12.285

EE25BTECH11064 - Yojit Manral

Question:
Let $\alpha = e^{2\pi i/5}$ and the matrix

$$\mathbf{M} = \begin{pmatrix} 1 & \alpha & \alpha^2 & \alpha^3 & \alpha^4 \\ 0 & \alpha & \alpha^2 & \alpha^3 & \alpha^4 \\ 0 & 0 & \alpha^2 & \alpha^3 & \alpha^4 \\ 0 & 0 & 0 & \alpha^3 & \alpha^4 \\ 0 & 0 & 0 & 0 & \alpha^4 \end{pmatrix} \tag{1}$$

Then the trace of the matrix $\mathbf{I} + \mathbf{M} + \mathbf{M}^2$ is

(a) 5 (b) 0 (c) 3 (d) -5

Solution:
$\rightarrow$ As α is the fifth root of unity, we have

$$\alpha^5 - 1 = 0 \implies 1 + \alpha + \alpha^2 + \alpha^3 + \alpha^4 = 0 \tag{2}$$

$\rightarrow$ According to the properties of trace

$$trace(\mathbf{A}_{n\times n}) \triangleq \sum_{i=1}^{n} a_{ii} \tag{3}$$

$$trace(\mathbf{I} + \mathbf{M} + \mathbf{M}^2) = trace(\mathbf{I}) + trace(\mathbf{M}) + trace(\mathbf{M}^2) \tag{4}$$

$$trace(\mathbf{I}_{5\times 5}) = \sum_{i=1}^{5} 1 = 5 \tag{5}$$

$\rightarrow$ From (1) and (3), we get

$$trace(\mathbf{M}) = \sum_{i=1}^{5} m_{ii} = 1 + \alpha + \alpha^2 + \alpha^3 + \alpha^4 \tag{6}$$

$\rightarrow$ Since $\mathbf{M}$ is an upper triangular matrix

$$trace(\mathbf{M}^2) = \sum_{i=1}^{5} m_{ii}^2 = (1)^2 + (\alpha)^2 + (\alpha^2)^2 + (\alpha^3)^2 + (\alpha^4)^2 \tag{7}$$

$\rightarrow$ On adding (5), (6), and (7), using (2) and (4), we get

$$trace(\mathbf{I} + \mathbf{M} + \mathbf{M}^2) = 5 + (1 + \alpha + \alpha^2 + \alpha^3 + \alpha^4) + (1 + \alpha^2 + \alpha^4 + \alpha\alpha^5 + \alpha^3\alpha^5) \tag{8}$$

$$= 5 + (0) + (0) = 5 \tag{9}$$

$\rightarrow$ Therefore, (a) 5 is the correct option.