§ 2 Vectors

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Vector

DEF | n-Vector
Let $\mathbb{F}$ be a field, we call an element

$$x_1, x_2, ...x_n \in \mathbb{F} \times \mathbb{F} \times \mathbb{F} ... \times \mathbb{F} = \mathbb{F}^n$$

an n-vector
You can think of a vector $\overrightarrow{v} \in \mathbb{F}^n$ as a function

$$1 \longrightarrow x_1$$

$$2 \longrightarrow x_2$$

$$n \longrightarrow x_n$$

OPERATIONS

1. Vector Addition: $\overrightarrow{v}, \overrightarrow{w} \in \mathbb{F}^n$
   $\overrightarrow{v} + \overrightarrow{w} = [v_1 + w_1 + v_2 + w_2 + ...v_n + w_n] = \overrightarrow{u}$
2. Scalar-vector multiplication: Let $\alpha \in \mathbb{F}$
   $\alpha \overrightarrow{v} = [av_1, av_2, ...\alpha vn]$

Example 1. Let $\overrightarrow{u} = [1, 2, 3]$, $\overrightarrow{v} = [2, 4, 6], \alpha = \frac{1}{2}$
Find $\overrightarrow{u} + \alpha \overrightarrow{v} = [1, 2, 3] + \frac{1}{2}[2, 4, 6]$
$= [1, 2, 3] + [1, 2, 3] = [2, 4, 6]$ **Example 2.**

(a) How can we span the line segment $(0, 0)$ to $(1, 1)$ using $\overrightarrow{v} = [1, 1]$? NOTE: Span means generate every point.
Answer. $\frac{1}{2}[1, 1] = [\frac{1}{2}, \frac{1}{2}] \longrightarrow$ generates $(\frac{1}{2}, \frac{1}{2})$
$0.01[1, 1] = [0.01, 0.01] \longrightarrow$ generates $(0.01, 0.01)$
$\alpha[1, 1] | o \leq \alpha \leq 1$

(b) How can we span the line segment from $(0.5, 1)$ to $(3.5, 3)$?
Answer. The slope of the line segment is going to be given by

$$m = \frac{3 - 1}{3.5 - 0.5} = \frac{2}{3}$$

This has the same slope as the line segment from $(0, 0)$ to $(3, 2)$. Therefore, the vector $[3, 2]$ runs parallel to the given line segment. This implies

$$\alpha[3, 2] \quad 0 \leq \alpha \leq 1$$

will generate the line segment $(0,0)$ to $(3,2)$.
Hence, $\alpha[3,2]+[0.5,1]$ will generate the line segment $(0.5,1)$ to $(3.5,3)$ for $0 \leq \alpha \leq 1$

$$[3\alpha, 2\alpha]+[0.5,1]=[3\alpha+0.5, 2\alpha+1$$

$$[3\alpha+0.5, 2\alpha+1]| \quad 0 \leq \alpha \leq 1]$$

DEF | Convex Combination
Let $\alpha, \beta \in \mathbb{F}$ and $\overrightarrow{u}, \overrightarrow{w} \in \mathbb{F}$
The vector $\overrightarrow{u} \in \mathbb{F}^n$

$$\overrightarrow{u}=\alpha\overrightarrow{u}+\beta\overrightarrow{w}$$

is called a convex combination if $\alpha+\beta=1$

Example 3. Express the line segment between $(0.5,1)$ and $(3.5,3)$ as a convex combination of $\overrightarrow{v}=[0.5,1]$ and $\overrightarrow{w}=[3.5,3]$
From Ex2(b), $\alpha[3,2] \longrightarrow [0.5,1]$ spans the segment.

$$\alpha([3.5,3]-[0.5,1])+[0.5,1]$$

$$\alpha[3.5,3]-\alpha[0.5,1]+[0.5,1]$$

$$\alpha[3.5,3]+(1-\alpha)[0.5,1]$$

$$\alpha[3.5,3]+\beta[0.5,1]$$

where $\beta=1-\alpha$ and $=1$

DEF | Dot Product
Let $\overrightarrow{v}, \overrightarrow{w} \in \mathbb{F}^n$

$$\overrightarrow{v} \cdot \overrightarrow{w}=v_1w_1+v_2w_2+...v_nw_n$$

Example 4. Find $\overrightarrow{u} \cdot \overrightarrow{w}$
$\overrightarrow{u}=[2,3,4]$, $\overrightarrow{2}=[-1,1]$
$\overrightarrow{u} \cdot \overrightarrow{w}=2-2+0+4=2$
Example 5. Express the equation

$$a_1x_1+a_2x_2...a_nx_n=c$$

as a dot product of two vectors. Let

$$\overrightarrow{a}=[a_1, a_2...a_n]$$

$$\overrightarrow{x}=[x_1, x_2...x_n]$$

$$\overrightarrow{a} \cdot \overrightarrow{x}=c$$

DEF | Vector Space
A set of V vectors is called vector space if it satisfies

1. that if contains a zero vector $\overrightarrow{0} \in \mathbb{V}$
   a) $A\overrightarrow{u} \in V \quad \overrightarrow{u}\,\overrightarrow{0} = \overrightarrow{u}$
   b) $\overrightarrow{u} - \overrightarrow{u} = \overrightarrow{0}$

2. For any $\alpha \in \mathbb{F}$ and for any $\overrightarrow{v} \in V$

$$\alpha\overrightarrow{v} \in V$$

   closed under scalar multiplication

3. A$\overrightarrow{u} + \overrightarrow{w} \in V$ closed under vector addition

Example 5. Is $\mathbb{R}^5$ a vector space?

$$\mathbb{R}^5 = [x_1, x_2, x_3, x_4, x_5]|x_i \in \mathbb{R}$$

1. Pick $\overrightarrow{x} \in \mathbb{R}^5$ $\overrightarrow{x} + \overrightarrow{0} = \overrightarrow{x} = \overrightarrow{0} + \overrightarrow{x}$ means $\overrightarrow{0} = [0, 0, 0, 0, 0,] \in \mathbb{R}^5$

2. Pick some $\alpha \in \mathbb{R}$ and some $\overrightarrow{v} \in \mathbb{R}^5$

$$\alpha\overrightarrow{v} = [\alpha v_1, ...\alpha v_5] \in \mathbb{R}^5$$

   because

3. Pick $\overrightarrow{u}, \overrightarrow{v} \in \mathbb{R}^5$

$$\overrightarrow{u} + \overrightarrow{v} = [u_1 + v_1..., u_5 + v_5] \in \mathbb{R}^5$$

   because ... and is closed under addition

DEF | Subspace
A subset W of a vector space V is a subspace of V if it is a vector space over the field $\mathbb{F}$. i.e. W is a subspace of V

1. $W \in V$

2. W is a vector space

Example 1. Show that $W = a[2, 1]|\alpha \in \mathbb{R}$

1. Pick a vector $\overrightarrow{w} \in \mathbb{W}$, then $\overrightarrow{w} = [2\alpha, \alpha]$ for some $\alpha \in \mathbb{R}$. Since $\alpha \in \mathbb{R}$, $2\alpha \in \mathbb{R}$ and $[2\alpha, \alpha] \in \mathbb{R}^2$
   i.e. A$\overrightarrow{w}(\overrightarrow{w} \in \mathbb{W} - >)$

2. NTS: W is a vector space
   $\overrightarrow{o} \in W$ because $\alpha = 0 \in \mathbb{R}$ therefore $0 \cdot [2,1] \in W$
   $\overrightarrow{0} + \overrightarrow{w} = [0,0] + [2\alpha, \alpha] = [0 + 2\alpha, o + \alpha] = [2\alpha, \alpha]$
   $\overrightarrow{w} + -(\overrightarrow{w}) = [2\alpha, \alpha] + [-2\alpha, -\alpha] = [0,0] = \overrightarrow{0}$
   For any $\overrightarrow{u}, \overrightarrow{w} \in W$, $\overrightarrow{u} = [2\alpha, \alpha]$ $\overrightarrow{w} = [2\beta, \beta]$
   $\overrightarrow{u} + \overrightarrow{w} = [2\alpha + 2\beta, \alpha + \beta] = [2(\alpha + \beta, \alpha, \beta)] = (\alpha + \beta[2,1] \in W)$ becase $\alpha + \beta \in \mathbb{R}$
   For some real number σ, $\sigma w = \sigma\alpha[2,1] \in W$ because $\sigma\alpha \in \mathbb{R} \Rightarrow$ W is a vector space
   $\Rightarrow$ W is a subspace of $\mathbb{R}^2$.

DEF | Linear Combination
Let V be a vector space and

$$U = \overrightarrow{u}, \overrightarrow{u}_2, ..., \overrightarrow{u}_k \in V$$

A vecotr $\overrightarrow{v} \in V$ is called a linear combination of vectors in U if

$$\overrightarrow{v} = \alpha_1 \overrightarrow{u}_1 + \alpha_2 \overrightarrow{u}_2 + ... + \alpha_x \overrightarrow{u}_x$$

where $\alpha_i \in \mathbb{F}$

Example 2. Find all linear combinations of $S = [1,0], [0,1] \in \mathbb{R}^2$
$\overrightarrow{v} = \alpha_1[1,0] + \alpha_2[0,1] \quad \alpha_1\alpha_2 \in \mathbb{R}$ is a linear combination.

$$\overrightarrow{v} = [\alpha_1, 0] + [0, \alpha_2] = [\alpha_1, \alpha_2]$$

All linear combinations are of this form. Hence all linear comb. $= [\alpha_1, \alpha_2] | \alpha_1\alpha_2 \in \mathbb{R} = \mathbb{R}^2$

DEF | Span
Let $U = \overrightarrow{u}_1, \overrightarrow{u}_2, ..., \overrightarrow{u}_k$. The set of all linear combinations of vectors in U is called the span of U.
$Span(U) = \sum \alpha_i u_i | \alpha_i \in \mathbb{F}$

Example 3. What is the span of T = $[1,0,0], [0,1,0]$ over $\mathbb{R}$?
Span(T) $= \alpha[1,0,0] + \beta[0,1,0] | \alpha, \beta \in \mathbb{R}$
$= [\alpha, 0, 0] + [0, \beta, 0] | \alpha, \beta \in \mathbb{R} = [\alpha, \beta, 0] | \alpha, \beta \in \mathbb{R} =$ vectors on xy-plane

THM | 1
Let V be a vector space and S $\leq$ V. Then

(i) Span(S) is a subspace of V

(ii) If T is a subspace of V and S $\leq$ T, then Span(S)$\leq$T

Proof
NTS: Span(S) $\leq$ V, Span(S) is a vector space

1. Pick $\overrightarrow{w}$ $\in$Span(S), then $\overrightarrow{w}$ must have the form $\overrightarrow{w} = \alpha_1\overrightarrow{u}_1 + \alpha_2\overrightarrow{u}_2 + ... + \alpha_k\overrightarrow{u}_k$ assuming $S = \overrightarrow{u}_1, ..., \overrightarrow{u}_v$
   Since V is a vector space, it is closed under scalar multiplication and vector addition. Hence, $\alpha_i u_i \in V$ (closed under scalar mult.) and $\sum \alpha_i u_i \in V$ (closed under vector addition), $\Rightarrow \overrightarrow{w} \in V$. Hence A$\overrightarrow{w}(\overrightarrow{w} \in Span(S) + \overrightarrow{w} \in V) \Rightarrow Span(S) \leq V$

2. To show Span(S) is a vector space:

$\overrightarrow{0}$ $\in$Span(S) because $\alpha = 0 \in \mathbb{R}$

1. $\overrightarrow{0} + \overrightarrow{w} = (0 \cdot \overrightarrow{u} + ... + 0 \cdot \overrightarrow{u}_k) = \sum \alpha_i u_i = \overrightarrow{w}$

2.

$\overrightarrow{w} = \sum \alpha_i \overrightarrow{u}_i$
$\overrightarrow{r} = \sum \beta_i \overrightarrow{u}_i$
$\overrightarrow{w}, \overrightarrow{r} \in SPan(S)$
$\rightarrow$ ......
   $\lambda \in \mathbb{R}$ $\quad \lambda\overrightarrow{w} = \lambda\alpha_1\overrightarrow{u}_1 + ... + \lambda\alpha_x\overrightarrow{u}_x \in Span(S)$ because $\lambda\alpha_i \in \mathbb{F}$
$\Rightarrow$ Span(S) is a subspace $\Rightarrow$ Span(S) is a subspace of V

DEF || Generates A subset S of vector space V generates (or spans) V if span(S) = V

Example 4. Find subset of $\mathbb{R}^2$ that generates $\mathbb{R}^2$

$$S = [1, 1], [-1, 1]$$

Span(S) = $\alpha[1, 1] + \beta[-1, 1]|\alpha, \beta$ =[$\alpha-\beta$,$\alpha+\beta$]$|\alpha, \beta = \mathbb{R}^2$ **Example 5.** Does S =[1, 2], [2, 4] span $\mathbb{R}^2$?
(Graphed) No, because [2,4] = 2[1,2]. Hence, any linear combination would result in the following: $\alpha[1, 2] + \beta[2, 4] = \alpha[1, 2] + 2\beta[1, 2] = (\alpha + 2\beta)[1, 2] <-$ the scaling of [1,2]; only generates a line.