§ 3 Matrices

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MOTIVATION

$$V = \left[\begin{bmatrix} cupcake \\ 0.00 \\ 0.05 \\ 0.06 \\ 0.30 \\ 0.00 \\ \overrightarrow{v_1} \end{bmatrix}, \begin{bmatrix} coconut \\ 0.00 \\ 0.02 \\ 0.02 \\ 0.40 \\ 0.00 \\ \overrightarrow{v_2} \end{bmatrix}, \begin{bmatrix} custardpie \\ 0.00 \\ 0.34 \\ 0.05 \\ 4.70 \\ 0.00 \\ \overrightarrow{v_3} \end{bmatrix}\right] \qquad \alpha_1 \overrightarrow{v}_1 + \alpha_1 \overrightarrow{v}_2 + \alpha_1 \overrightarrow{v}_3 = \begin{bmatrix} 0.00 \\ 0.45 \\ 0.63 \\ 6.20 \\ 0.00 \end{bmatrix}$$

What scalars satisfy this?

DEF | Matrix

A matrix $A_{m\times n}$ over a field $\mathbb{F}$ is a two dimensional array with m rows and n columns.

*Notations

$$A_{m\times n} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ . & & & & \\ . & & & & \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{bmatrix}$$

$$A_{m\times n} = \begin{bmatrix} a_{ij} \end{bmatrix} \qquad \text{Where } i \text{ represents row and } j \text{ represents column}$$

$$A_{m\times n} = \begin{bmatrix} | & & | \\ \overrightarrow{a_1} & \dots & \overrightarrow{a_n} \\ | & & | \end{bmatrix}$$

$$A_{m\times n} = \begin{bmatrix} -- & \overrightarrow{a_1} & -- \\ . & & \\ . & & \\ -- & \overrightarrow{a_n} & -- \end{bmatrix}$$

DEF | Row Space
Let $A \in \mathbb{F}_{m \times n}$ (the set of all m by n matrices over field $\mathbb{F}$).
* The column space of A is the set of column vectors of A

$$\text{Col(A)} = \left[\begin{bmatrix} a_{11} \\ a_{21} \\ . \\ . \\ a_{m1} \end{bmatrix}, \begin{bmatrix} a_{12} \\ a_{22} \\ . \\ . \\ a_{m2} \end{bmatrix}, ..., \begin{bmatrix} a_{1n} \\ a_{2n} \\ . \\ . \\ a_{mn} \end{bmatrix} \right]$$

* The row space of A is the set of row vectors

$$\text{Row(A)} = \left[\begin{bmatrix} a_{11} \\ a_{12} \\ . \\ . \\ a_{1n} \end{bmatrix}, \begin{bmatrix} a_{21} \\ a_{22} \\ . \\ . \\ a_{2n} \end{bmatrix}, ..., \begin{bmatrix} a_{m1} \\ a_{m2} \\ . \\ . \\ a_{mn} \end{bmatrix} \right]$$

Example.

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 3 & 1 & 1 \\ 4 & 0 & 1 \end{bmatrix}$$

$$\text{Col(A)} = \left[\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right]$$

$$\text{Row(A)} = \left[\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix} \right]$$

DEF | Transpose
Let $A \in \mathbb{F}_{m \times n}$ and $A = [a_{ij}]$. The transpose of A, A^T, is an $n \times m$ matrix $A^T = [a_{ji}]$ i.e. rows of A^T are columns of A.

Example. Consider

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{bmatrix}_{4 \times 3} \qquad \text{Find } A^T \qquad A^T = \begin{bmatrix} 1 & 4 & 7 & 10 \\ 2 & 5 & 8 & 11 \\ 3 & 6 & 9 & 12 \end{bmatrix}_{3 \times 4}$$

DEF | Symmetric
A matrix A is symmetric if $A = A^T$.
*Note: Symmetric matrices must be nxn i.e. be square matrices.

Example. Which of the following are symmetric?

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \quad B^T = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \quad C^T = \begin{bmatrix} 1 & 1 \\ 2 & 2 \\ 3 & 3 \end{bmatrix}$$

Only B is symmetric.

DEF | Matrix Addition
Let $A, B, \in \mathbb{F}_{m \times n}$, then

$$A + B = [a_{ij}]_{m\times n} + [m_{ij}]_{m\times n} = [a_{ij} + b_{ij}]_{m\times n}$$

e.g. $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1+0 & 2+1 \\ 3+1 & 4+0 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 4 & 4 \end{bmatrix}$

DEF | Matrix-Matrix Multiplication

Let $A = \begin{bmatrix} a_{11} & \dots & a_{1p} \\ . & & \\ . & & \\ a_{m1} & \dots & a_{mp} \end{bmatrix}_{m\times p} \quad B = \begin{bmatrix} b_{11} & \dots & b_{1n} \\ . & & \\ . & & \\ b_{p1} & \dots & b_{pn} \end{bmatrix}_{p\times n}$

$$A \cdot B = \begin{bmatrix} \sum_{k=1}^{p} a_{1k}bk1 & \dots & \sum_{k=1}^{p} a_{1k}bkn \\ . & & \\ . & & \\ \sum_{k=1}^{p} a_{mk}bk1 & \dots & \sum_{k=1}^{p} a_{mk}bkn \end{bmatrix}_{m\times n}$$

Example.
Let $A = \begin{bmatrix} 10 & 3 \\ 2 & 1 \\ 3 & 3 \end{bmatrix}_{3\times 2} \quad B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}_{2\times 3}$

$$AB = \begin{bmatrix} 10 & 3 \\ 2 & 1 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 10(1)+3(4) & 10(2)+3(5) & 10(3)+3(6) \\ 2(1)+1(4) & 2(2)+1(5) & 2(3)+1(6) \\ 3(1)+3(4) & 3(2)+3(5) & 3(3)+3(6) \end{bmatrix} = \begin{bmatrix} 22 & 35 & 48 \\ 6 & 9 & 12 \\ 15 & 21 & 27 \end{bmatrix}$$

DEF | Scalar-Matrix Multiplication
Let $\alpha \in \mathbb{F}, A \in \mathbb{F}_{m\times n}$

$$\alpha A = \alpha[a_{ij}]_{m\times n} = [\alpha a_{ij}]_{mxn}$$

e.g. $\frac{1}{2}\begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix} = \begin{bmatrix} 1.5 & 2 \\ 2.5 & 3 \end{bmatrix}$