8.2.32

EE25BTECH11064 - Yojit Manral

Question:
Find the conic equation that satisfies the given conditions: ends of the major axis $(0, \pm 5)$, ends of the minor axis $(\pm 1, 0)$.

Solution:
$\rightarrow$ The equation for any conic having directrix $\mathbf{n}^T\mathbf{x} = c$ and eccentricity e is given by

$$\mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{1}$$

$$\mathbf{V} = \|\mathbf{n}\|^2 \mathbf{I} - e^2\mathbf{n}\mathbf{n}^T \tag{2}$$

$\rightarrow$ As the major axis is along the $Y - axis$

$$\mathbf{n} = \mathbf{e_2} \implies \mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 1 - e^2 \end{pmatrix} \tag{3}$$

$\rightarrow$ Also, as the center of the ellipse is at origin

$$\mathbf{C} = 0 \implies \mathbf{u} = 0 \tag{4}$$

$\rightarrow$ Let $\mathbf{P}$ and $\mathbf{Q}$ be points on the ellipse

$$\mathbf{P} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \mathbf{Q} = \begin{pmatrix} 0 \\ 5 \end{pmatrix} \tag{5}$$

$\rightarrow$ Then, $\mathbf{P}$ and $\mathbf{Q}$ satisfy (1)

$$\mathbf{P}^T\mathbf{V}\mathbf{P} + 2\mathbf{u}^T\mathbf{P} + f = 0 \qquad \mathbf{Q}^T\mathbf{V}\mathbf{Q} + 2\mathbf{u}^T\mathbf{Q} + f = 0 \tag{6}$$

$$\begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 - e^2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + f = 0 \qquad \begin{pmatrix} 0 & 5 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 - e^2 \end{pmatrix}\begin{pmatrix} 0 \\ 5 \end{pmatrix} + f = 0 \tag{7}$$

$$1 + f = 0 \qquad 25(1 - e^2) + f = 0 \tag{8}$$

$$f = -1 \implies \qquad e^2 = 1 + \frac{f}{25} = \frac{24}{25} \tag{9}$$

$\rightarrow$ Thus, we get the equation of the conic as

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 1 - e^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1/25 \end{pmatrix}, \mathbf{u} = 0, \text{ and } f = -1 \tag{10}$$

$$\implies \mathbf{x}^T\begin{pmatrix} 1 & 0 \\ 0 & 1/25 \end{pmatrix}\mathbf{x} - 1 = 0 \tag{11}$$

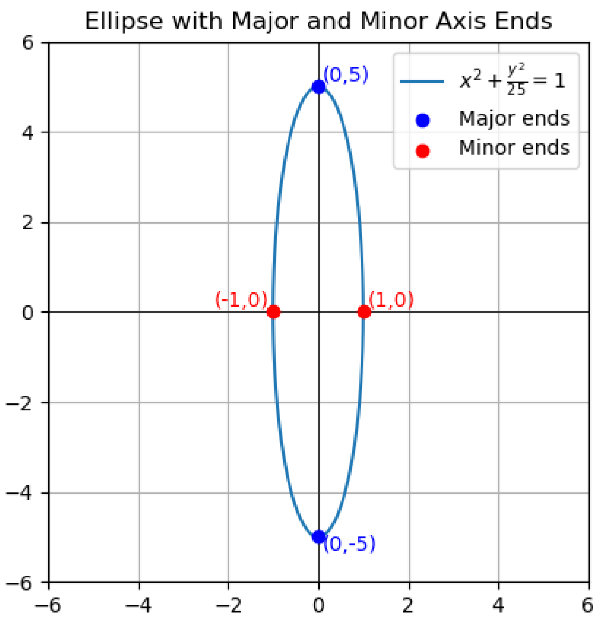

Fig. 0: Plot of required conic