Linear Transformations

So far we've been treating the matrix equation

$$A\mathbf{x} = \mathbf{b}$$

as simply another way of writing the vector equation

$$x_1\mathbf{a_1} + \cdots + x_n\mathbf{a_n} = \mathbf{b}.$$

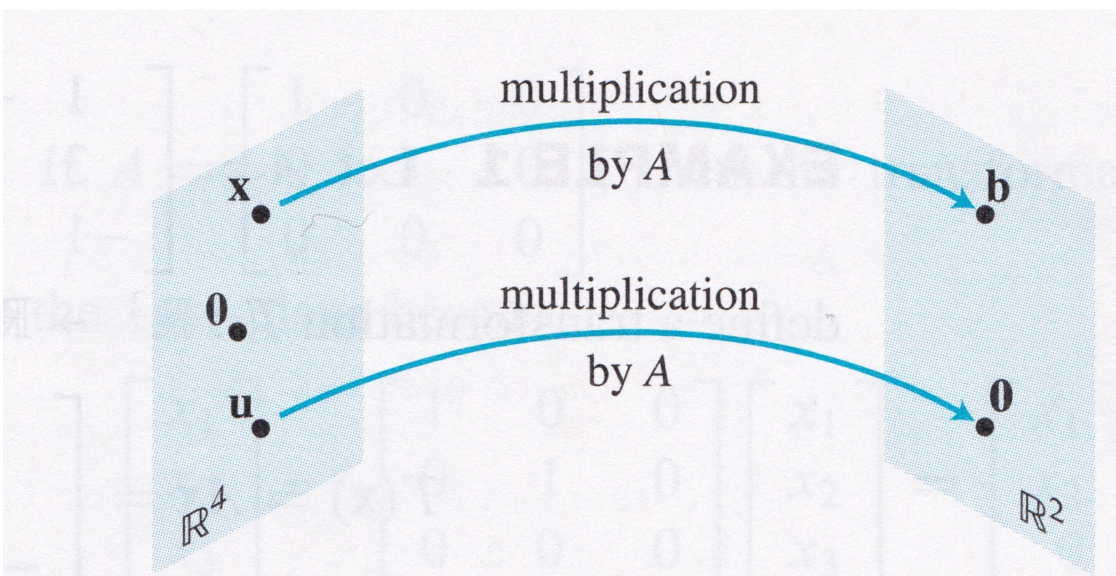
However, we'll now think of the matrix equation in a new way: we will think of A as "acting on" the vector $\mathbf{x}$ to form a new vector $\mathbf{b}$.

For example, let's let $A = \begin{bmatrix} 4 & -3 & 1 & 3 \\ 2 & 0 & 5 & 1 \end{bmatrix}$. Then we find:

$$A\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \end{bmatrix} \quad \text{and} \quad A\begin{bmatrix} 1 \\ 4 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

In other words, if $\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 5 \\ 8 \end{bmatrix}$, then A *transforms* $\mathbf{x}$ into $\mathbf{b}$.

Likewise, if $\mathbf{u} = \begin{bmatrix} 1 \\ 4 \\ -1 \\ 3 \end{bmatrix}$, then A transforms $\mathbf{u}$ into the $\mathbf{0}$ vector.



This gives a **new** way of thinking about solving $A\mathbf{x} = \mathbf{b}$. We are "searching" for the vectors $\mathbf{x}$ in $\mathbb{R}^4$ that are transformed into $\mathbf{b}$ in $\mathbb{R}^2$ under the "action" of A.

We have moved out of the familiar world of functions of one variable: we are now thinking about functions that transform a vector into a vector.

Or, put another way, functions that transform multiple variables into multiple variables.

Some terminology:

A **transformation** (or **function** or **mapping**) T from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a rule that assigns to each vector $\mathbf{x}$ in $\mathbb{R}^n$ a vector $T(\mathbf{x})$ in $\mathbb{R}^m$.

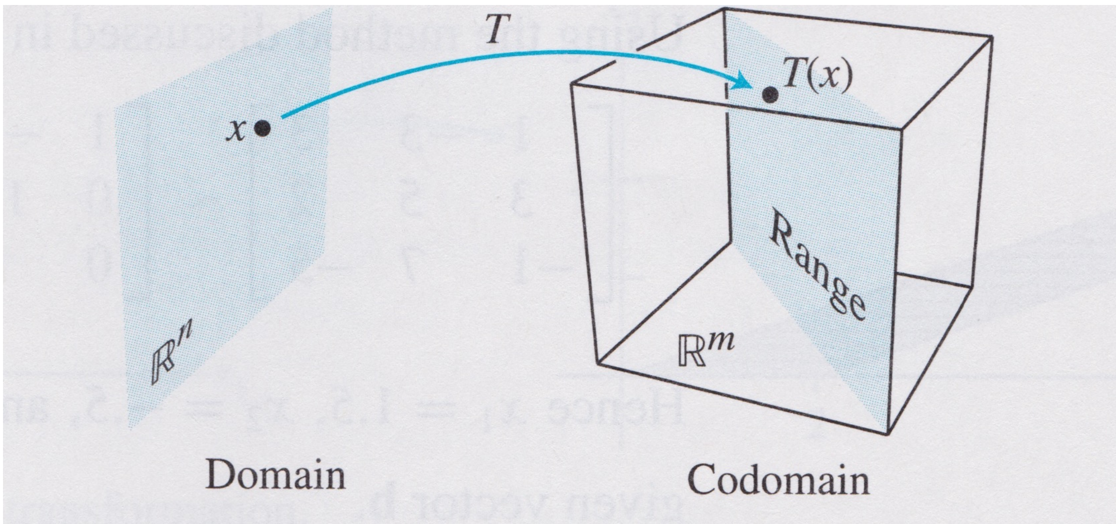
The set $\mathbb{R}^n$ is called the **domain** of T, and $\mathbb{R}^m$ is called the **codomain** of T.

The notation:

$$T : \mathbb{R}^n \to \mathbb{R}^m$$

indicates that the domain of T is $\mathbb{R}^n$ and the codomain is $\mathbb{R}^m$.

For $\mathbf{x}$ in $\mathbb{R}^n$, the vector $T(\mathbf{x})$ is called the **image** of $\mathbf{x}$ (under T). The set of all images $T(\mathbf{x})$ is called the **range** of T.



Question Time! Q7.1

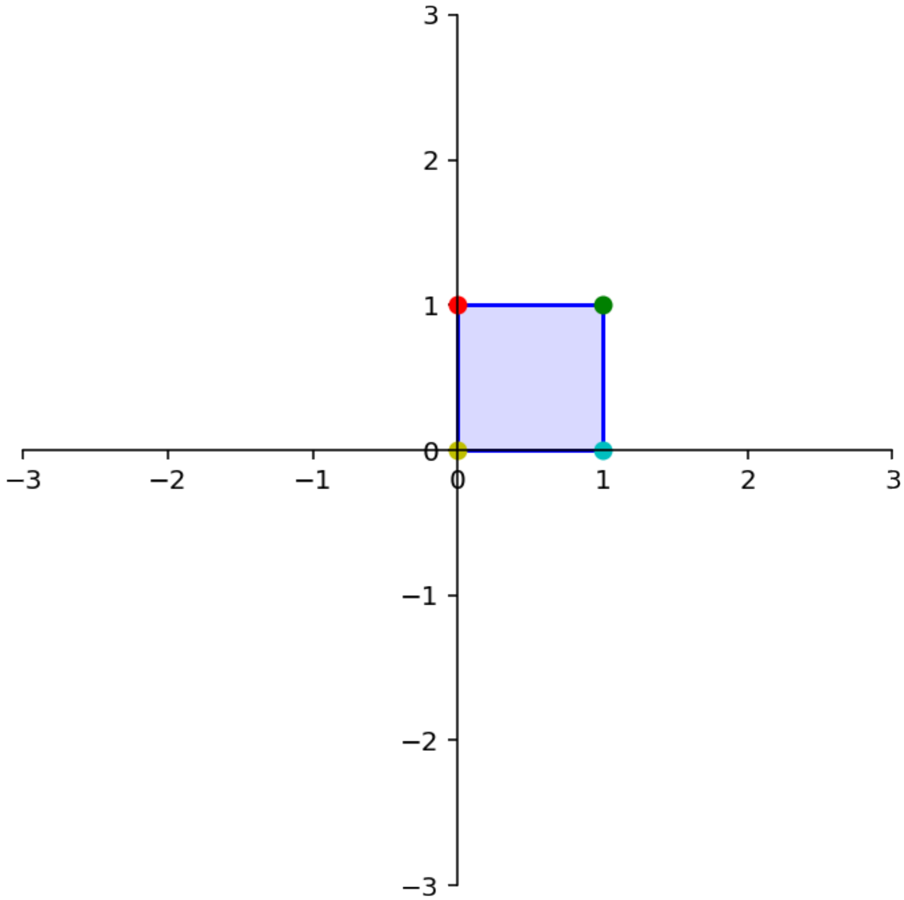
Let's do an example. Let's say I have these points in $\mathbb{R}^2$:

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Where are these points located?

```
[9]: square = np.array([[0.0,1,1,0],[1,1,0,0]])
     dm.plotSetup()
     print(square)
     dm.plotSquare(square)
```

```
[[ 0.  1.  1.  0.]
 [ 1.  1.  0.  0.]]
```



Now let's transform each of these points according to the following rule. Let

$$A = \begin{bmatrix} 1 & 1.5 \\ 0 & 1 \end{bmatrix}.$$

We define $T(\mathbf{x}) = A\mathbf{x}$. Then we have

$$T : \mathbb{R}^2 \to \mathbb{R}^2.$$

What is the image of each of these points under T?

$$A \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1.5 \\ 1 \end{bmatrix}$$

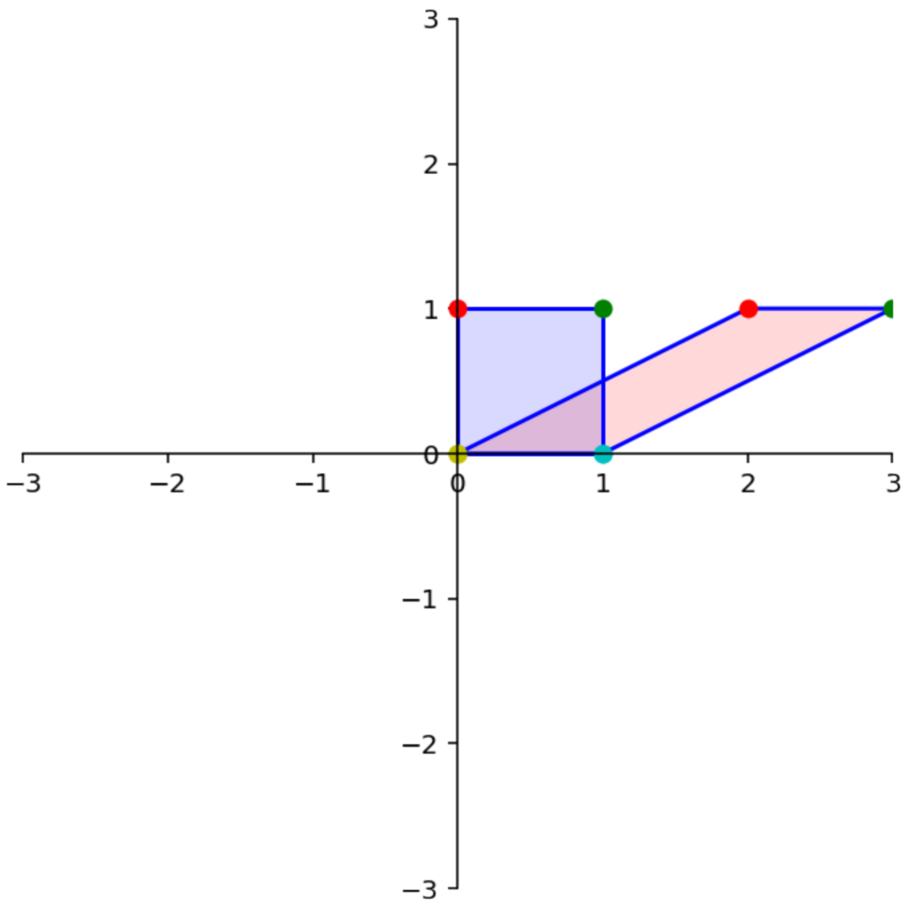
$$A \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2.5 \\ 1 \end{bmatrix}$$

$$A \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$A \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

[10]:
```
ax = dm.plotSetup()
print("square = "); print(square)
dm.plotSquare(square)
#
# create the A matrix
A = np.array([[1.0, 2],[0.0,1.0]])
print("A matrix = "); print(A)
#
# apply the shear matrix to the square
ssquare = np.zeros(np.shape(square))
for i in range(4):
    ssquare[:,i] = dm.AxVS(A,square[:,i])
print("sheared square = "); print(ssquare)
dm.plotSquare(ssquare,'r')
```

```
square =
[[ 0.  1.  1.  0.]
 [ 1.  1.  0.  0.]]
A matrix =
[[ 1.  2.]
 [ 0.  1.]]
sheared square =
[[ 2.  3.  1.  0.]
 [ 1.  1.  0.  0.]]
```



This sort of transformation, where points are successively slid sideways, is called a **shear** transformation.

Linear Transformations

By the properties of matrix-vector multiplication, we know that the transformation $\mathbf{x} \mapsto A\mathbf{x}$ has the properties that

$$A(\mathbf{u}+\mathbf{v}) = A\mathbf{u} + A\mathbf{v} \quad \text{and} \quad A(c\mathbf{u}) = cA\mathbf{u}$$

for all $\mathbf{u}, \mathbf{v}$ in $\mathbb{R}^n$ and all scalars c.

We are now ready to define one of the most fundamental concepts in the course: the concept of a *linear transformation*.

(You are now finding out why the subject is called linear algebra!)

Definition. A transformation T is **linear** if: 1. $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v}) \quad$ for all $\mathbf{u}, \mathbf{v}$ in the domain of T; and 2. $T(c\mathbf{u}) = cT(\mathbf{u}) \quad$ for all scalars c and all $\mathbf{u}$ in the domain of T.

Question Time! Q7.2

To fully grasp the significance of what a linear transformation is, don't think of just matrix-vector multiplication. Think of T as a function in more general terms.

The definition above captures a *lot* of functions that are not matrix-vector multiplication. For example, think of:

$$T(x) = \int_0^1 x(t)\, dt$$

Is T a linear function?

Properties of Linear Transformations

A key aspect of a linear transformation is that it **preserves the operations of vector addition and scalar multiplication.**

For example: for vectors $\mathbf{u}$ and $\mathbf{v}$, one can either: 1. Transform them both according to $T()$, then add them, or: 2. Add them, and then transform the result according to $T()$.

One gets the same result either way. The transformation does not affect the addition.

This leads to two important facts.

If T is a linear transformation, then

$$T(\mathbf{0}) = \mathbf{0}$$

and

$$T(c\mathbf{u} + d\mathbf{v}) = cT(\mathbf{u}) + dT(\mathbf{v})$$

In fact, if a transformation satisfies the second equation for all $\mathbf{u}, \mathbf{v}$ and c, d, then it must be a linear transformation. Both of the rules defining a linear transformation derive from this single equation.

Example.

Given a scalar r, define $T : \mathbb{R}^2 \to \mathbb{R}^2$ by $T(\mathbf{x}) = r\mathbf{x}$.

(T is called a **contraction** when $0 \leq r \leq 1$ and a **dilation** when $r > 1$.)

Let $r = 3$, and show that T is a linear transformation.

Solution.

Let $\mathbf{u}, \mathbf{v}$ be in $\mathbb{R}^2$ and let c, d be scalars. Then

$$
\begin{aligned}
T(c\mathbf{u} + d\mathbf{v}) &= 3(c\mathbf{u} + d\mathbf{v}) \\
&= 3c\mathbf{u} + 3d\mathbf{v} \\
&= c(3\mathbf{u}) + d(3\mathbf{v}) \\
&= cT(\mathbf{u}) + dT(\mathbf{v})
\end{aligned}
$$

Thus T is a linear transformation because it satisfies the rule $T(c\mathbf{u} + d\mathbf{v}) = cT(\mathbf{u}) + dT(\mathbf{v})$.