§ 4 Basis I & II

Jessica Wei

MOTIVATION: Earlier we saw $[1,0],[0,1]$ generates $\mathbb{R}^2$ i.e.

$$Span[1,0],[0,1] = \mathbb{R}^2$$

What does this mean? For any vector $\overrightarrow{v} = [a,b] \in \mathbb{R}^2$, we can define its location as

$$a[1,0] + b[0,1] = \overrightarrow{v}$$

> **DEF** | Coordinates
> Let G be a generating set of vectors and V be a vector space over a field $\mathbb{F}$
>
> $$Span(G) = V$$
>
> The coordinates of a vector $\overrightarrow{v} \in V$ with respect to the set G are the coefficients $\alpha_1, \alpha_2, ..., \alpha_n$ where
> $$\overrightarrow{v} = \alpha_1 \overrightarrow{g}_1 + \alpha_2 \overrightarrow{g}_2 + ... + \alpha_n \overrightarrow{g}_n$$
> and $\overrightarrow{g}_i \in G$.

Example. Find the coordinates of vector $\overrightarrow{v} = [1,2]$ with respect to the given generating set of $\mathbb{R}^2$.

(a) $G = [1,1],[-1,1]$
$[1,2] = \alpha[1,1], \beta[-1,1]$
$[\alpha,\alpha] + [-\beta,\beta]$
$[\alpha-\beta, \alpha+\beta]$
$[1,2] = \frac{3}{2}[1,1] + \frac{1}{2}[-1,1]$
Coordinates: $< \frac{3}{2}, \frac{1}{2} > G$

(b) $G_2 = [-2,0],[0,1]$
$[1,2] = \alpha[-2,0] + \beta[0,1]$
$[-2\alpha, 0] + [0,\beta]$
$-2\alpha = 1, \beta = 2$
$\alpha = \frac{-1}{2}$
Coordinates: $< \frac{-1}{2}, 2 >_{G_2}$

NOTE: We could also argue that $H = [1,0],[0,1],[2,0]$ also generates $\mathbb{R}^2$ e.g. $\overrightarrow{v} \in \mathbb{R}$

$$\overrightarrow{v} = \alpha[1,0] + \beta[0,1] + \lambda[2,0]$$

$$1,2 = [\alpha,0] + [0,\beta] + [2\lambda,0] = [\alpha+2\lambda, \beta]$$

$$\beta = 1 \qquad \alpha = -1 \qquad \lambda = 1 \qquad < -1,1,1 >_H$$

$$\beta = 1 \qquad \alpha = 1 \qquad \lambda = 0 \qquad < 1,1,0 >_H$$

$$\beta = 1 \qquad \alpha = 2 \qquad \lambda = -\frac{1}{2} \qquad < 1,2,-\frac{1}{2} >_H$$

We obtain more than one possible set of coordinates for the same vector under the same generating set. Confusing!

DEF | Linear Dependence
The vectors $\overrightarrow{v_1}, \overrightarrow{v_2}, ..., \overrightarrow{v_k}$ are linearly dependent if we can find scalars $\alpha_1, \alpha_2, ..., \alpha_k \in \mathbb{F}$ not all zero such that

$$\alpha_1 \overrightarrow{v}_1 + \alpha_2 \overrightarrow{v}_2 + ... + \alpha_k \overrightarrow{v}_k = \overrightarrow{0}$$

$$\Rightarrow \alpha_i \overrightarrow{v}_i = -\alpha_1 \overrightarrow{v}_1 - \alpha_2 \overrightarrow{v}_2 - ... - \alpha_k \overrightarrow{v}_k$$

$$\Rightarrow \overrightarrow{v}_i = \frac{\alpha_1}{\alpha_i} \overrightarrow{v}_1 - \frac{\alpha_2}{\alpha_i} \overrightarrow{v}_2 - ... - \frac{\alpha_k}{\alpha_i} \overrightarrow{v}_k$$

i.e. any vector in the set can be written as a linear combination of the rest of the vectors.

Example. Determine if the set of vectors is linearly dependent.

(a) $[1,0], [2,0]$
$\alpha_1[1,0] + \alpha_2[2,0] = [0,0]$
$alpha_1 = -2, \alpha_2 = 1 \Rightarrow$ linearly dependent

(b) [1,0,0],[0,2,0],[2,4,0],[0,1,0]
$\alpha_1[1,0,0] + \alpha_2[0,2,0] + \alpha_3[2,4,0] + \alpha_4[0,1,0] = \overrightarrow{0}$
$\alpha_1 = -2\alpha_2 = -2\alpha_3 = 1\alpha_4 = 0 \Rightarrow$ linearly dependent

(c) $[1,0,0], [0,2,0], [0,0,4]$
$\alpha_1[1,0,0] + \alpha_2[0,2,0] + \alpha_3[0,0,4] = [0,0,0]$
$\alpha_1 = 0\alpha_2 = 0\alpha_3 = 0 \Rightarrow$ linearly independent

LEMMA 1
Let $G = \overrightarrow{v}_1, ..., \overrightarrow{v}_k$ where $\overrightarrow{v}_i \in \mathbb{F}^2$. Then $\overrightarrow{x} \in Span(G)$ if and only if $\exists \alpha_i \in \mathbb{F}$ such that

$$\alpha_1 \overrightarrow{v}_1 + \alpha_2 \overrightarrow{v}_2 + ... + \alpha_k \overrightarrow{v}_k + \alpha_{k+1} \overrightarrow{x} = 0$$

and not all scalars are 0.

Proof. Assume

$$\alpha_1 \overrightarrow{v}_1 + ... + \alpha_k \overrightarrow{v}_k + \alpha_{k+1} \overrightarrow{x} = 0$$

and not all scalars are 0. Then,

$$\alpha_{k+1} \overrightarrow{x} = -\alpha_1 \overrightarrow{v}_1 - ... - \alpha_k \overrightarrow{v}_k$$

$$\overrightarrow{x} = \frac{-\alpha_1}{\alpha_{k+1}} \overrightarrow{v}_1 - ... - \frac{-\alpha_k}{\alpha_{k+1}} \overrightarrow{v}_k$$

$$= \beta_1 \overrightarrow{v}_1 + ... + \beta_k \overrightarrow{v}_k$$

where $\beta_i = \frac{-\alpha_i}{\alpha_{k+1}}$
$\Rightarrow \overrightarrow{x} \in Span(G)$
Now assume $\overrightarrow{x} tSpan(G)$. Then $\exists \lambda_i \in \mathbb{F}$ such that

$$\overrightarrow{x} = \lambda_1 \overrightarrow{v}_1 + ... + \lambda_k \overrightarrow{v}_k$$

$$\Rightarrow \lambda_1 \overrightarrow{v}_1 + ... + \lambda_k \overrightarrow{v}_k + (-1)\overrightarrow{x} = \overrightarrow{0}$$

Hence $\lambda_1 \overrightarrow{v}_1 + ... + \lambda_k \overrightarrow{v}_k + \lambda_{k+1} \overrightarrow{x} = \overrightarrow{0}$ where $\lambda_{k+1} = -1$

LEMMA 2
Let $S = \overrightarrow{\overrightarrow{v}_1, v_2, ..., \overrightarrow{v}_k, \overrightarrow{x}}$ such that $\overrightarrow{x} = \alpha_1 \overrightarrow{v} + ... + \alpha_k \overrightarrow{v}_k$. Then $Span(S) = Span(S - \overrightarrow{x})$

Proof NTS 1. $Span(S) \leq Span(S - \overrightarrow{x})$ and 2. $Span(S - \overrightarrow{x}) \leq Span S$

(1) Pick an arbitrary vector $\overrightarrow{u} \in Span(S)$. Then
$\overrightarrow{u} = \beta_1 \overrightarrow{v}_1 + ... + \beta_k \overrightarrow{v}_k + \beta_{k+1} \overrightarrow{x}$
$= \beta_1 \overrightarrow{v}_1 + ... + \beta_k \overrightarrow{v}_k + \beta_{k+1}(\alpha_1 \overrightarrow{v}_1 + ... + \alpha_k \overrightarrow{v}_k)$
$= \beta_1 \overrightarrow{v}_1 + ... + \beta_k \overrightarrow{v}_k + \beta_{k+1}\alpha_1 \overrightarrow{v}_1 + ... + \beta_{k+1}\alpha_k \overrightarrow{v}_k$
$= (\beta_1 + \beta_{k+1}\alpha_1)\overrightarrow{v}_1 + ... + (\beta_k + \beta_{k+1}\alpha_k)\overrightarrow{v}_k$
$= \lambda_1 \overrightarrow{v}_1 + ... + \lambda_k \overrightarrow{v}_k$
$\Rightarrow \overrightarrow{u} \in Span(S - \overrightarrow{x})$
$\Rightarrow Span(S) \leq Span(S - \overrightarrow{x})$

To show (2), pick an arbitrary $\overrightarrow{w} \in Span(S - \overrightarrow{x})$. Then,
$\overrightarrow{w} = w_1 \overrightarrow{v}_1 + ... + w_k \overrightarrow{v}_k = w_1 \overrightarrow{v}_1 + ... + w_k \overrightarrow{v}_k + 0\overrightarrow{x}$
$\Rightarrow \overrightarrow{w} \in Span(S)$
$Span(S - \overrightarrow{x}) \leq Span(S)$
By (1) and (2) it follows that $Span\ S = Span(S - \overrightarrow{x})$

DEF | Basis
Let G be a set of vectors. We call G a basis for a vector space V if

(i) $Span(G) = V$ i.e. G generates V

(ii) G is a linearly independent set

e.g $G = [1, 0], [0, 1]$ is a basis for $\mathbb{R}^2$
$T = [1, 0], [0, 1], [0, 2]$ is a generating set for $\mathbb{R}^2$ i.e. $R^2 = Span(T)$ but T is NOT a basis.

Size of a Basis

LEMMA 3 Let V be a vector space, S be a generating set such that $Span(S) = V$ and B be a subset V that is linearly independent. Then

$$|S| \geq |B|$$

size of a generating set $\geq$ size of a linearly independent set

THM Basis Theorem
Let V be a vector space. All bases for V have the same size.

Proof. Suppose B_1 and B_2 are bases for V.
*B_1 is a bases means (i) $Span(B_1) = V$ and (ii) B_1 is linearly independent.
*B_2 is a basis means (iii) $Span(B_2) = V$ and (iv) B_2 is linearly independent.
From Lemma 3.
(i) & (iv) $|B_2| \leq |B_1|$ assuming $B_2 \leq V$ (which is true b/c any vector in B_2 *)
(ii) & (iii) $|B_1| = |B_2| \Rightarrow$ bases have the same size.

THM Let V be a vector space, B a set of generators for V. B will be the smallest set of generators if and only if it is the basis for V.
Assume B is a basis for V. Consider some set of generators T. Since B is a basis, it is linearly independent & by Lemma 3

$$|B| \leq |T|$$

Since T is arbitrary, the result holds for all sets of generators. Hence, B is the smallest generating set.
Now, assume B is the smallest generating set but is not a basis. i.e. either
$Span(B) \neq V \quad \leftarrow \quad$ FALSE
or B is not linearly independent $\leftarrow$ TRUE
If B is not linearly independent, there is some vector $\overrightarrow{x} \in B$ that is a linear combination of the other vectors in B. By Lemma 2, then,

$$V = Span(B) = Span(B - \overrightarrow{x})$$

i.e. B is NOT the smallest generating set which is a contradiction. So B MUST be a basis.

DEF | Dimension
The dimension of a vector space is the size of a basis for $V = dim(V) = |B|$ where B is a basis.

Example. Find the dimension of each set.

a) $\mathbb{R}^3$
   $B = \begin{bmatrix} [1,0,0]^T \\ [0,1,0]^T \\ [0,0,1]^T \end{bmatrix}$
   is a basis because (i) B is linearly independent and (ii) $Span(B) = [\alpha, \beta, \lambda]^T = \mathbb{R}^3$
   $dim(\mathbb{R}^3) = |B| = 3$

b) $\mathbb{R}^2$
   $B = [1,0]^T, [0,1]^T$ is a basis; $|B| = 2$
   $dim(\mathbb{R}^2 = |B| = 2$

> **DEF** | Rank
> Let $A \in \mathbb{F}_{m \times n}$. The Row Rank of A, $row(A)$ or $rowRank(A)$, is the dimension of the row space of A. i.e. $row(A) = dim(rowsp(A_c))$
> Find a basis for the rowspace, then find the size of that basis. Similarly $colRank(A) = dim(colsp(A))$

Example. Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 2 & 4 & 0 \end{bmatrix}$ Find the ranks of A.

$*rowsp(A) = [1,0,0], [0,2,0], [2,4,0]$
$B_{row} = [1,0,0], [0,2,0]$ linearly indep. & $Span(B_{row}) = rowsp(B)$
$rowRank(A) = |B_{row}| = 2$

$*colsp(A) = [1,0,2]^T, [0,2,4]^T, [0,0,0]^T$
$B_{col} = [1,0,2]^T, [0,2,4]^T$
$colRank(A) = |B_{col}| = 2$

> **THM**
> Let $A \in \mathbb{F}_{n \times n}$. $rowRank(A) = colRank(A)$. We refer to any rank of A simply as $rank(A)$.

Q. Given a vector space V, how do we find a basis for V?

> **A1.** Shrink Algorithm
>
> ```
> def shrink(V):
>     B = V
>     for v in B:
>         if Span(B − v) == V:
>             B = B − v
>     return B
> ```

A2. Growth Algorithm

```
def grow(V):
    B = null
    for v in V:
        if v not in Span(B): #i.e. if v is linearly independent
            B = B union {v}  #add v to B
    return B
```

Proof. Show that the grow algorithm always returns a basis for V.
NTS (i) B is linearly independent and (ii) $Span(B) = V$
To show i) it suffices to consider the condition for the if statement if $\vec{v} \notin Span(B)$. This shows that a linear combination of vectors already in B. Hence, if none of the vectors in B are linear combinations of each other, they are linearly independent.

Example. Find a basis for $v = [4,1,2]^T, [0,0,1]^T, [5,4,3]^T, [2,0,1]^T$ using the growth algorithm: $B = \emptyset$
Check: $[4,1,2] \in Span(\emptyset) \rightarrow$ False
$B = \{[4,1,2]\}$
Check: $[0,0,1] \in Span([4,1,2]) \rightarrow$ False
$B = \{[4,1,2],[0,0,1]\}$
Check: $[5,4,3] \in Span(\{[4,1,2],[0,0,1]\})$
$\alpha[4,1,2] + \beta[0,0,1] = [5,4,3]$
$[4\alpha, \alpha, 2\alpha + \beta] = [5,4,3]$
$\alpha_1 = \frac{5}{3}$ and $\alpha_2 = 4 \rightarrow$ Contradiction!
$B = \{[4,1,2],[0,0,1],[5,4,3]\}$

Check. $[2,0,1] \in Span(B) \Rightarrow$ True

$$\alpha \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \lambda \begin{bmatrix} 5 \\ 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 4\alpha + 5\lambda \\ 1\alpha + 4\lambda \\ 2\alpha + \beta + 3\lambda \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

Returns $B = \{[4,1,2],[0,0,1],[5,4,3]\}$