partial_\xi^{\alpha_1+\cdots+\alpha_s} p_l)(x, \xi),$$

where the sum is taken in the same range as in equation (7.3). Note that $\operatorname{grad} f(x) = \left(\sum_\nu \frac{\partial \tilde{x}_\nu}{\partial x_i}\tilde{\xi}_\nu\right)_i = \xi$.

Calculating the first three terms, we have

$$
\begin{aligned}
(7.8)\quad & \tilde{p}_m(\tilde{x},\tilde{\xi}) \\
= \;& p_m(x,\xi) + \frac{1}{2}\sum_{i,j,\nu} \frac{\partial^2 \tilde{x}_\nu}{\partial x_i \partial x_j}\tilde{\xi}_\nu \frac{\partial^2 p_{m+1}}{\partial \xi_i \partial \xi_j}(x,\xi) \\
& + \frac{1}{6}\sum_{i,j,k,\nu} \frac{\partial^3 x_\nu}{\partial x_i \partial x_j \partial x_k}\tilde{\xi}_\nu \frac{\partial^3 p_{m+2}}{\partial \xi_i \partial \xi_j \partial \xi_k}(x,\xi) \\
& + \frac{1}{8}\sum_{i,j,k,l,\nu,\mu} \frac{\partial^2 x_\nu}{\partial x_i \partial x_j}\frac{\partial^2 x_\mu}{\partial x_k \partial x_l}\tilde{\xi}_\nu\tilde{\xi}_\mu \frac{\partial^4 p_{m+2}}{\partial \xi_i \partial \xi_j \partial \xi_k \partial \xi_l}(x,\xi) \\
& + (\text{terms with } p_{m+3},\cdots).
\end{aligned}
$$

In the following, we show that (7.7) gives a formula for coordinate change of $\widehat{\mathcal{E}}_X$. Let $x^{(1)} = (x_1^{(1)},\cdots,x_n^{(1)})$ and $x^{(2)} = (x_1^{(2)},\cdots,x_n^{(2)})$ be two coordinate systems of a manifold X. Then we have sheaves of formal microdifferential operators $\widehat{\mathcal{E}}_X^{(k)}$ on T^*X in the coordinate systems $x^{(k)}$ $(k=1,2)$. Equation (7.7) gives $\Phi_{21} : \widehat{\mathcal{E}}_X^{(1)} \to \widehat{\mathcal{E}}_X^{(2)}$. This satisfies the axiom of patching, i.e.,

$$
\begin{aligned}
(7.9)\quad & \Phi_{kk} &=\;& \mathrm{id}_{\widehat{\mathcal{E}}_X^{(k)}}, \\
(7.10)\quad & \Phi_{12}\circ\Phi_{21} &=\;& \mathrm{id}_{\widehat{\mathcal{E}}_X^{(1)}}, \\
(7.11)\quad & \Phi_{32}\circ\Phi_{21} &=\;& \Phi_{31}.
\end{aligned}
$$

To show these, it suffices to show that

$$\Phi_{12}\circ\Phi_{21} - \mathrm{id}_{\widehat{\mathcal{E}}_X^{(1)}}, \quad \Phi_{13}\circ\Phi_{32}\circ\Phi_{21} - \mathrm{id}_{\widehat{\mathcal{E}}_X^{(1)}}$$

equal 0. This follows from the following lemma and the fact that it holds for $\mathcal{D}_X$.

LEMMA 7.5. *Let*

$$a_{\alpha,\beta,\mu}(x,\xi) \quad (\alpha,\beta\in\mathbb{Z}^n_{\geq 0},\ \mu\in\mathbb{Z})$$

be homogeneous of degree μ in ξ. Suppose that $a_{\alpha,\beta,\mu}(x,\xi) = 0$ $(\mu > |\alpha| + c)$ for some $c\in\mathbb{Z}$. Let φ be the map sending $\sum p_k(x,\partial_x)\in\mathcal{D}_X$ to $\sum q_k(x,\partial_x)\in\widehat{\mathcal{E}}_X$, where

$$(7.12)\qquad q_k(x,\xi) = \sum_{k=l-|\alpha|+\mu} a_{\alpha,\beta,\mu}(x,\xi)\partial_\xi^\alpha\partial_x^\beta p_l(x,\xi).$$

If $\varphi = 0$, then all $a_{\alpha,\beta,\mu}(x,\xi)$ equal 0.

PROOF. We prove that $a_{\alpha_0,\beta_0,\mu} = 0$ by induction on $m = |\alpha_0| + |\beta_0|$. We assume that $a_{\alpha,\beta,\mu} = 0$ for $|\alpha| + |\beta| < m$, and prove that $a_{\alpha_0,\beta_0,\mu} = 0$.

Let $P(x,\partial_x) = x^{\beta_0}\partial_x^{\alpha_0}$, or

$$p_l(x,\xi) = \begin{cases} x^{\beta_0}\xi^{\alpha_0} & (l = |\alpha_0|), \\ 0 & (l \neq |\alpha_0|). \end{cases}$$

Then

$$\begin{aligned} 0 &= q_k(x,\xi) \\ &= \sum_{\substack{k=|\alpha_0|-|\alpha|+\mu \\ \alpha_0 \geq \alpha,\, \beta_0 \geq \beta}} a_{\alpha,\beta,\mu}(x,\xi)\frac{\alpha_0!}{(\alpha_0-\alpha)!}\frac{\beta_0!}{(\beta_0-\beta)!}x^{\beta_0-\beta}\xi^{\alpha_0-\alpha}, \end{aligned}$$

where $\alpha_0 \geq \alpha$ means $(\alpha_0)_\nu \geq \alpha_\nu$ for all ν. If $\alpha_0 \geq \alpha$ and $\beta_0 \geq \beta$, then $|\alpha_0| + |\beta_0| \geq |\alpha| + |\beta|$, and hence

$$a_{\alpha,\beta,\mu} = 0 \text{ for } (\alpha,\beta) \neq (\alpha_0,\beta_0).$$

Hence we obtain $0 = q_k(x,\xi) = a_{\alpha_0,\beta_0,k}(x,\xi)$. □

Similarly, reducing it to the case $\mathcal{D}_X$ by the above lemma, we can prove that Φ_{12} is a homomorphism of rings.

Since we have checked the axiom of patching, we have constructed a sheaf of rings $\widehat{\mathcal{E}}_X$ on the cotangent bundle T^*X for any manifold X. Similarly, patching $\widehat{\mathcal{E}}_{\mathbb{C}^n}(m)$ together, we can define a subsheaf $\widehat{\mathcal{E}}_X(m)$ of $\widehat{\mathcal{E}}_X$.

THEOREM 7.6. (1) $\widehat{\mathcal{E}}_X = \bigcup \widehat{\mathcal{E}}_X(m)$.

(2) *$\widehat{\mathcal{E}}_X$ contains $\pi_X^{-1}\mathcal{D}_X$ as a subsheaf.*

(3) *$i_X^{-1}\widehat{\mathcal{E}}_X \cong \mathcal{D}_X$, where $i_X : X \to T^*X$ is the zero section.*

(4) *$\widehat{\mathcal{E}}_X(m)/\widehat{\mathcal{E}}_X(m-1) \cong \mathcal{O}_{T^*X}(m)$.*

Similarly to the case $X = \mathbb{C}^n$, we define $\sigma_m : \widehat{\mathcal{E}}_X(m) \to \mathcal{O}_{T^*X}(m)$.

Similarly to the above, we define, for any $\lambda \in \mathbb{C}$,

$$\begin{aligned} \widehat{\mathcal{E}}_X(\lambda) &= \{\{p_{\lambda+k}(x,\xi)\}_{k\in\mathbb{Z}_{\leq 0}};\, p_{\lambda+k}(x,\xi) \text{ is homogeneous} \\ &\qquad\qquad \text{of degree } (\lambda+k)\}, \\ \widehat{\mathcal{E}}_X^{(\lambda)} &= \bigcup_{k\in\mathbb{Z}} \widehat{\mathcal{E}}_X(\lambda+k). \end{aligned}$$

Then $\widehat{\mathcal{E}}(\lambda)$ and $\widehat{\mathcal{E}}_X^{(\lambda)}$ are defined on the whole T^*X by patching. The multiplication $\widehat{\mathcal{E}}_X(\lambda) \times \widehat{\mathcal{E}}_X(\mu) \to \widehat{\mathcal{E}}_X(\lambda + \mu)$ is also defined. In particular, $\widehat{\mathcal{E}}_X(\lambda)$ is a two-sided $\widehat{\mathcal{E}}_X(0)$-module.

In § 1.4, we have defined the formal adjoints in the case $\mathcal{D}_X$. Similarly we define them in $\widehat{\mathcal{E}}_X$. Let $a : T^*X \to T^*X$ be the map multiplying ξ by -1, i.e., $(x, \xi) \mapsto (x, -\xi)$. For $P = \{p_k(x, \xi)\}_k \in \widehat{\mathcal{E}}_X$, we define $P^* \in a^{-1}\widehat{\mathcal{E}}_X$ in a coordinate system $x = (x_1, \cdots, x_n)$ by

$$(P^*)_k(x, \xi) = \sum_{k=l-|\alpha|} \frac{(-1)^{|\alpha|}}{\alpha!}(\partial_\xi^\alpha \partial_x^\alpha p_l)(x, -\xi).$$

THEOREM 7.7.

$$(\widehat{\mathcal{E}}_X)^{\mathrm{op}} \to \Omega_X \otimes a^{-1}\widehat{\mathcal{E}}_X \otimes \Omega_X^{\otimes -1} \quad (P^{\mathrm{op}} \mapsto dx \otimes P^* \otimes dx^{-1})$$

is an isomorphism of rings, independent of the choice of a coordinate system.

We can reduce this theorem to the corresponding theorem for $\mathcal{D}$ by Lemma 7.5.

7.3. Microdifferential Operators

Let $P(x, \partial_x) = \sum_{k \le m} P_k(x, \partial)$ be a formal microdifferential operator. For $\lambda \in \mathbb{C} \setminus \mathbb{Z}$, we formally apply Lemma 7.4 to $\varphi(t) = t^\lambda$ and $f(x) = \langle x, \xi \rangle + t$. Since $\varphi^{(k)}(t) = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-k+1)} t^{\lambda-k}$, we obtain

$$\begin{aligned} P(x, \partial_x)(\langle x, \xi \rangle + t)^\lambda \\ = \sum_{k \le m} \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-k+1)} P_k(x, \xi)(\langle x, \xi \rangle + t)^{\lambda-k}. \end{aligned} \tag{7.13}$$

The infinite series in (7.13) does not converge in general. If the condition

$$|p_k(x, \xi)| \le R^{-k}(-k)! \quad (k < 0) \tag{7.14}$$

is satisfied for some $R > 0$, then the series in (7.13) converges in $|\langle x, \xi \rangle + t| < 1/R$, since $\Gamma(\lambda+1)/\Gamma(\lambda+1-k) \sim 1/(-k)!$ for $k < 0$.

We denote by $\mathcal{E}(m)$ the sheaf on $T^*\mathbb{C}^n$ consisting of $\{p_k(x, \xi)\}_{k \le m}$ satisfying (7.14) locally, and call its section a *microdifferential operator*. Then $\mathcal{E} := \bigcup_m \mathcal{E}(m)$ is a subsheaf of $\widehat{\mathcal{E}}$. Although it is the ring $\mathcal{E}$ that is actually needed in the theory of linear differential equations, we deal with $\widehat{\mathcal{E}}$ in this book to skip the proof of convergence. Almost all properties of $\widehat{\mathcal{E}}$ described in this book hold for $\mathcal{E}$ as well. Only in properties that involve irregular singularities, which we do not treat in this book, a difference between $\mathcal{E}$ and $\widehat{\mathcal{E}}$ does appear. This difference occurs because convergent solutions and formal solutions are different at irregular singularities.

7.4. Algebraic Properties of $\widehat{\mathcal{E}}$

In this section, we show that $\widehat{\mathcal{E}}(0)$ and $\widehat{\mathcal{E}}$ are Noetherian rings (see §A. 1). Note that things are quite different on the zero section T_X^*X and on its complement $\overset{\circ}{T}{}^*X$. For example, as an $\widehat{\mathcal{E}}_X(0)$-module, $\widehat{\mathcal{E}}_X(m)$ is locally isomorphic to $\widehat{\mathcal{E}}_X(0)$ on $\overset{\circ}{T}{}^*X$, but not even locally finitely generated on T^*X when $m < 0$.

PROPOSITION 7.8. *Let $p \in T^*X$. Suppose that $P \in \widehat{\mathcal{E}}_X(m)_p$ satisfies $\sigma_m(P)(p) \neq 0$. Then there exists $Q \in \widehat{\mathcal{E}}_X(-m)_p$ such that $PQ = QP = 1$.*

PROOF. First we show the existence of $Q \in \widehat{\mathcal{E}}_X(-m)_p$ satisfying $PQ = 1$. Take $A \in \widehat{\mathcal{E}}_X(-m)_p$ such that $\sigma_{-m}(A) = \sigma_m(P)^{-1}$, and put $S = PA$. Write $S = 1 + R$ with $R \in \widehat{\mathcal{E}}_X(-1)_p$. Then $T = \sum(-1)^j R^j$ is clearly an element of $\widehat{\mathcal{E}}_X(0)_p$ satisfying $ST = 1$. Hence AT is the right inverse of P. Since P similarly has a left inverse, the right inverse and the left inverse coincide. □

In particular, for $P \in \widehat{\mathcal{E}}_X(-1)$, $1 + P \in \widehat{\mathcal{E}}_X(0)$ is invertible in $\widehat{\mathcal{E}}_X(0)$. Hence $\widehat{\mathcal{E}}_X(-1)_p$ is contained in the Jacobson radical of $\widehat{\mathcal{E}}_X(0)_p$ at every $p \in T^*X$. We thus have the following proposition by Nakayama's Lemma (see for example Theorem 2. 29 of [Hotta2]).

PROPOSITION 7.9 (Nakayama's Lemma). *Let $p \in T^*X$.*

(1) *Let M be a finitely generated $\widehat{\mathcal{E}}_X(0)_p$-module. Suppose that $M = \widehat{\mathcal{E}}_X(-1)_p M$. Then $M = 0$.*
(2) *Let M be a finitely generated $\widehat{\mathcal{E}}_X(0)_p$-module, and N a submodule of M. Suppose that $M = \widehat{\mathcal{E}}_X(-1)_p M + N$. Then $M = N$.*

The following lemma is obvious, since, for P with $\sigma_m(P) \neq 0$, $\widehat{\mathcal{E}}_X(m) = \widehat{\mathcal{E}}_X(0)P$ is a free $\widehat{\mathcal{E}}_X(0)$-module.

LEMMA 7.10. (1) *$\widehat{\mathcal{E}}_X$ and $\widehat{\mathcal{E}}_X(m)$ are flat over $\widehat{\mathcal{E}}_X(0)$ on $\overset{\circ}{T}{}^*X$.*

(2) *The multiplication gives an isomorphism*

$$\widehat{\mathcal{E}}_X(m_1) \otimes_{\widehat{\mathcal{E}}_X(0)} \widehat{\mathcal{E}}_X(m_2) \xrightarrow{\sim} \widehat{\mathcal{E}}_X(m_1 + m_2)$$

on $\overset{\circ}{T}{}^*X$ *for* $m_1, m_2 \in \mathbb{Z} \cup \{\infty\}$, *where we set* $\widehat{\mathcal{E}}_X(\infty) = \widehat{\mathcal{E}}_X$.

PROPOSITION 7.11. *$\widehat{\mathcal{E}}_X$ and $\widehat{\mathcal{E}}_X(m)$ are flat as right $\pi_X^{-1}\mathcal{O}_X$-modules and as left $\pi_X^{-1}\mathcal{O}_X$-modules.*

PROOF. By Theorem 7.7, it suffices to prove the statement for left modules. Since it is obvious on $T^*_X X$, we show it on $\overset{\circ}{T}{}^*X$. For this, we show that an exact sequence $\mathcal{O}_X^{N_2} \xrightarrow{f} \mathcal{O}_X^{N_1} \xrightarrow{g} \mathcal{O}_X^{N_0}$ gives rise to an exact sequence

$$\mathcal{O}_X^{N_2} \otimes_{\mathcal{O}_X} \widehat{\mathcal{E}}_X(m)_p \xrightarrow{f} \mathcal{O}_X^{N_1} \otimes_{\mathcal{O}_X} \widehat{\mathcal{E}}_X(m)_p \xrightarrow{g} \mathcal{O}_X^{N_0} \otimes_{\mathcal{O}_X} \widehat{\mathcal{E}}_X(m)_p \tag{7.15}$$

at every $p \in \overset{\circ}{T}{}^*X$. Let $p = (x_0, \xi_0) \in \overset{\circ}{T}{}^*X$. We may assume that $(\xi_0)_n \neq 0$. Put

$$U_\varepsilon = \left\{ (x, \xi); |x - x_0| < \varepsilon,\ \left| \frac{\xi_i}{\xi_n} - \frac{\xi_{0i}}{\xi_{0n}} \right| < \varepsilon,\ |\xi_n - \xi_{0n}| < \varepsilon \right\}. \tag{7.16}$$

By the general theory of Stein manifolds, $\Gamma(U_\varepsilon; \bullet)$ sends an exact sequence of coherent $\mathcal{O}_{\overset{\circ}{T}{}^*X}(0)|_{U_\varepsilon}$-modules to an exact sequence. Hence

$$\begin{aligned}\Gamma(U_\varepsilon; \mathcal{O}_X^{N_2} \otimes_{\mathcal{O}} \mathcal{O}_{T^*X}(k)) &\to \Gamma(U_\varepsilon; \mathcal{O}_X^{N_1} \otimes_{\mathcal{O}} \mathcal{O}_{T^*X}(k)) \\ &\to \Gamma(U_\varepsilon; \mathcal{O}_X^{N_0} \otimes_{\mathcal{O}} \mathcal{O}_{T^*X}(k))\end{aligned}$$

is exact. Since

$$\Gamma(U_\varepsilon; \mathcal{O}_X^{N_\nu} \otimes_{\mathcal{O}} \widehat{\mathcal{E}}_X(m)) = \prod_{k \le m} \Gamma(U_\varepsilon; \mathcal{O}_X^{N_\nu} \otimes_{\mathcal{O}} \mathcal{O}_{T^*X}(k))$$

as $\mathcal{O}$-modules, it follows that the sequence

$$\begin{aligned}\Gamma(U_\varepsilon; \mathcal{O}_X^{N_2} \otimes_{\mathcal{O}} \widehat{\mathcal{E}}_X(m)) &\to \Gamma(U_\varepsilon; \mathcal{O}_X^{N_1} \otimes_{\mathcal{O}} \widehat{\mathcal{E}}_X(m)) \\ &\to \Gamma(U_\varepsilon; \mathcal{O}_X^{N_0} \otimes_{\mathcal{O}} \widehat{\mathcal{E}}_X(m))\end{aligned}$$

is exact for arbitrary ε. We have thus proved the exactness of (7.15). □

By Proposition A.14, $\mathcal{O}_{T^*X}(0)|_{\overset{\circ}{T}{}^*X}$ is a Noetherian ring. Applying Lemma A.11 inductively to exact sequences

$$0 \to \mathcal{O}_{T^*X}(-k) \to \widehat{\mathcal{E}}_X(0)/\widehat{\mathcal{E}}_X(-k-1) \to \widehat{\mathcal{E}}_X(0)/\widehat{\mathcal{E}}_X(-k) \to 0,$$

we see that

(7.17) $\quad \widehat{\mathcal{E}}_X(0)/\widehat{\mathcal{E}}_X(-k)|_{\overset{\circ}{T}{}^*X}$ are Noetherian rings for all $k > 0$.

LEMMA 7.12. *For an open subset $\Omega \subset \overset{\circ}{T}{}^*X$, any coherent $(\widehat{\mathcal{E}}_X(0)/\widehat{\mathcal{E}}_X(-k))|_\Omega$-module is also coherent as an $\widehat{\mathcal{E}}_X(0)|_\Omega$-module.*

PROOF. It suffices to show that any coherent $(\widehat{\mathcal{E}}_X(0)/\widehat{\mathcal{E}}_X(-k))|_\Omega$-module $\mathcal{M}$ is a locally finitely presented $\widehat{\mathcal{E}}_X(0)$-module. Locally we have an exact sequence

$$(\widehat{\mathcal{E}}_X(0)/\widehat{\mathcal{E}}_X(-k))^{\oplus N_1} \longrightarrow (\widehat{\mathcal{E}}_X(0)/\widehat{\mathcal{E}}_X(-k))^{\oplus N_0} \longrightarrow \mathcal{M} \longrightarrow 0.$$

Since $\widehat{\mathcal{E}}_X(0)/\widehat{\mathcal{E}}_X(-k)$ is a locally finitely presented $\widehat{\mathcal{E}}_X(0)$-module, $\mathcal{M}$ is a locally finitely presented $\widehat{\mathcal{E}}_X(0)$-module by Proposition A.2. $\square$

From now on, for an $\widehat{\mathcal{E}}_X(0)$-module $\mathcal{M}$, we set

$$\mathcal{M}(k) = \widehat{\mathcal{E}}_X(k) \otimes_{\widehat{\mathcal{E}}_X(0)} \mathcal{M}.$$

There exist $\widehat{\mathcal{E}}_X(0)$-module homomorphisms $\mathcal{M}(k) \to \mathcal{M}$ for $k \leq 0$ and $\mathcal{M} \to \mathcal{M}(k)$ for $k \geq 0$.

LEMMA 7.13. *Let $\mathcal{M}$ be an $\widehat{\mathcal{E}}_X(0)$-module. Then the following two conditions are equivalent on $\overset{\circ}{T}{}^*X$:*

(1) *$\mathcal{M}(-1) \to \mathcal{M}$ is injective,*

(2) *$\mathcal{M} \to \widehat{\mathcal{E}}_X \otimes_{\widehat{\mathcal{E}}_X(0)} \mathcal{M}$ is injective.*

PROOF. (1)$\Longrightarrow$(2). Since $\mathcal{M}(k) \to \mathcal{M}(k+1)$ is injective for any k, $\mathcal{M} \to \mathcal{M}(k)$ is injective for $k \geq 0$. Hence we obtain (2) by taking the inductive limit with respect to k.

(2)$\Longrightarrow$(1). $\mathcal{M} \to \widehat{\mathcal{E}}_X \otimes_{\widehat{\mathcal{E}}_X(0)} \mathcal{M}$ can be factored into $\mathcal{M} \to \mathcal{M}(1) \to \widehat{\mathcal{E}}_X \otimes_{\widehat{\mathcal{E}}_X(0)} \mathcal{M}$. Hence $\mathcal{M} \to \mathcal{M}(1)$ is injective. Applying the exact functor $\widehat{\mathcal{E}}_X(-1) \otimes_{\widehat{\mathcal{E}}_X(0)} \bullet$ to this, we obtain (1). $\square$

The following lemma gives a criterion for the finiteness of $\widehat{\mathcal{E}}_X$-modules.

LEMMA 7.14. *Let Ω be an open set in $\overset{\circ}{T}{}^*X$, and $\mathcal{M}$ an $\widehat{\mathcal{E}}_X(0)|_\Omega$-module. Assume that a decreasing sequence $\{\mathcal{M}_k\}_{k\geq 0}$ of $\widehat{\mathcal{E}}_X(0)$-submodules of $\mathcal{M}$ satisfies the following conditions* (a)–(d)*:*

(a) $\mathcal{M}_0 = \mathcal{M}$,

(b) $\mathcal{M}_{k+1} = \widehat{\mathcal{E}}_X(-1)\mathcal{M}_k + \mathcal{M}_{k+2} \quad (k \geq 0)$,

(c) $\bigcap_k \mathcal{M}_k = 0$,

(d) $\mathcal{M}_k/\mathcal{M}_{k+1}$ *is a coherent* $\mathcal{O}_{T^*X}(0)$*-module.*

Then the following (1) *and* (2) *hold:*

(1) $\mathcal{M}$ *is a locally finitely presented* $\widehat{\mathcal{E}}_X(0)$*-module,*

(2) $\mathcal{M}_k = \widehat{\mathcal{E}}_X(-k)\mathcal{M}$.

PROOF. Take $u_1, \ldots, u_N \in \mathcal{M}$ so that

$$\mathcal{M}/\mathcal{M}_1 = \sum_i \mathcal{O}_{T^*X}(0)\bar{u}_i,$$

where $\bar{u}_i = u_i \mod \mathcal{M}_1$. First we show that

$$\mathcal{M}_{k_0} = \sum_i \widehat{\mathcal{E}}_X(-k_0)u_i \tag{7.18}$$

for all k_0.

Since the statement is local, it is enough to consider it in a neighborhood of $p = (x_0, \xi_0)$. We may assume $x_0 = 0$ and $\xi_0 = (0, \ldots, 0, 1)$. For a neighborhood U_ε of p small enough, say $U_\varepsilon = \{(x,\xi); |x - x_0| < \varepsilon,\ |\xi_\nu/\xi_n| < \varepsilon\ (\nu = 1, \ldots, n-1),\ |\xi_n - 1| < \varepsilon\}$, the functor $\Gamma(U_\varepsilon; \bullet)$ is exact on the category of coherent $\mathcal{O}_{T^*X}(0)|_{U_\varepsilon}$-modules by the theory of Stein manifolds. To show (7.18), it is enough to show that

$$\mathcal{M}_{k_0}(U_\varepsilon) = \sum_i \widehat{\mathcal{E}}_X(-k_0)(U_\varepsilon)u_i. \tag{7.19}$$

Using induction on k, we easily obtain from (b)

$$\mathcal{M}_k = \widehat{\mathcal{E}}_X(-k)\mathcal{M} + \mathcal{M}_{k+1} \quad (k \geq 0). \tag{7.20}$$

Hence the homomorphism defined by $\{u_i\}$

$$(\widehat{\mathcal{E}}_X(-k)/\widehat{\mathcal{E}}_X(-k-1))^{\oplus N} \to \mathcal{M}_k/\mathcal{M}_{k+1}$$

is an epimorphism of coherent $\mathcal{O}_{T^*X}(0)$-modules. By the remark above,

$$\Gamma(U_\varepsilon; \widehat{\mathcal{E}}_X(-k)/\widehat{\mathcal{E}}_X(-k-1))^{\oplus N} \to \Gamma(U_\varepsilon; \mathcal{M}_k/\mathcal{M}_{k+1})$$

is surjective. Since $\Gamma(U_\varepsilon; \widehat{\mathcal{E}}_X(-k)) \to \Gamma(U_\varepsilon; \widehat{\mathcal{E}}_X(-k)/\widehat{\mathcal{E}}_X(-k-1))$ is also surjective, we obtain

$$\Gamma(U_\varepsilon; \mathcal{M}_k) = \sum_i \Gamma(U_\varepsilon; \widehat{\mathcal{E}}_X(-k))u_i + \Gamma(U_\varepsilon; \mathcal{M}_{k+1}). \tag{7.21}$$

Using this, we show (7.19). Suppose $u \in \Gamma(U_\varepsilon; \mathcal{M}_{k_0})$. Applying (7.21) inductively, we see that there exist $a_i^{(k)} \in \Gamma(U_\varepsilon; \widehat{\mathcal{E}}_X(-k))$ $(k \geq k_0)$ such that

$$u - \sum_{\nu=k_0}^{k} \sum_i a_i^{(\nu)} u_i \in \Gamma(U_\varepsilon; \mathcal{M}_{k+1})$$

for all $k \geq k_0$. Put $a_i = \sum_{k=k_0}^{\infty} a_i^{(k)} \in \Gamma(U_\varepsilon; \widehat{\mathcal{E}}_X(-k_0))$. Then $a_i - \sum_{\nu=k_0}^{k} a_i^{(\nu)} \in \Gamma(U_\varepsilon; \widehat{\mathcal{E}}_X(-k-1))$ for all $k \geq k_0$. Hence

$$\begin{aligned} u - \sum a_i u_i &= \left(u - \sum_{\nu=k_0}^{k} \sum_i a_i^{(\nu)} u_i\right) - \sum_i \left(a_i - \sum_{\nu=k_0}^{k} a_i^{(\nu)}\right) u_i \\ &\in \Gamma(U_\varepsilon; \mathcal{M}_{k+1}) + \Gamma(U_\varepsilon; \widehat{\mathcal{E}}_X(-k-1)\mathcal{M}) \subset \Gamma(U_\varepsilon; \mathcal{M}_{k+1}) \end{aligned}$$

for all k. By (c), $u = \sum a_i u_i$. We have thus obtained (7.19) and hence (7.18). In particular, we have proved (2) and the locally finite generation of $\mathcal{M}$.

Next we show that $\mathcal{M}$ is locally finitely presented. Since $\mathcal{M}$ is locally finitely generated, we have an epimorphism from $\mathcal{L} := \widehat{\mathcal{E}}_X(0)^{\oplus m}$ onto $\mathcal{M}$. Let $\mathcal{N}$ be its kernel, and let $\mathcal{N}_k = \mathcal{N} \cap \mathcal{L}(-k)$. Then, by the exact sequence

$$0 \to \mathcal{N}_k/\mathcal{N}_{k+1} \to \mathcal{L}(-k)/\mathcal{L}(-k-1) \to \mathcal{M}_k/\mathcal{M}_{k+1} \to 0,$$

$\mathcal{N}_k/\mathcal{N}_{k+1}$ is a coherent $\mathcal{O}_{\mathring{T}^*X}(0)$-module. Hence the locally finite generation of $\mathcal{N}$ implies the locally finite presentation of $\mathcal{M}$. Thus (1) is reduced to a weaker form (the one with locally finite generation in place of locally finite presentation) of the following lemma. □

LEMMA 7.15. *Let Ω be an open set in $\mathring{T}^*X$, and $\mathcal{M}$ an $\widehat{\mathcal{E}}_X(0)|_\Omega$-submodule of $\widehat{\mathcal{E}}_X(0)^{\oplus m}|_\Omega$. Suppose that*

$$(\widehat{\mathcal{E}}_X(-k)^{\oplus m} \cap \mathcal{M})/(\widehat{\mathcal{E}}_X(-1-k)^{\oplus m} \cap \mathcal{M})$$

are coherent $\mathcal{O}_{\mathring{T}^*X}(0)|_\Omega$-modules for all $k \geq 0$. Then $\mathcal{M}$ is a locally finitely presented $\widehat{\mathcal{E}}_X(0)|_\Omega$-module.

PROOF. Put

$$\mathcal{L} = \widehat{\mathcal{E}}_X(0)^{\oplus m}|_\Omega, \quad \mathcal{M}_k = \mathcal{M} \cap \widehat{\mathcal{E}}_X(-k)\mathcal{L}.$$

Then $\mathcal{O}_{T^*X}(k) \otimes_{\mathcal{O}_{T^*X(0)}} (\mathcal{M}_k/\mathcal{M}_{k+1})$ is an increasing sequence of coherent $\mathcal{O}_{T^*X}(0)$-submodules of $\mathcal{L}/\mathcal{L}(-1)$. Take $k_0 \geq 0$ such that, PROOF. We prove the proposition only on $\overset{\circ}{T}{}^*X$. Along T^*_XX, the proof can be reduced to the case of $\mathcal{O}_X$ similarly to the proof of Proposition 7.16. Let $\mathcal{I}_1 \subset \mathcal{I}_2 \subset \cdots$ be an increasing sequence of coherent ideals of $\widehat{\mathcal{E}}_X(0)$. Put

$$\bar{\mathcal{I}}_{k,\nu} := \Big(\frac{\mathcal{I}_\nu \cap \widehat{\mathcal{E}}_X(-k)}{\mathcal{I}_\nu \cap \widehat{\mathcal{E}}_X(-k-1)}\Big) \otimes_{\mathcal{O}_{T^*X}(0)} \mathcal{O}_{T^*X}(k) \subset \mathcal{O}_{T^*X}(0).$$

Then

$$\bar{\mathcal{I}}_{k,\nu} \supset \bar{\mathcal{I}}_{k',\nu'} \quad (k \geq k',\ \nu \geq \nu').$$

Since $\mathcal{O}_{T^*X}(0)$ is a Noetherian ring, there exist k_0 and ν_0 such that

$$\bar{\mathcal{I}}_{k,\nu} = \bar{\mathcal{I}}_{k_0,\nu_0} \quad (k \geq k_0,\ \nu \geq \nu_0).$$

We have

$$\mathcal{I}_\nu \cap \widehat{\mathcal{E}}_X(-k-1) \subset \widehat{\mathcal{E}}_X(-1)(\mathcal{I}_\nu \cap \widehat{\mathcal{E}}_X(-k)) + \mathcal{I}_\nu \cap \widehat{\mathcal{E}}_X(-k-2).$$

Hence, applying Lemma 7.14 to $\{\mathcal{I}_\nu \cap \widehat{\mathcal{E}}_X(-k)\}_{k\geq k_0}$ for $\nu \geq \nu_0$, we obtain

$$\mathcal{I}_\nu \cap \widehat{\mathcal{E}}_X(-k) = \widehat{\mathcal{E}}_X(-k+k_0)(\mathcal{I}_\nu \cap \widehat{\mathcal{E}}_X(-k_0)) \quad (\nu \geq \nu_0,\ k \geq k_0).$$

Put $\mathcal{I}'_\nu := \mathcal{I}_\nu \cap \widehat{\mathcal{E}}_X(-k_0)$. Then for $\nu \geq \nu_0$ we have

$$\frac{\mathcal{I}'_\nu}{\widehat{\mathcal{E}}_X(-1)\mathcal{I}'_\nu} \simeq \frac{\mathcal{I}'_{\nu_0}}{\widehat{\mathcal{E}}_X(-1)\mathcal{I}'_{\nu_0}}.$$

By Nakayama's Lemma, $\mathcal{I}'_\nu = \mathcal{I}'_{\nu_0}$ $(\nu \geq \nu_0)$. Since $\mathcal{I}_\nu/\mathcal{I}'_\nu$ is an increasing sequence of coherent ideals of a Noetherian ring $\widehat{\mathcal{E}}_X(0)/\widehat{\mathcal{E}}_X(-k_0)$, we have $\mathcal{I}_\nu/\mathcal{I}'_\nu = \mathcal{I}_{\nu_1}/\mathcal{I}'_{\nu_1}$ $(\nu \geq \nu_1)$ for $\nu_1 \geq \nu_0$ large enough. Hence $\mathcal{I}_\nu = \mathcal{I}_{\nu_1}$.

Similarly we can prove that $\widehat{\mathcal{E}}_X(0)_p$ is a Noetherian ring. □

Next we prove that $\widehat{\mathcal{E}}_X$ is Noetherian.

LEMMA 7.18. *Let U be an open set in $\overset{\circ}{T}{}^*X$. If $\mathcal{I}$ is a locally finitely generated $\widehat{\mathcal{E}}_X|_U$-submodule of $\widehat{\mathcal{E}}_X^{\oplus l}|_U$, then $\mathcal{I} \cap (\widehat{\mathcal{E}}_X(k)^{\oplus l}|_U)$ are coherent $\widehat{\mathcal{E}}_X(0)|_U$-modules for all k.*

PROOF. Take local generators $u_1, \dots, u_N$ of $\mathcal{I}$. We may assume that all the u_ν belong to $\widehat{\mathcal{E}}_X(0)^{\oplus l}$. Then $\sum_i \widehat{\mathcal{E}}_X(m)u_i \subset \widehat{\mathcal{E}}_X(m)^{\oplus l}$ is a coherent $\widehat{\mathcal{E}}_X(0)$-module. Since $\{\widehat{\mathcal{E}}_X(k)^{\oplus l} \cap (\sum_i \widehat{\mathcal{E}}_X(m)u_i)\}_m$ is an increasing sequence of coherent modules, it follows that $\widehat{\mathcal{E}}_X(k)^{\oplus l} \cap \mathcal{I} = \sum_m (\widehat{\mathcal{E}}_X(k)^{\oplus l} \cap (\sum_i \widehat{\mathcal{E}}_X(m)u_i))$ is a coherent $\widehat{\mathcal{E}}_X(0)$-module. □

PROPOSITION 7.19. *$\widehat{\mathcal{E}}_X$ is a Noetherian ring.*

PROOF. We prove the proposition only on $\overset{\circ}{T}{}^*X$. Along T_X^*X, the proof can be reduced to the Noetherian property of $\mathcal{D}_X$ by the same argument as in the proof of Proposition 7.16.

(Coherence) Let $\mathcal{I}$ be a locally finitely generated left ideal of $\widehat{\mathcal{E}}_X$. Then $\mathcal{I}_0 = \widehat{\mathcal{E}}_X(0) \cap \mathcal{I}$ is a coherent left ideal of $\widehat{\mathcal{E}}_X(0)$. Since $\widehat{\mathcal{E}}_X$ is flat over $\widehat{\mathcal{E}}_X(0)$, $\mathcal{I} \simeq \widehat{\mathcal{E}}_X \otimes_{\widehat{\mathcal{E}}_X(0)} \mathcal{I}_0$ is a locally finitely presented $\widehat{\mathcal{E}}_X$-module.

(Noetherian property) Let $\{\mathcal{I}_\nu\}$ be a directed family of coherent ideals. Since $\{\widehat{\mathcal{E}}_X(0) \cap \mathcal{I}_\nu\}_\nu$ is a family of coherent left ideals of $\widehat{\mathcal{E}}_X(0)$, $\sum_\nu(\widehat{\mathcal{E}}_X(0) \cap \mathcal{I}_\nu)$ is a coherent $\widehat{\mathcal{E}}_X(0)$-module. Hence $\sum_\nu \mathcal{I}_\nu \simeq \widehat{\mathcal{E}}_X \otimes_{\widehat{\mathcal{E}}_X(0)} (\sum_\nu(\widehat{\mathcal{E}}_X(0) \cap \mathcal{I}_\nu))$ is a coherent $\widehat{\mathcal{E}}_X$-module. The Noetherian property of each stalk $(\widehat{\mathcal{E}}_X)_p$ can be similarly reduced to that of $\widehat{\mathcal{E}}_X(0)_p$. □

Now the following proposition is not difficult to prove.

PROPOSITION 7.20. *If $\mathcal{M}$ is a coherent $\widehat{\mathcal{E}}_X$-module defined on an open set in $\overset{\circ}{T}{}^*X$, then $\mathcal{M}$ is a pseudo-coherent $\widehat{\mathcal{E}}_X(0)$-module (see Definition A.5).*

PROOF. Let $\mathcal{N}$ be a locally finitely generated $\widehat{\mathcal{E}}_X(0)$-submodule of $\mathcal{M}$. Then there exists an epimorphism f from $\widehat{\mathcal{E}}_X(0)^{\oplus m}$ onto $\mathcal{N}$. The epimorphism f extends to a homomorphism of $\widehat{\mathcal{E}}_X$-modules $\bar{f}\colon \widehat{\mathcal{E}}_X^{\oplus m} \to \mathcal{M}$. By Lemma 7.18, $\mathrm{Ker}(f) = \mathrm{Ker}(\bar{f}) \cap \widehat{\mathcal{E}}_X(0)^{\oplus m}$ is a coherent $\widehat{\mathcal{E}}_X(0)$-module. Hence $f(\widehat{\mathcal{E}}_X(0)^{\oplus m})$ is also a coherent $\widehat{\mathcal{E}}_X(0)$-module. □

PROPOSITION 7.21. *Let $\mathcal{M}' \xrightarrow{f} \mathcal{M} \xrightarrow{g} \mathcal{M}''$ be a sequence of coherent $\widehat{\mathcal{E}}_X$-modules on an open set in $\overset{\circ}{T}{}^*X$ with $g \circ f = 0$. Let $\mathcal{M}_0'$, $\mathcal{M}_0$, and $\mathcal{M}_0''$ be coherent $\widehat{\mathcal{E}}_X(0)$-submodules of $\mathcal{M}'$, $\mathcal{M}$, and $\mathcal{M}''$ respectively such that $\mathcal{M}' = \widehat{\mathcal{E}}_X\mathcal{M}_0'$, $\mathcal{M} = \widehat{\mathcal{E}}_X\mathcal{M}_0$, $\mathcal{M}'' = \widehat{\mathcal{E}}_X\mathcal{M}_0''$, $f\mathcal{M}_0' \subset \mathcal{M}_0$, and $g\mathcal{M}_0 \subset \mathcal{M}_0''$. Assume that*

$$\frac{\mathcal{M}_0'}{\mathcal{M}_0'(-1)} \to \frac{\mathcal{M}_0}{\mathcal{M}_0(-1)} \to \frac{\mathcal{M}_0''}{\mathcal{M}_0''(-1)}$$

is exact. Then

(1) *$\mathcal{M}_0' \to \mathcal{M}_0 \to \mathcal{M}_0''$ is exact,*

(2) $\mathcal{M}' \to \mathcal{M} \to \mathcal{M}''$ *is exact,*
(3) $g(\mathcal{M}) \cap \mathcal{M}_0'' = g(\mathcal{M}_0)$.

PROOF. Put

$$\mathcal{L}_0 = \operatorname{Ker} g \cap \mathcal{M}_0 = \operatorname{Ker}(\mathcal{M}_0 \to \mathcal{M}_0'').$$

Then $\mathcal{L}_0$ is a coherent $\widehat{\mathcal{E}}_X(0)$-module.

By the assumption, $\mathcal{L}_0 \subset f(\mathcal{M}_0') + \mathcal{M}_0(-1)$. Since $\mathcal{L}_0 \cap \mathcal{M}_0(-1) = \mathcal{L}_0(-1)$, $\mathcal{L}_0 \subset f(\mathcal{M}_0') + \widehat{\mathcal{E}}_X(-1)\mathcal{L}_0$. We obtain $\mathcal{L}_0 = f(\mathcal{M}_0')$ from Nakayama's Lemma. Thus $\mathcal{M}_0' \to \mathcal{M}_0 \to \mathcal{M}_0''$ is exact. Applying the exact functor $\widehat{\mathcal{E}}_X \otimes_{\widehat{\mathcal{E}}_X(0)} \bullet$ to this, we obtain the exactness of $\mathcal{M}' \to \mathcal{M} \to \mathcal{M}''$.

Finally, we show (3). Since $g(\mathcal{M}) \cap \mathcal{M}_0'' = \bigcup_k g(\mathcal{M}_0(k)) \cap \mathcal{M}_0''$, we locally have $g(\mathcal{M}) \cap \mathcal{M}_0'' = g(\mathcal{M}_0(k)) \cap \mathcal{M}_0''$ for k large enough. Hence it is enough to show that $g(\mathcal{M}_0(k)) \cap \mathcal{M}_0'' \subset g(\mathcal{M}_0(k-1)) \cap \mathcal{M}_0''$ for all $k > 0$. To show this, it is enough to show that $g(\mathcal{M}_0(k)) \cap \mathcal{M}_0''(k-1) \subset g(\mathcal{M}_0(k-1))$.

By the assumption,

$$\mathcal{M}_0(k) \cap g^{-1}\mathcal{M}_0''(k-1) \subset f(\mathcal{M}_0'(k)) + \mathcal{M}_0(k-1).$$

Hence

$$\begin{aligned} g(\mathcal{M}_0(k)) \cap \mathcal{M}_0''(k-1) &= g(\mathcal{M}_0(k) \cap g^{-1}\mathcal{M}_0''(k-1)) \\ &\subset gf(\mathcal{M}_0'(k)) + g(\mathcal{M}_0(k-1)). \end{aligned}$$

□

The following lemma is proved similarly to Theorem A.30.

LEMMA 7.22. $(\bigoplus_{k\geq 0} \widehat{\mathcal{E}}_X(-k)T^k)|_{\overset{\circ}{T^*X}}$ *is a Noetherian ring, where T is an indeterminate.*

PROOF. Put $\mathcal{A} = \bigoplus_{k\geq 0} \widehat{\mathcal{E}}_X(-k)T^k$, and define a filtration by $F_l(\mathcal{A}) = \bigoplus_{0\leq k\leq l} \widehat{\mathcal{E}}_X(-k)T^k$. We shall apply Theorem A.20. The conditions (1) and (2) are obvious; we show (3). Suppose that $\mathcal{N} \subset \mathcal{A}^{\oplus m}$ is an $\mathcal{A}$-submodule, and that $\mathcal{N} \cap F_l(\mathcal{A})^{\oplus m}$ are coherent $F_0(\mathcal{A})$-modules. Set $F_l(\mathcal{A}^{\oplus m}) = F_l(\mathcal{A})^{\oplus m}$ and $F_l(\mathcal{N}) = \mathcal{N} \cap F_l(\mathcal{A}^{\oplus m})$. Then $\mathrm{Gr}_k^F(\mathcal{N}) \subset \mathrm{Gr}_k^F(\mathcal{A}^{\oplus m}) = \widehat{\mathcal{E}}_X(-k)^{\oplus m}T^k$. Hence $\widehat{\mathcal{E}}_X(k)T^{-k} \otimes_{\widehat{\mathcal{E}}_X(0)} \mathrm{Gr}_k^F(\mathcal{N})$ is an increasing sequence of coherent $\widehat{\mathcal{E}}_X(0)$-submodules of

$\widehat{\mathcal{E}}_X(0)^{\oplus m}$. Thus locally

$$\begin{aligned}&\widehat{\mathcal{E}}_X(k)T^{-k} \otimes_{\widehat{\mathcal{E}}_X(0)} \mathrm{Gr}_k^F(\mathcal{N}) \\ &= \widehat{\mathcal{E}}_X(k-1)T^{1-k} \otimes_{\widehat{\mathcal{E}}_X(0)} \mathrm{Gr}_{k-1}^F(\mathcal{N}) \quad (k \geq k_0)\,.\end{aligned}$$

Hence

$$F_k(\mathcal{N}) \subset \widehat{\mathcal{E}}_X(-1)TF_{k-1}(\mathcal{N}) + F_{k-1}(\mathcal{N}) \quad (k > k_0).$$

This induces $F_k(\mathcal{N}) \subset \sum_{\nu=0}^{k-k_0} \widehat{\mathcal{E}}_X(-\nu)T^\nu F_{k_0}(\mathcal{N})$. Hence $\mathcal{N}$ is locally finitely generated. □

The following proposition of Artin–Rees type immediately follows from this; we omit its proof.

PROPOSITION 7.23. *For any $p \in \overset{\circ}{T}{}^*X$ and any finitely generated $\widehat{\mathcal{E}}_X(0)_p$-module M,*

$$\bigcap_{k \geq 0} \widehat{\mathcal{E}}_X(-k)_p M = 0\,.$$

7.5. Relations between $\mathcal{D}_X$ and $\widehat{\mathcal{E}}_X$

We shall describe relations between $\mathcal{D}_X$ and $\widehat{\mathcal{E}}_X$. To a $\mathcal{D}_X$-module $\mathcal{M}$, we can associate an $\widehat{\mathcal{E}}_X$-module $\widehat{\mathcal{E}}_X \otimes_{\pi_X^{-1}\mathcal{D}_X} \pi_X^{-1}\mathcal{M}$ (denoted by $\widehat{\mathcal{E}}_X \otimes_{\mathcal{D}_X} \mathcal{M}$ for short). Let

$$\mathrm{Sp} : \pi_X^{-1}\mathcal{M} \to \widehat{\mathcal{E}}_X \otimes_{\mathcal{D}_X} \mathcal{M}$$

denote the canonical morphism $u \mapsto 1 \otimes u$. Given a filtered $\mathcal{D}_X$-module $(\mathcal{M}, F)$, denote

$$\sum_k \widehat{\mathcal{E}}_X(-k)\,\mathrm{Sp}(F_{k+l}(\mathcal{M})) \subset \widehat{\mathcal{E}}_X \otimes_{\mathcal{D}_X} \mathcal{M}$$

by $F_l(\widehat{\mathcal{E}}_X \otimes_{\mathcal{D}_X} \mathcal{M})$ for any l. This is an $\widehat{\mathcal{E}}_X(0)$-submodule of $\widehat{\mathcal{E}}_X \otimes_{\mathcal{D}_X} \mathcal{M}$. Clearly we have

$$\widehat{\mathcal{E}}_X(k)F_l(\widehat{\mathcal{E}}_X \otimes_{\mathcal{D}_X} \mathcal{M})|_{\overset{\circ}{T}{}^*X} = F_{l+k}(\widehat{\mathcal{E}}_X \otimes_{\mathcal{D}_X} \mathcal{M})|_{\overset{\circ}{T}{}^*X}.$$

As below, considering filtered $\mathcal{D}_X$-modules corresponds to considering $\widehat{\mathcal{E}}(0)$-modules.

If $\mathcal{M}$ is a coherent $\mathcal{D}_X$-module, then $\widehat{\mathcal{E}}_X \otimes_{\mathcal{D}_X} \mathcal{M}$ is locally finitely presented and hence a coherent $\widehat{\mathcal{E}}_X$-module. If $(\mathcal{M}, F)$ is a coherent filtered $\mathcal{D}_X$-module, then $F_0(\widehat{\mathcal{E}}_X \otimes_{\mathcal{D}_X} \mathcal{M})$ is a coherent $\widehat{\mathcal{E}}_X(0)$-module for all $k \geq k_0$,

$$\mathcal{O}_{T^*X}(k) \otimes (\mathcal{M}_k/\mathcal{M}_{k+1}) = \mathcal{O}_{T^*X}(k+1) \otimes (\mathcal{M}_{k+1}/\mathcal{M}_{k+2}),$$

and thus

$$\mathcal{M}_{k+1} = \widehat{\mathcal{E}}_X(-1)\mathcal{M}_k + \mathcal{M}_{k+2}.$$

Hence by a weaker form of Lemma 7.14 (the one with locally finite generation in place of locally finite presentation in (1), already proved) $\mathcal{M}_{k_0}$ is a locally finitely generated $\widehat{\mathcal{E}}_X(0)$-module. Since $\mathcal{M}/\mathcal{M}_{k_0}$ is a coherent $\widehat{\mathcal{E}}_X(0)$-module, we have proved the locally finite generation of $\mathcal{M}$. Thus we have completed the proof of the strong form of Lemma 7.14. Hence $\mathcal{M}_{k_0}$ is locally finitely presented, and in turn $\mathcal{M}$ is also locally finitely presented. □

Using the result above, we prove the coherence of $\widehat{\mathcal{E}}_X(0)$.

PROPOSITION 7.16. *$\widehat{\mathcal{E}}_X(0)$ is a coherent ring.*

PROOF. Suppose that $\mathcal{I} \subset \widehat{\mathcal{E}}_X(0)$ is locally finitely generated. We need to show that $\mathcal{I}$ is locally finitely presented. First we work in $\overset{\circ}{T}{}^*X$. For any $k \geq 0$, put $\mathcal{I}_k = \mathcal{I} \cap \widehat{\mathcal{E}}_X(-k)$. Since $\mathcal{I}/\mathcal{I}_{k+1} \subset \widehat{\mathcal{E}}_X(0)/\widehat{\mathcal{E}}_X(-k-1)$ is a locally finitely generated $\widehat{\mathcal{E}}_X(0)/\widehat{\mathcal{E}}_X(-k-1)$-module, $\mathcal{I}/\mathcal{I}_{k+1}$ is a coherent $\widehat{\mathcal{E}}_X(0)/\widehat{\mathcal{E}}_X(-k-1)$-module. Hence

$$\mathcal{I}_k/\mathcal{I}_{k+1} = (\mathcal{I}/\mathcal{I}_{k+1}) \cap (\widehat{\mathcal{E}}_X(-k)/\widehat{\mathcal{E}}_X(-k-1))$$

is a coherent $\widehat{\mathcal{E}}_X(0)/\widehat{\mathcal{E}}_X(-k-1)$-module and thus a coherent $\mathcal{O}_{T^*X}(0)$-module. By Lemma 7.15, $\mathcal{I}$ is locally finitely presented.

Next we work in a neighborhood of $p \in T^*_XX$. Since $\widehat{\mathcal{E}}_X(0)_p = (\mathcal{O}_X)_{\pi(p)}$, there exist a coherent ideal $\mathcal{J} \subset \mathcal{O}_X$ and an exact sequence of $\mathcal{O}_X$-modules $\mathcal{O}_X^{\oplus N_1} \to \mathcal{O}_X^{\oplus N_0} \to \mathcal{J} \to 0$ such that $\mathcal{J}_{\pi(p)} = \mathcal{I}_p$. Then $\mathcal{I} = \widehat{\mathcal{E}}_X(0)\mathcal{J}$ in a neighborhood of p. Since $\widehat{\mathcal{E}}_X(0)$ is flat over $\mathcal{O}_X$, we obtain exact sequences

$$\widehat{\mathcal{E}}_X(0)^{\oplus N_1} \to \widehat{\mathcal{E}}_X(0)^{\oplus N_0} \to \widehat{\mathcal{E}}_X(0) \otimes_{\mathcal{O}_X} \mathcal{J} \to 0,$$

$$0 \to \widehat{\mathcal{E}}_X(0) \otimes_{\mathcal{O}_X} \mathcal{J} \to \widehat{\mathcal{E}}_X(0).$$

Hence $\mathcal{I} \cong \widehat{\mathcal{E}}_X(0) \otimes_{\mathcal{O}_X} \mathcal{J}$, and this is locally finitely presented. □

Next we show that $\widehat{\mathcal{E}}_X(0)$ is Noetherian.

PROPOSITION 7.17. *$\widehat{\mathcal{E}}_X(0)$ is a Noetherian ring.*

on $\mathring{T}^*X$. To see this, take k_0 such that $F_{k_0+\nu}(\mathcal{M}) = F_\nu(\mathcal{D}_X)F_{k_0}(\mathcal{M})$ $(\forall \nu \geq 0)$. Then

$$\begin{aligned} F_0(\widehat{\mathcal{E}}_X \otimes_{\mathcal{D}_X} \mathcal{M}) &= \sum_\nu \widehat{\mathcal{E}}_X(-k_0-\nu)\,\mathrm{Sp}(F_{k_0+\nu}(\mathcal{M})) \\ &= \widehat{\mathcal{E}}_X(-k_0)\,\mathrm{Sp}(F_{k_0}(\mathcal{M})). \end{aligned}$$

Thus $F_0(\widehat{\mathcal{E}}_X \otimes_{\mathcal{D}_X} \mathcal{M})$ is a locally finitely generated $\widehat{\mathcal{E}}_X(0)$-submodule.

For an $\widehat{\mathcal{E}}_X(0)$-module $\mathcal{N}$, we define an $\mathcal{O}_{\mathring{T}^*X}$-module $G(\mathcal{N})$ by

$$G(\mathcal{N}) = \mathcal{O}_{\mathring{T}^*X} \otimes_{\mathcal{O}_{\mathring{T}^*X}(0)} \left(\frac{\widehat{\mathcal{E}}_X(0)}{\widehat{\mathcal{E}}_X(-1)} \otimes_{\widehat{\mathcal{E}}_X(0)} \mathcal{N} \right).$$

Since $\mathcal{N} \mapsto G(\mathcal{N})$ is a right exact functor, it sends a coherent $\widehat{\mathcal{E}}_X(0)$-module to a coherent $\mathcal{O}_{\mathring{T}^*X}$-module. For a filtered $\mathcal{D}_X$-module $(\mathcal{M}, F)$, we have a morphism on $\mathring{T}^*X$ as the following composition:

$$\begin{aligned} \mathring{\pi}_X^{-1} F_k(\mathcal{M}) &\to \widehat{\mathcal{E}}_X(k) \otimes_{\widehat{\mathcal{E}}_X(0)} \widehat{\mathcal{E}}_X(-k) \otimes_{\pi_X^{-1}\mathcal{O}_X} \pi_X^{-1} F_k(\mathcal{M}) \\ &\to \widehat{\mathcal{E}}_X(k) \otimes_{\widehat{\mathcal{E}}_X(0)} F_0(\widehat{\mathcal{E}}_X \otimes_{\mathcal{D}_X} \mathcal{M}) \\ &\to \mathcal{O}_{\mathring{T}^*X}(k) \otimes_{\widehat{\mathcal{E}}_X(0)} F_0(\widehat{\mathcal{E}}_X \otimes_{\mathcal{D}_X} \mathcal{M}) \\ &\to G(F_0(\widehat{\mathcal{E}}_X \otimes_{\mathcal{D}_X} \mathcal{M})). \end{aligned}$$

This gives rise to

$$\mathring{\pi}_X^{-1} \operatorname{Gr}^F(\mathcal{M}) \to G(F_0(\widehat{\mathcal{E}}_X \otimes_{\mathcal{D}_X} \mathcal{M})).$$

Since this is $\operatorname{Gr}^F(\mathcal{D}_X)$-linear, we obtain a homomorphism of $\mathcal{O}_{\mathring{T}^*X}$-modules

$$(\operatorname{Gr}^F \mathcal{M})^\sim|_{\mathring{T}^*X} \to G(F_0(\widehat{\mathcal{E}}_X \otimes_{\mathcal{O}_X} \mathcal{M})), \tag{7.22}$$

where $(\operatorname{Gr}^F \mathcal{M})^\sim := \mathcal{O}_{T^*X} \otimes_{\pi_X^{-1}\operatorname{Gr}^F(\mathcal{D}_X)} \pi_X^{-1} \operatorname{Gr}^F(\mathcal{M})$ as in § 2.2. This is an isomorphism if $\mathcal{M} = \mathcal{D}_X$, and hence an isomorphism if $\mathcal{M}$ is a free filtered module, or if it is isomorphic to $\bigoplus_j \mathcal{D}_X(m_j)$. The following lemma is a key to the subsequent results.

LEMMA 7.24. *Let $\mathcal{L}_\nu$ $(\nu = 1, 2, 3)$ be free filtered modules, and $\mathcal{L}_1 \xrightarrow{g} \mathcal{L}_2 \xrightarrow{f} \mathcal{L}_3$ a complex of filtered $\mathcal{D}_X$-modules. If*

$$\operatorname{Gr}^F(\mathcal{L}_1) \to \operatorname{Gr}^F(\mathcal{L}_2) \to \operatorname{Gr}^F(\mathcal{L}_3)$$

is exact, then the sequences on $\overset{\circ}{T^*}X$

$$F_0(\widehat{\mathcal{E}}_X \otimes_{\mathcal{D}_X} \mathcal{L}_1) \to F_0(\widehat{\mathcal{E}}_X \otimes_{\mathcal{D}_X} \mathcal{L}_2) \to F_0(\widehat{\mathcal{E}}_X \otimes_{\mathcal{D}_X} \mathcal{L}_3)\,, \tag{7.23}$$

$$\widehat{\mathcal{E}}_X \otimes_{\mathcal{D}_X} \mathcal{L}_1 \xrightarrow{\tilde{g}} \widehat{\mathcal{E}}_X \otimes_{\mathcal{D}_X} \mathcal{L}_2 \xrightarrow{\tilde{f}} \widehat{\mathcal{E}}_X \otimes_{\mathcal{D}_X} \mathcal{L}_3 \tag{7.24}$$

are exact. Moreover,

$$\tilde{f}(\widehat{\mathcal{E}}_X \otimes_{\mathcal{D}_X} \mathcal{L}_2) \cap F_0(\widehat{\mathcal{E}}_X \otimes_{\mathcal{D}_X} \mathcal{L}_3) = \tilde{f} F_0(\widehat{\mathcal{E}}_X \otimes_{\mathcal{D}_X} \mathcal{L}_2).$$

PROOF. In the commutative diagram

$$\begin{tikzcd}
(\mathrm{Gr}^F \mathcal{L}_1)^\sim \arrow[r] \arrow[d, "\wr"] & (\mathrm{Gr}^F \mathcal{L}_2)^\sim \arrow[r] \arrow[d, "\wr"] & (\mathrm{Gr}^F \mathcal{L}_3)^\sim \arrow[d, "\wr"] \\
G(F_0(\widehat{\mathcal{E}}_X \otimes_{\mathcal{D}_X} \mathcal{L}_1)) \arrow[r] & G(F_0(\widehat{\mathcal{E}}_X \otimes_{\mathcal{D}_X} \mathcal{L}_2)) \arrow[r] & G(F_0(\widehat{\mathcal{E}}_X \otimes_{\mathcal{D}_X} \mathcal{L}_3))\,,
\end{tikzcd}$$

the first row is exact, and the vertical homomorphisms are isomorphisms. Hence the second row is also exact. Since $\mathcal{O}_{\overset{\circ}{T^*}X}$ is faithfully flat over $\mathcal{O}_{\overset{\circ}{T^*}X}(0)$, we see the exactness of the sequence obtained by applying $(\widehat{\mathcal{E}}_X(0)/\widehat{\mathcal{E}}_X(-1)) \otimes_{\widehat{\mathcal{E}}_X(0)} \bullet$ to (7.23). By Proposition 7.21, we obtain the desired result. □

We prove the following theorem by using this lemma.

THEOREM 7.25. $\widehat{\mathcal{E}}_X$ *is flat over* $\pi_X^{-1}\mathcal{D}_X$.

PROOF. The assertion is trivial on the zero section $T^*_X X$; we prove it on $\overset{\circ}{T^*}X$. It is enough to show that

$$\mathcal{T}or_1^{\mathcal{D}_X}(\widehat{\mathcal{E}}_X, \mathcal{M}) = 0 \tag{7.25}$$

for any coherent $\mathcal{D}_X$-module $\mathcal{M}$. Take a coherent filtration of $\mathcal{M}$ and a resolution by free filtered modules

$$\mathcal{L}_2 \to \mathcal{L}_1 \to \mathcal{L}_0 \to \mathcal{M} \to 0.$$

Then by Lemma 7.24 the sequence

$$\widehat{\mathcal{E}}_X \otimes_{\mathcal{D}_X} \mathcal{L}_2 \to \widehat{\mathcal{E}}_X \otimes_{\mathcal{D}_X} \mathcal{L}_1 \to \widehat{\mathcal{E}}_X \otimes_{\mathcal{D}_X} \mathcal{L}_0$$

is exact on $\overset{\circ}{T^*}X$. Since its cohomology is $\mathcal{T}or_1^{\mathcal{D}_X}(\widehat{\mathcal{E}}_X, \mathcal{M})$, we obtain (7.25). □

PROPOSITION 7.26. *Let* $(\mathcal{M}, F)$ *be a coherent filtered* $\mathcal{D}_X$*-module. Then the homomorphism of* $\mathcal{O}_{\overset{\circ}{T^*}X}$*-modules*

$$(\mathrm{Gr}^F \mathcal{M})^\sim|_{\overset{\circ}{T^*}X} \to G(F_0(\widehat{\mathcal{E}}_X \otimes_{\mathcal{D}_X} \mathcal{M}))$$

is an isomorphism.

PROOF. Let

$$\mathcal{L}_2 \xrightarrow{\varphi_2} \mathcal{L}_1 \xrightarrow{\varphi_1} \mathcal{L}_0 \xrightarrow{\varphi_0} \mathcal{M} \longrightarrow 0$$

be a resolution of $(\mathcal{M}, F)$ by free filtered modules $\mathcal{L}_\nu$ ($\nu = 0, 1, 2$). Then we have a commutative diagram

$$\begin{array}{ccccccc}
F_0(\widehat{\mathcal{E}}_X \otimes_{\mathcal{D}_X} \mathcal{L}_1) & \longrightarrow & F_0(\widehat{\mathcal{E}}_X \otimes_{\mathcal{D}_X} \mathcal{L}_0) & \longrightarrow & F_0(\widehat{\mathcal{E}}_X \otimes_{\mathcal{D}_X} \mathcal{M}) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
\widehat{\mathcal{E}}_X \otimes_{\mathcal{D}_X} \mathcal{L}_1 & \xrightarrow{\tilde{\varphi}_1} & \widehat{\mathcal{E}}_X \otimes_{\mathcal{D}_X} \mathcal{L}_0 & \xrightarrow{\tilde{\varphi}_0} & \widehat{\mathcal{E}}_X \otimes_{\mathcal{D}_X} \mathcal{M} & \longrightarrow & 0.
\end{array}$$

The second row is exact. We have

$$\tilde{\varphi}_0(F_0(\widehat{\mathcal{E}}_X \otimes_{\mathcal{D}_X} \mathcal{L}_0)) = F_0(\widehat{\mathcal{E}}_X \otimes_{\mathcal{D}_X} \mathcal{M}),$$

and Lemma 7.24 implies

$$\tilde{\varphi}_1(\widehat{\mathcal{E}}_X \otimes_{\mathcal{D}_X} \mathcal{L}_1) \cap F_0(\widehat{\mathcal{E}}_X \otimes_{\mathcal{D}_X} \mathcal{L}_0) = \tilde{\varphi}_1(F_0(\widehat{\mathcal{E}}_X \otimes_{\mathcal{D}_X} \mathcal{L}_1)).$$

Hence

$$F_0(\widehat{\mathcal{E}}_X \otimes_{\mathcal{D}_X} \mathcal{L}_1) \to F_0(\widehat{\mathcal{E}}_X \otimes_{\mathcal{D}_X} \mathcal{L}_0) \to F_0(\widehat{\mathcal{E}}_X \otimes_{\mathcal{D}_X} \mathcal{M}) \to 0$$

is exact. This yields the exactness of the second row in the following diagram:

$$\begin{array}{ccccccc}
(\mathrm{Gr}^F \mathcal{L}_1)^\sim & \to & (\mathrm{Gr}^F \mathcal{L}_0)^\sim & \to & (\mathrm{Gr}^F \mathcal{M})^\sim & \to 0 \\
\wr\downarrow & & \wr\downarrow & & \downarrow & & \\
G(F_0(\widehat{\mathcal{E}}_X \otimes_{\mathcal{D}_X} \mathcal{L}_1)) & \to & G(F_0(\widehat{\mathcal{E}}_X \otimes_{\mathcal{D}_X} \mathcal{L}_0)) & \to & G(F_0(\widehat{\mathcal{E}}_X \otimes_{\mathcal{D}_X} \mathcal{M})) & \to 0.
\end{array}$$

Since the first row is also exact, we obtain the desired result. $\square$

Recall that for a coherent filtered $\mathcal{D}_X$-module $(\mathcal{M}, F)$ the characteristic variety $\mathrm{Ch}(\mathcal{M})$ of $\mathcal{M}$ is the support of the $\mathcal{O}_{\overset{\circ}{T^*}X}$-module $(\mathrm{Gr}^F \mathcal{M})^\sim$. We hence obtain the following theorem.

THEOREM 7.27. *If $\mathcal{M}$ is a coherent $\mathcal{D}_X$-module, then*

$$\mathrm{Supp}(\widehat{\mathcal{E}}_X \otimes_{\mathcal{D}_X} \mathcal{M}) = \mathrm{Ch}(\mathcal{M})\,.$$

Note that this theorem is trivial on the zero section $T^*_X X$, since $\widehat{\mathcal{E}}_X|_{T^*_X X} = \mathcal{D}_X$ and $\mathrm{Ch}(\mathcal{M}) \cap T^*_X X = \mathrm{Supp}(\mathcal{M})$.

LEMMA 7.28. *Let $(\mathcal{M}, F)$ be a coherent filtered $\mathcal{D}_X$-module. Put*

$$F'_k = \{u \in \mathcal{M}; \mathrm{Sp}(u)|_{\overset{\circ}{T^*}X} \in F_k(\widehat{\mathcal{E}}_X \otimes_{\mathcal{D}_X} \mathcal{M})\}.$$

Then

(1) *all F'_k are coherent $\mathcal{O}_X$-modules,*
(2) *locally on X we have $F'_k = F_k$ $(k \gg 0)$, and*
(3) *$F_k/F_{k-1} \to \overset{\circ}{\pi}_{X*}(F_k(\widehat{\mathcal{E}}_X \otimes_{\mathcal{D}_X} \mathcal{M})/F_{k-1}(\widehat{\mathcal{E}}_X \otimes_{\mathcal{D}_X} \mathcal{M}))$ is an isomorphism for $k \gg 0$, where $\overset{\circ}{\pi}_X$ is the projection $\overset{\circ}{T^*}X \to X$.*

PROOF. Let P^*X be the projective bundle associated with T^*X, namely $P^*X = \overset{\circ}{T^*}X/\mathbb{C}^*$.

Let $\bar{\pi}_X : P^*X \to X$ and $\gamma : \overset{\circ}{T^*}X \to P^*X$ be projections. Since $F_k(\widehat{\mathcal{E}}_X \otimes_{\mathcal{D}_X} \mathcal{M})$ is a constant sheaf on each fiber of γ, we have

$$F_k(\widehat{\mathcal{E}}_X \otimes_{\mathcal{D}_X} \mathcal{M}) = \gamma^{-1}\gamma_* F_k(\widehat{\mathcal{E}}_X \otimes_{\mathcal{D}_X} \mathcal{M}),$$
$$F_k(\widehat{\mathcal{E}}_X \otimes_{\mathcal{D}_X} \mathcal{M})/F_{k-1}(\widehat{\mathcal{E}}_X \otimes_{\mathcal{D}_X} \mathcal{M}) = \gamma^{-1}\mathcal{F}_k,$$

where

$$\mathcal{F}_k := \gamma_*(F_k(\widehat{\mathcal{E}}_X \otimes_{\mathcal{D}_X} \mathcal{M})/F_{k-1}(\widehat{\mathcal{E}}_X \otimes_{\mathcal{D}_X} \mathcal{M})).$$

Since $F_k(\widehat{\mathcal{E}}_X \otimes_{\mathcal{D}_X} \mathcal{M})$ is a coherent $\widehat{\mathcal{E}}_X(0)$-module, $\mathcal{F}_k$ is a coherent $\mathcal{O}_{P^*X}$-module, and $\mathcal{F}_k = \mathcal{O}_{P^*X}(k) \otimes_{\mathcal{O}_{P^*X}} \mathcal{F}_0$. Moreover, $\mathcal{F}_0$ is the coherent sheaf on P^*X associated with $\mathrm{Gr}^F(\mathcal{M})$. Hence by Serre's Theorem

$$\bar{\pi}_{X*}\mathcal{F}_k \text{ are coherent } \mathcal{O}_X\text{-modules for all } k, \tag{7.26}$$

$$\text{locally on } X,\ \ \mathrm{Gr}^F_k(\mathcal{M}) \simeq \bar{\pi}_{X*}\mathcal{F}_k \ (k \gg 0). \tag{7.27}$$

(3) is obtained from (7.27).

We have

$$\overset{\circ}{\pi}_{X*}\Big(\frac{F_k(\widehat{\mathcal{E}}_X \otimes_{\mathcal{D}_X} \mathcal{M})}{F_{k-1}(\widehat{\mathcal{E}}_X \otimes_{\mathcal{D}_X} \mathcal{M})}\Big) = \bar{\pi}_{X*}\mathcal{F}_k.$$

By the definition, $F'_k \supset F_k$, and the sequences

$$0 \to \frac{F'_k \cap F_{k+1}}{F_k} \to \frac{F_{k+1}}{F_k} \to \bar{\pi}_{X*}\mathcal{F}_{k+1}, \tag{7.28}$$

$$0 \to F'_{k-1} \to F'_k \to \bar{\pi}_{X*}\mathcal{F}_k \tag{7.29}$$

are exact. Take k_0 such that $\mathrm{Gr}^F_k(\mathcal{M}) \simeq \bar{\pi}_{X*}\mathcal{F}_k$ for all $k \geq k_0$. Then

$$F'_k \cap F_{k+1} = F_k \quad (k \geq k_0). \tag{7.30}$$

We now show that

$$F'_k = F_k \quad (k \geq k_0). \tag{7.31}$$

Since $F'_k \cap F_l \subset F'_{l-1} \cap F_l = F_{l-1}$ for $l > k \geq k_0$, by induction on l we obtain $F'_k \cap F_l = F_k$ for all $l > k \geq k_0$. Hence we obtain (7.31).

Finally we show that all F'_k are coherent $\mathcal{O}_X$-modules, by descending induction on k. For $k \geq k_0$, $F'_k = F_k$ is a coherent $\mathcal{O}_X$-module, and hence by (7.29) F'_{k-1} is also a coherent $\mathcal{O}_X$-module. □

REMARK 7.29. We cannot conclude that F' is a coherent filtration of $\mathcal{M}$, because $F'_k = 0$ ($k \ll 0$) does not hold in general. However, for any k_0, a filtration $F'^{\geq k_0}$ defined by

$$(F'^{\geq k_0})_k = \begin{cases} F'_k & (k \geq k_0), \\ 0 & (k < k_0) \end{cases}$$

is a coherent filtration.

The reason why $F_k = F'_k$ do not hold for all k is that the effect of the zero section $T^*_X X$ is not ignorable. For example, if $\mathcal{M} = \mathcal{O}_X$, then $\operatorname{Supp}(\widehat{\mathcal{E}}_X \otimes_{\mathcal{D}_X} \mathcal{M}) = T^*_X X$, and thus $F'_k = \mathcal{M}$ for all $k \in \mathbb{Z}$.

PROPOSITION 7.30. *Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module, and $\mathcal{N}_0$ a coherent $\widehat{\mathcal{E}}_X(0)$-submodule (defined on $\mathring{T}^*X$) of $\widehat{\mathcal{E}}_X \otimes_{\mathcal{D}_X} \mathcal{M}|_{\mathring{T}^*X}$ such that $\widehat{\mathcal{E}}_X \otimes_{\mathcal{D}_X} \mathcal{M}|_{\mathring{T}^*X} = \widehat{\mathcal{E}}_X \cdot \mathcal{N}_0|_{\mathring{T}^*X}$. Then*

$$F_k(\mathcal{M}) := \{\, u \in \mathcal{M}\,;\, \operatorname{Sp}(u)|_{\mathring{T}^*X} \in \mathcal{N}_0(k) \,\}$$

is a coherent $\mathcal{O}_X$-module, and $F^{\geq k_0}$ is a coherent filtration of $\mathcal{M}$ for any k_0. Furthermore, the coherent $\widehat{\mathcal{E}}_X(0)$-module $F_0(\widehat{\mathcal{E}}_X \otimes_{\mathcal{D}_X} \mathcal{M})$ corresponding to $(\mathcal{M}, F^{\geq k_0})$ coincides with $\mathcal{N}_0$.

PROOF. Take a coherent filtration ${}^0F$ of $\mathcal{M}$, and consider the $\widehat{\mathcal{E}}_X(0)$-module $\mathcal{M}_0 = {}^0F_0(\widehat{\mathcal{E}}_X \otimes_{\mathcal{D}_X} \mathcal{M})$ corresponding to $(\mathcal{M}, {}^0F)$. Put ${}^0F'_k = \{u \in \mathcal{M}; \operatorname{Sp}(u)|_{\mathring{T}^*X} \in \mathcal{M}_0(k)|_{\mathring{T}^*X}\}$. Then ${}^0F_k = {}^0F'_k$ for $k \gg 0$.

Taking N large enough, we may assume

$$\mathcal{M}_0 \subset \mathcal{N}_0 \subset \mathcal{M}_0(N).$$

We prove the proposition by induction on N. It has already been proved when $N = 0$.

(1) The case $N = 1$. Since $\mathcal{M}_0(k+1)/\mathcal{N}_0(k)$ is a coherent $\mathcal{O}_{\mathring{T}^*X}(0)$-module, $\mathring{\pi}_{X*}(\mathcal{M}_0(k+1)/\mathcal{N}_0(k))$ is a coherent $\mathcal{O}_X$-module,

and

$$0 \to F_k \to {}^0F'_{k+1} \to \overset{\circ}{\pi}_{X*}(\mathcal{M}_0(k+1)/\mathcal{N}_0(k)) \tag{7.32}$$

is exact. Hence all F_k are coherent $\mathcal{O}_X$-modules, and $F^{\geq k_0}$ is a coherent filtration, since it is a subfiltration of ${}^0F'(1)^{\geq k_0}$. Applying Serre's Theorem to the exact sequence of $\mathcal{O}_{\overset{\circ}{T^*}X}(0)$-modules

$$0 \to \frac{\mathcal{N}_0(k)}{\mathcal{M}_0(k)} \to \frac{\mathcal{M}_0(k+1)}{\mathcal{M}_0(k)} \to \frac{\mathcal{M}_0(k+1)}{\mathcal{N}_0(k)} \to 0\,,$$

we see that the second row of the following commutative diagram is exact:

$$\begin{array}{ccccccccc}
0\to & F_k/{}^0F_k & \to & {}^0F_{k+1}/{}^0F_k & \to & {}^0F_{k+1}/F_k & \to 0 \\
 & \downarrow & & \downarrow & & \downarrow & \\
0\to \overset{\circ}{\pi}_{X*} & \left(\dfrac{\mathcal{N}_0(k)}{\mathcal{M}_0(k)}\right) & \to \overset{\circ}{\pi}_{X*} & \left(\dfrac{\mathcal{M}_0(k+1)}{\mathcal{M}_0(k)}\right) & \to \overset{\circ}{\pi}_{X*} & \left(\dfrac{\mathcal{M}_0(k+1)}{\mathcal{N}_0(k)}\right) & \to 0\,.
\end{array}$$

Since the middle vertical arrow is an isomorphism for $k \gg 0$, the right vertical arrow ${}^0F_{k+1}/F_k \to \overset{\circ}{\pi}_{X*}\left(\frac{\mathcal{M}_0(k+1)}{\mathcal{N}_0(k)}\right)$ is also an isomorphism for $k \gg 0$. Accordingly the left vertical arrow $F_k/{}^0F_k \to \overset{\circ}{\pi}_{X*}(\mathcal{N}_0(k)/\mathcal{M}_0(k))$ is an isomorphism for $k \gg 0$ as well. Again applying Serre's Theorem to the exact sequence of $\mathcal{O}_{\overset{\circ}{T^*}X}(0)$-modules

$$0 \to \mathcal{M}_0(k)/\mathcal{N}_0(k-1) \to \mathcal{N}_0(k)/\mathcal{N}_0(k-1) \to \mathcal{N}_0(k)/\mathcal{M}_0(k) \to 0,$$

we see that the second row of the following diagram is exact:

$$\begin{array}{ccccccccc}
0\to & \dfrac{{}^0F_k}{F_{k-1}} & \to & \dfrac{F_k}{F_{k-1}} & \to & \dfrac{F_k}{{}^0F_k} & \to 0 \\
 & \downarrow\wr & & \downarrow & & \downarrow\wr & \\
0\to \overset{\circ}{\pi}_{X*} & \left(\dfrac{\mathcal{M}_0(k)}{\mathcal{N}_0(k-1)}\right) & \to \overset{\circ}{\pi}_{X*} & \left(\dfrac{\mathcal{N}_0(k)}{\mathcal{N}_0(k-1)}\right) & \to \overset{\circ}{\pi}_{X*} & \left(\dfrac{\mathcal{N}_0(k)}{\mathcal{M}_0(k)}\right) & \to 0\,.
\end{array}$$

Hence $F_k/F_{k-1} \to \overset{\circ}{\pi}_{X*}(\mathcal{N}_0(k)/\mathcal{N}_0(k-1))$, and accordingly

$$\overset{\circ}{\pi}_{X*}(F_k(\widehat{\mathcal{E}}_X \otimes_{\mathcal{D}_X} \mathcal{M})/F_{k-1}(\widehat{\mathcal{E}}_X \otimes_{\mathcal{D}_X} \mathcal{M})) \to \overset{\circ}{\pi}_{X*}(\mathcal{N}_0(k)/\mathcal{N}_0(k-1))$$

are isomorphisms for $k \gg 0$. Therefore

$$F_0(\widehat{\mathcal{E}}_X \otimes_{\mathcal{D}_X} \mathcal{M})/F_{-1}(\widehat{\mathcal{E}}_X \otimes_{\mathcal{D}_X} \mathcal{M}) \to \mathcal{N}_0/\mathcal{N}_0(-1)$$

is an isomorphism. From this, we obtain

$$\mathcal{N}_0 = F_0(\widehat{\mathcal{E}}_X \otimes_{\mathcal{D}_X} \mathcal{M}).$$

(2) The case $N > 1$. Put $\mathcal{N}_0' = \mathcal{N}_0 \cap \mathcal{M}_0(1)$. Then

$$\mathcal{M}_0 \subset \mathcal{N}_0' \subset \mathcal{M}_0(1).$$

Put

$$F_k' = \{\, u \in \mathcal{M}\,;\, \mathrm{Sp}(u) \in \mathcal{N}_0'(k)\,\}.$$

Then, by the case $N = 1$, $F_0'(\widehat{\mathcal{E}}_X \otimes_{\mathcal{D}_X} \mathcal{M})$ equals $\mathcal{N}_0'$. Furthermore it follows from the inclusions

$$\mathcal{N}_0' \subset \mathcal{N}_0 \subset \mathcal{N}_0'(N-1)$$

that we obtain the desired result by the induction hypothesis. □

Proposition 7.30 means that considering a coherent $\widehat{\mathcal{E}}_X(0)$-submodule of $\widehat{\mathcal{E}}_X \otimes_{\mathcal{D}_X} \mathcal{M}$ corresponds to considering a coherent filtration of $\mathcal{M}$.

Since many propositions about $\widehat{\mathcal{E}}_X(0)$-modules hold only on $\mathring{T}^*X$, the next proposition is often useful. Given a manifold X, put $Z = X \times \mathbb{C}$. Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module, and put $\mathcal{N} = \mathcal{M} \boxtimes \mathcal{B}_{\{0\}|\mathbb{C}}$. Let F be a coherent filtration of $\mathcal{M}$, and put

$$F_k(\mathcal{N}) = \sum_\nu F_{k-\nu}(\mathcal{M}) \boxtimes F_\nu(\mathcal{D}_\mathbb{C})\delta(t) \subset \mathcal{N}.$$

Then $(\mathcal{N}, F)$ is a coherent filtered $\mathcal{D}_Z$-module. As in the first paragraph of this section, we define $\mathrm{Sp} : \pi_Z^{-1}\mathcal{N} \to \widehat{\mathcal{E}}_Z \otimes_{\mathcal{D}_Z} \mathcal{N}$ and put $F_0(\widehat{\mathcal{E}}_Z \otimes_{\mathcal{D}_Z} \mathcal{N}) = \sum_k \widehat{\mathcal{E}}_Z(-k)\,\mathrm{Sp}(F_k(\mathcal{N}))$. Then we have the following result.

LEMMA 7.31.

$$F_k(\mathcal{M}) = \{\, u \in \mathcal{M}\,;\, \mathrm{Sp}(u \boxtimes \delta(t)) \in F_0(\widehat{\mathcal{E}}_Z \otimes_{\mathcal{D}_Z} \mathcal{N})(k)|_{\mathring{T}^*Z}\,\}$$

for all k.

PROOF. It is obvious that $\mathrm{Sp}(F_k(\mathcal{M}) \boxtimes \delta(t)) \subset F_k(\widehat{\mathcal{E}}_X \otimes_{\mathcal{D}_X} \mathcal{N})$; we prove the converse. We have

$$(\mathrm{Gr}^F \mathcal{N})^\sim = i_* p^* (\mathrm{Gr}^F \mathcal{M})^\sim,$$

where $p : T^*X \times T^*_{\{0\}}\mathbb{C} \to T^*X$ and $i : T^*X \times T^*_{\{0\}}\mathbb{C} \hookrightarrow T^*Z$. Hence, for $u \in \mathrm{Gr}^F \mathcal{M}$, $u \boxtimes \delta(t)$ equals 0 in $(\mathrm{Gr}^F \mathcal{N})^\sim|_{\mathring{T}^*Z}$ if and only if u equals 0. Thus

$$F_{k-1}(\mathcal{M}) = \{\, u \in F_k(\mathcal{M})\,;\, \mathrm{Sp}(u \boxtimes \delta(t)) \in F_{k-1}(\widehat{\mathcal{E}}_Z \otimes_{\mathcal{D}_Z} \mathcal{N})\,\}.$$

From this and $\mathcal{M} = \bigcup F_k(\mathcal{M})$, we obtain the desired result. □

7.6. Involutivity of the Supports of Coherent $\widehat{\mathcal{E}}$-modules

In this section, we shall prove the involutivity of the supports of $\widehat{\mathcal{E}}$-modules by Gabber's result. We prove it in a more general form and deduce the involutivity of the characteristic varieties of $\mathcal{D}$-modules. The strategy of the proof is basically the same as that in Chapter 3 of [T], but they look different.

Let Ω be an open set in T^*X, and S a closed analytic subset of Ω. Let $I_S := \{u \in \mathcal{O}_\Omega; u|_S = 0\}$ be the defining ideal of S. Recall that S is said to be involutive if $\{I_S, I_S\} \subset I_S$.

LEMMA 7.32. *For any closed analytic subset S, there exists the largest involutive closed analytic subset (denoted by $I(S)$ from now on) contained in S.*

PROOF. For an ideal I of $\mathcal{O}_{T^*X}$, we denote by $\{I, I\}$ the $\mathbb{Z}$-submodule of $\mathcal{O}_{T^*X}$ generated by $\{f, g\}$ $(f, g \in I)$. If $I = \sum_i \mathcal{O}_X u_i$, then $I + \{I, I\} = \sum \mathcal{O} u_i + \sum \mathcal{O}\{u_i, u_j\}$. Hence, if I is a coherent ideal, then $I + \{I, I\}$ is also a coherent ideal. Define an increasing sequence $\{I_k\}$ of coherent ideals of $\mathcal{O}_\Omega$ inductively by

$$I_k = \begin{cases} I_S & (k = 0)\,, \\ \sqrt{I_{k-1} + \{I_{k-1}, I_{k-1}\}} & (k > 0)\,, \end{cases}$$

where for a coherent ideal J

$$\sqrt{J} := \{\, u \in \mathcal{O}\,;\, u^n \in J \ (n \gg 0)\,\} = \{\, u \in \mathcal{O}\,;\, u|_{\mathrm{Supp}(\mathcal{O}/J)} = 0\,\}.$$

Hence, by the Noetherian property of $\mathcal{O}$, $\tilde{I} := \bigcup I_k$ is a coherent ideal, $\{\tilde{I}, \tilde{I}\} \subset \tilde{I}$, and $\sqrt{\tilde{I}} = \tilde{I}$. Conversely, any coherent ideal J satisfying $I_S \subset J$, $\{J, J\} \subset J$, and $\sqrt{J} = J$, also satisfies $\tilde{I} \subset J$. This means that $\mathrm{Supp}(\mathcal{O}_\Omega/\tilde{I})$ is the largest involutive analytic subset contained in S. $\square$

THEOREM 7.33 (Gabber). *Let Ω be an open set in $\overset{\circ}{T}{}^*X$, $\mathcal{M}$ a coherent $\widehat{\mathcal{E}}_X|_\Omega$-module, and $\mathcal{M}_0$ a coherent $\widehat{\mathcal{E}}_X(0)|_\Omega$-submodule of $\mathcal{M}$ satisfying $\mathcal{M} = \widehat{\mathcal{E}}_X \mathcal{M}_0$. If S is a homogeneous closed analytic subset of Ω satisfying $I(S) = \emptyset$, then*

$$\mathcal{M}_0' := \{\, u \in \mathcal{M}\,;\, u \text{ belongs to } \mathcal{M}_0 \text{ outside } S\,\}$$

is a coherent $\widehat{\mathcal{E}}_X(0)|_\Omega$-module.

This theorem can be variously restated. The following version is suitable for applications.

THEOREM 7.34. *Let Ω be an open set in $\overset{\circ}{T}{}^*X$, $\mathcal{M}$ a coherent $\widehat{\mathcal{E}}_X|_\Omega$-module, and $\mathcal{N} \subset \mathcal{M}$ an $\widehat{\mathcal{E}}_X(0)|_\Omega$-module. Assume that $\mathcal{N}$ is a sum of coherent $\widehat{\mathcal{E}}_X(0)|_\Omega$-modules. Let U be the set of $p \in \Omega$ in a neighborhood of which $\mathcal{N}$ is a coherent $\widehat{\mathcal{E}}_X(0)|_\Omega$-module. Then $\Omega \setminus U$ is an involutive analytic subset.*

PROOF. Assuming Theorem 7.33, we prove this theorem. Let $\mathcal{L} \subset \mathcal{M}$ be a coherent $\widehat{\mathcal{E}}_X(0)$-module satisfying $\widehat{\mathcal{E}}_X\mathcal{L} = \mathcal{M}$. Then, for any $k \in \mathbb{Z}$, $\mathcal{L}(k) \cap \mathcal{N}$ is a coherent $\widehat{\mathcal{E}}_X(0)$-module. Since

$$\begin{aligned}\widehat{\mathcal{E}}_X(1) \otimes_{\widehat{\mathcal{E}}_X(0)} \left(\frac{\mathcal{L}(k) \cap \mathcal{N}}{\mathcal{L}(k-1) \cap \mathcal{N}}\right) &= \frac{\mathcal{L}(k+1) \cap \mathcal{N}(1)}{\mathcal{L}(k) \cap \mathcal{N}(1)} \\ &\supset \frac{\mathcal{L}(k+1) \cap \mathcal{N}}{\mathcal{L}(k) \cap \mathcal{N}},\end{aligned}$$

$\mathrm{Supp}((\mathcal{L}(k) \cap \mathcal{N})/(\mathcal{L}(k-1) \cap \mathcal{N}))$ is a decreasing sequence. Hence for k_0 large enough

$$S = \mathrm{Supp}((\mathcal{L}(k) \cap \mathcal{N})/(\mathcal{L}(k-1) \cap \mathcal{N})) \quad (k \geq k_0).$$

Hence $\mathcal{N} = \mathcal{L}(k_0) \cap \mathcal{N}$ outside S, and $S = \Omega \setminus U$. We prove that S is involutive. Assume the contrary, and take p from $S \setminus I(S)$. Then $I(S)$ is empty in a neighborhood of p. Hence, by Theorem 7.33, $\mathcal{N}$ is a coherent $\widehat{\mathcal{E}}_X(0)$-module on a neighborhood of p, which is absurd. □

Before proving Theorem 7.33, we show the following corollary to Theorem 7.34.

THEOREM 7.35. *Let Ω be an open set in $\overset{\circ}{T}{}^*X$, and $\mathcal{M}$ a coherent $\widehat{\mathcal{E}}_X|_\Omega$-module. Then* $\mathrm{Supp}(\mathcal{M})$ *is involutive.*

PROOF. We apply Theorem 7.34 to $\mathcal{N} = \mathcal{M}$. Suppose that $\mathcal{N}$ is coherent over $\widehat{\mathcal{E}}_X(0)$. Then from $\widehat{\mathcal{E}}_X(-1)\mathcal{N} = \mathcal{N}$ we obtain $\mathcal{N} = 0$ by Nakayama's Lemma.

Hence $\mathrm{Supp}(\mathcal{M})$ corresponds to $\Omega \setminus U$ in Theorem 7.34. □

This leads to the following corollary, stated in § 2.3.

COROLLARY 7.36. *Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module. Then its characteristic variety* $\mathrm{Ch}(\mathcal{M})$ *is involutive.*

PROOF. By Theorems 7.27 and 7.35, $\mathrm{Ch}(\mathcal{M})$ is involutive in $\overset{\circ}{T}{}^*X$. From here, there are several ways to show the involutivity in the whole T^*X. For example, we can use the following trick. Put

$Z = X \times \mathbb{C}$ and $\mathcal{N} = \mathcal{M} \boxtimes \mathcal{B}_{\{0\}|\mathbb{C}}$. Then $\mathcal{N}$ is a coherent $\mathcal{D}_Z$-module, and $\mathrm{Ch}(\mathcal{N}) = \mathrm{Ch}(\mathcal{M}) \times T^*_{\{0\}}\mathbb{C}$. Hence $\mathrm{Ch}(\mathcal{M}) \times \mathring{T}^*_{\{0\}}\mathbb{C}$ is involutive in $T^*X \times \mathring{T}^*\mathbb{C}$. Therefore $\mathrm{Ch}(\mathcal{M})$ is involutive. $\square$

PROOF OF THEOREM 7.33. Define $\widehat{\mathcal{E}}_X(0)$-submodules $\mathcal{M}_k$ ($k \geq 0$) inductively by $\mathcal{M}_{k-1} \subset \mathcal{M}_k \subset \mathcal{M}_{k-1}(1)$ and

$$\frac{\mathcal{M}_k}{\mathcal{M}_{k-1}} = \Gamma_S\left(\frac{\mathcal{M}_{k-1}(1)}{\mathcal{M}_{k-1}}\right) \quad (k \geq 1). \tag{7.33}$$

If $\mathcal{M}_{k-1}$ is a coherent $\widehat{\mathcal{E}}_X(0)$-module, then $\mathcal{M}_k/\mathcal{M}_{k-1}$ is a coherent $\widehat{\mathcal{E}}_X(0)$-module, and hence $\mathcal{M}_k$ is also a coherent $\widehat{\mathcal{E}}_X(0)$-module. Hence, by induction on k, we see that all $\mathcal{M}_k$ are coherent $\widehat{\mathcal{E}}_X(0)$-modules. By (7.33), we have

$$\begin{aligned} &\mathrm{Supp}(\mathcal{M}_k/\mathcal{M}_{k-1}) \subset S, \\ &\Gamma_S(\mathcal{M}_{k-1}(1)/\mathcal{M}_k) = 0. \end{aligned} \tag{7.34}$$

Hence $\mathrm{Supp}(\mathcal{M}_k/\mathcal{M}_0) \subset S$ for all k. Next we show that

$$\mathcal{M}'_0 = \bigcup_k \mathcal{M}_k. \tag{7.35}$$

Obviously $\mathcal{M}'_0 \supset \bigcup_k \mathcal{M}_k$. For the reverse inclusion, it suffices to show that $\mathcal{M}'_0 \cap \mathcal{M}_0(k) \subset \mathcal{M}_k$ for all $k \geq 0$. Using induction on k, we may assume $\mathcal{M}'_0 \cap \mathcal{M}_0(k-1) \subset \mathcal{M}_{k-1}$. Thus

$$\mathcal{M}'_0 \cap \mathcal{M}_0(k) \subset \mathcal{M}'_0(1) \cap \mathcal{M}_0(k) \subset \mathcal{M}_{k-1}(1).$$

Since $\mathrm{Supp}(\mathcal{M}'_0 \cap \mathcal{M}_0(k)/\mathcal{M}_{k-1}) \subset S$, we obtain $\mathcal{M}'_0 \cap \mathcal{M}_0(k) \subset \mathcal{M}_k$ from (7.33). We have thus proved (7.35).

Now the proof of the theorem reduces to showing that $\mathcal{M}_k = \mathcal{M}_{k_0}$ ($k \geq k_0$) for k_0 large enough.

By (7.34),

$$\Gamma_S\left(\frac{\mathcal{M}_{k+1} \cap \mathcal{M}_{k-1}(1)}{\mathcal{M}_k}\right) = 0,$$

and $\mathrm{Supp}(\mathcal{M}_{k+1} \cap \mathcal{M}_{k-1}(1)/\mathcal{M}_k) \subset S$. Hence

$$\mathcal{M}_{k+1} \cap \mathcal{M}_{k-1}(1) = \mathcal{M}_k \quad (k \geq 1). \tag{7.36}$$

Thus

$$\frac{\mathcal{M}_{k+1}}{\mathcal{M}_k} \rightarrowtail \frac{\mathcal{M}_k(1)}{\mathcal{M}_{k-1}(1)} \text{ is injective.} \tag{7.37}$$

In particular, $\operatorname{Supp}(\mathcal{M}_k/\mathcal{M}_{k-1})$ is a decreasing sequence of closed analytic subsets. Hence locally

$$\operatorname{Supp}(\mathcal{M}_k/\mathcal{M}_{k-1}) = \operatorname{Supp}(\mathcal{M}_{k_0}/\mathcal{M}_{k_0-1}) \quad (k \geq k_0)$$

for k_0 large enough. We denote this by S_0. Since $S_0 \subset S$, S_0 also satisfies $I(S_0) = \emptyset$. Showing $S_0 = \emptyset$ finishes the proof of the theorem.

Supposing $S_0 \neq \emptyset$, we derive a contradiction. Considering a neighborhood of a nonsingular point p_0 of S_0, we may assume S_0 to be nonsingular. The multiplicities of $\mathcal{O}_{T^*X}(0)$-modules $\mathcal{M}_k/\mathcal{M}_{k-1}$ at a generic point of S_0 are monotonously decreasing by (7.37). Hence we may assume that the multiplicity of $\mathcal{M}_k/\mathcal{M}_{k-1}$ is constant for $k \geq k_0$. Then for $k \geq k_0$ the support of the cokernel of the homomorphism (7.37) is a nowhere dense closed analytic set. Hence, in a neighborhood of some $p \in S_0$,

$$\frac{\mathcal{M}_{k_0+1}}{\mathcal{M}_{k_0}} \xrightarrow{\sim} \frac{\mathcal{M}_{k_0}(1)}{\mathcal{M}_{k_0-1}(1)}. \tag{7.38}$$

Put

$$A = \widehat{\mathcal{E}}_X(0)_p/\widehat{\mathcal{E}}_X(-2)_p\,,$$
$$M = (\mathcal{M}_{k_0+1}/\mathcal{M}_{k_0-1})_p.$$

Then A is a Noetherian ring, and M is an A-module. Take $Q \in \widehat{\mathcal{E}}_X(-1)_p$ such that $\sigma_{-1}(Q)(p) \neq 0$, and let $T \in A$ be the image of Q. Since $[\widehat{\mathcal{E}}_X(0), Q] \subset \widehat{\mathcal{E}}_X(-2)$,

(7.39) T belongs to the center of A,

(7.40) $T^2 = 0$,

(7.41) $\operatorname{Ker}(T : A \to A) = TA \ (= \mathcal{E}(-1)_p/\mathcal{E}(-2)_p)$,

(7.42) A/TA is a commutative Noetherian ring.

The A-module M is finitely generated, and

$$\operatorname{Ker}(T : M \to M) = TM. \tag{7.43}$$

To show (7.43), it suffices to show that

$$\mathcal{M}_{k_0+1} \cap \mathcal{M}_{k_0-1}(1) \subset \mathcal{M}_{k_0+1}(-1) + \mathcal{M}_{k_0-1}.$$

This follows from (7.36) and the equality $\mathcal{M}_{k_0} = \mathcal{M}_{k_0+1}(-1) + \mathcal{M}_{k_0-1}$, which is the surjectivity of (7.38).

At this point, we apply Theorem 7.37 below (for the proof, see Chapter 3 of [T]).

THEOREM 7.37 (Gabber). *Let A be a ring satisfying the conditions* (7.39)–(7.42), *and M a finitely generated A-module satisfying* (7.43). *Put $\mathfrak{a} = \{ a \in A/TA ; a(M/TM) = 0 \}$. Then*

$$\{\sqrt{\mathfrak{a}}, \sqrt{\mathfrak{a}}\} \subset \sqrt{\mathfrak{a}}. \tag{7.44}$$

Here $\{\ ,\ \} : (A/TA) \times (A/TA) \to A/TA$ is given by $\{\bar{a}, \bar{b}\} = \bar{c}$ when $[a, b] = cT$, where $a, b, c \in A$, and $\bar{a}, \bar{b}, \bar{c}$ denote the corresponding elements of A/TA.

In our case, $\sqrt{\mathfrak{a}} = I_{S_0}(0)_p$, and (7.44) reads as the inclusion

$$\{I_{S_0}(0)_p, I_{S_0}(0)_p\} \subset I_{S_0}(-1)_p.$$

Hence $\{I_{S_{0p}}, I_{S_{0p}}\} \subset I_{S_{0p}}$, and S_0 is involutive in a neighborhood of p. This contradicts the fact $I(S_0) = \emptyset$. We have thus finished the proof of Theorem 7.33. $\square$

Note that $\mathcal{M}'_0$ in Theorem 7.33 can be characterized as a coherent $\widehat{\mathcal{E}}_X(0)$-module satisfying the following three conditions:

(7.45) $\mathcal{M}_0 \subset \mathcal{M}'_0$,

(7.46) $\Gamma_S(\mathcal{M}'_0/\mathcal{M}'_0(-1)) = 0$,

(7.47) $\mathcal{M}_0/\mathcal{M}_0(-1) \to \mathcal{M}'_0/\mathcal{M}'_0(-1)$ is an isomorphism outside S.

COROLLARY 7.38. *Let $(\mathcal{M}, F)$ be a coherent filtered $\mathcal{D}_X$-module. Let S be a homogeneous closed analytic subset of T^*X satisfying $I(S) = \emptyset$. Then there exists a unique coherent filtration F' of $\mathcal{M}$ satisfying the following three conditions:*

$$F'_k \supset F_k\,, \tag{7.48}$$

$$\Gamma_S((\mathrm{Gr}^{F'} \mathcal{M})^\sim) = 0\,, \tag{7.49}$$

$$(\mathrm{Gr}^F \mathcal{M})^\sim \to (\mathrm{Gr}^{F'} \mathcal{M})^\sim \text{ is an isomorphism outside } S. \tag{7.50}$$

PROOF. Let $I_S \subset \mathrm{Gr}^F(\mathcal{D})$ denote the defining ideal of S.

Define an increasing sequence of filtrations $F^{(k)}$ of $\mathcal{M}$ by $F^{(0)} = F$, and for $k \geq 1$

$$\begin{aligned} F_l^{(k)}(\mathcal{M}) \quad = \quad & \{ u \in F_{l+1}^{(k-1)}(\mathcal{M}) \,;\, \text{locally for some } n > 0, \\ & \qquad (I_S^n \cap \mathrm{Gr}_j^F \mathcal{D})u \in F_{j+l}^{(k-1)}(\mathcal{M}) \text{ for all } j \}. \end{aligned}$$

Then the $F^{(k)}$ are coherent. If we prove that $\bigcup F^{(k)}$ is coherent, then we can show, similarly to (7.35), that

$$F' = \bigcup_k F^{(k)} = F^{(k_0)} \quad (k_0 \gg 0)$$

is the unique filtration with the required conditions. Put

$$Z = X \times \mathbb{C}, \quad \mathcal{N} = \mathcal{M} \boxtimes \mathcal{B}_{\{0\}|\mathbb{C}},$$
$$F_l^{(k)}(\mathcal{N}) = \sum_\nu F_{l-\nu}^{(k)}(\mathcal{M}) \boxtimes F_\nu(\mathcal{D}_{\mathbb{C}})\delta(t).$$

Then $F^{(k)}(\mathcal{N})$ is an increasing sequence of coherent $\mathcal{D}_X$-submodules of $\mathcal{N}$, and

$$(\mathrm{Gr}^{F^{(k)}} \mathcal{N})^\sim = (\mathrm{Gr}^{F^{(k)}} \mathcal{M})^\sim \boxtimes \mathcal{O}_{T_0^*\mathbb{C}}.$$

Hence it suffices to show that $(\mathrm{Gr}^{F^{(k)}} \mathcal{N})^\sim \xrightarrow{\sim} (\mathrm{Gr}^{F^{(k+1)}} \mathcal{N})^\sim$ for $k \gg 0$ on $\mathring{T}^*Z$. Put $\mathcal{L}_k = \sum_\nu \widehat{\mathcal{E}}_Z(-\nu)\,\mathrm{Sp}(F_\nu^{(k)}(\mathcal{N}))$. Then by the definitions

$$(\mathrm{Gr}^{F^{(k)}} \mathcal{N})^\sim = \mathcal{O}_{T^*Z} \otimes_{\mathcal{O}_{T^*Z}(0)} (\mathcal{L}_k/\mathcal{L}_k(-1)),$$
$$\mathrm{Supp}(\mathcal{L}_k/\mathcal{L}_{k-1}) \subset S \times T^*\mathbb{C}.$$

Since $I(S \times T^*\mathbb{C}) = \emptyset$, we have $\mathcal{L}_k = \mathcal{L}_{k_0}$ $(k \geq k_0)$ on $\mathring{T}^*Z$ for k_0 large enough by Theorem 7.33. By Lemma 7.31,

$$F_m^{(k)}(\mathcal{M}) = \{\, u \in \mathcal{M}\,;\ \mathrm{Sp}(u \boxtimes \delta(t))|_{\mathring{T}^*Z} \subset \mathcal{L}_k(m)|_{\mathring{T}^*Z} \,\},$$

and hence $F^{(k)} = F^{(k_0)}$ $(k \geq k_0)$. $\square$

CHAPTER 8

Microlocal Analysis of Holonomic Systems

A coherent $\widehat{\mathcal{E}}_X$-module $\mathcal{M}$ is called a *holonomic system* if its support $\mathrm{Supp}(\mathcal{M})$ is Lagrangian. In this chapter, we study its properties.

8.1. Simple $\widehat{\mathcal{E}}$-modules

Let $\mathcal{M}$ be a coherent $\widehat{\mathcal{E}}_X$-module generated by $u \in \mathcal{M}$. Then $\mathcal{M} \simeq \widehat{\mathcal{E}}_X/\mathcal{J}$, where $\mathcal{J} = \{ P \in \widehat{\mathcal{E}}_X \, ; \, Pu = 0 \}$. Let $\bar{\mathcal{J}}$ denote the ideal of $\mathcal{O}_{T^*X}$ generated by $\{ \sigma_m(P) \, ; \, m \in \mathbb{Z}, \; P \in \mathcal{J} \cap \widehat{\mathcal{E}}_X(m) \}$. Then $\mathrm{Supp}(\mathcal{M}) = \mathrm{Supp}(\mathcal{O}_{T^*X}/\bar{\mathcal{J}})$. Letting $V = \mathrm{Supp}(\mathcal{M})$, we call $(\mathcal{M}, u)$ a *simple $\widehat{\mathcal{E}}_X$-module* if $\bar{\mathcal{J}} = \{ v \in \mathcal{O}_{T^*X} \, ; \, v|_V = 0 \}$. To check whether a module is simple, the following lemma is often useful. We omit its proof, since we can prove it similarly to Proposition 2.12.

LEMMA 8.1. *Let $P_i \in \widehat{\mathcal{E}}_X(m_i)$ $(i = 1, \dots, d)$. Suppose that the codimension of*

$$\{ p \in T^*X \, ; \, \sigma_{m_i}(P_i)(p) = 0 \;\; (i = 1, \dots, d) \}$$

equals d, and that

$$[P_i, P_j] \in \sum_k \widehat{\mathcal{E}}_X(m_i + m_j - m_k - 1) P_k.$$

Then for $\mathcal{J} = \sum \widehat{\mathcal{E}}_X P_k$ we have $\bar{\mathcal{J}} = \sum \mathcal{O}_{T^*X} \sigma_{m_i}(P_i)$.

8.2. Quantized Contact Transformation

We can define operations of $\widehat{\mathcal{E}}$-modules such as integration, pull-back, and product, but here we explain only those needed later.

Let X and Y be complex manifolds, and p_1 and p_2 the projections from $T^*(X \times Y) \cong T^*X \times T^*Y$ to T^*X and T^*Y respectively. Then we have the following lemma.

LEMMA 8.2. *There exists a canonical homomorphism*

$$p_1^{-1}\widehat{\mathcal{E}}_X \otimes_{\mathbb{C}} p_2^{-1}\widehat{\mathcal{E}}_Y \to \widehat{\mathcal{E}}_{X\times Y}$$

of rings on $T^*(X \times Y)$.

Let $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_m)$ be local coordinate systems of X and Y respectively. Let $(x, \xi) = (x_1, \dots, x_n; \xi_1, \dots, \xi_n)$ and $(y, \eta) = (y_1, \dots, y_m; \eta_1, \dots, \eta_m)$ be the corresponding coordinate systems of T^*X and T^*Y respectively. Then, using total symbols, we describe the homomorphism in Lemma 8.2 as follows: For $P = \{p_j(x, \xi)\} \in \widehat{\mathcal{E}}_X$ and $Q = \{q_j(y, \eta)\} \in \widehat{\mathcal{E}}_Y$, the total symbol $\{r_j(x, y, \xi, \eta)\}$ of the corresponding element $PQ \in \widehat{\mathcal{E}}_{X\times Y}$ is given by

$$r_j(x, y, \xi, \eta) = \sum_{j=j_1+j_2} p_{j_1}(x, \xi) q_{j_2}(y, \eta).$$

Hence the homomorphism in Lemma 8.2 gives a commutative diagram for any $m_1, m_2 \in \mathbb{Z}$:

$$\begin{array}{ccc}
p_1^{-1}\widehat{\mathcal{E}}_X \otimes p_2^{-1}\widehat{\mathcal{E}}_Y & \longrightarrow & \widehat{\mathcal{E}}_{X\times Y} \\
\cup & & \cup \\
p_1^{-1}\widehat{\mathcal{E}}_X(m_1) \otimes p_2^{-1}\widehat{\mathcal{E}}_Y(m_2) & \longrightarrow & \widehat{\mathcal{E}}_{X\times Y}(m_1 + m_2) \\
\Big\downarrow{\scriptstyle \sigma_{m_1} \otimes \sigma_{m_2}} & & \Big\downarrow{\scriptstyle \sigma_{m_1+m_2}} \\
p_1^{-1}\mathcal{O}_{T^*X}(m_1) \otimes p_2^{-1}\mathcal{O}_{T^*Y}(m_2) & \longrightarrow & \mathcal{O}_{T^*(X\times Y)}(m_1 + m_2),
\end{array}$$

where the arrow on the bottom row is given by multiplication. For an $\widehat{\mathcal{E}}_X$-module $\mathcal{M}$ and an $\widehat{\mathcal{E}}_Y$-module $\mathcal{N}$, we set

$$\mathcal{M} \boxtimes \mathcal{N} = \widehat{\mathcal{E}}_{X\times Y} \otimes_{p_1^{-1}\widehat{\mathcal{E}}_X \otimes_{\mathbb{C}} p_2^{-1}\widehat{\mathcal{E}}_Y} (p_1^{-1}\mathcal{M} \otimes_{\mathbb{C}} p_2^{-1}\mathcal{N})$$

and call it the *external tensor product* of $\mathcal{M}$ and $\mathcal{N}$. Similarly, for an $\widehat{\mathcal{E}}_X(0)$-module and an $\widehat{\mathcal{E}}_Y(0)$-module, the external tensor product is defined as an $\widehat{\mathcal{E}}_{X\times Y}(0)$-module. The following proposition can be easily reduced to the case of $\mathcal{O}$-modules; we omit its proof.

PROPOSITION 8.3. *For any* $\widehat{\mathcal{E}}_Y$*-module* $\mathcal{N}$*, the functor* $\mathcal{M} \mapsto \mathcal{M} \boxtimes \mathcal{N}$ *sends exact sequences of* $\widehat{\mathcal{E}}_X$*-modules to exact sequences of* $\widehat{\mathcal{E}}_{X\times Y}$*-modules.*

Let $\mathcal{M}_0$ be an $\widehat{\mathcal{E}}_{X\times Y}(0)$-module defined on an open set U in $\overset{\circ}{T}{}^*(X \times Y)$, and U_X an open set in $\overset{\circ}{T}{}^*X$ satisfying $p_1(U) \subset U_X$.

We denote by $\tilde{p} : U \to U_X$ the restriction of p_1. Then, through $p_1^{-1}\widehat{\mathcal{E}}_X(0) \to \widehat{\mathcal{E}}_{X\times Y}(0)$, $\tilde{p}_*\mathcal{M}_0$ has a structure of $\widehat{\mathcal{E}}_X(0)|_{U_X}$-module.

PROPOSITION 8.4. *If $\mathcal{M}_0$ is a coherent $\widehat{\mathcal{E}}_{X\times Y}(0)$-module defined on U, and if* $\mathrm{Supp}(\mathcal{M}_0)$ *is finite over U_X, then $\tilde{p}_*\mathcal{M}_0$ is a coherent $\widehat{\mathcal{E}}_X(0)$-module.*

PROOF. Put $\mathcal{N} = \tilde{p}_*\mathcal{M}_0$ and $\mathcal{N}_k = \tilde{p}_*\widehat{\mathcal{E}}_{X\times Y}(-k)\mathcal{M}_0$. Then the $\mathcal{O}_{T^*(X\times Y)}(0)|_\Omega$-module $\widehat{\mathcal{E}}_{X\times Y}(-k)\mathcal{M}_0/\widehat{\mathcal{E}}_{X\times Y}(-k-1)\mathcal{M}_0$ is coherent. Accordingly $\tilde{p}_*(\widehat{\mathcal{E}}_{X\times Y}(-k)\mathcal{M}_0/\widehat{\mathcal{E}}_{X\times Y}(-k-1)\mathcal{M}_0)$ is a coherent $\mathcal{O}_{T^*X}(0)$-module. Thus $\mathcal{N}_k/\mathcal{N}_{k+1}$ is a coherent $\mathcal{O}_{T^*X}(0)$-module. By Proposition 7.23, we have $\bigcap \mathcal{N}_k = 0$. Thus $\tilde{p}_*\mathcal{M}_0$ satisfies the conditions in Lemma 7.14, and hence it is a coherent $\widehat{\mathcal{E}}_X(0)$-module. □

As a corollary of this proposition, we obtain the following proposition.

PROPOSITION 8.5. *Let $(\mathcal{M}, u)$ be a simple $\widehat{\mathcal{E}}_{X\times Y}$-module defined on U. If* $\mathrm{Supp}(\mathcal{M}) \to U_X$ *is an isomorphism, then $(\tilde{p}_*\mathcal{M}, \tilde{p}_*u)$ is isomorphic to $(\widehat{\mathcal{E}}_X, 1)$.*

PROOF. The section $\tilde{p}_*u \in \tilde{p}_*\mathcal{M}$ gives a morphism $\widehat{\mathcal{E}}_X(0) \to \tilde{p}_*(\widehat{\mathcal{E}}_{X\times Y}(0)u)$. By the assumption,

$$\widehat{\mathcal{E}}_X(0)/\widehat{\mathcal{E}}_X(-1) \longrightarrow \widehat{\mathcal{E}}_X(0)/\widehat{\mathcal{E}}_X(-1) \otimes_{\widehat{\mathcal{E}}_X(0)} p_*(\widehat{\mathcal{E}}_{X\times Y}(0)u)$$

is an isomorphism. Hence $\widehat{\mathcal{E}}_X(0) \to \tilde{p}_*(\widehat{\mathcal{E}}_{X\times Y}(0)u)$ is an isomorphism by Proposition 7.21 and the preceding proposition. □

Let X and Y be complex manifolds of an equal dimension. Let $p_1 : T^*(X\times Y) \to T^*X$ and $p_2 : T^*(X\times Y) \to T^*Y$ denote the canonical projections, and $p_2^a : T^*(X\times Y) \to T^*Y$ the composite of p_2 and $a : T^*Y \to T^*Y$ $((y,\eta) \mapsto (y, -\eta))$.

Let U be an open set in $T^*(X\times Y)$, and U_X and U_Y open sets in T^*X and T^*Y satisfying $p_1(U) \subset U_X$ and $p_2^a(U) \subset U_Y$. Let $(\mathcal{M}, u)$ be a simple $\widehat{\mathcal{E}}_{X\times Y}$-module defined on U. For simplicity, we assume that there exists a generator of Ω_Y. Then we have an anti-isomorphism $* : a^{-1}\widehat{\mathcal{E}}_Y \xrightarrow{\sim} \widehat{\mathcal{E}}_Y$ (Theorem 7.7). Put $\Lambda := \mathrm{Supp}(\mathcal{M})$.

Assume that $\Lambda \xrightarrow{p_1} U_X$ and $\Lambda \xrightarrow{p_2^a} U_Y$ are isomorphisms. Since Λ is involutive, and $\dim\Lambda = \dim(X\times Y)$, it is Lagrangian. Define $\varphi : U_X \xrightarrow{\sim} U_Y$ by $(p_2^a|_\Lambda) \circ (p_1|_\Lambda)^{-1}$. Then by Proposition 8.5

$$p_{1*}\mathcal{M} \simeq \widehat{\mathcal{E}}_X|_{U_X}, \quad p_{2*}\mathcal{M} \simeq \widehat{\mathcal{E}}_Y|_{U_Y^a}.$$

Composing the latter with $*$, we have $p^a_{2*}\mathcal{M} \simeq \widehat{\mathcal{E}}_Y|_{U_Y}$. Hence we obtain $\Phi : \varphi^{-1}(\widehat{\mathcal{E}}_Y|_{U_Y}) \xrightarrow{\sim} \widehat{\mathcal{E}}_X|_{U_X}$, explicitly given by

$$\Phi(Q)u = Q^* u \quad (Q \in \widehat{\mathcal{E}}_Y). \tag{8.1}$$

THEOREM 8.6. (1) *φ is a homogeneous symplectic transformation (i.e., $\varphi^*\omega_Y = \omega_X$, where ω_Y and ω_X are the canonical 1-forms on T^*Y and T^*X respectively).*

(2) *Φ is an isomorphism of rings.*

PROOF. (1) Since $\Lambda = \operatorname{Supp}\mathcal{M}$ is Lagrangian, $\omega_{X\times Y}|_\Lambda = 0$. Since $\omega_{X\times Y} = p_1^*\omega_X - p_2^{a*}\omega_Y$, we have $p_1^*\omega_X|_\Lambda = p_2^{a*}\omega_Y|_\Lambda$. Hence $\varphi^*\omega_Y = \omega_X$. Here we note that $a^*\omega_Y = -\omega_Y$.

(2) It suffices to show that Φ preserves the multiplications. For $Q_1, Q_2 \in \widehat{\mathcal{E}}_Y$, we have

$$\Phi(Q_1Q_2)u = (Q_1Q_2)^*u = Q_2^*Q_1^*u = Q_2^*\Phi(Q_1)u.$$

Since $\widehat{\mathcal{E}}_X$ and $\widehat{\mathcal{E}}_Y$ commute with each other in $\widehat{\mathcal{E}}_{X\times Y}$, we have

$$Q_2^*\Phi(Q_1)u = \Phi(Q_1)Q_2^*u = \Phi(Q_1)\Phi(Q_2)u.$$

□

DEFINITION 8.7. The transformation $\Phi : \varphi^{-1}(\widehat{\mathcal{E}}_Y|_{U_Y}) \xrightarrow{\sim} \widehat{\mathcal{E}}_X|_{U_X}$ defined through a simple holonomic $\widehat{\mathcal{E}}_{X\times Y}$-module $(\mathcal{M}, u)$ and given by (8.1) is called a *quantized contact transformation* (with kernel u, associated with φ).

PROPOSITION 8.8. *For any m, we have a commutative diagram*

$$\begin{array}{ccc}
\varphi^{-1}\widehat{\mathcal{E}}_Y|_{U_Y} & \xrightarrow[\Phi]{\sim} & \widehat{\mathcal{E}}_X|_{U_X} \\
\cup & & \cup \\
\varphi^{-1}\widehat{\mathcal{E}}_Y(m)|_{U_Y} & \xrightarrow{\sim} & \widehat{\mathcal{E}}_X(m)|_{U_X} \\
\downarrow \sigma_m & & \downarrow \sigma_m \\
\varphi^{-1}\mathcal{O}_{T^*Y}(m)|_{U_Y} & \xrightarrow{\sim} & \mathcal{O}_{T^*X}(m)|_{U_X},
\end{array}$$

where the isomorphism in the bottom row is given by $\mathcal{O}_{T^*Y}(m) \ni g \mapsto g\circ\varphi \in \mathcal{O}_{T^*X}(m)$.

PROOF. Since $p_1^{-1}(\widehat{\mathcal{E}}_X(m))u = \widehat{\mathcal{E}}_{X\times Y}(m)u = p_2^{-1}(\widehat{\mathcal{E}}_Y(m))u$, we have $\Phi(\varphi^{-1}\widehat{\mathcal{E}}_Y(m)|_{U_Y}) = \widehat{\mathcal{E}}_X(m)|_{U_X}$. For $Q \in \widehat{\mathcal{E}}_Y(m)$, we have

$$\Phi(Q)u = Q^*u,$$

and thus $\sigma_m(\Phi(Q) - Q^*)|_\Lambda = 0$, or $\sigma_m(\Phi(Q))|_\Lambda = \sigma_m(Q^*)|_\Lambda$. From $p_2^*\sigma_m(Q^*)|_\Lambda = p_2^{a*}\sigma_m(Q)|_\Lambda$, we obtain $\sigma_m(\Phi(Q)) = \sigma_m(Q) \circ \varphi$. □

If $\varphi : T^*X \supset U_X \to U_Y \subset T^*Y$ is a homogeneous symplectic transformation, and if $\Phi : \varphi^{-1}\widehat{\mathcal{E}}_Y \to \widehat{\mathcal{E}}_X$ is a homomorphism of $\mathbb{C}$-algebras, then Φ is a quantized contact transformation. To show this, we first prove the following lemma.

LEMMA 8.9. *Let $X = \mathbb{C}^n$ and $p \in \mathring{T}^*X$. Let $\Phi : \widehat{\mathcal{E}}_{X,p} \to \widehat{\mathcal{E}}_{X,p}$ be a homomorphism of rings. Assume $\Phi(\widehat{\mathcal{E}}_X(m)_p) \subset \widehat{\mathcal{E}}_X(m)_p$ for all m. If $\Phi(x_i) = x_i$ and $\Phi(\partial_i) = \partial_i$ $(i = 1, \dots, n)$, then $\Phi = \mathrm{id}$.*

PROOF. We may assume $p = (0, dx_n)$. For every m, put

$$\begin{aligned} &I(m)\\ &= \{\, P = \{p_k(x,\xi)\}\,;\, p_k(x,\xi) \text{ and all of its partial derivatives}\\ &\qquad\qquad\qquad \text{of order up to } (m-1) \text{ equal } 0 \text{ at } p\,\}. \end{aligned}$$

Then

$$I(m) = \sum_{\substack{|\alpha|+|\beta|=m\\ \beta_n=0}} x^\alpha \widehat{\mathcal{E}}_{X,p}\partial^\beta,$$

and

$$\bigcap_m I(m) = 0.$$

Furthermore, $\Phi(I(m)) \subset I(m)$. For any m, l, and $P \in \widehat{\mathcal{E}}_{X,p}$, there exists $A \in \mathbb{C}[x_1, \dots, x_n, \partial_1, \dots, \partial_n, \partial_n^{-1}]$ such that $P - A \in I(m) + \widehat{\mathcal{E}}_X(-l)_p$.

Since $\Phi(A) = A$,

$$\Phi(P) - P = \Phi(P - A) - (P - A) \in I(m) + \widehat{\mathcal{E}}_X(-l)_p$$

for all m and l. Hence $\Phi(P) = P$. □

PROPOSITION 8.10. *Let $\varphi : U_X \to U_Y$ be a homogeneous symplectic transformation from an open set U_X in T^*X to an open set U_Y in T^*Y, and let $p_X \in U_X$ and $\varphi(p_X) = p_Y$. Assume that $\Phi : \varphi^{-1}\widehat{\mathcal{E}}_Y|_{U_X} \xrightarrow{\sim} \widehat{\mathcal{E}}_X|_{U_X}$ is an isomorphism of $\mathbb{C}$-algebras mapping $\varphi^{-1}\widehat{\mathcal{E}}_Y(m)$ to $\widehat{\mathcal{E}}_X(m)$, and that $\sigma_m(\Phi(Q)) = \varphi^*\sigma_m(Q)$ for all $Q \in \widehat{\mathcal{E}}_Y(m)$. Then there exists a simple holonomic module $(\mathcal{M}, u)$ defined on a neighborhood of (p_X, p_Y^a) such that Φ is the quantized contact transformation with kernel $(\mathcal{M}, u)$.*

PROOF. Let $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ be local coordinate systems of X and Y respectively, and (x, ξ) and (y, η) the corresponding local coordinate systems of T^*X and T^*Y.

Put $\Phi(y_i) = Q_i(x, \partial_x)$ and $\Phi(\partial_{y_i}) = P_i(x, \partial_x)$. Then $Q_i \in \widehat{\mathcal{E}}_X(0)$, $P_i \in \widehat{\mathcal{E}}_X(1)$, and φ is given by

$$(x, \xi) \mapsto (y, \eta) \quad (y_i = \sigma_0(Q_i)(x, \xi),\ \eta_i = \sigma_1(P_i)(x, \xi)).$$

We have commutation relations

$$[Q_i, Q_j] = [P_i, P_j] = 0,$$
$$[P_i, Q_j] = \delta_{ij}.$$

Hence $y_i - Q_i(x, \partial_x)$ and $\partial_{y_i} + P_i(x, \partial_x)$ commute with each other. Let $\mathcal{M} = \widehat{\mathcal{E}}_{X \times Y} u$ be an $\widehat{\mathcal{E}}_{X \times Y}$-module defined by

$$(y_i - Q_i(x, \partial_x))u = 0,$$
$$(\partial_{y_i} + P_i(x, \partial_x))u = 0.$$

Then, by Lemma 8.1, $\mathcal{M}$ is a simple holonomic $\widehat{\mathcal{E}}_{X \times Y}$-module with support $\Lambda = \{(p, p') \in T^*X \times T^*Y; p'^a = \varphi(p)\}$. Let $\Psi : \varphi^{-1}\widehat{\mathcal{E}}_Y \to \widehat{\mathcal{E}}_X$ be the quantized contact transformation with kernel $(\mathcal{M}, u)$. Then $\Psi(y_i) = Q_i$ and $\Psi(\partial_{y_i}) = P_i$. Hence $\Phi^{-1}\Psi \in \mathrm{Aut}(\widehat{\mathcal{E}}_Y)$ sends y_i and ∂_{y_i} to themselves. Therefore, by Lemma 8.9, we have $\Phi^{-1}\Psi = \mathrm{id}$, or $\Phi = \Psi$. □

As a corollary of this proposition, we conclude that the composite of quantized contact transformations is again a quantized contact transformation.

THEOREM 8.11. *Let $\varphi : U_X \to U_Y$ be a homogeneous symplectic transformation from an open set U_X in T^*X to an open set U_Y in T^*Y. Let $p_X \in U_X$ and $p_Y = \varphi(p_X)$. Then there exists a quantized contact transformation defined on a neighborhood W of p_X,*

$$\Phi : (\varphi^{-1}\widehat{\mathcal{E}}_Y)|_W \to \widehat{\mathcal{E}}_X|_W.$$

PROOF. (First Step)

Suppose that there exists a hyperplane Z of $X \times Y$ such that the graph $\Lambda := \{(p, p'); p'^a = \varphi(p)\}$ of φ coincides with $T^*_Z(X \times Y)$. Then there exists a simple holonomic system $(\mathcal{M}, u)$ with support Λ. Indeed, when $Z = f^{-1}(0)$, $\mathcal{M} = \widehat{\mathcal{E}}_{X \times Y} \otimes_{\mathcal{D}_{X \times Y}} \mathcal{B}_{Z|X \times Y}$ with $u = \delta(f)$ will do. Hence we may take the quantized contact transformation associated with $(\mathcal{M}, u)$ as Φ.

(Second Step) General Case.

Since φ can be written as the composite of homogeneous symplectic transformations of the form in the first step, we may take the composite of the corresponding quantized contact transformations as Φ. □

8.3. Subprincipal Symbols

Let $P \in \widehat{\mathcal{E}}_X(m)$. Taking a coordinate system $x = (x_1, \dots, x_n)$, we consider the adjoint operator P^* of $P = \{P_k(x,\xi)\}_k$. Then

$$\begin{aligned}
(P^*)_m(x,\xi) &= P_m(x,-\xi) = (-1)^m P_m(x,\xi), \\
(P^*)_{m-1}(x,\xi) &= P_{m-1}(x,-\xi) - \sum_i \frac{\partial^2 P_m}{\partial x_i \partial \xi_i}(x,-\xi) \\
&= (-1)^{m-1}\left(P_{m-1}(x,\xi) - \sum_i \frac{\partial^2 P_m}{\partial x_i \partial \xi_i}(x,\xi)\right).
\end{aligned}$$

Hence $P - (-1)^m P^* \in \widehat{\mathcal{E}}_X(m-1)$. We set

$$\begin{aligned}
\sigma_{m-1}(P) &:= \frac{1}{2}\sigma_{m-1}(P - (-1)^m P^*) \\
&= P_{m-1} - \frac{1}{2}\sum_i \frac{\partial^2 P_m}{\partial x_i \partial \xi_i}
\end{aligned}$$

and call it the *subprincipal symbol* of P.

The map $\sigma_{m-1} : \widehat{\mathcal{E}}_X(m) \to \mathcal{O}_{T^*X}$ defined above extends the principal symbol $\sigma_{m-1} : \widehat{\mathcal{E}}_X(m-1) \to \mathcal{O}_{T^*X}$. However, this depends on the choice of a coordinate system. Let $\tilde{P} = (\tilde{P}_k(\tilde{x},\tilde{\xi}))$ be the presentation of P in another coordinate system $\tilde{x} = (\tilde{x}_1, \dots, \tilde{x}_n)$. Then, by Theorem 7.7, we have $d\tilde{x}\tilde{P}^* d\tilde{x}^{-1} = dx P^* dx^{-1}$, or equivalently $\tilde{P}^* = \left(\frac{d\tilde{x}}{dx}\right)^{-1} P^* \left(\frac{d\tilde{x}}{dx}\right)$. Hence, letting $\tilde{\sigma}_{m-1}(\tilde{P})$ be the subprincipal symbol in $\tilde{x}$, we have

$$\begin{aligned}
2\tilde{\sigma}_{m-1}(\tilde{P}) &= \sigma_{m-1}(\tilde{P} - (-1)^m \tilde{P}^*) \\
&= \sigma_{m-1}\left(P - (-1)^m \left(\frac{d\tilde{x}}{dx}\right)^{-1} P^* \left(\frac{d\tilde{x}}{dx}\right)\right) \\
&= \sigma_{m-1}(P - (-1)^m P^*) \\
&\qquad -(-1)^m \sigma_{m-1}\left(\left(\frac{d\tilde{x}}{dx}\right)^{-1}\left[P^*, \frac{d\tilde{x}}{dx}\right]\right).
\end{aligned}$$

The first term equals $2\sigma_{m-1}(P)$, and the second equals

$$\begin{aligned}
&-(-1)^m \left(\frac{d\tilde{x}}{dx}\right)^{-1} \sigma_{m-1}\left(\left[P^*, \frac{d\tilde{x}}{dx}\right]\right) \\
&= -\left(\frac{d\tilde{x}}{dx}\right)^{-1} (-1)^m \left\{\sigma_m(P^*), \frac{d\tilde{x}}{dx}\right\} \\
&= -\left(\frac{d\tilde{x}}{dx}\right)^{-1} \left\{\sigma_m(P), \frac{d\tilde{x}}{dx}\right\}.
\end{aligned}$$

We have thus obtained

$$\tilde{\sigma}_{m-1}(\tilde{P}) = \sigma_{m-1}(P) - \frac{1}{2}\left\{\sigma_m(P), \log\frac{d\tilde{x}}{dx}\right\}. \tag{8.2}$$

We shall restate this in a form invariant under coordinate change. We consider an invertible sheaf $\Omega_X^{1/2}$ such that $(\Omega_X^{1/2})^{\otimes 2} \cong \Omega_X$. Such a sheaf $\Omega_X^{1/2}$ locally exists on X. Let $\sqrt{dx}$ denote a section of $\Omega_X^{1/2}$ such that $(\sqrt{dx})^{\otimes 2} = dx$, where $dx = dx_1 \wedge \cdots \wedge dx_n \in \Omega_X$. Define a differential operator $\mathcal{L}_P^{(m-1)}$ acting on $\mathcal{O}_{T^*X} \otimes \Omega_X^{-1/2}$ ($\Omega_X^{-1/2} := (\Omega_X^{1/2})^{\otimes -1}$) by

$$\mathcal{L}_P^{(m-1)}(a/\sqrt{dx}) = (H_{\sigma_m(P)}a + \sigma_{m-1}(P)a)/\sqrt{dx}.$$

PROPOSITION 8.12. *$\mathcal{L}_P^{(m-1)}$ does not depend on the choice of a coordinate system.*

PROOF. Let $x = (x_1, \ldots, x_n)$ and $\tilde{x} = (\tilde{x}_1, \ldots, \tilde{x}_n)$ be two coordinate systems. Let $\tilde{\mathcal{L}}_{\tilde{P}}^{(m-1)}$ be the operator acting on $\mathcal{O}_{T^*X} \otimes \Omega_X^{-1/2}$ in $\tilde{x}$. Then

$$\begin{aligned}
&\tilde{\mathcal{L}}_{\tilde{P}}^{(m-1)}(a/\sqrt{d\tilde{x}}) \\
&= (H_{\sigma_m(P)}a + \tilde{\sigma}_{m-1}(\tilde{P})a)/\sqrt{d\tilde{x}} \\
&= \left(H_{\sigma_m(P)}a + \sigma_{m-1}(P)a - \frac{1}{2}\left\{\sigma_m(P), \log\frac{d\tilde{x}}{dx}\right\}a\right)/\sqrt{d\tilde{x}} \\
&= \left(\frac{d\tilde{x}}{dx}\right)^{-1/2}\left(H_{\sigma_m(P)}a + \sigma_{m-1}(P)a - \frac{1}{2}\left\{\sigma_m(P), \log\frac{d\tilde{x}}{dx}\right\}a\right)/\sqrt{dx} \\
&= \left(H_{\sigma_m(P)}\left(a\left(\frac{d\tilde{x}}{dx}\right)^{-1/2}\right) + \sigma_{m-1}(P)a\left(\frac{d\tilde{x}}{dx}\right)^{-1/2}\right)/\sqrt{dx}.
\end{aligned}$$

Hence this coincides with

$$\mathcal{L}_P^{(m-1)}(a/\sqrt{d\tilde{x}}) = \mathcal{L}_P^{(m-1)}\left(a\left(\frac{d\tilde{x}}{dx}\right)^{-1/2}/\sqrt{dx}\right).$$

□

LEMMA 8.13. *For $P \in \widehat{\mathcal{E}}_X(m)$ and $Q \in \widehat{\mathcal{E}}_X(l)$, we have*

$$\begin{aligned}\sigma_{m+l-1}(PQ) &= \sigma_m(P)\sigma_{l-1}(Q) + \sigma_{m-1}(P)\sigma_l(Q) \\ &\qquad + \frac{1}{2}\{\sigma_m(P), \sigma_l(Q)\}, \\ \sigma_{m+l-2}([P,Q]) &= \{\sigma_m(P), \sigma_{l-1}(Q)\} + \{\sigma_{m-1}(P), \sigma_l(Q)\}.\end{aligned}$$

PROOF. These are the principal symbols of the following equations:

$$\begin{aligned}&PQ - (-1)^{m+l}(PQ)^* = PQ - (-1)^{m+l}Q^*P^* \\ &\quad = [P,Q] + (Q - (-1)^l Q^*)P + (-1)^l Q^*(P - (-1)^m P), \\ &[P,Q] - (-1)^{m+l-1}([P,Q])^* = [P,Q] - (-1)^{m+l}[P^*,Q^*] \\ &\quad = [P - (-1)^m P^*, Q] + [(-1)^m P^*, Q - (-1)^l Q^*].\end{aligned}$$

□

8.4. Preparation for the Classification of Simple Holonomic Systems

Next we shall classify simple holonomic $\widehat{\mathcal{E}}_X$-modules $(\mathcal{M}, u)$. In this section, we consider the case when the support of $\mathcal{M}$ is nonsingular. It is roundabout, but we prove the following theorem first. Assume that

$$\operatorname{Re}(c_1), \ldots, \operatorname{Re}(c_n) > 0, \tag{8.3}$$

and that holomorphic functions $a_i(z)$ $(i = 1, \ldots, n)$ defined on a neighborhood of the origin of $\mathbb{C}^n$ satisfy

$$a_i(z) = c_i z_i + (\text{terms of degree} \geq 2).$$

Let $B(z)$ be a square matrix of size l with coefficients of holomorphic functions defined on a neighborhood of the origin, $B(0) = S + N$ the Jordan decomposition (S is diagonalizable, N is nilpotent, and S and N commute with each other), and $\mathfrak{S}$ the set of eigenvalues of $B(0)$. Put $P = \sum_i a_i(z)\frac{\partial}{\partial z_i} - B(z)$ and, for $\lambda \in \mathbb{C}$, $\tilde{\mathfrak{S}}_\lambda = \{\alpha \in \mathbb{Z}^n_{\geq 0}; \sum_i c_i\alpha_i - \lambda \in \mathfrak{S}\}$. By the assumption, $\tilde{\mathfrak{S}}_\lambda$ is finite. Let $\mathfrak{m}$ denote the ideal of $\mathcal{O}_{\mathbb{C}^n}$ consisting of holomorphic functions vanishing at the origin. For any $\varepsilon > 0$, put $U_\varepsilon = \{z \in \mathbb{C}^n; |z| < \varepsilon\}$.

THEOREM 8.14. *Under the above assumption, we have the following:*

(1) *For any $\lambda \in \mathbb{C}$, integers $q \geq p \geq 0$, and $v(z) \in (\mathfrak{m}_0^p)^{\oplus l}$, there exist $u(z), w(z) \in (\mathbb{C}[z_1, \ldots, z_n] \cap \mathfrak{m}_0^p)^{\oplus l}$ such that*

$$(P - \lambda)u(z) \equiv v(z) + w(z) \quad \bmod \ \mathfrak{m}_0^q,$$

$$\left(\sum_{i=1}^n c_i z_i \frac{\partial}{\partial z_i} - S - \lambda\right) w(z) = 0.$$

(2) *There exists $\varepsilon_0 > 0$ satisfying the following condition: Let a positive integer p satisfy $\alpha \notin \tilde{\mathfrak{S}}_\lambda$ for $|\alpha| \geq p$. Then, for any $0 < \varepsilon < \varepsilon_0$ and for any $v(z) \in \mathfrak{m}^p(U_\varepsilon)^{\oplus l}$, there exists a unique $u(z) \in \mathfrak{m}^p(U_\varepsilon)^{\oplus l}$ such that*

$$(P - \lambda)u(z) = v(z).$$

This immediately leads to the following corollary.

COROLLARY 8.15. *There exists $\varepsilon_0 > 0$ satisfying the following condition: If $v(z) \in \mathcal{O}(U_\varepsilon)^{\oplus l}$ for all $0 < \varepsilon < \varepsilon_0$, then there exist $u(z) \in \mathcal{O}(U_\varepsilon)^{\oplus l}$ and $w(z) \in \mathbb{C}[z_1, \ldots, z_n]^{\oplus l}$ such that*

$$(P - \lambda)u(z) = v(z) + w(z),$$

$$\left(\sum c_i z_i \frac{\partial}{\partial z_i} - S - \lambda\right) w(z) = 0.$$

PROOF OF THEOREM 8.14. (1) easily follows by induction on q from the fact that

$$\left(\sum_i c_i z_i \frac{\partial}{\partial z_i} - B(0) - \lambda\right) z^\alpha = z^\alpha\left(\sum_i c_i \alpha_i - B(0) - \lambda\right)$$

and the fact that $\operatorname{Im}(\mu + S + N) + \operatorname{Ker}(\mu + S) = \mathbb{C}^{\oplus l}$ for all $\mu \in \mathbb{C}$.

Next we prove (2). Let $\widehat{\mathfrak{m}}$ denote the ideal of $\mathbb{C}[[z_1, \ldots, z_n]]$ consisting of series vanishing at the origin. Note that, by the assumption on p, $P_0 = \sum_i c_i z_i \frac{\partial}{\partial z_i} - B(0) - \lambda$ is bijective on $(\widehat{\mathfrak{m}}^p)^{\oplus l}$. It easily follows from this that $P - \lambda$ is bijective on $(\widehat{\mathfrak{m}}^p)^{\oplus l}$. Hence there exists a unique $u(z)$ in $(\widehat{\mathfrak{m}}^p)^{\oplus l}$ satisfying the condition in (2). We need to show that this converges in U_ε.

For $v(z) = \sum_{\alpha \in \mathbb{Z}^n_{\geq 0}} v_\alpha z^\alpha$ ($v_\alpha \in \mathbb{C}$), set

$$\|v\|_\varepsilon = \sum |v_\alpha| \varepsilon^{|\alpha|},$$

where $|v_\alpha|$ is a norm in $\mathbb{C}^l$. To prove (2), it suffices to show that, for $0 < \varepsilon < \varepsilon_0$ and $\|v\|_\varepsilon < \infty$, we can find u satisfying $(P - \lambda)u = v$ and $\|u\|_\varepsilon < \infty$. Put

$$\begin{aligned} a_i(z) &= c_i z_i + \tilde{a}_i(z)\,, \\ B(z) &= B(0) + \tilde{B}(z)\,, \\ \tilde{P} &= -\sum \tilde{a}_i(z)\partial_i + \tilde{B}(z). \end{aligned}$$

Then $P - \lambda = P_0 - \tilde{P}$. Define $u_n \in (\widehat{\mathfrak{m}}^p)^{\oplus l}$ $(n \geq 0)$ by the following equations:

$$\begin{aligned} u_0 &= 0\,, \\ P_0 u_n &= \tilde{P} u_{n-1} + v \quad (n \geq 1)\,. \end{aligned}$$

Put $w_n = u_n - u_{n-1}$ $(n \geq 1)$. Then $w_n = P_0^{-1}\tilde{P}w_{n-1}$ for $n \geq 2$. For $\delta > 0$ small enough, we have $\|\tilde{B}(z)\|_\delta, \|a_i(z)\|_\delta < R$ for some $R > 0$. Here $\|\tilde{B}(z)\|_\delta = \sum |B_\alpha| \delta^{|\alpha|}$ for $\tilde{B}(z) = \sum_{|\alpha| \geq 1} B_\alpha z^\alpha$, where $|B_\alpha|$ is a norm in $M_l(\mathbb{C})$ such that $|B_\alpha u| \leq |B_\alpha| \cdot |u|$. Since for any $w(z) = \sum w_\alpha z^\alpha \in \mathbb{C}[[z_1, \ldots, z_n]]$ we have $\tilde{B}(z)w(z) = \sum B_\alpha w_\beta z^{\alpha+\beta}$, for $\varepsilon < \delta$

$$\begin{aligned} \|\tilde{B}(z)w(z)\|_\varepsilon &\leq \sum |B_\alpha w_\beta| \varepsilon^{|\alpha+\beta|} \\ &\leq \sum |B_\alpha| \cdot |w_\beta| \varepsilon^{|\alpha+\beta|} \\ &\leq \sum_{|\alpha| \geq 1} |B_\alpha| \delta^{|\alpha|} |w_\beta| \varepsilon^{|\beta|} \left(\frac{\varepsilon}{\delta}\right)^{|\alpha|} \\ &\leq \frac{\varepsilon}{\delta} \|\tilde{B}\|_\delta \|w\|_\varepsilon. \end{aligned} \tag{8.4}$$

By (8.3), we can take R so that $|(\sum_i c_i\alpha_i - B(0) - \lambda)^{-1}| \leq R/|\alpha|$ for $|\alpha| \geq p$. Since $P_0^{-1}u = \sum_\alpha (\sum_i c_i \alpha_i - B(0) - \lambda)^{-1} u_\alpha z^\alpha$ for $u = \sum_{|\alpha| \geq p} u_\alpha z^\alpha \in (\widehat{\mathfrak{m}}^p)^{\oplus l}$, we obtain

$$\|P_0^{-1}u\|_\varepsilon \leq R\|u\|_\varepsilon. \tag{8.5}$$

Let $\tilde{a}_j(z) = \sum_{|\alpha| \geq 2} a_\alpha^{(j)} z^\alpha$. Then

$$\tilde{a}_j(z)\frac{\partial u(z)}{\partial z_j} = \sum_{\alpha,\beta} a_\alpha^{(j)} \beta_j u_\beta z^{\alpha+\beta-\delta_j}\,,$$

where δ_j is the unit vector with j-th component 1. Thus

$$\begin{aligned} &P_0^{-1}\tilde{a}_j(z)\frac{\partial}{\partial z_j}u(z) \\ &= \sum\Big(\sum_i c_i(\alpha_i+\beta_i-\delta_{ij}) - B(0) - \lambda\Big)^{-1}\beta_j a_\alpha^{(j)} u_\beta z^{\alpha+\beta-\delta_j}, \end{aligned}$$

and hence

$$\begin{aligned} (8.6)\quad \left\|P_0^{-1}\tilde{a}_j(z)\frac{\partial}{\partial z_j}u\right\|_\varepsilon &\le \sum_{|\alpha|\ge 2}\frac{R\beta_j}{|\alpha+\beta-\delta_j|}|a_\alpha^{(j)}u_\beta|\varepsilon^{|\alpha|+|\beta|-1} \\ &\le R\sum_{|\alpha|\ge 2}|a_\alpha^{(j)}|\delta^{|\alpha|}|u_\beta|\varepsilon^{|\beta|}\frac{\varepsilon^{|\alpha|-1}}{\delta^{|\alpha|}} \\ &\le \frac{\varepsilon R}{\delta^2}\|\tilde{a}_j\|_\delta\|u\|_\varepsilon . \end{aligned}$$

From (8.4), (8.5), and (8.6), we obtain

$$\|w_n\|_\varepsilon \le \left(\frac{\varepsilon R}{\delta^2}(\sum\|\tilde{a}_j\|_\delta) + R\frac{\varepsilon}{\delta}\|\tilde{B}\|_\delta\right)\|w_{n-1}\|_\varepsilon.$$

Taking ε_0 small enough, we have for all $\varepsilon<\varepsilon_0$

$$\|w_n\|_\varepsilon \le \frac{1}{2}\|w_{n-1}\|_\varepsilon.$$

If $v(z)\in \mathfrak{m}(U_\varepsilon)^p$, then, for any $\varepsilon_1<\varepsilon$, $w_1 = P_0^{-1}v$ satisfies $\|w_1\|_{\varepsilon_1}<\infty$. Hence u_n converges with respect to $\|\bullet\|_{\varepsilon_1}$, and $u=\sum_{n=1}^\infty w_n$ is holomorphic on U_ε and satisfies $(P-\lambda)u=v$. □

Let $X=\mathbb{C}^n$ and $p=(0,dx_n)\in T^*X$. Suppose that $a_1,\dots,a_n$ are real numbers satisfying

$$(8.7)\qquad 0<a_i<a_n \qquad (i=1,\dots,n-1).$$

Put $\vartheta=\sum_{\nu=1}^n a_\nu x_\nu\partial_\nu$. Our aim in the remaining part of this section is to study formal microdifferential operators with principal part ϑ. We say that $P\in(\widehat{\mathcal{E}}_X)_p$ is *ϑ-finite* if $\varphi(\operatorname{ad}\vartheta)P=0$ for some nonzero polynomial $\varphi(t)$, where $\operatorname{ad}\vartheta\in\operatorname{End}_{\mathbb{C}}((\widehat{\mathcal{E}}_X)_p)$ is the operator defined by $(\operatorname{ad}\vartheta)(P)=[\vartheta,P]$. We call P a *ϑ-homogeneous formal microdifferential operator* of degree λ if $[\vartheta,P]=\lambda P$. For $P=\sum P_i(x,\partial_x)$, the principal symbol of degree i of $(\operatorname{ad}\vartheta)P=[\vartheta,P]$ equals

$$\sum_{\nu=1}^n\left(a_\nu x_\nu\frac{\partial P_i}{\partial x_\nu} - a_\nu\xi_\nu\frac{\partial P_i}{\partial\xi_\nu}\right).$$

Furthermore, letting

$$P_i = \sum_{\substack{\alpha\in\mathbb{Z}^n_{\geq 0} \\ \beta\in\mathbb{Z}^{n-1}_{\geq 0}\times\mathbb{Z},\,|\beta|=i}} c_{\alpha\beta}x^\alpha(\xi_1/\xi_n)^{\beta_1}\cdots(\xi_{n-1}/\xi_n)^{\beta_{n-1}}\xi_n^i,$$

we have

$$\sum_\nu\left(a_\nu x_\nu\frac{\partial P_i}{\partial x_\nu} - a_\nu\xi_\nu\frac{\partial P_i}{\partial \xi_\nu}\right)$$
$$= \sum_{\alpha,\beta}\Big(\sum_{\nu=1}^{n} a_\nu\alpha_\nu + \sum_{\nu=1}^{n-1}\beta_\nu(a_n - a_\nu) - ia_n\Big)c_{\alpha\beta}x^\alpha\xi^\beta.$$

Thus

$$(\operatorname{ad}\vartheta)P = \sum_{\alpha,\beta}\Big(\sum_\nu a_\nu\alpha_\nu + \sum_\nu\beta_\nu(a_n - a_\nu) - a_n|\beta|\Big)c_{\alpha\beta}x^\alpha\partial_x^\beta,$$

and accordingly

$$\varphi(\operatorname{ad}\vartheta)P = \sum_{\alpha,\beta}\varphi\Big(\sum_\nu a_\nu\alpha_\nu + \sum_\nu\beta_\nu(a_n - a_\nu) - a_n|\beta|\Big)c_{\alpha\beta}x^\alpha\partial_x^\beta$$

for all polynomials $\varphi(t)$. Suppose $P \in \widehat{\mathcal{E}}_X(m)$. Then it follows from $a_\nu > 0$, $a_n - a_\nu > 0$, and $-|\beta| \geq -m$ that for any $\lambda \in \mathbb{C}$ there exist only finitely many α, β such that

$$\lambda = \sum_\nu a_\nu\alpha_\nu + \sum_\nu\beta_\nu(a_n - a_\nu) - a_n|\beta|.$$

Hence we obtain the following lemma.

LEMMA 8.16. (1)

$$\{\,P \in (\widehat{\mathcal{E}}_X)_p\,;\ P \text{ is } \vartheta\text{-finite}\,\}$$
$$= \mathbb{C}[x_1,\dots,x_n,\partial_1,\dots,\partial_{n-1},\partial_n,\partial_n^{-1}].$$

(2) *Any ϑ-finite formal microdifferential operator is a finite sum of ϑ-homogeneous ones.*

(3) *For any $m \in \mathbb{Z}$ and $\lambda \in \mathbb{C}$,*

$$\{\,P \in \widehat{\mathcal{E}}_X(m)_p\,;\ (\operatorname{ad}\vartheta)P = \lambda P\,\}$$

is finite-dimensional. If this is not 0, *then*

$$\lambda \in -a_n m + \sum_{\nu=1}^{n}\mathbb{Z}_{\geq 0}a_\nu + \sum_{\nu=1}^{n-1}\mathbb{Z}_{\geq 0}(a_n - a_\nu)\,.$$

In particular, $\lambda \geq 0$ for $m = 0$, and $\lambda > 0$ for $m < 0$.

Any formal microdifferential operator with principal part ϑ can be transformed into a normal form by the adjoint action, as follows.

THEOREM 8.17. *Let $A(x,\partial)$ be a square matrix of size l with coefficients in $\widehat{\mathcal{E}}_X(0)_p$, and $\sigma_0(A)(p) = S+N$ the Jordan decomposition. Then there exist square matrices $Q(x,\partial)$ and $C(x,\partial)$ of size l with coefficients in $\widehat{\mathcal{E}}(0)_p$ such that*

$$\begin{aligned}
&\sigma_0(Q)(p) = 1\,,\\
&\sigma_0(C)(p) = \sigma_0(A)(p)\,,\\
&\vartheta - A = Q(\vartheta - C)Q^{-1}\,,\\
&[\vartheta - S, C] = 0\,.
\end{aligned}$$

PROOF. We solve

$$(\vartheta - A)Q = Q(\vartheta - C),$$

or equivalently $[\vartheta, Q] = AQ - QC$. Let $C = \sum_{i\le 0} C_i(x,\partial)$ and $Q = \sum_{i\le 0} Q_i(x,\partial)$. Then, on the one hand,

$$\begin{aligned}
(8.8)\quad \sum a_\nu x_\nu \frac{\partial}{\partial x_\nu} Q_i - \sum a_\nu \xi_\nu \frac{\partial}{\partial \xi_\nu} Q_i
&= A_0 Q_i - Q_i C_0 - \delta(i \neq 0) Q_0 C_i\\
&\quad + (\text{terms containing } Q_0, \dots, Q_{i-1}, C_0, \dots, C_{i-1})\,.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\sum_{\nu=1}^{n} a_\nu x_\nu \frac{\partial}{\partial x_\nu} Q_i - \sum_{\nu=1}^{n} a_\nu \xi_\nu \frac{\partial}{\partial \xi_\nu} Q_i\\
&\quad = \sum_{\nu=1}^{n} a_\nu x_\nu \frac{\partial}{\partial x_\nu} Q_i + \sum_{\nu=1}^{n-1} (a_n - a_\nu)\xi_\nu \frac{\partial}{\partial \xi_\nu} Q_i - i a_n Q_i.
\end{aligned}$$

Put $L = \sum_{\nu=1}^{n} a_\nu x_\nu \frac{\partial}{\partial x_\nu} + \sum_{\nu=1}^{n-1} (a_n - a_\nu)\xi_\nu \frac{\partial}{\partial \xi_\nu}$. Then (8.8) is rewritten as

$$\begin{aligned}
(8.9)\quad LQ_i &= (i a_n Q_i + A_0 Q_i - Q_i C_0) - \delta(i \neq 0) Q_0 C_i\\
&\quad + (\text{terms containing } Q_0, \dots, Q_{i-1}, C_0, \dots, C_{i-1}).
\end{aligned}$$

When $i = 0$,

$$(8.10)\qquad LQ_0 = A_0 Q_0 - Q_0 C_0.$$

If this is solved with

$$LC_0 = [S, C_0] \text{ and } Q_0 \in M_l(\mathbb{C}[[x_1, \dots, x_n, \xi_1/\xi_n, \dots, \xi_{n-1}/\xi_n]]),$$

then

$$C_0 \in M_l(\mathbb{C}[x_1, \dots, x_n, \xi_1/\xi_n, \dots, \xi_{n-1}/\xi_n]),$$

and hence, by Theorem 8.14, Q_0 converges in a small neighborhood

$$U_\varepsilon = \{ (x,\xi) \,;\, |x_j| < \varepsilon,\ |\xi_j/\xi_n| < \varepsilon,\ |\xi_n - 1| < \varepsilon \}$$

of p. Such a solution exists under the condition that

$$\begin{aligned} C_0(p) &= A_0(p)\,, \\ Q_0(p) &= 1. \end{aligned}$$

Indeed, let $\widehat{\mathfrak{m}}$ denote the ideal of formal power series in $x_1, \dots, x_n$, $\xi_1/\xi_n, \dots, \xi_{n-1}/\xi_n$ vanishing at p. Then

$$L1 \equiv A_0 1 - 1A_0(p) \mod \widehat{\mathfrak{m}}\,.$$

Suppose

$$\begin{aligned} &LQ \equiv A_0Q - QC_0 \mod \widehat{\mathfrak{m}}^k\,, \\ &Q(p) = 1\,, \\ &C_0(p) = A_0(p)\,, \\ &LC_0 = [S, C_0]. \end{aligned}$$

The linear map $\operatorname{ad} A_0(p)$ on $M_l(\mathbb{C})$ has the Jordan decomposition $(\operatorname{ad} S) + (\operatorname{ad} N)$. Hence, by Theorem 8.14, there exist $Q' \in M_l(\widehat{\mathfrak{m}}^k)$ and $C' \in M_l(\widehat{\mathfrak{m}}^k)$ such that

$$LQ' - A_0Q' + Q'A_0(p) \equiv (LQ - A_0Q + QC_0) + C' \mod \widehat{\mathfrak{m}}^{k+1}$$

and $LC' = [S, C']$. Hence

$$\begin{aligned} L(Q - Q') &\equiv A_0(Q - Q') - QC_0 + Q'A_0(p) - C' \mod \widehat{\mathfrak{m}}^{k+1} \\ &\equiv A_0(Q - Q') - (Q - Q')(C_0 - C') + Q'(A_0(p) - C_0) \\ &\qquad\qquad - (Q - Q' + 1)C' \mod \widehat{\mathfrak{m}}^{k+1} \\ &\equiv A_0(Q - Q') - (Q - Q')(C_0 - C') \mod \widehat{\mathfrak{m}}^{k+1}\,. \end{aligned}$$

Repeating this process, we obtain a formal series solution Q_0 to (8.10). Repeating the similar argument, we obtain Q_i and C_i. By Theorem 8.14, Q_i converges in a small neighborhood (independent of i) of p. □

PROPOSITION 8.18. *Let $\mathcal{J}$ be a left ideal of $(\widehat{\mathcal{E}}_{\mathbb{C}^n})_p$. Suppose $P \in \widehat{\mathcal{E}}_{\mathbb{C}^n}(1)_p \cap \mathcal{J}$ and $\sigma_1(P) = \sigma_1(\vartheta)$. Then there exists an invertible $Q \in \widehat{\mathcal{E}}_{\mathbb{C}^n}(0)_p$ such that $\mathcal{J}Q \ni \vartheta - \lambda$ for some $\lambda \in \mathbb{C}$.*

PROOF. By Theorem 8.17, there exists an invertible $Q \in \widehat{\mathcal{E}}_{\mathbb{C}^n}(0)_p$ such that $P = Q(\vartheta - \lambda)Q^{-1}$. □

PROPOSITION 8.19. *Let $\mathcal{J}$ be a left ideal of $(\widehat{\mathcal{E}}_{\mathbb{C}^n})_p$. If $\mathcal{J} \ni \vartheta - \lambda$ for some $\lambda \in \mathbb{C}$, then $\widehat{\mathcal{E}}_{\mathbb{C}^n}(0)_p \cap \mathcal{J}$ is generated by ϑ-homogeneous operators.*

PROOF. Let $P_1, \dots, P_l$ generate $\widehat{\mathcal{E}}_{\mathbb{C}^n}(0)_p \cap \mathcal{J}$. Since $[\vartheta - \lambda, P_\nu] \in \widehat{\mathcal{E}}_{\mathbb{C}^n}(0)_p \cap \mathcal{J}$, there exist $A_{\nu\mu} \in \widehat{\mathcal{E}}_{\mathbb{C}^n}(0)_p$ such that

$$[\vartheta, P_\nu] = \sum_\mu A_{\nu\mu} P_\mu.$$

Let A be the matrix $(A_{\nu\mu})_{1 \le \nu,\mu \le l}$, and P the column vector with components $P_1, \dots, P_l$. Then the above equations are rewritten as

$$(\vartheta - A)P = P\vartheta.$$

By Theorem 8.17, there exist Q and C such that $\vartheta - A = Q(\vartheta - C)Q^{-1}$. Here, for the Jordan decomposition $\sigma_0(C)(p) = S + N$, $[\vartheta - S, C] = 0$. Hence

$$(\vartheta - C)Q^{-1}P = (Q^{-1}P)\vartheta.$$

Put $P' = Q^{-1}P$. Then the components of P' also generate $\mathcal{J}$. Hence, replacing P by P', we may assume $[\vartheta, P] = CP$. We may also assume that S is a diagonal matrix with diagonal components $\lambda_1, \dots, \lambda_n$ satisfying $\operatorname{Re}\lambda_1 \le \operatorname{Re}\lambda_2 \le \cdots$, and that $S + N$ is a lower triangular matrix. It follows from $[\vartheta, C] = [S, C]$ that the (i,j)-component C_{ij} of C is ϑ-homogeneous of degree $\lambda_i - \lambda_j$. Hence $C_{ij} \ne 0$ implies $\lambda_i - \lambda_j \ge 0$. If $C_{ij} \ne 0$ and $\lambda_i - \lambda_j > 0$, then $i > j$. If $C_{ij} \ne 0$ and $\lambda_i - \lambda_j = 0$, then C_{ij} is constant, and hence $C_{ij} = S_{ij} + N_{ij}$. Since N is lower triangular, we have $C_{ij} = 0$ $(i < j)$ and $C_{ii} = \lambda_i$. We can prove that the i-th component P_i of P is ϑ-homogeneous of degree λ_i by induction on i. Indeed, $(\operatorname{ad}\vartheta - \lambda_i)P_i = \sum_{j<i} C_{ij}P_j$ is ϑ-homogeneous of degree λ_i, and hence so is P_i. □

8.5. Classification of Simple Holonomic Systems (1)

With the preparation above, we shall locally classify simple holonomic $\widehat{\mathcal{E}}_X$-modules $(\mathcal{M}, u)$.

Let $(\mathcal{M}, u)$ be a simple holonomic $\widehat{\mathcal{E}}_X$-module, and Λ its support. In this section, we consider the case when Λ is nonsingular. Take a point $p \in \Lambda$, and work in its neighborhoods. By a homogeneous symplectic transformation, Λ can be transformed into

$$\Lambda = \{ (x, \xi) ; x_1 = \xi_2 = \cdots = \xi_n = 0 \} \ni p = (0, dx_1).$$

Hence, by a quantized contact transformation, we may assume that the support of $\mathcal{M}$ equals the above Λ. Let

$$\vartheta = 2x_1\partial_1 + x_2\partial_2 + \cdots + x_n\partial_n.$$

Since $\sigma_1(\vartheta)|_\Lambda = 0$, there exists $A \in \widehat{\mathcal{E}}_X(0)$ such that $(\vartheta - A)u = 0$. By Proposition 8.18, there exist an invertible $Q \in \widehat{\mathcal{E}}_X(0)$ and $\lambda \in \mathbb{C}$ such that $\vartheta - A = Q(\vartheta - \lambda)Q^{-1}$. Replacing u by $Q^{-1}u$, we may assume $(\vartheta - \lambda)u = 0$. Then $\{P \in \widehat{\mathcal{E}}_X(m); Pu = 0\}$ is generated by ϑ-homogeneous operators. Since ∂_j $(2 \le j \le n)$ are of degree -1, and $\xi_j|_\Lambda = 0$, there exist ϑ-homogeneous operators Q_j of degree -1 such that $Q_j \in \widehat{\mathcal{E}}_X(1)$, $\sigma_1(Q_j) = \xi_j$, and $Q_j u = 0$. Here note that x_1 and ∂_1^{-1} are of degree 2, and $x_2, \ldots, x_n, \partial_2\partial_1^{-1}, \ldots, \partial_n\partial_1^{-1}$ are of degree 1. Hence the degrees of nonconstant homogeneous operators in $\widehat{\mathcal{E}}_X(0)$ are positive. From the consideration above, we obtain $Q_j = \partial_j$. We have thus obtained the following theorem.

THEOREM 8.20. *For a simple holonomic module $(\mathcal{M}, u)$ defined on a neighborhood of $p = (0, dx_1)$ with support $\Lambda = \{(x, \xi); x_1 = \xi_2 = \cdots = \xi_n = 0\}$, there exist an invertible $P \in \widehat{\mathcal{E}}_X(0)$ and $\lambda \in \mathbb{C}$ such that $v = Pu$ has a system of defining equations*

$$\begin{aligned} &(x_1\partial_1 - \lambda)v = 0\,, \\ &\partial_2 v = \cdots = \partial_n v = 0. \end{aligned} \tag{8.11}$$

Let $(\mathcal{M}_\lambda, u_\lambda)$ be the simple holonomic $\widehat{\mathcal{E}}_X$-module defined by (8.11) with $v = u_\lambda$. Then

$$\widehat{\mathcal{E}}_X^{(\lambda-\mu)} \otimes_{\widehat{\mathcal{E}}_X} \mathcal{M}_\lambda \xrightarrow{\sim} \mathcal{M}_\mu \quad (\partial_1^{\lambda-\mu} \otimes u_\lambda \mapsto u_\mu).$$

Furthermore,

$$\mathcal{E}nd_{\widehat{\mathcal{E}}_X}(\mathcal{M}_\lambda) \simeq \mathbb{C}_\Lambda.$$

The proof is omitted; we leave it to the reader.

We immediately obtain the following theorem.

THEOREM 8.21. *Let Λ be a nonsingular Lagrangian submanifold of $\mathring{T}^*X$, and $(\mathcal{M}_\nu, u_\nu)$ simple holonomic $\widehat{\mathcal{E}}_X$-modules with support Λ $(\nu = 1, 2)$. Let $p \in \Lambda$. Then there exist $\lambda \in \mathbb{C}$, an invertible operator $P \in \widehat{\mathcal{E}}_X(\lambda)$, and an isomorphism on a neighborhood of p*

$$\varphi : \widehat{\mathcal{E}}_X^{(\lambda)} \otimes_{\widehat{\mathcal{E}}_X} \mathcal{M}_1 \xrightarrow{\sim} \mathcal{M}_2$$

such that $\varphi(P \otimes u_1) = u_2$. Moreover, $\lambda \in \mathbb{C}$ is uniquely determined by this condition, and $\sigma_\lambda(P)|_\Lambda$ is also unique up to scalar multiplication.

8.6. Principal Symbols of Simple Holonomic Modules

Let Λ be a Lagrangian submanifold of T^*X, and $(\mathcal{M}, u)$ a simple holonomic system with support Λ. If the rank of $\Lambda \to X$ is constant on a neighborhood of $p \in \Lambda$, then there exists a submanifold Z of X such that $\Lambda \simeq T^*_Z X$ in a neighborhood of p. Define a simple holonomic $\widehat{\mathcal{E}}_X$-module $\widehat{\mathcal{C}}_{Z|X}$ with support $T^*_Z X$ by $\widehat{\mathcal{C}}_{Z|X} = \widehat{\mathcal{E}}_X \otimes_{\mathcal{D}_X} \mathcal{B}_{Z|X}$. By Theorem 8.21, $\mathcal{M} \simeq \widehat{\mathcal{E}}_X^{(\lambda)} \otimes_{\widehat{\mathcal{E}}_X} \widehat{\mathcal{C}}_{Z|X}$. This isomorphism is unique up to scalar multiplication. Take a coordinate system $(x_1, \dots, x_n)$ so that $Z = \{x_1 = \cdots = x_l = 0\}$, and put $\delta_{Z|X} = \delta(x_1, \dots, x_l)$. Then $u = P\delta_{Z|X}$ for some $P \in \widehat{\mathcal{E}}_X(\lambda)$. Here $\sigma_\lambda(P)|_\Lambda$ is unique up to scalar multiplication. Thus we wish to define a principal symbol of u as $\sigma_\lambda(P)|_\Lambda$; however we need to modify this, since $\delta_{Z|X}$ depends on the choice of a coordinate system. It is $\delta_{Z|X} dx_1 \wedge \cdots \wedge dx_l \in \Omega^l_X \otimes \mathcal{C}_{Z|X}$ that is canonical, i.e., independent of the choice of a coordinate system. Hence $\sigma_\lambda(P)(dx_1 \cdots dx_l)^{-1}$ is invariant under coordinate change. This is considered as a section of $\mathcal{O}_\Lambda \otimes_{\mathcal{O}_X} \Omega_{Z/X}$ ($\Omega_{Z/X} = \Omega_Z \otimes \Omega_X^{\otimes -1}$). If p is an arbitrary point of Λ, then such a Z need not exist. Hence we need an expression of $\mathcal{O}_\Lambda \otimes_{\mathcal{O}_X} \Omega_{Z/X}$ free from Z. By the exact sequence

$$0 \to (T^*_Z X)_{\pi(p)} \to T_p\Lambda \to T_{\pi(p)}Z \to 0,$$

we obtain

$$\begin{aligned} \det T_p\Lambda &= \det T_{\pi(p)}Z \otimes \det(T^*_Z X)_{\pi(p)} \\ &= (\det(T^*_Z X)_{\pi(p)})^{\otimes 2} \otimes \det T_{\pi(p)}X\,, \end{aligned}$$

where, for a finite-dimensional vector space V, $\det V := \bigwedge^{\dim V} V$. Hence $\Omega^{\otimes 2}_{Z/X} \cong \Omega_\Lambda \otimes \Omega_X^{\otimes -1}$. When $T^*_Z X = \{(x, \xi); x_1 = \cdots = x_l = \xi_{l+1} = \cdots = \xi_n = 0\}$, this isomorphism is explicitly given by

$$(dx_1 \cdots dx_l)^{-2} \leftrightarrow \frac{d\xi_1 \wedge \cdots \wedge d\xi_l \wedge dx_{l+1} \wedge \cdots \wedge dx_n}{dx_1 \wedge \cdots \wedge dx_n}.$$

(Note that, roughly speaking, $d\xi_1 \wedge \cdots \wedge d\xi_l$ and $(dx_1 \wedge \cdots \wedge dx_l)^{-1}$ share the same coordinate changes.) Hence

$$\sigma_\lambda(P)\sqrt{\frac{d\xi_1 \wedge \cdots \wedge d\xi_l \wedge dx_{l+1} \wedge \cdots \wedge dx_n}{dx_1 \wedge \cdots \wedge dx_n}} \in (\Omega_{\Lambda/X})^{\otimes 1/2}$$

is invariant under coordinate change. We denote this by $\sigma(u)$ and call it the *principal symbol* of u, where $\Omega^{\otimes 1/2}_{\Lambda/X}$ denotes an invertible sheaf satisfying $(\Omega_{\Lambda/X}^{\otimes 1/2})^{\otimes 2} \simeq \Omega_{\Lambda/X} = \Omega_\Lambda \otimes \Omega_X^{\otimes -1}$. Such a sheaf $\Omega_{\Lambda/X}^{\otimes 1/2}$ locally exists. For $s \in \Omega_{\Lambda/X}$, we denote by $\sqrt{s}$ a section $t \in \Omega_{\Lambda/X}^{\otimes 1/2}$ such that $t \otimes t = s$. For invertible sheaves L and L' and for an isomorphism $\varphi : L^{\otimes 2} \xrightarrow{\sim} L'^{\otimes 2}$, an isomorphism $f : L \xrightarrow{\sim} L'$ satisfying $f^{\otimes 2} = \varphi$ is determined up to sign. Since $\sigma_\lambda(P)$ makes sense only up to scalar multiplication, $\sigma(u) \in (\Omega_{\Lambda/X})^{\otimes 1/2}$ is determined up to scalar multiplication. This is a holomorphic section at generic points of Λ. To see that $\sigma(u)$ is holomorphic at any point of Λ, let us find differential equations satisfied by the section $\sigma(u)$.

For a Lagrangian submanifold Λ of T^*X, put

$$\mathcal{J}_\Lambda(m) = \{\, P \in \widehat{\mathcal{E}}_X(m+1)\,;\, \sigma_{m+1}(P)|_\Lambda = 0 \,\}.$$

Note that $[\mathcal{J}_\Lambda(m), \mathcal{J}_\Lambda(m')] \subset \mathcal{J}_\Lambda(m+m')$. For $P \in \mathcal{J}_\Lambda(m)$, define an operator $\mathcal{L}_P^{(m)}$ on $(\Omega_{\Lambda/X})^{\otimes 1/2}$ by

$$\mathcal{L}_P^{(m)}(a(dx)^{-1/2}) = (H_{\sigma_{m+1}(P)}a + \sigma_m(P)a)(dx)^{-1/2} \quad (a \in \Omega_\Lambda^{\otimes 1/2}).$$

The meaning of this equation is as follows: $\sigma_m(P)$ is the subprincipal symbol introduced in § 8.3. Since $\sigma_{m+1}(P)|_\Lambda = 0$, $H_{\sigma_{m+1}(P)}$ is considered as a vector field on Λ. For any non-vanishing section s of Ω_Λ and any $\varphi \in \mathcal{O}_\Lambda$, set

$$H_{\sigma_{m+1}(P)}(\varphi\sqrt{s}) = (H_{\sigma_{m+1}(P)}\varphi)\sqrt{s} + \frac{1}{2}\left(\frac{H_{\sigma_{m+1}(P)}s}{s}\right)\varphi\sqrt{s}.$$

It is easy to see that this is well-defined. Thus $H_{\sigma_{m+1}(P)}$ acts on $\Omega_\Lambda^{\otimes 1/2}$.

By the results about subprincipal symbols in § 8.3, $\mathcal{L}_P^{(m)}$ is a differential operator on $\Omega_{\Lambda/X}^{\otimes 1/2}$ invariant under coordinate change.

PROPOSITION 8.22. (1) *If $P \in \mathcal{J}_\Lambda(m)$, and $A \in \widehat{\mathcal{E}}_X(l)$, then*

$$\begin{aligned} \mathcal{L}_{AP}^{(m+l)} &= \sigma_l(A)\mathcal{L}_P^{(m)}, \\ \mathcal{L}_{PA}^{(m+l)} &= \mathcal{L}_P^{(m)}\sigma_l(A). \end{aligned}$$

(2) *If $P \in \mathcal{J}_\Lambda(m)$, and $Q \in \mathcal{J}_\Lambda(l)$, then*

$$\mathcal{L}_{[P,Q]}^{(m+l)} = [\mathcal{L}_P^{(m)}, \mathcal{L}_Q^{(l)}].$$

PROOF. For a vector field v, we denote by $L(v)$ the corresponding Lie derivative. By Lemma 1.8, $L(av) = aL(v) + v(a)$ $(a \in \mathcal{O})$ as operators on Ω. Hence

$$L(av) = aL(v) + v(a)/2$$

as operators on $\Omega^{\otimes 1/2}$. Indeed, for $\omega \in \Omega$, we have

$$\begin{aligned} L(av)(\sqrt{\omega}) &= \frac{L(av)\omega}{2\omega}\sqrt{\omega} \\ &= \frac{aL(v)\omega + v(a)\omega}{2\omega}\sqrt{\omega} = aL(v)\sqrt{\omega} + \frac{v(a)}{2}\sqrt{\omega}\,. \end{aligned}$$

First we prove (1). We have $\mathcal{L}_{AP}^{(m+l)} = L(H_{\sigma_{m+l+1}(AP)}) + \sigma_{m+l}(AP)$. Since

$$H_{\sigma_{m+l+1}(AP)} = \sigma_l(A)H_{\sigma_{m+1}(P)} + \sigma_{m+1}(P)H_{\sigma_l(A)},$$

it follows that $H_{\sigma_{m+l+1}(AP)} = \sigma_l(A)H_{\sigma_{m+1}(P)}$ as vector fields on Λ. Hence

$$L(H_{\sigma_{m+l+1}(AP)}) = \sigma_l(A)L(H_{\sigma_{m+1}(P)}) + \{\sigma_{m+1}(P), \sigma_l(A)\}/2$$

as operators on $\Omega_\Lambda^{\otimes 1/2}$. By Lemma 8.13,

$$\begin{aligned} \sigma_{m+l}(AP) &= \sigma_l(A)\sigma_m(P) + \sigma_{l-1}(A)\sigma_{m+1}(P) \\ &\qquad + \{\sigma_l(A), \sigma_{m+1}(P)\}/2 \\ &= \sigma_l(A)\sigma_m(P) + \{\sigma_l(A), \sigma_{m+1}(P)\}/2\,. \end{aligned}$$

Hence we obtain

$$L(H_{\sigma_{m+l+1}(AP)}) + \sigma_{m+l}(AP) = \sigma_l(A)(L(H_{\sigma_{m+1}(P)}) + \sigma_m(P)).$$

We can similarly prove the second equation in (1).

(2) immediately follows from

$$\begin{aligned} H_{\sigma_{m+l+1}([P,Q])} &= H_{\{\sigma_{m+1}(P), \sigma_{l+1}(Q)\}} \\ &= [H_{\sigma_{m+1}(P)}, H_{\sigma_{l+1}(Q)}], \end{aligned}$$

$$L(H_{\sigma_{m+l+1}([P,Q])}) = [L(H_{\sigma_{m+1}(P)}), L(H_{\sigma_{l+1}(Q)})],$$

and $\sigma_{m+l}([P,Q]) = H_{\sigma_{m+1}(P)}(\sigma_l(Q)) - H_{\sigma_{l+1}(Q)}(\sigma_m(P))$ (Lemma 8.13). □

Let $\Lambda \subset T^*X$ be a Lagrangian submanifold, and $(\mathcal{M}, u)$ a simple holonomic system with support Λ. Let $\mathcal{I} = \{P \in \widehat{\mathcal{E}}_X; Pu = 0\}$. Consider the following system of differential equations for $\varphi \in \Omega_{\Lambda/X}^{\otimes 1/2}$: For all $m \in \mathbb{Z}$ and $P \in \mathcal{I} \cap \mathcal{J}_\Lambda(m)$,

$$\mathcal{L}_P^{(m)}\varphi = 0\,. \tag{8.12}$$

By Proposition 8.22, this system of differential equations is involutive, and locally its solution is uniquely determined up to scalar multiplication. We denote this by $\sigma(u)$ and call it the *principal symbol* of

u. When we emphasize the support Λ of $\mathcal{M}$, we denote the principal symbol by $\sigma_\Lambda(u)$.

Let $A \in \widehat{\mathcal{E}}_X(l)$ be invertible, and let $v = Au$. Then $\mathcal{I}' = \{P; Pv = 0\} = \mathcal{I}A^{-1}$. Since $\mathcal{L}_P^{(m)} = \mathcal{L}_{PA}^{(m+l)}\sigma_l(A)^{-1}$ for $P \in \mathcal{I}' \cap \mathcal{J}_\Lambda(m)$, we have

$$\mathcal{L}_P^{(m)}(\sigma_l(A)\sigma(u)) = 0 \qquad (P \in \mathcal{I}' \cap \mathcal{J}_\Lambda(m))\,.$$

Hence $\sigma(Au) = \sigma_l(A)\sigma(u)$.

LEMMA 8.23. *Let $Z = \{x_1 = \cdots = x_l = 0\}$ be a submanifold of X. Let $P \in \widehat{\mathcal{E}}_X(\lambda)$, and $u = P\delta_{Z|X}$. Then the solution to* (8.12) *equals*

$$\sigma_\lambda(P)\sqrt{\frac{d\xi_1 \wedge \cdots \wedge d\xi_l \wedge dx_{l+1} \wedge \cdots \wedge dx_n}{dx}}.$$

PROOF. We may assume $P = \partial_1^\lambda$. Hence the system of equations for u is

$$\begin{aligned} &x_j u = 0 && (2 \le j \le l)\,,\\ &\partial_j u = 0 && (l < j \le n)\,,\\ &(x_1\partial_1 + \lambda + 1)u = 0. \end{aligned}$$

Hence for $a = a(x_{l+1}, \dots, x_n, \xi_1, \dots, \xi_l) \in \mathcal{O}_\Lambda$ we have

$$\begin{aligned} \mathcal{L}_{x_j}^{(-1)} a\sqrt{\frac{d\xi' \wedge dx''}{dx}} &= -\frac{\partial a}{\partial \xi_j}\sqrt{\frac{d\xi' \wedge dx''}{dx}} && (2 \le j \le l)\,,\\ \mathcal{L}_{\partial_j}^{(0)} a\sqrt{\frac{d\xi' \wedge dx''}{dx}} &= \frac{\partial a}{\partial x_j}\sqrt{\frac{d\xi' \wedge dx''}{dx}} && (l < j \le n)\,,\\ \mathcal{L}_{x_1\partial_1+\lambda+1}^{(0)} a\sqrt{\frac{d\xi' \wedge dx''}{dx}} &= \left(-\xi_1\frac{\partial a}{\partial \xi_1} + \lambda a\right)\sqrt{\frac{d\xi' \wedge dx''}{dx}}, \end{aligned}$$

where $d\xi' = d\xi_1 \wedge \cdots \wedge d\xi_l$ and $dx'' = dx_{l+1} \wedge \cdots \wedge dx_n$. Hence its solution equals $a = \xi_1^\lambda$. □

Thus the definition of $\sigma(u)$ coincides with the one given in the beginning of this section. We have thus defined the principal symbol of u up to scalar multiplication. We call the homogeneous degree of $\sigma(u)$ in ξ the *order* of u and denote it by $\operatorname{ord}(u)$. When we emphasize the support Λ of $\mathcal{M}$, we denote it by $\operatorname{ord}_\Lambda(u)$. By the definition, for the Euler vector field $E = \sum \xi_i \frac{\partial}{\partial \xi_i}$,

$$E\sigma(u) = \operatorname{ord}(u) \cdot \sigma(u)\,,$$

where the action of E on $\Omega^{\otimes 1/2}_{\Lambda/X}$ is defined similarly to that of $H_{\sigma_{m+1}(P)}$ defined before. For example, for the u in the proof of Lemma 8.23, $\operatorname{ord}(u) = \lambda + l/2$. Note that the homogeneous degree of the section $\sqrt{\frac{d\xi_1 \cdots d\xi_l dx_{l+1} \cdots dx_n}{dx}}$ equals $l/2$.

Given a simple holonomic $\widehat{\mathcal{E}}_X$-module $(\mathcal{M}, u)$, we shall give a formula for calculating $\operatorname{ord}(u)$. Let $p \in T^*X$. Note that, if a vector field v on T^*X satisfies $v(p) = 0$, then $d\varphi \mapsto d(v(\varphi))$ induces an endomorphism of $T^*_p(T^*X)$ (the coisotropy representation of v). If w is a vector field, then $w(p) \mapsto [v, w](p)$ gives an endomorphism of $T_p(T^*X)$. This is called the isotropy representation of v. The coisotropy representation equals the adjoint operator of isotropy representation with sign change.

Let $(\mathcal{M}, u)$ be a simple holonomic $\widehat{\mathcal{E}}_X$-module. Let $\Lambda = \operatorname{Supp} \mathcal{M}$, $p \in \Lambda$, and

$$\mathcal{I} = \{\, u \in \widehat{\mathcal{E}}_X \,;\, Pu = 0 \,\}.$$

Suppose that $a \in \mathbb{C}$ and $P \in \widehat{\mathcal{E}}_X(m) \cap \mathcal{I}$ satisfy

$$d\sigma_m(P)(p) = a\omega_X(p) \tag{8.13}$$

in $T^*_p(T^*X)$. Then, for the Euler vector field E on T^*X ($E = \sum \xi_i \frac{\partial}{\partial \xi_i}$ in a local coordinate system), $H_{\sigma_m(P)} + aE$ is a vector field on Λ vanishing at p. Hence this acts on $T^*_p\Lambda$.

PROPOSITION 8.24. *Under the above assumptions,*

$$a \operatorname{ord}(u) = \sigma_{m-1}(P)(p) + \frac{1}{2} \operatorname{tr}(H_{\sigma_m(P)} + aE; T^*_p\Lambda). \tag{8.14}$$

PROOF. (1) The case when $d\sigma_m(P)(q) = \omega_X(q)$ for all $q \in \Lambda$. In this case, we have $H_{\sigma_m(P)} = -E$ as vector fields on Λ. Hence the right hand side of (8.14) equals $\sigma_{m-1}(P)(p)$. Since $(H_{\sigma_m(P)} + \sigma_{m-1}(P))(\sigma(u)\sqrt{dx}) = 0$, we have

$$\operatorname{ord}(u)(\sigma(u)\sqrt{dx}) = E(\sigma(u)\sqrt{dx}) = \sigma_{m-1}(P)(\sigma(u)\sqrt{dx}).$$

(2) The general case. Take Q such that $d\sigma(Q)(q) = \omega(q)$ for all $q \in \Lambda$. Then there exist A_j and R_j such that

$$P - aQ = \sum A_j R_j\,,$$
$$R_j u = 0, \quad \sigma(A_j)(p) = 0.$$

Hence it suffices to show that, if $P = AR$, $\sigma(A)(p) = 0$, and $Ru = 0$, then the right hand side of (8.14) equals 0. Let

$A \in \widehat{\mathcal{E}}_X(l)$ and $R \in \widehat{\mathcal{E}}_X(m)$. Put $A_l = \sigma_l(A)$, $R_m = \sigma_m(R)$, and $P_{l+m} = \sigma_{l+m}(P)$. Then

$$\begin{aligned} H_{P_{l+m}}|_\Lambda &= (A_l H_{R_m} + R_m H_{A_l})|_\Lambda \\ &= A_l H_{R_m}|_\Lambda . \end{aligned}$$

Hence $H_{P_{l+m}}$ maps df ($f \in \mathcal{O}_\Lambda$) to $\{R_m, f\} dA_l$. Thus $H_{P_{l+m}} : T_p^*\Lambda \to T_p^*\Lambda$ is a composite of $T_p^*\Lambda \ni \eta \mapsto \langle H_{R_m}, \eta\rangle \in \mathbb{C}$ with $\mathbb{C} \ni 1 \mapsto dA_l \in T_p^*\Lambda$. Hence

$$\operatorname{tr}(H_{P_{l+m}}; T_p^*\Lambda) = \{R_m, A_l\}(p).$$

By Lemma 8.13,

$$\begin{aligned} &\sigma_{l+m-1}(P) \\ &= \sigma_l(A)\sigma_{m-1}(R) + \sigma_{l-1}(A)\sigma_m(R) + \{\sigma_l(A), \sigma_m(R)\}/2. \end{aligned}$$

Since $A_l(p) = R_m(p) = 0$, we have

$$\sigma_{l+m-1}(P)(p) = \{A_l, R_m\}(p)/2 .$$

We have thus obtained

$$\sigma_{l+m-1}(P)(p) + \frac{1}{2}\operatorname{tr}(H_{\sigma_{l+m}(P)}; T_p^*\Lambda) = 0.$$

□

To end this section, we consider how principal symbols vary under quantized contact transformations.

Let X and Y be n-dimensional manifolds, $(\mathcal{M}, K)$ a simple holonomic system defined on an open set Ω in $T^*(X \times Y)$ with support Λ, and Φ the quantized contact transformation with kernel $(\mathcal{M}, K)$. Let $(\mathcal{M}_Y, u_Y)$ be a simple holonomic $\widehat{\mathcal{E}}_Y$-module with support Λ_Y, and $(\mathcal{M}_X, u_X)$ its image under Φ, a simple holonomic $\widehat{\mathcal{E}}_X$-module with support Λ_X. Let $\omega_Y^n \in \Omega_{T^*Y}$ be the canonical section on T^*Y.

PROPOSITION 8.25.

$$\begin{aligned} \sigma(u_X) &= \sigma(K)\sigma(u_Y) \cdot dy/\sqrt{\omega_Y^n} , \\ \operatorname{ord}(u_X) &= \operatorname{ord}(K) + \operatorname{ord}(u_Y) - n/2 . \end{aligned}$$

Since $\Lambda \to T^*Y$ is an isomorphism, $\Omega_\Lambda \simeq \Omega_{T^*Y}$. The first equation means that, under the identification

$$\begin{aligned} &\sigma(K)\sigma(u_Y)dy/\sqrt{\omega_Y^n} \\ &\in \ \Omega_\Lambda^{1/2} \otimes \Omega_{X\times Y}^{-1/2} \otimes \Omega_{\Lambda_Y}^{1/2} \otimes \Omega_Y^{-1/2} \otimes \Omega_Y \otimes \Omega_{T^*Y}^{-1/2} \\ &\simeq \ \Omega_{\Lambda_Y}^{1/2} \otimes \Omega_X^{-1/2} \simeq \Omega_{\Lambda_X}^{1/2} \otimes \Omega_X^{-1/2}, \end{aligned}$$

the section $\sigma(K)\sigma(u_Y)dy/\sqrt{\omega_Y^n}$ equals $\sigma(u_X)$.

PROOF. Since the second equation follows from the first one, we prove the first one. Suppose $Pu_X = 0$. Letting $P = \Phi(Q)$, we have $PK = Q^*K$.

It suffices to show that $p_{1*}(\sigma(K)p_2^{a*}\sigma(u_Y))$ is annihilated by the operator $H_{\sigma_m(P)}+\sigma_{m-1}(P)$. Since $p_{1*}(H_{\sigma_m(P)}-H_{\sigma_m(Q^*)}) = H_{\sigma_m(P)}$, we have

$$\begin{aligned}&(H_{\sigma_m(P)} + \sigma_{m-1}(P))p_{1*}(\sigma(K)p_2^{a*}\sigma(u_Y))\\ &\quad = \quad p_{1*}\left\{(H_{\sigma_m(P)} - H_{\sigma_m(Q^*)} + \sigma_{m-1}(P))(\sigma(K)p_2^{a*}\sigma(u_Y))\right\}.\end{aligned}$$

Since

$$(H_{\sigma_m(P)} - H_{\sigma_m(Q^*)} + \sigma_{m-1}(P) - \sigma_{m-1}(Q^*))\sigma(K) = 0,$$

we obtain

$$\begin{aligned}&(H_{\sigma_m(P)} - H_{\sigma_m(Q^*)} + \sigma_{m-1}(P))\sigma(K)p_2^{a*}\sigma(u_Y)\\ &\quad = \quad \left\{(H_{\sigma_m(P)} - H_{\sigma_m(Q^*)} + \sigma_{m-1}(P))\sigma(K)\right\}\cdot p_2^{a*}\sigma(u_Y)\\ &\qquad\qquad + \sigma(K)(H_{\sigma_m(P)} - H_{\sigma_m(Q^*)})p_2^{a*}\sigma(u_Y)\\ &\quad = \quad \sigma(K)\left\{\sigma_{m-1}(Q^*)p_2^{a*}\sigma(u_Y) - H_{\sigma_m(Q^*)}p_2^{a*}\sigma(u_Y)\right\}.\end{aligned}$$

Since $p^a_{2*}H_{\sigma_m(Q^*)} = -H_{\sigma_m(Q)}$ and $p^a_{2*}\sigma_{m-1}(Q^*) = \sigma_{m-1}(Q)$, the above equals 0. □

8.7. Regular Holonomic $\widehat{\mathcal{E}}_X$-modules

In § 5.2, we defined the notion of regular holonomic $\mathcal{D}_X$-modules. In this section, we generalize this to $\widehat{\mathcal{E}}_X$-modules and investigate their elementary properties.

For a homogeneous analytic subset V of T^*X, set

$$\mathcal{J}_V := \{\, P \in \widehat{\mathcal{E}}_X(1)\,;\, \sigma_1(P)|_V = 0\,\}.$$

Then $\mathcal{J}_V$ is a two-sided $\widehat{\mathcal{E}}_X(0)$-module. Furthermore, it is coherent as a left $\widehat{\mathcal{E}}_X(0)$-module and as a right $\widehat{\mathcal{E}}_X(0)$-module. Let $\mathcal{M}$ be a holonomic $\widehat{\mathcal{E}}_X$-module defined on an open set Ω in $\mathring{T}^*X$.

By definition, $\mathrm{Supp}(\mathcal{M})$ is Lagrangian.

PROPOSITION 8.26. *Let $\mathcal{M}$ be a holonomic $\widehat{\mathcal{E}}_X$-module defined on $\Omega \subset \overset{\circ}{T}{}^*X$. Then the following two conditions are equivalent:*

(1) *For any open subset $U \subset \Omega$, any coherent $\widehat{\mathcal{E}}_X(0)$-submodule $\mathcal{N}$ of $\mathcal{M}|_U$, and any Lagrangian analytic subset Λ containing* $\mathrm{Supp}(\mathcal{M}) \cap U$, *$\bigcup_m \mathcal{J}_\Lambda^m \mathcal{N}$ is a coherent $\widehat{\mathcal{E}}_X(0)|_U$-module.*

(2) *For any $p \in \Omega$, there exist a Lagrangian analytic subset Λ containing* $\mathrm{Supp}(\mathcal{M})$ *in a neighborhood of p and a finitely generated $\widehat{\mathcal{E}}_X(0)_p$-submodule N of $\mathcal{M}_p$ such that*

$$(\mathcal{J}_\Lambda)_p N = N, \quad \widehat{\mathcal{E}}_{X,p} N = \mathcal{M}_p. \tag{8.15}$$

PROOF. Since (1)$\Longrightarrow$(2) is obvious, we prove only (2)$\Longrightarrow$(1). Assume (2). Then, for any p, there exist a Lagrangian analytic subset Λ_1 containing $\mathrm{Supp}(\mathcal{M})$ in a neighborhood of p and a module N satisfying (8.15) for Λ_1. Let $\mathcal{N}_1$ be a coherent $\widehat{\mathcal{E}}_X(0)$-submodule of $\mathcal{M}$ defined on a neighborhood of p such that $N = (\mathcal{N}_1)_p$. Then $\mathcal{J}_{\Lambda_1}\mathcal{N}_1 \subset \mathcal{N}_1$, and $\widehat{\mathcal{E}}_X\mathcal{N}_1 = \mathcal{M}$ in a neighborhood of p. We shall prove that, for any coherent $\widehat{\mathcal{E}}_X(0)$-submodule $\mathcal{N}$ of $\mathcal{M}$ defined on a neighborhood of p and any Lagrangian analytic subset Λ containing $\mathrm{Supp}(\mathcal{M})$, $\bigcup_m \mathcal{J}_\Lambda^m \mathcal{N}$ is a coherent $\widehat{\mathcal{E}}_X(0)$-module.

Without loss of generality, we may assume $\mathcal{N} \subset \mathcal{N}_1$. By Theorem 7.34, the set S of points at which $\bigcup_m \mathcal{J}_\Lambda^m \mathcal{N}$ is not coherent over $\widehat{\mathcal{E}}_X(0)$ is an involutive closed analytic subset. Hence S is Lagrangian. We have $S \subset \mathrm{Supp}(\mathcal{M}) \subset \Lambda \cap \Lambda_1$, and S is a union of irreducible components of $\mathrm{Supp}(\mathcal{M})$. If $S \neq \emptyset$, then Λ and Λ_1 coincide with S in a neighborhood of some $q \in S$. Accordingly, $\mathcal{J}_\Lambda = \mathcal{J}_{\Lambda_1}$ in a neighborhood of q. Hence $\mathcal{N} \subset \mathcal{N}_1$ leads to $\bigcup_m \mathcal{J}_\Lambda^m \mathcal{N} \subset \mathcal{N}_1$. Thus $\bigcup_m \mathcal{J}_\Lambda^m \mathcal{N}$ is a coherent $\widehat{\mathcal{E}}_X(0)$-module on a neighborhood of q, which contradicts the fact that q belongs to S. Hence $S = \emptyset$. □

DEFINITION 8.27. A holonomic $\widehat{\mathcal{E}}_X$-module $\mathcal{M}$ defined on an open subset Ω of $\overset{\circ}{T}{}^*X$ is said to be *regular* if it satisfies the equivalent conditions (1) and (2) in Proposition 8.26.

Let $\mathrm{IR}(\mathcal{M})$ denote the set of points $p \in \Omega$ that do not have a neighborhood on which $\mathcal{M}$ is regular holonomic.

By the proof of Proposition 8.26, we have the following.

PROPOSITION 8.28. $\mathrm{IR}(\mathcal{M})$ *is a closed Lagrangian analytic subset of Ω. Hence it is a union of irreducible components of* $\mathrm{Supp}(\mathcal{M})$.

Next we classify regular holonomic $\widehat{\mathcal{E}}_X$-modules $\mathcal{M}$ locally at a nonsingular point of their supports. Let $\Lambda = \mathrm{Supp}(\mathcal{M})$. Then Λ is a Lagrangian submanifold (locally). Hence, by a contact transformation, we may assume

$$\Lambda = \{(x,\xi); x_1 = \xi_2 = \cdots = \xi_n = 0\},$$
$$p = (0, dx_1).$$

THEOREM 8.29. *Let $\mathcal{M}$ be a regular holonomic module defined on a neighborhood of $p = (0, dx_1)$ with support Λ above. Then there exist $u_1, \ldots, u_N$ such that $\mathcal{M} = \sum_{j=1}^{N} \widehat{\mathcal{E}}_X(0)u_j$, and such that the system of defining relations of $(u_1, \ldots, u_N)$ is given by*

$$(x_1\partial_1 - A)u = 0,$$
$$\partial_2 u = \cdots = \partial_n u = 0,$$

where u is the column vector with components $u_1, \ldots, u_N$, and $A \in M_N(\mathbb{C})$. Furthermore we can take A so that the differences of its distinct eigenvalues are not integers.

PROOF. We call a coherent $\widehat{\mathcal{E}}_X(0)$-submodule $\mathcal{N}$ of $\mathcal{M}$ a *lattice* of $\mathcal{M}$ if $\widehat{\mathcal{E}}_X\mathcal{N} = \mathcal{M}$. By the assumption, there exists a lattice $\mathcal{N}$ with $x_1\partial_1\mathcal{N} \subset \mathcal{N}$ and $\partial_j\mathcal{N} \subset \mathcal{N}$ $(j = 2, \ldots, n)$.

By the equality $\mathcal{O}_\Lambda(0) = \widehat{\mathcal{E}}_X(0)/\mathcal{J}_\Lambda(-1)$ and the assumption, $\mathcal{N}/\mathcal{N}(-1)$ is a coherent $\mathcal{O}_\Lambda(0)$-module. Moreover, since $\partial_2, \ldots, \partial_n$ act on it, $(\mathcal{N}/\mathcal{N}(-1))_p$ is a $\mathcal{D}_{\mathbb{C}^{n-1}}$-module and coherent over $\mathcal{O}_{\mathbb{C}^{n-1}}$. Thus $\mathcal{N}/\mathcal{N}(-1)$ is locally free over $\mathcal{O}_\Lambda(0)$, and the stalks of the sheaf $\mathcal{E}nd_{\mathcal{D}_{\mathbb{C}^{n-1}}}(\mathcal{N}/\mathcal{N}(-1))$ are finite-dimensional over $\mathbb{C}$. Since $\vartheta := x_1\partial_1$ is an element of $\mathcal{E}nd_{\mathcal{D}_{\mathbb{C}^{n-1}}}(\mathcal{N}/\mathcal{N}(-1))$, there exists a nonzero polynomial $b(s)$ such that $b(\vartheta)\mathcal{N} \subset \mathcal{N}(-1)$. First we claim the following:

(8.16) There exist a lattice $\mathcal{N}$, stable under $\mathcal{J}_\Lambda$, and a nonzero polynomial $b(s)$ such that $b(\vartheta)\mathcal{N} \subset \mathcal{N}(-1)$, and the differences of distinct roots of $b(s)$ are not integers.

To prove this, it suffices to show the following:

Let $\mathcal{J}_\Lambda\mathcal{N} \subset \mathcal{N}$ and $b(\vartheta)\mathcal{N} \subset \mathcal{N}(-1)$. Put $b(s) = (s-\lambda)c(s)$ and $\tilde{b}(s) = (s - \lambda - 1)c(s)$. Then there exists a lattice $\mathcal{N}'$ such that $\mathcal{J}_\Lambda\mathcal{N}' \subset \mathcal{N}'$ and $\tilde{b}(\vartheta)\mathcal{N}' \subset \mathcal{N}'(-1)$.

Note first

$$[\vartheta, \widehat{\mathcal{E}}(0)] \subset \mathcal{J}_\Lambda(-1).$$

Put

$$\mathcal{N}' = \mathcal{N}(-1) + (\vartheta - \lambda)\mathcal{N} = \partial_1^{-1}\mathcal{N} + (\vartheta - \lambda)\mathcal{N}.$$

Then, since

$$\begin{aligned}\widehat{\mathcal{E}}_X(0)(\vartheta-\lambda)\mathcal{N} &\subset [\widehat{\mathcal{E}}_X(0),\vartheta-\lambda]\mathcal{N}+(\vartheta-\lambda)\widehat{\mathcal{E}}_X(0)\mathcal{N}\\ &\subset \mathcal{J}_\Lambda(-1)\mathcal{N}+(\vartheta-\lambda)\mathcal{N}\\ &\subset \mathcal{N}',\end{aligned}$$

$\mathcal{N}'$ is a coherent $\widehat{\mathcal{E}}(0)$-module. Since $\partial_j(\vartheta-\lambda)\mathcal{N}\subset\mathcal{N}'$ $(j\neq 1)$, and $\vartheta(\vartheta-\lambda)\mathcal{N}\subset\mathcal{N}'$, we have $\mathcal{J}_\Lambda\mathcal{N}'\subset\mathcal{N}'$. Moreover

$$\begin{aligned}(\vartheta-\lambda-1)\mathcal{N}' &\subset (\vartheta-\lambda-1)\partial_1^{-1}\mathcal{N}+(\vartheta-\lambda)(\vartheta-\lambda-1)\mathcal{N}\\ &= \partial_1^{-1}(\vartheta-\lambda)\mathcal{N}+(\vartheta-\lambda)(\vartheta-\lambda-1)\mathcal{N}\\ &\subset \mathcal{N}'(-1)+(\vartheta-\lambda)(\vartheta-\lambda-1)\mathcal{N}.\end{aligned}$$

Hence

$$\begin{aligned}c(\vartheta)(\vartheta-\lambda-1)\mathcal{N}' &\subset \mathcal{N}'(-1)+c(\vartheta)(\vartheta-\lambda)(\vartheta-\lambda-1)\mathcal{N}\\ &\subset \mathcal{N}'(-1)+(\vartheta-\lambda-1)\partial_1^{-1}\mathcal{N}\\ &= \mathcal{N}'(-1)+\partial_1^{-1}(\vartheta-\lambda)\mathcal{N}\\ &\subset \mathcal{N}'(-1).\end{aligned}$$

We have thus proved (8.16).

Take an $\mathcal{N}$ satisfying (8.16), and take $u_1,\dots,u_N\in\mathcal{N}_p$ such that their images $[u_j]\in(\mathcal{N}/\mathcal{N}(-1))_p$ form a basis of an $\mathcal{O}_\Lambda(0)$-module $(\mathcal{N}/\mathcal{N}(-1))_p$. Then, by Nakayama's Lemma (Proposition 7.9), $\mathcal{N}=\sum\widehat{\mathcal{E}}(0)u_j$. Since $x_1\partial_1\mathcal{N}\subset\mathcal{N}$, there exist $A_{jk}\in\widehat{\mathcal{E}}(0)$ such that

$$x_1\partial_1u_j=\sum_k A_{jk}u_k.$$

Let u be the column vector with components $u_1,\dots,u_N$, and put $A=(A_{jk})\in M_N(\widehat{\mathcal{E}}(0))$. Then

$$(x_1\partial_1-A)u=0.$$

The action of $x_1\partial_1$ on $\mathcal{N}/\widehat{\mathcal{E}}_X(-1)\mathcal{N}=\mathcal{N}/(\widehat{\mathcal{E}}_X(-1)\mathcal{N}+x_1\mathcal{N})$ is given by $\sigma_0(A)$ with respect to the basis $[u_1],\dots,[u_N]$. Hence the differences of distinct eigenvalues of $\sigma_0(A)(p)$ are not integers. Furthermore, $\partial_ju=C_ju$ $(j=2,\dots,n)$ for some $C_j\in M_N(\widehat{\mathcal{E}}_X(0))$. We take an appropriate complex number λ and put $\vartheta:=x_1\partial_1+\lambda(x_2\partial_2+\cdots+x_n\partial_n)$. Then

$$\vartheta u=(A+\lambda\sum_{j=2}^n x_jC_j)u.$$

Put $P = A + \lambda \sum_{j=2}^{n} x_j C_j$. Take a generic λ. Then the differences of distinct eigenvalues of $\sigma_0(P)$ are not integers. Let $\sigma_0(A)(p) = S + N$ be the Jordan decomposition. If $0 < \lambda < 1$, then by Theorem 8.17 there exists $Q \in GL_N(\widehat{\mathcal{E}}(0))$ such that

$$\begin{aligned} \vartheta - P &= Q(\vartheta - C)Q^{-1}, \\ \sigma_0(C)(p) &= S + N, \\ [\vartheta - S, C] &= 0. \end{aligned}$$

Since λ is generic, $C \in M_N(\widehat{\mathcal{E}}(0))$ satisfying $[\vartheta - S, C] = 0$ necessarily belongs to $M_N(\mathbb{C})$. Indeed, the eigenvalues of $\mathrm{ad}(S)$ belong to $\{0\} \cup (\mathbb{C}\backslash\mathbb{Z})$, and the degrees of ϑ-homogeneous operators in $\sum_{j=1}^{n} x_j \mathcal{E}(0) + \sum_{j=2}^{n} \widehat{\mathcal{E}}_X(-1)\partial_j + \widehat{\mathcal{E}}_X(-1)$ belong to $\mathbb{Z}_{>0} + \mathbb{Z}\lambda$. Since $C \in M_N(\mathbb{C})$, we have $C = S + N$. The vector $v = Q^{-1}u$ generates $\mathcal{M}$ and satisfies $(\vartheta - C)v = 0$. By an argument similar to that in § 8.5, $\partial_j v = 0$ $(j = 2, \dots, n)$. It remains to show that the relations of v are given by $(x_1\partial_1 - C)v = \partial_2 v = \cdots = \partial_n v = 0$. We show that, if $Rv = 0$, then $R \in (\widehat{\mathcal{E}}_X)^N (x_1\partial_1 - C) + \sum_{j=2}^{n} \widehat{\mathcal{E}}_X^N \partial_j$. We can write R as

$$R = E_0(x_2, \dots, x_n, \partial_1) + E_1(x, \partial)(x_1\partial_1 - C) + \sum_{j=2}^{n} E_j(x, \partial)\partial_j.$$

Hence we may assume $R = R(x_2, \dots, x_n, \partial_1)$. Suppose $R \neq 0$. Let r be the order of R, and let $[v] \subset (\mathcal{N}/\mathcal{N}(-1))^{\oplus N}$ be the image of v. Then $R_r[v] = 0$ in $(\mathcal{N}(r)/\mathcal{N}(r-1))^{\oplus N}$. Since $[v]$ is a basis, we obtain $R_r = 0$, which is absurd. $\square$

Taking the Jordan normal form of A in Theorem 8.29, we obtain the following theorem.

THEOREM 8.30. *Let $X = \mathbb{C}^n$, and let $p = (0, dx_1)$ be a point of $\Lambda = \{(x, \xi); x_1 = \xi_2 = \cdots = \xi_n = 0\}$. Set*

$$\mathcal{N}_{\lambda,m} := \widehat{\mathcal{E}}_X / (\widehat{\mathcal{E}}_X (x_1\partial_1 - \lambda)^m + \sum_{j=2}^{n} \widehat{\mathcal{E}}_X \partial_j) \quad (\lambda \in \mathbb{C},\ m \in \mathbb{Z}_{\geq 1}).$$

If $\mathcal{M}$ is a regular holonomic $\widehat{\mathcal{E}}_X$-module defined on a neighborhood of p with support Λ, then there exists an isomorphism

$$\mathcal{M} \simeq \bigoplus_{j=1}^{N} \mathcal{N}_{\lambda_j, m_j} \quad (\lambda_j \in \mathbb{C},\ m_j \in \mathbb{Z}_{\geq 1}).$$

We have the following proposition; its proof is left to the reader. Let $u_{\lambda,m} \in \mathcal{N}_{\lambda,m}$ denote the image of $1 \in \widehat{\mathcal{E}}_X$.

PROPOSITION 8.31. (1) *$\mathcal{H}om_{\widehat{\mathcal{E}}_X}(\mathcal{N}_{\lambda,m}, \mathcal{N}_{\lambda',m'}) = 0$, unless $\lambda - \lambda' \in \mathbb{Z}$.*

(2) *$\widehat{\mathcal{E}}_X^{(\mu)} \otimes_{\widehat{\mathcal{E}}_X} \mathcal{N}_{\lambda,m} \simeq \mathcal{N}_{\lambda-\mu,m}$ $(\lambda, \mu \in \mathbb{C},\ m \in \mathbb{Z}_{\geq 1})$. In particular, $\mathcal{N}_{\lambda+k,m} \simeq \mathcal{N}_{\lambda,m}$ for $k \in \mathbb{Z}$.*

(3) *If $\mathcal{N}_{\lambda,m} \simeq \mathcal{N}_{\lambda',m'}$, then $\lambda - \lambda' \in \mathbb{Z}$ and $m = m'$.*

(4) *$\mathcal{H}om_{\widehat{\mathcal{E}}_X}(\mathcal{N}_{\lambda,m}, \mathcal{N}_{\lambda,m'}) = \bigoplus_{j=1}^{\min(m,m')} \mathbb{C}\varphi_j$. Here $\varphi_j : \mathcal{N}_{\lambda,m} \to \mathcal{N}_{\lambda,m'}$ is given by $\varphi_j(u_{\lambda,m}) = (x_1\partial_1 - \lambda)^{m'-j} u_{\lambda,m'}$.*

By the above, we can show that the category of regular holonomic $\widehat{\mathcal{E}}_X$-modules with support Λ, a nonsingular Lagrangian submanifold, is equivalent to the category of twisted locally constant sheaves on Λ. For details, see [K3].

From the above results of classification, we obtain the following proposition.

PROPOSITION 8.32. *Let $\mathcal{M}$ be a regular holonomic $\widehat{\mathcal{E}}_X$-module with support Λ, a nonsingular Lagrangian submanifold, and let u be a section of $\mathcal{M}$. Suppose that $\mathcal{M} = \widehat{\mathcal{E}}_X u$, and that $(\mathcal{M}, u)$ is simple on a dense open subset of Λ. Then $\sigma_\Lambda(u)$ is holomorphic on Λ.*

PROOF. By a quantized contact transformation, we may assume $\Lambda = \{(x,\xi); x_1 = \xi_2 = \cdots = \xi_n = 0\}$. In addition, we may assume that $\mathcal{M} = \widehat{\mathcal{E}}_X v$, and that v is defined by $(x_1\partial_1 - \lambda)v = \partial_2 v = \cdots = \partial_n v = 0$. Then u is written as $u = P(x_2, \ldots, x_n, \partial_1)v$. Since $\sigma_\Lambda(u) = \sigma(P)\sigma_\Lambda(v)$, $\sigma_\Lambda(u)$ is holomorphic on Λ. □

8.8. Classification of Simple Holonomic Systems (2)

Let $(\mathcal{M}, u)$ be a simple holonomic $\widehat{\mathcal{E}}_X$-module. In the case when its support Λ is nonsingular, we have made a microlocal classification of such a module in § 8.5. In this section, we consider the case when Λ is the union of two nonsingular Lagrangian submanifolds, $\Lambda_1 \cup \Lambda_2$.

We have the following theorem, though we do not prove it here.

THEOREM 8.33 ([KK]). *Let V_1 and V_2 be closed analytic subsets of T^*X, and $\mathcal{M}$ a coherent $\widehat{\mathcal{E}}_X$-module with support $V_1 \cup V_2$. If $\dim(V_1 \cap V_2) \leq \dim X - 2$, then*

$$\mathcal{M} \cong \mathcal{M}_1 \oplus \mathcal{M}_2 \quad \textit{with} \quad \mathrm{Supp}(\mathcal{M}_i) \subset V_i.$$

Moreover, this decomposition is unique.

Hence it is essential when the dimension of $\Lambda_1 \cap \Lambda_2$ equals $\dim X - 1$. We consider the following additional conditions:

$$\begin{aligned} &\Lambda_1 \cap \Lambda_2 \text{ is a nonsingular submanifold} \\ &\text{of dimension } \dim X - 1; \end{aligned} \tag{8.17}$$

$$T_p(\Lambda_1 \cap \Lambda_2) = T_p\Lambda_1 \cap T_p\Lambda_2 \text{ for all } p \in \Lambda_1 \cap \Lambda_2. \tag{8.18}$$

If these two conditions are satisfied, we say that Λ_1 and Λ_2 have a *good intersection*. First we classify good intersections up to contact transformation.

PROPOSITION 8.34. *Suppose that two Lagrangian submanifolds Λ_1 and Λ_2 have a good intersection in a neighborhood of $p \in \Lambda_1 \cap \Lambda_2 \cap \overset{\circ}{T}{}^*X$. Then, by a homogeneous symplectic transformation defined on a neighborhood of p, they can be transformed into $X = \mathbb{C}^n$,*

$$\Lambda_1 = \{\, (x, \xi)\,;\, x_1 = \xi_2 = \xi_3 = \cdots = \xi_n = 0 \,\}, \tag{8.19}$$

$$\Lambda_2 = \{\, (x, \xi)\,;\, x_1 = x_2 = \xi_3 = \cdots = \xi_n = 0 \,\}, \tag{8.20}$$

$$p = (0, dx_1). \tag{8.21}$$

PROOF. By a homogeneous symplectic transformation, we may assume that the ranks of $T_p\Lambda_1 \to T_{\pi(p)}X$ and $T_p\Lambda_2 \to T_{\pi(p)}X$ equal $n-1$. Hence there exist nonsingular hypersurfaces S_i $(i = 1, 2)$ of X such that $\Lambda_i = T^*_{S_i}X$ $(i = 1, 2)$ (Lemma A.52). Take a coordinate system so that $S_1 = \{x_1 = 0\}$. Since $\Lambda_1 \cap \Lambda_2$ is $\mathbb{C}^*$-stable, there exists a hypersurface R of S_1 such that $\Lambda_1 \cap \Lambda_2 = \pi^{-1}(R) \cap \Lambda_1$. Take the coordinate system so that $R = \{x_1 = x_2 = 0\}$, and $p = (0, dx_1)$ as well. Thus

$$\begin{aligned} \Lambda_1 &= \{\, (x, \xi)\,;\, x_1 = \xi_2 = \cdots = \xi_n = 0 \,\}, \\ \Lambda_1 \cap \Lambda_2 &= \{\, (x, \xi)\,;\, x_1 = x_2 = \xi_2 = \cdots = \xi_n = 0 \}. \end{aligned}$$

Since $T_0 S_2 = \{x_1 = 0\}$, we can write $S_2 = \{x_1 = f(x_2, \ldots, x_n)\}$. By $\Lambda_2 \cap \{x_2 = 0\} = \Lambda_1 \cap \Lambda_2$, we have

$$f(0, x') = 0, \quad f_{x_2}(0, x') = 0, \quad x' = (x_3, \ldots, x_n).$$

Hence we can write $f(x_2, x') = x_2^2 \varphi(x_2, x')$. Since

$$\Lambda_2 = \left\{(x, \xi); x_1 = f(x_2, x'),\ \xi_j/\xi_1 = -\frac{\partial}{\partial x_j} f(x_2, x')\ (j = 2, \ldots, n)\right\},$$

we have

$$
\begin{aligned}
&T_p\Lambda_2\\
&= \left\{(x,\xi); x_1 = 0,\ \xi_j = -\sum_{k=2}^{n} \frac{\partial^2 f}{\partial x_j \partial x_k}(0,0)x_k\ (j = 2,\dots,n)\right\}\\
&= \left\{(x,\xi); x_1 = 0,\ \xi_3 = \cdots = \xi_n = 0,\ \xi_2 = -\frac{\partial^2 f}{\partial x_2^2}(0,0)x_2\right\}.
\end{aligned}
$$

Since $T_p\Lambda_2 \cap T_p\Lambda_1 = \{x_1 = x_2 = \xi_2 = \xi_3 = \cdots = 0\}$, we have $\frac{\partial^2 f}{\partial x_2^2}(0,0) \neq 0$ and hence $\varphi(0,0) \neq 0$. Replacing x_2 by $x_2\varphi(x_2,x')^{1/2}$, we have

$$
\begin{aligned}
\Lambda_1 &= T^*_{\{x_1=0\}}X,\\
\Lambda_2 &= T^*_{\{x_1=x_2^2\}}X.
\end{aligned}
$$

Since the Lagrangian submanifolds given by (8.19) and (8.20) can also be transformed into the above form, we obtain the proposition. □

Let Λ_1 and Λ_2 be the Lagrangian submanifolds given by (8.19) and (8.20) respectively, and $(\mathcal{M}, u)$ a simple holonomic $\widehat{\mathcal{E}}_X$-module with support $\Lambda := \Lambda_1 \cup \Lambda_2$. Put $\vartheta = 2x_1\partial_1 + \sum_{j=2}^n x_j\partial_j$. By the same argument as in the case when $\operatorname{Supp}\mathcal{M}$ is nonsingular, we may assume $(\vartheta - \lambda)u = 0$ for some $\lambda \in \mathbb{C}$, replacing u by Pu with an invertible $P \in \widehat{\mathcal{E}}(0)$. In addition, we have $\partial_3 u = \cdots = \partial_n u = 0$. By Proposition 8.19, there exists $Q \in \widehat{\mathcal{E}}_X(1)$ such that

$$
(8.22)\qquad
\begin{aligned}
&Qu = 0, \quad \sigma_1(Q) = x_1\xi_1\,,\\
&Q \text{ is } \vartheta\text{-homogeneous of degree } 0.
\end{aligned}
$$

Since x_1 and ∂_1^{-1} are ϑ-homogeneous of degree 2, and x_j and $\partial_j\partial_1^{-1}$ $(j \neq 1)$ are of degree 1, Q must have a form $Q = x_1\partial_1 - \mu$ $(\mu \in \mathbb{C})$. Hence u satisfies the following:

$$
\begin{cases}
(x_1\partial_1 - \mu)u = 0\,,\\
(x_2\partial_2 - (\lambda - 2\mu))u = 0\,,\\
\partial_3 u = \cdots = \partial_n u = 0.
\end{cases}
$$

We have thus obtained the following theorem.

THEOREM 8.35. *Let Λ_1 and Λ_2 be Lagrangian submanifolds given in Proposition 8.34, and let $\Lambda = \Lambda_1 \cup \Lambda_2$. Then, given a simple* *holonomic system* $(\mathcal{M}, u)$ *with support* Λ, *there exists an invertible* $P \in \widehat{\mathcal{E}}(0)$ *such that* $v = Pu$ *satisfies*

$$\left\{\begin{array}{l}(x_1\partial_1 - \lambda)v = 0\,,\\ (x_2\partial_2 - \mu)v = 0\,,\\ \partial_3 v = \cdots = \partial_n v = 0.\end{array}\right. \tag{8.23}$$

In this case, the principal symbols of v along Λ_1, Λ_2 are

$$\sigma_{\Lambda_1}(v) = \frac{\xi_1^{-\lambda-1} x_2^{\mu} \sqrt{d\xi_1 dx_2 \cdots dx_n}}{\sqrt{dx}}\,,$$
$$\sigma_{\Lambda_2}(v) = \frac{\xi_1^{-\lambda-1}\xi_2^{-\mu-1} \sqrt{d\xi_1 d\xi_2 dx_3 \cdots dx_n}}{\sqrt{dx}}.$$

Their orders are

$$\operatorname{ord}_{\Lambda_1}(v) = -\frac{1}{2} - \lambda\,,$$
$$\operatorname{ord}_{\Lambda_2}(v) = -1 - \lambda - \mu.$$

Hence $\mu = \operatorname{ord}_{\Lambda_1}(v) - \operatorname{ord}_{\Lambda_2}(v) - 1/2$.

PROPOSITION 8.36. *Let* $\mathcal{M} = \widehat{\mathcal{E}}_X u$ *be a simple holonomic* $\widehat{\mathcal{E}}_X$*-module defined by*

$$\left\{\begin{array}{l}(x_1\partial_1 - \lambda)u = 0\,,\\ (x_2\partial_2 - \mu)u = 0\,,\\ \partial_3 u = \cdots = \partial_n u = 0.\end{array}\right.$$

Let $p = (0, dx_1)$, $\Lambda_1 = T^*_{\{x_1=0\}}\mathbb{C}^n$, *and* $\Lambda_2 = T^*_{\{x_1=x_2=0\}}\mathbb{C}^n$.

(1) *If* $\mu \notin \mathbb{Z}$, *then* $\mathcal{M}_p$ *is an irreducible* $\widehat{\mathcal{E}}_{X,p}$*-module.*

(2) *The following three conditions are equivalent:*
   (a) $\mathcal{M}$ *has a coherent* $\widehat{\mathcal{E}}_X$*-submodule* $\mathcal{N}$ *with support* Λ_1 *in a neighborhood of* p.
   (b) $\mathcal{M}$ *has a coherent quotient* $\widehat{\mathcal{E}}_X$*-module* $\mathcal{N}$ *with support* Λ_2 *in a neighborhood of* p.
   (c) $\mu \in \mathbb{Z}_{\leq -1}$.

(3) *The following three conditions are equivalent:*
   (a) $\mathcal{M}$ *has a coherent* $\widehat{\mathcal{E}}_X$*-submodule* $\mathcal{N}$ *with support* Λ_2 *in a neighborhood of* p.
   (b) $\mathcal{M}$ *has a coherent quotient* $\widehat{\mathcal{E}}_X$*-module* $\mathcal{N}$ *with support* Λ_1 *in a neighborhood of* p.
   (c) $\mu \in \mathbb{Z}_{\geq 0}$.

PROOF. We work in a neighborhood of p. Let $\mathcal{N} \subset \mathcal{M}$ be a coherent $\widehat{\mathcal{E}}_X$-submodule. At each point of $\Lambda_1 \backslash \Lambda_2$, $\mathcal{M}$ is irreducible. Hence one and only one of $\mathrm{Supp}(\mathcal{N}) \subset \Lambda_2$ and $\mathrm{Supp}(\mathcal{M}/\mathcal{N}) \subset \Lambda_2$ holds. Similarly $\mathrm{Supp}(\mathcal{N}) \subset \Lambda_1$, or $\mathrm{Supp}(\mathcal{M}/\mathcal{N}) \subset \Lambda_1$. Hence there are only four possibilities:

(i) $\mathrm{Supp}(\mathcal{N}) \subset \Lambda_1 \cap \Lambda_2$; in this case, $\mathcal{N} = 0$, since $\mathrm{Supp}(\mathcal{N})$ is Lagrangian.

(ii) $\mathrm{Supp}(\mathcal{M}/\mathcal{N}) \subset \Lambda_1 \cap \Lambda_2$; in this case, $\mathcal{N} = \mathcal{M}$.

(iii) $\mathrm{Supp}(\mathcal{N}) \subset \Lambda_2$, and $\mathrm{Supp}(\mathcal{M}/\mathcal{N}) \subset \Lambda_1$; in this case, in fact, $\mathrm{Supp}(\mathcal{N}) = \Lambda_2$, and $\mathrm{Supp}(\mathcal{M}/\mathcal{N}) = \Lambda_1$, because, for instance, $\mathrm{Supp}(\mathcal{N}) \subsetneqq \Lambda_2$ leads to $\mathcal{N} = 0$, which is absurd.

(iv) $\mathrm{Supp}(\mathcal{N}) \subset \Lambda_1$, and $\mathrm{Supp}(\mathcal{M}/\mathcal{N}) \subset \Lambda_2$; in this case, in fact, $\mathrm{Supp}(\mathcal{N}) = \Lambda_1$, and $\mathrm{Supp}(\mathcal{M}/\mathcal{N}) = \Lambda_2$.

We thus deduce the equivalence of (a) and (b) in (2) and (3). (1) follows from (2) and (3).

(2) (c)$\Longrightarrow$(b) Let $\mathcal{N}_2 = \widehat{\mathcal{E}}_X v_2$ be the simple holonomic $\widehat{\mathcal{E}}_X$-module with system of relations $(x_1\partial_1 - \lambda)v_2 = 0$, $x_2 v_2 = 0$, $\partial_j v_2 = 0$ $(j \geq 3)$. Define

$$\varphi_2 : \mathcal{M} \to \mathcal{N}_2$$

by $\varphi_2(u) = \partial_2^{-1-\mu} v_2$. Then φ_2 is a homomorphism of $\widehat{\mathcal{E}}_X$-modules. Since

$$x_2^{-1-\mu}\varphi_2(u) = x_2^{-1-\mu}\partial_2^{-1-\mu}v_2 = \prod_{k=0}^{-2-\mu}(x_2\partial_2 - k)v_2 = \prod_{k=0}^{-2-\mu}(-1-k)v_2,$$

φ_2 is surjective.

(3) (c)$\Longrightarrow$(b) The proof is similar to the above. Let $\mathcal{N}_1 = \widehat{\mathcal{E}}_X v_1$ be the simple holonomic $\widehat{\mathcal{E}}_X$-module with system of relations

$$(x_1\partial_1 - \lambda)v_1 = \partial_j v_1 = 0 \qquad (j \geq 2).$$

The support of $\mathcal{N}_1$ equals Λ_1. Define

$$\varphi_1 : \mathcal{M} \to \mathcal{N}_1$$

by $\varphi_1(u) = x_2^{\mu} v_1$. Then this is a homomorphism of $\widehat{\mathcal{E}}_X$-modules. Since $\partial_2^{\mu} x_2^{\mu} v_1 = \prod_{k=1}^{\mu}(x_2\partial_2 + k)v_1 = \mu! v_1$, it is surjective.

(2) (b)$\Longrightarrow$(c) Let $\mathcal{N}$ be an $\widehat{\mathcal{E}}_X$-module with support Λ_2, and $\varphi : \mathcal{M} \to \mathcal{N}$ an epimorphism. Then φ is an isomorphism on $\Lambda_2 \backslash \Lambda_1$.

Put $\varphi(u) = v$. Then on $\Lambda_2 \backslash \Lambda_1$

$$\sigma_{\Lambda_2}(v) = \sigma_{\Lambda_2}(u) = \frac{\xi_1^{-\lambda-1}\xi_2^{-\mu-1}\sqrt{d\xi_1 d\xi_2 dx_3 \cdots dx_n}}{\sqrt{dx}}$$

up to scalar multiplication. Since $\mathcal{N}$ is a regular holonomic $\widehat{\mathcal{E}}_X$-module with support Λ_2, $\sigma_{\Lambda_2}(v)|_{\Lambda_2}$ is holomorphic by Proposition 8.32. Hence we obtain $-\mu - 1 \in \mathbb{Z}_{\geq 0}$.

(3) (b)$\Longrightarrow$(c) similarly follows from the fact that

$$\sigma_{\Lambda_1}(u) = \frac{\xi_1^{-\lambda-1}x_2^{\mu}\sqrt{d\xi_1 dx_2 \cdots dx_n}}{\sqrt{dx}}$$

is holomorphic on a neighborhood of p. □

Noting that $\mathrm{ord}_{\Lambda_1}(u) - \mathrm{ord}_{\Lambda_2}(u)$ is invariant under quantized contact transformations (Proposition 8.25), we can restate Proposition 8.36 as follows:

THEOREM 8.37. *Let Λ_1 and Λ_2 be nonsingular Lagrangian submanifolds with a good intersection, and $(\mathcal{M}, u)$ a simple holonomic $\widehat{\mathcal{E}}_X$-module with support $\Lambda := \Lambda_1 \cup \Lambda_2$. Let $p \in \Lambda_1 \cap \Lambda_2$.*

(1) *If $\mathrm{ord}_{\Lambda_1}(u) - \mathrm{ord}_{\Lambda_2}(u) \notin \mathbb{Z} + 1/2$, then $\mathcal{M}_p$ is an irreducible $\widehat{\mathcal{E}}_{X,p}$-module.*

(2) *The following three conditions are equivalent:*

(a) *$\mathcal{M}$ has a coherent $\widehat{\mathcal{E}}_X$-submodule $\mathcal{N}$ with support Λ_1 in a neighborhood of p.*

(b) *$\mathcal{M}$ has a coherent quotient $\widehat{\mathcal{E}}_X$-module $\mathcal{N}$ with support Λ_2 in a neighborhood of p.*

(c) $\mathrm{ord}_{\Lambda_2}(u) - \mathrm{ord}_{\Lambda_1}(u) - \frac{1}{2} \in \mathbb{Z}_{\geq 0}$.

8.9. Subholonomic Systems

Let n be the dimension of X. Similarly to the proof of Theorem 3.7, we have

$$\mathrm{gl.dim}(\widehat{\mathcal{E}}_X)_p = n$$

for every $p \in T^*X$.

A coherent $\widehat{\mathcal{E}}_X$-module $\mathcal{M}$ is said to be *subholonomic* if the dimension of its support equals $n + 1$. In addition, if $\mathcal{M}$ does not contain a nonzero holonomic $\widehat{\mathcal{E}}_X$-submodule, $\mathcal{M}$ is said to be *purely* *subholonomic*. Similarly to the argument in § 2.4, we see that

$$\begin{aligned} T_n(\mathcal{M}) &= H^0\mathbb{R}\mathcal{H}om_{\widehat{\mathcal{E}}_X}(\tau^{\geq n}\mathbb{R}\mathcal{H}om_{\widehat{\mathcal{E}}_X}(\mathcal{M},\widehat{\mathcal{E}}_X),\widehat{\mathcal{E}}_X) \\ &= \mathcal{E}xt^n_{\widehat{\mathcal{E}}_X}(\mathcal{E}xt^n_{\widehat{\mathcal{E}}_X}(\mathcal{M},\widehat{\mathcal{E}}_X),\widehat{\mathcal{E}}_X) \end{aligned}$$

coincides with $\{\, u \in \mathcal{M}\,;\, \operatorname{codim}(\operatorname{supp}(u)) \geq n\,\}$. Hence a subholonomic system $\mathcal{M}$ is purely subholonomic if and only if

$$\mathcal{E}xt^n_{\widehat{\mathcal{E}}_X}(\mathcal{M},\widehat{\mathcal{E}}_X) = 0.$$

THEOREM 8.38. *Let $\mathcal{M}$ be a purely subholonomic $\widehat{\mathcal{E}}_X$-module, and $f : \mathcal{M} \to \mathcal{M}$ a homomorphism of $\widehat{\mathcal{E}}_X$-modules. Suppose that there exist $u \in \mathcal{M}$ and $P \in \widehat{\mathcal{E}}_X(m)$ such that $f(u) = Pu$, $\mathcal{M} = \widehat{\mathcal{E}}_X u$, and $\sigma_m(P)$ is not identically* 0 *on any irreducible component of* $\operatorname{Supp}\mathcal{M}$. *Then the support of* $\operatorname{Coker}(f)$ *equals* $\operatorname{Supp}\mathcal{M} \cap \sigma_m(P)^{-1}(0)$.

PROOF. We prove the theorem only on $\mathring{T}^*X$; the general case follows by replacing $\mathcal{M}$ by $\mathcal{M} \boxtimes \mathcal{B}_{\{0\}|\mathbb{C}}$. Put $\mathcal{N} = \widehat{\mathcal{E}}_X(0)u$. Since the $\mathcal{O}_{T^*X}(0)$-module $\mathcal{N}/\mathcal{N}(-1)$ may have a submodule supported by an n-dimensional analytic subset, we shall replace $\mathcal{N}$ by its bidual.

From the exact sequence

$$0 \to \widehat{\mathcal{E}}_X(-1) \to \widehat{\mathcal{E}}_X(0) \to \mathcal{O}_{T^*X}(0) \to 0,$$

we obtain an exact sequence

$$\begin{aligned} &\mathcal{E}xt^{k-1}_{\widehat{\mathcal{E}}_X(0)}(\mathcal{N},\mathcal{O}_{T^*X}(0)) \to \mathcal{E}xt^{k}_{\widehat{\mathcal{E}}_X(0)}(\mathcal{N},\widehat{\mathcal{E}}_X(-1)) \\ &\to \mathcal{E}xt^{k}_{\widehat{\mathcal{E}}_X(0)}(\mathcal{N},\widehat{\mathcal{E}}_X(0)) \to \mathcal{E}xt^{k}_{\widehat{\mathcal{E}}_X(0)}(\mathcal{N},\mathcal{O}_{T^*X}(0)). \end{aligned} \tag{8.24}$$

We have

$$\begin{aligned} &\mathcal{E}xt^{k}_{\widehat{\mathcal{E}}_X(0)}(\mathcal{N},\mathcal{O}_{T^*X}(0)) \\ &= \mathcal{E}xt^{k}_{\mathcal{O}_{T^*X}(0)}(\mathcal{O}_{T^*X}(0)\overset{\mathbb{L}}{\otimes}_{\widehat{\mathcal{E}}_X(0)}\mathcal{N},\mathcal{O}_{T^*X}(0)) \\ &= \mathcal{E}xt^{k}_{\mathcal{O}_{T^*X}(0)}(\mathcal{N}/\mathcal{N}(-1),\mathcal{O}_{T^*X}(0)) = 0. \end{aligned}$$

Since $\operatorname{codim}(\operatorname{Supp}(\mathcal{N}/\mathcal{N}(-1))) = n-1$, Proposition 2.20 implies

$$\mathcal{E}xt^{k}_{\widehat{\mathcal{E}}_X(0)}(\mathcal{N},\mathcal{O}_{T^*X}(0)) = 0 \qquad (k < n-1). \tag{8.25}$$

Proposition 2.20 also implies

$$\operatorname{codim}(\operatorname{Supp}\mathcal{E}xt^{k}_{\widehat{\mathcal{E}}_X(0)}(\mathcal{N},\mathcal{O}_{T^*X}(0))) \geq 0. \tag{8.26}$$

The exact sequence (8.24) yields

$$\begin{aligned}\mathcal{E}xt^k_{\widehat{\mathcal{E}}_X(0)}(\mathcal{N}, \widehat{\mathcal{E}}_X(0)) \otimes_{\widehat{\mathcal{E}}_X(0)} (\widehat{\mathcal{E}}_X(0)/\widehat{\mathcal{E}}_X(-1))& \\ \subset \mathcal{E}xt^k_{\widehat{\mathcal{E}}_X(0)}(\mathcal{N}, \mathcal{O}_{T^*X}(0)).&\end{aligned}$$

Hence

$$\operatorname{Supp} \mathcal{E}xt^k_{\widehat{\mathcal{E}}_X(0)}(\mathcal{N}, \widehat{\mathcal{E}}_X(0)) \subset \operatorname{Supp} \mathcal{E}xt^k_{\widehat{\mathcal{E}}_X(0)}(\mathcal{N}, \mathcal{O}_{T^*X}(0)).$$

Thus (8.25) and (8.26) give us

$$\mathcal{E}xt^k_{\widehat{\mathcal{E}}_X(0)}(\mathcal{N}, \widehat{\mathcal{E}}_X(0)) = 0 \qquad (k < n-1) \tag{8.27}$$

and

$$\operatorname{codim}(\operatorname{Supp} \mathcal{E}xt^k_{\widehat{\mathcal{E}}_X(0)}(\mathcal{N}, \widehat{\mathcal{E}}_X(0))) \geq k. \tag{8.28}$$

We have

$$\mathcal{M}^* := \mathcal{E}xt^{n-1}_{\widehat{\mathcal{E}}_X(0)}(\mathcal{N}, \widehat{\mathcal{E}}_X) = \mathcal{E}xt^{n-1}_{\widehat{\mathcal{E}}_X}(\mathcal{M}, \widehat{\mathcal{E}}_X).$$

Since

$$\begin{aligned}\mathcal{M} &= \mathbb{R}\mathcal{H}om_{\widehat{\mathcal{E}}_X}(\mathbb{R}\mathcal{H}om_{\widehat{\mathcal{E}}_X}(\mathcal{M}, \widehat{\mathcal{E}}_X), \widehat{\mathcal{E}}_X) \\ &= \mathbb{R}\mathcal{H}om_{\widehat{\mathcal{E}}_X}(\mathcal{M}^*, \widehat{\mathcal{E}}_X)[n-1],\end{aligned}$$

$\mathcal{M}^*$ is also purely subholonomic, and $\mathcal{M} = \mathcal{E}xt^{n-1}_{\widehat{\mathcal{E}}_X}(\mathcal{M}^*, \widehat{\mathcal{E}}_X)$. Put $\mathcal{N}_1 := \mathcal{E}xt^{n-1}_{\widehat{\mathcal{E}}_X(0)}(\mathcal{N}, \widehat{\mathcal{E}}_X(0))$. By the exact sequence (8.24) and equation (8.27) for $k = n-1$, $\mathcal{N}_1(-1) \to \mathcal{N}_1$ is injective. Accordingly, $\mathcal{N}_1 \to \mathcal{N}_1 \otimes_{\widehat{\mathcal{E}}_X(0)} \widehat{\mathcal{E}}_X = \mathcal{M}^*$ is also injective. Put $\bar{\mathcal{N}} = \mathcal{N}/\mathcal{N}(-1)$ and $\bar{\mathcal{N}_1} = \mathcal{N}_1/\mathcal{N}_1(-1)$. Then by (8.24)

$$\begin{aligned}\bar{\mathcal{N}_1} &\subset \mathcal{E}xt^{n-1}_{\widehat{\mathcal{E}}_X(0)}(\mathcal{N}, \mathcal{O}_{T^*X}(0)) \\ &= \mathcal{E}xt^{n-1}_{\mathcal{O}_{T^*X}(0)}(\bar{\mathcal{N}}, \mathcal{O}_{T^*X}(0)).\end{aligned}$$

For any n-dimensional homogeneous closed analytic subset Z, we have $H^k\mathbb{R}\Gamma_Z\mathcal{O}_{T^*X}(0) = 0$ $(k < n = \operatorname{codim} Z)$. Hence

$$H^k\mathbb{R}\mathcal{H}om_{\mathcal{O}_{T^*X}(0)}(\bar{\mathcal{N}}, \mathbb{R}\Gamma_Z\mathcal{O}_{T^*X}(0)) = 0 \quad (k < n),$$

and

$$\begin{aligned}&\Gamma_Z\mathcal{E}xt^{n-1}_{\mathcal{O}_{T^*X}(0)}(\bar{\mathcal{N}}, \mathcal{O}_{T^*X}(0)) \\ &= H^{n-1}\mathbb{R}\Gamma_Z\mathbb{R}\mathcal{H}om_{\mathcal{O}_{T^*X}(0)}(\bar{\mathcal{N}}, \mathcal{O}_{T^*X}(0)) \\ &= H^{n-1}\mathbb{R}\mathcal{H}om_{\mathcal{O}_{T^*X}}(\bar{\mathcal{N}}, \mathbb{R}\Gamma_Z\mathcal{O}_{T^*X}(0)) = 0.\end{aligned}$$

We thus have $\Gamma_Z(\bar{\mathcal{N}}_1) = 0$. Applying the same argument to $\mathcal{N}_2 = \mathcal{E}xt^{n-1}_{\widehat{\mathcal{E}}_X(0)}(\mathcal{N}_1, \widehat{\mathcal{E}}_X(0))$, we have $\mathcal{N}_2 \subset \mathcal{M}$ and $\Gamma_Z(\bar{\mathcal{N}}_2) = 0$ for any n-dimensional homogeneous closed analytic subset Z. Hence we obtain $\Gamma_Z(\mathcal{N}_2(k)/\mathcal{N}_2) = 0$ for all k, and thus $\Gamma_Z(\mathcal{M}/\mathcal{N}_2) = 0$. By the assumption, $f\mathcal{N} \subset \mathcal{N}(m)$, and $\bar{f} : \bar{\mathcal{N}} \to \bar{\mathcal{N}}(m) = \mathcal{N}(m)/\mathcal{N}(m-1)$ is given by the multiplication by $a := \sigma_m(P)$. We have

$$\mathcal{N} = \mathbb{R}\mathcal{H}om_{\widehat{\mathcal{E}}_X(0)}(\mathbb{R}\mathcal{H}om_{\widehat{\mathcal{E}}_X(0)}(\mathcal{N}, \widehat{\mathcal{E}}_X(0)), \widehat{\mathcal{E}}_X(0)).$$

By (8.27) and (8.28),

$$\mathbb{R}\mathcal{H}om_{\widehat{\mathcal{E}}_X(0)}(\mathcal{N}, \widehat{\mathcal{E}}_X(0)) = \mathcal{N}_1[1-n]$$

outside an n-codimensional analytic subset. Hence $\mathcal{N} = \mathcal{N}_2$ outside an n-codimensional analytic subset of $\operatorname{Supp}\mathcal{M}$. We have $f\mathcal{N}_2 \subset \mathcal{N}_2(m)$, and $\bar{\mathcal{N}}_2 \to \bar{\mathcal{N}}_2(m)$ is also given by the multiplication by a. Since $f^{-1}\mathcal{N}(m) = \mathcal{N}$ outside $a^{-1}(0)$, we have $f^{-1}\mathcal{N}_2(m) = \mathcal{N}_2$, or $\mathcal{N}_2(m)/f\mathcal{N}_2 \subset \mathcal{M}/f\mathcal{M}$. Hence

$$\begin{aligned}\operatorname{Supp}(\mathcal{M}/f\mathcal{M}) &= \operatorname{Supp}(\mathcal{N}_2/f\mathcal{N}_2(-m)) \\ &= \operatorname{Supp}(\bar{\mathcal{N}}_2/a\bar{\mathcal{N}}_2(-m)).\end{aligned}$$

This leads to $\operatorname{Supp}(\mathcal{M}/f\mathcal{M}) = a^{-1}(0) \cap \operatorname{Supp}(\bar{\mathcal{N}}_2)$. □

CHAPTER 9

Microlocal Calculus of b-functions

9.1. Microlocal b-functions

In this chapter, we explain how to calculate, microlocally, the b-function of a holomorphic function f on a manifold X. Let $\mathcal{N} = \mathcal{D}_X[s]f^s$. For simplicity, we assume that there exists a vector field v on X such that $v(f) = f$. Hence, letting $\mathcal{J} = \{P \in \mathcal{D}_X; Pf^s = 0\}$, we have $\{P \in \mathcal{D}_X[s]; Pf^s = 0\} = \mathcal{J} + \mathcal{D}_X[s](v - s)$. As we have seen in Theorem 6.8, the characteristic variety $\mathrm{Ch}(\mathcal{N})$ of $\mathcal{N}$ coincides with the closure W_f of $\{(x, sd\log f); f(x) \neq 0,\ s \in \mathbb{C}\}$. Since $\sigma(v)(x, sd\log f) = sv(\log f) = s$, s is a holomorphic function on W_f. Set

$$W_f^0 = s^{-1}(0) \subset W_f.$$

We have

$$W_f^0 \subset T_X^*X \cup \pi_X^{-1} f^{-1}(0).$$

For any $\lambda \in \mathbb{C}$, put

$$\mathcal{N}_\lambda = \mathcal{N}/(s - \lambda)\mathcal{N},$$

and symbolically denote its canonical generator by f^λ. Then

$$\mathcal{N}_\lambda \cong \mathcal{D}_X/\mathcal{J}_\lambda, \quad \text{where } \mathcal{J}_\lambda = \mathcal{J} + \mathcal{D}_X(v - \lambda).$$

Clearly we have a homomorphism $\mathcal{N}_{\lambda+1} \to \mathcal{N}_\lambda$ $(f^{\lambda+1} \mapsto f \cdot f^\lambda)$.

The next theorem immediately follows from Theorems 6.8 and 8.38.

THEOREM 9.1. $\mathrm{Ch}(\mathcal{N}_\lambda) = W_f^0$ *for every* $\lambda \in \mathbb{C}$.

In what follows, we use the notation $\mathcal{N}, \mathcal{N}_\lambda$, and $\mathcal{J}$ again for $\widehat{\mathcal{E}}_X \otimes_{\mathcal{D}_X} \mathcal{N}$, $\widehat{\mathcal{E}}_X \otimes_{\mathcal{D}_X} \mathcal{N}_\lambda$, and $\widehat{\mathcal{E}}_X \otimes_{\mathcal{D}_X} \mathcal{J}$ respectively.

Let $\bar{\mathcal{J}}$ be the symbol ideal of $\mathcal{J}$. Then W_f coincides with the zero set of $\bar{\mathcal{J}}$. Let Λ be an irreducible component of W_f^0. Then Λ is Lagrangian.

DEFINITION 9.2. We say that Λ is *simple Lagrangian* (with respect to f) if $\bar{\mathcal{J}} + \mathcal{O}\sigma(v)$ is the defining ideal of Λ at a generic point of Λ.

If Λ is simple Lagrangian, then $(\mathcal{N}_\lambda, f^\lambda)$ is a simple holonomic $\widehat{\mathcal{E}}_X$-module with support Λ (on a neighborhood of a generic point of Λ).

LEMMA 9.3. *If Λ is simple Lagrangian, then, in a neighborhood of a generic point p of Λ, W_f is nonsingular, s has a zero of order* 1 *along Λ, and Λ is the hyperplane of W_f defined by $s = 0$.*

PROOF. Since $\bar{\mathcal{J}}$ vanishes on W_f, we have $\dim\{d\varphi(p); \varphi \in \bar{\mathcal{J}}\} \leq \operatorname{codim} W_f = n - 1$. Hence we can take $\varphi_1, \cdots, \varphi_{n-1} \in \bar{\mathcal{J}}$ such that

$$d\varphi_1(p), \;\cdots,\; d\varphi_{n-1}(p),\; d\sigma(v)(p)$$

are linearly independent.

Since W_f is an analytic subset of codimension $(n-1)$ contained in $\varphi_1 = \cdots = \varphi_{n-1} = 0$, W_f equals $\varphi_1 = \cdots = \varphi_{n-1} = 0$ in a neighborhood of p. Furthermore, Λ is the hyperplane of W_f defined by $\sigma_1(v) = 0$. $\square$

In the case when Λ is simple Lagrangian, we shall compute the principal symbol of $f^\lambda \in \mathcal{N}_\lambda$ along Λ. Let $\pi_f : W_f \to X$ be the restriction of $\pi_X : T^*X \to X$. We take a generic point p of Λ and work in a neighborhood of p. Let m_Λ denote the order of zero along Λ of a holomorphic function $f \circ \pi_f$ on W_f. Set $f_\Lambda = (f \circ \pi_f)/s^{m_\Lambda}|_\Lambda$. Then f_Λ is a holomorphic function on Λ defined on a neighborhood of a generic point of Λ. Note that f_Λ does not vanish at a generic point of Λ, and is homogeneous of degree $(-m_\Lambda)$.

Take a nowhere vanishing n-form $\omega \in \Omega_X$ defined on a neighborhood of $\pi_f(p)$. Since (s, x) is a coordinate system of W_f outside $f^{-1}(0)$, $ds \wedge \pi_f^*(\omega)$ defines a nonzero $(n+1)$-form on W_f. Let μ_Λ denote its order of zero along Λ. Then $(ds \wedge \pi_f^*(\omega))/s^{\mu_\Lambda}$ is an element of Ω_{W_f}. Since $\mathcal{O}_\Lambda \otimes \Omega_{W_f} = \mathcal{O}_\Lambda ds \otimes \Omega_\Lambda$, there exists $\omega_\Lambda \in \Omega_\Lambda$ such that

$$(ds \wedge \pi_f^*(\omega))/s^{\mu_\Lambda} = ds \otimes \omega_\Lambda.$$

Then ω_Λ is invertible in a neighborhood of p, and ω_Λ/ω defines an element of $\Omega_\Lambda \otimes \Omega_X^{\otimes -1}$ independent of the choice of $\omega \in \Omega_X$, homogeneous of degree $(-\mu_\Lambda)$.

THEOREM 9.4. *For any $\lambda \in \mathbb{C}$, the principal symbol of $f^\lambda \in \mathcal{N}_\lambda$ equals*

$$\sigma_\Lambda(f^\lambda) = f_\Lambda^\lambda \sqrt{\omega_\Lambda/\omega} \in \Omega_\Lambda^{\otimes 1/2} \otimes \Omega_X^{\otimes(-1/2)}.$$

PROOF. We obtain $\mathcal{L}_{v-\lambda}^{(0)}(f_\Lambda^\lambda \sqrt{\omega_\Lambda/\omega}) = f_\Lambda^\lambda \mathcal{L}_v^{(0)}(\sqrt{\omega_\Lambda/\omega})$ from $v(f) = f$. We omit the proof of $\mathcal{L}_v^{(0)}(\sqrt{\omega_\Lambda/\omega}) = 0$, since we can prove it similarly to the case below.

We prove that $\mathcal{L}_P^{(m-1)}(f_\Lambda^\lambda \sqrt{\omega_\Lambda/\omega}) = 0$ for $P \in \mathcal{J} \cap F_m(\mathcal{D}_X)$. Put $P = \{P_k(x,\xi)\}_k$. Applying Lemma 7.4 to $\varphi(t) = e^{st}$ and $t = \log f(x)$, we obtain

$$\begin{aligned} &Pf^s \\ &= s^m P_m(x, d\log f) f^s \\ &\quad + s^{m-1}\Big(P_{m-1}(x, d\log f) + \frac{1}{2}\sum \frac{\partial^2 \log f}{\partial x_i \partial x_j}\frac{\partial^2 P_m}{\partial \xi_i \partial \xi_j}(x, d\log f)\Big) f^s \\ &\quad + (\text{terms of degree} \le m-2 \text{ in } s). \end{aligned}$$

We thus have on W_f

$$\begin{aligned} &P_m(x,\xi) = 0\,, \\ &P_{m-1}(x,\xi) + \frac{s}{2}\sum \frac{\partial^2 \log f}{\partial x_i \partial x_j}\frac{\partial^2 P_m}{\partial \xi_i \partial \xi_j}(x,\xi) = 0\,. \end{aligned} \tag{9.1}$$

Hence H_{P_m} is a vector field acting on $\mathcal{O}_{W_f}$, where H_{P_m} is the Hamiltonian of P_m (see §2. 1). From $\{\mathcal{J}, v\} \subset \mathcal{J}$, we obtain $H_{P_m}(s) = 0$. Since $\xi = s \operatorname{grad} \log f(x)$ on W_f,

$$\begin{aligned} \{P_m, \log f\} &= \sum \frac{\partial P_m}{\partial \xi_i}\frac{\partial}{\partial x_i}\log f(x) \\ &= \frac{1}{s}\sum \xi_i \frac{\partial P_m}{\partial \xi_i} = \frac{m}{s}P_m(x,\xi) = 0\,. \end{aligned}$$

Hence $H_{P_m}(f_\Lambda^\lambda) = 0$. We have thus obtained

$$\mathcal{L}_P^{(m-1)}(f_\Lambda^\lambda \sqrt{\omega_\Lambda/\omega}) = f_\Lambda^\lambda \mathcal{L}_P^{(m-1)}(\sqrt{\omega_\Lambda/\omega}).$$

Next we prove that $\mathcal{L}_P^{(m-1)}(\sqrt{\omega_\Lambda/\omega}) = 0$. Put $\omega = dx_1 \wedge \cdots \wedge dx_n$. The Hamiltonian H_{P_m} acts on Ω_Λ as its Lie derivative. Letting

$(s, x_1, \cdots, x_n)$ be a coordinate system of $W_f \backslash f^{-1}(0)$, we have

$$\begin{aligned}
&H_{P_m}(ds \wedge dx_1 \wedge \cdots \wedge dx_n) \\
&= \sum_{i=1}^{n} ds \wedge dx_1 \wedge \cdots \wedge d\left(\frac{\partial P_m}{\partial \xi_i}(x, sd\log f(x))\right) \wedge \cdots \wedge dx_n \\
&= \left(\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\frac{\partial P_m}{\partial \xi_i}(x, sd\log f(x))\right)\right) ds \wedge dx_1 \wedge \cdots \wedge dx_n .
\end{aligned}$$

We also have

$$\begin{aligned}
&\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\frac{\partial P_m}{\partial \xi_i}(x, sd\log f(x))\right) \\
&= \sum_{i} \frac{\partial^2 P_m}{\partial x_i \partial \xi_i}(x, sd\log f(x)) \\
&\qquad + s \sum_{i,j} \frac{\partial^2 \log f(x)}{\partial x_i \partial x_j} \frac{\partial^2 P_m}{\partial \xi_i \partial \xi_j}(x, sd\log f(x)) \\
&= \sum_{i} \frac{\partial^2 P_m}{\partial x_i \partial \xi_i}(x, \xi) + s \sum_{i,j} \frac{\partial^2 \log f}{\partial x_i \partial x_j} \frac{\partial^2 P_m}{\partial \xi_i \partial \xi_j}(x, \xi).
\end{aligned}$$

Hence $\eta = \sqrt{ds \wedge \pi_f^* \omega / s^{\mu_\Lambda}}$ satisfies

$$H_{P_m}(\eta) = \frac{1}{2}\left(\sum_{i} \frac{\partial^2 P_m}{\partial x_i \partial \xi_i}(x, \xi) + s \sum_{i,j} \frac{\partial^2 \log f}{\partial x_i \partial x_j} \frac{\partial^2 P_m}{\partial \xi_i \partial \xi_j}(x, \xi)\right) \eta.$$

We thus have

$$\begin{aligned}
&(H_{P_m} + \sigma_{m-1}(P))\eta \\
&= H_{P_m}(\eta) + \left(P_{m-1} - \frac{1}{2}\sum_{i} \frac{\partial^2 P_m}{\partial x_i \partial \xi_i}\right)\eta \\
&= \left(\frac{1}{2}\sum_{i} \frac{\partial^2 P_m}{\partial x_i \partial \xi_i} + \frac{s}{2}\sum_{i,j} \frac{\partial^2 \log f}{\partial x_i \partial x_j} \frac{\partial^2 P_m}{\partial \xi_i \partial \xi_j}\right)\eta \\
&\qquad + \left(P_{m-1} - \frac{1}{2}\sum_{i} \frac{\partial^2 P_m}{\partial x_i \partial \xi_i}\right)\eta \\
&= \left(P_{m-1} + \frac{s}{2}\sum_{i,j} \frac{\partial^2 \log f}{\partial x_i \partial x_j} \frac{\partial^2 P_m}{\partial \xi_i \partial \xi_j}\right)\eta .
\end{aligned}$$

This equals 0 by (9.1). From the equality $H_{P_m}(s) = 0$, we obtain $(H_{P_m} + \sigma_{m-1}(P))\sqrt{\omega_\Lambda} = 0$. □

COROLLARY 9.5. (1) $\mathrm{ord}_\Lambda(f^\lambda) = -m_\Lambda\lambda - \mu_\Lambda/2$.

(2) $\sigma_\Lambda(f^{\lambda+1}) = f_\Lambda\sigma_\Lambda(f^\lambda)$.

(3) $\mathrm{ord}_\Lambda(f^{\lambda+1}) - \mathrm{ord}_\Lambda(f^\lambda) = -m_\Lambda$.

REMARK 9.6. In the case when $\Lambda = T^*_X X$, we have $\sigma_\Lambda(f^\lambda) = f^\lambda$ and $\mathrm{ord}_\Lambda(f^\lambda) = 0$. If $\Lambda = T^*_{f^{-1}(0)}X$ and f has a simple zero at a generic point of $f^{-1}(0)$, then $\mathrm{ord}_\Lambda(f^\lambda) = -\lambda - 1/2$.

THEOREM 9.7. *If Λ is simple Lagrangian, then there exist a polynomial $b_\Lambda(s)$ of degree exactly m_Λ and an invertible formal microdifferential operator P_Λ of order $(-m_\Lambda)$ defined on a neighborhood of a generic point p of Λ such that*

(1) $f \cdot f^s = b_\Lambda(s)P_\Lambda f^s$,

(2) $\sigma_{-m_\Lambda}(P_\Lambda)|_{W_f} = f_\Lambda$.

Furthermore, $b_\Lambda(s)$ is the minimal polynomial of

$$s \in \mathrm{End}_{\widehat{\mathcal{E}}}(\widehat{\mathcal{E}} \otimes_{\mathcal{D}_X} (\mathcal{N}_f/t\mathcal{N}_f))$$

in a neighborhood of p.

We prove this theorem in the next section.

DEFINITION 9.8. We call $b_\Lambda(s)$ the *microlocal b-function* of f along Λ.

THEOREM 9.9. *If all irreducible components of W_f^0 are simple Lagrangian, then the b-function of f is the least common multiple of the microlocal b-functions $b_\Lambda(s)$.*

PROOF. Since the b-function $b(s)$ of f is the minimal polynomial of $s \in \mathrm{End}(\mathcal{N}_f/t\mathcal{N}_f)$, it is a multiple of all $b_\Lambda(s)$. Conversely, let $b(s)$ be the least common multiple of the $b_\Lambda(s)$. Then the characteristic variety of $\mathcal{L} := b(s)(\mathcal{N}_f/t\mathcal{N}_f) \subset \mathcal{N}_f/t\mathcal{N}_f$ cannot contain any irreducible component of W_f^0, and hence $\mathrm{Ch}(\mathcal{L})$ is of dimension $\leq (n-1)$. Since $\mathrm{Ch}(\mathcal{L})$ is involutive, we obtain $\mathcal{L} = 0$. □

By this theorem, we can compute the b-function of f from the microlocal b-functions, if all irreducible components of W_f^0 are simple Lagrangian.

PROPOSITION 9.10. *Let $c(s)$ be a polynomial. If*

$$f \cdot f^s \in c(s)\widehat{\mathcal{E}}_X f^s,$$

then $c(s) \mid b_\Lambda(s)$ (i.e., $c(s)$ divides $b_\Lambda(s)$).

PROOF. There exists $G \in \widehat{\mathcal{E}}_X$ such that $f \cdot f^s \in c(s)Gf^s$. As in Theorem 9.7, take P_Λ such that $f \cdot f^s = b_\Lambda(s)P_\Lambda f^s$. Let $d(s)$ be the greatest common divisor of $c(s)$ and $b_\Lambda(s)$. Put $c(s) = a_1(s)d(s)$ and $b_\Lambda(s) = a_0(s)d(s)$. If $c(s)$ does not divide $b_\Lambda(s)$, then there exists λ such that $a_1(\lambda) = 0$. Since $a_0(s)P_\Lambda f^s = a_1(s)Gf^s$, we obtain $f^\lambda = 0$ by taking account of $a_0(\lambda) \neq 0$ and the invertibility of P_Λ. This contradicts Theorem 9.1. $\square$

9.2. Existence of Microlocal b-functions

In this section, we give a proof of Theorem 9.7.

DEFINITION 9.11. Given $P(s) = \sum s^j P_j \in \widehat{\mathcal{E}}[s] = \widehat{\mathcal{E}} \otimes \mathbb{C}[s]$ ($P_j \in \widehat{\mathcal{E}}$), set

$$\begin{aligned} \operatorname{ord} P(s) &= \max_j(\operatorname{ord} P_j)\,, \\ \underline{\operatorname{ord}} P(s) &= \max_j(j + \operatorname{ord} P_j). \end{aligned}$$

Set $\sigma_m(P(s)) := \sum_j \sigma_m(P_j)s^j \in \mathcal{O}_{T^*X} \otimes \mathbb{C}[s]$ if $\operatorname{ord} P(s) \leq m$.

LEMMA 9.12. *Let Λ be simple Lagrangian, and p a generic point of Λ. If $G(s) \in \widehat{\mathcal{E}}[s]$ satisfies $\operatorname{ord} G(s) \leq m$, $\underline{\operatorname{ord}}\, G(s) \leq k$, and*

$$\sigma_m(G(s))|_\Lambda = 0 \quad (\text{in } \mathcal{O}_\Lambda \otimes \mathbb{C}[s]),$$

then there exists $P(s) \in \widehat{\mathcal{E}}[s]$ such that $\operatorname{ord} P(s) < m$, $\underline{\operatorname{ord}}\, P(s) \leq k$, and

$$G(s)f^s = P(s)f^s.$$

Here we define $\operatorname{ord} 0 = -\infty$.

PROOF. Considering each term $G_j s^j$ of $G(s) = \sum G_j s^j$ ($G_j \in \widehat{\mathcal{E}}$), we may assume $G(s) = G \in \widehat{\mathcal{E}}(m)$ and $m \leq k$ from the beginning. Since $\sigma_m(G)|_\Lambda = 0$, there exist $A \in \mathcal{J} \cap \widehat{\mathcal{E}}(m)$ and $R \in \widehat{\mathcal{E}}(m-1)$ such that

$$\sigma_m(G) = \sigma_m(A) + \sigma_{m-1}(R)\sigma_1(v).$$

Hence $P(s) = G - A - R(v - s)$ satisfies the required conditions. $\square$

Take $P \in \widehat{\mathcal{E}}(-m_\Lambda)$ such that

$$\sigma_{-m_\Lambda}(P) = f/s^{m_\Lambda}$$

on W_f. Then $f - Pv^{m_\Lambda} \in \widehat{\mathcal{E}}(0)$ satisfies $\sigma_0(f - Pv^{m_\Lambda})|_{W_f} = 0$. Hence there exists $Q \in \widehat{\mathcal{E}}(-1)$ such that $(f - Pv^{m_\Lambda})f^s = Qf^s$, or equivalently

$$f \cdot f^s = s^{m_\Lambda} P f^s + Q f^s \,.$$

LEMMA 9.13. *There exists $G(s)$ such that*

$$\begin{aligned} &\operatorname{ord} G(s) \leq -m_\Lambda \,, \\ &\underline{\operatorname{ord}}\, G(s) \leq -1 \,, \\ &Qf^s = G(s) f^s. \end{aligned}$$

PROOF. Suppose that we have, after applying Lemma 9.12 inductively,

$$\begin{aligned} &G(s) f^s = Qf^s \,, \\ &\underline{\operatorname{ord}}\, G(s) \leq -1 \,, \\ &m = \operatorname{ord} G(s) > -m_\Lambda \,, \\ &\sigma_m(G(s))|_\Lambda \neq 0. \end{aligned}$$

Take $\lambda \in \mathbb{C}$ such that $\sigma_m(G(\lambda))|_\Lambda \not\equiv 0$. Then

$$f \cdot f^\lambda = (\lambda^{m_\Lambda} P + G(\lambda)) f^\lambda.$$

Since $\sigma_m(\lambda^{m_\Lambda} P + G(\lambda)) = \sigma_m(G(\lambda))$ does not vanish on Λ, we obtain

$$\operatorname{ord}_\Lambda(f^{\lambda+1}) = \operatorname{ord}_\Lambda(f^\lambda) + m.$$

This contradicts Corollary 9.5. $\square$

By this lemma, there exists $G(s) \in \widehat{\mathcal{E}}(-m_\Lambda)[s]$ such that

$$\begin{aligned} &f \cdot f^s = (s^{m_\Lambda} P + G(s)) f^s \,, \\ &\operatorname{ord} G(s) \leq -m_\Lambda \,, \\ &\underline{\operatorname{ord}}\, G(s) \leq -1. \end{aligned} \tag{9.2}$$

Write $G(s) = \sum s^j G_j$. Then $\operatorname{ord} G_j \leq -1 - j$. Hence

$$\sigma_{-m_\Lambda}(G(s)) = \sum_{j=0}^{m_\Lambda - 1} \sigma_{-m_\Lambda}(G_j) s^j. \tag{9.3}$$

LEMMA 9.14. *There exist a polynomial* $b_\Lambda(s) = s^{m_\Lambda} + \cdots$ *and an operator* $P(s) \in \widehat{\mathcal{E}}(-m_\Lambda)[s]$ *such that*

$$\begin{aligned} &f \cdot f^s = P(s) f^s\,, \\ &\sigma_{-m_\Lambda}(P(s))|_\Lambda = b_\Lambda(s) f_\Lambda\,, \\ &\underline{\operatorname{ord}}\, P(s) \le 0\,. \end{aligned}$$

PROOF. Take a $G(s)$ as in (9.2), and put $P(s) = s^{m_\Lambda} P + G(s)$. Then $\operatorname{ord} P(s) \le -m_\Lambda$, $\underline{\operatorname{ord}}\, P(s) \le 0$, and $f \cdot f^s = P(s) f^s$. By (9.3),

$$\sigma_{-m_\Lambda}(P(s))|_\Lambda = f_\Lambda s^{m_\Lambda} + \sum_{0 \le j < m_\Lambda} \sigma_{-m_\Lambda}(G_j) s^j, \tag{9.4}$$

which is not 0. Hence for a generic $\lambda \in \mathbb{C}$

$$\sigma_\Lambda(f \cdot f^\lambda) = \sigma_{-m_\Lambda}(P(\lambda))|_\Lambda \sigma_\Lambda(f^\lambda)\,.$$

Since

$$\sigma_\Lambda(f \cdot f^\lambda) = \sigma_\Lambda(f^{\lambda+1}) = f_\Lambda \sigma_\Lambda(f^\lambda)$$

by Corollary 9.5, and since the principal symbol is unique up to scalar multiplication, there exists a constant $b_\Lambda(\lambda)$, depending only on λ, such that

$$\sigma_{-m_\Lambda}(P(\lambda))|_\Lambda = b_\Lambda(\lambda) f_\Lambda. \tag{9.5}$$

It follows from (9.4) that $b_\Lambda(s)$ is a monic polynomial of degree m_Λ. $\square$

LEMMA 9.15. *Let* $Q(s) \in \widehat{\mathcal{E}}(-m_\Lambda)[s]$. *Assume that* $Q(s) f^s$ *and* f^{s+1} *satisfy the same differential equations. Let* $\lambda \in \mathbb{C}$, *and suppose* $\sigma_{-m_\Lambda}(Q(\lambda))|_\Lambda = 0$. *Then* $Q(\lambda) f^\lambda = 0$.

PROOF. Suppose $Q(\lambda) f^\lambda \ne 0$. Since $\sigma_{-m_\Lambda}(Q(\lambda))|_\Lambda = 0$, we have $\operatorname{ord} Q(\lambda) f^\lambda < -m_\Lambda + \operatorname{ord} f^\lambda$. Since $Q(\lambda) f^\lambda$ satisfies the same differential equations as $f^{\lambda+1}$ does, its order equals $\operatorname{ord} f^{\lambda+1} = -m_\Lambda + \operatorname{ord} f^\lambda$. This is absurd. $\square$

LEMMA 9.16. *Let* $T(s) \in \widehat{\mathcal{E}}[s]$ *and* $\lambda \in \mathbb{C}$. *Assume* $T(\lambda) f^\lambda = 0$. *Then there exists* $R(s) \in \widehat{\mathcal{E}}[s]$ *such that*

$$T(s) f^s = (s - \lambda) R(s) f^s\,, \tag{9.6}$$

$$\operatorname{ord} R(s) \le \operatorname{ord} T(s)\,, \tag{9.7}$$

$$\sigma(T(s))|_\Lambda = (s - \lambda)\sigma(R(s))|_\Lambda\,, \tag{9.8}$$

$$\underline{\operatorname{ord}}\, R(s) < \underline{\operatorname{ord}}\, T(s)\,. \tag{9.9}$$

PROOF. Let $m = \operatorname{ord} T(s)$. By the assumption, there exist $A \in \mathcal{J} \cap \widehat{\mathcal{E}}(m)$ and $B \in \widehat{\mathcal{E}}(m-1)$ such that

$$T(\lambda) = A + B(v - \lambda).$$

Write $T(s) = T(\lambda) + (s - \lambda)R_1(s)$. Then

$$\begin{aligned}&\operatorname{ord} R_1(s) \leq m\,,\\ &\underline{\operatorname{ord}}\, R_1(s) < \underline{\operatorname{ord}}\, T(s).\end{aligned}$$

Furthermore, $T(s)f^s = ((s - \lambda)B + (s - \lambda)R_1(s))f^s$. Hence $R(s) = B + R_1(s)$ satisfies the required conditions. Here we note that

$$\operatorname{ord} B < \operatorname{ord} T(s) \leq \underline{\operatorname{ord}}\, T(s).$$

□

LEMMA 9.17. *Let $Q(s) \in \widehat{\mathcal{E}}(-m_\Lambda)[s]$. Assume that $Q(s)f^s$ and f^{s+1} satisfy the same differential equations, i.e.,*

$$(v - s - 1)Q(s)f^s = 0, \quad \mathcal{J}Q(s)f^s = 0.$$

In addition, assume that we have $\sigma_{-m_\Lambda}(Q(s))|_\Lambda \in \mathcal{O}_\Lambda[s]c(s)$ for a polynomial $c(s)$ in s. Then there exists $\tilde{Q}(s) \in \widehat{\mathcal{E}}(-m_\Lambda)[s]$ such that

(9.10) $$Q(s)f^s = c(s)\tilde{Q}(s)f^s\,,$$

(9.11) $$\operatorname{ord} \tilde{Q}(s) \leq -m_\Lambda\,,$$

(9.12) $$\sigma_{-m_\Lambda}(Q(s))|_\Lambda = c(s)\sigma_{-m_\Lambda}(\tilde{Q}(s))\,,$$

(9.13) $$\underline{\operatorname{ord}}\, \tilde{Q}(s) + \deg c \leq \underline{\operatorname{ord}}\, Q(s).$$

PROOF. Let $c(s) = (s - \lambda_1)\cdots(s - \lambda_l)$. We prove the lemma by induction on l.

(1) The case $l = 1$. By Lemma 9.15, we have $Q(\lambda_1)f^{\lambda_1} = 0$. Hence Lemma 9.16 states the existence of $\tilde{Q}(s)$.

(2) The general case. Apply the induction hypothesis to $c'(s) = (s - \lambda_2)\cdots(s - \lambda_{l-1})$, and then apply (1) to $s - \lambda_1$. □

PROOF OF THEOREM 9.7. By Lemmas 9.14 and 9.17, there exists $\tilde{P}(s)$ such that

$$\begin{aligned}&f \cdot f^s = b_\Lambda(s)\tilde{P}(s)f^s\,,\\ &\operatorname{ord} \tilde{P}(s) \leq -m_\Lambda\,,\\ &\underline{\operatorname{ord}}\, \tilde{P}(s) \leq -m_\Lambda\,,\\ &\sigma_{-m_\Lambda}(\tilde{P}(s))|_\Lambda = f_\Lambda.\end{aligned}$$

Letting $\tilde{P}(s) = \sum \tilde{P}_j s^j$, we have $\operatorname{ord} \tilde{P}_j \le -m_\Lambda - j$. Put $P_\Lambda = \sum \tilde{P}_j v^j$. Then

$$\begin{aligned} &f \cdot f^s = b_\Lambda(s) P_\Lambda f^s, \\ &\operatorname{ord}(P_\Lambda) \le -m_\Lambda, \\ &\sigma_{-m_\Lambda}(P_\Lambda) = \sum_j \sigma_{-m_\Lambda - j}(\tilde{P}_j)\sigma(v)^j. \end{aligned}$$

Hence we obtain $\sigma_{-m_\Lambda}(P_\Lambda)|_\Lambda = \sigma_{-m_\Lambda}(\tilde{P}_0)|_\Lambda = f_\Lambda$.

Finally we show that $b_\Lambda(s)$ is the minimal polynomial of $s \in \operatorname{End}(\widehat{\mathcal{E}} \otimes (\mathcal{N}_f/t\mathcal{N}_f))$. Suppose $b(s)f^s \in \widehat{\mathcal{E}} f \cdot f^s$. Then by the equality $f \cdot f^s = b_\Lambda(s)P_\Lambda f^s$, there exists $Q \in \widehat{\mathcal{E}}$ such that $b(s)f^s = b_\Lambda(s)QP_\Lambda f^s$. Let $c(s)$ be the greatest common divisor of $b(s)$ and $b_\Lambda(s)$. Let $b(s) = b_1(s)c(s)$, and $b_\Lambda(s) = b_0(s)c(s)$. Then $b_1(s)f^s = b_0(s)QP_\Lambda f^s$. Suppose that $b(s)$ is not a multiple of $b_\Lambda(s)$. Then $b_0(\lambda) = 0$ for some λ. For such λ, we have $b_1(\lambda)f^\lambda = 0$. From $b_1(\lambda) \neq 0$, we obtain $f^\lambda = 0$. This contradicts Theorem 9.1. We have thus completed the proof of Theorem 9.7. $\square$

9.3. Simple Intersection of Lagrangian Analytic Subsets

Let Λ_1 and Λ_2 be simple Lagrangian analytic subsets contained in W_f^0.

DEFINITION 9.18. We say that Λ_1 and Λ_2 have a *simple intersection* along $S \subset \Lambda_1 \cap \Lambda_2$ if

(9.14) S is $(n-1)$-dimensional,

(9.15) Λ_1 and Λ_2 are nonsingular in a neighborhood of a generic point p of S,

(9.16) $T_p\Lambda_1 \cap T_p\Lambda_2 = T_pS$ at a generic point p of S,

(9.17) $\bar{\mathcal{J}} + \mathcal{O}\sigma_1(v)$ is the defining ideal of $\Lambda_1 \cup \Lambda_2$ in a neighborhood of a generic point p of S.

The conditions (9.14)–(9.16) mean that Λ_1 and Λ_2 have a good intersection in a neighborhood of a generic point of S.

We give a sufficient condition for Λ_1 and Λ_2 to have a simple intersection.

For $p \in \Lambda_1 \cap \Lambda_2$, put

$$V = \left\{ d\varphi(p)\,;\, \varphi \in (\bar{\mathcal{J}} + \mathcal{O}\sigma_1(v))_p \right\} \subset T_p^*(T^*X) \cong T_p(T^*X).$$

Then V is an isotropic vector space. For $\varphi \in (\bar{\mathcal{J}} + \mathcal{O}\sigma_1(v))_p$ with $d\varphi(p) = 0$, its isotropy representation

$$H_\varphi \in \mathfrak{sp}(T_p(T^*X))$$

preserves V, where $\mathfrak{sp}$ denotes the Lie algebra of the symplectic group. Hence it preserves $V^\perp$, the orthogonal complement to V, as well.

LEMMA 9.19. *If the following conditions are satisfied, then Λ_1 and Λ_2 have a simple intersection in a neighborhood of p:*

(1) $T_p\Lambda_1 \neq T_p\Lambda_2$;
(2) V *is* $(n-1)$*-dimensional;*
(3) *there exists* $\varphi \in (\bar{\mathcal{J}} + \mathcal{O}\sigma_1(v))_p$ *such that* $d\varphi(p) = 0$ *and* $H_\varphi \in \mathfrak{sp}(V^\perp/V)$ *does not equal* 0.

PROOF. Since $V \subset T_p\Lambda_1 \cap T_p\Lambda_2$, we get $V = T_p\Lambda_1 \cap T_p\Lambda_2$ by (1) and (2). It follows from $\dim(V^\perp/V) = 2$ that we have $\mathfrak{sp}(V^\perp/V) = \mathfrak{sl}(V^\perp/V)$. Since $A = H_\varphi \in \mathfrak{sl}(V^\perp/V)$ has two eigenspaces $T_p\Lambda_1/V$ and $T_p\Lambda_2/V$, it is nondegenerate by (3). Take a basis $d\varphi_i(p)$ $(1 \leq i \leq n-1)$ of V such that $\varphi_i \in \bar{\mathcal{J}} + \mathcal{O}\sigma_1(v)$. Then $Z = \bigcap_i \varphi_i^{-1}(0)$ is nonsingular of dimension $(n+1)$. Put $S := \Lambda_1 \cap \Lambda_2$. Then $\dim S \leq n-1$. Since S is stable under $H_{\varphi_1}, \cdots, H_{\varphi_n}$, it contains an $(n-1)$-dimensional manifold $\{\exp(\sum_{j=1}^{n-1} x_j H_{\varphi_j})(p); |x_j| \ll 1\}$. Hence S is an $(n-1)$-dimensional manifold. Since the Hessian of $\varphi|_Z$ along S is given by $A = H_\varphi \in \mathfrak{sp}(V^\perp/V)$, it is nondegenerate. Noting $\varphi|_S = 0$, we can write $\varphi|_Z = f_1 f_2$ $(df_1(p) \wedge df_2(p) \neq 0)$. Hence $\bar{\mathcal{J}} + \mathcal{O}\sigma_1(v) \supset J := \sum_{i=1}^{n-1} \mathcal{O}\varphi_i + \mathcal{O}f_1 f_2$. Since the zero set of J contains $\Lambda_1 \cup \Lambda_2$ and equals the union of two n-dimensional manifolds, we have $J = I_{\Lambda_1 \cup \Lambda_2}$. $\square$

9.4. Relation of b-functions of Two Lagrangian Analytic Subsets with a Good Intersection

In this section, in the case when two Lagrangian analytic subsets Λ_0 and Λ_1 contained in W_f^0 have a simple intersection, we investigate a relation between $b_{\Lambda_0}(s)$ and $b_{\Lambda_1}(s)$.

THEOREM 9.20. *Assume that Λ_0 and Λ_1 have a simple intersection along S. In addition, assume $m_{\Lambda_0} \geq m_{\Lambda_1}$. Then, up to scalar multiplication, we have the following formula:*

$$\begin{aligned} \frac{b_{\Lambda_0}(s)}{b_{\Lambda_1}(s)} &= \left[\operatorname{ord}_{\Lambda_1} f^s - \operatorname{ord}_{\Lambda_0} f^s + \frac{1}{2}\right]^{m_{\Lambda_0}-m_{\Lambda_1}} \\ &= \left[(m_{\Lambda_0} - m_{\Lambda_1})s + \frac{1}{2}(\mu_{\Lambda_0} - \mu_{\Lambda_1})\right]^{m_{\Lambda_0}-m_{\Lambda_1}}, \end{aligned}$$

where $[\alpha]^k := \alpha(\alpha+1)\cdots(\alpha+k-1)$.

We precede the proof of this theorem with three lemmas.

LEMMA 9.21. *If $d(s)$ is a common divisor of $b_{\Lambda_0}(s)$ and $b_{\Lambda_1}(s)$, then there exists $G \in \widehat{\mathcal{E}}$ in a neighborhood of a generic point p of S such that $f \cdot f^s = d(s)Gf^s$.*

PROOF. Since we can write $f \cdot f^s = b_{\Lambda_\nu}(s)P_{\Lambda_\nu} f^s$ in a neighborhood of a generic point of Λ_ν, the support of the holonomic $\widehat{\mathcal{E}}_X$-module

$$\widehat{\mathcal{E}}(f \cdot f^s)/(\widehat{\mathcal{E}}(f \cdot f^s) \cap d(s)\widehat{\mathcal{E}} f^s)$$

does not contain Λ_0 or Λ_1. Since this is involutive, the above $\widehat{\mathcal{E}}_X$-module equals 0 in a neighborhood of a generic point p of S. Hence there exists $G \in \widehat{\mathcal{E}}$ such that $f \cdot f^s = d(s)Gf^s$. □

LEMMA 9.22. *Under the assumptions of Theorem 9.20, $b_{\Lambda_1}(s) \mid b_{\Lambda_0}(s)$ (i.e., $b_{\Lambda_1}(s)$ divides $b_{\Lambda_0}(s)$).*

PROOF. To show this lemma, it suffices to show that, if a polynomial $d(s)$ satisfies $d(s) \mid b_{\Lambda_0}(s)$ and $d(s)(s-\lambda) \mid b_{\Lambda_1}(s)$, then $d(s)(s-\lambda) \mid b_{\Lambda_0}(s)$.

By Lemma 9.21, there exists $G \in \widehat{\mathcal{E}}$ such that

$$f \cdot f^s = d(s)Gf^s.$$

Suppose $Gf^\lambda = 0$ in a neighborhood of a generic point p_0 of Λ_0. Then $Gf^s \in (s-\lambda)\widehat{\mathcal{E}} f^s$ in a neighborhood of p_0. By Proposition 9.10, we have $d(s)(s-\lambda) \mid b_{\Lambda_0}(s)$.

Now we assume $\Lambda_0 \subset \operatorname{supp}(Gf^\lambda)$. In a neighborhood of a generic point p_1 of Λ_1, there exists $P_{\Lambda_1} \in \widehat{\mathcal{E}}(-m_{\Lambda_1})_{p_1}$ such that $f^{s+1} = b_{\Lambda_1}(s)P_{\Lambda_1} f^s$, and hence

$$Gf^s = \frac{b_{\Lambda_1}(s)}{d(s)} P_{\Lambda_1} f^s.$$

Since $b_{\Lambda_1}(s)/d(s) = 0$ at $s = \lambda$, we have $Gf^\lambda = 0$ in a neighborhood of p_1. Thus $\operatorname{supp}(Gf^\lambda) \subset \Lambda_0$, and hence $\operatorname{supp}(Gf^\lambda) = \Lambda_0$.

Consider a homomorphism of $\widehat{\mathcal{E}}$-modules $\mathcal{N}_{\lambda+1} \twoheadrightarrow \widehat{\mathcal{E}}Gf^\lambda \subset \mathcal{N}_\lambda$ ($f^{\lambda+1} \mapsto Gf^\lambda$). On the one hand, since $\mathcal{N}_\lambda$ contains a submodule $\widehat{\mathcal{E}}Gf^\lambda$ with support Λ_0, we obtain from Theorem 8.37

$$\operatorname{ord}_{\Lambda_1} f^\lambda - \operatorname{ord}_{\Lambda_0} f^\lambda - \frac{1}{2} \in \mathbb{Z}_{\geq 0},$$

or

$$(m_{\Lambda_0} - m_{\Lambda_1})\lambda + \frac{1}{2}(\mu_{\Lambda_0} - \mu_{\Lambda_1} - 1) \in \mathbb{Z}_{\geq 0}\,. \tag{9.18}$$

On the other hand, since $\mathcal{N}_{\lambda+1}$ has a quotient module $\widehat{\mathcal{E}}Gf^\lambda$ with support Λ_0, we obtain from the same theorem

$$\operatorname{ord}_{\Lambda_0} f^{\lambda+1} - \operatorname{ord}_{\Lambda_1} f^{\lambda+1} - \frac{1}{2} \in \mathbb{Z}_{\geq 0},$$

or

$$(m_{\Lambda_1} - m_{\Lambda_0})(\lambda+1) + \frac{1}{2}(\mu_{\Lambda_1} - \mu_{\Lambda_0} - 1) \in \mathbb{Z}_{\geq 0}. \tag{9.19}$$

Adding (9.18) and (9.19), we obtain $m_{\Lambda_1} - m_{\Lambda_0} - 1 \in \mathbb{Z}_{\geq 0}$. This contradicts the assumption $m_{\Lambda_0} \geq m_{\Lambda_1}$. □

LEMMA 9.23. *Under the assumptions of Theorem 9.20, if $\gamma \in \mathbb{C}$ satisfies*

$$l = -\gamma - \frac{1}{2}(\mu_{\Lambda_0} - \mu_{\Lambda_1} + 1) \in \mathbb{Z}_{\geq 0}\,,$$
$$l - (m_{\Lambda_0} - m_{\Lambda_1}) \notin \mathbb{Z}_{\geq 0},$$

then $(m_{\Lambda_0} - m_{\Lambda_1})s - \gamma$ divides the polynomial $b_{\Lambda_0}(s)/b_{\Lambda_1}(s)$.

PROOF. We may assume $m_{\Lambda_0} > m_{\Lambda_1}$. Put $\lambda = \gamma/(m_{\Lambda_0} - m_{\Lambda_1})$. Since

$$\operatorname{ord}_{\Lambda_0} f^\lambda - \operatorname{ord}_{\Lambda_1} f^\lambda - \frac{1}{2} = l \in \mathbb{Z}_{\geq 0},$$

there exist a coherent $\widehat{\mathcal{E}}$-module $\mathcal{L}$ with support Λ_0 and an epimorphism $\varphi : \mathcal{N}_\lambda \twoheadrightarrow \mathcal{L}$ by Theorem 8.37. Put $w = \varphi(f^\lambda) \in \mathcal{L}$.

By Lemmas 9.21 and 9.22, there exists $G \in \widehat{\mathcal{E}}$ such that $f \cdot f^s = b_{\Lambda_1}(s)Gf^s$. Let $\psi : \mathcal{N}_{\lambda+1} \to \mathcal{N}_\lambda$ be the homomorphism of $\widehat{\mathcal{E}}$-modules defined by $f^{\lambda+1} \mapsto Gf^\lambda$. By the assumptions,

$$\operatorname{ord}_{\Lambda_0} f^{\lambda+1} - \operatorname{ord}_{\Lambda_1} f^{\lambda+1} - \frac{1}{2} = l - (m_{\Lambda_0} - m_{\Lambda_1}) \notin \mathbb{Z}_{\geq 0},$$

and hence $\mathcal{N}_{\lambda+1}$ has no quotient modules with support Λ_0. Thus $\varphi \circ \psi : \mathcal{N}_{\lambda+1} \to \mathcal{L}$ equals 0, and

$$\varphi\psi(f^{\lambda+1}) = Gw = 0.$$

Since $\varphi : \mathcal{N}_\lambda \to \mathcal{L}$ is an isomorphism in a neighborhood of a generic point p_0 of Λ_0, we have $Gf^\lambda = 0$ there. Hence $Gf^s \in (s-\lambda)\widehat{\mathcal{E}}_X f^s$, and $f \cdot f^s = b_{\Lambda_1}(s)Gf^s \in (s-\lambda)b_{\Lambda_1}(s)\widehat{\mathcal{E}}_X f^s$. From Proposition 9.10, we obtain $(s-\lambda)b_{\Lambda_1}(s) \mid b_{\Lambda_0}(s)$. □

We are now ready to prove Theorem 9.20. By Lemma 9.22, $h(s) := b_{\Lambda_0}(s)/b_{\Lambda_1}(s)$ is a polynomial of degree $(m_{\Lambda_0} - m_{\Lambda_1})$. Let ν be any integer satisfying $0 \le \nu < m_{\Lambda_0} - m_{\Lambda_1}$, and put

$$\gamma = -\frac{1}{2}(\mu_{\Lambda_0} - \mu_{\Lambda_1} + 1) - \nu.$$

Then

$$\begin{aligned} l &= -\gamma - \frac{1}{2}(\mu_{\Lambda_0} - \mu_{\Lambda_1} + 1) = \nu \in \mathbb{Z}_{\ge 0}, \\ l &- (m_{\Lambda_0} - m_{\Lambda_1}) = \nu - (m_{\Lambda_0} - m_{\Lambda_1}) \notin \mathbb{Z}_{\ge 0}. \end{aligned}$$

Hence, by Lemma 9.23, $h(s)$ contains

$$(m_{\Lambda_0} - m_{\Lambda_1})s - \gamma = (m_{\Lambda_0} - m_{\Lambda_1})s + \frac{1}{2}(\mu_{\Lambda_0} - \mu_{\Lambda_1} + 1) + \nu$$

as a factor. Since these factors are distinct, $h(s)$ is a multiple of

$$\begin{aligned} &\prod_{\nu=0}^{m_{\Lambda_0}-m_{\Lambda_1}-1} \left((m_{\Lambda_0} - m_{\Lambda_1})s + \frac{1}{2}(\mu_{\Lambda_0} - \mu_{\Lambda_1} + 1) + \nu \right) \\ &\qquad = \left[(m_{\Lambda_0} - m_{\Lambda_1})s + \frac{1}{2}(\mu_{\Lambda_0} - \mu_{\Lambda_1} + 1) \right]^{m_{\Lambda_0}-m_{\Lambda_1}}. \end{aligned}$$

Comparing the degrees of the two polynomials, we conclude that

$$h(s) = \left[(m_{\Lambda_0} - m_{\Lambda_1})s + \frac{1}{2}(\mu_{\Lambda_0} - \mu_{\Lambda_1} + 1) \right]^{m_{\Lambda_0}-m_{\Lambda_1}}$$

up to scalar multiplication.

We have thus completed the proof of Theorem 9.20.

9.5. b-functions of Relative Invariants

Unfortunately, the cases when every irreducible component of W_f^0 for $f(x)$ is simple Lagrangian are very special. Many such cases occur when $f(x)$ is a relative invariant of a prehomogeneous vector space. In such a situation, let us apply the microlocal method of computing b-functions developed in this chapter. Let V be a finite-dimensional vector space, and G a closed subgroup of $GL(V)$. If V contains an open G-orbit, it is called a prehomogeneous vector space. A polynomial $f(x)$ on a prehomogeneous vector space V is called a *relative invariant* if there exist scalars $\chi(g)$, depending on $g \in G$, such that

$$f(gx) = \chi(g)f(x). \tag{9.20}$$

Then $\chi : G \to \mathbb{C}^*$ is a group homomorphism. Since we can easily prove that $f(x)$ is homogeneous, we assume $G \supset \mathbb{C}^*$ from now on. Let $\mathfrak{g}$ denote the Lie algebra of G. Then $\mathfrak{g}$ is a Lie subalgebra of $\mathrm{End}(V)$. For $A \in \mathfrak{g}$, let L_A denote the vector field on V defined by

$$(L_A u)(x) = \frac{d}{dt}u(e^{-tA}x)|_{t=0} = \frac{d}{dt}u(x - tAx)|_{t=0}.$$

Then $[L_A, L_{A'}] = L_{[A,A']}$. Let $\delta\chi : \mathfrak{g} \to \mathbb{C}$ denote the differential of χ, i.e.,

$$\delta\chi(A) = \frac{d}{dt}\chi(e^{tA})|_{t=0}\,.$$

By (9.20), $f(x)$ satisfies the following differential equations:

$$L_A f = -\delta\chi(A)f\,. \tag{9.21}$$

Hence $f(x)^s$ satisfies the following:

$$L_A f(x)^s = -s\delta\chi(A)f(x)^s\,. \tag{9.22}$$

Thus, letting $\mathcal{N}_f' = \mathcal{D}[s]/(\sum_{A\in\mathfrak{g}} \mathcal{D}[s](L_A + s\delta\chi(A)))$, we obtain a $\mathcal{D}[s]$-module epimorphism $\mathcal{N}_f' \twoheadrightarrow \mathcal{N}_f := \mathcal{D}f^s$.

In what follows, we identify T^*V with $V \times V^*$. Then

$$\sigma_1(L_A) = -\langle Ax, y\rangle, \quad (x,y) \in V \times V^* = T^*V.$$

Define an action of $A \in \mathfrak{g}$ on V^* by $\langle x, Ay\rangle = -\langle Ax, y\rangle$. Put

$$W_f^{0\prime} := \{\,(x,y) \in V \times V^*\,;\, \langle Ax, y\rangle = 0 \ (A \in \mathfrak{g})\,\}.$$

Then $W_f^0 \subset W_f^{0\prime}$. The subset $W_f^{0\prime}$ is Lagrangian if and only if V has finitely many G-orbits. If this is the case, then

$$W_f^{0\prime} = \bigcup_{\mathcal{O}} T^*_{\mathcal{O}} V,$$

where $\mathcal{O}$ runs over all G-orbits in V. In what follows, we assume that V has finitely many G-orbits. Let $\Lambda \subset W_f^{0\prime}$ be an irreducible component. Suppose that $(x_0, y_0) \in \Lambda$ is a generic point of Λ. Then $\Lambda = \overline{T^*_{Gx_0} V}$. Since both T^*V and T^*V^* are isomorphic to $V \times V^*$, we identify them and have $\Lambda = \overline{T^*_{Gy_0} V^*}$ as well. The condition

$$\dim \mathfrak{g}(x_0, y_0) = \dim V \tag{9.23}$$

is sufficient for $\Lambda \subset W_f^0$ to be simple Lagrangian. Set $(\mathfrak{g}x_0)^\perp := \{y \in V^*; \langle \mathfrak{g}x_0, y\rangle = 0\} = T_{y_0}(\Lambda \cap \pi_V^{-1}(x_0))$ and $\mathfrak{g}_{x_0} = \{A \in \mathfrak{g}; Ax_0 = 0\}$. Then the condition (9.23) is equivalent to the condition

$$\mathfrak{g}_{x_0} y_0 = (\mathfrak{g}x_0)^\perp, \tag{9.24}$$

and it is also equivalent to the condition that $G(x_0, y_0)$ is an open set in Λ. If these equivalent conditions are satisfied, then $\widehat{\mathcal{E}}_V \otimes_{\mathcal{D}_V} \mathcal{N}_f' \to \widehat{\mathcal{E}}_V \otimes_{\mathcal{D}_V} \mathcal{N}_f$ is an isomorphism in a neighborhood of (x_0, y_0). Using Proposition 8.24, we can compute $\operatorname{ord}_\Lambda f^\lambda$ as follows:

LEMMA 9.24. *Assume that Λ is contained in W_f^0, and that (x_0, y_0) satisfies* (9.23). *If $A \in \mathfrak{g}_{x_0}$ satisfies $Ay_0 = y_0$, then*

$$\operatorname{ord}_\Lambda f^\lambda = \lambda\delta\chi(A) - \operatorname{tr}(A; (\mathfrak{g}x_0)^\perp) + \frac{1}{2}\dim(\mathfrak{g}x_0)^\perp.$$

PROOF. Put $p = (x_0, y_0)$. The operator

$$P = \langle Ax, \partial_x\rangle - \lambda\delta\chi(A)$$

satisfies $Pf^\lambda = 0$. We have

$$\sigma_1(P) = \langle Ax, y\rangle, \quad \sigma_0(P) = -\frac{1}{2}\operatorname{tr}(A; V) - \lambda\delta\chi(A)$$

and

$$d\sigma_1(P)(p) = \langle Ax_0, dy\rangle - \langle Ay_0, dx\rangle = -\omega_V(p).$$

Hence by Proposition 8.24

$$\begin{aligned} &-\operatorname{ord}_\Lambda f^\lambda \\ &= \sigma_0(P)(p) + \frac{1}{2}\operatorname{tr}(H_{\sigma_1(P)} - E; T_p^*\Lambda) \\ &= -\lambda\delta\chi(A) - \frac{1}{2}\operatorname{tr}(A; V) - \frac{1}{2}\operatorname{tr}(H_{\sigma_1(P)} - E; T_p\Lambda). \end{aligned} \tag{9.25}$$

Since

$$\begin{aligned} H_{\sigma_1(P)} - E &= \langle Ax, \partial_x\rangle + \langle Ay, \partial_y\rangle - \langle y, \partial_y\rangle \\ &= \langle Ax, \partial_x\rangle + \langle (A-1)y, \partial_y\rangle, \end{aligned}$$

this is tangent to $\pi^{-1}(x_0)$. From

$$\begin{tikzcd} 0 \arrow[r] & T_p(\pi_V^{-1}(x_0)\cap\Lambda) \arrow[r] \arrow[d, equal] & T_p\Lambda \arrow[r] & T_{x_0}(Gx_0) \arrow[r] \arrow[d, equal] & 0, \\ & (\mathfrak{g}x_0)^{\perp} & & \mathfrak{g}x_0 & \end{tikzcd}$$

we obtain

$$\begin{aligned} &\operatorname{tr}(H_{\sigma_1(P)} - E; T_p\Lambda) \\ &= \operatorname{tr}(H_{\sigma_1(P)} - E; T_p(\pi_\Lambda^{-1}(x_0)\cap\Lambda)) + \operatorname{tr}(H_{\sigma_1(P)} - E; T_{x_0}(Gx_0)) \\ &= -\operatorname{tr}(A-1; (\mathfrak{g}x_0)^{\perp}) - \operatorname{tr}(A; \mathfrak{g}x_0) \\ &= -\operatorname{tr}(A; (\mathfrak{g}x_0)^{\perp}) - \operatorname{tr}(A; \mathfrak{g}x_0) + \dim(\mathfrak{g}x_0)^{\perp}. \end{aligned}$$

Since

$$\begin{aligned} \operatorname{tr}(A; \mathfrak{g}x_0) &= -\operatorname{tr}(A; (\mathfrak{g}x_0)^*) \\ &= -\operatorname{tr}(A; V^*) + \operatorname{tr}(A; (\mathfrak{g}x_0)^{\perp}) \\ &= \operatorname{tr}(A; V) + \operatorname{tr}(A; (\mathfrak{g}x_0)^{\perp}), \end{aligned}$$

we obtain

$$\begin{aligned} &\operatorname{tr}(H_{\sigma_1(P)} - E; T_p\Lambda) \\ &\quad = -2\operatorname{tr}(A; (\mathfrak{g}x_0)^{\perp}) - \operatorname{tr}(A; V) + \dim(\mathfrak{g}x_0)^{\perp}. \end{aligned}$$

Hence by (9.25) we conclude that

$$-\operatorname{ord}_\Lambda f^\lambda = -\lambda\delta\chi(A) + \operatorname{tr}(A; (\mathfrak{g}x_0)^{\perp}) - \frac{1}{2}\dim(\mathfrak{g}x_0)^{\perp}.$$

□

We assume that $\Lambda_0 = T^*_{Gx_0}V$ satisfies (9.23), while we do not assume $\Lambda_0 \subset W^0_f$. Let $y_1 \in (T^*_{Gx_0}V)_{x_0} = (\mathfrak{g}x_0)^{\perp}$ satisfy

(9.26) $$\dim \mathfrak{g}_{x_0}y_1 = \dim(\mathfrak{g}x_0)^{\perp} - 1.$$

Then $\dim G(x_0, y_1) = \dim V - 1$. Let $\Lambda_1 = T^*_{Gy_1}V^*$. Assume that Λ_1 is simple Lagrangian and contained in W^0_f. We have $\Lambda_0 \cap \Lambda_1 \supset G(x_0, y_1)$.

LEMMA 9.25. *In addition to the conditions above, we assume the following conditions:*

(1) $(\mathfrak{g}x_0)^\perp \not\subset \mathfrak{g}y_1$.

(2) $\mathfrak{g}_{(x_0,y_1)}(\mathfrak{g}x_0)^\perp \not\subset \mathfrak{g}_{x_0}y_1$.

(3) $Ax_0 = 0,\ Ay_1 = y_1,\ \delta\chi(A) \neq -m_{\Lambda_1}$ *for some* $A \in \mathfrak{g}$.

Then Λ_0 *is contained in* W_f^0, *and* Λ_0 *and* Λ_1 *have a simple intersection.*

PROOF. Let $p = (x_0, y_1)$. Since $0 \oplus (\mathfrak{g}x_0)^\perp \subset T_p\Lambda_0$ and $(\mathfrak{g}y_1)^\perp \oplus 0 \subset T_p\Lambda_1$, the equality $T_p\Lambda_0 = T_p\Lambda_1$ implies $\langle(\mathfrak{g}y_1)^\perp, (\mathfrak{g}x_0)^\perp\rangle = 0$, which contradicts (1). Hence $T_p\Lambda_0 \neq T_p\Lambda_1$.

By this, the statements other than $\Lambda_0 \subset W_f^0$ immediately follow from Lemma 9.19. We derive a contradiction, supposing $\Lambda_0 \not\subset W_f^0$. By the assumptions, $\mathrm{Supp}(\mathcal{N}_\lambda) = \Lambda_1$ in a neighborhood of p. Hence, by Proposition 8.32, for any $\lambda \in \mathbb{C}$, $\sigma_{\Lambda_1}(f^\lambda) = f_{\Lambda_1}^\lambda \sqrt{\omega_{\Lambda_1}/\omega}$ is holomorphic on a neighborhood of p. Thus f_{Λ_1} is an invertible holomorphic function on a neighborhood of p. From the equality $f_{\Lambda_1}(gq) = \chi(g)f_{\Lambda_1}(q)$, we obtain $(L_Af_{\Lambda_1})(p) = -\delta\chi(A)f_{\Lambda_1}(p)$. For the A in (3), we have $(L_Af_{\Lambda_1})(p) = (Ef_{\Lambda_1})(p) = -m_{\Lambda_1}f_{\Lambda_1}(p)$. This contradicts the fact that $\delta\chi(A) \neq -m_{\Lambda_1}$. □

Consequently, we can compute $b_{\Lambda_0}(s)/b_{\Lambda_1}(s)$ by the formula in Theorem 9.20. We give two examples of such computation.

EXAMPLE 9.26. Let $V = \mathbb{C}^n$ and $f(x) = x_1^2 + \cdots + x_n^2$ $(n \geq 3)$. Then $f(x)$ is a relative invariant for

$$G := GO(n) = \mathbb{C}^* \cdot O(n) \subset GL(n),$$

and

$$\delta\chi(A) = \frac{2}{n}\operatorname{tr}_V A = -\frac{2}{n}\operatorname{tr}_{V^*} A\,.$$

The vector space V has three G-orbits, and

$$q_0 = (1, 0, \ldots, 0), \quad q_1 = (1, \sqrt{-1}, 0, \ldots, 0), \quad q_2 = 0$$

are generic points of them respectively. The corresponding orbits are

$$\mathcal{O}_0 = V \backslash f^{-1}(0), \quad \mathcal{O}_1 = f^{-1}(0)\backslash\{0\}, \quad \mathcal{O}_2 = \{0\}.$$

Put $\Lambda_i = T^*_{\mathcal{O}_i}V$. Since $\dim \mathfrak{g}q_0 = n$, $(\mathfrak{g}q_1)^\perp \ni q_1$, and $\dim \mathfrak{g}(q_1, q_1) = n$, the analytic subsets $\Lambda_0 = \overline{G(q_0, 0)}$, $\Lambda_1 = \overline{G(q_1, q_1)}$, and $\Lambda_2 = \overline{G(0, q_0)}$ are simple Lagrangian. Let

$$p_0 = (1, 0, \cdots, 0) \in (\mathfrak{g}q_2)^\perp.$$

Then $\mathfrak{g}_{q_2}p_0 = V^*$. When $A = -1$, we have $Ap_0 = p_0$, $\delta\chi(A) = -2$, and hence by Lemma 9.24 we obtain $\operatorname{ord}_{\Lambda_2} f^s = -2s - n/2$ from $\operatorname{tr}(A; (\mathfrak{g}q_2)^\perp) = n$ and $\dim(\mathfrak{g}q_2)^\perp = n$. Hence the holonomy diagram (a line means a simple intersection, the quotient of microlocal b-functions is written to the left of the corresponding line, and $\operatorname{ord}_\Lambda f^s$ to the right of the corresponding Λ) equals

$$\begin{array}{rcl} & \Lambda_0 & 0 \\ s+1 & \Big| & \\ & \Lambda_1 & -s-1/2 \\ s+n/2 & \Big| & \\ & \Lambda_2 & -2s-n/2. \end{array}$$

Thus $b_{\Lambda_0}(s) = 1$, $b_{\Lambda_1}(s) = s+1$, and $b_{\Lambda_2}(s) = (s+1)(s+n/2)$. Consequently, the b-function equals $(s+1)(s+n/2)$.

EXAMPLE 9.27. Let $V = M_n(\mathbb{C})$, and let $G = GL(n) \times GL(n)$ act on V by $G \ni (g_1, g_2) : M_n(\mathbb{C}) \ni X \mapsto g_1 X g_2^T$. Then $\mathfrak{g} = M_n(\mathbb{C}) \oplus M_n(\mathbb{C})$, and $f(X) = \det X$ is a relative invariant. We shall compute its b-function. We have $\delta\chi(A_1 \oplus A_2) = \operatorname{tr} A_1 + \operatorname{tr} A_2$ and $V^* = M_n(\mathbb{C})$, where we identify V^* with V through the inner product $\langle X, Y\rangle = \operatorname{tr} XY$. Thus the actions of $\mathfrak{g}$ on V and V^* are given by

$$\mathfrak{g} \ni (A_1, A_2) : V \ni X \mapsto A_1 X + X A_2^T,$$
$$\mathfrak{g} \ni (A_1, A_2) : V^* \ni Y \mapsto -A_2^T Y - Y A_1.$$

The vector space V has $(n+1)$ G-orbits:

$$\mathcal{O}_k = \{X \in V; \operatorname{rk} X = n-k\} \quad (0 \le k \le n),$$

where $\operatorname{rk} X$ denotes the rank of X. Take $q_k = \begin{pmatrix} \mathbb{E}_{n-k} & 0 \\ 0 & 0 \end{pmatrix}$ as a generic point of $\mathcal{O}_k$. Here $\mathbb{E}_{n-k}$ denotes the unit matrix of degree $(n-k)$. We have

$$\mathfrak{g}q_k = \left\{ \begin{array}{r|cc} & n-k & k \\ \hline n-k & * & * \\ k & * & 0 \end{array} \right\}, \qquad (\mathfrak{g}q_k)^\perp = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & * \end{bmatrix} \right\},$$

and $\dim(\mathfrak{g}q_k)^\perp = k^2$. Put

$$p_k = \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{E}_k \end{pmatrix} \in (\mathfrak{g}q_k)^\perp.$$

Then $\mathfrak{g}_{q_k} p_k = (\mathfrak{g} q_k)^\perp$. Hence $\Lambda_k := \overline{T^*_{\mathcal{O}_k} V}$ is simple Lagrangian. The element

$$A_k = \begin{pmatrix} 0 & 0 \\ 0 & -\mathbb{E}_k \end{pmatrix} \oplus 0 \in \mathfrak{g}$$

satisfies $A_k(q_k, p_k) = (0, p_k)$, and

$$\begin{aligned} \operatorname{tr}(A_k; (\mathfrak{g} q_k)^\perp) &= k^2 , \\ \delta\chi(A_k) &= -k . \end{aligned}$$

Hence by Lemma 9.24

$$\operatorname{ord}_{\Lambda_k} f^s = -ks - k^2/2 . \tag{9.27}$$

We have $p_{k-1} \in (\mathfrak{g} q_k)^\perp$ and $\dim \mathfrak{g}_{q_k} p_{k-1} = k^2 - 1$. We want to apply Lemma 9.25 to Λ_k and Λ_{k-1}. Let x_k be the matrix with $(n-k+1, n-k+1)$-component 1, and the others 0. The conditions (1) and (2) in Lemma 9.25 are satisfied, since $x_k \in \mathfrak{g}_{(q_k, p_{k-1})} p_k$ and $x_k \notin \mathfrak{g} p_{k-1}$; the condition (3) is satisfied, since $A_k(q_k, p_{k-1}) = (0, p_{k-1})$ and $\delta\chi(A_k) = -k \neq -m_{\Lambda_{k-1}}$. Hence Λ_{k-1} and Λ_k have a simple intersection. Furthermore, we can prove that $\Lambda_k \subset W_f^0$ by induction on k (note that Λ_0, $\Lambda_1 \subset W_f^0$).

Hence the holonomy diagram is

$$\begin{array}{rcl}
 & \Lambda_0 & 0 \\
s+1 & \Big| & \\
 & \Lambda_1 & -s - 1/2 \\
s+2 & \Big| & \\
 & \Lambda_2 & -2s - 2^2/2 \\
 & \Big| & \\
 & \vdots & \\
 & \Big| & \\
 & \Lambda_{n-1} & -(n-1)s - (n-1)^2/2 \\
s+n & \Big| & \\
 & \Lambda_n & -ns - n^2/2.
\end{array}$$

Consequently $b_{\Lambda_k}(s) = \prod_{\nu=1}^{k}(s+\nu)$, and the b-function of $f(X) = \det X$ equals $\prod_{\nu=1}^{n}(s+\nu)$.

Appendix

A.1. Finiteness of Filtered Rings and Their Modules

In this section, we discuss finiteness properties of modules, such as coherence. In particular, proofs of coherence of some $\mathcal{D}$-modules are in our mind.

(a) Coherence. Let $\mathcal{A}$ be a sheaf of rings on a topological space X.

DEFINITION A.1. (1) A (left) $\mathcal{A}$-module $\mathcal{F}$ is said to be *locally finitely generated* if for each $x \in X$ there exist an open neighborhood U of x, an integer $N \in \mathbb{Z}_{\geq 0}$, and an epimorphism of $\mathcal{A}|_U$-modules on U, $(\mathcal{A}|_U)^{\oplus N} \twoheadrightarrow \mathcal{F}|_U$.

(2) $\mathcal{F}$ is said to be *locally finitely presented* if for each $x \in X$ there exist an open neighborhood U of x, integers $N_0, N_1 \in \mathbb{Z}_{\geq 0}$, and an exact sequence of $\mathcal{A}|_U$-modules on U,

$$(\mathcal{A}|_U)^{\oplus N_1} \to (\mathcal{A}|_U)^{\oplus N_0} \to \mathcal{F}|_U \to 0.$$

The following proposition is easy to prove.

PROPOSITION A.2. *Let $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ be an exact sequence of $\mathcal{A}$-modules.*

(1) *If $\mathcal{F}$ is locally finitely generated, then so is $\mathcal{F}''$.*
(2) *If $\mathcal{F}'$ and $\mathcal{F}''$ are locally finitely generated, then so is $\mathcal{F}$.*
(3) *If $\mathcal{F}$ is locally finitely presented, and if $\mathcal{F}'$ is locally finitely generated, then $\mathcal{F}''$ is locally finitely presented.*
(4) *If $\mathcal{F}''$ is locally finitely presented, and if $\mathcal{F}$ is locally finitely generated, then $\mathcal{F}'$ is locally finitely generated.*
(5) *If $\mathcal{F}'$ and $\mathcal{F}''$ are locally finitely presented, then so is $\mathcal{F}$.*

The following proposition expressing stalks of $\mathcal{H}om(\mathcal{F}, \mathcal{G})$ in terms of those of $\mathcal{F}$ and $\mathcal{G}$ is often quite useful.

PROPOSITION A.3. *Let $\mathcal{F}$ and $\mathcal{G}$ be $\mathcal{A}$-modules. Let $x \in X$, and let*

$$\varphi_x : \mathcal{H}om_{\mathcal{A}}(\mathcal{F}, \mathcal{G})_x \to \mathcal{H}om_{\mathcal{A}_x}(\mathcal{F}_x, \mathcal{G}_x)$$

be the canonical homomorphism.

(1) *If $\mathcal{F}$ is locally finitely generated, then φ_x is injective.*

(2) *If $\mathcal{F}$ is locally finitely presented, then φ_x is an isomorphism.*

Applying the above proposition to $\mathrm{id}_{\mathcal{F}} \in \mathcal{H}om(\mathcal{F}, \mathcal{F})$ leads to the following proposition.

PROPOSITION A.4. *Let $\mathcal{F}$ be a locally finitely generated $\mathcal{A}$-module. Let $x \in X$, and suppose $\mathcal{F}_x = 0$. Then there exists a neighborhood U of x such that $\mathcal{F}|_U = 0$.*

DEFINITION A.5. (1) An $\mathcal{A}$-module $\mathcal{F}$ is said to be *pseudo-coherent* if for every open set U all locally finitely generated $\mathcal{A}|_U$-submodules of $\mathcal{F}|_U$ are locally finitely presented.

(2) A pseudo-coherent and locally finitely generated $\mathcal{A}$-module is said to be *coherent*.

The following proposition is also easy to prove.

PROPOSITION A.6. (1) *Let $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ be an exact sequence of $\mathcal{A}$-modules. If $\mathcal{F}'$ and $\mathcal{F}''$ are coherent, then so is $\mathcal{F}$.*

(2) *Let $\varphi : \mathcal{F} \to \mathcal{F}'$ be a homomorphism from a coherent $\mathcal{A}$-module $\mathcal{F}$ to a coherent $\mathcal{A}$-module $\mathcal{F}'$. Then* $\operatorname{Ker}\varphi$, $\operatorname{Im}\varphi$, *and* $\operatorname{Coker}\varphi$ *are coherent $\mathcal{A}$-modules.*

A ring $\mathcal{A}$ is called a (left) *coherent ring* if $\mathcal{A}$ itself is a coherent $\mathcal{A}$-module as a (left) $\mathcal{A}$-module.

When $\mathcal{A}$ is a coherent ring, Proposition A.6 implies the equivalence between the locally finite presentation and the coherence for an $\mathcal{A}$-module.

DEFINITION A.7. An $\mathcal{A}$-module $\mathcal{M}$ is called a (left) *Noetherian module* if it satisfies the following:

(1) $\mathcal{M}$ is a coherent $\mathcal{A}$-module,

(2) for any $x \in X$, $\mathcal{M}_x$ is a Noetherian $\mathcal{A}_x$-module, and

(3) for any open set U and any family $\{\mathcal{N}_i\}$ of coherent $\mathcal{A}|_U$-submodules of $\mathcal{M}|_U$, $\sum \mathcal{N}_i$ is a coherent $\mathcal{A}|_U$-module.

A ring $\mathcal{A}$ is called a (left) *Noetherian ring* if it is Noetherian as a (left) $\mathcal{A}$-module.

The following proposition can be proved similarly to the case of a classical Noetherian ring.

PROPOSITION A.8. *Let X be a locally compact topological space, and $\mathcal{M}$ an $\mathcal{A}$-module on X. Then condition* (3) *of Definition A.7 is equivalent to the following* (3)′ *under condition* (1).

(3)′ *For any open set U, any point $x \in U$, and any increasing sequence $\{\mathcal{N}_n\}_{n=1,2,\cdots}$ of coherent submodules of $\mathcal{M}|_U$, there exist a neighborhood $V \subset U$ of x and an integer m such that $\mathcal{N}_n|_V = \mathcal{N}_m|_V$ for all $n \geq m$.*

PROPOSITION A.9. *If $\mathcal{A}$ is a Noetherian ring, then all coherent $\mathcal{A}$-modules are Noetherian.*

The proof is easy, and left to the reader. The following proposition is almost obvious.

PROPOSITION A.10. *Let $\mathcal{A} \to \mathcal{B}$ be a homomorphism of rings. Suppose that $\mathcal{A}$ is a Noetherian ring, and that $\mathcal{B}$ is coherent as a left $\mathcal{A}$-module. Then $\mathcal{B}$ is also a Noetherian ring. Moreover, for a $\mathcal{B}$-module $\mathcal{N}$, being a coherent $\mathcal{B}$-module and being a coherent $\mathcal{A}$-module are equivalent.*

The following lemma is also easy to prove.

LEMMA A.11. *Let $\mathcal{A}$ be a ring, and $\mathcal{I}$ a two-sided ideal of $\mathcal{A}$ satisfying $\mathcal{I}^2 = 0$. If $\mathcal{A}/\mathcal{I}$ is a Noetherian ring, and if $\mathcal{I}$ is a coherent $\mathcal{A}/\mathcal{I}$-module, then $\mathcal{A}$ is a Noetherian ring, and $\mathcal{I}$ is a coherent ideal of $\mathcal{A}$.*

The following theorem (the *Oka–Cartan Theorem*) is well known.

THEOREM A.12. *Let X be a complex manifold. Then the sheaf $\mathcal{O}_X$ of holomorphic functions on X is a Noetherian ring.*

THEOREM A.13. *Let X be a scheme of finite type over a field. Then the structure sheaf $\mathcal{O}_X$ of X is a Noetherian ring.*

We used the following proposition in Chapter 7.

PROPOSITION A.14. *Let M be a topological manifold, and Y a topological space. Let $X = M \times Y$, and let $f : X \to Y$ be the canonical projection. If $\mathcal{A}$ is a Noetherian ring on Y, then $f^{-1}\mathcal{A}$ is a Noetherian ring on X.*

PROOF. Since this proposition is local on M, we may assume $M = \mathbb{R}^n$. Furthermore, we can reduce it to the case $M = \mathbb{R}$ by induction on n. We prove that $f^{-1}\mathcal{A}$ is a Noetherian ring in this case. Among the conditions for a Noetherian ring, (2) is obvious.

The following lemma is almost obvious.

LEMMA A.15. *Let $f : X \to Y$ be a continuous map of topological spaces, and $\mathcal{A}$ a ring on Y. If $\mathcal{M}$ is a locally finitely presented $f^{-1}\mathcal{A}$-module, then for any $x \in X$ there exist a neighborhood U of x, a neighborhood V of $f(x)$, and a locally finitely presented $\mathcal{A}|_V$-module $\mathcal{N}$ such that $f^{-1}\mathcal{N}|_{f^{-1}V\cap U} \simeq \mathcal{M}|_{f^{-1}V\cap U}$.*

Using this, we prove condition (1) for a Noetherian ring. It suffices to show that if $\mathcal{I}$ is a locally finitely generated submodule of $f^{-1}\mathcal{A}$, then it is locally finitely presented. By making X smaller if necessary, we may assume that the image of $\varphi : (f^{-1}\mathcal{A})^{\oplus N} \to f^{-1}\mathcal{A}$ equals $\mathcal{I}$. Then, by Lemma A.15, locally we have $\tilde{\varphi} : \mathcal{A}^{\oplus N} \to \mathcal{A}$ such that $\varphi = f^{-1}\tilde{\varphi}$. Hence $\mathcal{I} = f^{-1}\operatorname{Im}\tilde{\varphi}$. Since $\operatorname{Im}\tilde{\varphi}$ is locally finitely presented, so is $\mathcal{I}$.

Next we prove condition (3). Let $\mathcal{N}_i$ be a family of coherent submodules of $f^{-1}\mathcal{A}$. By Lemma A.15, $\mathcal{N}_i$ is a locally constant sheaf on every fiber of f. Since every fiber of f is simply connected, we have $\mathcal{N}_i \simeq f^{-1}f_*\mathcal{N}_i$. Since $f_*\mathcal{N}_i$ is a coherent $\mathcal{A}$-module, $\sum \mathcal{N}_i = f^{-1}(\sum f_*\mathcal{N}_i)$ is also a coherent $f^{-1}\mathcal{A}$-module. □

REMARK A.16. If $f : X \to Y$ is continuous, and if $\mathcal{A}$ is a coherent ring on Y, then $f^{-1}\mathcal{A}$ is a coherent ring on X by Lemma A.15. The similar statement for the Noetherian property, however, does not hold. For example, for a closed subset $X = \{1/n; n = 1, 2, \dots\} \cup \{0\}$ of $\mathbb{R}$, $\mathbb{C}_X$ is a coherent ring, but not a Noetherian ring. Indeed, for open subsets $U_n = \{1/m; 1 \leq m \leq n\}$ of X, $\mathcal{I}_n = \mathbb{C}_{U_n} \subset \mathbb{C}_X$ form an increasing sequence of coherent ideals of $\mathbb{C}_X$, but $\mathbb{C}_{X\setminus\{0\}} = \sum \mathcal{I}_n$ is not coherent.

(b) Filtration. A *filtered module* on a topological space X is a pair consisting of a sheaf of modules $\mathcal{M}$ and an increasing sequence of its submodules $\{F_n(\mathcal{M})\}_{n\in\mathbb{Z}}$ satisfying $\mathcal{M} = \bigcup_n F_n(\mathcal{M})$. A *homomorphism* $f : \mathcal{M} \to \mathcal{N}$ of filtered modules is a homomorphism of modules satisfying $f(F_n(\mathcal{M})) \subset F_n(\mathcal{N})$ for all n. For a filtered module $\mathcal{M}$, $\bigoplus_n F_n(\mathcal{M})$ is called the *Rees module* of $\mathcal{M}$. A sequence of filtered modules $\mathcal{M}_1 \xrightarrow{f} \mathcal{M}_2 \xrightarrow{g} \mathcal{M}_3$ is said to be *filteredly exact* if the sequence of the corresponding Rees modules

$$\bigoplus F_n(\mathcal{M}_1) \to \bigoplus F_n(\mathcal{M}_2) \to \bigoplus F_n(\mathcal{M}_3)$$

is exact. Any filteredly exact sequence is exact. A homomorphism of filtered modules $f : \mathcal{M} \to \mathcal{N}$ is said to be *filteredly surjective* if $\mathcal{M} \to \mathcal{N} \to 0$ is filteredly exact. We may say that $\mathcal{M} \to \mathcal{N}$ is filteredly injective if $0 \to \mathcal{M} \to \mathcal{N}$ is filteredly exact, but we do not use this terminology, since being filteredly injective is equivalent to being injective. For a filtered module $\mathcal{M}$, set

$$\begin{aligned} \mathrm{Gr}_n^F(\mathcal{M}) &= F_n(\mathcal{M})/F_{n-1}(\mathcal{M}), \\ \mathrm{Gr}^F(\mathcal{M}) &= \bigoplus_n \mathrm{Gr}_n^F(\mathcal{M}). \end{aligned}$$

The following proposition is fundamental.

PROPOSITION A.17. *Let $\mathcal{M}' \xrightarrow{f} \mathcal{M} \xrightarrow{g} \mathcal{M}''$ be a complex of filtered modules ($g \circ f = 0$). If $F_n(\mathcal{M}) = 0$ ($n \ll 0$), and if $\mathrm{Gr}_n^F(\mathcal{M}') \to \mathrm{Gr}_n^F(\mathcal{M}) \to \mathrm{Gr}_n^F(\mathcal{M}'')$ are exact for all n, then $\mathcal{M}' \to \mathcal{M} \to \mathcal{M}''$ is filteredly exact.*

PROOF. By induction on n, we shall prove that

$$F_n(\mathcal{M}') \to F_n(\mathcal{M}) \to F_n(\mathcal{M}'')$$

are exact for all n. The claim holds for $n \ll 0$. Supposing that the claim holds for $(n-1)$, we prove the claim for n.

In the commutative diagram

$$\begin{array}{ccccc}
0 & & 0 & & 0 \\
\downarrow & & \downarrow & & \downarrow \\
F_{n-1}(\mathcal{M}') & \longrightarrow & F_{n-1}(\mathcal{M}) & \longrightarrow & F_{n-1}(\mathcal{M}'') \\
\downarrow & & \downarrow & & \downarrow \\
F_n(\mathcal{M}') & \longrightarrow & F_n(\mathcal{M}) & \longrightarrow & F_n(\mathcal{M}'') \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Gr}_n^F(\mathcal{M}') & \longrightarrow & \mathrm{Gr}_n^F(\mathcal{M}) & \longrightarrow & \mathrm{Gr}_n^F(\mathcal{M}'') \\
\downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0,
\end{array}$$

all columns are exact, the first row is exact by the induction hypothesis, and the third row is exact by the assumption. Hence the second row is also exact. □

PROPOSITION A.18. *Let $\mathcal{M}_1 \to \mathcal{M}_2 \to \mathcal{M}_3 \to \mathcal{M}_4$ be a filteredly exact sequence of filtered modules. Then*

$$\mathrm{Gr}_n^F(\mathcal{M}_1) \to \mathrm{Gr}_n^F(\mathcal{M}_2) \to \mathrm{Gr}_n^F(\mathcal{M}_3)$$

are exact for all n.

PROOF. Chasing the following commutative diagram will do for the proof:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\
 & F_{n-1}(\mathcal{M}_2) \arrow[r] \arrow[d] & F_{n-1}(\mathcal{M}_3) \arrow[r] \arrow[d] & F_{n-1}(\mathcal{M}_4) \arrow[d] \\
F_n(\mathcal{M}_1) \arrow[r] \arrow[d] & F_n(\mathcal{M}_2) \arrow[r] \arrow[d] & F_n(\mathcal{M}_3) \arrow[r] \arrow[d] & F_n(\mathcal{M}_4) \\
\mathrm{Gr}_n^F(\mathcal{M}_1) \arrow[r] \arrow[d] & \mathrm{Gr}_n^F(\mathcal{M}_2) \arrow[r] \arrow[d] & \mathrm{Gr}_n^F(\mathcal{M}_3) \arrow[d] & \\
0 & 0 & 0. &
\end{tikzcd}$$

□

(c) Filtered Rings. Let X be a topological space, $\mathcal{A}$ a ring on X. We call $\mathcal{A}$ a *filtered ring* if $\mathcal{A}$ is equipped with an increasing sequence of submodules $\{F_n(\mathcal{A})\}_{n\in\mathbb{Z}}$ such that

(1) $\mathcal{A} = \bigcup_n F_n(\mathcal{A})$,
(2) $1 \in F_0(\mathcal{A})$,
(3) $F_n(\mathcal{A}) \cdot F_m(\mathcal{A}) \subset F_{n+m}(\mathcal{A})$.

For simplicity, we further assume

(4) $F_n(\mathcal{A}) = 0 \quad (n < 0)$

in this book. We call a filtered module $(\mathcal{M}, F)$ a *filtered $\mathcal{A}$-module* if $\mathcal{M}$ is an $\mathcal{A}$-module satisfying

$$F_n(\mathcal{A})F_m(\mathcal{M}) \subset F_{n+m}(\mathcal{M}). \tag{A.1}$$

When $\mathcal{A}$ is a filtered ring, $\bigoplus_{n\geq 0} F_n(\mathcal{A})$ also has a structure of ring. This is called the *Rees ring* of $\mathcal{A}$. When $(\mathcal{M}, F)$ is a filtered $\mathcal{A}$-module, $\bigoplus_{n\in\mathbb{Z}} F_n(\mathcal{M})$ has a structure of $\bigoplus F_n(\mathcal{A})$-module.

DEFINITION A.19. Let $\mathcal{A}$ be a filtered ring, and $\mathcal{M}$ a filtered $\mathcal{A}$-module. We say that $\mathcal{M}$ is a *locally finitely generated filtered $\mathcal{A}$-module* if $\bigoplus F_n(\mathcal{M})$ is locally finitely generated over $\bigoplus F_n(\mathcal{A})$. We similarly define a locally finitely presented filtered $\mathcal{A}$-module, a coherent filtered $\mathcal{A}$-module, and a Noetherian filtered $\mathcal{A}$-module. If $(\mathcal{M}, F)$ is a coherent filtered $\mathcal{A}$-module, we call F a *coherent filtration* of $\mathcal{M}$.

(d) Coherence of Filtered Rings. The following theorem is the goal of this subsection.

THEOREM A.20. *Let $\mathcal{A}$ be a filtered ring. Assume the following:*

(1) *$F_0(\mathcal{A})$ is a Noetherian ring.*
(2) *All $\mathrm{Gr}_k^F(\mathcal{A})$ are coherent $F_0(\mathcal{A})$-modules.*
(3) *For all integers $m \geq 1$ and all open subsets U, if a submodule $\mathcal{N}$ of $\mathcal{A}^{\oplus m}|_U$ has the property that $\mathcal{N} \cap F_k(\mathcal{A})^{\oplus m}$ are coherent $F_0(\mathcal{A})|_U$-modules for all k, then $\mathcal{N}$ is locally finitely generated over $\mathcal{A}$.*

Then $\mathcal{A}$ is a Noetherian ring.

Property (2) implies that all $F_k(\mathcal{A})$ are coherent $F_0(\mathcal{A})$-modules.
Property (3) holds even if we assume (3) only for $m = 1$, although we do not prove it in this book.
From now on, we assume the properties (1), (2), and (3) in Theorem A.20. Since $\mathcal{A} = \bigcup F_k(\mathcal{A})$, $\mathcal{A}$ is a pseudo-coherent $F_0(\mathcal{A})$-module.

LEMMA A.21. *Let $\mathcal{N}$ be a finitely generated $\mathcal{A}$-submodule of $\mathcal{A}^{\oplus m}$. Then $\mathcal{N} \cap F_k(\mathcal{A})^{\oplus m}$ is a coherent $F_0(\mathcal{A})$-module for every k.*

PROOF. $\mathcal{N}$ is generated by a coherent $F_0(\mathcal{A})$-submodule $\mathcal{N}_0$. The modules $\mathcal{N}_i := F_i(\mathcal{A})\mathcal{N}_0$ are coherent $F_0(\mathcal{A})$-submodules, and $\mathcal{N} = \bigcup \mathcal{N}_i$. Hence $\mathcal{N}_i \cap F_k(\mathcal{A})^{\oplus m}$ are also coherent $F_0(\mathcal{A})$-modules. By the Noetherian property of $F_k(\mathcal{A})^{\oplus m}$,

$$\mathcal{N} \cap F_k(\mathcal{A})^{\oplus m} = \bigcup_i (\mathcal{N}_i \cap F_k(\mathcal{A})^{\oplus m})$$

is also a coherent $F_0(\mathcal{A})$-module. $\square$

LEMMA A.22. *Let $\mathcal{M}$ be a locally finitely generated $\mathcal{A}$-module. Assume that $\mathcal{M}$ is a pseudo-coherent $F_0(\mathcal{A})$-module. Then $\mathcal{M}$ is a locally finitely presented $\mathcal{A}$-module.*

PROOF. There locally exists an epimorphism of $\mathcal{A}$-modules $\varphi : \mathcal{L} = \mathcal{A}^{\oplus m} \twoheadrightarrow \mathcal{M}$. Hence $F_k(\mathcal{A})^{\oplus m} \cap \mathrm{Ker}\varphi = \mathrm{Ker}(F_k(\mathcal{A})^{\oplus m} \to \mathcal{M})$ is a coherent $F_0(\mathcal{A})$-module. By assumption (3), $\mathrm{Ker}\varphi$ is a locally finitely generated $\mathcal{A}$-module. □

LEMMA A.23. *The ring $\mathcal{A}$ is coherent.*

PROOF. Let $\mathcal{M}$ be a finitely generated $\mathcal{A}$-submodule of $\mathcal{A}$. By Lemma A.21, $\mathcal{M}$ satisfies the assumption of Lemma A.22. Hence $\mathcal{M}$ is a locally finitely presented $\mathcal{A}$-module. □

We are now in position to prove the Noetherian property of $\mathcal{A}$. Among conditions (1), (2), and (3) in Definition A.7, we have already proved (1). We prove only (3), since (2) can be proved similarly to (3).

LEMMA A.24. *Let $\{\mathcal{N}_i\}$ be a coherent $\mathcal{A}$-submodule of $\mathcal{A}$. Then $\sum_i \mathcal{N}_i$ is also coherent.*

PROOF. We may assume that $\{\mathcal{N}_i\}$ is a directed family. Then for all k

$$(\sum_i \mathcal{N}_i) \cap F_k(\mathcal{A}) = \sum_i (\mathcal{N}_i \cap F_k(\mathcal{A})),$$

which are coherent over $F_0(\mathcal{A})$. Hence, by assumption (3) in Theorem A.20, $\sum \mathcal{N}_i$ is also coherent. □

We have thus finished the proof of Theorem A.20.

In the rest of this subsection, we continue to assume that $(\mathcal{A}, F)$ is a filtered ring satisfying properties (1), (2), and (3) in Theorem A.20.

LEMMA A.25. *Let $(\mathcal{M}, F)$ be a coherent filtered $\mathcal{A}$-module. Let $\mathcal{N}$ be an $\mathcal{A}$-submodule of $\mathcal{M}$, and $F(\mathcal{N})$ a filtration of $\mathcal{N}$. If $F_n(\mathcal{N}) \subset F_n(\mathcal{M})$, and if $F_n(\mathcal{N})$ is a locally finitely generated $F_0(\mathcal{A})$-module for every n, then $F(\mathcal{N})$ is a coherent filtration.*

PROOF. For $n \in \mathbb{Z}$ we define a filtration $F^{(n)}$ by

$$F_j^{(n)}(\mathcal{N}) = \begin{cases} F_j(\mathcal{N}) & \text{for } j \le n, \\ F_{j-n}(\mathcal{A}) F_n(\mathcal{N}) & \text{for } j \ge n. \end{cases}$$

Then $\{\bigoplus_j F_j^{(n)}(\mathcal{N})\}_n$ is an increasing sequence of locally finitely generated submodules of $\bigoplus_j F_j(\mathcal{M})$. Hence its union $\bigoplus_j F_j(\mathcal{N})$ is a coherent module over $\bigoplus_j F_j(\mathcal{A})$. □

LEMMA A.26. *Let $\mathcal{M}$ be a coherent $\mathcal{A}$-module.*

(1) *$\mathcal{M}$ locally has a coherent filtration.*
(2) *$\mathcal{M}$ is a pseudo-coherent $F_0(\mathcal{A})$-module.*

PROOF. We locally have an exact sequence

$$0 \to \mathcal{N} \to \mathcal{A}^{\oplus m} \to \mathcal{M} \to 0.$$

Then $F_n(\mathcal{N}) = \mathcal{N} \cap F_n(\mathcal{A}^{\oplus m})$ gives a coherent filtration of $\mathcal{N}$. Hence the image $F_n(\mathcal{M})$ of $F_n(\mathcal{A}^{\oplus m})$ gives a coherent filtration of $\mathcal{M}$. Since $\mathcal{M}$ is the union of coherent $F_0(\mathcal{A})$-modules $F_n(\mathcal{M})$, the $F_0(\mathcal{A})$-module $\mathcal{M}$ is pseudo-coherent. □

LEMMA A.27. *Let $\mathcal{M}$ be a coherent $\mathcal{A}$-module, and $\mathcal{N}$ an $\mathcal{A}$-submodule. If $F(\mathcal{N})$ is a locally finitely generated filtration of $\mathcal{N}$, then $F(\mathcal{N})$ is a coherent filtration.*

PROOF. Let $\tilde{F}$ be a coherent filtration of $\mathcal{M}$. Then there locally exists an integer a such that $F_n(\mathcal{N}) \subset \tilde{F}_{n+a}$ for every n. Hence $\bigoplus_n F_n(\mathcal{N})$ is a locally finitely generated submodule of the coherent module $\bigoplus_n \tilde{F}_{n+a}$. Hence it is coherent. □

LEMMA A.28. *Let $0 \to \mathcal{M}' \to \mathcal{M} \to \mathcal{M}'' \to 0$ be an exact sequence of coherent $\mathcal{A}$-modules, and let $F(\mathcal{M})$ be a coherent filtration of $\mathcal{M}$. Let $F(\mathcal{M}')$ and $F(\mathcal{M}'')$ be the induced filtrations of $\mathcal{M}'$ and $\mathcal{M}''$. Then $F(\mathcal{M}')$ and $F(\mathcal{M}'')$ are coherent filtrations.*

PROOF. The filtration $F(\mathcal{M}'')$ is locally finitely generated, and hence it is coherent by Lemma A.27. Hence $F(\mathcal{M}')$ is also coherent. □

(e) Noetherian Property of $\mathcal{D}$. By using Theorem A.20, we prove the following theorem.

THEOREM A.29. *Let $\mathcal{A}$ be a filtered ring. Assume that $F_0(\mathcal{A})$ and $\mathrm{Gr}^F(\mathcal{A})$ are Noetherian rings, and that $\mathrm{Gr}^F_k(\mathcal{A})$ are locally finitely generated $F_0(\mathcal{A})$-modules for all k. Then $\mathcal{A}$ is a Noetherian ring and satisfies property* (3) *in Theorem A.20.*

PROOF. It is enough to check properties (1), (2), and (3) in Theorem A.20. Since (1) is assumed, let us start with the proof of (2). We show that a locally finitely generated $F_0(\mathcal{A})$-submodule $\mathcal{M}$ of $\mathrm{Gr}^F_k(\mathcal{A})$ is a locally finitely presented $F_0(\mathcal{A})$-module. Suppose that locally we have an epimorphism of $F_0(\mathcal{A})$-modules $\varphi : F_0(\mathcal{A})^{\oplus m} \to \mathcal{M}$. This can be extended to a homomorphism of $\mathrm{Gr}^F(\mathcal{A})$-modules

$\tilde{\varphi} : \mathrm{Gr}^F \mathcal{A}^{\oplus m} \to \mathrm{Gr}^F(\mathcal{A})$. By the coherence of $\mathrm{Gr}^F(\mathcal{A})$, $\mathcal{N} = \mathrm{Ker}\tilde{\varphi}$ is a locally finitely generated $\mathrm{Gr}^F(\mathcal{A})$-module, and thus it is locally written as $\mathcal{N} = \sum_j (\mathrm{Gr}^F \mathcal{A}) u_j$ for finitely many $u_j \in \mathrm{Gr}^F \mathcal{A}^{\oplus m}$. Let $u_j = \sum u_{j,k}$ with $u_{j,k} \in \mathrm{Gr}_k^F \mathcal{A}^{\oplus m}$. Then $\mathrm{Ker}\varphi = \sum_j F_0(\mathcal{A}) u_{j,0}$, and hence $\mathrm{Ker}\varphi$ is locally finitely generated. We have thus proved (2).

Next we show (3). Suppose that $\mathcal{N}$ is an $\mathcal{A}$-submodule of $\mathcal{A}^{\oplus m}$, and that all $F_k(\mathcal{N}) := \mathcal{N} \cap F_k(\mathcal{A})^{\oplus m}$ are coherent $F_0(\mathcal{A})$-modules. Then $\mathrm{Gr}_k^F(\mathcal{N}) \subset \mathrm{Gr}_k^F \mathcal{A}^{\oplus m}$ are also coherent $F_0(\mathcal{A})$-modules. Hence $(\mathrm{Gr}^F \mathcal{A})(\mathrm{Gr}_k^F \mathcal{N})$ are coherent $(\mathrm{Gr}^F \mathcal{A})$-submodules of $\mathrm{Gr}^F \mathcal{A}^{\oplus m}$, and thus $\mathrm{Gr}^F(\mathcal{N}) = \sum_k (\mathrm{Gr}^F \mathcal{A})(\mathrm{Gr}_k^F \mathcal{N})$ is a coherent $\mathrm{Gr}^F(\mathcal{A})$-module. Hence there locally exist finitely many $u_j \in F_{m_j}(\mathcal{N})$ with $\mathrm{Gr}_k^F(\mathcal{N}) = \sum_j \mathrm{Gr}_{k-m_j}^F \mathcal{A}[u_j]$, where $[u_j] \in \mathrm{Gr}_{m_j}^F(\mathcal{N})$ is the image of u_j. Put $\mathcal{N}' = \sum \mathcal{A} u_j$ and $F_k(\mathcal{N}') = \sum_j F_{k-m_j}(\mathcal{A}) u_j$. Then $\mathcal{N}' \subset \mathcal{N}$, and $\mathrm{Gr}_k^F(\mathcal{N}') = \mathrm{Gr}_k^F(\mathcal{N})$. Hence $\mathcal{N}' = \mathcal{N}$ by Proposition A.17, and we have shown that $\mathcal{N}$ is a locally finitely generated $\mathcal{A}$-module. □

Let $\mathcal{A}$ be a ring on X, and T an indeterminate. Then $\mathcal{A}[T] = \bigoplus \mathcal{A} T^n$ is a ring with the multiplication $(aT^n)(a'T^{n'}) = aa'T^{n+n'}$. The following is a generalization of a well-known theorem about commutative Noetherian rings (Theorem 3.6 in [Hotta2]). Our strategy for the proof is the same as the one in the commutative case, although the terminologies are different.

THEOREM A.30. *If $\mathcal{A}$ is a Noetherian ring, then $\mathcal{A}[T]$ is also a Noetherian ring.*

PROOF. Set $F_n(\mathcal{A}[T]) = \bigoplus_{k=0}^n \mathcal{A} T^k$. Then $(\mathcal{A}[T], F)$ is a filtered ring, and $\mathrm{Gr}_k^F \mathcal{A}[T] = \mathcal{A}$ for all k. Among the properties (1), (2), and (3) in Theorem A.20, it remains to show (3).

Suppose that $\mathcal{N}$ is a submodule of $\mathcal{A}[T]^{\oplus m}$, and that all $F_k(\mathcal{N}) := \mathcal{N} \cap F_k(\mathcal{A}[T])^{\oplus m}$ are coherent $\mathcal{A}$-modules. We show that $\mathcal{N}$ is a locally finitely generated $\mathcal{A}[T]$-module. Since

$$\mathrm{Gr}_k^F(\mathcal{N}) \subset \mathrm{Gr}_k^F \mathcal{A}[T]^{\oplus m} \cong \mathcal{A}^{\oplus m} T^k ,$$

$\{T^{-k} \mathrm{Gr}_k^F(\mathcal{N})\}_k$ is an increasing sequence of coherent $\mathcal{A}|_U$-submodules of $\mathcal{A}^{\oplus m}$. Hence there locally exists $k_0 \geq 0$ such that

$$T^{-k} \mathrm{Gr}_k^F(\mathcal{N}) = T^{1-k} \mathrm{Gr}_{k-1}^F(\mathcal{N}) \qquad (k > k_0).$$

Hence

$$F_k(\mathcal{N}) \subset T F_{k-1}(\mathcal{N}) + F_{k-1}(\mathcal{N}) \qquad (k > k_0),$$

and thus

$$F_k(\mathcal{N}) \subset \mathcal{A}[T]F_{k_0}(\mathcal{N}) \qquad (k > k_0).$$

We thus have $\mathcal{N} = \mathcal{A}[T]F_{k_0}(\mathcal{N})$, and conclude that $\mathcal{N}$ is a locally finitely generated $\mathcal{A}[T]$-module. □

Applying this theorem inductively, we have the following theorem.

THEOREM A.31. *If $\mathcal{A}$ is a Noetherian ring, then $\mathcal{A}[T_1, \ldots, T_l] = \mathcal{A} \otimes_{\mathbb{Z}} \mathbb{Z}[T_1, \ldots, T_l]$ is also a Noetherian ring.*

We need the following theorem to study filtered modules.

THEOREM A.32. *Let $\mathcal{A}$ be a filtered ring. Suppose that $F_0(\mathcal{A})$ and $\mathrm{Gr}^F(\mathcal{A})$ are Noetherian rings, and that all $\mathrm{Gr}_k^F(\mathcal{A})$ are locally finitely generated $F_0(\mathcal{A})$-modules. Then the Rees ring $\bigoplus F_n(\mathcal{A})$ of $\mathcal{A}$ is Noetherian.*

PROOF. Put $\mathcal{B} := \bigoplus F_n(\mathcal{A})$. To make the description clearer, we consider $\mathcal{B}$ as a subring of $\mathcal{A}[T]$ by $\mathcal{B} = \bigoplus F_n(\mathcal{A})T^n \subset \mathcal{A}[T]$. Let

$$\begin{aligned} F_n(\mathcal{B}) &= \bigoplus_{k=0}^{n-1} F_k(\mathcal{A})T^k \oplus \bigoplus_{k=n}^{\infty} F_n(\mathcal{A})T^k \\ &= \sum_{j \le k,n} F_j(\mathcal{A})T^k. \end{aligned}$$

Then $(\mathcal{B}, F)$ is a filtered ring. We want to apply Theorem A.29 to $(\mathcal{B}, F)$. We have

$$\mathrm{Gr}_n^F(\mathcal{B}) = \bigoplus_{k=n}^{\infty} (\mathrm{Gr}_n^F \mathcal{A})T^k.$$

By Theorem A.30, $F_0(\mathcal{B}) = \bigoplus_{k=0}^{\infty}(\mathrm{Gr}_0^F \mathcal{A})T^k = F_0(\mathcal{A})[T]$ is a Noetherian ring. Since

$$\begin{aligned} \text{(A.2)} \qquad \mathrm{Gr}^F(\mathcal{B}) &= \bigoplus_n \mathrm{Gr}_n^F(\mathcal{B}) = \bigoplus_{k \ge n} (\mathrm{Gr}_n^F \mathcal{A})T^k \\ &= \bigoplus_{n,k \ge 0} (\mathrm{Gr}_n^F \mathcal{A})T^n \circ T^k \\ &\simeq (\mathrm{Gr}^F \mathcal{A})[T], \end{aligned}$$

$\mathrm{Gr}^F(\mathcal{B})$ is also a Noetherian ring, again by Theorem A.30.

Under the identification (A.2),

$$\mathrm{Gr}_n^F(\mathcal{B}) \simeq (\mathrm{Gr}_0^F \mathcal{B})(\mathrm{Gr}_n^F \mathcal{A}).$$

Hence $\mathrm{Gr}_n^F(\mathcal{B})$ is a locally finitely generated $F_0(\mathcal{B})$-module. By Theorem A.29, we conclude that $\mathcal{B}$ is a Noetherian ring. □

PROPOSITION A.33. *Let $(\mathcal{A}, F)$ be a filtered ring with the assumptions in Theorem A.32. Suppose that a filtered $(\mathcal{A}, F)$-module $(\mathcal{M}, F)$ satisfies $F_n(\mathcal{M}) = 0$ $(n \ll 0)$. If $\mathrm{Gr}^F(\mathcal{M})$ is a locally finitely generated $\mathrm{Gr}^F(\mathcal{A})$-module, then $\mathcal{M}$ is a locally finitely generated filtered $\mathcal{A}$-module, and similarly, if $\mathrm{Gr}^F(\mathcal{M})$ a coherent $\mathrm{Gr}^F(\mathcal{A})$-module, then $\mathcal{M}$ a coherent filtered $\mathcal{A}$-module.*

PROOF. Suppose that $\mathrm{Gr}^F(\mathcal{M})$ is finitely generated. Then there exist finitely many $u_j \in F_{m_j}(\mathcal{M})$ such that $\mathrm{Gr}^F(\mathcal{M})$ is generated by $[u_j] \in \mathrm{Gr}_{m_j}^F(\mathcal{M})$. Let $\mathcal{L} = \bigoplus_j \mathcal{A}(-m_j)$, where we set $\mathcal{A}(-m_j) = \mathcal{A}$ and $F_n(\mathcal{A}(-m_j)) = F_{n-m_j}(\mathcal{A})$. Then we have a filtered homomorphism $\varphi : \mathcal{L} \to \mathcal{M}$. Since $\mathrm{Gr}^F(\mathcal{L}) \to \mathrm{Gr}^F(\mathcal{M})$ is surjective, $\mathcal{L} \to \mathcal{M}$ is a filtered epimorphism by Proposition A.17. Hence $\mathcal{M}$ is a finitely generated filtered $\mathcal{A}$-module. In addition, suppose that $\mathrm{Gr}^F(\mathcal{M})$ is a coherent $\mathrm{Gr}^F(\mathcal{A})$-module. Then

$$0 \to \mathrm{Gr}^F(\mathcal{N}) \to \mathrm{Gr}^F(\mathcal{L}) \to \mathrm{Gr}^F(\mathcal{M}) \to 0$$

is exact, where $\mathcal{N} = \mathrm{Ker}\varphi$, and $F_n(\mathcal{N}) = \mathcal{N} \cap F_n(\mathcal{L})$. Hence $\mathrm{Gr}^F(\mathcal{N})$ is a locally finitely generated $\mathrm{Gr}^F(\mathcal{A})$-module. By the first part of the proof, $\mathcal{N}$ is a locally finitely generated filtered $\mathcal{A}$-module, and thus $\mathcal{M}$ is a coherent filtered $\mathcal{A}$-module. □

From these, we have the following theorem about the ring $\mathcal{D}$ of differential operators.

THEOREM A.34. *Let X be a complex manifold. Then:*

(1) *$\mathcal{D}_X$ is a Noetherian ring. In particular, $\mathcal{D}_X$ is a coherent ring.*
(2) *$\bigoplus F_n(\mathcal{D}_X)$ is a Noetherian ring.*

PROOF. Locally we have $\mathrm{Gr}^F(\mathcal{D}_X) \cong \mathcal{O}_X[\xi_1, \ldots, \xi_n]$. Since $\mathcal{O}_X$ is a Noetherian ring (Theorem A.12), $\mathrm{Gr}^F(\mathcal{D}_X)$ is also a Noetherian ring by Theorem A.31. Hence $\mathcal{D}_X$ is also a Noetherian ring by Theorem A.29. Moreover, $\bigoplus F_n(\mathcal{D}_X)$ is also a Noetherian ring by Theorem A.32. □

(f) Artin–Rees Theorem. As an application of the results in the previous subsection, we prove the following theorem about commutative rings (cf. Theorem 6.4 in [Hotta2], *Artin–Rees Theorem*).

THEOREM A.35. *Let $\mathcal{A}$ be a commutative Noetherian ring, and $\mathcal{I} \subset \mathcal{A}$ a coherent ideal of $\mathcal{A}$. Then $\bigoplus_{n\geq 0} \mathcal{I}^n T^n \subset \mathcal{A}[T]$ is a Noetherian ring.*

PROOF. Let $\mathcal{B} = \bigoplus \mathcal{I}^n T^n$. By taking generators $u_1, \dots, u_l$ of $\mathcal{I}$, we have an epimorphism of rings

$$\varphi : \mathcal{C} := \mathcal{A}[S_1, \dots, S_l] \twoheadrightarrow \mathcal{B} \quad (S_i \mapsto u_i T).$$

By Theorem A.31, $\mathcal{C}$ is a Noetherian ring. Define a filtration of $\mathcal{C}$ by $F_n(\mathcal{C}) = \sum_{|\alpha|\leq n} \mathcal{A} S_1^{\alpha_1} \cdots S_l^{\alpha_l}$. Then $(\mathcal{C}, F)$ satisfies conditions (1), (2), and (3) in Theorem A.20. Since $\mathcal{B}$ is the union of an increasing sequence of coherent $F_0(\mathcal{C})$-modules, it is a coherent $\mathcal{C}$-module by Lemma A.22. Hence $\mathcal{B}$ is a Noetherian ring by Proposition A.10. □

COROLLARY A.36 (Artin–Rees). *Let $\mathcal{A}$ be a commutative Noetherian ring on a topological space X, and $\mathcal{I}$ a coherent ideal of $\mathcal{A}$. If $\mathcal{F}$ is a coherent $\mathcal{A}$-module and $\mathcal{G}$ a coherent $\mathcal{A}$-submodule of $\mathcal{F}$, then locally there exists $m_0 > 0$ such that*

$$\mathcal{I}^m \mathcal{F} \cap \mathcal{G} = \mathcal{I}^{m-m_0}(\mathcal{I}^{m_0} \mathcal{F} \cap \mathcal{G})$$

for all $m \geq m_0$.

PROOF. Let

$$\begin{aligned} \mathcal{B} &= \bigoplus \mathcal{I}^n T^n \subset \mathcal{A}[T], \\ \mathcal{M} &= \bigoplus \mathcal{I}^n \mathcal{F} \otimes T^n \subset \mathcal{F} \otimes \mathbb{Z}[T], \\ \mathcal{N} &= \bigoplus (\mathcal{I}^n \mathcal{F} \cap \mathcal{G}) \otimes T^n \subset \mathcal{M}. \end{aligned}$$

Then $\mathcal{B}$ is a Noetherian ring. Since $\mathcal{M}$ is a finitely generated $\mathcal{B}$-module and a direct sum of coherent $\mathcal{A}$-modules, it is a coherent $\mathcal{B}$-module by Lemma A.22. Since $\mathcal{N}$ is a submodule of a coherent $\mathcal{B}$-module and a sum of coherent $\mathcal{A}$-modules, $\mathcal{N}$ is also a coherent $\mathcal{B}$-module. Hence, for m_0 large enough,

$$\mathcal{N} = \sum_{n \leq m_0} \mathcal{B}((\mathcal{I}^n \mathcal{F} \cap \mathcal{G}) \otimes T^n).$$

Consequently we have

$$\mathcal{I}^m \mathcal{F} \cap \mathcal{G} = \sum_{n\leq m_0} \mathcal{I}^{m-n}(\mathcal{I}^n \mathcal{F} \cap \mathcal{G}) = \mathcal{I}^{m-m_0}(\mathcal{I}^{m_0} \mathcal{F} \cap \mathcal{G})$$

for all $m \geq m_0$. □

We used the following proposition in § 3.4.

PROPOSITION A.37. *Take $\mathcal{A}$ and $\mathcal{I}$ as in Theorem A.35. Let $\mathcal{F}$ be a coherent $\mathcal{A}$-module. Then locally there exists $m_0 \geq 0$ such that for any $m \geq m_0$ the homomorphism*

$$\mathcal{I}^m \otimes_{\mathcal{A}} \mathcal{F} \to \mathcal{I}^{m-m_0} \otimes_{\mathcal{A}} \mathcal{F}$$

is uniquely factored as

$$\mathcal{I}^m \otimes_{\mathcal{A}} \mathcal{F} \to \mathcal{I}^m \mathcal{F} \to \mathcal{I}^{m-m_0} \otimes_{\mathcal{A}} \mathcal{F}.$$

PROOF. Locally we have an epimorphism $\mathcal{L} := \mathcal{A}^{\oplus m} \to \mathcal{F}$. Let $\mathcal{N}$ be its kernel. Then, by Corollary A.36, locally there exists $m_0 \geq 0$ such that

$$\mathcal{I}^m \mathcal{L} \cap \mathcal{N} \subset \mathcal{I}^{m-m_0} \mathcal{N}$$

for any $m \geq m_0$. Since

$$\begin{aligned} \mathcal{I}^m \otimes_{\mathcal{A}} \mathcal{F} &= \mathcal{I}^m \mathcal{L} / \mathcal{I}^m \mathcal{N}, \\ \mathcal{I}^m \mathcal{F} &= \mathcal{I}^m \mathcal{L} / (\mathcal{I}^m \mathcal{L} \cap \mathcal{N}), \end{aligned}$$

we obtain

$$\mathcal{I}^m \mathcal{F} = \mathcal{I}^m \mathcal{L} / (\mathcal{I}^m \mathcal{L} \cap \mathcal{N}) \to \mathcal{I}^{m-m_0} \mathcal{L} / \mathcal{I}^{m-m_0} \mathcal{N} = \mathcal{I}^{m-m_0} \otimes_{\mathcal{A}} \mathcal{F}.$$

□

A.2. Derived Categories

In this section, we give an exposition of the derived categories of abelian categories, needed in this book. For details, see [KS] or [H].

In classical homological algebra, we compute derived functors (such as Ext and Tor) as cohomologies of complexes such as a projective resolution and an injective resolution of a module. It has been recognized in the history of the study that cohomologies of the complexes do not carry all information; some important information is left in the complexes themselves. This is the reason why the notion of the derived category has been constructed. The notion has not yet reached its final formalization; for instance we do not have the uniqueness of a mapping cone.

(a) Homotopy Categories. In what follows, let $\mathcal{C}$ be an additive category. As $\mathcal{C}$, we have in mind the category $\mathrm{Mod}(\mathcal{A})$ of $\mathcal{A}$-modules for a ring $\mathcal{A}$ on a topological space X. Readers not used to additive categories or abelian categories may assume $\mathcal{C} = \mathrm{Mod}(\mathcal{A})$.

A *complex* M in $\mathcal{C}$ consists of a family $\{(M^n, d_M^n)\}_{n\in\mathbb{Z}}$, where M^n is an object of $\mathcal{C}$, and d_M^n is a morphism $d_M^n : M^n \to M^{n+1}$, called a differential of M, satisfying $d_M^{n+1} \circ d_M^n = 0$. Denote by $\mathrm{C}(\mathcal{C})$ the additive category of complexes in $\mathcal{C}$. A morphism $f : M \to N$ in $\mathrm{C}(\mathcal{C})$ consists of a family $\{f^n\}_{n\in\mathbb{Z}}$ of morphisms $f^n : M^n \to N^n$ $(n \in \mathbb{Z})$ satisfying $d_N^n \circ f^n = f^{n+1} \circ d_M^n$. For a complex M and an integer k, define a complex $M[k]$ by $M[k]^n = M^{k+n}$ and $d_{M[k]}^n = (-1)^k d_M^{k+n}$. Then the functor $M \mapsto M[k]$ is additive from $\mathrm{C}(\mathcal{C})$ to $\mathrm{C}(\mathcal{C})$ itself. For $f : M \to N$, define $f[k] : M[k] \to N[k]$ by $f[k]^n = f^{k+n}$. The sign $(-1)^k$ here is not important (since $M[k]$ is isomorphic to the complex with differentials with no signs), but it is useful to reduce the number of signs in many morphisms appearing later.

A complex M is said to be *bounded below* if $M^n = 0$ $(n \ll 0)$, *bounded above* if $M^n = 0$ $(n \gg 0)$, and *bounded* if it is bounded above and below. Denote by $\mathrm{C}^+(\mathcal{C})$, $\mathrm{C}^-(\mathcal{C})$, and $\mathrm{C}^b(\mathcal{C})$ the full subcategories of $\mathrm{C}(\mathcal{C})$ consisting of complexes bounded below, bounded above, and bounded, respectively.

To $X \in \mathcal{C}$, we associate a complex

$$X^n = \begin{cases} X & (n = 0), \\ 0 & (n \neq 0), \end{cases} \qquad d_X^n = 0.$$

We thus consider $\mathcal{C}$ as a full subcategory of $\mathrm{C}(\mathcal{C})$.

For complexes M and N in $\mathcal{C}$, define a complex $\mathrm{Hom}(M, N)$ of modules by

$$\mathrm{Hom}(M, N)^n = \prod_k \mathrm{Hom}_{\mathcal{C}}(M^k, N^{k+n}).$$

For $f = (f^k) \in \mathrm{Hom}(M, N)^n$ $(f^k \in \mathrm{Hom}(M^k, N^{k+n}))$, we define the differential $df \in \mathrm{Hom}_{\mathcal{C}}(M, N)^{n+1}$ by letting its k-th component $(df)^k : M^k \to N^{k+n+1}$ be the sum of $M^k \xrightarrow{f^k} N^{k+n} \xrightarrow{d_N^{k+n}} N^{k+n+1}$ and $M^k \xrightarrow{(-1)^{n+1} d_M^k} M^{k+1} \xrightarrow{f^{k+1}} N^{k+1+n}$.

The signs here are given so that the sign rule in a superalgebra (A.3)

$$d(f(u)) = (df)(u) + (-1)^n f(du) \quad (f \in \mathrm{Hom}(M, N)^n,\ u \in M^k)$$

holds.

Then $d^2 = 0$, and $\mathrm{Hom}(M, N)$ is a complex of modules. The 0-th cocycle

$$Z^0(\mathrm{Hom}(M, N)) := \mathrm{Ker}(\mathrm{Hom}(M, N)^0 \xrightarrow{d} \mathrm{Hom}(M, N)^1)$$

is nothing but $\mathrm{Hom}_{\mathrm{C}(\mathcal{C})}(M,N)$. We denote by $\mathrm{Ht}(M,N)$ the 0-th coboundary

$$B^0(\mathrm{Hom}(M,N)) = \mathrm{Im}(\mathrm{Hom}(M,N)^{-1} \xrightarrow{d} \mathrm{Hom}(M,N)^0);$$

its element is called a morphism from M to N homotopic to 0. By the definition, $f = (f^n) \in \mathrm{Hom}_{\mathrm{C}(\mathcal{C})}(M,N)$ is homotopic to 0 if and only if there exists $s = (s^n : M^n \to N^{n-1})_{n\in\mathbb{Z}}$ such that

$$f^n = d_N^{n-1}s^n + s^{n+1}d_M^n.$$

The following is immediate.

$$\text{(A.4)}\quad \begin{array}{l}\text{If one of the morphisms } f : M \to N,\ g : N \to L \text{ in } \mathrm{C}(\mathcal{C}) \text{ is}\\ \text{homotopic to 0, then } g \circ f : M \to L \text{ is also homotopic to 0.}\end{array}$$

Define a new additive category $K(\mathcal{C})$ by

$$\begin{aligned}&\mathrm{Ob}(K(\mathcal{C})) := \mathrm{Ob}(\mathrm{C}(\mathcal{C}))\,,\\ &\mathrm{Hom}_{K(\mathcal{C})}(M,N) := H^0(\mathrm{Hom}(M,N))\\ &\qquad\qquad = \mathrm{Hom}_{\mathrm{C}(\mathcal{C})}(M,N)/\,\mathrm{Ht}(M,N).\end{aligned}$$

The composition of morphisms is the one induced from $\mathrm{C}(\mathcal{C})$. Then $K(\mathcal{C})$ is an additive category by property (A.4), and $M \mapsto M[k]$ is a functor from $K(\mathcal{C})$ to $K(\mathcal{C})$ itself. Similarly to the case $\mathrm{C}(\mathcal{C})$, we consider $\mathcal{C}$ as a full subcategory of $K(\mathcal{C})$.

When $\mathcal{C}$ is an abelian category, define the n-th cohomology of $M \in \mathrm{C}(\mathcal{C})$ by

$$\begin{aligned}H^n(M) &:= \mathrm{Ker}(d_M^n : M^n \to M^{n+1})/\mathrm{Im}(d_M^{n-1} : M^{n-1} \to M^n)\\ &\simeq \mathrm{Coker}(M^{n-1} \to \mathrm{Ker}(d_M^n : M^n \to M^{n+1}))\\ &\simeq \mathrm{Ker}(\mathrm{Coker}(d_M^{n-1} : M^{n-1} \to M^n) \to M^{n+1}).\end{aligned}$$

Let

$$\begin{aligned}Z^n(M) &:= \mathrm{Ker}(d_M^n : M^n \to M^{n+1})\,,\\ Z^{*n}(M) &:= \mathrm{Coker}(d_M^{n-1} : M^{n-1} \to M^n).\end{aligned}$$

Define the *truncated complexes* $\tau^{\leq n}M$ and $\tau^{\geq n}M$ of M by

$$\text{(A.5)}\quad \begin{array}{l}\tau^{\leq n}M : \cdots \to M^{n-2} \to M^{n-1} \to Z^n(M) \to 0 \to 0 \to \cdots\,,\\ \tau^{\geq n}M : \cdots \to 0 \to 0 \to Z^{*n}(M) \to M^{n+1} \to M^{n+2} \to \cdots\,.\end{array}$$

Then we have morphisms

$$\tau^{\leq n}M \to M \to \tau^{\geq n+1}M$$

with their composition 0, and

$$H^k(\tau^{\leq n}M)\begin{cases} \xrightarrow{\sim} H^k(M) & (k \leq n), \\ = 0 & (k > n), \end{cases}$$

$$H^k(\tau^{\geq n}M)\begin{cases} = 0 & (k < n), \\ \xleftarrow{\sim} H^k(M) & (k \geq n). \end{cases}$$

Since $M \mapsto \tau^{\leq n}M$ and $M \mapsto \tau^{\geq n}M$ are additive functors from $\mathrm{C}(\mathcal{C})$ to $\mathrm{C}(\mathcal{C})$, and they map morphisms homotopic to 0 to morphisms homotopic to 0, we have functors

$$\tau^{\leq n}, \tau^{\geq n} : K(\mathcal{C}) \to K(\mathcal{C}).$$

The functor H^n from $\mathrm{C}(\mathcal{C})$ to $\mathcal{C}$ induces an additive functor

$$H^n : K(\mathcal{C}) \to \mathcal{C}.$$

(b) Triangulated Categories. If $\mathcal{C}$ is an abelian category, then $\mathrm{C}(\mathcal{C})$ is also an abelian category, but $K(\mathcal{C})$ is not. Instead, $K(\mathcal{C})$ has a structure called a triangulated category.

Let $\mathcal{C}$ be an additive category. For a morphism $f : M \to N$ in $\mathrm{C}(\mathcal{C})$, define a new complex $M(f)$, called a *mapping cone* of f, as follows: $M(f)^n = M^{n+1} \oplus N^n$, and its differential $d^n_{M(f)} : M(f)^n \to M(f)^{n+1}$ is the map such that $M^{n+1} \longrightarrow M(f)^n \xrightarrow{d^n_{M(f)}} M(f)^{n+1}$ is the sum of $M^{n+1} \xrightarrow{f^{n+1}} N^{n+1} \longrightarrow M(f)^{n+1}$ and $M^{n+1} \xrightarrow{-d^{n+1}_M} M^{n+2} \longrightarrow M(f)^{n+1}$, and $N^n \longrightarrow M(f)^n \xrightarrow{d^n_{M(f)}} M(f)^{n+1}$ equals $N^n \xrightarrow{d^n_N} N^{n+1} \longrightarrow M(f)^{n+1}$. By using a matrix, we can write

$$d^n_{M(f)} = \begin{bmatrix} -d^{n+1}_M & 0 \\ f^{n+1} & d^n_N \end{bmatrix}.$$

Then $d^{n+1}_{M(f)} \circ d^n_{M(f)} = 0$, and thus $M(f)$ is a complex.

Define morphisms $\alpha_f : N \to M(f)$ and $\beta_f : M(f) \to M[1]$ in $\mathrm{C}(\mathcal{C})$ by

$$\alpha^n_f : N^n \to M^{n+1} \oplus N^n = M(f)^n,$$
$$\beta^n_f : M(f)^n = M^{n+1} \oplus N^n \to M^{n+1} = M[1]^n.$$

It is easy to see that α_f and β_f commute with differentials. Clearly $\beta_f \circ \alpha_f = 0$ in $\mathrm{C}(\mathcal{C})$. Moreover, $\alpha_f \circ f$ and $f[1] \circ \beta_f$ are homotopic to 0, or equivalently $\alpha_f \circ f = 0$ and $f[1] \circ \beta_f = 0$ in $K(\mathcal{C})$.

A diagram $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$ in $K(\mathcal{C})$ is called a *triangle*. A morphism from a triangle $X \to Y \to Z \to X[1]$ to a triangle $X' \to Y' \to Z' \to X'[1]$ is a commutative diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "\xi"'] & Y \arrow[r] \arrow[d, "\eta"'] & Z \arrow[r] \arrow[d, "\zeta"'] & X[1] \arrow[d, "\xi[1]"] \\ X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & X'[1]. \end{tikzcd}$$

If ξ, η, and ζ are isomorphisms, these two triangles are said to be *isomorphic*. A triangle $X \to Y \to Z \to X[1]$ in $K(\mathcal{C})$ is said to be *distinguished* if there exists a morphism $X' \xrightarrow{f} Y'$ such that the triangle is isomorphic to $X' \xrightarrow{f} Y' \xrightarrow{\alpha_f} M(f) \xrightarrow{\beta_f} X'[1]$.

We have the following properties:

(TR1) A triangle isomorphic to a distinguished triangle is distinguished.

(TR2) $X \xrightarrow{\mathrm{id}_X} X \xrightarrow{0} 0 \xrightarrow{0} X[1]$ is distinguished.

(TR3) $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$ is distinguished if and only if $Y \xrightarrow{g} Z \xrightarrow{h} X[1] \xrightarrow{-f[1]} Y[1]$ is distinguished.

(TR4) For any morphism $f : X \to Y$, there exists a distinguished triangle $X \xrightarrow{f} Y \to Z \to X[1]$.

(TR5) If both rows are distinguished triangles in the following commutative diagram without ξ

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\varphi"'] & Y \arrow[r, "g"] \arrow[d, "\psi"'] & Z \arrow[r, "h"] \arrow[d, dashed, "\xi"'] & X[1] \arrow[d, "\varphi[1]"] \\ X' \arrow[r, "f'"] & Y' \arrow[r, "g'"] & Z' \arrow[r, "h'"] & X'[1], \end{tikzcd}$$

then there exists $\xi : Z \to Z'$ such that the diagram with ξ is commutative.

(TR6) If the triangles

$$X \xrightarrow{f} Y \longrightarrow Z' \longrightarrow X[1],$$

$$Y \xrightarrow{g} Z \longrightarrow X' \longrightarrow Y[1],$$

$$X \xrightarrow{g\circ f} Z \longrightarrow Y' \longrightarrow X[1]$$

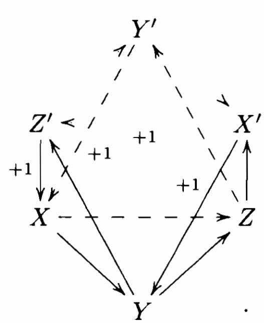
are distinguished, then there exists a distinguished triangle

$$Z' \to Y' \to X' \to Z'[1]$$

such that the following diagram commutes:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\mathrm{id}_X"'] & Y \arrow[r] \arrow[d, "g"'] & Z' \arrow[r] \arrow[d] & X[1] \arrow[d, "\mathrm{id}_X[1]"] \\
X \arrow[r, "g\circ f"] \arrow[d, "f"'] & Z \arrow[r] \arrow[d, "\mathrm{id}_Z"] & Y' \arrow[r] \arrow[d] & X[1] \arrow[d, "f[1]"] \\
Y \arrow[r, "g"] \arrow[d] & Z \arrow[r] \arrow[d] & X' \arrow[r] \arrow[d, "\mathrm{id}_{X'}"] & Y[1] \arrow[d] \\
Z' \arrow[r] & Y' \arrow[r] & X' \arrow[r] & Z'[1].
\end{tikzcd}$$

REMARK A.38. We often call (TR6) the *octahedral axiom* by viewing it as follows:



DEFINITION A.39. We call an additive category $\mathcal{D}$ a *triangulated category* if it is equipped with an isomorphism $[1] : \mathcal{D} \to \mathcal{D}$ of additive categories and a family of triangles $X \to Y \to Z \to X[1]$, called distinguished triangles, with the properties (TR1)–(TR6).

Write $[1]^n = [n]$ for $n \in \mathbb{Z}$.

For an additive category $\mathcal{C}$, the category $K(\mathcal{C})$ is triangulated.

Let $\mathcal{C}$ be an abelian category, and $X \to Y \to Z \to X[1]$ a distinguished triangle in $K(\mathcal{C})$. By applying the functor H^n, we have an exact sequence

$$\begin{tikzcd} \cdots \arrow[r] & H^n(X) \arrow[r] & H^n(Y) \arrow[r] & H^n(Z) \arrow[r] & H^n(X[1]) \arrow[r] \arrow[d, equal] & \cdots \\ & & & & H^{n+1}(X) & \end{tikzcd}$$

in $\mathcal{C}$.

(c) Derived Categories. Since cohomologies are what we are really interested in, we want to construct a new category by identifying two complexes that share the same cohomologies. In this way, the *derived category* is defined. Let us precisely formulate this idea.

Let $\mathcal{C}$ be an abelian category. A morphism $f : X \to Y$ in $K(\mathcal{C})$ is called a *quasi-isomorphism* if $H^n(X) \to H^n(Y)$ are isomorphisms for all n. This is often abbreviated to qis.

We obtain the derived category from $K(\mathcal{C})$ by regarding quasi-isomorphisms as isomorphisms. More precisely, we define the derived category $D(\mathcal{C})$ as follows:

Define the family $\mathrm{Ob}(D(\mathcal{C}))$ of objects in $D(\mathcal{C})$ to equal $\mathrm{Ob}(K(\mathcal{C}))$. For $X, Y \in \mathrm{Ob}(D(\mathcal{C})) = \mathrm{Ob}(K(\mathcal{C}))$, define the family $\mathrm{Hom}_{D(\mathcal{C})}(X, Y)$ of morphisms to be $\mathcal{S}(X,Y)/\sim$, where $\mathcal{S}(X,Y)$ is a family given in the following, and $\sim$ is its equivalence relation. $\mathcal{S}(X,Y)/\sim$ denotes the family of $\sim$-equivalence classes in $\mathcal{S}(X,Y)$.

(1) $\mathcal{S}(X,Y)$ is the set of pairs (s, f), where $s : X' \to X$ is a quasi-isomorphism and $f : X' \to Y$ is a morphism in $K(\mathcal{C})$.

(2) For $(s_1, f_1), (s_2, f_2) \in \mathcal{S}(X,Y)$ ($s_i : X'_i \underset{\text{qis}}{\to} X$, $f_i : X'_i \to Y$), we define $(s_1, f_1) \sim (s_2, f_2)$ if there exists the following commutative diagram:

$$\begin{tikzcd} & X'_1 \arrow[dl, "s_1"'] \arrow[dr, "f_1"] & \\ X & X'_0 \arrow[u, "t_1"'] \arrow[d, "t_2"] \arrow[r, "g"] & Y, \\ & X'_2 \arrow[ul, "s_2"] \arrow[ur, "f_2"'] & \end{tikzcd}$$

where $t_i : X'_0 \to X'_i$ $(i = 1, 2)$ are quasi-isomorphisms.

By the following properties of quasi-isomorphisms, $\sim$ is indeed an equivalence relation in $\mathcal{S}(X,Y)$.

(MS1) If $s : X \to Y$ and $t : Y \to Z$ are quasi-isomorphisms, then $t \circ s : X \to Z$ is also a quasi-isomorphism.

(MS2) If $s_i : X_i \to Y$ $(i = 1, 2)$ are quasi-isomorphisms, then there exist quasi-isomorphisms $t_i : Z \to X_i$ such that $t_1 \circ s_1 = t_2 \circ s_2$.

(MS3) Given a quasi-isomorphism $t : Y' \to Y$ and a morphism $f : X \to Y$, there exists a quasi-isomorphism $s : X' \to X$ with the following commutative diagram:

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "s"'] & Y' \arrow[d, "t"] \\ X \arrow[r, "f"'] & Y. \end{tikzcd}$$

(MS4) For $f : X \to Y$, the following two conditions are equivalent:
  (a) $f \circ s = 0$ for some quasi-isomorphism $s : X' \to X$.
  (b) $t \circ f = 0$ for some quasi-isomorphism $t : Y \to Y'$.

Hence

$$\mathrm{Hom}_{D(\mathcal{C})}(X, Y) = \varinjlim_{\substack{X' \to X \\ \text{qis}}} \mathrm{Hom}_{K(\mathcal{C})}(X', Y),$$

where the inductive limit is taken over all quasi-isomorphisms $X' \to X$.

Note that, for any $X, Y \in K(\mathcal{C})$,

$$\begin{aligned} &\varinjlim_{\substack{X' \to X \\ \text{qis}}} \mathrm{Hom}_{K(\mathcal{C})}(X', Y) \\ &\xrightarrow{\sim} \varinjlim_{\substack{X' \to X, Y \to Y' \\ \text{qis} \quad \text{qis}}} \mathrm{Hom}_{K(\mathcal{C})}(X', Y') \\ &\xleftarrow{\sim} \varinjlim_{\substack{Y \to Y' \\ \text{qis}}} \mathrm{Hom}_{K(\mathcal{C})}(X, Y'). \end{aligned}$$

For X, Y, $Z \in \mathrm{Ob}(D(\mathcal{C}))$, we define the composition of morphisms by

$$\begin{aligned}
&\mathrm{Hom}_{D(\mathcal{C})}(X,Y) \times \mathrm{Hom}_{D(\mathcal{C})}(Y,Z) \\
&\to \varinjlim_{\substack{X' \to X \\ \mathrm{qis}}} \mathrm{Hom}_{K(\mathcal{C})}(X',Y) \times \varinjlim_{\substack{Z \to Z' \\ \mathrm{qis}}} \mathrm{Hom}_{K(\mathcal{C})}(Y,Z') \\
&\to \varinjlim_{\substack{X' \to X, Z \to Z' \\ \mathrm{qis}\quad \mathrm{qis}}} \mathrm{Hom}_{K(\mathcal{C})}(X',Z') \simeq \mathrm{Hom}_{D(\mathcal{C})}(X,Z).
\end{aligned}$$

Then $D(\mathcal{C})$ is an additive category, and we have an additive functor $Q_{\mathcal{C}} : K(\mathcal{C}) \to D(\mathcal{C})$.

The additive functor $[n] : K(\mathcal{C}) \to K(\mathcal{C})$ induces a functor

$$[n] : D(\mathcal{C}) \to D(\mathcal{C}).$$

Since $H^n : K(\mathcal{C}) \to \mathcal{C}$ maps quasi-isomorphisms to isomorphisms, we have a functor

$$H^n : D(\mathcal{C}) \to \mathcal{C}.$$

Note that $\mathcal{C}$ is a full subcategory of $D(\mathcal{C})$, similarly to the cases $\mathrm{C}(\mathcal{C})$ and $K(\mathcal{C})$.

A triangle $X \to Y \to Z \to X[1]$ in $D(\mathcal{C})$ is said to be *distinguished* if it is isomorphic to the image of a distinguished triangle in $K(\mathcal{C})$. With this definition of distinguished triangles and [1], $D(\mathcal{C})$ is a triangulated category. If $X \to Y \to Z \to X[1]$ is a distinguished triangle in $D(\mathcal{C})$, then we have an exact sequence

$$\begin{tikzcd}
\cdots \arrow[r] & H^n(X) \arrow[r] & H^n(Y) \arrow[r] & H^n(Z) \arrow[r] & H^n(X[1]) \arrow[r] \arrow[d, equal] & \cdots \\
& & & & H^{n+1}(X) &
\end{tikzcd}$$

in $\mathcal{C}$.

We often use the following lemma.

LEMMA A.40. *Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be morphisms in $\mathrm{C}(\mathcal{C})$. If*

$$0 \to X^n \to Y^n \to Z^n \to 0$$

are exact for all n, then we can construct a morphism $Z \to X[1]$ in $D(\mathcal{C})$ such that $X \to Y \to Z \to X[1]$ is a distinguished triangle in $D(\mathcal{C})$.

PROOF. Since the morphisms $M(f)^n = X^{n+1} \oplus Y^n \longrightarrow Y^n \xrightarrow{g^n} Z^n$ commute with d, we have a morphism $M(f) \to Z$ in $\mathrm{C}(\mathcal{C})$. We can prove that this is a quasi-isomorphism. Hence $M(f)$ and Z are isomorphic in $D(\mathcal{C})$. This leads to the lemma. $\square$

Since the functors $\tau^{\geq n}$, $\tau^{\leq n} : K(\mathcal{C}) \to K(\mathcal{C})$ map quasi-isomorphisms to quasi-isomorphisms, they induce functors

$$\tau^{\leq n}, \tau^{\geq n} : D(\mathcal{C}) \to D(\mathcal{C})$$

in $D(\mathcal{C})$. For every $X \in D(\mathcal{C})$, we have a distinguished triangle

$$\tau^{\leq n} X \to X \to \tau^{\geq n+1} X \to \tau^{\leq n} X[1] \tag{A.6}$$

in $D(\mathcal{C})$.

(d) Derived Functors. Let $\mathcal{C}$ and $\mathcal{C}'$ be two abelian categories, and $F : \mathcal{C} \to \mathcal{C}'$ an additive functor. Then F naturally induces an additive functor $K(F) : K(\mathcal{C}) \to K(\mathcal{C}')$. If F is exact, then it maps quasi-isomorphisms to quasi-isomorphisms, and hence induces a functor

$$F : D(\mathcal{C}) \to D(\mathcal{C}').$$

Since F does not map quasi-isomorphisms to quasi-isomorphisms in general, we cannot define a functor $D(\mathcal{C}) \to D(\mathcal{C}')$ so simply. Roughly speaking, we construct a functor $D(\mathcal{C}) \to D(\mathcal{C}')$ by associating $X \in D(\mathcal{C})$ to the inductive limit $\varinjlim_{Q_{\mathcal{C}}(M) \simeq X} F(M)$ if it exists in $D(\mathcal{C}')$. The precise definition is as follows:

DEFINITION A.41. We say that a functor F is *right derivable* at $X \in \mathrm{Ob}(D(\mathcal{C}))$ if the functor

$$D(\mathcal{C}') \ni Z \mapsto \varinjlim_{\substack{X \to X' \\ \text{qis}}} \mathrm{Hom}_{D(\mathcal{C}')}(Z, F(X')) \tag{A.7}$$

from $D(\mathcal{C}')$ to $\mathrm{Mod}(\mathbb{Z})$ is representable (see for example §3. 2 in [U]), where the inductive limit is taken over all quasi-isomorphisms $X \to X'$ in $K(\mathcal{C})$. Then we denote by $\mathbb{R}F(X)$ the object, uniquely determined up to isomorphism, in $D(\mathcal{C}')$ that represents the functor (A.7). Namely,

$$\mathrm{Hom}_{D(\mathcal{C}')}(Z, \mathbb{R}F(X)) \simeq \varinjlim_{\substack{X \to X' \\ \text{qis}}} \mathrm{Hom}_{D(\mathcal{C}')}(Z, F(X'))$$

for all $Z \in D(\mathcal{C}')$.

The following (1) and (2) combined constitute an explicit restatement of the fact that $D(\mathcal{C}') \ni Z_0$ represents the functor (A.7):

(1) A quasi-isomorphism $s_0 : X \to X_0$ in $K(\mathcal{C})$ and a morphism $f : Z_0 \to F(X_0)$ in $D(\mathcal{C}')$ are given.

(2) For every quasi-isomorphism $s : X_0 \to X'$, there exist quasi-isomorphisms $t : X' \to X''$ and $F(X') \xrightarrow{g} Z_0$ such that the composition $Z_0 \xrightarrow{f} F(X_0) \xrightarrow{F(s)} F(X') \xrightarrow{g} Z_0$ equals id_{Z_0}, and the composition $F(X') \xrightarrow{g} Z_0 \xrightarrow{f} F(X_0) \xrightarrow{F(t \circ s)} F(X'')$ equals $F(t)$.

If F is right derivable at all $X \in D(\mathcal{C})$, then we can define a functor

$$\mathbb{R}F : D(\mathcal{C}) \to D(\mathcal{C}').$$

This commutes with $[n]$ and maps distinguished triangles to distinguished triangles. Hence it is a functor between triangulated categories.

Since it is hard to check the derivability at all objects in $D(\mathcal{C})$, we often restrict the domain of $\mathbb{R}F$ to a subcategory of $D(\mathcal{C})$. Define full subcategories $D^+(\mathcal{C})$, $D^-(\mathcal{C})$, and $D^b(\mathcal{C})$ of $D(\mathcal{C})$ by

$$\begin{aligned}
\mathrm{Ob}(D^+(\mathcal{C})) &= \{ X \,;\, H^n(X) = 0 \quad (n \ll 0) \}, \\
\mathrm{Ob}(D^-(\mathcal{C})) &= \{ X \,;\, H^n(X) = 0 \quad (n \gg 0) \}, \\
\mathrm{Ob}(D^b(\mathcal{C})) &= \{ X \,;\, H^n(X) = 0 \quad (|n| \gg 0) \} \\
&= \mathrm{Ob}(D^+(\mathcal{C})) \cap \mathrm{Ob}(D^-(\mathcal{C})).
\end{aligned}$$

A sufficient condition for F to be right derivable on $D^+(\mathcal{C})$ is given in the following.

DEFINITION A.42. Let $F : \mathcal{C} \to \mathcal{C}'$ be a left exact functor. We say that a subcategory $\mathcal{J}$ of $\mathcal{C}$ is *injective* with respect to F, or F-*injective*, if it satisfies the following:

(I. 1) If $0 \to X' \to X \to X'' \to 0$ is an exact sequence in $\mathcal{C}$ with $X', X \in \mathcal{J}$, then $X'' \in \mathcal{J}$, and

$$0 \to F(X') \to F(X) \to F(X'') \to 0$$

is also exact.

(I. 2) For every $X \in \mathrm{Ob}(\mathcal{C})$, there exist $Y \in \mathcal{J}$ and a monomorphism $X \rightarrowtail Y$.

(I. 3) $\mathcal{J}$ contains 0 and is closed under $\oplus$.

We have the following theorem.

THEOREM A.43. *Let $F : \mathcal{C} \to \mathcal{C}'$ be a left exact functor, and $\mathcal{J}$ F-injective.*

(1) *For every $X \in D^+(\mathcal{C})$, there exist a complex $I^\bullet$ in $\mathcal{J}$ bounded below and a quasi-isomorphism*
$$X \to I^\bullet.$$
(2) *If $I^\bullet \in K^+(\mathcal{C})$ is a complex in $\mathcal{J}$, then F is right derivable at $I^\bullet$, and $F(I^\bullet) \xrightarrow{\sim} \mathbb{R}F(I^\bullet)$ in $D(\mathcal{C}')$.*

Therefore if we have an F-injective subcategory $\mathcal{J}$, then we can define its right derived functor
$$\mathbb{R}F : D^+(\mathcal{C}) \to D^+(\mathcal{C}').$$
Furthermore,
$$\mathcal{J}_F := \{ X \in \mathcal{C}\,;\, H^i\mathbb{R}F(X) = 0 \quad (i > 0) \}$$
contains $\mathcal{J}$, and $\mathcal{J}_F$ is also F-injective.

DEFINITION A.44. We say that $\mathcal{C}$ has *enough injectives* if for every object X in $\mathcal{C}$ there exist an injective object I and a monomorphism $X \rightarrowtail I$.

If $0 \to X' \to X \to X'' \to 0$ is exact and X' is injective, then the sequence splits, or $X \simeq X' \oplus X''$. Hence $0 \to F(X') \to F(X) \to F(X'') \to 0$ is exact. Therefore the full subcategory $\mathcal{I}$ consisting of all injective objects in $\mathcal{C}$ satisfies conditions (I. 1)–(I. 3). Hence we obtain the following theorem.

THEOREM A.45. *Let $F : \mathcal{C} \to \mathcal{C}'$ be a left exact functor, and let $\mathcal{C}$ have enough injectives. Then F is right derivable at all objects in $D^+(\mathcal{C})$, and we can define*
$$\mathbb{R}F : D^+(\mathcal{C}) \to D^+(\mathcal{C}').$$

The following theorem is well known.

THEOREM A.46. *Let $\mathcal{A}$ be a ring on a topological space X. Then the category* $\mathrm{Mod}(\mathcal{A})$ *of $\mathcal{A}$-modules has enough injectives.*

Hence, given a left exact functor F from $\mathrm{Mod}(\mathcal{A})$ to any abelian category $\mathcal{C}'$, we can define $\mathbb{R}F : D^+(\mathrm{Mod}(\mathcal{A})) \to D^+(\mathcal{C}')$.

The following theorem is not hard to see.

THEOREM A.47. *Let $F : \mathcal{C} \to \mathcal{C}'$ and $G : \mathcal{C}' \to \mathcal{C}''$ be left exact functors, $\mathcal{J}$ an F-injective subcategory of $\mathcal{C}$, and $\mathcal{J}'$ a G-injective subcategory of $\mathcal{C}'$. If $F(\mathcal{J}) \subset \mathcal{J}'$, then:*

(1) *$\mathcal{J}$ is $(G \circ F)$-injective.*
(2) *We can define*

$$\begin{aligned} \mathbb{R}F : D^+(\mathcal{C}) &\to D^+(\mathcal{C}'), \\ \mathbb{R}G : D^+(\mathcal{C}') &\to D^+(\mathcal{C}''), \\ \mathbb{R}(G \circ F) : D^+(\mathcal{C}) &\to D^+(\mathcal{C}''). \end{aligned}$$

(3) $\mathbb{R}(G \circ F) \simeq \mathbb{R}G \circ \mathbb{R}F$.

Thus the right derived functor of a composition is the composition of right derived functors, which simplifies many equations. This is one advantage of the theory of derived categories.

Let $F : \mathcal{C} \to \mathcal{C}'$ be a left exact functor, and suppose that we have an F-injective subcategory of $\mathcal{C}$. For $X \in \mathcal{C}$, we usually abbreviate $H^i(\mathbb{R}F(X))$ to $R^iF(X)$. We say that the *cohomological dimension* of F is at most $d \in \mathbb{Z}_{\geq 0}$ if $R^iF(X) = 0$ for all $X \in \mathcal{C}$ and $i > d$. If the cohomological dimension of F is finite, then $\mathbb{R}F$ maps $D^b(\mathcal{C})$ to $D^b(\mathcal{C}')$.

By reversing the directions of arrows in the above definition of a right derived functor, we can define its dual notion, a *left derived functor*.

DEFINITION A.48. A right exact functor $F : \mathcal{C} \to \mathcal{C}'$ is said to be *left derivable* at $X \in D(\mathcal{C})$ if the functor

$$D(\mathcal{C}') \ni Z \mapsto \varinjlim_{\substack{X' \to X \\ \text{qis}}} \operatorname{Hom}_{D(\mathcal{C}')}(F(X'), Z) \tag{A.8}$$

is representable. We denote by $\mathbb{L}F(X)$ the representing object.

The dual notion to an injective subcategory is as follows:

DEFINITION A.49. Let $F : \mathcal{C} \to \mathcal{C}'$ be a right exact functor. A subcategory $\mathcal{P}$ of $\mathcal{C}$ is *F-projective* if it satisfies the following:

(P.1) If $0 \to X' \to X \to X'' \to 0$ is exact in $\mathcal{C}$, and if $X, X'' \in \mathcal{P}$, then $X' \in \mathcal{P}$, and

$$0 \to F(X') \to F(X) \to F(X'') \to 0$$

is also exact.

(P.2) For every $X \in \operatorname{Ob}(\mathcal{C})$, there exist $Y \in \mathcal{P}$ and an epimorphism $Y \twoheadrightarrow X$.

(P.3) $\mathcal{P}$ contains 0 and is closed under $\oplus$.

Stating the theorem dual to Theorem A.43 is not hard; it is left to the reader.

A.3. Derived Functors on Categories of Modules

Let A be a ring on a topological space, and $\mathrm{Mod}(A)$ the abelian category of left A-modules. We shall discuss functors concerning this category. From now on, we assume that A is a $\mathbb{C}$-algebra on X. We have similar results, when we replace $\mathbb{C}$ by other fields.

We have a canonical functor $\mathrm{Mod}(A) \to \mathrm{Mod}(\mathbb{C}_X)$, where $\mathbb{C}_X$ is the constant sheaf on X with stalk $\mathbb{C}$.

We abbreviate $K(\mathrm{Mod}(A))$ and $D(\mathrm{Mod}(A))$ to $K(A)$ and $D(A)$.

Let A^{op} denote the opposite ring of A (cf. §1.4). Then left A^{op}-modules and right A-modules are the same.

(a) Tensor Product. Let A_1 and A_2 be $\mathbb{C}$-algebras. The tensor product $A_1 \otimes_{\mathbb{C}} A_2$ (we omit $\mathbb{C}$ in $\otimes_{\mathbb{C}}$ in the sequel) is a $\mathbb{C}$-algebra with multiplication

$$(a_1 \otimes a_2)(a_1' \otimes a_2') = a_1 a_1' \otimes a_2 a_2' \quad (a_i, a_i' \in A_i,\ i = 1, 2),$$

and $\mathrm{Mod}(A_1 \otimes A_2^{\mathrm{op}})$ is the category of (A_1, A_2)-modules. Given $\mathbb{C}$-algebras A_1, A_2, and A_3, we have a functor

$$\bullet \otimes_{A_2} \bullet : \mathrm{Mod}(A_1 \otimes A_2^{\mathrm{op}}) \times \mathrm{Mod}(A_2 \otimes A_3^{\mathrm{op}}) \to \mathrm{Mod}(A_1 \otimes A_3^{\mathrm{op}}).$$

For $M \in K(A_1 \otimes A_2^{\mathrm{op}})$ and $N \in K(A_2 \otimes A_3^{\mathrm{op}})$, we define $M \otimes_{A_2} N \in K(A_1 \otimes A_3^{\mathrm{op}})$ by

$$\begin{aligned} (M \otimes_{A_2} N)^n &= \bigoplus_{i+j=n} M^i \otimes_{A_2} N^j , \\ d(x \otimes y) &= d_M x \otimes y + (-1)^i x \otimes d_N y \quad (x \in M^i,\ y \in N^j). \end{aligned}$$

If $M \in D^-(A_1 \otimes A_2^{\mathrm{op}})$ and $N \in D^-(A_2 \otimes A_3^{\mathrm{op}})$, then

$$L \mapsto \varinjlim_{\substack{M' \underset{\mathrm{qis}}{\to} M,\, N' \underset{\mathrm{qis}}{\to} N}} \mathrm{Hom}_{D(A_1 \otimes A_3^{\mathrm{op}})}(L,\, M' \otimes_{A_2} N')$$

is representable in $D(A_1 \otimes A_3^{\mathrm{op}})$, and we denote by $M \overset{\mathbb{L}}{\otimes}_{A_2} N$ the representing object. We thus obtain a left derived functor

$$\overset{\mathbb{L}}{\otimes} : D^-(A_1 \otimes A_2^{\mathrm{op}}) \times D^-(A_2 \otimes A_3^{\mathrm{op}}) \to D^-(A_1 \otimes A_3^{\mathrm{op}}) .$$

Indeed, for any $M \in D^-(A_1 \otimes A_2^{\mathrm{op}})$, there exist $M' \in K(A_1 \otimes A_2^{\mathrm{op}})$ consisting of A_2^{op}-flat modules and a quasi-isomorphism $M' \to M$.

Then we have

$$M \overset{\mathbb{L}}{\underset{A_2}{\otimes}} N \simeq M' \otimes_{A_2} N.$$

Similarly we can compute $M \overset{\mathbb{L}}{\underset{A_2}{\otimes}} N$, using an A_2-flat resolution of N. The cohomology $H^{-i}(M \overset{\mathbb{L}}{\underset{A_2}{\otimes}} N)$ is often denoted by $\mathcal{T}or_i^{A_2}(M, N)$. We have the following associative law: Let A_1, A_2, A_3, and A_4 be $\mathbb{C}$-algebras, and let ${}_iM_{i+1} \in D^-(A_i \otimes A_{i+1}^{\mathrm{op}})$ $(i = 1, 2, 3)$; then

$$({}_1M_2 \overset{\mathbb{L}}{\underset{A_2}{\otimes}} {}_2M_3) \overset{\mathbb{L}}{\underset{A_3}{\otimes}} {}_3M_4 \simeq {}_1M_2 \overset{\mathbb{L}}{\underset{A_2}{\otimes}} ({}_2M_3 \overset{\mathbb{L}}{\underset{A_3}{\otimes}} {}_3M_4)$$

in $D^-(A_1 \otimes A_4^{\mathrm{op}})$.

If A is commutative, then $M \overset{\mathbb{L}}{\otimes} N \simeq N \overset{\mathbb{L}}{\otimes} M$, where for complexes M and N the isomorphism $M \otimes N \xrightarrow{\sim} N \otimes M$ is given by

$$\text{(A.9)} \qquad x \otimes y \mapsto (-1)^{pq} y \otimes x \quad (x \in M^p,\ y \in N^q).$$

(b) Inner Hom. Let

$$M \in \mathrm{Mod}(A_1 \otimes A_2^{\mathrm{op}}), \quad N \in \mathrm{Mod}(A_1 \otimes A_3^{\mathrm{op}}).$$

Then $\mathcal{H}om_{A_1}(M, N)$ belongs to $\mathrm{Mod}(A_2 \otimes A_3^{\mathrm{op}})$.

For $M \in \mathrm{C}(A_1 \otimes A_2^{\mathrm{op}})$ and $N \in \mathrm{C}(A_1 \otimes A_3^{\mathrm{op}})$, we define a complex $\mathcal{H}om_{A_1}(M, N)$ of $(A_2 \otimes A_3^{\mathrm{op}})$-modules by

$$\mathcal{H}om_{A_1}(M, N)^n = \prod_{n=-i+j} \mathcal{H}om_{A_1}(M^i, N^j).$$

Its differentials are defined in the sentence just above (A.3). For $M \in D^-(A_1 \otimes A_2^{\mathrm{op}})$ and $N \in D^+(A_1 \otimes A_3^{\mathrm{op}})$, we set

$$\begin{aligned} \mathbb{R}\mathcal{H}om_{A_1}(M, N) &= \varinjlim_{\substack{M' \xrightarrow[\text{qis}]{} M,\ N \xrightarrow[\text{qis}]{} N'}} \mathcal{H}om_{A_1}(M', N') \\ &\simeq \varinjlim_{N \xrightarrow[\text{qis}]{} N'} \mathcal{H}om_{A_1}(M, N'). \end{aligned}$$

Using a complex $I \in \mathrm{C}^+(A_1 \otimes A_3^{\mathrm{op}})$ of injective A_1-modules and a quasi-isomorphism $N \to I$, we can compute $\mathbb{R}\mathcal{H}om_{A_1}(M, N)$ by

$$\mathbb{R}\mathcal{H}om_{A_1}(M, N) \simeq \mathcal{H}om_{A_1}(M, I).$$

We thus have a functor

$$\mathbb{R}\mathcal{H}om_{A_1} : D^-(A_1 \otimes A_2^{\mathrm{op}})^{\mathrm{op}} \times D^+(A_1 \otimes A_3^{\mathrm{op}}) \to D^+(A_2 \otimes A_3^{\mathrm{op}}).$$

Here $\mathcal{C}^{\mathrm{op}}$ denotes the *opposite category* of $\mathcal{C}$ defined by $\mathrm{Ob}(\mathcal{C}^{\mathrm{op}}) = \mathrm{Ob}(\mathcal{C})$ and $\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(X,Y) = \mathrm{Hom}_{\mathcal{C}}(Y,X)$. The cohomology module $H^i\mathbb{R}\mathcal{H}om_{A_1}(M,N)$ is often denoted by $\mathcal{E}xt^i_{A_1}(M,N)$.

We have the following "associative law": For ${}_iM_j \in D^-(A_i \otimes A_j^{\mathrm{op}})$ and ${}_3N_4 \in D^+(A_3 \otimes A_4^{\mathrm{op}})$, we have

$$\mathbb{R}\mathcal{H}om_{A_1}({}_1M_2, \mathbb{R}\mathcal{H}om_{A_3}({}_3M_1, {}_3N_4)) \simeq \mathbb{R}\mathcal{H}om_{A_3}({}_3M_1 \overset{\mathbb{L}}{\underset{A_1}{\otimes}} {}_1M_2, {}_3M_4)$$

in $D^+(A_2 \otimes A_4^{\mathrm{op}})$.

If the flat dimension of A_3 is finite, then we have a canonical morphism

$$\mathbb{R}\mathcal{H}om_{A_1}({}_1M_2, {}_1N_3) \overset{\mathbb{L}}{\underset{A_3}{\otimes}} {}_3N_4 \to \mathbb{R}\mathcal{H}om_{A_1}({}_1M_2, {}_1N_3 \overset{\mathbb{L}}{\underset{A_3}{\otimes}} {}_3N_4) \tag{A.10}$$

in $D^+(A_2 \otimes A_4^{\mathrm{op}})$. Indeed, we obtain the morphism by replacing ${}_3N_4$ by a complex of flat A_3-modules, and ${}_1N_3$ by a complex of injective modules. The morphism (A.10) is an isomorphism, if A_3 is coherent and all $H^i({}_3N_4)$ are coherent A_3-modules. If A_1 is coherent and ${}_1M_2$ is a coherent A_1-module, (A.10) is again an isomorphism.

If the cohomological dimension of A is finite, then in $D^b(A)$ we have a morphism

$$\varphi_M : M \to \mathbb{R}\mathcal{H}om_{A^{\mathrm{op}}}(\mathbb{R}\mathcal{H}om_A(M,A),A) \tag{A.11}$$

for every $M \in D^b(A)$. Using an injective resolution $A \to I$ of the $(A \otimes A^{\mathrm{op}})$-module A, we obtain the morphism (A.11) from the morphism in $C(A)$

$$M \to \mathcal{H}om_{A^{\mathrm{op}}}(\mathcal{H}om_A(M,I),I).$$

Set $M^* = \mathbb{R}\mathcal{H}om_A(M,A)$. Then the composition

$$M^* \xrightarrow{\varphi_{M^*}} M^{***} \xrightarrow{(\varphi_M)^*} M^*$$

equals id_{M^*}. Moreover, if A is a coherent ring, and if all $H^i(M)$ are coherent A-modules, then (A.11) is an isomorphism.

(c) Direct Image and Inverse Image. Let $f : X \to Y$ be a morphism of topological spaces, and let A be a ring on Y. If N is an A-module, then the inverse image $f^{-1}N$ is an $f^{-1}A$-module. Since $f^{-1} : \mathrm{Mod}(A) \to \mathrm{Mod}(f^{-1}A)$ is an exact functor, it induces a functor

$$f^{-1} : D(A) \to D(f^{-1}A).$$

The functor $f^{-1} : \mathrm{Mod}(A) \to \mathrm{Mod}(f^{-1}A)$ has a right adjoint functor

$$f_* : \mathrm{Mod}(f^{-1}A) \to \mathrm{Mod}(A),$$

where f_*M is defined by $\Gamma(U; f_*M) = \Gamma(f^{-1}U; M)$, and called the *direct image* of M. Namely, we have

$$f_*\mathcal{H}om_A(f^{-1}N, M) \cong \mathcal{H}om_A(N, f_*M) \qquad (N \in \mathrm{Mod}(A),\ M \in \mathrm{Mod}(f^{-1}A)).$$

Since f_* is left exact, we have its right derived functor

$$\mathbb{R}f_* : D^+(f^{-1}A) \to D^+(A).$$

A sheaf F on X is said to be *flabby* if $\Gamma(X; F) \to \Gamma(U; F)$ is surjective for every open subset U. Let

$$\mathrm{Fl}(A) = \{\, N \in \mathrm{Mod}(A)\,;\, N \text{ is flabby}\,\}.$$

Then $\mathrm{Fl}(f^{-1}A)$ is f_*-injective. Hence, for a flabby resolution $M \underset{\text{qis}}{\to} M'$ of $M \in D^+(f^{-1}A)$, we have

$$\mathbb{R}f_*M \simeq f_*M'.$$

The functor $f^{-1} : D^+(A) \to D^+(f^{-1}A)$ is left adjoint to $\mathbb{R}f_* : D^+(f^{-1}A) \to D^+(A)$, that is,

$$\text{(A.12)} \qquad \mathrm{Hom}_{D^+(f^{-1}A)}(f^{-1}N, M) \simeq \mathrm{Hom}_{D^+(A)}(N, \mathbb{R}f_*M) \qquad (N \in D^+(A),\ M \in D^+(f^{-1}A)).$$

In particular, we have $N \to \mathbb{R}f_*f^{-1}N$ and $f^{-1}\mathbb{R}f_*M \to M$. Since f_* preserves the flabbiness, we have the following lemma.

LEMMA A.50. *Let $f : X \to Y$ and $g : Y \to Z$ be continuous maps, and let A be a ring on Z. Then*

$$\mathbb{R}(g \circ f)_* \cong \mathbb{R}g_* \circ \mathbb{R}f_* : D^+(f^{-1}g^{-1}A) \to D^+(A).$$

For ${}_1N_2 \in D^-(A_1 \otimes A_2^{\mathrm{op}})$ and ${}_1M_3 \in D^+(f^{-1}A_1 \otimes f^{-1}A_3^{\mathrm{op}})$, we have the following formula in $D^+(A_2 \otimes A_3^{\mathrm{op}})$:

$$\mathbb{R}f_*\mathcal{H}om_{f^{-1}A_1}(f^{-1}{}_1N_2, {}_1M_3) \simeq \mathbb{R}\mathcal{H}om_{A_1}({}_1N_2, \mathbb{R}f_{*1}M_3).$$

(d) Direct Image with Proper Supports. In this subsection, we assume X and Y to be locally compact topological spaces (and hence Hausdorff). For $f : X \to Y$ and a sheaf F of modules on X, set

$$\Gamma(U; f_!F) = \{\, s \in \Gamma(f^{-1}U; F)\,;\, \mathrm{supp}(s) \text{ is proper over } U\,\}.$$

Clearly $f_!F \hookrightarrow f_*F$. We have the following isomorphism for every $y \in Y$:

$$\text{(A.13)} \qquad (f_!F)_y \xrightarrow{\sim} \Gamma_c(f^{-1}(y); F|_{f^{-1}(y)}),$$

where $\Gamma_c(f^{-1}(y); \bullet)$ denotes the module of sections with compact supports. For a ring A on Y, $f_!$ gives a left exact functor

$$f_! : \mathrm{Mod}(f^{-1}A) \to \mathrm{Mod}(A).$$

By (A.13), for a flat A^{op}-module N,

$$f_!(f^{-1}N \otimes_A M) \simeq N \otimes_A f_!M. \tag{A.14}$$

A sheaf F of modules on X is said to be *c-soft* if $\Gamma(X; F) \to \Gamma(K; F)$ are surjective for all compact subsets K. Let $\mathcal{S}(A)$ denote the set of c-soft A-modules. Then $\mathcal{S}(f^{-1}A)$ is $f_!$-injective. Hence we can write

$$\mathbb{R}f_! : D^+(f^{-1}A) \to D^+(A)$$

explicitly by using a c-soft resolution as

$$\mathbb{R}f_! \simeq f_!M', \quad M \underset{\mathrm{qis}}{\to} M' \quad (M' \text{ is } c\text{-soft}).$$

If the cohomological dimension of $f_!$ is finite (for instance, X is a finite-dimensional manifold), then we have

$$\mathbb{R}f_! : D^b(f^{-1}A) \to D^b(A).$$

LEMMA A.51 (Projection formula). *Let A_i be $\mathbb{C}$-algebras on Y $(i = 1, 2, 3)$. Suppose that the flat dimension of A_2 and the cohomological dimension of $f_!$ are finite. Then, for ${}_1N_2 \in D^b(A_1 \otimes A_2^{\mathrm{op}})$ and ${}_2M_3 \in D^b(f^{-1}A_2 \otimes f^{-1}A_3^{\mathrm{op}})$, we have*

$$\mathbb{R}f_!(f^{-1}{}_1N_2 \overset{\mathbb{L}}{\underset{f^{-1}A_2}{\otimes}} {}_2M_3) \simeq {}_1N_2 \overset{\mathbb{L}}{\underset{A_2}{\otimes}} \mathbb{R}f_!({}_2M_3)$$

in $D^b(A_1 \otimes A_3^{\mathrm{op}})$.

A.4. Symplectic Manifolds

In this section, let us briefly review symplectic manifolds in the framework of complex manifolds. Let X be a manifold of even dimension (say $2n$), and let θ be a closed 2-form on X. The pair (X, θ) is called a *symplectic manifold* if θ^n does not vanish anywhere. For a symplectic manifold X, as in § 2.1, we have $H : T^*X \xrightarrow{\sim} TX$, and we can define a Poisson bracket.

If a vector field v on X satisfying $L_v\theta = \theta$ is given, we call X a *homogeneous symplectic manifold*. Then, for $\omega = i_v\theta$, we have $\theta = d\omega$ by the equality $L_v = di_v + i_vd$, and $H(\omega) = -v$.

For any manifold Z, the cotangent bundle T^*Z is a homogeneous symplectic manifold, where the Euler vector field is taken as v. In coordinates, $\theta = \sum d\xi_i \wedge dx_i$, $v = \sum \xi_i \frac{\partial}{\partial \xi_i}$, and $\omega = \sum \xi_i dx_i$.

For every $p \in X$, the tangent space T_pX has a nondegenerate antisymmetric bilinear form $E(\bullet, \bullet)$, and hence $V = T_pX$ is a symplectic vector space.

Given a vector subspace W of a $(2n)$-dimensional symplectic vector space V, denote by $W^\perp$ its orthogonal complement, i.e.,

$$W^\perp = \{ v \in V ;\ E(v, W) = 0 \}.$$

Then $\dim W + \dim W^\perp = \dim V$, $(W^\perp)^\perp = W$, and $(W_1 \cap W_2)^\perp = W_1^\perp + W_2^\perp$. W is said to be *involutive* if $W^\perp \subset W$, *Lagrangian* if $W^\perp = W$, and *isotropic* if $W \subset W^\perp$. Then W satisfies $\dim W \geq n$, $\dim W = n$, and $\dim W \leq n$ respectively. If W is involutive, then $W/W^\perp$ has a natural symplectic structure.

Let Z be a closed analytic subset of a symplectic manifold X, Z_{reg} the set of nonsingular points of Z, and I_Z the ideal of holomorphic functions vanishing on Z. Then the following two conditions are equivalent:

(A.15) $\{I_Z, I_Z\} \subset I_Z$.

(A.16) T_pZ is an involutive subspace of T_pX for every $p \in Z_{\text{reg}}$.

If these two equivalent conditions are satisfied, Z is said to be *involutive*. If this is the case, then the dimension of every irreducible component of Z is greater than or equal to n.

The following two conditions are also equivalent:

(A.17) T_pZ is an isotropic subspace of T_pX for every $p \in Z_{\text{reg}}$.

(A.18) $f^*\theta = 0$ for every nonsingular manifold S and every morphism $f : S \to Z$.

If these two equivalent conditions are satisfied, Z is said to be *isotropic*. The dimension of an isotropic analytic set is less than or equal to n.

The following four conditions are also equivalent:

(A.19) Z is involutive and isotropic.

(A.20) T_pZ is Lagrangian for every $p \in Z_{\text{reg}}$.

(A.21) Z is involutive, and the dimension of each irreducible component equals n.

(A.22) Z is isotropic, and the dimension of each irreducible component equals n.

If these four equivalent conditions are satisfied, Z is said to be a *Lagrangian* analytic set.

Given a closed analytic subset S of X, we denote by T_S^*X the closure of $T_{S_{\mathrm{reg}}}^*X \subset \pi_X^{-1}(S_{\mathrm{reg}})$ in T^*X. Then T_S^*X is Lagrangian.

If X is a homogeneous symplectic manifold and Z is a homogeneous (i.e., $vI_Z \subset I_Z$) closed analytic subset, then Z is isotropic if and only if one of the following equivalent conditions is satisfied:

(A.23) $\omega|_{Z_{\mathrm{reg}}} = 0$.

(A.24) $f^*\omega = 0$ for every nonsingular manifold S and every morphism $f : S \to Z$.

(A.25) $f^*\omega = 0$ for every curve C and every morphism $f : C \to Z$.

We have the following lemma.

LEMMA A.52. *Let X be a manifold, and Λ a homogeneous Lagrangian submanifold of T^*X defined on a neighborhood of $p \in T^*X$. If the rank of $\Lambda \xrightarrow{\pi} X$ is constant, then $\Lambda = T_{\pi(\Lambda)}^*X$.*

PROOF. $Y = \pi(\Lambda)$ is a manifold in a neighborhood of $x = \pi(p)$. Take a coordinate system $(x_1, \ldots, x_n)$ of X so that $Y = \{x_{p+1} = \cdots = x_n = 0\}$. Then

$$\omega_X|_\Lambda = \sum_{i=1}^{p} \xi_i dx_i = 0 .$$

Since $dx_i|_\Lambda$ $(1 \le i \le p)$ are linearly independent, we have $\xi_1 = \cdots = \xi_p = 0$ on Λ. Hence $\Lambda \subset T_Y^*X$. Since their dimensions are equal, we obtain $\Lambda = T_Y^*X$. □

More generally, we have the following.

PROPOSITION A.53. *Let $\Lambda \subset T^*X$ be a closed homogeneous Lagrangian analytic subset, and let $Y := \pi_X(\Lambda)$. Then $T_Y^*X \subset \Lambda$, where $T_Y^*X = \overline{T_{Y_{\mathrm{reg}}}^*X}$. Moreover, $\pi_X\left(\overline{\Lambda \setminus T_Y^*X}\right)$ is a nowhere dense subset of Y.*

PROOF. Since the rank of $\Lambda \to X$ is constant at generic points of Λ, we obtain $T_Y^*X \subset \Lambda$ by applying Lemma A.52. Put $Y' = \pi_X\left(\overline{\Lambda \setminus T_Y^*X}\right)$. Then $\overline{\Lambda \setminus T_Y^*X} \supset T_{Y'}^*X$. Hence $Y' \subset Y$ is a nowhere dense subset of Y. Note that the dimension of $T_Y^*X \cap \overline{\Lambda \setminus T_Y^*X}$ is at most $\dim X - 1$. □

By this proposition, for a closed homogeneous Lagrangian analytic subset Λ, there exist locally finite closed analytic subsets $\{Y_j\}$ of X such that

$$\Lambda = \bigcup_j T^*_{Y_j} X.$$

Given a morphism $f : X \to Y$ of manifolds, let $f_\pi : X \underset{Y}{\times} T^*Y \to T^*Y$ and $f_d : X \underset{Y}{\times} T^*Y \to T^*X$ denote its associated morphisms (cf. §4.4).

PROPOSITION A.54. *Let S be a closed homogeneous isotropic analytic subset of T^*X. If $f_d^{-1}S \to T^*Y$ is proper, then $f_\pi f_d^{-1}S$ is also isotropic.*

PROOF. Since $f_\pi f_d^{-1}S$ is a closed homogeneous analytic subset of T^*Y, it suffices to show that $g^*\omega_Y = 0$ for every curve C and every $g : C \to f_\pi f_d^{-1}S$, where ω_Y is the canonical 1-form on T^*Y. For any $p \in C$, there exist a curve C' in $C \times_{T^*Y} f_d^{-1}S$ and a point $p' \in C'$ such that $\psi : C' \to C$ satisfies $\psi(p') = p$ and sends neighborhoods of p' to neighborhoods of p. Hence it suffices to show that $\psi^* g^* \omega_Y = 0$. This is obtained from the equality $f_\pi^*\omega_Y = f_d^*\omega_X$ and the commutative diagram

$$\begin{tikzcd}
C' \arrow[r] \arrow[d, "\psi"'] & f_d^{-1}S \arrow[r] \arrow[d] & X \underset{Y}{\times} T^*Y \arrow[r, "f_d"] \arrow[d, "f_\pi"] & T^*X \\
C \arrow[r] & f_\pi f_d^{-1}S \arrow[r] & T^*Y . &
\end{tikzcd}$$

□

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Index

Index of Notations