MatGeo Presentation - Problem 12.701

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Question

Let $(\cdot,\cdot) : R^n \times R^n \to R$ be the inner product. Consider

$$P : |(\mathbf{u},\mathbf{v})| \leq \frac{(\mathbf{u},\mathbf{u}) + (\mathbf{v},\mathbf{v})}{2} \forall \mathbf{u},\mathbf{v}.$$

$$Q : \text{If } (\mathbf{u},\mathbf{v}) = (2\mathbf{u},\mathbf{v}) \ \forall \mathbf{v}, \text{ then } \mathbf{u} = 0.$$

Then

(a) both P, Q are true

(b) P is true, Q is false

(c) P is false, Q is true

(d) both P, Q are false

Solution

→ We can take the inner product as the dot product, since all inner products have the same properties as satisfied by the dot product.

$$\implies (\mathbf{u}, \mathbf{v}) = \mathbf{u}^T\mathbf{v} \tag{0.1}$$

→ For Statement P

$$|(\mathbf{u}, \mathbf{v})| \leq \|\mathbf{u}\|\|\mathbf{v}\| \qquad \text{(Cauchy-Schwarz Inequality)} \tag{0.2}$$

$$\frac{(\mathbf{u}, \mathbf{u}) + (\mathbf{v}, \mathbf{v})}{2} = \frac{\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2}{2} \tag{0.3}$$

$$\|\mathbf{u}\|\|\mathbf{v}\| \leq \frac{\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2}{2} \qquad \text{(G.M. } \leq \text{ A.M.)} \tag{0.4}$$

$$\implies |(\mathbf{u}, \mathbf{v})| = \frac{(\mathbf{u}, \mathbf{u}) + (\mathbf{v}, \mathbf{v})}{2} \qquad \implies \textit{P is true} \tag{0.5}$$

Solution

→ For Statement Q

$$
\begin{aligned}
(\mathbf{u}, \mathbf{v}) &= (2\mathbf{u}, \mathbf{v}) && (\forall \mathbf{v} \in R^n) && (0.6)\\
(\mathbf{u}, \mathbf{v}) - (2\mathbf{u}, \mathbf{v}) &= 0 && && (0.7)\\
(\mathbf{u} - 2\mathbf{u}, \mathbf{v}) &= 0 && && (0.8)\\
(-\mathbf{u}, \mathbf{v}) &= 0 && (\forall \mathbf{v} \in R^n) && (0.9)\\
\implies \mathbf{u} &= 0 && \implies \textit{Q is true} && (0.10)
\end{aligned}
$$

⟶ Therefore, *(b) both P, Q are true* is the correct option.