Practice Test 2

1. Let $a=1$, $b=0$. Then
$\sqrt{a+b} = \sqrt{1+0} = 1$, and $\sqrt{a} + \sqrt{b} = \sqrt{1} + \sqrt{0} \neq 1$.

2. Let $a=-2$, $b=-1$.
Then $a<b$, but $a^2 = 4 > 1 = b^2$.

3. If a is any odd integer and b is any even integer, then $2a+3b$ is even.

Pf:
Suppose a is an odd integer and b an even integer.
Then $a=2r+1$ and $b=2s$ for some $r,s \in \mathbb{Z}$.
So $2a+3b = 2(2r+1)+3(2s) = 4r+2+6s$
$= 2(2r+1+3s)$.
Note $2r+1+3s \in \mathbb{Z}$, so $2a+3b$ is even by definition.

4. Counterexample:
Let $a=1$, $b=2$. Then $3 \mid (1+2)$, but $3 \nmid (1-2)$.

5. a) $28 \operatorname{div} 5 = 5$

b) $43 \bmod 9 = 7$

6. For any odd integer n, $\lceil n^2/4 \rceil = \frac{n^2+3}{4}$.

Pf:

Let n be an odd integer. Then $n = 2k+1$ for some $k \in \mathbb{Z}$. So

$$\lceil n^2/4 \rceil = \left\lceil \frac{(2k+1)^2}{4} \right\rceil = \left\lceil \frac{4k^2+4k+1}{4} \right\rceil = \lceil k^2+k+1/4 \rceil$$
$$= k^2+k+1.$$

and $\frac{(2k+1)^2+3}{4} = \frac{4k^2+4k+1+3}{4} = \frac{4k^2+4k+4}{4} = k^2+k+1$.

Since both sides are equal to k^2+k+1, they must be equal to each other.

7. If a and b are rational numbers, $b \neq 0$, and r an irrational number, then $a+br$ is irrational.

Pf:

Suppose not. That is, suppose $a+br$ is rational. Then $a+br = p/q$ for some $p, q \in \mathbb{Z}, q \neq 0$.

So $br = \frac{p}{q} - a = \frac{p}{q} - \frac{aq}{q} = \frac{p-aq}{q}$.

Thus $r = \frac{p-aq}{bq}$. Note that $p-aq, bq \in \mathbb{Z}$ and $bq \neq 0$ by zero product property. Thus, r is rational.

But this is a contradiction.

Hence, it must be that $a+br$ is irrational.

8. a) If one of m and n is odd and the other even, then $m+n$ is odd.

b)

Pf:

Without loss of generality, assume m is odd and n even. Then we can write $m=2r+1$ and $n=2s$ for some $r,s \in \mathbb{Z}$.

Then $m+n=(2r+1)+2s = 2r+2s+1 = 2(r+s)+1$.

Since $r+s \in \mathbb{Z}$, it must be that $m+n$ is odd.

9. $\displaystyle\sum_{n=2}^{6} \frac{(-1)^n \cdot n}{(n+1)(n+2)}$

10. Prove: $1+3n \le 4^n \ \forall\, n \in \mathbb{Z},\ n \ge 0$.

Pf:

Base case: let $n=0$.

Then $1+3(0)=1 \le 4^0 = 1$.

Induction step: let $n=k$.

Suppose $1+3k \le 4^k$. Then

$$\begin{aligned} 1+3(k+1) = 1+3k+3 &\le 4^k+3 \\ &\le 4^k + 3\cdot 4^k \quad (k \ge 0) \\ &= 4^k(1+3) \\ &= 4^k \cdot 4 \\ &= 4^{k+1}. \end{aligned}$$