10.7.8

EE25BTECH11064 - Yojit Manral

Question:
The area (in sq. units) of the quadrilateral formed by the tangents at the end points of the latera recta to the ellipse $\frac{x^2}{9} + \frac{y^2}{5} = 1$ is

(a) $\frac{27}{2}$ (b) 27 (c) $\frac{27}{4}$ (d) 18

Solution:
→ The parameters of the given conic are

$$\mathbf{V} = \begin{pmatrix} 5/9 & 0 \\ 0 & 1 \end{pmatrix}, \mathbf{u} = 0, f = -5 \tag{1}$$

→ Also, using eigenvalue decomposition

$$\mathbf{P} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \mathbf{D} = \begin{pmatrix} 5/9 & 0 \\ 0 & 1 \end{pmatrix} \implies \lambda_1 = 5/9 \text{ and } \lambda_2 = 1 \tag{2}$$

→ This gives us very useful information

$$e = \sqrt{1 - \frac{\lambda_1}{\lambda_2}} = \frac{2}{3} \tag{3}$$

$$\mathbf{F} = \pm e \sqrt{\frac{|\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f|}{\lambda_2(1 - e^2)}} \mathbf{e_1} = \pm \begin{pmatrix} 2 \\ 0 \end{pmatrix} \tag{4}$$

→ Given that the points of contact are the end points of the latera recta. In the first quadrant, we have

$$\mathbf{q} = \begin{pmatrix} 2 \\ 5/3 \end{pmatrix} \tag{5}$$

→ If the point of contact is $\mathbf{q}$, the equation of tangent to the conic is

$$(\mathbf{V}\mathbf{q} + \mathbf{u})^T \mathbf{x} + \mathbf{u}^T \mathbf{q} + f = 0 \tag{6}$$

$$\implies \begin{pmatrix} 2 & 3 \end{pmatrix} \mathbf{x} = 9 \tag{7}$$

→ Thus, we get the area of the quadrilateral in the 1^{st} quadrant to be

$$\Delta_1 = \text{Area of } \triangle = \frac{1}{2} x_{int} y_{int} = \frac{1}{2} \times \frac{9}{2} \times 3 \tag{8}$$

$$\Delta_1 = \frac{27}{4} sq.units \tag{9}$$

→ Since an ellipse has a two-fold symmetry, total area of the quadrilateral must be 4 times the area in the 1^{st} quadrant

$$\Delta_{Total} = 4 \times \Delta_1 = 27 sq.units \tag{10}$$

→ Therefore, (b) 27 is the correct option

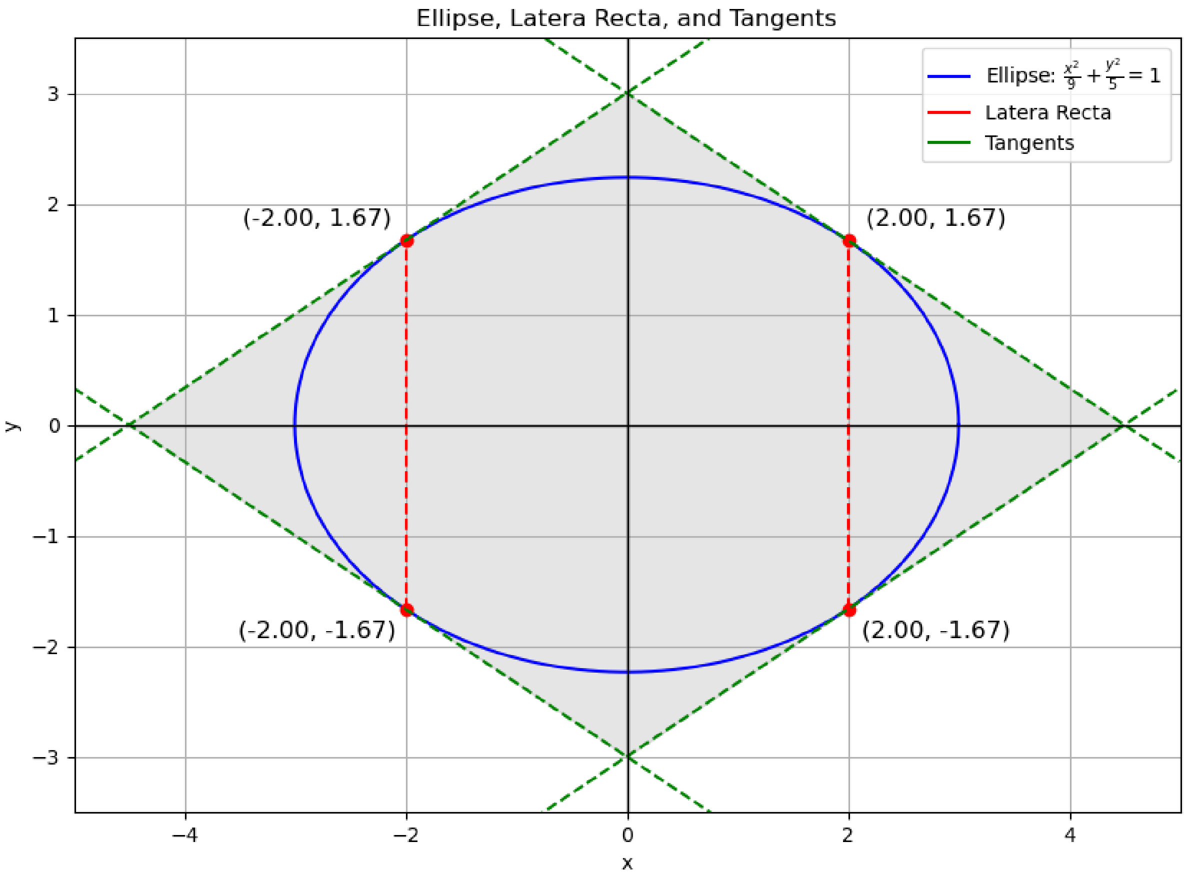

Fig. 4: Plot of ellipse and quadrilateral