§ 1.1 - 1.5 Intro to Fields

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Fields

DEF | Field
A field F is a set of values with defined operations for addition and multiplication. The operations satisfy the following:

(i) Closure: for $x, y \in \mathbb{Z}$
$x + y \in F$
$xy \in F$

(ii) Commutative
$x + y = y + x$
$xy = yx$

(iii) Associative; $z \in \mathbb{F}$
$x + y + z = x + (y + z)$
$(xy)z = x(yz)$

(iv) Identity element I
$x + I_+ = I_+ + x = x$ for any element x
$x \cdot I_+ = I_x \cdot x = x$

(v) Inverse Element
$x + \bar{x} = I_+$ for every $x \in F$
$x \cdot \hat{x} = I_x$

(vi) Distributive Identity: $x, y, z \in F$
$x(y + z) = xy + xz$

Example 1. Verify that the set of real numbers $\mathbb{R}$ is a field.

(i) For any two real numbers $x, y \in \mathbb{R}$
$x + y \in \mathbb{R}$, $xy \in \mathbb{R}$

(ii) $x + y = y + x$ is TRUE
$xy = yx$ is TRUE

(iii) Associativity
$(x + y) + z = x + (y + z)$

(iv) Identity
$x + 0 = 0 + x = x$ 0 is identity element under I_+
$x \cdot 1 = 1 \cdot x = x$ $I_x = 1$

(v) Inverse
$x + (-x) = 0$ where $-x \in \mathbb{R}$ is the additive inverse
$x(\frac{1}{x}) = 1$ $\frac{1}{x} \in \mathbb{R}$ is the multiplicative inverse as long as $x \neq 0$

(vi) Distributive
$x(y + z) = xy + xz$ i.e. $2(3 + 4) = 2(7) = 2 \cdot 3 + 2 \cdot 4 = 14$

Example 2. Is $\mathbb{Z}$ a field under the usual definitions of $+$ and $\cdot$?

No. For any integer $a \in \mathbb{Z}$, it is not possible to find a multiplicative inverse $a \in \mathbb{Z}$ such that $a\hat{a} = 1$.

DEF | Complex Numbers
The set of complex numbers C is the set of numbers of the form

$$a + bi$$

where a is the real part, bi is the imaginary part, and $i^2 = -1$.
*Note $\mathbb{C}$ is a field under the following operations:

For $z_1, z_2 \in \mathbb{C}$:
Addition: $z_1 + z_2 = (a_1 + b_i) + (a_2 + b_2 i) = (a_1 + a_2) + (b_1 + b_2)i$
Multiplication: $z_1 z_2 = (a_1 + b_1 i)(a_2 + b_2 i) = a_1 a_2 + a_1 b_2 i + a_2 b_1 i + b_1 b_2 i^2$
$= (a_1 a_2 - b_1 b_2) + (a_1 b_2 + a_2 b_1)i$

Example 3. Perform the operation

(a) $(3 + 2i) + (1 + i) = (3 + 1) + (2 + 1)i = 4 + 3i$

(b) $(3 + 2i)(7 - 5i) = 21 - 15i + 14i + 10 = 31 - i$

(c) $(3 + 2i)(3 - 2i) = 9 - 6i + 6i + 4 = 13$

(d) $\frac{31-i}{7-5i} \cdot \frac{7-5i}{7-5i} = \frac{(217+155i-7i-5)}{49+25} = \frac{222+148i}{74} = \frac{222}{74} + \frac{148}{74}i = 3 + 2i$

Visualizing Complex Numbers

Imaginary is on the y axis, real is on the x axis. We represent $3+2i$ as a vector $\overrightarrow{z}$ on the coordinate plane. Suppose that $z = a + bi$ is any complex number.

$\Rightarrow tan\theta = \frac{b}{a}$

$\Rightarrow |\overrightarrow{z}| = \sqrt{a^2+b^2} = \sqrt{z\bar{z}}$

Notice that $(a+bi)(a-bi) = a^2+b^2$ and $\bar{z}$ is conjugate

$cos\theta = \frac{a}{|\overrightarrow{z}|}$, $sin\theta = \frac{b}{|\overrightarrow{z}|}$

$cos^2\theta = \frac{a^2}{|\overrightarrow{z}|^2}$, $sin^2\theta = \frac{b^2}{|\overrightarrow{z}|^2}$

1. $1 = cos^2\theta + sin^2\theta = \frac{a^2}{|\overrightarrow{z}|^2} + \frac{b^2}{|\overrightarrow{z}|^2}$
   $\Rightarrow a^2+b^2 = |\overrightarrow{z}|^2$
2. $a = |\overrightarrow{z}|cos\theta$, $b = |\overrightarrow{z}|sin\theta$
   $z = a+bi = |\overrightarrow{z}|cos\theta + |\overrightarrow{z}|sin\theta i$
   $= |\overrightarrow{z}|(cos\theta + sin\theta)$

Euler's Identity: $e^{ie} = cos\theta + sin\theta$

$$\Rightarrow z = |\overrightarrow{z}|e^{i\theta}$$

where $|\overrightarrow{z}| = \sqrt{a^2+b^2}$ and $tan\theta = \frac{b}{a}$

$\Rightarrow z = re^{i\theta}$

Translations

DEF | Translations

Define $f(z) = z + z_1$ where $z_1 = a_1 + b_1 i$ is given (constant). Then $f(z)$ describes a translation:

$$f(z) = z + z_1$$
$$= (a+bi) + (a_1 b_1 i)$$
$$= (a+a_1) + (b+b_1)i$$

e.g. $a_1 = 2$, $b_1 = 3$

Add a_1, b_1 to original values of a, b on the graph. In this case, the point/vector representing $z = a+bi$ has shifted **2 units right** and **3 units up**.

In general for $z_1 = a_1 + b_1 i$, we have the following cases:

$a_1 > 0$ shift right
$a_1 = 0$ no horizontal shift
$a_1 < 0$ shift left

$b_1 > 0$ shift up
$b_1 = 0$ no vertical shift

$b_1 < 0$ shift down

Let $f_1(z) = z + z_1 \& f_2(z) = z + z_2, z_1, z_2 \in \mathbb{C}$ given"

$$f_1 o f_2 = f_1(f_2(z)) = f_1(z + z_2) = z + z_2 + z_1$$
$$= z + w$$

where $w = z_2 + z_1$. Also describes a translation and $f_2 o f_1 = f_1 o f_2$

Scaling & Rotation

DEF | Scaling Define $h(z) = \alpha z$ where $\alpha \in \mathbb{R}$
Notice $|h(z)| = |\alpha z| = |\alpha(a + bi)| = |a\alpha + b\alpha i|$

$$= \sqrt{(a\alpha)^2 + (b\alpha)^2}$$
$$= \sqrt{a^2\alpha^2 + b^2\alpha^2}$$
$$= \sqrt{a^2(\alpha^2 + b^2)}$$
$$= |\alpha||z|$$

Hence multiplying $z \in \mathbb{C}$ by a scalar α, scales the magnitude of z.
Cases:
$\alpha > 1$ extends the line straight along its original path (magnitude increases)
e.g. $z = 2.5 + 2i$ and $\alpha = 2$

$$2z = 5 + 4i$$

$0 < \alpha < 1$ shrinks the line straight along its original path (magnitude decreases)
e.g. $z = 2.5 + 2i$ and $\alpha = \frac{1}{2}$

$$\frac{1}{2}z = 1.25 + 1i$$

$-1 < \alpha < 0$ reverses direction and shrinks the line (rotation by 180 and shrinking in)
e.g. $z = 2 + 4i$ and $\alpha = \frac{1}{2}$

$$-\frac{1}{2}z = -1 - 2i$$

$\alpha < -1$ reverse direction and expands (rotation by 180 and magnitude increases)
e.g. $z = 2.5 + 2i$ and $\alpha = -1$

$$-1z = 2.5 + 2i$$

DEF | Rotations by τ rad.
Recall $z = re^{i\theta}$
To obtain a rotated point/vector,

$$w = re^{i(\theta+\tau)}$$

To scale out by a factor of 2 and rotate by $\frac{\pi}{3} rad$.

$$w = 2re^{i(\pi+\frac{\pi}{3})}$$

Example 4. What is the result of scaling $z = 1 + i$ by $\alpha = 3$ and rotating $\frac{pi}{2}$ rad?

$$\begin{gathered}
z = re^{i\theta} \\
r = \sqrt{1^2 + 1^2} = \sqrt{2} \\
\theta = tan^{-1}(1) = \frac{\pi}{4} \\
z = \sqrt{2}e^{i\frac{\pi}{4}} \\
w = 3\sqrt{2}e^{i(\frac{pi}{4})+\frac{\pi}{2}} \\
= 3\sqrt{2}e^{\frac{3\pi}{4}} \\
= 3\sqrt{2}(cos(\frac{3\pi}{4}) + sin(\frac{3\pi}{4})) \\
= -3 + 3i
\end{gathered}$$

Note: $\theta = tan^{-1}(\frac{b}{a})$