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Progressive iterative approximation and bases with the fastest convergence rates

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Abstract

All normalized totally positive bases satisfy the progressive iterative approximation property. The normalized B-basis has optimal shape preserving properties and we prove that it satisfies the progressive iterative approximation property with the fastest convergence rates. A similar result for tensor product surfaces is also derived.

Keywords: Progressive iterative approximation; B-basis; Total positivity

1. Introduction

Qi et al. as well as de Boor showed (see (Qi et al., 1975; de Boor, 1979)) that the uniform cubic B-spline basis satisfied a property known as *progressive iteration approximation* property. This property can be proved for more general blending bases. Let us recall that a basis $(u_0,\dots,u_m)$ defined on a set I formed by nonnegative functions and such that $\sum_{i=0}^m u_i(\mathbf{x})=1$ for all $\mathbf{x}\in I$ is called a *blending basis*. Given a sequence of points $\{P_i\}_{i=0}^m$ whose ith point is assigned to a parameter value t_i, $i=0,1,\dots,m$, and a blending basis $(u_0,\dots,u_m)$, we generate a starting curve as $\gamma^0(t)=\sum_{i=0}^m P_i^0 u_i(t)$ where $P_i^0=P_i$ for $i=0,1,\dots,m$. Then, by calculating the adjusting vector for each control point $\Delta_i^0=P_i^0-\gamma^0(t_i)$, and taking $P_i^1=P_i^0+\Delta_i^0$ for $i=0,1,\dots,m$ we get the curve $\gamma^1(t)=\sum_{i=0}^m P_i^1 u_i(t)$. Iterating this process we can get a sequence of curves $\{\gamma^k\}_{k=0}^\infty$. Then, when the curve sequence converges to a curve interpolating the given initial sequence of points the initial curve is said to have the *progressive iteration approximation* property.

In (Qi et al., 1975; de Boor, 1979), the progressive iteration approximation for the uniform cubic B-spline basis was considered. Lin et al. in (2003) showed that the nonuniform cubic B-spline basis also satisfied the progressive iteration approximation property and extended the property for surfaces, showing that the nonuniform cubic B-spline tensor product surface satisfied such property. In (Lin et al., 2005) it was proved that curves and tensor product surfaces

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generated by normalized totally positive bases satisfy the progressive iteration approximation property. Let us recall that the *collocation matrix* of a system $(u_0(t), \dots, u_m(t))$ at the points $(t_i)_{i=0}^r$ in $\mathbf{R}$ is given by

$$M\begin{pmatrix} u_0, \dots, u_m \\ t_0, \dots, t_r \end{pmatrix} = \big(u_j(t_i)\big)_{0\leqslant i\leqslant r}^{0\leqslant j\leqslant m}.$$

Obviously, $(u_0, \dots, u_m)$ is blending if and only if all its collocation matrices are stochastic (that is, nonnegative and such that the sum of the elements of each row is 1). A matrix is *totally positive* (TP) if all its minors are nonnegative and a basis $(u_0, \dots, u_m)$ is *totally positive* (TP) when all its collocation matrices are TP. A blending basis that is TP is said to be a normalized totally positive (NTP) basis. It is well known that (cf. (Goodman, 1989; Peña, 1999)) the bases providing shape preserving representations are precisely the NTP bases and that, by Theorem 4.2 of (Carnicer and Peña, 1994), a space with a normalized totally positive basis always has a unique normalized B-basis, which is the basis with optimal shape preserving properties. For instance the Bernstein basis is the normalized B-basis in the case of the space of polynomials of degree at most n on a compact interval (Carnicer and Peña, 1993) and the B-splines form the normalized B-basis of their corresponding space (Carnicer and Peña, 1994).

In this paper we prove that the normalized B-basis is the NTP basis with the fastest convergence rates and that the tensor product of normalized B-bases also present the fastest convergence rates. The proofs are given in Section 3 and they also show that the smallest eigenvalue of a collocation matrix of the normalized B-basis is always greater than or equal to the smallest eigenvalue of the corresponding collocation matrix of another NTP basis. In Section 4 we include some numerical experiments comparing the Bernstein basis with another well known NTP basis of the space of polynomials: the Said–Ball basis (see (Dejdumrong and Phien, 2000; Goodman and Said, 1991)). This basis generalizes the basis used by Ball (1974, 1975, 1977).

2. Bases with the fastest convergence rates

First let us consider the progressive iteration approximation property in the case of curves. Let $(u_0, \dots, u_m)$ be a normalized totally positive basis. Then, considering a sequence of points $\{P_i\}_{i=0}^m$ in $\mathbf{R}^2$ or $\mathbf{R}^3$ we can define a parametric curve as

$$\gamma(t) = \sum_{i=0}^m P_i u_i(t).$$

Now we parameterize the control points P_i with the real increasing sequence $t_0 < t_1 < \cdots < t_m$, where the parameter t_i is assigned to the control point P_i for $i = 0, 1, \dots, m$. Then we have the starting curve

$$\gamma^0(t) = \sum_{i=0}^m P_i^0 u_i(t)$$

of the sequence where $P_i^0 = P_i$ for $i = 0, 1, \dots, m$. The remaining curves of the sequence, $\gamma^{k+1}(t)$ for $k \geqslant 0$, can be calculated as follows

$$\gamma^{k+1}(t) = \sum_{i=0}^m P_i^{k+1} u_i(t),$$

with

$$\begin{aligned} \Delta_i^k &= P_i - \gamma^k(t_i), \\ P_i^{k+1} &= P_i^k + \Delta_i^k, \end{aligned} \tag{1}$$

for $i = 0, 1, \dots, m$. Developing formula (1) we get

$$\Delta_j^k = \Delta_j^{k-1} - \sum_{i=0}^n \Delta_i^{k-1} u_i(t_j),$$

for $j = 0, 1, \dots, m$. Hence, the iterative process can be written in matrix form in the following way:

$$\big[\Delta_0^k, \Delta_1^k, \dots, \Delta_m^k\big]^\top = (I - B)\big[\Delta_0^{k-1}, \Delta_1^{k-1}, \dots, \Delta_m^{k-1}\big]^\top, \tag{2}$$

where I is the identity matrix of $n+1$ order and B is the collocation matrix of the basis $(u_0, \dots, u_m)$ at $t_0 < t_1 < \cdots < t_m$. Given a square matrix A, we denote by $\rho(A)$ its *spectral radius*, that is, the maximal absolute value of its eigenvalues. Then the iterative process (2) converges when $\rho(I-B) < 1$ and its speed depends on $\rho(I-B)$. In particular B must be a nonsingular matrix. In Theorem 2.1 of (Lin et al., 2005) the authors proved the following result showing that NTP bases can be used for progressive iterative approximations.

Theorem 1. *The progressive iterative approximation process* (*see* (2)) *converges for any nonsingular collocation matrix B of an NTP basis.*

We can derive an algorithm for the progressive iterative approximation taking into account the following facts:

- Considering a tolerance *Tol* and a maximum number of iterations *Kmax*.
- Iterating the process of calculating the curves γ^k until the first k such that $\max_i \|\Delta_i^k\| < Tol$ or the maximum number of iterations *Kmax* is exceeded.

Now let us consider the case of surfaces. Let $(u_0, \dots, u_m)$ and $(\bar{u}_0, \dots, \bar{u}_n)$ be two normalized totally positive bases. Then, given a sequence of control points $\{P_{ij}\}_{0\leqslant i\leqslant m}^{0\leqslant j\leqslant n}$ in $\mathbf{R}^3$ we can define a parametric tensor product surface as

$$S(u,v) = \sum_{i=0}^{m}\sum_{j=0}^{n} P_{ij}\, u_i(u)\bar{u}_j(v).$$

Now, analogously to the curve case, we parameterize the control points P_{ij} with the real increasing sequences $x_0 < x_1 < \cdots < x_m$ and $y_0 < y_1 < \cdots < y_n$, where the parameter (x_i, y_j) is assigned to the control point P_{ij} for $i = 0, 1, \dots, m$ and $j = 0, 1, \dots, n$. Then we have the starting surface

$$S^0(x,y) = \sum_{i=0}^{m}\sum_{j=0}^{n} P_{ij}^0 u_i(x)\bar{u}_j(y)$$

of the sequence where $P_{ij}^0 = P_{ij}$ for $i = 0, 1, \dots, m$ and $j = 0, 1, \dots, n$. The remaining tensor product surfaces of the sequence, $S^{k+1}(x,y)$ for $k \geqslant 0$, can be obtained as follows

$$S^{k+1}(x,y) = \sum_{i=0}^{m}\sum_{j=0}^{n} P_{ij}^{k+1}\, u_i(x)\bar{u}_j(y),$$

with

$$\begin{aligned} \Delta_{ij}^k &= P_{ij} - S^k(x_i, y_j), \\ P_{ij}^{k+1} &= P_{ij}^k + \Delta_{ij}^k, \end{aligned} \tag{3}$$

for $i = 0, 1, \dots, m$ and $j = 0, 1, \dots, n$. Developing formula (3) we obtain

$$\Delta_{rs}^k = \Delta_{rs}^{k-1} - \sum_{i=0}^{m}\sum_{j=0}^{n} \Delta_{ij}^{k-1}\, u_i(x_r)\bar{u}_j(y_s), \tag{4}$$

for $r = 0, 1, \dots, m$ and $s = 0, 1, \dots, n$.

Let us recall that, given two matrices $A = (a_{ij})_{1\leqslant i\leqslant m}^{1\leqslant j\leqslant n}$, $B = (b_{ij})_{1\leqslant i\leqslant p}^{1\leqslant j\leqslant q}$, their *Kronecker product* is denoted by $A \otimes B$ and is defined to be the block matrix

$$A \otimes B = \begin{pmatrix} a_{11}B & \dots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \dots & a_{mn}B \end{pmatrix},$$

and so $A \otimes B$ is an $(mp) \times (nq)$ matrix. Taking into account that the collocation matrix of the tensor product basis $(u_i \bar{u}_j)_{0 \leqslant i \leqslant m}^{0 \leqslant j \leqslant n}$ is the Kronecker product of the corresponding collocation matrices, we can write the iterative process (4) in matrix form $\Delta^k = (I - B)\,\Delta^{k-1}$, where

$$\Delta^j = \left[\Delta_{00}^j, \Delta_{01}^j, \dots, \Delta_{0n}^j, \Delta_{10}^j, \Delta_{11}^j, \dots, \Delta_{1n}^j, \dots, \Delta_{m0}^j, \Delta_{m1}^j, \dots, \Delta_{mn}^j\right]^\top,$$

I is the identity matrix of $((m+1)\cdot(n+1)) \times ((m+1)\cdot(n+1))$ order, and B is the Kronecker product of the collocation matrices of the bases $(u_0, \dots, u_m)$ and $(\bar{u}_0, \dots, \bar{u}_n)$ at $x_0 < x_1 < \cdots < x_m$ and $y_0 < y_1 < \cdots < y_n$, respectively. Hence, an algorithm for the progressive iterative approximation of tensor product surfaces can be also derived analogously to the univariate case (see also (Lin et al., 2005)).

3. Main result

In this section we prove the optimality of the normalized B-basis with respect to the progressive iterative approximation. Previously, we recall some results of Linear Algebra that will be used later in order to get a paper as self-contained as possible.

By the well known Perron–Frobenius theorem (cf. Theorem 4.2 of Chapter I of (Minc, 1988)), a nonnegative matrix has a nonnegative eigenvalue $r = \rho(A)$. Let us recall that, if $C = (c_{ij})_{1 \leqslant i,j \leqslant n}$ is a complex matrix and $A = (a_{ij})_{1 \leqslant i,j \leqslant n}$ is a nonnegative matrix such that $|c_{ij}| \leqslant a_{ij}$ for all i, j, then A is said to *dominate* C. We shall also use the notation $|C| \leqslant A$: each entry of $|C| := (|c_{ij}|)_{1 \leqslant i,j \leqslant n}$ is less than or equal to the corresponding entry of A. The following result is due to Wienlandt (see Corollary 2.1 of Chapter II of (Minc, 1988)):

Theorem 2. *Let M be a nonnegative matrix with maximal eigenvalue r, and let C be a complex matrix dominated by M, then $r = \rho(M) \geqslant \rho(C)$.*

The following result collects two properties of TP matrices which will be used in the proof of the main theorem. The first part corresponds to Corollary 6.6 of (Ando, 1987) and the second part to Theorem 3.3 of (Ando, 1987).

Theorem 3. *Let A be a nonsingular TP $n \times n$ matrix. Then*:

(i) *All the eigenvalues of A are positive.*
(ii) *Given the $n \times n$ diagonal matrix $J := \mathrm{diag}(1, -1, 1, \dots, (-1)^{n-1})$, the matrix $JA^{-1}J$ is TP.*

The next theorem is the main result of the paper.

Theorem 4. *Given a space U with an NTP basis, the normalized B-basis of U provides a progressive iterative approximation with the fastest convergence rates among all NTP bases of U.*

Proof. The matrices associated to the progressive iterative approximation are of the form $I - B$, where B is a nonsingular collocation matrix of the NTP basis (see Section 2). Since the matrix B is TP, all its eigenvalues are positive by Theorem 3(i). Then $\rho(I - B)$ will be minimum when the smallest eigenvalue of B is maximum.

By Theorem 4.2(i) of (Carnicer and Peña, 1994) U has a unique normalized B-basis $(b_0, \dots, b_n)$. If $(v_0, \dots, v_n)$ is another NTP basis of U, then by Theorem 4.2(ii) of (Carnicer and Peña, 1994) there exists a stochastic TP matrix K such that

$$(v_0, \dots, v_n) = (b_0, \dots, b_n)K. \tag{5}$$

Let $t_0 < t_1 < \cdots < t_n$ in the domain of the functions of U such that $V := M\binom{v_0, \dots, v_n}{t_0, \dots, t_n}$ and $B := M\binom{b_0, \dots, b_n}{t_0, \dots, t_n}$ are nonsingular. By (5), $V = BK$. We shall prove that the smallest eigenvalue of B is greater than the smallest eigenvalue of V.

Let us consider the $(n+1) \times (n+1)$ matrix $J := \mathrm{diag}(1, -1, 1, \dots, (-1)^n)$. Since $J^{-1} = J$, $JV^{-1}J$ is similar to V^{-1} and $JB^{-1}J$ is similar to B^{-1}, it is sufficient to prove that

$$\rho\big(JV^{-1}J\big) \geqslant \rho\big(JB^{-1}J\big). \tag{6}$$

By Theorem 3(ii), the matrices $JV^{-1}J$ and $JB^{-1}J$ are also TP and so nonnegative. Then, by Theorem 2 and since $JV^{-1}J = J(BK)^{-1}J = (JK^{-1}J)(JB^{-1}J)$, in order to prove the theorem it is sufficient to see that

$$JB^{-1}J \leqslant JV^{-1}J = \big(JK^{-1}J\big)\big(JB^{-1}J\big). \tag{7}$$

Since K is TP and stochastic, by Theorem 2.6 of (Peña, 1999) we can write

$$K = F_{n-1}F_{n-2}\dots F_1G_1\dots G_{n-2}G_{n-1}, \tag{8}$$

with

$$F_i = \begin{pmatrix} 1 &&&&&& \\ 0 & 1 &&&&& \\ & \ddots & \ddots &&&& \\ && 0 & 1 &&& \\ &&& \alpha_{i+1,1} & 1-\alpha_{i+1,1} && \\ &&&& \ddots & \ddots & \\ &&&&& \alpha_{n,n-i} & 1-\alpha_{n,n-i} \end{pmatrix}$$

and

$$G_i = \begin{pmatrix} 1 & 0 &&&&& \\ & \ddots & \ddots &&&& \\ && 1 & 0 &&& \\ &&& 1-\alpha_{1,i+1} & \alpha_{1,i+1} && \\ &&&& \ddots & \ddots & \\ &&&&& 1-\alpha_{n-i,n} & \alpha_{n-i,n} \\ &&&&&& 1 \end{pmatrix},$$

where, $\forall(i,j)$, $0 \leqslant \alpha_{i,j} < 1$.

If we denote by $U(\lambda_i)$ (resp., $L(\lambda_i)$) the *bidiagonal*, nonsingular and upper (resp., lower) triangular matrix with at most one nonzero offdiagonal element

$$U(\lambda_i) = \begin{pmatrix} 1 & 0 &&&&& \\ & 1 & 0 &&&& \\ && \ddots & \ddots &&& \\ &&& 1-\lambda_i & \lambda_i && \\ &&&& \ddots & \ddots & \\ &&&&& 1 & 0 \\ &&&&&& 1 \end{pmatrix}, \quad 0 \leqslant \lambda_i < 1 \tag{9}$$

(resp.,

$$L(\lambda_i) = \begin{pmatrix} 1 &&&&&& \\ 0 & 1 &&&&& \\ & \ddots & \ddots &&&& \\ && \lambda_i & 1-\lambda_i &&& \\ &&& \ddots & \ddots && \\ &&&&& 1 & \\ &&&&& 0 & 1 \end{pmatrix}, \quad 0 \leqslant \lambda_i < 1), \tag{10}$$

then we can write

$$F_i = L(\alpha_{i+1,1})\dots L(\alpha_{n,n-i}) \tag{11}$$

and

$$G_i = U(\alpha_{n-i,n})\dots U(\alpha_{1,i+1}). \tag{12}$$

From (8), (11) and (12), we conclude that $K = \Pi_{i=1}^{r} E_i$, where r is a positive integer and E_i is a matrix of the form (9) or (10). Therefore

$$JK^{-1}J = \prod_{i=1}^{r} \left(JE_i^{-1}J\right).$$

We have seen that $JB^{-1}J$ is nonnegative and we can observe that all matrices $JE_i^{-1}J$ are also nonnegative. So, in order to prove (7), it is sufficient to see that if we multiply a nonnegative matrix A by a matrix of the form $JE_i^{-1}J$, we obtain a new matrix whose entries are greater than or equal to the entries of A. But this last property is an immediate consequence of the form of each $JE_i^{-1}J$,

$$\begin{pmatrix} 1 & 0 & & & & & & \\ & 1 & 0 & & & & & \\ & & \ddots & \ddots & & & & \\ & & & \frac{1}{1-\lambda_i} & \frac{\lambda_i}{1-\lambda_i} & & & \\ & & & & \ddots & \ddots & & \\ & & & & & & 1 & 0 \\ & & & & & & & 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 & & & & & & & \\ 0 & 1 & & & & & & \\ & \ddots & \ddots & & & & & \\ & & & \frac{\lambda_i}{1-\lambda_i} & \frac{1}{1-\lambda_i} & & & \\ & & & & \ddots & \ddots & & \\ & & & & & & 1 & \\ & & & & & & 0 & 1 \end{pmatrix},$$

with $0 \leqslant \lambda_i < 1$, and of the fact that $1/(1-\lambda_i) \geqslant 1$. □

Let us now consider the case of tensor product surfaces. The following well known result can be found in Theorem 4.2.12 of (Horn and Johnson, 1994; Stephen, 1979):

Theorem 5. *The mn eigenvalues of the Kronecker product of an $m \times m$ matrix and an $n \times n$ matrix are given by the mn products of an eigenvalue of one matrix by an eigenvalue of the other matrix.*

From Theorem 5 and taking into account that the collocation matrix of the tensor product of two bases is the Kronecker product of the corresponding collocation matrices, we can deduce the following consequence from the proof of Theorem 4: the smallest eigenvalue of the Kronecker product of the collocation matrices of two normalized B-bases is greater than or equal to the smallest eigenvalue of the Kronecker product of the corresponding collocation matrices of any two NTP bases. Hence we can derive the following result.

Corollary 6. *Given spaces U, V with NTP bases, the tensor product of the normalized B-bases of U, V provides a progressive iterative approximation with the fastest convergence rates among all bases which are tensor product of NTP bases of U, V.*

Although the Kronecker product of two matrices is a matrix of much higher size, it arises in a natural way as the collocation matrix of the tensor product basis and allows us to guarantee convergence of progressive iterative approximation for nonsingular collocation matrices of the tensor product basis (see also (Lin et al., 2005)). In contrast, it is an open problem to know convergence conditions for the total degree case of bivariate polynomials, that is, bivariate polynomials defined on a triangle.

4. Numerical tests

The Ball basis was introduced for cubic polynomials by Ball (see Ball (1974, 1975, 1977)). A generalization of this basis for higher degree polynomials has been called Said–Ball basis (see (Delgado and Peña, 2006)).

Definition 7. The Said–Ball basis $(s_0^m(t), \dots, s_m^m(t))$, $m \geqslant 1$, $t \in [0,1]$, is defined by:

$$s_i^m(t) = \binom{\lfloor \frac{m}{2} \rfloor + i}{i} t^i (1-t)^{\lfloor \frac{m}{2} \rfloor + 1}, \quad 0 \leqslant i \leqslant \left\lfloor \frac{m-1}{2} \right\rfloor,$$

$$s_i^m(t) = \binom{\lfloor \frac{m}{2} \rfloor + m - i}{m-i} t^{\lfloor \frac{m}{2} \rfloor + 1} (1-t)^{m-i}, \quad \left\lfloor \frac{m}{2} \right\rfloor + 1 \leqslant i \leqslant m,$$

and, if m is even

$$s^m_{\frac{m}{2}}(t) = \binom{m}{\frac{m}{2}} t^{\frac{m}{2}} (1-t)^{\frac{m}{2}}.$$

On one hand, in (Goodman and Said, 1991) it was proved that the Said–Ball basis of the space of polynomials of degree less than or equal to m (Π_m) is NTP when m is even and in Proposition 3 of (Delgado and Peña, 2006) we proved the same result when m is odd. On the other hand, in (Carnicer and Peña, 1993) it was proved that the Bernstein basis is the normalized B-basis of Π_m. So, by Theorem 4 the Bézier curves provide a progressive iterative approximation with faster convergence rate than the Said–Ball curves. In this section we compare numerically the convergence rates of the Bézier and Said–Ball curves sequences with some examples. In all the examples, we have used a uniform distribution of the parameters t_i ($i = 0, 1, \dots, m$), that is, $t_i - t_{i-1} = 1/m$ for all $i > 0$. With additional numerical experiments which are not included, we have checked that this choice is not the best in order to get faster convergence, but we have preferred it by its simplicity. The numerical results show a slow convergence. But if we consider other spaces more adequate to the examples instead of the space of polynomials (for instance, spline or rational spaces), the convergence may be much faster as shown in (Lin et al., 2005).

First let us consider the Lemniscate of Gerono given by the parametric equations

$$\begin{cases} x(t) = \cos t, \\ y(t) = \sin t \cdot \cos t, \end{cases}$$

$t \in [0, 2\pi]$. We took a sequence of 11 points $\{P_i\}_{i=0}^{10}$ from the Lemniscate of Gerono in the following way

$$P_i = \left(x\left(-\frac{\pi}{2} + i \cdot \frac{2\pi}{10}\right), y\left(-\frac{\pi}{2} + i \cdot \frac{2\pi}{10}\right)\right)^T, \quad i = 0, 1, \dots, 10.$$

Starting with these control points we fit the Lemniscate of Gerono by two sequences of curves, generated by progressive iteration approximation of a Bézier curve and a Said–Ball curve, respectively. In Fig. 1 we can see the starting, the 20th and the 40th curves of the sequence of Bézier curves, while in Fig. 2 we can see the starting, the 20th and the 40th curves of the sequence of Said–Ball curves. In these figures we can see that the sequence of Bézier curves adjust to the lemniscate of Gerono with a faster convergence rate than the sequence of Said–Ball curves as we already know theoretically (see Theorem 4 in the previous section). This fact can also be checked in Table 1 where we have listed the fitting errors of both curves sequences after specific iteration levels. The fitting error at the step $k \geqslant 0$ is taken as the maximum Euclidean norm of the adjusting vectors $\Delta_0^k, \Delta_1^k, \dots, \Delta_{10}^k$.

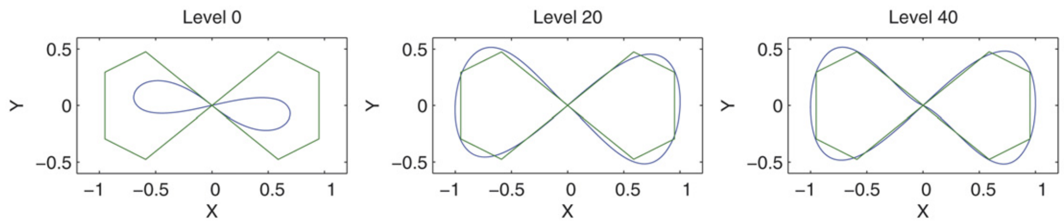

Fig. 1. Bézier basis.

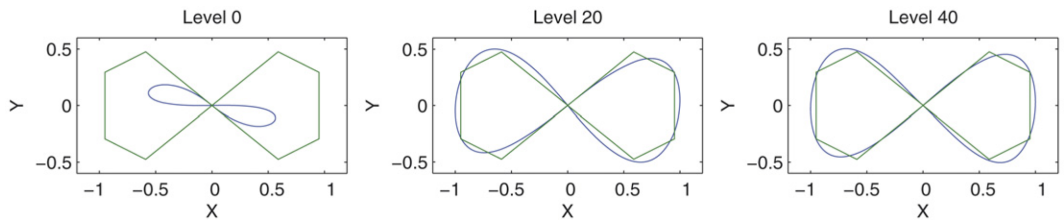

Fig. 2. Said–Ball basis.

Table 1
Euclidean norm when fitting a lemniscate of Gerono

Level	Bézier curve	Said–Ball curve
0	4.620135E–001	5.470408E–001
10	1.012377E–001	1.618460E–001
20	5.637195E–002	9.293990E–002
30	3.579771E–002	6.853643E–002
40	2.466991E–002	5.330010E–002

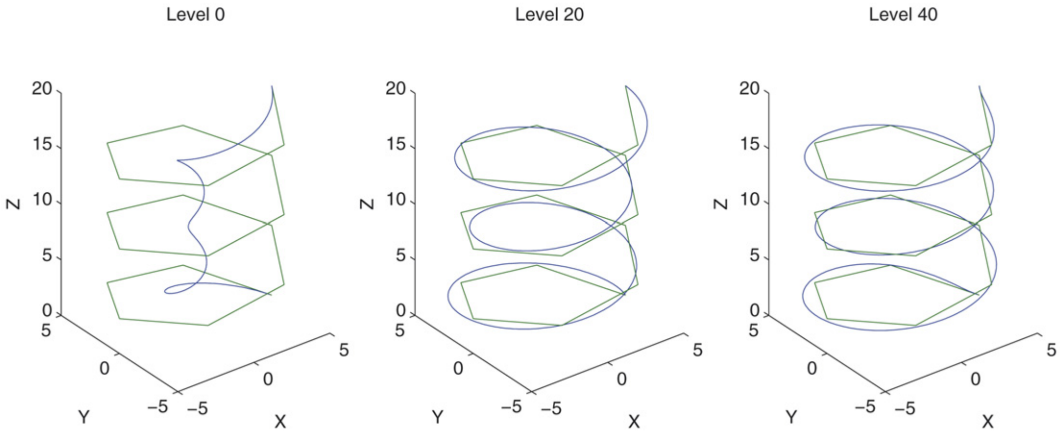

Fig. 3. Bézier basis.

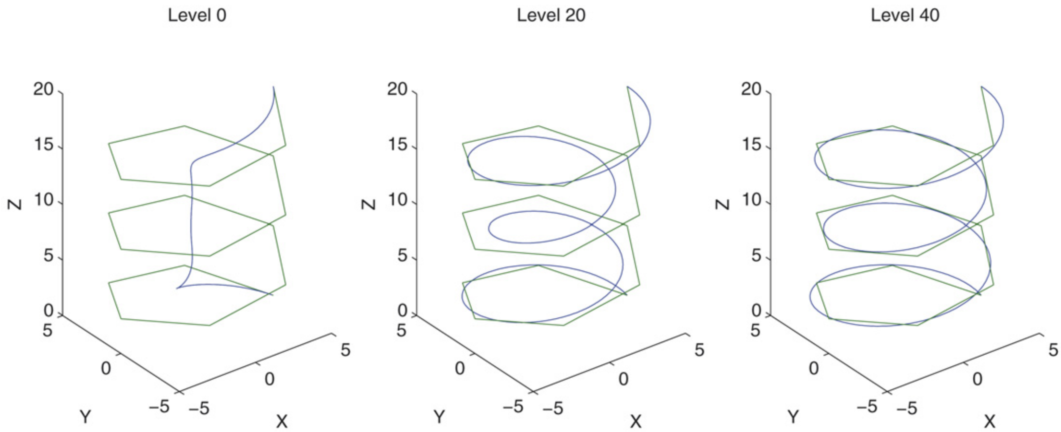

Fig. 4. Said–Ball basis.

Finally let us consider an helix of radius 5 given by the parametric equations

$$\begin{cases} x(t) = 5 \cdot \cos t, \\ y(t) = 5 \cdot \sin t, \\ z(t) = t, \end{cases}$$

$t \in [0, 6\pi]$. We took a sequence of 19 points $\{P_i\}_{i=0}^{18}$ from the helix previously defined in the following way

$$P_i = \left(x\left(i \cdot \frac{\pi}{3}\right), y\left(i \cdot \frac{\pi}{3}\right), z\left(i \cdot \frac{\pi}{3}\right)\right)^T, \quad i = 0, 1, \ldots, 18.$$

Starting with these control points we fit an helix of radius 5 by two sequences of curves, generated by progressive iteration approximation of a Bézier curve and a Said–Ball curve, respectively. In Fig. 3 we can see the starting, the

Table 2
Euclidean norm when fitting a helix

Level	Bézier curve	Said–Ball curve
0	4.624577E+000	5.698051E+000
10	1.691200E+000	3.574839E+000
20	8.971303E–001	1.901447E+000
30	5.968398E–001	1.209260E+000
40	4.376396E–001	8.576443E–001

20th and the 40th curves of the sequence of Bézier curves, while in Fig. 4 we can see the starting, the 20th and the 40th curves of the sequence of Said–Ball curves. Again, we can see that the sequence of Bézier curves adjust to the helix with a faster convergence rate than the sequence of Said–Ball curves as we have already proved in the previous section. This fact can also be checked in Table 2 where we have listed the fitting errors of both curves sequences after specific iteration levels. The fitting error at the step $k \geqslant 0$ is taken as the maximum Euclidean norm of the adjusting vectors $\Delta_0^k, \Delta_1^k, \ldots, \Delta_{18}^k$.

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