# Geometric structures on manifolds and Bieberbach theorem

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**Abstract**

This report presents the internship I completed at the mathematics laboratory of the ENS de Lyon (UMPA) during my first year of mathematics at the École normale supérieure de Lyon.

It studies geometric structures on manifolds using algebraic and topological tools. We prove Bieberbach theorems and give a full classification of bicrystallographic groups.

# Contents

## Introduction

Manifolds, whether viewed through the lens of topology or of smooth geometry, are in essence flexible objects—determined "up to deformation." To probe their deeper structure, one is naturally led to geometrize them: that is, to endow them with geometric structures whose local behavior mirrors global topological invariants. Among the many realizations of this vague program, the theory of Klein geometries [11] stands out by prescribing that a manifold be locally modeled on a homogeneous space $X = G/H$, with transition maps given by elements of the Lie group $G$. Such geometric structures not only unify classical geometries (Euclidean, affine, hyperbolic, lorentzian,..) but also give rise to the developing-map and holonomy-representation machinery that translate questions about manifolds into questions about group actions and linear representations.

In particular, we will study crystallographic groups, which are discrete subgroups of the Euclidean group. They derive their name from physics in three dimensions, where they appear as the symmetry groups of crystals (see [7]). We shall prove Bieberbach theorems that characterize the compact, flat Riemannian manifolds whose fundamental groups are torsion free crystallographic groups, called Bieberbach groups, and show that these fit into a short exact sequence

$$0 \longrightarrow \mathbb{Z}^n \longrightarrow \pi_1(M) \longrightarrow F \longrightarrow 0$$

where the translation lattice $\mathbb{Z}^n$ is the unique maximal abelian, torsion-free subgroup of finite index, and the quotient $F$ is a finite group of orthogonal transformations.

It is also notable that any compact flat (Riemannian) manifold is obtained as the orbit space of a Bieberbach group, so there is a one-to-one correspondence between the geometric world of flat manifolds and the algebraic world of Bieberbach groups.

In the complete version of this document, we also give a complete classification of 2-crystallographic groups, called wallpaper groups, using two different approaches (see [2] and [18]).

# 1 Preliminaries

In this section we shall give necessary definitions and results in group theory and topology to prove the algebraic version of 1st Bieberbach theorem.

*Definition* 1.1 (Semidirect product [19])
Let $H$ and $K$ denote groups with group multiplication $\circ$ and $\star$ respectively, and a homomorphism $f : H \longrightarrow \mathrm{Aut}(K)$. The semi-direct product $H \ltimes_f K$ of the groups $H$ and $K$ with respect to $f$ is the set of all pairs $(h, k)$, $h \in H, k \in K$, with the following multiplication

$$(h_1, k_1)\,(h_2, k_2) = (h_1 \circ h_2, k_1 \star f(h_1)(k_2))$$

In this document, we will consider $H$ as a subgroup of $\mathrm{Aut}(K)$ and $f$ will be the identity.

*Example* 1.2
We have the following examples of semidirect product:

- The Euclidean group $E(n) = O(n) \ltimes \mathbb{R}^n$
- The affine group $A(n) = GL(n, \mathbb{R}) \ltimes \mathbb{R}^n$
- The group of $\mathbb{C}$-biholomorphisms $\mathbb{C}^* \ltimes \mathbb{C}$

*Remark*
One may notice that we have the following sequence of groups:

$$E(n) \subset A(n) \subset GL(n+1, \mathbb{R})$$

From the definition $E(n) \subset A(n)$. And for $(A, a) \in A(n)$. We have the matrix $\begin{pmatrix} A & a \\ 0 & 1 \end{pmatrix}$, which defines an inclusion $A(n) \subset GL(n+1, \mathbb{R})$.

*Definition* 1.3
Let $\Gamma$ be a subgroup of the group $E(n)$. Then $\Gamma$ is discrete if it is a discrete subset of the Euclidean space $\mathbb{R}^{(n+1)^2}$. We say that $\Gamma$ acts properly discontinuously on $\mathbb{R}^n$ if for any $x \in \mathbb{R}^n$ there is an open neighbourhood $U_x$ such that the set

$$\{\gamma \in \Gamma \mid \gamma U_x \cap U_x \neq \emptyset\}$$

is finite. Moreover, $\Gamma$ acts freely, if for any $x \in \mathbb{R}^n$ we have

$$\{\gamma \in \Gamma \mid \gamma x = x\} = \{(I, 0)\}$$

**Lemma 1.4**
*Let $G$ be a topological group and suppose it is $T_1$. Then any discrete subgroup of $G$ is closed.*

*Proof.* If $G$ is $T_1$, then it is Hausdorff. Let $H$ be a discrete subgroup of $G$, then there exists a neighbourhood $U$ of the identity such that $U \cap H = \{1\}$.
Let $V$ be a neighbourhood of the identity such that $V^{-1}V \subset U$, let $g \in G \backslash H$. If $gV \cap H$ is empty then we are done, suppose that it is not empty. Let $h, k \in gV \cap H$ that is $k^{-1}h \in (gV)^{-1}gV = V^{-1}V \subset U$, hence $k^{-1}h = 1$, therefore $gV \cap H = \{h\}$. As $g \notin gV \cap H$ and $G$ is Hausdorff, there exists a neighbourhood $W$ such that $h \notin gW$ □

**Corollary 1.5**
*Any discrete subgroup of the group $E(n)$ is closed in $E(n)$.*

*Proof.* $E(n)$ is a metric group. The result follows. □

**Lemma 1.6** (see [17])
*If $\Gamma$ is a discrete subgroup of the group $E(n)$ and $V_0 = D(0, r) \subset \mathbb{R}^n$ is an open disk, then*

$$\{\gamma \in \Gamma \mid \gamma V_0 \cap V_0 \neq \emptyset\} \subset \Gamma \cap (O(n) \times V_0')$$

*where $V_0' = D(0, 2r)$ is an open disk.*

*Proof.* Let $\gamma = (A, a) \in \Gamma$ and $\gamma V_0 \cap V_0 \neq \emptyset$. Then there exist $x, x' \in V_0$, such that $\gamma x = Ax + a = x'$.
Hence, from the triangle inequality $\|a\| = \|x' - Ax\| \leq \|x'\| + \|Ax\| < 2r$ and $\gamma \in O(n) \times V_0'$. □

**Proposition 1.7** (see [17])
*Let $\Gamma$ be a subgroup of the group $E(n)$. The following conditions are equivalent:*

1. *$\Gamma$ acts properly discontinuously on $\mathbb{R}^n$;*
2. *$\forall x \in \mathbb{R}^n$, $\Gamma x$ is a discrete subset of $\mathbb{R}^n$;*
3. *$\Gamma$ is a discrete subgroup of $E(n)$.*

*Proof.* Let $\Gamma$ act properly discontinuously on $\mathbb{R}^n$. We claim that $\Gamma$ is discrete. Let elements $\{\gamma_n\} \subset \Gamma$ converge to the identity. By assumption there is a neighbourhood $U_0$ of $0$ such that the set $\{\gamma_i \mid U_0 \cap \gamma_i U_0 \neq \emptyset\}$ is finite. Hence $\gamma_i = (I, 0)$ for large $i$ and the sequence $\{\gamma_n\}$ is eventually constant. In general case if $\gamma_n \to \gamma$ then $\gamma_n \gamma^{-1} \to (I, 0)$ and from previous consideration, the implication (i) $\to$ (iii) is proved.
We shall use the previous lemma for the proof of the reverse implication. Let $x \in \mathbb{R}^n$ be any point and $V_x$ be a disk of radius $r$ centered at $x$. By definition of properly discontinuous action and the lemma we have

$$\{\gamma \in \Gamma \mid \gamma V_x \cap V_x \neq \emptyset\} = \{\gamma \in \Gamma \mid t_{-x}\gamma t_x V_0 \cap V_0 \neq \emptyset\} \subset t_{-x}\Gamma t_x \cap (O(n) \times V_0')$$

Since $\Gamma$ is discrete and hence also closed, the above set is finite and the implication (iii) $\to$ (i) is proved.

Let us assume the condition (i) (or equivalently (iii)). We have to prove that for any $x \in \mathbb{R}^n$ the set $\Gamma x$ is discrete. Suppose it is not. Then there is $y \in \mathbb{R}^n$ and a sequence $\{\gamma_i x = A_i x + a_i\}$, which is not eventually constant and converges to $y$. Since the group $O(n)$ is compact, the sequence $\{A_i\}$ converges to some $A \in O(n)$. We claim that the sequence $\{a_i\}$ converges to $-Ax + y$. In fact, the value

$$\|a_i + Ax - y\| \leq \|a_i + A_i x - y\| + \|Ax - A_i x\|$$

can be arbitrarily small for large $i$. Summing up, we showed that the sequence $\{\gamma_i\}$ converges to $\gamma = (A, -Ax + y)$ in $E(n)$. Hence, $\left\{\gamma_i \gamma_{i+1}^{-1}\right\}$ converges to the identity. Since $\Gamma$ is discrete, a sequence $\{\gamma_i x\}$ is eventually constant. This contradicts our assumptions and proves the implication (i) $\rightarrow$ (ii).
Finally, we prove that (i) follows from (ii). Let $\{\gamma_n\}$ be a convergent sequence in $\Gamma$. Then, for any $x \in \mathbb{R}^n$, $\{\gamma_n x\}$ is a convergent sequence in $\Gamma x$. By definition it is eventually constant. Hence a sequence $\{\gamma_n\}$ is also eventually constant. □

*Definition* 1.8 (see [17] and [19])
A group is said to be torsion-free if the only element of finite order is the identity.

**Proposition 1.9** (see [17])
*A discrete subgroup of $E(n)$ acts freely on $\mathbb{R}^n$ if and only if it is torsion-free.*

*Proof.* Assume that a group $\Gamma$ has an element $\gamma$ of order $k$. For any $x \in \mathbb{R}^n$ the element $x + \gamma x + \gamma^2 x + \cdots + \gamma^{k-1} x$ is invariant under the action of $\gamma$. Hence the action of $\Gamma$ is not free. The reverse implication follows from the equality of the sets

$$\{\gamma \in \Gamma \mid \gamma a = a\} = \Gamma \cap t_a (O(n) \times 0) t_{-a}$$

where $a \in \mathbb{R}^n$. In fact, since the orthogonal group $O(n)$ is compact and $\Gamma$ is discrete, the above set is always finite. □

*Definition* 1.10 (Lie group)
Let $G$ be a differential manifold and simultaneously a group, such that the group operation and inversion are smooth. Then $G$ is called a Lie group. The tangent space at the identity $1 \in G$ is called the Lie algebra and denoted $\mathfrak{g} = T_1 G$.
Let $\Psi : G \longrightarrow \mathrm{Aut}(G)$ the conjugation map $\Psi(g) : x \mapsto g x g^{-1}$, the adjoint representation is the map:

$$\begin{aligned} \mathrm{Ad} : & G \longrightarrow \mathrm{Aut}(\mathfrak{g}) \\ & g \longmapsto \mathrm{Ad}_{\mathrm{g}} = (d\Psi(g))_1 \end{aligned}$$

*Example* 1.11
The sphere $S^1 \subset \mathbb{C}$, with the multiplication of complex numbers, is a Lie group. The Cartesian product of $S^1$, i.e. $n$-dimensional torus $T^n = \left(S^1\right)^n$, is a Lie group.
The matrix groups $U(n), O(n), SL(n, \mathbb{R}), SL(n, \mathbb{C})$ are Lie groups.
The group $(\mathbb{Z}/2\mathbb{Z})^{\mathbb{N}}$ is not a Lie group.

*Definition* 1.12
Let $\Gamma$ be a subgroup of $E(n)$. The orbit space of the action of $\Gamma$ on $\mathbb{R}^n$ is defined to be the set of $\Gamma$-orbits $\mathbb{R}^n/\Gamma = \{\Gamma x \mid x \in \mathbb{R}^n\}$ topologized with the quotient topology from $\mathbb{R}^n$.

The next lemma follows from the quotient topology.

**Lemma 1.13**
*If $\Gamma$ is a subgroup of $E(n)$, then the natural projection map $p : E(n) \to E(n)/\Gamma$ and the projection on the orbit space $\pi : \mathbb{R}^n \to \mathbb{R}^n/\Gamma$ are open and closed.*

**Proposition 1.14** (see [17])
*Let $\Gamma$ be a subgroup of $E(n)$. Then the orbit space $\mathbb{R}^n/\Gamma$ is compact if and only if the space of cosets $E(n)/\Gamma$ is compact.*

*Proof.* By definition $E(n)/O(n) = \mathbb{R}^n$.
A group $\Gamma$ acts on the space $E(n)/O(n)$ by $gO(n) \mapsto (\gamma g)O(n)$, where $\gamma \in \Gamma, g \in E(n)$. The above action agrees with a standard action of $\Gamma$ on $\mathbb{R}^n$.
Next, let us note that the map $E(n)/\Gamma \to (E(n)/O(n))/\Gamma = \mathbb{R}^n/\Gamma$, given by

$$g^{-1}\Gamma \to \Gamma(gO(n))$$

is a continuous open map with compact fibers. Hence it follows that $E(n)/\Gamma$ is compact if and only if $\mathbb{R}^n/\Gamma$ is compact. $\square$

**Lemma 1.15** (see [17])
*A space $E(n)/\Gamma$ is compact if and only if there exists a compact subset $D \subset E(n)$, such that $E(n) = D\Gamma$.*

*Proof.* Since $E(n)$ is a subset of $\mathbb{R}^{(n+1)^2}$, there exists a family of open sets $U_k \cap E(n)$, such that the family of sets $p\,(U_k \cap E(n))$ covers the compact space $E(n)/\Gamma$.
Here $U_k$ is an open disk centered at origin and of radius $k$. Hence there exists $k_0$, such that $p\,(U_{k_0} \cap E(n)) = E(n)/\Gamma$.
Let $D$ be a closure of the set $(U_{k_0} \cap E(n))$, i.e. $D = \overline{(U_{k_0} \cap E(n))}$. Finally, we have

$$E(n) = D\Gamma$$

The proof of the reverse implication follows from the equality $D/\Gamma = E(n)/\Gamma$. $\square$

*Definition* 1.16
A subgroup $\Gamma \subset E(n)$ is cocompact, if the space $E(n)/\Gamma$ is compact.

*Definition* 1.17
Let $X$ be a metric space and $G$ a subgroup of a group of its isometries. An open, connected subset $F \subset X$ is a fundamental domain if

$$X = \bigcup_{g \in G} g\bar{F}$$

and $gF \cap g'F = \emptyset$, for $g \neq g' \in G$.

Roughly speaking a fundamental domain for a group $\Gamma$ of isometries in a metric space $X$ is a subset of $X$ which contains exactly one point from each of these orbits.

## 2 Bieberbach Theorem

The proof that we will give to the theorem 2.3 appears in [17] and it was given by Auslander (see [3]). We will also give a proof of P. Buser (see [4] and [17]) that was inspired by Gromov's almost flat manifolds.

*Definition* 2.1 (see [17] and [19] and [3])
A crystallographic group of dimension $n$ is a cocompact and discrete subgroup of $E(n)$. A Bieberbach group is a torsion free crystallographic group.

The above definition was a theorem given by Auslander to the definition of crystallographic groups in [3, Theorem 1, p. 1230-1231].

*Example* 2.2 (Klein bottle)
Let

$$\left(B, \begin{pmatrix} 0 \\ 1/2 \end{pmatrix}\right), \left(I, \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) \in E(2) \text{ where } B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Then the group $\Gamma \subset E(2)$ generated by the above elements is a crystallographic group of dimension two and the orbit space $\mathbb{R}^2/\Gamma$ is the Klein bottle.

The first part of the eighteenth Hilbert problem was about the description of discrete and cocompact groups of isometries of $\mathbb{R}^n$. The German mathematician L. Bieberbach answers this problem.

**Theorem 2.3** (1st Bieberbach theorem, see [17] and [19])
*If $\Gamma \subset E(n)$ is a crystallographic group then the set of translations $\Gamma \cap (I \times \mathbb{R}^n)$ is a torsion free and finitely generated abelian group of rank n, and is a maximal abelian and normal subgroup of finite index.*

We shall prove these lemmas first:

**Lemma 2.4** ([17])
*There exists a neighborhood of the identity $U \subset O(n)$ such that for any $h \in U$, if $g \in O(n)$ commutes with $[g, h] = ghg^{-1}h^{-1}$, then $g$ commutes with $h$.*

*Proof.* Let $\lambda_1, \lambda_2, \dots, \lambda_r$ be the eigenvalues of an orthogonal matrix $g : \mathbb{C}^n \to \mathbb{C}^n$, and let $\mathbb{C}^n = V_1 \oplus V_2 \oplus \cdots \oplus V_r$ be its invariant subspaces. Since $g[g, h] = [g, h]g$, we have

$$ghg^{-1}h^{-1} = hg^{-1}h^{-1}g.$$

Moreover, for $i = 1, \dots, r$ and $\forall x \in V_i$, we have $gx = \lambda_i x$. Hence,

$$ghg^{-1}h^{-1}x = hg^{-1}h^{-1}gx = hg^{-1}h^{-1}\lambda_i x = \lambda_i hg^{-1}h^{-1}x$$

and $hg^{-1}h^{-1}V_i \subset V_i$. Since $h$ and $g$ are isomorphisms, $h^{-1}V_i = gh^{-1}V_i$. This shows that $h^{-1}V_i$ is $g$-invariant, and so

$$h^{-1}V_i = (h^{-1}V_i \cap V_1) \oplus (h^{-1}V_i \cap V_2) \oplus \cdots \oplus (h^{-1}V_i \cap V_r),$$

where $h^{-1}V_i \cap V_j = \{x \in h^{-1}V_i \mid gx = \lambda_j x\}$.

Let $w, v \in \mathbb{C}^n$ be such that $\|w\| = \|v\| = 1$ and $w \perp v$ in the Hermitian inner product. Then $\|w - v\| = \sqrt{2}$. Moreover, let $\|h^{-1} - I\| < \epsilon = \sqrt{2} - 1$, $i \neq j$ and suppose $0 \neq x \in (h^{-1}V_i \cap V_j)$.
We can assume $\|x\| = 1$. By definition, there is $y \in V_i$ such that $h^{-1}y = x$. But $x \in V_j$ and $\langle x, y\rangle = 0$. Since

$$\sqrt{2} = \|x - y\| = \|h^{-1}y - y\| \leq \|(h^{-1} - I)y\| < \sqrt{2} - 1,$$

we obtain a contradiction. Hence $h^{-1}V_i = V_i$ for all $i = 1, \dots, r$, and $gh = hg|_{V_i}$. In fact, the matrix of $g$ is diagonal. Since any element of $\mathbb{C}^n$ is a sum of elements from $V_i$, it follows that $g$ and $h$ commute. Let

$$U = \{h \in O(n) \mid \|I - h^{-1}\| < \epsilon\}.$$

$\square$

**Lemma 2.5** (see [3])
*There exists a neighborhood of the identity $U \subset O(n)$ such that for any $g, h \in U$, the sequence*

$$[g, h],\ [g, [g, h]],\ [g, [g, [g, h]]],\ \dots$$

*converges to the identity.*

*Proof.* Let $U$ be a neighborhood of the identity with radius $\epsilon < 1/4$. By definition, we have

$$\|[g, h] - I\| = \|gh - hg - I + I\| = \|gh - g - h + I - hg + h + g - I\| \leq$$

$$\|(g - I)(h - I) - (h - I)(g - I)\| \leq 2\|g - I\|\|h - I\| < \frac{\|h - I\|}{2}$$

for $g, h \in U$. Hence $[g, h] \in U$, and by induction,

$$\|[g, [g, \dots, [g, h] \dots]] - I\| \leq \frac{\|h - I\|}{2^n}.$$

Hence the result. $\square$

*Remark*
There is a more general result of this lemma, the Zassenhaus neighborhood theorem that we shall prove later.

**Lemma 2.6** (see [17] and [3])
*There is an arbitrary small neighbourhood $V$ of $I \in O(n)$ such that $\forall g \in O(n)$, $gVg^{-1} = V$.*

*Proof.* Let $\epsilon$ be a positive number and $V = B(I, \epsilon)$ be an open disk. By definition we have $\forall g \in O(n)$ and $\forall h \in V$ we have

$$\|ghg^{-1} - I\| = \|g(h-I)g^{-1}\| = \|h - I\| < \epsilon.$$

Since $\forall g \in O(n)$, $gVg^{-1} \subset V$ and $g^{-1}Vg \subset V$, we get $V = g(g^{-1}Vg)g^{-1} \subset gVg^{-1}$. □

*Definition 2.7*
We call a neighbourhood $U$ satisfying the three previous lemmas a stable neighbourhood of identity.

**Lemma 2.8**
*Let $G \subset O(n)$ be a connected subgroup, and let $U$ be a neighborhood of the identity. Then the group $\langle G \cap U \rangle$ generated by $G \cap U$ is equal to $G$.*

*Proof.* $G$ is a topological group, $V = G \cap U$ is a neighborhood of the identity, $< V >$ is open, hence it is closed, since $G$ is connected and $< V >$ is non-empty, we have $< V >= G$ □

A different proof Lemma 2.8 is given in [17].

**Lemma 2.9** ([17] and [3])
*Let $\Gamma$ be a crystallographic group and $x \in \mathbb{R}^n$. Then the linear space generated by the set $\{\gamma(x)\}, \gamma \in \Gamma$ is equal to $\mathbb{R}^n$.*

*Proof.* Assume the lemma is false and that $x_0 \in \mathbb{R}^n$ exists such that $\{\gamma(x_0)\}$ lies in $W$, a proper linear subspace of $\mathbb{R}^n$. without loss of generalality, we may assume $O(n)$ leaves $x_0$ fixed as a new origin. In fact,

$$\Gamma(x_0) = \Gamma(I, x_0)(I, -x_0)(x_0) = \Gamma(I, x_0)(0).$$

Hence sets $(I, -x_0)\Gamma(I, x_0)(0)$ and $\Gamma(x_0)$ differ by translation $(I, -x_0)$ and define linear subspace of the same dimension. It follows that for $\gamma \in \Gamma, \gamma = (A, a)$ must have $a \in W$.

Since $\Gamma$ is a group, $A(W) = W$ for all $A \in p_1(\Gamma)$. Let $W^\perp$ be the orthogonal complement of $W$. Let $x \in W^\perp$ be an element at a distance $d$ from the origin. It is easy to see that for any $\gamma = (A, a) \in \Gamma$,

$$\langle \gamma(x), \gamma(x) \rangle = \langle x, x \rangle + \langle a, a \rangle.$$

Hence $\|x\| \leq \|\gamma(x)\|$. Summing up, points in $W^\perp$ at a distance $d$ from origin stay at least at a distance $d$ from 0. It follows that $\Gamma$ cannot have a compact fundamental domain. □

**Lemma 2.10** (see [17] and [3])
*Let $\Gamma$ be an abelian crystallographic group; then $\Gamma$ contains only pure translations.*

*Proof.* Let $(B, b) \in \Gamma$, where $B \neq I$. Then we can always choose an origin and a coordinate system in $\mathbb{R}^n$ such that

$$B = \begin{pmatrix} I & 0 \\ 0 & B' \end{pmatrix},$$

where $I$ is the $r \times r$ identity matrix, $B' - I$ is a nonsingular $s \times s$ matrix, $r + s = n$, and $r$ can be equal to zero.
Moreover, we can assume that $b = (b', 0, \ldots, 0)$, where $b' \in \mathbb{R}^r$.

Then, there exists an element $(C, (t_1, t_2)) \in \Gamma$, where $t_1 \in \mathbb{R}^r$, $t_2 \in \mathbb{R}^s$ and $t_2 \neq 0$. Then, since $\Gamma$ is abelian and $BCb = CBb = Cb$, we compute:

$$\begin{aligned} (B, b)(C, (t_1, t_2)) &= (BC,\ b' + t_1,\ B'(t_2)) \\ &= (CB,\ Cb + t_1,\ t_2) \\ &= (C, (t_1, t_2))(B, b). \end{aligned}$$

Hence $B'(t_2) = t_2$, which contradicts the nonsingularity of $B' - I$. □

**Lemma 2.11** (see [3] and [17])
*, Let $\Gamma$ be a crystallographic group. Let $p_1 : E(n) \to O(n)$ be the projection onto the first factor. Then $p_1(\Gamma)_0$ is an abelian group.*

*Proof.* Let $U = B(I, \epsilon)^3$, such that $\epsilon < \frac{1}{4}$
Let $\gamma_1 = (A_1, a_1), \gamma_2 = (A_2, a_2) \in (p_1^{-1}(U) \cap \Gamma)$. By recurrence we define for $i \geq 2$

$$\gamma_{i+1} = [\gamma_1, \gamma_i].$$

We have

$$\gamma_{i+1} = ([A_1, A_i], (I - A_1 A_i A_1^{-1})a_1 + A_1(I - A_i A_1^{-1} A_i^{-1})a_i).$$

Hence $A_{i+1} = [A_1, A_i]$ and

$$\|a_{i+1}\| \leq \|I - A_i\| \|a_1\| + \frac{1}{4}\|a_i\|.$$

From a previous lemma we have $\lim_{i\to\infty} A_i = I$. Hence $\lim_{i\to\infty} a_i = 0$. Since $\Gamma$ is discrete, $\gamma_i = (I, 0)$ for sufficiently large $i$. However, we have $A_1 A_2 = A_2 A_1$. Hence any elements of the set $p_1(\Gamma)_0 \cap U$ commute, hence we prove commutativity of the group $p_1(\Gamma)_0$. □

We finish the proof of the first Bieberbach Theorem (see [17]).

*proof of first Bieberbach theorem.* Assume first that $\Gamma \cap (I \times \mathbb{R}^n)$ is trivial. Then $p_1$ is an isomorphism of $\Gamma$ into $O(n)$. Since $O(n)$ is compact, the closure of $p_1(\Gamma)$ can have only a finite number of components. Since $p_1(\Gamma)_0$ is abelian by Lemma 2.11,$\Gamma$ contains a subgroup $\Gamma_1$ of finite index which is abelian. But then $\Gamma_1$, being of finite index in $\Gamma$, is also a crystallographic group. Hence, $\Gamma_1$ consists of pure translations. Thus we see that $\Gamma \cap (I \times \mathbb{R}^n)$ is nonempty.

Let $W \subset \mathbb{R}^n$ be the subspace of $\mathbb{R}^n$ spanned by the pure translations of $\Gamma$, i.e., by $\Gamma \cap (I \times \mathbb{R}^n)$. Then, $p_1(\Gamma)$ leaves $W$ invariant because $\Gamma \cap (I \times \mathbb{R}^n)$ is normal in $\Gamma$. Note further that $p_1(\Gamma)|_W$ is a finite group, for otherwise it would contain elements arbitrarily close to identity, which would, under inner automorphism with a basis of $\Gamma \cap (I \times \mathbb{R}^n)$, force $\Gamma$ to be nondiscrete.

In fact, let $(A_i, a_i) \in \Gamma, i \in \mathbb{N}$ be an infinite sequence of elements such that $A_i \to I$. Let

$$(B_i, b_i) = (I, e_k)(A_i, a_i)(I, -e_k) = (A_i, (I - A_i)e_k + a_i),$$

where $e_k \in \Gamma \cap (I \times \mathbb{R}^n)$. Then a sequence $(B_i, b_i)(A_i^{-1}, -A_i^{-1}(a_i)), i \in \mathbb{N}$ defines a nondiscrete subset of $\Gamma$. Moreover, we see that $\Gamma$ induces an action on $\mathbb{R}^n/W$ which is obviously cocompact. We claim that it is also properly discontinuous.

We have decomposition $\mathbb{R}^n = W \oplus W^\perp$, where $W^\perp \simeq \mathbb{R}^n/W$.
Let $pr_1 : \mathbb{R}^n \to W$, $pr_2 : \mathbb{R}^n \to W^\perp$ be projections. Let $X$ be any discrete subset of $\mathbb{R}^n$. It can happen that sets $pr_1(X)$ and $pr_2(X)$ are not discrete subsets of $W$ and $W^\perp$.

Since $p_1(\Gamma)|_W$ is finite, we can concentrate on elements $\gamma \in \Gamma$ such that $p_1(\gamma)$ acts as identity on $W$.
The orbit $\Gamma(0)$ is discrete in $\mathbb{R}^n$. By contradiction let us assume that $pr_2(\Gamma(0))$ is not discrete at $W^\perp$ and $y \in W^\perp$ is an accumulation point of $pr_2(\Gamma(0))$.
Let $pr_2(\gamma_i(0)) \to y$, where $\gamma_i \in \Gamma, i \in \mathbb{N}$. Using elements from $\Gamma \cap (I \times \mathbb{R}^n)$ we can define a sequence of elements of $\tilde{\gamma}_i \in \Gamma, i \in \mathbb{N}$ such that $\forall i \in \mathbb{N}, pr_1(\tilde{\gamma}_i(0)) \subset C \subset W$, where $C$ is a compact set.

Here we use the fact that $\Gamma \cap \mathbb{R}^n$ is a cocompact subgroup of $W$. We can see that a set $\{\tilde{\gamma}_i(0)\}, i \in \mathbb{N}$ has an accumulation point at a discrete set $\Gamma(0)$. We get contradiction and our claim is proved.

Hence $\Gamma$ is a crystallographic group on $\mathbb{R}^n/W$ with no pure translations. This implies the zero dimension of $\mathbb{R}^n/W$. Let $(A, a) \in \Gamma$ be an arbitrary element in $\Gamma$ and $(I_n, x)$ be an arbitrary element in the set of translations subgroup $\Gamma \cap (I_n \times \mathbb{R}^n)$. We have

$$(A, a)(I_n, x)(A, a)^{-1} = (I_n, Ax) \in \Gamma \cap (I_n \times \mathbb{R}^n)$$

Therefore $\Gamma \cap (I_n \times \mathbb{R}^n)$ is a normal subgroup of $\Gamma$. Let $T$ be a maximal abelian subgroup of $\Gamma$ and $(A, a)$ be an arbitrary element of $T$. If $(A, a)$ commutes with any translation of $\Gamma$, then we can see that $A = I_n$. Thus the set of translation subgroup $\Gamma \cap (I_n \times \mathbb{R}^n)$ is a maximal abelian normal subgroup of $\Gamma$. $\square$

## 2.1 Alternative Proof of Bieberbach Theorem

The proof we have presented is due to Auslander. In this section, we shall give a geometric proof given by Buser:

**Theorem 2.12**
*Let $\Gamma$ be a crystallographic group of dimension $n$. Then its translation subgroup has $n$ linearly independent elements.*

Suppose $A \in O(n)$. We define

$$m(A) = \max\left\{\frac{|Ax - x|}{|x|} \mid x \in \mathbb{R}^n \backslash \{0\}\right\}$$

Let us see that we always have $|Ax - x| \leq m(A)|x|$, for $x \in \mathbb{R}^n$. Moreover, the set

$$\text{(i)} \quad E_A = \{x \in \mathbb{R}^n \mid |Ax - x| = m(A)|x|\}$$

is a non-trivial, $A$-invariant linear subspace. This follows from the so-called parallelogram condition[1] and the sequences of equations

$$2m^2(A)\left(|x|^2 + |y|^2\right) = 2\left(|Ax - x|^2 + |Ay - y|^2\right) = |A(x+y) - (x+y)|^2$$

$$+|A(x-y) - (x-y)|^2 \leq m^2(A)\left(|x+y|^2 + |x-y|^2\right) = 2m^2(A)\left(|x|^2 + |y|^2\right)$$

where $x, y \in E_A$. Let $E_A^\perp$ be the $A$-orthogonal complement of $E_A$. We define

$$\text{(ii)} \quad m^\perp(A) = \max\left\{\frac{|Ax - x|}{|x|} \mid x \in E_A^\perp \backslash \{0\}\right\}$$

when $E_A^\perp \neq 0$, and $m^\perp(A) = 0$ in the opposite case. Hence

$$\text{(iii)} \quad m^\perp(A) < m(A)$$

when $A \neq \text{id}$. Let $x = x^E + x^\perp \in E_A \oplus E_A^\perp$. Then

$$\text{(iv)} \quad \left|Ax^E - x^E\right| = m(A)\left|x^E\right|, \quad \left|Ax^\perp - x^\perp\right| \leq m^\perp(A)\left|x^\perp\right|$$

After these elementary observations, we see that for all $A, B \in O(n)$ we have

$$m([A, B]) \leq 2m(A)m(B)$$

In fact, we have

$$[A, B] - \text{id} = (A - \text{id})(B - \text{id}) - (B - \text{id})(A - \text{id})A^{-1}B^{-1}$$

Since $\left|A^{-1}B^{-1}x\right| = |x|$, it follows that

$$|[A, B]x - x| \leq m(A)m(B)|x| + m(B)m(A)|x|$$

for all $x \in \mathbb{R}^n$.

**Lemma 2.13** (Mini Bieberbach)
*For each unit vector $u \in \mathbb{R}^n$ and for all $\epsilon, \delta > 0$ there exists $\beta = (B, b) \in \Gamma$, such that $b \neq 0$, $\angle(u, b) \leq \delta$, $m(B) \leq \epsilon$. (Here $\angle(u, b)$ denotes the angle between the vectors $u, b$ and $\cos(\angle(u, b)) = \frac{\langle u, b\rangle}{\|b\|}$.)*

*Proof.* From the definition of $\Gamma$ there exists $d$ and elements $\beta_k \in \Gamma$ such that for any natural number $k$, we have

$$|b_k - ku| \leq d$$

Moreover $|b_k| \to \infty$, $\angle(u, b_k) \to 0$ $(k \to \infty)$. Since $O(n)$ is compact, we find a subsequence such that for $i < j$ we have

$$m\left(B_j B_i^{-1}\right) \leq \epsilon, \quad \angle(u, b_j) \leq \delta/2, \quad |b_i| \leq \frac{\delta}{4}|b_j|$$

Finally, the element $\beta_j \beta_i^{-1}$ has the required properties. □

**Lemma 2.14**
*If $\alpha = (A, a) \in \Gamma$ and $m(A) \leq \frac{1}{2}$, then $\alpha$ is a translation.*

*Proof.* Suppose $\alpha = (A, a) \in \Gamma$ satisfies our assumptions and $m(A) \neq 0$. Since $\Gamma$ is a discrete group, we can assume that the number $|a|$ is minimal. From Lemma A, for $u \in E_A$, there exists $\beta = (B, b)$, such that

$$b \neq 0, \quad \left|b^{\perp}\right| \leq \left|b^{E}\right|, \quad m(B) \leq \frac{1}{8}\left(m(A) - m^{\perp}(A)\right)$$

Among these we consider again the one for which $|b|$ is a non-zero minimum. Observe that $|a| \leq |b|$, when $\beta$ is not a translation[2]. Let $\tilde{\beta} = [\alpha, \beta]$. From the considerations preceding Lemma A, we have

$$m(\tilde{B}) = m([A, B]) \leq 2m(A)m(B) \leq m(B)$$

and

$$\tilde{b} = (A - \mathrm{id})b^{E} + (A - \mathrm{id})b^{\perp} + r$$

where

$$r = (\mathrm{id} - \tilde{B})b + A(\mathrm{id} - B)A^{-1}a$$

If $\beta$ is a translation then $B = \mathrm{id} = \tilde{B}$ and $r = 0$. As we have already observed, from the choice of $\alpha$, in the case when $\beta$ is not a translation, we have an inequality

$$|a| \leq |b|$$

Hence

$$|r| \leq (m(\tilde{B}) + m(B))|b| \leq 2m(B)|b| < 4m(B)\left|b^E\right|$$

In each case we have

$$|r| < \frac{1}{2}\left(m(A) - m^\perp(A)\right)\left|b^E\right|$$

By definition and from (A.2) we have

$$\tilde{b}^\perp - (A - \mathrm{id})b^\perp - r^\perp = (A - \mathrm{id})b^E + r^E - \tilde{b}^E = 0$$

Hence, using (A.1) and the orthogonality of $r^E$ and $r^\perp$, we obtain

$$\left|\tilde{b}^\perp\right| \leq m^\perp(A)\left|b^\perp\right| + \left|r^\perp\right| < m^\perp(A)\left|b^E\right| + |r|$$

Summing up, with support of (A.3), we have an inequality

$$\left|\tilde{b}^\perp\right| < \frac{1}{2}\left(m(A) + m^\perp(A)\right)\left|b^E\right|$$

On the other hand

$$\left|\tilde{b}^E\right| = \left|(A - \mathrm{id})b^E + r^E\right| \geq m(A)\left|b^E\right| - \left|r^E\right|$$

$$> m(A)\left|b^E\right| - \frac{1}{2}\left(m(A) - m^\perp(A)\right)\left|b^E\right|$$

$$= \frac{1}{2}\left(m(A) + m^\perp(A)\right)\left|b^E\right|$$

Here we apply again (A.3) and

$$|x + y| \geq ||x| - |y||$$

Finally, we can see that $\tilde{\beta}$ also satisfies the condition (A.1) and because

$$|\tilde{b}| \leq m(A)|b| + |r| < m(A) + \frac{1}{2}\left(m(A) - m^\perp(A)\right)\left|b^E\right| < |b|$$

we have a contradiction.

By Lemma 2.13 it follows that there exist $n$ elements of $\Gamma$, such that their translation parts are linearly independent, and their rotation parts have norm smaller than $\frac{1}{2}$. Now, by Lemma 2.14 we can define $n$ linearly independent translations in $\Gamma$.

□

*Proof of* 1*st Bieberbach.* For the proof of the first Bieberbach Theorem (see [17, Exercise 2.3, p. 26]), since $\Gamma$ is discrete in $E(n)$, it is closed (by 1.5). Hence, the image $p_1(\Gamma)$ of the homomorphism $p_1 : \Gamma \to O(n)$ is a compact subgroup of the compact group $O(n)$. As $\Gamma$ is discrete, there exists a neighborhood $U \ni 1$ such that $U \cap \Gamma = \{1\}$, we prove that $V \doteq p_1(U \cap \Gamma) = p_1(U) \cap p_1(\Gamma)$.

The inclusion $V \subset p_1(U) \cap p_1(\Gamma)$ is satisfied.

Let $g \in p_1(U) \cap p_1(\Gamma)$, then there exist $h \in U$, $k \in \Gamma$ such that $p_1(k) = p_1(h) = g$. Therefore $k^{-1}h \in \ker(p_1)$. Shrinking $U$ enought (by taking a neighborhood $W$ around the identity such that $W \cap \ker(p_1)$ and replacing $U$ with $W \cap U$) we may assume $U \cap \ker(p_1) = \{1\}$. So $h = k$ and as $p_1$ is open $V = \{1\}$ is open.

$\square$

*Remark*

We also refer the reader to [19, Theorem 4.2.2, p. 222] for a different proof and other similar results.

# 3 Klein Geometries

Let $X$ be a space on which a Lie group $G$ acts transitively. Then in the spirit of Klein's Erlangen program, $(G, X)$ defines a geometry — that is, the objects in $X$ which are invariant under $G$. We want to impart this geometry to a manifold $M$ by a collection of coordinate charts that maps parts of $M$ to open subsets of $X$, ensuring the coordinate changes on overlapping patches are locally restrictions of the action of elements from $G$.

*Definition* 3.1 (Klein Geometry [16], [9])
A Klein geometry is defined by a pair $(G, X = G/H)$, where $G$, the principal group, is a Lie group, and $H$ is a closed subgroup (equivalently a Lie subgroup), such that the natural action of $G$ on $X$ is transitive.
The homogeneous space $X$ is called the space of the geometry, or by abuse of language, the Klein geometry. It follows from the definition and quotient topology that $X$ is connected.

*Remark*
Since $H$ is closed, $X$ inherits a smooth topological manifold structure.

*Definition* 3.2 (See [9])
Let $U \subset X$ be an open set. A morphism $f : U \to X$ is said to be locally-$G$ if for every connected component $C$ of $U$, there exists $g \in G$ such that $f_{|C} = g_{|C}$.

*Definition* 3.3
Given a geometry $(G, X)$ and a manifold $M$ of the same dimension as $X$, a $G/H$-structure on $M$ is a maximal atlas $\mathcal{U} = \{(U_i, \varphi_i)\}$ such that:

- $\{U_i\}$ is an open cover of $M$.
- The morphisms $\varphi_i : U_i \to X$ are open embeddings.
- The transition maps $\varphi_i \circ \varphi_j^{-1} : \varphi_j(U_i \cap U_j) \to \varphi_i(U_i \cap U_j)$ are locally-$G$.

A manifold equipped with such a structure is called a $(G, X)$-manifold or a manifold locally modeled on $(G, X)$

*Example* 3.4
For any geometry $(G, X)$, the space $X$ is tautologically a $(G, X)$-manifold such that the identity morphism is a global chart. More generally, for any open set $U$ of $X$, $U$ is a $(G, X)$-manifold with the identity as a global chart.

Classic examples of geometries include Euclidean geometry, affine geometry [19].

*Example* 3.5 (Euclidean manifolds)
Let $E(n) = O_n(\mathbb{R}) \ltimes \mathbb{R}^n$ the group of isometries of the Euclidean space $\mathbb{R}^n$, a $(E(n), \mathbb{R}^n)$-manifold is called a Euclidean manifold, or flat manifold. These manifolds are the main object of Bieberbach theorems. We will consider flat manifolds in more details after giving some main results in Klein geometry.

*Example* 3.6 (Affine manifolds)
Let $A(n) = GL_n(\mathbb{R}) \ltimes \mathbb{R}^n$ be the group of affine transformations of $\mathbb{R}^n$, a $(A(n), \mathbb{R}^n)$-manifold is called an affine manifold.
A Euclidean structure on a manifold automatically gives an affine structure. Bieberbach proved that closed Euclidean manifolds with the same fundamental group are equivalent as affine manifolds.

*Example* 3.7 (Elliptic manifolds)
The orthogonal group $O(n+1)$ acts on the sphere $\mathbb{S}^n \subset \mathbb{R}^{n+1}$. a $(O(n+1), \mathbb{S}^n)$-manifold is called spherical or elliptic manifold.

*Example* 3.8 (Hyperbolic manifold)
Let $G$ be the group of isometries of the Hyperbolic space:

$$\mathbb{H}^n = \{x = (x_0, \cdots, x_n) \in \mathbb{R}^{n+1} : -x_0^2 + x_1^2 + \cdots + x_n^2 = -1, x_0 > 0\}$$

with the metric induced by the quadratic form of signature $(1, n)$. G is the Lorenz group:

$$G = O(1,n) = \{A \in GL(n+1, \mathbb{R}) : A^T DA = D \text{ where } D = \mathrm{diag}(-1, 1, \cdots, 1)\}$$

Let $PO(1,n) = O(1,n)/O(1)$ the Orthochronous Lorentz group, then a Hyperbolic manifold is a $(PO(1,n), \mathbb{H}^n$-manifold. The subgroup $H$ is the group $O(1) \times O(n)$ so that we have:

*Example* 3.9 (Projective geometry)
If $G$ is the group $PGL(n+1)$, a projective manifold is a $(G, \mathbb{R}P^n)$-manifold. This geometry fixes projective lines and cross-ratio.

Let $M$ and $N$ be two $(G, X)$-manifolds and $f : M \to N$ a map. We say that $f$ is a $(G, X)$-map if for any two charts of $M$ and $N$:

$$\varphi_i : U_i \to X \quad , \quad \psi_j : V_j \to X$$

the restriction of $\psi_j \circ f \circ \varphi_i^{-1}$ to $\varphi_i(U_i \cap f^{-1}(V_j))$ is locally-$G$. In particular, we consider $(G, X)$-maps that are local diffeomorphisms. The set of $(G, X)$-automorphisms $M \to N$ is a group that we denote by $\mathrm{Aut}_{(G,X)}(M)$.

$$\varphi_\beta(U_\alpha \cap U_\beta) \xrightarrow{\varphi_\alpha \circ (\varphi_\beta)^{-1}} \varphi_\alpha(U_\alpha \cap U_\beta)$$

*Definition* 3.10
Given two geometries $(G_1, G_1/H_1)$ and $(G_2, G_2/H_2)$. We say that they are isomorphic if there is an isomorphism of Lie groups $\psi : G_1 \longrightarrow G_2$ such that $\psi(H_1) = H_2$. The group $\mathrm{Aut}(G_1)$ gives many such geometries.

A geometry $(G, X)$ may contain or refine another geometry $(G', X')$. In this way one can pass from $(G, X)$–structures to $(G', X')$–structures on the same manifold. More precisely, suppose there is a local diffeomorphism:

$$\Phi : X \longrightarrow X'$$

along with a group homomorphism:

$$\phi : G \longrightarrow G'$$

such that $\Phi$ is equivariant with respect to $\phi$, that is, for every $g \in G$ the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "\Phi"] \arrow[d, "g"'] & X' \arrow[d, "\phi(g)"] \\ X \arrow[r, "\Phi"] & X'. \end{tikzcd}$$

Then any $(G, X)$-atlas on a manifold $M$ pushes forward to a $(G', X')$–atlas as follows. If

$$(U_\alpha, \psi_\alpha) \quad \text{with} \quad \psi_\alpha : U_\alpha \longrightarrow X$$

is a chart in the $(G, X)$–structure, then

$$\left(U_\alpha, \Phi \circ \psi_\alpha\right) \quad \text{with} \quad \Phi \circ \psi_\alpha : U_\alpha \longrightarrow X'$$

is a chart in the induced $(G', X')$–structure. The transition functions on overlaps are still in the image of $G$ under $\phi$, so their consistency is preserved.

## 3.1 Developing map and holonomy

The following fact is essential in the study of $(G, X)$-structures.

**Proposition 3.11** (Unique Extension Property)
*Let $M$ and $N$ be two $(G, X)$-manifolds and $f, g : M \to N$ be two $(G, X)$-maps. If $M$ is connected, then $f$ and $g$ are equal if and only if they coincide locally.*

*Proof.* Define $S$ as the set of all points of $M$ that admit an open neighborhood in which $f$ and $g$ coincide.
We show that $S$ is both open and closed in $M$.
Let $x \in S$ and let $U$ be an open neighborhood of $x$, such that $f|_U = g|_U$. We have $U \subset S$, i.e $S$ is open.
Let $x \notin S$. If $f(x) \neq g(x)$, we can find a neighborhood $U$ of $x$ such that $f(U) \cap g(U) = \varnothing$. Thus $U \cap S = \varnothing$. Suppose that $f(x) = g(x)$. Let $(U, \varphi)$ be a local chart around $x$. By shrinking $U$ if necessary, we can assume that $f(U)$ and $g(U)$ are contained in the domain $W$ of a local chart $(W, \psi)$ of $N$. We have $U \cap S = \varnothing$. Suppose by contradiction

that there exists $x_0 \in U$ which has an open neighborhood $V$ in which $f$ and $g$ coincide. We can always reduce $V$ and assume that $V \subset U$. By construction, the charts $h_i = \psi \circ f_i \circ \varphi^{-1} : \varphi(V) \to \psi(W), i = 1, 2$ coincide, so they must extend to the same chart on $X$. Therefore, they are equal on the set $\varphi(U)$, which means that in particular $f$ and $g$ coincide on $U$. This contradicts the fact that $x \notin S$. Therefore, $U \cap S = \varnothing$ and $S$ is closed. $\square$

Let $M$ be a $(G, X)$-manifold. Let $p : \tilde{M} \to M$ be a universal cover with fundamental group $\pi = \pi_1(M)$. $p$ induces a $(G, X)$-structure on $\tilde{M}$ on which $\pi$ acts by $(G, X)$-automorphisms. The unique extension property implies:

**Proposition 3.12** (see [9])
*Let $M$ be a simply connected $(G, X)$-manifold. Then there exists a $(G, X)$-map $f : M \to X$.*

*Proof.* Choose a basepoint $x_0 \in M$ and a coordinate patch $U_0$ containing $x_0$. For $x \in M$, we define $f(x)$ as follows. Choose a path $\{x_t\}_{0 \le t \le 1}$ in $M$ from $x_0$ to $x = x_1$. Cover the path by coordinate patches $U_i$ (where $i = 0, \ldots, n$) such that $x_t \in U_i$ for $t \in (a_i, b_i)$ where

$$a_0 < 0 < a_1 < b_0 < a_2 < b_1 < a_3 < b_2 < \cdots < a_{n-1} < b_{n-2} < a_n < b_{n-1} < 1 < b_n$$

Let $U_i \xrightarrow{\psi_i} X$ be an $(G, X)$-chart and let $g_i \in G$ be the unique transformation such that $g_i \circ \psi_i$ and $\psi_{i-1}$ agree on the component of $U_i \cap U_{i-1}$ containing the curve $\{x_t\}_{a_i < t < b_{i-1}}$. Let

$$f(x) = g_1 g_2 \cdots g_{n-1} g_n \psi_n(x);$$

we show that $f$ is indeed well-defined. The map $f$ does not change if the cover is refined. Suppose that a new coordinate patch $U'$ is "inserted between" $U_{i-1}$ and $U_i$. Let $\{x_t\}_{a' < t < b'}$ be the portion of the curve lying inside $U'$:

$$a_{i-1} < a' < a_i < b_{i-1} < b' < b_i$$

Let $U' \xrightarrow{\psi'} X$ be the corresponding coordinate chart and let $h_{i-1}, h_i \in G$ be the unique transformations such that $\psi_{i-1}$ agrees with $h_{i-1} \circ \psi'$ on the component of $U' \cap U_{i-1}$ containing $\{x_t\}_{a' < t < b_{i-1}}$ and $\psi'$ agrees with $h_i \circ \psi_i$ on the component of $U' \cap U_i$ containing $\{x_t\}_{a_i < t < b'}$. By the unique extension property $h_{i-1} h_i = g_i$ and it follows that the corresponding developing map

$$f(x) = g_1 g_2 \cdots g_{i-1} h_{i-1} h_i g_{i+1} \cdots g_{n-1} g_n \psi_n(x)$$

$$= g_1 g_2 \cdots g_{i-1} g_i g_{i+1} \cdots g_{n-1} g_n \psi_n(x)$$

is unchanged. Thus the developing map as so defined is independent of the coordinate covering, since any two coordinate coverings have a common refinement.

Next we claim the developing map is independent of the choice of path. Since $M$ is simply connected, any two paths from $x_0$ to $x$ are homotopic. Every homotopy can be broken up into a succession of "small" homotopies, that is, homotopies such that there exists a partition

$$0 = c_0 < c_1 < \cdots < c_{m-1} < c_m = 1$$

such that during the course of the homotopy the segment $\{x_t\}_{c_i<t<c_{i+1}}$ lies in a coordinate patch. It follows that the expression defining $f(x)$ is unchanged during each of the small homotopies, and hence during the entire homotopy. Thus $f$ is independent of the choice of path.

Since $f$ is a composition of a coordinate chart with a transformation $X \to X$ coming from $G$, it follows that $f$ is an $(G, X)$-map. $\square$

For $M$ an arbitrary $(G, X)$-manifold, we shall apply this proposition to its universal cover $\tilde{M}$:

**Theorem 3.13** (see [9] and [19])
*Let $M$ be a $(G, X)$-manifold and $\pi : \tilde{M} \longrightarrow M$ its universal cover. Then there exists a pair* $(\mathrm{dev}, h)$ *consisting of a $(G, X)$-map* $\mathrm{dev} : \tilde{M} \longrightarrow X$ *and a homomorphism $h : \pi_1(M) \longrightarrow G$ such that for each $\gamma \in \pi_1(M)$, the following diagram commutes:*

$$\begin{tikzcd}
\tilde{M} \arrow[r, "\mathrm{dev}"] \arrow[d, "\gamma"'] & X \arrow[d, "h(\gamma)"] \\
\tilde{M} \arrow[r, "\mathrm{dev}"'] & X
\end{tikzcd}$$

*If* $(\mathrm{dev}', h')$ *is another pair, then there exists $g \in G$ such that* $\mathrm{dev}' = g \circ \mathrm{dev}$ *and* $h'(\gamma) = \mathrm{Inn}(g) \circ h(\gamma)$ *for all $\gamma \in \pi_1(M)$. That is, we have the following commutative diagram:*

$$\begin{tikzcd}
\tilde{M} \arrow[r, "\mathrm{dev}"] \arrow[d, "\gamma"'] & X \arrow[r, "g"] \arrow[d, "h(\gamma)"] & X \arrow[d, "h'(\gamma)"] \\
\tilde{M} \arrow[r, "\mathrm{dev}"'] & X \arrow[r, "g"'] & X
\end{tikzcd}$$

$\mathrm{dev}$ *is called the developing map and h the holonomy. The image of h is called the holonomy group.*

*Remark*
In literature, sometimes we call $h(\pi_1(M))$ the monodromy group and should not be confused from the holonomy representation in Riemannian geometry of parallel transport.

*Definition* 3.14 ([19])
We say that a $(G, X)$-manifold $M$ is complete if the developing map is a covering map.

*Remark*
Sometimes we will refer to the completeness defined above by geometric completeness to distinguish it with metric completeness. We will see later in 3.24 that they are equivalent.

The following proposition with no proof appear in Thurston [19]. We suggest the following proof:

**Proposition 3.15** (The holonomy group characterizes the manifold)
*If $G$ is a group that acts analytically by diffeomorphisms on a simply connected topological space $X$, any complete $(G, X)$-manifold can be constructed from its holonomy group $\Gamma$ as the quotient space $X/\Gamma$.*

*Proof.* Since $X$ is simply connected and $M$ is complete, $X$ is a universal cover of $M$, so that we have the following commutative diagram:

$$\begin{tikzcd} X \arrow[r, "\mathrm{dev}"] \arrow[d, "\gamma"'] & X \arrow[d, "h(\gamma)"] \\ X \arrow[r, "\mathrm{dev}"'] & X \end{tikzcd}$$

The development being a diffeomorphism, the action of the fundamental group is transported via dev to $X$ such that for all $x \in X, \gamma \in \pi_1(M)$, we have:

$$\gamma \cdot x = \mathrm{dev}(\gamma \cdot \mathrm{dev}^{-1}(x)) = \mathrm{hol}(\gamma) \cdot x$$

Thus the action of the holonomy group coincides with that of the fundamental group which is faithful. Therefore, the holonomy is injective, by the isomorphism theorem:

$$M \cong X/\pi_1(M) \cong X/\Gamma$$

$\square$

We recall the following theorem:

**Theorem 3.16** (Eckmann-Hilton Principle)
*We assume that a set $X$ is equipped with two unital magma structures $\star$ and $*$, and such that for $x, x', y, y' \in X$ we have:*

$$(x \star x') * (y \star y') = (x * y) \star (x' * y')$$

*then the two operations coincide and $X$ is a commutative monoid. In particular, the fundamental group of a topological group is abelian.*

*Proof.* Let $x = y' = 1_*$ and $x' = y = 1_\star$. This gives $1_\star = 1_* = 1$. Now let $x' = y = 1$, this gives:

$$x * y' = x \star y'$$

We then deduce associativity and commutativity:

$$x = 1 \Rightarrow x' * (y * y') = y * (x' * y')$$
$$x = y' = 1 \Rightarrow x' * y = y * x'$$

Thus $X$ is equipped with a commutative monoid structure. We show that the fundamental group $\pi = \pi_1(G)$ of a topological group $(G, \cdot)$ is abelian. The operation of $G$ induces a second operation on $\pi$ (concatenation $+$), for $\gamma, \delta$ two loops in $G$:

$$\gamma * \delta : t \mapsto \gamma(t) \cdot \delta(t)$$

This operation is well defined as the multiplication of $G$ is continuous. So that:

$$[\gamma * \delta] = [t \mapsto \gamma(t) \cdot \delta(t)] = [t \mapsto \gamma(t)] * [t \mapsto \delta(t)] = [\gamma] * [\delta]$$

Let $a, b, c, d \in \pi$, we have:

$$\begin{aligned}(a + b) * (c + d) &= t \mapsto (a + b)(t) \cdot (c + d)(t) \\ &= (t \mapsto a(t) \cdot c(t)) + (t \mapsto b(t) \cdot d(t))\end{aligned}$$

By Eckmann-Hilton, $\pi$ is abelian. □

We replace the space $X$ with its universal cover when $X$ is not simply connected. There exists a covering group $\tilde{G}$ acting on $\tilde{X}$ by homeomorphisms. We can then describe $\tilde{G}$ by the extension:

$$1 \to \pi_1(X) \to \tilde{G} \to G \to 1$$

We say that a group $G$ admits a local section (local cross-section [19]) with respect to a closed subgroup $H < G$ if we have the data:

- a neighborhood $U \subset G/H$ of the identity.
- a subset $S \subset G$ such that the canonical projection restricted to $S$ is a homeomorphism onto $U$. We call the local section the map $\sigma : U \to G$ such that $\pi(\sigma(gH)) = gH$ and $\sigma(1H) = 1$.

This means that locally around the neutral class, we can continuously choose a unique representative of each class $gH$. We then study the structure of $\pi_1(X)$ when $X$ is not simply connected:

- Let $X$ be a manifold and $G \subset \mathrm{Homeo}(X)$ transitive, let $x \in X$ and $G_x$ its stabilizer (closed in $G$), suppose that $G$ admits a local section with respect to $G_x$ and that the morphism $\rho : g \mapsto gx$ is open. We construct a homeomorphism $\phi : \pi_1(X) \to \pi_0(G_x)$. And we will show that the kernel of $\phi$ is central. In particular, if $G_x$ is arcwise connected then $\pi_1(X)$ is abelian.

**Lemma 3.17**

*If G is a Lie group, every closed subgroup (i.e., a Lie subgroup) has a local section. The map $\rho$ is open.*

*Proof.* If $G$ acts transitively on a manifold $X \ni x$, we have the identification $X \cong G/H$ where $H = \rho^{-1}(x)$ the stabilizer of $x$ which is closed. Therefore the projection is open, we then have:

$$G \xrightarrow{\quad \pi \quad} G/H \xrightarrow{\quad \phi \quad} X$$

so that the map $\rho = \phi \circ \pi$ is open.

Let $H$ be a closed subgroup, and $\pi$ the projection map as above. $\pi$ is a surjective submersion. Let $(U, \varphi)$ be a local chart of $G$ at 1 such that $\varphi(1) = 0$:

$$\varphi : U \longrightarrow V \subset \mathbb{R}^n$$

and $(W, \psi)$ a local chart of $G/H$ at $1H$ such that $\psi(1H) = 0$:

$$\psi : W \longrightarrow Z \subset \mathbb{R}^k$$

define

$$f = \psi \circ \pi \circ \varphi^{-1} : V \longrightarrow Z$$

The map

$$D_0 f : \mathbb{R}^n \longrightarrow \mathbb{R}^k$$

is surjective. Then, there exists $V_0 \subset V$ a neighborhood of 0 such that there exists

$$\beta : W_0 \longrightarrow V_0$$

Such that:

$$f \circ \beta = Id_{W_0}$$

that is $\beta$ is a local right inverse of $f$, define:

$$\alpha = \varphi^{-1} \circ \beta \circ \psi : W_1 = \psi^{-1}(W_0) \longrightarrow \varphi^{-1}(V_0)$$

We have for every $gH \in W_1$:

$$\pi(\alpha(gH)) = \psi^{-1}(f(\beta(\psi(gH)))) = \psi^{-1}(\psi(gH)) = gH$$

$\square$

**Corollary 3.18**
*Let $M$ be a manifold on which Lie group $G$ acts transitively with path-connected stabilizer. Then $\pi_1(M)$ is abelian.*

*Definition* 3.19 (Riemannian metric)
Let $M$ be a (topological) smooth manifold. A Riemannian metric $g$ on $M$ consists of a smooth function $q : TM \to \mathbb{R}$ whose restrictions $q_x : T_xM \to \mathbb{R}$ are all nondegenerate quadratic forms.

*Definition* 3.20 (Geodesic)
Let $M$ be a metric space with distance $d$ and let $I \subset \mathbb{R}$ be an interval. A curve (continuous map) $\gamma\colon I \to M$ is called a geodesic if there exists a constant $v \geq 0$ such that for every $t \in I$ there is a neighborhood $J \subset I$ of $t$ with the property

$$d\big(\gamma(t_1), \gamma(t_2)\big) = v\,|t_1 - t_2| \quad \text{for all } t_1, t_2 \in J.$$

If $v = 1$ the curve is said to be unit-speed.

**Proposition 3.21** (compact stabilizers imply completeness [19]))
*Let $G$ be a lie group acting analytically and transitively on a manifold $X$, and such that the stabilizer $G_x$ of $x$ is compact for some $x$ (hence all by transitivity). Then every closed $(G, X)$-manifold $M$ is complete.*

We shall prove the following lemma (see [19, Lemma 3.4.11, p. 144]).

**Lemma 3.22** (existence of invariant metric)
*Let $G$ act transitively on an analytic manifold $X$. Then $X$ admits a $G$-invariant Riemannian metric if and only if, for some $x \in X$, the image of $G_x$ in $GL(T_xX)$ has a compact closure.*

*Proof.* If $G$ preseves a metric $G_x$ maps to a subgroup of $O(T_xX)$, hence its closure is compact.
To prove the converse, fix $x$ and assume that the image of $G_x$ has a compact closure $H_x = \overline{\rho(G_x)}$ where $\rho$ is the tangent representation. Let $Q$ be any positive definite form on $T_xX$.

$$Q : T_xX \times T_xX \longrightarrow \mathbb{R}$$

$H_x$ is compact, hence unimododular, so it admits a bi-invariant Haar measure $\mu$, define a new inner product on $T_xX$ by:

$$\langle v, w \rangle_x = \int_{H_x} Q(h \cdot v, h \cdot w) d\mu(h)$$

Right-invariance of $\mu$ shows immediately that for every $k \in H_x$,

$$\langle k \cdot v, k \cdot w \rangle_x = \int_{h \in H_x} Q\big(hk \cdot v, hk \cdot w\big) d\mu(h) = \langle v, w \rangle_x,$$

so $\langle \cdot, \cdot \rangle_x$ is $H_x$-invariant.

Now, since the action is transitive, for an arbitrary point $y \in X$ there exists $g \in G$ with $g \cdot x = y$, and transport the inner product at $x$ to $T_x X$ by

$$\langle u, w \rangle_y = \langle d(g)_y^{-1}(u),\, d(g)_y^{-1}(w) \rangle_x .$$

If $g' \in G$ is another element with $g' \cdot x = y$, then $g' = g\, h$ for some $h \in G_x$, and the $G_x$-invariance of $\langle \cdot, \cdot \rangle_x$ guarantees that $\langle \cdot, \cdot \rangle_y$ is well-defined. □

*Remark*
Using a local cross section we can prove that this $G$-invariant metric is analytic.

*Proof of compact stabilizers imply completeness.* (see [19, Proposition 3.4.10, p. 144]) transitively imply that the given condition at one point $x$ is equivalent to the same condition everywhere. So fix $x \in X$ and let $T_x X$ be the tangen space to $X$ at $x$. There is an analytic homomorphism of $G_x$ to $GL(T_x X)$ whose image is compact.
If $M$ is any $(G, X)$-manifold, we can use charts to pull-back the $G$-invariant Riemannian metric from $X$ to $M$ (invariancy guarantees that such metric is well-defined), the resulting metric is a Riemannian metric on $M$ invariant under any $(G, X)$- morphism. In a manifold endowed with such metric, we can find for any point $y$ a ball $B_\epsilon(y)$ that is ball-like (homeomorphic image of the round ball) and convex. if $M$ is closed we can choose $\epsilon$ uniformly by compacteness. Without loss of generalality, we may assume that all $\epsilon$-balls in $X$ are contractible and convex since $G$ is a transitive group of isometries.
Then, for any $y \in \tilde{M}$, the ball $B_\epsilon(y)$ is mapped homeomorphically by dev, for if $\mathrm{dev}(y) = \mathrm{dev}(y')$ for $y \neq y'$ in the ball, the geodesic connecting $y$ and $y'$ maps to a self-intersecting geodesic in $X$ contradicting the convexity of the $\epsilon$-balls in $X$.
The map dev is an isometry between $B_\epsilon(y)$ and $B_\epsilon(\mathrm{dev}(y))$ by definition.
Take $x \in X$ and $y \in \mathrm{dev}^{-1}(B_{\epsilon/2})(x)$. The ball $B_\epsilon(y)$ maps isometrically, and thus contains a copy of $\mathrm{dev}^{-1}(B_{\epsilon/2})(x)$. The inverse image of $\mathrm{dev}^{-1}(B_{\epsilon/2})(x)$ is then the disjoint union of these copies. This proves that dev is indeed a covering map and $M$ is hence complete. □

*Example* 3.23
Let $\Gamma$ a finite subgroup of $O(4)$ acting freely on $\mathbb{S}^3$, an elliptic 3-manifold is the orbit space $M = \mathbb{S}^3/\Gamma$, that is a $(O(4), \mathbb{S}^3)$-manifold. Such manifold is a closed manifold by definition. The proposition says that the universal cover of $M$ is $\mathbb{S}^3$.

We shall give an equivalency between metric completeness and completeness of $(G, X)$-manifolds, which justifies the use of the word:

**Proposition 3.24** (completeness equivalency, [19])
*Let $G$ be a group acting transitively and analytically on $X$ with compact stabilizers $G_x$. Fix a $G$-invariant metric on $X$ and let $M$ be a $(G, X)$-manifold. The following conditions are equivalent:*

1. *$M$ is a complete $(G, X)$-manifold.*

2. *For some $\epsilon > 0$, every closed $\epsilon$-ball in $M$ is compact.*

3. *For every $a > 0$, every closed $a$-ball in $M$ is compact.*

4. *There is some family of compact subsets $S_t$ of $M$, for $t \geq 0$, such that $\cup_{t \geq 0} S_t = M$ and $S_{t+a}$ contains a neighborhood of radius $a$ about $S_t$.*

5. *$M$ is a complete metric space.*

*Proof.* $(1) \implies (2)$. if $p : Y \longrightarrow Z$ is a covering map between two manifolds endowed mith a Riemannian metric and $p$ preseves this metric, we have $\overline{B}_\epsilon(p(y)) = p(\overline{B}_\epsilon(y))$ for any $y \in Y$ and any $\epsilon > 0$, because distances are defined in terms of path lenghts and paths in $Z$ can be lifted to paths in $Y$. So the compacteness of balls in $Y$ implies the same in $Z$ and conversely. Fixing a point $x \in X$ and compact $\epsilon$-ball around $x$ by the local compacity of $X$, the transitive action of $G$ implies the same for all points in $X$. Idem for $\tilde{M}$ and $M$.

$(2) \implies (3)$. We show this by induction, suppose that $(3)$ is true for some $a > \epsilon$. Then $\overline{B}_a(x)$ can be covered with finitely mainy $\epsilon/2$-balls, and therefore $\overline{B}_{a+\epsilon/2}(x)$ can be covered with finitely mainy $\epsilon$-balls hence compact.

$(3) \implies (4)$. Let $S_t = \overline{B}_t(x)$ where $x$ is fixed.

$(4) \implies (5)$. Any Cauchy sequence must be conatined in some $S_t$ for some $t$, hence it converges.

$(5) \implies (1)$. Suppose $M$ metrically complete. We will prove that any path $\alpha_t$ in $X$ can be lifted to $\tilde{M}$, since local homeomorphisms with the path lifting property are covering maps.
The universal cover of a complete metric space is complete $M$ since the projection of Cauchy sequence converges to some point $x \in M$. Since $x$ has a compact neighborhood which is evenly covered and these are separated in the metric of $\tilde{M}$, the Cauchy sequence also converges in $\tilde{M}$.
Consider now any path $\alpha_t$ in $X$. If it has a lifting $\tilde{\alpha}_t$ for $t$ in $[0, s]$, then it has a lifting for $[0, s + \epsilon)$ for some $\epsilon > 0$ by the local homeomorphicity of dev. If it has a lifting for $t$ in a half-open interval $[0, s)$, the lifting extends by metric completeness. Thus, $M$ is complete. □

## 3.2 Flat manifolds

*Definition* 3.25 (see [19])
Let $M$ be a smooth manifold. A Euclidean geometry on $M$ is a Klein geometry on $M$ modeled on $(O_n(\mathbb{R}) \ltimes \mathbb{R}^n, \mathbb{R}^n)$.
We say that $M$ is a flat manifold.

*Definition* 3.26 (Riemannian manifold [16])
Let $M$ be a smooth manifold, $g$ a Riemannian metric on $M$. The pair $(M, g)$ is then a Riemannian manifold.

Given two Riemannian manifold $(M, g_M)$ and $(N, g_N)$, we define the product:

$$(M \times N, g_M \oplus g_N)$$

which is a Riemannian manifold.

Let $f : M \longrightarrow L$ a smooth immersion, where $M$ is a manifold and $(L, g_L)$ a Riemannian manifold, the pull-back metric

$$(f^* g_L)_p : (x, y) \mapsto (g_L)_{f(p)}(df_p(x), df_p(y))$$

is a Riemannian metric, and $M$ a Riemannian manifold.

*Example* 3.27
The space $\mathbb{R}^n$ equipped with its inner product is a Riemannian manifold.

We recall this famous result:

**Theorem 3.28** (The Whitney embedding theorem)
*Any smooth manifold of dimension $n$ can be embedded in $\mathbb{R}^{2n}$.*

*Remark*
This makes any smooth manifold, that is embedded in some $\mathbb{R}^n$ by Whitney theorem, a Riemannian manifold.

*Remark*
The above definition of a flat manifold $M$ is equivalent to saying that $M$ is locally isometric to some $\mathbb{R}^n$. In essence, this means that locally, every part of the manifold is indistinguishable from a region of Euclidean space in terms of its geometric properties like distances, angles, and straight lines.
$M$ inherits the usual Riemannian metric from $\mathbb{R}^n$ such that these manifolds are exactly Riemannian manifolds with zero curvature.

When $M$ is a flat and geometrically complete manifold (equivalent to being metrically complete), we have the identification:

$$M \cong \mathbb{R}^n / \pi_1(M)$$

Since $\mathbb{R}^n$ is simply-connected, using 3.15, we also have

$$M \cong \mathbb{R}^n / \operatorname{hol}(\pi_1(M))$$

The action of $\pi_1(M) < E(n)$ is free and properly discontinuous. Hence, it is discrete.

*Definition* 3.29
Let $\Gamma < \mathbb{R}$ be a lattice of translations. A flat torus is the orbit space $\mathbb{R}^n / \Gamma$

Now we can give the geometric version of Bieberbach's theorem is:

**Theorem 3.30** (1st Bieberbach)
*Let $M$ be a compact flat n-manifold. $\pi_1(M)$ contains a torsion free and finitely generated abelian group of rank n and is a maximal abelian and normal subgroup of finite index. i.e Every compact flat manifold is a quotient of a flat torus $\mathbb{R}^n/\Gamma$ by a finite subgroup that acts freely on $\mathbb{R}^n/\Gamma$.*

We then have the following exact sequence:

$$0 \longrightarrow T = \pi_1(M) \cap \mathbb{R}^n \longrightarrow \pi_1(M) \longrightarrow H = \pi_1(M)/T \longrightarrow 0$$

where $H$ is then a finite group. $\iota : \mathbb{Z}^n \hookrightarrow \Gamma$ is an inclusion map which maps $e_i$ to $(I_n, e_i)$ where $e_1, \ldots, e_n$ are the standard basis of $\mathbb{Z}^n$ and $p : \Gamma \to H$ is a projection map which maps $(A, a) \in \Gamma$ to $A$. Besides, the group $\mathbb{Z}^n$ is a maximal abelian subgroup. Given such a short exact sequence, it induces a representation $\rho : H \to GL_n(\mathbb{Z})$ given by $\rho(g)x = \bar{g}\iota(x)\bar{g}^{-1}$, where $x \in \mathbb{Z}^n$ and $\bar{g}$ is chosen arbitrarily such that $p(\bar{g}) = g$.

**Lemma 3.31**
*The induced representation $\rho : H \to GL_n(\mathbb{Z})$ is a faithful representation. In other words, the kernel of $\rho$ is trivial.*

*Remark*
Sometimes, in literature, one may found that $H$ is called the holonomy group, and $\rho$ the holonomy representation. But one must distinguish it from the holonomy in geometric structures sense for that in the case of complete $(G, X)$-manifolds, we have the identification $\mathrm{hol}(\pi_1(M)) \cong \pi_1(M)$ when $X$ is simply-connected.

*Proof.* Let $g \in \ker(\rho)$. We have $\rho(g) = id_{\mathbb{Z}^n}$. It follows that

$$x = \rho(g)x = \bar{g}\iota(x)\bar{g}^{-1}$$

where $x \in \mathbb{Z}^n$ and $\bar{g}$ is chosen arbitrarily such that $p(\bar{g}) = g$. Thus $\bar{g}$ commutes with any translation of $\Gamma$. Since $\mathbb{Z}^n$ is the maximal abelian subgroup, we can conclude that $g = I_n$. Therefore $\rho$ is a faithful representation. □

*Remark*
We recall that $O(n)$ is a maximal compact subgroup of $GL_n(\mathbb{R})$ up to conjugacy and any finite subgroup of $GL_n(\mathbb{R})$ is included in $O(n)$ up to conjugacy.

*Proof.* Let $\langle \cdot\,, \cdot \rangle$ be the standard inner product. Let $H \subset GL(n, \mathbb{R})$ a finite subgroup. Average over $H$ by the usual Haar measure and define the symmetric positive definite $H$-invariant bilinear form on $\mathbb{R}^n$

$$b(v, u) = \langle v, u \rangle_H = \int_H \langle hv, hu \rangle d\mu(h)$$

By Sylvester's Law of Inertia, there exists $P \in GL(n, \mathbb{R})$ such that $PO(b)P^{-1} = O(n)$ □

**Theorem 3.32** (Zassenhaus (Szczepanski [17]))
*A group $\Gamma$ is isomorphic to a crystallographic group of dimension $n$ if and only if $\Gamma$ contains a normal, free abelian subgroup $\mathbb{Z}^n$ of finite index which is a maximal abelian subgroup of $\Gamma$*

*Proof.* First, assume that $\Gamma$ is an $n$-crystallographic group. The first Bieberbach theorem guarantees that the group of translations $T = \Gamma \cap (I \ltimes \mathbb{R}^n)$ is a normal, free abelian subgroup of finite index which is a maximal abelian subgroup of $\Gamma$.

For the other direction, assume that $\Gamma$ contains a normal, free abelian subgroup $\mathbb{Z}^n$ of finite index which is a maximal abelian subgroup of $\Gamma$:

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z}^n \arrow[r, "\iota"] & \Gamma \arrow[r, "p"] & G \arrow[r] & 1 \end{tikzcd}$$

Where $G \cong \Gamma/\mathbb{Z}^n$ is a finite group. Given such short exact sequence, it induces a representation:

$$h_\Gamma : G \longrightarrow GL_n(\mathbb{Z})$$

. Since $\mathbb{Z}^n$ is maximal abelian group, by the previous lemma, $h_\Gamma$ is faithful. We can view the free abelian group $\mathbb{Z}^n$ as a subgroup of $\mathbb{R}^n$. Thus we have the following diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathbb{Z}^n \arrow[r, "i"] \arrow[d, "i'"] & \Gamma \arrow[r, "p"] \arrow[d] & G \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathbb{R}^n \arrow[r] \arrow[d, equal] & \Gamma' \arrow[r] \arrow[d] & G \arrow[r] \arrow[d, "h_\Gamma"] & 0 \\
0 \arrow[r] & \mathbb{R}^n \arrow[r] & GL_n(\mathbb{R}) \ltimes \mathbb{R}^n \arrow[r] & GL_n(\mathbb{R}) \arrow[r] & 0
\end{tikzcd}$$

Where $\Gamma'$ is the pushout of the monomorphisms $i : \mathbb{Z}^n \longrightarrow \Gamma$ and $i' : \mathbb{Z}^n \longrightarrow \mathbb{R}^n$. Notice that all vertical arrows are injective. By Proposition 6.19, we have $H^2(G, \mathbb{R}^n) = 0$. By Remark , $\Gamma'$ is isomorphic to $G \ltimes \mathbb{R}^n$ where the group action of $G$ on $\mathbb{R}^n$ is given by $h_\Gamma$. As we noted above, any finite subgroup of $GL_n(\mathbb{R}^n)$ is conjugate to a finite subgroup of $O(n)$. Therefore we can conclude that $\Gamma$ is an $n$-crystallographic group. $\square$

*Remark* (see [13] and [17])
If $\Gamma$ is torsion-free we can only assume that it is an abelian finitely generated subgroup of finite index.

**Proposition 3.33** (Szczepanski [17])
*For any given $n$, the number of isomorphism classes of finite subgroups of $GL(n, \mathbb{Z}$ is finite.*

*Proof.* We shall prove that the kernel of the homomorphism:

$$f : GL(n, \mathbb{Z}) \longrightarrow GL(n, \mathbb{Z}/p\mathbb{Z})$$

is torsion free for a prime number $p > 2$.
Let $A \in \ker(f)\backslash\{I\}$ and let $B \in M_n(\mathbb{Z})$ such that $A = I + pB$.
Assume that there exists a prime number $q$ such that:

$$A^q = (I + pB)^q = I + pqB + \binom{q}{2} p^2 B^2 + \cdots + p^q B^q = I$$

That is:

$$qB + \binom{q}{2} pB^2 + \cdots + p^{q-1} B^q = 0$$

Let $\alpha$ be a maximal number such that $p^\alpha$ divides all elements of $B$. Since any element of $pB^2$ is divisible by $p^{2\alpha+1}$, it follows that elements of $qB$ are divisible by $p^{2\alpha+1}$. Hence, if $p \neq q$, then from the maximality of $\alpha$ we get $2\alpha + 1 \leq \alpha$, what is absurd. Moreover, if $p = q$, then $2\alpha \leq \alpha$ and $\alpha = 0$. Hence , we have

$$B + \binom{p}{2} B^2 + \binom{p}{3} pB^3 + \cdots + p^{p-2} B^p = 0.$$

Since the prime number $p$ is odd, we conclude that $p \mid \binom{p}{2}$ and $p$ divides other elements of the above sum. Hence $p$ divides elements of $B$ and $\alpha \geq 1$. This proves our claim. Summing up, we can say that any finite subgroup of $GL(n, \mathbb{Z})$ has trivial intersection with the kernel of $\phi$ and is isomorphic to some finite subgroup of $GL(n, \mathbb{Z}/p\mathbb{Z})$.

$\square$

*Definition* 3.34 (Nilpotent group)
Let $G$ a group. For $A, B < G$, we define $[A, B] =< \{xyx^{-1}x^{-1} | x \in A, y \in B\} >$. The sequence $C^n(G)$ of subgroups of $G$ defined by $C^1(G)$ and $C^{n+1}(G) = [G, C^n(G)]$ is called decreasing central sequence. We say that $G$ is nilpotent if there exists $n$ such that $C^n(G) = \{1\}$. The smallest such $n$ is the class of nilpotency.

**Theorem 3.35** (Ado's theorem [1])
*Every finite-dimensional Lie algebra $\mathfrak{g}$ over a field $F$ of characteristic zero (e.g., over the real numbers) admits a faithful finite-dimensional representation. In particular, every Lie group locally embeds in $GL(V)$ for some finite-dimensional real vector space $V$.*

For more details on the above theorem and similar results we refer the reader to [5]. For the proof we refer to [8] and [15].

**Theorem 3.36** (Zassenhaus neighborhood theorem 1983)
*Let $G$ a lie group. There exists a neighborhood of the identity $U$ such that if $\Gamma \subset G$ is discrete, then $< \Gamma \cup U >$ is nilpotent.*
*Such neighborhood is called Zassenhaus neighborhood.*

*Proof.* We can locally embed a Lie group $G$ in some $GL(n, \mathbb{R})$, so can to show the theorem for $GL(n, \mathbb{R})$. The same proof in 2.5 applies, so that we get that taking commutators converges to the identity. let $U_r$ the $r$-ball around the identity in $GL(n, \mathbb{R})$. $\Gamma$ a discrete subgroup and

$$H = \langle \Gamma \cap U_r \rangle$$

Since $\Gamma$ is discrete and $U_r$ is an identity-neighborhood, there is a minimum distance $\delta > 0$ so that every non-trivial element of $\Gamma \cap U_r$ satisfies $\|A - I\| \geq \delta$. We choose $r$ small enough that

$$0 < \kappa < 1, \quad \kappa r < \delta.$$

Define the lower central series of $H$:

$$H^1 = H, \quad H^{k+1} = [H,\, H^k].$$

But if $h \in H^k \subset U_r$ and $g \in H \subset U_r$, then

$$[g, h] = f(g, h) \in U_{\kappa r},$$

so in fact

$$H^2 \subset U_{\kappa r}, \quad H^3 \subset U_{\kappa^2 r}, \quad \ldots, \quad H^k \subset U_{\kappa^{k-1} r}.$$

For large $k$, $\kappa^{k-1} r$ becomes smaller than $\delta$. But the only element of $\Gamma$ whose distance to the identity is less than $\delta$ is the identity itself. Hence for sufficiently large $k$,

$$H^k = \{e\}.$$

□

**Theorem 3.37** (2nd Bieberbach [17], [19])
*For any natural number $n$, there are only a finite number of isomorphism classes of crystallographic groups of dimension $n$.*

**Lemma 3.38** (Jordan Theorem)
*There is a number $\nu(n)$ such that any finite group $F \subset O(n)$ has an abelian normal subgroup $A(F)$ of index smaller than $\nu(n)$.*

*Proof.* Let $U = B(I, \varepsilon) \subset O(n)$ be a stable neighborhood and $U' = B\big(I, \frac{\varepsilon}{2}\big)$. Let us choose $\nu(n)$ such that $\mu(U') > \frac{1}{\nu(n)}$, where $\mu(\cdot)$ is the Haar measure on $O(n)$ with total measure 1. Moreover, let $A(F) = \langle U \cap F \rangle$. From Lemma 2.4 $A(F)$ is normal, and by Lemma 2.7 it is an abelian subgroup. From assumptions, $F/A(F) = \{[f_1], [f_2], \dots, [f_m]\}$, where $f_i \in F$, $i = 1, 2, \dots, m$. By definition, if $[f_i] \neq [f_j]$, then $f_i U' \cap f_j U' = \varnothing$. Hence

$$m\,\mu(U') \;=\; \sum_{i=1}^{m} \mu\big(f_i U'\big) \;\leq\; 1 \quad \text{and} \quad |F/A(F)| = m \leq \tfrac{1}{\mu(U')} < \nu(n).$$

□

**Theorem 3.39** (see [17])
*Let $G_l$, $l = 1, 2, \dots, k$, be the set of finite subgroups of $O(n)$ which can be expressed as integer matrices with determinant $\pm 1$ in $\mathrm{GL}(n, \mathbb{R})$. Then $k$ is finite.*

*Proof.* Let $A_l$ be the normal abelian subgroup of $G_l$ described in the Lemma 3.38. Since the order of $G_l/A_l$ is bounded, there exists only a finite number of distinct groups of the form $G_l/A_l$, $l = 1, 2, \dots, k$. If we can show there exist only a finite number of $A_l$, we will have proven our assertion as then the group extensions must also be finite. As a finite abelian subgroup of $O(n)$, $A_l$ is diagonalizable over $\mathbb{C}$. Hence it has a generating set consisting of at most $n$ elements; thus we must show that there are only a finite number of possibilities for the order of an element $g \in O(n)$ which is conjugate in $\mathrm{GL}(n, \mathbb{R})$ to an integer matrix. For this we observe that the coefficients of the characteristic polynomial of $g$ are integers which are elementary symmetric functions in the eigenvalues $e^{2\pi i \lambda}$ of $g$. This completes the proof.

□

From this 3.39 follows the theorem:

**Theorem 3.40** (Jordan–Zassenhaus)
*For any $n$, the number of conjugacy classes of finite subgroups of $\mathrm{GL}(n, \mathbb{Z})$ is finite.*

To finish the proof of the second Bieberbach Theorem let us observe that for any given $G$-module $\mathbb{Z}^n$ the number of short exact sequences

$$0 \longrightarrow \mathbb{Z}^n \longrightarrow \pi_1(M) \longrightarrow F \longrightarrow 0$$

is bounded by (see [12] p.117) the number of elements of the finite group $H^2(G, \mathbb{Z}^n)$. (See 6) This finishes the proof of the second Bieberbach Theorem.

Geometrically, since every compact, flat manifold $M$ arises as

$$M \cong \mathbb{R}^n/\Gamma$$

the Second Bieberbach Theorem implies:

Up to affine diffeomorphism, there are only finitely many compact flat $n$-manifolds.

**Theorem 3.41** (3rd Bieberbach, see [17] and [19])
*Two crystallographic groups of dimension $n$ are isomorphic if and only if they are conjugate in the group $A(n)$.*

*Proof.* Let $h : \Gamma_1 \to \Gamma_2$ be an isomorphism of $n$-dimensional crystallographic group. Let $T = \Gamma_1 \cap (I_n \times \mathbb{R}^n)$. The restriction $h|_T$ to the subgroup of translation defines a linear map $x \mapsto Mx$ where $M \in \mathrm{GL}_n(\mathbb{R})$. Let $(A, a) \in \Gamma_1$ and $h(A, a) = (B, b) \in \Gamma_2$. For any $i = 1, \dots, n$, we have

$$\begin{aligned} & h((A,a)(I_n, e_i)(A,a)^{-1}) = (B,b)(I_n, Me_i)(B,b)^{-1} = (I_n, BMe_i) \\ \text{and} \quad & h((A,a)(I_n, e_i)(A,a)^{-1}) = h(I_n, Ae_i) = (I_n, MAe_i) \end{aligned}$$

Hence we have $MAe_i = BMe_i$ for $i = 1, \dots, n$. Therefore $B = MAM^{-1}$. We can conjugate $h$ by some suitable matrix from $\mathrm{GL}_n(\mathbb{R})$ such that the matrix $M$ will be the identity. In other words, we define $h' : \Gamma_1 \to \Gamma_2$ as $h'(\gamma) = (M, 0)^{-1}h(\gamma)(M, 0)$. Let $h'(A, a) = (A, a_A) \in \Gamma_2$. We claim that there exists $x_0 \in \mathbb{R}^n$ such that

$$h'(\gamma) = (I_n, x_0)^{-1}\gamma(I_n, x_0)^{-1}$$

Define $\tilde{h} : \Gamma_1 \to A(n)$ which maps $(A, a)$ to $(A, a - a_A)$. We claim that $\tilde{h}$ is homomorphism. Let $(A, a), (B, b) \in \Gamma_1$, we have

$$\begin{aligned} h'(AB, a + Ab) &= h'((A,a)(B,b)) \\ &= h'((A,a))h'((B,b)) \\ &= (A, a_A)(B, b_B) \\ &= (AB, a_A + Ab_B) \end{aligned}$$

Hence,

$$\begin{aligned} \tilde{h}((A,a)(B,b)) &= \tilde{h}((AB, a + Ab)) \\ &= (AB, a + Ab - a_{AB}) \\ &= (A, a - a_A)(B, b - b_B) \\ &= \tilde{h}((A,a))\tilde{h}((B,b)) \end{aligned}$$

Hence $\tilde{h}$ is a homomorphism. It is clear that $\ker(\tilde{h}) = \Gamma_1 \cap \mathbb{R}^n$. Notice that $\tilde{h}(\Gamma) \cong \Gamma_1 / \ker \tilde{h} \cong \Gamma_1 / \Gamma_1 \cap (I_n \times \mathbb{R}^n)$ which is a finite group by the first Bieberbach theorem. By Proposition 2.1.18, there is a fixed point $x_0 \in \mathbb{R}^n$ of the action of the finite group $\tilde{h}(\Gamma_1)$. Thus we have

$$x_0 = (A, a - a_A)(x_0) = Ax_0 + a - a_A$$

Hence, $a = x_0 - Ax_0 + a_A$. Finally, for any $x \in \mathbb{R}^n$, we get

$$\begin{aligned}(I_n, x_0)(A, a_A)(I_n, -x_0)x &= (I_n, x_0)(A, a_A - Ax_0)x \\ &= (A, x_0 + a_A - Ax_0)x \\ &= (A, a)x\end{aligned}$$

Thus we have $h'(\gamma) = (I_n, x_0)^{-1}\gamma(I_n, x_0)$. Hence

$$\begin{aligned}h(\gamma) &= (M, 0)h'(\gamma)(M, 0)^{-1} \\ &= (M, 0)(I_n, x_0)^{-1}\gamma(I_n, x_0)(M, 0)^{-1} \\ &= (M, -Mx_0)\gamma(M^{-1}, x_0) \\ &= (M, -Mx)\gamma(M, -Mx)^{-1}\end{aligned}$$

Therefore we completed our proof. □

That is, geometrically, two compact flat $n$–manifolds are affine-equivalent if and only if their fundamental groups are isomorphic.

*Remark* (Lorentzian geometry)
We denote the Minkowski space $\mathbb{R}^{1,n-1}$ by the real vector space $\mathbb{R}^n$ endowed with the Lorentz quadratic form of signature $(n-1, 1)$. We call the Lorentz group $O(n-1, 1)$ the group of automorphisms that preseve Lorentz form.
Let define $E(n-1, 1) = O(n-1, 1) \ltimes \mathbb{R}^n$ the group of Lorentz isometries and we consider its crystallographic subgroups.

A flat Lorentzian manifold (a manifold of special relativity) is a manifold locally modeled on $(E(n-1, 1), \mathbb{R}^{1,n-1})$.
Unlike the Euclidean case, the Lorentz group $O(n-1, 1)$ is not compact so we cannot use the Proposition 3.24 to conclude that a compact, flat Lorentzian manifold is geometrically complete. Although, it is proven recently that such manifolds are orbit spaces of a Lorentzian crystallographic group (i.e. a cocompact, discrete subgroup) that are identitied with their fundamental groups. Goldman and Kamishima [10] proved that these groups are virtually polycyclic, and so we have the Theorem 3.43.

*Definition* 3.42
A Lie group $G$ is said to be solvable if there is a finite sequence of subalgebras $\mathfrak{g}_i$ of $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{g}_0 \supset \mathfrak{g}_1 \cdots \supset \mathfrak{g}_{\mathfrak{m}} = 0$, and $\mathfrak{g}_{i+1}$ is a coabelian ideal in $\mathfrak{g}_i$.

A solvmanifold $M$ is a homogeneous space of a connected solvable Lie group $G$, i.e $M$ is of the form $G/\Gamma$ where $\Gamma$ is a lattice.

**Theorem 3.43** (Goldman and Kamishima [10])
*A compact, flat Lorentzian manifold is a solvmanifold up to finite covering.*

**Theorem 3.44** (Bieberbach's theorem for flat Lorentzian manifolds [6])
*Let $M$ be a compact, $3$-dimensional, flat Lorentzian manifold. There exists a discrete subgroup $\Gamma \subset \mathbb{R}^{1,2}$ such that $M$ is isometric to the orbit space $\mathbb{R}^{1,2}/\Gamma$. Moreover, there exists a $3$-dimensional Lie subgroup $G \subset \mathbb{R}^{1,2}$ that acts simply transitively on $\mathbb{R}^{1,2}$ such that $[G \cap \Gamma : \Gamma]$ is finite and $\Gamma$ is a cocompact lattice in $G$*

# 4 Wallpaper groups

This section is essentially based on [2].
Define $\pi : E(2) \longrightarrow O(2)$ to be the canonical projection. Denote $T = \ker(\pi) \cong \mathbb{R}^2$ the translations subgroup, for a subrgoup $G$, write $H = T \cap G$ its translations group and $J = \pi(G)$ its point group.

*Definition* 4.1 (Wallpaper group)
A subgroup $G < E(2)$ is a wallpaper group if its translations subgroup is generated by two independent elements and its point subgroup is finite.

From now on $G$ will denote a wallpaper group with translation subgroup $H$ and point group $J$. Let $L$ be the orbit of the origin under the action of $H$ on $\mathbb{R}^2$. The set $L$ certainly contains two independent vectors because $H$ is generated by two independent translations. Select a non-zero vector $a$ of minimum length in $L$, then choose a second vector $b$ from $L$ which is skew to $a$ and whose length is as small as possible.

**Theorem 4.2**
*The set $L$ is the lattice spanned by $a$ and $b$. That is to say, $L = m\mathbb{Z} + n\mathbb{Z}$.*

*Proof.* The correspondence $(I, v) \to v$ is an isomorphism between $T$ and the additive group $\mathbb{R}^2$ which sends $H$ to $L$. Therefore $L$ is a subgroup of $\mathbb{R}^2$ and every point $ma + nb$ of the lattice spanned by $a$ and $b$ belongs to $L$. Using the points of this lattice we can divide up the plane into parallelograms. If $x$ belongs to $L$ yet is not in the lattice, choose a parallelogram which contains $x$ and a corner $c$ of this parallelogram which is as close as possible to $x$. Then the vector $x - c$ is not the zero vector, is not equal to $a$ or to $b$, and its length is less than $\|b\|$. But $x - c$ belongs to $L$ because $x$ and $c$ both lie in $L$. We cannot have $\|x - c\| < \|a\|$ since $a$ is supposed to be of minimum length in $L$. On the other hand, if $\|a\| \leq \|x - c\| < \|b\|$, then $x - c$ must be skew to $a$ and contradicts our choice of $b$. Therefore, no such point $x$ can exist and $L$ is the lattice spanned by $a$ and $b$. $\square$

We shall classify lattices into five different types according to the shape of the basic parallelogram determined by the vectors $a$ and $b$. From properties of the lattice of $G$ and the point group of $G$ we plan to build up information about $G$ itself. Replace $b$ by $-b$ if necessary to ensure that

$$\|a - b\| \leq \|a + b\|$$

With this assumption the different lattices are defined as follows:

(a) Oblique $\|a\| < \|b\| < \|a - b\| < \|a + b\|$

(b) Rectangular $\|a\| < \|b\| < \|a - b\| = \|a + b\|$

(c) Centred Rectangular $\|a\| < \|b\| = \|a - b\| < \|a + b\|$

(d) Square $\|a\| = \|b\| < \|a - b\| = \|a + b\|$

(e) Hexagonal $\|a\| = \|b\| = \|a - b\| < \|a + b\|$

*Remark*

Wallpaper groups occur in the literature under a variety of aliases, the most common being "plane crystallographic group".
If we imagine a similar scenario in three dimensions, the corresponding lattice is spanned by three independent vectors and gives a configuration of points which models the internal atomic structure found in crystals.

The point group $J$ is a subgroup of $O_2$. However, there may be no copy of $J$ inside $G$. Consider the wallpaper group which is generated by the translation $\tau(x, y) = (x + 1, y)$ and the glide reflection $h(x, y) = (-x, y + 1)$. The line of the glide is the $y$-axis and is perpendicular to the direction of the translation. Here the point group is the subgroup

$$J = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

of $O_2$. The group $G$, however, consists entirely of translations and glide reflections, all of which have infinite order. Therefore, $G$ cannot contain a copy of $J$. Carrying out the glide reflection $h$ twice gives the translation $(x, y) \to (x, y + 2)$ and the lattice in this example is spanned by the vectors $a = (1, 0)$, $b = (0, 2)$. Observe that not all the elements of $G$ send this lattice to itself. None the less the point group does preserve the lattice.

**Theorem 4.3**

*The point group $J$ acts on the lattice $L$.*

*Proof.* The point group, being a subgroup of $O_2$, acts on the plane in the usual way. If $M \in J$, and if $x \in L$, we must show that $f_M(x) \doteq xM^t$ belongs to $L$. Suppose $\pi(g) = M$ where $g = (M, v)$ and let $\tau$ denote the translation $(I, x)$. Since $H$ is the kernel of the homomorphism $\pi : G \to J$, it is a normal subgroup of $G$, and therefore $g\tau g^{-1}$ lies in $H$. But

$$\begin{aligned} g\tau g^{-1} &= (M, v)(I, x)\left(M^{-1}, -f_M^{-1}(v)\right) \\ &= (M, v)\left(M^{-1}, x - f_M^{-1}(v)\right) \\ &= \left(MM^{-1}, v + f_M\left(x - f_M^{-1}(v)\right)\right) \\ &= (I, v + f_M(x) - v) \\ &= (I, f_M(x)) \end{aligned}$$

consequently, $f_M(x)$ is a point of the lattice $L$ as required. $\square$

We recall from that finite subgroups of $O_2$ are either cyclic or dihedral. The next result tells us which of these subgroups can conceivably arise as the point group of a wallpaper group. It is referred to as the "crystallographic restriction".

**Theorem 4.4**
*The order of a rotation in a wallpaper group can only be 2, 3, 4, or 6.*

*Proof.* Every rotation in a wallpaper group $G$ has finite order because the point group is finite. If we have a rotation of order $q$, then a suitable power of this rotation is an anti-clockwise rotation through $2\pi/q$. Therefore the rotation matrix

$$A = \begin{bmatrix} \cos\left(\frac{2\pi}{q}\right) & -\sin\left(\frac{2\pi}{q}\right) \\ \sin\left(\frac{2\pi}{q}\right) & \cos\left(\frac{2\pi}{q}\right) \end{bmatrix}$$

belongs to $J$. As before, we use $a$ to denote a non-zero vector of shortest length in the lattice $L$ of $G$. Now $J$ acts on $L$, so $f_A(a)$ lies in $L$. Suppose $q$ is greater than 6. Then $2\pi/q$ is less than $\pi/3$ and $f_A(a) - a$ is a vector in $L$ which is shorter than $a$, contradicting our choice of $a$. If $q$ is equal to five the angle between $f_A^2(a)$ and $-a$ is $\pi/5$. This time $f_A^2(a) + a$ lies in $L$ and is shorter than $a$, and again we have a contradiction. □

From this theorem follows:

**Corollary 4.5**
*The point group of a wallpaper group is generated by a rotation through one of the angles* $0, \pi, 2\pi/3, \pi/2, \pi/3$ *and possibly a reflection.*

**Theorem 4.6**
*An isomorphism between wallpaper groups takes translations to translations, rotations to rotations, reflections to reflections and glide reflections to glide reflections.*

*Proof.* Let $\varphi : G \to G_1$ be an isomorphism between wallpaper groups, and let $\tau$ be a translation in $G$. Translations and glides have infinite order, whereas rotations and reflections are of finite order; therefore, $\varphi(\tau)$ must be either a translation or a glide. Assume $\varphi(\tau)$ is a glide and choose a translation $\tau_1$ from $G_1$ which does not commute with $\varphi(\tau)$. (Any translation whose direction is not parallel to the line of the glide will do.) If $\varphi(g) = \tau_1$, then $g$ has to be a translation or a glide. So $g^2$ is a translation, and hence commutes with $\tau$, contradicting the fact that $\varphi\left(g^2\right) = \tau_1^2$ does not commute with $\varphi(\tau)$. Therefore, translations correspond to translations and glides to glides.

Reflections have order 2, consequently the image of a reflection under an isomorphism is either a reflection or a half-turn. Let $g \in G$ be a reflection whose image $\varphi(g)$ is a half-turn, and choose a translation $\tau$ from $G$ in a direction which is not perpendicular to the mirror of $g$. Then $\tau g$ is a glide. But $\varphi(\tau g) = \varphi(\tau)\varphi(g)$ is the product of a translation and a half-turn, which is another half-turn. Therefore, we have a contradiction and reflections must correspond to reflections. Finally, rotations are now forced to correspond to rotations. □

**Corollary 4.7**
*If two wallpaper groups are isomorphic then their point groups are also isomorphic.*

*Proof.* Let $G, G_1$ be wallpaper groups with translation subgroups $H, H_1$ and point groups $J, J_1$ respectively. If $\varphi : G \to G_1$ is an isomorphism we have $\varphi(H) = H_1$. Therefore, $\varphi$ induces an isomorphism from $G/H$ to $G_1/H_1$. The result now follows because $J$ is isomorphic to $G/H$ and $J_1$ is isomorphic to $G_1/H_1$. □

*Remark*

There are seventeen different wallpaper groups. To see why, we shall examine each of the five possible types of lattice in turn. Given a lattice $L$ we first work out which orthogonal transformations preserve $L$. Such transformations form a group and, the point group of any wallpaper group which has $L$ as its lattice must be a subgroup of this group. This limitation on the point group is then sufficient to allow us to enumerate the different wallpaper groups with lattice $L$. An exhaustive analysis of every case would take up too much space. So we concentrate our attention on a small number of examples, and defer the remaining calculations to the exercises. That all the groups we find are genuinely different, in other words that no two are isomorphic, will be shown at the end of the chapter.

Before beginning the classification we add a word or two about notation. Each wallpaper group has a name made up of several (internationally recognised) symbols $p$, $c$, $m$, $g$ and the integers $1, 2, 3, 4, 6$. The letter $p$ refers to the lattice and stands for the word primitive. When we view a lattice as being made up of primitive cells (copies of the basic parallelogram which do not contain any lattice points in their interiors) we call it a primitive lattice. In one case (the centred rectangular lattice) we take a non-primitive cell together with its centre as the basic building block, and use the letter $c$ to denote the resulting centred lattice. The symbol for a reflection is $m$ (for mirror) and $g$ denotes a glide reflection. Finally, $1$ is used for the identity transformation and the numbers $2, 3, 4, 6$ indicate rotations of the corresponding order. Rotations of order two are usually called half turns.

We show the centres of rotations and the positions of mirrors and glide lines relative to a basic parallelogram. The symbols $\circ, \Delta, \square, \bullet$ mean that the stabilizer of the corresponding point is cyclic of order two, three, four, or six, respectively. Mirrors are drawn as thick lines and glides are indicated by broken lines.

We now proceed with our case-by-case analysis. As usual, $G$ is a wallpaper group with translation subgroup $H$, point group $J$, and lattice $L$. Vectors $a$ and $b$ which span the lattice are selected as previously. There is no harm in assuming that $a$ lies along the positive $x$-axis and that $b$ is in the first quadrant. Finally, $A_\theta$ is the matrix which represents an anticlockwise rotation of $\theta$ about the origin, while $B_\varphi$ represents reflection in the line through the origin which subtends an angle of $\varphi/2$ with the positive $x$-axis.

**Case (a)**

The lattice of $G$ is oblique. Then the only orthogonal transformations which preserve $L$ are the identity and rotation through $\pi$ about the origin. Therefore, the point group of $G$ is a subgroup of $\{\pm I\}$.

(p1) If $J$ only contains the identity, then $G$ is the simplest of all wallpaper groups; that

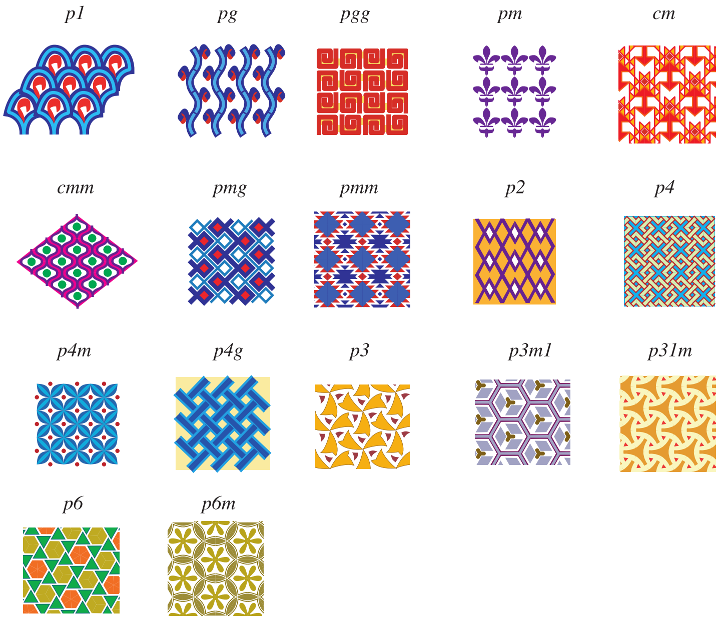


Figure 1: Wallpaper groups corresponding to the symmetries of each tile [20]

generated by two independent translations. Its elements have the form $(I, ma + nb)$, where $m, n \in \mathbb{Z}$.

(p2) Here $J$ is $\{\pm I\}$. Therefore, $G$ contains a half turn, and we may as well take the fixed point of this half turn as origin, so that $(-I, 0)$ belongs to $G$. The union of the two right cosets $H$ and $H(-I, 0)$ is a subgroup of $E_2$ which must be our group $G$. Those elements of $G$ which are not translations lie in $H(-I, 0)$ and have the form

$$(I, ma + nb)(-I, 0) = (-I, ma + nb)$$

where $m, n \in \mathbb{Z}$. In other words, we have all the half turns about the points $\frac{1}{2}ma + \frac{1}{2}nb$.

**Case (b)**

The lattice of $G$ is rectangular. There are now four orthogonal transformations which preserve $L$; namely, the identity, a half turn about $0$, reflection in the $x$-axis, and reflection in the $y$-axis. Therefore, the point group of $G$ is a subgroup of $\{I, -I, B_0, B_\pi\}$. We look for wallpaper groups which we have not seen before, ignoring the possibilities p1, p2 found above.

(pm) $J$ is $\{I, B_0\}$ and $G$ contains a reflection in a horizontal mirror.

(pg) Suppose $J$ is $\{I, B_0\}$, yet there are no reflections in $G$. Then $G$ has to contain a glide reflection whose line is horizontal, and we choose a point of this line as origin. Applying a glide reflection twice gives a translation, hence our glide has the form $\left(B_0, \frac{1}{2}ka\right)$ for some integer $k$. If $k$ is even, then $\left(I, -\frac{1}{2}ka\right)$ is a translation in $G$, and the reflection

$$(B_0, 0) = (I, -\frac{1}{2}ka)(B_0, \frac{1}{2}ka)$$

belongs to $G$, contradicting our initial assumption. Therefore, $k$ is odd and

$$(B_0, \frac{1}{2}a) = (I, -\frac{1}{2}(k-1)a)(B_0, \frac{1}{2}ka)$$

lies in $G$. The elements of $G$ which are not translations have the form

$$(I, ma + nb)(B_0, \frac{1}{2}a) = (B_0, \left(m + \frac{1}{2}\right) a + nb)$$

where $m, n \in \mathbb{Z}$. These are all glides along horizontal lines which either pass through lattice points or lie midway between lattice points. The length of each glide is an odd multiple of $\frac{1}{2}a$.

Taking $\{I, B_\pi\}$ as point group instead of $\{I, B_0\}$ is tantamount to interchanging the roles of "horizontal" and "vertical" in the preceding discussion, and does not lead to anything new. From now on we assume that the point group is all of $\{I, -I, B_0, B_\pi\}$. There are three possibilities according as both, just one, or neither of $B_0, B_\pi$ can be realised by reflections in $G$.

(p2mm) In this case $G$ contains a reflection in a horizontal mirror and a reflection in a vertical mirror.

(p2mg) Suppose $G$ contains a reflection in a horizontal mirror but does not contain a reflection in a vertical mirror. Then $B_\pi$ must be realised in $G$ by a vertical glide reflection. A judicious choice of origin, at the intersection of the horizontal mirror and the vertical glide line, plus the argument used for pg, allow us to assume that $(B_0, 0)$ and $(B_\pi, \frac{1}{2}b)$ lie in $G$. The product

$$(B_\pi, \frac{1}{2}b)(B_0, 0) = (-I, \frac{1}{2}b)$$

is the half turn about $\frac{1}{4}b$. The right cosets

$$H, \quad H(B_0, 0), \quad H(B_\pi, \frac{1}{2}b), \quad H(-I, \frac{1}{2}b)$$

fill out $G$. In the first of these we have the translations. A typical element of the second has the form

$$(I, ma + nb)(B_0, 0) = (B_0, ma + nb)$$

where $m, n \in \mathbb{Z}$. When $m = 0$, this isometry is reflection in a horizontal mirror which either passes through lattice points or lies midway between them. If $m$ is not zero, the mirrors change to glide lines and the translation part of the glide is $ma$. The third coset contains the elements

$$(B_\pi, ma + \left(n + \frac{1}{2}\right) b)$$

which are all vertical glides whose lines pass through lattice points or lie midway between them. The translation part of each of these glides is an odd multiple of $\frac{1}{2}b$. Finally, $H(-I, \frac{1}{2}b)$ consists of the half turns centred at the points $\frac{1}{2}ma + \frac{1}{2}\left(n + \frac{1}{2}\right) b$.

Interchanging horizontal and vertical in the preceding discussion leads to a group which is isomorphic to p2mg.

(p2gg) Here there are no reflections in $G$.

**Case (c)**

The lattice of $G$ is centred rectangular. The orthogonal transformations which preserve $L$ are the same as in the rectangular case. Therefore, the point group must again be a subgroup of $\{I, -I, B_0, B_\pi\}$. We discover two new groups.

(cm) Suppose $J$ is $\{I, B_0\}$ and that $(B_0, v)$ realises $B_0$ in $G$. This isometry is either a reflection in a horizontal mirror or a glide along a horizontal line. Choose a point on the mirror or glide line as origin, so that $2v$ is a multiple of $a$, and remember that the vertical direction is determined by the vector $2b - a$.

(i) If $2v = ka$ and $k$ is even, the reflection

$$(B_0, 0) = (I, -\frac{1}{2}ka)(B_0, \frac{1}{2}ka)$$

belongs to $G$. The elements of $G$ which are not translations have the form

$$(B_0, ma + nb) = \left(B_0, \left(m + \frac{1}{2}n\right) a + \frac{1}{2}n(2b - a)\right)$$

where $m, n \in \mathbb{Z}$. Taking $n$ to be even and $m = -\frac{1}{2}n$ produces all the reflections in horizontal mirrors which pass through lattice points. If $n$ is even but $m \neq -\frac{1}{2}n$, these mirrors change to glide lines, the translation part of each glide being a multiple of $a$. Finally, if $n$ is odd, we have glides along lines which lie midway between lattice points. The translation part of each of these glides is an odd multiple of $\frac{1}{2}a$.

(ii) If $k$ is odd, then

$$\left(B_0, \frac{1}{2}(2b-a)\right) = \left(I, -\frac{1}{2}(k+1)a + b\right)\left(B_0, \frac{1}{2}ka\right)$$

lies in $G$. This is again a reflection and shifting the origin onto its mirror leads back to the previous case.

Substituting $\{I, B_\pi\}$ as point group instead of $\{I, B_0\}$ leads to a group which is isomorphic to cm.

(c2mm) $J$ is $\{I, -I, B_0, B_\pi\}$. The type of calculation carried out above shows that both $B_0$ and $B_\pi$ can be realised by reflections in $G$.

**Case (d)**

The lattice of $G$ is square. Then the group of orthogonal transformations which preserves $L$ is the dihedral group of order 8 generated by $A_{\frac{\pi}{2}}$ and $B_0$. The point group $J$ is a subgroup of this group and, to obtain something new, we must include $A_{\frac{\pi}{2}}$ in $J$. (The other cases are dealt with in Exercise 26.9.)

(p4) Here $J$ is generated by $A_{\frac{\pi}{2}}$.

(p4mm) $J$ is generated by $A_{\frac{\pi}{2}}$ and $B_0$, and $B_0$ can be realised by a reflection in $G$.

(p4gm) Suppose $J$ is generated by $A_{\frac{\pi}{2}}$ and $B_0$, but $B_0$ cannot be realised by a reflection in $G$. Choose the fixed point of a rotation of order 4 as origin, so that $(A_{\frac{\pi}{2}}, 0)$ belongs to $G$, and let $(B_0, \lambda a + \mu b)$ realise $B_0$ in $G$. Squaring $(B_0, \lambda a + \mu b)$ gives $(I, 2\lambda a)$, so $2\lambda$ is an integer. If $2\lambda$ is even the reflection

$$(B_0, \mu b) = (I, -\lambda a)(B_0, \lambda a + \mu b)$$

lies in $G$ and we have a contradiction. Therefore, $2\lambda$ must be odd and

$$\left(B_0, \frac{1}{2}a + \mu b\right) = \left(I, \left(\frac{1}{2} - \lambda\right)a\right)(B_0, \lambda a + \mu b)$$

is an element of $G$. Also

$$(A_{\frac{\pi}{2}}, 0)\left(B_0, \frac{1}{2}a + \mu b\right) = \left(B_{\frac{\pi}{2}}, \frac{1}{2}b - \mu a\right)$$

and

$$\left(B_{\frac{\pi}{2}}, \frac{1}{2}b - \mu a\right)^2 = \left(I, \left(\frac{1}{2} - \mu\right)(a+b)\right)$$

showing $\frac{1}{2} - \mu$ to be an integer. We conclude that the glide

$$\left(B_0, \frac{1}{2}a + \frac{1}{2}b\right) = \left(I, \left(\frac{1}{2} - \mu\right) b\right)\left(B_0, \frac{1}{2}a + \mu b\right)$$

belongs to $G$. The right cosets

$$\begin{array}{ll} H(I,0), & H\left(A_{\frac{\pi}{2}}, 0\right) \\ H(-I,0), & H\left(A_{\frac{3\pi}{2}}, 0\right) \\ H\left(B_0, \frac{1}{2}a + \frac{1}{2}b\right) & H\left(B_{\frac{\pi}{2}}, \frac{1}{2}a + \frac{1}{2}b\right) \\ H\left(B_\pi, \frac{1}{2}a + \frac{1}{2}b\right) & H\left(B_{\frac{3\pi}{2}}, \frac{1}{2}a + \frac{1}{2}b\right) \end{array}$$

fill out $G$, and it is easy to recognise their elements geometrically. For example, a typical member of $H\left(B_{\frac{\pi}{2}}, \frac{1}{2}a + \frac{1}{2}b\right)$ has the form

$$\left(B_{\frac{\pi}{2}}, \left(m + \frac{1}{2}\right) a + \left(n + \frac{1}{2}\right) b\right)$$

where $m, n \in \mathbb{Z}$. Taking $m + n + 1 = 0$ gives all the reflections in mirrors tilted at $45^\circ$ to the horizontal which pass midway between lattice points. When $m+n+1$ is non-zero and $m-n$ is odd, these mirrors change to glide lines. Finally, if $m+n+1$ is non-zero and $m-n$ is even, we have glides along lines of gradient one which pass through lattice points. The coset $H(-I,0)$ on the other hand contains all the half turns

$$(-I, ma + nb)$$

centered at the points $\frac{1}{2}ma + \frac{1}{2}nb$. We leave the reader to work through the remaining cases.

### Case (e)

The lattice of $G$ is hexagonal. Then the point group must be contained in the dihedral group of order 12 generated by $A_{\frac{\pi}{3}}$ and $B_0$. We are led to new wallpaper groups when $J$ contains rotations of order 3 or 6. (The other cases are dealt with in Exercise 26.10.)

(p3) $J$ is generated by $A_{\frac{2\pi}{3}}$.

(p3m1) $J$ is generated by $A_{\frac{2\pi}{3}}$ and $B_0$.

(p31m) Suppose $J$ is generated by $A_{\frac{2\pi}{3}}$ and $B_{\frac{\pi}{3}}$. Choose the fixed point of a rotation of order 3 as origin, so that $(A_{\frac{2\pi}{3}}, 0)$ belongs to $G$, and let $(B_{\frac{\pi}{3}}, \lambda a + \mu b)$ realise $B_{\frac{\pi}{3}}$ in $G$. Now

$$\left(B_{\frac{\pi}{3}}, \lambda a + \mu b\right)^2 = ((\lambda + \mu)(a + b), I)$$

so $\lambda + \mu$ is an integer. Also

$$\left(A_{\frac{2\pi}{3}}, 0\right)\left(B_{\frac{\pi}{3}}, \lambda a + \mu b\right) = (B_\pi, \lambda(b - a) - \mu a)$$

and

$$(B_\pi, \lambda(b - a) - \mu a)^2 = (I, \lambda(2b - a))$$

showing that $\lambda$ is an integer. Therefore, both $\lambda$ and $\mu$ are integers and the reflection

$$\left(B_{\frac{\pi}{3}}, 0\right) = (I, -\lambda a - \mu b)\left(B_{\frac{\pi}{3}}, \lambda a + \mu b\right)$$

belongs to $G$. The elements of $G$ have the form $(M, ma + nb)$ where $m, n \in \mathbb{Z}$ and $M$ is one of the matrices $I, A_{\frac{2\pi}{3}}, A_{\frac{4\pi}{3}}, B_{\frac{\pi}{3}}, B_\pi, B_{\frac{2\pi}{3}}$. We ask the reader to interpret these elements geometrically. For example

$$(B_\pi, ma + nb) = \left(B_\pi, \left(m + \frac{1}{2}n\right) a + \frac{1}{2}n(2b - a)\right)$$

is a reflection in a vertical mirror when $n = 0$ and a vertical glide otherwise.

(p6) $J$ is generated by $A_{\frac{\pi}{3}}$.

(p6mm) $J$ is generated by $A_{\frac{\pi}{3}}$ and $B_0$.

Are all these seventeen groups genuinely different? The answer is yes, as we shall see below. By Corollary 4.7 we need only concern ourselves with groups whose point groups are isomorphic, so we begin with a summary of the point groups.

Remember that an isomorphism between wallpaper groups sends translations to translations, rotations to rotations, reflections to reflections, and glides to glides.

**Theorem 4.8**
*No two of p2, pm, pg, cm are isomorphic.*

| $G$ | $J$ | $G$ | $J$ |
|---|---|---|---|
| p1 | trivial | p4 | $\mathbb{Z}_4$ |
| p2 | $\mathbb{Z}_2$ | p4mm | $D_4$ |
| pm | $\mathbb{Z}_2$ | p4gm | $D_4$ |
| pg | $\mathbb{Z}_2$ | p3 | $\mathbb{Z}_3$ |
| p2mm | $\mathbb{Z}_2 \times \mathbb{Z}_2$ | p3m1 | $D_3$ |
| p2mg | $\mathbb{Z}_2 \times \mathbb{Z}_2$ | p31m | $D_3$ |
| p2gg | $\mathbb{Z}_2 \times \mathbb{Z}_2$ | p6 | $\mathbb{Z}_6$ |
| cm | $\mathbb{Z}_2$ | p6mm | $D_6$ |
| c2mm | $\mathbb{Z}_2 \times \mathbb{Z}_2$ | | |

*Proof.* Among these only p2 contains rotations, so it cannot be isomorphic to any of the others. Of the three remaining groups, pg is the only one which does not contain a reflection; consequently, pg is not isomorphic to pm or cm. Finally, we note that if we take a glide in pm and write it as a reflection followed by a translation, then both the reflection and the translation belong to pm. However, cm contains glides whose constituent parts do not lie in cm. For example, consider the glide

$$\left(B_0, \frac{1}{2}a + \frac{1}{2}(2b-a)\right) = \left(I, \frac{1}{2}a\right)\left(B_0, \frac{1}{2}(2b-a)\right)$$

Therefore, pm is not isomorphic to cm. □

**Theorem 4.9**
*No two of p2mm, p2mg, p2gg, c2mm are isomorphic.*

*Proof.* Among these, p2gg is the only one which does not contain a reflection, so it cannot be isomorphic to any of the others. Of the three remaining groups, only p2mm contains the constituent parts of each of its glides, consequently p2mm is not isomorphic to p2mg or c2mm. Finally, we note that the mirrors of all the reflections in p2mg are horizontal, so the product of two reflections is always a translation. But in c2mm there are reflections with horizontal mirrors and reflections with vertical mirrors, and the product of one of each is a half turn. Therefore, p2mg is not isomorphic to c2mm. □

**Theorem 4.10**
*p4mm is not isomorphic to p4gm.*

*Proof.* Each rotation of order 4 in p4mm can be written as the product of two reflections which both belong to p4mm. The corresponding statement is not true for p4gm. For example, $(A_{\frac{\pi}{2}}, a)$ cannot be factorised in p4gm as the product of two reflections. Therefore, p4mm is not isomorphic to p4gm. □

**Theorem 4.11**
*p3m1 is not isomorphic to p31m.*

*Proof.* In p31m each rotation of order 3 can be written as the product of two reflections, but this is not the case in p3m1. For example, $(A_{\frac{2\pi}{3}}, a)$ cannot be factorised in p3m1 as the product of two reflections. Therefore, p31m is not isomorphic to p3m1. $\square$

This completes our classification of wallpaper groups. We have adopted a "hands on" approach, and deliberately so, only by working out the elements of these groups do we gain any understanding of their structure.

# 5 Orbifolds

Orbifolds were introduced by Thurston in [19] and they are essential in the study of wallpaper groups. In this section we shall give a short introduction to orbifolds and classify wallpaper groups using this theory:

*Definition* 5.1 (Orbifold)
An orbifold is a Hausdorff topological space $X$ with a collection of open sets $\{U_j\}$ closed under finite intersections such that:

- $\bigcup_j U_j = X$.
- For every $U_j$ there is a homeomorphism $\varphi_j : U_j \to \tilde{U}_j/\Gamma_j$ where $\tilde{U}_j$ is a neighbourhood of $\mathbb{R}^n$ and $\Gamma_j$ is a finite group acting on $\tilde{U}_j$.
- For every $U_i \subset U_j$ there is an injective homomorphism $\pi_{ij} : \Gamma_i \to \Gamma_j$ and an embedding $\tilde{\varphi}_{ij} : \tilde{U}_i \to \tilde{U}_j$ such that:
  - $\tilde{\varphi}_{ij}(\gamma x) = \pi_{ij}(\gamma)\tilde{\varphi}_{ij}(x), \gamma \in \Gamma_i$.
  - The following diagram commutes:

$$\begin{tikzcd}
\tilde{U}_i \arrow[r, "\tilde{\varphi}_{ij}"] \arrow[d] & \tilde{U}_j \arrow[d] \\
\tilde{U}_i/\Gamma_i \arrow[r, "\varphi_{ij}=\tilde{\varphi}_{ij}/\Gamma_i"] & \tilde{U}_j/\Gamma_i \arrow[d, "f_{ij}"] \\
 & \tilde{U}_j/\Gamma_j \\
U_i \arrow[uu, "\varphi_i"] & U_j \arrow[u, "\varphi_j"]
\end{tikzcd}$$

*Example* 5.2
A simple way of constructing orbifolds is by taking certain quotients of smooth actions.

**Theorem 5.3**
*Let $M$ be a differentiable manifold and $\Gamma$ a group acting properly discontinuously and smoothly on $M$, then $M/\Gamma$ has an orbifold structure.*

*Remark*
Recall that an action of a group $G$ is properly discontinuous if for every $x$ in $M$ there is a neighbourhood $U$ of $x$ such that $g(U) \cap U \neq \emptyset$ for only a finite number of $g \in G$.

*Proof.* Let $x \in M/\Gamma$ and $\tilde{x} \in M$ which projects to $x$. Let $I_{\tilde{x}}$ be the isotropy group of $\tilde{x}$. We'll start by finding a neighbourhood of $\tilde{x}$ which is disjoint from its translates by elements not in $I_{\tilde{x}}$ and invariant by elements in $I_{\tilde{x}}$.

As the action is properly discontinuous we pick a neighbourhood $\tilde{V}$ of $\tilde{x}$ such that $\gamma(\tilde{V}) \cap \tilde{V} \neq \emptyset$ for only a finite number of elements in $\Gamma$. Let $\{\gamma_1, \ldots, \gamma_n\}$ be those elements which are not in $I_{\tilde{x}}$. Notice that this also implies that $I_{\tilde{x}}$ is finite.

For each $\gamma_j$, let $V_1$ and $V_2$ be open sets such that $\tilde{x} \in V_1, \gamma_j \cdot \tilde{x} \in V_2$ and $V_1 \cap V_2 = \emptyset$ (they exist because $M$ is Hausdorff). Let $W_j = \tilde{V} \cap V_1 \cap \gamma_j^{-1}(V_2)$ be a neighbourhood of $\tilde{x}$. We can easily check that $W_j \cap \gamma_j(W_j) = \emptyset$. Taking $W = \bigcap W_j$ we get a neighbourhood of $\tilde{x}$ such that $W \cap \gamma(W) \neq \emptyset$ if and only if $\gamma \in I_{\tilde{x}}$. On the other hand, if $\sigma \in \Gamma$ then $\sigma(\tilde{U}) = \bigcap_{\gamma \in I_{\tilde{x}}} \sigma(\gamma(W)) = \bigcap_{\gamma \in I_{\tilde{x}}} (\sigma \cdot \gamma)(W) = \bigcap_{\gamma \in I_{\tilde{x}}} \gamma(W)$. We have found a neighbourhood of $\tilde{x}$ with the desired properties. We can suppose that $\tilde{U}$ is contained in some coordinate chart, thus it homeomorphic to some open set in $\mathbb{R}^n$.

Let $Z = \cup_{\gamma \in \Gamma} \gamma(\tilde{U})$ and let $U_x = Z/\Gamma$. By restricting this projection to $\tilde{U}$ we get a homeomorphism between $U_x$ and $\tilde{U}/I_{\tilde{x}}$ where the action of $I_{\tilde{x}}$ is the restriction of the action of $\Gamma$ on $\tilde{U}$. We'll show that $U_x$ and its finite intersections form a cover of $M/\Gamma$.

Let $U_{x_1}, \ldots, U_{x_k}$ such that $U_{x_1} \cap \ldots \cap U_{x_k} \neq \emptyset$. This means that there are $\gamma_1, \ldots, \gamma_k \in \Gamma$ such that $\gamma_1\left(\tilde{U}_{x_1}\right) \cap \ldots \cap \gamma_k\left(\tilde{U}_{x_k}\right) \neq \emptyset$.

Consider the following subgroup of $\Gamma$, $G = \gamma_1 I_{x_1} \gamma_1^{-1} \cap \ldots \cap \gamma_k I_{x_k} \gamma_k^{-1}$.

Let $g \in G$, then $g = \gamma_i \sigma_i \gamma_i^{-1}$ where $\sigma_i \in I_{x_i}$. So,

$$
\begin{aligned}
& g\left(\gamma_1\left(\tilde{U}_{x_1}\right) \cap \ldots \cap \gamma_k\left(\tilde{U}_{x_k}\right)\right) = (g \cdot \gamma_1)\left(\tilde{U}_{x_1}\right) \cap \ldots \cap (g \cdot \gamma_k)\left(\tilde{U}_{x_k}\right) \\
& = \left(\gamma_1 \cdot \sigma_1 \cdot \gamma_1^{-1} \cdot \gamma_1\right)\left(\tilde{U}_{x_1}\right) \cap \ldots \cap \left(\gamma_k \cdot \sigma_k \cdot \gamma_k^{-1} \cdot \gamma_1\right)\left(\tilde{U}_{x_k}\right) \\
& = \gamma_1\left(\tilde{U}_{x_1}\right) \cap \ldots \cap \gamma_k\left(\tilde{U}_{x_k}\right).
\end{aligned}
$$

By similar calculations we can show that for $g \notin G$, we have

$$
\left(\gamma_1\left(\tilde{U}_{x_1}\right) \cap \ldots \cap \gamma_k\left(\tilde{U}_{x_k}\right)\right) \cap g\left(\gamma_1\left(\tilde{U}_{x_1}\right) \cap \ldots \cap \gamma_k\left(\tilde{U}_{x_k}\right)\right) = \emptyset
$$

thus $\gamma_1\left(\tilde{U}_{x_1}\right) \cap \ldots \cap \gamma_k\left(\tilde{U}_{x_k}\right)/G$ is homeomorphic to $U_{x_1} \cap \ldots \cap U_{x_k}$ with the homeomorphism given by the projection. □

*Example* 5.4
Every quotient of a differentiable manifold by a smooth action of a finite group.

*Definition* 5.5 (Singular point)
For every $x \in M$, let $\Gamma_x$ be the smallest group such that there is a neighbourhood of $x$, $U = \tilde{U}/\Gamma_x$. A point $x$ is called singular if $\Gamma_x$ is not the trivial group. The group $\Gamma_x$ is called the isotropy group of $x$.

*Definition* 5.6 (Covering of orbifolds)
A covering orbifold of an orbifold $O$ is an orbifold $\tilde{O}$ with a projection $p : X_{\tilde{O}} \to X_O$ between the underlying spaces with the following property: for every $x \in X_O$ there is a neighbourhood $U \cong \tilde{U}/\Gamma$ ($\tilde{U}$ is an open set in $\mathbb{R}^n$) such that each connected component of $p^{-1}(U)$ is homeomorphic to $\tilde{U}/\Gamma_i$ for some subgroup $\Gamma_i$ of $\Gamma$.

In addition, the homeomorphism has to respect both projection ($p$ and the canonic one between $\tilde{U}/\Gamma_i$ and $\tilde{U}/\Gamma$).

*Definition* 5.7 (Good orbifold)
An orbifold is good if it is covered by a manifold. Otherwise, it is a bad orbifold.

*Example* 5.8
Every manifold is trivially a good orbifold as it is covered by itself.

In a similar way to the case of topological spaces we can define the universal cover of an orbifold. As we don't have defined here paths in orbifold (although that is also possible!) we'll rely on the universal property to define universal covers:

*Definition* 5.9 (Universal cover of an orbifold)
A connected orbifold $\tilde{O}$ is a universal cover of $O$ if there is a projection $p : X_{\tilde{O}} \to X_O$ with non-singular base points $\tilde{x}$ and $x = p(\tilde{x})$ which has the universal property (i.e. if $p' : X_{O'} \to X_O$ is a covering of orbifold with $x = p'(x')$ then there is a covering $q : X_{\tilde{O}} \to X_{O'}$ such that $q(\tilde{x}) = x'$ and $q \circ p' = p$).

**Theorem 5.10** (Universal cover, see [18])
*Every connected orbifold $O$ has a universal cover.*

## 5.1 Wallpaper patterns classification

In this section we shall classify the 17 wallpaper patterns by making use of our knowledge about orbifolds. Notice that for each wallpaper pattern we can construct an orbifold, the quotient of $\mathbb{R}^2$ by the discrete group of symmetries of the pattern. As this group contains both a vertical and a horizontal translation the constructed orbifold is compact.

We'll start by noting that the singular points belong to one of three classes.

**Theorem 5.11**
*Every singular point of a 2-orbifold has its neighbourhood modeled by one of these classes:*

- *Mirror:* $\mathbb{R}^2/\mathbb{Z}_2$ *where* $\mathbb{Z}_2$ *acts by reflection in the* $x$*-axis.*
- *Elliptic point:* $\mathbb{R}^2/\mathbb{Z}_n$ *where* $\mathbb{Z}_n$ *acts by rotations.*

- *Corner reflector:* $\mathbb{R}^2/D_n$ *where* $D_n$ *is the dihedral group of order* $2n$*, which is generated by reflections in the lines which meet the* $x$*-axis at angle* $\frac{2k\pi}{n}$.

*Proof.* Let $O$ be a 2-orbifold and $x \in O$ a singular point. Let $U$ be a neighbourhood of $x$ which is diffeomorphic to $\tilde{U}/\Gamma$, $\tilde{U} \subset \mathbb{R}^2$. We can choose $U$ small enough such that $\tilde{x}$ is the only fixed point of the action of $\Gamma$ in $\tilde{U}$.

Let $g$ be a Riemannian metric on $\tilde{U}$, it can be the usual Euclidean metric. We construct a metric $g'$, invariant by the action of $\Gamma$, by averaging it under $\Gamma$:

$$g'(v,w) = \sum_{\gamma\in\Gamma} g(D\gamma \cdot v, D\gamma \cdot w)$$

Indeed,

$$\sigma_* g'(v,w) = \sum_{\gamma\in\Gamma} \sigma_* g(D\gamma{\cdot}v, D\gamma{\cdot}w) = \sum_{\gamma\in\Gamma} g(D(\sigma{\cdot}\gamma){\cdot}v, D(\sigma{\cdot}\gamma){\cdot}w) = \sum_{\gamma\in\Gamma} g(D\gamma{\cdot}v, D\gamma{\cdot}w)$$

for some $\sigma \in \Gamma$.

Note that we can define action of $\Gamma$ in $T_{\tilde{x}}\tilde{U}$ by considering the function $D\sigma : T_{\tilde{x}}\tilde{U} \to T_{\tilde{x}}\tilde{U}$. They form a group of isometries for the metric $g'_{\tilde{x}}$.

Let $\sigma \in \Gamma$, $v \in T_{\tilde{x}}\tilde{U}$ and $f$ a geodesic such that $\dot{f}(0) = v$. As $\sigma : \tilde{U} \to \tilde{U}$ is an isometry then $\sigma \circ f$ is also a geodesic and $(\sigma \circ f)(0) = D\sigma \cdot v$. This means that $\sigma\left(\exp_{\tilde{x}}(v)\right) = \sigma(f(1)) = (\sigma \circ f)(1) = \exp_{\tilde{x}}(D\sigma \cdot v)$. In conclusion, the actions of $\Gamma$ commute with the diffeomorphism given by the exponential map. This gives a diffeomorphism between a neighbourhood of $x$ in $O$ and $V/\Gamma$ where $V$ is a neighbourhood of the origin of $\mathbb{R}^2$. As $\Gamma$ acts by isometries then $\Gamma$ is a finite subgroup of $O_2$, the orthogonal group of order 2. Finally, we conclude that the neighbourhood of $x$ is modeled by the three classes defined above. □

Note that an open neighbourhood in each of these models is homeomorphic to an open neighbourhood of the half-plane. This means that every 2-orbifold has a topological surface with boundary (2-manifold with boundary) as its underlying space.

An important tool for our classification of (some) 2-orbifolds is the orbifold Euler number, generalizing the usual Euler number.

*Definition* 5.12 (Euler number)
Let $O$ be an orbifold and consider a cell-division (triangulation in the 2-dimensional case) of $X_O$ such that the isotropy group of points in the interior of each cell is constant. We define the Euler number, $\chi(O)$, by the following formula:

$$\chi(O) = \sum_{\text{cells } c} \frac{(-1)^{\dim(c)}}{|\Gamma(c)|}$$

Notice that this definition equals the original one for manifolds if $O$ is a manifold.

**Theorem 5.13**
*We say that $p : \tilde{O} \to O$ is a covering of $k$ sheets if the number of preimages of a non-singular point is $k$. In that case, we have*

$$\chi(\tilde{O}) = k\chi(O)$$

*Proof.* Let $x \in O$ and let $U \cong V/\Gamma_x$ be a well-covered neighbourhood of $x$. Let $y$ be a nonsingular point in $U$, which corresponds to $|\Gamma_x|$ points in $V$. Each preimage of $y$ by $p$ lies in a neighbourhood of a preimage of $x$ of the form $V/\Gamma_{\tilde{x}}$. Thus, in each of those neighbourhoods there are $\left|\frac{\Gamma_x}{\Gamma_{\tilde{x}}}\right|$ preimages of $y$. Computing the total number of preimages of $y$ gives:

$$k = \sum_{p^{-1}(x)} \frac{|\Gamma_x|}{|\Gamma_{\tilde{x}}|} \Leftrightarrow \frac{k}{|\Gamma_x|} = \sum_{p^{-1}(x)} \frac{1}{|\Gamma_{\tilde{x}}|}$$

We can construct a cell-division in $\tilde{O}$ by taking the preimage of a cell-division in $O$. Therefore,

$$k\chi(O) = \sum_{\text{cells } c} \frac{k(-1)^{\dim(c)}}{|\Gamma(c)|} = \sum_{\tilde{c}\in p^{-1}(c)} \frac{(-1)^{\dim(c)}}{|\Gamma(\tilde{c})|} = \chi(\tilde{O})$$

□

**Theorem 5.14** (Classification of compact surfaces)
*Any connected compact surface $M$ is either homeomorphic to a sphere, a connected sum of tori or a connected sum of projective planes. Moreover, $\chi\left(S^2\right) = 2$, $\chi(\underbrace{T\#\ldots\#T}_{n \text{ times}}) = 2-2n$ and $\chi(\underbrace{\mathbb{R}P^2\#\ldots\#\mathbb{R}P^2}_{n \text{ times}}) = 1-n$.*

**Corollary 5.15**
*The teardrop is a bad manifold.*

*Proof.* The teardrop has a single elliptic point (of order $n \geq 2$) so $\chi(\text{teardrop}) = \chi\left(S^2\right) - \left(1-\frac{1}{n}\right) = \frac{n+1}{n}$. If the teardrop is covered by a manifold $M$, then $\chi(M) = \frac{k(n+1)}{n}$ and $M$ has to be compact. As the Euler characteristic is an integer for manifolds then $n$ divides $k$. Thus, $\chi(M) \geq n+1 \geq 3$ which is impossible, due to the classification theorem. □

**Theorem 5.16** (Classification of compact surfaces with boundary)
*thm:classbound] Any bordered connected compact surface with boundary $N$ is of the form $M\setminus(D_1 \sqcup \ldots \sqcup D_k)$ where $M$ is a connected compact surface and $D_i$ are disjoint disks in $M$. Moreover, $\chi(N) = \chi(M) - k$.*

See [14] for the proof.

With these four theorems we are ready to classify the 17 wallpaper patterns. Each pattern corresponds to a 2-orbifold which is covered by $\mathbb{R}^2$. As the Euler number for orbifold matches the one for manifold when the orbifold is actually a manifold then $\chi\left(\mathbb{R}^2\right) = 0$. By theorem ??. we conclude that an orbifold $O$ covered by $\mathbb{R}^2$ also has $\chi(O) = 0$. We shall classify those orbifolds.

**Theorem 5.17**

*The compact 2-orbifolds O with $\chi(O) = 0$ are the following:*

| *Underlying space* $(X_O)$ | *Orders of elliptic points* | *Orders of corner reflectors* |
|---|---|---|
| $S^2$ | *2, 3, 6* | - |
| | *2, 4, 4* | - |
| | *3, 3, 3* | - |
| | *2, 2, 2, 2* | - |
| $D^2$ | - | *2, 3, 6* |
| | - | *2, 4, 4* |
| | - | *3, 3, 3* |
| | - | *2, 2, 2, 2* |
| | *2* | *2, 2* |
| | *3* | *3* |
| | *4* | *2* |
| | *2, 2* | - |
| $\mathbb{R}P^2$ | *2, 2* | - |
| $T^2$ | - | - |
| *Klein bottle* | - | - |
| *Annulus* | - | - |
| *Möbius band* | - | - |

*Proof.* First note that $0 = \chi(O) \leq \chi\left(X_O\right)$ where $X_O$ is the base space of $O$. By the first classification theorem, we know that the only compact surfaces with that property are $S^2$, $T$, $\mathbb{R}P^2$ and the Klein bottle (which is $\mathbb{R}P^2 \# \mathbb{R}P^2$). By removing disks from these surfaces we can construct some more appropriate manifolds: $D^2$ and the annulus arise from removing disks from $S^2$ and the Möbius band is constructed by taking a disk from $\mathbb{R}P^2$.

We'll now classify the orbifolds by considering the possible base spaces.

<u>$X_O = S^2$</u>:

Notice that compact surfaces cannot have corner reflectors as singular points because they always lie in the border of the orbifold. With this restriction, by theorem ??:

$$\chi(O) = \chi\left(S^2\right) - \sum_{i=1}^{n}\left(1 - \frac{1}{a_i}\right) \Leftrightarrow \sum_{i=1}^{n}\left(1 - \frac{1}{a_i}\right) = 2$$

As $1 - \frac{1}{a_i} \in \left[\frac{1}{2}, 1\right)$ then $n = 3, 4$.

If $n=3$,

$$\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}=1$$

which has the solutions $(a_1,a_2,a_3)=(2,3,6),(2,4,4),(3,3,3)$.

If $n=4$,

$$\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}+\frac{1}{a_4}=2$$

which has the solution $(a_1,a_2,a_3,a_4)=(2,2,2,2)$.

$\underline{X_O=D^2}$:

By theorem **??**:

$$\chi(O)=\chi\left(D^2\right)-\sum_{i=1}^{n}\left(1-\frac{1}{a_i}\right)-\frac{1}{2}\sum_{j=1}^{m}\left(1-\frac{1}{b_j}\right)\Leftrightarrow\sum_{i=1}^{n}\left(1-\frac{1}{a_i}\right)+\frac{1}{2}\sum_{j=1}^{m}\left(1-\frac{1}{b_j}\right)=1$$

By the argument used before, $n\leq 2$.

If $n=0$,

$$\sum_{j=1}^{m}\left(1-\frac{1}{b_j}\right)=2$$

We have already found the solutions to this equation: $(b_1,b_2,b_3)=(2,3,6),(2,4,4),(3,3,3)$ and $(b_1,b_2,b_3,b_4)=(2,2,2,2)$.

If $n=1$, then $m\geq 1$ and $\frac{1}{2}\sum_{j=1}^{m}\left(1-\frac{1}{b_j}\right)\geq\frac{1}{4}$. This means that $1-\frac{1}{a_1}\leq\frac{3}{4}\Leftrightarrow a_1\leq 4$.

If $a_1=2$, then $\sum_{j=1}^{m}\left(1-\frac{1}{b_j}\right)=1$. As $1-\frac{1}{a_i}\in\left[\frac{1}{2},1\right)$ then $m=2$. Thus we have,

$$\frac{1}{b_1}+\frac{1}{b_2}=1$$

which has the solution $(b_1,b_2)=(2,2)$.

If $a_1=3$, then $\sum_{j=1}^{m}\left(1-\frac{1}{b_j}\right)=\frac{2}{3}$. As $1-\frac{1}{a_i}\in\left[\frac{1}{2},1\right)$ then $m=1$ and the solution is $b_1=3$.

If $a_1=4$, then $\sum_{j=1}^{m}\left(1-\frac{1}{b_j}\right)=\frac{1}{2}$. As $1-\frac{1}{a_i}\in\left[\frac{1}{2},1\right)$ then $m=1$ and the solution is $b_1=2$.

If $n=2$, then $\sum_{i=1}^{n}\left(1-\frac{1}{a_i}\right)\geq 1$ with equality when $a_1=a_2=2$. Thus the only solution for $n=2$ is with $m=0$ and $a_1=a_2=2$.

$\underline{X_O=\mathbb{R}P^2}$:

By Theorem **??**:,

$$\chi(O) = \chi\left(D^2\right) - \sum_{i=1}^{n}\left(1-\frac{1}{a_i}\right) - \frac{1}{2}\sum_{j=1}^{m}\left(1-\frac{1}{b_j}\right)$$

is equivalent to:

$$\sum_{i=1}^{n}\left(1-\frac{1}{a_i}\right) + \frac{1}{2}\sum_{j=1}^{m}\left(1-\frac{1}{b_j}\right) = 1$$

This is the same equation we had for the disk. However, $\mathbb{R}P^2$ is compact so it cannot have corner reflectors. Therefore, the only solution is $n = 2$, $m = 0$ and $a_1 = a_2 = 2$.

$X_O = T$, Klein bottle, annulus, Möbius band:

As $\chi\left(X_O\right) = 0$, the orbifold $O$ cannot have singular points. $\square$

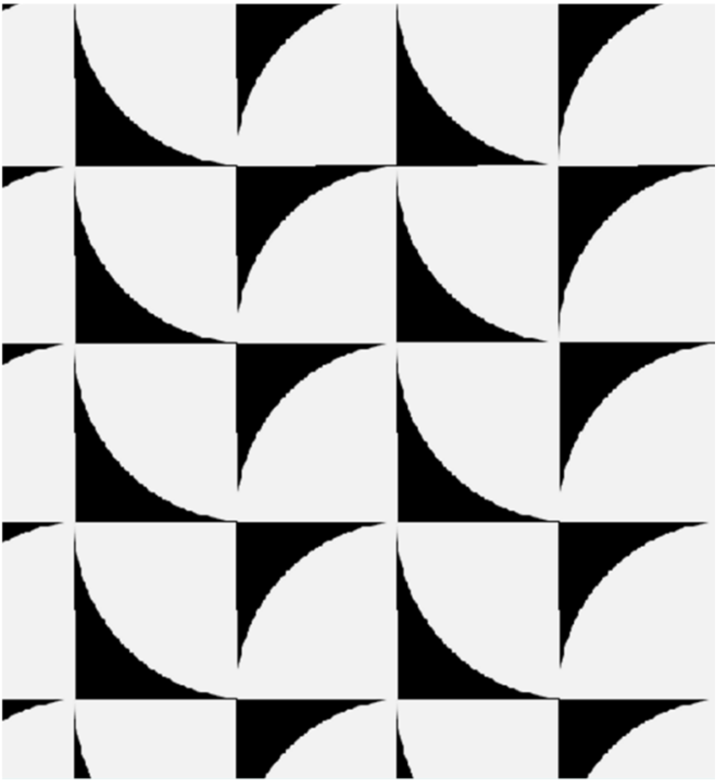

Figure 2: Klein bottle

# 6 Group Cohomology and Group Extension

We will introduce some cohomology theory (see [17] and [12]) that we use to prove 2nd and 3rd Bieberbach theorems.

Let $G$ be a group. The integral group ring $\mathbb{Z}G$ is defined to be the free $\mathbb{Z}$-module generated by the elements of $G$, that is, the group $\mathbb{Z}[G]$.
Therefore, an element of $\mathbb{Z}G$ can be uniquely expressed as $\sum_{g\in G} a(g)(g)$ where $a(g) \in \mathbb{Z}$ and $a(g) = 0$ for almost all $g \in G$.
We call $M$ a $G$-module if $M$ is an abelian group and there exists a homomorphism $\phi : G \to \mathrm{Aut}(M)$ such that the group $G$ acts on $M$ by $g \cdot m = \phi(g)(m)$. Since all abelian groups can be viewed as a module over $\mathbb{Z}$, a $G$-module $M$ is the same as a $\mathbb{Z}G$-module.
Throughout this section, we denote $R$ to be a unity ring.

*Definition 6.1*
Let $M_i$ be $R$-modules and $f_i$ be $R$-module homomorphisms for all $i > 0$. Consider the sequence,

$$M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_3 \xrightarrow{f_3} M_4 \to \cdots$$

- We say the sequence is exact at $M_n$ if and only if $\ker(f_n) = \mathrm{im}(f_{n-1})$ for $n > 0$.
- We say the sequence is exact if and only if it is exact at $M_n$ for all $n > 0$.

*Definition 6.2*
Let $M_i$ be $R$-modules and $d_i$ be $R$-module homomorphisms for $i \geq 0$. Then

$$0 \to M_0 \xrightarrow{d_0} M_1 \xrightarrow{d_1} \cdots \xrightarrow{d_n} M_{n+1} \to \cdots$$

is a cochain complex if the composition of any two successive maps $d_{n+1} \circ d_n$ is the zero map.

*Definition 6.3*
A short exact sequence is a 5-term exact sequence where the first and last terms are identity. In other words, the exact sequence

$$0 \to A \xrightarrow{\psi} B \xrightarrow{\phi} C \to 0$$

where $A, B, C$ are $R$-modules. We say the above short exact sequence splits if there is an $R$-module homomorphism $s : C \to B$ such that $\phi \circ s$ is the identity map on $C$. In this case, we call the map $s : C \to B$ a splitting homomorphism for the sequence and $B \cong A \oplus C$.

*Definition 6.4*
Let $M$ be an $R$-module. We say $M$ is a free module if there exists a subset $A \subset M$ such that

for any non-zero element $x \in M$, there exist unique non-zero elements $r_1, \ldots, r_n \in R$ and unique $a_1, \ldots, a_n \in A$ for some $n \in \mathbb{N}$ such that

$$x = \sum_{i=1}^{n} r_i a_i.$$

In this case, we say $A$ is a basis or set of generators of $M$.

*Definition* 6.5

Let $P$ be an $R$-module. We say $P$ is a projective module if $P$ has the following property. For any $R$-modules $M$ and $N$, if we have a surjection map $\phi : M \to N$, then for every $R$-module homomorphism from $P$ to $N$ lifts to an $R$-module homomorphism into $M$. In other words, given $f \in \mathrm{Hom}_R(P, N)$, there exists a lift $F \in \mathrm{Hom}_R(P, M)$ making the following diagram commute:

$$\begin{tikzcd} & P \arrow[dl, dashed, "F"'] \arrow[d, "f"] & \\ M \arrow[r, "\phi"'] & N \arrow[r] & 0 \end{tikzcd}$$

**Proposition 6.6**

*Let $P$ be an $R$-module. $P$ is a projective module if and only if $P$ is a direct summand of a free $R$-module.*

*Proof.* First, we assume $P$ is a projective module. Notice that $P$ is the quotient of a free module. Thus we always have a short exact sequence

$$0 \to \ker(\phi) \to \mathcal{F} \xrightarrow{\phi} P \to 0.$$

By definition of projective module, the identity map $\mathrm{id} : P \to P$ lifts to a homomorphism $\mu$ making the following diagram commute:

$$\begin{tikzcd} & & & P \arrow[dl, dashed, "\mu"'] \arrow[d, "id"] & \\ 0 \arrow[r] & ker(\phi) \arrow[r] & \mathcal{F} \arrow[r, "\phi"] & P \arrow[r] & 0 \end{tikzcd}$$

Since the above diagram commutes, we have $\phi \circ \mu = \mathrm{id}$. Thus $\mu$ is a splitting homomorphism for the above sequence and therefore $\mathcal{F} \cong \ker(\phi) \oplus P$.

Next, we assume $P$ is a direct summand of a free $R$-module. Let $\mathcal{F}(S) = P \oplus K$ where $\mathcal{F}(S)$ is a free $R$-module on some set $S$ and $K$ is an $R$-module. Let $M$ and $N$ be any $R$-modules and $\phi : M \to N$ be a surjection. Let $\pi : \mathcal{F}(S) \to P$ be the natural projection and let $f : P \to N$ be any $R$-module homomorphism. Our aim is to lift the map $f$ to an $R$-module homomorphism into $M$. Consider the map $f \circ \pi : \mathcal{F}(S) \to N$. For any $s \in S$, we define $n_s = f \circ \pi(s) \in N$. Since $\phi$ is surjective, we let $m_s \in M$ be any element of $M$ satisfying $\phi(m_s) = n_s$. By the universal property for free modules, there exists a unique $R$-module homomorphism $F' : \mathcal{F}(S) \to M$ such that $F'(s) = m_s$. We have the following diagram:

$$\begin{tikzcd}
 & \mathcal{F}(S) = P \oplus K \arrow[ddl, dashed, "F"'] \arrow[d, "\pi"] & \\
 & P \arrow[d, "f"] & \\
M \arrow[r, "\phi"] & N \arrow[r] & 0
\end{tikzcd}$$

for any $s \in S$, we have

$$\phi \circ F'(s) = \phi(m_s) = n_s = f \circ \pi(s).$$

It follows that $\phi \circ F' = f \circ \pi$. In other words, the above diagram commutes. We define a map $F : P \to M$ where $F(d) = F'((d, 0))$. Since $F$ is a composition of an injection $P \to \mathcal{F}(S)$ and the homomorphism $F'$, the map $F$ is an $R$-module homomorphism. Then

$$\phi \circ F(d) = \phi \circ F'((d, 0)) = f \circ \pi((d, 0)) = f(d).$$

Thus the below diagram commutes:

$$\begin{tikzcd}
 & P \arrow[dl, dashed, "F"'] \arrow[d, "f"] & \\
M \arrow[r, "\phi"] & N \arrow[r] & 0
\end{tikzcd}$$

and the proof is complete. $\square$

**Corollary 6.7**

*If $P$ is a free module, then $P$ is a projective module.*

*Definition* 6.8

Let $C^i$ be $R$-modules for all $i \geq 0$. Consider the following sequence

$$0 \to C^0 \xrightarrow{d^0} C^1 \xrightarrow{d^1} C^2 \xrightarrow{d^2} C^3 \to \cdots \to C^n \xrightarrow{d^n} C^{n+1} \to \cdots$$

where $d^n : C^n \to C^{n+1}$ is a homomorphism. We say the above sequence is a cochain complex if the composition of any two consecutive maps is the zero map. We define the $n$-th cohomology group of that cochain complex to be

$$H^n(\mathcal{C}) = \ker(d^n)/\operatorname{im}(d^{n-1})$$

where $\mathcal{C}$ is the cochain complex above.

*Definition* 6.9
Let $A$ be an $R$-module. A projective resolution of $A$ is an exact sequence

$$\cdots \to P_n \xrightarrow{d_n} \cdots \xrightarrow{d_1} P_0 \xrightarrow{d_0} A \to 0$$

where $P_i$ are projective $R$-modules for all $i \geq 0$.

**Lemma 6.10**
*Let $A$ be an $R$-module. There always exists a projective resolution of $A$.*

*Proof.* Choose a free module $P_0$ with a surjection $d_0 : P_0 \to A$ and define $\ker(d_0) = K_0$. Inductively, for $n \geq 1$, we choose a free module $P_n$ with surjection $P_n \to K_{n-1}$ and define $K_n$ to be the kernel of the surjection. We define $d_n$ to be the composition $P_n \to K_{n-1} \to P_{n-1}$. It is clear that $\ker(d_n) = \ker(P_n \to K_{n-1}) = K_n$. By the above construction, we have a surjection $d_n : P_n \to K_{n-1}$ and $\ker(d_n) = \operatorname{im}(d_{n+1})$. It follows that the sequence

$$\cdots \to P_n \to \cdots \to P_1 \to P_0 \to A \to 0$$

is exact and $P_i$ are projective $R$-modules for all $i \geq 0$ by Corollary 6.7 $\square$

Given the projective resolution, we can form a cochain complex by taking homomorphisms of each of the terms into an $R$-module $D$. In other words, we apply the functor $\operatorname{Hom}_R(-, D)$ to the projective resolution in the definition above and get the below sequence,

$$0 \to \operatorname{Hom}_R(A, D) \xrightarrow{d_0'} \operatorname{Hom}_R(P_0, D) \xrightarrow{d_1'} \operatorname{Hom}_R(P_1, D) \xrightarrow{d_2'} \operatorname{Hom}_R(P_2, D) \xrightarrow{d_3'} \cdots$$

Let $f \in \operatorname{Hom}_R(A, D)$, we define $d_0'(f) = f \circ d_0$. For $n \geq 0$, and let $f \in \operatorname{Hom}_R(P_n, D)$, we define $d_{n+1}'(f) = f \circ d_{n+1}$. For any $n \geq 0$ and let $f \in \operatorname{Hom}_R(P_{n-1}, D)$, where $P_{-1} = A$, we have

$$d_{n+1}' \circ d_n'(f) = d_{n+1}'(f \circ d_n) = f \circ d_n \circ d_{n+1}$$

Since the sequence of projective resolution is an exact sequence, the composition $d_{n+1}' \circ d_n'$ is zero map for all $n \geq 0$. Therefore the sequence that we have defined is a cochain complex.

We define

$$\mathrm{Ext}^n_R(A, D) = \ker(d'_{n+1})/\mathrm{im}(d'_n)$$

for $n \geq 1$ and $\mathrm{Ext}^0_R(A, D) = \ker(d'_1)$.

For $n \geq 0$, the $n$th cohomology group of group $G$ with $R$-module $M$ as coefficient is defined as

$$H^n(G, M) = \mathrm{Ext}^n_{\mathbb{Z}G}(\mathbb{Z}, M)$$

We define the standard resolution of $\mathbb{Z}$ as

$$\cdots \to F_n \xrightarrow{d_n} \cdots \xrightarrow{d_1} F_0 \xrightarrow{\epsilon} \mathbb{Z} \to 0$$

where $F_n$ is defined to be $\mathbb{Z}G^{(n+1)\otimes_{\mathbb{Z}}}$. Notice that $F_n$ is a $G$-module where the $G$-action is given by $g \cdot (g_0 \otimes \cdots \otimes g_n) = (gg_0) \otimes g_1 \otimes \cdots \otimes g_n$. Notice that $F_n$ is a free $\mathbb{Z}G$-module of rank $|G|^n$ and the set $\{1 \otimes g_1 \otimes \cdots \otimes g_n | g_i \in G \text{ for } 1 \leq i \leq n\}$ is a set of basis of $F_n$. We denote the basis element $1 \otimes g_1 \otimes \cdots \otimes g_n$ to be $(g_1, \cdots, g_n)$. We define the map $d_1(1 \otimes g) = g - 1$ and for $n \geq 2$, we define

$$\begin{aligned} d_n(g_1, \cdots, g_n) &= g_1 \cdot (g_2, \cdots, g_n) \\ &\quad + \sum_{i=1}^{n-1} (-a)^i (g_1, \cdots, g_{i-1}, g_i g_{i+1}, g_{n+2}, \cdots, g_n) \\ &\quad + (-1)^n (g_1, \cdots, g_{n-1}) \end{aligned}$$

Now, we apply the functor $\mathrm{Hom}_{\mathbb{Z}G}(-, M)$ to the sequence of standard resolution and obtain the below cochain complex

$$0 \to \mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}, M) \xrightarrow{\epsilon^t} \mathrm{Hom}_{\mathbb{Z}G}(F_0, M) \xrightarrow{d'_1} \mathrm{Hom}_{\mathbb{Z}G}(F_1, M) \xrightarrow{d'_2} \cdots$$

Notice that the elements of $\mathrm{Hom}_{\mathbb{Z}G}(F_n, M)$ can be uniquely determined by their values on the $\mathbb{Z}G$ basis elements of $F_n$. In other words, the group $\mathrm{Hom}_{\mathbb{Z}G}(F_n, M)$ can be identify with the set of functions from $G \times \cdots \times G$ ($n$ copies of $G$) to $M$ and $\mathrm{Hom}_{\mathbb{Z}G}(F_0, M) = M$. Now, we can give a definition of cohomology of group $G$ with coefficient $M$ as follow.

*Definition* 6.11

Let $G$ be a finite group and $M$ be a $G$-module. Define $C^0(G, M) = M$, $C^n(G, M)$ to be the collection of all functions from $G^n$ to $M$ for $n \geq 1$ and $C^n(G, M) = 0$ for $n < 0$. We define the coboundary operator $\delta^n : C^n(G, M) \to C^{n+1}(G, M)$ as follows,

$$\begin{aligned}\delta^n f(g_0, g_1, \ldots, g_n) = g_0 \cdot f(g_1, \ldots, g_n) \\ &+ \sum_{j=1}^{n} (-1)^j f(g_0, \ldots, g_{j-2}, g_{j-1}g_j, g_{j+1}, \ldots, g_n) \\ &+ (-1)^{n+1} f(g_0, \ldots, g_{n-1})\end{aligned}$$

for $n \geq 1$, $\delta^0 m(g_1) = g_1 \cdot m - m$ and $\delta^n = 0$ for $n < 0$. Then

$$0 \to C^0(G, M) \xrightarrow{\delta^0} C^1(G, M) \xrightarrow{\delta^1} \cdots \xrightarrow{\delta^n} H^{n+1}(G, M) \to \cdots$$

is a cochain complex. We define $Z^n(G, M) = \ker(\delta^n)$ and the elements of $Z^n(G, M)$ are called $n$-cocycles. We define $B^n(G, M) = \operatorname{im}(\delta^{n-1})$ and the elements of $B^n(G, M)$ are called $n$-coboundaries. We define the $n$-th cohomology group of $G$ with coefficients in $M$ to be

$$H^n(G, M) = Z^n(G, M)/B^n(G, M).$$

*Remark*
Using the same notation as above, if $G$ is a finite group and $M$ is a finitely generated $G$-module, then $\operatorname{Hom}_{\mathbb{Z}G}(F_n, M)$ is a finitely generated abelian group for all $n \geq 0$. Therefore $H^n(G, M)$ is a finitely generated abelian group for all $n \geq 0$.

*Definition* 6.12
Let $N, G, \Gamma$ be groups. We say $\Gamma$ is an extension of $G$ by $N$ if it fits in the short exact sequence,

$$1 \to N \xrightarrow{\iota} \Gamma \xrightarrow{p} G \to 1.$$

Let $\Gamma$ and $\Gamma'$ be both extensions of $G$ by $N$. We say two extensions are equivalent via $f$ if there exists a homomorphism $f : \Gamma \to \Gamma'$ such that the following diagram commutes:

$$\begin{tikzcd} & & \Gamma \arrow[dr, "p_1"] \arrow[dd, "f"] & & \\ 1 \arrow[r] & N \arrow[ur, "\iota_1"] \arrow[dr, "\iota_2"'] & & G \arrow[r] & 1 \\ & & \Gamma' \arrow[ur, "p_2"'] & & \end{tikzcd}$$

**Lemma 6.13**
*Using the same notations as above, the homomorphism $f$ is indeed an isomorphism.*

*Proof.* Let $\gamma \in \ker(f)$. Since $p_2 \circ f(\gamma) = p_2(1) = 1$ and the diagram commutes, we have $p_1(\gamma) = p_2 \circ f(\gamma) = 1$. Thus $\gamma \in \ker(p_1)$. By exactness at $\Gamma$, there exists $x \in N$

such that $\iota_1(x) = \gamma$. Hence $\iota_2(x) = f \circ \iota_1(x) = f(\gamma) = 1$. Since $\iota_1$ is injective, $x = 1$ and therefore $\gamma = \iota_1(x) = 1$. It follows that $f$ is injective.

Let $\gamma' \in \Gamma'$. Since $p_1$ is surjective, there exists $\gamma \in \Gamma$ such that $p_1(\gamma) = p_2(\gamma')$. We have $p_2 \circ f(\gamma) = p_1(\gamma) = p_2(\gamma')$ and therefore $p_2(\gamma'(f(\gamma))^{-1}) = 1$. By exactness at $\Gamma'$, there exists $x \in N$ such that $\iota_2(x) = \gamma'(f(\gamma))^{-1}$. It follows that $f(\iota_1(x)\gamma) = f(\iota_1(x))f(\gamma) = \iota_2(x)f(\gamma) = \gamma'(f(\gamma))^{-1}f(\gamma) = \gamma'$. Thus $f$ is surjective. Therefore $f$ is an isomorphism. $\square$

**Lemma 6.14**
*Given the short exact sequence*

$$0 \to N \xrightarrow{\iota} \Gamma \xrightarrow{p} G \to 1$$

*where $N$ is an abelian group. Then it induces a $G$-action on $N$. In other words, we can view $N$ as a $G$-module.*

*Proof.* Since $N$ is an abelian normal subgroup in $\Gamma$, $G$ acts on $N$ by conjugation. Explicitly, let $g \in G, x \in N$ and pick $\bar{g}$ be an element such that $p(\bar{g}) = g$. We define the action as below,

$$\iota(g \cdot x) = \bar{g}\iota(x)\bar{g}^{-1}.$$

Let $\bar{g}'$ be another element such that $p(\bar{g}') = g$. Since $\Gamma/N \cong G$, there exists $x_1 \in N$ such that $\bar{g}' = \bar{g}\iota(x_1)$. Since $N$ is an abelian group, we have

$$\bar{g}'\iota(x)\bar{g}'^{-1} = \bar{g}\iota(x_1)\iota(x)\iota(x_1)^{-1}\bar{g}^{-1} = \bar{g}\iota(x)\bar{g}^{-1}.$$

Hence the action is independent of the choice of $\bar{g}$. Therefore the action given by $\iota(g \cdot \dot{\ })$ is a well-defined $G$-action on $N$. $\square$

**Lemma 6.15**
*Equivalent extensions of $G$ by $N$ define the same $G$-module structure on $N$.*

*Proof.* Let $\Gamma$ and $\Gamma'$ be equivalent extensions and consider the below commutative diagram.

$$\begin{tikzcd}
 & & \Gamma \arrow[dd, "f"] \arrow[dr, "p_1"] & & \\
1 \arrow[r] & N \arrow[ur, "\iota_1"] \arrow[dr, "\iota_2"'] & & G \arrow[r] & 1 \\
 & & \Gamma' \arrow[ur, "p_2"'] & &
\end{tikzcd}$$

Let $g \in G$ be an arbitrary element of $G$. Let $\bar{g}$ be an element such that $p_1(\bar{g}) = g$. The $G$-module structure on $N$ induced from $\Gamma$ is

$$\iota_1(g \cdot x) = \bar{g}\iota_1(x)\bar{g}^{-1}$$

where $x \in N$. Let $\bar{g}' = f(\bar{g})$. Since the above diagram is a commutative diagram, we have $p_2(\bar{g}') = p_1(\bar{g}) = g$. Thus the $G$-module structure on $N$ induced from $\Gamma'$ is

$$\iota_2(g \cdot x) = \bar{g}' \iota_2(x) \bar{g}'^{-1}$$

where $x \in N$. Since $\iota_1, \iota_2$ and $f$ are injective, we have

$$f(\bar{g}\iota_1(x)\bar{g}^{-1}) = \bar{g}'\iota_2(x)\bar{g}'^{-1} = \iota_2(g \cdot x).$$

Therefore, equivalent extensions of $G$ by $N$ define the same $G$-action on $N$. □

Let $G$ be a group and $A$ be a $G$-module. We would like to study the relation between $H^2(G, A)$ and group extensions of $G$ by $A$. Roughly speaking, given a group extension of $G$ by $A$, we would like to define a class in $H^2(G, A)$. Next, we are going to reverse the procedure. Given a class in $H^2(G, A)$, we want to construct a group extension of $G$ by $A$ corresponding to that given class. Therefore, we conclude that there is a bijection between the set of all extensions of $G$ by $A$ and the group $H^2(G, A)$.

Let $G$ be a group and $A$ be a $G$-module. We first want to show that the below extension

$$0 \to A \xrightarrow{\iota} \Gamma \xrightarrow{p} G \to 1$$

defines a 2-cocycle in $Z^2(G, A)$. We want to study the above exact sequence by choosing a set-theoretic cross-section $s : G \to \Gamma$ such that $ps : G \to G$ is an identity map. We call the map $s$ a cross-section of $p$. We say the map $s$ is normalized, or we say $s$ satisfies the normalization condition if it satisfies the below condition

$$s(1) = 1$$

In general, $s$ is not necessarily a homomorphism. We would like to define a function $f : G \times G \to A$ to measure the failure of $s$ to be a homomorphism. Since for any $g_1, g_2 \in G$, the elements $s(g_1 g_2) \in \Gamma$ and $s(g_1) s(g_2) \in \Gamma$ both map to $g_1 g_2 \in G$, they differ by an element $\iota(a)$ for some $a \in A$. Therefore, we define $f : G \times G \to A$ by the below equation

$$s(g_1) s(g_2) = \iota(f(g_1, g_2)) s(g_1 g_2)$$

In particular, for any $g \in G$, we can view $s(g)$ as a set of coset representatives for $\iota(A)$ in $\Gamma$. Thus each element of $\Gamma$ can be written uniquely in the form $\iota(a)s(g)$ for some $a \in A$ and $g \in G$. Besides, we say $f$ is normalized if it satisfies the below condition

$$f(g, 1) = f(1, g) = 0$$

for all $g \in G$. It is easy to check that if $s$ is normalized, then $f$ is also normalized. We call the function $f$ the factor set associated with the short exact sequence of extension above and the section $s$.

Next, we are going to show that $f$ is an element in $Z^2(G, A)$. Let $\iota\left(a_1\right) s\left(g_1\right)$ and $\iota\left(a_2\right) s\left(g_2\right)$ be two arbitrary elements of $\Gamma$. By the relation $\iota(g\dot{x}) = \overline{g}\iota(x)\overline{g}^{-1}$, we have

$$\iota\left(g_1 \cdot a_2\right) = s\left(g_1\right) \iota\left(a_2\right) s\left(g_1\right)^{-1}$$

By the definition of $f$, we have

$$\begin{aligned}
\iota\left(a_1\right) s\left(g_1\right) \iota\left(a_2\right) s\left(g_2\right) &= \iota\left(a_1\right) \iota\left(g_1 \cdot a_2\right) s\left(g_1\right) s\left(g_2\right) \\
&= \iota\left(a_1 + g_1 \cdot a_2\right) \iota\left(f\left(g_1, g_2\right)\right) s\left(g_1 g_2\right) \\
&= \iota\left(a_1 + g_1 \cdot a_2 + f\left(g_1, g_2\right)\right) s\left(g_1 g_2\right)
\end{aligned}$$

Next, we compute the triple product $\left[s\left(g_1\right) s\left(g_2\right)\right] s\left(g_3\right)$ and $s\left(g_1\right)\left[s\left(g_2\right) s\left(g_3\right)\right]$, we have

$$\begin{aligned}
&\left[\iota\left(a_1\right) s\left(g_1\right) \iota\left(a_2\right) s\left(g_2\right)\right] \iota\left(a_3\right) s\left(g_3\right) \\
= \ &\iota\left(a_1 + g_1 \cdot a_2 + f\left(g_1, g_2\right) + \left(g_1 g_2\right) \cdot a_3 + f\left(g_1 g_2, g_3\right)\right) s\left(g_1 g_2 g_3\right)
\end{aligned}$$

and

$$\begin{aligned}
&\iota\left(a_1\right) s\left(g_1\right)\left[\iota\left(a_2\right) s\left(g_2\right) \iota\left(a_3\right) s\left(g_3\right)\right] \\
= \ &\iota\left(a_1 + g_1 \cdot a_2 + \left(g_1 g_2\right) \cdot a_3 + g_1 \cdot f\left(g_2, g_3\right) + f\left(g_1, g_2 g_3\right)\right) s\left(g_1 g_2 g_3\right)
\end{aligned}$$

Since $\Gamma$ is a group and therefore it satisfies the associative law, $f$ does satisfy the following condition

$$g_1 \cdot f\left(g_2, g_3\right) - f\left(g_1 g_2, g_3\right) + f\left(g_1, g_2 g_3\right) - f\left(g_1, g_2\right) = 0$$

Using the same notations in Definition 6.11, $f \in \ker\left(\delta^2\right)$. Thus $f$ is a 2-cocycle. Therefore, we can conclude that the factor set $f$ associated with the extensiond and a choice of section $s$ is an element of $Z^2(G, A)$. Let $f'$ be a factor set associated with the extension and a different choice of section $s'$. We are going to show that $f$ and $f'$ differ by a 2-coboundary. For all $g \in G$, the element $s(g)$ and $s'(g)$ lie in the same coset $Ag$. Thus there exists a function $\phi : G \to A$ such that $s'(g) = \iota(\phi(g))s(g)$ for all $g \in G$. For arbitrary elements $\iota\left(a_1\right) s'\left(g_1\right)$ and $\iota\left(a_2\right) s'\left(g_2\right)$ in $\Gamma$, we have

$$\begin{aligned}
s'\left(g_1\right) s'\left(g_2\right) &= \iota\left(f'\left(g_1, g_2\right)\right) s'\left(g_1 g_2\right) \\
&= \iota\left(f'\left(g_1, g_2\right)\right) \iota\left(\phi\left(g_1 g_2\right)\right) s\left(g_1 g_2\right) \\
&= \iota\left(f'\left(g_1, g_2\right) + \phi\left(g_1 g_2\right)\right) s\left(g_1 g_2\right)
\end{aligned}$$

and

$$\begin{aligned}
s'\left(g_1\right) s'\left(g_2\right) &= \iota\left(\phi\left(g_1\right)\right) s\left(g_1\right) \iota\left(\phi\left(g_2\right)\right) s\left(g_2\right) \\
&= \iota\left(\phi\left(g_1\right) + g_1 \cdot \phi\left(g_2\right) + f\left(g_1, g_2\right)\right) s\left(g_1 g_2\right)
\end{aligned}$$

Therefore we have

$$f'(g_1, g_2) = f(g_1, g_2) + \phi(g_1) + g_1 \cdot \phi(g_2) - \phi(g_1 g_2)$$

It follows that $f$ and $f'$ differ by the 2-coboundary $\phi$. Thus we can conclude that the factor sets associated with the extension corresponding to different choices of section give a 2-cocycle in $Z^2(G, A)$ that differ by a coboundary in $B^2(G, A)$. Hence associated with the extension is a well-defined cohomology class in $H^2(G, A)$ determined by the factor set for any choice of section $s$.

*Remark*
In particular, if the group extension is a split extension, then there is a homomorphism section $s : G \to \Gamma$. Therefore the factor set satisfies $f(g_1, g_2) = 0$ for all $g_1, g_2 \in G$. Hence the trivial cohomology class in $H^2(G, A)$ defines a split extension. In other words, $\Gamma = A \rtimes G$.

Next, we want to prove that equivalent extensions define the same cohomology class in $H^2(G, A)$. Let $\Gamma$ and $\Gamma'$ be two equivalent group extensions of $G$ by $A$. Consider the below commutative diagram:

$$\begin{tikzcd}
& & \Gamma \arrow[dd, "f"] \arrow[dr, "p_1"] & & \\
1 \arrow[r] & N \arrow[ur, "\iota_1"] \arrow[dr, "\iota_2"'] & & G \arrow[r] & 1 \\
& & \Gamma' \arrow[ur, "p_2"'] & &
\end{tikzcd}$$

Let $s$ be a section of $p$, then $s' = \psi \circ s$ is a section of $p'$. Let $f : G \times G \to A$ be a factor set of the extension corresponding to $\Gamma$ and section $s$. Recall that $f$ satisfies the condition

$$s(g_1) s(g_2) = \iota(f(g_1, g_2)) s(g_1 g_2)$$

for all $g_1, g_2 \in G$. Applying $\psi$ to the above condition, we have

$$\begin{aligned}
s'(g_1) s'(g_2) &= \psi(s(g_1)) \psi(s(g_2)) \\
&= \psi(\iota(f(g_1, g_2))) \psi(s(g_1 g_2)) \\
&= \iota'(f(g_1, g_2)) s'(g_1 g_2)
\end{aligned}$$

for all $g_1, g_2 \in G$. It follows that the factor set for $\Gamma'$ associated with $s'$ is the same as the factor set for $\Gamma$ associated with $s$. Thus equivalent extensions define the same cohomology class in $H^2(G, A)$.

Next, we want to show that given a class in $H^2(G, A)$, we could construct an extension $E_f$ such that its corresponding factor set is in the given class in $H^2(G, A)$. Using the notations in Definition 6.11 .Let $f \in Z^2(G, A) \subseteq C^2(G, A)$ be a 2-cocycle. Define

$f_1 \in C^1(G, A)$ which maps $g \in G$ to $f(1,1)$ for all $g \in G$. We claim that $f - \delta^1(f_1)$ is a normalized 2-cocycle. By definition of $\delta^1$, we have

$$\delta^1(f_1)(g,1) = g \cdot f_1(1) - f_1(g) + f_1(g) = g \cdot f(1,1)$$

and

$$\delta^1(f_1)(1,g) = 1 \cdot f_1(g) - f_1(g) + f_1(g) = f(1,1)$$

for all $g \in G$. Since $f$ is a 2-cocycle, we have

$$f(g,h) + f(gh,k) = gf(h,k) + f(g,hk)$$

for all $g, h, k \in G$. By setting $g = h = 1$, we have

$$f(1,k) - f(1,1) = 0$$

for all $k \in G$. By combining, we have

$$f(1,g) - \delta^1(f_1)(1,g) = 0$$

On the other hand, by setting $h = k = 1$, we have

$$f(g,1) - g \cdot f(1,1) = 0$$

By combining this with the first equation, we get

$$f(g,1) - \delta^1(f_1)(g,1) = 0$$

for all $g \in G$. It follows that

$$\left(f - \delta^1(f_1)\right)(g,1) = \left(f - \delta^1(f_1)\right)(1,g) = 0$$

Thus, we can conclude that $f - \delta^1(f_1)$ is a normalized 2-cocycle.

Let $f$ be a cohomology class representative in $H^2(G, A)$ where $f$ is a normalized 2-cocycle. Define $E_f$ to be the set $A \times G$ with a binary operation on $E_f$ as below

$$(a_1, g_1)(a_2, g_2) = (a_1 + g_1 \cdot a_2 + f(g_1, g_2), g_1 g_2)$$

where $(a_1, g_1), (a_2, g_2) \in A \times G$. We claim that $E_f$ is indeed a group. Since $f$ is a normalized 2-cocycle, we have

$$(a,g)(0,1) = (a + f(g,1), g) = (a,g)$$

and

$$(0,1)(a,g) = (a + f(1,g), g) = (a,g)$$

Thus $(0,1)$ is a 2-sided identity. Next, we check for associativity. By simple calculation, we have

$$\begin{aligned} & \left[\left(a_1, g_1\right)\left(a_2, g_2\right)\right]\left(a_3, g_3\right) \\ = & \left(a_1+g_1 \cdot a_2+f\left(g_1, g_2\right)+\left(g_1 g_2\right) \cdot a_3+f\left(g_1 g_2, g_3\right), g_1 g_2 g_3\right) \end{aligned}$$

and

$$\begin{aligned} & \left(a_1, g_1\right)\left[\left(a_2, g_2\right)\left(a_3, g_3\right)\right] \\ = & \left(a_1+g_1 \cdot a_2+\left(g_1 g_2\right) \cdot a_3+g_1 \cdot f\left(g_2, g_3\right)+f\left(g_1, g_2 g_3\right), g_1 g_2 g_3\right) \end{aligned}$$

Since $f$ satisfies $f(g,h)+f(gh,k)=gf(h,k)+f(g,hk)$, we have $\left[\left(a_1, g_1\right)\left(a_2, g_2\right)\right]\left(a_3, g_3\right)=\left(a_1, g_1\right)\left[\left(a_2, g_2\right)\left(a_3, g_3\right)\right]$. Thus the operation satisfies the associativity law. By simple calculation, we get

$$(0, g)\left[\left(0, g^{-1}\right)(0, g)\right]=\left(g \cdot f\left(g^{-1}, g\right)+f(g, 1), g\right)$$

and

$$\left[(0, g)\left(0, g^{-1}\right)\right](0, g)=\left(f\left(g, g^{-1}\right), g\right)$$

Since $f$ is normalized and the binary operation on $A \times E$ satisfies the associativity law, we have

$$g \cdot f\left(g^{-1}, g\right)=f\left(g, g^{-1}\right)$$

Besides, by simple calculation, we get

$$(a, g)\left(-g^{-1} a-g^{-1} f\left(g, g^{-1}\right), g^{-1}\right)=(0,1)=\left(-g^{-1} a-g^{-1} f\left(g, g^{-1}\right), g^{-1}\right)(a, g)$$

Thus for any $(a,g) \in A \times G$, there exists an inverse

$$(a, g)^{-1}=\left(-g^{-1} a-g^{-1} f\left(g, g^{-1}\right), g^{-1}\right)$$

Thus $E_f$ is a group. Define

$$A^{\prime}=\{(a, 1) \mid a \in A\}$$

Since $f$ is a normalized 2-cocycle, $A'$ is a subgroup of $E_f$, and the map $\iota' : a \mapsto (a,1)$ is an isomorphism from $A$ to $A'$. It is clear that $A'$ is a normal subgroup of $E_f$ and the map $p' : (a,g) \mapsto g$ is a surjective homomorphism from $E_f$ to $G$ with kernel $A'$. Thus we have

$$0 \rightarrow A \xrightarrow{\iota^{\prime}} E_f \xrightarrow{p^{\prime}} G \rightarrow 1$$

By simple calculation, we check that the action of $G$ on $A$ by conjugation in the above extension is the module action specified in determining the 2-cocycle $f \in H^2(G,A)$. The extension above has a normalized section $s : G \to E_f$ which maps $g \in G$ to

$(0, g) \in E_f$ whose corresponding normalized factor set is $f$. Thus we can conclude that every normalized 2-cocycle arises as the normalized factor set of some extension.

Finally, suppose $f'$ is another normalized 2-cocycle in the same cohomology class in $H^2(G, A)$ as $f$ and let $E_{f'}$ be the corresponding extension. Since $f$ and $f'$ are in the same cohomology class, they differ by the coboundary $f_1 : G \to A$. Explicitly, for all $g, h \in G$, we have

$$f(g,h) - f'(g,h) = g \cdot f_1(h) - f_1(gh) + f_1(g)$$

By setting $g = h = 1$, we get $f_1(1) = 0$. Define $\phi : E_f \to E_{f'}$ given by

$$\phi((a, g)) = (a + f_1(g), g)$$

It is clear that $\phi$ is a bijection. Next, we want to show that $\phi$ is a homomorphism.

$$\begin{aligned}
\phi\left(\left(a_1, g_1\right)\left(a_2, g_2\right)\right) &= \phi\left(\left(a_1 + g_1 \cdot a_2 + f\left(g_1, g_2\right), g_1 g_2\right)\right) \\
&= \left(a_1 + g_1 \cdot a_2 + f\left(g_1, g_2\right) + f_1\left(g_1 g_2\right), g_1 g_2\right) \\
&= \left(a_1 + f_1\left(g_1\right) + g_1 \cdot \left(a_2 + f_1\left(g_2\right)\right) + f'\left(g_1, g_2\right), g_1 g_2\right) \\
&= \left(a_1 + f_1\left(g_1\right), g_1\right)\left(a_2 + f_1\left(g_2\right), g_2\right) \\
&= \phi\left(\left(a_1, g_1\right)\right) \phi\left(\left(a_2, g_2\right)\right)
\end{aligned}$$

for all $\left(a_1, g_1\right), \left(a_2, g_2\right) \in E_f$. It follows that $\phi$ is an isomorphism. Consider the restriction of $\phi$ to $A$, we have

$$\phi((a, 1)) = (a + f_1(1), 1) = (a, 1)$$

for all $a \in A$. Therefore $\left.\phi\right|_A$ is the identity map on $A$. Similarly, $\phi$ is the identity map on the second component of $(a, g)$, so $\phi$ induces the identity map on the quotient of $G$. It follows that $\phi$ defines an equivalence between the extensions $E_f$ and $E_{f'}$. This shows that the equivalence class of the extension $E_f$ depends only on the cohomology class of $f \in H^2(G, A)$.

We summarize all the above discussion in the following theorem:

**Theorem 6.16**

*Let $G$ be a group and $A$ be a $G$-module. Let $\mathcal{E}(G, A)$ be the set of equivalence classes of extensions of $G$ by $A$ giving rise to the given action of $G$ on $A$. Then there is a bijection between the set $\mathcal{E}(G, A)$ and the group $H^2(G, A)$.*

*Remark*

Let $G$ be a group and $A$ be a $G$-module. The trivial class $[0] \in H^2(G, A)$ is correspond to a the below split extension

$$0 \to A \to A \rtimes G \to G \to 1$$

**Proposition 6.17**

*Let $G$ be a group and $A$ be a $G$-module. If $|G| = k$, then every element of $H^n(G, A)$ has order divisible by $k$ for $n > 0$.*

*Proof.* Let $f \in C^n(G, A)$ be an arbitrary $n$-cochain.
Define

$$g(x_1, \ldots, x_{n-1}) = \sum_{x \in G} f(x_1, \ldots, x_{n-1}, x).$$

By definition of $\delta^n$ for $n > 0$, we have

$$\begin{aligned}\delta^{n-1} g(x_1, \ldots, x_n) = &x_1 g(x_2, \ldots, x_n) \\ &+ \sum_{j=2}^{n} (-1)^{j+1} g(x_1, \ldots, x_{j-2}, x_{j-1}x_j, x_{j+1}, \ldots, x_n) \\ &+ (-1)^n g(x_1, \ldots, x_{n-1})\end{aligned}$$

and

$$\begin{aligned}\sum_{x \in G} \delta^n f(x_1, \ldots, x_n, x) =&x_1 g(x_2, \ldots, x_n) \\ &+ \sum_{j=2}^{n} (-1)^{j+1} g(x_1, \ldots, x_{j-1}x_j, x_{j+1}, \ldots, x_n) \\ &+ (-1)^n g(x_1, \ldots, x_{n-1}) + |G|(-1)^{n+1} f(x_1, \ldots, x_n)\end{aligned}$$

It follows that

$$\sum_{x \in G} \delta^n f(x_1, \ldots, x_n, x) = \delta^{n-1} g(x_1, \ldots, x_{n-1}, x_n) + |G|(-1)^{n+1} f(x_1, \ldots, x_n).$$

If $f \in Z^n(G, A)$, then we have $\delta^n f = 0$. Hence we have

$$\delta^{n-1} g(x_1, \ldots, x_{n-1}, x_n) = \pm |G| f(x_1, \ldots, x_n).$$

Thus the order of $f$ is divisible by $k$. $\square$

### Corollary 6.18

*Let $G$ be a group and $\mathbb{Z}^n$ be a $G$-module for any $n \geq 1$. If $G$ is finite, then so is $H^2(G, \mathbb{Z}^n)$.*

### Proposition 6.19

*Let $G$ be a group and $M$ be a $G$-module. If $|G| = m$ is invertible in $M$, then $H^n(G, M) = 0$ for all $n > 0$.*

*Proof.* Let $\phi : C^q(G, M) \to C^q(G, M)$ be a homomorphism that sends $f \in C^q(G, M)$ to $m \cdot f$. It suffices to show that the induced homomorphism $\phi^q : H^q(G, M) \to H^q(G, M)$ is the trivial homomorphism for $q \geq 1$. In other words, we want to show $\phi^q(H^q(G, M)) = 0$. We know that all $q$-cocycles have order divisible by $k$. Hence $\phi^q(H^q(G, M)) = 0$. $\square$

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