§ 5 Gaussian Elimination

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MOTIVATION: Grow Algorithm needs a solution to a system

$$A\overrightarrow{v} = \overrightarrow{b} \qquad A \in \mathbb{F}_{m\times n} \qquad \overrightarrow{b} \in \mathbb{F}^m$$

$\overrightarrow{x} = A^{-1}\overrightarrow{b}$ works only if $m = n$ (square matrix)

> **DEF** | Augmented Matrix
> Given the problem $A\overrightarrow{x} = \overrightarrow{b}, A \in \mathbb{F}_{m\times n}$ the augmented matrix for this system
> $A_G = \begin{bmatrix} A & . & 1 \end{bmatrix} = \begin{bmatrix} 1 \end{bmatrix}$

Example. Form A_G for the system

$$x + 2y + 3z = 7$$

$$4x + 5y + 8z = 8$$

$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ 8 \end{bmatrix}$

$A_G = \begin{bmatrix} 1 & 2 & 3 & 7 \\ 4 & 5 & 8 & 8 \end{bmatrix}$

> **DEF** | Reduced Row-Echelon Form (RREF)
> Let $A_G, B \in \mathbb{F}_{m\times(n+1)}$ where A_G is the augmented matrix of the system $A\overrightarrow{x} = \overrightarrow{b}, A \in \mathbb{F}_{m\times n}$

We call matrix B the RREF(A_G) if

(1) $\begin{bmatrix} b_{11} & ... & b_{1n} \\ . & & \\ . & & \\ b_{m1} & ... & b_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ . \\ . \\ x_n \end{bmatrix} = \begin{bmatrix} b_{1n+1} \\ . \\ . \\ b_{mn+1} \end{bmatrix}$ and

(2) if $n(i)$ =column index of first nonzero entry of row i
then $n(i) < n(k)$ whenever $i < k$.

Example. Find the RREF for the augmented matrix of the system

$$4x + y = 5$$

$$2x - y = 1$$

$\begin{bmatrix} 4 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$

$A_G = \begin{bmatrix} 4 & 1 & 5 \\ 2 & -1 & 1 \end{bmatrix}$

GOAL: $RREF(A_G) = \begin{bmatrix} * & * & * \\ 0 & * & * \end{bmatrix}$

Notice if vector $[x, y]^T$ that solves the system

$$[4, 1] \cdot [x, y] = 5$$

$$[2, -1] \cdot [x, y] = 1$$

Then,

$$[4, 1] \cdot [x, y] - 2[2, -1] \cdot [x, y] = 5 - 2$$

In such case,

$$4x + y - 2(2x - y) = 3$$

$$4x + y - 4x + 2y = 3$$

$$3y = 3$$

$$[0, 3] \cdot [x, y] = 3$$

i.e. Solving the system

$$[4, 1] \cdot [x, y] = 5$$

$$[2, -1] \cdot [x, y] = 1$$

is equivalent to solving the system

$$[4, 1] \cdot [x, y] = 5 \Rightarrow 4x + y = 5$$

$$[0, 3] \cdot [x, y] = 3 \Rightarrow 3y = 3$$

$\Rightarrow B = \begin{bmatrix} 4 & 1 & 5 \\ 0 & 3 & 3 \end{bmatrix} \Rightarrow \text{RREF}(A_G) = B$

***Remarks**

(i) In general, we can construct RREF(A_G) by adding multiples of row vectors to form rows with zeros in the appropriate columns.

(ii) If a row contains all zero entries in a RREF matrix, then all rows below it must be zeros as well: $n(i) < n(k)$ for $i < k$

(iii) The nonzero rows of a RREF matrix form a basis for the for the row space of A_G.

(iv) If $A_G = \begin{bmatrix} a_{11} & ... & a_{1n} & b_1 \\ . & & & \\ . & & & \\ a_{m1} & ... & a_{mn} & b_n \end{bmatrix}$ has RREF(A_G)= $\begin{bmatrix} r_{11} & ... & r_{1n} & \hat{b}_1 & \\ 0 & r_{22} & ... & r_{2n} & \hat{b}_2. \\ . & & & & \\ 0 & ... & ... & r_{nn} & \hat{b})n \end{bmatrix}$

Then we can obtain the solution vector $\overrightarrow{x}$ to the system

$$a_{11}x_1 + ... + a_{1n}x_n = b_1$$

...

$$a_{m1}x_1 + ... + a_{mn}x_n = b_n$$

Example. Find the RREF of the system & use it to find a solution

$$x + 3y = 4$$

$$2x - y = 1$$

$$3x + 2y = 5$$

$$5x + 15y = 20$$

$$A_G = \begin{bmatrix} 1 & 3 & 4 \\ 2 & -1 & 1 \\ 3 & 2 & 5 \\ 5 & 15 & 20 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 3 \\ 2 & -1 \\ 3 & 2 \\ 5 & 15 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 5 \\ 20 \end{bmatrix}$$

$$A_G = \begin{bmatrix} 1 & 3 & 4 \\ 2 & -1 & 1 \\ 3 & 2 & 5 \\ 5 & 15 & 20 \end{bmatrix} \Rightarrow \begin{bmatrix} r_1 \\ r_2 = r_2 - 2r_1 \\ r_3 = r_3 - 3r_1 \\ r_4 = r_4 - 5r_1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 3 & 4 \\ 0 & -7 & -7 \\ 0 & -7 & -7 \\ 0 & 0 & 0 \end{bmatrix}$$

$$r_3 = r_3 - r_2 \Rightarrow \begin{bmatrix} 1 & 3 & 4 \\ 0 & -7 & -7 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{matrix} \rightarrow & x + 3y = 4 \\ \rightarrow -7y = -7 \Rightarrow y = 1, x = 4 - 3y = 1 & \end{matrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Procedure. Gaussian Elimination
IN: Matrix $A \in \mathbb{F}_{m \times n}$, vector $\overrightarrow{b} \in \mathbb{F}^n$
OUT: Vector $\overrightarrow{x} \in \mathbb{F}^m$ solution to $A\overrightarrow{x} = \overrightarrow{b}$
Preliminary Step: Form A_G.
Step 1: For every row, r_i, $i = 2, 3, ..., m$, replace $\overrightarrow{r_i} = \overrightarrow{r_i} - (\frac{r_{i1}}{11})\overrightarrow{r_1}$
For every row, $\overrightarrow{r}_i$, $i = 2, 3, ..., m$, replace $\overrightarrow{r}_i = \overrightarrow{r}_i - (\frac{r_{i1}}{a_{11}})\overrightarrow{r}_1$

$$\begin{bmatrix} a_{11} & a_{12} & ... & a_{1n} & b_1 \\ 0 & a_{22} & ... & a_{2n}^{(1)} & b_2^{(1)} \\ 0 & a_{32}^{(1)} & ... & a_{3n}^{(1)} & b_3^{(1)} \\ . & & & & \\ . & & & & \\ 0 & a_{m2} & ... & a_{mn}^{(1)} & b_m^{(1)} \end{bmatrix} \qquad \begin{matrix} \overrightarrow{r}_2 = \overrightarrow{r}_2 - (\frac{a_{21}}{a_11})\overrightarrow{r}_1 \\ \overrightarrow{r}_3 = \overrightarrow{r}_3 - (\frac{a_{21}}{a_11})\overrightarrow{r}_1 \\ \overrightarrow{r}_m = \overrightarrow{r}_m - (\frac{a_{m1}}{a_11})\overrightarrow{r}_1 \end{matrix}$$

Step 2: Reducing at 3rd row. For every row, $\overrightarrow{r}_i$, $i = 3, 4, ..., m$, replace $\overrightarrow{r}_i = \overrightarrow{r}_i - (\frac{a_{i2}^{(1)}}{a_{22}^{(1)}})\overrightarrow{r}_2$

Step k: Reducing at row $(k+1)$. For every row, $\overrightarrow{r}_i$, $i = k+1, k+2...m$, replace $\overrightarrow{r}_i = \overrightarrow{r}_i - (\frac{a_{ik}^{(k-1)}}{a_{kk}^{(k-1)}})$

Repeat until step $k = m-1$.

Final Step: Use the RREF(A_G) to solve for $\overrightarrow{x}$ using **Backward Substitution**.

$$\begin{bmatrix} a_{11} & a_{12} & ... & ... & a_{1n} & b_1 \\ 0 & a_{22} & ... & ... & a_{2n}^{(1)} & b_2^{(1)} \\ 0 & 0 & a_{33}^{(1)} & ... & a_{3n}^{(1)} & b_3^{(1)} \\ . & & & & & \\ . & & & & & \\ 0 & 0 & ... & ... & a_{mn}^{(k)} & b_m^{(k)} \end{bmatrix}$$

If the last nonzero row is row n, then (i.e. $rank(A_G) = rank(A) = n = \#$ of columns in A)

$$*a_{nn}x_n = b_n \Rightarrow x_n = \frac{b_n}{a_{mn}}$$

$$*a_{n-1n-1}x_{n-1} + a_{n-1n}x_n = b_{n-1} \Rightarrow x_{n-1} = \frac{1}{a_{n-1n-1}}(b_{n-1} - a_{n-1n}x_n)$$

$$*a_{kk}x_k + a_{kk-1}x_{k-1} + ... + a_{kn}x_n = b_k$$

$$\Rightarrow x_k = \frac{1}{a_{kk}}[b_k - a_{kk-1}x_{k-1} - ... - a_{kn}x_n]$$