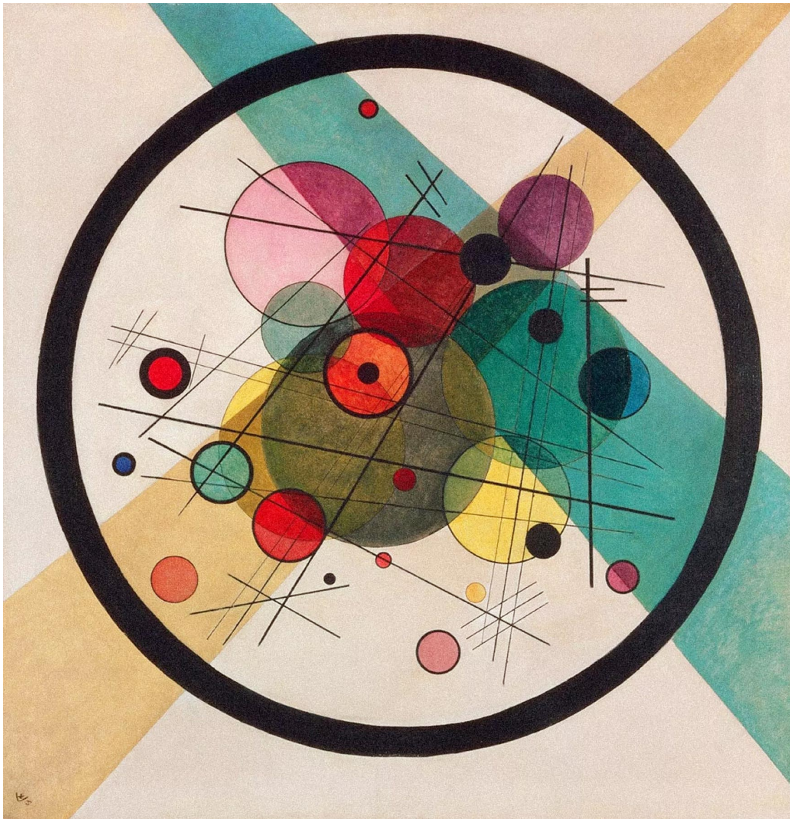


Teach Yourself Logic 2020: A Study Guide

Peter Smith
University of Cambridge



Version of December 11, 2019

Pass it on, That's the game I want you to learn. Pass it on.

Alan Bennett, *The History Boys*

©Peter Smith 2019

The Guide's layout is optimized for reading on-screen: try, for example, reading in iBooks on an iPad, or in double-page mode in Adobe Reader on a laptop.

URLs in [blue](#) are live links to external web-pages or PDF documents. Internal cross-references to sections etc. are also live. But to avoid the text being covered in ugly red rashes internal cross-references are not coloured. If you are reading on a computer, your cursor should change as you hover over such live links: *try here*.

Against the entry for a book, one star means (relatively) cheap, two stars means (legally) free: see §2.3.

The latest version of this Guide can always be downloaded from logicmatters.net/tyl/

The cover image is Wassily Kandinsky, *Circles in a Circle*, 1923.

Contents

A very quick introduction	iii
I Preliminaries	1
1 Using this Guide	2
1.1 Why this Guide for philosophers?	2
1.2 Why this Guide for mathematicians too?	3
1.3 A strategy for reading logic books (and another reason why this Guide is so long)	4
1.4 On the question of exercises	4
1.5 Assumed background	5
1.6 Philosophers: do you <i>really</i> need more logic?	6
1.7 How to prove it	7
2 How the Guide is structured	9
2.1 Mapping the field	9
2.2 Three comments on the Guide's structure	12
2.3 Choices, choices	13
3 'Intro logic'	16
II Basic mathematical logic – and some other topics	18
4 First order logic	19
4.1 FOL: the basic topics	19
4.2 The main recommendations on FOL	21
4.3 Some parallel reading	24
4.4 Further into FOL	26
4.5 Other treatments?	28
5 Continuing Mathematical Logic	32

5.1	From first-order logic to elementary model theory	33
5.2	Computability and Gödelian incompleteness	36
5.3	Beginning set theory	40
5.4	Extras: two variant logics	46
5.4.1	Second-order logic	46
5.4.2	Intuitionist logic	48
6	Modal and other logics	52
6.1	Modal logic	52
6.2	Free logic, plural logic	56
6.2.1	Free Logic	56
6.2.2	Plural logic	57
6.3	Relevance logics (and wilder logics too)	58
III	Beyond the basics	61
	About Part III	62
7	More advanced reading on some core topics	63
7.1	Proof theory	63
7.2	Beyond the model-theoretic basics	67
7.3	Computability	71
7.3.1	Computable functions	71
7.3.2	Computational complexity	73
7.4	Incompleteness and related matters	74
7.5	Theories of arithmetic	76
8	Serious set theory	78
8.1	ZFC, with all the bells and whistles	79
8.1.1	A first-rate overview	79
8.1.2	Rather more slowly, towards forcing	79
8.1.3	Pausing for problems	81
8.1.4	Pausing for more descriptive set-theory	81
8.1.5	Forcing further explored	82
8.1.6	The higher infinite	84
8.2	The Axiom of Choice	84
8.3	Other set theories?	85
9	What else?	90
	Index of authors	92

A very quick introduction

Before I retired from the University of Cambridge, it was my greatest good fortune to have secure, decently paid, university posts for forty years in leisurely times, with almost total freedom to follow my interests wherever they led. Like many of my contemporaries, for most of that time I didn't really appreciate how *very* lucky I was. This Study Guide to logic textbooks is my attempt to give a little back by way of heartfelt thanks.

As the first chapter explains, this Guide is aimed at philosophers who want to go on beyond their first introductory logic course to learn some more serious logic. It is also equally aimed at mathematicians wanting to get to grips with an under-taught but exciting area of mathematics. Why not two separate Guides for the two audiences? Mainly because it would be difficult to decide what should go where. After all, a number of philosophers develop serious interests in more mathematical corners of the broad field of logic; and a number of mathematicians find themselves becoming interested in more foundational/conceptual issues. Rather than impose artificial divisions, I provide here a single but wide-ranging menu for everyone to choose from as their interests dictate. So ...

Don't be scared off by the Guide's length! This is due both to its breadth of coverage and also to its starting just half a step beyond 'baby logic' and then going a long way down the road towards state-of-the-art stuff. Different readers will want to jump on and off the bus at different stops. Simply choose the sections which are most relevant to your background and your interests, and you will be able to cut the Guide down to more manageable proportions. There should be enough signposting to enable you to pick your way through.

However, if you are hoping for help with *very* elementary logic (e.g. as typically encountered by philosophers in their first-year courses), then let me say straight away that this Guide is not really designed for you. The only part that directly pertains logic at that level is the short Chapter 3; all the rest is about rather more advanced – and eventually *very* much more advanced – material.

Part I

Preliminaries

Chapter 1

Using this Guide

This chapter outlines the Guide's aims, and says something about how to use it (noting in passing another reason why it is so long). Then the following chapter describes the Guide's overall structure.

I imagine readers to be mostly either philosophy students or mathematics students, at various levels from upper undergraduate to postgraduate. (To be sure, computer scientists and some theoretical linguists can be very interested in logic too: but I haven't written with them in mind.) So I'll begin by explaining the role of the Guide for the two main constituencies.

1.1 Why this Guide for philosophers?

It is an odd phenomenon, and a very depressing one too. Logic, beyond the most elementary introduction, is taught less and less, at least in UK philosophy departments. Fewer and fewer philosophers with serious logical interests seem to get appointed to teaching posts.

Yet logic itself is, of course, no less exciting and rewarding a subject than it ever was, and the amount of good formally-informed work in philosophy is ever greater as time goes on – to mention just one currently very active area, consider work on theories of truth. Moreover, logic is far too important to be left entirely to the mercies of technicians from maths or computer science departments with different agendas (who often reveal an insouciant casualness about basic conceptual issues that will matter to the philosophical reader).

So how is a real competence in logic to be passed on if there are not enough courses, or indeed if there are none at all?

It seems that many beginning graduate students in philosophy – if they are

not to be quite dismally uneducated in logic and therefore cut off from working in some of the most exciting areas of their discipline – will need to teach themselves from books, either solo or (better, but not always possible) by organizing their own study groups.

In a way, that's perhaps no real hardship; there are some wonderful books written by great expositors out there. But *what* to read and work through? Logic books can have a very long shelf life, and you shouldn't at all dismiss older texts when starting out on some topic area: so there's more than a sixty year span of publications to select from. Without having tried very hard, I have accumulated on my own shelves well over two hundred formal logic books that might feature somewhere in a Guide such as this – and these are still only a selection of what's available.

Philosophy students evidently need a Study Guide if they are to find their way around the very large literature old and new, with the aim of teaching themselves enjoyably and effectively. So this is my on-going attempt to provide one.

1.2 Why this Guide for mathematicians too?

The situation of logic teaching in mathematics departments can also be pretty dire. Students will of course pick up a passing acquaintance with some very basic notions about sets and some logical symbolism. But there are full university maths courses in good UK universities with precisely zero courses offered on logic, computability theory, or serious set theory. And I believe that the situation can be equally patchy elsewhere.

So if you want to teach yourself some logic or related areas, where should you start? What are the topics you might want to cover? What textbooks are likely to prove accessible, engaging, and rewarding to work through? Again, this Guide – or at least, the sections on the core mathematical logic curriculum – will give you pointers.

True, this is written by someone who, apart from a few guest mini-courses, has taught in philosophy departments and who has never been a research mathematician. Which no doubt gives a distinctive tone to the Guide (and also explains why it occasionally ranges into areas of logic of special interest to philosophers). Still, mathematics remains my first love, and these days it is mathematicians whom I mostly get to hang out with. Most of the books I recommend on core topics are very definitely paradigm *mathematics* texts. So I won't be leading you astray!

1.3 A strategy for reading logic books (and another reason why this Guide is so long)

We cover a great deal of ground in this Guide then, which is one reason for its initially daunting length. But there is another reason, connected to a point which I now want to highlight:

I very strongly recommend tackling an area of logic by reading a series of books which *overlap* in level (with the next one covering some of the same ground and then pushing on from the previous one), rather than trying to proceed by big leaps.

In fact, I probably can't stress this advice too much (which, in my experience, applies equally to getting to grips with any new area of mathematics). This approach will really help to reinforce and deepen understanding as you re-encounter the same material at different levels, coming at it from different angles, with different emphases.

Exaggerating only a little, there are many instructors who say 'This is the textbook we are using/here is my set of notes: take it or leave it'. But you will *always* gain from looking at some overlapping texts. (When responding to student queries on a question-and-answer internet site, I'm repeatedly struck by how much puzzlement would be quickly resolved by taking the occasional quick look outside the course textbook/lecturer's notes.)

The multiple overlaps in coverage in the reading lists below, which help make the Guide as long as it is, are therefore fully intended. They also mean that you should always be able to find options that suit your degree of mathematical competence.

To repeat: you will certainly miss a lot if you concentrate on just one text in a given area, especially at the outset. Yes, do very carefully read one or two central texts, choosing books at a level that works for you. But do also cultivate the crucial further habit of judiciously skipping and skimming through a number of other works so that you can build up a good overall picture of an area seen from various somewhat different angles of approach.

1.4 On the question of exercises

While we are talking about strategies for reading logic books, I should say something very briefly on the question of doing exercises.

Mathematics is, as they say, not merely a spectator sport: so you should try some of the exercises in the books as you read along to check and reinforce comprehension. On the other hand, don't obsess about doing exercises if you are a philosopher – understanding proof ideas is very much the crucial thing, not the ability to roll-your-own proofs. And even mathematicians shouldn't get too hung up on routine additional exercises beyond those needed to initially fix ideas (unless you have specific exams to prepare for!): concentrate on the exercises that look interesting and/or might deepen understanding.

Do note however that some authors have the irritating(?) habit of burying quite important results among the exercises, mixed in with routine homework. It is therefore always a good policy to skim through the exercises in a book even if you don't plan to work on answers to very many of them.

1.5 Assumed background

So what do you need to bring to the party, if you are going to tackle some of the books recommended in the main body of this Guide?

If you are a mathematician, there is no specific assumed background you need before tackling the entry-level books mentioned in Chs 4 and 5. They don't presuppose much 'mathematical maturity', so you can really just dive in – and here in this Guide, now skip on to §1.7.

If, however, you are a philosopher without any significant mathematical background, then how to proceed will depend on how much logic you have already encountered. Let's distinguish three levels you might have reached:

- L1. If you have only done an 'informal logic' or 'critical reasoning course', then you'll probably need to read a good introductory formal logic text before tackling the more advanced work covered in the body of this Guide. See Chapter 3.
- L2. If you *have* taken a logic course with a formal element, but it was based on some really, *really*, elementary text book like Sam Guttenplan's *The Languages of Logic*, Howard Kahane's *Logic and Philosophy*, or Patrick Hurley's *Concise Introduction to Logic* (to mention some frequently used texts), then you *might* still struggle with the initial suggestions in the main part of the Guide which starts with Chapter 4.

But this will of course vary a lot from person to person. So the best advice is probably just to make a start and see how you go, and if the

going gets too tough, backtrack to try one of the more introductory books I mention in Chapter 3, skipping quickly over what you already know.

- L3. If you have taken an introductory formal logic course based on a more substantial text like the ones mentioned in Chapter 3, then you should be well prepared.

1.6 Philosophers: do you *really* need more logic?

This section is again for philosophers; mathematicians should skip on to the next section.

It is perhaps worth pausing to ask whether you, as a budding philosopher, really *do* want or need to pursue your logical studies much further if you have already worked through a book like mine or Nick Smith's. Far be it from me to put people off doing more logic: perish the thought! But for many philosophical purposes, you might well survive by just reading this:

Eric Steinhart, *More Precisely: The Math You Need to Do Philosophy** (Broadview 2009) The author writes: 'The topics presented ... include: basic set theory; relations and functions; machines; probability; formal semantics; utilitarianism; and infinity. The chapters on sets, relations, and functions provide you with all you need to know to apply set theory in any branch of philosophy. The chapter of machines includes finite state machines, networks of machines, the game of life, and Turing machines. The chapter on formal semantics includes both extensional semantics, Kripkean possible worlds semantics, and Lewisian counterpart theory. The chapter on probability covers basic probability, conditional probability, Bayes theorem, and various applications of Bayes theorem in philosophy. . . . The chapters on infinity cover recursive definitions, limits, countable infinity, Cantor's diagonal and power set arguments, uncountable infinities, the aleph and beth numbers, and definitions by transfinite recursion. *More Precisely* is designed both as a text book and reference book to meet the needs of upper level undergraduates and graduate students. It is also useful as a reference book for any philosopher working today.'

Steinhart's book is admirable, and will give many philosophers a handle on some technical ideas going well beyond 'baby logic' and which they really should know just a little about, without all the hard work of doing a full mathematical logic course. What's not to like? It could be enough for you. And then, if there indeed

turns out to be some particular area (modal logic, for example) that seems especially germane to your particular philosophical interests, you always can go to the relevant section of this Guide for more.

1.7 How to prove it

Before getting down to the main business let me mention a rather special book, which is aimed at mathematicians though philosophers should find it very useful too:

Daniel J. Velleman, *How to Prove It: A Structured Approach** (CUP, 3rd edition, 2019). From the Preface: ‘Students of mathematics ... often have trouble the first time that they’re asked to work seriously with mathematical proofs, because they don’t know ‘the rules of the game’. What is expected of you if you are asked to prove something? What distinguishes a correct proof from an incorrect one? This book is intended to help students learn the answers to these questions by spelling out the underlying principles involved in the construction of proofs.’

There are chapters on the propositional connectives and quantifiers, and *informal* proof-strategies for using them, and chapters on relations and functions, a chapter on mathematical induction, and a final chapter on infinite sets (countable vs. uncountable sets). This is a truly excellent student text.

Yes, if you are a mathematician who has got to the point of wanting to find out something about mathematical logic, you will probably have already mastered nearly all the content of Velleman’s splendidly clear book. However, a few hours speed-reading through this text (except perhaps for the very final section), pausing over anything that doesn’t look very comfortably familiar, could still be time extremely well spent.

What if you are a philosophy student who (as we are now assuming) has done some elementary logic? Well, experience shows that being able to handle e.g. natural deduction proofs in a formal system doesn’t always translate into being able to construct good *informal* proofs. For example, one of the few meta-theoretic results that might be met in a first logic course is the expressive completeness of the set of formal connectives $\{\wedge, \vee, \neg\}$. The proof of this result is really easy, based on a simple proof-idea. But many students who will ace the part of the end-of-course exam asking for quite complex *formal* proofs inside a deductive system can find themselves all at sea when asked to replicate this *informal* bookwork proof *about* a formal system.

Another example: it is only too familiar to find philosophy students introduced to set notation not even being able to make a start on giving a correct informal proof that $\{\{a\}, \{a, b\}\} = \{\{a'\}, \{a', b'\}\}$ if and only if $a = a'$ and $b = b'$.

Well, if you *are* one of those students who jumped through the formal hoops but were unclear about how to set out elementary mathematical proofs (e.g. from the ‘metatheory’ of baby logic, or from very introductory set theory), then again working through Velleman’s book from the beginning could be exactly what you need to get you prepared for the serious study of logic. And even if you were one of those comfortable with the informal proofs, you will probably still profit from skipping and skimming through (perhaps paying especial attention to the chapter on mathematical induction).

Chapter 2

How the Guide is structured

This chapter explains how the field of logic is being carved up into subfields in this Guide. It also explains how many of these subfields are visited twice, once to give entry-level readings, and then again later when we consider some more advanced texts.

2.1 Mapping the field

‘Logic’, in the broad sense, is a big field. Its technical development is of concern to philosophers and mathematicians, not to mention computer scientists (and others). Different constituencies will be particularly interested in different areas and give different emphases. Core ‘classical first-order logic’ is basic by anyone’s lights. But after that, interests can diverge. For example, modal logic is of considerable interest to some philosophers, not so much to mathematicians, though parts of this sub-discipline are of great concern to some computer scientists. Set theory (which falls within the purview of mathematical logic, broadly understood) is an active area of research interest in mathematics, but – because of its (supposed) foundational status – even quite advanced results can be of interest to philosophers too. Type theory started off as a device of philosophy-minded logicians looking to avoid the paradoxes: it has become primarily the playground of computer scientists. The incompleteness theorems are relatively elementary results of the theory of computable functions, but are of particular conceptual interest to philosophers. Finite model theory is of interest to mathematicians and computer scientists, but perhaps not so much to philosophers. And so it goes.

In this Guide, as I said, I’m going to have to let the computer scientists and others largely look after themselves. Our main focus is going to be on the core

mathematical logic curriculum of most concern to philosophers and mathematicians, together with some extras of particular interest to philosophers – if only because that’s what I know a little about. Here, then, is an overview map of the territory we cover in the Guide.

We are going to spend very little time, here in **Part I** of the Guide, on

- **‘Intro logic’ (Ch. 3)** I mean the sort of material that is covered in elementary formal logic courses for philosophers (what is often somewhat dismissively labelled ‘baby logic’!). Topics covered might include some truth-functional propositional logic (introducing the truth-table test for validity), a light-weight introduction to quantifiers and identity (giving perhaps little more than a familiarity with the use of the notation). Longer introductory courses may also have time for an introduction to a proof system (e.g. natural deduction or ‘trees’).

Our brief Ch. 3 gives some quick pointers to books for philosophers covering elementary material at a level that should provide an adequate background for starting work on topics covered in the **Part II** of the Guide, beginning with

- **First-order logic (Ch. 4)** The serious study of mathematical logic always starts with a reasonably rigorous treatment of quantification theory, covering both a proof-system for classical first-order logic (FOL), and the standard classical semantics, getting at least as far as a soundness and completeness proof for your favourite proof system, and perhaps taking a first look at a few easy corollaries. I will both give the headline news about the topics that need to be covered and also suggest a number of different ways of covering them (I’m not suggesting you read all the texts mentioned!). This part of the Guide is therefore quite long even though it doesn’t cover a lot of ground: it does, however, provide the essential foundation for ...
- **Continuing the basic ‘Mathematical Logic’ curriculum (Ch. 5)** Mathematical logic programmes typically comprise – in one order or another and in various proportions – three or perhaps four elements in addition to a serious treatment of FOL:

1. **A little model theory**, i.e. a little more exploration of the fit between theories cast framed in formal languages and the structures they are supposed to be ‘about’. This will start with the compactness theorem and Löwenheim-Skolem theorems (if these aren’t already covered in your basic FOL reading), and then will push on just a bit further.

You will need to know a very little set theory as background, mainly ideas about cardinality; so you might need to interweave beginning model theory with the very beginnings of your work on set theory if those cardinality ideas are new to you.

2. **Computability and decidability**, and proofs of epochal results such as Gödel’s incompleteness theorems. This is perhaps the most readily approachable area of mathematical logic.
3. **Some introductory set theory**. Informal set theory, basic notions of cardinals and ordinals, constructions in set theory, the role of the axiom of choice, etc. The formal axiomatization of ZFC.
4. **Extras: variants of standard FOL** The additional material that you could (should?) meet in a first serious encounter with mathematical logic includes:
 - (a) Second Order Logic (where we can quantify over properties as well as objects — what difference does this make?), and
 - (b) Intuitionistic Logic (which drops the law of excluded middle, motivated by a non-classical understanding of the significance of the logical operators).

These two variants are of both technical and philosophical interest.

- **Modal and other logics (Ch. 6)** In this chapter, we consider a number of logical topics of particular concern to philosophers (many mathematicians will want to skip – though perhaps you should at least know that there *is* a logical literature on these topics). These topics are actually quite approachable even if you know little other logic. So the fact that this material is mentioned after the heavier-duty mathematical topics in Ch. 5 doesn’t mean that there is a jump in difficulty from Ch. 4. We look at:

1. **Modal logic** Even before encountering a full-on treatment of first-order logic, philosophers are often introduced to modal logic (the logic of necessity and possibility) and to its ‘possible world semantics’. You can indeed do an amount of propositional modal logic with little more than ‘intro logic’ as background.
2. **Other classical variations** and extensions of standard logic which are of conceptual interest but still classical in spirit, e.g. free logic, plural logic.

3. **Further non-classical variations** The most important non-classical logic – and the one of real interest to mathematicians too – is intuitionist logic which we’ve already mentioned. But here we might also consider e.g. relevant logics which drop the classical rule that a contradiction entails anything.

Part II forms the basic core of the Guide. **Part III** moves on to consider more advanced treatments of many of the areas introduced in Part II, and also adds coverage of some further topics.

- **Exploring further in core mathematical logic (Ch. 7)** The introductory texts mentioned in Chapters 4 and 5 will already contain numerous pointers onwards to further books, more than enough to put you in a position to continue exploring solo. However, Chapter 7 offers my own suggestions for more advanced reading on model theory, computability, formal arithmetic, and now on proof theory too. This chapter is for specialist graduate students (among philosophers) and for final year undergraduate/beginning graduate students (among mathematicians).
- **Serious set theory (Ch 8)** We continue the exploration beyond the basics of set theory covered in Ch. 5 with a chapter on more advanced set theory. This gets seriously tough!
- **Other topics (Ch. 9)** The very brief final chapter mentions a number of additional topics which, given world enough and time, this Guide might eventually cover, with a pointer to my web-page on category theory.

Don’t be alarmed if (some of) the descriptions of topics above are at the moment opaque to you: we will be explaining things rather more as we go through the Guide.

2.2 Three comments on the Guide’s structure

Three comments on all this:

1. The Guide divides up the broad field of logic into subfields in a pretty conventional way. But of course, even the ‘horizontal’ divisions into different areas can in places be a little arbitrary.

And the ‘vertical’ division between the entry-level readings on mathematical logic in Chapter 5 and the further explorations in Chapters 7 and 8

is necessarily going to be a lot more arbitrary. I think that everyone will agree (at least in retrospect!) that e.g. the elementary theory of ordinals and cardinals belongs to the basics of set theory, while explorations of ‘large cardinals’ or independence proofs via forcing are decidedly advanced. But in most areas, there are far fewer natural demarcation lines between the entry-level basics and more advanced work. Still, it is surely very much better to have *some* such structuring than to heap everything together.

2. Within sections in the coming chapters, I have usually put the main recommendations into what strikes me as a sensible reading order of increasing difficulty (without of course supposing you will want to read everything – those with stronger mathematical backgrounds might sometimes want to try starting in the middle of a list). Some further books are listed in asides or postscripts.
3. The Guide used also to have a substantial final part considering some of ‘The Big Books on mathematical logic’ (meaning typically broader-focus books that cover first-order logic together with one or more subfields from the further menu of mathematical logic). These books vary a lot in level and coverage, but can provide very useful consolidating/amplifying reading. This supplement to the main Guide is now available online as a separate [Appendix](#) which I hope to continue adding to. Alternatively, for individual webpages on those texts and a number of additional reviews, visit the Logic Matters [Book Notes](#).

2.3 Choices, choices

(a) So what has guided my choices of what to recommend within the sections of this Guide?

Different people find different expository styles congenial. For example, what is agreeably discursive for one reader is irritatingly verbose and slow-moving for another. For myself, I do particularly like books that are good on conceptual details and good at explaining the motivation for the technicalities while avoiding needless complications, excessive hacking through routine detail, or misplaced ‘rigour’. Given the choice, I tend to prefer a treatment that doesn’t rush too fast to become too general, too abstract, and thereby obscures intuitive motivation: this is surely what we want in books to be used for self-study. (There’s a certain tradition of masochism in older maths writing, of going for brusque formal abstraction from the outset with little by way of explanatory chat: this is quite

unnecessary in other areas, and just because logic is all about formal theories, that doesn't make it any more necessary here.)

The selection of books in the following chapters no doubt reflects these tastes. But overall, I don't think that I have been very idiosyncratic: indeed, in many respects I have probably been somewhat conservative in my choices. So nearly all the books I recommend will very widely be agreed to have significant virtues (even if other logicians would have different preference-orderings).

(b) Most of the books mentioned here should be held by any larger university library which has been paying reasonable attention to maintaining core collections in mathematics and philosophy, and every book should be borrowable through your local inter-library loans system.

Since it is assumed that you will by default be using library copies of books, I have *not* made cost or being currently in print a significant consideration. However, I have marked with one star* books that are available new at a reasonable price (or at least are unusually good value for the length and/or importance of the book).

I have marked with two stars** those books for which e-copies are freely and legally available, and links are provided. Most articles, encyclopaedia entries, etc., can also be downloaded, again with links supplied.

(We must pass over in silence the question of using the well-known copyright-infringing PDF repositories. I could not possibly comment ...)

(c) And yes, the references here *are* very largely to published books and articles rather than to on-line lecture notes etc. Many such notes are excellent, but they do tend to be terse, and quite often *very* terse (as entirely befits material intended to support a lecture course). And so they are perhaps not as helpful as fully-worked-out book-length treatments for students needing to teach themselves. But I'm sure that there is an increasing number of excellent e-resources out there which do amount, more or less, to free stand-alone textbooks: I mention a couple of more recent arrivals, and I would be very happy to get recommendations about others.

(d) Finally, the earliest versions of this Guide kept largely to positive recommendations: I didn't pause to explain the reasons for the then absence of some well-known books. This was partly due to considerations of length which have now quite gone by the wayside; but also I wanted to keep the tone enthusiastic, rather than to start criticizing or carping.

However, enough people kept asking what I think about an alternative X , or asking why the old warhorse Y wasn't mentioned, for me to change my mind. So I have just occasionally added some reasons why I *don't* particularly recommended certain books.

Chapter 3

‘Intro logic’

This is by way of a very short prequel to the main Guide. But I keep being asked to recommend a *really* introductory formal logic book – the sort of text suitable for a first formal logic course for philosophers.

The world isn’t short of introductory logic texts, and there are a number of very respectable options. Though some books are, frankly, just not as good as they should be at explaining what’s going on and why; and some others are more reliable but to my mind are likely to prove dull or off-putting. Hopefully my own book avoids these two shortcomings! – I’ll mention it in a moment. First though, here are two other books that I like and which can be warmly recommend:

1. Paul Teller’s *A Modern Formal Logic Primer*** (Prentice Hall 1989) has been out of print for a while, but the scanned pages are freely available online at [the book’s website](#), which makes it unbeatable value! The book (in fact, two slim volumes) is in many ways excellent, and had I known about it at the time (or listened to Paul’s good advice, when I got to know him, about how long it takes to write an intro book), I’m not sure that I’d have ever written my own book! As well as introducing trees, Teller also covers a version of ‘Fitch-style’ natural deduction. (He also goes significantly beyond the really elementary, getting as far as a so-called completeness proof.) Notably user-friendly. Answers to exercises are available at the author’s website.
2. Nicholas Smith’s more recent *Logic: The Laws of Truth* (Princeton UP 2012) is again very clearly written and has many virtues (particularly if you like your texts to go rather slowly and discursively). The first two parts of the book focus on logic by trees. But the third part ranges wider, including a brisk foray into natural deduction – and are there some extras too, again

going well beyond ‘baby logic’. It is a particularly readable addition to the introductory literature. I have [commented further here](#). Answers to exercises can be found at [the book’s website](#).

Another book, which I am often asked about, is

3. Dave Barker-Plummer, Jon Barwise and John Etchemendy’s *Language, Proof and Logic* (CSLI Publications, 1999; 2nd edition 2011). The unique selling point for this widely used book is that it comes as part of a ‘courseware package’, which includes software such as a famous program called ‘Tarski’s World’ in which you build model worlds and can query whether various first-order sentences are true of them. Some students really like it, but at least equally many don’t find this kind of thing particularly useful. There is an associated online course from Stanford, with video lectures by the authors which you can watch for free, though you have to buy the courseware package to complete the course with a ‘statement of accomplishment’. For more details, see [the book’s website](#).

This is another book which is user-friendly, goes slowly, and does Fitch-style natural deduction. It is a very respectable option. But Teller is perhaps rather snappier, I think no less clear, and certainly wins on price!

Unsurprisingly, though, the book that I would most strongly recommend to philosophical beginners wanting a way into formal logic has to be

4. Peter Smith *An Introduction to Formal Logic** (CUP second edition 2020). This is intended for complete beginners, and the first edition was the first year text in Cambridge for a decade. It was written as an accessible teach-yourself book, covering basic propositional and predicate logic. The first edition did logic ‘by trees’. This new second edition instead focuses on natural deduction (with an online supplement on trees). For more details, including an extended table of contents for the new edition, see the [IFL pages](#), where there are also answers to the exercises.

Part II

Basic mathematical logic – and some other topics

Chapter 4

First order logic

So, at last, let's get down to business! This chapter starts with a checklist of the topics we will be treating as belonging to the basics of first-order logic (predicate logic, quantificational logic, call it what you will: we'll use 'FOL' for short). There are then some main recommendations for texts covering these topics, followed by some suggestions for parallel and further reading. The chapter ends with some additional comments, mostly responding to frequently asked questions.

A note to philosophers. If you have carefully read and mastered a substantial introductory logic text for philosophers such as Nick Smith's, or even my own, you will already be familiar with (versions of) much of the material covered in this chapter. However, the big change is that you will now begin to see perhaps familiar material being re-presented in the sort of mathematical style and with the sort of rigorous detail that you will necessarily encounter more and more as you progress in logic. You do need to start feeling entirely comfortable with this mode of presentation at an early stage. So it will be well worth working through even familiar topics again, now with more mathematical precision.

4.1 FOL: the basic topics

Without further ado, here in headline terms is what you need to get to know about – though don't worry, of course, if at present you don't fully grasp the import of every point: the next section gives recommended reading to help you reach enlightenment!

- Starting with *syntax*, you need to know how first-order *languages* are constructed. If you have already encountered such languages, you should now get to understand how to prove various things about them that might

seem obvious and that you perhaps previously took for granted – for example, that ‘bracketing works’ to avoid ambiguities, meaning that every well-formed formula has a unique parsing.

(By the way, it is worth remarking in passing that introductory logic courses for philosophers very often ignore *functions*. But given that FOL is deployed to regiment everyday mathematical reasoning, and that functions are of course crucial to mathematics, function expressions now become important, even if there are tricks that make them in principle eliminable.)

- On the *semantic* side, you need to understand the idea of a *structure* (a domain of objects equipped with some relations and/or functions, and having some particular objects especially picked out, ready to be the denotations of constants in the language). And, crucially, you need to grasp the idea of an interpretation of a language in such a structure. You’ll need to understand how such an interpretation generates a unique assignment of truth-values to every sentence of the interpreted language – this means grasping a proper formal semantic story with the bells and whistles required to cope with quantifiers adequately.

With these ideas to hand, you now can define the relation of semantic entailment, where the sentences Γ semantically entail φ when no interpretation in any appropriate structure can make all the sentences Γ true without making φ true too. You’ll need to know some of the basic properties of this relation.

- Back to syntax: you need to get to know a deductive *proof-system* for FOL reasonably well.

Now, a variety of styles of proof-system feature in texts at various levels. For a start, you will find

1. Old-school ‘axiomatic’ systems,
2. Natural deduction done Gentzen-style,
3. Natural deduction done Fitch-style,
4. ‘Truth trees’ or ‘semantic tableaux’,
5. Sequent calculi.

These types of proof-system have various strengths and shortcomings. To put it very roughly, the third and fourth are easiest for beginners to actually *use* to produce working formal proofs (that’s why elementary logic books for philosophers usually introduce one or other or both systems).

While the first and second types of system are rather easier to prove results *about*. The fifth type really comes into its own in more advanced work about so-called proof theory.

At some point, the educated logician will want to find about *all* these proof styles (at the very least, you should get a general sense of how they respectively work, and appreciate the interrelations between them – the book by Bostock mentioned in §4.3 is very helpful on this). However, standard mathematical logic texts concentrate on the first two systems on our list, as in my recommendations below.

- Next – and for mathematicians this is probably the point at which things actually start getting really interesting! – you need to know how to prove a *soundness* and a *completeness* theorem for your favourite deductive system for first-order logic. That is to say, you need to be able to show that there’s a deduction in your chosen system of a conclusion from given premisses only if the premisses do indeed semantically entail the conclusion (the system doesn’t give false positives), and whenever an inference is semantically valid there’s a formal deduction of the conclusion from the premisses (the system captures all the semantical entailments).
- Depending on where you get to in the text you read, you might at this stage catch a first glimpse of some initial results of model theory which flow quickly from the proofs of soundness and completeness (e.g. the so-called compactness theorem). However, let’s return to consider these results in §5.1 when we consider model theory proper.

4.2 The main recommendations on FOL

There is, unsurprisingly, a *very* long list of possible texts. But the point of this Guide is to choose! So here are four of the best:

Let’s start with a couple of stand-out books which, taken together, make an excellent introduction to the serious study of FOL.

1. Ian Chiswell and Wilfrid Hodges, *Mathematical Logic* (OUP 2007). This very nicely written text is only one notch up in actual difficulty from some introductory texts for philosophers like mine or Paul Teller’s or Nick Smith’s. However – as its title might suggest – it does have a notably

more mathematical ‘look and feel’, being indeed written by mathematicians for mathematicians. Despite that, it remains particularly friendly and approachable and should be entirely manageable for self study by philosophers and mathematicians alike. It is also pleasingly short. Indeed, I’m rather tempted to say that if you *don’t* like this lovely book then serious logic might not be for you!

The briefest headline news is that authors explore a (Gentzen-style) natural deduction system. But by building things up in three stages – so after propositional logic, they consider an interesting fragment of first-order logic before turning to the full-strength version – they make proofs of e.g. the completeness theorem for first-order logic quite unusually comprehensible. For a more detailed description see my [book note](#) on C&H.

Very warmly recommended, then. For the moment, you only *need* read up to and including §7.7; but having got that far, you might as well read the final couple of sections and the Postlude too! (The book has brisk solutions to some exercises. A demerit mark to OUP for not publishing C&H more cheaply.)

Next, you should complement C&H by reading the first half of

2. Christopher Leary and Lars Kristiansen’s *A Friendly Introduction to Mathematical Logic*** (1st edn by Leary alone, Prentice Hall 2000; inexpensive 2nd edn Milne Library 2015; and now [freely available for download](#)).

There is a great deal to like about this book. Chs. 1–3, in either edition, do make a very friendly and helpful introduction to first-order logic, this time done in axiomatic style. At this stage you could stop reading after §3.2, which means you will be reading just over 100 pages. Unusually, L&K dive straight into a full treatment of FOL without spending an introductory chapter or two on propositional logic: but that happily means (in the present context) that you won’t feel that you are labouring through the very beginnings of logic one more time than is really necessary – so this book dovetails very nicely with C&H.

The book *is* again written by mathematicians for a mostly mathematical audience so some illustrations of ideas can presuppose a smattering of elementary background mathematical knowledge; but you will miss very little if you occasionally have to skip an example (and curious philoso-

phers can always resort to Wikipedia, which is quite reliable in this area, for explanations of some mathematical terms). I like the tone very much indeed, and say more about this admirable book in [another book note](#).

Next, here's an alternative to the C&H/L&K pairing which is also wonderfully approachable and can be warmly recommended:

3. Derek Goldrei's *Propositional and Predicate Calculus: A Model of Argument** (Springer, 2005) is explicitly designed for self-study. Read Chs. 1 to 5 (you could skip §§4.4 and 4.5, leaving them until you turn to elementary model theory). While C&H and the first half of L&K together cover overlapping material twice, Goldrei – in much the same total number of pages – covers very similar ground once. So this is a somewhat more gently-paced book, allowing Goldrei to be more expansive about fundamentals, and to give a lot of examples and exercises to test comprehension along the way. A very great deal of thought has gone into making this text as helpful as possible. So if you struggle slightly with the alternative reading, or just want a comfortingly manageable additional text, you should find this exceptionally accessible and useful. Or you might just warm more to Goldrei's style anyway.

Like L&K, Goldrei uses an axiomatic system (which is one reason why, on balance, I still recommend starting with C&H instead). As with C&H and L&K, I like the tone and approach a great deal.

Fourthly, even though it is giving a second bite to an author we've already met, I must mention a rather different discussion of FOL:

4. Wilfrid Hodges 'Elementary Predicate Logic', in the *Handbook of Philosophical Logic*, Vol. 1, ed. by D. Gabbay and F. Guenther, (Kluwer 2nd edition 2001). This is a slightly expanded version of the essay in the first edition of the *Handbook* (read that earlier version if this one isn't available), and is written with Hodges's usual enviable clarity and verve. As befits an essay aimed at philosophically minded logicians, it is full of conceptual insights, historical asides, comparisons of different ways of doing things, etc., so it very nicely complements the more conventional textbook presentations of C&H, L&K and/or Goldrei.

Read at this stage the first twenty sections (70 pp.): they are wonderfully illuminating.

4.3 Some parallel reading

The material covered in the last section is so very fundamental, and the alternative options so very many, that I really do need to say at least something about a few other books and note some different approaches.

So in this section I will list – in rough order of difficulty/sophistication – a small handful of further texts which could well make for useful parallel reading at different levels. Then in the following section I'll note just three books which push on the discussion of FOL in interestingly different ways. In the final section of this chapter of the Guide, I will then mention some other books I've been asked about.

If you've read C&H and L&K you will know about both natural deduction and axiomatic approaches to logic. If you are a philosopher, you may also have already encountered 'truth trees' or 'semantic tableaux', which are often used in introductory logic courses. If you don't know about tableaux, it is worth at some point pausing to find out at least the general principle behind this approach.

The first edition of my *Introduction to Formal Logic** (CUP 2003) has chapters on trees for both propositional and predicate logic. For the second edition, I am re-writing these chapters and making them available as an on-line supplement to what is now a natural-deduction-based text. The first batch of the rewritten trees chapters is now here:

1. Peter Smith, [Propositional Truth Trees](#) (28 pp., 2019).

Alternatively, you could jump straight to the next recommendation which covers trees but also a considerable amount more. This is a text by a philosopher, aimed at philosophers, though mathematicians interested in conceptual foundations could still profit from a quicker browse given the excellent coverage:

2. David Bostock's *Intermediate Logic* (OUP 1997) ranges more widely but not as deeply as Goldrei, for example, and in a much more discursive style. From the preface: 'The book is confined to ... what is called first-order predicate logic, but it aims to treat this subject in very much more detail than a standard introductory text. In particular, whereas an introductory text will pursue just one style of semantics, just one method of proof, and so on, this book aims to create a wider and a deeper understanding by showing how several alternative approaches are possible, and by introducing comparisons between them.' So Bostock does indeed usefully introduce you to tableaux (trees) *and* an Hilbert-style axiomatic proof system *and* natural deduction *and* even a so-called sequent calculus as well (as noted,

it is important eventually to understand what is going on in these different kinds of proof-system). Anyone could profit from at least a quick browse of his Part II to pick up the headline news about the various approaches.

Bostock eventually touches on issues of philosophical interest such as free logic which are not often dealt with in other books at this level. Still, the discussions mostly remain at much the same level of conceptual/mathematical difficulty as the later parts of Teller's book and my own. He proves completeness for tableaux in particular, which I always think makes the needed construction seem particularly natural. *Intermediate Logic* should therefore be, as intended, particularly accessible to philosophers who haven't done much formal logic before and should, if read in parallel, help ease the transition to coping with the more mathematical style of the books recommended in the last section.

Next let me mention a freely available alternative presentation of logic via natural deduction:

3. Neil Tennant, *Natural Logic*** (Edinburgh UP 1978, 1990). Now out of print, but downloadable as [a scanned PDF](#).

All credit to the author for writing the first textbook aimed at an introductory level which does Gentzen-style natural deduction. Tennant thinks that this approach to logic is philosophically highly significant, and in various ways this shows through in his textbook. Although not as conventionally mathematical in look-and-feel as some alternatives, it is in fact *very* careful about important details.

This is not always an easy read, however, despite its being intended as a first logic text for philosophers, which is why I didn't mention it in Chap. 3. However the book is there to freely sample, and some may well find it highly illuminating parallel reading on natural deduction.

Now, I recommended L&K's *A Friendly Introduction* as a follow-up to C&H which uses an axiomatic system. As an alternative, here is an older and much used text which should certainly be very widely available:

4. Herbert Enderton's *A Mathematical Introduction to Logic* (Academic Press 1972, 2002) also focuses on a Hilbert-style axiomatic system, and is often regarded as a classic of exposition.

However, it does strike me as somewhat more difficult than L&K, so I'm not surprised that some students report finding it a bit challenging *if used by itself as a first text*. Still, it is an admirable and very reliable piece

of work which you should be able to cope with as a second text, e.g. after you have tackled C&H. Read up to and including §2.5 or §2.6 at this stage. Later, you can finish the rest of that chapter to take you a bit further into model theory. For more about this classic, see [this book note](#).

Lastly in this section I'll mention – though this time with some hesitation – another much used text. This has gone through multiple editions and should also be in any library, making it a useful natural-deduction based alternative to C&H. Later chapters of this book are also mentioned below in the Guide as possible reading for more advanced work, so it could be worth making early acquaintance with ...

5. Dirk van Dalen, *Logic and Structure** (Springer, 1980; 5th edition 2012). The chapters up to and including §3.2 provide an introduction to FOL via natural-deduction. The treatment *is* often approachable and written with a relatively light touch. However – and this explains my hesitation – it has to be said that the book isn't without its quirks and flaws and inconsistencies of presentation (though perhaps you have to be an alert and pernickety reader to notice and be bothered by them). Still, the coverage and general approach is good.

Mathematicians should be able to cope readily. I suspect, however, that the book would occasionally be tougher going for philosophers if taken from a standing start – which is another reason why I have recommended beginning with C&H instead. (See my [more extended review](#) of the whole book.)

A final aside: The treatments of FOL I have mentioned so far here and in the last section include exercises, of course. But for many more exercises – and this time with extensive worked solutions in the book – you could also look at the ends of Chs. 1, 3 and 4 of René Cori and Daniel Lascar, *Mathematical Logic: A Course with Exercises* (OUP, 2000). I can't, however, particularly recommend the main bodies of those chapters.

4.4 Further into FOL

In this section for enthusiasts, I mention three books, very different from each other, which push on the story about FOL in various ways.

1. Raymond Smullyan, *First-Order Logic** (Springer 1968, Dover Publications 1995) recommends itself. It is an absolute classic, packed with good things.

This is the tersest, most sophisticated, book I'm mentioning in this chapter, but those with a taste for mathematical elegance can certainly try reading Parts I and II, just a hundred pages, after C&H. This beautiful little book is the source and inspiration of many modern treatments of logic based on tree/tableau systems.

Not always easy, especially as the book progresses, but a delight for the mathematically minded.

2. Jan von Plato's *Elements of Logical Reasoning** (CUP, 2014) is based on the author's introductory lectures. A lot of material is touched on in a relatively short compass as von Plato talks about a range of different natural deduction and sequent calculi. I suspect that, without any classroom work to round things out, this might not be easy as a first introduction to logic. But suppose you have already met one system of natural deduction (e.g., as in C&H), and now want to know more about 'proof-theoretic' aspects of this and related systems. Suppose, for example, that you want to know about variant ways of setting up ND systems, about proof-search, about the relation with so-called sequent calculi, etc. Then this is a very clear, approachable and interesting book. Experts will see that there are some novel twists, with deductive systems tweaked to have some very nice features: beginners will be put on the road towards understanding some of the initial concerns and issues in proof theory.
3. Don't be put off by the title of Melvin Fitting's *First-Order Logic and Automated Theorem Proving* (Springer, 1990, 2nd ed. 1996). This is a wonderfully lucid book by a renowned expositor. (Yes, at various places in the book there are illustrations of how to implement various algorithms in Prolog. But either you can easily pick up the very small amount of background knowledge about Prolog that's needed to follow everything that is going on (and that's quite a fun thing to do anyway) or you can just skip those implementation episodes.)

As anyone who has tried to work inside an axiomatic system knows, proof-discovery for such systems is often hard. Which axiom schema should we instantiate with which wffs at any given stage of a proof? Natural deduction systems are nicer: but since we can, in effect, make any new temporary assumption at any stage in a proof, again we still need to keep our wits about us if we are to avoid going off on useless diversions. By contrast, tableau proofs (a.k.a. tree proofs, as in my book) can pretty much write themselves even for quite complex FOL arguments, which is why I used to introduce formal proofs to students that way (in teaching

tableaux, we can largely separate the business of getting across the idea of formality from the task of teaching heuristics of proof-discovery). And because tableau proofs very often write themselves, they are also good for automated theorem proving. Fitting explores both the tableau method and the related so-called resolution method in this exceptionally clearly written book.

This book's emphasis is, then, rather different from most of the other recommended books. So I initially hesitated to mention it here in this Guide. However, I think that the fresh light thrown on first-order logic makes the detour through this book *vaut le voyage*, as the Michelin guides say. (If you don't want to take the full tour, however, there's a nice introduction to proofs by resolution in Shawn Hedman, *A First Course in Logic* (OUP 2004): §1.8, §§3.4–3.5.)

4.5 Other treatments?

Obviously, I have *still* only touched on a small proportion of books that cover first-order logic. Most of the Big Books on Mathematical Logic have treatments of the standard material – though not so often at a level and pace that makes them suitable for a first encounter with this area. See some of the entries in the Guide's [Appendix](#) for more detailed discussion of some options.

I end this chapter, however, by responding to some Frequently Asked Questions, mostly questions raised in response to earlier versions of the Guide.

A blast from the past: What about Mendelson? Somewhat to my surprise, perhaps the most frequent question I used to get asked in response to early versions of the Guide is 'But what about Mendelson, Chs. 1 and 2'? Well, Elliot Mendelson's *Introduction to Mathematical Logic* (Chapman and Hall/CRC 6th edn 2015) was first published in 1964 when I was a student and the world was a great deal younger. The book was I think the first modern textbook of its type (so immense credit to Mendelson for that), and I no doubt owe my whole career to it – it got me through tripos!

It seems that some others who learnt using the book are in their turn still using it to teach from. But let's not get sentimental! It has to be said that the book in its first incarnation was often brisk to the point of unfriendliness, and the basic look-and-feel of the book hasn't changed a great deal as it has run through successive editions. Mendelson's presentation of axiomatic systems of logic are quite tough going, and as the book progresses in later chapters through

formal number theory and set theory, things if anything get somewhat less reader-friendly. Which certainly doesn't mean the book won't repay battling with. But unsurprisingly, fifty years on, there are many rather more accessible and more amiable alternatives for beginning serious logic. Mendelson's book is a landmark worth visiting one day, but I can't recommend starting there. For a little more about it, [see here](#).

(As an aside, if you *do* want an old-school introduction from roughly the same era, I'd recommend instead Geoffrey Hunter, *Metalogic** (Macmillan 1971, University of California Press 1992). This is not groundbreaking in the way e.g. Smullyan's *First-Order Logic* is, nor is it as comprehensive as Mendelson: but it is still an exceptionally good student textbook from a time when there were few to choose from, and I still regard it with admiration. Read Parts One to Three at this stage. And if you are enjoying it, then do eventually finish the book: it goes on to consider formal arithmetic and proves the undecidability of first-order logic, topics we revisit in §5.2. Unfortunately, the typography – from pre-L^AT_EX days – isn't at all pretty to look at: this can make the book's pages appear rather unappealing. But in fact the treatment of an axiomatic system of logic is extremely clear and accessible. It might be worth blowing the dust off your library's copy!)

The very latest thing: What about the Open Logic Text? This is a collaborative, open-source, enterprise, and very much work in progress. [You can download the latest full version from this page](#).

Although this is referred to as a textbook, it is perhaps better regarded as a set of souped-up lecture notes, written at various degrees of sophistication and with various degrees of more book-like elaboration. The chapters on propositional and quantificational logic have expanded very considerably since the 2017 version of this Guide. They still, however, strike me as rather idiosyncratic (why *start* with the LK system of sequent calculus? doesn't the exposition of Gentzen-style natural deduction just go too fast for anyone new to this?).

These notes could be very useful for revision work, – and various selections in a different order could no doubt could well work if accompanied by a lot of lecture-room chat around and about, and more blackboard illustrations. But I do still rather doubt that this Text would work as a purely stand-alone introduction for initial self-study. Your mileage may vary.

Designed for philosophers: What about Sider? Theodore Sider – a very well-known philosopher – has written a text called *Logic for Philosophy** (OUP, 2010) aimed at philosophers, which I've repeatedly been asked to comment on. The

book in fact falls into two halves. The second half (about 130 pages) is on modal logic, and I will return to that in §6.1. The first half of the book (almost exactly the same length) is on propositional and first-order logic, together with some variant logics, so is very much on the topic of this chapter. But while the coverage of modal logic is quite good, I can't at all recommend the first half of this book: I explain why [here](#).

True, a potentially attractive additional feature of this part of Sider's book is that it does contain brief discussions about e.g. some non-classical propositional logics, and about descriptions and free logic. But remember all this is being done in 130 pages, which means that things are whizzing by very fast, so the breadth of Sider's coverage here goes with far too much superficiality. If you want some breadth, Bostock is still much to be preferred, plus perhaps some reading from §6.3 below.

For philosophers again: What about Bell, DeVidi and Solomon? As I've said before, if you concentrate at the outset on a one-proof-style book, you would do well to widen your focus at an early stage to look at other logical options. And one good thing about Bostock's book is that it tells you about different styles of proof-system. A potential alternative to Bostock at about the same level, and which can initially look promising, is John L. Bell, David DeVidi and Graham Solomon's *Logical Options: An Introduction to Classical and Alternative Logics* (Broadview Press 2001). This book covers a lot pretty snappily – for the moment, just Chapters 1 and 2 are relevant – and some years ago I used it as a text for second-year seminar for undergraduates who had used my own tree-based book for their first year course. But many students found it quite hard going, as the exposition is terse, and I found myself having to write very extensive seminar notes. If you want some breadth, you'd again do better sticking with Bostock.

Puzzles galore: What about some of Smullyan's other books? Let's end the chapter on a more positive note!

I have already warmly recommended Smullyan's terse 1968 classic *First-Order Logic*. He went on to write some classic and very accessible texts on Gödel's theorem and on recursive functions, which we'll be mentioning later. But as well as these, Smullyan wrote many 'puzzle' based-books aimed at a wider audience, including the justly famous 1981 *What is the Name of This Book?*²* (Dover Publications reprint, 2011).

More recently, he wrote *Logical Labyrinths* (A. K. Peters, 2009). From the blurb: "This book features a unique approach to the teaching of mathematical logic by putting it in the context of the puzzles and paradoxes of common lan-

guage and rational thought. It serves as a bridge from the author's puzzle books to his technical writing in the fascinating field of mathematical logic. Using the logic of lying and truth-telling, the author introduces the readers to informal reasoning preparing them for the formal study of symbolic logic, from propositional logic to first-order logic, . . . The book includes a journey through the amazing labyrinths of infinity, which have stirred the imagination of mankind as much, if not more, than any other subject."

Smullyan starts, then, with puzzles of the kind where you are visiting an island where there are Knights (truth-tellers) and Knaves (persistent liars) and then in various scenarios you have to work out what's true from what the inhabitants say about each other and the world. And, without too many big leaps, he ends with first-order logic (using tableaux), completeness, compactness and more. This is no substitute for standard texts, but – for those with a taste for being led up to the serious stuff via sequences of puzzles – an entertaining and illuminating supplement.

Smullyan's later *A Beginner's Guide to Mathematical Logic** (Dover Publications, 2014) is more conventional. The first 170 pages are relevant to FOL. A rather uneven read, it seems to me, but again perhaps an interesting supplement to the texts recommended above.

Chapter 5

Continuing Mathematical Logic

We next press on from an initial look at first-order logic to consider other core elements of mathematical logic. Recall from our map of the territory in §2.1 that we'll want to look next at:

- Some elements of the model theory for first-order theories.
- Formal arithmetic, theory of computation, Gödel's incompleteness theorems.
- Elements of set theory.

But also at some point we'll need to touch briefly on two standard 'extras'

- Second-order logic and second-order theories.
- Intuitionistic logic.

Now, as I explained in §1.3, I do very warmly recommend reading a series of books on a topic which overlap in coverage and difficulty, rather than leaping immediately from an 'entry level' text to a really advanced one. Of course, you don't have to follow this excellent advice! But I mention it again here to remind you of one reason why the list of recommendations in most sections is quite extensive and why the increments in coverage/difficulty between successive recommendations are often quite small. So let me stress that *this level of logic really isn't as daunting as the overall length of this chapter might superficially suggest*. Promise!

5.1 From first-order logic to elementary model theory

The completeness theorem is the first high point – the first mathematically serious result – in a course in first-order logic; and some elementary treatments more or less stop there. Many introductory texts, however, continue just a little further with some first steps into model theory. It is clear enough what needs to come next: discussions of the so-called compactness theorem (also called the ‘finiteness theorem’), of the downward and upward Löwenheim-Skolem theorems, and of their implications. There’s less consensus, however, about what other introductory model-theoretic topics you should tackle at an early stage.

As you will see when you dive into the reading, you will very quickly meet claims that involve infinite cardinalities, and there also occasional references to the axiom of choice. Now in fact, even if you haven’t yet done an official set theory course, you may well have picked up all you need to know at least in order to begin model theory. If you have met Cantor’s proof that infinite collections come in different sizes, and if you have been warned to take note when a proof involves making an infinite series of choices, you will probably know enough. And anyway, Goldrei’s chapter recommended in a moment in fact has a brisk section on the ‘Set theory background’ needed at this stage. (If that’s *too* brisk, then perhaps do a skim read of e.g. Paul Halmos’s very short *Naive Set Theory**, or one of the other books mentioned at the beginning of §5.3 below.)

What to read on model theory, then? The very first volume in the prestigious and immensely useful Oxford Logic Guides series is Jane Bridge’s very compact *Beginning Model Theory: The Completeness Theorem and Some Consequences* (Clarendon Press, 1977) which neatly takes the story onwards just a few steps from the reading on FOL mentioned in our §4.2 above. The coverage strikes me as exemplary for a short first introduction. But the writing, though very clear, is rather terse in an old-school way; and the book looks like photo-reproduced typescript, and which makes it rather unpleasant to read. What, then, are the alternatives?

Two of the introductions to FOL that I mentioned in §4.3 have treatments of some elementary model theory. Thus there are fragments of model theory in §2.6 of Herbert Enderton’s *A Mathematical Introduction to Logic* (Academic Press 1972, 2002), followed by a discussion in §2.8 of non-standard analysis: but this, for our purposes here, is perhaps too little done too fast. Dirk van Dalen’s *Logic and Structure** (Springer, 1980; 5th edition 2012) covers rather more model-theoretic material in more detail in his Ch. 3. You could read the first section for revision

on the completeness theorem, then tackle §3.2 on compactness, the Löwenheim-Skolem theorems and their implications, before moving on to the action-packed §3.3 which covers more model theory including non-standard analysis again, and indeed touches on slightly more advanced topics like ‘quantifier elimination’. However, my top votes for help with the first steps in elementary model theory go elsewhere:

1. I have already sung the praises of Derek Goldrei’s *Propositional and Predicate Calculus: A Model of Argument** (Springer, 2005) for the accessibility of its treatment of FOL in the first five chapters. You can now read his §§4.4 and 4.5 (which I previously said you could skip) and then Ch. 6 on ‘Some uses of compactness’ to get a very clear introduction to some model theoretic ideas.

In a little more detail, §4.4 introduces some axiom systems describing various mathematical structures (partial orderings, groups, rings, etc.): this section could be particularly useful to philosophers who haven’t really met the notions before. Then §4.5 introduces the notions of substructures and structure-preserving mappings. After proving the compactness theorem in §6.1 (as a corollary of his completeness proof), Goldrei proceeds to use it in §§6.2 and 6.3 to show various theories can’t be finitely axiomatized, or can’t be nicely axiomatized at all. §6.4 introduces the Löwenheim-Skolem theorems and some consequences, and the following section introduces the notion of diagrams and puts it to work. The final section, §6.6 considers issues about categoricity, completeness and decidability. All this is done with the same admirable clarity as marked out Goldrei’s earlier chapters.

If you then want to go just a bit further, what you probably need next is the rather more expansive

2. Maria Manzano, *Model Theory*, Oxford Logic Guides 37 (OUP, 1999). This book aims to be an introduction at the kind of intermediate level we are currently concerned with. And standing back from the details, I do very much like the way that Manzano structures her book. The sequencing of chapters makes for a very natural path through her material, and the coverage seems very appropriate for a book at her intended level. After chapters about structures (and mappings between them) and about first-order languages, she proves the completeness theorem again,

and then has a sequence of chapters on core model-theoretic notions and proofs. This is all done tolerably accessibly (just half a step up from Goldrei, perhaps).

True, the discussions at some points would have benefitted from rather more informal commentary, motivating various choices. And there are some infelicities. But overall, Manzano's text should work well and there is no evident competitor book at this level. See this [Book Note on Manzano](#) for more details.

And now the path forks. Philosophers will now certainly want to tackle the following recently published book, which strikes me as a particularly impressive achievement:

3. Tim Button and Sean Walsh, *Philosophy and Model Theory** (OUP, 2018). This both explains technical results in model theory, and also explores the appeals to model theory in various branches of philosophy, particularly philosophy of mathematics, but in metaphysics more generally (recall 'Putnam's model-theoretic argument'), the philosophy of science, philosophical logic and more. So that's a very scattered literature that is being expounded, brought together, examined, inter-related, criticised and discussed. Button and Walsh don't pretend to be giving the last word on the many and varied topics they discuss; but they are offering us a very generous helping of first words and second thoughts. It's a large book because it is to a significant extent self-contained: model-theoretic notions get defined as needed, and many of the most philosophically significant results are proved.

The expositions of the technical stuff are usually exemplary (the authors have a good policy of shuffling some extended proofs into chapter appendices), and the philosophical discussion is done with vigour and a very engaging style. The breadth and depth of knowledge brought to the enterprise is remarkable. Philosophical readers of this Guide should find the book fascinating, then. And indeed, with judicious skimming/skipping (the signposting in the book is excellent), mathematicians with an interest in some foundational questions should find much of interest here too.

And that might already be about as far as philosophers may want or need to go with model theory. Many mathematicians, however, will want to take the story about model theory further: the story resumes in §7.2.

Postscript, mostly for mathematicians Thanks to the efforts of the respective authors to write very accessibly, the suggested path through Chiswell & Hodges → (part of) Leary & Kristiansen → (excerpts from) Goldrei → Manzano is not at all a hard road to follow, yet we end up at least in the foothills of model theory. We can climb up to the same foothills by routes involving rather tougher scrambles, taking in some additional side-paths and new views along the way. Here are two suggestions for the more mathematical reader:

4. Shawn Hedman’s *A First Course in Logic* (OUP, 2004) covers a surprising amount of model theory. Ch. 2 tells you about structures and relations between structures. Ch. 4 starts with a nice presentation of a Henkin completeness proof, and then pauses (as Goldrei does) to fill in some background about infinite cardinals etc., before going on to prove the Löwenheim-Skolem theorems and compactness theorems. Then the rest of Ch. 4 and the next chapter covers more introductory model theory, already touching on some topics beyond the scope of Manzano’s book, and Hedman so far could serve as a rather tougher alternative to her treatment. (Then Ch. 6 takes the story on a lot further, quite a way beyond what I’d regard as ‘entry level’ model theory.) For more, see this [Book Note on Hedman](#).
5. Peter Hinman’s weighty *Fundamentals of Mathematical Logic* (A. K. Peters, 2005) is not for the faint-hearted, and I wouldn’t at all recommend using this book as your guide in your *first* outing into this territory. But if you are mathematically minded and have already made a first foray along a gentler route, you could now try reading Ch. 1 – skipping material that is now familiar – and then carefully working through Ch. 2 and Ch. 3 (leaving the last two sections, along with a further chapter on model theory, for later). This should significantly deepen your knowledge of FOL, or at least of its semantic features, and of the beginnings of model theory. For more, see this [Book Note on Hinman](#).

5.2 Computability and Gödelian incompleteness

The standard mathematical logic curriculum, as well as looking at some elementary general results about formalized theories and their models, looks at two particular instances of non-trivial, rigorously formalized, axiomatic systems – arithmetic (a paradigm theory about finite whatnots) and set theory (a paradigm theory about infinite whatnots). We’ll take arithmetic first.

In more detail, there are three inter-related topics here: (a) the elementary (informal) theory of arithmetic computations and of *computability* more generally, (b) an introduction to *formal* theories of arithmetic, leading up to (c) Gödel's epoch-making proof of the *incompleteness* of any sufficiently nice formal theory that can 'do' enough arithmetical computations (a result of profound interest to philosophers).

Now, Gödel's 1931 proof of his incompleteness theorem uses facts in particular about so-called *primitive recursive* functions: these functions are a subclass (but only a subclass) of the computable numerical functions, i.e. only a subclass of the functions which a suitably programmed computer could evaluate (abstracting from practical considerations of time and available memory). A more general treatment of the effectively computable functions (arguably capturing *all* of them) was developed a few years later, and this in turn throws more light on the incompleteness phenomenon.

So there's a choice to be made. Do you look at things in roughly the historical order, first introducing just the primitive recursive functions and theories of formal arithmetic and learning how to prove initial versions of Gödel's incompleteness theorem before moving on to look at the general treatment of computable functions? Or do you do some of the general theory of computation first, turning to the incompleteness theorems later?

Here then to begin with are a couple of introductory books, one taking the first route, one the other route:

1. Peter Smith, *An Introduction to Gödel's Theorems** (CUP 2007, 2nd edition 2013) takes things in something like the historical order. (Note to mathematicians: don't be put off by the series title 'Cambridge Introductions to Philosophy' – putting it in that series was the price I happily paid for cheap paperback publication! This is still quite a meaty logic book, with a lot of theorems and a lot of proofs, but I hope rendered very accessibly.)

The book starts by exploring various ideas such as effective computability informally, and proving two correspondingly informal versions of the incompleteness theorem. The next part of the book gets down to work talking about formal arithmetics, primitive recursive functions, and establishes more formal versions of Gödel's theorems. The third part of the book widens the discussion by exploring the idea of a computable function more generally.

The book's website is at <http://godelbook.net>, where there are supple-

mentary materials of various kinds, including a freely available cut-down version of a large part of the book, *Gödel Without (Too Many) Tears*.

2. Richard Epstein and Walter Carnielli, *Computability: Computable Functions, Logic, and the Foundations of Mathematics* (Wadsworth 2nd edn. 2000: Advanced Reasoning Forum 3rd edn. 2008) does computability theory first. This is a very nicely introductory book on the standard basics, particularly clearly and attractively done, with lots of interesting and illuminating historical information too in Epstein's 28 page timeline on 'Computability and Undecidability' at the end of the book.

Those first two books should be very accessible to those without much mathematical background: but even more experienced mathematicians should appreciate the careful introductory orientation which they provide. And as you'll immediately see, this really is a delightful topic area. Elementary computability theory is conceptually very neat and natural, and the early Big Results are proved in quite remarkably straightforward ways. Just get the hang of the basic 'diagonalization' construction, the idea of Gödel-style coding and one or two other tricks, and off you go

To consolidate your understanding at this level, you might usefully look at

3. Herbert Enderton's *A Mathematical Introduction to Logic* (Academic Press 1972, 2002), Ch. 3. This is a very good short treatment of different strengths of formal theories of arithmetic, and then proves the incompleteness theorem first for a formal arithmetic with exponentiation and then – after touching on other issues – shows how to use the β -function trick to extend the theorem to apply to arithmetic without exponentiation.

Then, going up just one step in mathematical sophistication we arrive at a beautiful book:

4. George Boolos and Richard Jeffrey, *Computability and Logic* (CUP 3rd edn. 1990). This is a modern classic, wonderfully lucid and engaging.

There are in fact later editions – heavily revised and considerably expanded – with John Burgess as a third author. But I know that I am not the only reader to think that the later versions (excellent though they are) do lose something of the original book's famed elegance and individuality. Still, whichever edition comes to hand, do read it! – you will learn a great deal in an enjoyable way.

And finally in this section, two more presentations of computability and the incompleteness phenomenon, again particularly clear and helpful:

5. Herbert E. Enderton's short book *Computability Theory: An Introduction to Recursion Theory* (Associated Press, 2011) is written with attractive zip and lightness of touch (this is a more relaxed book than his *Logic*); it makes a nice alternative to Epstein and Carnielli, though perhaps just a step or two more challenging/sophisticated/abstract. The first chapter is on the informal Computability Concept. There are then chapters on general recursive functions and on register machines (showing that the register-computable functions are exactly the recursive ones), and a chapter on recursive enumerability. Chapter 5 makes 'Connections to Logic' (including proving Tarski's theorem on the undefinability of arithmetical truth and a semantic incompleteness theorem). The final two chapters push on to say something about 'Degrees of Unsolvability' and 'Polynomial-time Computability'. This is all very nicely and accessibly done, and in under 150 pages too.
6. I have already warmly recommended Christopher Leary and Lars Kristiansen's *A Friendly Introduction to Mathematical Logic*** for its coverage of first-order logic. Chs. 4 to 7 now give a very illuminating double treatment of matters related to incompleteness (you don't have to have read the previous chapters, other than noting the arithmetical system N introduced in their §2.8). In headline terms that you'll only come fully to understand in retrospect:
 - (a) L&K's first approach doesn't go overtly via computability. Instead of showing that certain syntactic properties are primitive recursive and showing that all primitive recursive properties can be 'represented' in theories like N (as I do in *IGT*), L&K rely on more directly showing that some key syntactic properties can be represented. This representation result then leads to, inter alia, the incompleteness theorem.
 - (b) L&K follow this, however, with a general discussion of computability, and then use the introductory results they obtain to prove various further theorems, including incompleteness again.

This is all presented with the same admirable clarity as the first part of the book on FOL.

Postscript There are many other introductory treatments covering aspects of computability and/or incompleteness. Indeed there is no sharp boundary to be

drawn between the entry-level accounts mentioned in this section and some of the more sophisticated books on computability and Gödelian incompleteness discussed in Ch. 7. So we can leave discussion of further options until then.

Except that I will mention that there are a couple of relevant texts in the American Mathematical Society’s ‘Student Mathematical Library’. One is Rebecca Weber’s *Computability Theory* (AMA, 2012); but that strikes me as rather too uneven. For present purposes I much prefer

7. A. Shen and N. K. Vereshchagin, *Computable Functions*, (AMA, 2003). This is a lovely, elegant, little book, whose opening chapters can be recommended for giving a differently-structured quick tour through some of the Big Ideas, and hinting at ideas to come.

And finally, if only because I’ve been asked about it a good number of times, I suppose I should also mention the (in)famous

8. Douglas Hofstadter, *Gödel, Escher, Bach* (Penguin, first published 1979). When students enquire about this, I helpfully say that it is the sort of book that you might well like if you like that kind of book, and you won’t if you don’t. It is, to say the least, quirky and distinctive. As far as I recall, though, the parts of the book which touch on techie logical things are pretty reliable and won’t lead you astray. Which is a great deal more than can be said about many popularizing treatments of Gödel’s theorems.

5.3 Beginning set theory

Let’s say that the *elements of set theory* – the beginnings that any logician really ought to know about – will comprise enough to explain how numbers (natural, rational, real) are constructed in set theory (so enough to give us a glimmer of understanding about why it is said that set theory provides a foundation for mathematics). The elements also include the development of ordinal numbers and transfinite induction over ordinals, ordinal arithmetic, and something about the role of the axiom(s) of choice and its role in the arithmetic of cardinals. These initial ideas and constructions can (and perhaps should) be presented fairly informally: but something else that also belongs here at the beginning is an account of the development of ZFC as the now standard way of formally encapsulating and regimenting the key principles involved in the informal development of set theory.

Going beyond these elements we then have e.g. the exploration of ‘large cardinals’, proofs of the consistency and independence of e.g. the Continuum Hypothesis, and a lot more besides. But readings on these further delights are for Ch. 8: this present section is, as advertised, about the first steps for beginners in set theory. Even here, however, there are a large number of books to choose from, so an annotated Guide should (I hope!) again be particularly welcome.

I’ll start by mentioning again a famous ‘bare minimum’ book (only 104 pp. long), which could well be very useful for someone making a start on exploring basic set-theoretic notation and some fundamental concepts.

1. Paul Halmos, *Naive Set Theory** (Originally published 1960, and now available very inexpensively from Martino Fine Books). Informally written in an unusually conversational style for a maths book – though that won’t be a recommendation for everyone! And mostly exceptionally clear, though sometimes (e.g. on Zorn’s Lemma) there are perhaps lapses.

However, Halmos doesn’t cover even all of what I just called the elements of set theory, and most readers will want to look at one or more of the following equally admirable ‘entry level’ treatments which cover a little more in a bit more depth but still very accessibly:

2. Herbert B. Enderton’s, *The Elements of Set Theory* (Academic Press, 1977) forms a trilogy along with the author’s *Logic* and *Computability* which we have already mentioned. The book has exactly the coverage we need at this stage. But more than that, it is particularly clear in marking off the informal development of the theory of sets, cardinals, ordinals etc. (guided by the conception of sets as constructed in a cumulative hierarchy) from the ensuing formal axiomatization of ZFC. It is also particularly good and non-confusing about what is involved in (apparent) talk of classes which are too big to be sets – something that can mystify beginners. It is written with a certain lightness of touch and proofs are often presented in particularly well-signposted stages. The last couple of chapters or so perhaps do get a bit tougher, but overall this really is quite exemplary exposition.
3. Derek Goldrei, *Classic Set Theory* (Chapman & Hall/CRC 1996) has the subtitle ‘For guided independent study’. It is as you might expect – especially if you looked at Goldrei’s FOL text mentioned in §4.2 – extremely clear, and is indeed very well-structured for independent reading.

And moreover, it is fairly attractively written (as set theory books go!). The coverage is very similar to Enderton's, and either book makes a fine introduction (for what little it is worth, I slightly prefer Enderton).

Still starting from scratch, and initially also only half a notch or so up in sophistication from Enderton and Goldrei, we find two more really nice books:

4. Karel Hrbacek and Thomas Jech, *Introduction to Set Theory* (Marcel Dekker, 3rd edition 1999). This eventually goes a bit further than Enderton or Goldrei (more so in the 3rd edition than earlier ones), and you could – on a first reading – skip some of the later material. Though do look at the final chapter which gives a remarkably accessible glimpse ahead towards large cardinal axioms and independence proofs. Again this is a very nicely put together book, and recommended if you want to consolidate your understanding by reading a second presentation of the basics and want then to push on just a bit. (Jech is of course a major author on set theory, and Hrbacek once won a AMA prize for maths writing.)
5. Yiannis Moschovakis, *Notes on Set Theory* (Springer, 2nd edition 2006). A slightly more individual path through the material than the books previously mentioned, again with glimpses ahead and again, to my mind, attractively written.

Of these four books, I'd strongly advise reading one of the first pair and then one of the second pair.

I will add two more firm recommendations at this level. The first might come as a bit of surprise, as it is something of a 'blast from the past'. But we shouldn't ignore old classics – they can have a lot to teach us even if we have read the modern books.

6. Abraham Fraenkel, Yehoshua Bar-Hillel and Azriel Levy, *Foundations of Set-Theory* (North-Holland, originally 1958; but you want the revised 2nd edition 1973). Both philosophers and mathematicians should appreciate the way this puts the development of our canonical ZFC set theory into some context, and also discusses alternative approaches. Standard textbooks can present our canonical theory in a way that makes it seem that ZFC has to be the One True Set Theory, so it is worth understanding from the outset more of how it is arrived and where some choice points are. This book

really is attractively readable, and should be very largely accessible at this early stage. I'm not myself an enthusiast for history for history's sake: but it is very much worth knowing the stories that unfold here.

One thing that every set-theory novice now acquires is the picture of the universe of sets as built up in a hierarchy of stages or levels, each level containing all the sets at previous levels plus new ones (so the levels are cumulative). It is significant that, as the last book makes clear, the picture wasn't firmly in place from the beginning. But the hierarchical conception of the universe of sets is brought to the foreground in

7. Michael Potter, *Set Theory and Its Philosophy* (OUP, 2004). For philosophers (and for mathematicians concerned with foundational issues) this surely is a 'must read', a unique blend of mathematical exposition (mostly about the level of Enderton, with a few glimpses beyond) and extensive conceptual commentary. Potter is presenting not straight ZFC but a very attractive variant due to Dana Scott whose axioms more directly encapsulate the idea of the cumulative hierarchy of sets. However, it has to be said that there are passages which are harder going, sometimes because of the philosophical ideas involved, and just sometimes because of unnecessary expository compression (e.g. at p. 41 where a trick is used to define the notion of a level). However, if you have already read one or two set theory books from earlier in the list, you should have no problems.

Mathematicians who get intrigued by set theory done for its own sake will want to continue the story in Ch. 8. But it is a nice question how much more technical knowledge of results in set theory a philosophy student interested in logic and the philosophy of maths *needs* (if she is not specializing in the technical philosophy of set theory). But getting this far will certainly be a useful start for both mathematicians and philosophers, so let's pause here.

Postscript: some more options at this level There are so many good set theory books that I've been asked about, that I should certainly pause to mention some more.

Books by Ciesielski and by Hajnal and Hamburger, although in the LMS Student Text series and starting from scratch, are not really suitable for the present list (they go too far and probably too fast). But the next four still-introductory books, listed in order of publication, each have many things to recommend them for beginners. One is freely available online, and good libraries should have the others: so browse through and see which might suit your interests and mathematical level.

8. D. van Dalen, H.C. Doets and H. de Swart, *Sets: Naive, Axiomatic and Applied* (Pergamon, 1978). The first chapter covers the sort of elementary (semi)-naive set theory that any mathematician needs to know, up to an account of cardinal numbers, and then takes a first look at the paradox-avoiding ZF axiomatization. This is very attractively and illuminatingly done (or at least, the conceptual presentation is attractive – sadly, and a sign of its time of publication, the book seems to have been photo-typeset from original pages produced on electric typewriter, and the result is visually not attractive at all).

The second chapter carries on the presentation axiomatic set theory, with a lot about ordinals, and getting as far as talking about higher infinities, measurable cardinals and the like. The final chapter considers some applications of various set theoretic notions and principles. Well worth seeking out, if you don't find the typography off-putting.
9. Keith Devlin, *The Joy of Sets* (Springer, 1979: 2nd edn. 1993). The opening chapters of this book are remarkably lucid and attractively written. The opening chapter explores 'naive' ideas about sets and some set-theoretic constructions, and the next chapter introduces axioms for ZFC pretty gently (indeed, non-mathematicians could particularly like Chs 1 and 2, omitting §2.6). Things then speed up a bit, and by the end of Ch. 3 – some 100 pages into the book – we are pretty much up to the coverage of Goldrei's much longer first six chapters, though Goldrei says more about (re)constructing classical maths in set theory. Some will prefer Devlin's fast-track version. (The rest of the book then covers non-introductory topics in set theory, of the kind we take up again in Ch. 8.)
10. Judith Roitman, *Introduction to Modern Set Theory*** (Wiley, 1990: a 2011 version is [freely downloadable](#)). This relatively short, and very engagingly written, book manages to cover quite a bit of ground – we've reached the constructible universe by p. 90 of the downloadable pdf version, and there's even room for a concluding chapter on 'Semi-advanced set theory' which says something about large cardinals and infinite combinatorics. A few quibbles aside, this could make excellent revision material as Roitman is particularly good at highlighting key ideas without getting bogged down in too many details.
11. Winfried Just and Martin Weese, *Discovering Modern Set Theory I: The Basics* (American Mathematical Society, 1996). This covers overlapping

ground to Enderton, patchily but perhaps more zestfully and with a little more discussion of conceptually interesting issues. It is at some places more challenging – the pace can be uneven. But this is evidently written by enthusiastic teachers, and the book is very engaging. (The story continues in a second volume.)

I like the style a lot, and think it works very well. I don't mean the occasional (slightly laboured?) jokes: I mean the in-the-classroom feel of the way that proofs are explored and motivated, and also the way that teach-yourself exercises are integrated into the text. For instance there are exercises that encourage you to produce proofs that are in fact non-fully-justified, and then the discussion explores what goes wrong and how to plug the gaps.

I guess I should mention another book for beginners since it has been warmly recommended to me a number of times:

12. George Tourlakis, *Lectures in Logic and Set Theory, Volume 2: Set Theory* (CUP, 2003). Although this is the second of two volumes, it is a stand-alone text. Indeed Tourlakis goes as far as giving a 100 page outline of the logic covered in the first volume as the long opening chapter in this volume. Assuming you have already studied FOL, you can initially skip this chapter, consulting if/when needed. That still leaves over 400 pages on basic set theory, with long chapters on the usual axioms, on the Axiom of Choice, on the natural numbers, on order and ordinals, and on cardinality. (The final chapter on forcing should be omitted at this stage, and strikes me as less clear than what precedes it.)

As the title suggests, Tourlakis aims to retain something of the relaxed style of the lecture room, complete with occasional asides and digressions. And as the length suggests, the pace is quite gentle and expansive, with room to pause over questions of conceptual motivation etc. However, there is a certain quite excessive and unnecessary formalism that many will find off-putting (e.g. it takes Tourlakis a page to prove that if x is a set and $x \subseteq \{\emptyset\}$, then $x = \emptyset$ or $x = \{\emptyset\}$). And simple constructions and results take a long time to arrive. We don't meet the von Neumann ordinals for three hundred pages, and we don't get to Cantor's theorem on the uncountability of $\mathcal{P}(\omega)$ until p. 455!

So while this book might be worth dipping into for some of the motivational explanations, I can't myself recommend it.

Let's end though on a positive note. So here, to conclude, are two notably

short books aimed at beginners that are also excellent and could also be very useful additional reading to add to the boxed main recommendations above:

13. A. Shen and N. K. Vereshchagin, *Basic Set Theory* (American Mathematical Society, 2002), just over 100 pages, and mostly about ordinals. But very readable, with 151 ‘Problems’ as you go along to test your understanding. Potentially *very* helpful by way of revision/consolidation.
14. Ernest Schimmerling, *A Course on Set Theory* (CUP, 2011) is slightly mistitled: it is just 160 pages, again introductory but with some rather different emphases and occasional forays into what Roitman would call ‘semi-advanced’ material. Quite an engaging supplementary read at this level.

5.4 Extras: two variant logics

At some fairly early point in your logical education, you ought to get to know something about the motivation for one familiar *extension* of FOL, namely second-order logic, where we now also allow generalizations which quantify into predicate position. You also should get to know something about the motivation for rejecting the classical understanding of negation built into standard FOL, and then adopting a *rival*, so-called intuitionist, logic.

5.4.1 Second-order logic

Consider the intuitive principle of arithmetical induction. Take any property X ; if 0 has it, and for any n it is passed down from n to $n + 1$, then *all* numbers must have X . It is very natural to regiment this as follows:

$$\forall X[(X0 \wedge \forall n(Xn \rightarrow X(n+1))] \rightarrow \forall n Xn]$$

where the so-called second-order quantifier $\forall X$ quantifies ‘into predicate position’ and supposedly runs over all *properties* of numbers. But this formalization is illegitimate in standard first-order logic, where the first-order quantifiers can only run over *objects* in the relevant domain.

Important historical aside Note that the earliest presentations of quantificational logic, in Frege and in *Principia Mathematica*, were of logics that did allow

this kind of higher-order quantification. The now standard concentration on first-order logic was a later development, and its historical development is a rather tangled story.

It is, indeed, well worth exploring this history at some point. For two wonderfully helpful guides, see first José Ferreiros, ‘[The road to modern logic – an interpretation](#)’. *Bulletin of Symbolic Logic* 7 (2001): 441–484. And then for the broader setting see also Paolo Mancosu, Richard Zach and Calixto Badesa, ‘[The Development of Mathematical Logic from Russell to Tarski: 1900–1935](#)’ in Leila Haaparanta, ed., *The History of Modern Logic* (OUP, 2009, pp. 318–471).

To resume How should we handle apparent second-order quantifiers? One option is to keep your logic first-order but go set-theoretic and write the induction principle instead as

$$\forall X[(0 \in X \wedge \forall n(n \in X \rightarrow (n+1) \in X)] \rightarrow \forall n n \in X]$$

where the variable ‘ X ’ is now a sorted *first-order* variable running over sets. But arguably this changes the subject (our ordinary principle of arithmetical induction doesn’t *seem* to be about sets), and there are other issues too. So why not take things at face value and allow that the ‘natural’ logic of informal mathematical discourse often deploys second-order quantifiers that range over properties (expressed by predicates) as well as first-order quantifiers that range over objects (denoted by names), i.e. why not allow quantification into predicate position as well as into name position?

For a brief but very informative overview of second-order logic, see the article

1. Herbert Enderton, ‘[Second-order and Higher-order Logic](#)’, *The Stanford Encyclopedia of Philosophy*. (NB, this is the archived original SEP article from 2007/2009, which has been replaced by a much longer and much less ‘entry-level’ piece.)

You could then try one or both of

2. Herbert Enderton *A Mathematical Introduction to Logic* (Academic Press 1972, 2002), Ch. 4.
3. Dirk van Dalen, *Logic and Structure**, (Springer, 1980; 5th edition 2012). Ch. 4,

Any logician ought to know at least what these recommendations together explain – i.e. the contrast between ‘full’ vs ‘Henkin’ semantics, why the com-

pactness and Löwenheim-Skolem theorems fail for second-order logic with full semantics, and some important consequences of this.

And having got this far, philosophers in particular should want to dive into the self-recommending

4. Stewart Shapiro, *Foundations without Foundationalism: A Case for Second-Order Logic*, Oxford Logic Guides 17 (Clarendon Press, 1991), Chs. 3–5.

Indeed it would be a pity, while you have Shapiro’s wonderfully illuminating book in your hands, to skip the initial philosophical/methodological discussion in the first two chapters here. This whole book is a modern classic, remarkably accessible, and important too for the contrasting side-light it throws on FOL.

If, after tackling Shapiro, you want an up-to-the-minute sophisticated review of second order logic with many further pointers to an extensive literature then you can return to the *Stanford Encyclopedia* to look at the [2019 article on second-order and higher-order logic](#) by Jouko Väänänen.

5.4.2 Intuitionist logic

(a) Ask yourself: Why should we endorse the principle that $\varphi \vee \neg\varphi$ is always true no matter the domain? Even prescindng from issues of vagueness, does the world have to cooperate to determine any given proposition to be true or false? Could there perhaps be domains – mathematics, for example – where truth is in some good sense a matter of provability-in-principle, and falsehood a matter of refutability-in-principle? And if so, would every proposition from such a domain be either true or false, i.e. provable-in-principle or refutable-in-principle? Why so?

Perhaps then we *shouldn’t* suppose that the principle that $\varphi \vee \neg\varphi$ always holds true no matter the domain, as a matter of logic. Maybe the law of excluded middle, even when it does hold for some domain, doesn’t hold as a matter of pure *logic* but e.g. as a matter of metaphysics or of the specific nature of truth for that domain.

Thoughts like this give rise to one kind of challenge to classical two-valued logic, which of course does assume excluded middle across the board. For more first words on this ‘intuitionist’ challenge, see e.g. the brief remarks in Theodore Sider’s *Logic for Philosophy* (OUP, 2010), §3.5, or in John L. Bell, David DeVidi and Graham Solomon’s *Logical Options: An Introduction to Classical and Alternative Logics* (Broadview Press 2001), pp. 192–196.

Then here is an introductory treatment which gives both motivation and a first account of intuitionist logic in a natural deduction framework.

1. Richard Zach et al., *Open Logic Text*, §§49.1–49.4.

Now, in this natural deduction framework, the syntactic shape of intuitionist logic is straightforward. Assuming uncontroversial background, it is easily seen that having all instances of $\varphi \vee \neg\varphi$ as a theorem is equivalent to allowing all instances of the inference rule from $\neg\neg\varphi$, *infer* φ . So, putting it crudely, we just have to drop the usual double negation rule from the rules of your favourite proof system so as to block the derivation of excluded middle.

Things get more complicated when we turn to look at a revised *semantics* apt for a non-classical understanding of the logical operators. The now-standard version due to Kripke is a brand of ‘possible-world semantics’ of a kind that is also used in modal logic. Philosophers might prefer, therefore, to cover intuitionism after first looking at modal logic more generally. Still, you should have no great difficulty diving straight into

2. Dirk van Dalen, *Logic and Structure**, (Springer, 1980; 5th edition 2012), §§5.1–5.3.

This might usefully be read in tandem with the first 29 pages of

3. Nick Bezhanishvili and Dick de Jongh, *Intuitionistic Logic*.

If, however, you want to approach intuitionistic logic *after* looking at some modal logic, then you could read the appropriate chapters of the terrific

4. Graham Priest, *An Introduction to Non-Classical Logic** (CUP, much expanded 2nd edition 2008). Chs. 6 and 20 on intuitionistic logic respectively flow on naturally from Priest’s treatment in that book of modal logics, first propositional and then predicate.

Oddly, Priest’s tableaux-based book seems to be one of the few introductory texts covering modal logic which take the natural sideways step of discussing intuitionistic logic too.

(b) One theme not highlighted in these initial readings is that intuitionistic logic, from a more proof-theoretic point of view, seemingly has a certain naturalness compared with classical logic. Suppose we think of the natural deduction introduction rule for a logical operator as fixing the meaning of the operator

(rather than a prior semantics fixing what is the appropriate rule). Then the corresponding elimination rule surely ought to be in harmony with the introduction rule, in the sense of just ‘undoing’ its effect, i.e. giving us back from a wff φ with O as its main operator no more than what an application of the O -introduction rule to justify φ would have to be based on. For this idea of harmony see e.g. Neil Tennant’s *Natural Logic*, §4.12. From this perspective the characteristically classical double negation rule is seemingly an outlier, not ‘harmonious’. There’s now a significant literature on this idea: but for some initial discussion, and pointers to other discussions, you could still usefully start with Peter Milne, ‘Classical harmony: rules of inference and the meaning of the logical constants’, *Synthese* vol. 100 (1994), pp. 49–94.

For an introduction to intuitionistic logic in a related spirit, see

5. Stephen Pollard, *A Mathematical Prelude to the Philosophy of Mathematics* (Springer, 2014), Ch. 7, ‘Intuitionist Logic’.

(c) If you want to pursue things further, both of the following range widely and have a large number of further references:

6. Joan Moschovakis, ‘[Intuitionistic Logic](#)’, *The Stanford Encyclopedia of Philosophy*,
7. Dirk van Dalen, ‘Intuitionistic Logic’, in the *Handbook of Philosophical Logic*, Vol. 5, ed. by D. Gabbay and F. Guenther, (Kluwer 2nd edition 2002). This, however, rachets up the level of difficulty, and has parts you will probably want to/need to skip.

Note, by the way, that what we’ve been talking about is intuitionist *logic* not intuitionist *mathematics*. For more on the relation, see both the SEP entry by Moschovakis and

8. Rosalie Iemhoff, ‘[Intuitionism in the Philosophy of Mathematics](#)’, *The Stanford Encyclopedia of Philosophy*.

And then the stand-out recommendation is

9. Michael Dummett, *Elements of Intuitionism*, Oxford Logic Guides 39 (OUP 2nd edn. 2000). A classic text. But (it has to be said) this is quite hard going in parts, and some of the formal aspects are perhaps tougher than they need be. Note though that the final chapter, ‘Concluding philosophical remarks’, is very well worth looking at, even if you bale out from reading all the formal work that precedes it.

But further investigation of intuitionistic mathematics really needs to be set in the context of a wider engagement with varieties of constructive mathematics.

Chapter 6

Modal and other logics

Here's the menu for this chapter, which is probably of particular interest to philosophers rather than mathematicians:

- 6.1 We start with modal logic – like second-order logic, an *extension* of classical logic – for two reasons. First, the basics of modal logic don't involve anything mathematically more sophisticated than the elementary first-order logic covered in Chiswell and Hodges (indeed to make a start on modal logic you don't even need as much as that). Second, and much more importantly, philosophers working in many areas surely *ought* to know a little modal logic.
- 6.2 Classical logic demands that all terms denote one and one thing – i.e. it doesn't countenance empty terms which denote nothing, or plural terms which may denote more than one thing. In this section, we look at logics which remain classical in spirit (retaining the usual sort of definition of logical consequence) but which do allow empty and/or plural terms.
- 6.3 Among variant logics which are non-classical in spirit, we have already mentioned intuitionist logic. Here we consider some other deviations from the classical paradigm, starting with those which require that conclusions be related to their premisses by some connection of *relevance* (so the classical idea that a contradiction entails anything is dropped).

6.1 Modal logic

- (a) Basic modal logic is the logic of the one-place propositional operators ' \Box ' and ' \Diamond ' (read these as 'it is necessarily true that' and 'it is possibly true that');

it adopts new principles like $\Box\varphi \rightarrow \varphi$ and $\varphi \rightarrow \Diamond\varphi$, and investigates more disputable principles like $\Diamond\varphi \rightarrow \Box\Diamond\varphi$. Different readings of the box and diamond generate different modal logics, though initially you can concentrate on just three main systems, known as T, S4 and S5.

There are some nice introductory texts written for philosophers, though I think the place to start is clear:

1. Rod Girle, *Modal Logics and Philosophy* (Acumen 2000; 2nd edn. 2009). Girle's logic courses in Auckland, his enthusiasm and abilities as a teacher, are justly famous. Part I of this book provides a particularly lucid introduction, which in 136 pages explains the basics, covering both trees and natural deduction for some propositional modal logics, and extending to the beginnings of quantified modal logic. Philosophers may well want to go on to read Part II of the book, on applications of modal logic.

Also introductory, though perhaps a little brisker than Girle at the outset, is

2. Graham Priest, *An Introduction to Non-Classical Logic** (CUP, much expanded 2nd edition 2008): read Chs 2–4 for propositional modal logics, Chs 14–18 for quantified logics. This book – which is a terrific achievement and enviably clear and well-organized – systematically explores logics of a wide variety of kinds, using trees throughout in a way that can be very illuminating indeed. Although it starts from scratch, however, it would be better to come to the book with a prior familiarity with non-modal logic via trees, as in [my chapters available here](#). We will be mentioning Priest's book again in later sections for its excellent coverage of other non-classical themes.

If you do start with Priest's book, then at some point you will want to supplement it by looking at a treatment of natural deduction proof systems for modal logics. One option is to dip into Tony Roy's comprehensive '[Natural Derivations for Priest, *An Introduction to Non-Classical Logic*](#)' which presents natural deduction systems corresponding to the propositional logics presented in tree form in the first edition of Priest (so the first half of the new edition). Another possible way in to ND modal systems would be via the opening chapters of

3. James Garson, *Modal Logic for Philosophers** (CUP, 2006; 2nd end. 2014). This again is certainly intended as a gentle introductory book: it deals with

both ND and semantic tableaux (trees), and covers quantified modal logic. It is reasonably accessible, but not – I think – as attractive as Girle. But that’s a judgement call.

(b) We now go a step up in sophistication (and the more mathematical might want to start here):

4. Melvin Fitting and Richard L. Mendelsohn, *First-Order Modal Logic* (Kluwer 1998). This book starts again from scratch, but then does go rather more snappily, with greater mathematical elegance (though it should certainly be accessible to anyone who is modestly on top of non-modal first-order logic). It still also includes a good amount of philosophically interesting material. Recommended.

A few years ago, I would have said that getting as far as Fitting and Mendelsohn will give most philosophers a good enough grounding in basic modal logic. But e.g. Timothy Williamson’s book *Modal Logic as Metaphysics* (OUP, 2013) calls on rather more, including e.g. second-order modal logics. If you need to sharpen your knowledge of the technical background here, I guess there is nothing for it but to tackle

5. Nino B. Cocchiarella and Max A. Freund, *Modal Logic: An Introduction to its Syntax and Semantics* (OUP, 2008). The blurb announces that “a variety of modal logics at the sentential, first-order, and second-order levels are developed with clarity, precision and philosophical insight”. However, when I looked at this book with an eye to using it for a graduate seminar, I confess I didn’t find it very appealing: so I do suspect that many philosophical readers will indeed find the treatments in this book rather relentless. However, the promised wide coverage could make the book of particular interest to determined philosophers concerned with the kind of issues that Williamson discusses.

Finally, I should certainly draw your attention to the classic book by Boolos mentioned at the end of §7.4, where modal logic gets put to use in exploring results about provability in arithmetic, Gödel’s Second Incompleteness Theorem, and more.

Postscript for philosophers Old hands learnt their modal logic from G. E. Hughes and M. J. Cresswell *An Introduction to Modal Logic* (Methuen, 1968). This was at the time of original publication a unique book, enormously helpfully

bringing together a wealth of early work on modal logic in an approachable way. Nearly thirty years later, the authors wrote a heavily revised and updated version, *A New Introduction to Modal Logic* (Routledge, 1996). This newer version like the original one concentrates on *axiomatic* versions of modal logic, which doesn't make it always the most attractive introduction from a modern point of view. But it is still an admirable book at an introductory level (and going beyond), and a book that enthusiasts can still learn from.

I didn't recommend the first part of Theodore Sider's *Logic for Philosophy** (OUP, 2010). However, the second part of the book which is entirely devoted to modal logic (including quantified modal logic) and related topics like Kripke semantics for intuitionistic logic is significantly better. Compared with the early chapters with their inconsistent levels of coverage and sophistication, the discussion here develops more systematically and at a reasonably steady level of exposition. There is, however, a lot of (acknowledged) straight borrowing from Hughes and Cresswell, and – like those earlier authors – Sider also gives axiomatic systems. But if you just want a brisk and pretty clear explanation of Kripke semantics, and want to learn e.g. how to search systematically for countermodels, Sider's treatment in his Ch. 6 could well work as a basis. And then the later treatments of quantified modal logic in Chs 9 and 10 (and some of the conceptual issues they raise) are also brief, lucid and approachable.

Postscript for the more mathematical Here are a couple of good introductory modal logic books with a mathematical flavour:

6. Sally Popkorn, *First Steps in Modal Logic* (CUP, 1994). The author is, at least in this possible world, identical with the late mathematician Harold Simmons. This book, which entirely on propositional modal logics, is written for computer scientists. The Introduction rather boldly says 'There are few books on this subject and even fewer books worth looking at. None of these give an acceptable mathematically correct account of the subject. This book is a first attempt to fill that gap.' This considerably oversells the case: but the result is illuminating and readable.
7. Also just on propositional logic, I'd recommend Patrick Blackburn, Maarten de Rijke and Yde Venema's *Modal Logic* (CUP, 2001). This is one of the Cambridge Tracts in Theoretical Computer Science: but again don't let that provenance put you off – it is (relatively) accessibly and agreeably written, with a lot of signposting to the reader of possible routes through the book, and interesting historical notes. I think it works pretty well, and

will certainly give you an idea about how non-philosophers approach modal logic.

Going in a different direction, if you are particularly interested in the relation between modal logic and intuitionistic logic (see §5.4.2), then you might want to look at

Alexander Chagrov and Michael Zakharyashev *Modal Logic* (OUP, 1997). This is a volume in the Oxford Logic Guides series and again concentrates on propositional modal logics. Written for the more mathematically minded reader, it tackles things in an unusual order, starting with an extended discussion of intuitionistic logic, and is pretty demanding. But enthusiasts should find this enlightening.

Finally, if you want to explore even more, there's the giant *Handbook of Modal Logic*, edited by van Bentham et al. (Elsevier, 2005). You can get an idea of what's in the volume by looking at [this page of links to the opening pages of the various contributions](#).

6.2 Free logic, plural logic

We next look at what happens if you stay first-order (keep your variables running over objects), stay classical in spirit (keep the same basic notion of logical consequence) but allow terms that fail to denote (free logic) or allow terms that refer to more than one thing (plural logic).

6.2.1 Free Logic

Classical logic assumes that any term denotes an object in the domain of quantification, and in particular assumes that all functions are total, i.e. defined for every argument – so an expression like ' $f(c)$ ' always denotes. But mathematics cheerfully countenances partial functions, which may lack a value for some arguments. Should our logic accommodate this, by allowing terms to be free of existential commitment? In which case, what would such a 'free' logic look like?

For some background and motivation, see the gently paced and accessible

1. David Bostock, *Intermediate Logic* (OUP 1997), Ch. 8.

Then for more detail, we have a helpful overview article in the ever-useful

Stanford Encyclopedia:

2. John Nolt, ‘[Free Logic](#)’, *The Stanford Encyclopedia of Philosophy*.

For formal treatments in, respectively, natural deduction and tableau settings, see:

3. Neil Tennant, *Natural Logic*** (Edinburgh UP 1978, 1990), §7.10.
4. Graham Priest, *An Introduction to Non-Classical Logic** (CUP, 2nd edition 2008), Ch. 13.

If you want to explore further (going rather beyond the basics), you could make a start on

5. Ermanno Bencivenga, ‘Free Logics’, in D. Gabbay and F. Guentner (eds.), *Handbook of Philosophical Logic, vol. III: Alternatives to Classical Logic* (Reidel, 1986). Reprinted in D. Gabbay and F. Guentner (eds.), *Handbook of Philosophical Logic*, 2nd edition, vol. 5 (Kluwer, 2002).

Postscript Rolf Schock’s *Logics without Existence Assumptions* (Almqvist & Wiskell, Stockholm 1968) is still well worth looking at on free logic after all this time. And for a collection of articles of interest to philosophers, around and about the topic of free logic, see Karel Lambert, *Free Logic: Selected Essays* (CUP 2003).

6.2.2 Plural logic

In ordinary mathematical English we cheerfully use plural denoting terms such as ‘2, 4, 6, and 8’, ‘the natural numbers’, ‘the real numbers between 0 and 1’, ‘the complex solutions of $z^2 + z + 1 = 0$ ’, ‘the points where line L intersects curve C ’, ‘the sets that are not members of themselves’, and the like. Such locutions are entirely familiar, and we use them all the time without any sense of strain or logical impropriety. We also often generalize by using plural quantifiers like ‘any natural numbers’ or ‘some reals’ together with linked plural pronouns such as ‘they’ and ‘them’. For example, here is a version of the Least Number Principle: given any natural numbers, one of them must be the least. By contrast, there are some reals – e.g. those strictly between 0 and 1 – such that no one of *them* is the least.

Plural terms and plural quantifications appear all over the place in everyday mathematical argument. It is surely a project of interest to logicians to regiment

and evaluate the informal modes of argument involving such constructions. This is the business of plural logic, a topic of much recent discussion.

For an introduction, see

1. Øystein Linnebo, ‘[Plural Quantification](#)’, *The Stanford Encyclopedia of Philosophy*.

And do read at least these two of the key papers listed in Linnebo’s expansive bibliography:

2. Alex Oliver and Timothy Smiley, ‘Strategies for a Logic of Plurals’, *Philosophical Quarterly* (2001) pp. 289–306.
3. George Boolos, ‘To Be Is To Be a Value of a Variable (or to Be Some Values of Some Variables)’, *Journal of Philosophy* (1984) pp. 430–50. Reprinted in Boolos, *Logic, Logic, and Logic* (Harvard University Press, 1998).

(Oliver and Smiley give reasons why there is indeed a real subject here: you can’t readily eliminate all plural talk in favour e.g. of singular talk about sets. Boolos’s classic will tell you something about the possible relation between plural logic and second-order logic.) Then, for much more about plurals, you can follow up more of Linnebo’s bibliography, or could look at

4. Thomas McKay, *Plural Predication* (OUP 2006),

which is clear and approachable. However:

Real enthusiasts for plural logic will want to dive into the philosophically argumentative and formally rich

5. Alex Oliver and Timothy Smiley, *Plural Logic* (OUP 2013: revised and expanded second edition, 2016).

6.3 Relevance logics (and wilder logics too)

(a) Classically, if $\varphi \vdash \psi$, then $\varphi, \chi \vdash \psi$ (read ‘ \vdash ’ as ‘entails’: irrelevant premisses can be added without making a valid entailment invalid). And if $\varphi, \chi \vdash \psi$ then $\varphi \vdash \chi \rightarrow \psi$ (that’s the Conditional Proof rule in action, a rule that seems

to capture something essential to our understanding of the conditional). Presumably we have $P \vdash P$. So we have $P, Q \vdash P$. Whence $P \vdash Q \rightarrow P$. It seems then that classical logic's carefree attitude to questions of relevance in deduction and its dubious version of the conditional are tied closely together.

Classically, we also have $\varphi, \neg\varphi \vdash \psi$. But doesn't the inference from P and $\neg P$ to Q commit another fallacy of relevance? And again, if we allow it and also allow conditional proof, we will have $P \vdash \neg P \rightarrow Q$, another seemingly unhappy result about the conditional.

Can we do better? What does a more relevance-aware logic look like?

For useful introductory reading, see

1. Edwin Mares, '[Relevance Logic](#)', *The Stanford Encyclopedia of Philosophy*.
2. Graham Priest, Koji Tanaka, Zach Weber, '[Paraconsistent logic](#)', *The Stanford Encyclopedia of Philosophy*.

These two articles have many pointers for further reading across a range of topics, so I can be brief here. But I will mention two wide-ranging introductory texts:

3. Graham Priest, *An Introduction to Non-Classical Logic** (CUP, much expanded 2nd edition 2008). Look now at Chs. 7–10 for a treatment of propositional logics of various deviant kinds. Priest starts with relevance logic and goes on to also treat logics where there are truth-value gaps, and – more wildly – logics where a proposition can be both true and false (there's a truth-value glut). Then, if this excites you, carry on to look at Chs. 21–24 where the corresponding quantificational logics are presented. This book really is a wonderful resource.
4. J. C. Beall and Bas van Fraassen's *Possibilities and Paradox* (OUP 2003), also covers a range of logics. In particular, Part III of the book covers relevance logic and also non-standard logics involving truth-value gaps and truth-value gluts. (It is worth looking too at the earlier parts of the book on logical frameworks generally and on modal logic.)

The obvious next place to go is then the very lucid

5. Edwin Mares, *Relevant Logic: A Philosophical Interpretation* (CUP 2004). As the title suggests, this book has very extensive conceptual discussion

alongside the more formal parts elaborating what might be called the mainstream tradition in relevance logics.

(b) Note, however, that although they get discussed in close proximity in the books by Priest and by Beall and van Fraassen, there's no tight connection between (i) the reasonable desire to have a more relevance-aware logic (e.g. without the principle that a contradiction implies everything) and (ii) the highly revisionary proposal that there can be propositions which are both true and false at the same time.

At the risk of corrupting the youth, if you are interested in exploring the latter immodest proposal further, then I can point you to

6. Graham Priest, '[Dialetheism](#)', *The Stanford Encyclopedia of Philosophy*.

(c) There is however a minority tradition on relevance that I myself find extremely appealing, developed by Neil Tennant, initially in scattered papers.

Classically, we can unrestrictedly paste proofs together – so can e.g. paste together an uncontroversial proof that for the inference $P \therefore P \vee Q$ and a proof for the inference $\neg P, P \vee Q \therefore Q$ to give us a proof for the (dubious) inference $P, \neg P \therefore Q$. But maybe what is getting us into trouble is pasting together proofs with premisses which explicitly contradict each other. What if we restrict that? You will need to know a little proof theory to appreciate how Tennant handles this thought – though you can get a flavour of the approach from the early programmatic paper

6. Neil Tennant, '[Entailment and proofs](#)', *Proceedings of the Aristotelian Society* LXXIX (1979) 167–189.

Tennant wrote a sequence of interesting follow-up papers over the next decades, but he has now brought everything neatly together in his argumentative and technically engaging

7. Neil Tennant, *Core Logic* (OUP, 2017).

Part III

Beyond the basics

About Part III

In this third part of the Guide there are suggestions for more advanced reading on the various areas of logic we have already touched on in Part II. As noted in Chapter 9, there are yet further areas of logic which the Guide does not (yet!) cover: e.g. type theory, the lambda calculus, infinitary logics. But sufficient unto the day!

Three points before we begin:

- Before tackling the more difficult material in the next two chapters, it could be very well worth first taking the time to look at one or two of the wider-ranging Big Books on mathematical logic which will help consolidate your grip on the basics at the level of Chapter 5 and/or push things on just a bit. See the slowly growing set of [Book Notes](#) for some guidance on what's available.
- I did try to be fairly systematic in Chapter 5, aiming to cover the different core areas at comparable levels of detail, depth and difficulty. The coverage of various topics from here on is more varied: the recommendations can be many or few (or non-existent!) depending on my own personal interests and knowledge.
- I do, however, still aim to cluster suggestions within sections or subsections in rough order of difficulty. In this Part, boxes are used set off a number of acknowledged classics that perhaps any logician ought to read one day, whatever their speciality.

And a warning to those philosophers still reading: some of the material I point to is inevitably mathematically quite demanding!

Chapter 7

More advanced reading on some core topics

In this chapter, there are some suggestions for more advanced reading on a selection of topics in and around the core mathematical logic curriculum we looked at in Chs. 4 and 5 – other than set theory, which we return to at length in the next chapter.

7.1 Proof theory

Proof theory has been (and continues to be) something of a poor relation in the standard Mathematical Logic curriculum: the usual survey textbooks don't discuss it. Yet this is a fascinating area, of interest to philosophers, mathematicians, and computer scientists who all *ought* to be concerned with the notion of proof! So let's start to fill this gap next.

(a) I mentioned in §4.3 the introductory book by Jan von Plato, *Elements of Logical Reasoning** (CUP, 2014), which approaches elementary logic with more of an eye on proof theory than is at all usual: you might want to take a look at that book if you didn't before. However, you should start serious work by reading the same author's extremely useful encyclopaedia entry:

1. Jan von Plato, '[The development of proof theory](#)', *The Stanford Encyclopedia of Philosophy*.

This will give you orientation and introduce you to some main ideas: there is also an excellent bibliography which you can use to guide further exploration.

That biblio perhaps makes the rest of this section a bit redundant; but for what they are worth, here are my less informed suggestions. Everyone will agree that you should certainly read the little hundred-page classic

2. Dag Prawitz, *Natural Deduction: A Proof-Theoretic Study** (Originally published in 1965: reprinted Dover Publications 2006).

And if you want to follow up in more depth Prawitz's investigations of the proof theory of various systems of logic, the next place to look is surely

3. Sara Negri and Jan von Plato, *Structural Proof Theory* (CUP 2001). This is a modern text which is neither too terse, nor too laboured, and is generally very clear. When we read it in a graduate-level reading group, we did find we needed to pause sometimes to stand back and think a little about the motivations for various technical developments. So perhaps a few more 'classroom asides' in the text would have made a rather good text even better. But this is still *extremely* helpful.

Then in a more mathematical style, there is the editor's own first contribution to

4. Samuel R. Buss, ed., *Handbook of Proof Theory* (North-Holland, 1998). Later chapters of this very substantial handbook do get pretty hard-core; but the 78 pp. opening chapter by Buss himself, a 'Introduction to Proof Theory'**, is readable, and [freely downloadable](#). (Student health warning: there are, I am told, some confusing misprints in the cut-elimination proof.)

(b) And now the path through proof theory forks. In one direction, the path cleaves to what we might call classical themes (I don't mean themes simply concerning classical logic, as intuitionistic logic was also treated as central from the start: I mean themes explicit in the early classic papers in proof theory, in particular in Gentzen's work). It is along this path that we find e.g. Gentzen's famous proof of the consistency of first-order Peano Arithmetic using proof-theoretic ideas. One obvious text on these themes remains

5. Gaisi Takeuti, *Proof Theory** (North-Holland 1975, 2nd edn. 1987: reprinted Dover Publications 2013). This is a true classic – if only because for a while it was about the only available book on most of its topics. Later chapters won't really be accessible to beginners. But you could/should try reading Ch. 1 on logic, §§1–7 (and perhaps the beginnings of §8, pp. 40–45, which is easier than it looks if you compare how

you prove the completeness of a tree system of logic). Then on Gentzen's proof, read Ch. 2, §§9–11 and §12 up to at least p. 114. This isn't exactly plain sailing – but if you skip and skim over some of the more tedious proof-details you can pick up a very good basic sense of what happens in the consistency proof.

Gentzen's proof of the consistency of depends on transfinite induction along ordinals up to ε_0 ; and the fact that it requires just so much transfinite induction to prove the consistency of first-order PA is an important characterization of the strength of the theory. The project of 'ordinal analysis' in proof theory aims to provide comparable characterizations of other theories in terms of the amount of transfinite induction that is needed to prove *their* consistency. Things do get quite hairy quite quickly, however.

6. For a glimpse ahead, you could look at (initial segments of) these useful notes for mini-courses by Michael Rathjen, on '[The Realm of Ordinal Analysis](#)' and '[Proof Theory: From Arithmetic to Set Theory](#)'.

Turning back from these complications, however, let's now glance down the other path from the fork, where we investigate not the proof theory of theories constructed in familiar logics but rather investigate non-standard logics themselves. Reflection on the structural rules of classical and intuitionistic proof systems naturally raises the question of what happens when we tinker with these rules. We noted before the inference which takes us from the trivial $P \vdash P$ by 'weakening' to $P, Q \vdash P$ and on, via 'conditional proof', to $P \vdash Q \rightarrow P$. If we want a conditional that conforms better to intuitive constraints of relevance, then we need to block that proof: is 'weakening' the culprit? The investigation of what happens if we tinker with standard structural rules such as weakening belongs to substructural logic, outlined in

7. Greg Restall, '[Substructural Logics](#)', *The Stanford Encyclopedia of Philosophy*.

(which again has an admirable bibliography). And the place to continue exploring these themes at length is the same author's splendid

8. Greg Restall, *An Introduction to Substructural Logics* (Routledge, 2000), which will also teach you a lot more about proof theory generally in a very accessible way. Do read at least the first seven chapters.

(You could note again here the work on Neil Tennant mentioned at the very end of §6.3.)

(c) For the more mathematically minded, here are a few more books of considerable interest. I'll start with a couple that in fact aim to be accessible to beginners. They wouldn't be my recommendations of texts to start from, but they could be very useful if you already know a bit of proof theory.

9. Jean-Yves Girard, *Proof Theory and Logical Complexity. Vol. I* (Bibliopolis, 1987) is intended as an introduction [Vol. II was never published]. With judicious skipping, which I'll signpost, this is readable and insightful, though some proofs are a bit arm-waving.

So: skip the 'Foreword', but do pause to glance over 'Background and Notations' as Girard's symbolic choices need a little explanation. Then the long Ch. 1 is by way of an introduction, proving Gödel's two incompleteness theorems and explaining 'The Fall of Hilbert's Program': if you've read some of the recommendations in §5.2 above, you can probably skim this fairly quickly, though noting Girard's highlighting of the notion of 1-consistency.

Ch. 2 is on the sequent calculus, proving Gentzen's *Hauptsatz*, i.e. the crucial cut-elimination theorem, and then deriving some first consequences (you can probably initially omit the forty pages of annexes to this chapter). Then also omit Ch. 3 whose content isn't relied on later. But Ch. 4 on 'Applications of the *Hauptsatz*' is crucial (again, however, at a first pass you can skip almost 60 pages of annexes to the chapter). Take the story up again with the first two sections of Ch. 6, and then tackle the opening sections of Ch. 7. A rather bumpy ride but very illuminating.

10. A. S. Troelstra and H. Schwichtenberg's *Basic Proof Theory* (CUP 2nd ed. 2000) is a volume in the series Cambridge Tracts in Computer Science. Now, one theme that runs through the book indeed concerns the computer-science idea of formulas-as-types and invokes the lambda calculus: however, it is in fact possible to skip over those episodes in you aren't familiar with the idea. The book, as the title indicates, is intended as a first foray into proof theory, and it *is* reasonably approachable. However it is perhaps a little cluttered for my tastes because it spends quite a bit of time looking at slightly different ways of doing natural deduction and slightly different ways of doing the sequent calculus, and the differences may matter more for computer scientists with implementation concerns than for others. You could, however, with a bit of skipping, very usefully read just Chs. 1–3, the first halves of Chs. 4 and 6, and then Ch. 10 on arithmetic again.

And now for three more advanced offerings, worth commenting on:

11. I have already mentioned the compendium edited by Samuel R. Buss, *Handbook of Proof Theory* (North-Holland, 1998), and the fact that you can download its substantial first chapter. You can also freely access Ch. 2 on ‘[First-Order Proof-Theory of Arithmetic](#)’. Later chapters of the Handbook are of varying degrees of difficulty, and cover a range of topics (though there isn’t much on ordinal analysis).
12. Wolfram Pohlers, *Proof Theory: The First Step into Impredicativity* (Springer 2009). This book has introductory ambitions, to say something about so-called ordinal analysis in proof theory as initiated by Gentzen. But in fact I would judge that it requires quite an amount of mathematical sophistication from its reader. From the blurb: “As a ‘warm up’ Gentzen’s classical analysis of pure number theory is presented in a more modern terminology, followed by an explanation and proof of the famous result of Feferman and Schütte on the limits of predicativity.” The first half of the book is probably manageable if (but only if) you already have done some of the other reading. But then the going indeed gets pretty tough.
13. H. Schwichtenberg and S. Wainer, *Proofs and Computations* (Association of Symbolic Logic/CUP 2012) “studies fundamental interactions between proof-theory and computability”. The first four chapters, at any rate, will be of wide interest, giving another take on some basic material and should be manageable given enough background. Sadly, I found the book to be not particularly well written and it sometimes makes heavier weather of its material than seems really necessary. Still worth the effort though.

There is a recent more introductory text by Katalin Bimbó, *Proof Theory: Sequential Calculi and Related Formalisms* (CRC Press, 2014); but having looked at it, I’m not minded to recommend this.

7.2 Beyond the model-theoretic basics

(a) If you want to explore model theory beyond the entry-level material in §5.1, why not start with a quick warm-up, with some reminders of headlines and some useful pointers to the road ahead:

1. Wilfrid Hodges, ‘[First-order model theory](#)’, *The Stanford Encyclopedia of Philosophy*.

Now, we noted before that e.g. the wide-ranging texts by Hedman and Hinman eventually cover a substantial amount of model theory. But you will do even better with two classic stand-alone treatments of the area which really choose themselves. Both in order of first publication and of eventual difficulty we have:

2. C. Chang and H. J. Keisler, *Model Theory** (originally North Holland 1973: the third edition has been inexpensively republished by Dover Books in 2012). This is the Old Testament, the first systematic text on model theory. Over 550 pages long, it proceeds at an engagingly leisurely pace. It is particularly lucid and is extremely nicely constructed with different chapters on different methods of model-building. A really fine achievement that still remains a good route in to the serious study of model theory.
3. Wilfrid Hodges, *A Shorter Model Theory* (CUP, 1997). The New Testament is Hodges's encyclopedic original *Model Theory* (CUP 1993). This shorter version is half the size but still really full of good things. It does get tougher as the book progresses, but the earlier chapters of this modern classic, written with this author's characteristic lucidity, should certainly be readily manageable.

My suggestion would be to read the first three long chapters of Chang and Keisler, and then perhaps pause to make a start on

4. J. L. Bell and A. B. Slomson, *Models and Ultraproducts** (North-Holland 1969; Dover reprint 2006). Very elegantly put together: as the title suggests, the book focuses particularly on the ultra-product construction. At this point read the first five chapters for a particularly clear introduction.

You could then return to Ch. 4 of C&K to look at (some of) their treatment of the ultra-product construction, before perhaps putting the rest of their book on hold and turning to Hodges.

(b) A level up again, here are two more books. The first has been around long enough to have become regarded as a modern standard text. The second is more recent but also comes well recommended. Their coverage is significantly different – so those wanting to get seriously into model theory should probably take a look at both:

5. David Marker, *Model Theory: An Introduction* (Springer 2002). Despite its title, this book would surely be hard going if you haven't already tackled

some model theory (at least read Manzano first). But despite being sometimes a rather bumpy ride, this highly regarded text will teach you a great deal. Later chapters, however, probably go far over the horizon for all except those most enthusiastic readers of this Guide who are beginning to think about specializing in model theory – it isn't published in the series 'Graduate Texts in Mathematics' for nothing!

6. Katrin Tent and Martin Ziegler, *A Course in Model Theory* (CUP, 2012). From the blurb: "This concise introduction to model theory begins with standard notions and takes the reader through to more advanced topics such as stability, simplicity and Hrushovski constructions. The authors introduce the classic results, as well as more recent developments in this vibrant area of mathematical logic. Concrete mathematical examples are included throughout to make the concepts easier to follow." Again, although it starts from the beginning, it could be a bit of challenge to readers without any prior exposure to the elements of model theory – though I, for one, find it more approachable than Marker's book.

(c) So much for my principal suggestions. Now for an assortment of additional/alternative texts. Here are two more books which aim to give general introductions:

7. Philipp Rothmaler's *Introduction to Model Theory* (Taylor and Francis 2000) is, overall, comparable in level of difficulty with, say, the first half of Hodges. As the blurb puts it: "This text introduces the model theory of first-order logic, avoiding syntactical issues not too relevant to model theory. In this spirit, the compactness theorem is proved via the algebraically useful ultraproduct technique (rather than via the completeness theorem of first-order logic). This leads fairly quickly to algebraic applications," Now, the opening chapters are indeed very clear: but oddly the introduction of the crucial ultraproduct construction in Ch. 4 is done very briskly (compared, say, with Bell and Slomson). And thereafter it seems to me that there is some unevenness in the accessibility of the book. But others have recommended this text, so I mentioned it as a possibility worth checking out.
8. Bruno Poizat's *A Course in Model Theory* (English edition, Springer 2000) starts from scratch and the early chapters give an interesting and helpful account of the model-theoretic basics, and the later chapters form a rather comprehensive introduction to stability theory. This often-recommended

book is written in a rather distinctive style, with rather more expansive class-room commentary than usual: so an unusually engaging read at this sort of level.

Another book which is often mentioned in the same breath as Poizat, Marker, and now Tent and Ziegler as a modern introduction to model theory is *A Guide to Classical and Modern Model Theory*, by Annalisa Marcja and Carlo Toffalori (Kluwer, 2003) which also covers a lot: but I prefer the previously mentioned books.

(d) The next two suggestions are of books which are helpful on particular aspects of model theory:

9. Kees Doets's short *Basic Model Theory** (CSLI 1996) highlights so-called Ehrenfeucht games. This is enjoyable and very instructive.
10. Chs. 2 and 3 of Alexander Prestel and Charles N. Delzell's *Mathematical Logic and Model Theory: A Brief Introduction* (Springer 1986, 2011) are brisk but clear, and can be recommended if you want a speedy review of model theoretic basics. The key feature of the book, however, is the sophisticated final chapter on applications to algebra, which might appeal to mathematicians with special interests in that area. For a very little more on this book, see my [Book Note](#).

Indeed, as we explore model theory, we quickly get entangled with algebraic questions. And as well as going (so to speak) in the direction from logic to algebra, we can make connections the other way about, starting from algebra. For something on this approach, see the following short, relatively accessible, and illuminating book:

11. Donald W. Barnes and John M. Mack, *An Algebraic Introduction to Mathematical Logic* (Springer, 1975).

(e) Let me also briefly allude to the sub-area of Finite Model Theory which arises particularly from consideration of problems in the theory of computation (where, of course, we are interested in *finite* structures – e.g. finite databases and finite computations over them). What happens, then, to model theory if we restrict our attention to finite models? Trakhtenbrot's theorem, for example, tells that the class of sentences true in any finite model is not recursively enumerable. So there is no deductive theory for capturing such finitely valid sentences (that's a surprise, given that there's a complete deductive system for the valid

sentences!). It turns out, then, that the study of finite models is surprisingly rich and interesting (at least for enthusiasts!). So why not dip into one or other of

12. Leonard Libkin, *Elements of Finite Model Theory* (Springer 2004).
13. Heinz-Dieter Ebbinghaus and Jörg Flum, *Finite Model Theory* (Springer 2nd edn. 1999).

Either is a very good standard text to explore the area with, though I prefer Libkin's.

(f) Two afterthoughts. First, it is illuminating to read something about the history of model theory: there's a good, and characteristically lucid, unpublished piece by a now-familiar author here:

14. W. Hodges, 'Model Theory'.

Second, one thing you will have noticed if you tackle a few texts beyond the level of Manzano's is that the absolutely key compactness theorem (for example) can be proved in a variety of ways – indirectly via the completeness proof, via a more direct Henkin construction, via ultraproducts, etc. How do these proofs inter-relate? Do they generalize in different ways? Do they differ in explanatory power? For a quite excellent essay on this – on the borders of mathematics and philosophy (and illustrating that there is indeed very interesting work to be done in that border territory), see

15. Alexander Paseau, 'Proofs of the Compactness Theorem', *History and Philosophy of Logic* 31 (2001): 73–98.

7.3 Computability

In §5.2 we took a first look at the related topics of computability, Gödelian incompleteness, and theories of arithmetic. In this and the next two main sections, we return to these topics, taking them separately (though this division is necessarily somewhat artificial).

7.3.1 Computable functions

(a) Among the Big Books on mathematical logic, the one with the most useful treatment of computability is probably

1. Peter G. Hinman, *Fundamentals of Mathematical Logic* (A. K. Peters, 2005). Chs. 4 and 5 on recursive functions, incompleteness etc. strike me as the best written, most accessible (and hence most successful) chapters in this very substantial book. The chapters could well be read after my *IGT* as somewhat terse revision for mathematicians, and then as sharpening the story in various ways. Ch. 8 then takes up the story of recursion theory (the author's home territory).

However, good those these chapters are, I'd still recommend starting your more advanced work on computability with

2. Nigel Cutland, *Computability: An Introduction to Recursive Function Theory* (CUP 1980). This is a rightly much-reprinted classic and is beautifully lucid and well-organized. This *does* have the look-and-feel of a traditional maths text book of its time (so with fewer of the classroom asides we find in some modern, more discursive books). However, if you got through most of e.g. Boolos, Burgess and Jeffrey without too much difficulty, you ought certainly to be able to tackle this as the next step. Very warmly recommended.

And of more recent books covering computability this level (i.e. a step up from the books mentioned in §5.2, I also particularly like

3. S. Barry Cooper, *Computability Theory* (Chapman & Hall/CRC 2003). This is a very nicely done modern textbook. Read at least Part I of the book (about the same level of sophistication as Cutland, but with some extra topics), and then you can press on as far as your curiosity takes you, and get to excitements like the Friedberg-Muchnik theorem.

(b) The inherited literature on computability is huge. But, being *very* selective, let me mention three classics from different generations:

4. Rózsa Péter, *Recursive Functions* (originally published 1950: English translation Academic Press 1967). This is by one of those logicians who was 'there at the beginning'. It has that old-school slow-and-steady un-flashy lucidity that makes it still a considerable pleasure to read. It remains very worth looking at.
5. Hartley Rogers, Jr., *Theory of Recursive Functions and Effective Computability* (McGraw-Hill 1967) is a heavy-weight state-of-the-art-then classic, written at the end of the glory days of the initial development of the

logical theory of computation. It quite speedily gets advanced. But the opening chapters are still excellent reading and are action-packed. At least take it out of the library, read a few chapters, and admire!

6. Piergiorgio Odifreddi, *Classical Recursion Theory*, Vol. 1 (North Holland, 1989) is well-written and discursive, with numerous interesting asides. It's over 650 pages long, so it goes further and deeper than other books on the main list above (and then there is Vol. 2). But it certainly starts off quite gently paced and very accessible and can be warmly recommended for consolidating and extending your knowledge.

(c) A number of books we've already mentioned say something about the fascinating historical development of the idea of computability: as we noted before, Richard Epstein offers a very helpful 28 page timeline on 'Computability and Undecidability' at the end of the 2nd edn. of Epstein/Carnielli (see §5.2). Cooper's short first chapter on 'Hilbert and the Origins of Computability Theory' also gives some of the headlines. Odifreddi too has many historical details. But here are two more good essays on the history:

7. Robert I. Soare, 'The History and Concept of Computability', in E. Griffor, ed., *Handbook of Computability Theory* (Elsevier 1999).
8. Robin Gandy, 'The Confluence of Ideas in 1936' in R. Herken, ed., *The Universal Turing Machine: A Half-century Survey* (OUP 1988). Seeks to explain why so many of the absolutely key notions all got formed in the mid-thirties.

7.3.2 Computational complexity

Computer scientists are – surprise, surprise! – interested in the theory of feasible computation, and any logician should be interested in finding out at least a little about the topic of computational complexity.

1. Shawn Hedman *A First Course in Logic* (OUP 2004): Ch. 7 on 'Computability and complexity' has a nice review of basic computability theory before some lucid sections discussing computational complexity.
2. Michael Sipser, *Introduction to the Theory of Computation* (Thomson, 2nd edn. 2006) is a standard and very well regarded text on computation aimed at computer scientists. It aims to be very accessible and to take its time

giving clear explanations of key concepts and proof ideas. I think this is very successful as a general introduction and I could well have mentioned the book before. But I'm highlighting the book in this subsection because its last third is on computational complexity.

3. Ofed Goldreich, *P, NP, and NP-Completeness* (CUP, 2010). Short, clear, and introductory stand-alone treatment of computational complexity.
4. Ashley Montanaro, *Computational Complexity*. Excellent 2012 lecture notes, lucid and detailed and over 100 pages (also include a useful quick guide to further reading).
5. You could also look at the opening chapters of the pretty encyclopaedic Sanjeev Arora and Boaz Barak *Computational Complexity: A Modern Approach*** (CUP, 2009). The authors say 'Requiring essentially no background apart from mathematical maturity, the book can be used as a reference for self-study for anyone interested in complexity, including physicists, mathematicians, and other scientists, as well as a textbook for a variety of courses and seminars.' And it at least starts very readably. [A late draft of the book](#) can be freely downloaded.

7.4 Incompleteness and related matters

(a) If you have looked at my book and/or Boolos and Jeffrey you should now be in a position to appreciate the terse elegance of

1. Raymond Smullyan, *Gödel's Incompleteness Theorems*, Oxford Logic Guides 19 (Clarendon Press, 1992). This is delightfully short – under 140 pages – proving some beautiful, slightly abstract, versions of the incompleteness theorems. This is a modern classic which anyone with a taste for mathematical elegance will find rewarding.

2. Equally short and equally elegant is Melvin Fitting's, *Incompleteness in the Land of Sets** (College Publications, 2007). This approaches things from a slightly different angle, relying on the fact that there is a simple correspondence between natural numbers and 'hereditarily finite sets' (i.e. sets which have a finite number of members which in turn have a finite number of members which in turn ... where all downward membership chains bottom out with the empty set).

In terms of difficulty, these two lovely brief books could easily have appeared among our introductory readings in Chapter 5. I have put them here because (as I see it) the simpler, more abstract, stories they tell can probably only be fully appreciated if you've first met the basics of computability theory and the incompleteness theorems in a more conventional treatment.

You ought also at some stage read an even briefer, and still officially introductory, treatment of the incompleteness theorems,

3. Craig Smoryński, '[The incompleteness theorems](#)' in J. Barwise, editor, *Handbook of Mathematical Logic*, pp. 821–865 (North-Holland, 1977), which covers a lot very compactly.

After these, where should you go if you want to know more about matters more or less directly to do with the incompleteness theorems?

4. Raymond Smullyan's *Diagonalization and Self-Reference*, Oxford Logic Guides 27 (Clarendon Press 1994) is an investigation-in-depth around and about the idea of diagonalization that figures so prominently in proofs of limitative results like the unsolvability of the halting problem, the arithmetical undefinability of arithmetical truth, and the incompleteness of arithmetic. Read at least Part I.
5. Torkel Franzén, *Inexhaustibility: A Non-exhaustive Treatment* (Association for Symbolic Logic/A. K. Peters, 2004). The first two-thirds of the book gives another take on logic, arithmetic, computability and incompleteness. The last third notes that Gödel's incompleteness results have a positive consequence: 'any system of axioms for mathematics that we recognize as correct can be properly extended by adding as a new axiom a formal statement expressing that the original system is consistent. This suggests that our mathematical knowledge is inexhaustible, an essentially philosophical topic to which this book is devoted.' Not always easy (you will need to know something about ordinals before you read this), but very illuminating.
6. Per Lindström, *Aspects of Incompleteness* (Association for Symbolic Logic/A. K. Peters, 2nd edn., 2003). This is probably for enthusiasts. A terse book, not always reader-friendly in its choices of notation and the brevity of its argument, but the more mathematical reader will find that it again repays the effort.

(b) Going in a rather different direction, you will recall from my *IGT2* or other reading on the second incompleteness theorem that we introduced the so-called

derivability conditions on $\Box\varphi$ where this is an abbreviation for (or at any rate, is closely tied to) $\text{Prov}(\ulcorner\varphi\urcorner)$, which expresses the claim that the wff φ , whose Gödel number is $\ulcorner\varphi\urcorner$, is provable in some given theory. The ‘ \Box ’ here functions rather like a modal operator: so what is its modal logic? This is investigated in:

7. George Boolos, *The Logic of Provability* (CUP, 1993). From the blurb: “What [the author] does is to show how the concepts, techniques, and methods of modal logic shed brilliant light on the most important logical discovery of the twentieth century: the incompleteness theorems of Kurt Gödel and the ‘self-referential’ sentences constructed in their proof. The book explores the effects of reinterpreting the notions of necessity and possibility to mean provability and consistency.” This is a wonderful modern classic.

7.5 Theories of arithmetic

The readings in §5.2 will have introduced you to the canonical first-order theory of arithmetic, first-order Peano Arithmetic, as well as to some subsystems of PA (in particular, Robinson Arithmetic) and second-order extensions. And you will already know that first-order PA has non-standard models (in fact, it even has uncountably many non-isomorphic models which can be built out of natural numbers!).

So what to read next? You should get to more about models of PA. For a taster, you could look at these nice lecture notes:

1. Jaap van Oosten, ‘[Introduction to Peano Arithmetic: Gödel Incompleteness and Nonstandard Models](#)’. (1999)

But for a fuller story, you need

2. Richard Kaye’s *Models of Peano Arithmetic* (Oxford Logic Guides, OUP, 1991) which tells us a great deal about non-standard models of PA. This will reveal more about what PA can and can’t prove, and will introduce you to some non-Gödelian examples of incompleteness. This does get pretty challenging in places, and it is probably best if you’ve already done a little model theory. Still, this is a terrific book, and deservedly a modern classic.

(There’s also another volume in the Oxford Logic Guides series which can be thought of as a sequel to Kaye’s for real enthusiasts with more background in

model theory, namely Roman Kossak and James Schmerl, *The Structure of Models of Peano Arithmetic*, OUP, 2006. But this is much tougher.)

Next, going in a rather different direction, and explaining a lot about arithmetics weaker than full PA, here's another modern classic:

3. Petr Hájek and Pavel Pudlák, *Metamathematics of First-Order Arithmetic*** (Springer 1993). Now freely available from projecteuclid.org. This is pretty encyclopaedic, but the long first three chapters, say, actually do remain surprisingly accessible for such a work. This is, eventually, a must-read if you have a serious interest in theories of arithmetic and incompleteness.

And what about going beyond first-order PA? We know that full second-order PA (where the second-order quantifiers are constrained to run over all possible sets of numbers) is unaxiomatizable, because the underlying second-order logic is unaxiomatizable. But there are axiomatizable subsystems of second order arithmetic. These are wonderfully investigated in another encyclopaedic modern classic:

4. Stephen Simpson, *Subsystems of Second-Order Logic* (Springer 1999; 2nd edn CUP 2009). The focus of this book is the project of 'reverse mathematics' (as it has become known): that is to say, the project of identifying the weakest theories of numbers-and-sets-of-numbers that are required for proving various characteristic theorems of classical mathematics.

We know that we can reconstruct classical analysis in pure set theory, and rather more neatly in set theory with natural numbers as unanalysed 'urelemente'. But just *how much* set theory is needed to do the job, once we have the natural numbers? The answer is: stunningly little. The project of exploring what's needed is introduced very clearly and accessibly in the first chapter, which is a must-read for anyone interested in the foundations of mathematics. This introduction is freely available [at the book's website](#).

Chapter 8

Serious set theory

In §5.3, we gave suggestions for readings on the elements of set theory. These will have introduced you to the standard set theory ZFC, and the iterative hierarchy it seeks to describe. They also explained e.g. how we can construct the real number system in set theoretic terms (so giving you a sense of what might be involved in saying that set theory can be used as a ‘foundation’ for another mathematical theory). You will have in addition learnt something about the role of the axiom of choice, and about the arithmetic of infinite cardinal and ordinal numbers.

If you looked at the books by Fraenkel/Bar-Hillel/Levy or by Potter, however, you will also have noted that while standard ZFC is the market leader, it is certainly not the only set theory on the market.

So where do we go next? We’ll divide the discussion into three.

- We start by focusing again on our canonical theory, ZFC. The exploration eventually becomes seriously hard mathematics – and, to be honest, it becomes of pretty specialist interest (very well beyond ‘what every logician ought to know’). But it isn’t clear where to stop in a Guide like this, even if I have no doubt overdone it!
- So next we backtrack from those excursions towards the frontiers to consider old questions about the Axiom of Choice (as this is of particular conceptual interest).
- Then we will say something about non-standard set theories, rivals to ZFC (again, the long-recognised possibility of different accounts, with different degrees of departure from the canonical theory, is of considerable conceptual interest and you don’t need a huge mathematical background to understand some of the options).

8.1 ZFC, with all the bells and whistles

8.1.1 A first-rate overview

One option is immediately to go for broke and dive in to the modern bible, which is highly impressive not just for its size:

1. Thomas Jech, *Set Theory*, The Third Millennium Edition, Revised and Expanded (Springer, 2003). The book is in three parts: the first, Jech says, every student should know; the second part every budding set-theorist should master; and the third consists of various results reflecting ‘the state of the art of set theory at the turn of the new millennium’. Start at page 1 and keep going to page 705 (or until you feel glutted with set theory, whichever comes first).

This is indeed a masterly achievement by a great expositor. And if you’ve happily read e.g. the introductory books by Enderton and then Moschovakis mentioned earlier in the Guide, then you should be able to cope pretty well with Part I of the book while it pushes on the story a little with some material on small large cardinals and other topics. Part II of the book starts by telling you about independence proofs. The Axiom of Choice is consistent with ZF and the Continuum Hypothesis is consistent with ZFC, as proved by Gödel using the idea of ‘constructible’ sets. And the Axiom of Choice is independent of ZF, and the Continuum Hypothesis is independent with ZFC, as proved by Cohen using the much more tricky idea of ‘forcing’. The rest of Part II tells you more about large cardinals, and about descriptive set theory. Part III is indeed for enthusiasts.

Now, Jech’s book is wonderful, but let’s face it, the sheer size makes it a trifle daunting. It goes quite a bit further than many will need, and to get there it does in places speed along a bit faster than some will feel comfortable with. So what other options are there for if you want to take things more slowly?

8.1.2 Rather more slowly, towards forcing

Why not start with some preliminary historical orientation. If you looked at the old book by Fraenkel/Bar-Hillel/Levy which was recommended earlier in the Guide, then you will already know something of the early days. Alternatively, there is a nice short introductory overview

2. José Ferreirós, ‘[The early development of set theory](#)’, *The Stanford Encycl. of Philosophy*. (Ferreirós has also written a terrific book *Labyrinth of Thought: A History of Set Theory and its Role in Modern Mathematics* (Birkhäuser 1999), which at some stage in the future you might well want to read.)

You could also browse through the substantial article

3. Akhiro Kanamori, ‘The Mathematical Development of Set Theory from Cantor to Cohen’, *The Bulletin of Symbolic Logic* (1996) pp. 1-71, a revised version of which is [downloadable here](#). (You will very probably need to skip chunks of this at a first pass: but even a partial grasp will help give you a good sense of the lie of the land for when you work on the technicalities.)

Then to start filling in details, a much admired older book (still a fine first treatment of its topic) is

4. Frank R. Drake, *Set Theory: An Introduction to Large Cardinals* (North-Holland, 1974), which – at a gentler pace? – overlaps with Part I of Jech’s bible, but also will tell you about Gödel’s Constructible Universe and some more about large cardinals.

But the crucial next step – that perhaps marks the point where set theory gets really challenging – is to get your head around Cohen’s idea of forcing used in independence proofs. However, there is not getting away from it, this is tough. In the admirable

5. Timothy Y. Chow, ‘[A beginner’s guide to forcing](#)’,

(and don’t worry if initially even this beginner’s guide looks puzzling), Chow writes

All mathematicians are familiar with the concept of *an open research problem*. I propose the less familiar concept of *an open exposition problem*. Solving an open exposition problem means explaining a mathematical subject in a way that renders it totally perspicuous. Every step should be motivated and clear; ideally, students should feel that they could have arrived at the results themselves. The proofs should be ‘natural’ . . . [i.e., lack] any ad hoc constructions or brilliancies. I believe that it is an open exposition problem to explain forcing.

In short: if you find that expositions of forcing tend to be hard going, then join the club.

Here though is a very widely used and much reprinted textbook, which nicely complements Drake's book and which has (inter alia) a pretty good first presentation of forcing:

6. Kenneth Kunen, *Set Theory: An Introduction to Independence Proofs* (North-Holland, 1980). If you have read (some of) the introductory set theory books mentioned in the Guide, you should actually find much of this text now pretty accessible, and can probably speed through some of the earlier chapters, slowing down later, until you get to the penultimate chapter on forcing which you'll need to take slowly and carefully. This is a rightly admired classic text.

Kunen has since published another, totally rewritten, version of this book as *Set Theory** (College Publications, 2011). This later book is quite significantly longer, covering an amount of more difficult material that has come to prominence since 1980. Not just because of the additional material, my current sense is that the earlier book may remain the slightly more approachable read.

8.1.3 Pausing for problems

At this point mathematicians could very usefully dip into the problem sets in the excellent

7. Péter Komjáti and Vilmos Totik, *Problems and Theorems in Classical Set Theory* (Springer, 2006). From the blurb: "Most of classical set theory is covered, classical in the sense that independence methods are not used, but classical also in the sense that most results come from the period between 1920–1970. Many problems are also related to other fields of mathematics Rather than using drill exercises, most problems are challenging and require work, wit, and inspiration." Look at the problems that pique your interest: the authors give answers, often very detailed.

8.1.4 Pausing for more descriptive set-theory

Early on, it was discovered that the Axiom of Choice implied the existence of 'pathological' subsets of the reals, sets lacking desirable properties like being

measurable. In reaction, there developed the study of nice, well-behaved, ‘definable’ sets – the topic of descriptive set theory. This has already been touched on in e.g. Kunen’s book. For more see e.g.

8. Yiannis Moschovakis, *Descriptive Set Theory* (North Holland, 1980: second edition AMS, 2009, [freely available from the author’s website](#)).
9. Alexander Kechris, *Classical Descriptive Set Theory* (Springer-Verlag, 1994).

8.1.5 Forcing further explored

To return, though, to the central theme of independence proofs and other results that can be obtained by forcing: Kunen’s classic text takes a ‘straight down the middle’ approach, starting with what is basically Cohen’s original treatment of forcing, though he does relate this to some variant approaches. Here are two of them:

10. Raymond Smullyan and Melvin Fitting, *Set Theory and the Continuum Problem* (OUP 1996, Dover Publications 2010). This medium-sized book is divided into three parts. Part I is a nice introduction to axiomatic set theory (in fact, officially in its NBG version – see §8.3). The shorter Part II concerns matters round and about Gödel’s consistency proofs via the idea of constructible sets. Part III gives a different take on forcing (a variant of the approach taken in Fitting’s earlier *Intuitionistic Logic, Model Theory, and Forcing*, North Holland, 1969). This is beautifully done, as you might expect from two writers with a quite enviable knack for wonderfully clear explanations and an eye for elegance.
11. Keith Devlin, *The Joy of Sets* (Springer 1979, 2nd edn. 1993) Ch. 6 introduces the idea of Boolean-Valued Models and their use in independence proofs. The basic idea is fairly easily grasped, but details perhaps get hairy. For more on this theme, see John L. Bell’s classic *Set Theory: Boolean-Valued Models and Independence Proofs* (Oxford Logic Guides, OUP, 3rd edn. 2005). The relation between this approach and other approaches to forcing is discussed e.g. in Chow’s paper and the last chapter of Smullyan and Fitting.

Here are three further, more recent, books which highlight forcing ideas, one very short, the others much more wide-ranging:

10. Nik Weaver, *Forcing for Mathematicians* (World Scientific, 2014) is less than 150 pages (and the first applications of the forcing idea appear after just 40 pages: you don't have to read the whole book to get the basics). From the blurb: "Ever since Paul Cohen's spectacular use of the forcing concept to prove the independence of the continuum hypothesis from the standard axioms of set theory, forcing has been seen by the general mathematical community as a subject of great intrinsic interest but one that is technically so forbidding that it is only accessible to specialists ... This is the first book aimed at explaining forcing to general mathematicians. It simultaneously makes the subject broadly accessible by explaining it in a clear, simple manner, and surveys advanced applications of set theory to mainstream topics." And this does strike me as a clear and very helpful attempt to solve Chow's basic exposition problem.
11. Lorenz J. Halbeisen, *Combinatorial Set Theory, With a Gentle Introduction to Forcing* (Springer 2011, with a late draft freely downloadable [from the author's website](#)). This is particularly attractively written for a set theory book. From the blurb "This book provides a self-contained introduction to modern set theory and also opens up some more advanced areas of current research in this field. The first part offers an overview of classical set theory wherein the focus lies on the axiom of choice and Ramsey theory. In the second part, the sophisticated technique of forcing, originally developed by Paul Cohen, is explained in great detail. With this technique, one can show that certain statements, like the continuum hypothesis, are neither provable nor disprovable from the axioms of set theory. In the last part, some topics of classical set theory are revisited and further developed in the light of forcing." True, this book gets quite hairy towards the end: but the first two parts of the book ('Topics in Combinatorial Set Theory' and 'From Martin's Axiom to Cohen Forcing') should be within reach to any mathematician. I can well imagine this book being spoken of as a modern classic of exposition to put alongside Kunen's and Jech's books: certainly, it has also been strongly recommended by much more reliable judges than me.
12. Ralf Schindler, *Set Theory: Exploring Independence and Truth* (Springer, 2014). The book's theme is "the interplay of large cardinals, inner models, forcing, and descriptive set theory". It doesn't presume you already know any set theory, though it does proceed at a cracking pace in a brisk style. But, if you already have some knowledge of set theory, this seems a

clear and interesting exploration of some themes highly relevant to current research.

8.1.6 The higher infinite

And then what next? You want more?? Back to finish Jech's doorstep of a book, perhaps. And then – oh heavens! – there is another blockbuster awaiting you:

13. Akihiro Kanamori, *The Higher Infinite: Large Cardinals in Set Theory from Their Beginnings* (Springer, 1997, 2nd edn. 2003).

8.2 The Axiom of Choice

But now let's leave the Higher Infinite and get back down to earth, or at least to less exotic mathematics! In fact, to return to the beginning, we might wonder: is ZFC the 'right' set theory?

Start by thinking about the Axiom of Choice in particular. It is comforting to know from Gödel that AC is consistent with ZF (so adding it doesn't lead to contradiction). But we also know from Cohen's forcing argument that AC is independent with ZF (so accepting ZF doesn't commit you to accepting AC too). So why buy AC? Is it an optional extra?

Some of the readings already mentioned will have touched on the question of AC's status and role. But for an overview/revision of some basics, see

1. John L. Bell, '[The Axiom of Choice](#)', *The Stanford Encyclopedia of Philosophy*.

And for a short book also explaining some of the consequences of AC (and some of the results that you need AC to prove), see

2. Horst Herrlich, *Axiom of Choice* (Springer 2006), which has chapters rather tantalisingly entitled 'Disasters without Choice', 'Disasters with Choice' and 'Disasters either way'.

That already probably tells you more than enough about the impact of AC: but there's also a famous book by H. Rubin and J.E. Rubin, *Equivalents of the Axiom of Choice* (North-Holland 1963; 2nd edn. 1985) worth browsing through: it gives over two hundred equivalents of AC! Then next there is the nice short classic

3. Thomas Jech, *The Axiom of Choice** (North-Holland 1973, Dover Publications 2008). This proves the Gödel and Cohen consistency and independence results about AC (without bringing into play everything needed to prove the parallel results about the Continuum Hypothesis). In particular, there is a nice presentation of the so-called Fraenkel-Mostowski method of using ‘permutation models’. Then later parts of the book tell us something about mathematics without choice, and about alternative axioms that are inconsistent with choice.

And for a more recent short book, taking you into new territories (e.g. making links with category theory), enthusiasts might enjoy

4. John L. Bell, *The Axiom of Choice** (College Publications, 2009).

8.3 Other set theories?

From earlier reading you should have picked up the idea that, although ZFC is the canonical modern set theory, there are other theories on the market. I mention just a selection here (I’m not suggesting you follow up all these – the point is to stress that set theory is not quite the monolithic edifice that some presentations might suggest).

For a brisk overview, putting many of the various set theories we’ll consider below into some sort of order (and mentioning yet further alternatives) see

1. M. Randall Holmes, ‘[Alternative axiomatic set theories](#)’, *The Stanford Encyclopedia of Philosophy*.

At this stage, you might well find this *too* brisk and allusive, but it is useful to give you a preliminary sense of the range of possibilities here.

NBG You will have come across mention of this already (e.g. even in the early pages of Enderton’s set theory book). And in fact – in many of the respects that matter – it isn’t really an ‘alternative’ set theory. So let’s get it out of the way first. We know that the universe of sets in ZFC is not itself a set. But we might think that this universe is a *sort* of big collection. Should we explicitly recognize, then, two sorts of collection, sets and (as they are called in the trade) proper classes which are too big to be sets? NBG (named for von Neumann, Bernays, Gödel: some say VBG) is one such theory of collections. So NBG in *some* sense recognizes proper classes, objects having ‘members’ but that cannot be members

of other entities: but in some sense, these classes are merely virtual objects. NBG's principle of class comprehension is predicative; i.e. quantified variables in the defining formula can't range over proper classes but range only over sets, and we get a conservative extension of ZFC (nothing in the language of sets can be proved in NBG which can't already be proved in ZFC). See

1. Abraham Fraenkel, Yehoshua Bar-Hillel and Azriel Levy, *Foundations of Set-Theory* (North-Holland, 2nd edition 1973), Ch. II §7 is a classic general discussion of the role of classes in set theory.
2. Michael Potter, *Set Theory and Its Philosophy* (OUP 2004) Appendix C is a brisker account of NBG and on other theories with classes as well as sets.

For detailed presentations of set-theory via NBG, you can see either or both of

3. Elliott Mendelson, *Introduction to Mathematical Logic* (CRC, 4th edition 1997), Ch.4. is a classic and influential textbook presentation.
4. Raymond Smullyan and Melvin Fitting, *Set Theory and the Continuum Problem* (OUP 1996, Dover Publications 2010), Part I is another development of set theory in its NBG version.

SP This again is by way of reminder. Recall, earlier in the Guide, we very warmly recommended Michael Potter's book which we just mentioned again. This presents a version of an axiomatization of set theory due to Dana Scott (hence 'Scott-Potter set theory'). This axiomatization is consciously guided by the conception of the set theoretic universe as built up in levels (the conception that, supposedly, also warrants the axioms of ZF). What Potter's book aims to reveal is that we can get a rich hierarchy of sets, more than enough for mathematical purposes, without committing ourselves to *all* of ZFC (whose extreme richness comes from the full Axiom of Replacement). If you haven't read Potter's book before, now is the time to look at it.

ZFA (i.e. $\text{ZF} - \text{AF} + \text{AFA}$) Here again is the now-familiar hierarchical conception of the set universe: We start with some non-sets (maybe zero of them in the case of pure set theory). We collect them into sets (as many different ways as we can). Now we collect what we've already formed into sets (as many as we can). Keep on going, as far as we can. On this 'bottom-up' picture, the Axiom of Foundation is compelling (any downward chain linked by set-membership will bottom out, and won't go round in a circle). But now here's another alternative

conception of the set universe. Think of a set as a gadget that points you at some things, its members. And those members, if sets, point to *their* members. And so on and so forth. On this ‘top-down’ picture, the Axiom of Foundation is not so compelling. As we follow the pointers, can’t we for example come back to where we started? It is well known that in much of the usual development of ZFC the Axiom of Foundation AF does little work. So what about considering a theory of sets which drops AF and instead has an Anti-Foundation Axiom (AFA), which allows self-membered sets? To explore this idea,

1. Start with Lawrence S. Moss, ‘[Non-wellfounded set theory](#)’, *The Stanford Encycl. of Philosophy*.
2. Keith Devlin, *The Joy of Sets* (Springer, 2nd edn. 1993), Ch. 7. The last chapter of Devlin’s book, added in the second edition of his book, starts with a very lucid introduction, and develops some of the theory.
3. Peter Aczel’s, *[Non-well-founded sets](#)*, (CSLI Lecture Notes 1988) is a very readable short classic book.
4. Luca Incurvati, ‘The graph conception of set’ *Journal of Philosophical Logic* (2014) pp. 181-208, very illuminatingly explores the motivation for such set theories.

NF Now for a much more radical departure from ZF. Standard set theory lacks a universal set because, together with other standard assumptions, the idea that there is a set of all sets leads to contradiction. But by tinkering with those other assumptions, there are coherent theories with universal sets. For very readable presentations concentrating on Quine’s NF (‘New Foundations’), and explaining motivations as well as technical details, see

1. T. F. Forster, *Set Theory with a Universal Set* Oxford Logic Guides 31 (Clarendon Press, 2nd edn. 1995). A classic: very worth reading even if you are a committed ZF-iste.
2. M. Randall Holmes, *Elementary Set Theory with a Universal Set*** (Cahiers du Centre de Logique No. 10, Louvain, 1998). Now [freely available here](#).

ETCS Famously, Zermelo constructed his theory of sets by gathering together some principles of set-theoretic reasoning that seemed actually to be used by working mathematicians (engaged in e.g. the rigorization of analysis or the development of point set topology), hoping to get a theory strong enough for mathematical use while weak enough to avoid paradox. But does he overshoot? We’ve

already noted that SP is a weaker theory which may suffice. For a more radical approach, see

1. Tom Leinster, ‘[Rethinking set theory](#)’, gives an advertising pitch for the merits of Lawvere’s Elementary Theory of the Category of Sets, and ...
2. F. William Lawvere and Robert Rosebrugh, *Sets for Mathematicians* (CUP 2003) gives a presentation which in principle doesn’t require that you have already done any category theory. But I suspect that it won’t be an easy ride if you know no category theory (and philosophers will find it conceptually puzzling too – what *are* these ‘abstract sets’ that we are supposedly theorizing about?). In my judgement, to really appreciate what’s going on, you will have to start engaging with more category theory.

IZF, CZF ZF/ZFC has a classical logic: what if we change the logic to intuitionistic logic? what if we have more general constructivist scruples? The place to start exploring is

1. Laura Crosilla, ‘[Set Theory: Constructive and Intuitionistic ZF](#)’, *The Stanford Encyclopedia of Philosophy*.

Then for one interesting possibility, look at the version of constructive ZF in

2. Peter Aczel and Michael Rathjen, [Constructive Set Theory](#) (Draft, 2010).

IST Leibniz and Newton invented infinitesimal calculus in the 1660s: a century and a half later we learnt how to rigorize the calculus without invoking infinitely small quantities. Still, the idea of infinitesimals retains a certain intuitive appeal, and in the 1960s, Abraham Robinson created a theory of hyperreal numbers: this yields a rigorous formal treatment of infinitesimal calculus (you will have seen this mentioned in e.g. Enderton’s *Mathematical Introduction to Logic*, §2.8, or van Dalen’s *Logic and Structure*, p. 123). Later, a simpler and arguably more natural approach, based on so-called Internal Set Theory, was invented by Edward Nelson. As put it, ‘IST is an extension of Zermelo-Fraenkel set theory in that alongside the basic binary membership relation, it introduces a new unary predicate ‘standard’ which can be applied to elements of the mathematical universe together with some axioms for reasoning with this new predicate.’ Starting in this way we can recover features of Robinson’s theory in a simpler framework.

1. Edward Nelson, ‘[Internal set theory: a new approach to nonstandard analysis](#)’ *Bulletin of The American Mathematical Society* 83 (1977), pp. 1165–1198.
2. Nader Vakin, *Real Analysis through Modern Infinitesimals* (CUP, 2011). A monograph developing Nelson’s ideas whose early chapters are quite approachable and may well appeal to some.

Yet more? Well yes, we *can* keep on going. Take a look, for example, at [SEAR](#). But we must call a halt! Though you could round things out by taking a look at a piece that could be thought of as an expanded version of Randall Holmes’s *Stanford Encyclopedia* piece that we mentioned at the the beginning of this section:

2. M. Randall Holmes, Thomas Forster and Thierry Libert. ‘Alternative Set Theories’. In Dov Gabbay, Akihiro Kanamori, and John Woods, eds. *Handbook of the History of Logic, vol. 6, Sets and Extensions in the Twentieth Century*, pp. 559-632. (Elsevier/North-Holland 2012).

Chapter 9

What else?

Mathematical logicians and philosophers interested in the philosophy of maths will want to know about yet more areas that fall outside the traditional math logic curriculum. For example:

- It is worth knowing about at least core aspects of *type theory*;
- Relatedly, we should explore something of the *the lambda calculus*.
- Even in elementary model theory we relax the notion of a language to allow e.g. for uncountably many names: what if we further relax and allow for e.g. sentences which are infinite conjunctions? Pursuing such questions leads us to consider *infinitary logics*.

These areas may or may not get a section in a later version of this Guide. But there is one more topic which *does* get its own supplementary webpage, including a reading list for philosophers, and links to a lot of freely available material, including my own notes:

- If set theory traditionally counts as part of mathematical logic, because of its generality, breadth and foundational interest, then there is an argument for including some *category theory* too.

Meanwhile, those who feel that – even with the Big Books mentioned in the [Appendix](#) – they haven’t got enough to read or think that the mainstream diet so far is a bit restricted can always consult the long, wide-ranging, articles in the seventeen(!) volume *Handbook of Philosophical Logic*. This collection, edited by the indefatigable Dov Gabbay together with F. Guenther, is arguably mistitled. A lot of the articles are straight mathematics, which might be of interest to

some technically minded philosophers but could be of at least equal interest to mathematicians too.

But enough already!

Index of authors

- Aczel, P., 88, 89
Arora, S., 74

Badesa, C., 47
Bar-Hillel, Y., 42
Barak, B., 74
Barker-Plummer, D., 17
Barnes, D., 70
Barwise, J., 17
Beall, J., 59
Bell, J., 30, 48, 68, 85, 86
Bencivenga, E., 57
Bezhanishvili, N., 49
Blackburn, P., 55
Boolos, G., 38, 54, 58, 76
Bostock, D., 24–25, 56
Bridge, J., 33
Burgess, J., 38
Buss, S., 64, 67

Carnielli, W., 38
Chagrov, A., 56
Chang, C., 68
Chiswell, I., 21–22
Chow, T., 81
Ciesielski, K., 43
Cocchiarella, N., 54
Cooper, S. B., 72, 73
Core, R., 26
Creswell, M., 54–55
Crosilla, L., 89

Cutland, N., 72

Dalen, D. van, 26, 33, 44, 47, 50
de Jongh, D., 49
Delzell, C., 70
DeVidi, D., 30, 48
Devlin, K., 44, 83, 87
Doets, H., 44
Doets, K., 70
Drake, F., 81

Ebbinghaus, H., 71
Enderton, H., 25–26, 33, 38, 39, 41, 47
Epstein, R., 38, 73
Etchemendy, J., 17

Ferreirós, J., 81
Ferreiros, J., 47
Fitting, M., 27–28, 54, 75, 83
Flum, J., 71
Forster, T., 88, 90
Fraassen, B. van, 59
Fraenkel, A., 42
Franzén, T., 75
Freund, M., 54

Gandy, R., 73
Garson, J., 53
Girard, J-Y., 66
Girle, R., 53
Goldrei, D., 23, 34, 41

Goldreich, O., 74
 Guttenplan, S., 5

 Hájek, P., 77
 Hajnal, A., 43
 Halbeisen, L., 84
 Halmos, P., 33, 41
 Hamburger, P., 43
 Hedman, S., 28, 36, 68, 74
 Herrlich, H., 85
 Hinman, P., 36, 68, 72
 Hodges, W., 21–23, 67, 68, 71
 Hofstadter, D., 40
 Holmes, M. R., 86, 88, 90
 Hrbacek, K., 42
 Hughes, G., 54–55
 Hunter, G., 29
 Hurley, P., 5

 Iemhoff, R., 50
 Incurvati, L., 88

 Jech, T., 42, 80, 85
 Jeffrey, R., 38
 Just, W., 44–45

 Kahane, H., 5
 Kanamori, A., 81, 85
 Kaye, R., 77
 Kechris, A., 83
 Keisler, H., 68–69
 Komjáte, P., 82
 Kossak, K., 77
 Kristiansen, L., 22, 39
 Kunen, K., 82

 Lambert, K., 57
 Lascar, D., 26
 Lawvere, F. W., 88

 Leary, C., 22, 39
 Leinster, T., 88
 Levy, A., 42
 Libert, T., 90
 Libkin, L., 71
 Lindström, P., 76
 Linnebo, Ø., 58

 Mack, J., 70
 Mancosu, P., 47
 Manzano, M., 34
 Mares, E., 59
 Marker, D., 68
 McKay, T., 58
 Mendelsohn, R., 54
 Mendelson, E., 28–29, 87
 Milne, P., 50
 Montanaro, A., 74
 Moschovakis, J., 50
 Moschovakis, Y., 42, 83
 Moss, L., 87

 Negri, J., 64
 Nelson, E., 89
 Nolt, J., 57

 Odifreddi, P., 73
 Oliver, A., 58

 Péter, R., 73
 Paseau, A., 71
 Plato, J. von, 27, 63, 64
 Pohlers, W., 67
 Poizat, B., 70
 Pollard, S., 50
 Popkorn, S., 55
 Potter, M., 43, 87
 Prawitz, D., 64
 Prestel, A., 70

Priest, G., 49, 53, 57, 59, 60
 Pudlák, P., 77

 Rathjen, M., 65, 89
 Restall, G., 63, 65
 Ricke, M. de, 55
 Rogers, H., 73
 Rosebrugh, R., 88
 Rothmaler, P., 69
 Roy, T., 53
 Rubin, H., 85
 Rubin, J., 85

 Schimmerling, E., 46
 Schindler, Ralf, 84
 Schmerl, J., 77
 Schock, R., 57
 Schwichtenberg, H., 66, 67
 Shapiro, S., 48
 Shen, A., 40, 46
 Sider, T., 29–30, 48, 55
 Simmons, H., 55
 Simpson, S., 77–78
 Sipser, M., 74
 Slomson, A., 68
 Smiley, T., 58
 Smith, N., 16
 Smith, P., 17, 24, 37–38
 Smoryński, C., 75
 Smullyan, R., 26–27, 30–31, 74, 75, 83
 Soare, R., 73
 Solomon, G., 30, 48
 Steinhart, E., 6
 Swart, H. de, 44

 Takeuti, G., 64
 Tanaka, K., 59
 Teller, P., 16
 Tennant, N., 25, 50, 57, 60, 66

 Tent, K., 69
 Totik, V., 82
 Tourlakis, G., 45
 Troelstra, A., 66

 Väänänen, J., 48
 Vakin, N., 89
 Velleman, D., 7–8
 Venema, Y., 55
 Vereshchagin, N., 40, 46

 Wainer, S., 67
 Weaver, N., 83
 Weber, R., 40
 Weber, Z., 59
 Weese, M., 44–45
 Williamson, T., 54

 Zach, R., 29, 47, 49
 Zakharyashev, M., 56
 Ziegler, M., 69