Linear Transformations

So far we've been treating the matrix equation

$$A\mathbf{x} = \mathbf{b}$$

as simply another way of writing the vector equation

$$x_1\mathbf{a_1} + \cdots + x_n\mathbf{a_n} = \mathbf{b}.$$

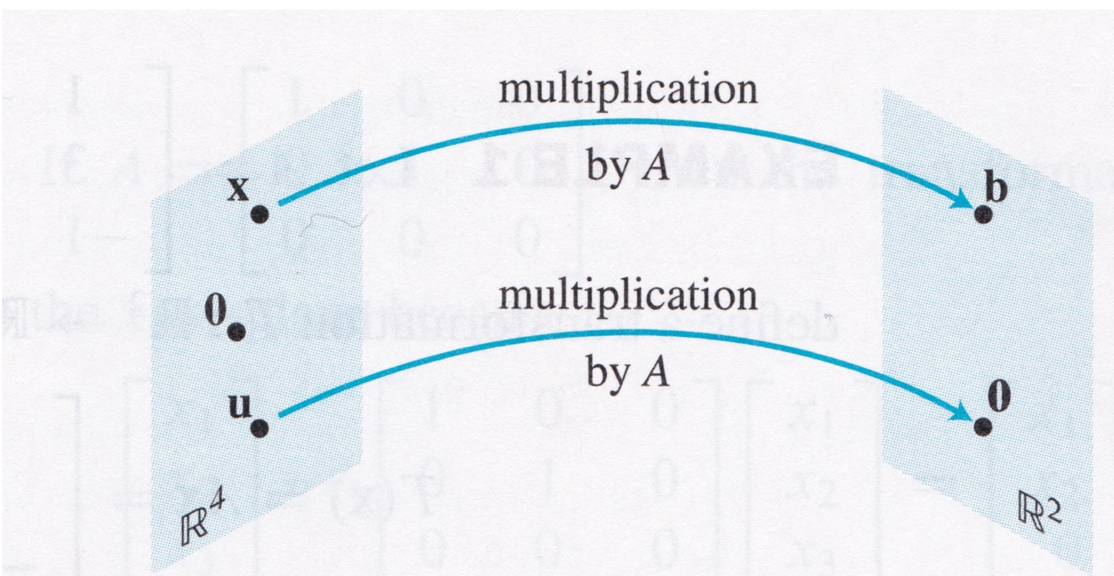
However, we'll now think of the matrix equation in a new way: we will think of A as "acting on" the vector $\mathbf{x}$ to form a new vector $\mathbf{b}$.

For example, let's let $A = \begin{bmatrix} 4 & -3 & 1 & 3 \\ 2 & 0 & 5 & 1 \end{bmatrix}$. Then we find:

$$A \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \end{bmatrix} \quad \text{and} \quad A \begin{bmatrix} 1 \\ 4 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

In other words, if $\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 5 \\ 8 \end{bmatrix}$, then A *transforms* $\mathbf{x}$ into $\mathbf{b}$.

Likewise, if $\mathbf{u} = \begin{bmatrix} 1 \\ 4 \\ -1 \\ 3 \end{bmatrix}$, then A transforms $\mathbf{u}$ into the $\mathbf{0}$ vector.



This gives a **new** way of thinking about solving $A\mathbf{x} = \mathbf{b}$. We are "searching" for the vectors $\mathbf{x}$ in $\mathbb{R}^4$ that are transformed into $\mathbf{b}$ in $\mathbb{R}^2$ under the "action" of A.

We have moved out of the familiar world of functions of one variable: we are now thinking about functions that transform a vector into a vector.

Or, put another way, functions that transform multiple variables into multiple variables.

Some terminology:

A **transformation** (or **function** or **mapping**) T from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a rule that assigns to each vector $\mathbf{x}$ in $\mathbb{R}^n$ a vector $T(\mathbf{x})$ in $\mathbb{R}^m$.

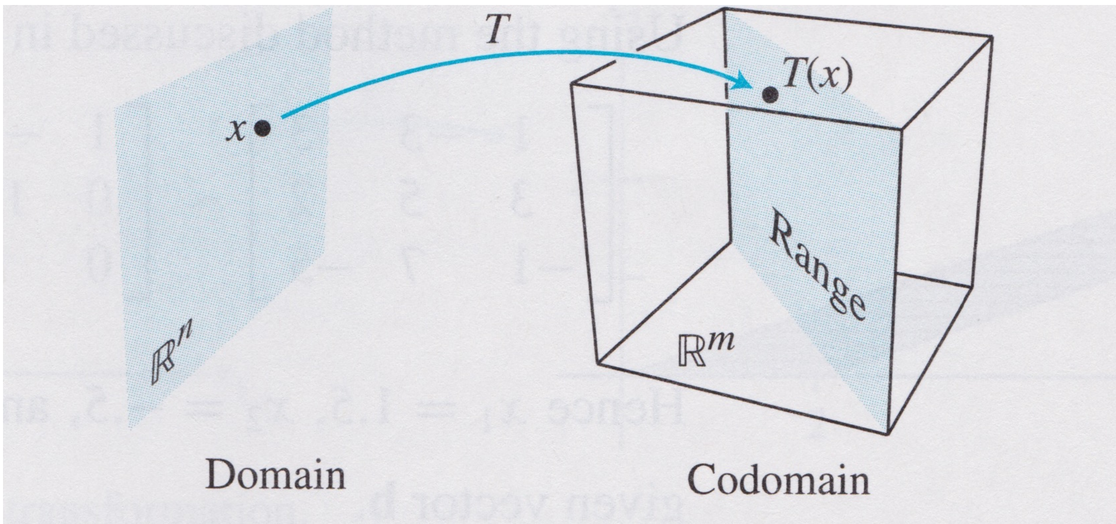
The set $\mathbb{R}^n$ is called the **domain** of T, and $\mathbb{R}^m$ is called the **codomain** of T.

The notation:

$$T : \mathbb{R}^n \rightarrow \mathbb{R}^m$$

indicates that the domain of T is $\mathbb{R}^n$ and the codomain is $\mathbb{R}^m$.

For $\mathbf{x}$ in $\mathbb{R}^n$, the vector $T(\mathbf{x})$ is called the **image** of $\mathbf{x}$ (under T). The set of all images $T(\mathbf{x})$ is called the **range** of T.



Question Time! Q7.1

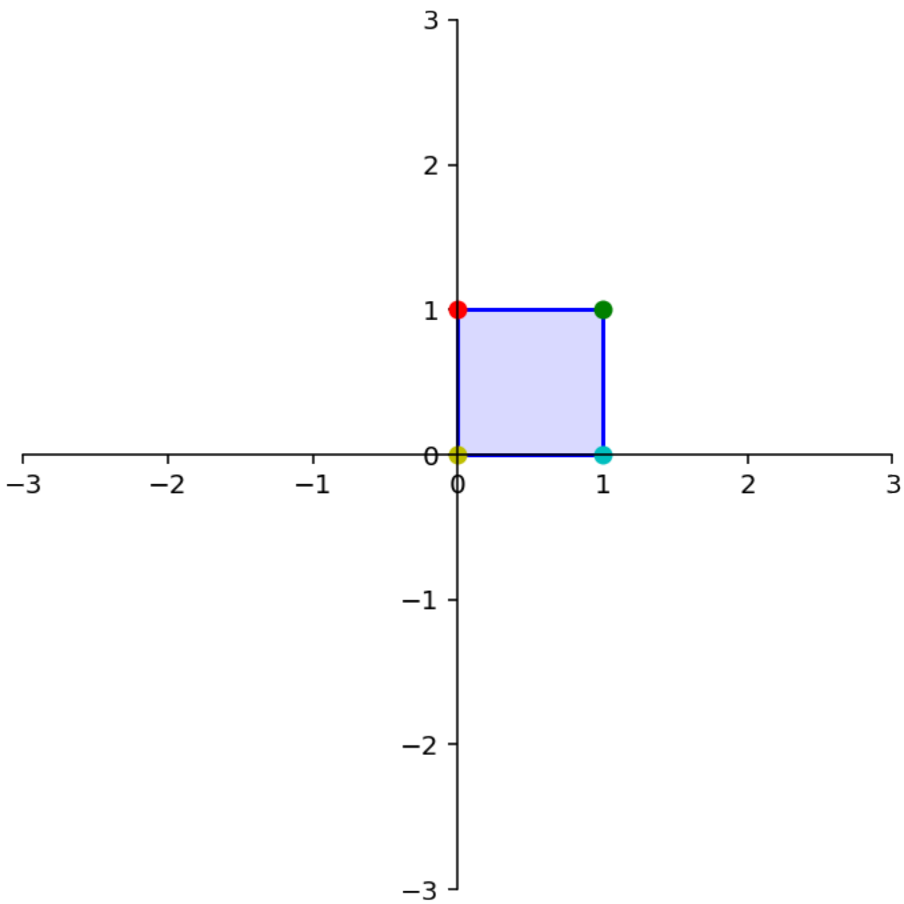
Let's do an example. Let's say I have these points in $\mathbb{R}^2$:

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Where are these points located?

```
[4]: square = np.array([[0.0,1,1,0],[1,1,0,0]])
     dm.plotSetup()
     print(square)
     dm.plotSquare(square)
```

```
[[0. 1. 1. 0.]
 [1. 1. 0. 0.]]
```



Now let's transform each of these points according to the following rule. Let

$$A = \begin{bmatrix} 1 & 1.5 \\ 0 & 1 \end{bmatrix}.$$

We define $T(\mathbf{x}) = A\mathbf{x}$. Then we have

$$T : \mathbb{R}^2 \to \mathbb{R}^2.$$

What is the image of each of these points under T?

$$A \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1.5 \\ 1 \end{bmatrix}$$

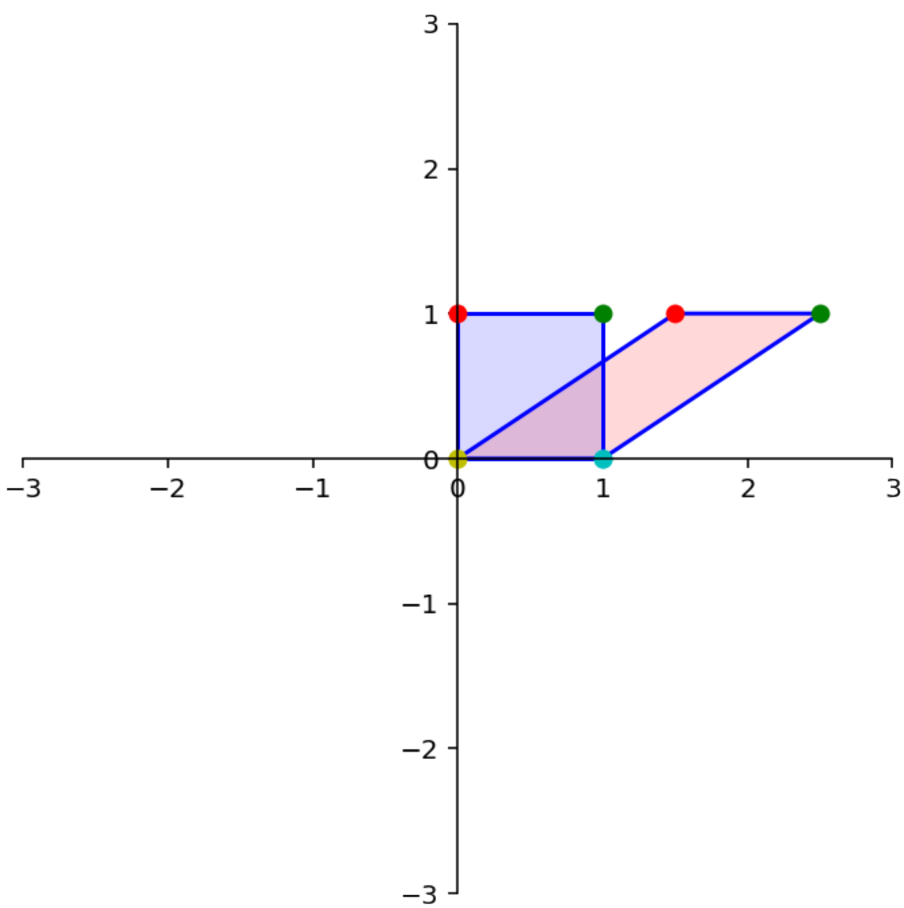
$$A \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2.5 \\ 1 \end{bmatrix}$$

$$A \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$A \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

```
[7]: ax = dm.plotSetup()
     print("square = "); print(square)
     dm.plotSquare(square)
     #
     # create the A matrix
     A = np.array([[1.0, 1.5],[0.0,1.0]])
     print("A matrix = "); print(A)
     #
     # apply the shear matrix to the square
     ssquare = np.zeros(np.shape(square))
     for i in range(4):
         ssquare[:,i] = dm.AxVS(A,square[:,i])
     print("sheared square = "); print(ssquare)
     dm.plotSquare(ssquare,'r')
```

```
square =
[[0. 1. 1. 0.]
 [1. 1. 0. 0.]]
A matrix =
[[1.  1.5]
 [0.  1. ]]
sheared square =
[[1.5 2.5 1.  0. ]
 [1.  1.  0.  0. ]]
```



This sort of transformation, where points are successively slid sideways, is called a **shear** transformation.

Linear Transformations

By the properties of matrix-vector multiplication, we know that the transformation $\mathbf{x} \mapsto A\mathbf{x}$ has the properties that

$$A(\mathbf{u}+\mathbf{v}) = A\mathbf{u} + A\mathbf{v} \quad \text{and} \quad A(c\mathbf{u}) = cA\mathbf{u}$$

for all $\mathbf{u}, \mathbf{v}$ in $\mathbb{R}^n$ and all scalars c.

We are now ready to define one of the most fundamental concepts in the course: the concept of a *linear transformation*.

(You are now finding out why the subject is called linear algebra!)

Definition. A transformation T is **linear** if: 1. $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v}) \quad$ for all $\mathbf{u}, \mathbf{v}$ in the domain of T; and 2. $T(c\mathbf{u}) = cT(\mathbf{u}) \quad$ for all scalars c and all $\mathbf{u}$ in the domain of T.

Question Time! Q7.2

To fully grasp the significance of what a linear transformation is, don't think of just matrix-vector multiplication. Think of T as a function in more general terms.

The definition above captures a *lot* of functions that are not matrix-vector multiplication. For example, think of:

$$T(x) = \int_0^1 x(t)\,dt$$

Is T a linear function?

Properties of Linear Transformations

A key aspect of a linear transformation is that it **preserves the operations of vector addition and scalar multiplication.**

For example: for vectors $\mathbf{u}$ and $\mathbf{v}$, one can either: 1. Transform them both according to $T()$, then add them, or: 2. Add them, and then transform the result according to $T()$.

One gets the same result either way. The transformation does not affect the addition.

This leads to two important facts.

If T is a linear transformation, then

$$T(\mathbf{0}) = \mathbf{0}$$

and

$$T(c\mathbf{u} + d\mathbf{v}) = cT(\mathbf{u}) + dT(\mathbf{v})$$

In fact, if a transformation satisfies the second equation for all $\mathbf{u}, \mathbf{v}$ and c, d, then it must be a linear transformation. Both of the rules defining a linear transformation derive from this single equation.

Example.

Given a scalar r, define $T : \mathbb{R}^2 \to \mathbb{R}^2$ by $T(\mathbf{x}) = r\mathbf{x}$.

(T is called a **contraction** when $0 \leq r \leq 1$ and a **dilation** when $r > 1$.)

Let $r = 3$, and show that T is a linear transformation.

Solution.

Let $\mathbf{u}, \mathbf{v}$ be in $\mathbb{R}^2$ and let c, d be scalars. Then

$$T(c\mathbf{u} + d\mathbf{v}) = 3(c\mathbf{u} + d\mathbf{v})$$

$$= 3c\mathbf{u} + 3d\mathbf{v}$$

$$= c(3\mathbf{u}) + d(3\mathbf{v})$$

$$= cT(\mathbf{u}) + dT(\mathbf{v})$$

Thus T is a linear transformation because it satisfies the rule $T(c\mathbf{u} + d\mathbf{v}) = cT(\mathbf{u}) + dT(\mathbf{v})$.

Example.

Let $T(\mathbf{x}) = \mathbf{x} + \mathbf{b}$ for some $\mathbf{b} \neq 0$.

What sort of operation does T implement?

Answer: **translation.**

Is T a linear transformation?

Solution.

We only need to compare

$$T(\mathbf{u} + \mathbf{v})$$

to

$$T(\mathbf{u}) + T(\mathbf{v}).$$

$$T(\mathbf{u} + \mathbf{v}) = \mathbf{u} + \mathbf{v} + \mathbf{b}$$

and

$$T(\mathbf{u}) + T(\mathbf{v}) = (\mathbf{u} + \mathbf{b}) + (\mathbf{v} + \mathbf{b})$$

If $\mathbf{b} \neq 0$, then the above two expressions are not equal.

So T is **not** a linear transformation.

A non-geometric example.

A company manufactures two products, B and C. To do so, it requires materials, labor, and overhead.

For \1.00*worthofproductB*, *itspends*
.45 on materials, \.25*onlabor*, *and*
.15 on overhead.

For \1.00*worthofproductC*, *itspends*
.40 on materials, \.30*onlabor*, *and*
.15 on overhead.

Let us construct a "unit cost" matrix:

$$U = \begin{array}{c r} \begin{array}{cc} \text{B} & \text{C} \end{array} & \\ \begin{bmatrix} .45 & .40 \\ .25 & .30 \\ .15 & .15 \end{bmatrix} & \begin{array}{r} \text{Materials} \\ \text{Labor} \\ \text{Overhead} \end{array} \end{array}$$

Let $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ be a production vector, corresponding to x_1 dollars of product B and x_2 dollars of product C.

Then define $T : \mathbb{R}^2 \to \mathbb{R}^3$ by

$$T(\mathbf{x}) = U\mathbf{x} = x_1 \begin{bmatrix} .45 \\ .25 \\ .15 \end{bmatrix} + x_2 \begin{bmatrix} .40 \\ .30 \\ .15 \end{bmatrix} = \begin{bmatrix} \text{Total cost of materials} \\ \text{Total cost of labor} \\ \text{Total cost of overhead} \end{bmatrix}$$

The mapping T transforms a list of production quantities into a list of total costs.
The linearity of this mapping is reflected in two ways:

1. If production is increased by a factor of, say, 4, ie, from $\mathbf{x}$ to $4\mathbf{x}$, then the costs increase by the same factor, from $T(\mathbf{x})$ to $4T(\mathbf{x})$.
2. If $\mathbf{x}$ and $\mathbf{y}$ are production vectors, then the total cost vector associated with combined production of $\mathbf{x} + \mathbf{y}$ is precisely the sum of the cost vectors $T(\mathbf{x})$ and $T(\mathbf{y})$.