# § 6 Gram-Schimdt Process

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**DEF** | Inner Product
Let $\overrightarrow{x}, \overrightarrow{y}, \in \mathbb{R}^n$, the inner product of $\overrightarrow{x}$ with $\overrightarrow{y}$

$$< \overrightarrow{x}, \overrightarrow{y} >= x_1y_1 + x_2y_2 + ... + x_ny_n = \overrightarrow{x} \cdot \overrightarrow{y}$$

**DEF** | Normalized
A vector $\overrightarrow{x} \in \mathbb{R}^n$ is normalized if $||\overrightarrow{x}||_2 = 1$.

**Example.** Find $< \overrightarrow{x}, \overrightarrow{x} >$ where $\overrightarrow{x} \in \mathbb{R}^n$. $\overrightarrow{x} = [x_1, x_2, ..., x_n]$

$$\overrightarrow{x}, \overrightarrow{x} >= x_1 \cdot x_1 + x_2 \cdot x_2 + ... + x_n \cdot x_n = x_1^2 + x_2^2 + ... + x_n^2$$

$$= ||\overrightarrow{x}||_2^2$$

*NOTE:

1. $||\overrightarrow{x}||_2 = \sqrt{< \overrightarrow{x}, \overrightarrow{x} >}$
2. For any vector $\overrightarrow{x} \in \mathbb{R}^n$, we can form the normalized vector $\overrightarrow{y}$ of $\overrightarrow{x}$ by $\overrightarrow{y = \frac{1}{||\overrightarrow{x}||_2}}\overrightarrow{x}$

i.e. any vector of the form $\overrightarrow{y = \frac{1}{||\overrightarrow{x}||_2}}\overrightarrow{x}$ is a normalized vector.

**Example.** Normalize $\overrightarrow{x} = [1, 1]^T$

$$||\overrightarrow{x}||_2 = \sqrt{1^2 + 1^2} = \sqrt{2}$$

$$\Rightarrow \overrightarrow{y} = \frac{1}{\sqrt{2}\overrightarrow{x}}$$

$$= [\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}]^T$$

Verifying:

$$||\overrightarrow{y}||_2 = \sqrt{(\frac{\sqrt{2}}{2})^2 + (\frac{\sqrt{2}}{2})^2}$$

$$= \sqrt{\frac{2}{4} + \frac{2}{4}}$$

$$= \sqrt{1} = 1$$

**Properties of Inner Product.** Let $\overrightarrow{x}, \overrightarrow{y}, \overrightarrow{z} \in \mathbb{R}^n$

1. $< \overrightarrow{x}, \overrightarrow{x} >> 0$
2. $< \overrightarrow{x}, \overrightarrow{x} >= 0$ if and only if $\overrightarrow{x} = \overrightarrow{0}$
3. $<> \overrightarrow{x} m, \overrightarrow{y} =< \overrightarrow{y}, \overrightarrow{x} >$
4. $\lambda < \overrightarrow{x}, \overrightarrow{y} >=< \lambda \overrightarrow{x}, \overrightarrow{y} >$ Proof E.C.
5. $< \overrightarrow{x} + \overrightarrow{z}, \overrightarrow{y} >=< \overrightarrow{x} + \overrightarrow{y} > + < \overrightarrow{z}, \overrightarrow{y} >$ Proof E.C.
6. $< \overrightarrow{0}, \overrightarrow{y} >= 0$

**CLAIM.** Let $\overrightarrow{u}, \overrightarrow{v} \in \mathbb{R}^n$

$$< \overrightarrow{u}, \overrightarrow{v} = ||\overrightarrow{u}||_2 \cdot ||\overrightarrow{v}||_2 \cos\theta >$$

Where $\theta$ is the angle between the vectors.

**Proof.** E.C. (Hint: Consider Law of Cosines)

**Gram-Schimdt Process**

> **DEF** | Orthogonal
> Let $\overrightarrow{x}, \overrightarrow{y} \in \mathbb{R}^n$. We say $\overrightarrow{x}$ is orthogonal to $\overrightarrow{y}$ if $< \overrightarrow{x}, \overrightarrow{y} >= 0$ i.e. graphically $\overrightarrow{x} \perp \overrightarrow{y}$.
> A set of vectors is orthogonal if the vectors are orthogonal to each other.

**Example.** Determine if the set of vectors are orthogonal.

1. $M = [\frac{1}{\sqrt{2}}, [\frac{1}{\sqrt{2}}, 0]^T, [[\frac{1}{\sqrt{2}}, -[\frac{1}{\sqrt{2}}, 0]^T, [0, 0, 1]$
   $< \overrightarrow{m_1}, \overrightarrow{m_2} >= ([\frac{1}{\sqrt{2}})^2 - \frac{1}{2} + 0 = \frac{1}{2} - \frac{1}{2} + 0 = 0$
   $< \overrightarrow{m_1}, \overrightarrow{m_3} >= 0 + 0 + 0 = 0$
   $< m_2, m_3 >= 0 + 0 + 0 = 0$

$\Rightarrow$ The vectors are orthogonal, the set is orthogonal

> **DEF** | Orthonormal
> A set of vectors in $\mathbb{R}^n$ is orthonormal if the vectors are orthogonal and they are normalized. i.e. they have a magnitude of 1

**Example.** Determine if the set $M$ from the previous example is orthonormal.
Ans:

1. The vectors are orthogonal
2. $||\overrightarrow{m}_1||_2 = \sqrt{([\frac{1}{\sqrt{2}})^2 + ([\frac{1}{\sqrt{2}})^2 + 0^2} = \sqrt{\frac{1}{2} + \frac{1}{2} + 0} = 1 = ||\overrightarrow{m}_2||_2$
   $||\overrightarrow{m_3}||_2 = \sqrt{0^2 + 0^2 + 1^2} = \sqrt{1} = 1$

Hence, M is orthonormal.

> **DEF** | Kronecker Delta
> The Kronecker delta, $d_{ij}$, is a relation defined by
> $d_{ij} = 1$ if $i = j$
> $d_{ij} = 0$ if otherwise *Remarks:
>
> 1. The set $\overrightarrow{x}_1, \overrightarrow{x}_2, ..., \overrightarrow{x}_n$ is orthonormal if
>    (a) $< \overrightarrow{x}_i, \overrightarrow{x}_j >= d_{ij}$ for all $i, j \in [1, n]$
>    (b) $||\overrightarrow{x}_i|| = 1 i = 1, 2, ..., n$
>    (c) $=< \overrightarrow{x_i}, \overrightarrow{x_i} >= ||x_i||^2 = 1^2 = 1$ when $i - j$
>    (d) $< \overrightarrow{x_i}, \overrightarrow{x_j} >= 0$ when $i \neq j$
>    (e) $= d_{ij}$

2. A set of orthogonal vectors can be made orthonormal by normalizing each vector ($\frac{1}{||\overrightarrow{x}||}\overrightarrow{x}$)
3. Any orthogonal/orthonormal set that spans a vector space $V$ is a bases for $V$. Proof: E.C.

**Motivation for Gram-Schimdt Process**

* The Grow Algorithm allows us to find a basis from a set of vectors $T$, for some vector space.
* What if we want the basis to be orthonormal?
  (a) i.e. $B = \overrightarrow{u}, \overrightarrow{x}$ is a basis for $\mathbb{R}^2$
  (b) Notice, $\overrightarrow{x} = \overrightarrow{x}_{\|} + \overrightarrow{x}_{\perp}$ where $\overrightarrow{x}_{\|} = \alpha\overrightarrow{u}$ for $\alpha \in \mathbb{R}$
  (c) If we can find $\alpha$, then $\overrightarrow{x}_{\perp} = \overrightarrow{x} - \overrightarrow{x}_{\|}$ and $\overrightarrow{x}_{\perp} = \overrightarrow{x} - \alpha\overrightarrow{u}$
  (d) In such a case $\overrightarrow{x}_{\perp}, \overrightarrow{u}$ would be an orthogonal basis
  (e) and if we normalize the vectors, we obtain an orthonormal basis: $\frac{1}{||\overrightarrow{x}_{\perp}||}\overrightarrow{x}_{\perp}, \frac{1}{||\overrightarrow{u}_{\perp}||}\overrightarrow{u}$

**Goal.** Find $\alpha$.
Notice:

$$< \overrightarrow{x}_{\perp}, \overrightarrow{x}_{\|} >= 0$$
$$< \overrightarrow{x} - \overrightarrow{x}_{\|}, \overrightarrow{x}_{\|} >= 0$$
$$< \overrightarrow{x} - \alpha\overrightarrow{u}, \alpha\overrightarrow{u} >= 0$$
$$< \overrightarrow{x} + (-\alpha\overrightarrow{u}, \alpha\overrightarrow{u}), \alpha u >= 0$$
$$< \overrightarrow{x}, \alpha\overrightarrow{u} > + < (-\alpha\overrightarrow{u}), \alpha\overrightarrow{u} >= 0$$
$$< \overrightarrow{x}, \alpha\overrightarrow{u} > + < (-\alpha\overrightarrow{u}), \alpha\overrightarrow{u} >= 0$$
$$< \overrightarrow{x}, \alpha\overrightarrow{u} > -\alpha < \overrightarrow{u, \alpha\overrightarrow{u}} >= 0$$
$$\alpha < \overrightarrow{x}, \overrightarrow{u} > -\alpha^2 < \overrightarrow{u}, \overrightarrow{u} >= 0$$
$$\alpha[< \overrightarrow{x}, \overrightarrow{u} > -\alpha < \overrightarrow{u}, \overrightarrow{u} >] = 0$$

**Case:** If $\alpha = 0$

$$\overrightarrow{x} = \overrightarrow{x}_{\|} + \overrightarrow{x}_{\perp} = \alpha\overrightarrow{u} + \overrightarrow{x}_{\perp}$$
$$\overrightarrow{x} = \overrightarrow{x}_{\perp}$$

$\overrightarrow{x}$ is already to $\overrightarrow{u}$
**Case:** $< \overrightarrow{x}, \overrightarrow{u} > -\alpha < \overrightarrow{u}, \overrightarrow{u} >= 0$

$$\alpha < \overrightarrow{u}, \overrightarrow{u} >=< \overrightarrow{x}, \overrightarrow{u} >$$

$$\alpha = \frac{< \overrightarrow{x}, \overrightarrow{u} >}{< \overrightarrow{u}, \overrightarrow{u} >}$$

Hence, given $\overrightarrow{x}, \overrightarrow{u}$

$$\overrightarrow{x} = \overrightarrow{x}_{\parallel} + \overrightarrow{x}_{\perp}$$

$$\overrightarrow{x}_{\parallel} = \alpha \overrightarrow{u} = \frac{< \overrightarrow{x}, \overrightarrow{u} >}{< \overrightarrow{u}, \overrightarrow{u} >} \overrightarrow{u} = proj_{\overrightarrow{u}\overrightarrow{x}}$$

$\frac{<\overrightarrow{x},\overrightarrow{u}>}{<\overrightarrow{u},\overrightarrow{u}>}\overrightarrow{u}$ is also known as the projection of $\overrightarrow{x}$ onto $\overrightarrow{u}$

**Gram Schimdt Process in $\mathbb{R}^2$**

* INPUT: Basis $B = \overrightarrow{b}_1, \overrightarrow{b_2}$
* OUTPUT: orthonormal basis $N = \overrightarrow{v}_1, \overrightarrow{v}_2$
* Normalize the first vector
  $\overrightarrow{v}_1 = \frac{1}{||\overrightarrow{b_1}||}\overrightarrow{b_1}$
* Form the vector
  $\overrightarrow{y} = \overrightarrow{b_1} - proj_{\overrightarrow{b_1}}\overrightarrow{b_2} \Rightarrow \overrightarrow{v_2} = \frac{1}{||\overrightarrow{y}||}\overrightarrow{y}$

**Example.** Find an orthonormal basis given the basis for $\mathbb{R}^2$ where $B = \{[2, 7], [-3, 4]\}$.

$$||\overrightarrow{b_1}|| = \sqrt{-3^2 + 4^2} = \sqrt{9 + 19} = 5$$

$$= \frac{1}{||\overrightarrow{b_1}||}\overrightarrow{b_1} = \frac{1}{5}[-3, 4] = [\frac{3}{5}, \frac{4}{5}]$$

$$* \overrightarrow{y} = \overrightarrow{b_1} - proj_{\overrightarrow{b_2}}\overrightarrow{b_1}$$

$$proj_{\overrightarrow{b_2}}\overrightarrow{b_1} = \frac{< \overrightarrow{b_1}, \overrightarrow{b_2} >}{< \overrightarrow{b_2}, \overrightarrow{b2} >}\overrightarrow{b_2} = \frac{-6 + 28}{9 + 16}\overrightarrow{b_2}$$

$$= \frac{22}{25}[3, 2] = [\frac{-66}{25}, \frac{88}{25}]$$

$$\Rightarrow \overrightarrow{y} = [2, 7] - [\frac{-66}{25}, \frac{88}{25}] = [\frac{116}{25}, \frac{87}{25}]$$

$$\overrightarrow{v_2} = \frac{1}{||\overrightarrow{y}||}\overrightarrow{y} = \frac{1}{||\overrightarrow{y}||}[\frac{116}{25}, \frac{87}{25}]$$

$\Rightarrow$ Orthonomral basis $N = \{\overrightarrow{v_1}, \overrightarrow{v_2}\} = \{[\frac{3}{5}, \frac{4}{5}], \frac{1}{||\overrightarrow{y}||}[\frac{116}{25}, \frac{87}{25}]\}$