4.13.68

EE25BTECH11064 - Yojit Manral

Question:
Prove or disprove: The straight line $5x + 4y = 0$ passes through the point of intersection of the straight lines $x + 2y - 10 = 0$ and $2x + y + 5 = 0$.
Solution:
→ Let

$$\mathbf{n_1}^T\mathbf{x} = c_1 \qquad \mathbf{n_2}^T\mathbf{x} = c_2 \qquad \mathbf{n_3}^T\mathbf{x} = c_3$$

where

$$\mathbf{n_1} = \begin{pmatrix}5\\4\end{pmatrix} \qquad c_1 = 0 \tag{1}$$

$$\mathbf{n_2} = \begin{pmatrix}1\\2\end{pmatrix} \qquad c_2 = 10 \tag{2}$$

$$\mathbf{n_3} = \begin{pmatrix}2\\1\end{pmatrix} \qquad c_3 = -5 \tag{3}$$

→ The three lines are concurrent if there exists some unique $\mathbf{x}$ such that

$$\begin{pmatrix}\mathbf{n_1}^T\\\mathbf{n_2}^T\\\mathbf{n_3}^T\end{pmatrix}\mathbf{x} = \begin{pmatrix}c_1\\c_2\\c_3\end{pmatrix} \tag{4}$$

$$\mathbf{A} \quad \mathbf{x} = \quad \mathbf{B} \tag{5}$$

→ A unique $\mathbf{x}$ exists iff the augmented matrix $(\mathbf{A}|\mathbf{B})$ is a full-rank matrix (i.e., rank 2 for a system of 2 variables).

$$(\mathbf{A}|\mathbf{B}) = \left(\begin{array}{cc|c}5 & 4 & 0\\1 & 2 & 10\\2 & 1 & -5\end{array}\right) \xrightarrow[R_2\leftrightarrow R_2-R_1]{R_1\leftrightarrow(1/5)R_1} \left(\begin{array}{cc|c}1 & 4/5 & 0\\0 & 6/5 & 10\\2 & 1 & -5\end{array}\right) \tag{6}$$

$$\xrightarrow[R_2\leftrightarrow(5/6)R_2]{R_3\leftrightarrow R_3-2R_1} \left(\begin{array}{cc|c}1 & 4/5 & 0\\0 & 1 & 25/3\\0 & -3/5 & -5\end{array}\right) \tag{7}$$

$$\xrightarrow[R_1\leftrightarrow R_1-(4/5)R_2]{R_3\leftrightarrow R_3+(3/5)R_2} \left(\begin{array}{cc|c}1 & 0 & -20/3\\0 & 1 & 25/3\\0 & 0 & 0\end{array}\right) \tag{8}$$

$$\implies \mathbf{x} = \begin{pmatrix}-20/3\\25/3\end{pmatrix} \tag{9}$$

$$\begin{aligned}\text{Only 2 nonzero rows in the matrix in (8)} &\longrightarrow \text{Rank of augmented matrix } (\mathbf{A}|\mathbf{B}) \text{ is 2}\\ &\longrightarrow \text{The matrix } (\mathbf{A}|\mathbf{B}) \text{ is a full rank matrix}\\ &\longrightarrow \text{A unique } \mathbf{x} \text{ exists satisfying all equations}\\ &\longrightarrow \text{The 3 lines are concurrent at } \mathbf{x} = \begin{pmatrix}-20/3\\25/3\end{pmatrix}\end{aligned}$$

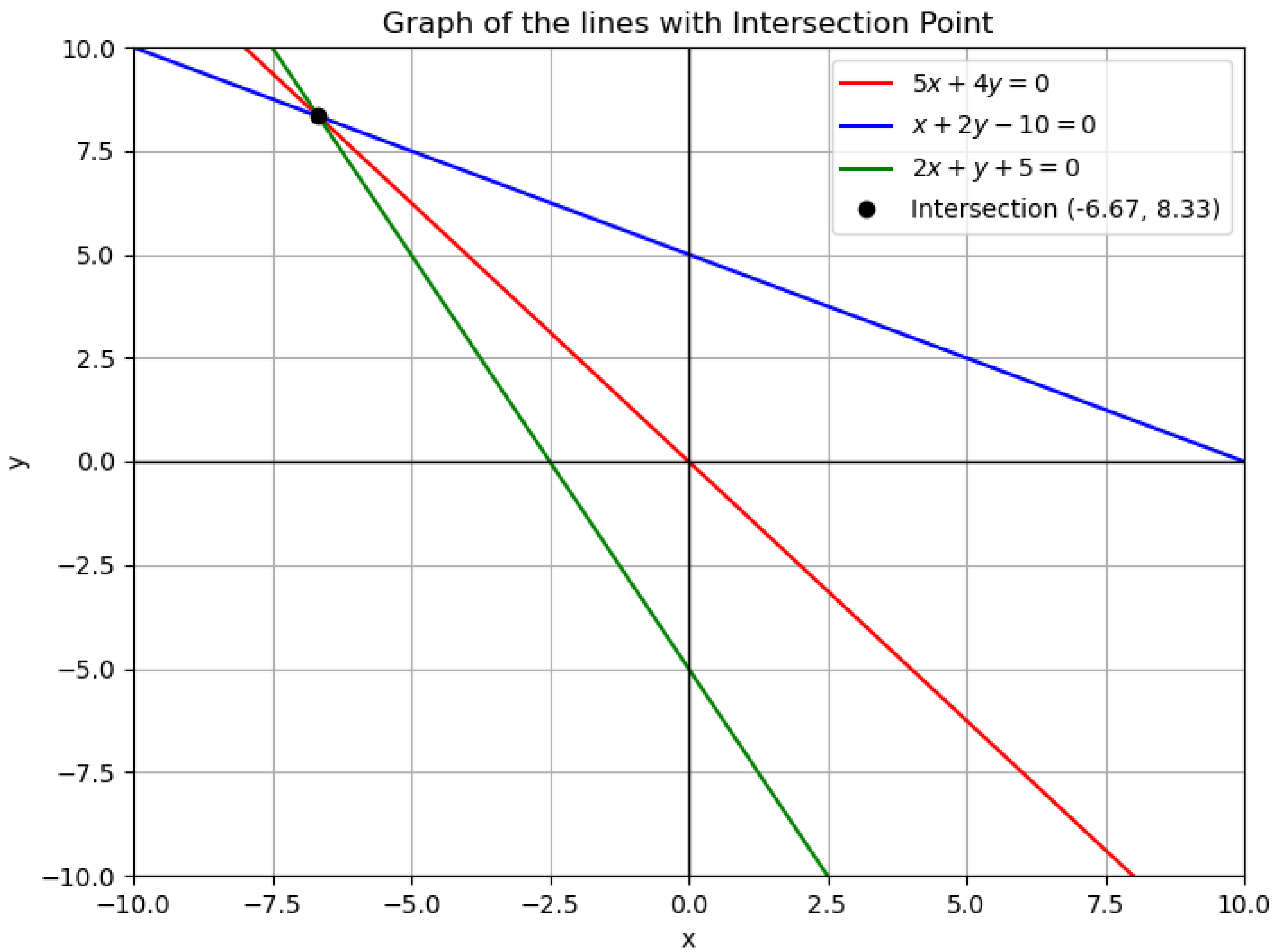

Fig. 0: Plot of the three lines