MDIQRNG based on continuous-variable bell state measurement

Abstract

This note derives the conditional probability distribution of discretized measurement outcomes for continuous-variable (CV) Bell measurement on two-mode coherent states. The standard normalized definitions of quadrature operators $\hat{X}$ and $\hat{P}$ are adopted, and the Bell measurement is implemented via the commuting joint quadratures $X_+ = \hat{X}_1 + \hat{X}_2$ and $P_- = \hat{P}_1 - \hat{P}_2$. The conditional probabilities are derived using Gaussian distribution properties and the error function, with explicit formulas provided.

1 Preliminaries

1.1 Standard Normalized Quadrature Operators

The single-mode position and momentum quadrature operators are defined as:

$$\hat{X} = \frac{\hat{a} + \hat{a}^\dagger}{\sqrt{2}}, \quad \hat{P} = \frac{\hat{a} - \hat{a}^\dagger}{i\sqrt{2}}$$

where $\hat{a}$ and $\hat{a}^\dagger$ are the annihilation and creation operators, respectively. Key properties for a coherent state $|\alpha\rangle$: - Expectation values: $\langle \hat{X} \rangle = \sqrt{2}\,\mathrm{Re}(\alpha)$, $\langle \hat{P} \rangle = \sqrt{2}\,\mathrm{Im}(\alpha)$ - Variances: $\mathrm{Var}(\hat{X}) = \mathrm{Var}(\hat{P}) = \frac{1}{2}$ (satisfies Heisenberg uncertainty principle $\Delta X \Delta P \geq \frac{1}{2}$)

1.2 Input Two-Mode Coherent States

The input states are two-mode coherent states $|\psi_\sigma\rangle = |\alpha_1\rangle \otimes |\alpha_2\rangle$, where: - $\alpha_1 = \sqrt{\mu_1} e^{i\phi_1}$, $\alpha_2 = \sqrt{\mu_2} e^{i\phi_2}$ - Phase angles $\phi_1, \phi_2 \in \{0, \pi\}$, leading to sign parameters $s_1 = \cos\phi_1 \in \{+1, -1\}$, $s_2 = \cos\phi_2 \in \{+1, -1\}$ - Four distinct input states are indexed by $\sigma = (s_1, s_2) \in \{(+1,+1), (+1,-1), (-1,+1), (-1,-1)\}$ - Since $\phi_1, \phi_2 \in \{0, \pi\}$, $\mathrm{Im}(\alpha_1) = \mathrm{Im}(\alpha_2) = 0$, so $\langle \hat{P}_1 \rangle = \langle \hat{P}_2 \rangle = 0$

1.3 CV Bell Measurement

A complete CV Bell measurement requires two commuting joint quadratures. We choose:

$$X_+ = \hat{X}_1 + \hat{X}_2, \quad P_- = \hat{P}_1 - \hat{P}_2$$

where $[X_+, P_-] = 0$ (commuting) and the measurement is informationally complete.

1.4 Discretization of Measurement Outcomes

- Continuous output of X_+: $x_+ \in \mathbb{R}$, discretized into n intervals $I_{+k} = [c_{k-1}, c_k)$ for $k = 1, 2, ..., n$ (with $c_0 \to -\infty$, $c_n \to +\infty$) - Continuous output of P_-: $p_- \in \mathbb{R}$, discretized into n intervals $I_{-l} = [d_{l-1}, d_l)$ for $l = 1, 2, ..., n$ (with $d_0 \to -\infty$, $d_n \to +\infty$) - Discretized measurement results: Joint intervals (I_{+k}, I_{-l}) denoted as (k, l), totaling n^2 distinct outcomes

2 Joint Probability Density of Continuous Outcomes

For input state $\sigma = (s_1, s_2)$, the joint probability density of continuous outcomes (x_+, p_-) follows a Gaussian distribution (linear combinations of Gaussian variables are Gaussian).

2.1 Mean Values

- Mean of X_+: $\mu_{+\sigma} = \langle X_+ \rangle_\sigma = \langle \hat{X}_1 \rangle + \langle \hat{X}_2 \rangle = \sqrt{2}(s_1\sqrt{\mu_1} + s_2\sqrt{\mu_2})$ - Mean of P_-: $\mu_{-\sigma} = \langle P_- \rangle_\sigma = \langle \hat{P}_1 \rangle - \langle \hat{P}_2 \rangle = 0$

2.2 Variances

Since the two modes are independent: - Variance of X_+: $\mathrm{Var}(X_+) = \mathrm{Var}(\hat{X}_1) + \mathrm{Var}(\hat{X}_2) = \frac{1}{2} + \frac{1}{2} = 1$ - Variance of P_-: $\mathrm{Var}(P_-) = \mathrm{Var}(\hat{P}_1) + \mathrm{Var}(\hat{P}_2) = \frac{1}{2} + \frac{1}{2} = 1$

2.3 Joint Gaussian Density

The joint probability density is separable (due to independence of X_+ and P_-):

$$f(x_+, p_- | \sigma) = f_+(x_+ | \sigma) \cdot f_-(p_-)$$

where: - $f_+(x_+|\sigma) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(x_+ - \mu_{+\sigma})^2}{2}\right)$ (marginal density of X_+) - $f_-(p_-) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{p_-^2}{2}\right)$ (marginal density of P_-)

Thus, the joint density simplifies to:

$$f(x_+, p_- | \sigma) = \frac{1}{2\pi} \exp\left(-\frac{(x_+ - \mu_{+\sigma})^2}{2} - \frac{p_-^2}{2}\right)$$

3 Derivation of Conditional Probabilities for Discretized Outcomes

The conditional probability $P((k, l)|\sigma)$ is the integral of the joint density over the discrete joint interval (I_{+k}, I_{-l}).

3.1 Definition of Conditional Probability

$$P((k, l)|\sigma) = \iint_{I_{+k} \times I_{-l}} f(x_+, p_- | \sigma)\, dx_+ dp_-$$

3.2 Separable Integration

Due to the separability of $f(x_+, p_-|\sigma)$, the double integral splits into a product of single integrals:

$$P((k,l)|\sigma) = \left(\int_{c_{k-1}}^{c_k} f_+(x_+|\sigma)\,dx_+\right)\cdot\left(\int_{d_{l-1}}^{d_l} f_-(p_-)\,dp_-\right)$$

3.3 Evaluation via Error Function

The error function is defined as $\mathrm{erf}(z) = \frac{2}{\sqrt{\pi}}\int_0^z \exp(-t^2)dt$, which simplifies Gaussian integrals.

3.3.1 Integral for X_+

Let $t = \frac{x_+ - \mu_{+\sigma}}{\sqrt{2}}$ (so $dx_+ = \sqrt{2}dt$). The integral becomes:

$$\int_{c_{k-1}}^{c_k} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(x_+ - \mu_{+\sigma})^2}{2}\right) dx_+ = \frac{1}{2}\left[\mathrm{erf}\left(\frac{c_k - \mu_{+\sigma}}{\sqrt{2}}\right) - \mathrm{erf}\left(\frac{c_{k-1} - \mu_{+\sigma}}{\sqrt{2}}\right)\right]$$

3.3.2 Integral for P_-

Let $t = \frac{p_-}{\sqrt{2}}$ (so $dp_- = \sqrt{2}dt$). The integral becomes:

$$\int_{d_{l-1}}^{d_l} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{p_-^2}{2}\right) dp_- = \frac{1}{2}\left[\mathrm{erf}\left(\frac{d_l}{\sqrt{2}}\right) - \mathrm{erf}\left(\frac{d_{l-1}}{\sqrt{2}}\right)\right]$$

3.4 Final Conditional Probability Formula

Substitute $\mu_{+\sigma} = \sqrt{2}(s_1\sqrt{\mu_1} + s_2\sqrt{\mu_2})$ and simplify the argument of the error function:

$$\begin{aligned} P((k,l)|s_1,s_2) = \frac{1}{4}\cdot &\left[\mathrm{erf}\left(\frac{c_k}{\sqrt{2}} - s_1\sqrt{\mu_1} - s_2\sqrt{\mu_2}\right) - \mathrm{erf}\left(\frac{c_{k-1}}{\sqrt{2}} - s_1\sqrt{\mu_1} - s_2\sqrt{\mu_2}\right)\right] \\ &\cdot\left[\mathrm{erf}\left(\frac{d_l}{\sqrt{2}}\right) - \mathrm{erf}\left(\frac{d_{l-1}}{\sqrt{2}}\right)\right] \end{aligned} \tag{1}$$

where: - $(s_1, s_2) \in \{(+1,+1),(+1,-1),(-1,+1),(-1,-1)\}$ (four input states) - $k, l \in \{1, 2, ..., n\}$ (discretized outcome indices) - c_{k-1}, c_k and d_{l-1}, d_l are user-defined discretization boundaries.

Remark: The mean photon number μ_1 and μ_2 can be the same, and need to be optimized, for example $[0, 10]$. The boundary $|c_1|$, $|c_{n-1}|$, $|d_1|$, $|d_{n-1}|$ can be set to be 10.

4 Matrix Representation of Input Two-Mode Coherent States

This section derives the vector (matrix) representations of the four input two-mode coherent states under the assumption $\mu_1 = \mu_2 = \mu$ (denoted $\alpha = \sqrt{\mu}$ for simplicity). We first define a two-dimensional subspace basis for each single mode, then extend it to the two-mode space.

4.1 Single-Mode Basis and State Expansion

For each mode (Mode 1 and Mode 2, symmetric due to $\mu_1 = \mu_2$), we define a two-dimensional orthonormal basis $\{|0\rangle, |1\rangle\}$ where: - Basis vector $|0\rangle = |\alpha\rangle$ (coherent state with phase $\phi = 0$) - Basis vector $|1\rangle$: Normalized vector orthogonal to $|0\rangle$, i.e., $\langle 0|1\rangle = 0$ and $\langle 1|1\rangle = 1$

The complementary coherent state $|-\alpha\rangle$ (phase $\phi = \pi$) lies in this two-dimensional subspace and can be expanded as:

$$|-\alpha\rangle = \delta|0\rangle + \sqrt{1-\delta^2}|1\rangle$$

where $\delta = \langle 0|-\alpha\rangle = \langle\alpha|-\alpha\rangle$ is the inner product of $|\alpha\rangle$ and $|-\alpha\rangle$.

4.2 Inner Product δ Calculation

The inner product of two coherent states $|\alpha\rangle$ and $|\beta\rangle$ is given by:

$$\langle\alpha|\beta\rangle = \exp\left(-\frac{1}{2}\left(|\alpha|^2 + |\beta|^2 - 2\alpha^*\beta\right)\right)$$

For $\beta = -\alpha$ and $|\alpha|^2 = \mu$ (real α), substitute into the formula:

$$\delta = \langle\alpha|-\alpha\rangle = \exp\left(-\frac{1}{2}\left(\mu + \mu - 2\alpha\cdot(-\alpha)\right)\right) = \exp\left(-\frac{1}{2}(2\mu + 2\mu)\right) = \exp(-2\mu)$$

Thus, $\delta = e^{-2\mu}$, a real positive parameter dependent only on the average photon number μ.

4.3 Two-Mode Space Basis

The two-mode Hilbert space is the tensor product of the two single-mode subspaces. We adopt the orthonormal basis for the two-mode space as:

$$\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$$

where $|ij\rangle = |i\rangle_1 \otimes |j\rangle_2$ (subscripts 1 and 2 denote Mode 1 and Mode 2, respectively). The vector representation of a two-mode state is a 4-dimensional column vector, with components ordered by $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$.

4.4 Vector Representations of Four Input States

All four input states are tensor products of single-mode coherent states $|\phi_{s_1}\rangle_1 \otimes |\phi_{s_2}\rangle_2$, where $|\phi_{+1}\rangle = |\alpha\rangle = |0\rangle$ and $|\phi_{-1}\rangle = |-\alpha\rangle = \delta|0\rangle + \sqrt{1-\delta^2}|1\rangle$. We derive their vector representations using tensor product expansion.

4.4.1 Input State 1: $\sigma = (+1, +1) \to |\alpha\rangle_1 \otimes |\alpha\rangle_2$

Substitute $|\phi_{+1}\rangle = |0\rangle$ for both modes:

$$|\alpha\rangle_1 \otimes |\alpha\rangle_2 = |0\rangle_1 \otimes |0\rangle_2 = |00\rangle$$

Vector representation (components ordered by $|00\rangle, |01\rangle, |10\rangle, |11\rangle$):

$$\boxed{|\psi_{(+1,+1)}\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}}$$

4.4.2 Input State 2: $\sigma = (+1, -1) \to |\alpha\rangle_1 \otimes |-\alpha\rangle_2$

Substitute $|\phi_{+1}\rangle_1 = |0\rangle_1$ and $|\phi_{-1}\rangle_2 = \delta|0\rangle_2 + \sqrt{1-\delta^2}|1\rangle_2$, then expand the tensor product:

$$|\alpha\rangle_1 \otimes |-\alpha\rangle_2 = |0\rangle_1 \otimes \left(\delta|0\rangle_2 + \sqrt{1-\delta^2}|1\rangle_2\right) = \delta|00\rangle + \sqrt{1-\delta^2}|01\rangle$$

Vector representation:

$$\boxed{|\psi_{(+1,-1)}\rangle = \begin{pmatrix} \delta \\ \sqrt{1-\delta^2} \\ 0 \\ 0 \end{pmatrix}}$$

4.4.3 Input State 3: $\sigma = (-1, +1) \to |-\alpha\rangle_1 \otimes |\alpha\rangle_2$

Substitute $|\phi_{-1}\rangle_1 = \delta|0\rangle_1 + \sqrt{1-\delta^2}|1\rangle_1$ and $|\phi_{+1}\rangle_2 = |0\rangle_2$, then expand the tensor product:

$$|-\alpha\rangle_1 \otimes |\alpha\rangle_2 = \left(\delta|0\rangle_1 + \sqrt{1-\delta^2}|1\rangle_1\right) \otimes |0\rangle_2 = \delta|00\rangle + \sqrt{1-\delta^2}|10\rangle$$

Vector representation:

$$\boxed{|\psi_{(-1,+1)}\rangle = \begin{pmatrix} \delta \\ 0 \\ \sqrt{1-\delta^2} \\ 0 \end{pmatrix}}$$

4.4.4 Input State 4: $\sigma = (-1, -1) \to |-\alpha\rangle_1 \otimes |-\alpha\rangle_2$

Substitute $|\phi_{-1}\rangle_1 = \delta|0\rangle_1 + \sqrt{1-\delta^2}|1\rangle_1$ and $|\phi_{-1}\rangle_2 = \delta|0\rangle_2 + \sqrt{1-\delta^2}|1\rangle_2$, then expand the tensor product using distributivity:

$$\begin{aligned} |-\alpha\rangle_1 \otimes |-\alpha\rangle_2 &= \left(\delta|0\rangle_1 + \sqrt{1-\delta^2}|1\rangle_1\right) \otimes \left(\delta|0\rangle_2 + \sqrt{1-\delta^2}|1\rangle_2\right) \\ &= \delta^2|00\rangle + \delta\sqrt{1-\delta^2}|01\rangle + \delta\sqrt{1-\delta^2}|10\rangle + (1-\delta^2)|11\rangle \end{aligned}$$

Vector representation:

$$\boxed{|\psi_{(-1,-1)}\rangle = \begin{pmatrix} \delta^2 \\ \delta\sqrt{1-\delta^2} \\ \delta\sqrt{1-\delta^2} \\ 1-\delta^2 \end{pmatrix}}$$