PARTIAL SOLUTIONS TO REAL ANALYSIS, FOLLAND

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Abstract. This following are partial solutions to exercises on Real Analysis, Folland, written concurrently as I took graduate real analysis at the University of California, Los Angeles. Last Updated: November 18, 2019

Contents

1. Chapter 1-Measures

Folland 1.10

Given a measure space $(X, \mathcal{M}, \mu)$ and $E \in \mathcal{M}$, define $\mu_E(A) = \mu(A \cup E)$ for $A \in \mathcal{M}$. Then μ_E is a measure.

Proof. First of all, $\mu_E(\emptyset) = \mu(\emptyset \cap E) = \mu(\emptyset) = 0$. Also, suppose $A_1, A_2, ... \in \mathcal{M}$, $\mu_E(\bigcup_{i=1}^{\infty}) = \mu(E \cap (\bigcup_{i=1}^{\infty} A_i)) = \mu(\bigcup_{i=1}^{\infty}(A_i \cap E)) = \sum_{i=1}^{\infty} \mu(A_i \cap E) = \sum_{i=1}^{\infty} \mu_E(A_i)$. Then μ_E is a measure. $\square$

2. Chapter 2-Integration

Folland 2.6

The supremum of an uncountable family of measurable $\bar{\mathbb{R}}$-valued functions on X can fail to be measurable (unless the σ-algebra is really special).

Proof. Let $X = \mathbb{R}$, $\mathcal{M} = \mathcal{L}$. We know that there is $A \subset \mathbb{R}$ such that $A \notin \mathcal{L}$. Then we define $f_x : \mathbb{R} \to \bar{\mathbb{R}}$ by $f_x = \chi_{\{x\}}$ for each $x \in \mathbb{R}$. Notice that $f := \sup_{x \in A} f_x = \sup_{x \in A} \chi_{\{x\}} = \chi_A$, and f is not measurable since $f^{-1}([1, \infty)) = A$ is non-measurable while $[1, \infty)$ is Borel. $\square$

Folland 2.7

Suppose that for each $\alpha \in \mathbb{R}$ we are given a set $E_\alpha \subset E_\beta$ whenever $\alpha < \beta$, $\bigcup_{\alpha \in \mathbb{R}} E_\alpha = X$ and $\bigcap_{\alpha \in \mathbb{R}} E_\alpha = \emptyset$. Then there is a measurable function $f : X \to \mathbb{R}$ such that $f(x) \leq \alpha$ on E_α and $f(x) \geq \alpha$ on E_α^c for every α.

Proof Sketch. Show that $f(x) := \inf\{\alpha \in \mathbb{R} : x \in E_\alpha\}$ works. $\square$

Folland 2.8

If $f : \mathbb{R} \to \mathbb{R}$ is monotone, then f is Borel measurable.

Proof. We shall show the conclusion by showing that for every $a \in \mathbb{R}$, $f^{-1}([a, \infty))$ is an interval, which is Borel measurable. Given $x, y \in f^{-1}([a, \infty))$, for any $z \in [x, y]$, since f is monotone, $f(z) \in [f(x), f(y)]$ and thus $z \in f^{-1}([a, \infty))$. Thus $f^{-1}([a, \infty))$ is an interval and we finish the proof. $\square$

Folland 2.9

Let $f : [0, 1] \to [0, 1]$ be the Cantor function, and let $g(x) = f(x) + x$.

(a) g is a bijection from $[0, 1]$ to $[0, 2]$, and $h = g^{-1}$ is continuous from $[0, 2]$ to $[0, 1]$.
(b) If C is the Cantor set, $m(g(C)) = 1$.
(c) By Exercise 29 Chapter 1, $g(C)$ contains a Lebesgue non-measurable set A. Let $B = g^{-1}(A)$. Then B is Lebesgue measurable but not Borel.
(d) There exists a Lebesgue measurable function F and a continuous function G on $\mathbb{R}$ such that $F \circ G$ is not Lebesgue measurable.

Proof. (a) Since f is monotone increasing, if $y \neq x$, without loss of generality we assume $y > x$ and $f(y) \geq f(x)$. Then $g(y) = f(y) + y > f(x) + x = g(x)$, so g is injective and monotone increasing. g is continuous as a sum of two continuous functions, and $g(0) = 0$ and $g(1) = f(1) + 1 = 2$, by intermediate value theorem the whole $[0, 2]$ is mapped by g and g is surjective. To show that h is continuous, it suffices to show that g is open map, and to that end it suffices to show that g maps open intervals to open intervals since every open set is a disjoint union of them. But that is apparent since g is monotone and thus $g(a, b) = (g(a), g(b))$. Thus h is continuous on $[0, 1]$.

(b) Since g is surjective, $[0, 2] = g([0, 1] \setminus C) \sqcup g(C)$, so it suffices to show that $m(g([0, 1] \setminus C)) = 1$. $[0, 1] \setminus C$ is open since $[0, 1]$ and C are both closed, $[0, 1] \setminus C$ is just a disjoint union of open intervals on which f is constant. Therefore, $g([0, 1] \setminus C)$ is a disjoint union of open intervals of the form $(f(a) + a, f(b) + b) = (f(a) + a, f(a) + b)$ and thus $m(a, b) = m(f(a)+a, f(b)+b)$. By countable additivity of m we get $1 = m([0, 1] \setminus C) = g([0, 1] \setminus C)$ and thus $m(g(C)) = 1$.

(c) Notice that $B = g^{-1}(A) \subset g^{-1}(g(C)) = C$ since g is a bijection. Also $m(B) \leq m(C) = 0$. By completeness of Lebesgue measure B is Lebesgue measurable. If B is Borel, $h^{-1}(B) = g(B) = A$ is Borel by continuity of h, a contradiction. Hence B is not Borel.

(d) Let $F = \chi_B$ and $G = h$. Then $F \circ G : [0, 2] \to [0, 1]$ such that F is Lebesgue measurable (since $B \in \mathcal{L}$) and h is continuous. Notice that

$$(F \circ G)^{-1}([1, \infty)) = G^{-1} \circ F^{-1}([1, \infty)) = G^{-1}(B) = g(B) = A \notin \mathcal{L}$$

so $F \circ G$ is not Lebesgue measurable. This finishes the proof.

□

Folland 2.11

Suppose that f is a function on $\mathbb{R} \times \mathbb{R}^k$ such that $f(x, \cdot)$ is Borel measurable for each $x \in \mathbb{R}$ and $f(\cdot, y)$ is continuous for each $y \in \mathbb{R}^k$. For $n \in \mathbb{N}$, define f_n as follows. For $i \in \mathbb{Z}$ let $a_i = i/n$, and for $a_i \leq x \leq a_{i+1}$ let

$$f_n(x, y) = \frac{f(a_{i+1}, y)(x - a_i) - f(a_i, y)(x - a_{i+1})}{a_{i+1} - a_i}$$

Then f_n is Borel measurable on $\mathbb{R} \times \mathbb{R}^k$ and $f_n \to f$ pointwise; hence f is Borel measurable on $\mathbb{R} \times \mathbb{R}^k$. Conclude by induction that every function $\mathbb{R}^n$ that is continuous in each variable separately is Borel measurable.

Proof. Observe that

$$|f_n(x, y) - f(x, y)| = \frac{f(a_{i+1}, y)(x - a_i) - f(a_i, y)(x - a_{i+1}) - f(x, y)(x - a_{i+1}) + f(x, y)(x - a_i)}{a_{i+1} - a_i}$$

□

Folland 2.19

Suppose $\{f_n\} \subset L^1(\mu)$ and $f_n \to f$ uniformly.

(a) If $\mu(X) < \infty$, then $f \in L^1(\mu)$ and $\int f_n \to \int f$

(b) If $\mu(X) = \infty$, the conclusions of (a) can fail.

Proof.

(a) First of all, since $f_n \to f$ uniformly, and f_n is measurable for each n, f is measurable. Let $\epsilon > 0$, since $f_n \to f$ uniformly, there is some $N \in \mathbb{N}$ such that when $n \geq N$, $|f_n(x) - f(x)| < \epsilon$ for all x. Then $|f| < |f_N| + \epsilon$, and

$$\int |f| < \int |f_N| + \epsilon\mu(X) < \infty$$

which implies that $f \in L^1$. Fix ϵ and N above, let $g = \max\{|f_1|, ..., |f_N|, |f| + \epsilon\}$, we can see that g is clearly measurable and $|f_n| \leq g$ for all n. Then we apply the dominated convergence theorem and get that $\int f_n \to \int f$.

(b) Let $f_n = \dfrac{1}{n}\chi_{[-n,n]}$. First we show that $f_n \to 0$ uniformly. Let $\epsilon > 0$ and $N = \lceil \frac{1}{\epsilon} \rceil$. When $n > N$, $|f_n - 0| = |f_n| < \epsilon$, so $f_n \to 0$ uniformly. However, $\int f_n = 2 \neq 0$ for all n, showing that conclusions of (a) can fail.

$\square$

Folland 2.21

Suppose $f_n, f \in L^1$ and $f_n \to f$ a.e., then $\int |f_n - f| \to 0$ iff $\int |f_n| \to \int |f|$.

Proof. Suppose $\int |f_n - f| \to 0$, $|\int |f_n - f|\,| \to 0$. Since $|\int |f_n| - \int |f|\,| \leq |\int |f_n - f|\,| \to 0$, $\int |f_n| \to \int |f|$.
Conversely, suppose $\int |f_n| \to \int |f|$, then $\int |f_n| + |f| \to \int 2|f|$. Notice that $f_n, f \in L^1$, $f_n - f \in L^1$, and thus $|f_n|$, $|f|$, $|f_n| + |f|$, and $|f_n - f| \in L^1$. Also, $f_n \to f$ a.e., it is clear that $|f_n - f| \to 0$ and $|f_n| + |f| \to 2f$ a.e.. Since $||f_n| - |f|| \leq |f_n| + |f|$ and $\int |f_n| + |f| \to \int 2f$, by previous problem $\int |f_n - f| \to 0$. Then the proof is complete. $\square$

Folland 2.25

Let $f(x) = x^{-1/2}$ if $0 < x < 1$, $f(x) = 0$ otherwise. Let $\{r_n\}_1^\infty$ be an enumeration of the rationals, and set $g(x) = \sum_1^\infty 2^{-n} f(x - r_n)$.

(a) $g \in L^1(m)$, and in particular $g < \infty$ a.e.

(b) g is discontinuous at every point and unbounded on every interval, and it remains so after any modification on a Lebesgue null set.

(c) $g^2 < \infty$ a.e. but g^2 is not integrable on any interval.

Proof. (a) For each n, we denote $f_n = 2^{-n} f(x - r_n)$ and have

$$\int |f_n(x)|dx = \int 2^{-n}|f(x - r_n)|dx = 2^{-n} \int_0^1 x^{-1/2}dx = 2^{-n} \cdot 2 = 2^{-(n-1)}$$

and thus

$$\sum_1^\infty \int |f_n(x)|dx = 1 + 2^{-1} + 2^{-2} + ... = 2 < \infty$$

Then

$$\int |g| = \int \sum_1^\infty |2^{-n} f(x - r_n)| = \sum_1^\infty \int |f_n(x)| < \infty$$

and thus $g \in L^1(m)$. In particular $g < \infty$ a.e. since otherwise suppose $g = \infty$ on U such that $m(U) > 0$, we have $\int |g| \geq \infty \cdot m(U) = \infty$, a contradiction.

(b) We directly prove the result for slightly modifyed g, and the previous statement easily follows.

Now suppose we modify g on a $m-$null set F, and let h be the modified function such that $h = g$ on $\mathbb{R} \setminus F$. Pick $x_0 \in \mathbb{R}$, for all $\delta > 0$, $B_\delta(x_0)$ contains some r_n. Since $B_\delta(x_0)$ is open, we choose k so large that $I_k := (r_n - \frac{1}{k}, r_n + \frac{1}{k}) \subset B_\delta(x_0)$. Notice that $I_k \setminus I_{k+1}$ contains some $x_1 \in \mathbb{R} \setminus F$ since it has positive measure. Then we choose such x_i inductively such that $x_i \in I_{k+i-1} \setminus I_{k+i}$. Then $x_i \to r_n$. Then observe that $2^{-n} f(x_i - r_n) \to \infty$ by our knowledge about the function $x^{-1/2}$. Then since we pick x_i such that $x_i \notin F$, $h(x_i) = g(x_i) \to \infty$ as $i \to \infty$. Then h is unbounded on $B_\delta(x_0)$ and cannot be continuous at x_0 (unbounded ocscillation). Also for an arbitrary interval I, pick $x \in I$ and $\delta > 0$ small enough such that $B_\delta(x) \in I$. Then f is unbounded on $B_\delta(x)$ and thus unbounded on I, and we show that the conclusions for g remains true after modifying g on a null set.

(c) Since $g < \infty$ a.e., $g^2 < \infty$ a.e. Now we show that g is not integrable on any interval. Given an interval, we can assume it to be $[a, b]$ since removing a null set won't change the result of integration. Then since g is non-negative,

$$\begin{aligned}
\int_a^b |g^2| &\geq \int_a^b \sum_1^\infty 2^{-2n} f^2(x - r_n) \\
&\geq \int_a^b 2^{-2n} f^2(x - r_n) \text{ where } r_n \in [a, b] \\
&\geq 2^{-2n} \int_{r_n}^b f^2(x - r_n) dx \\
&= 2^{-2n} \int_0^{b - r_n} f^2(x) dx = 2^{-2n} \int_0^{b - r_n} x^{-1} dx
\end{aligned}$$

which fails to converge and thus tends to infinity. Then g^2 is not integrable on I and our proof is complete.

□

Folland 2.32

Suppose $\mu(X) < \infty$. If f and g are complex-valued measurable functions on X, define

$$\rho(f, g) = \int \frac{|f - g|}{1 + |f - g|} d\mu$$

Then ρ is a metric on the space of measurable functions if we identify functions that are equal a.e. and $f_n \to f$ with respect to this metric iff $f_n \to f$ in measure.

Proof. We first verify that ρ is a well-defined metric.

(a) $\rho(f, g) = \int \frac{|f - g|}{1 + |f - g|} d\mu \geq \int 0 \; d\mu = 0$

(b) If $f = g$ a.e.,

$$\rho(f, g) = \int \frac{|f - g|}{1 + |f - g|} d\mu = \int 0 \; d\mu = 0$$

since a measure zero set doesn't change the result of integration.

(c) $\rho(f, g) = \int \frac{|f - g|}{1 + |f - g|} = \int \frac{|g - f|}{1 + |g - f|} = \rho(g, f)$

(d) Notice that

$$\begin{aligned}
&\frac{|f-g|}{1+|f-g|}+\frac{|g-h|}{1+|g-h|}-\frac{|f-h|}{1+|f-h|}\\
=&\frac{|f-g|+|g-h|-|f-h|+|f-g||g-h|+|f-g||g-h||f-h|+|g-h||f-g|}{(1+|f-g)(1+|g-h|)(1+|f-h|)}\\
\geq&\frac{|f-g|+|g-h|-|f-h|}{(1+|f-g)(1+|g-h|)(1+|f-h|)}\\
\geq&\, 0
\end{aligned}$$

so $\rho(f,g)+\rho(g,h)\geq\rho(f,h)$ and thus $\rho(f,g)+\rho(g,h)\geq\rho(f,h)$

Then ρ is a metric.

We then show that $f_n\to f$ in this metric iff $f_n\to f$ in measure. Suppose $f_n\to f$ in measure. Let $\epsilon>0$, and $E_n:=\{x:|f_n(x)-f(x)|\geq\frac{\epsilon}{\mu(X)}\}$, which is possible since $\mu(X)<\infty$.

$$\begin{aligned}
\rho(f_n,f)&=\int\frac{|f_n-f|}{1+|f_n-f|}d\mu\\
&=\int\frac{|f_n-f|}{1+|f_n-f|}\chi_{E_n}d\mu+\int\frac{|f_n-f|}{1+|f_n-f|}\chi_{(E_n)^c}d\mu\\
&=\mu(E_n)+\frac{\epsilon}{\mu(X)}\mu(E_n^c)
\end{aligned}$$

Then $\lim_{n\to\infty}\rho(f_n,f)\leq 0+\epsilon=\epsilon$. Since ϵ is arbitrary, this shows that actually $\lim_{n\to\infty}\rho(f_n,f)=0$ and this measn that $f_n\to f$ in this metric.

Conversely, suppose $f_n\to f$ with respect this metric. If $f_n\not\to f$ in measure, there are some δ,ϵ such that there are infinitely many n such that $\mu\{x:|f_n(x)-f(x)|\geq\epsilon\}\geq\delta$, and we denote this set E_n. Thus

$$\rho(f_n,f)\geq\int_{E_n}1-\frac{1}{1+|f-f_n|}\geq\int_{E_n}1-\frac{1}{1+\epsilon}d\mu=\frac{\epsilon}{1+\epsilon}\mu(E_n)\geq\frac{\epsilon\delta}{1+\epsilon}$$

for infinitely many n, contradiction to convergence in this metric and we finish our proof. □

Folland 2.34

Suppose $|f_n|\leq g\in L^1$ and $f_n\to f$ in measure.

(a) $\int f\to\lim\int f_n$

(b) $f_n\to f$ in L^1

Proof. I will slightly reverse the order of the two parts of the problem. Observe that if $f_n\to f$ in L^1, $\int|f_n-f|\to 0$ and thus $|\int f_n-\int f|=|\int(f_n-f)|\leq\int|f_n-f|\to 0$ and $\int f\to\lim\int f_n$. So at this point it suffices to prove (b) to show both (a) and (b).

$f_n\to f$ in measure, so we can find a subsequence $\{f_{n_j}\}$ that converges to f a.e. Since $f_{n_j}\to f$ a.e. and $|f_{n_j}|\leq g\in L^1$, by dominated convergence theorem (also paragraph 2 sentence 1 of page 61) $f_{n_j}\to f$ in L^1. We now claim that $\{f_n\}$ actually converges to f in L^1. Suppose in the contrary that there exists $\epsilon>0$ such that there are infinitely many n_k with $\int|f_{n_k}-f|\geq\epsilon$, then we arrange them into a subsequence f_{n_k}. Since $f_n\to f$ in measure, $f_{n_k}\to f$ in measure, and there is a subsequence of f_{n_k} which we call $\{g_n\}$ for convenience that converges to f a.e. Then $|g_n-f|\to 0$ a.e. Since $|f_n|\leq g$, $|g_n-g|\leq 2g\in L^1$. By dominated convergence theorem $g_n\to g$

in L^1. However, $\int |g_n - f| \geq \epsilon$, contradiction. This contradiction shows our claim that $f_n \to f$ in L^1 and we are done. $\square$

Folland 2.38

Suppose $f_n \to f$ in measure and $g_n \to g$ in measure.
(a) $f_n + g_n \to f + g$ in measure
(b) $f_n g_n \to fg$ in measure if $\mu(X) < \infty$, but not necessarily if $\mu(X) = \infty$

Proof.

(a) Let $\epsilon > 0$. Since $f_n \to f$ in measure and $g_n \to g$ in measure, as $n \to \infty$,

$$\mu\{|f_n - f| \geq \frac{\epsilon}{2}\} \to 0$$
$$\mu\{|g_n - g| \geq \frac{\epsilon}{2}\} \to 0$$

Notice that

$$\begin{aligned}\{|f_n + g_n - f_n - g_n| \geq \epsilon\} &\subset \{|f_n - f| + |g_n - g| \geq \epsilon\} \\ &\subset \{|f_n - f| \geq \frac{\epsilon}{2}\} \cup \{|g_n - g| \geq \frac{\epsilon}{2}\}\end{aligned}$$

and thus

$$\mu\{|f_n + g_n - f_n - g_n| \geq \epsilon\} \leq \mu\{|f_n - f| \geq \frac{\epsilon}{2}\} + \mu\{|g_n - g| \geq \frac{\epsilon}{2}\}$$

Which tends to 0 as $n \to \infty$. Then $f_n + g_n \to f + g$ in measure.

(b) We first prove a technical lemma:

Lemma 1. *If $f_n \to f$ in measure and $\mu(X) < \infty$, $f_n^2 \to f$ in measure.*

Proof of lemma. Since $f_n \to f$ in measure, some subsequence $\{f_{n_j}\} \to f$ a.e. Then $\{f_{n_j}^2\} \to f^2$ a.e. Since $\mu(X) < \infty$, $f_{n_j}^2 \to f^2$ in measure by Egoroff's theorem[1]. We now show that the whole sequence actually converge to f^2 in measure by contradiction. Let $\epsilon > 0$. Suppose that there is some $\delta > 0$ such that there are infinitely many n such that

$$\mu\{x : |f_n^2 - f^2| \geq \epsilon\} \geq \delta$$

then we arrange such f_n^2 into a sequence $f_{n_i}^2$. Notice that we still have $f_{n_i} \to f$ in measure and thus $f_{n_i'} \to f$ a.e for a further subsequence $\{f_{n_i'}\}$. Thus $f_{n_i'}^2 \to f^2$ a.e. Since $\mu(X) < \infty$, we should have $f_{n_i'}^2 \to f^2$ in measure by our argument above, but by our construction this cannot happen, a contradition. Then $f_n^2 \to f^2$ in measure. $\square$

By our lemma and part (a), $(f_n + g_n)^2 \to (f + g)^2$ in measure (i.e. $f_n^2 + 2f_n g_n + g_n^2 \to f^2 + 2fg + g^2$ in measure) and $-f_n^2 \to -f$ and $-g_n^2 \to -g$ in measure. Then by part (a) again we have $f_n g_n \to fg$ in measure.

If $\mu(X) = \infty$, we give a counterexample of functions $\mathbb{R} \to \mathbb{R}$ and the measure is Lebesgue measure. let $f(x) = g(x) = x$ and $f_n(x) = g_n(x) = x + \frac{1}{n}\chi_{[n,n+1)}$. Then clearly $f_n \to f$ and $g_n \to g$ in measure. However,

$$f_n(x)g_n(x) = (x + \frac{1}{n}\chi_{[n,n+1)})^2 = x^2 + \frac{2x}{n}\chi_{[n,n+1)} + \frac{1}{n^2}\chi_{[n,n+1)}$$

[1]This is an easy corollary of Egoroff's theorem and is also mentioned as a side remark in Folland P62.

and however large n is, for $x \in [n, n+1)$, we have $f(x)g(x) \geq x^2 + 2$, and $\mu[n, n+1) = 1$. This shows that $f_n g_n$ doesn't converge to fg in measure.

$\square$

Folland 2.40

In Egoroff's theorem, the hypothesis "$\mu(X) < \infty$" can be replaced by "$|f_n| \leq g$ for all n, where $g \in L^1(\mu)$."

Proof. Let $E_1(k) := \bigcup_{m=1}^{\infty}\{x : |f_m(x) - f(x)| \geq \frac{1}{k}\}$, as defined in the proof of Egoroff's theorem. The condition $\mu(X) < \infty$ is used to justify $\mu(E_1(k)) < \infty$ for any fixed k, so if we can show that $\mu(E_1(k)) < \infty$, we are done. Observe that since $|f_n - f| \leq 2g$[2],

$$E_1(k) \subset \bigcup_{m=1}^{\infty}\{x : g(x) \geq \frac{1}{2k}\} = \{x : g(x) \geq \frac{1}{2k}\}$$

Then if $\mu(E_1) = \infty$ for some k, $\int |g| \geq \frac{1}{2k} \cdot \infty = \infty$, a contradiction. $\square$

Folland 2.41

If μ is σ-finite and $f_n \to f$ a.e. there exists measurable $E_1, E_2, \ldots \subset X$ such that $\mu((\bigcup_1^{\infty} E_j)^c) = 0$ and $f_n \to f$ uniformly on each E_j.

Proof. We first suppose that $\mu(X)$ is actually finite. By Egoroff's theorem, for each k there is some E_k such that $\mu(E_k)^c < 2^{-k}$ and $f_n \to f$ uniformly on E_k. Set $F_n = \bigcup_1^n E_i$, then $\{F_n\}$ is an increasing sequence and $\{(F_n)^c\}$ is a decreasing sequence. Also $\mu(F_n^c) \leq \mu(E_n^c) < 2^{-k}$. Since $\mu(F_1^c) \leq \mu(X) < \infty$, by continuity from above,

$$\mu(\bigcup_1^{\infty} E_j)^c = \mu(\bigcup_1^{\infty} F_j)^c = \mu(\bigcap_1^{\infty} F_j^c) = \lim_{j \to \infty} \mu(F_j^c) = 0$$

and $f_n \to f$ uniformly on each E_j.
If X is σ-finite, then $X = X_1 \sqcup X_2 \sqcup \ldots$ such that $\mu(X_i) < \infty$ for each i. On each i there are $\{E_k^i\}_{k=1}^{\infty}$ such that $\mu(X_i - (\bigcup_{k=1}^{\infty} E_k^i)) = 0$ and $f_n \to f$ uniformly on each E_k^i. Then consider $\{E_k^i\}_{k,i=1}^{\infty}$, $f_n \to f$ uniformly on each E_k^i, and

$$\mu(\bigcup_{i,k} E_k^i)^c = \mu[\bigcup_{i=1}^{\infty}(X_i - \bigcup_{k=1}^{\infty} E_k^i)] = \sum_{i=1}^{\infty} \mu(X_i - \bigcup_{k=1}^{\infty} E_k^i) = 0$$

$\square$

Folland 2.43

Suppose that $\mu(X) < \infty$ and $f : X \times [0,1] \to \mathbb{C}$ is a function such that $f(\cdot, y)$ is measurable for each $y \in [0,1]$ and $f(x, \cdot)$ is continuous for each $x \in X$.

(a) If $0 < \epsilon$, $\delta < 1$ then $E_{\epsilon,\delta} = \{x : |f(x,y) - f(x,0)| \leq \epsilon$ for all $y < \delta\}$ is measurable.
(b) For any $\epsilon > 0$ there is a set $E \subset X$ such that $\mu(E) < \epsilon$ and $f(\cdot, y) \to f(\cdot, 0)$ uniformly on E^c as $y \to 0$.

[2] We adopt the assumption in Folland that $f_n \to f$ everywhere.

Proof. (a) Let $\mathbb{Q} \cap [0, \delta)$ be enumerated as $\{y_n\}$. Then we define

$$E_{\epsilon,n} := \{x : |f(x, y_n) - f(x, 0)| \leq \epsilon\}$$

Notice that $f(x, y_n)$ and $f(x, 0)$ are both measurable, $|f(x, y_n) - f(x, 0)|$ is measurable. Thus $E_{\epsilon,n}$ is measurable. Consider

$$E := \bigcap_{n=1}^{\infty} E_{\epsilon,n} = \bigcap_{n=1}^{\infty} \{x : |f(x, y_n) - f(x, 0)| \leq \epsilon\}$$

Then E is measurable and clearly $E_{\epsilon,\delta} \subset E$. Conversely, suppose $x \in E$. Then $|f(x, y_n) - f(x, 0)| \leq \epsilon$ for all y_n. Let $y < \delta$, then we can find $\{y_{n_k}\}$ that converges to y. Since $|f(x, y_{n_k}) - f(x, 0)| \leq \epsilon$ for all y_{n_k}, send n_k to infinity we get $|f(x, y) - f(x, 0)| \leq \epsilon$ and thus $x \in E_{\epsilon,\delta}$. Then $E = E_{\epsilon,\delta}$ and thus $E_{\epsilon,\delta}$ is measurable.

(b) Let $\epsilon > 0$. Choose a monotone decreasing sequence $\{\delta_n\} \in [0, 1)$ such that $\delta_n \to 0$ from above, notice that $E_{\epsilon,\delta_1} \subset E_{\epsilon,\delta_2} \subset E_{\epsilon,\delta_3}....$ Consider $F_{\epsilon,i} = (E_{\epsilon,\delta_i})^c$, then $F_{\epsilon,1} \supset F_{\epsilon,2} \supset F_{\epsilon,3} \supset F_{\epsilon,4}...$ and $\mu(F_{\epsilon,1}) \leq \mu(X) < \infty$. Thus by continuity from above we have

$$\mu(\bigcap_{i=1}^{\infty} F_{\epsilon,i}) = \lim_{i\to\infty} \mu(F_{\epsilon,i}) = 0$$

Therefore, for any $\gamma > 0$, there is some $N \in \mathbb{N}$ such that when $n > N$, $\mu(F_{\epsilon,n}) < \gamma$, $|f(x, y) - f(x, 0)| < \epsilon$ for $x \in (F_{\epsilon,n})^c$ and $y \leq \delta_n$. Therefore given $\gamma > 0$ and $m \in \mathbb{N}$ we can choose $N(\delta, m)$ such that $\mu(F_{\frac{1}{N(\delta,m)},m}) < \gamma$ and $|f(x, y) - f(x, 0)| < \frac{1}{N(\delta,m)}$ for $x \in F_{\frac{1}{N(\delta,m)},m}$ and $y \leq \delta_m$. Let $E = \bigcap_{m=1}^{\infty} F_{\frac{1}{N(\delta,m)},m}$, then $\mu(E) < \dfrac{1}{m}$ for all m and for all m, $|f(x, y) - f(x, 0)| < \dfrac{1}{m}$ for all x provided $y \leq \delta_m$, which means that $f(\cdot, y) \to f(\cdot, 0)$ uniformly as $y \to 0$.

$\square$

Folland 2.46

Let $X = Y = [0, 1]$, $\mathcal{M} = \mathcal{N} = \mathcal{B}_{[0,1]}$. μ = Lebesgue meaure, and ν = counting measure. If $D = \{(x, x) : x \in [0, 1]\}$ is the diagonal in $X \times Y$, then $\iint \chi_D \, d\mu d\nu$, $\iint \chi_D \, d\nu d\mu$, and $\int \chi_D \, d(\mu \times \nu)$ are all unequal.

Proof. Notice that

$$\begin{aligned}\iint \chi_D \, d\mu d\nu &= \int \Big[\int \chi_D \, d\mu(x)\Big] d\nu(y) = \int \Big[\int (\chi_D)^y \, d\mu(x)\Big] d\nu(y) \\ &= \int \Big[\int \chi_{D^y} \, d\mu(x)\Big] d\nu(y) = \int \Big[\int \chi_{\{(y,y)\}} \, d\mu(x)\Big] d\nu(y) = \int 0 \, d\nu(y) = 0\end{aligned}$$

and

$$\begin{aligned}\iint \chi_D \, d\nu d\mu &= \int \Big[\int \chi_D \, d\nu(y)\Big] d\mu(x) = \int \Big[\int (\chi_D)_x \, d\nu(y)\Big] d\mu(x) = \int \Big[\int \chi_{D_x} \, d\nu(y)\Big] d\mu(x) \\ &= \int \Big[\int \chi_{\{(x,x)\}} d\nu(y)\Big] d\mu(x) = \int d\mu(x) = 1\end{aligned}$$

Now notice that $\int \chi_D \, d(\mu\times\nu) = \mu\times\nu(D)$, and that

$$\mu\times\nu(D) = \inf\{\sum_1^\infty \mu(A_i)\nu(B_i) : A_i, B_i \in \mathcal{B}_{[0,1]}, D \subset \bigcup_{i=1}^\infty (A_i\times B_i)\}$$

We want to show that there is some i such that $\mu(A_i) > 0$ and $\nu(B_i) = \infty$, since then $\sum_1^\infty \mu(A_i)\nu(B_i) > \mu(A_i)\nu(B_i) = \infty$. And taking infimum we get $\mu\times\nu(D) = \infty$. By definition of counting measure, $\nu(B_i) < \infty$ if and only if $|B_i| < \infty$, and thus $\mu(B_i) = 0$. Similaly, if $\mu(A_i) > 0$, $\nu(A_i) = \infty$ since if $|A_i| < \infty$ there should be $\mu(A_i) = 0$. Observe that $\bigcup_i A_i = [0,1]$ and $\bigcup_i B_i = [0,1]$, so $\bigcup_i (A_i\cap B_i) = [0,1]$ and thus $\mu(\bigcup_i (A_i\cap B_i)) = \sum_{i=1}^\infty \mu(A_i\cap B_i) = 1$. Then there is some i such that $\mu(A_i\cap B_i) > 0$, and this means that $\mu(A_i) \geq \mu(A_i\cap B_i) > 0$ and $\nu(B_i) \geq \nu(A_i\cap B_i) = \infty$, so we show what we want to show and finishes the proof. □

Folland 2.55

Let $E = [0,1]\times[0,1]$. Investigate the existence and equality of $\int_E f dm^2$, $\int_0^1\int_0^1 f(x,y)dxdy$, and $\int_0^1\int_0^1 f(x,y)dydx$ for $f(x,y) = (x-\dfrac{1}{2})^{-3}$ if $0 < y < |x-\frac{1}{2}|$ and $f(x,y) = 0$ otherwise.

Proof. Notice that $\int_E f\, dm^2 = \int f\chi_E \, dm^2$, and that $f \leq 0$ on $E_1 = [0, \dfrac{1}{2}]$ and $f \geq 0$ on $E_2 = [\dfrac{1}{2}, 1]$. Now

$$\int_E f\, dm^2 = \int f\chi_E\, dm^2 = \int (f\chi_E)^+ \, dm^2 - \int (f\chi_E)^- \, dm^2 = \int f\chi_{E_2}\, dm^2 - \int f\chi_{E_1}\, dm^2$$

Clearly $f\chi_{E_2} \in L^+(m^2)$. Then by Tonelli,

$$\begin{aligned}
\int f\chi_{E_2}\, dm^2 &= \int_{\frac{1}{2}}^1 \left[\int_0^1 f(x,y)\, dy\right] dx = \int_{\frac{1}{2}}^1 \left[\int_0^{x-\frac{1}{2}} (x-\frac{1}{2})^{-3}\, dy\right] dx \\
&= \int_{\frac{1}{2}}^1 (x-\frac{1}{2})^{-2}\, dx = \lim_{n\to\infty} \int_{\frac{1}{2}+\frac{1}{n}}^1 (x-\frac{1}{2})^{-2}\, dx \quad \text{(Monotone Convergence)} \\
&= \lim_{n\to\infty} n - 2 = \infty
\end{aligned}$$

And therefore $\int_E f\, dm^2$ doesn't exist.
Then we examine the existence and equality of $\int_0^1\int_0^1 f(x,y)\, dydx$. We have

$$\begin{aligned}
\int_0^1\int_0^1 f(x,y)\, dydx &= \int_0^1 \left[\int_0^1 f(x,y)\, dy\right] dx = \int_0^1 \left[\int f_x\chi_{[0,1]}\, dy\right] dx \\
&= \int_0^1 \left[\int (f_x\chi_{[0,1]})^+\, dy - \int (f_x\chi_{[0,1]})^-\, dy\right] dx \quad (1)
\end{aligned}$$

Notice that if $x \geq \dfrac{1}{2}$, $(f_x\chi_{[0,1]})^+ = (x-\dfrac{1}{2})^{-3}\chi_{[0,x-\frac{1}{2}]}$ and $(f_x\chi_{[0,1]})^- = 0$; if $x < \dfrac{1}{2}$, $(f_x\chi_{[0,1]})^+ = 0$ and $(f_x\chi_{[0,1]})^- = (\dfrac{1}{2}-x)^{-3}\chi_{[0,\frac{1}{2}-x]}$. Also

$$\int (x-\frac{1}{2})^{-3}\chi_{[0,x-\frac{1}{2}]}\, dy = \int_0^{x-\frac{1}{2}} (x-\frac{1}{2})^{-3}\, dy = (x-\frac{1}{2})^{-2}$$

and

$$-\int (\frac{1}{2}-x)^{-3}\chi_{[0,\frac{1}{2}-x]}\ dy = -\int_0^{\frac{1}{2}-x}(\frac{1}{2}-x)^{-3}\ dy = (x-\frac{1}{2})^{-2}$$

Therefore,

$$\begin{aligned}(1) = \int_0^1 (x-\frac{1}{2})^{-2}\ dx &= \int ((x-\frac{1}{2})^{-2}\chi_{[0,1]})^{+}\ dx - \int ((x-\frac{1}{2})^{-2}\chi_{[0,1]})^{-}\ dx \\ &= \int_{\frac{1}{2}}^1 (x-\frac{1}{2})^{-2}\ dx - \int_0^{\frac{1}{2}}(\frac{1}{2}-x)^{-2}\ dx\end{aligned}$$

However, by our work of part one, $\int_{\frac{1}{2}}^1 (x-\dfrac{1}{2})^{-2}\ dx = \infty$ and therefore $\int_0^1\int_0^1 f(x,y)\ dydx$ doesn't exist as well.
We eventually investigate the existence and equality of $\int_0^1\int_0^1 f(x,y)\ dxdy$. Notice that $f(x,y) = 0$ for $y \geq \dfrac{1}{2}$. Then

$$\begin{aligned}\int_0^1\int_0^1 f(x,y)\ dxdy = \int_0^1\left[\int_0^1 f(x,y)\ dx\right]dy &= \int_0^{\frac{1}{2}}\left[\int_0^1 f^{+}\ dx - \int_0^1 f^{-}\ dx\right]dy \\ &= \int_0^{\frac{1}{2}}\left[\int_{\frac{1}{2}+y}^1 f^{+}\ dx - \int_0^{\frac{1}{2}-y} f^{-}\ dx\right]dy \\ &= \int_0^{\frac{1}{2}}\left[\int_{\frac{1}{2}+y}^1 (x-\frac{1}{2})^{-3}\ dx - \int_0^{\frac{1}{2}-y}(\frac{1}{2}-x)^{-3}\ dx\right]dy \\ &= \int_0^{\frac{1}{2}} -2\cdot(4-y^{-2}) - 2(y^{-2}-4)\ dy = 0\end{aligned}$$

and both $\int_0^{\frac{1}{2}} -2\cdot(4-y^{-2})\ dy$ and $\int_0^{\frac{1}{2}} 2(y^{-2}-4)\ dy < \infty$. Then the integral exists and equals 0. Now we complete the proof. □

3. Chapter 3-Signed Measures and Differentiation

Folland 3.4

If ν is a signed measure and λ, μ are positive measures such that $\mu = \lambda - \nu$, then $\lambda \geq v^{+}$ and $\mu \geq \nu^{-}$.

Proof. Observe that $\nu = \lambda - \nu = \nu^{+} - \nu^{-}$, so $\lambda - \nu^{+} = \mu - \nu^{-}$. Let $\rho := \lambda - \nu^{+} = \mu - \nu^{-}$, then clearly ρ is a signed measure. Notice that we finish our proof if we can show that ρ is a positive measure. Since $v^{+} \perp v^{-}$, there are E and F such that $X = E \sqcup F$ and $\nu^{+}(F) = 0$ and $\nu^{-}(E) = 0$. Then for any measurable $A \subset E$, $\rho(A) = \mu(A) - \nu^{-}(A) = \mu(A) \geq 0$, so E is positive; also for any measurable $B \subset F$, $\rho(B) = \lambda(B) - \nu^{+}(B) = \lambda(B) \geq 0$, so F is also positive. Thus take any $C \in \mathcal{M}$, $\rho(C) = \rho(C \cap E) + \rho(C \cap F) \geq 0$, so ρ is positive and we finish our proof. □

Folland 3.7

Suppose that ν is a signed measure on $(X, \mathcal{M})$ and $E \in \mathcal{M}$.

(a) $\nu^{+}(E) = \sup\{\nu(F) : F \in \mathcal{M}, F \subset E\}$ and $\nu^{-}(E) = -\inf\{\nu(F) : F \in \mathcal{M}, F \subset E\}$.

(b) $|\nu|(E) = \sup\{\sum_1^n |\nu(E_j)| : n \in \mathbb{N}, E_1, ..., E_n$ are disjoint, and $\cup_1^n E_j = E\}$

Proof. (a) Let $\mu = \sup\{\nu(F) : F \in \mathcal{M}, F \subset E\}$ and $\lambda = -\inf\{\nu(F) : F \in \mathcal{M}, F \subset E\}$. The idea is to show that μ, λ are well-defined (positive) measures such that $\mu \perp \lambda$ and $\nu = \mu - \lambda$. Then by uniqueness of Jordan decomposition we get the result. We first show that μ and λ are positive. This is true since for every measurable E, $\mu(E) \geq \nu(\emptyset)$ for every E since $\emptyset \in \mathcal{M}$ and $\emptyset \subset E$. Similarly we can show $\lambda(E) \geq 0$. We then show μ is a well-defined measure. $\mu(\emptyset) = \nu(\emptyset) = 0$. For countable additivity, suppose $E_1, E_2, \ldots$ disjoint, we denote $E = \bigcup_i E_i$ for convenience. Then

$$\begin{aligned}
\mu(E) &= \sup\{\nu(F) : F \in \mathcal{M}, F \subset E\} = \sup\{\nu(F \cap E) : F \in \mathcal{M}, F \subset E\} \\
&= \sup\{\nu(\bigcup_{i=1}^{\infty}(F \cap E_i)) : F \in \mathcal{M}, F \subset E\} = \sup\{\sum_{i=1}^{\infty} \nu(F \cap E_i) : F \in \mathcal{M}, F \subset E\} \\
&= \sup\{\sum_{i=1}^{\infty} \nu(F_i) : F_i \in \mathcal{M}, F_i \subset E_i\} = \sum_{i=1}^{\infty} \sup\{\nu(F_i) : F_i \in \mathcal{M}, F_i \subset E_i\}
\end{aligned}$$

if we notice $F \subset E$ iff $F_i = F \cap E_i \subset E_i$ for each i. Similarly we can show that λ is a well-defined positive measure. We then show that $\mu \perp \lambda$. By Hahn decomposition, we have $X = P \sqcup N$ where P is positive and N is negative. Then for any $E \in \mathcal{M}$, we write it as $E = (E \cap P) \sqcup (E \cap N)$. Notice that $F \subset E$ iff $F = F_1 \sqcup F_2$, where $F_1 \subset E \cap P$ and $F_2 \subset E \cap N$, so

$$\begin{aligned}
\mu(E) &= \sup\{\nu(F) : F \in \mathcal{M}, F \subset E\} = \sup\{\nu(F_1 \sqcup F_2) : F_1, F_2 \in \mathcal{M}, F_1 \subset E \cap P, F_2 \subset E \cap N\} \\
&= \sup\{\nu(F_1) + \nu(F_2) : F_1, F_2 \in \mathcal{M}, F_1 \subset E \cap P, F_2 \subset E \cap N\} \\
&= \sup\{\nu(F_1) : F_1 \in \mathcal{M}, F_1 \subset E \cap P\} + \sup\{\nu(F_2) : F_2 \in \mathcal{M}, F_2 \subset E \cap N\} \\
&= \sup\{\nu(F_1) : F_1 \in \mathcal{M}, F_1 \subset E \cap P\} = \sup\{\nu(F \cap P) : F \in \mathcal{M}, F \subset E\} \\
&= \nu(E \cap P)
\end{aligned}$$

and similarly we have $\lambda(E) = \nu(E \cap N)$ and $\mu(E) + \lambda(E) = \nu(E)$. The result follows by uniqueness of Jordan decomposition.

(b) First observe that for $E \in \mathcal{M}$,

$$|\nu(E)| = |\nu^+(E) - \nu^-(E)| \leq |\nu^+(E)| + |\nu^-(E)| = |\nu|(E)$$

Then by countable additivity,

$$\begin{aligned}
|\nu|(E) &= \sup\{\sum_1^n |\nu|(E_j) : n \in \mathbb{N}, E_1, ..., E_n \text{ are disjoint, and } \cup_1^n E_j = E\} \\
&\geq \sup\{\sum_1^n |\nu(E_j)| : n \in \mathbb{N}, E_1, ..., E_n \text{ are disjoint, and } \cup_1^n E_j = E\}
\end{aligned}$$

Conversely, if we apply the Hahn decomposition used above, we get $|\nu|(E) = |\nu|(E \cap P) + |\nu|(E \cap N)$ where P is positive and N is negative. Notice that

$$0 \leq |\nu|(E \cap P) = \nu^+(E \cup P) + \nu^-(E \cup P) = \nu^+(E \cup P) = \nu^+(E \cup P) - \nu^-(E \cup P) = v(E \cup P)$$

So $|\nu|(E \cap P) = |\nu(E \cup P)|$. Similarly $|\nu|(E \cap N) = |\nu(E \cup N)|$. Therefore

$$\begin{aligned}
|\nu|(E) &= |\nu(E \cap P)| + |\nu(E \cap N)| \quad (\text{where } (E \cap P) \cup (E \cap N) = E) \\
&\leq \sup\{\sum_1^n |\nu(E_j)| : n \in \mathbb{N}, E_1, ..., E_n \text{ are disjoint, and } \cup_1^n E_j = E\}
\end{aligned}$$

Two directions combined, we prove the equality and finish the proof.

□

Folland 3.12

For $j = 1, 2$, let μ_j, ν_j be σ-finite measures on $(X_j, \mathcal{M}_j)$ such that $v_j \ll \mu_j$. Then $\nu_1 \times \nu_2 \ll \mu_1 \times \mu_2$ and

$$\frac{d(\nu_1 \times \nu_2)}{(\mu_1 \times \mu_2)}(x_1, x_2) = \frac{d\nu_1}{d\mu_1}(x_1)\frac{d\nu_2}{d\mu_2}(x_2)$$

Proof. First we show that $\nu_1 \times \nu_2 \ll \mu_1 \times \mu_2$. Suppose $\mu_1 \times \mu_2(E) = 0$ for some $E \in \mathcal{M}_1 \otimes \mathcal{M}_2$, then $\chi_E \in L^+$. By Tonelli,

$$0 = \mu_1 \times \mu_2(E) = \iint \chi_E \; d\mu_1 d\mu_2 = \int \mu_1(E^y) \; d\mu_2(y)$$

Then $\mu_1(E^y) = 0$ for μ_1-a.e. y and since $\nu_1 \ll \mu_1$, $\nu_1(E^y) = 0$ for μ_2-a.e. y. Since $\nu_2 \ll \mu_2$, $\nu_1(E^y) = 0$ for ν_2-a.e. y. By Tonelli again,

$$\begin{aligned}(\nu_1 \times \nu_2)(E) &= \int \chi_E \; d(\nu_1 \times \nu_2) = \int \left(\int \chi_E \; d\nu_1(x) \right) d\nu_2(y) \\ &= \int \nu_1(E^y) \; d\nu_2(y) = 0\end{aligned}$$

Thus $\nu_1 \times \nu_2 \ll \mu_1 \times \mu_2$.
Then we show that

$$\frac{d(\nu_1 \times \nu_2)}{d(\mu_1 \times \mu_2)}(x_1, x_2) = \frac{d\nu_1}{d\mu_1}(x_1)\frac{d\nu_2}{d\mu_2}(x_2)$$

We claim that $\frac{d\nu_i}{d\mu_i} \geq 0$ a.e. Suppose in the contrary that $\frac{d\nu_i}{d\mu_i} < 0$ on some E with $\mu_i(E) > 0$. Then

$$\nu_i(E) = \int_E f \; d\mu_i < 0$$

contradicting to our assumption that ν_i is positive, so the claim is true. Then $\frac{d\nu_1}{d\mu_1}(x_1)\frac{d\nu_2}{d\mu_2}(x_2) \in L^+(\mathcal{M}_1 \times \mathcal{M}_2)$, and by Tonelli, we get for any $E \in \mathcal{M}_1 \otimes \mathcal{M}_2$,

$$\begin{aligned}\int \chi_E \frac{d(\nu_1 \times \nu_2)}{d(\mu_1 \times \mu_2)} d(\mu_1 \times \mu_2) &= \int \chi_E \; d(\nu_1 \times \nu_2) = \iint \chi_E \; d\nu_1 d\nu_2 = \iint \chi_E \; (\frac{d\nu_1}{d\mu_1} d\mu_1)(\frac{d\nu_2}{d\mu_2} d\mu_2) \\ &= \int \chi_E \; \frac{d\nu_1}{d\mu_1}(x_1)\frac{d\nu_2}{d\mu_2}(x_2) \; d(\mu_1 \times \mu_2)\end{aligned}$$

Then we have shown that

$$\frac{d(\nu_1 \times \nu_2)}{d(\mu_1 \times \mu_2)} = \frac{d\nu_1}{d\mu_1}(x_1)\frac{d\nu_2}{d\mu_2}(x_2)$$

a.e. and thus prove the result since we identify the derivative functions with their equivalence classes. □

Folland 3.16

μ, ν are measures on $(X, \mathcal{M})$ with $\nu \ll \mu$, and let $\lambda = \mu + \nu$. If $f = \frac{d\nu}{d\lambda}$, show that $0 \leq f < 1$ μ-a.e. and $\frac{d\nu}{d\mu} = \frac{f}{1-f}$.

Proof. We first show that $0 \leq f < 1$ μ-a.e. Suppose in the contrary that $f \geq 1$ on some E with $\mu(E) > 0$,

$$\nu(E) = \int_E f \, d\lambda \geq \lambda(E) = \mu(E) + \nu(E)$$

But this implies that $\mu(E) \leq 0$, a contradiction.

We then show that $\frac{d\nu}{d\mu} = \frac{f}{1-f}$. We claim that $\mu \ll \lambda$ and $\lambda \ll \nu$. If $\lambda(E) = \mu(E) + \nu(E) = 0$, $\mu(E)$. Conversely, if $\mu(E) = 0$, since $\nu \ll \mu$, $\nu(E) = 0$ and thus $\lambda(E) = \nu(E) + \mu(E) = 0$. So our claim is true and $(\frac{d\mu}{d\lambda})(\frac{d\lambda}{d\mu}) = 1$. By additivity of derivatives, we have $\frac{d\mu}{d\lambda} + \frac{d\nu}{d\lambda} = \frac{d\lambda}{d\lambda} = 1$ and therefore $\frac{d\mu}{d\lambda} = 1 - \frac{d\nu}{d\lambda}$. Then

$$\frac{f}{1-f} = \frac{d\nu/d\lambda}{1 - d\nu/d\lambda} = \frac{d\nu/d\lambda}{d\mu/d\lambda} = \frac{d\nu}{d\lambda} \cdot \frac{d\lambda}{d\mu}$$

Since we have $\nu \ll \lambda$ and $\lambda \ll \mu$,

$$\frac{f}{1-f} = \frac{d\nu}{d\lambda} \cdot \frac{d\lambda}{d\mu} = \frac{d\nu}{d\mu}$$

and we finish our proof. □

Folland 3.18

Let ν be a complex measure on $(X, \mathcal{M})$. $L^1(\nu) = L^1(|\nu|)$, and if $f \in L^1(\nu)$, then $|\int f d\nu| \leq \int |f| d|\nu|$.

Proof. We first show that $L^1(\nu) = L^1(|\nu|)$. By Lebesgue-Radon-Nikodym, we have $d\nu = f d\mu$ where μ is a positive measure, and thus $d|\nu| = |g| d\mu$. If $f \in L^1(|\nu|)$,

$$\infty > \int |f| d|\nu| = \int |f||g| d\mu = \int |fg| d\mu \geq \left| \int |f| g d\mu \right| = \int |f| d\nu \quad (2)$$

showing that $f \in L^1(\nu)$ and that $L^1(|\nu|) \subset L^1(\nu)$. Conversely if $f \in L^1(\nu)$, $f \in L^1(|\nu_r| + |\nu_i|)$. Then since $|\nu| \leq |v_r| + |v_i|$ are all positive measures we have

$$\infty > \int |f| d(|v_r| + |v_i|) \geq \int |f| d|\nu|$$

showing that $f \in L^(|\nu|)$. Thus $L^1(\nu) = L^1(|\nu|)$. Moreover by (1) we have

$$\left| \int f d\nu \right| \leq \int |f| d\nu \leq \int |f| d|\nu|$$

which finishes the proof. □

Remark. This is basically a formal check using definitions.

Folland 3.20

If ν is a complex measure on $(X, \mathcal{M})$ and $\nu(X) = |\nu|(X)$, then $\nu = |\nu|$.

Proof. By Lebesgue-Randon-Nikodym we have $d\nu = f d\mu$ for some positive measure μ, and thus $d|\nu| = |f| d\mu$. Then

$$\int f d\mu = \int |f| d\mu \implies \int (|f| - f) d\mu = 0 \quad (3)$$

Since $|f| - f \geq 0$, (2) implies that $|f| - f = 0$ μ-a.e., and thus $|f| = f$ since we identify f as equivalence class in L^1. Thus $d|\nu| = d\nu$ and thus $|\nu| = \nu$, as desired. ☐

Folland 3.21

Let ν be a complex measure on $(X, \mathcal{M})$. If $E \in \mathcal{M}$, define

$$\mu_1(E) = \sup\left\{\sum_1^n |\nu(E_j)| : n \in \mathbb{N}, E_1, ..., E_n \text{ disjoint and } E = \bigcup_{j=1}^n E_j\right\}$$

$$\mu_3(E) = \sup\left\{\left|\int_E f d\nu\right| : |f| \leq 1\right\}$$

Proof. We first show that $\mu_1 \leq \mu_3$. Define $f := \sum_{j=1}^n \operatorname{sgn}(\nu(E_i))\chi_{E_i}$, and since $|\operatorname{sgn}(\nu(E_i))| \leq 1$, $|f| \leq 1$. Thus we have

$$\left|\int_E f d\nu\right| = \left|\sum_{j=1}^n \int_{E_i} \operatorname{sgn}(\nu(E_i)) d\nu\right| = \left|\sum_{j=1}^n \operatorname{sgn}(\nu(E_i))\nu(E_i)\right| = \sum_{j=1}^n |\nu(E_i)|$$

And taking supremum over $\{E_n\}_n$ satisfying the conditions of μ_1 we get $\mu_1(E) \leq \mu_3(E)$ and thus $\mu_1 \leq \mu_3$. We then show that $\mu_3 = |\nu|$. Define $f = \overline{d\nu/d|\nu|}$ and by proposition 3.13b $|f| = |\overline{f}| = 1 \leq 1$. Moreover, Lebesgue-Radon-Nikodym gives $d\nu = g d\mu$ for some positive μ, and thus $\overline{d\nu} = \overline{f d\nu} = \overline{f} d\nu$ and $d|\nu| = |g| d\nu$. Then

$$\overline{d\nu/d|\nu|} d\nu = \frac{\overline{f} d\mu \cdot f d\mu}{|f| d\mu} = \frac{|f|^2 (d\mu)^2}{|f| d\mu} = |f| d\mu = d|\nu|$$

and thus $\int_E f d\nu = \int_E d|\nu| = |\nu|(E)$, showing that $\mu_3 \geq |\nu|$. On the other hand $\mu_3(E) \leq \sup\{\int_E |f| d|\nu| : |f| \leq 1\} \leq \int_E d|\nu| = |\nu|(E)$ and thus $\mu_3 = |\nu|$. Eventually we show that $\nu_3 \leq \nu_1$. Let $\phi := \sum_1^n c_k \chi_{E_k}$ where $|c_k| \leq 1$ for all k, E_is are disjoint and $\bigcup_{i=1}^n E_i = E$. We have

$$\left|\int_E \phi d\nu\right| \leq \sum_{k=1}^n \left|c_k \int_{E_k} \chi_{E_k} d\nu\right| = \sum_{k=1}^n |c_k||\nu(E_k)| \leq \sum_{k=1}^n |\nu(E_k)| \leq \mu_1(E)$$

Let $|f| \leq 1$, choose $\langle \phi_n \rangle_n$ simple functions that approximate f from below (meaning that $|\phi_n| \leq 1$ for all n) in L^1 since simple functions are dense in L^1 and apply dominated convergence theorem (since $|\phi_n| \leq \chi_E \in L^1$ for all n) we have $|\int_E f d\nu| \leq \mu_1(E)$. Taking supremum over f we have $\mu_3(E) \leq \mu_1(E)$. Eventually we have $\mu_1 = \mu_3 = \nu$, as desired.

☐

Folland 3.23

A useful variant of the Hardy-Littlewood maximal function is

$$H^* f(x) = \sup\left\{\frac{1}{m(B)} \int_B |f(y)|\, dy : B \text{ is a ball and } x \in B\right\}$$

Show that $Hf \leq H^* f \leq 2^n Hf$.

Proof. First inequality: We first observe that $x \in B(r, x)$ for any $r > 0$. Thus

$$\{B(r, x) : r > 0\} \subset \{B : B \text{ is a ball and } x \in B\}$$

and

$$\left\{\frac{1}{m(B(r,x))}\int_{B(r,x)}|f(y)|\,dy : r>0\right\}\subset\left\{\frac{1}{m(B)}\int_B|f(y)|\,dy : B \text{ is a ball and } x\in B\right\}$$

Therefore

$$\sup\left\{\frac{1}{m(B(r,x))}\int_{B(r,x)}|f(y)|\,dy : r>0\right\}\le\sup\left\{\frac{1}{m(B)}\int_B|f(y)|\,dy : B \text{ is a ball and } x\in B\right\}$$

which means $Hf\le H^*f$.

Second inequality: We observe that

$$\begin{aligned}
H^*f(x) &= \sup_{r>0}\left\{\frac{1}{m(B_r)}\int_{B_r}|f(y)|\,dy : B_r \text{ is a ball of radius } r \text{ such that } x\in B_r\right\}\\
&= \sup_{r>0}\left\{\frac{1}{m(B(r,x))}\int_{B_r}|f(y)|\,dy : B_r \text{ is a ball of radius } r \text{ such that } x\in B_r\right\}\\
&\le \sup_{r>0}\frac{1}{m(B(r,x))}\int_{B(2r,x)}|f(y)|\,dy \quad (\text{since } B(2r,x) \text{ contains all } B_r\ni x)\\
&= \sup_{r>0}\frac{2^n}{m(B(2r,x))}\int_{B(2r,x)}|f(y)|\,dy\\
&= 2^nHf(x)
\end{aligned}$$

so we finish the proof. □

Folland 3.24

If $f\in L^1_{loc}$ and f is continuous at x, then x is in the Lebesgue set of f.

Proof. Let $\epsilon>0$. Since f is continuous at x, there is $\delta>0$ such that whenever $|y-x|<\delta$, $|f(y)-f(x)|<\epsilon$. Then when $r<\delta$,

$$\frac{1}{m(B(r,x))}\int_{B(r,x)}|f(y)-f(x)|\,dx<\frac{\epsilon\, m(B(r,x))}{m(B(r,x))}=\epsilon$$

and thus

$$\lim_{r\to0}\frac{1}{m(B(r,x))}\int_{B(r,x)}|f(y)-f(x)|\,dy=0$$

which means that $x\in L_f$ and we finish the proof. □

Folland 3.25

If E is a Borel set in $\mathbb{R}^n$, the density $D_E(x)$ of E at x is defined as

$$D_E(x)=\lim_{r\to0}\frac{m(E\cap B(r,x))}{m(B(r,x))}$$

whenever the limit exists.

(a) Show that $D_E(x)=1$ for a.e. $x\in E$ and $D_E(x)=0$ for a.e. $x\in E^c$.
(b) Find examples of E and x such that $D_E(x)$ is a given number $\alpha\in(0,1)$, or such that $D_E(x)$ does not exist.

Proof. (a) We define $f := \chi_E$. Then clearly $f \in L^1_{loc}$ and thus $m((L_f)^c) = 0$. Therefore, for a.e. $x \in E$, $x \in L_f$ and thus

$$\lim_{r\to 0} \frac{1}{m(B(r,x))} \int_{B(r,x)} |f(y) - f(x)| \, dy = \lim_{r\to 0} \frac{1}{m(B(r,x))} \int_{B(r,x)} |f(y) - 1| \, dy = 0$$

This implies

$$\begin{aligned} 0 &= \lim_{r\to 0} \frac{1}{m(B(r,x))} \int_{B(r,x)} (f(y) - 1) \, dy \\ &= \lim_{r\to 0} \frac{1}{m(B(r,x))} \int_{B(r,x)} f(y) - \lim_{r\to 0} \frac{1}{m(B(r,x))} \int_{B(r,x)} 1 \, dy \\ &= \lim_{r\to 0} \frac{1}{m(B(r,x))} \int \chi_{E\cap B(r,x)} - \lim_{r\to 0} \frac{1}{m(B(r,x))} m(B(r,x)) \\ &= \lim_{r\to 0} \frac{m(E \cap B(r,x))}{m(B(r,x))} - 1 \end{aligned}$$

for a.e $x \in E$ and thus $\dfrac{m(E\cap B(r,x))}{m(B(r,x))} = 1$ for a.e. $x \in E$.

For a.e. $x \in E^c$, $x \in L_f$ and thus

$$\lim_{r\to 0} \frac{1}{m(B(r,x))} \int_{B(r,x)} |f(y) - f(x)| \, dy = \lim_{r\to 0} \frac{1}{m(B(r,x))} \int_{B(r,x)} |f(y) - 0| \, dy = 0$$

Using an argument similar to the one above we can show that $\dfrac{m(E\cap B(r,x))}{m(B(r,x))} = 0$ for a.e. $x \in E^c$.

(b) Let $E = [0,1]$ and $x = 0$, then

$$\lim_{r\to 0} \frac{m(E\cap B(r,x))}{B(r,x)} = \lim_{r\to 0} \frac{m[0,r)}{m(-r,r)} = \frac{1}{2} \in (0,1)$$

For the other example,

$$E = \{0\} \sqcup \bigsqcup_{n=1}^{\infty} \left(\frac{1}{2^{2n+1}}, \frac{1}{2^{2n}} \right)$$

We want to find two subsequences converging to different limit, and thus show that the limit doesn't exist. First we let $\{r_k\} := \{\frac{1}{2^{2k}}\}$, $x = 0$, then $r_k \to 0$, and $m(E\cap B(r_k,x)) = \dfrac{1/2^{2k+1}}{1-1/4} = \dfrac{4}{3}\cdot\dfrac{1}{2^{2k+1}} = \dfrac{1}{3}\cdot\dfrac{1}{2^{2k-1}}$. So

$$\lim_{k\to\infty} \frac{m(E\cap B(r_k,x))}{m(B(r_k,x))} = \lim_{k\to\infty} \frac{\frac{1}{3}\cdot\frac{1}{2^{2k-1}}}{\frac{1}{2^{2k-1}}} = \frac{1}{3}$$

Then we let $\{r'_k\}$ be $\dfrac{1}{2^{2k+1}}$, and we have $m(E\cap B(r'_k,x)) = \dfrac{1/2^{2k+2}}{1-1/4} = \dfrac{4}{3}\cdot\dfrac{1}{2^{2k+2}} = \dfrac{1}{6}\cdot\dfrac{1}{2^{2k-1}}$. So

$$\lim_{k\to\infty} \frac{m(E\cap B(r'_k,x))}{m(B(r'_k,x))} = \lim_{k\to\infty} \frac{\frac{1}{6}\cdot\frac{1}{2^{2k-1}}}{\frac{1}{2^{2k-1}}} = \frac{1}{6}$$

Then $\{r_k\}$ and $\{r'_k\}$ both converge to 0, but

$$\lim_{k\to\infty} \frac{m(E\cap B(r_k,x))}{m(B(r_k,x))} \neq \lim_{k\to\infty} \frac{m(E\cap B(r'_k,x))}{m(B(r'_k,x))}$$

showing that $D_E(0)$ doesn't exist and we finish the proof.

$\square$

Folland 3.28

If $F \in NBV$, let $G(x) = |\mu_F|((-\infty, x])$. Prove that $|\mu_F| = \mu_{T_F}$ by showing that $G = T_F$ via the following steps.

(a) From the definition of T_F, $T_F \le G$

(b) $|\mu_F(E)| \le \mu_{T_F}(E)$ when E is an interval, and hence when E is a Borel set.

(c) $|\mu_F| \le \mu_{T_F}$, and hence $G \le T_F$.

Proof. (a) By definition,

$$\begin{aligned} T_F &= \sup\{\sum_{j=1}^n |F(x_j) - F(x_{j-1})| : n \in \mathbb{N}, -\infty < x_0 < ... < x_n = x\} \\ &= \sup\{\sum_{j=1}^n |\mu_F(-\infty, x_j] - \mu_F(-\infty, x_{j-1}]| : n \in \mathbb{N}, -\infty < x_0 < ... < x_n = x\} \\ &= \sup\{\sum_{j=1}^n |\mu_F(x_j, x_{j-1}]| : n \in \mathbb{N}, -\infty < x_0 < ... < x_n = x\} \quad (1) \end{aligned}$$

Since $(x_{j-1}, x_j]$ $(j = 1, ..., n)$ are disjoint, and $\bigcup_{j=1}^n (x_{j-1}, x_j] = (x_0, x]$, $(1) \le |\mu_F|(x_0, x] \le |\mu_F|(-\infty, x]$ and thus $T_F \le G$.

(b) We first suppose $E = (a, b]$ is an h-interval, then

$$\begin{aligned} |\mu_F(E)| = |\mu_F(a, b]| &= |\mu_F(-\infty, b] - \mu_F(-\infty, a]| = |F(b) - F(a)| \\ &\le \sup\{\sum_{j=1}^n |F(x_j) - F(x_{j-1})| : n \in \mathbb{N}, a = x_0 < ... < x_n = b\} \\ &= T_F(b) - T_F(a) = \mu_{T_F}(-\infty, b] - \mu_{T_F}(-\infty, a] \\ &= \mu_{T_F}(a, b] = \mu_{T_F}(E) \end{aligned}$$

In general, for $E \in \mathcal{B}_{\mathbb{R}}$, by regularity

$$|\mu_F(E)| = |\inf\{\sum_{j=1}^\infty \mu_F(a_j, b_j] : E \subset \bigcup_{i=1}^\infty (a_j, b_j]\}| \le \inf\{\sum_{j=1}^\infty \big|\mu_F(a_j, b_j]\big| : E \subset \bigcup_{i=1}^\infty (a_j, b_j]\} \quad (2)$$

But $|\mu_F(a_j, b_j)| \le \mu_{T_F}(a_j, b_j]$ for every j, so

$$(2) \le \inf\{\sum_{j=1}^\infty \mu_{T_F}(a_j, b_j] : E \subset \bigcup_{i=1}^\infty (a_j, b_j]\} = \mu_{T_F}(E)$$

and we prove the result.

(c) By exercise 21, for $E \in \mathcal{B}_{\mathbb{R}}$,

$$\begin{aligned} |\mu_F(E)| &\le \sup\{\sum_{i=1}^n |\mu_F(E_i)| : E = \bigsqcup_{i=1}^n E_i\} \le \sup\{\sum_{i=1}^n \mu_{T_F}(E_i) : E = \bigsqcup_{i=1}^n E_i\} \\ &= \sup\{\mu_{T_F}(\bigsqcup_{i=1}^n E_i) : E = \bigsqcup_{i=1}^n E_i\} = \mu_{T_F}(E) \end{aligned}$$

In particular,

$$G(x) = |\mu_F|(-\infty, x] \le \mu_{T_F}(-\infty, x] = T_F(x) - T_F(-\infty) = T_F(x)$$

Since $G(x) = |\mu_F|(-\infty, x]$ for a complex Borel measure $|\mu_F|$, $G \in NBV$ and $|\mu_F| = \mu_G$. Then $T_F = G \in NBV$ and $\mu_{T_F} = \mu_G = |\mu_F|$. $\square$

Folland 3.31

Let $F(x) = x^2 \sin(x^{-1})$ and $G(x) = x^2 \sin(x^{-2})$ for $x \neq 0$, and $F(0) = G(0) = 0$.
(a) F and G are differentiable everywhere (including $x = 0$)
(b) $F \in BV([-1, 1])$, but $G \notin BV([-1, 1])$.

Proof. (a) When $x \neq 0$, both F and G are compositions of elementary functions and are thus differentiable, so it suffices to verify for $x = 0$ in both cases.

$$\lim_{h\to 0} \frac{F(x+h) - F(0)}{h} = \lim_{h\to 0} \frac{h^2 \sin(1/h)}{h} = \lim_{h\to 0} h \sin(1/h) = 0$$

since $|\sin(1/h)| \le 1$. Also,

$$\lim_{h\to 0} \frac{G(x+h) - G(0)}{h} = \lim_{h\to 0} \frac{h^2 \sin(1/h^2)}{h} = \lim_{h\to 0} h \sin(1/h^2) = 0$$

since $|\sin(1/h^2)| \le 1$. Thus F and G are differentiable everywhere.

(b) $F'(x) = 2x \sin(x^{-1}) - \cos(x^{-1})$ when $x \neq 0$ and $F'(0) = 0$. Then $|F'| \le 3$ in $[-1, 1]$ and thus $F \in BV[-1, 1]$. For G, choose $x_j = \sqrt{\dfrac{2}{(n-j+1)\pi}}$ $(j = 1, 2, ..., n)$, notice that $x_0 > 0$ and $x_n < 1$. Then

$$\begin{aligned} T_F(1) - T_F(-1) &\ge \sum_{j=0}^{n} |G(x_j) - G(x_{j-1})| \\ &= \sum_{j=0}^{n} \left| \frac{2}{(n-j+1)\pi} \sin \frac{n-j}{2}\pi - \frac{2}{(n-j+2)\pi} \sin \frac{n-j+1}{2}\pi \right| \ge \sum_{j=2}^{n+2} \frac{2}{j\pi} \end{aligned}$$

This partition gets finer as n increses, but $T_F(1) - T_F(-1) \ge \sum_{j=2}^{n+2} \frac{2}{j\pi} \to \infty$ as $n \to \infty$ since harmonic series diverge. Thus $G \notin BV([-1, 1])$.

$\square$

Folland 3.32

If $F_1, F_2, ..., F \in NBV$ and $F_j \to F$ pointwise, then $T_F \le \liminf T_{F_j}$.

Proof. Fix $x \in \mathbb{R}$, let $N \in \mathbb{N}$ and $(x_0, x_1, ..., x_N)$ be a partition such that $-\infty < x_0 < ... < x_N = x$. Then

$$\begin{aligned} \sum_{i=1}^{n} |F(x_i) - F(x_{i-1})| &= \liminf_{j\to\infty} \sum_{i=1}^{N} |F_j(x_i) - F_j(x_{i-1})| \\ &\le \liminf_{j\to\infty} \left\{ \sup\{\sum_{i=1}^{n} |F_j(x_i) - F_j(x_{i-1})| : n \in \mathbb{N}, -\infty < x_0 < ... < x_n = x\} \right\} \\ &\le \liminf_{j\to\infty} T_{F_j} \end{aligned}$$

Since N and the partition are arbitrary, taking supremum over them we get $T_F \le \liminf T_{F_j}$. $\square$

Folland 3.36

Let G be a continuous increasing function on $[a,b]$ and let $G(a)=c$, $G(b)=d$.

(a) If $E \subset [c,d]$ is a Borel set, then $m(E) = \mu_G(G^{-1}(E))$.

(b) If f is a Borel measure and integrable function on $[c,d]$, then $\int_c^d f(y)dy = \int_a^b f(G(x))dG(x)$. In particular, $\int_c^d f(y)dy = \int_a^b f(G(x))G'(x)dx$ if G is absolutely continuous.

(c) The validity of (b) may fail if G is merely right continuous rather than continuous.

Proof. (a) We first consider the case where $E = (c_1, d_1]$ is an h-interval in $[c,d]$. Since G is continuous increasing, we can conclude using intermediate value theorem that $[c,d] = G[a,b]$.

Claim 1. For any interval I, $G^{-1}(I)$ is also an interval.

Proof of Claim. Suppose I is an interval. For $x < y$ in $G^{-1}(I)$, if $z \in [x,y]$, $G(z) \in [f(x), f(y)] \subset I$ since G is continuous increasing. Then $z \in G^{-1}(I)$ and I is an interval. $\square$

Claim 2. $G^{-1}(c_1,d_1] = (a_1,b_1]$ where $G(a_1)=c_1$ and $G(b_1)=d_1$.

Proof of Claim. $[a,b] = G^{-1}(-\infty,c_1) \sqcup G^{-1}(c_1,d_1] \sqcup G^{-1}(d_1,+\infty)$. Since G is continuous, G pulls back open (closed) sets to open (closed) sets. Combined with claim 1, we conclude that $G^{-1}(-\infty,c_1] = [a,a_1]$ and $G^{-1}(d_1,+\infty) = (b_1,b]$. This forces $G^{-1}(c_1,d_1] = (a_1,b_1]$. Observe that $G^{-1}(c_1) \neq \emptyset$, and G is continuous increasing, $c_1 = \sup G[a,a_1] = G(a_1)$. Similarly, $d_1 = \sup G(a_1,b_1] = G(b_1)$. This establishes the claim. $\square$

Then $\mu_G(G^{-1}(E)) = \mu_G(a_1,b_1] = G(b_1) - G(a_1) = d_1 - c_1 = m(E)$.

For $E \in \mathcal{B}_{\mathbb{R}}$, by regularity,

$$m(E) = \inf\{\sum_{j=1}^{\infty}(c_j,d_j] : E \subset \bigcup_{j=1}^{\infty}(c_j,d_j]\} = \inf\{\sum_{j=1}^{\infty}\mu_G(G^{-1}(c_j,d_j]) : E \subset \bigcup_{j=1}^{\infty}(c_j,d_j]\} \quad (1)$$

Shrinking the intervals if necessary, we may assume $(c_j,d_j] \subset [c,d]$. By above claim,

$$(1) = \inf\{\sum_{j=1}^{\infty}\mu_G(a_j,b_j] : E \subset \bigcup_{i=1}^{\infty}(G(a_j),G(b_j)]\} \quad (2)$$

Claim 3. $E \subset \bigcup_{j=1}^{\infty}(G(a_j),G(b_j)]$ if and only if $G^{-1}(E) \subset \bigcup_{j=1}^{\infty}(a_j,b_j]$.

Proof of Claim. Suppose $G^{-1}(E) \subset \bigcup_{j=1}^{\infty}(a_j,b_j]$,

$$E \subset G(\bigcup_{j=1}^{\infty}(a_j,b_j]) \subset \bigcup_{i=1}^{\infty}G(a_j,b_j] = \bigcup_{j=1}^{\infty}(c_j,d_j]\bigcup_{j=1}^{\infty}(G(a_j),G(b_j)]$$

by claim 2. Conversely, suppose $E \subset \bigcup_{j=1}^{\infty}(G(a_j),G(b_j)]$,

$$G^{-1}(E) \subset G^{-1}(\bigcup_{j=1}^{\infty}(c_j,d_j]) = \bigcup_{j=1}^{\infty}G^{-1}(c_j,d_j] = \bigcup_{j=1}^{\infty}(a_j,b_j]$$

Then we prove the claim. $\square$

Then by claim 2 and regularity of μ_G,

$$(2) = \inf\{\sum_{j=1}^{\infty} \mu_G(a_j, b_j] : G^{-1}(E) \subset \bigcup_{j=1}^{\infty}(a_j, b_j]\} = \mu_G(G^{-1}(E))$$

and we prove the result.

(b) Dealing with f^+ and f^- separately, we may assume $f \in L^+$. Choose a sequence of simple functions $\{\phi_n\} \uparrow f$. We may assume $\phi_n = \sum_{i=1}^m a_i\chi_{E_i}$ vanishes outside $[c,d]$, i.e. $E_i \subset [c,d]$ for all i. Then

$$\begin{aligned}\int_c^d \phi_n(y)dy &= \sum_{i=1}^m a_i m(E_i) = \sum_{i=1}^m a_i \mu_G(G^{-1}(E_i)) \\ &= \sum_{i=1}^m a_i \int_a^b \chi_{G^{-1}(E_i)} dG \quad (\text{since } G^{-1}(E_i) \subset [a,b] \ \forall \ i) \\ &= \int_a^b \sum_{i=1}^m a_i \chi_{G^{-1}(E_i)} dG \quad (3)\end{aligned}$$

Observation 4. $x \in G^{-1}(E_i)$ if and only if $G(x) \in E_i$.

Then $\chi_{G^{-1}(E_i)} = \chi_{E_i}(G(x))$ and $(3) = \int \phi_n(G(x))dG(x)$. Sending $n \to \infty$, by monotone convergence theorem, we get $\int_c^d f(y)dy = \int_a^b f(G(x))dG(x)$. In particular, if G is absolutely continuous, G is differentiable a.e., so $dG(x) = G'(x)dx$ a.e. and $\int_c^d f(y)dy = \int_a^b f(G(x))G'(x)dx$.

(c) We define $G : [0,3] \to [1,3]$ such that $G(x) = 0$ on $[0,1)$, 1 on $[1,2)$, and 2 on $[2,3]$. By extending G to be 0 at $(-\infty, 0)$ and 2 at $(3, +\infty)$ we may assume $G \in NBV$. Then suppose $f \equiv 1$, $\int_{[0,3]} f(x)dx = 3$. However,

$$\begin{aligned}\int_{[0,3]} f(G(x))dG(x) &= \int_{(-\infty,3]} f(G(x))dG(x) \\ &= \int_{(-\infty,1]} f(G(x))dG(x) + \int_{(1,2]} f(G(x))dG(x) + \int_{(2,3]} f(G(x))dG(x) \\ &= \mu_G(-\infty,1] + \mu_G(1,2] + \mu_G(2,3] \\ &= G(1) + (G(2) - G(1)) + (G(3) - G(2)) \\ &= 1 + 1 + 0 = 2 \neq 3\end{aligned}$$

showing that the conclusion of (b) may fail.

□

Folland 3.39

If $\{F_j\}$ is a sequence of nonnegative functions on $[a,b]$ such that $F(x) = \sum_1^\infty F_j(x) < \infty$ for all $x \in [a,b]$, then $F'(x) = \sum_1^\infty F_j'(x)$ for a.e. $x \in [a,b]$.

Proof. Without loss of generality we may assume that $F_j \in NBV$ for all j. This is because we can extend F_j to $\mathbb{R}$ such that $F(x) = F(b)$ for all $x \geq b$ and $F(x) = F(a)$ for all $x \leq a$, and the resulted function is still bounded increasing and thus in BV. Consider $G(x) = F(x+) - F(-\infty)$, then $G \in NBV$ and $G' = F'$ m-a.e. Then there is a Borel measure μ_{F_j} such that $F_j(x) =$

$\mu_{F_j}(-\infty,x]$. Consider the Leabesgue-Radon-Nikodym representation of μ_{F_j}

$$d\mu_{F_j} = d\lambda_j + g_j dm \iff \mu_{F_j} = \lambda_j + \int g_j dm \quad (1)$$

Here λ_j is a positive measure since μ and $\int g_j$ are both positive, and $g_j \in L^1(m)$. We set $\mu = \sum_{j=1}^{\infty} \mu_{F_j}$, $\lambda = \sum_{j=1}^{\infty} \lambda_j$, and $g = \sum_{j=0}^{\infty} g_j$. Now

$$d\mu_F = d\lambda + g\ dm \iff \mu_F = \lambda + \int g\ dm \quad (2)$$

Observe that $\lambda \perp m$ since $m(E) = 0$ implies $\lambda_j(E) = 0$ (since $\lambda_j \perp m$) and thus $\lambda(E) = \sum_{j=1}^{\infty} \lambda_j(E) = 0$. Also $g \in L^1(m)$ since

$$\int |g|\ dm = \int g\ dm \leq \mu(X) = \sum_{j=1}^{\infty} \mu_{F_j}(X) = \sum_{j=1}^{\infty} F_j(b) = F(b) < \infty$$

Thus (2) is the a.e. unique LRN representation of μ. On the other hand,

$$F(x) = \sum_{j=1}^{\infty} \mu_j(-\infty,x] = (\sum_{j=1}^{\infty} \mu_j)(-\infty,x] = \mu(-\infty,x] < \infty$$

for some finite Borel measure μ. Then $F \in NBV$ and (2) is the a.e. unique LRN representation of μ. Then $F_j'(x) = \lim_{r\to 0} \dfrac{\mu_{F_j}(E_r)}{m(E_r)} = g_j(x)$ a.e. for $E_r = (x, x+r]$ or $(x-r, x]$ which shrink nicely to x. Also we have $F'(x) = \lim_{r\to 0} \dfrac{\mu_F(E_r)}{m(E_r)} = g(x)$ a.e. This means that $F' = g = \sum_{j=1}^{\infty} g_j = \sum_{j=1}^{\infty} F_j$ a.e. and this finishes the proof. $\square$

Folland 3.41

Let $A \subset [0,1]$ be a Borel set such that $0 < m(A \cap I) < m(I)$ for every subinterval I of $[0,1]$.

(a) Let $F(x) = m([0,x] \cap A)$. Then F is absolutely continuous and strictly increasing on $[0,1]$, but $F' = 0$ on a set of positive measure.

(b) Let $G(x) = m([0,x] \cap A) - m([0,x] \setminus A)$. Then G is absolutely continuous on $[0,1]$, but G is not monotone on any subinterval of $[0,1]$.

Proof. (a) As we did in the previous problem, we may assume that $F \in NBV$. Let $\epsilon > 0$. Take $\delta = \epsilon$. For finite disjoint $(a_1,b_1),...,(a_N,b_N)$, if $\sum_1^N (b_j - a_j) < \delta$,

$$\sum_1^N |F(b_j) - F(a_j)| = \sum_1^N m((a_j,b_j] \cap A) < \sum_1^N m((a_j,b_j]) = \sum_1^N (b_j - a_j) < \epsilon$$

so F is absolutely continuous. For $x_1 < x_2$ in $[0,1]$,

$$F(x_1) - F(x_1) = m([0,x_2] \cap A) - m([0,x_1] \cap A) = m((x_1,x_2] \cap A) > 0$$

so F is strictly increasing. To establish the last part, note first that since F is absolutely continuous, $F(1) = F(1) - F(0) = \int_0^1 F'(t)dt$. Also

$$F(1) = m([0,1] \cap A) = \int_0^1 \chi_A\ dm$$

Therefore $F' = \chi_A$ a.e. on $[0,1]$. Since $m([0,1] \setminus A) = m[0,1] - m([0,1] \cap A) > 0$, $\chi_A = 0$ on a set of positive measure and thus $F' = 0$ on a set of positive measure.

(b) Let $\epsilon > 0$ and $\delta = \epsilon$, then for finite disjoint $(a_1, b_1), ..., (a_N, b_N)$ in $[0,1]$, if $\sum_1^N (b_j - a_j) < \delta$,

$$\begin{aligned}|G(b_j) - G(a_j)| &= |m([0, b_j] \cap A) - m([0, a_j] \cap A) - m([0, b_j] \setminus A) + m([0, a_j] \setminus A)| \\ &= |m((a_j, b_j] \cap A) - m((a_j, b_j] \setminus A)| \leq |m((a_j, b_j] \cap A) + m((a_j, b_j] \setminus A)| \\ &= m(a_j, b_j) = (b_j - a_j)\end{aligned}$$

and thus $\sum_1^N |G(b_j) - G(a_j)| \leq \sum_1^N (b_j - a_j) < \delta = \epsilon$, showing that G is absolutely continuous on $[0,1]$. Suppose G is mototone on some interval I which can be assumed to (a, b), either $G' \geq 0$ or $G' \leq 0$. However, since G is absolutely continuous,

$$\int_a^b G'(x) dm = G(b) - G(a) = m((a, b] \cap A) - m((a, b] \setminus A) = \int_a^b \chi_A - \chi_{A^c}$$

Then $G'(x) = \chi_A - \chi_{A^c}$ m-a.e. However since $m(A \cap I) < m(I)$, I contains a positive measure of points in both A and A^c and G' must assume both -1 and 1, contradiction. So G is not monotone in any interval.

□

4. Chapter 4-Point Set Topology

Folland 4.8

If X is an infinite set with the cofinite topology and $\{x_j\}$ is a sequence of distinct points in X, then $x_j \to x$ for every $x \in X$.

Proof. Let $x \in X$. We take a neighborhood U of x. Since U° is open, $(U^\circ)^c$ contains only finitely many points in X. Therefore, since $\{x_j\}$ is a sequence of distinct points, starting at some sufficiently large N we have $x_n \in U^\circ \subset U$. Thus $x_j \to x$. Since x is arbitrary, this finishes the proof. □

Folland 4.13

If X is a topological space, U is open in X and A is dense in X, then $\overline{U} = \overline{U \cap A}$.

Proof. Clearly $\overline{U \cap A} \subset \overline{U}$. Conversely, suppose $x \in \overline{U}$, then for any neighborhood N of x, since N° is also a neighborhood of x, $N^\circ \cap U \neq \emptyset$. Since U is open, $N^\circ \cap U$ is open in X. Since A is dense in X, $N^\circ \cap U$ contains some element of A. Then $\emptyset \neq N^\circ \cap (U \cap A) \subset N \cap (U \cap A)$. Since N is arbitrary, this means that $x \in \overline{U \cap A}$. Thus $\overline{U} \subset \overline{U \cap A}$. Two directions combined, we prove that $\overline{U} = \overline{U \cap A}$. □

Folland 4.15

If X is a topological space, $A \subset X$ is closed, and $g \in C(A)$ satisfies $g = 0$ on ∂A, then the extension of g to X defined by $g(x) = 0$ for $x \in A^c$ is continuous.

Proof. Let's call the extension f. By considering real and imaginary parts seperately, we may assume that g and f are $\mathbb{R}$-valued. Since $\{(a, b)\}$ generate the usual topology on $\mathbb{R}$, it suffices to verify that $f^{-1}[(a, b)]$ is open in X for each (a, b). We split up to two cases:

(a) If $0 \notin (a, b)$, clearly $f^{-1}[(a, b)] = g^{-1}[(a, b)] \subset A^\circ$. Since g is continuous, $g^{-1}[(a, b)]$ is open in A. Then $g^{-1}[(a, b)] = U \cap A = (U \cap A^\circ) \cup (U \cap \partial A)$ for some U open in X. Notice that

$\partial A \cap g^{-1}[(a,b)] = \emptyset$ since $0 \notin (a,b)$, $(U \cap \partial A)$ must be empty and $g^{-1}[(a,b)] = U \cap A^\circ$ is open in X. Then $f^{-1}[(a,b)]$ is open in X.

(b) Suppose $0 \in (a,b)$, then

$$\begin{aligned} f^{-1}(a,b) &= f^{-1}(a,0) \cup f^{-1}(\{0\}) \cup f^{-1}(0,b) \\ &= g^{-1}(a,0) \cup f^{-1}(\{0\}) \cup g^{-1}(0,b) \\ &= g^{-1}(a,0) \cup \partial A \cup A^c \cup g^{-1}(0,b) \\ &= g^{-1}(a,0) \cup g^{-1}(\{0\}) \cup g^{-1}(0,b) \cup A^c \\ &= g^{-1}(a,b) \cup A^c \quad (*) \end{aligned}$$

Since g is continuous, $g^{-1}(a,b)$ is open in A, meaning that $g^{-1}(a,b) = U \cap A$ for some U open in X. Then

$$(*) = (U \cap A) \cup A^c = U \cup A^c$$

which is open in X. Thus $f^{-1}(a,b)$ is open in X.

In both cases we have $f^{-1}(a,b)$ open in X, so f is continuous. This finishes the proof. □

Folland 4.20

If A is a countable set and X_α is a first (resp. second) countable space for each $\alpha \in A$, then $\prod_{\alpha \in A} X_\alpha$ is first (resp. second) countable.

Proof. (a) X_α is first countable for all $\alpha \in A$. Suppose $\mathbf{x} = \langle x_\alpha \rangle_{\alpha \in A} \in X := \prod_{\alpha \in A} X_\alpha$, then for each x_α there is a countable neighborhood base $\mathcal{N}_\alpha$. We claim that finite intersections of sets in $\pi_\alpha^{-1}(\mathcal{N}_\alpha)$, where $\alpha \in A$, form a countable neighborhood base of $\mathbf{x}$. We denote this countable neighborhood base $\mathcal{N}$ and first show that it's countable. First of all $\pi_\alpha^{-1}(\mathcal{N}_\alpha)$ is countable, and A is countable, so $\mathcal{C} := \bigcup_{\alpha \in A} \pi_\alpha^{-1}(\mathcal{N}_\alpha)$ is countable, and we enumerate them as $C_1, C_2, \ldots$. We use $\mathcal{C}_n$ to denote the collection of finite intersections of sets in $\{C_1, \ldots, C_n\}$, so each $\mathcal{C}_n$ is finite. $\mathcal{N} \subset \bigcup_{n \in N} \mathcal{C}_n$, and the latter is a countable union of finite elements, and is therefore countable. Thus $\mathcal{N}$ is countable. We then show that $\mathcal{N}$ is indeed a neighborhood base of $\mathbf{x}$. First of all, since $\mathcal{N}_\alpha$ is a neighborhood base of x_α, for every $N_\alpha \in \mathcal{N}_\alpha$, $x_\alpha \in N_\alpha$ and thus $\mathbf{x} = \pi_\alpha^{-1}(N_\alpha)$. Any finite intersections of sets like $\pi_\alpha^{-1}(N_\alpha)$ must still contain $\mathbf{x}$. Thus every element of $\mathcal{N}$ contains $\mathbf{x}$. Suppose U is open the product topology on $\prod_{\alpha \in A} X_\alpha$, and $\mathbf{x} \in U$. U must take form $\prod_{\alpha \in A} U_\alpha$, where $U_\alpha = X_\alpha$ for all but finitely many α. Thus we suppose $U_\alpha \neq X_\alpha$ for $\alpha_1, \ldots, \alpha_n$. Since $\mathcal{N}_{\alpha_i}$ is a neighborhood base of x_{α_i}, there is some $V_{\alpha_i} \in \mathcal{N}_{\alpha_i}$ such that $x \in V_{\alpha_i} \subset U_{\alpha_i}$. Then we have $\bigcap_1^n \pi_{\alpha_i}^{-1}(V_{\alpha_i}) \in \mathcal{N}$ such that $x \in \pi_{\alpha_i}^{-1}(V_{\alpha_i}) \subset U$. Then our claim is true. Since $\mathcal{N}$ is countable and $\mathbf{x}$ is arbitrary, we show that $X := \prod_{\alpha \in A} X_\alpha$ is first countable.

(b) X_α is second countable for all $\alpha \in A$. Then for each α there is a countable base $\mathcal{N}_\alpha$ of X_α. We claim that finite intersections of sets in $\pi_\alpha^{-1}(\mathcal{N}_\alpha)$ is a countable base of $X := \prod_{\alpha \in A} X_\alpha$. First of all, using exactly the same technique as above can we prove that the collection, which we name $\mathcal{N}$, is countable. We then show that $\mathcal{N}$ is actually a base. First of all, let $\mathbf{x} = \langle x_\alpha \rangle_{\alpha \in A} \in X$, each $x_\alpha \in V_\alpha \in \mathcal{N}_\alpha$ for some V_α. Then we just randomly pick an $\alpha \in A$ and we have $x \in \pi_\alpha^{-1}(V_\alpha)$. Next, suppose we have $U = \bigcap_1^n \pi_{\alpha_i}^{-1}(U_{\alpha_i})$ and $V = \bigcap_1^m \pi_{\beta_i}^{-1}(V_{\beta_i})$ in $\mathcal{N}$ and $\mathbf{x} = \langle x_\alpha \rangle_{\alpha \in A} \in U \cap V$. For our convenience, we denote $U = \prod_{\alpha \in A} U_\alpha$ and $V = \prod_{\alpha \in A} V_\alpha$, where $U_\alpha \neq X_\alpha$ only for $\{\alpha_i\}$ and $V_\alpha \neq X_\alpha$ only for $\{\beta_i\}$. We construct a family $\{W_\alpha\}$ the following way:

1. If $U_\alpha = V_\alpha = X_\alpha$, let $W_\alpha = X_\alpha$.

2. If $U_\alpha = X_\alpha \neq V_\alpha$, we know that there is $W_\alpha \in \mathcal{N}_\alpha$ such that $x_\alpha \in W_\alpha \subset V_\alpha = V_\alpha \cap U_\alpha$. The case where $U_\alpha \neq X_\alpha = V_\alpha$ is similar.
3. If $U_\alpha \neq X_\alpha$ and $V_\alpha \neq X_\alpha$, since $\mathcal{N}_\alpha$ is a base, there is some $W_\alpha \in \mathcal{N}_\alpha$ such that $x \in W_\alpha \subset U_\alpha \cap V_\alpha$.

Then $W := \prod_{\alpha\in A} W_\alpha \subset \prod_{\alpha\in A} U_\alpha \cap \prod_{\alpha\in A} V_\alpha = U \cap V$, and $W \in \mathcal{N}$ since $W_\alpha \neq X_\alpha$ only for finitely many α, and for every such α $W_\alpha \in N_\alpha$. Thus $\mathcal{N}$ is a base. Since $\mathcal{N}$ is countable, X is second countable.

□

Folland 4.22

Let X be a topological space, (Y, ρ) a complete metric space, and $\{f_n\}$ a sequence in Y^X such that $\sup_{x\in X} \rho(f_n(x), f_m(x)) \to 0$ as $m, n \to \infty$. There is a unique $f \in Y^X$ such that $\sup_{x\in X} \rho(f_n(x), f(x)) \to 0$ as $n \to \infty$. If each f_n is continuous, so is f.

Proof. (a) Define f such that $f(x) = \lim_{n\to\infty} f_n(x)$, and we claim that $\sup_{x\in X} \rho(f_n(x), f(x)) \to 0$ as $n \to \infty$. Let $\epsilon > 0$. Since $\sup_{x\in X} \rho(f_n(x), f_m(x)) \to 0$ as $m, n \to \infty$, we can pick N large enough such that $\sup_{x\in X} \rho(f_n(x), f_m(x)) < \epsilon/2$ if $m, n > N$. Also, by our definition of $f(x)$, for every x we can pick large enough $m_x > N$ such that $\rho(f_{m_x}(x), f(x)) < \epsilon/2$. Then for every x, if $n > N$,

$$\rho(f_n(x), f(x)) \leq \rho(f_n(x), f_{m_x}(x)) + \rho(f_{m_x}(x), f(x)) < \epsilon$$

and thus $\sup(f_n(x), f(x)) < \epsilon$. This proves that $\sup \rho(f_n(x), f(x)) \to 0$. Moreover, suppose g has the property that $\sup_{x\in X} \rho(f_n(x), g(x)) \to 0$, for every x we have $\rho(f_n(x), g(x)) \to 0 \iff f_n(x) \to g(x)$. Since (Y, ρ) is a metric space, the limit is unique and $g(x) = f(x)$. Then $g = f$ and f is unique.

(b) Suppose each f_n is continuous. Let $x \in X$, then f_n is continuous at x for all n. Let $\epsilon > 0$. Then there is some $N > 0$ such that $\sup_{x\in X} \rho(f_N(x), f(x)) < \epsilon/3$. By continuity of f_N, $A := f_n^{-1}[B(\epsilon/3, f_n(x))]$ is an open neighborhood of x. Let $y \in A$, then

$$\rho(f(y), f(x)) \leq \rho(f(y), f_n(y)) + \rho(f_n(y), f_n(x)) + \rho(f_n(x), f(x)) < \epsilon$$

and thus $y \in A \subset (f^{-1}(B(\epsilon, f(x))))^\circ$ since A is open. To show that f is continuous at x, since open balls generate the topology on Y, it suffices to show that $f^{-1}(B)$ is a neighborhood of x for every open ball B containing $f(x)$. Since $B \ni f(x)$ is open, we can take a small enough $\epsilon > 0$ such that $B(\epsilon, f(x)) \subset B$. Then by what we did above $f^{-1}(B(\epsilon, f(x)))$ is a neighborhood containing x. Then $f^{-1}(B) \supset f^{-1}(B(\epsilon, f(x)))$ is also a neighborhood containing x. Thus f is continuous at x. Since x is arbitrary, this shows that f is continuous.

□

Folland 4.24

A Hausdorff space X is normal iff X satisfies the conclusion of Urysohn's lemma iff X satisfies the conclusion of the Tietze extension theorem.

Proof. We use (1), (2), (3) to denote these three statements respectively.
We first show (1) $\iff$ (2). The fact that (1) $\implies$ (2) is trivial. Conversely, first we notice that X is T_1 since X is Hausdorff. Given disjoint closed sets A and B, by Urysohn's lemma there is a continuous $f : X \to [0,1]$ such that $f \equiv 0$ on A and $f \equiv 1$ on B. Then take $U = f^{-1}[0, 1/2)$ and $V = f^{-1}(1/2, 1]$. Since f is continuous, U and V are open, and clearly U and V are disjoint

since $[0, 1/2)$ and $(1/2, 1]$ are disjoint. Then U is an open set containing A and V is an open set containing B such that $U \cap V = \emptyset$. Then X is normal.
We then show that (2) $\iff$ (3). Suppose we have (2). Since X is Hausdorff, by the previous paragraph we have X is normal. Then (3) holds automatically. Conversely, suppose we have (3), given disjoint closed sets A and B, define $f : A \cup B \to [0, 1]$ to be $f|_A \equiv 0$ and $f|_B \equiv 1$, then $f \in C(A \cup B, [0, 1])$. By (3) we have $F \in C(X, [0, 1])$ such that $F = f$ on $A \cup B$, i.e. $F \equiv 0$ on A and $F \equiv 1$ on B. Then (2) holds. □

Folland 4.38

Suppose that $(X, \mathcal{T})$ is a compact Hausdorff space and $\mathcal{T}'$ is another topology on X. If $\mathcal{T}'$ is strictly stronger than $\mathcal{T}$, then $(X, \mathcal{T}')$ is Hausdorff but not compact. If $\mathcal{T}'$ is strictly weaker than $\mathcal{T}$, then $(X, \mathcal{T}')$ is compact but not Hausdorff.

Proof. (a) Suppose $\mathcal{T}'$ is stricly stronger than $\mathcal{T}$. For $x \neq y \in X$, since $(X, \mathcal{T})$ is Hausdorff, there are disjoint closed A, B such that $x \in A$ and $y \in B$. But $(X, \mathcal{T}') \supsetneq (X, \mathcal{T})$, so $A, B \in \mathcal{T}'$. Thus $(X, \mathcal{T}')$ is Hausdorff. Suppose in the contrary that $(X, \mathcal{T}')$ is compact. Consider mapping $f : (X, \mathcal{T}') \to (X, \mathcal{T})$ defined by $x \mapsto x$, which is clearly bijective. For U open in $(X, \mathcal{T})$, $f^{-1}(U) = U$ open in $(X, \mathcal{T}')$ since $\mathcal{T}'$ is strictly stronger than $\mathcal{T}$, so f is continuous. If $(X, \mathcal{T}')$ is compact, then f is a continuous bijection mapping from a compact space to a Hausdorff space and is thus a homeomorphism. But $\mathcal{T}' \supsetneq \mathcal{T}$, a contradiction. Thus $(X, \mathcal{T}')$ cannot be compact.

(b) Suppose $\mathcal{T}'$ is strictly weaker than $\mathcal{T}$. Take an open cover $\mathcal{U}$ of X in $\mathcal{T}'$, $\mathcal{U}$ is also an open cover in $\mathcal{T}$ and thus has a finite subcover. Thus $(X, \mathcal{T}')$ is compact. Suppose $(X, \mathcal{T}')$ is Hausdorff, consider $g : (X, \mathcal{T}) \to (X, \mathcal{T}')$ defined by $x \mapsto x$, then g is bijective. For U open in $(X, \mathcal{T}')$, $f^{-1}(U) = U \in \mathcal{T}' \subset \mathcal{T}$ and is therefore open. Then g is continuous bijection between a compact space and a Hausdorff space and is therefore a homeomorphism. which is not possible since T' stricly weaker than $\mathcal{T}$. Thus $(X, \mathcal{T}')$ is not compact.

□

Folland 4.43

For $x \in [0, 1)$, let $\sum_1^\infty a_n(x) 2^{-n}$ ($a_n(x) = 0$ or 1) be the base-2 decimal expansion of x. (If x is a dyadic rational, choose the expansion such that $a_n(x) = 0$ for n large.) Then the sequence $\langle a_n \rangle$ in $\{0, 1\}^{[0,1)}$ has no pointwise convergent subsequence.

Proof. Take any subsequence $\langle a_{n_k} \rangle$ of $\langle a_n \rangle$, consider $x = \sum_{k=1}^\infty 2^{-n_{2k}}$. It is clear that x is not a dyadic rational (since there are no consecutive 1s or 0s), so the expression is uniquely determined here. Now we have $a_{n_{2k}} = 1$ and other $a_{n_k} = 0$. Since we have infinitely many alternating terms, $\langle a_{n_k}(x) \rangle$ fails to converge. Thus $\langle a_n \rangle$ has no pointwise convergent subsequence. □

Folland 4.56

Define $\phi : [0, \infty] \to [0, 1]$ by $\phi(t) = t/(t + 1)$ for $t \in [0, \infty]$ and $\phi(\infty) = 1$.

(a) ϕ is strictly increasing and $\phi(t + s) \le \phi(t) + \phi(s)$.

(b) If (Y, ρ) is a metric space, then $\phi \circ \rho$ is a bounded metric on Y that defines the same topology as ρ.

(c) If X is a topological space, the function $\rho(f, g) = \phi(\sup_{x\in X} ||f(x) - g(x)|)$ is a metric on $\mathbb{C}^X$ whose associated topology is the topology of uniform convergence.

(d) If $X = \mathbb{R}^n$ and $U_n = B(n, 0)$ for all n, the function

$$\rho(f, g) = \sum_1^\infty 2^{-n}\phi(\sup_{x\in \overline{U}_n} |f(x) - g(x)|)$$

is a metric on $\mathbb{C}^X$ whose associated topology is the topology of locally uniform convergence.

Proof. (a) We first show that ϕ is strictly increasing. Suppose $t_1 < t_2 < \infty$, then

$$\phi(t_2) - \phi(t_1) = \frac{t_2}{t_2 + 1} - \frac{t_1}{t_1 + 1} = \frac{t_2(t_1 + 1) - t_1(t_2 + 1)}{(t_2 + 1)(t_1 + 1)} = \frac{t_2 - t_1}{(t_2 + 1)(t_1 + 1)} > 0$$

Suppose $t_1 < t_2 = \infty$, then since $t_1 < \infty$, $\phi(t_1) < 1 = \phi(t_2)$. We next show that $\phi(t + s) \le \phi(t) + \phi(s)$. If one of t, s is ∞, which we may assume to be t, then

$$\phi(t + s) = \phi(\infty) \le \phi(\infty) + 1 = \phi(t) + \phi(s)$$

If $t, s < \infty$, then

$$\begin{aligned}
&\phi(t + s) - (\phi(t) + \phi(s)) = \frac{t + s}{t + s + 1} - \frac{t}{t + 1} - \frac{s}{s + 1} \\
=&1 - \frac{1}{t + s + 1} - 1 + \frac{1}{t + 1} - 1 + \frac{1}{s + 1} = \frac{t + 1 + s + 1}{(t + 1)(s + 1)} - \frac{t + s + 2}{t + s + 1} \\
\le&\frac{t + 1 + s + 1}{(t + 1)(s + 1)} - \frac{t + s + 2}{t + s + ts + 1} = \frac{t + 1 + s + 1}{(t + 1)(s + 1)} - \frac{t + s + 2}{(t + 1)(s + 1)} = 0
\end{aligned}$$

showing the result.

(b) For convenience, define $\rho' := \phi \circ \rho = \dfrac{\rho}{\rho + 1}$. It is easy to see ρ' is bounded by 1. We denote the topology generated by ρ and ρ' using $\mathcal{T}$ and $\mathcal{T}'$, respectively, and the collection of open balls in $\mathcal{T}$ and $\mathcal{T}'$ are named $\mathcal{E}$ and $\mathcal{E}'$, respectively. We know that $\mathcal{T}(\mathcal{E}) = \mathcal{T}$, and $\mathcal{T}(\mathcal{E}') = \mathcal{T}'$. Then it suffices to show $\mathcal{E} \subset \mathcal{T}'$ and $\mathcal{E}' \subset \mathcal{T}$. Let $B_\rho(x, r) \in \mathcal{E}$, and we claim that $B_\rho(x, r) = B_{\rho'}(x, \frac{r}{r+1})$. To show the claim, suppose we have $y \in Y$, then $\rho(x, y) < r$ if and only if $\rho'(x, y) = \phi \circ \rho(x, y) < \phi(r) = r/(r + 1)$ since ϕ is strictly increasing. In other words, $y \in B_\rho(x, r)$ if and only if $y \in B_{\rho'}(x, \frac{r}{r+1})$ and thus the claim is true. Thus $B_\rho(x, r) = B_{\rho'}(x, \frac{r}{r+1}) \in \mathcal{E}' \subset \mathcal{T}'$ as desired. Similarly we can show that $B_{\rho'}(x, r) = B_\rho(x, \frac{r}{1-r})$ since $\phi(r/1 - r) = r$. Thus $\mathcal{E}' \subset \mathcal{E} \subset \mathcal{T}$. Now we have $\mathcal{T} = \mathcal{T}(\mathcal{E}) \subset \mathcal{T}'$ and $\mathcal{T}' = \mathcal{T}(\mathcal{E}') \subset \mathcal{T}$, showing that ρ and ρ' generate the same topology on Y.

(c) We first verify that ρ is a metric on $\mathbb{C}^X$.

Non-Negativity: $\rho(f, g) \ge 0$ since $\phi \ge 0$

Identity of Indiscernibles: $\rho(f, g) = 0$ iff $\phi(\sup_{x\in X} |f(x)-g(x)|) = 0$ iff $\sup_{x\in X} |f(x)-g(x)| = 0$ iff $|f(x) - g(x)|$ for every x iff $f = g$.

Symmetry: $\rho(f, g) = \phi(\sup_{x\in X} |f(x) - g(x)|) = \phi(\sup_{x\in X} |g(x) - f(x)|) = \rho(g, f)$

Triangular Inequality: For $f, g, h \in \mathbb{C}^X$, $\rho(f,g) + \rho(g,h) = \phi(\sup_{x\in X}|f(x) - g(x)|) + \phi(\sup_{x\in X}|g(x) - h(x)|) \geq \phi(\sup_{x\in X}|f(x) - g(x)| + \sup_{x\in X}|g(x) - h(x)|) \geq \phi(\sup_{x\in X}|f(x) - h(x)| = \rho(f,h)$.

We know that $\rho_u(f,g) = \sup_{x\in X}||f(x) - g(x)||$, and ρ_u is a metric that generates the topology of uniform convergence, and by (b) we know that ρ_u and $\rho = \phi \circ \rho_u$ generate the same topology, so the associated topology of ρ is also the topology of uniform convergence.

(d) Suppose $\langle f_n \rangle_n$ converges locally uniformly to f. By definition

$$\rho(f_n, f) = \sum_{i=1}^{\infty} 2^{-i}\phi\left(\sup_{x\in\overline{U}_i}|f_n(x) - f(x)|\right)$$

Let $\epsilon > 0$. Choose N large enough such that $2^{-N+1} < \epsilon$. Since $\overline{U}_{N-1}$ is compact, by locally uniform convergence $f_n|\overline{U}_{N-1} \to f|\overline{U}_{N-1}$ uniformly (and automatically converges uniformly on $\overline{U}_i$ for all $i < N$). Then

$$\begin{aligned}
\rho(f_n, f) &= \sum_{i=1}^{N-1} 2^{-i}\phi\left(\sup_{x\in\overline{U}_i}|f_n(x) - f(x)|\right) + \sum_{i=N}^{\infty} 2^{-i}\phi\left(\sup_{x\in\overline{U}_i}|f_n(x) - f(x)|\right) \\
&\leq \sum_{i=1}^{N-1} 2^{-i}\phi\left(\sup_{x\in\overline{U}_i}|f_n(x) - f(x)|\right) + \sum_{i=N}^{\infty} 2^{-i} \\
&\leq \sum_{i=1}^{N-1} 2^{-i}\phi\left(\sup_{x\in\overline{U}_i}|f_n(x) - f(x)|\right) + \epsilon
\end{aligned}$$

Sending n to infinity, we get $\lim_{n\to\infty}\rho(f_n, f) < \epsilon$. Since ϵ is arbitrary, $\lim_{n\to\infty}\rho(f_n, f) = 0$, implying convergence in ρ. Conversely, suppose $\lim_{n\to\infty}\rho(f_n, f) = 0$ and $K \subset \mathbb{R}^n$ compact. We can take N large enough such that U_N contains K, so it suffices to show that $f_n|\overline{U}_N \to f|\overline{U}_N$. Actually, we prove a stronger claim: for all $n \in \mathbb{N}$, $f_k|\overline{U}_n \to f|U_n$ uniformly. Suppose not, there is some $N \in \mathbb{N}$ such that $f_k|\overline{U}_N \not\to f|\overline{U}_N$ uniformly. Then there is some $\delta > 0$ such that for infinitely many k we have $\sup_{x\in\overline{U}_n}|f_k(x) - f(x)| \geq \delta$. Notice that this means that for infinitely many k, we have $\sup_{x\in\overline{U}_n}|f_k(x) - f(x)| \geq \delta$ for $n \geq N$. Then

$$\rho(f, f_k) \geq \sum_{n=N}^{\infty} 2^{-n}\phi(\delta) = \frac{2^{-N}}{1 - 1/2} \cdot \frac{\delta}{\delta + 1} = 2^{-N+1}\frac{\delta}{\delta + 1}$$

for infinitely many k so f_k doesn't converge to f in ρ, a contradiction. This finishes the proof.

□

Folland 4.61

Theorem 4.43 remains valid for maps from a compact Hausdorff space X into a complete metric space Y provided the hypothesis of pointwise boundedness is replaced by pointwise total boundedness.

Proof. Let $\epsilon > 0$. Since $\mathcal{F}$ is equicontinuous, for each $x \in X$ there is open U_x of x such that $\rho(f(x), f(y)) < \dfrac{\epsilon}{4}$ for all $y \in U_x$ and $f \in \mathcal{F}$. Since X is compact, we can choose $x_1, ..., x_n \in X$ such that $\bigcup_1^n U_{x_j} = X$. By pointwise total boundedness, $\{f(x_j) : f \in F, 1 \leq j \leq n\}$ is a totally bounded subset of Y being a finite union of totally bounded subsets of Y. Then we can pick finite $\{z_1, ..., z_m\}$ that is $\frac{\epsilon}{4}$-dense in it. Let $A = \{x_1, ..., x_n\}$ and $B = \{z_1, ..., z_m\}$, then B^A is finite. For each $\phi \in B^A$ let

$$\mathcal{F}_\phi = \{f \in \mathcal{F} : \rho(f(x_j), \phi(x_j)) < \frac{\epsilon}{4} \text{ for } 1 \leq j \leq n\}$$

clearly $\cup_{\phi \in B^A} = \mathcal{F}$, and we claim that each $\mathcal{F}_\phi$ has diameter $\leq \epsilon$, so we obtain a finite ϵ-dense subset of $\mathcal{F}$ by picking one f from each $\mathcal{F}_\phi$ that is non-empty. To prove this claim, suppose we have $f, g \in \mathcal{F}_\phi$. Then $\rho(f, \phi) < \frac{\epsilon}{4}$ and $\rho(g, \phi) < \frac{\epsilon}{4}$ on A and we have $\rho(f, g) < \frac{\epsilon}{2}$ on A. If $x \in X$, we have $x \in U_{x_j}$ for some j, and then

$$\rho(f(x), g(x)) \leq \rho(f(x), f(x_j)) + \rho(f(x_j), g(x_j)) + \rho(g(x_j), g(x)) < \epsilon$$

This shows that $\mathcal{F}$ is totally bounded. Then $\overline{\mathcal{F}}$ is totally bounded. Since Y is complete, by exercise 4.22 $C(X, Y)$ is complete. Being a closed and totally bounded subset of complete metric space $C(X, Y)$, $\overline{\mathcal{F}}$ is compact. □

Folland 4.63

Let $K \in C([0,1] \times [0,1])$. For $f \in C([0,1])$, let $Tf(x) = \int_0^1 K(x,y)f(y)dy$. Then $Tf \in C([0,1])$, and $\{Tf : ||f||_u \leq 1\}$ is precompact in $C([0,1])$.

Proof. We first show that $Tf \in C([0,1])$. Let $\epsilon > 0$. Since $K \in C([0,1] \times [0,1])$ and $[0,1] \times [0,1]$ is compact, K is uniformly continuous on $[0,1] \times [0,1]$. Thus there is a $\delta > 0$ such that when $|\mathbf{x} - \mathbf{y}| < \delta$, $|K(\mathbf{x}) - K(\mathbf{y})| < \epsilon$. Then for $x_1, x_2 \in [0,1]$ such that $|x_1 - x_2| < \delta$,

$$|Tf(x_1) - Tf(x_2)| \leq \int_0^1 |K(x_1, y) - K(x_2, y)| f(y)dy < \epsilon \int_0^1 f(y)dy$$

Observe that $f \in C([0,1])$, so by extreme value theorem $|f| \leq M$ for some $M > 0$. Then $|Tf(x_1) - Tf(x_2)| < M\epsilon$. Since ϵ is arbitrary, Tf is uniformly continuous on $[0,1]$ and thus $Tf \in C([0,1])$.
We then show that $\{Tf : ||f||_u \leq 1\}$ is precompact in $C([0,1])$. For convenience, we use $\mathcal{F}$ to denote the family $\{Tf : ||f||_u \leq 1\}$. Let $\epsilon > 0$. Since $K \in C([0,1] \times [0,1])$ and $[0,1] \times [0,1]$ is compact, K is uniformly continuous on $[0,1] \times [0,1]$. By uniform continuity, there is a $\delta > 0$ such that $|K(\mathbf{x}) - K(\mathbf{y})| < \epsilon$ whenever $\mathbf{x}, \mathbf{y} \in [0,1] \times [0,1]$ satisfy $|\mathbf{x} - \mathbf{y}| < \delta$. Let $x_1 \in [0,1]$. Consider $U_{x_1} := (x_1 - \delta, x_1 + \delta)$ and by shrinking δ if necessary we may assume $U_{x_1} \subset [0,1]$. When $x_2 \in U_{x_1}$,

$$|Tf(x_2) - Tf(x_1)| \leq \int_0^1 |K(x_2, y) - K(x_1, y)| f(y)dy < \epsilon \int_0^1 f(y)dy \leq \epsilon \cdot 1 = \epsilon$$

The last inequality is validated since $|f| \leq 1$. Then $\mathcal{F}$ is equicontinuous at x_1 and thus equicontinuous. Since K is continuous on a compact set, $|K| \leq M$ for some $M > 0$ by extreme value theorem. Then for $x \in [0,1]$, $|Tf(x)| \leq \int_0^1 |K(x,y)| f(y)dy \leq M$ for all f such that $||f||_u \leq 1$ and is thus pointwise bounded. Since $[0,1]$ is compact Hausdorff, by Arzela-Ascoli, $\mathcal{F}$ is precompact as desired. □

Folland 4.64

Let (X, ρ) be a metric space. A function $f \in C(X)$ is called Holder continuous of exponent α if the quantity

$$N_\alpha(f) = \sup_{x \neq y} \frac{|f(x) - f(y)|}{\rho(x, y)^\alpha}$$

is finite. If X is compact, $\{f \in C(X) : ||f||_u \leq 1 \text{ and } N_\alpha \leq 1\}$ is compact in $C(X)$.

Proof. We first notice that X is a compact metric space and is therefore compact Hausdorff. For our convenience, we define $\mathcal{F} := \{f \in C(X) : ||f||_u \leq 1 \text{ and } N_\alpha \leq 1\}$. Let $\epsilon > 0$, $x \in X$ consider $U_x = B(\epsilon^{1/\alpha}, x)$.[3] Thus for any $f \in \mathcal{F}$, $y \in U_x$,

$$|f(y) - f(x)| \leq N_\alpha(f)\rho(x, y)^\alpha \leq \rho(x, y)^\alpha < \epsilon$$

Thus $\mathcal{F}$ is equicontinuous at x and is thus equicontinuous. Furthermore $\mathcal{F}$ is clearly pointwise bounded since it is uniformly bounded by 1. By Arzela-Ascoli, $\mathcal{F}$ is precompact, so it suffices to show that $\mathcal{F}$ is closed. Suppose $f \in \overline{\mathcal{F}}$, then for every $\epsilon > 0$, there is $f' \in \mathcal{F}$ such that $||f' - f||_u < \epsilon$ and $N_\alpha(f') \leq 1$. Then $||f||_u < ||f'||_u + \epsilon \leq 1 + \epsilon$. Since ϵ is arbitrary, $||f||_u \leq 1$. Also we have $N_\alpha(f') = \sup_{x \neq y} \dfrac{|f'(x) - f'(y)|}{\rho(x, y)^\alpha} \leq 1$. For $x \neq y \in X$, $|f(x) - f(y)| \leq |f(x) - f'(x)| + |f'(x) - f'(y)| + |f'(y) - f(y)| < 2\epsilon + \rho(x, y)^\alpha$. But ϵ is arbitrary, so $|f(x) - f(y)| \leq \rho(x, y)^\alpha$ and thus $N_\alpha(f) = \sup_{x \neq y} \dfrac{|f(x) - f(y)|}{\rho(x, y)^\alpha} \leq 1$. Thus $f \in \mathcal{F}$ and thus $\mathcal{F}$ is closed. This means that $\mathcal{F} = \overline{\mathcal{F}}$ is compact and this finishes the proof. □

Folland 4.68

Let X and Y be compact Hausdorff spaces. The algebra generated by functions of the form $f(x, y) = g(x)h(y)$, where $g \in C(X)$ and $h \in C(Y)$, is dense in $C(X \times Y)$.

Proof. We denote the algebra using $\mathcal{A}$. For $f \in \mathcal{A}$, $f = gh$ for $g \in C(X)$ and $f \in C(Y)$. Then $\overline{f} = \overline{gh} = \overline{g}\overline{h}$. Continuity is componentwise and therefore preserved under conjugation, so $\overline{g} \in C(X)$ and $\overline{h} \in C(Y)$ and $\overline{f} \in \mathcal{A}$. Thus $\mathcal{A}$ is closed under conjugation. Suppose $(x_1, y_1) \neq (x_2, y_2)$. Then $x_1 \neq x_2$ or $y_1 \neq y_2$ and we may assume $x_1 \neq x_2$. Remember that X is compact Hausdorff and therefore normal, and $\{x_1\}$ and $\{x_2\}$ are disjoint closed sets in X since singletons are closed in Hausdorff spaces. By Urysohn's lemma, there is a $g \in C(X)$ such that $g(x_1) = 0$ and $g(x_2) = 1$. Let $h \equiv 1$ in $C(Y)$. Then $f := gh \in \mathcal{A}$ satisfies

$$f(x_1, y_1) = g(x_1)h(y_1) = 0 \neq 1 = g(x_2)h(y_2) = f(x_2, y_2)$$

and thus $\mathcal{A}$ separate points. Notice that $f \equiv 1 \cdot 1 = 1 \in \mathcal{A}$ so f doesn't vanish at any x_0. By Stone-Weierstrass, $\mathcal{A}$ is dense in $C(X \times Y)$. □

Folland 4.69

Let A be a non-empty set, and let $X = [0, 1]^A$. The algebra generated by the coordinate maps $\pi_\alpha : X \to [0, 1]$ $(\alpha \in A)$ and the constant function 1 is dense in $C(X)$.

[3] Here $1/\alpha$ is defined since $\alpha > 0$.

Proof. We denote the algebra using $\mathcal{A}$. For $\pi_\alpha \in \mathcal{A}$, $\overline{\pi_\alpha} = \pi_\alpha$ since it is a function to $[0,1] \subset \mathbb{R}$. Thus $\mathcal{A}$ is closed under conjugation. Suppose we have distinct $\mathbf{x} = \langle x_\alpha \rangle_{\alpha \in A}$ and $\mathbf{y} = \langle y_\alpha \rangle_{\alpha \in A}$ in $[0,1]^A$, then $x_\alpha \neq y_\alpha$ at some α_0. Then $\pi_{\alpha_0}(\mathbf{x}) = x_{\alpha_0} \neq y_{\alpha_0} = \pi_{\alpha_0}(\mathbf{y})$. Thus $\mathcal{A}$ separates points. Since $1 \in \mathcal{A}$, $\mathcal{A}$ doesn't vanish at any $\mathbf{x} \in [0,1]^A$. By Stone-Weierstrass, $\mathcal{A}$ is dense in $C(X)$. □

Folland 4.76

If X is normal and second countable, there is a countable family $\mathcal{F} \subset C(X, I)$ that separates points and closed sets.

Proof. Since X is second countable, let $\mathcal{B}$ be a countable base of X. For each pair $(U,V) \in \mathcal{B} \times \mathcal{B}$ such that $\overline{U} \subset V$, by Urysohn's lemma, there is some function $f : X \to I$ such that $f(\overline{U}) = 0$ and $f(V^c) = 1$ since $\overline{U} \cap V^c \subset V \cap V^c = \emptyset$ and $\overline{U}, V^c$ are closed. For each such pair (U,V), we pick one particular function f that satisfies the conditions above, and let $\mathcal{F}$ be defined as the collection of such functions. $|\mathcal{F}| \leq |\mathcal{B} \times \mathcal{B}|$, and the latter set is countable since B is countable, so $\mathcal{F}$ is countable. We proceed to show that $\mathcal{F}$ separates points and closed sets. Given $E \subset X$ closed and $x \in E^c$, since E^c is open, there is some $V \in \mathcal{B}$ such that $x \in V \subset E^c$. Then $E = (E^c)^c \subset V^c$. We claim that there is some U' open such that $x \in U' \subset \overline{U'} \subset V$. To show the claim, we notice that $\{x\}$ and V^c are disjoint closed sets, so by normality there are disjoint open $V' \supset V^c$ and $U' \ni x$. Given $U' \cap V' = \emptyset$, $U' \subset (V')^c$ and thus $\overline{U'} \subset (V')^c$ since $(V')^c$ is closed. Then $x \in U' \subset \overline{U'} \subset (V')^c \subset (V^c)^c = V$, as desired. Since U' is open, we have $U \in \mathcal{B}$ such that $x \in U \subset U'$. Then $x \in \overline{U} \subset \overline{U'} \subset V$ for $U, V \in \mathcal{B}$. By definition there is some $f \in \mathcal{F}$ such that $f(\overline{U}) = 0$ and $f(V^c) = 1$. Remember that $x \in \overline{U}$ and $E \subset V^c$, so $f(x) = 0 \notin \{1\} = f(E)$, as desired. □

5. Chapter 5-Elements of Functional Analysis

Folland 5.3

If $\mathcal{Y}$ is complete, so is $L(\mathcal{X}, \mathcal{Y})$.

Proof. Pick a Cauchy sequence $\{T_n\}_n$ in $L(\mathcal{X}, \mathcal{Y})$ under the norm metric. Then $\{T_n x\}_n$ is Cauchy for each x since $\|T_n x - T_m x\| \leq \|T_n - T_m\| \|x\| \to 0$ as $n, m \to \infty$ since $\|T_n - T_m\| \to 0$ as $n, m \to \infty$. Thus $\{T_n x\}$ converges. Define $Tx := \lim_{n\to\infty} T_n x$. We first show that $T \in L(\mathcal{X}, \mathcal{Y})$. Let $\epsilon > 0$. Since $\{T_n\}$ is Cauchy, there is some $N > 0$ such that when $m, n \geq N$, $||T_n - T_m|| < \epsilon$. Then $||T_n x - T_m x||/||x|| < \epsilon$ for all $x \neq 0$. Sending m to infinity, we have $||T_n(x) - T(x)||/||x|| < \epsilon$ for all $x \neq 0$. In particular, $||T_N(x) - T(x)|| < \epsilon ||x||$ for all x. Since T_N is bounded, suppose $||T_N x|| \leq C_N ||x||$ for all x. Then

$$||T(x)|| \leq ||T_N(x)|| + ||T(x) - T_N(x)|| \leq (C_N + \epsilon)||x||$$

for all x and thus $T \in L(\mathcal{X}, \mathcal{Y})$. Furthermore, for the ϵ and N given above, when $n > N$, $||T_n(x) - T(x)||/||x|| < \epsilon$ for all $x \neq 0$ and thus $||T_n - T|| = \sup\{||T_n(x) - T(x)||/||x|| : x \neq 0\} < \epsilon$, showing that $||T_n - T|| \to 0$, as desired. □

Folland 5.8

Let $(X, \mathcal{M})$ be a measurable space, and let $M(X)$ be the space of finite signed measures on $(X, \mathcal{M})$. Then $||\mu|| = |\mu|(X)$ is a norm on $M(X)$ that makes $M(X)$ into a Banach space.

Proof. We first verify that $||\mu|| := |\mu|(X)$ is a norm.

- $||\mu_1 + \mu_2|| = |\mu_1 + \mu_2|(X) \leq |\mu_1|(X) + |\mu_2|(X) = ||\mu_1|| + ||\mu_2||$
- By Lebesgue-Randon-Nikodym, there is some positive measure ν such that $d\mu = f d\nu$. Then $d(\lambda\mu) = \lambda d\mu = \lambda f d\nu$ Then $||\lambda\mu|| = |\lambda\mu|(X) = \int |\lambda f| d\nu = |\lambda| \int |f| d\nu = |\lambda||\mu|(X) = |\lambda| \cdot ||\mu||$
- If $||\mu|| = 0$, then $\mu(X) = 0$ and $\mu = 0$ since any subset of a measure zero set (in this case, every measurable set) has measure zero.

We then show that $M(X)$ is a Banach space under the given norm by showing that every absolutely convergent series in $M(X)$ converges under this norm. Suppose we have $\mu_1, \mu_2, ... \in M(X)$ such that $\sum_{n=1}^{\infty} ||\mu_n|| < \infty$. Then $\sum_{n=1}^{\infty} |\mu_n|(X) < \infty$. Define $\nu := \sum_{n=1}^{\infty} \mu_n$. Notice that $\sum_{n=1}^{m} \mu_n(X) \leq \sum_{n=1}^{m} |\mu|(X)$ for all m and thus sending $m \to \infty$ gives $\sum_{n=1}^{\infty} \mu_n(X) \leq \sum_{n=1}^{m} |\mu|(X) < \infty$, showing that ν is a finite signed measure. Thus $\lim_{m\to\infty} \|\nu - \sum_{n=1}^{m} \mu_n\| = \|v - \sum_{n=1}^{\infty} \nu_n\| = 0$, showing that $\sum_{n=1}^{\infty}$ converges to $\nu \in M(X)$, as desired. Thus $M(X)$ is a Banach space. ∎

Folland 5.9

Let $C^k([0,1])$ be the space of functions on $[0,1]$ possessing continuous derivatives up to order k on $[0,1]$, including one-sided derivatives at endpoints.

(a) If $f \in C([0,1])$, then $f \in C^k([0,1])$ iff f is k times continuously differentiable on $(0,1)$ and $\lim_{x\downarrow 0} f^{(j)}(x)$ and $\lim_{x\uparrow 1} f^{(j)}(x)$ exist for $j \leq k$.

(b) $\|f\| = \sum_0^k \|f^{(j)}\|_u$ is a norm on $C^k([0,1])$ that makes $C^k([0,1])$ into a Banach space.

Proof. (a) If $f \in C^k([0,1])$, then clearly $f \in C^k(0,1)$ and $\lim_{x\downarrow 0} f^{(j)}(x) = f^{(j)}(0)$ and $\lim_{x\uparrow 1} f^{(j)}(x) = f^{(j)}(1)$ for all $j \leq k$. Conversely, suppose f is k times continuously differentiable on $(0,1)$ and $\lim_{x\downarrow 0} f^{(j)}(x)$ and $\lim_{x\uparrow 1} f^{(j)}(x)$ exist for $j \leq k$, we want to show that $f \in C^k[0,1]$. Since $f \in C^k(0,1)$, it suffices to show that f is continuously differentiable at 0 and 1. So we proceed by induction. The base case where $j = 0$ is true since $f \in C[0,1]$. Suppose f is j times continuously differentiable at 0. Notice that $\lim_{x\downarrow 0} \dfrac{f^{(j)}(x) - f^{(j)}(0)}{x} = \lim_{x\downarrow 0} f^{(j)}(c)$ for some $c \in (0,x]$ by mean value theorem, and thus $c \downarrow 0$ as $x \downarrow 0$. Thus $\lim_{x\downarrow 0} \dfrac{f^{(j)}(x) - f^{(j)}(0)}{x} = \lim_{c\downarrow 0} f^{(j)}(c)$ which is assumed to exist. Thus $f^{(j+1)}(0) = \lim_{c\downarrow 0} f^{(j)}(c)$, showing that f is $j+1$ times continuously differentiable at 0. Then it follows by induction that f is C^k at 0. Similarly we can show f is C^k at 1. Then $f \in C^k([0,1])$, as desired.

(b) We first show that $\|f\| = \sum_0^k \|f^{(j)}\|_u$ is a norm. $\|f+g\| = \sum_0^k \|(f+g)^{(j)}\|_u = \sum_0^k \|f^{(j)} + g^{(j)}\|_u \leq \sum_0^k (\|f^{(j)}\|_u + \|g^{(j)}\|_u) = \sum_0^k \|f^{(j)}\|_u + \sum_0^k \|g^{(j)}\|_u = \|f\| + \|g\|$. Also $\|\lambda f\| = \sum_0^k \|(\lambda f)^{(j)}\|_u = \sum_0^k \|\lambda f^{(j)}\|_u = |\lambda| \sum_0^k \|f^{(j)}\|_u = |\lambda|\|f\|$. $\|f\| = 0$ implies $\|f\|_u \leq \sum_0^k \|f^{(j)}\|_u = 0$ and thus $f \equiv 0$ since f is continuous. We then show that this norm makes $C^k([0,1])$ into a Banach space. Pick a Cauchy sequence $\{f_n\}$ in $C^k([0,1])$ and let $\epsilon > 0$. Then there is $N > 0$ such that $m, n > N$ implies $\|f_n - f_m\| = \sum_{j=0}^k \|f_n^{(j)} - f_m^{(j)}\|_u < \epsilon$. In particular, $\|f_n - f_m\| < \epsilon$ for $n, m > N$ and thus $\{f_n\}$ is uniformly Cauchy. Since f_n is continuous and $C([0,1])$ is complete, $f_n \to f$ for some $f \in C([0,1])$. We now claim that $f \in C^k([0,1])$ and $f_n^{(j)} \to f^{(j)}$ uniformly for all $j \leq k$. We prove the claim by induction. For $k = 0$, $f \in C([0,1])$ and $f_n \to f$ uniformly. Suppose $f \in C^l([0,1])$ and

$f_n^{(j)} \to f^{(i)}$ uniformly for all $j \leq l$, we try to show the result for $l+1$. Fix the ϵ and m,n above, $\|f_n^{l+1} - f_m^{l+1}\| < \epsilon$ for all $m,n > N$. Thus $\{f_n^{l+1}\}$ is uniformly Cauchy. Since $f_n, f_m \in C^k([0,1])$, f_n^{l+1} and f_m^{l+1} are continuous and thus $f_n^{l+1} \to g$ for some $g \in C([0,1])$. Notice that $f_n^l(x) - f_n^l(0) = \int_0^x f_n^{l+1}(t)dt$. Since $f_n^{l+1} \to g$ uniformly and f_n^{l+1} is continuous, by undergraduate analysis, sending $n \to \infty$ we get $f^l(x) - f^l(0) = \int_0^x g(t)dt$. By fundamental theorem of calculus, $f^{l+1}(x) = g(x)$, showing the result, and the claim follows by induction. By the claim, $\|f_n - f\| = \sum_{j=0}^k \|f_n^{(j)} - f_m^{(j)}\|_u \to 0$ as $n \to \infty$ and $f \in C^k([0,1])$, showing that $C([0,1])$ is complete under this norm and is thus a Banach space, as desired.

□

Folland 5.15

Suppose that $\mathcal{X}$ and $\mathcal{Y}$ are normed vector spaces and $T \in L(\mathcal{X}, \mathcal{Y})$. Let $\mathcal{N}(T) = \{x \in \mathcal{X} : Tx = 0\}$.

(a) $\mathcal{N}(T)$ is a closed subspace of $\mathcal{X}$.

(b) There is a unique $S \in L(\mathcal{X}/\mathcal{N}(T), \mathcal{Y})$ such that $T = S \circ \pi$ where $\pi : \mathcal{X} \to \mathcal{X}/\mathcal{M}$ is the projection. Moreover, $||S|| = ||T||$.

Proof. (a) Since $\mathcal{T} \in L(\mathcal{X}, \mathcal{Y})$, T is continuous. Since $\{0\}$ is closed in $\mathcal{Y}$, $\mathcal{N}(T) = T^{-1}(\{0\})$ is closed in $\mathcal{X}$.

(b) We first show that such S exists. We define $S : \mathcal{X}/\mathcal{N}(T) \to \mathcal{Y}$ by $S(x + \mathcal{N}(T)) := T(x)$, and claim that $S \in L(\mathcal{X}/\mathcal{N}(T), \mathcal{Y})$. To show the claim, we first show that S is well-defined. If $x + \mathcal{N}(T) = x' + \mathcal{N}(T)$, $x' = x + y$ for some $y \in \mathcal{N}(T)$. Thus

$$S(x' + \mathcal{N}(T)) = S(x + y + \mathcal{N}(T)) = T(x+y) = Tx + Ty = Tx = S(x + \mathcal{N}(T))$$

showing that S is well-defined. We then show that S is linear. This is true since

$$S[(x + \mathcal{N}(T)) + (y + \mathcal{N}(T))] = S(x + y + \mathcal{N}(T))$$
$$= T(x+y) = T(x) + T(y) = S(x + \mathcal{N}(T)) + S(y + \mathcal{N}(T))$$

Now we show that S is bounded. We claim a stronger result that any constant that bounds T also bounds S. To show this claim, we suppose $||Tx||_{\mathcal{Y}} \leq C||x||_{\mathcal{X}}$ for all $x \in \mathcal{X}$. Then for any $y \in \mathcal{N}(T)$, $||S(x + \mathcal{N}(T))||_{\mathcal{Y}} = ||Tx||_{\mathcal{Y}} = ||Tx + Ty||_{\mathcal{Y}} = ||T(x+y)||_{\mathcal{Y}} \leq C||x+y||_{\mathcal{X}}$ and thus $||S(x + \mathcal{N}(T))||_{\mathcal{Y}} \leq C \cdot \inf\{||x+y|| : y \in \mathcal{N}(T)\} = C||x + \mathcal{N}(T)||$, proving the claim. Next we show that such S is unique. Suppose $S_1 \circ \pi = S_2 \circ \pi = T$, then for any $x + \mathcal{N}(T) \in \mathcal{X}/\mathcal{N}(T)$, there is some x such that $\pi(x) = x + \mathcal{N}(T)$. Then $S_1(x + \mathcal{N}(T)) = S_1 \circ \pi(x) = S_2 \circ \pi(x) = S_2(x + \mathcal{N}(T))$, showing that $S_1 \equiv S_2$ and that S is unique. Eventually we show that $||S|| = ||T||$. By our claim above, $||S|| = \inf\{C : ||S(x + \mathcal{N}(T))||_{\mathcal{Y}} \leq C||x + \mathcal{N}(T)||$ for all $x\} \leq \inf\{C : ||Tx||_{\mathcal{Y}} \leq C||x||_{\mathcal{X}}$ for all $x\} = ||T||$. Conversely we have $||T|| = ||S \circ \pi|| \leq ||S|| \cdot ||\pi|| = ||S||$ and thus $||T|| = ||S||$, finishing the proof.

□

Folland 5.20

If $\mathcal{M}$ is a finite-dimensional subspace of a normed vector space $\mathcal{X}$, there is a closed subspace $\mathcal{N}$ such that $\mathcal{M} \cap \mathcal{N} = \{0\}$ and $\mathcal{M} + \mathcal{N} = \mathcal{X}$.

Proof. First consider the case where $\mathcal{M}$ is one dimensional. Then we may write $\mathcal{M} = Kx_1$, where K is $\mathbb{C}$ or $\mathbb{R}$ and x_1 is a non-zero vector in $\mathcal{M}$. We may assume $\|x_1\| = 1$, otherwise we just do

some scaling to make this happen. Then we define $f : \mathcal{M} \to K$ by $f(\lambda x_1) = \lambda$. Since $\|f\| = \sup\{||f(x)|| : \|x\| = 1\} = \sup\{\|f(\lambda x_1)\| : \|\lambda x_1\| = 1\} = \sup\{|\lambda| : |\lambda|\|x_1\| = 1\} = \sup\{|\lambda| : |\lambda| = 1\} = 1$, f is a bounded linear functional. By Hahn-Banach theorem we extend f to an $F \in \mathcal{X}^*$, and claim that $F^{-1}(\{0\})$ is a desired subspace $\mathcal{N}$. To prove the claim, we first notice that F is bounded linear by assumption and is thus continuous. Since $\{0\}$ is closed, $\mathcal{N} = F^{-1}(\{0\})$ is also closed. Moreover, for any λx_1 in $\mathcal{M}$, $\lambda x_1 \in \mathcal{N}$ iff $F(\lambda x_1) = f(\lambda x_1) = \lambda = 0$ iff $\lambda x_1 = 0$, so $\mathcal{M} \cap \mathcal{N} = \{0\}$. Eventually, for $x \in \mathcal{X}$, we write $x = F(x)x_1 + (x - F(x)x_1)$. Clearly $F(x)x_1 \in \mathcal{M}$, and $F(x - F(x)x_1) = F(x) - F(x)F(x_1) = F(x) - F(x) = 0$, so $x - F(x)x_1 \in \mathcal{N}$. Then $x \in \mathcal{M} + \mathcal{N}$ and thus $\mathcal{M} + \mathcal{N} = \mathcal{X}$. Then our claim is true.
We now finish the proof by induction. Suppose the results holds for dimension $\leq n$, we show that it holds for dimension $n + 1$. Suppose we have $\mathcal{M}'$ with dimension $n + 1$, we may choose $\mathcal{M} \subset \mathcal{M}'$ an n-dimensional subspace of $\mathcal{M}'$. By induction hypothesis we can choose a closed $\mathcal{N} \subset X$ such that $\mathcal{M} \cap \mathcal{N} = \{0\}$ and $\mathcal{M} + \mathcal{N} = \mathcal{X}$. Let $y \in \mathcal{M}' \setminus \mathcal{M}$, we have $y = m + x$ for $m \in \mathcal{M}$ and $x \in \mathcal{N}$. Clearly $x \notin \mathcal{M}$, and thus $\dim(\mathcal{M} + Kx) = \dim(\mathcal{M}) + \dim(Kx) = n + 1 = \dim(\mathcal{M}')$. Since $\mathcal{M} + Kx \subset \mathcal{M}'$, $\mathcal{M} + Kx = \mathcal{M}'$. Since Kx is a one-dimensional subspace of $\mathcal{N}'$, we use the same technique as above to choose some closed $\mathcal{N}' \subset \mathcal{N}$ such that $Kx \cap \mathcal{N}' = \{0\}$ and $Kx + \mathcal{N}' = \mathcal{N}$. Since $\mathcal{N}'$ is closed in $\mathcal{N}$ and $\mathcal{N}$ is closed, $\mathcal{N}'$ is closed in $\mathcal{X}$. Now we claim that $\mathcal{N}'$ is a closed subspace such that $\mathcal{M}' \cap \mathcal{N}' = \{0\}$ and $\mathcal{M}' + \mathcal{N}' = \mathcal{X}$. $\mathcal{M}' \cap \mathcal{N}' = (\mathcal{M} + Kx) \cap \mathcal{N}'$. If $m + kx \in \mathcal{N}' \subset \mathcal{N}$ for $m \in \mathcal{M}$, since $kx \in \mathcal{N}$, $m \in \mathcal{N}$ and thus $m = 0$. Then $kx \in \mathcal{N}'$ and $kx = 0$. Thus $m + kx = 0$ and $\mathcal{M}' \cap \mathcal{N}' = (\mathcal{M} + Kx) \cap \mathcal{N}' = \{0\}$. Moreover, $\mathcal{M}' + \mathcal{N}' = \mathcal{M} + Kx + \mathcal{N}' = \mathcal{M} + (Kx + \mathcal{N}') = \mathcal{M} + \mathcal{N} = \mathcal{X}$. The result follows by induction. $\square$

Folland 5.21

If $\mathcal{X}$ and $\mathcal{Y}$ are normed vector spaces, define $\alpha : \mathcal{X}^* \times \mathcal{Y}^* \to (\mathcal{X} \times \mathcal{Y})^*$ by $\alpha(f, g)(x, y) = f(x) + g(y)$. Then α is an isomorphism which is isometric if we use the norm $||(x, y)|| = \max(||x||, ||y||)$ on $\mathcal{X} \times \mathcal{Y}$, the corresponding operator norm on $(\mathcal{X} \times \mathcal{Y})^*$, and the norm $||(f, g)|| = ||f|| + ||g||$ on $\mathcal{X}^* \times \mathcal{Y}^*$.

Proof. We first show that α is an isomorphism. Let $h \in (\mathcal{X} \times \mathcal{Y})^*$, then we claim that $\beta : h \mapsto (f, g)$, where f and g are in $\mathcal{X}^* \times \mathcal{Y}^*$ such that $f(x) = h(x, 0)$ and $g(y) = h(0, y)$, is the inverse of α. First of all, $\beta \circ \alpha(f, g) = \beta(h)$, where $h(x, y) = f(x) + g(y)$. Suppose $\beta \circ \alpha(f, g) = (\beta \circ \alpha(f, g)_1, \beta \circ \alpha(f, g)_2)$, then $(\beta \circ \alpha(f, g)_1(x), \beta \circ \alpha(f, g)_2(y)) = (\beta(h)_1(x), \beta(h)_2(y)) = (h(x, 0), h(0, y)) = (f(x), g(y))$ for any $x \in \mathcal{X}$ and $y \in \mathcal{Y}$. Thus $\beta \circ \alpha(f, g) = (f, g)$. On the other hand, for $h \in (\mathcal{X} \times \mathcal{Y})^*$, $\alpha \circ \beta(h)(x, y) = \alpha(f, g)(x, y) = f(x) + g(y) = h(x, 0) + h(0, y) = h(x, y)$ where f and g are defined at the very beginning. Then $\alpha \circ \beta(h) = h$. Thus $\beta = \alpha^{-1}$ is the two-sided inverse. The inverse of a linear map is linear, so it remains to verify that β is bounded. Observe that $\sup\{||\beta(h)|| : ||h|| = 1\} = \sup\{||(f, g)|| : ||h|| = 1\} = \sup\{||f|| + ||g|| : ||h|| = 1\} < \infty$ since $||f||$ and $||g||$ are bounded. Then α is an isomorphism.

We then show that α is an isometry. First of all,

$$\begin{aligned}
\|\alpha(f,g)\| &= \sup\{\|\alpha(f,g)(x,y)\| : \|(x,y)\| = 1\} = \sup\{\|\alpha(f,g)(x,y)\| : \|(x,y)\| = 1\} \\
&= \sup\{\|f(x)+g(y)\| : \|(x,y)\| = 1\} \geq \sup\{\|f(\text{sgn}f \cdot x) + g(\text{sgn}g \cdot y)\| : \|(x,y)\| = 1\} \\
&= \sup\{\|(\text{sgn}f)f(x) + (\text{sgn}g)g(y)\| : \max(\|x\|, \|y\|) = 1\} \\
&= \sup\{\|(\text{sgn}f)f(x) + (\text{sgn}g)g(y)\| : \max(\|x\|, \|y\|) = 1\} \\
&= \sup\{(\text{sgn}f)f(x) + (\text{sgn}g)g(y) : \max(\|x\|, \|y\|) = 1\} \\
&= \sup\{\|f(x)\| + \|g(y)\| : \max(\|x\|, \|y\|) = 1\} \geq \sup\{\|f(x)\| + \|g(y)\| : \|x\| = 1, \|y\| = 1\} \\
&= \sup\{\|f(x)\| : \|x\| = 1\} + \sup\{\|g(y)\| : \|y\| = 1\} = \|f\| + \|g\| = \|(f,g)\|
\end{aligned}$$

Conversely, notice that if $||x|| \leq 1$, then there is some $|\lambda| \geq 1$ such that $||\lambda x|| = 1$ and thus $||f(x)|| \leq |\lambda| \cdot ||f(x)|| = ||\lambda f(x)|| = ||f(\lambda x)||$. Therefore, for any $||x|| \leq 1$, we can find a corresponding x' such that $||x'|| = 1$ and $||f(x)|| \leq ||f(x')||$. This means that $\sup\{||f(x)|| + ||g(y)|| : \max(||x||, ||y||) = 1\} \leq \sup\{||f(x)|| + ||g(y)|| : ||x|| = 1, ||y|| = 1\} = \sup\{||f(x)|| : ||x|| = 1\} + \sup\{||g(y)|| : ||y|| = 1\} = ||f|| + ||g|| = ||(f,g)||$. And $\sup\{||f(x)|| + ||g(y)|| : \max(||x||, ||y||) = 1\} = \sup\{||\alpha(f,g)(x,y)|| : ||(x,y)|| = 1\} = ||\alpha(f,g)||$. This shows that $||\alpha(f,g)|| = ||(f,g)||$ and thus α is an isometry. □

A Better Proof Idea. We first show that α is isometric and then show that it is bijective. If α is isometric, α^{-1} is also isometric and automatically bounded and thus α is an isomorphism. This avoids constructing an explicit inverse. □

Folland 5.22

Suppose $\mathcal{X}$ and $\mathcal{Y}$ are normed vector spaces and $T \in L(\mathcal{X}, \mathcal{Y})$.

(a) Define $T^+ : \mathcal{Y}^* \to \mathcal{X}^*$ by $T^+ f = f \circ T$. Then $T^+ \in L(\mathcal{Y}^*, \mathcal{X}^*)$ and $\|T^+\| = \|T\|$. T^+ is called the adjoint or transpose of T.

(b) Applying the construction in (a) twice, one obtains $T^{++} \in L(\mathcal{X}^{**}, \mathcal{Y}^{**})$. If $\mathcal{X}$ and $\mathcal{Y}$ are identified with their natural images $\hat{cX}$ and $\hat{\mathcal{Y}}$ in $\mathcal{X}^{**}$ and $\mathcal{Y}^{**}$, then $T^{++}|\mathcal{X} = T$.

(c) T^+ is injective iff the range of T is dense in $\mathcal{Y}$.

(d) If the range of T^+ is dense in $\mathcal{X}^*$, then T is injective; the converse is true if $\mathcal{X}$ is reflexive.

Proof. (a) $\|T^+ f\| \leq \|T\|\|f\|$ and thus $\|T^+\| = \sup\{\|T^+ f\| : \|f\| = 1\} \leq \|T\| < \infty$, meaning that T^+ is bounded. It remains to show that $\|T^+\| \geq \|T\|$. Let $\epsilon > 0$, choose $x \in \mathcal{X}$ such that $\|x\| = 1$ and $\|Tx\| > \|T\| - \epsilon$. If $\|Tx\| = 0$, then automatically $\|T^+\| \geq \|T\|$. Otherwise we can choose $f \in \mathcal{Y}^*$ such that $\|f\| = 1$ and $f(Tx) = \|Tx\|$. Then

$$\|T^+\| \geq \|T^+ f\| \geq \|T^+ f x\| = \|f(Tx)\| = \|Tx\| \geq \|T\| - \epsilon$$

and $\|T^+\| \geq \|T\|$ since ϵ is arbitrary.

(b) $T^{++} f = f \circ T^+$. Given $\hat{x} \in \hat{\mathcal{X}}$, $(T^{++}\hat{x})(f) = \hat{x} \circ T^+(f) = \hat{x}(T^+ f) = T^+ f(x) = f(Tx)$. Then $T^{++}\hat{x} = (\hat{Tx})$ and thus $T^{++}x = Tx$ if we identify Tx with its canonical image $(\hat{Tx})$.

(c) Suppose $T(\mathcal{X})$ is dense in $\mathcal{Y}$ and $T^+(f) = T^+(g)$. Then $f \circ T = g \circ T$. Let $y \in \mathcal{Y}$, by denseness choose a sequence $\{y_n\}_n \in T(\mathcal{X})$ that converges to y. Since $y_n \in T(\mathcal{X})$, $f(y_n) = g(y_n)$ for all n. Notice that f and g are both bounded linear and thus continuous. Sending $n \to \infty$, we get $f(y) = g(y)$. Thus $f = g$ and we show that T^+ is injective. Conversely, suppose T^+ is injective. If $T(\mathcal{X})$ is not dense in $\mathcal{Y}$, pick $y \notin \overline{T(\mathcal{X})}$. Then there is some $f \in \mathcal{Y}^*$ such that $f(y) := \inf_{z \in \overline{T(\mathcal{X})}} \|y - z\|$. Clearly $f \not\equiv 0$ since $f(y) \neq 0$, and

$f|\overline{T(\mathcal{X})} \equiv 0$. However,

$$T^+ f = f \circ T \equiv 0 = 0 \circ T = T^+ 0$$

contradicting injectivity. This shows the conclusion.

(d) Suppose $T^+(\mathcal{Y}^*)$ is dense in $\mathcal{X}^*$. By (c), T^{++} is injective and since $T^{++}|\mathcal{X} = T$, T is injective. Conversely, suppose T is injective, since $\hat{\mathcal{X}} = \mathcal{X}^{**}$, $T^{++} = T$, and thus T^{++} is injective. By (c) again the range of T^+ is dense in $\mathcal{X}^*$.

□

Remark. This problem is quite standard. You just follow your inituition and everything is clear. Recognizing the relationship between (c) and (d) will save a lot of work. Nevertheless, the result is important being an analog of the one in finite-dimensional linear algebra.

Folland 5.25

If $\mathcal{X}$ is a Banach space and $\mathcal{X}^*$ is separable, then $\mathcal{X}$ is separable.

Proof. Let $\{f_n\}_1^\infty$ be a countable dense subset of $\mathcal{X}^*$. For each n choose $x_n \in \mathcal{X}$ with $||x_n|| = 1$ and $|f_n(x_n)| \geq \frac{1}{2}||f_n||$. We claim that the linear combinations of $\{x_n\}_1^\infty$ are dense in $\mathcal{X}$. To prove the claim, we first define $\mathcal{M}$ to be the closure of linear combinations of $\{x_n\}_1^\infty$. Then $\mathcal{M}$ is a closed subsapce of $\mathcal{X}$. Suppose there is $x \in X \setminus \mathcal{M}$, then there is some $f \in \mathcal{X}^*$ such that $f(x) := \inf_{y \in \mathcal{M}} ||x-y||$. Let $\epsilon > 0$. By denseness of $\{f_n\}_1^\infty$ we have some f_n such that $||f - f_n|| < \epsilon$. In particular $|f_n(x_n) - f(x_n)| < \epsilon$ for the corresponding x_n and thus $|f_n(x_n)| < \epsilon$ since $x_n \in \mathcal{M}$ and thus $f(x_n) = 0$. Since $|f_n(x_n)| \geq \frac{1}{2}||f_n||$, $||f_n|| < 2\epsilon$ and $||f|| < ||f_n|| + \epsilon < 3\epsilon$. Since ϵ is arbitrary, $||f|| = 0$ and thus $f \equiv 0$. But this means that $0 = f(x) = \inf_{y \in \mathcal{M}} ||x - y||$ and thus $x \in \overline{\mathcal{M}} = \mathcal{M}$, a contradiction. Thus $\mathcal{M} = X$ and linear combinations of $\{x_n\}_1^\infty$ is dense in $\mathcal{X}$. We know that linear combinations of $\{x_n\}_1^\infty$ with coefficients whose real and imaginary parts are both rational, which we call $\mathcal{N}$, is dense in linear combinations of $\{x_n\}_1^\infty$ and is thus dense in $\mathcal{X}$. $\mathcal{N}$ is countable if we identify it as a countable union (union over n) of countable sets (the set of coefficients). Then $\mathcal{N}$ is a countable dense subset of $\mathcal{X}$ and thus $\mathcal{X}$ is separable. □

Remark. The key to solving this problem is finding the correct definition of denseness to be used. I started off tring to use the neighborhood definition of denseness, but I didn't find a way to use "linear combination" as suggested by the hint of the book. I then realized that linear combination endows a space structure, so I should consider the whole space spanned by $\{x_n\}_1^\infty$. The solution naturally follows.

Folland 5.27

There exists meager subsets of $\mathbb{R}$ whose complements have Lebesgue measure zero.

Proof. For our convenience, we define $I_m = [m, m+1]$ for all $m \in \mathbb{Z}$. We first show that there is a meager subset of I_m for all m whose complement in I_m has Lebesgue measure zero. To show this, we first claim that for every $n > 1$, there is a generalized Cantor set K_n on I_m such that $m(I_m \setminus K_n) = 1/n$. To show the claim we define K_n this way: $K_{n,0} = I_m$, and suppose we have defined $K_{n,j}$, define $K_{n,j+1}$ by removing the middle α_jth content from every interval that makes up $K_{n,j}$, where $\alpha_j = 1/(n+j-1)^2$ for $j \geq 1$. Thus

$$\begin{aligned} m(K_n) &= \prod_{i=1}^{\infty}(1-\alpha_j) = \left(1 - \frac{1}{n^2}\right)\left(1 - \frac{1}{(n+1)^2}\right)\cdots \\ &= \frac{1-1/n}{1-1/(n+1)} \cdot \frac{1-1/(n+1)}{1-1/(n+2)} \cdots \\ &= \lim_{k\to\infty} \frac{1-1/n}{1-1/(n+k)} = 1 - \frac{1}{n} \end{aligned}$$

and thus $m(I_m \setminus K_n) = m(I_m) - m(K_n) = 1 - (1 - 1/n) = 1/n$, showing that the claim is true. We know that K_n is nowhere dense, so $K := \bigcup_{n=1}^{\infty} K_n$ is meager. Meanwhile $1 \geq m(K) \geq 1 - 1/n$ for all n and thus $m(K) = 1$. Thus $m(I_m \setminus K) = 0$. Therefore, on every I_m we have $M_m \subset I_m$ meager such that $m(I_m \setminus M_m) = 0$. Let $M = \bigcup_{m\in\mathbb{Z}} M_m$. Then M is meager and

$$m(\mathbb{R} \setminus M) = m\left[\bigcup_{m\in\mathbb{Z}} (I_m \setminus M_m)\right] \leq \bigcup_{m\in\mathbb{Z}} m(I_m \setminus M_m) = 0$$

and thus $m(\mathbb{R} \setminus M) = 0$, as desired. □

Another Proof. Equivalently we prove that there is a residual set of measure 0. Let $\{q_n\}_n$ be an enumeration of $\mathbb{Q}$. Let $\epsilon > 0$, define $B_{n,\epsilon} := B(\epsilon 2^{-n}, q_n)$. It is clear that $B_\epsilon := \bigcup_n B_{n,\epsilon}$ is an open dense subset of $\mathbb{R}$. Then $(B_\epsilon)^c$ is nowhere dense and thus B_ϵ is residual. Define $B := \bigcap_n B_{1/n}$, then B is residual as a countable intersection of residual sets. Moreover $m(B) \leq m(B_{1/n}) = 2/n$ for every n and thus $m(B) = 0$, as desired. □

Folland 5.29

Let $\mathcal{Y} \in L^1(\mu)$, where μ is counting measure on $\mathbb{N}$, and let $\mathcal{X} = \{f \in \mathcal{Y} : \sum_1^\infty n|f(n)| < \infty\}$, equipped with the L^1 norm.

(a) $\mathcal{X}$ is proper dense subspace of $\mathcal{Y}$, hence $\mathcal{X}$ is not complete.
(b) Define $T : \mathcal{X} \to \mathcal{Y}$ by $Tf(n) = nf(n)$. Then T is closed but not bounded.
(c) Let $S = T^{-1}$. Then $S : \mathcal{Y} \to \mathcal{X}$ is bounded and surjective but not open.

Proof. (a) Let $\epsilon > 0$, $g \in \mathcal{Y}$. We want to find some $f \in \mathcal{X}$ such that $||g - f||_1 < \epsilon$. Recall that simple functions on $\mathbb{N}$ are dense in $L^1(\mu)$, so we can pick some simple function $f := \sum_{i=1}^n c_k \chi_{E_k}$, where E_k is a measurable subset of $\mathbb{N}$ and $c_k < \infty$, such that $||f - g||_1 < \epsilon$. We claim that each E_k is finite. Suppose not, there is some E_k that has infinite cardinality. Then $\int |f| d\mu \geq |c_k|\mu(E_k) = \infty$, contradicting $f \in L^1(\mu)$ and showing the claim. Thus $f(n) \neq 0$ for only finitely many $n \in \mathbb{N}$, and clearly $\sum_1^\infty n|f(n)| < \infty$. Then $f \in \mathcal{X}$ and $\|g - f\|_1 < \epsilon$, as desired. This shows that $\mathcal{X}$ is a dense subspace of $\mathcal{Y}$. Now consider f on $\mathbb{N}$ such that $f(n) = 1/n^2$, we know that $\int |f| d\mu = \sum_1^\infty 1/n^2 < \infty$ and thus $f \in \mathcal{Y}$. However, $\sum_1^\infty n|f(n)| = \sum_1^\infty 1/n = \infty$, so $f \notin \mathcal{X}$. Thus $\mathcal{X}$ is a proper dense subspace of $\mathcal{Y}$. Then $\overline{\mathcal{X}} = \mathcal{Y} \neq \mathcal{X}$, so $\mathcal{X}$ is not closed. Since a complete subspace of a metric space must be closed, $\mathcal{X}$ is not complete.

(b) We first show that T is closed, i.e. $\Gamma(T)$ is closed in $\mathcal{X} \times \mathcal{Y}$, i.e. $\overline{\Gamma(T)} = \Gamma(T)$. Let (f, g) be a limit point in $\Gamma(T)$, we have $\{(f_n, g_n)\} \in \gamma(T)$ such that $(f_n, g_n) \to (f, g)$ in the product norm. We want to show $g = Tf$. Let $\epsilon > 0$, by convergence there is some large N such that when $n > N$, $\|(f_n, g_n) - (f, g)\| < \epsilon$. This means that $\max(\|f_n - f\|_1, \|g_n - g\|_1) < \epsilon$

for large enough n. Then

$$\begin{aligned}\|g - Tf\|_1 &\leq \|g - g_n\|_1 + \|g_n - Tf_n\|_1 + \|Tf_n - Tf\|_1 \\ &\leq \epsilon + 0 + \int m|f_n(m) - f(m)|d\mu\end{aligned}$$

We define $h_n(m) := m(|f_n(m)|+|f(m)|)$, and $\int |h_n|d\mu = \int m|f_n(m)|d\mu + \int m|f(m)|d\mu < \infty$ since $f_n, f \in \mathcal{X}$. Thus $h_n \in L^1$ and $m|f_n(m) - f(m)| \leq h_n(m)$. By dominated convergence, sending $n \to \infty$ we get $\|g - Tf\| < \epsilon$. Since ϵ is arbitrary, $\|g - Tf\| = 0$. Then $\int |g - Tf|d\mu = \sum_1^\infty |g(n) - Tf(n)| = 0$ and thus $g(n) = Tf(n)$ for all n, from which we conclude $g = Tf$, as desired. Then $(f, g) \in \Gamma(T)$, showing that it is closed.

We then show that it is not bounded. Notice that $f_n := \chi_{\{n\}}$ satisfied $\|f_n\|_1 = |f(n)| = 1$ for all n. Then $\sup\{\|Tf\| : \|f\|_1 = 1\} \geq \sup_n \|Tf_n\| = \sup_n nf(n) \to \infty$ and thus T is unbounded.

(c) To make S well-defined, we need to show that T is bijective. For $g \in \mathcal{Y}$, define $f(n) := g(n)/n$ for all n. (Here we assume $0 \notin \mathbb{N}$) Then $\sum_1^\infty n|f(n)| = \sum_1^\infty |g(n)| = \int |g|d\mu < \infty$ and thus $f \in \mathcal{X}$. Also the most importantly $Tf(n) = nf(n) = g(n)$ and thus $g = Tf$, showing that T is surjective. Suppose $f_1 \neq f_2$, $f_1(n) \neq f_2(n)$ for some n. Then $Tf_1(n) = nf_1(n) \neq nf_2(n) = Tf_2(n)$ and thus $Tf_1 \neq Tf_2$, showing that T is injective. Then T is bijective as desired and S is well-defined. We now claim that S is defined such that $Sg(n) = g(n)/n$. To show the claim, observe that $TSg(n) = ng(n)/n = g(n)$ and thus TS is the identity. Similarly we can show that ST is also the identity. Then the claim is true. We need to show that S is bounded. This is true since

$$\begin{aligned}&\sup\{\|Sg\| : \|g\| = 1\} = \sup\left\{\int \left|\frac{g(n)}{n}\right| d\mu : \int |g|d\mu = 1\right\} \\ &\leq \sup\left\{\int |g|d\mu : \int |g|d\mu = 1\right\} = 1\end{aligned}$$

Also S is surjective since T is bijective. Eventually, if $S = T^{-1}$ is open, T is continuous and thus bounded, a contradiction, so S is not open, as desired. This finishes the proof. □

Folland 5.37

Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces. If $T : \mathcal{X} \to \mathcal{Y}$ is a linear map such that $f \circ T \in \mathcal{X}^*$ for every $f \in \mathcal{Y}^*$, then T is bounded.

Proof. Let $f \in \mathcal{Y}^*$. $f \circ T$ is bounded and thus continuous. Then $f \circ T$ is closed and thus $\Gamma(f \circ T) = \{(x, f \circ T(x)) : x \in \mathcal{X}\}$ is closed. We define $h_f(x, y) := (x, f(y))$ and it is continuous since continuity is component-wise. (In particular f is bounded and thus continuous) Then $h_f^{-1}(\Gamma(f \circ T)) = \{(x, y) : (x, f(y)) = (x, f \circ T(x))\}$ is closed. We claim that $\bigcap_{f \in \mathcal{Y}^*} h_f^{-1}(\Gamma(f \circ T)) = \Gamma(T)$. To show this claim, we first observe that $\Gamma(T) \subset h_f^{-1}(\Gamma(f \circ T))$ for every f and thus $\Gamma(T) \subset \bigcap_{f \in \mathcal{Y}^*} h_f^{-1}(\Gamma(f \circ T))$. Conversely, suppose $(x, y) \in \bigcap_{f \in \mathcal{Y}^*} h_f^{-1}(\Gamma(f \circ T))$, then $f(y) = f \circ T(x)$ for all $f \in \mathcal{Y}^*$. If $y \neq x$, since bounded linear functionals on $\mathcal{Y}$ separate points, there is some g such that $g(y) \neq g \circ T(x)$, a contradiction. Then $(x, y) \subset \Gamma(T)$, showing our claim. $\bigcap_{f \in \mathcal{Y}^*} h_f^{-1}(\Gamma(f \circ T))$ is closed since each $h_f^{-1}(\Gamma(f \circ T))$ is closed, so $\Gamma(T)$ is closed by the claim. Then T is closed and thus bounded by closed graph theorem. □

Remark. The condition $\mathcal{X}$ and $\mathcal{Y}$ are Banach spaces hints using the closed graph theorem. Therefore the goal is reduced to showing that $\Gamma(T)$ is closed. The key step here is expressing $\Gamma(T)$ as a (potentially) huge intersection of closed sets. A immature observation: sometimes there might not be a single object that satisfies the desired property, but considering a (huge) arbitrary union (mostly for open sets) or intersection (mostly for open sets) may work. In many other situations, if a union is still not clear enough, further expressing a union as a huge cartesian product and apply nice theorems like Tychonoff gives desired results.

Folland 5.38

Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces, and let $\{T_n\}$ be a sequence in $L(\mathcal{X}, \mathcal{Y})$ such that $\lim T_n x$ exists for every $x \in X$. Let $Tx = \lim T_n x$; then $T \in L(\mathcal{X}, \mathcal{Y})$.

Proof. Since addition and multiplication respect limits, T is linear. $\{T_n x\}$ converges and in particular bounded for each x. Since $\mathcal{X}$ is a Banach space, by uniform boundedness $\sup_m \|T_m\| < M$ for some $M > 0$. Then

$$\|Tx\| = \lim_{n\to\infty} \|T_m x\| \le \sup_m \|T_m\|\|x\| \le M\|x\|$$

and thus T is bounded, as desired. $\square$

Remark. This is a direct application of uniform boundedness principle. Uniform boundedness principle is proved cleverly, but its applications seem to be straightforward at most times.

Folland 5.42

Let E_n be the set of all $f \in C([0,1])$ for which there exists $x_0 \in [0,1]$ such that $|f(x) - f(x_0)| \le n|x - x_0|$ for all $x \in [0,1]$.

(a) E_n is nowhere dense in $[0,1]$.

(b) The set of nowhere differentiable functions is residual in $C([0,1])$.

Proof. (a) We claim that given E_n, every $f \in E_n$ can be uniformly approximated by a piecewise linear function, whose linear pieces, finite in number, have slope $\ge 2n$ or $\le -2n$. To show the claim, we first observe that f is continouous on $[0,1]$ compact and therefore uniformly continuous on $[0,1]$. Let $\epsilon > 0$, we know that there is some $\delta > 0$ such that when $|x - y| < \delta$, $|f(x) - f(y)| < \frac{\epsilon}{2}$. We define $\delta' := \min(\frac{\epsilon}{4n}, \delta)$. Then consider a partition $0 = x_0 \le x_1 \le \cdots \le x_n = 1$ such that $x_j - x_{j-1} < \delta'$ for all j. We construct a piecewise linear function g by connecting $f(x_{j-1})$ and $f(x_j)$ for every j, and we may assume that every such linear segment has slope whose absolute value $\ge 2n$, since if any linear piece has slope whose absolute value $< 2n$, we can replace it with a bell-shaped graph, i.e. a wedge such that the ascending piece has slope $2n$ and descending piece has slope $-2n$. The refined partition still satisfies all the assumptions mentioned above. Now notice that for the g we just constructed, in every $[x_{j-1}, x_j]$, for any $x, y \in [x_{j-1}, x_j]$, $|g(x) - g(y)| \le \min(\frac{\epsilon}{2}, 2n|x - y|) \le \min(\frac{\epsilon}{2}, 2n\delta') = \frac{\epsilon}{2}$. It remains to show that $\sup_{x\in[0,1]} |f(x) - g(x)| < \epsilon$. For any $x \in [0,1]$, we can find a j such that $x \in [x_{j-1}, x_j]$, then

$$|g(x) - f(x)| \le |g(x) - g(x_j)| + |g(x_j) - f(x_j)| + |f(x) - f(x_j)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

showing our claim. Notice that $|g(x) - g(x_0)| \ge 2n|x - x_0| > n|x - x_0|$ (since $n \ge 1$) for some $x \ne x_0$ lying in the same line segment as x_0 and thus $g \notin E_n$ Therefore, by our claim, for any $\epsilon > 0$ and $f \in E_n$, we can find $g \notin E_n$ such that $\sup_{x\in[0,1]} |f(x) - g(x)| < \epsilon$ and thus E_n is nowhere dense in $[0,1]$, as desired.

(b) We denote the set of nowhere differentiable functions using $\mathcal{C}$, and want to show that $\mathcal{C}^c$ is meager. Suppose $f \in \mathcal{C}^c$, then f is differentiable at some $x_0 \in [0,1]$. We define $\phi(x) := \dfrac{f(x)-f(x_0)}{x-x_0}$ where $\phi(x_0) := f'(x_0)$. and claim that it is continuous on $[0,1]$. It is clear that ϕ is continuous at $x \in [0,1] \setminus \{x_0\}$, so it suffices to show continuity at x_0. This is true since

$$\lim_{x\to x_0} \phi(x) = \lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0} = f'(x_0) = \phi(x_0)$$

Since ϕ is continuous, it is bounded on $[0,1]$ by extreme value theorem. Then $f \in E_m$ for some m. Then $\mathcal{C}^c \subset \bigcup_n E_n$ and $\mathcal{C}^c = \bigcup_n (E_n \cap \mathcal{C}^c)$. Since each E_n is nowhere dense, $\overline{E}_n^\circ = \emptyset$ and thus $\overline{E_n \cap \mathcal{C}^c}^\circ \subset \overline{E}_n^\circ = \emptyset$. Then $E_n \cap \mathcal{C}^c$ is nowhere dense and $\mathcal{C}^c$ is meager. Thus $\mathcal{C}$ is residual, finishing the proof.

□

Folland 5.45

The space $C^\infty(\mathbb{R})$ of all infinitely differentiable functions on $\mathbb{R}$ has a Frechet space topology with respect to which $f_n \to f$ iff $f_n^{(k)} \to f^{(k)}$ uniformly on compact sets for all $k \geq 0$.

Proof. We define $p_{k,l}(f) = \sup_{|x|\leq k} |f^{(l)}(x)|$ and verify that it is a semi-norm. This is true because $p_{k,l}(f+g) = \sup_{|x|\leq k} |(f+g)^{(l)}(x)| = \sup_{|x|\leq k} |f^{(l)}(x) + g^{(l)}(x)| \leq \sup_{|x|\leq k} |f^{(l)}(x)| + \sup_{|x|\leq k} |g^{(l)}(x)| = p_{k,l}(f) + p_{k,l}(g)$, and $p_{k,l}(rf) = \sup_{|x|\leq k} |(rf)^{(l)}(x)| = \sup_{|x|\leq k} |rf^{(l)}(x)| = |r| \sup_{|x|\leq k} |f^{(l)}(x)| = |r| p_{k,l}(f)$. Also $\{p_{k,l}\}_{k,l\in\mathbb{N}}$ is clearly countable since the index set $\mathbb{N}\times\mathbb{N}$ is countable. By theorem 5.14(b), $f_n \to f$ in the topology generated by these seminorms iff $p_{k,l}(f_n - f) = \sup_{|x|\leq k} |f_n^{(l)}(x) - f^{(l)}(x)| \to 0$ iff $f_n^{(l)} \to f^{(l)}$ uniformly on all compact subsets since any compact subset of $\mathbb{R}$ is closed and bounded and is eventually contained in some larger enough $[-k,k]$. Eventually we verify that $\{p_{k,l}\}_{k,l\in\mathbb{N}}$ makes makes C^∞ a Frechet space, i.e. a complete Hausdorff topological vector space. Let $f \neq 0$, there is some $x_0 \in R$ such that $f(x_0) \neq 0$. Then consider a large enough k such that $x_0 \in [-k,k]$ and thus $p_{k,0}(f) = \sup_{|x|\leq k} |f(x)| \geq f(x_0) > 0$. Thus by 5.16 (a) C^∞ is Hausdorff. It remains to show that it is complete. Suppose we have a Cauchy sequence $\langle f_n \rangle_n$ in $C^\infty(\mathbb{R})$, then $p_{k,l}(f_n - f_m) = \sup_{|x|\leq k} |f_n^l(x) - f_m^{(l)}(x)| \to 0$ as $m,n\to\infty$. Then $\langle f_n^{(l)}|_{[-k,k]} \rangle$ is uniformly Cauchy for all l and k. Since each $f_n^{(l)}|_{[-k,k]}$ is continuous and $C[k,k]$ is complete, $f_n^{(l)}|_{[-k,k]}$ uniformly converges to some $g_{k,l} \in C[-k,k]$. Now we define g_l on $\mathbb{R}$ such that $g_l(x) = g_{k,l}(x)$ where $x \in [-k,k]$. This function is well-defined since $g_{k,l}(x) = \lim_{n\to\infty} f_n^{(l)}(x)$ for all k which means that the definition of $g_{k,l}(x)$ is independent of k. Eventually we observe that $f_n^{(l)} \to g_l$ locally uniformly by our definition of g_l, and we claim that $g_l = g_0^{(l)}$. To show this claim, we shall do an induction on l as in problem 5.9. The base case is trivial since $g_0 = g_0$. Suppose we have $g_l = g_0^{(l)}$,

$$g_l(x) = \lim_{n\to\infty} f_n^{(l)}(x) = \lim_{n\to\infty} \int_0^x f_n^{(l+1)}(t)dt$$

Notice that $f_n^{(l+1)}$ is continuous and therefore bounded on $[0,x]$, so by dominated convergence theorem,

$$g_l(x) = \lim_{n\to\infty} \int_0^x f_n^{(l+1)}(t)dt = \int_0^x \lim_{n\to\infty} f_n^{(l+1)}(t)dt = \int_0^x g_{l+1}(t)dt$$

and by fundamental theorem of calculus we get $g_l'(x) = g_0^{l+1}(x) = g_{l+1}$, and the claim follows by induction. Thus $f_n^{(k)} \to g_0^{(k)}$ locally uniformly for every k and thus $f_n \to g_0$ by the previous part of the problem. $g_0^{(l)} = g_l$ is continuous for all l, so $g_0 \in C^\infty(\mathbb{R})$ and thus the space is complete. This finishes the proof. □

Folland 5.48

Suppose that $\mathcal{X}$ is a Banach space.

(a) The norm-closed unit ball $B = \{x \in \mathcal{X} : \|x\| \leq 1\}$ is also weakly closed.
(b) If $E \subset \mathcal{X}$ is bounded with respect to the norm, so is its weak closure.
(c) If $F \subset \mathcal{X}^*$ is bounded with respect to the norm, so is its weak* closure.
(d) Every weak*-Cauchy sequence in $\mathcal{X}^*$ converges.

Proof. (a) We show the result by showing that the complement of B is weakly open. Suppose $y \in B^c$. Since B is norm closed, by Hahn-Banach there is a linear functional f such that $f(y) = \delta$ where $\delta := \inf_{x \in B} \|x - y\|$. Then $f^{-1}[B(y, \delta)]$ is a weakly open ball excluding B. Hence B^c is weakly open and B is weakly closed.

(b) Since E is bounded with respect to norm, $E \subset B = \{x \in \mathcal{X} : \|x\| \leq M\}$ for some $M \geq 0$. By appropriate scaling if necessary, we may assume $M = 1$. Then by a) B is weakly closed and hence the weak closure of E is contained in B. It follows that the weak closure of E is bounded in norm as well.

(c) Suppose $F \subset \mathcal{X}^*$I s bounded in norm. Similar to above, without loss of generality we may assume $F \subset B$, where B is the norm closed unit ball as defined in a). By Alaoglu, B is compact in the weak* topology on $\mathcal{X}$ and hence closed. Then the weak* closure of F is also contained in B and it follows that it is weak* bounded.

(d) Let $\langle f_n \rangle_n$ be a weak* Cauchy sequence in $\mathcal{X}^*$, then $f_n - f_m \to 0$ as $n, m \to \infty$ and thus $(f_n - f_m)(x) \to 0$ as $n, m \to \infty$ and eventually $\|f_n(x) - f(x)\| \to 0$ as $n, m \to \infty$. Thus $\langle f_n(x) \rangle_n$ is Cauchy for each x and thus converges since $K = \{\mathbb{R}, \mathbb{C}\}$ is complete. We set $f(x) := \lim_n f_n(x)$, then $f \in \mathcal{X}*$ and $f_n(x) \to f(x)$ for all $x \in \mathcal{X}$, showing that $f_n \to f$ in weak* topology. This finishes the proof.

□

Folland 5.51

A vector subspace of a normed vector space $\mathcal{X}$ is norm-closed iff it is weakly closed.

Proof. Suppose $V \subset \mathcal{X}$ is norm closed. We show that $\mathcal{X}$ is weakly closed by showing that the complement of V is open. Suppose $x \in V^c$, then since V is norm closed, it follows by Hahn-Banach that there is a linear functional f such that $f(x) = \delta$ where $\delta = \inf_{y \in V} \|x - y\|$. Then $f^{-1}[B(\delta, f(x))]$ is a weakly open neighborhood of x excluding V [weakly open because f as a linear functional is assumed to be continuous in weak topology and $B(\delta, f(x))$ is open]. Thus V^c is weakly open and V is weakly closed. Conversely, suppose $V \subset X$ is weak closed. Let $x \in \overline{V}$ in the norm topology, then there is $\langle x_n \rangle_n$ such that $x_n \to x$ in the norm topology, i.e. $\|x_n - x\| \to 0$ as $n \to \infty$. For any $f \in \mathcal{X}^*$, $|f(x_n - x)| \leq \|f\| \cdot \|x_n - x\| \to 0$ as $n \to \infty$, so $x_n \to x$ in weak topology and thus $x \in V$ since V is weak-closed. □

Folland 5.53

Suppose that $\mathcal{X}$ is a Banach space and $\{T_n\}$, $\{S_n\}$ are sequences in $L(\mathcal{X},\mathcal{X})$ such that $T_n \to T$ strongly and $S_n \to S$ strongly.

(a) If $\{x_n\} \subset \mathcal{X}$ and $\|x_n - x\| \to 0$, then $\|T_n x_n - Tx\| \to 0$.

(b) $T_n S_n \to TS$ strongly.

Proof. (a) Notice that

$$\begin{aligned}\|T_n x_n - Tx\| &\le \|T_n x_n - T_n x\| + \|T_n x - Tx\| \\ &= \|T_n(x_n - x)\| + \|T_n x - Tx\| \\ &\le \|T_n\|\|x_n - x\| + \|T_n x - Tx\| \quad (*)\end{aligned}$$

For any $x \in \mathcal{X}$, since $\{T_n\}$ converges strongly, $\{T_n x\}$ converges and in particular bounded in the norm metric, i.e. $\sup_n \|Tx\| < \infty$ for every x. Since $\mathcal{X}$ is a Banach space, by uniform boundedness we have $\sup_n \|T_n\| < M$ for some large $M > 0$. Then $(*) \le M\|x_n - x\| + \|T_n x - Tx\|$. Sending n to infinity, $\|x_n - x\| \to 0$ and $\|T_n x - Tx\| \to 0$ (strong convergence). Thus $\|T_n x_n - Tx\| \to 0$, as desired.

(b) For every $x \in \mathcal{X}$, $T_n x \to Tx$ and $S_n \to Sx$ since $T_n \to T$ and $S_n \to S$ strongly. Then $T_n x S_n x \to TxSx$ or equivalently $T_n S_n x \to TSx$. Since x is arbitrary, $T_n S_n \to TS$ strongly. This finishes the proof.

□

6. Chapter 6-L^p Spaces

Folland 6.5

Suppose $0 < p < q < \infty$. Then $L^p \not\subset L^q$ iff X contains sets of arbitrarily small positive measure, and $L^q \not\subset L^p$ iff X contains sets of arbitrarily large finite measure. What about the case $q = \infty$?

Proof. **$L^p \not\subset L^q$:** Suppose $L^p \not\subset L^q$, then there is some $f \in L^p \setminus L^q$. Consider $E_n := \{x : |f(x)| > n\}$ for $n \in \mathbb{N}$, we have

$$\infty > \|f\|_p^p \ge \|f\chi_{E_n}\|_p^p = \int |f\chi_{E_n}|^p d\mu > n^p \mu(E_n)$$

and thus $\mu(E_n) < \|f\|_p^p / n^p$. Therefore $\mu(E_n) \to 0$ as $n \to \infty$ since $f \in L^p$ and thus $\|f\|_p^p < \infty$. Now it suffices to show that $\mu(E_n) > 0$ for each n. If in the contrary $\mu(E_n) = 0$, let $F_n := E_n^c$, we have

$$\|f\|_p^p = \int |f|^p d\mu = \int |f|^p \chi_{F_n} d\mu < \infty$$

notice that also we have

$$\|f\|_q^q = \int |f|^p |f|^{q-p} d\mu = \int |f|^p |f|^{q-p} \chi_{F_n} d\mu \le n^{q-p} \int |f|^p \chi_{F_n} d\mu = n^{q-p} \|f\|_p^p < \infty$$

contradicting $f \notin L^q$. Conversely, suppose X contains sets of arbitrarily small positive measure, then we can pick a disjoint family of sets $\{E_n\}_n$ such that $0 < \mu(E_n) < 2^{-n}$. This is because we can take a family $\{F_n\}_n$ of sets such that $\mu(F_n) < 2^{-n}$, and defining $E_n := F_n \setminus \bigcup_{n+1}^{\infty} E_i$ creates the desired subsets $\{E_n\}_n$. Now we define $f := \sum_1^{\infty} a_n \chi_{E_n}$, where $a_n = \mu(E_n)^{-1/p}$. Notice that

$$\|f\|_p^p = \int |f|^p d\mu = \int \left|\sum_{n=1}^{\infty} \mu(E_n)^{-1/p}\chi_{E_n}\right|^p d\mu = \int \left(\sum_{n=1}^{\infty} \mu(E_n)^{-1/p}\chi_{E_n}\right)^p d\mu$$

$$= \lim_{m\to\infty} \int \left(\sum_{n=1}^{m} \mu(E_n)^{-1/p}\chi_{E_n}\right)^p d\mu = \lim_{m\to\infty} \int \sum_{n=1}^{m} \chi_{E_n} d\mu = \lim_{m\to\infty} \sum_{n=1}^{m} \mu(E_n) \quad (1)$$

since E_ns are disjoint. Then by our assumption $0 < (1) < \lim_{m\to\infty} \sum_{n=1}^{m} 2^{-n} < \infty$, showing that $f \in L^p$. The same thing won't happen on $\|f\|_q$, since

$$\|f\|_q^q = \int |f|^q d\mu = \int \left|\sum_{n=1}^{\infty} \mu(E_n)^{-1/p}\chi_{E_n}\right|^q d\mu = \int \sum_{n=1}^{\infty} \mu(E_n)^{-q/p}\chi_{E_n} d\mu$$

$$\geq \lim_{m\to\infty} \sum_{n=1}^{m} (1/\mu(E_n))^{q/p-1} \geq \lim_{m\to\infty} \sum_{n=1}^{m} (2^{q/p-1})^n \geq \lim_{m\to\infty} m = \infty$$

which fails to converge. Thus $\|f\|_q = \infty$, showing that $f \notin L^q$.

$L^q \not\subset L^p$: Suppose X has finite measure, then by proposition 6.12 $L^p \subset L^q$, a contradiction. Conversely, suppose X contains sets of arbitrarily large measure, then we can find disjoint $\{E_n\}_n$ such that $1 \leq \mu(E_n) < \infty$ for all n: First we chose F_1 with $\infty > \mu(F_1) \geq 1$, and then suppose we have chosen F_n, choose F_{n+1} such that $\mu(F_{n+1}) \geq 2\sum_1^n \mu(F_n)$. Thus by considering $E_n := F_n \setminus \bigcup_1^{n-1} E_i$ we get the desired subsets. Consider $f := \sum_1^\infty a_n\chi_{E_n}$, where $a_n = \mu(E_n)^{-1/q}$. The rest of the proof is similar to above.

$q = \infty$: In the case $q = \infty$, we claim that $L^p \not\subset L^q$ iff X contains subsets of arbitrarily small measure, and that $L^q \not\subset L^p$ only if X has infinite measure. For the first part of the statement, we use the same proof as above, except setting $f = \sum_1^\infty a_n\chi_{E_n}$, where $a_n = \mu(E_n)^{-1/(p+1)}$. For the second part of the statement, if X has infinite measure, we can consider the same f as in the proof of $L^q \not\subset L^p$ above and that f is clearly in L^∞. This finishes the proof. Note that for the second part, the if direction fails. Consider the following silly example: let $X = \{0, 1\}$ such that 0 has infinite measure and 1 has zero measure. For every $f \in L^\infty$, f must be actually bounded, and thus $\|f\|^p = \int |f|^p = |f(1)|^p < \infty$, showing that $f \in L^p$.

□

Folland 6.9

Suppose $1 \leq p < \infty$. If $\|f_n - f\|_p \to 0$, then $f_n \to f$ in measure, and hence some subsquence converges to f a.e. On the other hand, if $f_n \to f$ in measure and $|f_n| \leq g \in L^p$ for all n, then $\|f_n - f\|_p \to 0$.

Proof. First of all suppose $\|f_n \to f\|_p \to 0$ for some $1 \leq p < \infty$. Let $\epsilon > 0$. We define $E_{n,\epsilon} := \{x : |f_n(x) - f(x)| \geq \epsilon\}$. Then

$$\|f_n - f\|_p = \left(\int |f_n - f|^p\right)^{1/p} \geq \left(\int_{E_{n,\epsilon}} |f_n - f|^p\right)^{1/p} \geq (\epsilon^p\mu(E_{n,\epsilon}))^{1/p} = \epsilon\mu(E_{n,\epsilon})^{1/p}$$

and thus $\mu(E_{n,\epsilon}) \leq \epsilon^{-p}(\|f_n - f\|_p)^p \to 0$ as $n \to \infty$ since $\|f_n \to f\|_p \to 0$ as $n \to \infty$. This means that $f_n \to f$ in measure, as desired. In particular, since $f_n \to f$ in measure, there is a subsequence of $\langle f_n\rangle_n$ that converges to f pointwise a.e. On the other hand, suppose $f_n \to f$ in measure and $|f_n| \leq g \in L^p$ for all n. We first make the observation that $g \in L^p$ implies $(\int |g|^p)^{1/p} <$

∞ and thus $\int |g|^p < \infty$, meaning that $g^p \in L^1$. Now since $f_n \to f$ in measure, there is some subsequence $\langle f_{n_k}\rangle_k$ that converges to f a.e. Since $\langle f_{n_k}\rangle_k$ is a subsequence, we also have $|f_{n_k}| < g$ for all k. Sending $k \to \infty$ we also obtain $|f| \leq g$ a.e. Therefore, $|f_{n_k} - f|^p \leq (|f_{n_k}| + |f|)^p \leq (2g)^p$. Remember our observation that $g^p \in L^1$, we have $|f_{n_k} - f|^p \leq (2g)^p = 2^p g^p \in L^1$. By dominated convergence theorem, we have

$$\lim_{n\to\infty} \int |f_{n_k} - f|^p = \int \lim_{n\to\infty} |f_{n_k} - f|^p = 0$$

and hence taking $1/p$th power on each side we get $\|f_{n_k} - f\|_p \to 0$. We now claim that actually $\|f_n - f\|_p \to 0$. Suppose not, there is some $\delta > 0$ such that $\|f_n - f\|_p \geq \delta$ for infinitely many f_n. We arrange them to a new sequence, which for our convenience we call $\langle g_n\rangle_n$. Since $\langle g_n\rangle_n$ is essentially a subsequence of $\langle f_n\rangle_n$, we still have $g_n \to f$ in measure and $|g_n| \leq g \in L^p$ for all n. By exactly the same reasoning as above we can show that there is a further subsequence $\langle g_{n_k}\rangle_n$ such that $\|g_{n_k} - f\|_p \to 0$, meaning that for large enough k we have $\|g_{n_k} - f\|_p < \delta$, contradicting our assumption. Thus our claim is true, meaning that $\|f_n - f\|_p \to 0$, as desired. □

Folland 6.13

$L^p(\mathbb{R}^n, m)$ is separable for $1 \leq p < \infty$. However, $L^\infty(\mathbb{R}^n, m)$ is not separable.

Proof. We first show that $L^p(\mathbb{R}, m)$ is separable for $1 \leq p < \infty$. Let $\mathcal{F}$ be the family of simple functions of form $\sum_{j=1}^n a_j \chi_{F_j}$ where $a_j \in \mathbb{Q}$ and F_j is a finite union of measurable rectangles with rational-length sides. We claim that $\mathcal{F}$ is a countable dense subset of $L^p(\mathbb{R}, m)$. First of all we need to show that $\mathcal{F}$ is countable. Notice that there are countably many such sets F_j since there are countably many such measurable rectangles, and countably many such a_j, so there are countably many characteristic functions of the form $a_j\chi_{R_j}$. Identifying each sum $\sum_{j=1}^n a_j\chi_{F_j}$ as $(a_1\chi_{F_1}, ...)$ we know that there are countably many of them. Thus $\mathcal{F}$ is countable. We proceed to show that $\mathcal{F}$ is dense. Let $f \in L^p(\mathbb{R}^n, m)$ and $\epsilon > 0$. Since simple functions are dense in $L^p(\mathbb{R}^n, m)$, we may assume that f is simple and can be written as $\sum_1^n b_j\chi_{E_j}$. Fix the ϵ. By denseness of rationals, we have some a_j such that $|a_j - b_j| < [\epsilon/m(E_j)]^{1/p}$ for each j. Moreover, by regularity of Lebesgue measure, for each E_j there is a finite union of measurable rectangles, which may be taken to have rational coordinates by denseness of rationals and which we call F_j, such that $m(E_j \triangle F_j) < \epsilon/|a_j|^p$. Then

$$\left(\int \left|\sum_{j=1}^n (b_j\chi_{E_j} - a_j\chi_{F_j})\right|^p dm\right)^{1/p} = \left(\sum_{j=1}^n \int |b_j\chi_{E_j} - a_j\chi_{F_j}|^p dm\right)^{1/p} \quad (1)$$

For each j,

$$\begin{aligned}&\int |b_j\chi_{E_j} - a_j\chi_{F_j}|^p dm \leq \int |(b_j - a_j)\chi_{E_j}|^p dm + \int |a_j(\chi_{E_j} - \chi_{F_j})|^p dm\\ \leq& |b_j - a_j|^p m(E_j) + |a_j|^p m(E_j \triangle F_j) < 2\epsilon\end{aligned}$$

and thus (1) $< (2n\epsilon)^{1/p}$. Since ϵ is arbitrary, this shows that $\mathcal{F}$ is dense, as desired.
We then show that $L^\infty(\mathbb{R}, m)$ is not separable. It suffices to give an uncountable family $\mathcal{F} \subset L^\infty$ such that $\|f - g\|_\infty \geq 1$ for all $f, g \in \mathcal{F}$ with $f \neq g$. This is because if we take an open ball of radius 1 around each $f \in \mathcal{F}$, we obtain an uncountable disjoint collection of open balls. Then for any countable subset of $L^\infty(\mathbb{R}, m)$, we must have some open ball in this collection not containing any of the points, meaning that this subset cannot be dense. Now we give the desired subset: consider $\mathcal{F} := \{\chi_{B_r}\}_{r\in\mathbb{R}_{>0}}$, where B_r is the open ball of radius r. Suppose $f = \chi_{B_r} \neq g = \chi_{B_{r'}}$, we must have $r \neq r'$ and we may assume $r < r'$. Then for $x \in B_{r'} \setminus B_r$, a positive measure

set, we have $|f(x) - g(x)| = 1$, and thus $\|f - g\|_\infty \geq 1$. The family is clearly uncountable, as desired. $\square$

Folland 6.19

Define $\phi_n \in (l^\infty)^*$ by $\phi_n(f) = n^{-1}\sum_1^n f(j)$. Then the sequence $\{\phi_n\}$ has a weak* cluster point ϕ, and ϕ is an element of $(l^\infty)^*$ that does not arise from an element of l^1.

Proof. First of all, since $f \in l^\infty$, f is essentially bounded by some $M > 0$, and in the case of counting measure this means that f is actually bounded by M. $\phi_n(f)$ takes the arithmetic mean of the first n terms, and is thus bounded by M for all n. Therefore

$$\|\phi_n\| = \sup\{|\phi_n(f)| : \|f\|_\infty = 1\} \leq 1$$

by above reasoning. This means that $\phi_n \in B^*$, the unit ball of $(l^\infty)^*$. By Alaoglu, B^* is compact in the weak* topology. We define $K_m := \{\phi_n : n \geq m\}$, and notice that $\langle \overline{K_m} \rangle_m$ is a family of closed sets (here it means weak* closure) with finite intersection property. Since each $\overline{K_m} \subset \overline{B^*} = \overline{B^*}$, we have $\bigcap_m \overline{K_m} \neq \emptyset$. Pick $\phi \in \bigcap_m \overline{K_m}$, we claim that ϕ is a weak* cluster point of $\langle \phi_n \rangle_n$. To show this claim, take a weak* neighborhood of ϕ, since $\phi \in \overline{K_m}$ for each m, $U \cap K_m \neq \emptyset$ for each m. This means that for each m, there is some $\phi_n \in U$ with $n \geq m$. Then $\langle \phi \rangle_n$ is frequently in U and thus ϕ is a weak* cluster point. This shows the claim.
We then show that ϕ doesn't arise from l^1. Suppose in the contrary that it does, then $\phi(f) = \varphi_g(f) := \int fg d\mu$ for some $g \in l^1$. Consider a special family of functions $\{f_k\}_k := \{\chi_{\{k\}}\}_k$, then $\phi(f_k) = \int \chi_{\{k\}} g d\mu = g(k)$. Notice that $\phi_n(f_k) = 0$ for $n < k$, and $\phi_n(f_k) = 1/n$ for $n \geq k$. Since ϕ is a weak* cluster point of $\langle \phi_n \rangle_n$, $\phi(f_k)$ is a cluster point of $\langle \phi_n(f_k) \rangle_n$. Then a subsequence of $\langle \phi_n(f_k) \rangle_n$ converges to $\phi(f_k)$ and $\phi(f_k) = 0$. Then $g(k) = 0$ for all k, meaning that $\phi \equiv 0$. However, consider $f \equiv 1$ which lies in l^∞ since it is uniformly bounded, $\phi_n(f) \equiv 1$ for all n and therefore $\phi(f) = 1 \geq 0$, a contrasiction. Then ϕ doesn't arise from l^1. $\square$

Folland 6.20

Suppose $\sup_n \|f_n\|_p < \infty$ and $f_n \to f$ a.e.

(a) If $1 < p < \infty$, then $f_n \to f$ weakly in L^p

(b) The result of (a) is false in general for $p = 1$. It is, however, true for $p = \infty$ if μ is σ-finite and weak convergence is replaced by weak* convergence.

Proof. (a) Let $1 < p < \infty$ and q the conjugated of p. By Riesz representation theorem, to prove $f_n \to f$ weakly in L^p, it suffices to prove that $\varphi_g(f_n) \to \varphi_g(f)$ for all $g \in L^q$, where $\varphi_g(f) = \int fg$. Given $g \in L^q$ and let $\epsilon > 0$, we have the following observations, which we shall prove at the end of this question:

1. There is some $\delta > 0$ such that $\int_E |g|^q < \epsilon$ whenever $\mu(E) < \delta$.
2. There is an $A \subset X$ such that $\mu(A) < \infty$ and $\int_{X \setminus A} |g|^q < \epsilon$.
3. There is $B \subset A$ such that $\mu(A \setminus B) < \epsilon$ and $f_n \to f$ uniformly on B.

Based on the observations, we have

$$|\varphi_{f_n}(g) - \varphi_f(g)| = \left|\int (f_n - f)g\right| \le \left|\int_B (f_n - f)g\right| + \left|\int_{A \setminus B} (f_n - f)g\right| + \left|\int_{X \setminus A} (f_n - f)g\right|$$

$$\le \int_B |(f_n - f)g| + \int_{A \setminus B} |(f_n - f)g| + \int_{X \setminus A} |(f_n - f)g|$$

$$\le \left(\int_B |(f_n - f)|^p\right)^{1/p} \left(\int_B |g|^q\right)^{1/q} + \left(\int_{A \setminus B} |g|^q\right)^{1/q} \|f - f_n\|_p + \left(\int_{X \setminus A} |g|^q\right)^{1/q} \|f_n - f\|_p$$

$$< \left(\int_B |f_n - f|^p\right)^{1/p} \|g_n - g\|_q + 4M\epsilon^{1/q} \le \|f_n\chi_B - f\chi_B\|_u \mu(B) + 4M\epsilon^{1/q} \quad (1)$$

since $f_n \to f$ uniformly on B, $\|f_n\chi_B - f\chi_B\|_u \to 0$. Also notice that $\mu(B) \le \mu(A) < \infty$. Therefore, sending $n \to \infty$, we get $\lim_{n\to\infty} |\varphi_{f_n}(g) - \varphi_f(g)| < 4M\epsilon^{1/q}$. Since ϵ is arbitrary, this shows that $\lim_{n\to\infty} |\varphi_{f_n}(g) - \varphi_f(g)| = 0$, as desired. Thus it remains to show the observations to finish the proof.

1. Since $|g|^q$ is measurable, $\nu(E) := \int_E |g|^q d\mu$ is a well-defined measure, and it is clear that ν is absolutely continuous to μ since integrating on a measure-zero set we get 0. Notice that $g \in L^1$, so $\nu(E) \le \|g\|_q$ is a finite signed measure. Then the result follows by the $\epsilon - \delta$ definition of absolute continuity.
2. Since $|g|^q \in L^+$ and $\int |g|^q < \infty$, $K := \{x : |g|^q > 0\}$ is σ-finite. Then we can write $K = \bigcup_1^\infty E_n$ where each E_n has finite measure, and define $K_n := \bigcup_1^n E_n$ for our convenience. Notice that $\nu(K) = \int_K |g|^q d\mu = \lim_{n\to\infty} \nu(K_n)$ by continuity from below, where ν is defined as above. Since $\nu(K) < \infty$, this means that there is some $N \in \mathbb{N}$ such that $|\nu(K) - \nu(K_N)| < \epsilon$. Take $A = K_N$, we have
$$|\nu(X) - \nu(A)| = |\nu(K) - \nu(A)| < \epsilon \implies |\nu(X \setminus A)| < \epsilon \implies \int_{X \setminus A} |g|^p d\mu < \epsilon$$
as desired.
3. Since $\mu(A) < \infty$ and $f_n\chi_A \to f\chi_A$ a.e., this result is immediate by Egoroff's theorem.

Now the proof is complete.

(b) First of all we show that the result of (a) is false by giving two counterexamples. First of all, in $L^1(\mathbb{R}, m)$ consider $f_n = \frac{1}{n}\chi_{(0,n)}$.

$$\int |f_n| dm = \int \frac{1}{n}\chi_{(0,n)} = \frac{1}{n} m(0, n) = 1$$

for all n and thus $\sup_n \|f_n\|_1 < \infty$. Also it is clear that $f_n \to 0$ a.e. However, for $g \equiv 1 \in L^\infty$ (since ∞ is the conjugate of 1), $\varphi_g(f_n) = \int f_n dm = 1$ for all n and thus $\varphi_g(f_n) \not\to 0 = \varphi_g(0)$. Thus $f_n \not\to f$ weakly in L^1. Next we consider $f_n = \frac{1}{n}\chi_{\{1,\ldots,n\}} \in l^1$, and we use μ for counting measure. Notice that

$$\int f_n d\mu = \int \frac{1}{n}\chi_{\{1,\ldots,n\}} = \frac{1}{n} \cdot n = 1$$

for all n and thus $\sup_n \|f\|_1 < \infty$. Also $f_n \to 0$ a.e. However, for $g \equiv 1 \in l^\infty$, we have $\varphi_g(f_n) = \int f_n d\mu = 1$ for all n. Thus $\varphi_g(f_n) \not\to 0 = \varphi_g(0)$ and hence $f_n \not\to 0$ weakly. We then show that the result is true for $p = \infty$ im the context of μ σ-finite and weak* convergence. Since μ is σ-finite, $L^\infty = (L^1)^*$. By Riesz representation, it suffices to show that $\varphi_{f_n} \to \varphi_f$, and this yields to show that $\varphi_{f_n}(g) \to \varphi_f(g)$ for all $g \in L^1$.

$$|\varphi_{f_n}(g) - \varphi_f(g)| = \left|\int f_n g - \int fg\right| \le \int |f_n g - fg| \quad (1)$$

Suppose $\sup_n \|f_n\|_\infty < M$, we have $|f_n g - fg| \le 2M|g| \in L^1$ for all n. Also $f_n \to f$ a.e. and g essentially bounded implies $f_n g \to fg$ a.e. Applying dominated convergence we get (1) $\to 0$, as desired.

$\square$

Folland 6.22

Let $X = [0,1]$ with the Lebesgue measure.

(a) Let $f_n(x) = \cos 2\pi nx$. Then $f_n \to 0$ weakly in L^2, but $f_n \not\to 0$ a.e. or in measure.

(b) Let $f_n(x) = n\chi_{(0,1/n)}$. Then $f_n \to 0$ a.e. and in measure, but $f_n \not\to 0$ weakly in L^p for any p.

Proof. (a) We first show that $f_n \to 0$ weakly in L^2. By Riesz representation, it suffices to show that $\varphi_g(f_n) := \int f_n g$ converges to $\varphi_g(0) = 0$ for all $g \in L^2$. To this end, let $g \in L^2$ and $\epsilon > 0$. By denseness of simple functions, we can choose a simple function $\phi := \sum_1^n a_i \chi_{E_i}$ such that $\|g - \phi\|_2 < \epsilon$. Furthermore, since each E_i is measurable and $m(E_i) \le m([0,1]) = 1 < \infty$, by regularity we can find a finite union of intervals, which we call F_i, such that $m(F_i \triangle E_i) < \epsilon/(n|a_i|)$ for each i. For convenience, let's call the redefined function $\phi' := \sum_1^m b_i \chi_{U_i} (= \sum_1^n a_i \chi_{F_i})$, where U_i are intervals with end points a_i and b_i.

Now

$$|\varphi_{\phi'}(f_n)| = \left| \int \phi' \cos 2\pi nx \right| = \left| \sum_{i=1}^m b_i \int_{U_i} \cos 2\pi nx \right| \le \sum_{i=1}^m b_i \left| \int_{a_i}^{b_i} \cos 2\pi nx \right|$$

$$\le \sum_{i=1}^m \left| \frac{b_i}{2\pi n} \sin 2\pi nx \Big|_{a_i}^{b_i} \right| \le \sum_{i=1}^m \frac{|b_i|}{\pi n} \to 0 \quad \text{as } n \to \infty$$

By assumption we have $\|g - \phi\|_2 \le \epsilon$, and

$$\|\phi - \phi'\|_2 = \int |\phi - \phi'| \le \sum_{i=1}^n |a_i| m(F_i \triangle E_i) < \epsilon$$

so by Minkowski inequality we have $\|g - \phi'\| < 2\epsilon$. Since the assignment $g \mapsto \varphi_g$ is isometric, we have $\|\varphi_{\phi'} - \varphi_g\| < 2\epsilon$. Since $\|f_n\|_2 = \int_{[0,1]} \cos 2\pi nx \le 1$, it follows that

$$|\varphi_{\phi'}(f_n) - \varphi_g(f_n)| \le \|\varphi_{\phi'} - \varphi_g\| \|f_n\|_2 \le \|\varphi_{\phi'} - \varphi_g\| < \epsilon$$

for all n and sending $n \to \infty$ we have $\lim_{n\to\infty} |\varphi_g(f_n)| < \epsilon$. Since ϵ is arbitrary, we have $\lim_{n\to\infty} |\varphi_g(f_n)| = 0$. Since g is arbitrary, this shows that $f_n \to 0$ weakly. Moreover, we show that no subsequence of $\langle f_n \rangle_n$ converges to 0 a.e. Let $\langle f_{n_k} \rangle_k$ be a subsequence of $\langle f_n \rangle_n$, suppose $f_{n_k} \to 0$ a.e., we have $|f_{n_k}^2| = |\cos^2(2\pi n_k x)| \le \chi_{[0,1]} \in L^1([0,1], m)$. Applying dominated convergence we should get $\int \cos^2 2\pi n_k x \to 0$ as $k \to \infty$. However,

$$\int \cos^2(2\pi n_k x) = \int \frac{\cos 4\pi nx + 1}{2} = \frac{1}{8\pi n} \sin 4\pi nx \Big|_0^1 + \frac{1}{2} = \frac{1}{2} \ne 0$$

a contradiction. It immediately follows that $f_n \not\to 0$ a.e. If $f_n \to 0$ in measure, there is a subsequence $f_{n_k} \to 0$ a.e., hence $f_n \not\to 0$ in measure as well.

(b) First of all $f_n \to 0$ a.e. since for every $x \in [0,1]$, when n is large enough $\frac{1}{n} < x$ and hence $f_n(x) = 0$. $f_n \to 0$ in measure since for every $\epsilon \ge 0$, $\mu\{|f_n| \ge \epsilon\} \le \frac{1}{n} \to 0$ as $n \to \infty$. We then show that $f_n \not\to 0$ weakly in L^p for any p. Let $p \in (1, \infty)$ consider its conjugate q. Clearly $g \equiv 1 \in L^q$ since $[0,1]$ has finite measure. Then $\varphi_g(f_n) = |\int f_n| = \int n\chi_{(0,1/n)} = 1$

for all n, and hence $\varphi_g(f_n)$ doesn't converge. Since we already showed in class that φ_g is a well-defined linear functional, this shows that $f_n \not\rightharpoonup 0$ weakly.

$\square$

Folland 6.26

Complete the proof of Theorem 6.18 for $p = 1$.

Proof. Suppose $p = 1$, and $q = \infty$ is its Holder conjugate. Since $|K(x,y)f(y)| \in L^+$, by Tonelli,

$$\int \left[\int |K(x,y)f(y)| d\nu(y) \right] d\mu(x) \le \iint |K(x,y)f(y)| d\mu(x) d\nu(y)$$
$$= \int \left[\int |K(x,y)| d\mu(x) \right] |f(y)| d\nu(y) \le C \int |f(y)| d\nu(y) \quad (1)$$

Since $f \in L^1(\nu)$, the last integral is finite, and thus $\int |K(x,y)f(y)| d\nu(y)$ is finite for a.e.-x. This shows that $Tf(x)$ converges absoluetely for a.e.-x and also

$$\int |Tf(x)| d\mu(x) \le \int \left[\int |K(x,y)f(y)| d\nu(y) \right] d\mu(x) \le C \int |f(y)| d\nu(y) < \infty (2)$$

by (1), showing that Tf is defined in $L^1(\mu)$. Eventually, we can read from (2) that $\|Tf\|_1 \le C\|f\|_1$. This finishes the proof. $\square$

Remark. This is a analogous but simpler argument conpared to the proof of Theorem 6.18.

Folland 6.36

If $f \in \text{weak}L^p$ and $\mu(\{x : f(x) \neq 0\}) < \infty$, then $f \in L^q$ for all $q < p$. On the other hand, if $f \in (\text{weak}L^p) \cap L^\infty$, then $f \in L^q$ for all $q > p$.

Proof. **Part 1.** First of all, since $\mu\{x : f(x) \neq 0\} < \infty$, we may assume $\mu\{x : f(x) \neq 0\} \le C$. It follows that $\lambda_f(\alpha) \le C$ for all α. Also, since $f \in \text{weak}L^p$, $[f]_p = \sup_{\alpha > 0} \alpha^p \lambda_f(\alpha) < \infty$. We may assume $\alpha^p \lambda_f(\alpha) \le M$ for some $M > 0$. Given that $\alpha > 0$, we have $\lambda_f(\alpha) \le M/\alpha^p$. Since $0 < p < \infty$, by proposition 6.24,

$$\|f\|_q^q = \int_0^\infty \alpha^{q-1} \lambda_f(\alpha) d\alpha = q \int_0^1 \alpha^{q-1} \lambda_f(\alpha) d\alpha + q \int_1^\infty \alpha^{q-1} \lambda_f(\alpha) d\alpha$$
$$= qC \int_0^1 \alpha^{q-1} d\alpha + qM \int_1^\infty \alpha^{q-p-1} d\alpha$$
$$= C\alpha^q \Big|_0^1 + qM \left(\frac{1}{q-p} \alpha^{q-p} \Big|_1^\infty \right) \quad (1)$$

Since $q > 0$, $C\alpha^q\big|_0^1 = C$, and for $q < p$, $q - p < 0$ and thus $qM(\frac{1}{q-p}\alpha^{q-p}\big|_1^\infty) = -\frac{qM}{q-p}$. It follows that $\|f\|_q^q = (1) < \infty$ for all $q < p$, showing that $f \in L^q$ for all $q < p$.
Part 2. Suppose $f \in \text{weak}L^p \cap L^\infty$. Similar to above, we have $\lambda_f(a) \le M/\alpha^p$ for some $M > 0$. Since $f \in L^\infty$, there is some β such that $\lambda(\alpha) = 0$ when $\alpha > \beta$. Then

$$\|f\|_q^q = q \int_0^\infty \alpha^{q-1} \lambda_f(\alpha) d\alpha = qM \int_0^\beta \alpha^{q-p-1} d\alpha = qM \frac{1}{q-p} \alpha^{q-p} \Big|_0^\beta = qM \frac{1}{q-p} \beta^{q-p} < \infty$$

for all $q \in (p, \infty)$. Since we already assume $f \in L^\infty$, $f \in L^q$ for all $q > p$. This finishes the proof. $\square$

Remark. The key technique in this problem is splitting up an integral into different scales, which is a standard technique when integrals behave differently at different scales. In part 1, the integral is nice in large scale, so we bound it in small scale; in part 2 the integral is nice in small scale, so we bound it in large scale.

Folland 6.45

If $0 < \alpha < n$, define an operator T_α on functions on $\mathbb{R}^n$ by

$$T_\alpha f(x) = \int |x-y|^{-\alpha} f(y) dy$$

Then T_α is weak type $(1, q)$ where $q = n/\alpha$, and strong type (p, r) where $r^{-1} + 1 = p^{-1} + na^{-1}$.

Proof. We define $K(x, y) := |x-y|^{-\alpha}$, which is clearly $X \times Y$ measurable, and let q denote the same thing as in the problem. Observe that

$$\begin{aligned}\beta^q \lambda_{K(x,\cdot)}(\beta) &= \beta^q m\{y : |x-y|^{-\alpha} > \beta\} = \beta^q m\{y : |x-y| < \beta^{-1/\alpha}\} \\ &\leq \beta^q m B(x, \beta^{-1/\alpha}) \leq \beta^q C_1 + \beta^{-n/\alpha} = \beta^q C_1 \beta^{-q} = C_1\end{aligned}$$

for some constant $C_1 > 0$. Therefore, $[\beta^q \lambda_{K(x,\cdot)}(\beta)]^{1/q} \leq C_1^{1/q}$ for all β and thus $[K(x,\cdot)]_q \leq C_1^{1/q}$ for some $C_1 > 0$. By symmetry of $K(x, y)$, we can similarly prove that $[K(\cdot, y)]_q \leq C_2$ for some $C_2 > 0$ for all y. By setting $C = \max(C_1^{1/q}, C_2)$ we obtain $[K(x,\cdot)]_q \leq C$ for all x and $[K(\cdot, y)]_q \leq C$ for all y. Now applying theorem 6.36 to q and p as given in the problem we conclude the proof. $\square$

Folland 6.37

If f is a measurable function and $A > 0$, let $E(A) = \{x : |f(x)| > A\}$, and set

$$h_A = f\chi_{X \setminus E(A)} + A(\operatorname{sgn} f)\chi_{E(A)}, \quad g_A = f - h_A = (\operatorname{sgn} f)(|f| - A)\chi_{E(A)}$$

Then

$$\lambda_{g_A}(\alpha) = \lambda_f(\alpha + A), \ \lambda_{h_A}(\alpha) = \begin{cases} \lambda_f(\alpha) & \alpha < A \\ 0 & \alpha \geq A \end{cases}$$

Proof. First of all,

$$\lambda_{g_A}(\alpha) = \mu\{|g_A| > \alpha\} = \mu\{|(\operatorname{sgn} f)(|f| - A)\chi_{E(A)}| > \alpha\} = \mu\{|(|f| - A)\chi_{E(A)}| > \alpha\} \quad (1)$$

On $E(A)$ we have $|f| > A$ and therefore $(|f| - A)\chi_{E(A)} > 0$. It follows that

$$(1) = \mu\{(|f| - A)\chi_{E(A)} > \alpha\} = \mu\{x \in E(A) : |f(x)| - A > \alpha\} = \mu[E(A) \cap \{|f| > A + \alpha\}] \quad (2)$$

Observe that $|f(x)| > A + \alpha$ implies $|f(x)| > A$ and hence $x \in E(A)$, so $\{|f| > A + \alpha\} \subset E(A)$ and therefore $(2) = \mu\{|f| > A + \alpha\} = \lambda_f(\alpha + A)$. For the second part of the problem,

$$\lambda_{h_A}(\alpha) = \mu\{|h_A| > \alpha\} = \mu\{|f\chi_{X \setminus E(A)} + A(\operatorname{sgn} f)\chi_{E(A)}| > \alpha\} \quad (3)$$

We now claim that $\{|f\chi_{X \setminus E(A)} + A(\operatorname{sgn} f)\chi_{E(A)}| > \alpha\} = \{|f\chi_{X \setminus E(A)}| > \alpha\} \cup \{|A(\operatorname{sgn} f)\chi_{E(A)}| > \alpha\}$. To prove the claim, we first prove the forward inclusion. $x \in \{|f\chi_{X \setminus E(A)} + A(\operatorname{sgn} f)\chi_{E(A)}| > \alpha\}$ means that $|f\chi_{X \setminus E(A)}(x) + A(\operatorname{sgn} f)\chi_{E(A)}(x)| > \alpha$. Since $X \setminus E(A)$ and $E(A)$ are disjoint, we have either

$$|f\chi_{X \setminus E(A)}(x) + A(\operatorname{sgn} f)\chi_{E(A)}(x)| = |f\chi_{X \setminus E(A)}(x) + 0| = |f\chi_{X \setminus E(A)}(x)| > \alpha$$

or

$$|f\chi_{X \setminus E(A)}(x) + A(\text{sgn}f)\chi_{E(A)}(x)| = |0 + A(\text{sgn}f)\chi_{E(A)}(x)| = |A(\text{sgn}f)\chi_{E(A)}(x)| > \alpha$$

meaning that $x \in \{|f\chi_{X \setminus E(A)}| > \alpha\} \cup \{|A(\text{sgn}f)\chi_{E(A)}| > \alpha\}$. Conversely, suppose $x \in \{|f\chi_{X \setminus E(A)}| > \alpha\} \cup \{|A(\text{sgn}f)\chi_{E(A)}| > \alpha\}$. If $x \in \{|f\chi_{X \setminus E(A)}| > \alpha\}$, $x \in X \setminus E(A)$ and hence $A(\text{sgn}f)\chi_{E(A)}(x) = 0$. It follows that

$$|f\chi_{X \setminus E(A)}(x) + A(\text{sgn}f)\chi_{E(A)}(x)| = |f\chi_{X \setminus E(A)}(x) + 0| = |f\chi_{X \setminus E(A)}(x)| > \alpha$$

and $x \in \{|f\chi_{X \setminus E(A)} + A(\text{sgn}f)\chi_{E(A)}| > \alpha\}$. Similarly we can show the result for the case where $x \in \{|A(\text{sgn}f)\chi_{E(A)}| > \alpha\}$. Therefore we have shown the claim. By the claim and the fact that $\{|f\chi_{X \setminus E(A)}| > \alpha\}$ and $\{|A(\text{sgn}f)\chi_{E(A)}| > \alpha\}$ are disjoint by the disjointness of $X \setminus E(A)$ and $E(A)$, we obtain

$$(3) = \mu\{|f\chi_{X \setminus E(A)}| > \alpha\} + \mu\{|A(\text{sgn}f)\chi_{E(A)}| > \alpha\}$$

If $\alpha < A$, $x \in \{|f\chi_{X \setminus E(A)}| > \alpha\}$ iff $x \in X \setminus E(A)$ and $|f(x)| > \alpha$ iff $\alpha < |f(x)| \leq A$, and

$$\begin{aligned}(3) &= \mu\{|f\chi_{X \setminus E(A)}| > \alpha\} + \mu\{|A(\text{sgn}f)\chi_{E(A)}| > \alpha\} = \mu\{\alpha < |f| \leq A\} + \mu\{|\chi_{E(A)}| > \alpha/A\} \\ &= \mu\{\alpha < |f| \leq A\} + \mu(E(A)) = \mu\{\alpha < |f| \leq A\} + \mu\{|f| > A\} = \mu\{|f| > \alpha\} = \lambda_f(\alpha)\end{aligned}$$

If $\alpha \geq A$, $|f(x)| \leq A \leq \alpha$ for all $x \in X \setminus E(A)$ and hence $\{|f\chi_{X \setminus E(A)}| > \alpha\} = \emptyset$. Moreover

$$\{|A(\text{sgn}f)\chi_{E(A)}| > \alpha\} = \{\chi_{E(A)} > \alpha/A \geq 1\} = \emptyset$$

so $(3) = \mu\{|f\chi_{X \setminus E(A)}| > \alpha\} + \mu\{|A(\text{sgn}f)\chi_{E(A)}| > \alpha\} = 0$. This finishes the proof. □

Folland 6.38

$f \in L^p$ iff $\sum_{-\infty}^{\infty} 2^{kp}\lambda_f(2^k) < \infty$.

Proof. Suppose $f \in L^p$. Then

$$\sum_{-\infty}^{\infty} 2^{kp}\lambda_f(2^k) = \sum_{-\infty}^{\infty} 2^k 2^{k(p-1)}\lambda_f(2^k) = 2^p \sum_{-\infty}^{\infty} 2^{k-1}2^{(k-1)(p-1)}\lambda_f(2^k) \quad (1)$$

Notice that on $[2^{k-1}, 2^k)$, $\alpha \geq 2^{k-1}$ and $\lambda_f(\alpha) \geq \lambda_f(2^k)$ (since λ_f is a decreasing function), so $\alpha^{p-1}\lambda_f(\alpha) \geq (2^{k-1})^{p-1}\lambda_f(2^k)$. Hence

$$\int_{[2^{k-1},2^k)} \alpha^{p-1}\lambda_f(\alpha)d\alpha \geq \int_{[2^{k-1},2^k)} (2^{k-1})^{p-1}\lambda_f(2^k) = 2^{k-1}2^{(k-1)(p-1)}\lambda_f(2^k) \quad (2)$$

and it follows by additivity that

$$(1) \leq 2^p \sum_{-\infty}^{\infty} \int_{[2^{k-1},2^k)} \alpha^{p-1}\lambda_f(\alpha)d\alpha \quad (3)$$

Notice that the collection of intervals $\{[2^{k-1}, 2^k)\}_k$ are pairwise disjoint, and that $\bigcup_{-\infty}^{\infty}[2^{k-1}, 2^k) = (0, \infty)$. We have

$$(3) \leq 2^p \sum_{-\infty}^{\infty} \int \alpha^{p-1}\lambda_f(\alpha)\chi_{[2^{k-1},2^k)}d\alpha \quad (4)$$

Since $|\alpha^{p-1}\lambda_f(\alpha)\chi_{[2^{k-1},2^k)}| \leq \alpha^{p-1}\lambda_f(\alpha) \in L^1$ by proposition 6.24, by dominated convergence theorem,

$$\begin{aligned}(4) &= 2^p \int \sum_{-\infty}^{\infty} \alpha^{p-1}\lambda_f(\alpha)\chi_{[2^{k-1},2^k)} d\alpha = 2^p \int \alpha^{p-1}\lambda_f(\alpha)\chi_{\bigcup_{-\infty}^{\infty}[2^{k-1},2^k)} d\alpha \\ &= 2^p \int \alpha^{p-1}\lambda_f(\alpha) \leq \frac{2^p}{p}\|f\|_p^p < \infty\end{aligned}$$

and the result follows. Conversely, suppose $\sum_{-\infty}^{\infty} 2^{kp}\lambda_f(2^k) < \infty$, $\sum_{-\infty}^{\infty} 2^{kp}\mu\{|f| > 2^k\} < \infty$. For our convenience, define $K_n := \{2^n < |f| \leq 2^{n+1}\}$,

$$\int |f|^p \leq \sum_{-\infty}^{\infty} 2^{(n+1)p}\mu(K_n) \leq \sum_{-\infty}^{\infty} 2^{(n+1)p}\lambda_f(2^n) \leq 2^p \sum_{-\infty}^{\infty} 2^{np}\lambda_f(2^n) < \infty$$

showing that $f \in L^p$, as desired. This finishes the proof. □

Folland 6.41

Suppose $1 < p \leq \infty$ and $p^{-1} + q^{-1} = 1$. If T is a bounded operator on L^p such that $\int (Tf)g = \int f(Tg)$ for all $f, g \in L^p \cap L^q$, then T extends uniquely to a bounded operator on L^r for all r in $[p, q]$ (if $p < q$) or $[q, p]$ (if $q < p$).

Proof. First of all we notice that $1 \leq q < \infty$. We use Σ to denote the space of simple functions that vanish outside a set of finite measure. For $f \in L^p \cap L^q$, $Tf \in L^p$ and is thus measurable. Moreover, for any $g \in \Sigma$, clearly $g \in L^p \cap L^q$, and thus $\|g(Tf)\|_1 \leq \|g\|_q\|Tf\|_p < \infty$ by Holder's inequality, meaning that $g(Tf) \in L^1$. Also, by assumption

$$\begin{aligned}M_q(Tf) &:= \sup\left\{\left|\int g(Tf)\right| : g \in \Sigma, \|g\|_p = 1\right\} \\ &\leq \sup\left\{\left|\int f(Tg)\right| : g \in \Sigma, \|g\|_p = 1\right\} \\ &\leq \sup\{\|f(Tg)\|_1 : g \in \Sigma, \|g\|_p = 1\} \\ &\leq \sup\{\|f\|_q\|Tg\|_p : g \in \Sigma, \|g\|_p = 1\} \quad \text{(Holder)} \\ &\leq \sup\{\|f\|_q\|T\|_{op}\|g\|_p : g \in \Sigma, \|g\|_p = 1\} \\ &\leq \sup\{\|f\|_q\|T\|_{op} : g \in \Sigma, \|g\|_p = 1\} \\ &= \|T\|_{op}\|f\|_q < \infty\end{aligned}$$

We also notice that since $f \in L^p$ and T is a bounded operator on L^p, $Tf \in L^p$. If $p < \infty$, $\int |Tf|^p < \infty$. Hence $\{|Tf|^p \neq 0\} = \{Tf \neq 0\}$ is σ-finite. If $p = \infty$, μ the background measure is assumed to be semifinite. Now applying theorem 6.14 on Folland we obtain that $Tf \in L^q$. That is, we prove that $Tf \in L^q$ for any $f \in L^p \cap L^q$.

Observe that $L^p \cap L^q$ is dense in L^q since simple functions, which are dense in L^p, are contained in $L^p \cap L^q$. Therefore for $g \in L^q$ we can choose $\{f_n\}_n \subset L^p \cap L^q$ such that $f_n \to g$ in L^q. Now with a little abuse of notation we extend T to L^q by defining

$$Tg = \lim_{n\to\infty} Tf_n$$

where the limit here refers to limit in L^q.

Claim. T is well-defined. That is, T is a bounded linear operator on L^q and it is independent of the sequence $\{f_n\}$.

Proof of Claim. Linearity of T easily follows from linearity of T on $L^p \cap L^q$ and linearity of limit. Notice that $\|Tf_m - Tf_n\|_q \leq C_q\|f_m - f_n\|_q \to 0$ as $m, n \to \infty$, so $\{Tf_n\}_n$ is Cauchy and converges since L^q is complete. Since the limit is unique, it must be Tg. Hence $Tg \in L^q$ and $\|Tg - Tf_n\|_q \to 0$. This shows that for any $f_n \to g$ in L^q, $Tf_n \to Tg$ in L^q. Then this definition indeed defines a linear operator on L^q and the definition is independent of the choice of the sequence. (Since for any such sequence $\{f_n\}_n$ the limit in L^q will be Tg. ☐

Hence for $f + g \in L^p + L^q$ where $f \in L^p$ and $g \in L^q$, $T(f + g) = Tf + Tg \in L^p + L^q$ and thus T is a linear operator on $L^p + L^q$. Moreover, we showed above that T is of strong type (p, p) (by assumption) and strong type (q, q) (by claim). By Riesz-Thorin applied with p and q as given in the problem, T can be extended to a bounded operator for r where r is given as in the problem. It remains to show that the extention is unique. Suppose there is another extension T', $h \in L^r$. Without loss of generality suppose $r \in [p, q]$, $h = f + g$ for some $f \in L^p$ and $g \in L^q$. Then

$$T'h = T'(f + g) = T'f + T'g = Tf + T'g$$

Choose $g_n \in L^p \cap L^q$ such that $g_n \to g \in L^q$. Then $T'g = \lim_{n\to\infty} Tg_n = Tg$ and hence $T'h = Tf + Tg = Th$. This shows that T is unique. ☐

7. Chapter 7-Radon Measures

Folland 7.21

Let $\{f_\alpha\}_{\alpha\in A}$ be a subset of $C(X)$ where X is compact and $\{c_\alpha\}_{\alpha\in A}$ be a family of complex numbers. If for each finite set $B \subset A$ there is $\mu_B \in M(X)$ such that $\|\mu_B\| \leq 1$ and $\int f_\alpha d\mu_B = c_\alpha$ for $\alpha \in B$, then there is $\mu \in M(X)$ such that $\|\mu\| \leq 1$ and $\int f_\alpha d\mu = c_\alpha$ for all $\alpha \in A$.

Proof. [4] First of all since X is compact the uniform norm on $C(X)$ makes sense, making $C(X)$ a normed vector space. Then by Alaoglu's theorem $B^* := \{I_\mu \in C(X)^* : \|I_\mu\| \leq 1\}$ is compact in the weak* topology of $C(X)$. We define M_α to be the set of measures μ such that $\|I_\mu\| \leq 1$ and $\int f_\alpha d\mu = c_\alpha$. M_α is non-empty since $\{\alpha\}$ is a finite subset of A. We claim that $\{M_\alpha\}_{\alpha\in A}$ has finite intersection property. To show the claim, we first show that M_α is closed for each α. Let $B := \{\alpha_1, ..., \alpha_n\}$ be a finite subset of A and by assumptions in the problem there is some $\mu_B \in B^*$ such that $\|I_{\mu_B}\| \leq 1$ and $\int f_\alpha d\mu_B = c_\alpha$ for each $\alpha \in B$, meaning that $\mu_B \in \bigcap_{\alpha\in B} M_\alpha$ and thus showing that $\bigcap_{\alpha\in B} M_\alpha$ is non-empty. Thus the claim is true. Notice that $M_\alpha \subset B^*$ for each α, and that $M_\alpha = B^* \cap \{\mu \in M(X) : \mu(f_\alpha) = c_\alpha\} = B^* \cap \{\mu \in M(X) : \mu(f_\alpha) = c_\alpha\} = B^* \cap \hat{f_\alpha}^{-1}(\{c_\alpha\})\}$ where $\hat{f_\alpha} : \mu \mapsto \mu(f_\alpha)$ is the canonical embedding. Since $\hat{f_\alpha}$ is bounded and thus continuous, $\hat{f_\alpha}^{-1}(\{c_\alpha\})$ is closed and therefore each M_α is closed as a closed subset of a compact set. Then $\{M_\alpha\}_{\alpha\in A}$ is a family of closed subsets of B^* compact with finite intersection property and thus $\bigcap_{\alpha\in A} M_\alpha \neq \emptyset$. Let $\mu \in \bigcap_{\alpha\in A} M_\alpha$, $\|\mu\| \leq 1$ and $\int f_\alpha d\mu = c_\alpha$ for all $\alpha \in A$, as desired. ☐

8. Chapter 8-Elements of Fourier Analysis

Folland 8.4

If $f \in L^\infty$ and $\|T_y f - f\|_\infty \to 0$ as $y \to 0$, then f agrees a.e. with a uniformly continuous function.

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[4]In this proof I may use μ and I_μ interchangably, which is common in this situation.

Proof. The main goal of this proof is to show that $h(x) := \lim_{n\to\infty} A_{1/n}f(x)$ (where $A_rf(x) = \frac{1}{m(B(r,x))}\int_{B(r,x)} f(y)dy$) is well-defined, and that it is the desired function. Before establishing this result, we prove some important claims.

Claim 1. A_rf is uniformly continuous for any $r > 0$.

Proof of Claim. Let $r > 0$. For our convenience we use g_r to denote A_rf, then

$$\begin{aligned}
\|\tau_y g_r - g_r\|_u &= \sup_x |g_r(x) - g_r(x-y)| = \sup_x \frac{1}{m(B(r,x))}\left|\int_{B(r,x)} f(z)dz - \int_{B(r,x-y)} f(z)dz\right| \\
&= \sup_x \frac{1}{m(B(r,x)}\left|\int_{B(r,x-y)} f(z-y) - f(z)dz\right| \\
&\le \sup_x \frac{1}{m(B(r,x))}\int_{B(r,x-y)} |\tau_y f(z) - f(z)|dz \\
&\le \sup_x \frac{1}{m(B(r,x))}\int_{B(r,x-y)} \|\tau_y f - f\|_\infty dz \\
&= \sup \|\tau_y f - f\|_\infty = \|\tau_y f - f\|_\infty
\end{aligned}$$

which tends to 0 as $y \to 0$ and thus $g_r = A_rf$ is uniformly continuous. □

Claim 2. A_rf is uniformly Cauchy as $r \to 0$.

Proof of Claim. Let $\epsilon > 0$. Since $\|\tau_y f - f\|_\infty \to 0$ as $y \to 0$, there is some $\delta > 0$ such that when $|y| < \delta$, $\|\tau_y f - f\|_\infty < \epsilon$. Therefore, if $r_1, r_2 < \delta$,

$$\begin{aligned}
\left|\frac{1}{m(B(r_1,x))}\int_{B(r_1,x)} f(y)dy - f(x)\right| &\le \frac{1}{m(B(r_1,x))}\int_{B(r_1,x)} |f(y) - f(x)|dy \\
&= \frac{1}{m(B(r_1,x))}\int_{B(r_1,x)} |f(x-(x-y)) - f(x)|dy < \epsilon
\end{aligned}$$

Since $|x - y| \le r_1 < \delta$. Analogously we can prove that

$$\left|\frac{1}{m(B(r_2,x))}\int_{B(r_2,x)} f(y)dy - f(x)\right| < \epsilon$$

and therefore

$$\begin{aligned}
\|A_{r_1}f - A_{r_2}f\|_u &= \sup_x \left|\frac{1}{m(B(r_1,x))}\int_{B(r_1,x)} f(y)dy - \frac{1}{m(B(r_2,x))}\int_{B(r_2,x)} f(y)dy\right| \\
&\le \sup_x \left|\frac{1}{m(B(r_1,x))}\int_{B(r_1,x)} f(y)dy - f(x)\right| + \left|\frac{1}{m(B(r_2,x))}\int_{B(r_2,x)} f(y)dy - f(x)\right| \\
&< \sup_x 2\epsilon = 2\epsilon
\end{aligned}$$

and thus A_rf is uniformly Cauchy as $r \to 0$. □

We now prove that h is uniformly continuous. Since $\{A_{1/n}f\}_n$ is uniformly Cauchy by claim 2, $\{A_{1/n}f(x)\}_n$ is a Cauchy sequence for each x and therefore converges since $\mathbb{R}$ is complete. Let $\epsilon > 0$, there is some $N \in \mathbb{N}$ such that when $n, m > N$, $|A_{1/n}f(x) - A_{1/m}f(x)| < \epsilon$ for all x. Sending $m \to \infty$ gives us $|A_{1/n}f(x) - h(x)| < \epsilon$, and since x is arbitrary $\{A_{1/n}\}_n$ must converges uniformly to h.

Claim 3. h is uniformly continuous.

Proof of Claim. Let $\epsilon > 0$. Since $\{A_{1/n}f\}_n \to h$ uniformly, there is some N large enough such that $\|A_{1/N}f - h\|_u < \epsilon/3$. By claim 1 $A_{1/N}f$ is uniformly continuous, so there is some $\delta > 0$ such that when $|y| < \delta$, $|A_{1/N}f(x-y) - A_{1/N}f(x)| < \epsilon/3$ for all x. Now for all x and $|y| < \epsilon$,

$$\begin{aligned}|h(x-y) - h(x)| &\le |h(x-y) - A_{1/n}(x-y)| + |A_{1/n}(x-y) - A_{1/n}(x)| + |A_{1/n}(x) - h(x)| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon\end{aligned}$$

showing that h is uniformly continuous and finishes the claim. □

Since $f \in L^\infty$, f integrated on any bounded measurable set must admit finite value and thus f is locally integrable. By theorem 3.18 on Folland, h agrees with f a.e. Since h is uniformly continuous, this finishes the proof. □

Folland 8.8

Suppose that $f \in L^p(\mathbb{R})$. If there exists $h \in L^p(\mathbb{R})$ such that

$$\lim_{y \to 0} \|y^{-1}(\tau_{-y}f - f) - h\|_p = 0$$

we call h the strong L^p derivative of f. If $f \in L^p(\mathbb{R}^n)$, L^p partial derivatives of f are defined similarly. Suppose that p and q are conjugate exponents, $f \in L^p$, $g \in L^q$, and the L^p derivative $\partial_j f$ exists. Then $\partial_j(f * g)$ exists (in the ordinary sense) and equals $(\partial_j f) * g$.

Proof. First of all we show that the definition $(\partial_j f) * g$ makes sense. Since $\partial_j f$ is the L^p derivative, it lies in L^p, and $g \in L^q$ by assumtion. It follows from proposition 8.8 that $\|\partial_j f * g\|_u \le \|\partial_j f\|_p \|g\|_q < \infty$, showing that $(\partial_j f) * g$ is well defined. Behold that

$$|(h^{-1}(\tau_{-h}(f*g) - f*g) - \partial_j f * g)(x)| = \int \left| \frac{f(x-y+he_j) - f(x-y)}{h} - \partial_j f(x-y) \right| \left| g(y) \right| dy \quad (*)$$

where e_j is the unit vector on the jth component. For our convenience, we denote $\frac{f(x+he_j)-f(x)}{h}$ as $F_h(x)$ and $\partial_j f(x)$ as $F(x)$. Note that $F_h \in L^p$ for each h since $f \in L^p$, and $F = \partial_j f \in L^p$, so $F - F_h \in L^p$. Now

$$(*) = |(F_h - F) * g(x)| \le \|(F_h - F) * g\|_u \le \|F_h - F\|_p \|g\|_q$$

and sending $h \to 0$, our assumption $\lim_{y\to 0} \|y^{-1}(\tau_{-y}f - f) - h\|_p = 0$ tells us exactly that $\|F_h - F\|_p$ tends to 0 and hence

$$0 = \lim_{h\to 0} |(h^{-1}(\tau_{-h}(f*g) - f*g) - \partial_j f * g)(x)| = \lim_{h \to 0} \left| \frac{f*g(x+h) - f*g(x)}{h} - \partial_j f * g \right|$$

showing that $\partial_j(f*g)$ exists in an ordinary sense and that it equals $\partial_j f * g$, as desired. □

Folland 8.9

If $f \in L^p(\mathbb{R})$, the L^p derivative of f (which we call h) exists iff f is absolutely continuous on every bounded interval (perhaps after modification on a null set) and its pointwise derivatice f' is in L^p, in which case $h = f'$ a.e.

Proof. **Only If.** Let $[a,b]$ be a bounded interval since addition or deletion of one single point doesn't affect the result of integration. It is well known that we can construct a $g \in C_c^\infty$ such that $\text{Supp} g = [a,b]$ and $\int g = 1$ using the bump function.
Claim 1. For each $t > 0$, $|g(x)| \le C(1+|x|)^{-2}$ for some $C > 0$.

Proof of Claim. Since $g \in C_c$, by extreme value theorem we have $|g| \leq M$ for some $M > 0$. Consider $C := M(1 + \max(|a|, |b|))^2$. Then

$$C(1+|x|)^{-2} = M\left(\frac{1+\max(|a|,|b|)}{1+|x|}\right)^2 \geq M \geq |g(x)|$$

showing the claim. $\square$

Observation 2. Since $f \in L^p$ and claim 1 holds, theorem 8.15 gives $f * g_t \to f$ a.e. as $t \to 0$.
Observation 3. Let $t > 0$. Since $g \in C_c$, so is g_t, and thus $g_t \in L^q$ where q is the conjugate of p, and by problem 8.8 we have $(f * g_t)$ exists and

$$(f * g_t)' = h * g_t \to h \text{ a.e. as } t \to 0$$

where the limit is attained by applying theorem 8.15 since $h \in L^p$.
For convenience, we use $\{k_n\}_n$ to denote the sequence $\{h * g_{1/n}\}_n$. Since $h \in L^p[a, b]$, by theorem 8.14 a), $h * g_t \to h$ in $L^p[a, b]$, and in particular $\{k_n\}$ is bounded in $L^p[a, b]$. Since $[a, b]$ has finite measure $b - a$, by proposition 6.12 we have

$$\|k_n \chi_{[a,b]}\|_1 \leq \|k_n \chi_{[a,b]}\|_p (b-a)^{1-1/p}$$

and hence $\{k_n\}$ is bounded in $L^1[a, b]$. Notice that

$$f * g_{1/n}(x) - f * g_{1/n}(a) = \int_a^x (f * g_{1/n})' = \int_a^x h * g_{1/n}' = \int_a^b k_n \chi_{[a,x]}$$

and sending n to infinity and apply dominated convergence theorem (since $|k_n \chi_{[a,x]}| \leq |k_n|$ is bounded in $L^1[a, b]$) we obtain

$$f(x) - f(b) = \int_b^x h(t)dt$$

showing that f is abosolutely continuous on $[a, b]$ with a possible modification on a null set and that $f' = h$ a.e.
If. For $y > 0$, notice that

$$\frac{f(x+y) - f(x)}{y} - f'(x) = \frac{1}{y}\int_0^y [f'(x+t) - f'(x)]dt$$

Therefore

$$\begin{aligned}
\left\|\frac{f(x+y) - f(x)}{y} - f'(x)\right\|_p &= \left(\left[\int_0^y \frac{1}{y}[f'(x+t) - f'(x)]dt\right]^p dx\right)^{\frac{1}{p}} \\
&\leq \int_0^y \left(\int \left|\frac{f'(x+t) - f'(x)}{y}\right|^p dx\right)^{\frac{1}{p}} dt \\
&\leq \frac{1}{y}\int_0^y \left(\int |f'(x+t) - f'(x)|^p dx\right)^{\frac{1}{p}} dt \\
&= \frac{1}{y}\int_0^y \|\tau_{-t} f' - f'\|_p dt
\end{aligned}$$

where the first inequality is obtained by Minkowski's inequality for integrals. Since $\|\tau_{-t} f'\|_p = \|f'\|_p$, by triangular inequality we have $\|\tau_{-t} f' - f'\|_p \chi_{[0,y]} \leq 2\|f'\|_p \chi_{[0,y]} \in L^1$ for all $y > 0$ since $f' \in L^p$. Moreover, by proposition 8.5,

$$\lim_{t \to 0} \|\tau_{-t} f' - f'\|_p = 0 \quad (1)$$

so applying dominated convergence theorem, we get

$$\begin{aligned}\lim_{y\to 0^+}\left\|\frac{f(x+y)-f(x)}{y}-f'(x)\right\|_p &= \lim_{y\to 0^+}\frac{1}{y}\int_0^y \|\tau_{-t}f'-f'\|_p dt\\ &= \frac{1}{y}\int_0^y \lim_{y\to 0^+}\|\tau_{-t}f'-f'\|_p dt \quad (2)\end{aligned}$$

Let $\epsilon > 0$. By (1) we know that there is some $\delta > 0$ such that when $|t| < \delta$, $\|\tau_{-t}f' - f'\|_p < \epsilon$. Thus when $y < \delta$,

$$(2) < y\int_0^y \epsilon dt = \epsilon$$

showing that

$$\lim_{y\to 0^+}\left\|\frac{f(x+y)-f(x)}{y}-f'(x)\right\|_p = 0$$

The case where $y < 0$ is the same is we replace every $[0, y]$ with $[y, 0]$. It follows that

$$\lim_{y\to 0}\left\|\frac{f(x+y)-f(x)}{y}-f'(x)\right\|_p = 0$$

meaning that the L^p derivative of f exists and equals to its usual derivative. □

Folland 8.14

(Wirtinger's Inequality) If $f \in C^1([a, b])$ and $f(a) = f(b) = 0$, then

$$\int_a^b |f(x)|^2 dx \le \left(\frac{b-a}{\pi}\right)^2 \int_a^b |f'(x)|^2 dx$$

Proof. **Step 1.** We show that by change of variable it suffices to assume that $a = 0$, $b = \frac{1}{2}$. To see this, suppose the inequality holds for $a = 0$, $b = \frac{1}{2}$, i.e. for $f \in C^1([0, \frac{1}{2}])$ and $f(0) = f(\frac{1}{2}) = 0$,

$$\int_0^{1/2} |f(x)|^2 dx \le \left(\frac{1}{2\pi}\right)^2 \int_0^{1/2} |f'(x)|^2 dx \quad (1)$$

Now, given an $f \in C^1([a, b])$ with $f(a) = f(b) = 0$, we consider $g(x) := f(2(b-a)x + a)$. Since $2(b-a)x + a \in [a, b]$, $x \in [0, \frac{1}{2}]$, meaning that g is defined on $[0, \frac{1}{2}]$. Also since $2(b-a)x + a$ is clearly a C^1 function g is still C^1 as f. Thus $g \in C^1([0, \frac{1}{2}])$ and $g(0) = g(\frac{1}{2}) = 0$. Applying our assumption (1) we get

$$\int_0^{1/2} |g(x)|^2 dx \le \left(\frac{1}{2\pi}\right)^2 \int_0^{1/2} |g'(x)|^2 dx$$

Notice that by change of variable we have

$$LHS = \int_0^{1/2} |f(2(b-a)x + a)|^2 dx = \frac{1}{2(b-a)}\int_a^b |f(x)|^2 dx$$

and since $g'(x) = [f(2(b-a)x + a)]' = 2(b-a)f'(2(b-a)x + a)$, we also have

$$\begin{aligned}RHS &= \left(\frac{1}{2\pi}\right)^2 \int_0^{1/2} [2(b-a)]^2 f'(2(b-a)x + a) dx\\ &= \left(\frac{1}{2\pi}\right)^2 [2(b-a)]^2 \frac{1}{2(b-a)}\int_a^b |f'(x)|^2 dx\end{aligned}$$

and putting them together we get

$$\int_a^b |f(x)|^2 dx \le \left(\frac{b-a}{\pi}\right)^2 \int_a^b |f'(x)|^2 dx$$

as desired.
Step 2. Now that we have assume without loss of generality that f is defined on $[0, \frac{1}{2}]$, we extend f to $[-\frac{1}{2}, \frac{1}{2}]$ by setting $f(-x) = -f(x)$, and then extend f to be periodic on $\mathbb{R}$ with period 1. This is extension is well-defined at the overlapping points since $f(\frac{n}{2}) = 0$ for $n \in \mathbb{Z}$. We check that $f \in C^1(\mathbb{T})$.
We use $f'(\frac{1}{2})$ to denote the left one-sided derivative of f at $\frac{1}{2}$, and use $f'(-\frac{1}{2})$ to denote the right one-sided derivatice of f at $-\frac{1}{2}$. For $h > 0$, we have

$$\left|\frac{f(\frac{1}{2}) - f(\frac{1}{2} - h)}{h} - f'\left(\frac{1}{2}\right)\right| = \left|\frac{-f(-\frac{1}{2}) + f(-\frac{1}{2} + h)}{h} - f'\left(\frac{1}{2}\right)\right|$$

and thus

$$0 = \lim_{h\to 0}\left|\frac{f(\frac{1}{2}) - f(\frac{1}{2} - h)}{h} - f'\left(\frac{1}{2}\right)\right| = \lim_{h\to 0}\left|\frac{f(-\frac{1}{2} + h) - f(-\frac{1}{2})}{h} - f'\left(\frac{1}{2}\right)\right|$$

showing that $f'(-\frac{1}{2}) = f'(\frac{1}{2})$, and we denote this common quantity by L. Since f is C^1,

$$\lim_{x\to 1/2^-} f'(x) = L = \lim_{x\to -1/2^+} f'(x)$$

and the way we extend f makes sure that this result holds for $[\frac{2n-1}{2}, \frac{2n+1}{2}]$ for every $n \in \mathbb{Z}$. It follows that f is C^1 at the endpoint of $\mathbb{T}$. f is C^1 in the interior of $\mathbb{T}$ by assumption, so $f \in C^1(\mathbb{T})$.
Step 3. We want to use Parseval's identity to conclude the result.
By proposition 7.9, $C(\mathbb{T})$ is dense in $L^2(\mathbb{T})$ and in particular $C(\mathbb{T}) \subset L^2(\mathbb{T})$. Hence $f, f' \in L^2(\mathbb{T})$. By Parseval's identity, $\|f\|_2 = \|\hat{f}\|_2$ and $\|f'\|_2 = \|\hat{f}'\|_2$. Thus

$$\int_0^{1/2} |f(x)|^2 dx = \int_{\mathbb{T}} |f(x)|^2 dx = \|\hat{f}\|_2^2 = \sum_{k\in\mathbb{Z}} |\hat{f}(k)|^2$$

and using integration by parts we have

$$\begin{aligned}\hat{f}(k) &= \int_{\mathbb{T}} f(x)e^{-2\pi ikx}dx = \int_0^{1/2} f(x)e^{-2\pi ikx} \\ &= \frac{1}{2\pi ik} f(x)e^{-2\pi ikx}\Big|_0^{1/2} + \int_0^{1/2} \frac{1}{2\pi ik} f'(x)e^{-2\pi ik}dx \\ &= 0 + \frac{1}{2\pi ik}\int_0^{1/2} f'(x)e^{-2\pi ikx} = \frac{1}{2\pi ik}\hat{f}'(k)\end{aligned}$$

implying $|\hat{f}(k)| = |\frac{1}{2\pi k}||\hat{f}'(k)|$ and thus

$$\begin{aligned}\int_0^{1/2} |f(x)|^2 dx = \|f\|_2^2 = \|\hat{f}\|_2^2 = \sum_{k\in\mathbb{Z}} |\hat{f}(k)|^2 \le \sum_{k\in\mathbb{Z}} \left(\frac{1}{2\pi k}\right)^2 |\hat{f}'(k)| \le \sum_{k\in\mathbb{Z}} \left(\frac{1}{2\pi}\right)^2 |\hat{f}'(k)|^2 \\ = \left(\frac{1}{2\pi}\right)^2 \|\hat{f}'\|_2^2 = \left(\frac{1}{2\pi}\right)^2 \|f'\|_2^2 = \left(\frac{1}{2\pi}\right)^2 \int_0^{1/2} |f'(x)|^2 dx\end{aligned}$$

showing the result as desired. This completes the proof. $\square$

Folland 8.26

The aim of this exercise is to show that the inverse Fourier transform of $e^{-2\pi|\xi|}$ on $\mathbb{R}^n$ is

$$\phi(x) = \frac{\Gamma(\frac{1}{2}(n+1))}{\pi^{(n+1)/2}(1+|x|^2)^{-(n+1)/2}}$$

(a) If $\beta > 0$, $e^{-\beta} = \pi^{-1}\int_{-\infty}^{\infty}(1+t^2)^{-1}e^{-i\beta t}dt$.
(b) If $\beta \geq 0$, $e^{-\beta} = \int_0^{\infty}(\pi s)^{-1/2}e^{-s}e^{-\beta^2/4s}ds$.
(c) Let $\beta = 2\pi|\xi|$ where $\xi \in \mathbb{R}^n$; the the formula in (b) expresses $e^{-2\pi|\xi|}$ as a superposition of dilated Gauss kernels. Use proposition 8.24 again to derive the asserted formula for ϕ.

Proof. (a) By (8.37), we have $\phi(x) = \frac{1}{\pi(1+x^2)}$ and therefore

$$e^{-2\pi|\xi|} = \Phi(\xi) = \hat{\phi}(\xi) = \int_{\mathbb{R}} \frac{e^{-2\pi i\xi t}}{\pi(1+t^2)}dt \quad (1)$$

If $\beta > 0$, $\beta = 2\pi\xi$ for some $\xi > 0$, and then plugging in (1) we get

$$e^{-\beta} = \int_{\mathbb{R}} \frac{e^{-i\beta t}}{\pi(1+t^2)}dt = \pi^{-1}\int_{-\infty}^{\infty}(1+t^2)^{-1}e^{-i\beta t}dt$$

as desired.

(b) Notice that

$$\begin{aligned} e^{-\beta} &= \frac{1}{\pi}\int \frac{1}{1+t^2}e^{-i\beta t} = \frac{1}{\pi}\int_0^{\infty} e^{-i\beta t}\left(\int_0^{\infty} e^{-(1+t^2)s}ds\right)dt \\ &= \frac{1}{\pi}\int_0^{\infty}\int_{-\infty}^{\infty} e^{-(1+t^2)s}e^{-i\beta t}dtds \quad (1) \end{aligned}$$

where the last step holds since $e^{-(1+t^2)s}e^{-i\beta t} \in L^+$ and Tonelli's theorem justifies the interchange of order of integrals. Now substituting $z = \beta t/2\pi$, we obtain

$$\begin{aligned} (1) &= \frac{1}{\pi}\int_0^{\infty} e^{-s}\int_{-\infty}^{\infty} \frac{2\pi}{\beta}e^{-s\frac{4\pi^2 z^2}{\beta^2}}e^{-2\pi iz}dtds = \frac{2}{\beta}\int_0^{\infty} e^{-s}\int_{-\infty}^{\infty} e^{\frac{4\pi^2 s}{\beta^2}z^2}e^{-2\pi iz}dzds \\ &= \frac{2}{\beta}\int_0^{\infty} e^{-s}\mathcal{F}(e^{-\frac{4\pi^2 s}{\beta^2}z^2})(1)ds = \frac{2}{\beta}\int_0^{\infty} e^{-s}\left(\frac{4\pi s}{\beta^2}\right)^{-1/2}e^{-\pi\frac{\beta^2}{4\pi s}}ds \quad \text{(Prop 8.24)} \\ &= \frac{2}{\beta}\int_0^{\infty} e^{-s}\left(\frac{\beta^2}{4\pi s}\right)^{1/2}e^{-\frac{\beta^2}{4s}} = \int_0^{\infty}(\pi s)^{-1/2}e^{-s}e^{-\beta^2/4s}ds \end{aligned}$$

showing the result as desired.

(c) Since $\beta = 2\pi|\xi|$, now

$$e^{-\beta} = e^{-2\pi|\xi|} = \int_0^{\infty}(\pi s)^{-1/2}e^{-s}e^{-\frac{4\pi^2|\xi|^2}{4s}}ds = \int_0^{\infty}(\pi s)^{-1/2}e^{-s}e^{-\frac{\pi^2|\xi|^2}{s}}ds$$

And computing the inverse Fourier transform we have

$$(e^{-2\pi|\xi|})^{\vee}(x) = \int_{\mathbb{R}^n} e^{-2\pi|\xi|}e^{2\pi i\xi\cdot x}d\xi = \int_{\mathbb{R}^n}\int_0^{\infty}(\pi s)^{-1/2}e^{-s}e^{-\frac{\pi^2|\xi|^2}{s}}e^{2\pi i\xi\cdot x}dsd\xi \quad (2)$$

Note that $|(\pi s)^{-1/2}e^{-s}e^{-\frac{\pi^2|\xi|^2}{s}}e^{2\pi i\xi\cdot x}| \in L^+$, so by Tonelli's theorem we have

$$\begin{aligned}
&\int_{\mathbb{R}^n\times[0,\infty)} |(\pi s)^{-1/2}e^{-s}e^{-\frac{\pi^2|\xi|^2}{s}}e^{2\pi i\xi\cdot x}|ds\otimes\xi \\
=&\int_0^\infty\int_{\mathbb{R}^n} |(\pi s)^{-1/2}e^{-s}e^{-\frac{\pi^2|\xi|^2}{s}}e^{2\pi i\xi\cdot x}|d\xi ds \\
=&\int_0^\infty\int_{\mathbb{R}^n} |(\pi s)^{-1/2}e^{-s}e^{-\frac{\pi^2|\xi|^2}{s}}|d\xi ds \\
=&\int_0^\infty |(\pi s)^{-1/2}e^{-s}|\left(\int_{\mathbb{R}^n}|e^{-\frac{\pi^2|\xi|^2}{s}}|d\xi\right)ds \\
=&\int_0^\infty |(\pi s)^{-1/2}e^{-s}|(\pi\cdot\frac{s}{\pi^2})^{2/n}ds \\
=&\int_0^\infty |\pi s|^{-1/2}|e^{-s}|\left(\frac{s}{\pi}\right)^{n/2}ds \quad (\text{since } s>0) \\
=&\pi^{-\frac{1+n}{2}}\int_0^\infty s^{\frac{n+1}{2}-1}e^{-s}ds = \pi^{-\frac{1+n}{2}}\Gamma(\frac{n+1}{2})<\infty
\end{aligned}$$

where the last step is by proposition 2.55. Then $(\pi s)^{-1/2}e^{-s}e^{-\frac{\pi^2|\xi|^2}{s}}e^{2\pi i\xi\cdot x}\in L^1$, and we can apply Fubini-Tonelli to interchange the order of integral in (2). Thus we get

$$\begin{aligned}
(2) &= \int_0^\infty (\pi s)^{-1/2}e^{-s}\left(\int_{\mathbb{R}^n} e^{-\frac{\pi^2|\xi|^2}{s}}e^{2\pi i\xi\dot{x}}d\xi\right)ds \\
&= \int_0^\infty (\pi s)^{-1/2}e^{-s}(e^{-\frac{\pi^2|\xi|^2}{s}})^\vee(x)ds \\
&= \int_0^\infty (\pi s)^{-1/2}e^{-s}\left(\frac{s}{\pi}\right)^{n/2}e^{-s|x|^2}ds \quad (\text{Prop } 8.24) \\
&= \frac{1}{\pi^{\frac{n+1}{2}}}\int_0^\infty s^{\frac{n-1}{2}}e^{-s(1+|x|^2)}ds \quad (3)
\end{aligned}$$

Substituting $z=-s(1+|x|^2)$, we get

$$(3) = \frac{1}{\pi^{\frac{n+1}{2}}}\left(\frac{1}{1+|x|^2}\right)^{\frac{n+1}{2}}\int_0^\infty z^{\frac{n+1}{2}-1}e^{-s}dz = \frac{\Gamma(\frac{1}{2}(n+1))}{\pi^{(n+1)/2}(1+|x|^2)^{-(n+1)/2}}$$

as desired. This finishes the proof.

$\square$

Folland 8.30

If $f\in L^1(\mathbb{R}^n)$, f is continuous at 0, and $\hat{f}\geq 0$, then $\hat{f}\in L^1$.

Proof. Observe that

$$\begin{aligned}\|\hat{f}\|_1 &= \int |\hat{f}(\xi)|d\xi = \int \hat{f}(\xi)d\xi \\ &= \int \lim_{t\to 0} \hat{f}(\xi)e^{-\pi|t\xi|^2}e^{2\pi i\xi\cdot 0}d\xi \\ &\leq \lim_{t\to 0}\int \hat{f}(\xi)e^{-\pi|t\xi|^2}e^{2\pi i\xi\cdot 0}d\xi \quad \text{(Fatou's Lemma)} \\ &= f(0) < \infty\end{aligned}$$

The last equality holds by Theorem 8.35 and the fact that Gauss kernel fits the theorem; the last inequality holds since f is continuous and thus bounded at 0. ☐

9. Chapter 9-Elements of Distribution Theory

Folland 9.6

If f is absolutely continuous on compact subsets of an interval $U \subset \mathbb{R}$, the distribution derivative $f' \in \mathcal{D}'(U)$ coincides with the pointwise (a.e.-defined) derivative of f.

Proof. Let f' be the distribution derivative and g pointwise a.e.-defined derivative. For any $\phi \in C_c^\infty(U)$, $\mathrm{Supp}\phi = K$ for some K compact and thus f is absolutely continuous on K. It follows that

$$\int_K \phi f' = -\int_K f\phi' = -\int_K f d\phi = \int \phi df = \int_K \phi g \quad (1)$$

by absolute continuity of f and properties of distribution derivative.
Since U is an open interval, we may assume $U = (a,b)$, and let $K_n := [a+\frac{1}{n}, b-\frac{1}{n}]$. Taking $\phi = \chi_{K_n}$ in (1) we obtain $f' = g$ a.e. on K_n. We may suppose $E_n \subset K_n$ is the set on which f' and g don't agree, and hence E_n is a null set for each n. Since $U = \bigcup_1^\infty K_n$, f' and g disagree on at most $\bigcup_1^\infty E_n$ which is still a null set. Thus f' and g agree a.e. on U, as desired. ☐