MatGeo Presentation - Problem 12.181

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Question

Consider the matrix $\mathbf{P} = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}$. The value of $e^{\mathbf{P}}$ is

(a) $\begin{pmatrix} 2e^{-2} - 3e^{-1} & e^{-1} - e^{-2} \\ 2e^{-2} - 2e^{-1} & 5e^{-2} - e^{-1} \end{pmatrix}$ (c) $\begin{pmatrix} 5e^{-2} - 6e^{-1} & 3e^{-1} - e^{-2} \\ 2e^{-2} - 6e^{-1} & 4e^{-2} - e^{-1} \end{pmatrix}$

(b) $\begin{pmatrix} e^{-1} + e^{-2} & 2e^{-2} - e^{-1} \\ 2e^{-1} - 4e^{-2} & 3e^{-1} + e^{-2} \end{pmatrix}$ (d) $\begin{pmatrix} 2e^{-1} - e^{-2} & e^{-1} - e^{-2} \\ -2e^{-1} + 2e^{-2} & -e^{-1} + 2e^{-2} \end{pmatrix}$

Solution

$\rightarrow$ To find $e^{\mathbf{P}}$, we first need to diagonalize the matrix $\mathbf{P}$ by finding the eigenvalues and their corresponding eigenvectors. So, we use

$$|\mathbf{P} - \lambda \mathbf{I}| = 0 \tag{0.1}$$

$$\begin{vmatrix} -\lambda & 1 \\ -2 & -3-\lambda \end{vmatrix} = 0 \tag{0.2}$$

$$\lambda^2 + 3\lambda + 2 = 0 \tag{0.3}$$

$$\implies \lambda_1 = -1 \text{ and } \lambda_2 = -2 \tag{0.4}$$

$\rightarrow$ To find the eigenvector for λ_1, we use the augmented matrix

$$\left(\begin{array}{cc|c} 1 & 1 & 0 \\ -2 & -2 & 0 \end{array}\right) \xrightarrow{R_2 \leftrightarrow R_2 + 2R_1} \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right) \tag{0.5}$$

$$\implies \mathbf{v_1} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{0.6}$$

Solution

→ To find the eigenvector for λ_2, we use the augmented matrix

$$\left(\begin{array}{cc|c} 2 & 1 & 0 \\ -2 & -1 & 0 \end{array}\right) \xrightarrow{R_2 \leftrightarrow R_2 + R_1} \left(\begin{array}{cc|c} 2 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right) \tag{0.7}$$

$$\implies \mathbf{v_2} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \tag{0.8}$$

→ Thus, after diagonalization, we get

$$\mathbf{P} = \mathbf{QDQ}^{-1} \tag{0.9}$$

$$\mathbf{Q} = \begin{pmatrix} \mathbf{v_1} & \mathbf{v_2} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix} \tag{0.10}$$

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix} \tag{0.11}$$

$$\mathbf{Q}^{-1} = \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix} \tag{0.12}$$

Solution

→ Now, using property of matrix exponential

$$e^{\mathbf{P}} = \mathbf{Q}e^{\mathbf{D}}\mathbf{Q}^{-1} \tag{0.13}$$
$$= \begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} e^{-1} & 0 \\ 0 & e^{-2} \end{pmatrix} \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix} \tag{0.14}$$
$$= \begin{pmatrix} e^{-1} & e^{-2} \\ -e^{-1} & -2e^{-2} \end{pmatrix} \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix} \tag{0.15}$$
$$e^{\mathbf{P}} = \begin{pmatrix} 2e^{-1} - e^{-2} & e^{-1} - e^{-2} \\ -2e^{-1} + 2e^{-2} & -e^{-1} + 2e^{-2} \end{pmatrix} \tag{0.16}$$

→ Therefore, (d) $\begin{pmatrix} 2e^{-1} - e^{-2} & e^{-1} - e^{-2} \\ -2e^{-1} + 2e^{-2} & -e^{-1} + 2e^{-2} \end{pmatrix}$ is the correct option.