Dimension and Rank

<IPython.core.display.HTML object>

Rene Descartes (1596-1650) was a French philosopher, mathematician, and writer. He is often credited with developing the idea of a coordinate system, although versions of coordinate systems had been seen in Greek mathematics since 300BC.

As a young man, Descartes had health problems and generally stayed in bed late each day. The story goes that one day as he lay in bed, he observed a fly on the ceiling of his room. He thought about how to describe the movement of the fly, and realized that he could completely describe it by measuring its distance from the walls of the room. This gave birth to the so-called *Cartesian coordinate system*.

What is certain is that Descartes championed the idea that geometric problems could be cast into algebraic form and solved in that fashion. This was an important shift in thinking; the mathematical tradition begun by the Greeks held that geometry, as practiced by Euclid with compass and straightedge, was a more fundamental approach. For example, the problem of copying a triangle was one that the Greeks had studied and solved using non-numeric methods. Descartes argued that a triangle could be copied exactly by simply knowing the coordinates of its vertices.

The study of curves and shapes in algebraic form laid important groundwork for calculus, and Newton was strongly influenced by Descartes' ideas.

Why is a coordinate system so useful?

The value of a coordinate system is that it gives a *unique name* to each point in the plane (or in any vector space).

Now, here is a key question: what if the walls of Descartes' room had not been square (perpendicular)? Would his system still have worked?

In other words, would he still have been able to precisely specify the path of the fly?

We'll explore this question today and use it to further deepen our understanding of linear operators.

Coordinate Systems

In the last lecture we developed the idea of a *basis* – a minimal spanning set for a subspace H.

Today we'll emphasize this aspect: a key value of a basis is that it provides a *coordinate system* for H.

In other words: if we are given a basis for H, then **each vector in H can be written in only one way** as a linear combination of the basis vectors.

Let's see this convincingly.

Suppose $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_p\}$ is a basis for H, and suppose a vector $\mathbf{x}$ in H can be generated in two ways, say

$$\mathbf{x} = c_1\mathbf{b}_1 + \cdots + c_p\mathbf{b}_p \quad \text{and} \quad \mathbf{x} = d_1\mathbf{b}_1 + \cdots + d_p\mathbf{b}_p.$$

Then, subtracting gives

$$\mathbf{0} = \mathbf{x} - \mathbf{x} = (c_1 - d_1)\mathbf{b}_1 + \cdots + (c_p - d_p)\mathbf{b}_p.$$

Since $\mathcal{B}$ is a basis, we know that it is linearly independent.

So by the definition of linear independence, the weights in the above expression must all be zero.
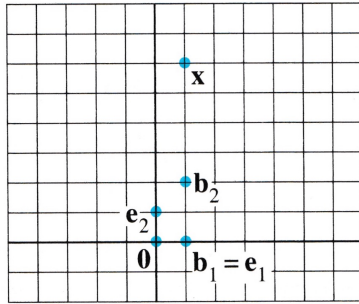
That is, $c_j = d_j$ for all j – which shows that the two representations must be the same.

Definition. Suppose the set $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_p\}$ is a basis for the subspace H. For each $\mathbf{x}$ in H, the **coordinates of x relative to the basis** $\mathcal{B}$ are the weights $c_1, \ldots, c_p$ such that $\mathbf{x} = c_1\mathbf{b}_1 + \cdots + c_p\mathbf{b}_p$.
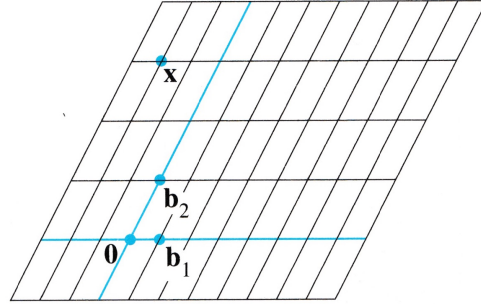
The vector in $\mathbb{R}^p$

$$[\mathbf{x}]_\mathcal{B} = \begin{bmatrix} c_1 \\ \vdots \\ c_p \end{bmatrix}$$

is called the **coordinate vector of x (relative to $\mathcal{B}$)** or the $\mathcal{B}$**-coordinate vector of x.**

FIGURE 1 Standard graph

FIGURE 2 $\mathcal{B}$-graph paper.

```
[12]: # idea: put a simple button here that switches between the two plots
      xmin = -6.0
      xmax = 6.0
      ymin = -2.0
      ymax = 8.0
      fig = ut.two_d_figure('Figure 15.1', xmin = xmin, xmax = xmax, ymin = ymin, ymax = ymax, size=(6,5
      for x in np.linspace(ymin, ymax, int(ymax-ymin+1)):
          fig.plotLinEqn(0., 1., x, alpha=0.3)
      for y in np.linspace(xmin, xmax, int(xmax-xmin+1)):
          fig.plotLinEqn(1., 0., y, alpha=0.3)
      fig.centerAxes()
```

```
<IPython.core.display.Javascript object>
```

```
<IPython.core.display.HTML object>
```

Example. Let $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} 3 \\ 12 \\ 7 \end{bmatrix}$, and $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2\}$.

Then $\mathcal{B}$ is a basis for $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$ because $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent. Determine if $\mathbf{x}$ is in H, and if it is, find the coordinate vector of $\mathbf{x}$ relative to $\mathcal{B}$.

Solution. If $\mathbf{x}$ is in H, then the following vector equation is consistent:

$$c_1 \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 12 \\ 7 \end{bmatrix}.$$

The scalars c_1 and c_2, if they exist, are the $\mathcal{B}$-coordinates of $\mathbf{x}$.
Row operations show that

$$\begin{bmatrix} 3 & -1 & 3 \\ 6 & 0 & 12 \\ 2 & 1 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus $c_1 = 2, c_2 = 3$, and $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$. The basis $\mathcal{B}$ determines a "coordinate system" on H, which can be visualized like this:

3

```
<IPython.core.display.Javascript object>
```

```
<IPython.core.display.HTML object>
```

Important: Don't be confused by the fact that the "coordinate axes" are not purpendicular.

The whole idea here is that **they don't need to be.**

As long as the independent vectors span the space, there is only **one** way to express any point in terms of them. Thus, every point has a unique coordinate.

Isomorphism.

Another important idea is that, although points in H are in $\mathbb{R}^3$, they are completely determined by their coordinate vectors, which belong to $\mathbb{R}^2$.

The grid in the figure above makes H "look like" $\mathbb{R}^2$.

The correspondence $\mathbf{x} \mapsto [\mathbf{x}]_{\mathcal{B}}$ is one-to-one correspondence between H and $\mathbb{R}^2$ that preserves linear combinations.

In our example, $\begin{bmatrix} 3 \\ 12 \\ 7 \end{bmatrix} \mapsto \begin{bmatrix} 2 \\ 3 \end{bmatrix}_{\mathcal{B}}$.

When we have a one-to-one correspondence between two subspaces that preserves linear combinations, we call such a correspondence an *isomorphism,* and we say that H is *isomorphic* to $\mathbb{R}^2$.

In general, if $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_p\}$ is a basis for H, then the mapping $\mathbf{x} \mapsto [\mathbf{x}]_{\mathcal{B}}$ is a one-to-one correspondence that makes H look and act the same as $\mathbb{R}^p$. This is *even through the vectors in H themselves may have more than p entries.*

The Dimension of a Subspace

It can be shown that if a subspace H has a basis of p vectors, then every basis of H must consist of exactly p vectors. Thus we can make this definition:

Definition. The *dimension* of a nonzero subspace H, denoted by dim H, is the number of vectors in any basis for H.

The dimension of the zero subspace $\{\mathbf{0}\}$ is defined to be zero.

So now we can say **with precision** things we've previous said informally. For example, a plane through $\mathbf{0}$ is two-dimensional, and a line through $\mathbf{0}$ is one-dimensional.

Question: What is the dimension of a line not through the origin?

Answer: It is undefined, because a line not through the the origin is not a subspace.

Example. We previously determined that the null space of this matrix:

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$$

had a basis of 3 vectors. So the dimension of Nul A, (or dim Nul A) in this case is 3.

Remember that we constructed an explicit description of the nullspace of this matrix, as:

$$= x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

Each basis vector corresponds to a free variable in the equation $A\mathbf{x} = \mathbf{0}$. So, to find the dimension of Nul A, simply identify and count the number of free variables in $A\mathbf{x} = \mathbf{0}$.

4

Question Time! Q15.1

Definition. The **rank** of a matrix, denoted by $\mathbb{R}ank\ A$, is the dimension of the column space of A.

Since the pivot columns of A form a basis for Col A, the rank of A is just the number of pivot columns in A.

Example. Determine the rank of the matrix

$$A = \begin{bmatrix} 2 & 5 & -3 & -4 & 8 \\ 4 & 7 & -4 & -3 & 9 \\ 6 & 9 & -5 & 2 & 4 \\ 0 & -9 & 6 & 5 & -6 \end{bmatrix}.$$

Solution Reduce A to an echelon form:

$$A = \begin{bmatrix} 2 & 5 & -3 & -4 & 8 \\ 0 & -3 & 2 & 5 & -7 \\ 0 & -6 & 4 & 14 & -20 \\ 0 & -9 & 6 & 5 & -6 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} 2 & 5 & -3 & -4 & 8 \\ 0 & -3 & 2 & 5 & -7 \\ 0 & 0 & 0 & 4 & -6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The matrix A has 3 pivot columns, so Rank $A = 3$.

Question Time! Q15.2

The Rank Theorem.

Consider a matrix A.

In the last lecture we showed the following: one can construct a basis for Nul A using the columns corresponding to free variables in the solution of $A\mathbf{x} = \mathbf{0}$.

This shows that dim Nul A = the number of free variables in $A\mathbf{x} = \mathbf{0}$, which is the number of non-pivot columns in A.

We also saw that the number of columns in any basis for Col A is the number of pivot columns.

So we can now make this important connection:

$$\begin{aligned} \dim \text{Nul } A + \dim \text{Col } A \ &= \ \text{number of non-pivot columns of } A + \text{number of pivot columns of } A \\ &= \ \text{number of columns of } A. \end{aligned}$$

This leads to the following theorem:

If a matrix A has n columns, then Rank $A + \dim$ Nul $A = n$.

This is a terrifically important fact!

Here is an intuitive way to understand it. Let's think about a matrix A and the associated linear transformation $T(x) = Ax$.

If the matrix A has n columns, then A's columnspace *could* have dimension as high as n.

In other words, T's range *could* have dimension as high as n.

However, if A "throws away" a nullspace of dimension p, then that *reduces* the columnspace of A to $n - p$.

Meaning, the dimension of T's range is reduced to $n - p$.

Question Time! Q15.3

Extending the Invertible Matrix Theorem.

The above arguments show that when A has n columns, then the "larger" that the column space is, the "smaller" that the null space is.

(Where "larger" means "has more dimensions.")

This is particularly important when A is square $(n \times n)$.

Let's consider the extreme, in which the column space of A has maximum dimension – i.e., dim Col $A = n$.

Recall that the IMT said that an $n \times n$ matrix is invertible if and only if its columns are linearly independent, and if and only if its columns span $\mathbb{R}^n$.

Hence we now can see that an $n \times n$ matrix is invertible if and only if the columns of A form a basis for $\mathbb{R}^n$.

This leads to the following facts, which further extend the IMT:

Let A be an $n \times n$ matrix. Then the following statements are each equivalent to the statement that A is an invertible matrix:

1. The columns of A form a basis for $\mathbb{R}^n$.
2. $\text{Col } A = \mathbb{R}^n$.
3. $\dim \text{Col } A = n$.
4. $\text{Rank } A = n$.
5. $\text{Nul } A = \{\mathbf{0}\}$.
6. $\dim \text{Nul } A = 0$.