Klein geometric structures on manifolds and Bieberbach theorem

Houdayfa Achemlal

Supervised by Dr Abdelghani Zeghib

19/05/2025 to 11/07/2025

Abstract

This document is the result of an internship completed at the mathematics laboratory of the ENS de Lyon (UMPA) at the end of the first year of mathematics at the École normale supérieure de Lyon.

Contents

1 Preliminary

In this section we shall give necessary definitions and lemmas for our topic.

Definition 1. Let H and K denote groups with group multiplication $\circ$ and $\star$ respectively and assume that $H < \operatorname{Aut}(K)$ and that K is abelian. The semi-direct product $H \ltimes K$ of the groups H and K is the set of all pairs $(h, k), h \in H, k \in K$, with the following multiplication

$$(h_1, k_1)(h_2, k_2) = (h_1 \circ h_2, k_1 \star h_1(k_2))$$

Example 1.

- The isometries group $E(n) = O(n) \ltimes \mathbb{R}^n$
- The affine group $A(n) = GL(n, \mathbb{R}) \ltimes \mathbb{R}^n$

Proposition 1. There is the following sequence of subgroups

$$E(n) \subset A(n) \subset GL(n+1, \mathbb{R})$$

Proof. From the definition $E(n) \subset A(n)$.
Let $(A, a) \in A(n)$. We have an $(n+1) \times (n+1)$ matrix $\begin{pmatrix} A & a \\ 0 & 1 \end{pmatrix}$, which defines an inclusion $A(n) \subset GL(n+1, \mathbb{R})$. $\square$

Definition 2. Let Γ be a subgroup of the group $E(n)$. Then Γ is discrete if it is a discrete subset of the Euclidean space $\mathbb{R}^{(n+1)^2}$. We say that Γ acts properly discontinuously on $\mathbb{R}^n$ if for any $x \in \mathbb{R}^n$ there is an open neighbourhood U_x such that the set

$$\{\gamma \in \Gamma \mid \gamma U_x \cap U_x \neq \emptyset\}$$

is finite. Moreover, Γ acts freely, if for any $x \in \mathbb{R}^n$ we have

$$\{\gamma \in \Gamma \mid \gamma x = x\} = \{(I, 0)\}$$

Lemma 1. *Any discrete subgroup of the group $E(n)$ is closed in $E(n)$.*

Proof. Let Γ be a discrete subgroup of $E(n)$ and suppose that $E(n) \backslash \Gamma$ is not open. Then there is a γ in $E(n) \backslash \Gamma$ and γ_n in $B(\gamma, 1/n) \cap \Gamma$.
As $\gamma_n \to \gamma$ in $E(n)$, we have $\gamma_n \gamma_{n+1}^{-1} \to 1$ in Γ. However $\{\gamma_n \gamma_{n+1}^{-1}\}$ is not eventually constant, which is a contradiction for that Γ is a discrete metric space. Therefore, $E(n) \backslash \Gamma$ is open, and so Γ is closed in $E(n)$. $\square$

Lemma 2. *If Γ is a discrete subgroup of the group $E(n)$ and $V_0 = D(0, r) \subset \mathbb{R}^n$ is an open disk, then*

$$\{\gamma \in \Gamma \mid \gamma V_0 \cap V_0 \neq \emptyset\} \subset \Gamma \cap (O(n) \times V_0')$$

where $V_0' = D(0, 2r)$ is an open disk.

Proof. Let $\gamma = (A, a) \in \Gamma$ and $\gamma V_0 \cap V_0 \neq \emptyset$. Then there exist $x, x' \in V_0$, such that $\gamma x = Ax + a = x'$.
Hence, from the triangle inequality $\|a\| = \|x' - Ax\| \leq \|x'\| + \|Ax\| < 2r$ and $\gamma \in O(n) \times V_0'$. $\square$

Proposition 2. Let Γ be a subgroup of the group $E(n)$. The following conditions are equivalent:

1. Γ acts properly discontinuously on $\mathbb{R}^n$;
2. $\forall x \in \mathbb{R}^n$, Γx is a discrete subset of $\mathbb{R}^n$;
3. Γ is a discrete subgroup of $E(n)$.

Proof. Let Γ act properly discontinuously on $\mathbb{R}^n$. We claim that Γ is discrete. Let elements $\{\gamma_n\} \subset \Gamma$ converge to the identity. By assumption there is a neighbourhood U_0 of 0 such that the set $\{\gamma_i \mid U_0 \cap \gamma_i U_0 \neq \emptyset\}$ is finite. Hence $\gamma_i = (I, 0)$ for large i and the sequence $\{\gamma_n\}$ is eventually constant. In general case if $\gamma_n \to \gamma$ then $\gamma_n \gamma^{-1} \to (I, 0)$ and from previous consideration, the implication (i) $\to$ (iii) is proved.
We shall use the previous lemma for the proof of the reverse implication. Let $x \in \mathbb{R}^n$ be any point and V_x be a disk of radius r centered at x. By definition of properly discontinuous action and the lemma we have

$$\{\gamma \in \Gamma \mid \gamma V_x \cap V_x \neq \emptyset\} = \{\gamma \in \Gamma \mid t_{-x} \gamma t_x V_0 \cap V_0 \neq \emptyset\} \subset t_{-x} \Gamma t_x \cap (O(n) \times V_0')$$

Since Γ is discrete and hence also closed, the above set is finite and the implication (iii) $\to$ (i) is proved.

Let us assume the condition (i) (or equivalently (iii)). We have to prove that for any $x \in \mathbb{R}^n$ the set Γx is discrete. Suppose it is not. Then there is $y \in \mathbb{R}^n$ and a sequence $\{\gamma_i x = A_i x + a_i\}$, which is not eventually constant and converges to y. Since the group $O(n)$ is compact, the sequence $\{A_i\}$ converges to some $A \in O(n)$. We claim that the sequence $\{a_i\}$ converges to $-Ax + y$. In fact, the value

$$\|a_i + Ax - y\| \leq \|a_i + A_i x - y\| + \|Ax - A_i x\|$$

can be arbitrarily small for large i. Summing up, we showed that the sequence $\{\gamma_i\}$ converges to $\gamma = (A, -Ax + y)$ in $E(n)$. Hence, $\{\gamma_i \gamma_{i+1}^{-1}\}$ converges to the identity. Since Γ is discrete, a sequence $\{\gamma_i x\}$ is eventually constant. This contradicts our assumptions and proves the implication (i) $\to$ (ii).
Finally, we prove that (i) follows from (ii). Let $\{\gamma_n\}$ be a convergent sequence in Γ. Then, for any $x \in \mathbb{R}^n$, $\{\gamma_n x\}$ is a convergent sequence in Γx. By definition it is eventually constant. Hence a sequence $\{\gamma_n\}$ is also eventually constant. $\square$

Proposition 3. A discrete subgroup of $E(n)$ acts freely on $\mathbb{R}^n$ if and only if it is torsion free (i.e it has no elements of finite order).

Proof. Assume that a group Γ has an element γ of order k. For any $x \in \mathbb{R}^n$ the element $x + \gamma x + \gamma^2 x + \cdots + \gamma^{k-1} x$ is invariant under the action of γ. Hence the action of Γ is not free. The reverse implication follows from the equality of the sets

$$\{\gamma \in \Gamma \mid \gamma a = a\} = \Gamma \cap t_a(O(n) \times 0)t_{-a}$$

where $a \in \mathbb{R}^n$. In fact, since the orthogonal group $O(n)$ is compact and Γ is discrete, the above set is always finite. □

Definition 3. Let G be a differential manifold and simultaneously a group, such that the group operation and inversion are differential maps of the manifolds. Then G is called a Lie group.

Example 2. The sphere $S^1 \subset \mathbb{C}$, with the multiplication of complex numbers, is a Lie group. The Cartesian product of S^1, i.e. n-dimensional torus $T^n = \left(S^1\right)^n$, is a Lie group.
The matrix groups $U(n), O(n), SL(n,\mathbb{R}), SL(n,\mathbb{C})$ are Lie groups.
The group $(\mathbb{Z}/2\mathbb{Z})^{\mathbb{N}}$ is not a Lie group.

Definition 4. Let Γ be a subgroup of $E(n)$. The orbit space of the action of Γ on $\mathbb{R}^n$ is defined to be the set of Γ-orbits $\mathbb{R}^n/\Gamma = \{\Gamma x \mid x \in \mathbb{R}^n\}$ topologized with the quotient topology from $\mathbb{R}^n$. The quotient map will be denoted by $\pi : \mathbb{R}^n \to \mathbb{R}^n/\Gamma$.

Lemma 3. *If Γ is a subgroup of $E(n)$, then the natural projection map $p : E(n) \to E(n)/\Gamma$ and the projection on the orbit space $\pi : \mathbb{R}^n \to \mathbb{R}^n/\Gamma$ are open and closed.*

Proposition 4. Let Γ be a subgroup of $E(n)$. Then the orbit space $\mathbb{R}^n/\Gamma$ is compact if and only if the space of cosets $E(n)/\Gamma$ is compact.

Proof. By definition $E(n)/O(n) = \mathbb{R}^n$.
A group Γ acts on the space $E(n)/O(n)$ by $gO(n) \mapsto (\gamma g)O(n)$, where $\gamma \in \Gamma, g \in E(n)$. The above action agrees with a standard action of Γ on $\mathbb{R}^n$.
Next, let us note that the map $E(n)/\Gamma \to (E(n)/O(n))/\Gamma = \mathbb{R}^n/\Gamma$, given by

$$g^{-1}\Gamma \to \Gamma(gO(n))$$

is a continuous open map with compact fibers. Hence it follows that $E(n)/\Gamma$ is compact if and only if $\mathbb{R}^n/\Gamma$ is compact. □

Lemma 4. *A space $E(n)/\Gamma$ is compact if and only if there exists a compact subset $D \subset E(n)$, such that $E(n) = D\Gamma$.*

Proof. Since $E(n)$ is a subset of $\mathbb{R}^{(n+1)^2}$, there exists a family of open sets $U_k \cap E(n)$, such that the family of sets $p\left(U_k \cap E(n)\right)$ covers the compact space $E(n)/\Gamma$.
Here U_k is an open disk centered at origin and of radius k. Hence there exists k_0, such that $p\left(U_{k_0} \cap E(n)\right) = E(n)/\Gamma$.
Let D be a closure of the set $(U_{k_0} \cap E(n))$, i.e. $D = \overline{(U_{k_0} \cap E(n))}$. Finally, we have

$$E(n) = D\Gamma$$

The proof of the reverse implication follows from the equality $D/\Gamma = E(n)/\Gamma$. $\square$

Definition 5. A subgroup $\Gamma \subset E(n)$ is cocompact, if the space $E(n)/\Gamma$ is compact.

Roughly speaking a fundamental domain for a group Γ of isometries in a metric space X is a subset of X which contains exactly one point from each of these orbits.

Definition 6. Let X be a metric space and G a subgroup of a group of its isometries. An open, connected subset $F \subset X$ is a fundamental domain if

$$X = \bigcup_{g \in G} g\bar{F}$$

and $gF \cap g'F = \emptyset$, for $g \neq g' \in G$.

2 Bieberbach Theorems

Definition 7. A crystallographic group of dimension n is a cocompact and discrete subgroup of $E(n)$.
A Bieberbach group is a torsion free crystallographic group.

Example 3. If

$$\left(B, \begin{pmatrix} 0 \\ 1/2 \end{pmatrix}\right), \left(I, \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) \in E(2) \text{ where } B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Then the group $\Gamma \subset E(2)$ generated by the above elements is a crystallographic group of dimension two and the orbit space $\mathbb{R}^2/\Gamma$ is the Klein bottle.

The first part of the eighteenth Hilbert problem was about the description of discrete and cocompact groups of isometries of $\mathbb{R}^n$. The German mathematician L. Bieberbach answers this problem.

Theorem 1 (Bieberbach)**.**

1. *If $\Gamma \subset E(n)$ is a crystallographic group then the set of translations $\Gamma \cap (I \times \mathbb{R}^n)$ is a torsion free and finitely generated abelian group of rank n, and is a maximal abelian and normal subgroup of finite index.*
2. *For any natural number n, there are only a finite number of isomorphism classes of crystallographic groups of dimension n.*
3. *Two crystallographic groups of dimension n are isomorphic if and only if they are conjugate in the group $A(n)$.*

2.1 Proof of the first Bieberbach theorem

We shall prove these lemmas first:

Lemma 5. *There exists a neighborhood of the identity $U \subset O(n)$ such that for any $h \in U$, if $g \in O(n)$ commutes with $[g, h] = ghg^{-1}h^{-1}$, then g commutes with h.*

Proof. Let $\lambda_1, \lambda_2, \ldots, \lambda_r$ be the eigenvalues of an orthogonal matrix $g : \mathbb{C}^n \to \mathbb{C}^n$, and let $\mathbb{C}^n = V_1 \oplus V_2 \oplus \cdots \oplus V_r$ be its invariant subspaces. Since $g[g, h] = [g, h]g$, we have

$$ghg^{-1}h^{-1} = hg^{-1}h^{-1}g.$$

Moreover, for $i = 1, \ldots, r$ and $\forall x \in V_i$, we have $gx = \lambda_i x$. Hence,

$$ghg^{-1}h^{-1}x = hg^{-1}h^{-1}gx = hg^{-1}h^{-1}\lambda_i x = \lambda_i hg^{-1}h^{-1}x$$

and $hg^{-1}h^{-1}V_i \subset V_i$. Since h and g are isomorphisms, $h^{-1}V_i = gh^{-1}V_i$. This shows that $h^{-1}V_i$ is g-invariant, and so

$$h^{-1}V_i = (h^{-1}V_i \cap V_1) \oplus (h^{-1}V_i \cap V_2) \oplus \cdots \oplus (h^{-1}V_i \cap V_r),$$

where $h^{-1}V_i \cap V_j = \{x \in h^{-1}V_i \mid gx = \lambda_j x\}$.

Let $w, v \in \mathbb{C}^n$ be such that $\|w\| = \|v\| = 1$ and $w \perp v$ in the Hermitian inner product. Then $\|w - v\| = \sqrt{2}$. Moreover, let $\|h^{-1} - I\| < \epsilon = \sqrt{2} - 1$, $i \neq j$ and suppose $0 \neq x \in (h^{-1}V_i \cap V_j)$.
We can assume $\|x\| = 1$. By definition, there is $y \in V_i$ such that $h^{-1}y = x$. But $x \in V_j$ and $\langle x, y \rangle = 0$. Since

$$\sqrt{2} = \|x - y\| = \|h^{-1}y - y\| \leq \|(h^{-1} - I)y\| < \sqrt{2} - 1,$$

we obtain a contradiction. Hence $h^{-1}V_i = V_i$ for all $i = 1, \dots, r$, and $gh = hg|_{V_i}$. In fact, the matrix of g is diagonal. Since any element of $\mathbb{C}^n$ is a sum of elements from V_i, it follows that g and h commute. Let

$$U = \{h \in O(n) \mid \|I - h^{-1}\| < \epsilon\}.$$

$\square$

Lemma 6. *There exists a neighborhood of the identity $U \subset O(n)$ such that for any $g, h \in U$, the sequence*

$$[g, h],\ [g, [g, h]],\ [g, [g, [g, h]]],\ \dots$$

converges to the identity.

Proof. Let U be a neighborhood of the identity with radius $\epsilon < 1/4$. By definition, we have

$$\|[g, h] - I\| = \|gh - hg - I + I\| = \|gh - g - h + I - hg + h + g - I\| \leq$$

$$\|(g - I)(h - I) - (h - I)(g - I)\| \leq 2\|g - I\|\|h - I\| < \frac{\|h - I\|}{2}$$

for $g, h \in U$. Hence $[g, h] \in U$, and by induction,

$$\|[g, [g, \dots, [g, h] \dots]] - I\| \leq \frac{\|h - I\|}{2^n}.$$

Hence the result. $\square$

Lemma 7. *There is an arbitrary small neighbourhood V of $I \in O(n)$ such that $\forall g \in O(n)$, $gVg^{-1} = V$.*

Proof. Let ϵ be a positive number and $V = B(I, \epsilon)$ be an open disk. By definition we have $\forall g \in O(n)$ and $\forall h \in V$ we have

$$\|ghg^{-1} - I\| = \|g(h - I)g^{-1}\| = \|h - I\| < \epsilon.$$

Since $\forall g \in O(n)$, $gVg^{-1} \subset V$ and $g^{-1}Vg \subset V$, we get $V = g(g^{-1}Vg)g^{-1} \subset gVg^{-1}$. $\square$

Definition 8. We shall call a neighbourhood U satisfying the three previous lemmas a stable neighbourhood of identity.

Lemma 8. *Let $G \subset O(n)$ be a connected subgroup, and let U be a neighborhood of the identity. Then the group $\langle G \cap U \rangle$ generated by $G \cap U$ is equal to G.*

Proof. G is a topological group, $V = G \cap U$ is a neighborhood of the identity, $< V >$ is open, hence it is closed, since G is connected and $< V >$ is non-empty, we have $< V >= G$ □

Lemma 9. *Let Γ be a crystallographic group and $x \in \mathbb{R}^n$. Then the linear space generated by the set $\{\gamma(x)\}, \gamma \in \Gamma$ is equal to $\mathbb{R}^n$.*

Proof. Assume the lemma is false and that $x_0 \in \mathbb{R}^n$ exists such that $\{\gamma(x_0)\}$ lies in W, a proper linear subspace of $\mathbb{R}^n$. without loss of generalality, we may assume $O(n)$ leaves x_0 fixed as a new origin. In fact,

$$\Gamma(x_0) = \Gamma(I, x_0)(I, -x_0)(x_0) = \Gamma(I, x_0)(0).$$

Hence sets $(I, -x_0)\Gamma(I, x_0)(0)$ and $\Gamma(x_0)$ differ by translation $(I, -x_0)$ and define linear subspace of the same dimension. It follows that for $\gamma \in \Gamma, \gamma = (A, a)$ must have $a \in W$.

Since Γ is a group, $A(W) = W$ for all $A \in p_1(\Gamma)$. Let $W^{\perp}$ be the orthogonal complement of W. Let $x \in W^{\perp}$ be an element at a distance d from the origin. It is easy to see that for any $\gamma = (A, a) \in \Gamma$,

$$\langle \gamma(x), \gamma(x) \rangle = \langle x, x \rangle + \langle a, a \rangle.$$

Hence $\|x\| \leq \|\gamma(x)\|$. Summing up, points in $W^{\perp}$ at a distance d from origin stay at least at a distance d from 0. It follows that Γ cannot have a compact fundamental domain. □

Lemma 10. *Let Γ be an abelian crystallographic group; then Γ contains only pure translations.*

Proof. Let $(B, b) \in \Gamma$, where $B \neq I$. Then we can always choose an origin and a coordinate system in $\mathbb{R}^n$ such that

$$B = \begin{pmatrix} I & 0 \\ 0 & B' \end{pmatrix},$$

where I is the $r \times r$ identity matrix, $B' - I$ is a nonsingular $s \times s$ matrix, $r + s = n$, and r can be equal to zero.

Moreover, we can assume that $b = (b', 0, \ldots, 0)$, where $b' \in \mathbb{R}^r$.

Then, there exists an element $(C, (t_1, t_2)) \in \Gamma$, where $t_1 \in \mathbb{R}^r, t_2 \in \mathbb{R}^s$ and $t_2 \neq 0$. Then, since Γ is abelian and $BCb = CBb = Cb$, we compute:

$$\begin{aligned}
(B, b)(C, (t_1, t_2)) &= (BC,\ b' + t_1,\ B'(t_2)) \\
&= (CB,\ Cb + t_1,\ t_2) \\
&= (C, (t_1, t_2))(B, b).
\end{aligned}$$

Hence $B'(t_2) = t_2$, which contradicts the nonsingularity of $B' - I$. $\square$

Lemma 11. *Let Γ be a crystallographic group. Let $p_1 : E(n) \to O(n)$ be the projection onto the first factor. Then $p_1(\Gamma)_0$ is an abelian group.*

Proof. Let $U = B(I, \epsilon)^3$, such that $\epsilon < \frac{1}{4}$
Let $\gamma_1 = (A_1, a_1), \gamma_2 = (A_2, a_2) \in (p_1^{-1}(U) \cap \Gamma)$. By recurrence we define for $i \geq 2$

$$\gamma_{i+1} = [\gamma_1, \gamma_i].$$

We have

$$\gamma_{i+1} = ([A_1, A_i], (I - A_1 A_i A_1^{-1})a_1 + A_1(I - A_i A_1^{-1} A_i^{-1})a_i).$$

Hence $A_{i+1} = [A_1, A_i]$ and

$$\|a_{i+1}\| \leq \|I - A_i\| \|a_1\| + \frac{1}{4}\|a_i\|.$$

From a previous lemma we have $\lim_{i\to\infty} A_i = I$. Hence $\lim_{i\to\infty} a_i = 0$. Since Γ is discrete, $\gamma_i = (I, 0)$ for sufficiently large i. However, we have $A_1 A_2 = A_2 A_1$. Hence any elements of the set $p_1(\Gamma)_0 \cap U$ commute, hence we prove commutativity of the group $p_1(\Gamma)_0$. $\square$

We finish the proof of the first Bieberbach Theorem.

Assume first that $\Gamma \cap (I \times \mathbb{R}^n)$ is trivial. Then p_1 is an isomorphism of Γ into $O(n)$. Since $O(n)$ is compact, the closure of $p_1(\Gamma)$ can have only a finite number of components. Hence, since $p_1(\Gamma)_0$ is abelian, Γ contains a subgroup Γ_1 of finite index which is abelian. But then Γ_1, being of finite index in Γ, is also a crystallographic group. Hence, Γ_1 consists of pure translations. Thus we see that $\Gamma \cap (I \times \mathbb{R}^n)$ is nonempty.

Let $W \subset \mathbb{R}^n$ be the subspace of $\mathbb{R}^n$ spanned by the pure translations of Γ, i.e., by $\Gamma \cap (I \times \mathbb{R}^n)$. Then, $p_1(\Gamma)$ leaves W invariant because $\Gamma \cap (I \times \mathbb{R}^n)$ is normal in Γ. Note further that $p_1(\Gamma)|_W$ is a finite group, for otherwise it would contain elements arbitrarily close to identity, which would, under inner automorphism with a basis of $\Gamma \cap (I \times \mathbb{R}^n)$, force Γ to be nondiscrete.

In fact, let $(A_i, a_i) \in \Gamma, i \in \mathbb{N}$ be an infinite sequence of elements such that $A_i \to I$. Let

$$(B_i, b_i) = (I, e_k)(A_i, a_i)(I, -e_k) = (A_i, (I - A_i)e_k + a_i),$$

where $e_k \in \Gamma \cap (I \times \mathbb{R}^n)$. Then a sequence $(B_i, b_i)(A_i^{-1}, -A_i^{-1}(a_i)), i \in \mathbb{N}$ defines a nondiscrete subset of Γ. Moreover, we see that Γ induces an action on $\mathbb{R}^n/W$ which is obviously cocompact. We claim that it is also properly discontinuous.

We have decomposition $\mathbb{R}^n = W \oplus W^\perp$, where $W^\perp \simeq \mathbb{R}^n/W$.
Let $pr_1 : \mathbb{R}^n \to W, pr_2 : \mathbb{R}^n \to W^\perp$ be projections. Let X be any discrete subset of $\mathbb{R}^n$. It can happen that sets $pr_1(X)$ and $pr_2(X)$ are not discrete subsets of W and $W^\perp$.

Since $p_1(\Gamma)|_W$ is finite, we can concentrate on elements $\gamma \in \Gamma$ such that $p_1(\gamma)$ acts as identity on W.
The orbit $\Gamma(0)$ is discrete in $\mathbb{R}^n$. By contradiction let us assume that $pr_2(\Gamma(0))$ is not discrete at $W^\perp$ and $y \in W^\perp$ is an accumulation point of $pr_2(\Gamma(0))$.
Let $pr_2(\gamma_i(0)) \to y$, where $\gamma_i \in \Gamma, i \in \mathbb{N}$. Using elements from $\Gamma \cap (I \times \mathbb{R}^n)$ we can define a sequence of elements of $\tilde{\gamma}_i \in \Gamma, i \in \mathbb{N}$ such that $\forall i \in \mathbb{N}, pr_1(\tilde{\gamma}_i(0)) \subset C \subset W$, where C is a compact set.

Here we use the fact that $\Gamma \cap \mathbb{R}^n$ is a cocompact subgroup of W. We can see that a set $\{\tilde{\gamma}_i(0)\}, i \in \mathbb{N}$ has an accumulation point at a discrete set $\Gamma(0)$. We get contradiction and our claim is proved.

Hence Γ is a crystallographic group on $\mathbb{R}^n/W$ with no pure translations. By the above, this implies the zero dimension of $\mathbb{R}^n/W$. □

2.2 Alternative Proof of Bieberbach Theorem

The proof we have presented is due to Auslander. In this section, we shall give a geometric proof given by Buser:

Theorem 1 (A.1). *Let Γ be a crystallographic group of dimension n. Then its translation subgroup has n linearly independent elements.*

Suppose $A \in O(n)$. We define

$$m(A) = \max\left\{\frac{|Ax - x|}{|x|} \mid x \in \mathbb{R}^n \backslash \{0\}\right\}$$

Let us see that we always have $|Ax - x| \leq m(A)|x|$, for $x \in \mathbb{R}^n$. Moreover, the set

$$\text{(i)} \quad E_A = \{x \in \mathbb{R}^n \mid |Ax - x| = m(A)|x|\}$$

is a non-trivial, A-invariant linear subspace. This follows from the so-called parallelogram condition[1] and the sequences of equations

$$2m^2(A)\left(|x|^2 + |y|^2\right) = 2\left(|Ax - x|^2 + |Ay - y|^2\right) = |A(x+y) - (x+y)|^2$$

$$+|A(x-y) - (x-y)|^2 \leq m^2(A)\left(|x+y|^2 + |x-y|^2\right) = 2m^2(A)\left(|x|^2 + |y|^2\right)$$

where $x, y \in E_A$. Let $E_A^\perp$ be the A-orthogonal complement of E_A. We define

$$\text{(ii)} \quad m^\perp(A) = \max\left\{\frac{|Ax - x|}{|x|} \mid x \in E_A^\perp \backslash \{0\}\right\}$$

when $E_A^\perp \neq 0$, and $m^\perp(A) = 0$ in the opposite case. Hence

$$\text{(iii)} \quad m^\perp(A) < m(A)$$

when $A \neq \mathrm{id}$. Let $x = x^E + x^\perp \in E_A \oplus E_A^\perp$. Then

$$\text{(iv)} \quad \left|Ax^E - x^E\right| = m(A)\left|x^E\right|, \quad \left|Ax^\perp - x^\perp\right| \leq m^\perp(A)\left|x^\perp\right|$$

After these elementary observations, we see that for all $A, B \in O(n)$ we have

$$m([A, B]) \leq 2m(A)m(B)$$

In fact, we have

$$[A, B] - \mathrm{id} = (A - \mathrm{id})(B - \mathrm{id}) - (B - \mathrm{id})(A - \mathrm{id})A^{-1}B^{-1}$$

Since $\left|A^{-1}B^{-1}x\right| = |x|$, it follows that

$$|[A, B]x - x| \leq m(A)m(B)|x| + m(B)m(A)|x|$$

for all $x \in \mathbb{R}^n$.

Lemma 12 (A. ("Mini Bieberbach")). *For each unit vector $u \in \mathbb{R}^n$ and for all $\epsilon, \delta > 0$ there exists $\beta = (B, b) \in \Gamma$, such that $b \neq 0$, $\angle(u, b) \leq \delta$, $m(B) \leq \epsilon$. (Here $\angle(u, b)$ denotes the angle between the vectors u, b and $\cos(\angle(u, b)) = \frac{\langle u, b\rangle}{\|b\|}$.)*

Proof. From the definition of Γ there exists d and elements $\beta_k \in \Gamma$ such that for any natural number k, we have

$$|b_k - ku| \leq d$$

Moreover $|b_k| \to \infty$, $\angle(u, b_k) \to 0$ $(k \to \infty)$. Since $O(n)$ is compact, we find a subsequence such that for $i < j$ we have

$$m\left(B_j B_i^{-1}\right) \leq \epsilon, \quad \angle(u, b_j) \leq \delta/2, \quad |b_i| \leq \frac{\delta}{4}|b_j|$$

Finally, the element $\beta_j \beta_i^{-1}$ has the required properties. □

Lemma 13 (B.). *If $\alpha = (A, a) \in \Gamma$ and $m(A) \leq \frac{1}{2}$, then α is a translation.*

Proof. Suppose $\alpha = (A, a) \in \Gamma$ satisfies our assumptions and $m(A) \neq 0$. Since Γ is a discrete group, we can assume that the number $|a|$ is minimal. From Lemma A, for $u \in E_A$, there exists $\beta = (B, b)$, such that

$$b \neq 0, \quad \left|b^\perp\right| \leq \left|b^E\right|, \quad m(B) \leq \frac{1}{8}\left(m(A) - m^\perp(A)\right)$$

Among these we consider again the one for which $|b|$ is a non-zero minimum. Observe that $|a| \leq |b|$, when β is not a translation[2]. Let $\tilde{\beta} = [\alpha, \beta]$. From the considerations preceding Lemma A, we have

$$m(\tilde{B}) = m([A, B]) \leq 2m(A)m(B) \leq m(B)$$

and

$$\tilde{b} = (A - \mathrm{id})b^E + (A - \mathrm{id})b^\perp + r$$

where

$$r = (\mathrm{id} - \tilde{B})b + A(\mathrm{id} - B)A^{-1}a$$

If β is a translation then $B = \mathrm{id} = \tilde{B}$ and $r = 0$. As we have already observed, from the choice of α, in the case when β is not a translation, we have an inequality

$$|a| \leq |b|$$

Hence

$$|r| \leq (m(\tilde{B}) + m(B))|b| \leq 2m(B)|b| < 4m(B)\left|b^E\right|$$

In each case we have

$$|r| < \frac{1}{2}\left(m(A) - m^\perp(A)\right)\left|b^E\right|$$

By definition and from (A.2) we have

$$\tilde{b}^\perp - (A - \mathrm{id})b^\perp - r^\perp = (A - \mathrm{id})b^E + r^E - \tilde{b}^E = 0$$

Hence, using (A.1) and the orthogonality of r^E and $r^\perp$, we obtain

$$\left|\tilde{b}^\perp\right| \leq m^\perp(A)\left|b^\perp\right| + \left|r^\perp\right| < m^\perp(A)\left|b^E\right| + |r|$$

Summing up, with support of (A.3), we have an inequality

$$\left|\tilde{b}^\perp\right| < \frac{1}{2}\left(m(A) + m^\perp(A)\right)\left|b^E\right|$$

On the other hand

$$\left|\tilde{b}^E\right| = \left|(A - \mathrm{id})b^E + r^E\right| \geq m(A)\left|b^E\right| - \left|r^E\right|$$

$$> m(A)\left|b^E\right| - \frac{1}{2}\left(m(A) - m^\perp(A)\right)\left|b^E\right|$$

$$= \frac{1}{2}\left(m(A) + m^\perp(A)\right)\left|b^E\right|$$

Here we apply again (A.3) and

$$|x+y| \geq ||x| - |y||$$

Finally, we can see that $\tilde{\beta}$ also satisfies the condition (A.1) and because

$$|\tilde{b}| \leq m(A)|b| + |r| < m(A) + \frac{1}{2}\left(m(A) - m^{\perp}(A)\right)\left|b^E\right| < |b|$$

we have a contradiction.

By Lemma A it follows that there exist n elements of Γ, such that their translation parts are linearly independent, and their rotation parts have norm smaller than $\frac{1}{2}$. Now, by Lemma B we can define n linearly independent translations in Γ.

For the proof of the first Bieberbach Theorem it is enough to observe that the image $p_1(\Gamma)$ of the homomorphism $p_1 : \Gamma \to O(n)$ is a discrete subgroup of the compact group $O(n)$. (exercice 2.3 et lemme 2.7) $\square$

3 Klein Geometry

Definition 9 (Klein Geometry). A Klein geometry is defined by a pair $(G, G/H)$, where G, the principal group, is a Lie group, and H is a closed subgroup (equivalently a Lie subgroup), such that the natural action of G on $X = G/H$ is transitive.
The homogeneous space X is called the space of the geometry, or by abuse of language, the Klein geometry. It follows from the definition and quotient topology that X is connected.

Since H is closed, X inherits a topological manifold structure.

Definition 10. Let $U \subset X$ be an open set. A morphism $f : U \to X$ is said to be locally-G if for every connected component C of U, there exists $g \in G$ such that $f_{|C} = g_{|C}$.

Definition 11. Given a geometry (G, X) and a manifold M of the same dimension as X, a G/H-structure on M is a maximal atlas $\mathcal{U} = \{(U_i, \varphi_i)\}$ such that:

- $\{U_i\}$ is an open cover of M.
- The morphisms $\varphi_i : U_i \to X$ are open embeddings.
- The transition maps $\varphi_i \circ \varphi_j^{-1} : \varphi_j(U_i \cap U_j) \to \varphi_i(U_i \cap U_j)$ are locally-G.

A manifold equipped with such a structure is called a (G, X)-manifold.

Example 4. For any geometry (G, X), the space X is tautologically a (G, X)-manifold such that the identity morphism is a global chart. More generally, for any open set U of X, U is a (G, X)-manifold with the identity as a global chart.

Classic examples of geometries include Euclidean geometry, affine geometry.

Example 5 (Euclidean manifolds). Let $E(n) = O_n(\mathbb{R}) \ltimes \mathbb{R}^n$ is the group of isometries of the Euclidean space $\mathbb{R}^n$, a $(E(n), \mathbb{R}^n)$-manifold is called a Euclidean manifold, or flat manifold. These manifolds are the main object of Bieberbach theorems. We will consider flat manifolds in more details after giving some main results in Klein geometry.

Example 6 (Affine manifolds). Let $A(n) = GL_n(\mathbb{R}) \ltimes \mathbb{R}^n$ be the group of affine transformations of $\mathbb{R}^n$, a $(A(n), \mathbb{R}^n)$-manifold is called an affine manifold.
A Euclidean structure on a manifold automatically gives an affine structure. Bieberbach proved that closed Euclidean manifolds with the same fundamental group are equivalent as affine manifolds.

Example 7 (Elliptic manifolds). The orthogonal group $O(n+1)$ acts on the sphere $\mathbb{S}^n \subset \mathbb{R}^{n+1}$. a $(O(n+1), \mathbb{S}^n)$-manifold is called spherical or elliptic manifold.

Example 8 (Hyperbolic manifold).

Example 9 (Complex torus). Consider the group of biholomorphisms $\mathbb{C}^\times \ltimes \mathbb{C}$. Let Λ be a lattice, for example $\mathbb{Z}[i]$. We define the complex torus $T_\mathbb{C} = \mathbb{C}/\Lambda$. Then $\mathbb{C}$ is a universal cover over $T_\mathbb{C}$, which gives the torus a $(\mathbb{C}^\times \ltimes \mathbb{C}, \mathbb{C})$-structure.

Let M and N be two (G, X)-manifolds and $f : M \to N$ a map. We say that f is a (G, X)-map if for any two charts of M and N:

$$\varphi_i : U_i \to X \quad , \quad \psi_j : V_j \to X$$

the restriction of $\psi_j \circ f \circ \varphi_i^{-1}$ to $\varphi_i(U_i \cap f^{-1}(V_j))$ is locally-G. In particular, we consider (G, X)-maps that are local diffeomorphisms. The set of (G, X)-automorphisms $M \to N$ is a group that we denote by $\mathrm{Aut}_{(G,X)}(M)$.

$$\varphi_\beta(U_\alpha \cap U_\beta) \xrightarrow{\varphi_\alpha \circ (\varphi_\beta)^{-1}} \varphi_\alpha(U_\alpha \cap U_\beta)$$

3.1 (G, X)-Automorphisms

If Ω is a non-empty connected open set, a (G, X)-automorphism $f : \Omega \to \Omega$ is the restriction of a unique element $g \in G$ preserving Ω:

$$\mathrm{Aut}(\Omega) \cong \mathrm{Stab}(\Omega) = \{g \in G : g \cdot \Omega = \Omega\}$$

We now assume $f : M \to \Omega$ is a local diffeomorphism. There exists a homomorphism:

$$f_* : \mathrm{Aut}_{(G,X)}(\Omega) \to \mathrm{Aut}_{(G,X)}(\Omega)$$

whose kernel is the set of maps $h : M \to M$ such that the following diagram commutes:

3.2 Developing map and holonomy

The following fact is essential in the study of (G, X)-structures.

Proposition 5 (Unique Extension Property). Let M and N be two (G, X)-manifolds and $f_1, f_2 : M \to N$ be two (G, X)-morphisms. If M is connected, then f_1 and f_2 are equal if and only if they coincide locally.

Proof. Let S be the set of all points of M that have an open neighborhood in which f_1 and f_2 coincide. We will show that S is both open and closed in M, which will imply the proposition. Let $x \in S$ and let U be an open neighborhood of x, such that $f_1|_U = f_2|_U$. We have $U \subset S$ which means that S is a neighborhood of each of its points, i.e., S is open. Let $x \notin S$. If $f_1(x) \neq f_2(x)$, we can find a neighborhood U of x such that $f_1(U) \cap f_2(U) = \varnothing$. Thus $U \cap S = \varnothing$ and S is closed. Suppose that $f_1(x) = f_2(x)$. Let (U, φ) be a local chart around x. By shrinking U if necessary, we can assume that $f_1(U)$ and $f_2(U)$ are contained in the domain W of a local chart (W, ψ) of N. We have $U \cap S = \varnothing$. Suppose by contradiction that there exists $x_0 \in U$ which has an open neighborhood V in which f_1 and f_2 coincide. We can always reduce V and assume that $V \subset U$. By construction, the charts $g_i = \psi \circ f_i \circ \varphi^{-1} : \varphi(V) \to \psi(W), i = 1, 2$ coincide, so they must extend to the same chart on X. Therefore, they are equal on the set $\varphi(U)$, which means that in particular f_1 and f_2 coincide on U. This contradicts the fact that $x \notin S$. Therefore, $U \cap S = \varnothing$ and S is closed. $\square$

Let M be a (G, X)-manifold. Let $p : \tilde{M} \to M$ be a universal cover with fundamental group $\pi = \pi_1(M)$. p induces a (G, X)-structure on $\tilde{M}$ on which π acts by (G, X)-automorphisms. The unique extension property implies:

Proposition 6. Let M be a simply connected (G, X)-manifold. Then there exists a (G, X)-application $f : M \to X$.

Proof. Choose a basepoint $x_0 \in M$ and a coordinate patch U_0 containing x_0. For $x \in M$, we define $f(x)$ as follows. Choose a path $\{x_t\}_{0 \le t \le 1}$ in M from x_0 to $x = x_1$. Cover the path by coordinate patches U_i (where $i = 0, \dots, n$) such that $x_t \in U_i$ for $t \in (a_i, b_i)$ where

$$a_0 < 0 < a_1 < b_0 < a_2 < b_1 < a_3 < b_2 < \cdots < a_{n-1} < b_{n-2} < a_n < b_{n-1} < 1 < b_n$$

Let $U_i \xrightarrow{\psi_i} X$ be an (G, X)-chart and let $g_i \in G$ be the unique transformation such that $g_i \circ \psi_i$ and ψ_{i-1} agree on the component of $U_i \cap U_{i-1}$ containing the curve $\{x_t\}_{a_i < t < b_{i-1}}$. Let

$$f(x) = g_1 g_2 \cdots g_{n-1} g_n \psi_n(x);$$

we show that f is indeed well-defined. The map f does not change if the cover is refined. Suppose that a new coordinate patch U' is "inserted between" U_{i-1} and U_i. Let $\{x_t\}_{a' < t < b'}$ be the portion of the curve lying inside U':

$$a_{i-1} < a' < a_i < b_{i-1} < b' < b_i$$

Let $U' \xrightarrow{\psi'} X$ be the corresponding coordinate chart and let $h_{i-1}, h_i \in G$ be the unique transformations such that ψ_{i-1} agrees with $h_{i-1} \circ \psi'$ on the component of $U' \cap U_{i-1}$ containing $\{x_t\}_{a' < t < b_{i-1}}$ and ψ' agrees with $h_i \circ \psi_i$ on the component of $U' \cap U_i$ containing $\{x_t\}_{a_i < t < b'}$. By the unique extension property $h_{i-1} h_i = g_i$ and it follows that the corresponding developing map

$$f(x) = g_1 g_2 \cdots g_{i-1} h_{i-1} h_i g_{i+1} \cdots g_{n-1} g_n \psi_n(x)$$

$$= g_1 g_2 \cdots g_{i-1} g_i g_{i+1} \cdots g_{n-1} g_n \psi_n(x)$$

is unchanged. Thus the developing map as so defined is independent of the coordinate covering, since any two coordinate coverings have a common refinement.

Next we claim the developing map is independent of the choice of path. Since M is simply connected, any two paths from x_0 to x are homotopic. Every homotopy can be broken up into a succession of "small" homotopies, that is, homotopies such that there exists a partition

$$0 = c_0 < c_1 < \cdots < c_{m-1} < c_m = 1$$

such that during the course of the homotopy the segment $\{x_t\}_{c_i<t<c_{i+1}}$ lies in a coordinate patch. It follows that the expression defining $f(x)$ is unchanged during each of the small homotopies, and hence during the entire homotopy. Thus f is independent of the choice of path.

Since f is a composition of a coordinate chart with a transformation $X \to X$ coming from G, it follows that f is an (G,X)-map. $\square$

For M an arbitrary (G,X)-manifold, we shall apply this proposition to its universal cover $\tilde{M}$:

Theorem 2. *Let M be a (G,X)-manifold and $\pi : \tilde{M} \longrightarrow M$ its universal cover. Then there exists a pair (dev, h) consisting of a (G,X)-map $\mathrm{dev} : \tilde{M} \longrightarrow X$ and a homomorphism $h : \pi_1(M) \longrightarrow G$ such that for each $\gamma \in \pi_1(M)$, the following diagram commutes:*

$$\begin{tikzcd} \tilde{M} \arrow[r, "\mathrm{dev}"] \arrow[d, "\gamma"'] & X \arrow[d, "h(\gamma)"] \\ \tilde{M} \arrow[r, "\mathrm{dev}"'] & X \end{tikzcd}$$

If (dev', h') is another pair, then there exists $g \in G$ such that $\mathrm{dev}' = g \circ \mathrm{dev}$ and $h'(\gamma) = \mathrm{nn}(g) \circ h(\gamma)$ for all $\gamma \in \pi_1(M)$. That is, we have the following commutative diagram:

$$\begin{tikzcd} \tilde{M} \arrow[r, "\mathrm{dev}"] \arrow[d, "\gamma"'] & X \arrow[r, "g"] \arrow[d, "h(\gamma)"] & X \arrow[d, "h'(\gamma)"] \\ \tilde{M} \arrow[r, "\mathrm{dev}"'] & X \arrow[r, "g"'] & X \end{tikzcd}$$

Definition 12. We say that a (G,X)-manifold M is complete if the developing map is a covering map.

Proposition 7 (The holonomy group characterizes the manifold). If G is a group that acts analytically by diffeomorphisms on a simply connected topological space X, any complete (G,X)-manifold can be constructed from its holonomy group Γ as the quotient space X/Γ.

Proof. Since X is simply connected and M is complete, by the uniqueness of the universal cover:

$$\tilde{M} \cong X$$

The development being a diffeomorphism, the action of the fundamental group on $\tilde{M}$ is transported via dev to X such that for all $x \in X, \gamma \in \pi_1(M)$, we have:

$$\gamma \cdot x = \mathrm{dev}(\gamma \cdot \mathrm{dev}^{-1}(x)) = \mathrm{hol}(\gamma) \cdot x$$

By the commutative diagram:

Thus the action of the holonomy group coincides with that of the fundamental group which is faithful. Thus the holonomy is injective, by the isomorphism theorem:

$$M \cong X/\pi_1(M) \cong X/\Gamma$$

□

Lemma 14 (Eckmann-Hilton Principle). *We assume that a set X is equipped with two unital magma structures $*$ and $\cdot$, and such that for $x, x', y, y' \in X$ we have:*

$$(x \cdot x') * (y \cdot y') = (x * y) \cdot (x' * y')$$

then the two laws coincide and X is a commutative monoid. In particular, the fundamental group of a topological group is abelian.

Proof. Let $x = y' = 1_*$ and $x' = y = 1_.$. This gives $1_. = 1_* = 1$. Now let $x' = y = 1$, this gives:

$$x * y' = x \cdot y'$$

Thus $* = \cdot$, we then deduce associativity and commutativity:

$$\begin{gathered} x = 1 \Rightarrow x' * (y * y') = y * (x' * y') \\ x = y' = 1 \rightarrow x' * y = y * x' \end{gathered}$$

Thus X is equipped with a commutative monoid structure. We show that the fundamental group $\pi = \pi_1(G)$ of a topological group $(G, \cdot)$ is abelian. The law of G induces a second law on π (concatenation $+$), for γ, δ two loops in G:

$$\gamma * \delta : t \mapsto \gamma(t) \cdot \delta(t)$$

This law is well defined as the multiplication of G is continuous. So that:

$$[\gamma * \delta] = [t \mapsto \gamma(t) \cdot \delta(t)] = [t \mapsto \gamma(t)] * [t \mapsto \delta(t)] = [\gamma] * [\delta]$$

Let $a, b, c, d \in \pi$, we have:

$$\begin{aligned} (a + b) * (c + d) &= t \mapsto (a + b)(t) \cdot (c + d)(t) \\ &= (t \mapsto a(t) \cdot c(t)) + (t \mapsto b(t) \cdot d(t)) \end{aligned}$$

By Eckmann-Hilton, π is abelian. □

We replace the space X with its universal cover when X is not simply connected. There exists a covering group $\tilde{G}$ acting on $\tilde{X}$ by homeomorphisms. We can then describe $\tilde{G}$ by the extension:

$$1 \to \pi_1(X) \to \tilde{G} \to G \to 1$$

We say that a group G admits a local section (local cross-section) with respect to a closed subgroup $H < G$ if we have the data:

- a neighborhood $U \subset G/H$ of the identity.
- a subset $S \subset G$ such that the canonical projection restricted to S is a homeomorphism onto U. We call the local section the map $\sigma : U \to G$ such that $\pi(\sigma(gH)) = gH$ and $\sigma(1H) = 1$.

This means that locally around the neutral class, we can continuously choose a unique representative of each class gH. We then study the structure of $\pi_1(X)$ when X is not simply connected:

- Let X be a manifold and $G \subset \mathrm{Homeo}(X)$ transitive, let $x \in X$ and G_x its stabilizer (closed in G), suppose that G admits a local section with respect to G_x and that the morphism $\rho : g \mapsto gx$ is open. We construct a homeomorphism $\phi : \pi_1(X) \to \pi_0(G_x)$. And we will show that the kernel of ϕ is central. In particular, if G_x is arcwise connected then $\pi_1(X)$ is abelian.

Lemma 15. *If G is a Lie group, every closed subgroup (i.e., a Lie subgroup) has a local section. The map ρ is open.*

Definition 13 (Riemannian metric)*.* Let M be a (topological) smooth manifold. A Riemannian metric g on M consists of a smooth function $q : TM \to \mathbb{R}$ whose restrictions $q_x : T_xM \to \mathbb{R}$ are all nondegenerate quadratic forms.

Definition 14 (Geodesic)*.* Let M be a metric space with distance d and let $I \subset \mathbb{R}$ be an interval. A curve (continuous map) $\gamma : I \to M$ is called a geodesic if there exists a constant $v \geq 0$ such that for every $t \in I$ there is a neighborhood $J \subset I$ of t with the property

$$d\big(\gamma(t_1), \gamma(t_2)\big) = v\,|t_1 - t_2| \quad \text{for all } t_1, t_2 \in J.$$

If $v = 1$ the curve is said to be unit-speed.

Proposition 1 (compact stabilizers imply completeness)**.** *Let G be a lie group acting analytically and transitively on a manifold X, and such that the stabilizer G_x of x is compact for some x (hence all by transitivity). Then every closed (G, X)-manifold M is complete.*

We shall prove the following lemma:

Lemma 16 (existence of invariant metric)**.** *Let G act transitively on an analytic manifold X. Then X admits a G-invariant Riemannian metric if and only if, for some $x \in X$, the image of G_x in $GL(T_xX)$ has a compact closure.*

Proof. If G preseves a metric G_x maps to a subgroup of $O(T_xX)$, hence its closure is compact.
To prove the converse, fix x and assume that the image of G_x has a compact closure $H_x = \overline{\rho(G_x)}$ where ρ is the tangent representation. Let Q be any positive definite form on T_xX.

$$Q : T_xX \times T_xX \longrightarrow \mathbb{R}$$

H_x is compact, hence unimododular, so it admits a bi-invariant Haar measure μ, define a new inner product on T_xX by:

$$\langle v, w \rangle_x = \int_{H_x} Q(h \cdot v, h \cdot w) d\mu(h)$$

Right-invariance of μ shows immediately that for every $k \in H_x$,

$$\langle k \cdot v, k \cdot w \rangle_x = \int_{h \in H_x} Q\big(hk \cdot v, hk \cdot w\big) d\mu(h) = \langle v, w \rangle_x,$$

so $\langle \cdot, \cdot \rangle_x$ is H_x-invariant.

Now, since the action is transitive, for an arbitrary point $y \in X$ there exists $g \in G$ with $g \cdot x = y$, and transport the inner product at x to T_xX by

$$\langle u, w \rangle_y = \langle d(g)_y^{-1}(u),\, d(g)_y^{-1}(w) \rangle_x.$$

If $g' \in G$ is another element with $g' \cdot x = y$, then $g' = g\,h$ for some $h \in G_x$, and the G_x-invariance of $\langle \cdot, \cdot \rangle_x$ guarantees that $\langle \cdot, \cdot \rangle_y$ is well-defined. □

Remark. Using a local cross section we can prove that this G-invariant metric is analytic.

Proof of compact stabilizers imply completeness. transitively imply that the given condition at one point x is equivalent to the same condition everywhere. So fix $x \in X$ and let T_xX be the tangen space to X at x. There is an analytic homomorphism of G_x to $GL(T_xX)$ whose image is compact.
If M is any (G, X)-manifold, we can use charts to pull-back the G-invariant Riemannian metric from X to M (invariancy guarantees that such metric is well-defined), the resulting metric is a Riemannian metric on M invariant under any (G, X)- morphism. In a manifold endowed with such metric, we can find for any point y a ball $B_\epsilon(y)$ that is ball-like (homeomorphic image of the round ball) and convex. if M is closed we can choose ϵ uniformly by compacteness. Without loss of generalality, we may assume that all ϵ-balls in X are contractible and convex since G is a transitive group of isometries.
Then, for any $y \in \tilde{M}$, the ball $B_\epsilon(y)$ is mapped homeomorphically by dev, for if $\mathrm{dev}(y) = \mathrm{dev}(y')$ for $y \neq y'$ in the ball, the geodesic connecting y and y' maps to a self-intersecting geodesic in X contradicting the convexity of the ϵ-balls in X.
The map dev is an isometry between $B_\epsilon(y)$ and $B_\epsilon(\mathrm{dev}(y))$ by definition.
Take $x \in X$ and $y \in \mathrm{dev}^{-1}(B_{\epsilon/2})(x)$. The ball $B_\epsilon(y)$ maps isometrically, and thus

contains a copy of $\text{dev}^{-1}(B_{\epsilon/2})(x)$. The inverse image of $\text{dev}^{-1}(B_{\epsilon/2})(x)$ is then the disjoint union of these copies. This proves that dev is indeed a covering map and M is hence complete. □

Example 10. Let Γ a finite subgroup of $O(4)$ acting freely on $\mathbb{S}^3$, an elliptic 3-manifold is the orbit space $M = \mathbb{S}^3/\Gamma$, that is a $(O(4), \mathbb{S}^3)$-manifold. Such manifold is a closed manifold by definition. The proposition says that the universal cover of M is $\mathbb{S}^3$.

We shall give an equivalency between metric completeness and completeness of (G, X)-manifolds, which justifies the use of the word:

Proposition 2 (completeness equivalency)**.** *Let G be a group acting transitively and analytically on X with compact stabilizers G_x. Fix a G-invariant metric on X and let M be a (G, X)-manifold. The following conditions are equivalent:*

1. *M is a complete (G, X)-manifold.*
2. *For some $\epsilon > 0$, every closed ϵ-ball in M is compact.*
3. *For every $a > 0$, every closed a-ball in M is compact.*
4. *There is some family of compact subsets S_t of M, for $t \geq 0$, such that $\cup_{t\geq 0} S_t = M$ and S_{t+a} contains a neighborhood of radius a about S_t.*
5. *M is a complete metric space.*

Proof. $(1) \implies (2)$. if $p : Y \longrightarrow Z$ is a covering map between two manifolds endowed mith a Riemannian metric and p preseves this metric, we have $\overline{B}_\epsilon(p(y)) = p(\overline{B}_\epsilon(y))$ for any $y \in Y$ and any $\epsilon > 0$, because distances are defined in terms of path lenghts and paths in Z can be lifted to paths in Y. So the compacteness of balls in Y implies the same in Z and conversely. Fixing a point $x \in X$ and compact ϵ-ball around x by the local compacity of X, the transitive action of G implies the same for all points in X. Idem for $\tilde{M}$ and M.

$(2) \implies (3)$. We show this by induction, suppose that (3) is true for some $a > \epsilon$. Then $\overline{B}_a(x)$ can be covered with finitely mainy $\epsilon/2$-balls, and therefore $\overline{B}_{a+\epsilon/2}(x)$ can be covered with finitely mainy ϵ-balls hence compact.

$(3) \implies (4)$. Let $S_t = \overline{B}_t(x)$ where x is fixed.

$(4) \implies (5)$. Any Cauchy sequence must be conatined in some S_t for some t, hence it converges.

$(5) \implies (1)$. Suppose M metrically complete. We will prove that any path α_t in X can be lifted to $\tilde{M}$, since local homeomorphisms with the path lifting property are covering maps.
The universal cover of a complete metric space is complete M since the projection of

Cauchy sequence converges to some point $x \in M$. Since x has a compact neighborhood which is evenly covered and these are separated in the metric of $\tilde{M}$, the Cauchy sequence also converges in $\tilde{M}$.
Consider now any path α_t in X. If it has a lifting $\tilde{\alpha}_t$ for t in $[0, s]$, then it has a lifting for $[0, s + \epsilon)$ for some $\epsilon > 0$ by the local homeomorphicity of dev. If it has a lifting for t in a half-open interval $[0, s)$, the lifting extends by metric completeness. Thus, M is complete. □

3.3 Flat manifolds

Definition 15. Let M be a smooth manifold. A Euclidean geometry on M is a Klein geometry on M modeled on $(O_n(\mathbb{R}) \ltimes \mathbb{R}^n, \mathbb{R}^n)$.
We say that M is a flat manifold.

Definition 16 (Riemannian manifold). Let M be a smooth manifold, g a Riemannian metric on M. The pair (M, g) is then a Riemannian manifold.

Given two Riemannian manifold (M, g_M) and (N, g_N), we define the product:

$$(M \times N, g_M \oplus g_N)$$

which is a Riemannian manifold.

Let $f : M \longrightarrow L$ a smooth immersion, where M is a manifold and (L, g_L) a Riemannian manifold, the pull-back metric

$$(f^* g_L)_p : (x, y) \mapsto (g_L)_{f(p)}(df_p(x), df_p(y))$$

is a Riemannian metric, and M a Riemannian manifold.

Example 11. The space $\mathbb{R}^n$ equipped with its inner product is a Riemannian manifold.

Remark. This makes any smooth manifold, that is embedded in some $\mathbb{R}^n$ by Whitney theorem, a Riemannian manifold.

Remark. The above definition of a flat manifold M is equivalent to saying that M is locally isometric to some $\mathbb{R}^n$. In essence, this means that locally, every part of the manifold is indistinguishable from a region of Euclidean space in terms of its geometric properties like distances, angles, and straight lines.
M inherits the usual Riemannian metric from $\mathbb{R}^n$ such that these manifolds are exactly Riemannian manifolds with zero curvature.

When M is a flat and geometrically complete manifold (equivalent to being metrically complete), we have the identification:

$$M \cong \mathbb{R}^n / \pi_1(M)$$

The action of $\pi_1(M) < E(n)$ is free and properly discontinuous. Hence, it is discrete.

Definition 1. Let $\Gamma < \mathbb{R}$ be a lattice of translations. A flat torus is the orbit space $\mathbb{R}^n / \Gamma$

Now we can give the geometric version of Bieberbach's theorem is:

Theorem 3 (1st Bieberbach)**.** *Let M be a compact flat n-manifold. $\pi_1(M)$ contains a torsion free and finitely generated abelian group of rank n and is a maximal abelian and normal subgroup of finite index. i.e Every compact flat manifold is a quotient of a flat torus $\mathbb{R}^n/\Gamma$ by a finite subgroup that acts freely on $\mathbb{R}^n/\Gamma$.*

We then have the following exact sequence:

$$0 \longrightarrow T = \pi_1(M) \cap \mathbb{R}^n \longrightarrow \pi_1(M) \longrightarrow H = \pi_1(M)/T \longrightarrow 0$$

where H is then a finite group. $\iota : \mathbb{Z}^n \hookrightarrow \Gamma$ is an inclusion map which maps e_i to (I_n, e_i) where $e_1, \ldots, e_n$ are the standard basis of $\mathbb{Z}^n$ and $p : \Gamma \to H$ is a projection map which maps $(A, a) \in \Gamma$ to A. Besides, the group $\mathbb{Z}^n$ is a maximal abelian subgroup. Given such a short exact sequence, it induces a representation $\rho : H \to GL_n(\mathbb{Z})$ given by $\rho(g)x = \bar{g}\iota(x)\bar{g}^{-1}$, where $x \in \mathbb{Z}^n$ and $\bar{g}$ is chosen arbitrarily such that $p(\bar{g}) = g$.

Lemma 17. *The induced representation $\rho : H \to GL_n(\mathbb{Z})$ is a faithful representation. In other words, the kernel of ρ is trivial.*

Proof. Let $g \in \ker(\rho)$. We have $\rho(g) = id_{\mathbb{Z}^n}$. It follows that

$$x = \rho(g)x = \bar{g}\iota(x)\bar{g}^{-1}$$

where $x \in \mathbb{Z}^n$ and $\bar{g}$ is chosen arbitrarily such that $p(\bar{g}) = g$. Thus $\bar{g}$ commutes with any translation of Γ. Since $\mathbb{Z}^n$ is the maximal abelian subgroup, we can conclude that $g = I_n$. Therefore ρ is a faithful representation. □

We recall that $O(n)$ is a maximal compact subgroup of $GL_n(\mathbb{R})$ up to conjugacy. Hence any finite subgroup of $GL_n(\mathbb{R})$ is included in $O(n)$ up to conjugacy.

Theorem 4 (Zassenhaus)**.** *A group Γ is isomorphic to a crystallographic group of dimension n if and only if Γ contains a normal, free abelian subgroup $\mathbb{Z}^n$ of finite index which is a maximal abelian subgroup of Γ*

Proof. First, assume that Γ is an n-crystallographic group. The first Bieberbach theorem guarantees that the group of translations $T = \Gamma \cap (I \ltimes \mathbb{R}^n$ is a normal, free abelian subgroup of finite index which is a maximal abelian subgroup of Γ.

For the other direction, assume that Γ contains a normal, free abelian subgroup $\mathbb{Z}^n$ of finite index which is a maximal abelian subgroup of Γ:

$$0 \longrightarrow \mathbb{Z}^n \xrightarrow{\iota} \Gamma \xrightarrow{p} G \longrightarrow 1$$

Where $G \cong \Gamma/\mathbb{Z}^n$ is a finite group. Given such short exact sequence, it induces a representation:

$$h_\Gamma : G \longrightarrow GL_n(\mathbb{Z})$$

. Since $\mathbb{Z}^n$ is maximal abelian group, by the previous lemma, h_Γ is faithful. We can view the free abelian group $\mathbb{Z}^n$ as a subgroup of $\mathbb{R}^n$. Thus we have the following diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathbb{Z}^n \arrow[r, "i"] \arrow[d, "i'"] & \Gamma \arrow[r, "p"] \arrow[d] & G \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathbb{R}^n \arrow[r] \arrow[d, equal] & \Gamma' \arrow[r] \arrow[d] & G \arrow[r] \arrow[d, "h_\Gamma"] & 0 \\
0 \arrow[r] & \mathbb{R}^n \arrow[r] & GL_n(\mathbb{R}) \ltimes \mathbb{R}^n \arrow[r] & GL_n(\mathbb{R}) \arrow[r] & 0
\end{tikzcd}$$

Where Γ' is the pushout of the monomorphisms $i : \mathbb{Z}^n \longrightarrow \Gamma$ and $i' : \mathbb{Z}^n \longrightarrow \mathbb{R}^n$. Notice that all vertical arrows are injective. By proposition (cf Cohomology), we have $H^2(G, \mathbb{R}^n) = 0$. That is (see remark) Γ' is isomorphic to $G \ltimes \mathbb{R}^n$ where the group action of G on $\mathbb{R}^n$ is given by h_Γ. As we noted above, any finite subgroup of $GL_n(\mathbb{R}^n)$ is conjugate to a finite subgroup of $O(n)$. Therefore we can conclude that Γ is an n-crystallographic group. $\square$

4 Wallpaper groups

Define $\pi : E(2) \longrightarrow O(2)$ to be the canonical projection. Denote $T = \ker(\pi) \cong \mathbb{R}^2$ the translations subgroup, for a subrgoup G, write $H = T \cap G$ its translations group and $J = \pi(G)$ its point group.

Definition 17 (Wallpaper group). A subgroup $G < E(2)$ is a wallpaper group if its translations subgroup is generated by two independent elements and its point subgroup is finite.

From now on G will denote a wallpaper group with translation subgroup H and point group J. Let L be the orbit of the origin under the action of H on $\mathbb{R}^2$. The set L certainly contains two independent vectors because H is generated by two independent translations. Select a non-zero vector a of minimum length in L, then choose a second vector b from L which is skew to a and whose length is as small as possible.

Theorem 2. *The set L is the lattice spanned by a and b. That is to say, $L = m\mathbb{Z} + n\mathbb{Z}$.*

Proof. The correspondence $(I, v) \to v$ is an isomorphism between T and the additive group $\mathbb{R}^2$ which sends H to L. Therefore L is a subgroup of $\mathbb{R}^2$ and every point $ma+nb$ of the lattice spanned by a and b belongs to L. Using the points of this lattice we can divide up the plane into parallelograms. If x belongs to L yet is not in the lattice, choose a parallelogram which contains x and a corner c of this parallelogram which is as close as possible to x. Then the vector $x - c$ is not the zero vector, is not equal to a or to b, and its length is less than $\|b\|$. But $x - c$ belongs to L because x and c both lie in L. We cannot have $\|x - c\| < \|a\|$ since a is supposed to be of minimum length in L. On the other hand, if $\|a\| \leq \|x - c\| < \|b\|$, then $x - c$ must be skew to a and contradicts our choice of b. Therefore, no such point x can exist and L is the lattice spanned by a and b. $\square$

We shall classify lattices into five different types according to the shape of the basic parallelogram determined by the vectors a and b. From properties of the lattice of G and the point group of G we plan to build up information about G itself. Replace b by $-b$ if necessary to ensure that

$$\|a - b\| \leq \|a + b\|$$

With this assumption the different lattices are defined as follows:

(a) Oblique $\|a\| < \|b\| < \|a - b\| < \|a + b\|$

(b) Rectangular $\|a\| < \|b\| < \|a - b\| = \|a + b\|$

(c) Centred Rectangular $\|a\| < \|b\| = \|a - b\| < \|a + b\|$

(d) Square $\|a\| = \|b\| < \|a - b\| = \|a + b\|$

(e) Hexagonal $\|a\| = \|b\| = \|a - b\| < \|a + b\|$

Remark. Wallpaper groups occur in the literature under a variety of aliases, the most common being "plane crystallographic group".
If we imagine a similar scenario in three dimensions, the corresponding lattice is spanned by three independent vectors and gives a configuration of points which models the internal atomic structure found in crystals.

The point group J is a subgroup of O_2. However, there may be no copy of J inside G. Consider the wallpaper group which is generated by the translation $\tau(x, y) = (x+1, y)$ and the glide reflection $h(x, y) = (-x, y+1)$. The line of the glide is the y-axis and is perpendicular to the direction of the translation. Here the point group is the subgroup

$$J = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

of O_2. The group G, however, consists entirely of translations and glide reflections, all of which have infinite order. Therefore, G cannot contain a copy of J. Carrying out the glide reflection h twice gives the translation $(x, y) \to (x, y+2)$ and the lattice in this example is spanned by the vectors $a = (1, 0)$, $b = (0, 2)$. Observe that not all the elements of G send this lattice to itself. None the less the point group does preserve the lattice.

Theorem 3. *The point group J acts on the lattice L.*

Proof. The point group, being a subgroup of O_2, acts on the plane in the usual way. If $M \in J$, and if $x \in L$, we must show that $f_M(x) \doteq xM^t$ belongs to L. Suppose $\pi(g) = M$ where $g = (M, v)$ and let τ denote the translation (I, x). Since H is the kernel of the homomorphism $\pi : G \to J$, it is a normal subgroup of G, and therefore $g\tau g^{-1}$ lies in H. But

$$\begin{aligned} g\tau g^{-1} &= (M, v)(I, x)\left(M^{-1}, -f_M^{-1}(v)\right) \\ &= (M, v)\left(M^{-1}, x - f_M^{-1}(v)\right) \\ &= \left(MM^{-1}, v + f_M\left(x - f_M^{-1}(v)\right)\right) \\ &= (I, v + f_M(x) - v) \\ &= (I, f_M(x)) \end{aligned}$$

consequently, $f_M(x)$ is a point of the lattice L as required. □

We recall from that finite subgroups of O_2 are either cyclic or dihedral. The next result tells us which of these subgroups can conceivably arise as the point group of a wallpaper group. It is referred to as the "crystallographic restriction".

Theorem 4. *The order of a rotation in a wallpaper group can only be 2, 3, 4, or 6.*

Proof. Every rotation in a wallpaper group G has finite order because the point group is finite. If we have a rotation of order q, then a suitable power of this rotation is an anti-clockwise rotation through $2\pi/q$. Therefore the rotation matrix

$$A = \begin{bmatrix} \cos\left(\frac{2\pi}{q}\right) & -\sin\left(\frac{2\pi}{q}\right) \\ \sin\left(\frac{2\pi}{q}\right) & \cos\left(\frac{2\pi}{q}\right) \end{bmatrix}$$

belongs to J. As before, we use a to denote a non-zero vector of shortest length in the lattice L of G. Now J acts on L, so $f_A(a)$ lies in L. Suppose q is greater than 6. Then $2\pi/q$ is less than $\pi/3$ and $f_A(a) - a$ is a vector in L which is shorter than a, contradicting our choice of a. If q is equal to five the angle between $f_A^2(a)$ and $-a$ is $\pi/5$. This time $f_A^2(a) + a$ lies in L and is shorter than a, and again we have a contradiction. □

From this theorem follows:

Corollary. The point group of a wallpaper group is generated by a rotation through one of the angles $0, \pi, 2\pi/3, \pi/2, \pi/3$ and possibly a reflection.

Theorem 5. *An isomorphism between wallpaper groups takes translations to translations, rotations to rotations, reflections to reflections and glide reflections to glide reflections.*

Proof. Let $\varphi : G \to G_1$ be an isomorphism between wallpaper groups, and let τ be a translation in G. Translations and glides have infinite order, whereas rotations and reflections are of finite order; therefore, $\varphi(\tau)$ must be either a translation or a glide. Assume $\varphi(\tau)$ is a glide and choose a translation τ_1 from G_1 which does not commute with $\varphi(\tau)$. (Any translation whose direction is not parallel to the line of the glide will do.) If $\varphi(g) = \tau_1$, then g has to be a translation or a glide. So g^2 is a translation, and hence commutes with τ, contradicting the fact that $\varphi\left(g^2\right) = \tau_1^2$ does not commute with $\varphi(\tau)$. Therefore, translations correspond to translations and glides to glides.

Reflections have order 2, consequently the image of a reflection under an isomorphism is either a reflection or a half-turn. Let $g \in G$ be a reflection whose image $\varphi(g)$ is a half-turn, and choose a translation τ from G in a direction which is not perpendicular to the mirror of g. Then τg is a glide. But $\varphi(\tau g) = \varphi(\tau)\varphi(g)$ is the product of a translation and a half-turn, which is another half-turn. Therefore, we have a contradiction and reflections must correspond to reflections. Finally, rotations are now forced to correspond to rotations. □

Corollary. If two wallpaper groups are isomorphic then their point groups are also isomorphic.

Proof. Let G, G_1 be wallpaper groups with translation subgroups H, H_1 and point groups J, J_1 respectively. If $\varphi : G \to G_1$ is an isomorphism we have $\varphi(H) = H_1$. Therefore, φ induces an isomorphism from G/H to G_1/H_1. The result now follows because J is isomorphic to G/H and J_1 is isomorphic to G_1/H_1. □

Remark. There are seventeen different wallpaper groups. To see why, we shall examine each of the five possible types of lattice in turn. Given a lattice L we first work out which orthogonal transformations preserve L. Such transformations form a group and, the point group of any wallpaper group which has L as its lattice must be a subgroup of this group. This limitation on the point group is then sufficient to allow us to enumerate the different wallpaper groups with lattice L. An exhaustive analysis of every case would take up too much space. So we concentrate our attention on a small number of examples, and defer the remaining calculations to the exercises. That all the groups we find are genuinely different, in other words that no two are isomorphic, will be shown at the end of the chapter.

Before beginning the classification we add a word or two about notation. Each wallpaper group has a name made up of several (internationally recognised) symbols p, c, m, g and the integers $1, 2, 3, 4, 6$. The letter p refers to the lattice and stands for the word primitive. When we view a lattice as being made up of primitive cells (copies of the basic parallelogram which do not contain any lattice points in their interiors) we call it a primitive lattice. In one case (the centred rectangular lattice) we take a non-primitive cell together with its centre as the basic building block, and use the letter c to denote the resulting centred lattice. The symbol for a reflection is m (for mirror) and g denotes a glide reflection. Finally, 1 is used for the identity transformation and the numbers $2, 3, 4, 6$ indicate rotations of the corresponding order. Rotations of order two are usually called half turns.

We show the centres of rotations and the positions of mirrors and glide lines relative to a basic parallelogram. The symbols $\circ, \Delta, \square, \bullet$ mean that the stabilizer of the corresponding point is cyclic of order two, three, four, or six, respectively. Mirrors are drawn as thick lines and glides are indicated by broken lines.

We now proceed with our case-by-case analysis. As usual, G is a wallpaper group with translation subgroup H, point group J, and lattice L. Vectors a and b which span the lattice are selected as previously. There is no harm in assuming that a lies along the positive x-axis and that b is in the first quadrant. Finally, A_θ is the matrix which represents an anticlockwise rotation of θ about the origin, while B_φ represents reflection in the line through the origin which subtends an angle of $\varphi/2$ with the positive x-axis.

Case (a)

The lattice of G is oblique. Then the only orthogonal transformations which preserve L are the identity and rotation through π about the origin. Therefore, the point group of G is a subgroup of $\{\pm I\}$.

(p1) If J only contains the identity matrix, then G is the simplest of all wallpaper groups; that generated by two independent translations. Its elements have the form $(I, ma+nb)$, where $m, n \in \mathbb{Z}$.

(p2) Here J is $\{\pm I\}$. Therefore, G contains a half turn, and we may as well take the fixed point of this half turn as origin, so that $(-I, 0)$ belongs to G. The union of the two right cosets H and $H(-I, 0)$ is a subgroup of E_2 which must be our group G. Those elements of G which are not translations lie in $H(-I, 0)$ and have the form

$$(I, ma + nb)(-I, 0) = (-I, ma + nb)$$

where $m, n \in \mathbb{Z}$. In other words, we have all the half turns about the points $\frac{1}{2}ma + \frac{1}{2}nb$.

Case (b)

The lattice of G is rectangular. There are now four orthogonal transformations which preserve L; namely, the identity, a half turn about 0, reflection in the x-axis, and reflection in the y-axis. Therefore, the point group of G is a subgroup of $\{I, -I, B_0, B_\pi\}$. We look for wallpaper groups which we have not seen before, ignoring the possibilities p1, p2 found above.

(pm) J is $\{I, B_0\}$ and G contains a reflection in a horizontal mirror.

(pg) Suppose J is $\{I, B_0\}$, yet there are no reflections in G. Then G has to contain a glide reflection whose line is horizontal, and we choose a point of this line as origin. Applying a glide reflection twice gives a translation, hence our glide has the form $\left(B_0, \frac{1}{2}ka\right)$ for some integer k. If k is even, then $\left(I, -\frac{1}{2}ka\right)$ is a translation in G, and the reflection

$$(B_0, 0) = (I, -\frac{1}{2}ka)(B_0, \frac{1}{2}ka)$$

belongs to G, contradicting our initial assumption. Therefore, k is odd and

$$(B_0, \frac{1}{2}a) = (I, -\frac{1}{2}(k-1)a)(B_0, \frac{1}{2}ka)$$

lies in G. The elements of G which are not translations have the form

$$(I, ma + nb)(B_0, \frac{1}{2}a) = (B_0, \left(m + \frac{1}{2}\right) a + nb)$$

where $m, n \in \mathbb{Z}$. These are all glides along horizontal lines which either pass through lattice points or lie midway between lattice points. The length of each glide is an odd multiple of $\frac{1}{2}a$.

Taking $\{I, B_\pi\}$ as point group instead of $\{I, B_0\}$ is tantamount to interchanging the roles of "horizontal" and "vertical" in the preceding discussion, and does not lead to anything new. From now on we assume that the point group is all of $\{I, -I, B_0, B_\pi\}$. There are three possibilities according as both, just one, or neither of B_0, B_π can be realised by reflections in G.

(p2mm) In this case G contains a reflection in a horizontal mirror and a reflection in a vertical mirror.

(p2mg) Suppose G contains a reflection in a horizontal mirror but does not contain a reflection in a vertical mirror. Then B_π must be realised in G by a vertical glide reflection.

A judicious choice of origin, at the intersection of the horizontal mirror and the vertical glide line, plus the argument used for pg, allow us to assume that $(B_0, 0)$ and $(B_\pi, \frac{1}{2}b)$ lie in G. The product

$$(B_\pi, \frac{1}{2}b)(B_0, 0) = (-I, \frac{1}{2}b)$$

is the half turn about $\frac{1}{4}b$. The right cosets

$$H, \quad H(B_0, 0), \quad H(B_\pi, \frac{1}{2}b), \quad H(-I, \frac{1}{2}b)$$

fill out G. In the first of these we have the translations. A typical element of the second has the form

$$(I, ma + nb)(B_0, 0) = (B_0, ma + nb)$$

where $m, n \in \mathbb{Z}$. When $m = 0$, this isometry is reflection in a horizontal mirror which either passes through lattice points or lies midway between them. If m is not zero, the mirrors change to glide lines and the translation part of the glide is ma. The third coset contains the elements

$$(B_\pi, ma + \left(n + \frac{1}{2}\right) b)$$

which are all vertical glides whose lines pass through lattice points or lie midway between them. The translation part of each of these glides is an odd multiple of $\frac{1}{2}b$. Finally, $H(-I, \frac{1}{2}b)$ consists of the half turns centred at the points $\frac{1}{2}ma + \frac{1}{2}\left(n + \frac{1}{2}\right) b$.

Interchanging horizontal and vertical in the preceding discussion leads to a group which is isomorphic to p2mg.

(p2gg) Here there are no reflections in G.

Case (c)

The lattice of G is centred rectangular. The orthogonal transformations which preserve L are the same as in the rectangular case. Therefore, the point group must again be a subgroup of $\{I, -I, B_0, B_\pi\}$. We discover two new groups.

(cm) Suppose J is $\{I, B_0\}$ and that (B_0, v) realises B_0 in G. This isometry is either a reflection in a horizontal mirror or a glide along a horizontal line. Choose a point on the mirror or glide line as origin, so that $2v$ is a multiple of a, and remember that the vertical direction is determined by the vector $2b - a$.

(i) If $2v = ka$ and k is even, the reflection

$$(B_0, 0) = (I, -\frac{1}{2}ka)(B_0, \frac{1}{2}ka)$$

belongs to G. The elements of G which are not translations have the form

$$(B_0, ma + nb) = \left(B_0, \left(m + \frac{1}{2}n\right) a + \frac{1}{2}n(2b - a)\right)$$

where $m, n \in \mathbb{Z}$. Taking n to be even and $m = -\frac{1}{2}n$ produces all the reflections in horizontal mirrors which pass through lattice points. If n is even but $m \neq -\frac{1}{2}n$, these mirrors change to glide lines, the translation part of each glide being a multiple of a. Finally, if n is odd, we have glides along lines which lie midway between lattice points. The translation part of each of these glides is an odd multiple of $\frac{1}{2}a$.

(ii) If k is odd, then

$$\left(B_0, \frac{1}{2}(2b - a)\right) = \left(I, -\frac{1}{2}(k+1)a + b\right)\left(B_0, \frac{1}{2}ka\right)$$

lies in G. This is again a reflection and shifting the origin onto its mirror leads back to the previous case.

Substituting $\{I, B_\pi\}$ as point group instead of $\{I, B_0\}$ leads to a group which is isomorphic to cm.

(c2mm) J is $\{I, -I, B_0, B_\pi\}$. The type of calculation carried out above shows that both B_0 and B_π can be realised by reflections in G.

Case (d)

The lattice of G is square. Then the group of orthogonal transformations which preserves L is the dihedral group of order 8 generated by $A_{\frac{\pi}{2}}$ and B_0. The point group J is a subgroup of this group and, to obtain something new, we must include $A_{\frac{\pi}{2}}$ in J. (The other cases are dealt with in Exercise 26.9.)

(p4) Here J is generated by $A_{\frac{\pi}{2}}$.

(p4mm) J is generated by $A_{\frac{\pi}{2}}$ and B_0, and B_0 can be realised by a reflection in G.

(p4gm) Suppose J is generated by $A_{\frac{\pi}{2}}$ and B_0, but B_0 cannot be realised by a reflection in G. Choose the fixed point of a rotation of order 4 as origin, so that $(A_{\frac{\pi}{2}}, 0)$ belongs to G, and let $(B_0, \lambda a + \mu b)$ realise B_0 in G. Squaring $(B_0, \lambda a + \mu b)$ gives $(I, 2\lambda a)$, so 2λ is an integer. If 2λ is even the reflection

$$(B_0, \mu b) = (I, -\lambda a)(B_0, \lambda a + \mu b)$$

lies in G and we have a contradiction. Therefore, 2λ must be odd and

$$\left(B_0, \frac{1}{2}a + \mu b\right) = \left(I, \left(\frac{1}{2} - \lambda\right) a\right)(B_0, \lambda a + \mu b)$$

is an element of G. Also

$$(A_{\frac{\pi}{2}}, 0)\left(B_0, \frac{1}{2}a + \mu b\right) = \left(B_{\frac{\pi}{2}}, \frac{1}{2}b - \mu a\right)$$

and

$$\left(B_{\frac{\pi}{2}}, \frac{1}{2}b - \mu a\right)^2 = \left(I, \left(\frac{1}{2} - \mu\right)(a + b)\right)$$

showing $\frac{1}{2} - \mu$ to be an integer. We conclude that the glide

$$\left(B_0, \frac{1}{2}a + \frac{1}{2}b\right) = \left(I, \left(\frac{1}{2} - \mu\right) b\right)\left(B_0, \frac{1}{2}a + \mu b\right)$$

belongs to G. The right cosets

$$\begin{array}{ll} H(I, 0), & H\left(A_{\frac{\pi}{2}}, 0\right) \\ H(-I, 0), & H\left(A_{\frac{3\pi}{2}}, 0\right) \\ H\left(B_0, \frac{1}{2}a + \frac{1}{2}b\right) & H\left(B_{\frac{\pi}{2}}, \frac{1}{2}a + \frac{1}{2}b\right) \\ H\left(B_\pi, \frac{1}{2}a + \frac{1}{2}b\right) & H\left(B_{\frac{3\pi}{2}}, \frac{1}{2}a + \frac{1}{2}b\right) \end{array}$$

fill out G, and it is easy to recognise their elements geometrically. For example, a typical member of $H\left(B_{\frac{\pi}{2}}, \frac{1}{2}a + \frac{1}{2}b\right)$ has the form

$$\left(B_{\frac{\pi}{2}}, \left(m + \frac{1}{2}\right) a + \left(n + \frac{1}{2}\right) b\right)$$

where $m, n \in \mathbb{Z}$. Taking $m + n + 1 = 0$ gives all the reflections in mirrors tilted at 45° to the horizontal which pass midway between lattice points. When $m+n+1$ is non-zero and $m - n$ is odd, these mirrors change to glide lines. Finally, if $m+n+1$ is non-zero and $m - n$ is even, we have glides along lines of gradient one which pass through lattice points. The coset $H(-I, 0)$ on the other hand contains all the half turns

$$(-I, ma + nb)$$

centered at the points $\frac{1}{2}ma + \frac{1}{2}nb$. We leave the reader to work through the remaining cases.

Case (e)

The lattice of G is hexagonal. Then the point group must be contained in the dihedral group of order 12 generated by $A_{\frac{\pi}{3}}$ and B_0. We are led to new wallpaper groups when J contains rotations of order 3 or 6. (The other cases are dealt with in Exercise 26.10.)

(p3) J is generated by $A_{\frac{2\pi}{3}}$.

(p3m1) J is generated by $A_{\frac{2\pi}{3}}$ and B_0.

(p31m) Suppose J is generated by $A_{\frac{2\pi}{3}}$ and $B_{\frac{\pi}{3}}$. Choose the fixed point of a rotation of order 3 as origin, so that $(A_{\frac{2\pi}{3}}, 0)$ belongs to G, and let $(B_{\frac{\pi}{3}}, \lambda a + \mu b)$ realise $B_{\frac{\pi}{3}}$ in G. Now

$$\left(B_{\frac{\pi}{3}}, \lambda a + \mu b\right)^2 = ((\lambda + \mu)(a + b), I)$$

so $\lambda + \mu$ is an integer. Also

$$\left(A_{\frac{2\pi}{3}}, 0\right)\left(B_{\frac{\pi}{3}}, \lambda a + \mu b\right) = (B_\pi, \lambda(b - a) - \mu a)$$

and

$$(B_\pi, \lambda(b - a) - \mu a)^2 = (I, \lambda(2b - a))$$

showing that λ is an integer. Therefore, both λ and μ are integers and the reflection

$$\left(B_{\frac{\pi}{3}}, 0\right) = (I, -\lambda a - \mu b)\left(B_{\frac{\pi}{3}}, \lambda a + \mu b\right)$$

belongs to G. The elements of G have the form $(M, ma + nb)$ where $m, n \in \mathbb{Z}$ and M is one of the matrices $I, A_{\frac{2\pi}{3}}, A_{\frac{4\pi}{3}}, B_{\frac{\pi}{3}}, B_\pi, B_{\frac{2\pi}{3}}$. We ask the reader to interpret these elements geometrically. For example

$$(B_\pi, ma + nb) = \left(B_\pi, \left(m + \frac{1}{2}n\right)a + \frac{1}{2}n(2b - a)\right)$$

is a reflection in a vertical mirror when $n = 0$ and a vertical glide otherwise.

(p6) J is generated by $A_{\frac{\pi}{3}}$.

(p6mm) J is generated by $A_{\frac{\pi}{3}}$ and B_0.

Are all these seventeen groups genuinely different? The answer is yes, as we shall see below. By (25.6) we need only concern ourselves with groups whose point groups are isomorphic, so we begin with a summary of the point groups.

G	J	G	J
p1	trivial	p4	$\mathbb{Z}_4$
p2	$\mathbb{Z}_2$	p4mm	D_4
pm	$\mathbb{Z}_2$	p4gm	D_4
pg	$\mathbb{Z}_2$	p3	$\mathbb{Z}_3$
p2mm	$\mathbb{Z}_2 \times \mathbb{Z}_2$	p3m1	D_3
p2mg	$\mathbb{Z}_2 \times \mathbb{Z}_2$	p31m	D_3
p2gg	$\mathbb{Z}_2 \times \mathbb{Z}_2$	p6	$\mathbb{Z}_6$
cm	$\mathbb{Z}_2$	p6mm	D_6
c2mm	$\mathbb{Z}_2 \times \mathbb{Z}_2$		

Remember that an isomorphism between wallpaper groups sends translations to translations, rotations to rotations, reflections to reflections, and glides to glides.

Theorem 6. *No two of p2, pm, pg, cm are isomorphic.*

Proof. Among these only p2 contains rotations, so it cannot be isomorphic to any of the others. Of the three remaining groups, pg is the only one which does not contain a reflection; consequently, pg is not isomorphic to pm or cm. Finally, we note that if we take a glide in pm and write it as a reflection followed by a translation, then both the reflection and the translation belong to pm. However, cm contains glides whose constituent parts do not lie in cm. For example, consider the glide

$$\left(B_0, \frac{1}{2}a + \frac{1}{2}(2b-a)\right) = \left(I, \frac{1}{2}a\right)\left(B_0, \frac{1}{2}(2b-a)\right)$$

Therefore, pm is not isomorphic to cm. $\square$

Theorem 7. *No two of p2mm, p2mg, p2gg, c2mm are isomorphic.*

Proof. Among these, p2gg is the only one which does not contain a reflection, so it cannot be isomorphic to any of the others. Of the three remaining groups, only p2mm contains the constituent parts of each of its glides, consequently p2mm is not isomorphic to p2mg or c2mm. Finally, we note that the mirrors of all the reflections in p2mg are horizontal, so the product of two reflections is always a translation. But in c2mm there are reflections with horizontal mirrors and reflections with vertical mirrors, and the product of one of each is a half turn. Therefore, p2mg is not isomorphic to c2mm. $\square$

Theorem 8. *p4mm is not isomorphic to p4gm.*

Proof. Each rotation of order 4 in p4mm can be written as the product of two reflections which both belong to p4mm. The corresponding statement is not true for p4gm. For example, $(A_{\frac{\pi}{2}}, a)$ cannot be factorised in p4gm as the product of two reflections. Therefore, p4mm is not isomorphic to p4gm. □

Theorem 9. *p3m1 is not isomorphic to p31m.*

Proof. In p31m each rotation of order 3 can be written as the product of two reflections, but this is not the case in p3m1. For example, $(A_{\frac{2\pi}{3}}, a)$ cannot be factorised in p3m1 as the product of two reflections. Therefore, p31m is not isomorphic to p3m1. □

This completes our classification of wallpaper groups. We have adopted a "hands on" approach, and deliberately so, only by working out the elements of these groups do we gain any understanding of their structure.

5 Group Cohomology and Group Extension

Let G be a group. The integral group ring $\mathbb{Z}G$ is defined to be the free $\mathbb{Z}$-module generated by the elements of G, that is, the group $\mathbb{Z}[G]$.
Therefore, an element of $\mathbb{Z}G$ can be uniquely expressed as $\sum_{g\in G} a(g)(g)$ where $a(g) \in \mathbb{Z}$ and $a(g) = 0$ for almost all $g \in G$.
We call M a G-module if M is an abelian group and there exists a homomorphism $\phi : G \to \mathrm{Aut}(M)$ such that the group G acts on M by $g \cdot m = \phi(g)(m)$. Since all abelian groups can be viewed as a module over $\mathbb{Z}$, a G-module M is the same as a $\mathbb{Z}G$-module.
Throughout this section, we denote R to be a unity ring.

Definition 18. Let M_i be R-modules and f_i be R-module homomorphisms for all $i > 0$. Consider the sequence,

$$M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_3 \xrightarrow{f_3} M_4 \to \cdots$$

- We say the sequence is exact at M_n if and only if $\ker(f_n) = \mathrm{im}(f_{n-1})$ for $n > 0$.
- We say the sequence is exact if and only if it is exact at M_n for all $n > 0$.

Definition 19. Let M_i be R-modules and d_i be R-module homomorphisms for $i \geq 0$. Then

$$0 \to M_0 \xrightarrow{d_0} M_1 \xrightarrow{d_1} \cdots \xrightarrow{d_n} M_{n+1} \to \cdots$$

is a cochain complex if the composition of any two successive maps $d_{n+1} \circ d_n$ is the zero map.

Definition 20. A short exact sequence is a 5-term exact sequence where the first and last terms are identity. In other words, the exact sequence

$$0 \to A \xrightarrow{\psi} B \xrightarrow{\phi} C \to 0$$

where A, B, C are R-modules. We say the above short exact sequence splits if there is an R-module homomorphism $s : C \to B$ such that $\phi \circ s$ is the identity map on C. In this case, we call the map $s : C \to B$ a splitting homomorphism for the sequence and $B \cong A \oplus C$.

Definition 21. Let M be an R-module. We say M is a free module if there exists a subset $A \subset M$ such that for any non-zero element $x \in M$, there exist unique non-zero elements $r_1, \ldots, r_n \in R$ and unique $a_1, \ldots, a_n \in A$ for some $n \in \mathbb{N}$ such that

$$x = \sum_{i=1}^{n} r_i a_i.$$

In this case, we say A is a basis or set of generators of M.

Definition 22. Let P be an R-module. We say P is a projective module if P has the following property. For any R-modules M and N, if we have a surjection map $\phi : M \to N$, then for every R-module homomorphism from P to N lifts to an R-module homomorphism into M. In other words, given $f \in \mathrm{Hom}_R(P, N)$, there exists a lift $F \in \mathrm{Hom}_R(P, M)$ making the following diagram commute:

$$\begin{tikzcd} & P \arrow[dl, dashed, "F"'] \arrow[d, "f"] & \\ M \arrow[r, "\phi"'] & N \arrow[r] & 0 \end{tikzcd}$$

Proposition 8. Let P be an R-module. P is a projective module if and only if P is a direct summand of a free R-module.

Proof. First, we assume P is a projective module. Notice that P is the quotient of a free module. Thus we always have a short exact sequence

$$0 \to \ker(\phi) \to \mathcal{F} \xrightarrow{\phi} P \to 0.$$

By definition of projective module, the identity map $\mathrm{id} : P \to P$ lifts to a homomorphism μ making the following diagram commute:

$$\begin{tikzcd} & & & P \arrow[dl, dashed, "\mu"'] \arrow[d, "id"] & \\ 0 \arrow[r] & ker(\phi) \arrow[r] & \mathcal{F} \arrow[r, "\phi"] & P \arrow[r] & 0 \end{tikzcd}$$

Since the above diagram commutes, we have $\phi \circ \mu = \mathrm{id}$. Thus μ is a splitting homomorphism for the sequence and therefore $\mathcal{F} \cong \ker(\phi) \oplus P$.

Next, we assume P is a direct summand of a free R-module. Let $\mathcal{F}(S) = P \oplus K$ where $\mathcal{F}(S)$ is a free R-module on some set S and K is an R-module. Let M and N be any R-modules and $\phi : M \to N$ be a surjection. Let $\pi : \mathcal{F}(S) \to P$ be the natural projection and let $f : P \to N$ be any R-module homomorphism. Our aim is to lift the map f to an R-module homomorphism into M. Consider the map $f \circ \pi : \mathcal{F}(S) \to N$. For any $s \in S$, we define $n_s = f \circ \pi(s) \in N$. Since ϕ is surjective, we let $m_s \in M$ be any element of M satisfying $\phi(m_s) = n_s$. By the universal property for free modules, there exists a unique R-module homomorphism $F' : \mathcal{F}(S) \to M$ such that $F'(s) = m_s$.

Thus we have the following diagram:

$$\begin{tikzcd}
 & \mathcal{F}(S) = P \oplus K \arrow[dl, dashed, "F"'] \arrow[d, "\pi"] & \\
 & P \arrow[d, "f"] & \\
M \arrow[r, "\phi"] & N \arrow[r] & 0
\end{tikzcd}$$

for any $s \in S$, we have

$$\phi \circ F'(s) = \phi(m_s) = n_s = f \circ \pi(s).$$

It follows that $\phi \circ F' = f \circ \pi$. In other words, the above diagram commutes. We define a map $F : P \to M$ where $F(d) = F'((d, 0))$. Since F is a composition of an injection $P \to \mathcal{F}(S)$ and the homomorphism F', the map F is an R-module homomorphism. Then

$$\phi \circ F(d) = \phi \circ F'((d, 0)) = f \circ \pi((d, 0)) = f(d).$$

Thus the below diagram commutes:

$$\begin{tikzcd}
P \arrow[r, "F"] \arrow[dr, "f"'] & M \arrow[d, "\phi"] \\
 & N
\end{tikzcd}$$

and we complete the proof. $\square$

Corollary. If P is a free module, then P is a projective module.

Definition 23. Let C^i be R-modules for all $i \geq 0$. Consider the following sequence

$$0 \to C^0 \xrightarrow{d^0} C^1 \xrightarrow{d^1} C^2 \xrightarrow{d^2} C^3 \to \cdots \to C^n \xrightarrow{d^n} C^{n+1} \to \cdots$$

where $d^n : C^n \to C^{n+1}$ is a homomorphism. We say the sequence is a cochain complex if the composition of any two consecutive maps is the zero map. We define the n-th cohomology group of that cochain complex to be

$$H^n(\mathcal{C}) = \ker(d^n)/\operatorname{im}(d^{n-1})$$

where $\mathcal{C}$ is the cochain complex.

Definition 24. Let A be an R-module. A projective resolution of A is an exact sequence

$$\cdots \to P_n \xrightarrow{d_n} \cdots \xrightarrow{d_1} P_0 \xrightarrow{d_0} A \to 0$$

where P_i are projective R-modules for all $i \geq 0$.

Lemma 18. *Let A be an R-module. There always exists a projective resolution of A.*

Proof. Choose a free module P_0 with a surjection $d_0 : P_0 \to A$ and define $\ker(d_0) = K_0$. Inductively, for $n \geq 1$, we choose a free module P_n with surjection $P_n \to K_{n-1}$ and define K_n to be the kernel of the surjection. We define d_n to be the composition $P_n \to K_{n-1} \to P_{n-1}$. It is clear that $\ker(d_n) = \ker(P_n \to K_{n-1}) = K_n$. By the above construction, we have a surjection $d_n : P_n \to K_{n-1}$ and $\ker(d_n) = \operatorname{im}(d_{n+1})$. It follows that the sequence

$$\cdots \to P_n \to \cdots \to P_1 \to P_0 \to A \to 0$$

is exact and P_i are projective R-modules for all $i \geq 0$ by Corollary 2.3.7. □

Definition 25. Let G be a finite group and M be a G-module. Define $C^0(G, M) = M$, $C^n(G, M)$ to be the collection of all functions from G^n to M for $n \geq 1$ and $C^n(G, M) = 0$ for $n < 0$. We define the coboundary operator $\delta^n : C^n(G, M) \to C^{n+1}(G, M)$ as follows,

$$\delta^n f(g_0, g_1, \ldots, g_n) = g_0 \cdot f(g_1, \ldots, g_n) + \sum_{j=1}^{n} (-1)^j f(g_0, \ldots, g_{j-2}, g_{j-1}g_j, g_{j+1}, \ldots, g_n)$$

$$+(-1)^{n+1} f(g_0, \ldots, g_{n-1})$$

for $n \geq 1$, $\delta^0 m(g_1) = g_1 \cdot m - m$ and $\delta^n = 0$ for $n < 0$. Then

$$0 \to C^0(G, M) \xrightarrow{\delta^0} C^1(G, M) \xrightarrow{\delta^1} \cdots \xrightarrow{\delta^n} H^{n+1}(G, M) \to \cdots$$

is a cochain complex. We define $Z^n(G, M) = \ker(\delta^n)$ and the elements of $Z^n(G, M)$ are called n-cocycles. We define $B^n(G, M) = \operatorname{im}(\delta^{n-1})$ and the elements of $B^n(G, M)$ are called n-coboundaries. We define the n-th cohomology group of G with coefficients in M to be

$$H^n(G, M) = Z^n(G, M)/B^n(G, M).$$

Remark. Using the notation as in (2.15), if G is a finite group and M is a finitely generated G-module, then $\operatorname{Hom}_{\mathbb{Z}G}(F_n, M)$ is a finitely generated abelian group for all $n \geq 0$. Therefore $H^n(G, M)$ is a finitely generated abelian group for all $n \geq 0$.

Definition 26. Let N, G, Γ be groups. We say Γ is an extension of G by N if it fits in the short exact sequence,

$$1 \to N \xrightarrow{\iota} \Gamma \xrightarrow{p} G \to 1.$$

Let Γ and Γ' be both extensions of G by N. We say two extensions are equivalent via f if there exists a homomorphism $f : \Gamma \to \Gamma'$ such that the following diagram commutes:

Lemma 19. *Using the same notations as above, the homomorphism f is indeed an isomorphism.*

Proof. Let $\gamma \in \ker(f)$. Since $p_2 \circ f(\gamma) = p_2(1) = 1$ and the diagram commutes, we have $p_1(\gamma) = p_2 \circ f(\gamma) = 1$. Thus $\gamma \in \ker(p_1)$. By exactness at Γ, there exists $x \in N$ such that $\iota_1(x) = \gamma$. Hence $\iota_2(x) = f \circ \iota_1(x) = f(\gamma) = 1$. Since ι_1 is injective, $x = 1$ and therefore $\gamma = \iota_1(x) = 1$. It follows that f is injective.

Let $\gamma' \in \Gamma'$. Since p_1 is surjective, there exists $\gamma \in \Gamma$ such that $p_1(\gamma) = p_2(\gamma')$. We have $p_2 \circ f(\gamma) = p_1(\gamma) = p_2(\gamma')$ and therefore $p_2(\gamma'(f(\gamma))^{-1}) = 1$. By exactness at Γ', there exists $x \in N$ such that $\iota_2(x) = \gamma'(f(\gamma))^{-1}$. It follows that $f(\iota_1(x)\gamma) = f(\iota_1(x))f(\gamma) = \iota_2(x)f(\gamma) = \gamma'(f(\gamma))^{-1}f(\gamma) = \gamma'$. Thus f is surjective. Therefore f is an isomorphism. $\square$

Lemma 20. *Given the short exact sequence*

$$0 \to N \xrightarrow{\iota} \Gamma \xrightarrow{p} G \to 1$$

where N is an abelian group. Then it induces a G-action on N. In other words, we can view N as a G-module.

Proof. Since N is an abelian normal subgroup in Γ, G acts on N by conjugation. Explicitly, let $g \in G, x \in N$ and pick $\bar{g}$ be an element such that $p(\bar{g}) = g$. We define the action as below,

$$\iota(g \cdot x) = \bar{g}\iota(x)\bar{g}^{-1}.$$

Let $\bar{g}'$ be another element such that $p(\bar{g}') = g$. Since $\Gamma/N \cong G$, there exists $x_1 \in N$ such that $\bar{g}' = \bar{g}\iota(x_1)$. Since N is an abelian group, we have

$$\bar{g}'\iota(x)\bar{g}'^{-1} = \bar{g}\iota(x_1)\iota(x)\iota(x_1)^{-1}\bar{g}^{-1} = \bar{g}\iota(x)\bar{g}^{-1}.$$

Hence the action is independent of the choice of $\bar{g}$. Therefore the action given by (2.18) is a well-defined G-action on N. $\square$

Lemma 21. *Equivalent extensions of G by N define the same G-module structure on N.*

Proof. Let Γ and Γ' be equivalent extensions and consider the below commutative diagram.

$$\begin{tikzcd}
1 \arrow[r] & N \arrow[r] \arrow[d, "\iota_1"] & \Gamma \arrow[r] \arrow[d, "f"] & G \arrow[r] \arrow[d, "="] & 1 \\
1 \arrow[r] & N \arrow[r, "\iota_2"'] & \Gamma' \arrow[r] & G \arrow[r] & 1
\end{tikzcd}$$

Let $g \in G$ be an arbitrary element of G. Let $\bar{g}$ be an element such that $p_1(\bar{g}) = g$. The G-module structure on N induced from Γ is

$$\iota_1(g \cdot x) = \bar{g}\iota_1(x)\bar{g}^{-1}$$

where $x \in N$. Let $\bar{g}' = f(\bar{g})$. Since the diagram (2.19) is a commutative diagram, we have $p_2(\bar{g}') = p_1(\bar{g}) = g$. Thus the G-module structure on N induced from Γ' is

$$\iota_2(g \cdot x) = \bar{g}'\iota_2(x)\bar{g}'^{-1}$$

where $x \in N$. Since ι_1, ι_2 and f are injective, we have

$$f(\bar{g}\iota_1(x)\bar{g}^{-1}) = \bar{g}'\iota_2(x)\bar{g}'^{-1} = \iota_2(g \cdot x).$$

Therefore, equivalent extensions of G by N define the same G-action on N. □

Theorem 10. *Let G be a group and A be a G-module. Let $\mathcal{E}(G, A)$ be the set of equivalence classes of extensions of G by A giving rise to the given action of G on A. Then there is a bijection between the set $\mathcal{E}(G, A)$ and the group $H^2(G, A)$.*

Proposition 9. Let G be a group and A be a G-module. If $|G| = k$, then every element of $H^n(G, A)$ has order divisible by k for $n > 0$.

Proof. Let $f \in C^n(G, A)$ be an arbitrary n-cochain.
Define

$$g(x_1, \ldots, x_{n-1}) = \sum_{x \in G} f(x_1, \ldots, x_{n-1}, x).$$

By definition of δ^n for $n > 0$, we have

$$\delta^{n-1}g(x_1, \ldots, x_n) = x_1 g(x_2, \ldots, x_n) + \sum_{j=2}^{n}(-1)^{j+1}g(x_1, \ldots, x_{j-2}, x_{j-1}x_j, x_{j+1}, \ldots, x_n)$$
$$+(-1)^n g(x_1, \ldots, x_{n-1})$$

and

$$\sum_{x \in G} \delta^n f(x_1, \ldots, x_n, x) = x_1 g(x_2, \ldots, x_n) + \sum_{j=2}^{n}(-1)^{j+1}g(x_1, \ldots, x_{j-1}x_j, x_{j+1}, \ldots, x_n)$$
$$+(-1)^n g(x_1, \ldots, x_{n-1}) + |G|(-1)^{n+1}f(x_1, \ldots, x_n)$$

It follows that

$$\sum_{x \in G} \delta^n f(x_1, \ldots, x_n, x) = \delta^{n-1}g(x_1, \ldots, x_{n-1}, x_n) + |G|(-1)^{n+1}f(x_1, \ldots, x_n).$$

If $f \in Z^n(G, A)$, then we have $\delta^n f = 0$. Hence we have

$$\delta^{n-1}g(x_1, \ldots, x_{n-1}, x_n) = \pm|G|f(x_1, \ldots, x_n).$$

Thus the order of f is divisible by k. □

Corollary. Let G be a group and $\mathbb{Z}^n$ be a G-module for any $n \geq 1$. If G is finite, then so is $H^2(G, \mathbb{Z}^n)$.

Proposition 10. Let G be a group and M be a G-module. If $|G| = m$ is invertible in M, then $H^n(G, M) = 0$ for all $n > 0$.

Proof. Let $\phi : C^q(G, M) \to C^q(G, M)$ be a homomorphism that sends $f \in C^q(G, M)$ to $m \cdot f$. It suffices to show that the induced homomorphism $\phi^q : H^q(G, M) \to H^q(G, M)$ is the trivial homomorphism for $q \geq 1$. In other words, we want to show $\phi^q(H^q(G, M)) = 0$. We know that all q-cocycles have order divisible by k. Hence $\phi^q(H^q(G, M)) = 0$. $\square$