12.701

EE25BTECH11064 - Yojit Manral

Question:
Let $(\cdot, \cdot) : R^n \times R^n \to R$ be the inner product. Consider

$$P : |(\mathbf{u}, \mathbf{v})| \leq \frac{(\mathbf{u}, \mathbf{u}) + (\mathbf{v}, \mathbf{v})}{2} \forall \mathbf{u}, \mathbf{v}.$$
$$Q : \text{If } (\mathbf{u}, \mathbf{v}) = (2\mathbf{u}, \mathbf{v}) \ \ \forall \mathbf{v}, \text{ then } \mathbf{u} = 0.$$

Then

(a) both P, Q are true
(b) P is true, Q is false
(c) P is false, Q is true
(d) both P, Q are false

Solution:
→ We can take the inner product as the dot product, since all inner products have the same properties as satisfied by the dot product.

$$\implies (\mathbf{u}, \mathbf{v}) = \mathbf{u}^T \mathbf{v} \tag{1}$$

→ For Statement P

$$|(\mathbf{u}, \mathbf{v})| \leq ||\mathbf{u}|| \, ||\mathbf{v}|| \qquad \text{(Cauchy-Schwarz Inequality)} \tag{2}$$
$$\frac{(\mathbf{u}, \mathbf{u}) + (\mathbf{v}, \mathbf{v})}{2} = \frac{||\mathbf{u}||^2 + ||\mathbf{v}||^2}{2} \tag{3}$$
$$||\mathbf{u}|| \, ||\mathbf{v}|| \leq \frac{||\mathbf{u}||^2 + ||\mathbf{v}||^2}{2} \qquad \text{(G.M. } \leq \text{ A.M.)} \tag{4}$$
$$\implies |(\mathbf{u}, \mathbf{v})| = \frac{(\mathbf{u}, \mathbf{u}) + (\mathbf{v}, \mathbf{v})}{2} \qquad \implies \textit{P is true} \tag{5}$$

→ For Statement Q

$$(\mathbf{u}, \mathbf{v}) = (2\mathbf{u}, \mathbf{v}) \qquad (\forall \mathbf{v} \in R^n) \tag{6}$$
$$(\mathbf{u}, \mathbf{v}) - (2\mathbf{u}, \mathbf{v}) = 0 \tag{7}$$
$$(\mathbf{u} - 2\mathbf{u}, \mathbf{v}) = 0 \tag{8}$$
$$(-\mathbf{u}, \mathbf{v}) = 0 \qquad (\forall \mathbf{v} \in R^n) \tag{9}$$
$$\implies \mathbf{u} = 0 \qquad \implies \textit{Q is true} \tag{10}$$

⟶ Therefore, *(b) both P, Q are true* is the correct option.