MatGeo Presentation - Problem 12.77

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Question

Let $\mathbf{M}$ be a 2×2 real matrix such that

$$(\mathbf{I} + \mathbf{M})^{-1} = \mathbf{I} - \alpha \mathbf{M} \tag{0.1}$$

where α is a nonzero real number and $\mathbf{I}$ is the 2×2 identity matrix. If the trace of the matrix $\mathbf{M}$ is 3, then the value of α is

(a) $\frac{3}{4}$ (b) $\frac{1}{3}$ (c) $\frac{1}{2}$ (d) $\frac{1}{4}$

Solution

$\rightarrow$ Post-multiplying by $(\mathbf{I}+\mathbf{M})$ on both sides of (1), we get

$$(\mathbf{I}+\mathbf{M})^{-1}(\mathbf{I}+\mathbf{M}) = (\mathbf{I}-\alpha\mathbf{M})(\mathbf{I}+\mathbf{M}) \tag{0.2}$$

$$\mathbf{I} = \mathbf{I}+\mathbf{M}-\alpha\mathbf{M}-\alpha\mathbf{M}^2 \tag{0.3}$$

$$\alpha\mathbf{M}^2-(1-\alpha)\mathbf{M} = 0 \tag{0.4}$$

$\rightarrow$ Since (4) is a degree 2 equation for a 2×2 matrix, we can use the Cayley-Hamilton Theorem to get the characteristic equation for the matrix $\mathbf{M}$ to be

$$\alpha\lambda^2-(1-\alpha)\lambda = 0 \tag{0.5}$$

Solution

$\rightarrow$ In the characteristic equation for any 2×2 matrix $\mathbf{A}$, we know that

$$\lambda^2 - trace(\mathbf{A})\lambda + det(\mathbf{A}) = 0 \tag{0.6}$$

$\rightarrow$ Thus using (5) and (6), we get

$$trace(\mathbf{M}) = \frac{1-\alpha}{\alpha} = 3 \tag{0.7}$$

$$1 - \alpha = 3\alpha \tag{0.8}$$

$$\alpha = \frac{1}{4} \tag{0.9}$$

$\rightarrow$ Therefore, (d) $\frac{1}{4}$ is the correct option.