MatGeo Presentation - Problem 6.3.4

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Question

Find the shortest distance between the lines

$$\mathbf{r} = (\mathbf{i} + 2\mathbf{j} + \mathbf{k}) + \kappa_1(\mathbf{i} - \mathbf{j} + \mathbf{k}) \tag{0.1}$$

$$\mathbf{r} = (2\mathbf{i} - \mathbf{j} - \mathbf{k}) + \kappa_2(2\mathbf{i} + \mathbf{j} + 2\mathbf{k}) \tag{0.2}$$

Solution

$\longrightarrow$ The lines can be represented in vector form as

$$L_1 \equiv \mathbf{x} = \mathbf{A} + \kappa_1 \mathbf{m_1} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + \kappa_1 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \tag{0.3}$$

$$L_2 \equiv \mathbf{x} = \mathbf{B} + \kappa_2 \mathbf{m_2} = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} + \kappa_2 \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \tag{0.4}$$

$\rightarrow$ To check whether the given lines are skewed

$$\mathbf{M} = \begin{pmatrix} \mathbf{m_1} & \mathbf{m_2} \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ -1 & 1 \\ 1 & 2 \end{pmatrix}, \mathbf{B} - \mathbf{A} = \begin{pmatrix} 1 \\ -3 \\ -2 \end{pmatrix} \tag{0.5}$$

$$\begin{pmatrix} 1 & 2 & 1 \\ -1 & 1 & -3 \\ 1 & 2 & -2 \end{pmatrix} \xrightarrow[R_3 \leftrightarrow R_3 - R_1]{R_2 \leftrightarrow R_2 + R_1} \begin{pmatrix} 1 & 2 & 1 \\ 0 & 3 & -2 \\ 0 & 0 & -3 \end{pmatrix} \tag{0.6}$$

$$rank \begin{pmatrix} \mathbf{M} & \mathbf{B} - \mathbf{A} \end{pmatrix} = 3 \implies \text{The given lines are skew} \tag{0.7}$$

Solution

$\longrightarrow$ Let $x_1(\mu_1)$ and $x_2(\mu_2)$ be the points closest to each other from the lines L_1 and L_2, respectively

$$\mathbf{x_1} = \mathbf{A} + \mu_1 \mathbf{m_1} \qquad \mathbf{x_2} = \mathbf{B} + \mu_2 \mathbf{m_2} \tag{0.8}$$

$\rightarrow$ Now, we have

$$(\mathbf{x_1} - \mathbf{x_2})^T \mathbf{m_1} = (\mathbf{x_1} - \mathbf{x_2})^T \mathbf{m_2} = 0 \tag{0.9}$$

$$(\mathbf{x_1} - \mathbf{x_2})^T \begin{pmatrix} \mathbf{m_1} & \mathbf{m_2} \end{pmatrix} = 0 \tag{0.10}$$

$$\mathbf{M}^T (\mathbf{x_1} - \mathbf{x_2}) = 0 \tag{0.11}$$

$\rightarrow$ Using the *least squares method*

$$\mathbf{x_1} - \mathbf{x_2} = \mathbf{A} - \mathbf{B} + \begin{pmatrix} \mathbf{m_1} & \mathbf{m_2} \end{pmatrix} \begin{pmatrix} \mu_1 \\ -\mu_2 \end{pmatrix} \tag{0.12}$$

$$0 = \mathbf{M}^T (\mathbf{A} - \mathbf{B}) + \mathbf{M}^T \mathbf{M} \begin{pmatrix} \mu_1 \\ -\mu_2 \end{pmatrix} \tag{0.13}$$

$$\mathbf{M}^T \mathbf{M} \mu = \mathbf{M}^T (\mathbf{B} - \mathbf{A}), \;\; \mu = \begin{pmatrix} \mu_1 \\ -\mu_2 \end{pmatrix} \tag{0.14}$$

Solution

$\longrightarrow$ To perform *singular value decomposition*, we do the following eigen-decompositions

$$\mathbf{M}^T\mathbf{M} = \begin{pmatrix} 3 & 3 \\ 3 & 9 \end{pmatrix} = \mathbf{V}\mathbf{D_2}\mathbf{V}^T \tag{0.15}$$

$$\mathbf{M}\mathbf{M}^T = \begin{pmatrix} 5 & 1 & 5 \\ 1 & 2 & 1 \\ 5 & 1 & 5 \end{pmatrix} = \mathbf{U}\mathbf{D_1}\mathbf{U}^T \tag{0.16}$$

$\rightarrow$ For $\mathbf{M}^T\mathbf{M}$, the characteristic polynomial is

$$char(\mathbf{M}^T\mathbf{M}) = \begin{vmatrix} 3-\lambda & 3 \\ 3 & 9-\lambda \end{vmatrix} = \lambda^2 - 12\lambda + 18 \tag{0.17}$$

$$\implies \lambda_1 = 6 + 3\sqrt{2}, \lambda_2 = 6 - 3\sqrt{2} \tag{0.18}$$

$\rightarrow$ For λ_1, the augmented matrix formed using the eigenvalue-eigenvector equation gives

$$\begin{pmatrix} -3-3\sqrt{2} & 3 \\ 3 & 3-3\sqrt{2} \end{pmatrix} \xleftrightarrow{\text{which simplifies to}} \begin{pmatrix} 1 & 1-\sqrt{2} \\ 0 & 0 \end{pmatrix} \tag{0.19}$$

Solution

$$\implies \mathbf{v_1} = \frac{1}{\sqrt{4-2\sqrt{2}}}\begin{pmatrix}-1+\sqrt{2}\\ 1\end{pmatrix} \tag{0.20}$$

$\rightarrow$ For λ_2, the augmented matrix formed using the eigenvalue-eigenvector equation gives

$$\begin{pmatrix}-3+3\sqrt{2} & 3\\ 3 & 3+3\sqrt{2}\end{pmatrix} \xleftrightarrow{\text{which simplifies to}} \begin{pmatrix}1 & 1+\sqrt{2}\\ 0 & 0\end{pmatrix} \tag{0.21}$$

$$\implies \mathbf{v_2} = \frac{1}{\sqrt{4+2\sqrt{2}}}\begin{pmatrix}-1-\sqrt{2}\\ 1\end{pmatrix} \tag{0.22}$$

$\rightarrow$ Using (15), we get

$$\mathbf{V} = \begin{pmatrix}\mathbf{v_1} & \mathbf{v_2}\end{pmatrix} = \begin{pmatrix}\frac{-1+\sqrt{2}}{\sqrt{4-2\sqrt{2}}} & \frac{-1-\sqrt{2}}{\sqrt{4+2\sqrt{2}}}\\ \frac{1}{\sqrt{4-2\sqrt{2}}} & \frac{1}{\sqrt{4+2\sqrt{2}}}\end{pmatrix} \tag{0.23}$$

$$\mathbf{D_2} = \begin{pmatrix}6+3\sqrt{2} & 0\\ 0 & 6-3\sqrt{2}\end{pmatrix} \tag{0.24}$$

Solution

→ For $\mathbf{MM}^T$, the characteristic polynomial is

$$char(\mathbf{MM}^T) = \begin{vmatrix} 5-\lambda & 1 & 5 \\ 1 & 2-\lambda & 1 \\ 5 & 1 & 5-\lambda \end{vmatrix} = \lambda(\lambda^2 - 12\lambda + 18) \quad (0.25)$$

$$\implies \lambda_1 = 6 + 3\sqrt{2}, \lambda_2 = 6 - 3\sqrt{2}, \lambda_3 = 0 \quad (0.26)$$

→ For λ_1, the augmented matrix formed using the eigenvalue-eigenvector equation gives

$$\begin{pmatrix} -1-3\sqrt{2} & 1 & 5 \\ 1 & -4-3\sqrt{2} & 1 \\ 5 & 1 & -1-3\sqrt{2} \end{pmatrix} \xleftrightarrow{\text{simplifies to}} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 4-3\sqrt{2} \\ 0 & 0 & 0 \end{pmatrix} \quad (0.27)$$

$$\implies \mathbf{u_1} = \frac{1}{\sqrt{36-24\sqrt{2}}} \begin{pmatrix} 1 \\ -4+3\sqrt{2} \\ 1 \end{pmatrix} = \frac{1}{\sqrt{12}} \begin{pmatrix} 1+\sqrt{2} \\ 2-\sqrt{2} \\ 1+\sqrt{2} \end{pmatrix} \quad (0.28)$$

Solution

$\rightarrow$ For λ_2, the augmented matrix formed using the eigenvalue-eigenvector equation gives

$$\begin{pmatrix} -1+3\sqrt{2} & 1 & 5 \\ 1 & -4+3\sqrt{2} & 1 \\ 5 & 1 & -1+3\sqrt{2} \end{pmatrix} \xleftrightarrow{\text{simplifies to}} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 4+3\sqrt{2} \\ 0 & 0 & 0 \end{pmatrix} \tag{0.29}$$

$$\implies \mathbf{u_2} = \frac{-1}{\sqrt{36+24\sqrt{2}}} \begin{pmatrix} 1 \\ -4-3\sqrt{2} \\ 1 \end{pmatrix} = \frac{1}{\sqrt{12}} \begin{pmatrix} 1-\sqrt{2} \\ 2+\sqrt{2} \\ 1-\sqrt{2} \end{pmatrix} \tag{0.30}$$

$\rightarrow$ For λ_3, the augmented matrix formed using the eigenvalue-eigenvector equation gives

$$\begin{pmatrix} 5 & 1 & 5 \\ 1 & 2 & 1 \\ 5 & 1 & 5 \end{pmatrix} \xleftrightarrow{\text{simplifies to}} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{0.31}$$

Solution

$$\implies \mathbf{u_3} = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \tag{0.32}$$

$\rightarrow$ Using (16), we get the following

$$\mathbf{U} = \begin{pmatrix} \mathbf{u_1} & \mathbf{u_2} & \mathbf{u_3} \end{pmatrix} = \begin{pmatrix} \frac{1+\sqrt{2}}{\sqrt{12}} & \frac{1-\sqrt{2}}{\sqrt{12}} & -\frac{1}{\sqrt{2}} \\ \frac{2-\sqrt{2}}{\sqrt{12}} & \frac{2+\sqrt{2}}{\sqrt{12}} & 0 \\ \frac{1+\sqrt{2}}{\sqrt{12}} & \frac{1-\sqrt{2}}{\sqrt{12}} & \frac{1}{\sqrt{2}} \end{pmatrix} \tag{0.33}$$

$$\implies \mathbf{U_R} = \begin{pmatrix} \frac{1+\sqrt{2}}{\sqrt{12}} & \frac{1-\sqrt{2}}{\sqrt{12}} \\ \frac{2-\sqrt{2}}{\sqrt{12}} & \frac{2+\sqrt{2}}{\sqrt{12}} \\ \frac{1+\sqrt{2}}{\sqrt{12}} & \frac{1-\sqrt{2}}{\sqrt{12}} \end{pmatrix} \tag{0.34}$$

$$\mathbf{D_1} = \begin{pmatrix} 6+3\sqrt{2} & 0 & 0 \\ 0 & 6-3\sqrt{2} & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{0.35}$$

Solution

$\longrightarrow$ Then, for using *singular value decomposition*, we define

$$\mathbf{\Sigma} \triangleq \begin{pmatrix} \sqrt{6+3\sqrt{2}} & 0 \\ 0 & \sqrt{6-3\sqrt{2}} \\ 0 & 0 \end{pmatrix} \implies \mathbf{\Sigma_R} \triangleq \begin{pmatrix} \sqrt{6+3\sqrt{2}} & 0 \\ 0 & \sqrt{6-3\sqrt{2}} \end{pmatrix} \tag{0.36}$$

$\rightarrow$ Using *singular value decomposition* and substituting in (14)

$$\mathbf{M} = \mathbf{U_R \Sigma_R V}^T \tag{0.37}$$

$$\mathbf{V\Sigma_R}^T\mathbf{U_R}^T\mathbf{U_R\Sigma_R V}^T\mu = \mathbf{V\Sigma_R}^T\mathbf{U_R}^T(\mathbf{B}-\mathbf{A}) \tag{0.38}$$

$$\mathbf{V\Sigma_R}^2\mathbf{V}^T\mu = \mathbf{V\Sigma_R U_R}^T(\mathbf{B}-\mathbf{A}) \tag{0.39}$$

$$\mu = (\mathbf{V\Sigma_R}^2\mathbf{V}^T)^{-1}\mathbf{V\Sigma_R U_R}^T(\mathbf{B}-\mathbf{A}) \tag{0.40}$$

$$\mu = \mathbf{V\Sigma_R}^{-2}\mathbf{V}^T\mathbf{V\Sigma_R U_R}^T(\mathbf{B}-\mathbf{A}) \tag{0.41}$$

$$\mu = \mathbf{V\Sigma_R}^{-1}\mathbf{U_R}^T(\mathbf{B}-\mathbf{A}) \tag{0.42}$$

Solution

$\rightarrow$ Putting the required values in (42)

$$\mu = \begin{pmatrix} \frac{-1+\sqrt{2}}{\sqrt{4-2\sqrt{2}}} & \frac{-1-\sqrt{2}}{\sqrt{4+2\sqrt{2}}} \\ \frac{1}{\sqrt{4-2\sqrt{2}}} & \frac{1}{\sqrt{4+2\sqrt{2}}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{6+3\sqrt{2}}} & 0 \\ 0 & \frac{1}{\sqrt{6-3\sqrt{2}}} \end{pmatrix}$$

$$\begin{pmatrix} \frac{1+\sqrt{2}}{\sqrt{12}} & \frac{2-\sqrt{2}}{\sqrt{12}} & \frac{1+\sqrt{2}}{\sqrt{12}} \\ \frac{1-\sqrt{2}}{\sqrt{12}} & \frac{2+\sqrt{2}}{\sqrt{12}} & \frac{1-\sqrt{2}}{\sqrt{12}} \end{pmatrix} \begin{pmatrix} 1 \\ -3 \\ -2 \end{pmatrix} \tag{0.43}$$

$$\begin{pmatrix} \mu_1 \\ -\mu_2 \end{pmatrix} = \begin{pmatrix} \frac{-1+\sqrt{2}}{\sqrt{12}} & \frac{-1-\sqrt{2}}{\sqrt{12}} \\ \frac{1}{\sqrt{12}} & \frac{1}{\sqrt{12}} \end{pmatrix} \begin{pmatrix} \frac{-7+2\sqrt{2}}{\sqrt{12}} \\ \frac{-7-2\sqrt{2}}{\sqrt{12}} \end{pmatrix} = \begin{pmatrix} 11/6 \\ -7/6 \end{pmatrix} \tag{0.44}$$

$$\implies \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} = \begin{pmatrix} 11/6 \\ 7/6 \end{pmatrix} \tag{0.45}$$

Solution

$\longrightarrow$ This gives us the vector coordinates of x_1 and x_2, as

$$\mathbf{x_1} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + \left(\frac{11}{6}\right) \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 17/6 \\ 1/6 \\ 17/6 \end{pmatrix} \tag{0.46}$$

$$\mathbf{x_2} = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} + \left(\frac{7}{6}\right) \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 26/6 \\ 1/6 \\ 8/6 \end{pmatrix} \tag{0.47}$$

$\rightarrow$ And the least distance as

$$\|\mathbf{x_1} - \mathbf{x_2}\| = \left\| \begin{pmatrix} -3/2 \\ 0 \\ 3/2 \end{pmatrix} \right\| \tag{0.48}$$

$$= \frac{3\sqrt{2}}{2} \tag{0.49}$$

Solution

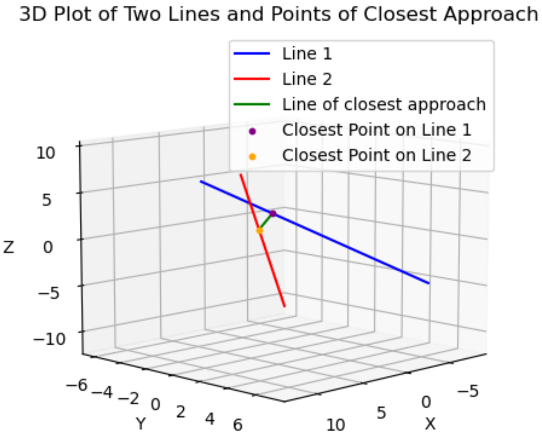

Figure: Plot of given lines and shortest distance between them

File: points.c

```c
#include <stdio.h>

int main() {
  FILE *fp;

  // -------------------
  // Question 6.3.4
  // -------------------


  fp = fopen("points.dat", "w");
  fprintf(fp, "%d,%d,%d\n", 1, 2, 1); // A
  fprintf(fp, "%d,%d,%d\n", 2, -1, -1); // B
  fprintf(fp, "%d,%d,%d\n", 1, -1, 1); // m1
  fprintf(fp, "%d,%d,%d\n", 2, 1, 2); // m2
  fprintf(fp, "%f,%f,%f\n", 17/6, 1/6, 17/6); // c1
  fprintf(fp, "%f,%f,%f\n", 26/6, 1/6, 8/6); // c2
  fclose(fp);
  return 0;
}
```

File: call_c.py

```
import subprocess

# Compile the C program
subprocess.run(["gcc", "points.c", "-o", "points"])

# Run the compiled C program
result = subprocess.run(["./points"], capture_output=True, text=True)

# Print the output from the C program
print(result.stdout)
```

File: plot.py

```
import numpy as np
import matplotlib.pyplot as plt
from mpl_toolkits.mplot3d import Axes3D

# Define the lines with their parametric equations
# Define the parametric values for kappa_1 and kappa_2
kappa_1_vals = np.linspace(-5, 5, 50)
kappa_2_vals = np.linspace(-5, 5, 50)
# Parametric equations of the lines
r1 = np.array([1, 2, 1]) + kappa_1_vals[:, None] * np.array([1, -1, 1])
r2 = np.array([2, -1, -1]) + kappa_2_vals[:, None] * np.array([2, 1, 2])

# Plot the lines
fig = plt.figure()
ax = fig.add_subplot(111, projection='3d')
# Plot line 1
ax.plot(r1[:, 0], r1[:, 1], r1[:, 2], label='Line 1', color='blue')
# Plot line 2
ax.plot(r2[:, 0], r2[:, 1], r2[:, 2], label='Line 2', color='red')

# Points of closest approach on both lines
closest_point_1 = np.array([17/6,1/6,17/6])
closest_point_2 = np.array([26/6,1/6,8/6])
```

File: plot.py

```
# Mark the vector along points of closest approach
closest_line_vals = np.linspace(0, 1, 10)
closest_line = np.array([17/6,1/6,17/6]) + closest_line_vals[:, None] * np.array([3/2,0,-3/2])
ax.plot(closest_line[:, 0], closest_line[:, 1], closest_line[:, 2], label='Line of closest approach', color
      ='green')

# Mark the points of closest approach
ax.scatter(closest_point_1[0], closest_point_1[1], closest_point_1[2], color='purple', s=10, label='Closest
      Point on Line 1')
ax.scatter(closest_point_2[0], closest_point_2[1], closest_point_2[2], color='orange', s=10, label='Closest
      Point on Line 2')

# Labels and title
ax.set_xlabel('X')
ax.set_ylabel('Y')
ax.set_zlabel('Z')
ax.set_title('3D Plot of Two Lines and Points of Closest Approach')

# Set the camera angle
ax.view_init(elev=10, azim=45)

# Show the plot
ax.legend()
plt.show()
```