Symmetric Matrices

Today we'll study a very important class of matrices: **symmetric** matrices.

We'll see that symmetric matrices have properties that relate to both eigendecomposition, and orthogonality.

Furthermore, symmetric matrices open up a broad class of problems we haven't yet touched on: constrained optimization.

As a result, symmetric matrices arise very often in applications.

Definition. A symmetric matrix is a matrix A such that $A^T = A$.

Clearly, such a matrix is square.

Furthermore, the entries that are not on the diagonal come in pairs, on opposite sides of the diagonal.

Example. Here are three **symmetric** matrices:

$$\begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix}, \quad \begin{bmatrix} 0 & -1 & 0 \\ -1 & 5 & 8 \\ 0 & 8 & -7 \end{bmatrix}, \quad \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix}$$

Here are three **nonsymmetric** matrices:

$$\begin{bmatrix} 1 & -3 \\ 3 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & -4 & 0 \\ -6 & 1 & -4 \\ 0 & -6 & 1 \end{bmatrix}, \quad \begin{bmatrix} 5 & 4 & 3 & 2 \\ 4 & 3 & 2 & 1 \\ 3 & 2 & 1 & 0 \end{bmatrix}$$

Orthogonal Diagonalization

First, we'll look at a remarkable property of symmatric matrices: their eigenvectors are **orthogonal**.

Example. Diagonalize the following symmetric matrix:

$$A = \begin{bmatrix} 6 & -2 & -1 \\ -2 & 6 & -1 \\ -1 & -1 & 5 \end{bmatrix}$$

Solution.

The characteristic equation of A is

$$0 = -\lambda^3 + 17\lambda^2 - 90\lambda + 144$$

$$= -(\lambda - 8)(\lambda - 6)(\lambda - 3)$$

So the eigenvalues are 8, 6, and 3.

We construct a basis for each eigenspace (using our standard method of finding the nullspace of $A - \lambda I$):

$$\lambda_1 = 8 : \mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}; \quad \lambda_2 = 6 : \mathbf{v}_2 = \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}; \quad \lambda_3 = 3 : \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

These three vectors form a basis for $\mathbb{R}^3$.

More interestingly, these three vectors are **mutually orthogonal.**

For example,

$$\mathbf{v}_1^T \mathbf{v}_2 = (-1)(-1) + (1)(-1) + (0)(2) = 0$$

That is $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is an *orthogonal* basis for $\mathbb{R}^3$.

Let's normalize these vectors so they each have length 1:

$$\mathbf{u}_1 = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}; \quad \mathbf{u}_2 = \begin{bmatrix} -1/\sqrt{6} \\ -1/\sqrt{6} \\ 2/\sqrt{6} \end{bmatrix}; \quad \mathbf{u}_3 = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$$

Now let's write the diagonalization of A in terms of these eigenvectors and eigenvalues:

$$P = \begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} & 1/\sqrt{3} \end{bmatrix}, \quad D = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

Then, $A = PDP^{-1}$, as usual.

But, here is the interesting thing: P is square and has orthonormal columns. So P is an **orthogonal** matrix.

So, that measns that $P^{-1} = P^T$.

So, $A = PDP^T$.

Here is a theorem that shows that this **always** happens when we diagonalize a symmetric matrix:

Theorem. If A is symmetric, then any two eigenvectors of A from different eigenspaces are orthogonal.

Proof.

Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be eigenvectors that correspond to distinct eigenvalues, say, λ_1 and λ_2.

To show that $\mathbf{v}_1^T\mathbf{v}_2 = 0$, compute

$$\lambda_1\mathbf{v}_1^T\mathbf{v}_2 = (\lambda_1\mathbf{v}_1)^T\mathbf{v}_2$$

$$= (A\mathbf{v}_1)^T\mathbf{v}_2$$

$$= (\mathbf{v}_1^T A^T)\mathbf{v}_2$$

$$= \mathbf{v}_1^T(A\mathbf{v}_2)$$

$$= \mathbf{v}_1^T(\lambda_2\mathbf{v}_2)$$

$$= \lambda_2\mathbf{v}_1^T\mathbf{v}_2$$

So we conclude that $\lambda_1(\mathbf{v}_1^T\mathbf{v}_2) = \lambda_2(\mathbf{v}_1^T\mathbf{v}_2)$.

But $\lambda_1 \neq \lambda_2$, so this can only happen if $\mathbf{v}_1^T\mathbf{v}_2 = 0$.

So $\mathbf{v}_1$ is orthogonal to $\mathbf{v}_2$.

We can now introduce a special kind of diagonalizability:

An $n \times n$ matrix is said to be **orthogonally diagonalizable** if there are an orthogonal matrix P (with $P^{-1} = P^T$) and a diagonal matrix D such that

$$A = PDP^T = PDP^{-1}$$

Such a diagonalization rquires n linearly independent and orthonormal eigenvectors.

When is this possible?

If A is orthogonally diagonalizable, then

$$A^T = (PDP^T)^T = (P^T)^T D^T P^T = PDP^T = A$$

So A is symmetric!

That is, whenever A is orthogonally diagonalizable, it is symmetric.

It turns out the converse is true (though we won't prove it). This leads to the following theorem:

Theorem. An $n \times n$ matrix is orthogonally diagonalizable if and only if it is a symmetric matrix.

Remember that when we studied diagonalization, we found that it was a difficult process to determine if an arbitrary matrix was diagonalizable.

But here, we have a very nice rule: **every symmetric matrix is (orthogonally) diagonalizable.**

Quadratic Forms

Up until now, we have mainly focused on linear equations – equations in which the x_i terms occur only to the first power.

Actually, though, we have looked at some quadratic expressions when we considered least-squares problems.

For example, we looked at expressions such as $\|x\|^2$ which is $\sum x_i^2$.

We'll now look at quadratic expressions generally. We'll see that there is a natural and useful connection to symmetric matrices.

Definition. A **quadratic form** is a function of variables, eg, $x_1, x_2, \ldots, x_n$, in which every term has degree two.

Example:

$4x_1^2 + 2x_1x_2 + 3x_2^2$ is a quadratic form.

$4x_1^2 + 2x_1$ is not a quadratic form.

Quadratic forms arise in many settings, including optimization, signal processing, physics, economics, and statistics.

Essentially, what an expression like x^2 is to a scalar, a quadratic form is to a vector.

Fact. Every quadratic form can be expressed as $\mathbf{x}^T A\mathbf{x}$, where A is a symmetric matrix.

To see this, let's look at some examples.

Example. Let $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. Compute $\mathbf{x}^T A\mathbf{x}$ for the matrix $A = \begin{bmatrix} 4 & 0 \\ 0 & 3 \end{bmatrix}$.

Solution.

$$\mathbf{x}^T A\mathbf{x} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 4x_1 \\ 3x_2 \end{bmatrix}$$

$$= 4x_1^2 + 3x_2^2.$$

Example. Compute $\mathbf{x}^T A\mathbf{x}$ for the matrix $A = \begin{bmatrix} 3 & -2 \\ -2 & 7 \end{bmatrix}$.

Solution.

$$\mathbf{x}^T A\mathbf{x} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 3 & -2 \\ -2 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= x_1(3x_1 - 2x_2) + x_2(-2x_1 + 7x_2)$$

$$= 3x_1^2 - 2x_1x_2 - 2x_2x_1 + 7x_2^2$$

$$= 3x_1^2 - 4x_1x_2 + 7x_2^2$$

Example. For $\mathbf{x}$ in $\mathbb{R}^3$, let

$$Q(\mathbf{x}) = 5x_1^2 + 3x_2^2 + 2x_3^2 - x_1x_2 + 8x_2x_3.$$

Write this quadratic form $Q(\mathbf{x})$ as $\mathbf{x}^T A\mathbf{x}$.

Solution.

The coefficients of x_1^2, x_2^2, x_3^2 go on the diagonal of A.

Based on the previous example, we can see that the coefficient of each cross term x_ix_j is the sum of two values in symmetric positions on opposite sides of the diagonal of A.

So to make A symmetric, the coefficient of x_ix_j for $i \neq j$ must be split evenly between the (i, j)- and (j, i)-entries of A.

You can check that

$$Q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x} = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 5 & -1/2 & 0 \\ -1/2 & 3 & 4 \\ 0 & 4 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Question Time! Q24.1

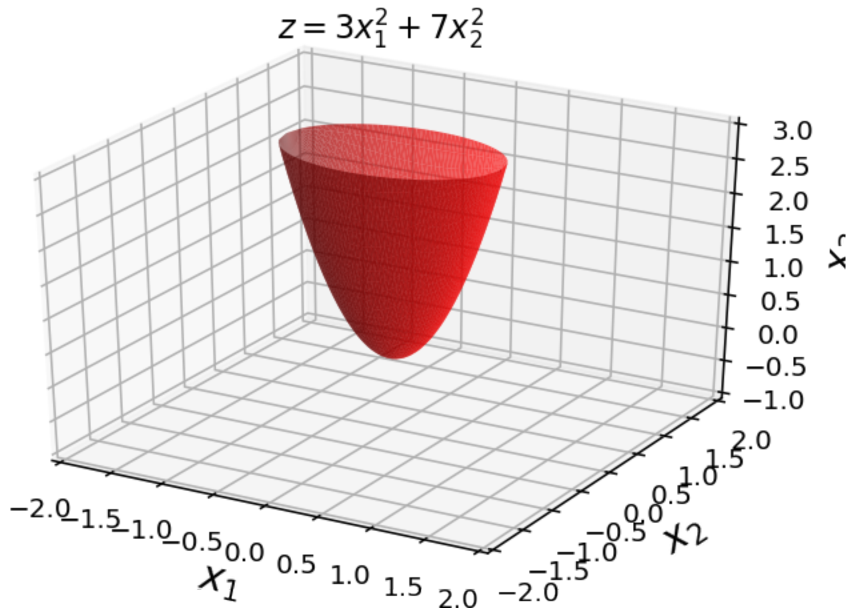
Classifying Quadratic Forms

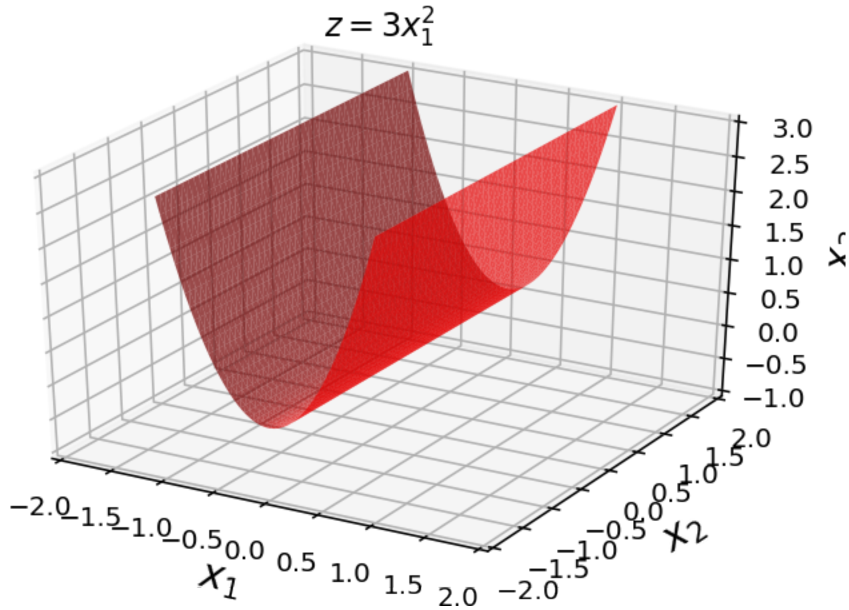
Notice that $\mathbf{x}^T A\mathbf{x}$ is a **scalar**.

In other words, when A is an $n \times n$ matrix, the quadratic form $Q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$ is a real-valued function with domain $\mathbb{R}^n$.

Here are four quadratic forms with domain $\mathbb{R}^2$.

Notice that except at $\mathbf{x} = \mathbf{0}$, the values of $Q(\mathbf{x})$ are all positive in the first case, and all negative in the last case.



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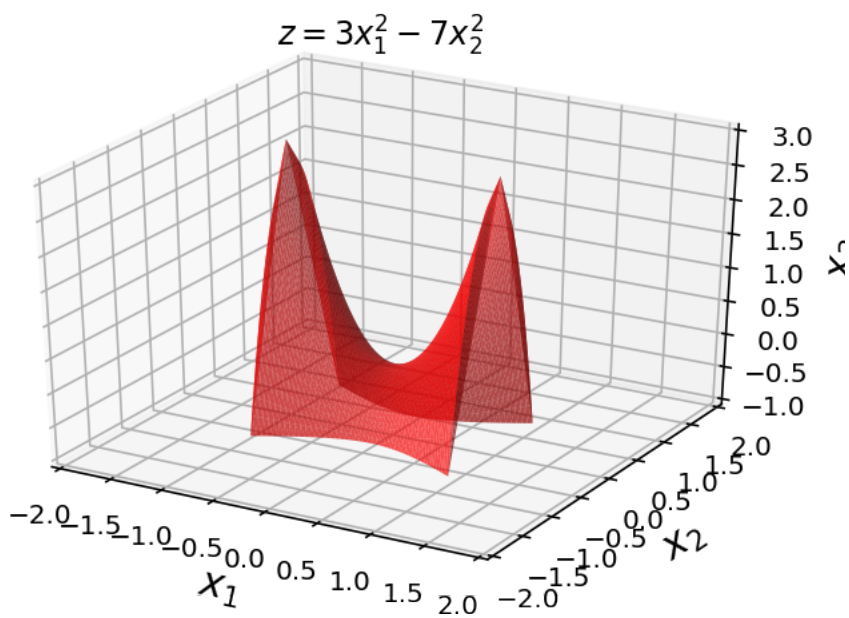


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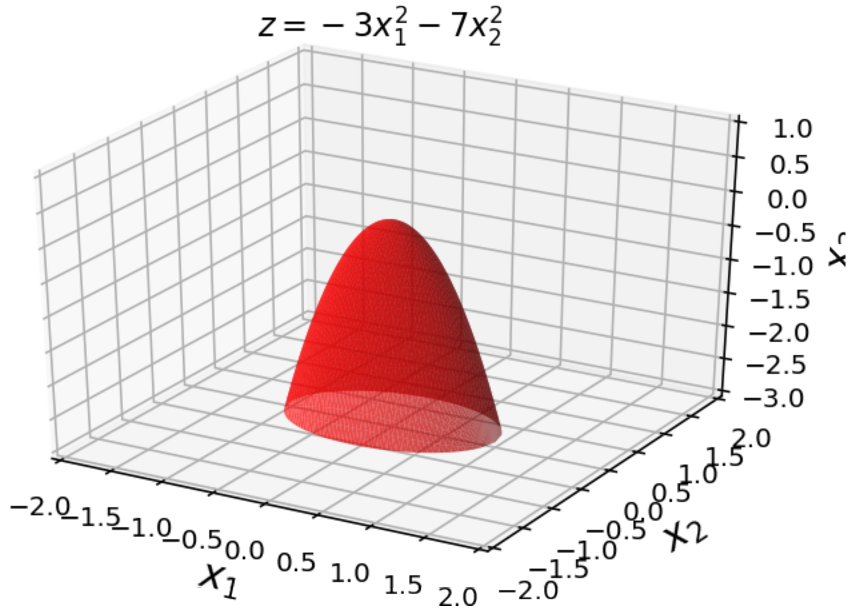


The differences between these surfaces is important for problems such as **optimization**.

In an optimization problem, one seeks the minimum or maximum value of a function (perhaps over a subset of its domain).

Definition. A quadratic form Q is:

1. **positive definite** if $Q(\mathbf{x}) > 0$ for all $\mathbf{x} \neq 0$.
2. **negative definite** if $Q(\mathbf{x}) < 0$ for all $\mathbf{x} \neq 0$.
3. **indefinite** if $Q(\mathbf{x})$ assumes both positive and negative values.
4. **positive semidefinite** if $Q(\mathbf{x}) \geq 0$ for all $\mathbf{x} \neq 0$.

There is a remarkably simple way to determine, for any quadratic form, which class it falls into.

Theorem. Let A be an $n \times n$ symmetric matrix. Then a quadratic form $\mathbf{x}^T A\mathbf{x}$ is

1. positive definite if and only if the eigenvalues of A are all positive.
2. negative definite if and only if the eigenvalues of A are all negative.
3. indefinite if and only if A has both postive and negative eigenvalues.
4. positive semidefinite if and only if the eigenvalues of A are all nonnegative.

Proof.

A proof sketch for the postive definite case.

Let's consider $\mathbf{u}_i$, an eigenvector of A. Then

$$\mathbf{u}_i^T A\mathbf{u}_i = \lambda_i \mathbf{u}_i^T \mathbf{u}_i.$$

If all eigenvalues are positive, then all such terms are positive.

Since A is symmetric, it is diagonalizable and so its eigenvectors span $\mathbb{R}^n$.

So any $\mathbf{x}$ can be expressed as a weighted sum of A's eigenvectors.

Writing out the expansion of $\mathbf{x}^T A\mathbf{x}$ in terms of A's eigenvectors, we get only positive terms.

Example. Let's look at the four quadratic forms above. Their associated matrices are

$$(a) \begin{bmatrix} 3 & 0 \\ 0 & 7 \end{bmatrix} \quad (b) \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix} \quad (c) \begin{bmatrix} 3 & 0 \\ 0 & -7 \end{bmatrix} \quad (d) \begin{bmatrix} -3 & 0 \\ 0 & -7 \end{bmatrix}$$

Example. Is $Q(\mathbf{x}) = 3x_1^2 + 2x_2^2 + x_3^2 + 4x_1x_2 + 4x_2x_3$ positive definite?

Solution. Because of all the plus signs, this form "looks" positive definite. But the matrix of the form is

$$\begin{bmatrix} 3 & 2 & 0 \\ 2 & 2 & 2 \\ 0 & 2 & 1 \end{bmatrix}$$

and the eigenvalues of this matrix turn out to be 5, 2, and -1. So Q is an indefinite quadratic form.

Question Time! Q24.2

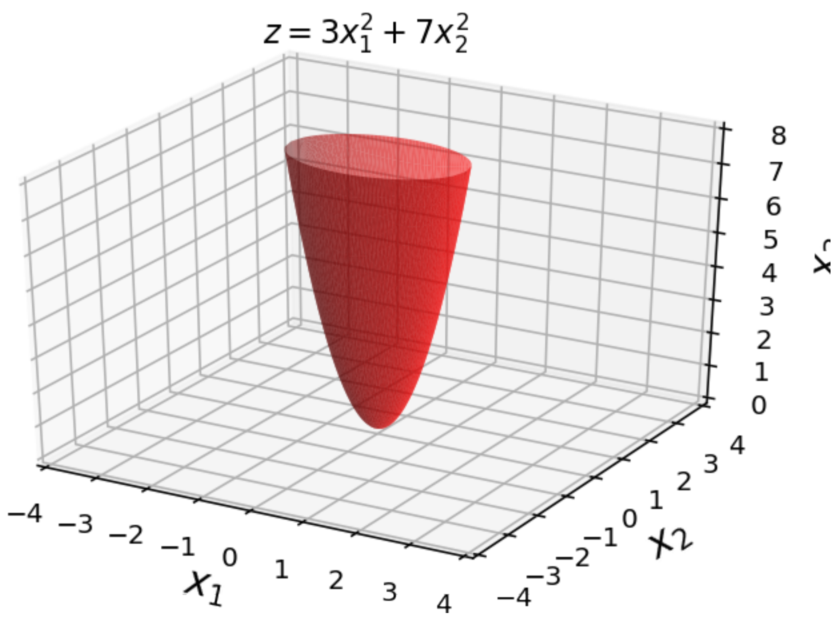
Constrained Optimization

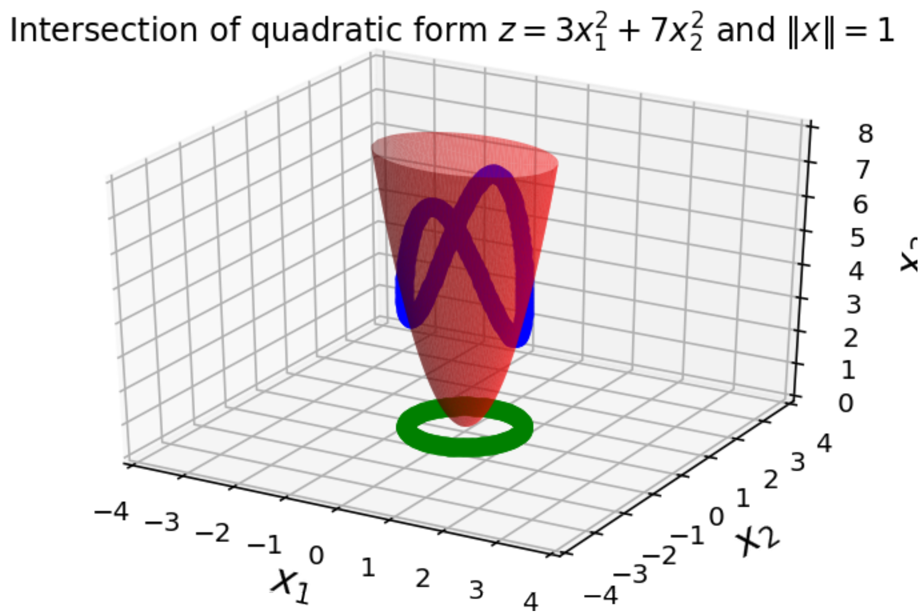
A common problem is to find the maximum or the minimum value of a quadratic form $Q(\mathbf{x})$ for $\mathbf{x}$ in some specified set. Typically the problem can be arranged so that $\mathbf{x}$ varies over the set of unit vectors.

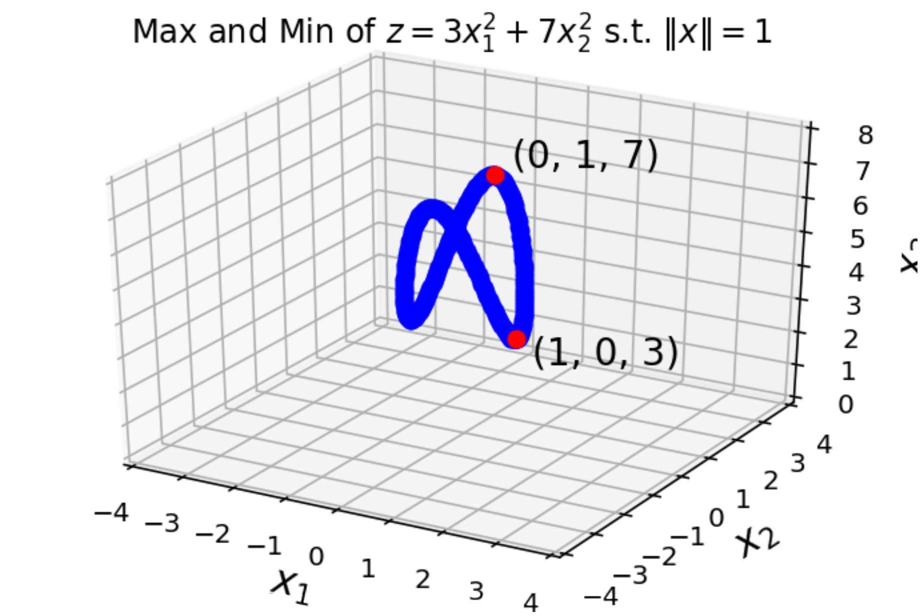
This is called **constrained optimization.** While it can be a difficult problem in general, for quadratic forms it has a particularly elegant solution.

The requirement that a vector $\mathbf{x}$ in $\mathbb{R}^n$ be a unit vector can be stated in several equivalent ways:

$$\|\mathbf{x}\| = 1, \quad \|\mathbf{x}\|^2 = 1, \quad \mathbf{x}^T\mathbf{x} = 1.$$







When a quadratic form has no cross-product terms, it is easy to find the maximum and minimum of $Q(\mathbf{x})$ for $\mathbf{x}^T\mathbf{x} = 1$.

Example. Find the maximum and minimum values of $Q(\mathbf{x}) = 9x_1^2 + 4x_2^2 + 3x_3^2$ subject to the constraint $\mathbf{x}^T\mathbf{x} = 1$.

Since x_2^2 and x_3^2 are nonnegative, we know that

$$4x_2^2 \le 9x_2^2 \quad \text{and} \quad 3x_3^2 \le 9x_3^2.$$

So

$$Q(\mathbf{x}) = 9x_1^2 + 4x_2^2 + 3x_3^2$$

$$\leq 9x_1^2 + 9x_2^2 + 9x_3^2$$

$$= 9(x_1^2 + x_2^2 + x_3^2)$$

$$= 9$$

Whenever $x_1^2 + x_2^2 + x_3^2 = 1$. So the maximum value of $Q(\mathbf{x})$ cannot exceed 9 when $\mathbf{x}$ is a unit vector.
Furthermore, $Q(\mathbf{x}) = 9$ when $\mathbf{x} = (1, 0, 0)$.
Thus 9 is the maximum value of $Q(\mathbf{x})$ for $\mathbf{x}^T\mathbf{x} = 1$.
A similar argument shows that the minimum value of $Q(\mathbf{x})$ when $\mathbf{x}^T\mathbf{x} = 1$ is 3.
Observation.
Note that the matrix of the quadratic form in the example is

$$A = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

So the eigenvalues of A are 9, 4, and 3.
We note that the greatest and least eigenvalues equal, respectively, the (constrained) maximum and minimum of $Q(\mathbf{x})$.
In fact, this is true for any quadratic form.
Theorem. Let A be a symmetric matrix, and let

$$M = \max_{\mathbf{x}^T\mathbf{x}=1} \mathbf{x}^T A\mathbf{x}.$$

Then M is the greatest eigenvalue λ_1 of A.
The value of $Q(\mathbf{x})$ is λ_1 when $\mathbf{x}$ is a unit eigenvector corresponding to M.
A similar theorem holds for the constrained minimum of $Q(\mathbf{x})$ and the least eigenvector λ_n.

Question Time! Q24.3