Klein geometric structures on manifolds and Bieberbach theorem

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Abstract

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Contents

1 Preliminary

In this section we shall give necessary definitions and lemmas for our topic.

Definition 1. Let H and K denote groups with group multiplication $\circ$ and $\star$ respectively and assume that $H < \mathrm{Aut}(K)$ and that K is abelian. The semi-direct product $H \ltimes K$ of the groups H and K is the set of all pairs $(h,k), h \in H, k \in K$, with the following multiplication

$$(h_1, k_1)(h_2, k_2) = (h_1 \circ h_2, k_1 \star h_1(k_2))$$

Example 1.

- The isometries group $E(n) = O(n) \ltimes \mathbb{R}^n$
- The affine group $A(n) = GL(n, \mathbb{R}) \ltimes \mathbb{R}^n$

Proposition 1. There is the following sequence of subgroups

$$E(n) \subset A(n) \subset GL(n+1, \mathbb{R})$$

Proof. From the definition $E(n) \subset A(n)$.
Let $(A, a) \in A(n)$. We have an $(n+1) \times (n+1)$ matrix $\begin{pmatrix} A & a \\ 0 & 1 \end{pmatrix}$, which defines an inclusion $A(n) \subset GL(n+1, \mathbb{R})$. □

Definition 2. Let Γ be a subgroup of the group $E(n)$. Then Γ is discrete if it is a discrete subset of the Euclidean space $\mathbb{R}^{(n+1)^2}$. We say that Γ acts properly discontinuously on $\mathbb{R}^n$ if for any $x \in \mathbb{R}^n$ there is an open neighbourhood U_x such that the set

$$\{\gamma \in \Gamma \mid \gamma U_x \cap U_x \neq \emptyset\}$$

is finite. Moreover, Γ acts freely, if for any $x \in \mathbb{R}^n$ we have

$$\{\gamma \in \Gamma \mid \gamma x = x\} = \{(I, 0)\}$$

Lemma 1. *Any discrete subgroup of the group $E(n)$ is closed in $E(n)$.*

Proof. Let Γ be a discrete subgroup of $E(n)$ and suppose that $E(n) \backslash \Gamma$ is not open. Then there is a γ in $E(n) \backslash \Gamma$ and γ_n in $B(\gamma, 1/n) \cap \Gamma$.
As $\gamma_n \to \gamma$ in $E(n)$, we have $\gamma_n \gamma_{n+1}^{-1} \to 1$ in Γ. However $\{\gamma_n \gamma_{n+1}^{-1}\}$ is not eventually constant, which is a contradiction for that Γ is a discrete metric space. Therefore, $E(n) \backslash \Gamma$ is open, and so Γ is closed in $E(n)$. □

Lemma 2. *If Γ is a discrete subgroup of the group $E(n)$ and $V_0 = D(0, r) \subset \mathbb{R}^n$ is an open disk, then*

$$\{\gamma \in \Gamma \mid \gamma V_0 \cap V_0 \neq \emptyset\} \subset \Gamma \cap (O(n) \times V_0')$$

where $V_0' = D(0, 2r)$ is an open disk.

Proof. Let $\gamma = (A, a) \in \Gamma$ and $\gamma V_0 \cap V_0 \neq \emptyset$. Then there exist $x, x' \in V_0$, such that $\gamma x = Ax + a = x'$.
Hence, from the triangle inequality $\|a\| = \|x' - Ax\| \leq \|x'\| + \|Ax\| < 2r$ and $\gamma \in O(n) \times V_0'$. $\square$

Proposition 2. Let Γ be a subgroup of the group $E(n)$. The following conditions are equivalent:

1. Γ acts properly discontinuously on $\mathbb{R}^n$;
2. $\forall x \in \mathbb{R}^n$, Γx is a discrete subset of $\mathbb{R}^n$;
3. Γ is a discrete subgroup of $E(n)$.

Proof. Let Γ act properly discontinuously on $\mathbb{R}^n$. We claim that Γ is discrete. Let elements $\{\gamma_n\} \subset \Gamma$ converge to the identity. By assumption there is a neighbourhood U_0 of 0 such that the set $\{\gamma_i \mid U_0 \cap \gamma_i U_0 \neq \emptyset\}$ is finite. Hence $\gamma_i = (I, 0)$ for large i and the sequence $\{\gamma_n\}$ is eventually constant. In general case if $\gamma_n \to \gamma$ then $\gamma_n \gamma^{-1} \to (I, 0)$ and from previous consideration, the implication (i) $\to$ (iii) is proved.
We shall use the previous lemma for the proof of the reverse implication. Let $x \in \mathbb{R}^n$ be any point and V_x be a disk of radius r centered at x. By definition of properly discontinuous action and the lemma we have

$$\{\gamma \in \Gamma \mid \gamma V_x \cap V_x \neq \emptyset\} = \{\gamma \in \Gamma \mid t_{-x}\gamma t_x V_0 \cap V_0 \neq \emptyset\} \subset t_{-x}\Gamma t_x \cap (O(n) \times V_0')$$

Since Γ is discrete and hence also closed, the above set is finite and the implication (iii) $\to$ (i) is proved.

Let us assume the condition (i) (or equivalently (iii)). We have to prove that for any $x \in \mathbb{R}^n$ the set Γx is discrete. Suppose it is not. Then there is $y \in \mathbb{R}^n$ and a sequence $\{\gamma_i x = A_i x + a_i\}$, which is not eventually constant and converges to y. Since the group $O(n)$ is compact, the sequence $\{A_i\}$ converges to some $A \in O(n)$. We claim that the sequence $\{a_i\}$ converges to $-Ax + y$. In fact, the value

$$\|a_i + Ax - y\| \leq \|a_i + A_i x - y\| + \|Ax - A_i x\|$$

can be arbitrarily small for large i. Summing up, we showed that the sequence $\{\gamma_i\}$ converges to $\gamma = (A, -Ax + y)$ in $E(n)$. Hence, $\left\{\gamma_i \gamma_{i+1}^{-1}\right\}$ converges to the identity. Since Γ is discrete, a sequence $\{\gamma_i x\}$ is eventually constant. This contradicts our assumptions and proves the implication (i) $\to$ (ii).
Finally, we prove that (i) follows from (ii). Let $\{\gamma_n\}$ be a convergent sequence in Γ. Then, for any $x \in \mathbb{R}^n$, $\{\gamma_n x\}$ is a convergent sequence in Γx. By definition it is eventually constant. Hence a sequence $\{\gamma_n\}$ is also eventually constant. $\square$

Proposition 3. A discrete subgroup of $E(n)$ acts freely on $\mathbb{R}^n$ if and only if it is torsion free (i.e it has no elements of finite order).

Proof. Assume that a group Γ has an element γ of order k. For any $x \in \mathbb{R}^n$ the element $x + \gamma x + \gamma^2 x + \cdots + \gamma^{k-1} x$ is invariant under the action of γ. Hence the action of Γ is not free. The reverse implication follows from the equality of the sets

$$\{\gamma \in \Gamma \mid \gamma a = a\} = \Gamma \cap t_a(O(n) \times 0)t_{-a}$$

where $a \in \mathbb{R}^n$. In fact, since the orthogonal group $O(n)$ is compact and Γ is discrete, the above set is always finite. $\square$

Definition 3. Let G be a differential manifold and simultaneously a group, such that the group operation and inversion are differential maps of the manifolds. Then G is called a Lie group.

Example 2. The sphere $S^1 \subset \mathbb{C}$, with the multiplication of complex numbers, is a Lie group. The Cartesian product of S^1, i.e. n-dimensional torus $T^n = \left(S^1\right)^n$, is a Lie group. The matrix groups $U(n), O(n), SL(n, \mathbb{R}), SL(n, \mathbb{C})$ are Lie groups.
The group $(\mathbb{Z}/2\mathbb{Z})^{\mathbb{N}}$ is not a Lie group.

Definition 4. Let Γ be a subgroup of $E(n)$. The orbit space of the action of Γ on $\mathbb{R}^n$ is defined to be the set of Γ-orbits $\mathbb{R}^n/\Gamma = \{\Gamma x \mid x \in \mathbb{R}^n\}$ topologized with the quotient topology from $\mathbb{R}^n$. The quotient map will be denoted by $\pi : \mathbb{R}^n \to \mathbb{R}^n/\Gamma$.

Lemma 3. *If Γ is a subgroup of $E(n)$, then the natural projection map $p : E(n) \to E(n)/\Gamma$ and the projection on the orbit space $\pi : \mathbb{R}^n \to \mathbb{R}^n/\Gamma$ are open and closed.*

Proposition 4. Let Γ be a subgroup of $E(n)$. Then the orbit space $\mathbb{R}^n/\Gamma$ is compact if and only if the space of cosets $E(n)/\Gamma$ is compact.

Proof. By definition $E(n)/O(n) = \mathbb{R}^n$.
A group Γ acts on the space $E(n)/O(n)$ by $gO(n) \mapsto (\gamma g)O(n)$, where $\gamma \in \Gamma, g \in E(n)$. The above action agrees with a standard action of Γ on $\mathbb{R}^n$.
Next, let us note that the map $E(n)/\Gamma \to (E(n)/O(n))/\Gamma = \mathbb{R}^n/\Gamma$, given by

$$g^{-1}\Gamma \to \Gamma(gO(n))$$

is a continuous open map with compact fibers. Hence it follows that $E(n)/\Gamma$ is compact if and only if $\mathbb{R}^n/\Gamma$ is compact. $\square$

Lemma 4. *A space $E(n)/\Gamma$ is compact if and only if there exists a compact subset $D \subset E(n)$, such that $E(n) = D\Gamma$.*

Proof. Since $E(n)$ is a subset of $\mathbb{R}^{(n+1)^2}$, there exists a family of open sets $U_k \cap E(n)$, such that the family of sets $p\left(U_k \cap E(n)\right)$ covers the compact space $E(n)/\Gamma$.
Here U_k is an open disk centered at origin and of radius k. Hence there exists k_0, such that $p\left(U_{k_0} \cap E(n)\right) = E(n)/\Gamma$.
Let D be a closure of the set $(U_{k_0} \cap E(n))$, i.e. $D = \overline{(U_{k_0} \cap E(n))}$. Finally, we have

$$E(n) = D\Gamma$$

The proof of the reverse implication follows from the equality $D/\Gamma = E(n)/\Gamma$. $\square$

Definition 5. A subgroup $\Gamma \subset E(n)$ is cocompact, if the space $E(n)/\Gamma$ is compact.

Roughly speaking a fundamental domain for a group Γ of isometries in a metric space X is a subset of X which contains exactly one point from each of these orbits.

Definition 6. Let X be a metric space and G a subgroup of a group of its isometries. An open, connected subset $F \subset X$ is a fundamental domain if

$$X = \bigcup_{g \in G} g\bar{F}$$

and $gF \cap g'F = \emptyset$, for $g \neq g' \in G$.

2 Bieberbach Theorems

Definition 7. A crystallographic group of dimension n is a cocompact and discrete subgroup of $E(n)$. We also refer to a such group as a Bieberbach group.

Example 3. If

$$\left(B, \begin{pmatrix} 0 \\ 1/2 \end{pmatrix}\right), \left(I, \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) \in E(2) \text{ where } B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Then the group $\Gamma \subset E(2)$ generated by the above elements is a crystallographic group of dimension two and the orbit space $\mathbb{R}^2/\Gamma$ is the Klein bottle.

The first part of the eighteenth Hilbert problem was about the description of discrete and cocompact groups of isometries of $\mathbb{R}^n$. The German mathematician L. Bieberbach answers this problem.

Theorem 1 (Bieberbach)**.**

1. *If $\Gamma \subset E(n)$ is a crystallographic group then the set of translations $\Gamma \cap (I \times \mathbb{R}^n)$ is a torsion free and finitely generated abelian group of rank n, and is a maximal abelian and normal subgroup of finite index.*

2. *For any natural number n, there are only a finite number of isomorphism classes of crystallographic groups of dimension n.*

3. *Two crystallographic groups of dimension n are isomorphic if and only if they are conjugate in the group $A(n)$.*

We shall prove these lemmas first:

Lemma 5. *There exists a neighborhood of the identity $U \subset O(n)$ such that for any $h \in U$, if $g \in O(n)$ commutes with $[g, h] = ghg^{-1}h^{-1}$, then g commutes with h.*

Proof. Let $\lambda_1, \lambda_2, \ldots, \lambda_r$ be the eigenvalues of an orthogonal matrix $g : \mathbb{C}^n \to \mathbb{C}^n$, and let $\mathbb{C}^n = V_1 \oplus V_2 \oplus \cdots \oplus V_r$ be its invariant subspaces. Since $g[g, h] = [g, h]g$, we have

$$ghg^{-1}h^{-1} = hg^{-1}h^{-1}g.$$

Moreover, for $i = 1, \dots, r$ and $\forall x \in V_i$, we have $gx = \lambda_i x$. Hence,

$$ghg^{-1}h^{-1}x = hg^{-1}h^{-1}gx = hg^{-1}h^{-1}\lambda_i x = \lambda_i hg^{-1}h^{-1}x$$

and $hg^{-1}h^{-1}V_i \subset V_i$. Since h and g are isomorphisms, $h^{-1}V_i = gh^{-1}V_i$. This shows that $h^{-1}V_i$ is g-invariant, and so

$$h^{-1}V_i = (h^{-1}V_i \cap V_1) \oplus (h^{-1}V_i \cap V_2) \oplus \cdots \oplus (h^{-1}V_i \cap V_r),$$

where $h^{-1}V_i \cap V_j = \{x \in h^{-1}V_i \mid gx = \lambda_j x\}$.

Let $w, v \in \mathbb{C}^n$ be such that $\|w\| = \|v\| = 1$ and $w \perp v$ in the Hermitian inner product. Then $\|w - v\| = \sqrt{2}$. Moreover, let $\|h^{-1} - I\| < \epsilon = \sqrt{2} - 1$, $i \neq j$ and suppose $0 \neq x \in (h^{-1}V_i \cap V_j)$.
We can assume $\|x\| = 1$. By definition, there is $y \in V_i$ such that $h^{-1}y = x$. But $x \in V_j$ and $\langle x, y \rangle = 0$. Since

$$\sqrt{2} = \|x - y\| = \|h^{-1}y - y\| \leq \|(h^{-1} - I)y\| < \sqrt{2} - 1,$$

we obtain a contradiction. Hence $h^{-1}V_i = V_i$ for all $i = 1, \dots, r$, and $gh = hg|_{V_i}$. In fact, the matrix of g is diagonal. Since any element of $\mathbb{C}^n$ is a sum of elements from V_i, it follows that g and h commute. Let

$$U = \{h \in O(n) \mid \|I - h^{-1}\| < \epsilon\}.$$

□

Lemma 6. *There exists a neighborhood of the identity $U \subset O(n)$ such that for any $g, h \in U$, the sequence*

$$[g, h],\ [g, [g, h]],\ [g, [g, [g, h]]],\ \dots$$

converges to the identity.

Proof. Let U be a neighborhood of the identity with radius $\epsilon < 1/4$. By definition, we have

$$\|[g, h] - I\| = \|gh - hg - I + I\| = \|gh - g - h + I - hg + h + g - I\| \leq$$

$$\|(g - I)(h - I) - (h - I)(g - I)\| \leq 2\|g - I\|\|h - I\| < \frac{\|h - I\|}{2}$$

for $g, h \in U$. Hence $[g, h] \in U$, and by induction,

$$\|[g, [g, \dots, [g, h] \dots]] - I\| \leq \frac{\|h - I\|}{2^n}.$$

Hence the result. □

Lemma 7. *There is an arbitrary small neighbourhood V of $I \in O(n)$ such that $\forall g \in O(n)$, $gVg^{-1} = V$.*

Proof. Let ϵ be a positive number and $V = B(I, \epsilon)$ be an open disk. By definition we have $\forall g \in O(n)$ and $\forall h \in V$ we have

$$\|ghg^{-1} - I\| = \|g(h - I)g^{-1}\| = \|h - I\| < \epsilon.$$

Since $\forall g \in O(n)$, $gVg^{-1} \subset V$ and $g^{-1}Vg \subset V$, we get $V = g(g^{-1}Vg)g^{-1} \subset gVg^{-1}$. $\square$

Definition 8. We shall call a neighbourhood U satisfying the three previous lemmas a stable neighbourhood of identity.

Lemma 8. *Let $G \subset O(n)$ be a connected subgroup, and let U be a neighborhood of the identity. Then the group $\langle G \cap U\rangle$ generated by $G \cap U$ is equal to G.*

Proof. G is a topological group, $V = G \cap U$ is a neighborhood of the identity, $< V >$ is open, hence it is closed, since G is connected and $< V >$ is non-empty, we have $< V >= G$ $\square$

Lemma 9. *Let Γ be a crystallographic group and $x \in \mathbb{R}^n$. Then the linear space generated by the set $\{\gamma(x)\}, \gamma \in \Gamma$ is equal to $\mathbb{R}^n$.*

Proof. Assume the lemma is false and that $x_0 \in \mathbb{R}^n$ exists such that $\{\gamma(x_0)\}$ lies in W, a proper linear subspace of $\mathbb{R}^n$. without loss of generalality, we may assume $O(n)$ leaves x_0 fixed as a new origin. In fact,

$$\Gamma(x_0) = \Gamma(I, x_0)(I, -x_0)(x_0) = \Gamma(I, x_0)(0).$$

Hence sets $(I, -x_0)\Gamma(I, x_0)(0)$ and $\Gamma(x_0)$ differ by translation $(I, -x_0)$ and define linear subspace of the same dimension. It follows that for $\gamma \in \Gamma, \gamma = (A, a)$ must have $a \in W$.

Since Γ is a group, $A(W) = W$ for all $A \in p_1(\Gamma)$. Let $W^\perp$ be the orthogonal complement of W. Let $x \in W^\perp$ be an element at a distance d from the origin. It is easy to see that for any $\gamma = (A, a) \in \Gamma$,

$$\langle \gamma(x), \gamma(x)\rangle = \langle x, x\rangle + \langle a, a\rangle.$$

Hence $\|x\| \leq \|\gamma(x)\|$. Summing up, points in $W^\perp$ at a distance d from origin stay at least at a distance d from 0. It follows that Γ cannot have a compact fundamental domain. $\square$

Lemma 10. *Let Γ be an abelian crystallographic group; then Γ contains only pure translations.*

Proof. Let $(B, b) \in \Gamma$, where $B \neq I$. Then we can always choose an origin and a coordinate system in $\mathbb{R}^n$ such that

$$B = \begin{pmatrix} I & 0 \\ 0 & B' \end{pmatrix},$$

where I is the $r \times r$ identity matrix, $B' - I$ is a nonsingular $s \times s$ matrix, $r + s = n$, and r can be equal to zero.
Moreover, we can assume that $b = (b', 0, \dots, 0)$, where $b' \in \mathbb{R}^r$.

Then, there exists an element $(C, (t_1, t_2)) \in \Gamma$, where $t_1 \in \mathbb{R}^r$, $t_2 \in \mathbb{R}^s$ and $t_2 \neq 0$. Then, since Γ is abelian and $BCb = CBb = Cb$, we compute:

$$\begin{aligned}(B, b)(C, (t_1, t_2)) &= (BC,\ b' + t_1,\ B'(t_2)) \\ &= (CB,\ Cb + t_1,\ t_2) \\ &= (C, (t_1, t_2))(B, b).\end{aligned}$$

Hence $B'(t_2) = t_2$, which contradicts the nonsingularity of $B' - I$. $\square$

Lemma 11. *Let Γ be a crystallographic group. Let $p_1 : E(n) \to O(n)$ be the projection onto the first factor. Then $p_1(\Gamma)_0$ is an abelian group.*

Proof. Let $U = B(I, \epsilon)^3$, such that $\epsilon < \frac{1}{4}$
Let $\gamma_1 = (A_1, a_1), \gamma_2 = (A_2, a_2) \in (p_1^{-1}(U) \cap \Gamma)$. By recurrence we define for $i \geq 2$

$$\gamma_{i+1} = [\gamma_1, \gamma_i].$$

We have

$$\gamma_{i+1} = ([A_1, A_i], (I - A_1 A_i A_1^{-1}) a_1 + A_1 (I - A_i A_1^{-1} A_i^{-1}) a_i).$$

Hence $A_{i+1} = [A_1, A_i]$ and

$$\|a_{i+1}\| \leq \|I - A_i\| \|a_1\| + \frac{1}{4} \|a_i\|.$$

From a previous lemma we have $\lim_{i \to \infty} A_i = I$. Hence $\lim_{i \to \infty} a_i = 0$. Since Γ is discrete, $\gamma_i = (I, 0)$ for sufficiently large i. However, we have $A_1 A_2 = A_2 A_1$. Hence any elements of the set $p_1(\Gamma)_0 \cap U$ commute, hence we prove commutativity of the group $p_1(\Gamma)_0$. $\square$

We finish the proof of the first Bieberbach Theorem.

Assume first that $\Gamma \cap (I \times \mathbb{R}^n)$ is trivial. Then p_1 is an isomorphism of Γ into $O(n)$. Since $O(n)$ is compact, the closure of $p_1(\Gamma)$ can have only a finite number of components. Hence, since $p_1(\Gamma)_0$ is abelian, Γ contains a subgroup Γ_1 of finite index which is abelian. But then Γ_1, being of finite index in Γ, is also a crystallographic group. Hence, by Lemma 2.6, Γ_1 consists of pure translations. Thus we see that $\Gamma \cap (I \times \mathbb{R}^n)$ is nonempty.

Let $W \subset \mathbb{R}^n$ be the subspace of $\mathbb{R}^n$ spanned by the pure translations of Γ, i.e., by $\Gamma \cap (I \times \mathbb{R}^n)$. Then, $p_1(\Gamma)$ leaves W invariant because $\Gamma \cap (I \times \mathbb{R}^n)$ is normal in Γ. Note further that $p_1(\Gamma)|_W$ is a finite group, for otherwise it would contain elements arbitrarily close to identity, which would, under inner automorphism with a basis of $\Gamma \cap (I \times \mathbb{R}^n)$, force Γ to be nondiscrete.

In fact, let $(A_i, a_i) \in \Gamma, i \in \mathbb{N}$ be an infinite sequence of elements such that $A_i \to I$. Let

$$(B_i, b_i) = (I, e_k)(A_i, a_i)(I, -e_k) = (A_i, (I - A_i)e_k + a_i),$$

where $e_k \in \Gamma \cap (I \times \mathbb{R}^n)$. Then a sequence $(B_i, b_i)(A_i^{-1}, -A_i^{-1}(a_i)), i \in \mathbb{N}$ defines a nondiscrete subset of Γ. Moreover, we see that Γ induces an action on $\mathbb{R}^n/W$ which is obviously cocompact. We claim that it is also properly discontinuous.

We have decomposition $\mathbb{R}^n = W \oplus W^\perp$, where $W^\perp \simeq \mathbb{R}^n/W$.
Let $pr_1 : \mathbb{R}^n \to W$, $pr_2 : \mathbb{R}^n \to W^\perp$ be projections. Let X be any discrete subset of $\mathbb{R}^n$. It can happen that sets $pr_1(X)$ and $pr_2(X)$ are not discrete subsets of W and $W^\perp$.

Since $p_1(\Gamma)|_W$ is finite, we can concentrate on elements $\gamma \in \Gamma$ such that $p_1(\gamma)$ acts as identity on W.
The orbit $\Gamma(0)$ is discrete in $\mathbb{R}^n$. By contradiction let us assume that $pr_2(\Gamma(0))$ is not discrete at $W^\perp$ and $y \in W^\perp$ is an accumulation point of $pr_2(\Gamma(0))$.
Let $pr_2(\gamma_i(0)) \to y$, where $\gamma_i \in \Gamma, i \in \mathbb{N}$. Using elements from $\Gamma \cap (I \times \mathbb{R}^n)$ we can define a sequence of elements of $\tilde{\gamma}_i \in \Gamma, i \in \mathbb{N}$ such that $\forall i \in \mathbb{N}, pr_1(\tilde{\gamma}_i(0)) \subset C \subset W$, where C is a compact set.

Here we use the fact that $\Gamma \cap \mathbb{R}^n$ is a cocompact subgroup of W. We can see that a set $\{\tilde{\gamma}_i(0)\}, i \in \mathbb{N}$ has an accumulation point at a discrete set $\Gamma(0)$. We get contradiction and our claim is proved.

Hence Γ is a crystallographic group on $\mathbb{R}^n/W$ with no pure translations. By the above, this implies the zero dimension of $\mathbb{R}^n/W$. □

3 Klein Geometry

Definition 9 (Klein Geometry). A Klein geometry is defined by a pair $(G, G/H)$, where G, the principal group, is a Lie group, and H is a closed subgroup (equivalently a Lie subgroup), such that the natural action of G on $X = G/H$ is transitive.
The homogeneous space X is called the space of the geometry, or by abuse of language, the Klein geometry. It follows from the definition and quotient topology that X is connected.

Since H is closed, X inherits a topological manifold structure.

Definition 10. Let $U \subset X$ be an open set. A morphism $f : U \to X$ is said to be locally-G if for every connected component C of U, there exists $g \in G$ such that $f_{|C} = g_{|C}$.

Classic examples of geometries include Euclidean geometry $(O_n(\mathbb{R}) \ltimes \mathbb{R}^n, \mathbb{R}^n)$, affine geometry $(GL_n(\mathbb{R}) \ltimes \mathbb{R}^n, \mathbb{R}^n)$.

Definition 11. Given a geometry (G, X) and a manifold M of the same dimension as X, a G/H-structure on M is a maximal atlas $\mathcal{U} = \{(U_i, \varphi_i)\}$ such that:

- $\{U_i\}$ is an open cover of M.
- The morphisms $\varphi_i : U_i \to X$ are open embeddings.
- The transition maps $\varphi_i \circ \varphi_j^{-1} : \varphi_j(U_i \cap U_j) \to \varphi_i(U_i \cap U_j)$ are locally-G.

A manifold equipped with such a structure is called a (G, X)-manifold.

Example 4. For any geometry (G, X), the space X is tautologically a (G, X)-manifold such that the identity morphism is a global chart. More generally, for any open set U of X, U is a (G, X)-manifold with the identity as a global chart.

Example 5. Consider the group of biholomorphisms $\mathbb{C}^\times \ltimes \mathbb{C}$. Let Λ be a lattice, for example $\mathbb{Z}[i]$. We define the complex torus $T_\mathbb{C} = \mathbb{C}/\Lambda$. Then $\mathbb{C}$ is a universal cover over $T_\mathbb{C}$, which gives the torus a $(\mathbb{C}^\times \ltimes \mathbb{C}, \mathbb{C})$-structure.

Let M and N be two (G, X)-manifolds and $f : M \to N$ a map. We say that f is a (G, X)-map if for any two charts of M and N:

$$\varphi_i : U_i \to X \quad , \quad \psi_j : V_j \to X$$

the restriction of $\psi_j \circ f \circ \varphi_i^{-1}$ to $\varphi_i(U_i \cap f^{-1}(V_j))$ is locally-G. In particular, we consider (G, X)-maps that are local diffeomorphisms. The set of (G, X)-automorphisms $M \to N$ is a group that we denote by $\mathrm{Aut}_{(G,X)}(M)$.

$$\varphi_\beta(U_\alpha \cap U_\beta) \xrightarrow{\varphi_\alpha \circ (\varphi_\beta)^{-1}} \varphi_\alpha(U_\alpha \cap U_\beta)$$

3.1 (G, X)-Automorphisms

If Ω is a non-empty connected open set, a (G, X)-automorphism $f : \Omega \to \Omega$ is the restriction of a unique element $g \in G$ preserving Ω:

$$\mathrm{Aut}(\Omega) \cong \mathrm{Stab}(\Omega) = \{g \in G : g \cdot \Omega = \Omega\}$$

We now assume $f : M \to \Omega$ is a local diffeomorphism. There exists a homomorphism:

$$f_* : \mathrm{Aut}_{(G,X)}(\Omega) \to \mathrm{Aut}_{(G,X)}(\Omega)$$

whose kernel is the set of maps $h : M \to M$ such that the following diagram commutes:

3.2 Developing map and holonomy

The following fact is essential in the study of (G, X)-structures.

Proposition 5 (Unique Extension Property). Let M and N be two (G, X)-manifolds and $f_1, f_2 : M \to N$ be two (G, X)-morphisms. If M is connected, then f_1 and f_2 are equal if and only if they coincide locally.

Proof. Let S be the set of all points of M that have an open neighborhood in which f_1 and f_2 coincide. We will show that S is both open and closed in M, which will imply the proposition. Let $x \in S$ and let U be an open neighborhood of x, such that $f_1|_U = f_2|_U$. We have $U \subset S$ which means that S is a neighborhood of each of its points, i.e., S is open. Let $x \notin S$. If $f_1(x) \neq f_2(x)$, we can find a neighborhood U of x such that $f_1(U) \cap f_2(U) = \varnothing$. Thus $U \cap S = \varnothing$ and S is closed. Suppose that $f_1(x) = f_2(x)$. Let (U, φ) be a local chart around x. By shrinking U if necessary, we can assume that $f_1(U)$ and $f_2(U)$ are contained in the domain W of a local chart (W, ψ) of N. We have $U \cap S = \varnothing$. Suppose by contradiction that there exists $x_0 \in U$ which has an open neighborhood V in which f_1 and f_2 coincide. We can always reduce V and assume that $V \subset U$. By construction, the charts $g_i = \psi \circ f_i \circ \varphi^{-1} : \varphi(V) \to \psi(W), i = 1, 2$ coincide, so they must extend to the same chart on X. Therefore, they are equal on the set $\varphi(U)$, which means that in particular f_1 and f_2 coincide on U. This contradicts the fact that $x \notin S$. Therefore, $U \cap S = \varnothing$ and S is closed. $\square$

Let M be a (G, X)-manifold. Let $p : \tilde{M} \to M$ be a universal cover with fundamental group $\pi = \pi_1(M)$. p induces a (G, X)-structure on $\tilde{M}$ on which π acts by (G, X)-automorphisms. The unique extension property implies:

Proposition 6. Let M be a simply connected (G, X)-manifold. Then there exists a (G, X)-application $f : M \to X$.

Proof. Choose a basepoint $x_0 \in M$ and a coordinate patch U_0 containing x_0. For $x \in M$, we define $f(x)$ as follows. Choose a path $\{x_t\}_{0 \leq t \leq 1}$ in M from x_0 to $x = x_1$. Cover the path by coordinate patches U_i (where $i = 0, \dots, n$) such that $x_t \in U_i$ for $t \in (a_i, b_i)$ where

$$a_0 < 0 < a_1 < b_0 < a_2 < b_1 < a_3 < b_2 < \cdots < a_{n-1} < b_{n-2} < a_n < b_{n-1} < 1 < b_n$$

Let $U_i \xrightarrow{\psi_i} X$ be an (G, X)-chart and let $g_i \in G$ be the unique transformation such that $g_i \circ \psi_i$ and ψ_{i-1} agree on the component of $U_i \cap U_{i-1}$ containing the curve $\{x_t\}_{a_i < t < b_{i-1}}$. Let

$$f(x) = g_1 g_2 \cdots g_{n-1} g_n \psi_n(x);$$

we show that f is indeed well-defined. The map f does not change if the cover is refined. Suppose that a new coordinate patch U' is "inserted between" U_{i-1} and U_i. Let

$\{x_t\}_{a'<t<b'}$ be the portion of the curve lying inside U':

$$a_{i-1} < a' < a_i < b_{i-1} < b' < b_i$$

Let $U' \xrightarrow{\psi'} X$ be the corresponding coordinate chart and let $h_{i-1}, h_i \in G$ be the unique transformations such that ψ_{i-1} agrees with $h_{i-1} \circ \psi'$ on the component of $U' \cap U_{i-1}$ containing $\{x_t\}_{a'<t<b_{i-1}}$ and ψ' agrees with $h_i \circ \psi_i$ on the component of $U' \cap U_i$ containing $\{x_t\}_{a_i<t<b'}$. By the unique extension property $h_{i-1}h_i = g_i$ and it follows that the corresponding developing map

$$f(x) = g_1 g_2 \cdots g_{i-1} h_{i-1} h_i g_{i+1} \cdots g_{n-1} g_n \psi_n(x)$$

$$= g_1 g_2 \cdots g_{i-1} g_i g_{i+1} \cdots g_{n-1} g_n \psi_n(x)$$

is unchanged. Thus the developing map as so defined is independent of the coordinate covering, since any two coordinate coverings have a common refinement.

Next we claim the developing map is independent of the choice of path. Since M is simply connected, any two paths from x_0 to x are homotopic. Every homotopy can be broken up into a succession of "small" homotopies, that is, homotopies such that there exists a partition

$$0 = c_0 < c_1 < \cdots < c_{m-1} < c_m = 1$$

such that during the course of the homotopy the segment $\{x_t\}_{c_i<t<c_{i+1}}$ lies in a coordinate patch. It follows that the expression defining $f(x)$ is unchanged during each of the small homotopies, and hence during the entire homotopy. Thus f is independent of the choice of path.

Since f is a composition of a coordinate chart with a transformation $X \to X$ coming from G, it follows that f is an (G, X)-map. The proof of Proposition 5.2.1 is complete. $\square$

Definition 12. We say that a (G, X)-manifold M is complete if the developing map is a covering.

Proposition 7 (The holonomy group characterizes the manifold). If G is a group that acts analytically by diffeomorphisms on a simply connected topological space X, any complete (G, X)-manifold can be constructed from its holonomy group Γ as the quotient space X/Γ.

Proof. Since X is simply connected and M is complete, by the uniqueness of the universal cover:

$$\tilde{M} \cong X$$

The development being a diffeomorphism, the action of the fundamental group on $\tilde{M}$ is transported via dev to X such that for all $x \in X, \gamma \in \pi_1(M)$, we have:

$$\gamma \cdot x = \mathrm{dev}(\gamma \cdot \mathrm{dev}^{-1}(x)) = \mathrm{hol}(\gamma) \cdot x$$

By the commutative diagram:

Thus the action of the holonomy group coincides with that of the fundamental group which is faithful. Thus the holonomy is injective, by the isomorphism theorem:

$$M \cong X/\pi_1(M) \cong X/\Gamma$$

□

Lemma 12 (Eckmann-Hilton Principle). *We assume that a set X is equipped with two unital magma structures $*$ and $\cdot$, and such that for $x, x', y, y' \in X$ we have:*

$$(x \cdot x') * (y \cdot y') = (x * y) \cdot (x' * y')$$

then the two laws coincide and X is a commutative monoid. In particular, the fundamental group of a topological group is abelian.

Proof. Let $x = y' = 1_*$ and $x' = y = 1$

$.Thisgives$1. $= 1_* = 1$. Now let $x' = y = 1$, this gives:

$$x * y' = x \cdot y'$$

Thus $* = \cdot$, we then deduce associativity and commutativity:

$$x = 1 \Rightarrow x' * (y * y') = y * (x' * y')$$
$$x = y' = 1 \to x' * y = y * x'$$

Thus X is equipped with a commutative monoid structure. We show that the fundamental group $\pi = \pi_1(G)$ of a topological group $(G, \cdot)$ is abelian. The law of G induces a second law on π (concatenation $+$), for γ, δ two loops in G:

$$\gamma * \delta : t \mapsto \gamma(t) \cdot \delta(t)$$

This law is well defined as the multiplication of G is continuous. So that:

$$[\gamma * \delta] = [t \mapsto \gamma(t) \cdot \delta(t)] = [t \mapsto \gamma(t)] * [t \mapsto \delta(t)] = [\gamma] * [\delta]$$

Let $a, b, c, d \in \pi$, we have:

$$\begin{aligned}(a + b) * (c + d) &= t \mapsto (a + b)(t) \cdot (c + d)(t) \\ &= (t \mapsto a(t) \cdot c(t)) + (t \mapsto b(t) \cdot d(t))\end{aligned}$$

By Eckmann-Hilton, π is abelian. □

We replace the space X with its universal cover when X is not simply connected. There exists a covering group $\tilde{G}$ acting on $\tilde{X}$ by homeomorphisms. We can then describe $\tilde{G}$ by the extension:

$$1 \to \pi_1(X) \to \tilde{G} \to G \to 1$$

We say that a group G admits a local section (local cross-section) with respect to a closed subgroup $H < G$ if we have the data:

- a neighborhood $U \subset G/H$ of the identity.
- a subset $S \subset G$ such that the canonical projection restricted to S is a homeomorphism onto U. We call the local section the map $\sigma : U \to G$ such that $\pi(\sigma(gH)) = gH$ and $\sigma(1H) = 1$.

This means that locally around the neutral class, we can continuously choose a unique representative of each class gH. We then study the structure of $\pi_1(X)$ when X is not simply connected:

- Let X be a manifold and $G \subset \text{Homeo}(X)$ transitive, let $x \in X$ and G_x its stabilizer (closed in G), suppose that G admits a local section with respect to G_x and that the morphism $\rho : g \mapsto gx$ is open. We construct a homeomorphism $\phi : \pi_1(X) \to \pi_0(G_x)$. And we will show that the kernel of ϕ is central. In particular, if G_x is arcwise connected then $\pi_1(X)$ is abelian.

Lemma 13. *If G is a Lie group, every closed subgroup (i.e., a Lie subgroup) has a local section. The map ρ is open.*

Definition 13 (Riemannian metric). Let M be a (topological) smooth manifold. A Riemannian metric g on M consists of a smooth function $q : TM \to \mathbb{R}$ whose restrictions $q_x : T_xM \to \mathbb{R}$ are all nondegenerate quadratic forms.

Definition 14 (Geodesic). Let M be a metric space with distance d and let $I \subset \mathbb{R}$ be an interval. A curve (continuous map) $\gamma : I \to M$ is called a geodesic if there exists a constant $v \geq 0$ such that for every $t \in I$ there is a neighborhood $J \subset I$ of t with the property

$$d\big(\gamma(t_1), \gamma(t_2)\big) = v\,|t_1 - t_2| \quad \text{for all } t_1, t_2 \in J.$$

If $v = 1$ the curve is said to be unit-speed.

Proposition 1 (compact stabilizers imply completeness)**.** *Let G be a lie group acting analytically and transitively on a manifold X, and such that the stabilizer G_x of x is compact for some x (hence all by transitivity). Then every closed (G, X)-manifold M is complete.*

We shall prove the following lemma:

Lemma 14 (existence of invariant metric)**.** *Let G act transitively on an analytic manifold X. Then X admits a G-invariant Riemannian metric if and only if, for some $x \in X$, the image of G_x in $GL(T_xX)$ has a compact closure.*

Proof. If G preseves a metric G_x maps to a subgroup of $O(T_xX)$, hence its closure is compact.
To prove the converse, fix x and assume that the image of G_x has a compact closure $H_x = \overline{\rho(G_x)}$ where ρ is the tangent representation. Let Q be any positive definite form on T_xX.

$$Q : T_xX \times T_xX \longrightarrow \mathbb{R}$$

H_x is compact, hence unimododular, so it admits a bi-invariant Haar measure μ, define a new inner product on T_xX by:

$$\langle v, w\rangle_x = \int_{H_x} Q(h \cdot v, h \cdot w) d\mu(h)$$

Right-invariance of μ shows immediately that for every $k \in H_x$,

$$\langle k \cdot v, k \cdot w\rangle_x = \int_{h \in H_x} Q\big(hk \cdot v, hk \cdot w\big) d\mu(h) = \langle v, w\rangle_x,$$

so $\langle \cdot, \cdot\rangle_x$ is H_x-invariant.

Now, since the action is transitive, for an arbitrary point $y \in X$ there exists $g \in G$ with $g \cdot x = y$, and transport the inner product at x to T_xX by

$$\langle u, w\rangle_y = \langle d(g)_y^{-1}(u),\, d(g)_y^{-1}(w)\rangle_x.$$

If $g' \in G$ is another element with $g' \cdot x = y$, then $g' = g\,h$ for some $h \in G_x$, and the G_x-invariance of $\langle \cdot, \cdot\rangle_x$ guarantees that $\langle \cdot, \cdot\rangle_y$ is well-defined. □

Remark. Using a local cross section we can prove that this G-invariant metric is analytic.

Proof of compact stabilizers imply completeness. transitively imply that the given condition at one point x is equivalent to the same condition everywhere. So fix $x \in X$ and let T_xX be the tangen space to X at x. There is an analytic homomorphism of G_x to $GL(T_xX)$ whose image is compact.
If M is any (G, X)-manifold, we can use charts to pull-back the G-invariant Riemannian metric from X to M (invariancy guarantees that such metric is well-defined), the resulting metric is a Riemannian metric on M invariant under any (G, X)- morphism. In a manifold endowed with such metric, we can find for any point y a ball $B_\epsilon(y)$ that is ball-like (homeomorphic image of the round ball) and convex. if M is closed we can choose ϵ uniformly by compacteness. Without loss of generalality, we may assume that all ϵ-balls in X are contractible and convex since G is a transitive group of isometries.
Then, for any $y \in \tilde{M}$, the ball $B_\epsilon(y)$ is mapped homeomorphically by dev, for if $\mathrm{dev}(y) = \mathrm{dev}(y')$ for $y \neq y'$ in the ball, the geodesic connecting y and y' maps to a self-intersecting geodesic in X contradicting the convexity of the ϵ-balls in X.
The map dev is an isometry between $B_\epsilon(y)$ and $B_\epsilon(\mathrm{dev}(y))$ by definition.
Take $x \in X$ and $y \in \mathrm{dev}^{-1}(B_{\epsilon/2})(x)$. The ball $B_\epsilon(y)$ maps isometrically, and thus

contains a copy of $\text{dev}^{-1}(B_{\epsilon/2})(x)$. The inverse image of $\text{dev}^{-1}(B_{\epsilon/2})(x)$ is then the disjoint union of these copies. This proves that dev is indeed a covering map and M is hence complete. $\square$

Example 6. Let Γ a finite subgroup of $O(4)$ acting freely on $\mathbb{S}^3$, an elliptic 3-manifold is the orbit space $M = \mathbb{S}^3/\Gamma$, that is a $(O(4), \mathbb{S}^3)$-manifold. Such manifold is a closed manifold by definition. The proposition says that the universal cover of M is $\mathbb{S}^3$.

We shall give an equivalency between metric completeness and completeness of (G, X)-manifolds, which justifies the use of the word:

Proposition 2 (completeness equivalency)**.** *Let G be a group acting transitively and analytically on X with compact stabilizers G_x. Fix a G-invariant metric on X and let M be a (G, X)-manifold. The following conditions are equivalent:*

1. *M is a complete (G, X)-manifold.*
2. *For some $\epsilon > 0$, every closed ϵ-ball in M is compact.*
3. *For every $a > 0$, every closed a-ball in M is compact.*
4. *There is some family of compact subsets S_t of M, for $t \geq 0$, such that $\cup_{t\geq 0} S_t = M$ and S_{t+a} contains a neighborhood of radius a about S_t.*
5. *M is a complete metric space.*

Proof. $(1) \implies (2)$. if $p : Y \longrightarrow Z$ is a covering map between two manifolds endowed mith a Riemannian metric and p preseves this metric, we have $\overline{B}_\epsilon(p(y)) = p(\overline{B}_\epsilon(y))$ for any $y \in Y$ and any $\epsilon > 0$, because distances are defined in terms of path lenghts and paths in Z can be lifted to paths in Y. So the compacteness of balls in Y implies the same in Z and conversely. Fixing a point $x \in X$ and compact ϵ-ball around x by the local compacity of X, the transitive action of G implies the same for all points in X. Idem for $\tilde{M}$ and M.

$(2) \implies (3)$. We show this by induction, suppose that (3) is true for some $a > \epsilon$. Then $\overline{B}_a(x)$ can be covered with finitely mainy $\epsilon/2$-balls, and therefore $\overline{B}_{a+\epsilon/2}(x)$ can be covered with finitely mainy ϵ-balls hence compact.

$(3) \implies (4)$. Let $S_t = \overline{B}_t(x)$ where x is fixed.

$(4) \implies (5)$. Any Cauchy sequence must be conatined in some S_t for some t, hence it converges.

$(5) \implies (1)$. Suppose M metrically complete. We will prove that any path α_t in X can be lifted to $\tilde{M}$, since local homeomorphisms with the path lifting property are covering maps.
The universal cover of a complete metric space is complete M since the projection of

Cauchy sequence converges to some point $x \in M$. Since x has a compact neighborhood which is evenly covered and these are separated in the metric of $\tilde{M}$, the Cauchy sequence also converges in $\tilde{M}$.
Consider now any path α_t in X. If it has a lifting $\tilde{\alpha}_t$ for t in $[0, s]$, then it has a lifting for $[0, s + \epsilon)$ for some $\epsilon > 0$ by the local homeomorphicity of dev. If it has a lifting for t in a half-open interval $[0, s)$, the lifting extends by metric completeness. Thus, M is complete. □

3.3 Riemannian Geometry

Definition 15. Let M be a smooth manifold. A Euclidean geometry on M is a Klein geometry on M modeled on $(O_n(\mathbb{R}) \ltimes \mathbb{R}^n, \mathbb{R}^n)$.

Definition 16 (Riemannian manifold). Let M be a smooth manifold, g a Riemannian metric on M. The pair (M, g) is then a Riemannian manifold.

Given two Riemannian manifold (M, g_M) and (N, g_N), we define the product:

$$(M \times N, g_M \oplus g_N)$$

which is a Riemannian manifold.

Let $f : M \longrightarrow L$ a smooth immersion, where M is a manifold and (L, g_L) a Riemannian manifold, the pull-back metric

$$(f^* g_L)_p : (x, y) \mapsto (g_L)_{f(p)}(df_p(x), df_p(y))$$

is a Riemannian metric, and M a Riemannian manifold.

Remark. This makes any smooth manifold, that is embedded in some R^n by Whitney theorem, a Riemannian manifold.

Definition 17 (Flat manifolds). Let M be a Riemannian manifold, and assume it to be a geometrically complete. We say that M is (locally) flat if it is locally isometric to a Euclidean space.

Definition 18. Let (M, g) be a connected Riemannian manifold which is complete as a metric space. We say that (M, g) is *flat* if and only if the following two conditions hold:

1. The universal cover $\widetilde{M}$, equipped with the pull-back metric $\tilde{g}$, is isometric to the Euclidean space
$$(\mathbb{R}^n, \langle \cdot, \cdot \rangle_{\text{eucl}}).$$

2. The deck-transformation group $\Gamma = \pi_1(M)$ can be identified with a discrete subgroup of the Euclidean affine group
$$\mathrm{E}(n) = O(n) \ltimes \mathbb{R}^n,$$
acting freely and properly discontinuously on $\mathbb{R}^n$.

In this case we have the global isometry

$$(M, g) \;\simeq\; (\mathbb{R}^n, \langle\cdot,\cdot\rangle_{\mathrm{eucl}})\big/\Gamma,$$