Mathematical Economics

Convex Analysis (II)

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Concave and Convex Functions

For any nonempty convex set $X \subseteq \mathbb{R}^n$, a function f: $X \to \mathbb{R}$ is

- ***concave*** if $\forall x, y \in X, \forall \lambda \in [0, 1]$,
$$f(\lambda x + (1-\lambda)y) \geq \lambda f(x) + (1-\lambda)f(y).$$
- ***convex*** if $\forall x, y \in X, \forall \lambda \in [0, 1]$,
$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y).$$
- ***strictly concave*** if $\forall x, y \in X$ with $x \neq y$, $\forall \lambda \in (0, 1)$,
$$f(\lambda x + (1-\lambda)y) > \lambda f(x) + (1-\lambda)f(y).$$
- ***strictly convex*** if $\forall x, y \in X$ with $x \neq y$, $\forall \lambda \in (0, 1)$,
$$f(\lambda x + (1-\lambda)y) < \lambda f(x) + (1-\lambda)f(y).$$
- Obviously, f is concave iff $-f$ is convex, and f is strictly concave iff $-f$ is strictly convex.

Quasi-concave and Quasi-convex Functions

For any nonempty convex set $X \subseteq \mathbb{R}^n$, a function $f: X \to \mathbb{R}$ is

- ***quasi-concave*** if $\forall x, y \in X$, $\forall \lambda \in [0, 1]$,

$$f(\lambda x + (1-\lambda)y) \geq \min\{f(x), f(y)\};$$

- ***quasi-convex*** if $\forall x, y \in X$, $\forall \lambda \in [0, 1]$,

$$f(\lambda x + (1-\lambda)y) \leq \max\{f(x), f(y)\};$$

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- ***quasi-convex*** if $\forall x, y \in X$, $\forall \lambda \in [0, 1]$,
$$f(\lambda x + (1-\lambda)y) \leq \max\{f(x), f(y)\};$$
- ***strictly quasi-concave*** if $\forall x, y \in X$ with $x \neq y$, $\forall \lambda \in (0, 1)$,
$$f(\lambda x + (1-\lambda)y) > \min\{f(x), f(y)\};$$
- ***strictly quasi-convex*** if $\forall x, y \in X$ with $x \neq y$, $\forall \lambda \in (0, 1)$,
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$$f(\lambda x + (1-\lambda)y) < \max\{f(x), f(y)\}.$$

Obviously, f is quasi-concave iff $-f$ is quasi-convex, and f is strictly quasi-concave iff $-f$ is strictly quasi-convex.

Note that a (strictly) concave (convex) function must be (strictly) quasi-concave (quasi-convex), but not vice versa.

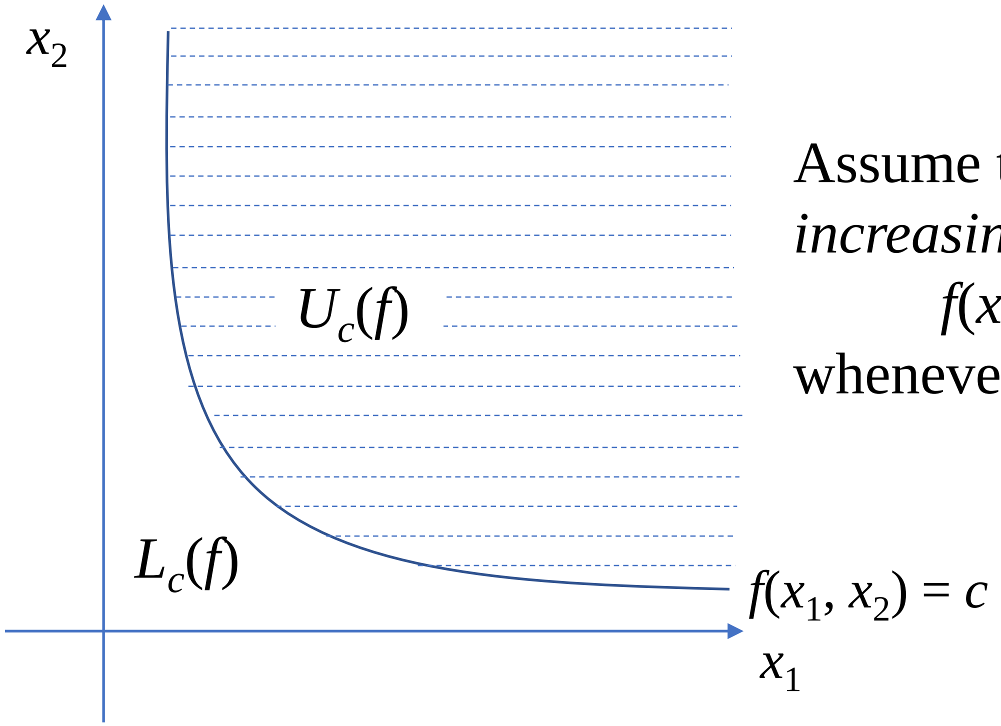
For any set $X \subseteq \mathbb{R}^n$ and function $f: X \longrightarrow \mathbb{R}$,

- the ***upper-contour set*** of f for $c \in \mathbb{R}$ is defined to be

$$U_c(f) = \{x \in X \mid f(x) \geq c\};$$

- the ***lower-contour set*** of f for $c \in \mathbb{R}$ is defined to be

$$L_c(f) = \{x \in X \mid f(x) \leq c\}.$$



Assume that f is *monotonically increasing* in the sense that

$$f(x_1, x_2) \geq f(a_1, a_2)$$

whenever $(x_1, x_2) \geq (a_1, a_2)$.

Theorem 7: For any convex set $X \subseteq \mathbb{R}^n$, a function $f\colon X \to \mathbb{R}$ is

- quasi-concave iff $\forall c \in \mathbb{R}$, the upper-contour set $U_c(f)$ is convex;
- quasi-convex iff $\forall c \in \mathbb{R}$, the lower-contour set $L_c(f)$ is convex.

Theorem 7: For any convex set $X \subseteq \mathbb{R}^n$, a function $f: X \to \mathbb{R}$ is
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- quasi-convex iff $\forall c \in \mathbb{R}$, the lower-contour set $L_c(f)$ is convex.

- If $f: X \to \mathbb{R}$ is quasi-concave, then $\forall x, y \in X$ with $f(x) \geq c$ and $f(y) \geq c$, $\forall \lambda \in [0, 1]$,
$$f(\lambda x + (1-\lambda)y) \geq \min\{f(x), f(y)\} \geq c.$$
- On the other hand, suppose that $\forall c \in \mathbb{R}$, $\{x \in X \mid f(x) \geq c\}$ is convex. Note that $\forall x, y \in X$, $f(x) \geq \min\{f(x), f(y)\}$ and $f(y) \geq \min\{f(x), f(y)\}$, and hence $\forall \lambda \in [0, 1]$, $f(\lambda x + (1-\lambda)y) \geq \min\{f(x), f(y)\}$.

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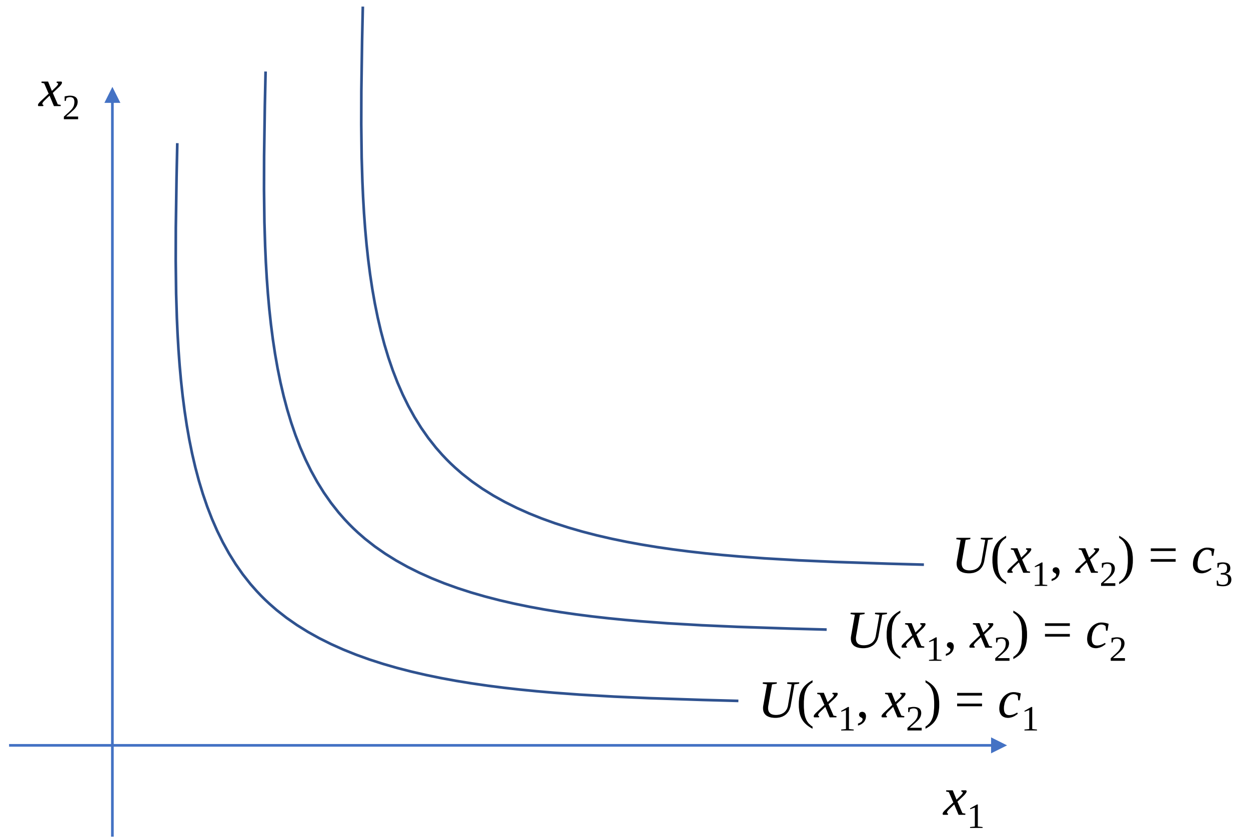

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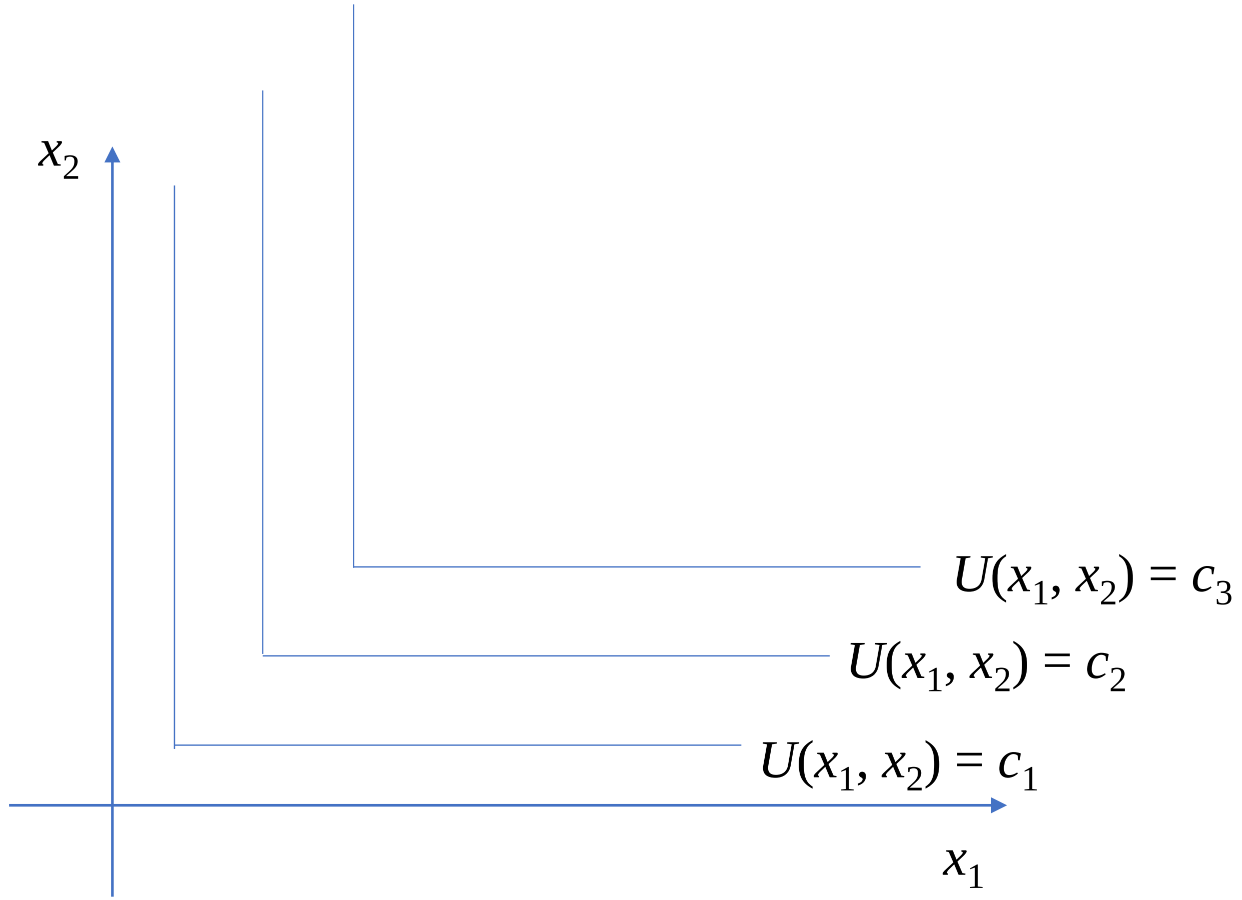

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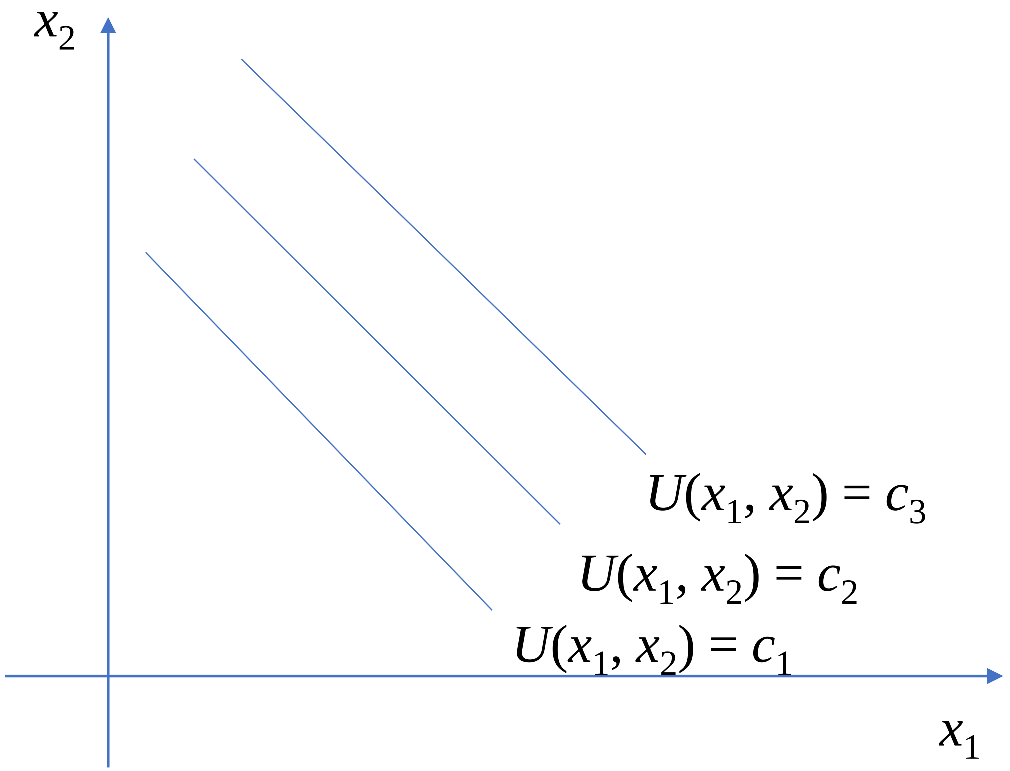

$U(x_1, x_2) = x_1 + x_2$

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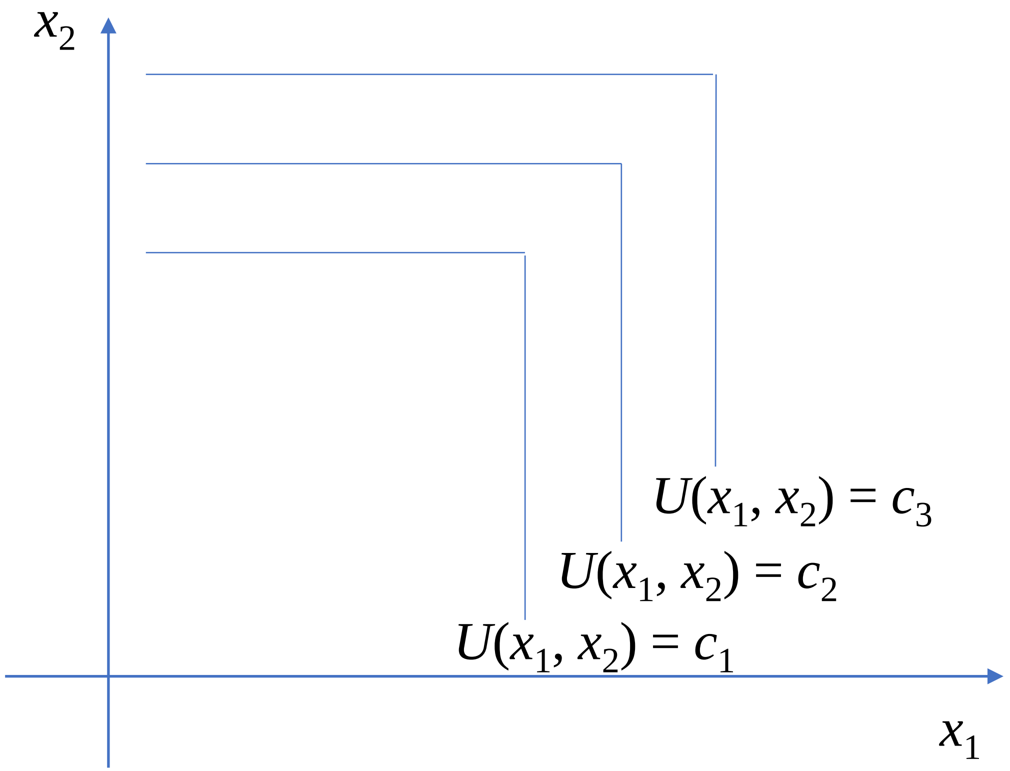

$U(x_1, x_2) = \max\{x_1, x_2\}$

Theorem 8: Let $X \subseteq \mathbb{R}^n$ be open and convex and f: $X \longrightarrow \mathbb{R}$ continuously differentiable. Then f is

- quasi-concave iff $Df(x) \cdot (y - x) \geq 0$ whenever $f(y) \geq f(x)$ for all $x, y \in X$;
- quasi-convex iff $Df(x) \cdot (y - x) \leq 0$ whenever $f(y) \leq f(x)$ for all $x, y \in X$;
- strictly quasi-concave if $Df(x) \cdot (y - x) > 0$ whenever $f(y) \geq f(x)$ for all $x, y \in X$ with $x \neq y$;
- strictly quasi-convex if $Df(x) \cdot (y - x) < 0$ whenever $f(y) \leq f(x)$ for all $x, y \in X$ with $x \neq y$.

Theorem 8: Let $X \subseteq \mathbb{R}^n$ be open and convex and $f\colon X \to \mathbb{R}$ continuously differentiable. Then f is

- quasi-concave iff $Df(x) \cdot (y - x) \geq 0$ whenever $f(y) \geq f(x)$ for all $x, y \in X$;
- quasi-convex iff $Df(x) \cdot (y - x) \leq 0$ whenever $f(y) \leq f(x)$ for all $x, y \in X$;
- strictly quasi-concave if $Df(x) \cdot (y - x) > 0$ whenever $f(y) \geq f(x)$ for all $x, y \in X$ with $x \neq y$;
- strictly quasi-convex if $Df(x) \cdot (y - x) < 0$ whenever $f(y) \leq f(x)$ for all $x, y \in X$ with $x \neq y$.

Example: $f(x, y) = xy$, $x > 0$, $y > 0$.

- Let $f(x, y) \geq f(a, b)$, i.e., $xy \geq ab$. Then

$$\begin{aligned}
Df(a, b) \cdot ((x, y) - (a, b)) &= (b, a) \cdot ((x, y) - (a, b)) \\
&= (b, a) \cdot (x - a, y - b) \\
&= b(x - a) + a(y - b) \\
&= bx + ay - 2ab \\
&\geq 2\sqrt{xyab} - 2ab \\
&\geq 2\sqrt{abab} - 2ab \\
&= 0.
\end{aligned}$$

Theorem 9: Let $X \subseteq \mathbb{R}^n$ be open and convex, and f: $X \rightarrow \mathbb{R}$ twice continuously differentiable.

(i) If f is quasi-concave, then $(-1)^k|\bar{H}_k(x)| \geq 0$ $\forall x \in X$ $\forall k \in \{1, 2, \cdots, n\}$;

(ii) If f is quasi-convex, then $|\bar{H}_k(x)| \leq 0$ $\forall x \in X$ $\forall k \in \{1, 2, \cdots, n\}$;

(iii) If $(-1)^k|\bar{H}_k(x)| > 0$ $\forall x \in X$ $\forall k \in \{1, 2, \cdots, n\}$, then f is strictly quasi-concave;

(iv) If $|\bar{H}_k(x)| < 0$ $\forall x \in X$ $\forall k \in \{1, 2, \cdots, n\}$, then f is strictly quasi-convex.

$\forall x \in X$, $\forall k \in \{1, 2, \cdots, n\}$, let

$$\bar{H}_k(x) = \begin{pmatrix} 0 & \frac{\partial f}{\partial x_1} & \cdots & \frac{\partial f}{\partial x_k} \\ \frac{\partial f}{\partial x_1} & \frac{\partial^2 f}{\partial x_1^2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_k} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial f}{\partial x_k} & \frac{\partial^2 f}{\partial x_k \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_k^2} \end{pmatrix}$$

and $\bar{H}(x) = \bar{H}_n(x)$.

Theorem 9: Let $X \subseteq \mathbb{R}^n$ be open and convex, and f: $X \to \mathbb{R}$ twice continuously differentiable.

(i) If f is quasi-concave, then $(-1)^k|\overline{H}_k(x)| \geq 0$ $\forall x \in X$ $\forall k \in \{1, 2, \cdots, n\}$;

(ii) If f is quasi-convex, then $|\overline{H}_k(x)| \leq 0$ $\forall x \in X$ $\forall k \in \{1, 2, \cdots, n\}$;

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and $\overline{H}(x) = \overline{H}_n(x)$.

Example: $f(x, y) = xy$, $x > 0$, $y > 0$.

- $\overline{H}(x,y) = \begin{pmatrix} 0 & y & x \\ y & 0 & 1 \\ x & 1 & 0 \end{pmatrix}$.
- $|\overline{H}_1(x,y)| = \begin{vmatrix} 0 & y \\ y & 0 \end{vmatrix} = -y^2 < 0$, $|\overline{H}_2(x,y)| = 2xy > 0$.

Theorem 10: Let $X \subseteq \mathbb{R}^n$ and $A \subseteq \mathbb{R}$ be convex.

(i) If $f\colon X \to \mathbb{R}$ is quasi-concave and $g\colon A \to \mathbb{R}$ increasing with $f(X) \subseteq A$, then $g \circ f$ is quasi-concave; if f is quasi-concave and g decreasing, then $g \circ f$ is quasi-convex;

(ii) If $f\colon X \to \mathbb{R}$ is quasi-convex and $g\colon A \to \mathbb{R}$ increasing with $f(X) \subseteq A$, then $g \circ f$ is quasi-convex; if f is quasi-convex and g decreasing, then $g \circ f$ is quasi-concave;

(iii) If $f\colon X \to \mathbb{R}$ is strictly quasi-concave and $g\colon A \to \mathbb{R}$ strictly increasing with $f(X) \subseteq A$, then $g \circ f$ is strictly quasi-concave;

(iv) If $f\colon X \to \mathbb{R}$ is strictly quasi-convex and $g\colon A \to \mathbb{R}$ strictly increasing with $f(X) \subseteq A$, then $g \circ f$ is strictly quasi-convex;

(v) If $f\colon X \to \mathbb{R}$ is concave and $g\colon A \to \mathbb{R}$ increasing and concave with $f(X) \subseteq A$, then $g \circ f$ is concave;

(vi) If $f\colon X \to \mathbb{R}$ is convex and $g\colon A \to \mathbb{R}$ increasing and convex with $f(X) \subseteq A$, then $g \circ f$ is convex.

- See Exercises 1 and 3.

Theorem 11: Let $X \subseteq \mathbb{R}^n$ be convex.

(i) *Nonnegative linear combination of concave functions must be concave*, i.e., for any concave functions $f_1, f_2, \cdots, f_n$ defined on X, $\forall a_1, a_2, \cdots a_n \in \mathbb{R}_+$, $\sum_{k=1}^{n} a_k f_k$ is concave.

(ii) *Nonnegative linear combination of convex functions must be convex*, i.e., for any convex functions $f_1, f_2, \cdots, f_n$ defined on X, $\forall a_1, a_2, \cdots a_n \in \mathbb{R}_+$, $\sum_{k=1}^{n} a_k f_k$ is convex.

- See Exercise 1.

Theorem 12: Let $X \subseteq \mathbb{R}^n$ be nonempty and convex.

- A strictly concave function cannot be convex, and a strictly convex function cannot be concave (unless $|X| = 1$).
- Let $n = 1$. If $f: X \to \mathbb{R}$ is increasing or decreasing, then f is both quasi-concave and quasi-convex; if f is strictly increasing or strictly decreasing, then f is both strictly quasi-concave and strictly quasi-convex.

- See Exercise 1.

Example 3:

(1) $f(x) = x$, $g(x) = -x$, $h(x) = x^3$.

(2) $u(x, y) = x^a y^b$, $x > 0$, $y > 0$, $a > 0$, $b > 0$.

(3) $f(x, y) = (x - 1)^2(y - 1)^2$.

Example 3:

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(2) $u(x, y) = x^a y^b$, $x > 0$, $y > 0$, $a > 0$, $b > 0$.

(3) $f(x, y) = (x - 1)^2(y - 1)^2$.

(1) f and g are concave, convex, strictly quasi-concave, and strictly quasi-convex, but neither strictly concave nor strictly convex; h is strictly quasi-convex and strictly quasi-convex, but neither concave nor convex.

Example 3:

(1) $f(x) = x$, $g(x) = -x$, $h(x) = x^3$.

(2) $u(x, y) = x^a y^b$, $x > 0$, $y > 0$, $a > 0$, $b > 0$.

(3) $f(x, y) = (x - 1)^2(y - 1)^2$.

(2) $u(x, y)$ is strictly quasi-concave since $\bar{H}(x, y) =$

$$\begin{pmatrix} 0 & ax^{a-1}y^b & bx^a y^{b-1} \\ ax^{a-1}y^b & a(a-1)x^{a-2}y^b & abx^{a-1}y^{b-1} \\ bx^a y^{b-1} & abx^{a-1}y^{b-1} & b(b-1)x^a y^{b-2} \end{pmatrix},$$

$|\bar{H}_1(x, y)| = -a^2 x^{2a-2} y^{2b} < 0$,

$|\bar{H}_2(x, y)| = (a + b)abx^{3a-2}y^{3b-2} > 0$.

Example 3:

(1) $f(x) = x$, $g(x) = -x$, $h(x) = x^3$.

(2) $u(x, y) = x^a y^b$, $x > 0$, $y > 0$, $a > 0$, $b > 0$.

(3) $f(x, y) = (x - 1)^2(y - 1)^2$.

(3) $\bar{H}(x, y) =$

$$\begin{pmatrix} 0 & 2(x-1)(y-1)^2 & 2(x-1)^2(y-1) \\ 2(x-1)(y-1)^2 & 2(y-1)^2 & 4(x-1)(y-1) \\ 2(x-1)^2(y-1) & 4(x-1)(y-1) & 2(x-1)^2 \end{pmatrix},$$

$|\bar{H}_1(x, y)| = -4(x-1)^2(y-1)^4 \le 0$,

$|\bar{H}_2(x, y)| = 16(x-1)^4(y-1)^4 \ge 0$.

However, $f(x, y)$ is not quasi-concave since

$f(0.5(0, 0) + 0.5(2, 2)) = f(1, 1) = 0 < 1 = \min\{f(0, 0), f(2, 2)\}$.

- Even if the domain is restricted to $\mathbb{R}^2_{++}$, we have

$$f(0.5(0.2, 0.2) + 0.5(1.8, 1.8)) = f(1, 1) = 0$$
$$< 0.64^2 = \min\{f(0.2, 0.2), f(1.8, 1.8)\}.$$

Convex Preference Relations

Let $X \subseteq \mathbb{R}^n$ be a set of alternatives and $\succsim$ a rational preference relation on X. We say that $\succsim$ is

- ***convex*** if $\forall x \in X$, $\{y \in X \mid y \succsim x\}$ is convex; i.e., $\forall x, y, z \in X$, if $y \succsim x$ and $z \succsim x$, then $\forall t \in [0, 1]$, $ty + (1-t)z \succsim x$;
- ***strictly convex*** if $\forall x, y, z \in X$ such that $y \succsim x$, $z \succsim x$, and $y \neq z$, we have $ty + (1-t)z \succ x$ $\forall t \in (0, 1)$.
- ***weakly convex*** if $\forall x, y \in X$ with $x \succ y$, we have $tx + (1-t)y \succ y$ $\forall t \in (0, 1)$.
- ***explicitly convex*** if it is both convex and weakly convex.
- ***weakly monotone*** if $\forall x, y \in X$, $x > y$ implies $x \succsim y$;
- ***monotone*** if $\forall x, y \in X$, $x \geq y$ implies $x \succsim y$;
- ***strictly monotone*** if $\forall x, y \in X$, $x > y$ implies $x \succ y$;
- ***strongly monotone*** if $\forall x, y \in X$, $x \geq y$ and $x \neq y$ imply $x \succ y$.

Theorem 13: Let $X \subseteq \mathbb{R}^n$ be convex. If $\succsim$ is represented by a utility function $u: X \rightarrow \mathbb{R}$, then

- $\succsim$ is convex iff u is quasi-concave,
- $\succsim$ is strictly convex iff u is strictly quasi-concave.

- See Exercise 1.

Homogenous and Homothetic Functions:

Let $X \subseteq \mathbb{R}^n$ be a cone. For any $k \in \mathbb{R}$, we say that

- $f: X \to \mathbb{R}$ is ***homogeneous*** of degree k if for any $x \in X$ and $r > 0$, $f(rx) = r^k f(x)$;
- $h: X \to \mathbb{R}$ is ***homothetic*** if h is a strictly increasing transformation of a homogeneous function; that is, $h = g \circ f$, where $g: f(X) \to \mathbb{R}$ is strictly increasing and $f: X \to \mathbb{R}$ homogeneous of degree k.
- **Example:** The utility function $v(x, y) = x^3y^3 + e^{xy} + 10$ is homothetic, since the utility function $u(x, y) = xy$ is homogeneous of degree 2 and $g(z) = z^3 + e^z + 10$ is an increasing transformation. Note that u and v represent the same preference.

- A set $X \subseteq \mathbb{R}^n$ is a ***cone*** if $\forall x \in X$, $\forall r > 0$, $rx \in X$.

Theorem 14: Let $X \subseteq \mathbb{R}^n$ be a convex cone. If f: $X \to \mathbb{R}$ is quasi-concave and homogenous of degree $k \in (0, 1]$ such that $f(0) = 0$ (if $0 \in X$) and $f(x) > 0$ $\forall x \in X \setminus \{0\}$, then f is concave.

- See Exercise 1.

Example 4:

(1) The Cobb-Douglas Utility Function:

$$u(x_1, \cdots, x_n) = a\prod_{i=1}^{n} x_i^{b_i},$$

where $x_1, \cdots, x_n, a, b_1, \cdots, b_n > 0$.

(2) The Generalized CES (Constant Elasticity of Substitution) Utility Function:

$$u(x_1, \cdots, x_n) = a\left(\sum_{i=1}^{n} b_i x_i^{-\rho}\right)^{-\frac{c}{\rho}},$$

where $x_1, \cdots, x_n, a, b_1, \cdots, b_n, c > 0$, and $\rho \neq 0$.

Example 4:

(1) The Cobb-Douglas Utility Function:

$$u(x_1, \cdots, x_n) = a\prod_{i=1}^{n} x_i^{b_i},$$

where $x_1, \cdots, x_n, a, b_1, \cdots, b_n > 0$.

$$H(x) = u(x)\begin{pmatrix} \frac{b_1(b_1-1)}{x_1^2} & \frac{b_1 b_2}{x_1 x_2} & \cdots & \frac{b_1 b_n}{x_1 x_n} \\ \frac{b_1 b_2}{x_1 x_2} & \frac{b_2(b_2-1)}{x_2^2} & \cdots & \frac{b_2 b_n}{x_2 x_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{b_1 b_n}{x_1 x_n} & \frac{b_2 b_n}{x_2 x_n} & \cdots & \frac{b_n(b_n-1)}{x_n^2} \end{pmatrix}.$$

$\forall k = 1, 2, \cdots, n$, let $s_k = \sum_{i=1}^{k} b_i$, and then $|H_k(x)| =$

$$(u(x))^k \frac{b_1 b_2 \cdots b_k}{(x_1 x_2 \cdots x_k)^2}\begin{vmatrix} b_1 - 1 & b_1 & \cdots & b_1 \\ b_2 & b_2 - 1 & \cdots & b_2 \\ \cdots & \cdots & \cdots & \cdots \\ b_k & b_k & \cdots & b_k - 1 \end{vmatrix} =$$

$$(u(x))^k \frac{b_1 b_2 \cdots b_k}{(x_1 x_2 \cdots x_k)^2}\begin{vmatrix} s_k - 1 & s_k - 1 & \cdots & s_k - 1 \\ b_2 & b_2 - 1 & \cdots & b_2 \\ \cdots & \cdots & \cdots & \cdots \\ b_k & b_k & \cdots & b_k - 1 \end{vmatrix} =$$

$$(u(x))^k \frac{b_1 b_2 \cdots b_k (s_k - 1)}{(x_1 x_2 \cdots x_k)^2}\begin{vmatrix} 1 & 1 & \cdots & 1 \\ b_2 & b_2 - 1 & \cdots & b_2 \\ \cdots & \cdots & \cdots & \cdots \\ b_k & b_k & \cdots & b_k - 1 \end{vmatrix} =$$

Example 4:

(1) The Cobb-Douglas Utility Function:

$$u(x_1, \cdots, x_n) = a\prod_{i=1}^{n} x_i^{b_i},$$

where $x_1, \cdots, x_n, a, b_1, \cdots, b_n > 0$.

$$H(x) = u(x)\begin{pmatrix} \frac{b_1(b_1-1)}{x_1^2} & \frac{b_1 b_2}{x_1 x_2} & \cdots & \frac{b_1 b_n}{x_1 x_n} \\ \frac{b_1 b_2}{x_1 x_2} & \frac{b_2(b_2-1)}{x_2^2} & \cdots & \frac{b_2 b_n}{x_2 x_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{b_1 b_n}{x_1 x_n} & \frac{b_2 b_n}{x_2 x_n} & \cdots & \frac{b_n(b_n-1)}{x_n^2} \end{pmatrix}.$$

$\forall k = 1, 2, \cdots, n$, let $s_k = \sum_{i=1}^{k} b_i$, and then $|H_k(x)| =$

$$(u(x))^k\frac{b_1 b_2 \cdots b_k}{(x_1 x_2 \cdots x_k)^2}\begin{vmatrix} b_1 - 1 & b_1 & \cdots & b_1 \\ b_2 & b_2 - 1 & \cdots & b_2 \\ \cdots & \cdots & \cdots & \cdots \\ b_k & b_k & \cdots & b_k - 1 \end{vmatrix} =$$

$$(u(x))^k\frac{b_1 b_2 \cdots b_k(s_k-1)}{(x_1 x_2 \cdots x_k)^2}\begin{vmatrix} 1 & 0 & \cdots & 0 \\ b_2 & -1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ b_k & 0 & \cdots & -1 \end{vmatrix} =$$

$$(-1)^{k-1}(s_k - 1)(u(x))^k\frac{b_1 b_2 \cdots b_k}{(x_1 x_2 \cdots x_k)^2},$$

having the same sign as $(-1)^k$ if $\sum_{i=1}^{n} b_i < 1$.

Hence u is strictly concave if $\sum_{i=1}^{n} b_i < 1$.

Example 4:

(1) The Cobb-Douglas Utility Function:

$$u(x_1, \cdots, x_n) = a\prod_{i=1}^{n} x_i^{b_i},$$

where $x_1, \cdots, x_n, a, b_1, \cdots, b_n > 0$.

- u is strictly concave if $\sum_{i=1}^{n} b_i < 1$, and not concave if $\sum_{i=1}^{n} b_i > 1$.
- Note that $\ln u(x) = \ln a + \sum_{i=1}^{n} b_i \ln x_i$, which is strictly concave since

$$D^2(\ln u(x)) = \begin{pmatrix} -\frac{b_1}{x_1^2} & 0 & \cdots & 0 \\ 0 & -\frac{b_2}{x_2^2} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & -\frac{b_n}{x_n^2} \end{pmatrix}$$

is negative definite. Hence $u(x) = e^{\ln u(x)}$ is strictly quasi-concave.

Example 4:

(1) The Cobb-Douglas Utility Function:

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where $x_1, \cdots, x_n, a, b_1, \cdots, b_n > 0$.

- u is strictly concave if $\sum_{i=1}^{n} b_i < 1$, and not concave if $\sum_{i=1}^{n} b_i > 1$.
- u is strictly quasi-concave.
- When $\sum_{i=1}^{n} b_i = 1$, note that u is quasi-concave and homogeneous with degree 1 s.t. $u(x) > 0$ for all $x \in \mathbb{R}^n_{++}$, implying that u is concave.

Example 4:

(2) The Generalized CES (Constant Elasticity of Substitution) Utility Function:

$$u(x_1, \cdots, x_n) = a\left(\sum_{i=1}^{n} b_i x_i^{-\rho}\right)^{-\frac{c}{\rho}},$$

where $x_1, \cdots, x_n, a, b_1, \cdots, b_n, c > 0$, and $\rho \neq 0$.

- If $\rho = -1$, then v is linear.
- Note that $D^2(v(x)) =$

$$\begin{pmatrix} \frac{b_1\rho(\rho+1)}{x_1^{\rho+2}} & 0 & \cdots & 0 \\ 0 & \frac{b_2\rho(\rho+1)}{x_2^{\rho+2}} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \frac{b_n\rho(\rho+1)}{x_n^{\rho+2}} \end{pmatrix}.$$

- Let $v = \sum_{i=1}^{n} b_i x_i^{-\rho}$, then $u = av^{-\frac{c}{\rho}}$.
- If $\rho \leq -1$, then v is quasi-convex (Indeed, v is convex) and u increasing in v. Thus u is quasi-convex.
- If $-1 \leq \rho < 0$, then v is quasi-concave (Indeed, v is concave) and u increasing in v. Thus u is quasi-concave.
- If $\rho > 0$, then v is quasi-convex (Indeed, v is convex) and u decreasing in v. Thus u is quasi-concave.

Theorem 15: Let $X \subseteq \mathbb{R}^n$ be a cone. If $f\colon X \to \mathbb{R}$ is differentiable and homogeneous of degree k, then $\forall i \in \{1, 2, \cdots, n\}$, the partial derivative function $\partial f/\partial x_i$ is homogenous of degree $k - 1$.

Theorem 15: Let $X \subseteq \mathbb{R}^n$ be a cone. If f: $X \to \mathbb{R}$ is differentiable and homogeneous of degree k, then $\forall i \in \{1, 2, \cdots, n\}$, the partial derivative function $\partial f/\partial x_i$ is homogenous of degree $k - 1$.

- Differentiating $f(rx^*) = r^k f(x^*)$ with respect to x_i^*, we have

$$r\,\frac{\partial f(rx^*)}{\partial x_i} = r^k\,\frac{\partial f(x^*)}{\partial x_i},$$

i.e.,

$$\frac{\partial f(rx^*)}{\partial x_i} = r^{k-1}\,\frac{\partial f(x^*)}{\partial x_i}.$$

Theorem 16 (Euler's Formula): Let $X \subseteq \mathbb{R}^n$ be a cone. If $f: X \to \mathbb{R}$ is differentiable and homogeneous of degree k, then $\forall x^* \in X$,

$$kf(x^*) = \sum_{i=1}^{n} \frac{\partial f(x^*)}{\partial x_i} x_i^* .$$

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$$\sum_{i=1}^{n} \frac{\partial f(rx^*)}{\partial x_i} x_i^* = kr^{k-1} f(x^*),$$

and let $r = 1$.

Theorem 16 (Euler's Formula): Let $X \subseteq \mathbb{R}^n$ be a cone. If $f: X \to \mathbb{R}$ is differentiable and homogeneous of degree k, then $\forall x^* \in X$,

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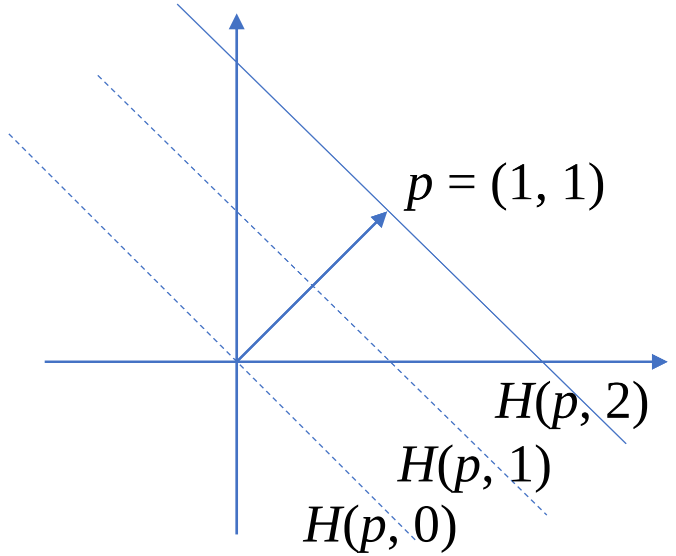
$$\sum_{i=1}^{n} \frac{\partial f(rx^*)}{\partial x_i} x_i^* = kr^{k-1} f(x^*),$$
and let $r = 1$.
- **Example:** Let the production function $F(K, L)$ be differentiable and homogeneous of degree 1, then
$$F(K, L) = \frac{\partial F(K,L)}{\partial K} K + \frac{\partial F(K,L)}{\partial L} L.$$

Supporting and Separating Hyperplanes

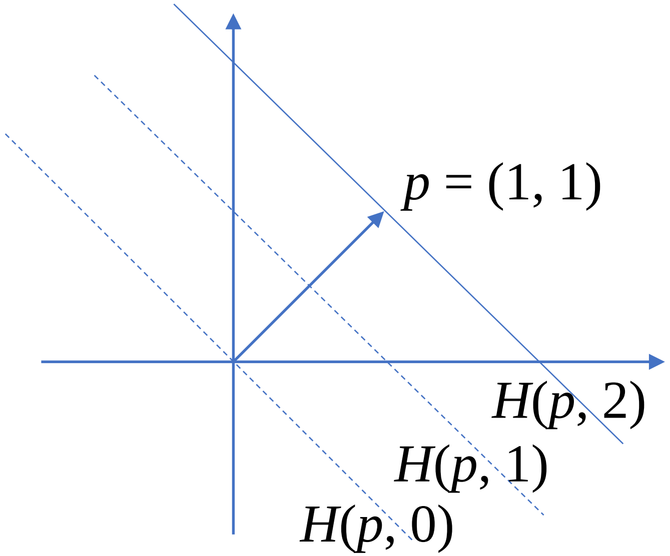
Let $c \in \mathbb{R}$ and $p \in \mathbb{R}^n \backslash \{0\}$.

- The set $H(p, c) = \{x \in \mathbb{R}^n \mid p \cdot x = c\}$ is a ***hyperplane*** in $\mathbb{R}^n$ determined by (p, c), where p is called the ***normal vector*** to the hyperplane $H(p, c)$.
- **Example:** Let $p = (1, 1)$ and $c = 2$. Then

$$H(p, c) = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 + x_2 = 2\}.$$



Supporting and Separating Hyperplanes



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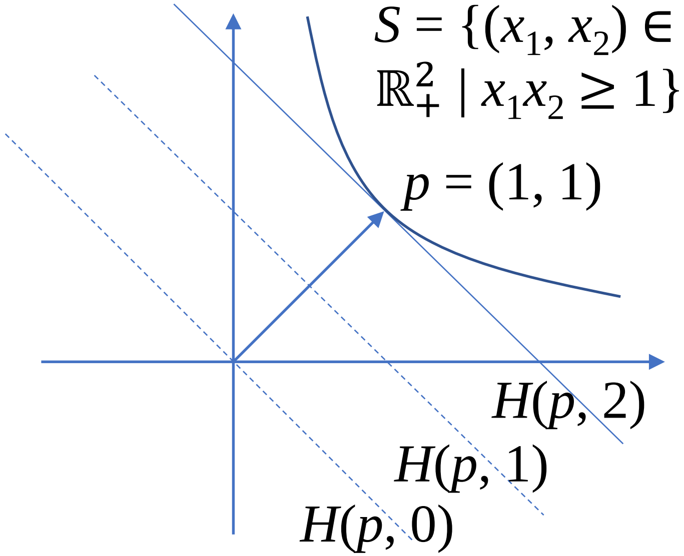
$$H(p, c) = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 + x_2 = 2\}.$$
- A ***closed half-space*** determined by $H(p, c)$ is either the set $\{x \in \mathbb{R}^n \mid p \cdot x \leq c\}$ or the set $\{x \in \mathbb{R}^n \mid p \cdot x \geq c\}$.
- An ***open half-space*** determined by $H(p, c)$ is either the set $\{x \in \mathbb{R}^n \mid p \cdot x < c\}$ or the set $\{x \in \mathbb{R}^n \mid p \cdot x > c\}$.

Supporting and Separating Hyperplanes



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- An ***open half-space*** determined by $H(p, c)$ is either the set $\{x \in \mathbb{R}^n \mid p \cdot x < c\}$ or the set $\{x \in \mathbb{R}^n \mid p \cdot x > c\}$.
- A ***bounding hyperplane*** $H(p, c)$ to a set $S \subseteq \mathbb{R}^n$ is a hyperplane such that $S \subseteq \{x \in \mathbb{R}^n \mid p \cdot x \leq c\}$ or $S \subseteq \{x \in \mathbb{R}^n \mid p \cdot x \geq c\}$, i.e., S lies entirely on one side of $H(p, c)$.
- A ***supporting hyperplane*** $H(p, c)$ for a set $S \subseteq \mathbb{R}^n$ is a bounding hyperplane to S that shares a point in common with the boundary of S.

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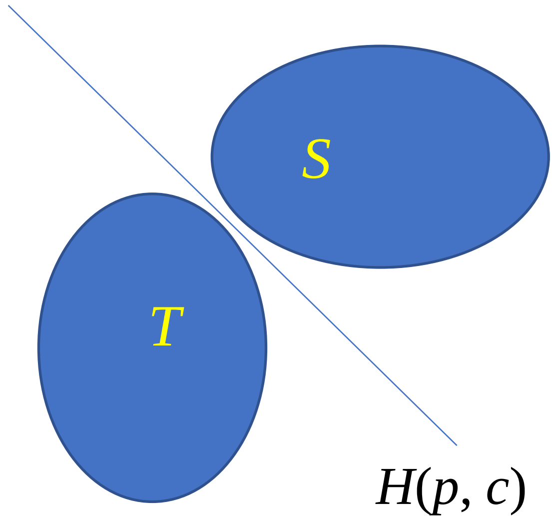
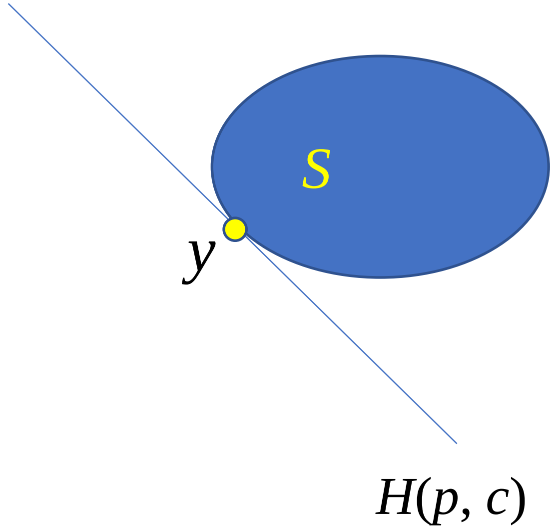
- An ***open half-space*** determined by $H(p, c)$ is either the set $\{x \in \mathbb{R}^n \mid p \cdot x < c\}$ or the set $\{x \in \mathbb{R}^n \mid p \cdot x > c\}$.
- A ***bounding hyperplane*** $H(p, c)$ to a set $S \subseteq \mathbb{R}^n$ is a hyperplane such that $S \subseteq \{x \in \mathbb{R}^n \mid p \cdot x \leq c\}$ or $S \subseteq \{x \in \mathbb{R}^n \mid p \cdot x \geq c\}$, i.e., S lies entirely on one side of $H(p, c)$.
- A ***supporting hyperplane*** $H(p, c)$ for a set $S \subseteq \mathbb{R}^n$ is a bounding hyperplane to S that shares a point in common with the boundary of S.
- A hyperplane $H(p, c)$ ***separates*** two sets $S, T \subseteq \mathbb{R}^n$ if $p \cdot s \leq c$ for all $s \in S$ and $p \cdot t \geq c$ for all $t \in T$, i.e., S and T lie on opposite sides of $H(p, c)$.



Supporting and Separating Hyperplanes

The Supporting Hyperplane Theorem: If $S \subseteq \mathbb{R}^n$ is convex and $y \in \partial S$, then there exists a supporting hyperplane for S that passes through y.



Supporting and Separating Hyperplanes

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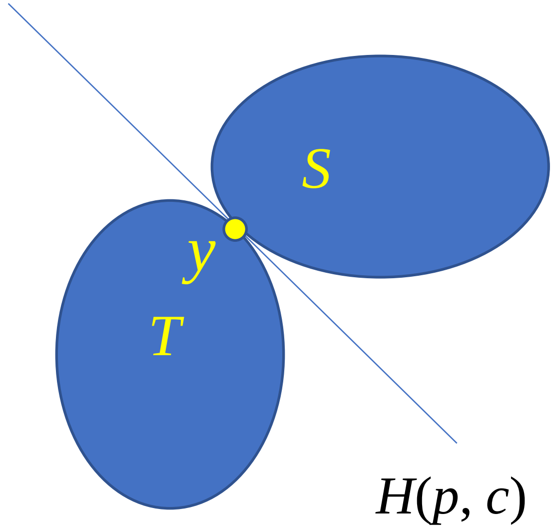

Minkowski's Separating Hyperplane Theorem: For any two convex sets $S, T \subseteq \mathbb{R}^n$, if $\mathrm{Int}(S) \cap \mathrm{Int}(T) = \emptyset$, then there exists a hyperplane separating S and T.

Exercise 1

(1) Prove Theorem 10.

(2) Prove Theorem 11.

(3) Prove Theorem 12.

(4) Prove Theorem 13.

(5) (Optional) Prove Theorem 14.

(6) Determine if the following functions are quasi-concave:

(a) $f(x) = -\frac{x^2}{1+x^2}$;

(b) $f(x, y) = ye^x$, $y > 0$;

(c) $f(x_1, \cdots, x_n) = \sum_{k=1}^{n} a_k x_k^b$, where $x_1, \cdots, x_n, a_1, \cdots, a_n, b > 0$;

(d) $f(x) = 1 + x^2 + x^3$ if $x < 0$, $f(x) = 1$ if $x \geq 0$.

(7) Show that the sum of two quasi-concave functions is not necessarily quasi-concave.

Exercise 2 (Jensen's Inequality)

(1) For any nonempty convex set $X \subseteq \mathbb{R}^n$, show that a function $f\colon X \to \mathbb{R}$ is

- concave iff $f(\sum_{k=1}^{m} \lambda_k x_k) \geq \sum_{k=1}^{m} \lambda_k f(x_k)$ for all $x_1, \cdots, x_m \in X$ and $\lambda_1, \cdots, \lambda_m \in [0, 1]$ with $\sum_{k=1}^{m} \lambda_k = 1$;
- convex iff $f(\sum_{k=1}^{m} \lambda_k x_k) \leq \sum_{k=1}^{m} \lambda_k f(x_k)$ for all $x_1, \cdots, x_m \in X$ and $\lambda_1, \cdots, \lambda_m \in [0, 1]$ with $\sum_{k=1}^{m} \lambda_k = 1$.

(2) (Optional) Let $\phi\colon [a, b] \to \mathbb{R}$ and $\lambda\colon [a, b] \to \mathbb{R}_+$ be continuous and $\int_a^b \lambda(t)dt = 1$. Show that if f is a concave function defined on the range of ϕ, then

$$f\left(\int_a^b \lambda(t)\phi(t)dt\right) \geq \int_a^b \lambda(t)f(\phi(t))dt.$$

Exercise 3

(1) Let $X \subseteq \mathbb{R}^n$ and $A \subseteq \mathbb{R}$ be convex. Show that:

(i) If $f: X \to \mathbb{R}$ is concave and $g: A \to \mathbb{R}$ decreasing and convex with $f(X) \subseteq A$, then $g \circ f$ is convex;

(ii) If $f: X \to \mathbb{R}$ is convex and $g: A \to \mathbb{R}$ decreasing and concave with $f(X) \subseteq A$, then $g \circ f$ is concave;

(iii) If $f: X \to \mathbb{R}$ is strictly concave and $g: A \to \mathbb{R}$ strictly increasing and concave with $f(X) \subseteq A$, then $g \circ f$ is strictly concave;

(iv) If $f: X \to \mathbb{R}$ is strictly convex and $g: A \to \mathbb{R}$ strictly increasing and convex with $f(X) \subseteq A$, then $g \circ f$ is strictly convex;

(v) If $f: X \to \mathbb{R}$ is strictly quasi-concave and $g: A \to \mathbb{R}$ strictly decreasing with $f(X) \subseteq A$, then $g \circ f$ is strictly quasi-convex;

(vi) If $f: X \to \mathbb{R}$ is strictly quasi-convex and $g: A \to \mathbb{R}$ strictly decreasing with $f(X) \subseteq A$, then $g \circ f$ is strictly quasi-concave.

(2) Can a strictly concave function be strictly quasi-convex? Can a strictly convex function be strictly quasi-concave?

Exercise 4

Let $X \subseteq \mathbb{R}^n_{++}$ be open and f: $X \to \mathbb{R}$ differentiable at $x \in X$ with $f(x) \neq 0$.

The ***partial elasticity*** of f with respect to x_i is defined by

$$\mathrm{El}_{x_i}\big(f(x)\big) = \frac{x_i}{f(x)}\frac{\partial f(x)}{\partial x_i} = \frac{\partial \ln|f(x)|}{\partial \ln x_i}.$$

The ***marginal rate of substitution*** of x_j for x_i is defined by

$$\mathrm{MRS}_{x_j,x_i} = \frac{\frac{\partial f(x)}{\partial x_i}}{\frac{\partial f(x)}{\partial x_j}}.$$

The ***elasticity of substitution*** of x_j for x_i ($i \neq j$) is defined by

$$\sigma_{x_j,x_i} = \mathrm{El}_{\mathrm{MRS}_{x_j,x_i}}\left(\frac{x_j}{x_i}\right) = \frac{\partial \ln\frac{x_j}{x_i}}{\partial \ln|\mathrm{MRS}_{x_j,x_i}|}.$$

- Let $f(x) = a\left(\sum_{i=1}^n b_i x_i^{-\rho}\right)^{-\frac{c}{\rho}}$, a, b_1, $\cdots$, b_n, $c > 0$, $\rho > -1$, and $\rho \neq 0$. Find MRS_{x_j,x_i} and σ_{x_j,x_i} for each $i \neq j \in \{1, 2, \cdots, n\}$.

Exercise 5 (L'Hospital's Rule)

(1) (Optional) Let $X \subseteq \mathbb{R}$ and $f\colon X \to \mathbb{R}$ and $g\colon X \to \mathbb{R}$ be differentiable in some $(a, b) \subseteq X$, $-\infty \leq a < b \leq \infty$, and $g'(x) \neq 0\ \forall x \in (a, b)$.

- Suppose that $\lim_{x\to a} \frac{f'(x)}{g'(x)} = c \in \overline{\mathbb{R}}$. Show that if $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$, or if $\lim_{x\to a} |g(x)| = \infty$, then $\lim_{x\to a} \frac{f(x)}{g(x)} = c$.
- Suppose that $\lim_{x\to b} \frac{f'(x)}{g'(x)} = d \in \overline{\mathbb{R}}$. Show that if $\lim_{x\to b} f(x) = \lim_{x\to b} g(x) = 0$, or if $\lim_{x\to b} |g(x)| = \infty$, then $\lim_{x\to b} \frac{f(x)}{g(x)} = d$.

(2) Let $f(x) = x + \cos(x)\sin(x)$ and $g(x) = e^{\sin(x)}(x + \cos(x)\sin(x))$. Show that $\lim_{x\to\infty} \frac{f'(x)}{g'(x)} = 0$ but $\lim_{x\to\infty} \frac{f(x)}{g(x)}$ does not exist.

(3) Let $X \subseteq \mathbb{R}^n_{++}$ and $f\colon X \to \mathbb{R}$ be defined by $f(x) = a\left(\sum_{i=1}^n b_i x_i^{-\rho}\right)^{-\frac{c}{\rho}}$, where a, $b_1, \cdots, b_n$, $c > 0$, and $\rho \neq 0$. Find

- $\lim_{\rho\to 0} a\left(\sum_{i=1}^n b_i x_i^{-\rho}\right)^{-\frac{c}{\rho}}$ when $\sum_{i=1}^n b_i = 1$;
- $\lim_{\rho\to \infty} a\left(\sum_{i=1}^n b_i x_i^{-\rho}\right)^{-\frac{c}{\rho}}$.

Thank you!