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Inflection points on 3D curves ☆

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ABSTRACT

Existing definitions for inflection point on 3D curves lack the direct relation to local shape-characteristics of the 3D curve that the corresponding definition for planar curves has. This paper presents a new generalization of the definition of planar-curve inflection to the case of a 3D curve that is directly related to the local convexity of the 3D curve and to that of orthogonal projections of this 3D curve on planes of interest. Various properties of this new "3D-curve inflection" concept, employing standard Differential-Geometric properties of the 3D curve, are proved, establishing its usefulness for 3D-curve interrogation within current CAD/CAGD curve/surface-design systems.

Dedicated to the memory of Gerald Farin, whose writings and teachings greatly affected and inspired also this research

1. Introduction

A point of a planar curve, at which the curve changes from convex to concave or vice-versa is called *inflection point* (see, e.g., Patrikalakis and Maekawa, 2002, p.41, Piegl and Tiller, 1996, p.8, Gray et al., 2006, p.116, Banchoff and Lovett, 2010, p.26). Only when talking about planar curves, the curvature can be signed and this sign quantifies locally the convexity of the curve (see Stoker, 1969, p.55, Tapp, 2016, p.33). Thus, a curve presents an inflection at a point where the curvature changes sign. Assuming that the curve has continuous curvature, the curvature vanishes at inflection points, while the inverse does not hold.

Existing definitions of the inflection point for a spatial curve are restrained to the zeros of curvature, and fail to relate to any "shape characteristics" (beyond the tangent) of the curve. Indicatively, Sasaki in (1957) proposes that: "*by a point of inflexion of the curve* **C** *we mean a point* **M** *of* **C** *such that the tangent to* **C** *at* **M** *is stationary*" and Lipschultz in (1969, p.63) suggests that: "*a point on a curve where the curvature vector is zero is called a point of inflection*" (see also Struik, 1950, p.12, Willmore, 1959, p.9, Manocha and Canny, 1992, Li and Cripps, 1997 and Abate and Tovena, 2012, p.51). Evidently, the definitions in Sasaki (1957); Lipschultz (1969); Struik (1950); Willmore (1959); Manocha and Canny (1992); Li and Cripps (1997); Abate and Tovena (2012) for the 3D curves, fail to characterize correctly many planar curves, disregarding convexity. For example,

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they incorrectly characterize $y = x^4$ as inflectional at $x = 0$, despite the fact that this curve is everywhere convex. In this respect, that use of the term "inflection point" goes against its commonly adopted notion in Calculus (see, e.g., Strang, 1991, p.107). On the other hand, if one adopts the usual definition for 2D curves and the one of Sasaki (1957) and Lipschultz (1969) for the 3D curves, one ends up with two incompatible definitions in 2D and 3D for the same concept (see also Farin, 2016).

The above incompatibility may be the reason that many authors in the area of Differential Geometry avoid using the term inflection (see, e.g., Weatherburn, 1988, Pogorelov, 1959, Stoker, 1969, Spivak, 1999, Kreyszig, 1991, Animov, 2001, Tapp, 2016) while others refer indirectly to inflections by presenting them in the context of curve singularities; see, e.g., do Carmo (1976), Bruce and Giblin (1984). This paper employs classical Differential Geometry methods to generalize the definition of the inflection point of 2D curves (see Definition 5) to the case of 3D curves (see Definition 6) in a manner that the two concepts are fully compatible.

Any discussion about inflection points passes inevitably through the notion of convexity. Moreover, since the topic of this work refers to spatial curves, for which convexity is not self-evident, comprehension of this notion offers a solid ground to build on. A *convex set* in an affine space is a set of points such that any straight line segment connecting any two points of the set is completely contained within the set (see Farin, 2002, App.A). The *convex hull of a set* in an affine space is the intersection of all half-spaces containing the given set (see Sedykh, 1984). This implies that the convex hull of any set is the smallest convex set containing the given set, thus any convex set in an affine space coincides with its convex hull. In the next two sub-sections the notion of convexity of planar and spatial curves is given, on the basis of the above definitions.

1.1. The notion of convexity for curves in 2D space

Schoenberg (1954) defines the notion of convexity for planar curves, giving the following two alternative descriptions: "*A closed convex curve in the plane is usually defined as the boundary of a compact convex set. Alternatively, if the curve is given in parametric form we could say that the curve is convex provided it never crosses a straight line more than twice.*" The first description correlates the convexity of the curve with set theory, while in the second alternative, the convexity is a direct result of the variation diminishing property (see Liu and Traas, 1997). Farin (2002, App.A) defines, in a similar manner the open convex planar curve as: "*A planar curve that is a subset of the boundary of its convex hull*".

If a closed curve is at least tangent vector continuous and bounded, then it defines a set, the convex hull of which is determined by the tangents of the curve at the points of it (see Spivak, 1999, Vol.2, p.15). In Ren (1994, p.1), one finds the following definition:

Definition 1. Suppose a convex set in $\mathbb{E}^2$ and a point $\mathbf{P}$ on the boundary of it are given. A line $\mathbf{L}$ through $\mathbf{P}$ is called support line of the set if the set lies completely in one of the closed half spaces, into which $\mathbf{L}$ divides $\mathbb{E}^2$. The point $\mathbf{P}$ is called the contact point of $\mathbf{L}$ with the set.

Since a convex set coincides with its convex hull and this is determined by the tangents of its boundary, the definition of the support lines and contact points of a convex set is extended to the boundary of the set, i.e. to the curve. Supposing that the boundary of the set is tangent vector continuous, Definition 1 correlates the convexity of a planar curve with the support lines and the contact points of it, since for any point on a curve the only candidate support line is its tangent line at this point. If the tangent line at some point is not a support line, then it splits the curve into two segments and the curve necessarily presents an inflection at that point.

1.2. The notion of convexity for curves in 3D space

In 3D space, the concept "convex curve" is defined in the following way: if the given set is any spatial curve and the affine space, in which the curve is embedded, is the Euclidean space $\mathbb{E}^3$, then the half-spaces, which contain the curve, are those determined by the so-called *support planes*. The support planes generalize the notion of support lines of Definition 1. Based on the definition given in Luenberger (1969, p. 133) and the discussion in Fuster and Sedykh (1995, p.169), we adopt the following definition of the support plane:

Definition 2. Given any tangent vector continuous closed and simple curve $\mathbf{C}(s) : [a, b] \longrightarrow \mathbb{R}^3$, we say that a plane is a support plane of $\mathbf{C}$, if the curve lies entirely in one of the two closed half-spaces defined by this plane and has a common point with it.

Then, Fuster and Sedykh (1995) introduces the following definition for a convex curve in 3D space:

Definition 3. A curve is said to be convex, if it lies in the boundary of its convex hull, or equivalently, each one of its points is contained in a support plane.

In this respect, one wonders: *What happens if at some point of a 3D curve no support plane exists? Could such a point be characterized as an inflection point of the 3D curve?* These two questions concern this paper.

The next section introduces the notation to be used from classical Differential Geometry. Section 3, first investigates the consequences of zero curvature and zero curvature-derivatives (up to a certain order ℓ), and then Sections 3.1 and 3.2 deal with curves being G^1 piecewise analytic with continuous Frenet planes. Section 4 adopts one of the standard definitions of inflection point on 2D curves (see Definition 5), and it elaborates on how this particular definition leads, in a natural way, to a definition of the inflection point for 3D curves (see Definition 6). Section 4.1 is devoted to the development of necessary and sufficient conditions for inflections on 3D curves (see Theorem 1) and Section 4.2 analyzes the effect of an inflection point on a 3D curve, on the projections of that curve onto any arbitrary plane. After this, Sections 4.3 and 4.4 discuss the consequences of the proposed definition, regarding the contact order of a curve with the tangent line, and the tangent indicatrix in the neighborhood of a 3D inflection, respectively. Section 5 presents examples illustrating the results of Section 4 and it is followed by Section 6, which presents an algorithm to detect inflection points in a spatial curve and discusses potential applications of it. Finally, Sections 7 and 8 give proposals for future research and the conclusions of this work. All curve models, discussed in this paper, have been produced and analyzed using the CAD-CAE software package "Genie" DNV-GL Digital Solutions (2020), which also has been used for the creation of all images in this publication.

2. Preliminaries from differential geometry

Following Pogorelov (1959, p.20) and Willmore (1959, p.3), a curve is a k-times (k can be infinite) differentiable-vector valued function:

$$\mathbf{C}(s) : [a, b] \longrightarrow \mathbb{R}^3, \text{ with } \mathbf{C}(s) := (x(s), y(s), z(s)), \tag{2.1}$$

when $x(s), y(s), z(s) \in C^k$- continuous in $[a, b]$. The set of functions $x(s), y(s), z(s)$ determine the so-called *parameterization* of the curve. The curve is *regular* if and only if:

$$\|\mathbf{C}'(s)\|^2 = x'(s)^2 + y'(s)^2 + z'(s)^2 > 0 \quad (a \le s \le b). \tag{2.2}$$

A single point $\mathbf{C}(s_0)$ is called a *regular point*, if the curve permits a regular parametrization in a neighborhood of s_0, satisfying (2.2).

A real function of a real variable in $[a, b]$ is called *analytic*, if at each point in (a, b) the function can be represented by a convergent power series. An analytic function is C^∞-continuous in (a, b); see Markushevich (1965, Vol.1, p.5). Let us begin our investigation with curves which are regular and analytic, in the sense that each component of them is an analytic function in the open interval (a, b). From Section 3.1 onwards, the continuity conditions are relaxed to piecewise analytic curves with tangent vector continuity at a finite set of internal nodes.

If (2.2) holds for all $s \in (a, b)$, then the *unit tangent vector* at the point $\mathbf{C}(s)$ is defined by:

$$\mathbf{t}(s) = \frac{\mathbf{C}'(s)}{\|\mathbf{C}'(s)\|}. \tag{2.3}$$

Taking the derivative of the tangent vector, then dividing it by $\|\mathbf{C}'(s)\|$, and after this taking the cross product of both sides with $\mathbf{t}(s)$, the *normal curvature vector* $\mathbf{k}(s)$ is defined as (see Xu and Shi, 2001, p.818):

$$\mathbf{k}(s) = \frac{\mathbf{t}(s) \times \mathbf{t}'(s)}{\|\mathbf{C}'(s)\|} = \frac{\mathbf{C}'(s) \times \mathbf{C}''(s)}{\|\mathbf{C}'(s)\|^3}. \tag{2.4}$$

The (unsigned) *curvature* of the curve $\mathbf{C}(s)$, is given by:

$$k(s) = \frac{\|\mathbf{C}'(s) \times \mathbf{C}''(s)\|}{\|\mathbf{C}'(s)\|^3}. \tag{2.5}$$

Noting that the scalar product $\mathbf{t}(s) \cdot \mathbf{t}'(s)$ always vanishes, the identity $\|\mathbf{t}(s)\|^2 \|\mathbf{t}'(s)\|^2 = \|\mathbf{t}(s) \times \mathbf{t}'(s)\|^2 + (\mathbf{t}(s) \cdot \mathbf{t}'(s))^2$, for $\|\mathbf{t}(s)\|^2 = 1$ and $\mathbf{t}(s) \cdot \mathbf{t}'(s) = 0$, gives: $\|\mathbf{t}'(s)\| = \|\mathbf{t}(s) \times \mathbf{t}'(s)\|$. Then, taking the length of $\mathbf{k}(s)$ in (2.4), one can easily obtain $\|\mathbf{t}'(s)\| = \|\mathbf{C}'(s)\| k(s)$. For $k(s) \neq 0$, the normalized derivative of the tangent vector defines the *principle unit normal vector*:

$$\mathbf{n}(s) = \frac{\mathbf{t}'(s)}{\|\mathbf{t}'(s)\|} = \frac{\mathbf{t}'(s)}{\|\mathbf{C}'(s)\| k(s)}, \tag{2.6}$$

which is perpendicular to the unit tangent of the curve. If $\mathbf{n}(s)$ exists, then the cross product of $\mathbf{t}$ and $\mathbf{n}$ defines the *principle binormal vector*:

$$\mathbf{b}(s) = \mathbf{t}(s) \times \mathbf{n}(s) = \frac{\mathbf{C}'(s) \times \mathbf{C}''(s)}{\|\mathbf{C}'(s) \times \mathbf{C}''(s)\|}. \tag{2.7}$$

The normal curvature vector (2.4) now can be written with the aid of (2.5) and (2.7) as:

$$\mathbf{k}(s) = \frac{\mathbf{C}'(s) \times \mathbf{C}''(s)}{\|\mathbf{C}'(s)\|^3} = k(s)\mathbf{b}(s). \tag{2.8}$$

The orthogonal system of the vectors $\mathbf{t}(s)$, $\mathbf{n}(s)$ and $\mathbf{b}(s)$, is the so-called *Frenet frame*. They determine the *Frenet planes*, which are given implicitly by the following equations (see Kreyszig, 1991, p.37, do Carmo, 1976, p.17):

$$\text{the osculating plane} \quad (\mathbf{q} - \mathbf{C}(s)) \cdot \mathbf{b}(s) = 0,\ \mathbf{q} \in \mathbb{R}^3,\ s \in [a,b], \tag{2.9}$$

$$\text{the normal plane} \quad (\mathbf{q} - \mathbf{C}(s)) \cdot \mathbf{t}(s) = 0,\ \mathbf{q} \in \mathbb{R}^3,\ s \in [a,b], \tag{2.10}$$

$$\text{the rectifying plane} \quad (\mathbf{q} - \mathbf{C}(s)) \cdot \mathbf{n}(s) = 0,\ \mathbf{q} \in \mathbb{R}^3,\ s \in [a,b]. \tag{2.11}$$

The definition of the Frenet frame at any point $s_0 \in (a,b)$ relies on one prerequisite: the curvature numerator must not vanish at s_0. If $\mathbf{C}'(s_0) \times \mathbf{C}''(s_0) = \mathbf{0}$, then the normal and the binormal, given by (2.6) and (2.7), respectively, cannot be defined at s_0. However, the Frenet planes can be defined at points, where the normal curvature vector vanishes. As Spivak points out in Spivak (1999, Vol.2, p. 25): "*unlike the osculating circle, the osculating plane may exist even if the curvature vanishes at some point (for example, if a curve is planar but not straight, then the osculating plane certainly exists)*".

3. Differential-geometric analysis of piecewise analytic curves

In the beginning of this section the curve $\mathbf{C}(s)$ is assumed to be analytic in (a,b), i.e., it can be represented as a power series about $s_0 \in (a,b)$:

$$\mathbf{C}(s) = \mathbf{C}(s_0) + \sum_{j=1}^{\infty} \frac{\Delta s^j}{j!} \mathbf{C}^{(j)}(s_0), \tag{3.12}$$

where $\Delta s = s - s_0$. In order to simplify the notation, let us assume that $\mathbf{C}$ is arc-length parameterized, i.e. $\|\mathbf{C}'(s_0)\| = 1$ and $\mathbf{C}'(s_0) = \mathbf{t}(s_0)$ from (2.3). Assuming that $\mathbf{C}^{(j+2)}(s_0) = \mathbf{0}$, for $j = 0, \ldots, \ell - 1$, but $\mathbf{C}^{(\ell+2)}(s_0) \neq \mathbf{0}$, for $\ell \geq 0$, the first derivative of $\mathbf{C}$ about s_0 is given by:

$$\mathbf{C}'(s) = \mathbf{t}(s_0) + \sum_{j=\ell+2}^{\infty} \frac{\Delta s^{j-1}}{(j-1)!} \mathbf{C}^{(j)}(s_0),$$

and the second derivative of $\mathbf{C}$ about s_0 becomes:

$$\mathbf{C}''(s) = \frac{\Delta s^{\ell}}{\ell!} \mathbf{C}^{(\ell+2)}(s_0) + \sum_{j=\ell+3}^{\infty} \frac{\Delta s^{j-2}}{(j-2)!} \mathbf{C}^{(j)}(s_0), \text{ for } \ell > 0.$$

Then, the normal curvature vector numerator about s_0 is given by:

$$\mathbf{C}'(s) \times \mathbf{C}''(s) = \frac{\Delta s^{\ell}}{\ell!} \mathbf{t}(s_0) \times \mathbf{C}^{(\ell+2)}(s_0) + O(\Delta s^{\ell+1}).$$

Let us investigate the continuity of the binormal about s_0, i.e. the limits of $\mathbf{b}(s) = \frac{\mathbf{C}'(s)\times\mathbf{C}''(s)}{\|\mathbf{C}'(s)\times\mathbf{C}''(s)\|}$, as $s \to s_0^{\pm}$. Disregarding the higher order terms, the numerator of the binormal vector about s_0 can be written as:

$$\mathbf{C}'(s) \times \mathbf{C}''(s) = \begin{cases} (-1)^{\ell} \frac{e^{\ell}}{\ell!} \mathbf{t}(s_0) \times \mathbf{C}^{(\ell+2)}(s_0), & s < s_0 \\ \frac{e^{\ell}}{\ell!} \mathbf{t}(s_0) \times \mathbf{C}^{(\ell+2)}(s_0), & s_0 < s \end{cases}$$

where $e = |\Delta s|$, with $\Delta s = s - s_0$. Thus, one may equivalently consider the limits of both branches of $\mathbf{b}(s)$, as $e \to 0$. These are:

$$\mathbf{b}(s_0^-) = (-1)^{\ell} \frac{\mathbf{t}(s_0) \times \mathbf{C}^{(\ell+2)}(s_0)}{\|\mathbf{t}(s_0) \times \mathbf{C}^{(\ell+2)}(s_0)\|} \quad \text{and} \quad \mathbf{b}(s_0^+) = \frac{\mathbf{t}(s_0) \times \mathbf{C}^{(\ell+2)}(s_0)}{\|\mathbf{t}(s_0) \times \mathbf{C}^{(\ell+2)}(s_0)\|} \tag{3.13}$$

For ℓ even, the above two limits are equal, but for ℓ odd they return two vectors of opposite direction, which implies that the binormal vector is continuous (or discontinuous) at s_0 if and only if ℓ is even (or odd, respectively). If ℓ is not finite, then $(-1)^{\ell}$ is indeterminate and consequently the osculating plane is indeterminate as well (see Willmore, 1959, p.9-10 and Spivak, 1999, Vol.2, p.26). We do not go further into this case.

Combining the expressions of $\mathbf{b}(s_0^-)$ and $\mathbf{b}(s_0^+)$ in (3.13) into one, let us denote the binormal vector about s_0 by:

$$\mathbf{b}(s_0^{\pm}) = (\pm 1)^{\ell} \frac{\mathbf{t}(s_0) \times \mathbf{C}^{(\ell+2)}(s_0)}{\|\mathbf{t}(s_0) \times \mathbf{C}^{(\ell+2)}(s_0)\|}, \tag{3.14}$$

indicating by $\pm$ the possible change in direction at s_0. Consequently, the normal vector of $\mathbf{C}(s)$ about s_0 (being given by $\mathbf{b}(s_0^{\pm}) \times \mathbf{t}(s_0)$) becomes:

$$\mathbf{n}(s_0^{\pm}) = (\pm 1)^{\ell} \frac{\mathbf{t}(s_0) \times \mathbf{C}^{(\ell+2)}(s_0)}{\|\mathbf{t}(s_0) \times \mathbf{C}^{(\ell+2)}(s_0)\|} \times \mathbf{t}(s_0). \tag{3.15}$$

Thus, either if ℓ is odd or even, the osculating plane and the rectifying plane can still be defined implicitly by (2.9) and (2.11) respectively. Noting that for any vector $\mathbf{w}$, the equations $(\mathbf{q} - \mathbf{C}(s_0)) \cdot \mathbf{w} = 0$ and $(\mathbf{q} - \mathbf{C}(s_0)) \cdot (-\mathbf{w}) = 0$ refer to the same plane, one concludes that the osculating plane and the rectifying plane before and after s_0 are the same. In other words, the osculating and the rectifying planes are continuous at s_0 (see also Willmore, 1959, p.8-9 and Spivak, 1999, Vol.2, p.24).

Since the osculating plane and the rectifying plane are continuous, for any finite ℓ, the binormal and the normal vectors defined at s_0 by (3.14) and (3.15) are well defined unit vectors. Thus, expression (2.6) at s_0 becomes

$$\mathbf{t}'(s_0) = \mathbf{C}''(s_0) = k(s_0^{\pm})\mathbf{n}(s_0^{\pm}),$$

and indicates that, since $\mathbf{n}(s_0^{\pm})$ is a unit vector, $k(s_0^-) = k(s_0^+) = 0$ if and only if $\mathbf{C}''(s_0) = \mathbf{0}$. Differentiating (2.6) one obtains at s_0:

$$\mathbf{C}'''(s_0) = k'(s_0^{\pm})\mathbf{n}(s_0^{\pm}) + k(s_0^{\pm})\mathbf{n}'(s_0^{\pm}).$$

Then, for $k(s_0^-) = k(s_0^+) = 0$, the above can be written as:

$$\mathbf{C}'''(s_0) = k'(s_0^{\pm})\mathbf{n}(s_0^{\pm}),$$

implying that $\mathbf{C}'''(s_0) = \mathbf{0}$ if and only if $k'(s_0^-) = k'(s_0^+) = 0$. Differentiating repeatedly (2.6) and taking into account the zeros of curvature derivatives up to $(j-1)$-order, the conclusion is that:

$$\mathbf{C}^{(j+2)}(s_0) = \mathbf{0} \Longleftrightarrow k^{(j)}(s_0^-) = k^{(j)}(s_0^+) = 0, \quad j = 0, \ldots, \ell - 1. \tag{3.16}$$

For $j = \ell$ the following holds:

$$\mathbf{C}^{(\ell+2)}(s_0) = k^{(\ell)}(s_0^{\pm})\mathbf{n}(s_0^{\pm}). \tag{3.17}$$

It is convenient to define the number ℓ, since it is important in what follows.

Definition 4. Let $\mathbf{C}(s) : [a, b] \longrightarrow \mathbb{R}^3$ be analytic in (a, b). The integer $\ell \geq 0$, for which:

$$\mathbf{C}^{(j+2)}(s_0) = \mathbf{0},\ j = 0, 1, \ldots, \ell - 1 \text{ and } \mathbf{C}^{(\ell+2)}(s_0) \neq \mathbf{0}, \tag{3.18}$$

and necessarily

$$k^{(j)}(s_0) = 0,\ j = 0, 1, \ldots, \ell - 1 \text{ and } k^{(\ell)}(s_0^{\pm}) \neq 0, \tag{3.19}$$

is called the critical derivative order of $\mathbf{C}(s)$ at $s_0 \in (a, b)$.

The critical derivative order is the lowest order non-zero derivative of the curvature of $\mathbf{C}(s)$. It is defined uniquely at any point of an analytic curve. The above analysis proves the following:

Lemma 1. *Let $\mathbf{C}(s) : [a, b] \longrightarrow \mathbb{R}^3$ be analytic in (a, b). The normal, the binormal and the normal curvature vectors are continuous at any point $s_0 \in (a, b)$, if and only if the critical derivative order, ℓ, of $\mathbf{C}(s)$ at s_0 is even, and they are discontinuous if and only if ℓ is odd. In any case, the Frenet planes are defined and they are continuous at s_0.*

3.1. Frenet planes at the nodes of G^1 piecewise analytic curves

Definitions 1 and 2 imply that the discussion about inflections in planar as well as in spatial curves are meaningful as long as the curve is at least G^1-continuous with continuous Frenet planes. In what follows, we are concerned with curves satisfying the following hypothesis:

Hypothesis 1. Assume that $\mathbf{C}(s)$ is tangent vector continuous (or G^1-continuous) at $s_0 \in (a, b)$, while it is analytic in the vicinity of s_0, i.e. there exists a fixed positive number, δ, such that the curve is analytic in $(s_0 - \delta, s_0) \subset (a, b)$ and $(s_0, s_0 + \delta) \subset (a, b)$.

The condition for G^1 continuity at s_0, is necessary in what follows, allowing the definitions of inflections in 2D and 3D to be applicable to piecewise analytic curves, such as splines. The next lemma clarifies when the Frenet planes are continuous in the vicinity of a point of a piecewise analytic curve, where Hypothesis 1 holds.

Lemma 2. *Let $\mathbf{C}(s) : [a, b] \longrightarrow \mathbb{R}^3$ satisfy Hypothesis 1 in the vicinity of $s_0 \in (a, b)$. The curve $\mathbf{C}(s)$ has continuous Frenet-planes at s_0, if and only if the normal and the binormal vectors around s_0 are collinear, i.e. when:*

$$\mathbf{n}(s_0^-) \cdot \mathbf{n}(s_0^+) = -1 \quad \text{and} \quad \mathbf{b}(s_0^-) \cdot \mathbf{b}(s_0^+) = -1 \tag{3.20}$$

or

$$\mathbf{n}(s_0^-) \cdot \mathbf{n}(s_0^+) = 1 \quad \text{and} \quad \mathbf{b}(s_0^-) \cdot \mathbf{b}(s_0^+) = 1. \tag{3.21}$$

Proof. The vectors $\mathbf{n}(s_0^-)$ and $\mathbf{n}(s_0^+)$ are collinear, if and only if $\mathbf{n}(s_0^-) \cdot \mathbf{n}(s_0^+) = \pm 1$. Moreover, since $\mathbf{b}(s_0^\pm) = \mathbf{t}(s_0) \times \mathbf{n}(s_0^\pm)$ and $\mathbf{t}(s)$ is continuous at s_0, it is straightforward to see that $\mathbf{n}(s_0^-) \cdot \mathbf{n}(s_0^+) = 1$ if and only if $\mathbf{b}(s_0^-) \cdot \mathbf{b}(s_0^+) = 1$, and $\mathbf{n}(s_0^-) \cdot \mathbf{n}(s_0^+) = -1$ if and only if $\mathbf{b}(s_0^-) \cdot \mathbf{b}(s_0^+) = -1$. Assume, now, that the vectors $\mathbf{n}(s_0^-)$ and $\mathbf{n}(s_0^+)$ are collinear, i.e., either (3.20) or (3.21) holds. Then, the osculating plane and the rectifying plane are continuous at s_0. Since the curve has continuous tangent vector at s_0, its normal plane (2.10) is continuous, i.e. the Frenet planes are continuous at s_0. Inversely, if the Frenet planes are continuous at s_0, then the normal and the binormal vectors around s_0 are necessarily collinear, and by $\mathbf{b}(s_0^\pm) = \mathbf{t}(s_0) \times \mathbf{n}(s_0^\pm)$ either (3.20) or (3.21) holds. □

Lemma 2 introduces the most important restriction to be applied in the sequel, that of the existence of unique Frenet-planes. This restriction is a vital prerequisite for the analysis of inflection points on spatial curves, and it always holds for G^1 piecewise analytic planar curves.

Since (3.16) and (3.17) hold for the one-sided derivatives near the ends of (a,b), i.e. for $s_0 = a^+$ and $s_0 = b^-$, the critical derivative order can be defined for a^+ and b^-, referring only to the one-sided derivatives at them, as the integer, for which (3.18) and (3.19) hold for $s_0 = a^+$ and the integer for which (3.18) and (3.19) hold for $s_0 = b^-$. Thus, if a curve satisfies Hypothesis 1, it has two (not necessarily equal) critical derivative orders near s_0; one for s_0^- and one for s_0^+. Let them be denoted by $n \geq 0$ and $m \geq 0$, respectively. Note that n and m are related to the continuity of the curve at s_0 in the following way: If the curve has continuous curvature derivatives up to n order, then $n = m$ and $k^{(n)}(s_0^-) = k^{(n)}(s_0^+)$. Otherwise, the equality $n = m$ does not imply that $k^{(n)}(s_0^-) = k^{(n)}(s_0^+)$ and if $k^{(n)}(s_0^-) = k^{(m)}(s_0^+)$ this does not imply that $n = m$. In other words, for a curve, which is only G^1 continuous at s_0, no relation between n and m is implied.

3.2. Power series expansion of G^1 piecewise analytic curves

In what follows, only curves, which satisfy Hypothesis 1 in the vicinity of $s_0 \in (a, b)$, and have continuous Frenet planes at s_0, i.e. either (3.20) or (3.21) holds, are considered. Also, n and m denote the critical derivative orders at s_0^- and at s_0^+, respectively, i.e. the derivatives of $\mathbf{C}(s)$ at $s_0^\pm$ satisfy (3.18), for $\ell = n$, at $s = s_0^-$ and for $\ell = m$, at $s = s_0^+$. Under these prerequisites, (3.17) gives near s_0:

$$\mathbf{C}^{(n+2)}(s_0^-) = k^{(n)}(s_0^-)\mathbf{n}(s_0^-) \quad \text{and} \quad \mathbf{C}^{(m+2)}(s_0^+) = k^{(m)}(s_0^+)\mathbf{n}(s_0^+).$$

Substituting the above expressions into (3.14) the binormal vector in the vicinity of s_0 becomes:

$$\mathbf{b}(s_0^-) = (-1)^n \frac{k^{(n)}(s_0^-)}{|k^{(n)}(s_0^-)|}\mathbf{t}(s_0) \times \mathbf{n}(s_0^-) \quad \text{and} \quad \mathbf{b}(s_0^+) = \frac{k^{(m)}(s_0^+)}{|k^{(m)}(s_0^+)|}\mathbf{t}(s_0) \times \mathbf{n}(s_0^+)$$

Taking the inner product of both sides of the left equation above with $\mathbf{b}(s_0^-)$ and the inner product of both sides of the right equation above with $\mathbf{b}(s_0^+)$, since the triple products $(\mathbf{t}(s_0) \times \mathbf{n}(s_0^\pm)) \cdot \mathbf{b}(s_0^\pm) = 1$, the above equations imply that the coefficients $(-1)^n k^{(n)}(s_0^-)$ and $k^{(m)}(s_0^+)$ are both positive:

$$(-1)^n k^{(n)}(s_0^-) > 0 \quad \text{and} \quad k^{(m)}(s_0^+) > 0 \tag{3.22}$$

The curve $\mathbf{C}(s)$ can be written in power series, around s_0 as:

$$\mathbf{C}(s) = \begin{cases} \mathbf{C}(s_0) + \sum_{j=1}^{\infty} (-1)^j \frac{e^j}{j!}\mathbf{C}^{(j)}(s_0^-), & s \in (s_0 - \delta, s_0) \\ \mathbf{C}(s_0) + \sum_{j=1}^{\infty} \frac{e^j}{j!}\mathbf{C}^{(j)}(s_0^+), & s \in (s_0, s_0 + \delta) \end{cases}$$

where $e = |\Delta s|$, with $\Delta s = s - s_0$. The use of the positive e instead of Δs transfers the different sign of Δs when $s \in (s_0 - \delta, s_0)$, and when $s \in (s_0, s_0 + \delta)$, in the above representations, to the coefficient $(-1)^j$. Thus, the curve is given by:

$$\mathbf{C}(s) = \begin{cases} \mathbf{C}(s_0) - e\mathbf{t}(s_0) + \frac{(-1)^{n+2}e^{n+2}}{(n+2)!}\mathbf{C}^{(n+2)}(s_0^-) + O(e^{n+3}), & s \in (s_0 - \delta, s_0) \\ \mathbf{C}(s_0) + e\mathbf{t}(s_0) + \frac{e^{m+2}}{(m+2)!}\mathbf{C}^{(m+2)}(s_0^+) + O(e^{m+3}), & s \in (s_0, s_0 + \delta) \end{cases} \tag{3.23}$$

Substituting the derivatives of $\mathbf{C}(s)$ at s_0^- and s_0^+ with their equivalent from (3.17) one obtains:

$$\mathbf{C}(s)=\begin{cases}\mathbf{C}(s_0)-e\mathbf{t}(s_0)+(-1)^{n+2}\frac{e^{n+2}}{(n+2)!}k^{(n)}(s_0^-)\mathbf{n}(s_0^-)+O(e^{n+3}),\ s\in(s_0-\delta,s_0)\\ \mathbf{C}(s_0)+e\mathbf{t}(s_0)+\frac{e^{m+2}}{(m+2)!}k^{(m)}(s_0^+)\mathbf{n}(s_0^+)+O(e^{m+3}),\ s\in(s_0,s_0+\delta)\end{cases}$$

Then, writing $\mathbf{C}(s)$ with respect to the orthogonal systems of vectors $\{\mathbf{t}(s_0),\mathbf{n}(s_0^-),\mathbf{b}(s_0^-)\}$, for $s\in(s_0-\delta,s_0)$ and $\{\mathbf{t}(s_0),\mathbf{n}(s_0^+),\mathbf{b}(s_0^+)\}$, for $s\in(s_0,s_0+\delta)$, the dominant part of $\mathbf{C}(s)$, component-wise, becomes:

$$\tilde{\mathbf{C}}(s)=\begin{cases}\mathbf{C}(s_0)-e\mathbf{t}(s_0)+(-1)^{n+2}k^{(n)}(s_0^-)\frac{e^{n+2}}{(n+2)!}\mathbf{n}(s_0^-)+(-1)^p\chi(s_0^-)\frac{e^p}{p!}\mathbf{b}(s_0^-),\ s\in(s_0-\delta,s_0)\\ \mathbf{C}(s_0)+e\mathbf{t}(s_0)+k^{(m)}(s_0^+)\frac{e^{m+2}}{(m+2)!}\mathbf{n}(s_0^+)+\psi(s_0^+)\frac{e^q}{q!}\mathbf{b}(s_0^+),\ s\in(s_0,s_0+\delta)\end{cases}$$

where $(-1)^p\chi(s_0^-)\frac{e^p}{p!}$, with $p\geq n+3$, is the non-zero coefficient of the binormal vector $\mathbf{b}(s_0^-)$ and, analogously, $\psi(s_0^+)\frac{e^q}{q!}$, with $q\geq m+3$, is the non-zero coefficient of the binormal vector $\mathbf{b}(s_0^+)$. Note that, since n (resp. m) is the critical derivative order of $\mathbf{C}(s)$ at s_0^- (resp. s_0^+), p is greater than $n+2$ (resp. $q>m+2$) and finite as long as $\mathbf{C}(s)$ is non-planar. Without any loss of generality, one may apply a rigid-body motion to $\mathbf{C}(s)$, such that:

$$\mathbf{C}(s_0)=\mathbf{0},\ \mathbf{t}(s_0)=(1,\ 0,\ 0)^T \text{ and } \mathbf{n}(s_0^-)=(0,\ -1,\ 0)^T. \tag{3.24}$$

Then, necessarily, the binormal vector at s_0^- is $\mathbf{b}(s_0^-)=(0,\ 0,\ -1)^T$, and the normal and binormal vectors at s_0^+ are $\mathbf{n}(s_0^+)=(0,\ n_0,\ 0)^T$ and $\mathbf{b}(s_0^+)=(0,\ 0,\ b_0)^T$, respectively, with $n_0=b_0=\pm1$, so that either (3.20) or (3.21) holds. In view of the assumption (3.24), the dominant part of $\mathbf{C}(s)$ near s_0 becomes:

$$\tilde{\mathbf{C}}(s)=\begin{cases}\left(-e,\ -(-1)^{n+2}k^{(n)}(s_0^-)\frac{e^{n+2}}{(n+2)!},\ -(-1)^p\chi(s_0^-)\frac{e^p}{p!}\right)^T,\ s\in(s_0-\delta,s_0)\\ \left(e,\ k^{(m)}(s_0^+)\frac{e^{m+2}}{(m+2)!}n_0,\ \psi(s_0^+)\frac{e^q}{q!}b_0\right)^T,\ s\in(s_0,s_0+\delta)\end{cases} \tag{3.25}$$

The proofs of Lemmas 3, 4, 5 and 6, in the next section, are based on (3.22) and (3.25). The representation (3.23) is used in Section 4.3.

4. Definition of the inflection point

A standard definition for the inflection point of a planar curve is the following (see Liu and Traas, 1997):

Definition 5. (2D) Let $\mathbf{C}(s):[a,b]\longrightarrow\mathbb{R}^2$ satisfy Hypothesis 1 in the vicinity of $s_0\in(a,b)$. If the curve does not have a support line at $s_0\in(a,b)$, then $\mathbf{C}(s_0)$ is called inflection point.

Alternatively, an inflection point on a planar curve can be identified using:

Lemma 3. *Let the planar curve $\mathbf{C}(s):[a,b]\longrightarrow\mathbb{R}^2$ satisfy Hypothesis 1 in the vicinity of $s_0\in(a,b)$. The point $\mathbf{C}(s_0)$ is an inflection point of the curve if and only if the condition (3.20) holds at the parametric value s_0.*

Proof. Without any loss of generality, adopting the assumption (3.24), the dominant part of $\mathbf{C}(s)$ becomes:

$$\tilde{\mathbf{C}}(s)=\begin{cases}\left(-e,\ -(-1)^{n+2}k^{(n)}(s_0^-)\frac{e^{n+2}}{(n+2)!},\ 0\right)^T,\ s\in(s_0-\delta,s_0)\\ \left(e,\ k^{(m)}(s_0^+)\frac{e^{m+2}}{(m+2)!}n_0,\ 0\right)^T,\ s\in(s_0,s_0+\delta)\end{cases}$$

Then, the signed distance from the curve to the tangent line at it, i.e. x-axis, is given by:

$$D(\tilde{\mathbf{C}}(s);x):=\begin{cases}-(-1)^{n+2}k^{(n)}(s_0^-)\frac{e^{n+2}}{(n+2)!},\quad s\in(s_0-\delta,s_0)\\ k^{(m)}(s_0^+)\frac{e^{m+2}}{(m+2)!}n_0,\quad s\in(s_0,s_0+\delta)\end{cases}$$

Since (3.22) holds, it becomes obvious that if the curve has inflection at s_0, then $n_0=1$, i.e. (3.20) holds, and, inversely, if (3.20) holds, then $n_0=1$, thus the curve has inflection at s_0. □

A direct consequence of Definition 5 is the following property of the curves in the vicinity of an inflection point:

Corollary 1. *Let the planar curve $\mathbf{C}(s):[a,b]\longrightarrow\mathbb{R}^3$, which is embedded in the 3D space, satisfy Hypothesis 1 in the vicinity of $s_0\in(a,b)$. If the curve at $s=s_0$ has inflection point in the sense of Definition 5, then it does not have support plane at this position.*

Proof. For the sake of simplicity, let us assume that $\mathbf{C}(s) = \left(C_x(s),\ C_y(s),\ 0\right)^T$ with $\mathbf{C}(s_0) = \mathbf{0}$ and the tangent vector at this position is $\mathbf{t}(s_0) = (1,\ 0,\ 0)^T$, i.e. the tangent line of the curve at s_0 coincides with the x-axis. Then, any candidate support plane at s_0 is given by

$$\mathcal{P}(\alpha, \beta) := \alpha y + \beta z = 0,$$

where $\alpha = \cos(\theta)$, $\beta = \sin(\theta)$, with θ being the angle between the normal of the plane and the y-axis. Note that the range of the angle θ that one needs to test is $[0, \pi/2]$; the results for the remaining angles can be produced in a similar manner. The fact that $\mathbf{C}(s)$ has inflection at s_0 implies that the signed distance from $\mathbf{C}(s)$ to the x-axis is such that it changes sign at s_0, i.e.:

$$D(\mathbf{C}(s); x) := C_y(s),\ s \in (s_0 - \delta, s_0 + \delta), \quad \text{with} \quad C_y(s_0^-)C_y(s_0^+) < 0.$$

The signed distance of a point to a plane indicates if the point lies in one of the half-spaces, defined by the plane, or if it lies onto that plane, in case that distance is zero. The signed distance from $\mathbf{C}(s)$ to the plane $\mathcal{P}$ is given by:

$$D(\mathbf{C}(s); \mathcal{P}(\alpha, \beta)) := \alpha C_y(s).$$

The signed distance $D(\mathbf{C}(s); \mathcal{P}(\alpha, \beta))$ changes sign around s_0, because $\alpha^2 C_y(s_0^-)C_y(s_0^+) < 0$. Thus, for $\alpha = \cos(\theta) > 0$, i.e. $\theta \in [0, \pi/2)$, the plane $\mathcal{P}(\alpha, \beta)$ cannot be a support plane of $\mathbf{C}(s)$ in the vicinity of s_0, since it splits the curve into two segments, each one lying in one of the closed half-spaces defined by this plane. For $\alpha = 0$, i.e. $\theta = \pi/2$, the plane $\mathcal{P}(0, \beta)$ is actually the plane of the curve, which cannot be a support plane of it, according to Definition 2. Thus, $\mathbf{C}(s)$ does not have support plane at s_0. $\square$

Corollary 1 formulates exactly the observation, which suggests the following definition for the inflection point in a 3D curve:

Definition 6. (3D) Let the spatial curve $\mathbf{C}(s) : [a, b] \longrightarrow \mathbb{R}^3$ satisfy Hypothesis 1 in the vicinity of $s_0 \in (a, b)$. Assume also that the curve $\mathbf{C}(s)$ has continuous Frenet-planes at s_0. If $\mathbf{C}(s)$ does not have support plane at the parametric value s_0, then $\mathbf{C}(s_0)$ is called an inflection point of the curve.

The above definition is a direct generalization of Definition 5 to the case of spatial curves and in this sense it overcomes the incompatibility issue (see the second and third paragraphs of Section 1) associated with all standard definitions for the inflection point of a spatial curve (see Fig. 1).

4.1. Necessary and sufficient conditions for an inflection point on a 3D curve

The equivalence between Definition 5 and Lemma 3, raises the following question: can one prove that a 3D curve has inflection (i.e. Definition 6 is satisfied) at a point, if and only if the normal and the binormal vectors are discontinuous at this point, thus generalizing Lemma 3 to 3D curves? This is the subject of this sub-section. This question actually splits into the following two:

- Assume that the curve has an inflection at s_0 according to Definition 6. Do (3.20) hold at this point?
- Inversely, assume that (3.20) hold for a curve $\mathbf{C}(s)$ at s_0. Is $\mathbf{C}(s_0)$ an inflection point?

The answer to the first question is in the affirmative, as detailed in Lemma 4 below. The second question has, in general, a negative answer, unless in addition to (3.20), also the orthogonal projections of the curve on its osculating plane and on its rectifying plane at s_0 have inflection at it; see Lemma 5

Lemma 4. *Let the non-planar curve $\mathbf{C}(s) : [a, b] \longrightarrow \mathbb{R}^3$ satisfy Hypothesis 1 in the vicinity of $s_0 \in (a, b)$. If $\mathbf{C}(s)$ has inflection at s_0, in the sense of Definition 6, then the orthogonal projections of $\mathbf{C}(s)$ onto its osculating plane and onto its rectifying plane at s_0 have inflection at that point, in the sense of Definition 5. Moreover, the normal and the binormal vectors at $s_0 \in (a, b)$ satisfy (3.20).*

Proof. Without any loss of generality, adopting the assumption (3.24), one begins with the representation (3.25) of $\tilde{\mathbf{C}}(s)$. Since $\mathbf{C}(s)$ satisfies Definition 6 at $\mathbf{C}(s_0)$, any plane passing through the tangent line of the curve at s_0 divides the curve into two pieces, each one lying in one of the two half-spaces defined by that plane. Let us consider the rectifying plane of $\mathbf{C}(s)$ at s_0, which is $y = 0$. The signed distance from $\tilde{\mathbf{C}}(s)$ to $y = 0$ is given by:

$$D(\tilde{\mathbf{C}}(s); y = 0) = \begin{cases} -(-1)^{n+2} \frac{e^{n+2}}{(n+2)!} k^{(n)}(s_0^-),\ s \in (s_0 - \delta, s_0) \\ \frac{e^{m+2}}{(m+2)!} k^{(m)}(s_0^+) n_0,\ s \in (s_0, s_0 + \delta) \end{cases}$$

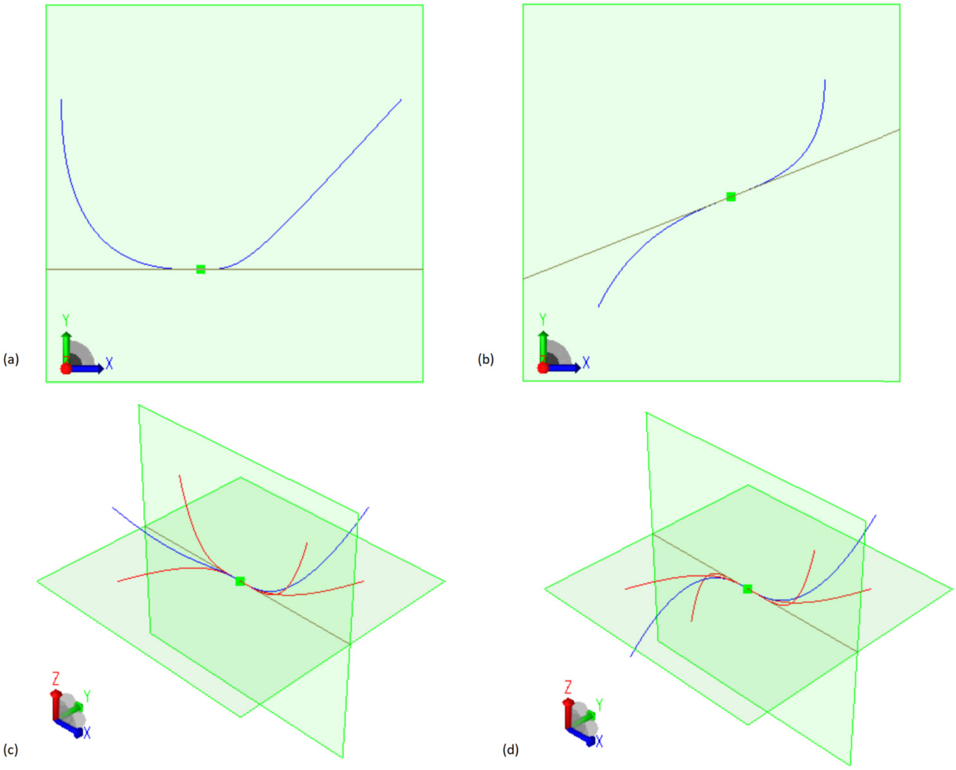

Fig. 1. Inflection points and non-inflection points on 2D and 3D curves, at points where Hypothesis 1 holds. (a) A 2D curve $\mathbf{C}(s)$ with a support line at $\mathbf{C}(s_0)$, implying that no inflection exists at $\mathbf{C}(s_0)$ according to Definition 5. (b) A 2D curve $\mathbf{C}(s)$ with no support line at $\mathbf{C}(s_0)$, implying that $\mathbf{C}(s)$ has an inflection at $\mathbf{C}(s_0)$ according to Definition 5. (c) A 3D curve $\mathbf{C}(s)$ with a support plane at $\mathbf{C}(s_0)$, implying that no inflection exists at $\mathbf{C}(s_0)$ according to Definition 6. (d) A 3D curve $\mathbf{C}(s)$ with no support plane at $\mathbf{C}(s_0)$, implying that $\mathbf{C}(s)$ has a 3D inflection at $\mathbf{C}(s_0)$ according to Definition 6. In Case (c), despite the fact that the 3D curve's projection on the osculating plane presents an infection at $\mathbf{C}(s_0)$, the projection on the rectifying plane does not have infection at $\mathbf{C}(s_0)$, and Theorem 1 characterizes $\mathbf{C}(s)$ as not inflectional at $\mathbf{C}(s_0)$, implying the existence of a support plane at $\mathbf{C}(s_0)$. In Case (d), both projections (on the osculating and rectifying planes) are inflectional at $\mathbf{C}(s_0)$, implying that also the 3D curve $\mathbf{C}(s)$ is inflectional, i.e., this curve has no support plane at $\mathbf{C}(s_0)$.

Since the sign of the above function changes at s_0, and taking into account (3.22), one obtains: $n_0 > 0$, i.e. $\mathbf{n}(s_0^+) = (0,\ 1,\ 0)^T$. Then, $\mathbf{b}(s_0^+) = (0,\ 0,\ 1)^T$, i.e. $b_0 = 1$ and (3.20) holds. Moreover, the projection of $\tilde{\mathbf{C}}(s)$ onto its osculating plane has inflection at s_0.

The osculating plane of $\mathbf{C}(s)$ at s_0 is $z = 0$. The signed distance from $\tilde{\mathbf{C}}(s)$ to $z = 0$ is given by:

$$D(\tilde{\mathbf{C}}(s); z = 0) = \begin{cases} -(-1)^p \chi(s_0^-)\frac{e^p}{p!}, & s \in (s_0 - \delta, s_0) \\ \psi(s_0^+)\frac{e^q}{q!}, & s \in (s_0, s_0 + \delta) \end{cases}$$

Since $z = 0$ divides also $\tilde{\mathbf{C}}(s)$ into two pieces, the sign of the above function changes at s_0, which implies:

$$(-1)^p \chi(s_0^-)\psi(s_0^+) > 0, \tag{4.26}$$

thus the projection of $\tilde{\mathbf{C}}(s)$ onto its rectifying plane has inflection at s_0. □

Lemma 5. *Let the non-planar curve $\mathbf{C}(s) : [a, b] \longrightarrow \mathbb{R}^3$, satisfy Hypothesis 1 in the vicinity of $s_0 \in (a, b)$. Assume also that the curve $\mathbf{C}(s)$ has continuous Frenet-planes at s_0. If the orthogonal projections, of the curve onto its osculating plane and onto its rectifying plane at s_0, have inflection at s_0 in the sense of Definition 5, then the curve has inflection at s_0, in the sense of Definition 6. Moreover, the normal and the binormal vectors at $s_0 \in (a, b)$ satisfy (3.20).*

Proof. Without any loss of generality, adopting the assumption (3.24) the representation of the dominant part of $\mathbf{C}(s)$ is given by (3.25). Based on Definition 6, one can prove that if the orthogonal projections of the curve onto its osculating plane and onto its rectifying plane at s_0 have inflection, then the curve does not have support plane at s_0, i.e. any plane, which passes through its tangent line at s_0, splits the curve into two pieces, each one lying in one of the half-spaces defined by that plane. The projection on the osculating plane is given by:

$$\tilde{\mathbf{C}}_{osc}(s)=\begin{cases}\left(-e,\ -(-1)^{n+2}\frac{e^{n+2}}{(n+2)!}k^{(n)}(s_0^-),\ 0\right)^T, & s\in(s_0-\delta,s_0)\\ \left(e,\ \frac{e^{m+2}}{(m+2)!}k^{(m)}(s_0^+)n_0,\ 0\right)^T, & s\in(s_0,s_0+\delta)\end{cases}$$

Since it has inflection, the inequalities in (3.22) hold and $n_0>0$, i.e. $\mathbf{n}(s_0^+)=(0,\ 1,\ 0)^T$ and, consequently, $\mathbf{b}(s_0^+)=(0,\ 0,\ 1)^T$, i.e. $b_0=1$, which imply that (3.20) holds. Then, the projection on the rectifying plane becomes:

$$\tilde{\mathbf{C}}_{rect}(s)=\begin{cases}\left(-e,\ 0,\ -(-1)^p\chi(s_0^-)\frac{e^p}{p!}\right)^T, & s\in(s_0-\delta,s_0)\\ \left(e,\ 0,\ \psi(s_0^+)\frac{e^q}{q!}\right)^T, & s\in(s_0,s_0+\delta)\end{cases}$$

and the curve $\tilde{\mathbf{C}}_{rect}(s)$ has inflection at s_0, if and only if (4.26) holds. Consequently, the family of planes, which pass through the tangent line at s_0, i.e. the x-axis, are:

$$\mathcal{P}(\alpha,\beta):=\alpha y+\beta z=0,\quad \alpha=\cos(\theta),\quad \beta=\sin(\theta),\quad 0\le\theta\le\frac{\pi}{2}.$$

The signed distance from $\tilde{\mathbf{C}}(s)=\left(\tilde{C}_x(s),\ \tilde{C}_y(s),\ \tilde{C}_z(s)\right)^T$ to $\mathcal{P}(\alpha,\beta)$ is given by:

$$D(\tilde{\mathbf{C}}(s);\mathcal{P}(\alpha,\beta))=\alpha\tilde{C}_y(s)+\beta\tilde{C}_z(s),$$

which in view of (3.25) (for $n_0=b_0=1$) becomes:

$$D(\tilde{\mathbf{C}};\mathcal{P}(\alpha,\beta))=\begin{cases}-\alpha(-1)^{n+2}k^{(n)}(s_0^-)\frac{e^{n+2}}{(n+2)!}-\beta(-1)^p\chi(s_0^-)\frac{e^p}{p!}, & s\in(s_0-\delta,s_0)\\ \alpha k^{(m)}(s_0^+)\frac{e^{m+2}}{(m+2)!}+\beta\psi(s_0^+)\frac{e^q}{q!}, & s\in(s_0,s_0+\delta)\end{cases}$$

For sufficiently small e, the dominant part of $D(\tilde{\mathbf{C}};\mathcal{P}(\alpha,\beta))$ depends on the sign of $-\alpha(-1)^{n+2}k^{(n)}(s_0^-)$, when $s\in(s_0-\delta,s_0)$ and on the sign of $\alpha k^{(m)}(s_0^+)$, when $s\in(s_0,s_0+\delta)$, since $p<n+2$ and $q<m+2$. Taking into account (3.22), the signed distance $D(\tilde{\mathbf{C}};\mathcal{P}(\alpha,\beta))$ is negative, for $s\in(s_0-\delta,s_0)$ and positive, for $s\in(s_0,s_0+\delta)$, for all angles $\theta\in[0,\pi/2]$, hence the curve has no support plane at s_0, thus the point $\mathbf{C}(s_0)$ satisfies Definition 6 i.e. it is an inflection point of the spatial curve $\mathbf{C}(s)$. $\square$

Lemmas 4 and 5 lead directly to the following necessary and sufficient condition for the existence of an inflection point on a 3D curve, which can be regarded as an equivalent to Definition 6.

Theorem 1. *Let the non-planar curve $\mathbf{C}(s):[a,b]\longrightarrow\mathbb{R}^3$ satisfy Hypothesis 1 in the vicinity of $s_0\in(a,b)$. Suppose also that the curve has continuous Frenet planes at $s_0\in(a,b)$. The curve has inflection at $s_0\in(a,b)$, in the sense of Definition 6, if and only if the orthogonal projections, of the curve onto its osculating plane and onto its rectifying plane at s_0, have inflection at s_0, in the sense of Definition 5.*

Theorem 1 (accompanying the proposal of Definition 6 and accompanied by Theorems 2 and 3, proved in Section 4.2) is the central contribution of this paper. Not only it gives a necessary and sufficient condition for the existence of an inflection on a spatial curve, but reduces this problem to (a) calculating the points at which the curvature vanishes, and (b) testing if the two orthogonal projections of the curve onto its osculating and rectifying plane have inflection in the usual 2D sense of Definition 5 (see Fig. 1). Theorem 1 cannot be applied to planar curves, since the projection of a planar curve at any point onto its rectifying plane is a line, thus no inflection can be detected. Inversely, if the projections of a curve onto its osculating plane and onto its rectifying planes, have inflection point at some parametric value, i.e. they are not lines, then this curve cannot be planar. Finally, one should note that Lemmas 4 and 5 imply, in addition, that the relation (3.20) holds as a necessary and sufficient condition only when the curve is planar and not when the curve is non-planar, thus answering the questions posed in the beginning of this subsection.

4.2. Orthogonal projections of a 3D curve near a 3D inflection point

This subsection is devoted to the justification of the characterization "inflection" for 3D curves by considering the behavior near such a point of all the orthogonal projections of the curve.

Goodman (1991) introduces the notion of *inflection count* for a 3D curve as "*the maximum number of inflections that the curve can appear to have when viewed from any direction*". He actually connects a vaguely implied notion of "convexity of a 3D curve", to the convexity of projections of the curve onto any plane. In this section we deal with 3D inflections, when viewed from any viewpoint. It is to be proved that, if a point satisfies Definition 6, then the orthogonal projections of the curve onto any plane satisfy the usual definition of 2D inflection point at the same position. Let us begin with the projection of a non-planar curve onto its normal plane, at the position of an inflection point.

Lemma 6. *Let a non-planar* $\mathbf{C}(s):[a,b]\longrightarrow\mathbb{R}^3$ *satisfy Hypothesis 1 in the vicinity of* $s_0\in(a,b)$*, where it has continuous Frenet planes. Suppose also that* $\mathbf{C}(s_0)$ *is an inflection of* $\mathbf{C}(s)$*, in the sense of Definition 6. Then, the orthogonal projection of* $\mathbf{C}(s)$ *onto its normal plane at* $s=s_0$ *has an inflection point in the sense of Definition 5 at this position.*

Proof. Without any loss of generality, adopting the assumption (3.24), the projection on the normal plane at s_0 of the dominant part of $\mathbf{C}(s)$, from (3.25), becomes:

$$\tilde{\mathbf{C}}_{norm}(s)=\begin{cases}\left(0,\ -(-1)^{n+2}k^{(n)}(s_0^-)\frac{e^{n+2}}{(n+2)!},\ -(-1)^p\chi(s_0^-)\frac{e^p}{p!}\right)^T, & s\in(s_0-\delta,s_0)\\ \left(0,\ k^{(m)}(s_0^+)\frac{e^{m+2}}{(m+2)!},\ \psi(s_0^+)\frac{e^q}{q!}\right)^T, & s\in(s_0,s_0+\delta)\end{cases}$$

Taking into account (3.22) and (4.26), if $(-1)^p\chi(s_0^-)>0$ and $\psi(s_0^+)>0$, then $\tilde{\mathbf{C}}_{norm}(s)$ lies in the third quadrant of the yz-plane, when $s\in(s_0-\delta,s_0)$, and in the first quadrant of the yz-plane, when $s\in(s_0,s_0+\delta)$, whereas if $(-1)^p\chi(s_0^-)<0$ and $\psi(s_0^+)<0$, then the $\tilde{\mathbf{C}}_{norm}(s)$ lies in the second quadrant of the yz-plane, when $s\in(s_0-\delta,s_0)$, and in the fourth quadrant of the yz-plane, when $s\in(s_0,s_0+\delta)$. Consequently, $\tilde{\mathbf{C}}_{norm}(s)$ and thus the projection of $\mathbf{C}(s)$ on the normal plane at s_0 has an inflection point at this position in the sense of Definition 5. □

Theorem 2. *Let* $\mathbf{C}(s):[a,b]\longrightarrow\mathbb{R}^3$ *satisfy Hypothesis 1 in the vicinity of* $s_0\in(a,b)$*, where it has continuous Frenet planes. Suppose also that* $\mathbf{C}(s_0)$ *is an inflection of* $\mathbf{C}(s)$*, in the sense of Definition 6. Then, the orthogonal projection of* $\mathbf{C}(s)$ *onto any plane with normal vector* $\mathbf{w}$*, such that* $\mathbf{w}\cdot\mathbf{b}(s_0^\pm)\neq 0$*, has an inflection point at* s_0*.*

Proof. Let us introduce an orthonormal system of vectors $\{\mathbf{v}_0,\mathbf{v}_1,\mathbf{w}\}$, with $\mathbf{w}=\mathbf{v}_0\times\mathbf{v}_1$, and consider the projection of $\mathbf{C}(s)$ onto a plane with normal equal to $\mathbf{w}$, i.e. a plane spanned by $\mathbf{v}_0$ and $\mathbf{v}_1$. This projection is given by:

$$\mathbf{P}(s)=(\mathbf{C}(s)\cdot\mathbf{v}_0)\mathbf{v}_0+(\mathbf{C}(s)\cdot\mathbf{v}_1)\mathbf{v}_1$$

On that plane, the normal curvature vector of $\mathbf{P}(s)$ is parallel to $(\mathbf{v}_0\times\mathbf{v}_1)$. Indeed, the numerator of the normal curvature vector of $\mathbf{P}(s)$ is:

$$\begin{aligned}\mathbf{P}'(s)\times\mathbf{P}''(s)&=\left((\mathbf{C}'(s)\cdot\mathbf{v}_0)\mathbf{v}_0+(\mathbf{C}'(s)\cdot\mathbf{v}_1)\mathbf{v}_1\right)\times\left((\mathbf{C}''(s)\cdot\mathbf{v}_0)\mathbf{v}_0+(\mathbf{C}''(s)\cdot\mathbf{v}_1)\mathbf{v}_1\right)\\&=\left((\mathbf{C}'(s)\cdot\mathbf{v}_0)(\mathbf{C}''(s)\cdot\mathbf{v}_1)\ -\ (\mathbf{C}'(s)\cdot\mathbf{v}_1)(\mathbf{C}''(s)\cdot\mathbf{v}_0)\right)(\mathbf{v}_0\times\mathbf{v}_1)\end{aligned}$$

Recalling the standard quadruple vector product identity we have:

$$\mathbf{P}'(s)\times\mathbf{P}''(s)=\left[\left(\mathbf{C}'(s)\times\mathbf{C}''(s)\right)\cdot(\mathbf{v}_0\times\mathbf{v}_1)\right](\mathbf{v}_0\times\mathbf{v}_1)$$

On the other hand, the length of the tangent vector remains the same, since:

$$\begin{aligned}\|\mathbf{P}'(s)\|^2&=\|(\mathbf{C}'(s)\cdot\mathbf{v}_0)\mathbf{v}_0+(\mathbf{C}'(s)\cdot\mathbf{v}_1)\mathbf{v}_1\|^2\\&=(\mathbf{C}'(s)\cdot\mathbf{v}_0)^2+(\mathbf{C}'(s)\cdot\mathbf{v}_1)^2=\mathbf{C}'(s)^2=\|\mathbf{C}'(s)\|^2\end{aligned}$$

So, the normal curvature vector of the projection becomes:

$$\mathbf{k}_{proj}(s)=\frac{\mathbf{P}'(s)\times\mathbf{P}''(s)}{\|\mathbf{P}'(s)\|^3}=\left[\frac{\mathbf{C}'(s)\times\mathbf{C}''(s)}{\|\mathbf{C}(s)\|^3}\cdot(\mathbf{v}_0\times\mathbf{v}_1)\right](\mathbf{v}_0\times\mathbf{v}_1)=(\mathbf{k}(s)\cdot\mathbf{w})\mathbf{w},$$

from which one obtains: $\mathbf{k}_{proj}(s)\cdot\mathbf{w}=\mathbf{k}(s)\cdot\mathbf{w}$ and finally $k_{proj}(s)\mathbf{b}_{proj}(s)\cdot\mathbf{w}=k(s)\mathbf{b}(s)\cdot\mathbf{w}$, by employing (2.8), where $\mathbf{b}_{proj}(s)$ is the binormal of the projection $\mathbf{P}(s)$ (equal to $\pm\mathbf{w}$). The last equation implies the following:

$$sign\left(\mathbf{b}_{proj}(s)\cdot\mathbf{w}\right)=sign\left(\mathbf{b}(s)\cdot\mathbf{w}\right).$$

The above relation, in the vicinity of s_0, becomes:

$$\begin{aligned}&sign\left(\mathbf{b}_{proj}(s_0^-)\cdot\mathbf{w}\right)=sign\left(\mathbf{b}(s_0^-)\cdot\mathbf{w}\right)\\&sign\left(\mathbf{b}_{proj}(s_0^+)\cdot\mathbf{w}\right)=sign\left(\mathbf{b}(s_0^+)\cdot\mathbf{w}\right)\end{aligned}\tag{4.27}$$

Since the curve $\mathbf{C}(s)$ satisfies Definition 6 at s_0, Lemma 4 implies that the binormal of the curve changes direction at s_0, i.e. $\mathbf{b}(s_0^-)\cdot\mathbf{b}(s_0^+)=-1$. Thus, if $\mathbf{b}(s_0^-)\cdot\mathbf{w}=\cos(\psi)$, then $\mathbf{b}(s_0^+)\cdot\mathbf{w}=\cos(\pi-\psi)$, i.e. the signs of these two inner products are opposite, for $\psi\neq\pi/2$, and so the signs of the inner products on the left hand sides of (4.27) are opposite as well. This implies that necessarily $\mathbf{b}_{proj}(s_0^-)\cdot\mathbf{b}_{proj}(s_0^+)=-1$, thus the binormal vector of the projection is discontinuous at s_0, i.e., the projection has inflection point at s_0, according to Lemma 3. □

When $\mathbf{b}(s_0^{\pm})\cdot\mathbf{w}=0$, the vector $\mathbf{w}$ lies in the osculating plane of the curve at s_0. If the curve is planar, then the projection on a plane with normal vector equal to $\mathbf{w}$ fails to detect unambiguously any inflection, since the projection of curve degenerates to a line. The next theorem proves that if the curve is non-planar, then the projection on any plane with normal vector on the osculating plane of the curve, has inflection at that point.

Theorem 3. *Let a non-planar $\mathbf{C}(s):[a,b]\longrightarrow\mathbb{R}^3$ satisfy Hypothesis 1 in the vicinity of $s_0\in(a,b)$, where it has continuous Frenet planes. Suppose also that $\mathbf{C}(s_0)$ is an inflection of $\mathbf{C}(s)$, in the sense of Definition 6. Then, the orthogonal projection of $\mathbf{C}(s)$ onto any plane with normal vector $\mathbf{w}$, such that $\mathbf{b}(s_0^{\pm})\cdot\mathbf{w}=0$, has an inflection point at s_0.*

Proof. If $\mathbf{b}(s_0^{\pm})\cdot\mathbf{w}=0$, then

$$\mathbf{w}=\gamma\mathbf{t}(s_0)+\delta\mathbf{n}(s_0^{\pm}),\quad \text{where } |\gamma|+|\delta|>0.$$

The plane with normal equal to $\mathbf{w}$ is spanned by the axis with unit vector $\mathbf{b}(s_0^{\pm})$ and the axis with unit vector:

$$\mathbf{w}^{\perp}=\mathbf{b}(s_0^{\pm})\times\mathbf{w}=\gamma\mathbf{n}(s_0^{\pm})-\delta\mathbf{t}(s_0).$$

Then, adopting the assumption

$$\mathbf{t}(s_0)=(1,\ 0,\ 0)^T,\quad \mathbf{n}(s_0^{\pm})=(0,\ \pm1,\ 0)^T,\quad \mathbf{b}(s_0^{\pm})=(0,\ 0,\ \pm1)^T,$$

the projection becomes:

$$\mathbf{P}(s)=\Big(-\gamma\delta C_y(s)+\delta^2Cx(s),\ \gamma^2C_y(s)-\gamma\delta C_x(s),\ Cz(s)\Big)^T,$$

where $\mathbf{C}(s)=\big(C_x(s),\ C_y(s),\ C_z(s)\big)^T$. Calculating the curvature numerator of $\mathbf{P}(s)$, leads to:

$$\mathbf{P}'(s)\times\mathbf{P}''(s)=\begin{pmatrix}\gamma\left[\gamma D_{yz}(s)-\delta D_{xz}(s)\right]\\ \delta\left[\gamma D_{yz}(s)-\delta D_{xz}(s)\right]\\ 0\end{pmatrix},$$

where the determinants

$$D_{xz}(s)=\begin{vmatrix}C'_x(s) & C'_z(s)\\ C''_x(s) & C''_x(s)\end{vmatrix}\quad\text{and}\quad D_{yz}(s)=\begin{vmatrix}C'_y(s) & C'_z(s)\\ C''_y(s) & C''_z(s)\end{vmatrix}$$

are actually equal to the signed curvature numerator of the projections of $\mathbf{C}(s)$ onto its rectifying plane and onto its normal plane, respectively. At $s=s_0$ the sign of $D_{xz}(s)$ changes, since the projection on the rectifying plane has inflection. Also, due to Lemma 6, since the projection of $\mathbf{C}(s)$ onto the normal plane at $s=s_0$ has inflection, and the determinant $D_{yz}(s)$ also changes sign at $s=s_0$. In other words, the binormal of the projection $\mathbf{P}(s)$, which is collinear to $\mathbf{P}'(s)\times\mathbf{P}''(s)$, flips direction at $s=s_0$ and consequently $\mathbf{P}(s)$ has inflection point at this position, due to Lemma 3. □

Theorems 2 and 3 establish that the projection of a spatial curve, with an inflection point, onto any plane has inflection at that point; the only exception being the case when the curve is planar and the projection is on a plane perpendicular to its osculating plane. These theorems lead directly to the following:

Corollary 2. *Let $\mathbf{C}(s):[a,b]\longrightarrow\mathbb{R}^3$ satisfy Hypothesis 1 in the vicinity of $s_0\in(a,b)$, where it has continuous Frenet planes. Suppose also that $\mathbf{C}(s_0)$ is an inflection of $\mathbf{C}(s)$, in the sense of Definition 6. Then, the inflection count of $\mathbf{C}(s)$ (as defined in Goodman, 1991) in the vicinity of s_0 is infinite.*

4.3. Contact with the tangent line at a 3D inflection point

In the introduction we commented on the definition of inflection for 3D curves adopted by Sasaki (1957), Lipschultz (1969), Manocha and Canny (1992) and Li and Cripps (1997). That definition is based, essentially, on the $(\ell+2)$-contact order of the curve with the tangent line at this point (see Definition 1 and also Bruce and Giblin, 1984, p.16-19 and Banchoff and Lovett, 2010, p.28-29). If the curve $\mathbf{C}(s)$ possesses sufficiently-high continuity order, and presents, according to Definition 6, an inflection at $\mathbf{C}(s_0)$, then the tangent line at s_0 has $(\ell+2)$-point contact, for $\ell>0$, with this curve.

In the vicinity of s_0, the tangent line of $\mathbf{C}(s)$ is given by

$$\mathbf{L}(s)=\begin{cases}\mathbf{C}(s_0)-e\mathbf{t}(s_0),\ s\in(s_0-\delta,s_0)\\ \mathbf{C}(s_0)+e\mathbf{t}(s_0),\ s\in(s_0,s_0+\delta)\end{cases}$$

Then, with the aid of (3.23), one can easily obtain:

$$\mathbf{C}(s) - \mathbf{L}(s) = \begin{cases} \frac{(-1)^{n+2} e^{n+2}}{(n+2)!} \mathbf{C}^{(n+2)}(s_0^-) + O(e^{n+3}), \ s \in (s_0 - \delta, s_0) \\ \frac{e^{m+2}}{(m+2)!} \mathbf{C}^{(m+2)}(s_0^+) + O(e^{m+3}), \ s \in (s_0, s_0 + \delta) \end{cases}$$

where $e = |s - s_0|$. Supposing that $\ell = n = m$, the tangent line and has exactly an $(\ell + 2)$-contact with the curve $\mathbf{C}(s)$ at s_0. On the other hand, the existence of an $(\ell + 2)$-contact of the curve $\mathbf{C}(s)$ at s_0 with its tangent line is not sufficient for Definition 6 to hold, since Definition 6 restricts the characterization of 3D inflections to those points the curve has no support plane at them. Definition 6 holds at s_0 if and only if ℓ is odd. Indeed, Lemma 1 ensures that the normal and the binormal vectors flip direction at s_0. Then, it is straightforward to see that the projections of $\mathbf{C}(s)$ on the osculating plane and on the rectifying plane have inflection, thus the curve has inflection, by Theorem 1.

4.4. The tangent indicatrix in the neighborhood of a 3D inflection point

For a regular planar curve, the image of the unit tangent $\mathbf{t}(s)$ lies on the unit circle, thus it can be parameterized with the polar angle, ω, of its head. This relation can be expressed by a function $\omega = \omega(s)$. At an inflection point, $\mathbf{t}'$ changes direction, and so the monotonicity of the polar angle changes. This observation can be generalized for spatial curves, at the points, which satisfy Definition 6.

Assuming that $\mathbf{C}(s)$ is regular, with natural parameterization, the head of its tangent vector, $\mathbf{t}$, lies always on the unit sphere (see Fig. 12). This is the so-called *spherical indicatrix of the tangent of the curve* (see Weatherburn, 1988 p.28). The image of the normal plane of the curve coincides with the tangent plane of the unit sphere at the head of $\mathbf{t}$. The exact position of the image of the osculating plane is determined by the expression $\mathbf{t}' = k\mathbf{b} \Longrightarrow \mathbf{b} = \mathbf{t}'/k$ and the image of the rectifying plane is also perpendicular to the other two image planes. Both the images of the osculating plane and of the rectifying plane of $\mathbf{C}(s)$ are passing through the vector $\mathbf{t}$, and so pass through the origin and the head of $\mathbf{t}$, splitting the sphere into two hemispheres.

It has been shown in Section 3.2 that the Frenet planes can be determined at those points, where conditions (3.20) hold. Then, the image of the osculating plane cuts the head of $\mathbf{t}$ and the image of the rectifying plane leaves the head of $\mathbf{t}$ on one half space of it (an example is presented in Fig. 12).

5. Examples

Example 1. Inflections in $\left(s,\ s^3,\ s^p\right)^T$ The first example considers existence of an inflection point on the one-parameter family of curves $\left(s,\ s^3,\ s^p\right)^T$, for p positive integer greater than 3 and $s \in [-1, 1]$, at $s = 0$. (For $p = 1, 2, 3$ see Example 2.) The tangent at $s = 0$ is $(1,\ 0,\ 0)^T$. The normal curvature vector numerator is

$$\left(3p(p-3)s^p,\ -p(p-1)s^{p-2},\ 6s\right)^T = s\mathbf{v}(s),$$

where $\mathbf{v}(s) = \left(3p(p-3)s^{p-1},\ -p(p-1)s^{p-3},\ 6\right)^T$, thus the binormal vector is given by:

$$\mathbf{b}(s) = \frac{s}{|s|}\frac{\mathbf{v}(s)}{\|\mathbf{v}(s)\|}, \quad s \neq 0.$$

Then, it is straightforward to see that, near $s = 0$:

$$\mathbf{b}(0^-) = (-1)\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{b}(0^+) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

i.e. (3.20) holds for all $p > 3$. Then, the projection on the osculating plane at $s = 0$ is $\left(s,\ s^3,\ 0\right)^T$ and the projection on the rectifying plane at $s = 0$ is $\left(s,\ 0,\ s^p\right)^T$. Thus, Definition 6 holds at $s = 0$ for odd p, i.e., the corresponding curves have an inflection at $s = 0$. For even p, the curve has no inflection point at $s = 0$, because its projection on the rectifying plane (xz-plane) lies entirely on the positive-z half-plane, i.e. the xy-plane is a support plane of the curve at that point. Figs. 2 and 3 illustrate the results of Theorem 1 for $p = 5$ and Figs. 4 and 5 illustrate the results of Theorem 1 for $p = 4$, by presenting a number of projections of the corresponding curves on some of the planes passing through the tangent line of the curve, which coincides with x-axis. The fact that for p even, the curve does not have an inflection, although all but one projections (in Figs. 4 and 5) present inflection at this point, sounds like a paradox. However, existence of at-least one support plane, at a point of a curve characterizes it as 3D convex at that point, according to Definition 3. Indeed, all curves of the family $\left(s,\ s^3,\ s^p\right)^T$, with p even, are convex near $s = 0$ and they lie on their convex hulls.

Example 2. Spatial & planar generalized cubic curves The generalized cubic curves in $\mathbb{R}^3$ (see Gabrielides and Sapidis, 2020) are defined by:

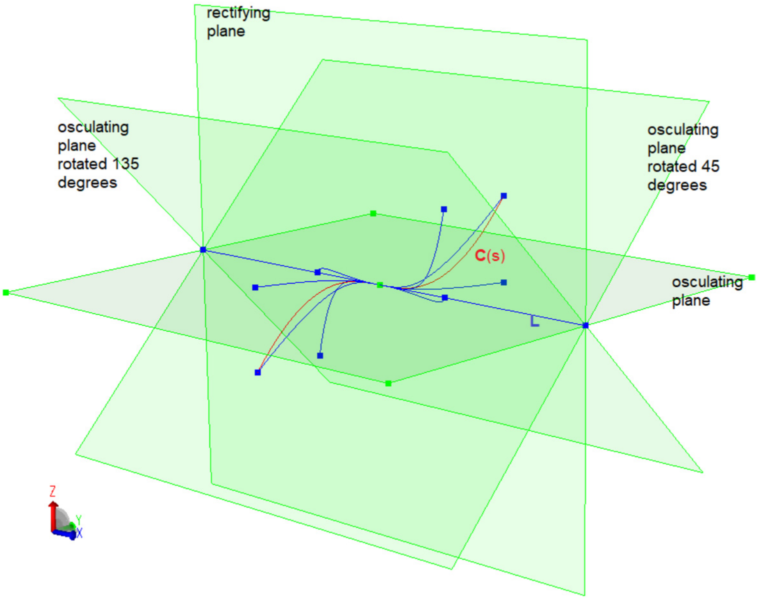

Fig. 2. Example 1(a): the red curve is $\mathbf{C}(s) = \left(s,\ s^3,\ s^5\right)^T$, $s \in [-1, 1]$ and the blue straight-line **L** is the tangent line of $\mathbf{C}(s)$ at $s = 0$, where curvature vanishes, along with a number of orthogonal projections of $\mathbf{C}(s)$ on planes passing through **L**. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

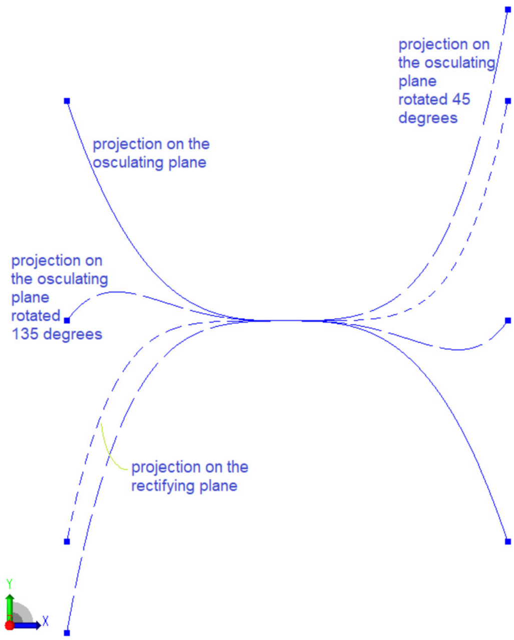

Fig. 3. Example 1(a): the projections of the curve $\mathbf{C}(s)$ in Fig. 2 presented on the same plane. All the projections of $\mathbf{C}(s)$ present an inflection at $s = 0$, thus $\mathbf{C}(s)$ has a 3D inflection at this point.

$$\mathbf{F}(s) = \mathbf{P}(1-t) + \mathbf{Q}t + \mathbf{U}h^2u(t) + \mathbf{V}h^2v(t), \quad t = \frac{s-a}{h},\ h = b-a,\ s \in [a,b],$$

where $\mathbf{P}, \mathbf{Q}, \mathbf{U}, \mathbf{V} \in \mathbb{R}^3$ and the basis functions $u(t), v(t)$ are C^2-continuous and such that their second order derivatives form a Chebyshev system. Since generalized cubic curves are C^2 continuous, their curvature numerator should vanish at any inflection point. However, this does not happen, when **P**, **Q**, **U** and **V** are not co-planar (see Gabrielides and Sapidis, 2020, Theorem 7), thus they never have inflection points.

Obviously, for $u(t) = t^2$ and $v(t) = t^3$, $\mathbf{F}(s)$ represents a cubic polynomial curve in the 3D space. In this respect, the curve $\left(s,\ s^3,\ s^p\right)^T$, for $p = 2$ of Example 1 is a non-planar cubic polynomial curve, thus it cannot have an inflection point.

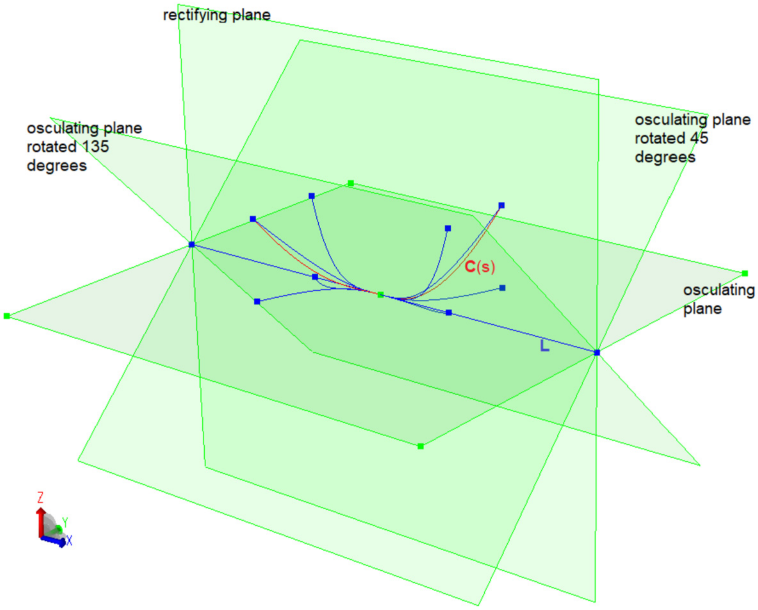

Fig. 4. Example 1(b): the red curve is $\mathbf{C}(s) = \left(s,\ s^3,\ s^4\right)^T,\ s \in [-1, 1]$ and the blue straight-line $\mathbf{L}$ is the tangent line of $\mathbf{C}(s)$ at $s = 0$, where curvature vanishes, along with a number of orthogonal projections of $\mathbf{C}(s)$ on planes passing through $\mathbf{L}$.

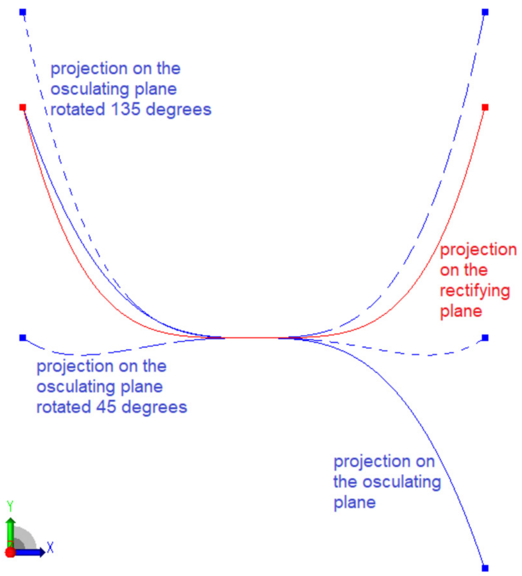

Fig. 5. Example 1(b): the projections of the curve $\mathbf{C}(s)$ in Fig. 4 presented on the same plane. The red one, which is the projection of $\mathbf{C}(s)$ on its rectifying plane at $s = 0$, does not present an inflection, thus $\mathbf{C}(s)$ has a support plane at $s = 0$.

The curves $\left(s,\ s^3,\ s^p\right)^T$, for $p = 1$ and $p = 3$ are planar cubics. If a curve is planar, then the plane, in which the curve lies, is its osculating plane. It is straightforward to verify that for $p = 1$ the curve lies on $x - z = 0$ while for $p = 3$ it lies on $y + z = 0$. Moreover, one can easily see that the curvature numerator in both cases vanishes at $s = 0$. In both cases that point is actually an inflection point, since both curves are split by their tangent line at $s = 0$, or by their rectifying planes, which are $y = 0$ and $y + z = 0$, respectively. Alternatively, applying the following affine transformations, one ends up with the cubic polynomial s^3:

$$\text{for } p = 1: \begin{bmatrix} s \\ s^3 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} s \\ s^3 \\ s \end{bmatrix} \quad \text{and for } p = 3: \begin{bmatrix} s \\ s^3 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} s \\ s^3 \\ s^3 \end{bmatrix}$$

which has an inflection point at $s = 0$.

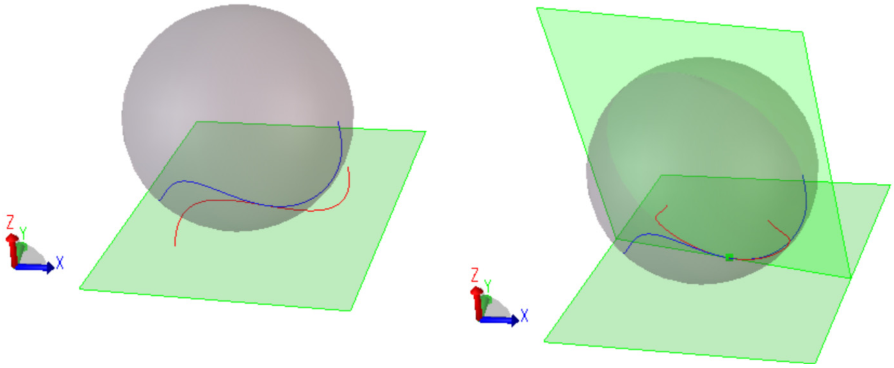

Fig. 6. Example 3: The blue curve lies on a sphere. In the left image, the red curve is the projection of the blue one on its osculating plane at the inflection candidate. Clearly this projection has inflection at that point. In the right image, the red curve is the projection of the blue curve on the rectifying plane at the inflection candidate. This projection does not have inflection. The osculating plane at the inflection-candidate is the support plane of the curve at this point.

Example 3. G^1-continuous "S"-shaped curve on a sphere (We thank very much an anonymous reviewer for recommending this important example case). This example presents an "S"-shaped curve on a sphere. Such a curve does not have any inflection point. The curve is actually convex, since it has a support plane at all its points, which is tangential to the sphere. The support planes of such a curve are its osculating planes. At the parametric value, where the curvature vanishes, the projection of the curve onto its osculating plane has an inflection point but not on the rectifying plane (see Fig. 6).

Example 4. Inflection at a node of a G^1 continuous cubic spline This example presents a G^1 continuous cubic spline, consisting of two segments, with the following control points:

$$\begin{pmatrix} 0.0000000000 \\ 0.0000000000 \\ 0.0000000000 \end{pmatrix}, \begin{pmatrix} 3.3128834356 \\ 0.3680981595 \\ -0.1840490798 \end{pmatrix}, \begin{pmatrix} 6.6871165644 \\ 0.1840490798 \\ -0.3680981595 \end{pmatrix}, \begin{pmatrix} 10.0000000000 \\ 0.0000000000 \\ 0.0000000000 \end{pmatrix}$$

and

$$\begin{pmatrix} 10.0000000000 \\ 0.0000000000 \\ 0.0000000000 \end{pmatrix}, \begin{pmatrix} 13.3460122699 \\ -0.1858895706 \\ 0.3717791411 \end{pmatrix}, \begin{pmatrix} 16.7539877301 \\ -0.3717791411 \\ 0.1858895706 \end{pmatrix}, \begin{pmatrix} 20.1000000000 \\ 0.0000000000 \\ 0.0000000000 \end{pmatrix}.$$

At the intermediate node $\mathbf{p} = (10,\ 0,\ 0)^T$, it is straightforward to verify that the curve is G^1 continuous, and its unit tangent vector at it is equal to $(0.9938,\ -0.055,\ 0.1104)^T$. At the same point, the normal and the binormal vectors of the first segment are $(-0.1,\ 0,\ 0.9)^T$ and $(-0.0548,\ -0.9985,\ -0.0060)^T$, respectively, while of the second segment are $(0.1,\ 0,\ -0.9)^T$ and $(0.0548,\ 0.9985,\ 0.0060)^T$, respectively. Thus, the osculating plane and the rectifying plane are continuous at $\mathbf{p}$, i.e. Hypothesis 1 is satisfied. The curvature is discontinuous at $\mathbf{p}$, being equal to 0.0332 and 0.0329 before and after it. Taking the projections of both segments on the osculating plane and on the rectifying plane, proves that the curve presents an inflection at this position, satisfying Theorem 1 (see Figs. 7 and 8). None of the segments has an inflection in its interior, since the curvature of a non-planar cubic polynomial never vanishes. Thus, this non-planar curve presents an inflection at a point, where the curvature is discontinuous and not zero.

Now, if one rotates the second segment of the curve around the axis with unit vector equal to the tangent at the middle point, then the new composite curve has not an inflection any more, because the osculating and the rectifying planes are discontinuous at this point. Fig. 9 illustrates this.

Example 5. Inflection at a node of a C^2 continuous quintic spline The last example presents a C^2 quintic spline of two segments, which has an inflection at the intermediate node. The spline segments satisfy Hermite boundary conditions. The data points are: $(0,\ 0,\ 0)^T$, $(3,\ 2,\ 0)^T$, $(6,\ 3,\ 1)^T$, the tangent vectors on them are: $(0,\ 5,\ 0.5)^T$, $(5,\ 0,\ 0)^T$, $(0,\ 2.5,\ 2.5)^T$, and the second order derivative vectors are: $(10,\ -6,\ 4)^T$, $(0,\ 0,\ 0)^T$, $(0,\ 10,\ 0)^T$. The curvature of the resulting curve vanishes at the intermediate node and the osculating plane at it is the xy-plane. The curve has no support plane at the intermediate node, i.e. it presents an inflection. Figs. 10 and 11 illustrate the results. Fig. 12 illustrates the spherical indicatrix of the tangent of the curve.

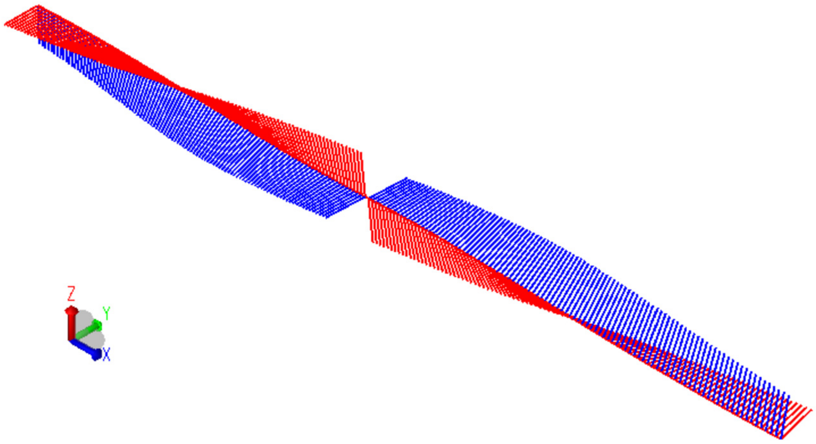

Fig. 7. Example 4: The G^1 continuous cubic spline of two segments of Example 4, along with the distributions of the normal (red) and the binormal (blue) vectors. None of the segments has an inflection in its interior. The spline has continuous osculating and rectifying planes at the intermediate node.

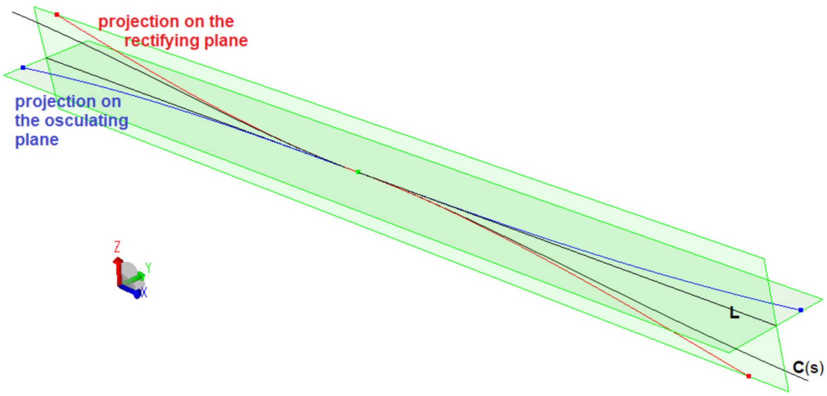

Fig. 8. Example 4: The black curve is the G^1 continuous cubic spline of two segments, $\mathbf{C}(s)$, of Fig. 7 and $\mathbf{L}$ is the tangent line at the intermediate node. In blue is the projection of $\mathbf{C}(s)$ on its osculating plane at the intermediate node and red is the projection of $\mathbf{C}(s)$ on its rectifying plane at the intermediate node. Both projections have inflection at the node of the spline, thus the curve has no support planes at this point, i.e. it presents a 3D inflection at its intermediate node.

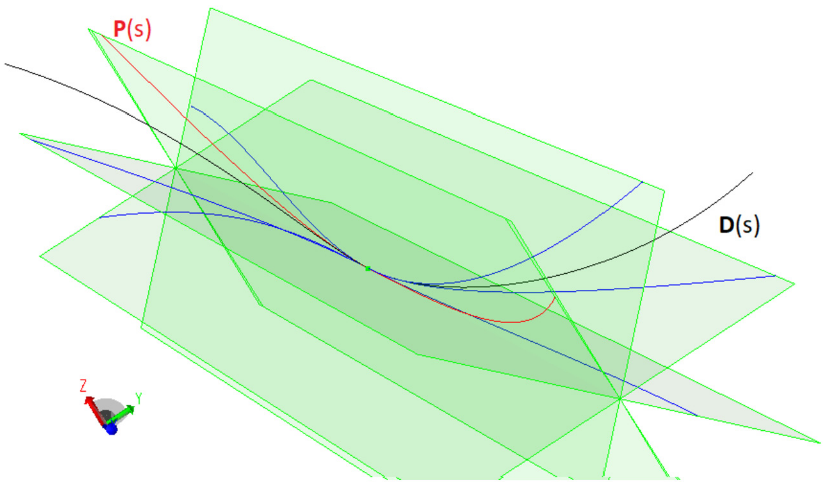

Fig. 9. Example 4: Rotating the second segment of the spline shown in Fig. 7 one creates the G^1 spline $\mathbf{D}(s)$ (black). The curve $\mathbf{D}(s)$ has not continuous osculating and rectifying planes at the intermediate node and it has support planes at it. The projection $\mathbf{P}(s)$ (red) on one of the planes passing through the tangent line of the curve at the intermediate node is convex, implying that $\mathbf{D}(s)$ has a support plane at that point.

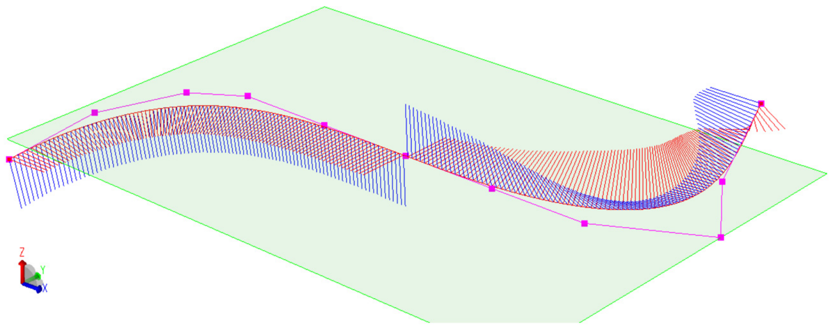

Fig. 10. Example 5: The C^2 continuous quintic spline of two segments of Example 5, along with the distributions of the normal (red) and the binormal(blue) vectors. None of the segments has an inflection in its interior. The spline has continuous osculating and rectifying planes at the intermediate node.

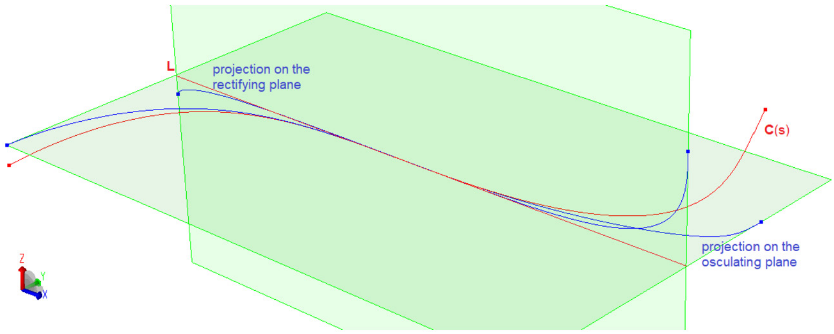

Fig. 11. Example 5: The C^2 continuous quintic spline of Fig. 10, along with its two projections on the osculating and on the rectifying planes at the intermediate node. Both present an inflection, thus the curve has a 3D inflection at that node.

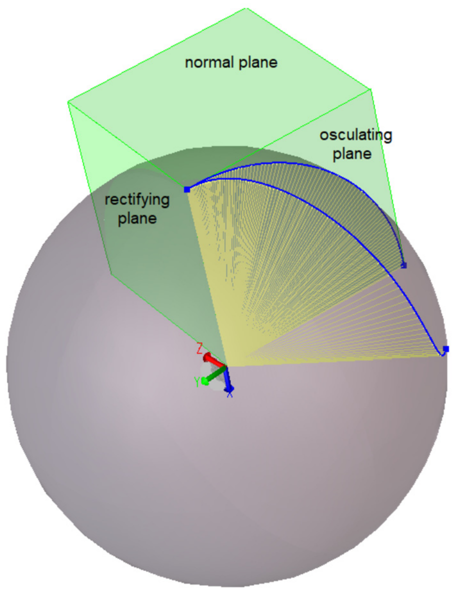

Fig. 12. Example 5: The unit tangents (yellow) and the spherical indicatrix of the tangent (blue) of the curve $\mathbf{C}(s)$ in Fig. 10 on the unit sphere. The images of the Frenet-planes (green) are shown at the point of inflection of the curve.

6. An algorithm to detect inflection points in a spatial curve

Assume that one has an algorithm to identify if a given point on a planar curve is an inflection or not, i.e. if conditions (3.20) hold, according to Definition 5. Given a non-planar curve $\mathbf{C}(s)$, the following algorithm detects inflections on this curve.

Step 1: Identify points, where continuity of the curve is finite and split the curve into segments at those points.
Step 2: At the common point of any two consecutive segments, check if the curve is at least G^1-continuous. If it is, then check if (3.20) holds. If this is true, then test if the conditions of Theorem 1 hold. If this is true, then this is an inflection point.
Step 3: In the interior of every segment of the curve, compute the zeros of the curvature numerator. At each of these points check if (3.20) holds. If this is true, then test if the conditions of Theorem 1 hold. If this is true, then this point is an inflection point.

The conditions of Theorem 1 can be verified at the position $s = s_0$, by checking if the projections of the curve onto its osculating plane and onto its rectifying plane (at s_0) have inflection point at s_0.

6.1. Potential applications of the concept of 3D inflection

The field of applications of the definition of inflection for 3D curves and consequently of the above algorithm, is mainly an unexplored territory. However, in the literature there exists a large number of research-papers, the topics of which are related directly or indirectly to the present research. They come from various disciplines. For example:

- interrogation techniques based on local Differential Geometry of curves, e.g., Patrikalakis and Maekawa (2002, p.200), Piegl and Tiller (1996, p.481),
- inflections of curves on surfaces, e.g., Hoitsma (1996), Pottmann and Paukowitsch (1997),
- definition and calculation of frames along a curve, such as rotation minimizing frames, e.g., Shah et al. (2010), Farouki et al. (2020),
- global Differential Geometry of curves, e.g., extensions of the 4-vertex theorem and related topics, in works such as Sedykh (1994), Fuster and Sedykh (1995), Shiba and Umehara (2012), Ghomi (2012), Ghomi (2019), and Dias and Tari (2018),
- spatial motion in kinematic geometry of mechanisms, e.g., Skreiner (1966), Skreiner (1967), Bokelberg et al. (1992) and Condurache (2018)
- 3D curve evolution by the curvature vector, e.g., Altschuler (1991) and Altschuler and Grayson (1992)
- calculation of the convex hull of a spatial curve, e.g., Seong et al. (2004) and Ranestad and Sturmfels (2012), and finally
- 3D curve construction, e.g., by shape-preserving interpolation, Kaklis and Karavelas (1997), Goodman (2001), and Gabrielides (2012) or by shape-preserving approximation, e.g. Kong and Ong (2009).

7. Future work

Let us briefly pose a number of probable open questions related to this work, which would worth research effort in the future.

i Is it possible to generalize the idea of employing support (hyper)planes in order to define inflections in curves of higher dimensions?

ii Existence of an inflection in a 3D curve implies that the curve is non-convex, since at the inflection point there exist no support plane. Inversely, if a 3D curve is not convex, does this have any inflection point?

iii As presented in Example 2 a spatial generalized cubic curve cannot have an inflection point. Spatial generalized cubic curves belong to a wider class of curves called "*true 3D curves*", meaning curves with the property that no four points on them belong to the same plane (see Gabrielides and Sapidis, 2020, Boehm, 1982). Is it possible for a true 3D curve to have inflection point?

iv Given an ordered set of points on the plane and a corresponding sequence of increasing parametric values, a shape-preserving curve interpolation algorithm detects parametric sub-intervals between couples of consecutive points, in which the curve has necessarily inflection. Then, the algorithm tries to limit the number of inflection points to one per such segment. In 3D shape-preserving curve interpolation, is it possible to detect such intervals? In other words, is it possible to detect inflectional segments on a 3D polygon of data points? Then, what should an interpolation curve do in order to preserve the inflectional character of the corresponding segments, but limit the number of inflection points to at-most one per segment?

v Let us consider the case of a curve $\mathbf{C}(s) : [a, b] \longrightarrow \mathbb{R}^3$, where, for the interval $[c, d] \in (a, b)$, the two end normal curvature vectors (and the normal and binormal vectors) are such that:

$$\mathbf{k}(c) \cdot \mathbf{k}(d) < 0, \quad \mathbf{b}(c) \cdot \mathbf{b}(d) < 0, \quad \mathbf{n}(c) \cdot \mathbf{n}(d) < 0$$

Obviously, the above inequalities do not imply that there exists a point in (c, d) such that $\mathbf{k}(s)$ vanishes and $\mathbf{b}(s)$, $\mathbf{n}(s)$ are discontinuous and, moreover, that there exists an inflection point in (c, d) satisfying Definition 6. However, it is interesting to investigate the consequences of the above, and especially the case where $[c, d]$ is very small. Such an "*inflection sub-interval*" may cause abrupt changes in the binormal distribution and "visual inflections" in some view directions. Similar to the above case, is the situation $\mathbf{b}(c) \cdot \mathbf{b}(d) = -1$ and $\mathbf{n}(c) \cdot \mathbf{n}(d) = -1$. Research on these cases is of interest for 3D curve interpolation/approximation and fairing.

8. Conclusions

This study presents a definition of the inflection point for curves in 3D space. This is based on considering the non-existence of any support plane at a point of the curve, which generalizes the non-existence of a support line at inflection points in 2D curves. Although past works did present a concept of inflection for 3D curves, the new definition avoids the drawbacks of the existing ones described in the Introduction.

Theorem 1 provides a necessary and sufficient condition for the existence of an inflection in a 3D curve, which permits a rigorous calculation of the inflection points of a 3D curve. The analysis has been done for regular curves, in the vicinity of points, where the curve is G^1 continuous and analytic in sufficiently small parametric sub-intervals near them. This analysis covers the case of G^1-continuous piecewise analytic curves, such as splines, with continuous osculating and rectifying planes. The results have been illustrated in three examples, where an inflection appears at a point, at which the curve is analytic (Example 1 and Example 3), the curve is G^1 continuous (Example 4) and the curve is C^2 continuous (Example 5). Also Example 2 illustrates non-existence of inflection points on generalized cubic curves.

Corollary 2, a direct consequence of Theorems 2 and 3, establishes that, at any inflection point of a 3D curve, the curve corresponds to an infinite "inflection count" (as this concept is defined in Goodman, 1991), i.e. a viewer sees an inflection at this point, viewing the curve from any direction. In other words, "*an inflection point on a 3D curve, in the sense of Definition 6, can hardly be hidden from viewing*", as this induces inflections also to all planar projections of this 3D curve.

CRediT authorship contribution statement

Nikolaos Gabrielides developed the theoretical formalism, performed the analytic calculations and performed the numerical simulations. Both Nikolaos Gabrielides and Nickolas Sapidis contributed to the final version of the manuscript. Nickolas Sapidis supervised the project.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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