6.3.4

EE25BTECH11064 - Yojit Manral

Question:
Find the shortest distance between the lines

$$\mathbf{r} = (\mathbf{i} + 2\mathbf{j} + \mathbf{k}) + \kappa_1(\mathbf{i} - \mathbf{j} + \mathbf{k}) \tag{1}$$
$$\mathbf{r} = (2\mathbf{i} - \mathbf{j} - \mathbf{k}) + \kappa_2(2\mathbf{i} + \mathbf{j} + 2\mathbf{k}) \tag{2}$$

Solution:
$\longrightarrow$ The lines can be represented in vector form as

$$L_1 \equiv \mathbf{x} = \mathbf{A} + \kappa_1\mathbf{m_1} = \begin{pmatrix}1\\2\\1\end{pmatrix} + \kappa_1\begin{pmatrix}1\\-1\\1\end{pmatrix} \tag{3}$$
$$L_2 \equiv \mathbf{x} = \mathbf{B} + \kappa_2\mathbf{m_2} = \begin{pmatrix}2\\-1\\-1\end{pmatrix} + \kappa_2\begin{pmatrix}2\\1\\2\end{pmatrix} \tag{4}$$

$\rightarrow$ To check whether the given lines are skewed

$$\mathbf{M} = \begin{pmatrix}\mathbf{m_1} & \mathbf{m_2}\end{pmatrix} = \begin{pmatrix}1 & 2\\-1 & 1\\1 & 2\end{pmatrix}, \mathbf{B} - \mathbf{A} = \begin{pmatrix}1\\-3\\-2\end{pmatrix} \tag{5}$$
$$\begin{pmatrix}1 & 2 & 1\\-1 & 1 & -3\\1 & 2 & -2\end{pmatrix} \xrightarrow[R_3\leftrightarrow R_3-R_1]{R_2\leftrightarrow R_2+R_1} \begin{pmatrix}1 & 2 & 1\\0 & 3 & -2\\0 & 0 & -3\end{pmatrix} \tag{6}$$
$$rank\begin{pmatrix}\mathbf{M} & \mathbf{B}-\mathbf{A}\end{pmatrix} = 3 \implies \text{The given lines are skew} \tag{7}$$

$\longrightarrow$ Let $x_1(\mu_1)$ and $x_2(\mu_2)$ be the points closest to each other from the lines L_1 and L_2, respectively

$$\mathbf{x_1} = \mathbf{A} + \mu_1\mathbf{m_1} \qquad\qquad \mathbf{x_2} = \mathbf{B} + \mu_2\mathbf{m_2} \tag{8}$$

$\rightarrow$ Now, we have

$$(\mathbf{x_1} - \mathbf{x_2})^T\mathbf{m_1} = (\mathbf{x_1} - \mathbf{x_2})^T\mathbf{m_2} = 0 \tag{9}$$
$$(\mathbf{x_1} - \mathbf{x_2})^T\begin{pmatrix}\mathbf{m_1} & \mathbf{m_2}\end{pmatrix} = 0 \tag{10}$$
$$\mathbf{M}^T(\mathbf{x_1} - \mathbf{x_2}) = 0 \tag{11}$$

$\rightarrow$ Using the *least squares method*

$$\mathbf{x_1} - \mathbf{x_2} = \mathbf{A} - \mathbf{B} + \begin{pmatrix}\mathbf{m_1} & \mathbf{m_2}\end{pmatrix}\begin{pmatrix}\mu_1\\-\mu_2\end{pmatrix} \tag{12}$$
$$0 = \mathbf{M}^T(\mathbf{A} - \mathbf{B}) + \mathbf{M}^T\mathbf{M}\begin{pmatrix}\mu_1\\-\mu_2\end{pmatrix} \tag{13}$$
$$\mathbf{M}^T\mathbf{M}\mu = \mathbf{M}^T(\mathbf{B} - \mathbf{A}),\ \ \mu = \begin{pmatrix}\mu_1\\-\mu_2\end{pmatrix} \tag{14}$$

$\longrightarrow$ To perform *singular value decomposition*, we do the following eigen-decompositions

$$\mathbf{M}^T\mathbf{M} = \begin{pmatrix}3 & 3\\3 & 9\end{pmatrix} = \mathbf{V}\mathbf{D_2}\mathbf{V}^T \tag{15}$$
$$\mathbf{M}\mathbf{M}^T = \begin{pmatrix}5 & 1 & 5\\1 & 2 & 1\\5 & 1 & 5\end{pmatrix} = \mathbf{U}\mathbf{D_1}\mathbf{U}^T \tag{16}$$

$\rightarrow$ For $\mathbf{M}^T\mathbf{M}$, the characteristic polynomial is

$$char(\mathbf{M}^T\mathbf{M}) = \begin{vmatrix}3-\lambda & 3\\3 & 9-\lambda\end{vmatrix} = \lambda^2 - 12\lambda + 18 \implies \lambda_1 = 6 + 3\sqrt{2}, \lambda_2 = 6 - 3\sqrt{2} \tag{17}$$

$\rightarrow$ For λ_1, the augmented matrix formed using the eigenvalue-eigenvector equation gives

$$\begin{pmatrix}-3-3\sqrt{2} & 3\\3 & 3-3\sqrt{2}\end{pmatrix} \xleftrightarrow{\text{which simplifies to}} \begin{pmatrix}1 & 1-\sqrt{2}\\0 & 0\end{pmatrix} \tag{18}$$
$$\implies \mathbf{v_1} = k\begin{pmatrix}-1+\sqrt{2}\\1\end{pmatrix} = \frac{1}{\sqrt{4-2\sqrt{2}}}\begin{pmatrix}-1+\sqrt{2}\\1\end{pmatrix} \tag{19}$$

$\rightarrow$ For λ_2, the augmented matrix formed using the eigenvalue-eigenvector equation gives

$$\begin{pmatrix} -3+3\sqrt{2} & 3 \\ 3 & 3+3\sqrt{2} \end{pmatrix} \xleftrightarrow{\text{which simplifies to}} \begin{pmatrix} 1 & 1+\sqrt{2} \\ 0 & 0 \end{pmatrix} \tag{20}$$

$$\implies \mathbf{v_2} = k\begin{pmatrix} -1-\sqrt{2} \\ 1 \end{pmatrix} = \frac{1}{\sqrt{4+2\sqrt{2}}}\begin{pmatrix} -1-\sqrt{2} \\ 1 \end{pmatrix} \tag{21}$$

$\rightarrow$ Using (15), we get

$$\mathbf{V} = \begin{pmatrix} \mathbf{v_1} & \mathbf{v_2} \end{pmatrix} = \begin{pmatrix} \frac{-1+\sqrt{2}}{\sqrt{4-2\sqrt{2}}} & \frac{-1-\sqrt{2}}{\sqrt{4+2\sqrt{2}}} \\ \frac{1}{\sqrt{4-2\sqrt{2}}} & \frac{1}{\sqrt{4+2\sqrt{2}}} \end{pmatrix} \tag{22}$$

$$\mathbf{D_2} = \begin{pmatrix} 6+3\sqrt{2} & 0 \\ 0 & 6-3\sqrt{2} \end{pmatrix} \tag{23}$$

$\rightarrow$ For $\mathbf{MM}^T$, the characteristic polynomial is

$$char(\mathbf{MM}^T) = \begin{vmatrix} 5-\lambda & 1 & 5 \\ 1 & 2-\lambda & 1 \\ 5 & 1 & 5-\lambda \end{vmatrix} = \lambda(\lambda^2 - 12\lambda + 18) \implies \lambda_1 = 6+3\sqrt{2}, \lambda_2 = 6-3\sqrt{2}, \lambda_3 = 0 \tag{24}$$

$\rightarrow$ For λ_1, the augmented matrix formed using the eigenvalue-eigenvector equation gives

$$\begin{pmatrix} -1-3\sqrt{2} & 1 & 5 \\ 1 & -4-3\sqrt{2} & 1 \\ 5 & 1 & -1-3\sqrt{2} \end{pmatrix} \xleftrightarrow{\text{which simplifies to}} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 4-3\sqrt{2} \\ 0 & 0 & 0 \end{pmatrix} \tag{25}$$

$$\implies \mathbf{u_1} = k\begin{pmatrix} 1 \\ -4+3\sqrt{2} \\ 1 \end{pmatrix} = \frac{1}{\sqrt{36-24\sqrt{2}}}\begin{pmatrix} 1 \\ -4+3\sqrt{2} \\ 1 \end{pmatrix} = \frac{1}{\sqrt{12}}\begin{pmatrix} 1+\sqrt{2} \\ 2-\sqrt{2} \\ 1+\sqrt{2} \end{pmatrix} \tag{26}$$

$\rightarrow$ For λ_2, the augmented matrix formed using the eigenvalue-eigenvector equation gives

$$\begin{pmatrix} -1+3\sqrt{2} & 1 & 5 \\ 1 & -4+3\sqrt{2} & 1 \\ 5 & 1 & -1+3\sqrt{2} \end{pmatrix} \xleftrightarrow{\text{which simplifies to}} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 4+3\sqrt{2} \\ 0 & 0 & 0 \end{pmatrix} \tag{27}$$

$$\implies \mathbf{u_2} = k\begin{pmatrix} 1 \\ -4-3\sqrt{2} \\ 1 \end{pmatrix} = \frac{-1}{\sqrt{36+24\sqrt{2}}}\begin{pmatrix} 1 \\ -4-3\sqrt{2} \\ 1 \end{pmatrix} = \frac{1}{\sqrt{12}}\begin{pmatrix} 1-\sqrt{2} \\ 2+\sqrt{2} \\ 1-\sqrt{2} \end{pmatrix} \tag{28}$$

$\rightarrow$ For λ_3, the augmented matrix formed using the eigenvalue-eigenvector equation gives

$$\begin{pmatrix} 5 & 1 & 5 \\ 1 & 2 & 1 \\ 5 & 1 & 5 \end{pmatrix} \xleftrightarrow{\text{which simplifies to}} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{29}$$

$$\implies \mathbf{u_3} = k\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \tag{30}$$

$\rightarrow$ Using (16), we get the following

$$\mathbf{U} = \begin{pmatrix} \mathbf{u_1} & \mathbf{u_2} & \mathbf{u_3} \end{pmatrix} = \begin{pmatrix} \frac{1+\sqrt{2}}{\sqrt{12}} & \frac{1-\sqrt{2}}{\sqrt{12}} & -\frac{1}{\sqrt{2}} \\ \frac{2-\sqrt{2}}{\sqrt{12}} & \frac{2+\sqrt{2}}{\sqrt{12}} & 0 \\ \frac{1+\sqrt{2}}{\sqrt{12}} & \frac{1-\sqrt{2}}{\sqrt{12}} & \frac{1}{\sqrt{2}} \end{pmatrix} \tag{31}$$

$$\implies \mathbf{U_R} = \begin{pmatrix} \frac{1+\sqrt{2}}{\sqrt{12}} & \frac{1-\sqrt{2}}{\sqrt{12}} \\ \frac{2-\sqrt{2}}{\sqrt{12}} & \frac{2+\sqrt{2}}{\sqrt{12}} \\ \frac{1+\sqrt{2}}{\sqrt{12}} & \frac{1-\sqrt{2}}{\sqrt{12}} \end{pmatrix} \tag{32}$$

$$\mathbf{D_1} = \begin{pmatrix} 6+3\sqrt{2} & 0 & 0 \\ 0 & 6-3\sqrt{2} & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{33}$$

$\longrightarrow$ Then, for using *singular value decomposition*, we define

$$\mathbf{\Sigma} \triangleq \begin{pmatrix} \sqrt{6+3\sqrt{2}} & 0 \\ 0 & \sqrt{6-3\sqrt{2}} \\ 0 & 0 \end{pmatrix} \implies \mathbf{\Sigma_R} \triangleq \begin{pmatrix} \sqrt{6+3\sqrt{2}} & 0 \\ 0 & \sqrt{6-3\sqrt{2}} \end{pmatrix} \tag{34}$$

→ Using *singular value decomposition* and substituting in (14)

$$\mathbf{M} = \mathbf{U_R}\mathbf{\Sigma_R}\mathbf{V}^T \tag{35}$$

$$\mathbf{V}\mathbf{\Sigma_R}^T\mathbf{U_R}^T\mathbf{U_R}\mathbf{\Sigma_R}\mathbf{V}^T\mu = \mathbf{V}\mathbf{\Sigma_R}^T\mathbf{U_R}^T(\mathbf{B}-\mathbf{A}) \tag{36}$$

$$\mathbf{V}\mathbf{\Sigma_R}^2\mathbf{V}^T\mu = \mathbf{V}\mathbf{\Sigma_R}\mathbf{U_R}^T(\mathbf{B}-\mathbf{A}) \tag{37}$$

$$\mu = (\mathbf{V}\mathbf{\Sigma_R}^2\mathbf{V}^T)^{-1}\mathbf{V}\mathbf{\Sigma_R}\mathbf{U_R}^T(\mathbf{B}-\mathbf{A}) \tag{38}$$

$$\mu = \mathbf{V}\mathbf{\Sigma_R}^{-2}\mathbf{V}^T\mathbf{V}\mathbf{\Sigma_R}\mathbf{U_R}^T(\mathbf{B}-\mathbf{A}) \tag{39}$$

$$\mu = \mathbf{V}\mathbf{\Sigma_R}^{-1}\mathbf{U_R}^T(\mathbf{B}-\mathbf{A}) \tag{40}$$

→ Putting the required values in (40)

$$\mu = \begin{pmatrix} \frac{-1+\sqrt{2}}{\sqrt{4-2\sqrt{2}}} & \frac{-1-\sqrt{2}}{\sqrt{4+2\sqrt{2}}} \\ \frac{1}{\sqrt{4-2\sqrt{2}}} & \frac{1}{\sqrt{4+2\sqrt{2}}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{6+3\sqrt{2}}} & 0 \\ 0 & \frac{1}{\sqrt{6-3\sqrt{2}}} \end{pmatrix} \begin{pmatrix} \frac{1+\sqrt{2}}{\sqrt{12}} & \frac{2-\sqrt{2}}{\sqrt{12}} & \frac{1+\sqrt{2}}{\sqrt{12}} \\ \frac{1-\sqrt{2}}{\sqrt{12}} & \frac{2+\sqrt{2}}{\sqrt{12}} & \frac{1-\sqrt{2}}{\sqrt{12}} \end{pmatrix} \begin{pmatrix} 1 \\ -3 \\ -2 \end{pmatrix} \tag{41}$$

$$\begin{pmatrix} \mu_1 \\ -\mu_2 \end{pmatrix} = \begin{pmatrix} \frac{-1+\sqrt{2}}{\sqrt{12}} & \frac{-1-\sqrt{2}}{\sqrt{12}} \\ \frac{1}{\sqrt{12}} & \frac{1}{\sqrt{12}} \end{pmatrix} \begin{pmatrix} \frac{-7+2\sqrt{2}}{\sqrt{12}} \\ \frac{-7-2\sqrt{2}}{\sqrt{12}} \end{pmatrix} = \begin{pmatrix} 11/6 \\ -7/6 \end{pmatrix} \implies \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} = \begin{pmatrix} 11/6 \\ 7/6 \end{pmatrix} \tag{42}$$

⟶ This gives us the vector coordinates of x_1 and x_2, as

$$\mathbf{x_1} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + \left(\frac{11}{6}\right) \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 17/6 \\ 1/6 \\ 17/6 \end{pmatrix} \qquad \mathbf{x_2} = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} + \left(\frac{7}{6}\right) \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 26/6 \\ 1/6 \\ 8/6 \end{pmatrix} \tag{43}$$

→ And the least distance as

$$\|\mathbf{x_1} - \mathbf{x_2}\| = \left\| \begin{pmatrix} -3/2 \\ 0 \\ 3/2 \end{pmatrix} \right\| = \frac{3\sqrt{2}}{2} \tag{44}$$

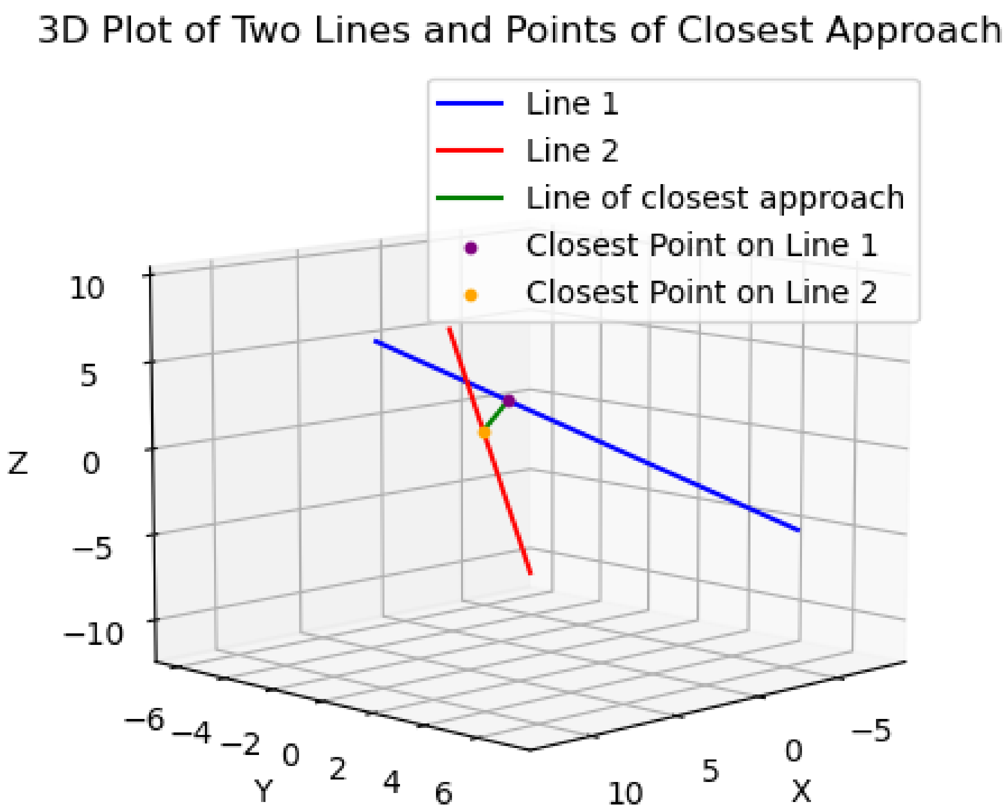

Fig. 0: Plot of given lines and shortest distance between them