§ 4 Basis I & II

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MOTIVATION: Earlier we saw $[1,0],[0,1]$ generates $\mathbb{R}^2$ i.e.

$$Span[1,0],[0,1] = \mathbb{R}^2$$

What does this mean? For any vector $\overrightarrow{v} = [a,b] \in \mathbb{R}^2$, we can define its location as

$$a[1,0] + b[0,1] = \overrightarrow{v}$$

DEF | Coordinates
Let G be a generating set of vectors and V be a vector space over a field $\mathbb{F}$

$$Span(G) = V$$

The coordinates of a vector $\overrightarrow{v} \in V$ with respect to the set G are the coefficients $\alpha_1, \alpha_2, ..., \alpha_n$ where

$$\overrightarrow{v} = \alpha_1 \overrightarrow{g}_1 + \alpha_2 \overrightarrow{g}_2 + ... + \alpha_n \overrightarrow{g}_n$$

and $\overrightarrow{g}_i \in G$.

Example. Find the coordinates of vector $\overrightarrow{v} = [1,2]$ with respect to the given generating set of $\mathbb{R}^2$.

(a) $G = [1,1],[-1,1]$
$[1,2] = \alpha[1,1], \beta[-1,1]$
$[\alpha,\alpha] + [-\beta,\beta]$
$[\alpha-\beta, \alpha+\beta]$
$[1,2] = \frac{3}{2}[1,1] + \frac{1}{2}[-1,1]$
Coordinates: $< \frac{3}{2}, \frac{1}{2} > G$

(b) $G_2 = [-2,0],[0,1]$
$[1,2] = \alpha[-2,0] + \beta[0,1]$
$[-2\alpha, 0] + [0,\beta]$
$-2\alpha = 1, \beta = 2$
$\alpha = \frac{-1}{2}$
Coordinates: $< \frac{-1}{2}, 2 >_{G_2}$

NOTE: We could also argue that $H = [1,0],[0,1],[2,0]$ also generates $\mathbb{R}^2$ e.g. $\overrightarrow{v} \in \mathbb{R}$

$$\overrightarrow{v} = \alpha[1,0] + \beta[0,1] + \lambda[2,0]$$

$$1,2 = [\alpha,0] + [0,\beta] + [2\lambda,0] = [\alpha+2\lambda, \beta]$$

$$\beta = 1 \quad \alpha = -1 \quad \lambda = 1 \quad < -1,1,1 >_H$$

$$\beta = 1 \quad \alpha = 1 \quad \lambda = 0 \quad < 1,1,0 >_H$$

$$\beta = 1 \quad \alpha = 2 \quad \lambda = -\frac{1}{2} \quad < 1,2,-\frac{1}{2} >_H$$

We obtain more than one possible set of coordinates for the same vector under the same generating set. Confusing!

DEF | Linear Dependence
The vectors $\overrightarrow{v_1}, \overrightarrow{v_2}, ..., \overrightarrow{v_k}$ are linearly dependent if we can find scalars $\alpha_1, \alpha_2, ..., \alpha_k \in \mathbb{F}$ not all zero such that

$$\alpha_1 \overrightarrow{v}_1 + \alpha_2 \overrightarrow{v}_2 + ... + \alpha_k \overrightarrow{v}_k = \overrightarrow{0}$$

$$\Rightarrow \alpha_i \overrightarrow{v}_i = -\alpha_1 \overrightarrow{v}_1 - \alpha_2 \overrightarrow{v}_2 - ... - \alpha_k \overrightarrow{v}_k$$

$$\Rightarrow \overrightarrow{v}_i = \frac{\alpha_1}{\alpha_i} \overrightarrow{v}_1 - \frac{\alpha_2}{\alpha_i} \overrightarrow{v}_2 - ... - \frac{\alpha_k}{\alpha_i} \overrightarrow{v}_k$$

i.e. any vector in the set can be written as a linear combination of the rest of the vectors.

Example. Determine if the set of vectors is linearly dependent.

(a) $[1, 0], [2, 0]$
$\alpha_1[1, 0] + \alpha_2[2, 0] = [0, 0]$
$alpha_1 = -2, \alpha_2 = 1 \Rightarrow$ linearly dependent

(b) [1,0,0],[0,2,0],[2,4,0],[0,1,0]
$\alpha_1[1, 0, 0] + \alpha_2[0, 2, 0] + \alpha_3[2, 4, 0] + \alpha_4[0, 1, 0] = \overrightarrow{0}$
$\alpha_1 = -2\alpha_2 = -2\alpha_3 = 1\alpha_4 = 0 \Rightarrow$ linearly dependent

(c) $[1, 0, 0], [0, 2, 0], [0, 0, 4]$
$\alpha_1[1, 0, 0] + \alpha_2[0, 2, 0] + \alpha_3[0, 0, 4] = [0, 0, 0]$
$\alpha_1 = 0\alpha_2 = 0\alpha_3 = 0 \Rightarrow$ linearly independent

LEMMA 1
Let $G = \overrightarrow{v}_1, ..., \overrightarrow{v}_k$ where $\overrightarrow{v}_i \in \mathbb{F}^2$. Then $\overrightarrow{x} \in Span(G)$ if and only if $\exists \alpha_i \in \mathbb{F}$ such that

$$\alpha_1 \overrightarrow{v}_1 + \alpha_2 \overrightarrow{v}_2 + ... + \alpha_k \overrightarrow{v}_k + \alpha_{k+1} \overrightarrow{x} = 0$$

and not all scalars are 0.

Proof. Assume

$$\alpha_1 \overrightarrow{v}_1 + ... + \alpha_k \overrightarrow{v}_k + \alpha_{k+1} \overrightarrow{x} = 0$$

and not all scalars are 0. Then,

$$\alpha_{k+1} \overrightarrow{x} = -\alpha_1 \overrightarrow{v}_1 - ... - \alpha_k \overrightarrow{v}_k$$

$$\overrightarrow{x} = \frac{-\alpha_1}{\alpha_{k+1}} \overrightarrow{v}_1 - ... - \frac{-\alpha_k}{\alpha_{k+1}} \overrightarrow{v}_k$$

$$= \beta_1 \overrightarrow{v}_1 + ... + \beta_k \overrightarrow{v}_k$$

where $\beta_i = \frac{-\alpha_i}{\alpha_{k+1}}$
$\Rightarrow \overrightarrow{x} \in Span(G)$
Now assume $\overrightarrow{x} tSpan(G)$. Then $\exists \lambda_i \in \mathbb{F}$ such that

$$\overrightarrow{x} = \lambda_1 \overrightarrow{v}_1 + ... + \lambda_k \overrightarrow{v}_k$$

$$\Rightarrow \lambda_1 \overrightarrow{v}_1 + ... + \lambda_k \overrightarrow{v}_k + (-1) \overrightarrow{x} = \overrightarrow{0}$$

Hence $\lambda_1 \overrightarrow{v}_1 + ... + \lambda_k \overrightarrow{v}_k + \lambda_{k+1} \overrightarrow{x} = \overrightarrow{0}$ where $\lambda_{k+1} = -1$

LEMMA 2
Let $S = \overrightarrow{\vec{v}_1, v_2, ..., \vec{v}_k, \vec{x}}$ such that $\vec{x} = \alpha_1 \vec{v} + ... + \alpha_k \vec{v}_k$. Then $Span(S) = Span(S - \vec{x})$

Proof NTS 1. $Span(S) \leq Span(S - \vec{x})$ and 2. $Span(S - \vec{x}) \leq Span S$

(1) Pick an arbitrary vector $\vec{u} \in Span(S)$. Then
$\vec{u} = \beta_1 \vec{v}_1 + ... + \beta_k \vec{v}_k + \beta_{k+1} \vec{x}$
$= \beta_1 \vec{v}_1 + ... + \beta_k \vec{v}_k + \beta_{k+1}(\alpha_1 \vec{v}_1 + ... + \alpha_k \vec{v}_k)$
$= \beta_1 \vec{v}_1 + ... + \beta_k \vec{v}_k + \beta_{k+1}\alpha_1 \vec{v}_1 + ... + \beta_{k+1}\alpha_k \vec{v}_k$
$= (\beta_1 + \beta_{k+1}\alpha_1) \vec{v}_1 + ... + (\beta_k + \beta_{k+1}\alpha_k) \vec{v}_k$
$= \lambda_1 \vec{v}_1 + ... + \lambda_k \vec{v}_k$
$\Rightarrow \vec{u} \in Span(S - \vec{x})$
$\Rightarrow Span(S) \leq Span(S - \vec{x})$

To show (2), pick an arbitrary $\vec{w} \in Span(S - \vec{x})$. Then,
$\vec{w} = w_1 \vec{v}_1 + ... + w_k \vec{v}_k = w_1 \vec{v}_1 + ... + w_k \vec{v}_k + 0 \vec{x}$
$\Rightarrow \vec{w} \in Span(S)$
$Span(S - \vec{x}) \leq Span(S)$
By (1) and (2) it follows that $Span\ S = Span(S - \vec{x})$

DEF | Basis
Let G be a set of vectors. We call G a basis for a vector space V if

(i) $Span(G) = V$ i.e. G generates V

(ii) G is a linearly independent set

e.g $G = [1, 0], [0, 1]$ is a basis for $\mathbb{R}^2$
$T = [1, 0], [0, 1], [0, 2]$ is a generating set for $\mathbb{R}^2$ i.e. $R^2 = Span(T)$ but T is NOT a basis.

Size of a Basis

LEMMA 3 Let V be a vector space, S be a generating set such that $Span(S) = V$ and B be a subset V that is linearly independent. Then

$$|S| \geq |B|$$

size of a generating set $\geq$ size of a linearly independent set

THM Basis Theorem
Let V be a vector space. All bases for V have the same size.

Proof. Suppose B_1 and B_2 are bases for V.
*B_1 is a bases means (i) $Span(B_1) = V$ and (ii) B_1 is linearly independent.
*B_2 is a basis means (iii) $Span(B_2) = V$ and (iv) B_2 is linearly independent.
From Lemma 3.
(i) & (iv) $|B_2| \leq |B_1|$ assuming $B_2 \leq V$ (which is true b/c any vector in B_2 *)
(ii) & (iii) $|B_1| = |B_2| \Rightarrow$ bases have the same size.

THM Let V be a vector space, B a set of generators for V. B will be the smallest set of generators if and only if it is the basis for V.
Assume B is a basis for V. Consider some set of generators T. Since B is a basis, it is linearly independent & by Lemma 3

$$|B| \leq |T|$$

Since T is arbitrary, the result holds for all sets of generators. Hence, B is the smallest generating set.
Now, assume B is the smallest generating set but is not a basis. i.e. either
$Span(B) \neq V$ $\leftarrow$ FALSE
or B is not linearly independent $\leftarrow$ TRUE
If B is not linearly independent, there is some vector $\overrightarrow{x} \in B$ that is a linear combination of the other vectors in B. By Lemma 2, then,

$$V = Span(B) = Span(B - \overrightarrow{x})$$

i.e. B is NOT the smallest generating set which is a contradiction. So B MUST be a basis.

DEF | Dimension
The dimension of a vector space is the size of a basis for $V = dim(V) = |B|$ where B is a basis.