The class WT

Sandro Claudio Lera*[1]

[1]Singapore-ETH Centre, ETH Zurich, 1 CREATE Way, #06-01 CREATE Tower, 138602 Singapore

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We document the wavelet transform class `WT` implemented in Python. You may use this code for your own applications. Note that this code is written for personal use and may contain bugs or inefficiencies (please report to slera@ethz.ch). Acknowledgment is appreciated.

1 The wavelet transform

A ψ-wavelet transform W_ψ in continuous time and frequency is simply a projection of a signal $X(t)$ onto b-translated and a-dilated versions of ψ [1]:

$$W_\psi(a,b) = \int_{-\infty}^{\infty} \mathrm{d}t\ \psi(t-b;a)X(t). \tag{1}$$

Here, ψ is the analyzing function, called the wavelet, that has to be a localized function both in time and frequency domain, among with some other properties, see [2] for details. Here, we are careful to not write any pre-factor in front of the transform. The user of the class `WT` is responsible to incorporate proper overall normalization into the wavelet ψ (if one of the predefined wavelets `Gaussian,Gaussian_1st_derivative` or `Ricker` are used, this is already taken care of).

Because the analyzing wavelet is localized in time, the wavelet coefficient $W_\psi(a,b)$ has the following, intuitive interpretation: it approximates a (weighted) average of the signal $X(t)$, averaged over the interval $[b-a, b+a]$. This interpretation is important to understand the influence of numerical finite size effects.

Replacing ψ in (1) by its n-th derivative $\psi^{(n)}$ corresponds to a ψ-analysis of the n-th derivative of the time series $X(t)$ (up to a normalisation factor!), as a simple integration by parts argument shows. For fixed n, the overall statistical characterization of complex structures depends only weakly on the choice of the mother wavelet [3].

Typical wavelets are, for example, the Gaussian

$$\psi(t;a) = \frac{1}{2\sqrt{\pi}a} \exp\left(-\frac{1}{2}\left(\frac{t}{a}\right)^2\right), \tag{2}$$

or the Ricker ('Mexican-hat') wavelet

$$\psi(t;a) = \frac{1}{\sqrt{2\pi a^3}} \left(1-\left(\frac{x}{a}\right)^2\right) \exp\left(-\frac{1}{2}\left(\frac{x}{a}\right)^2\right). \tag{3}$$

*slera@ethz.ch

2 Implementation of the wavelet transform in Python

In this section I describe the implementation of the class `WT`. Conceptually, the implementation is simple. We only have to calculate (1) numerically and depict the result, for instance as a color map, in the $(\log_2(a), b)$ plane. Additionally, we can extract the skeleton defined as the structure, determined by local extrema (or just maxima/minima) if requested.

Numerically, there are some additional details to consider. Here is a brief overview of what the class does:

- An object `O` of class `WT` takes the following input arguments:
  - A signal X, i.e. the data in form of a one-dimensional array.
  - The points at which X is sampled (usually "the time", "the position", or just "arguments", a 1d-array of same size as "signal").
  - A minimum and maximum scale, $s_{\min} and s_{\max}$, respectively.
  - A positive integer N, denoting the number of scales considered. The interval $[s_{\min}, s_{\max}]$ is then partitioned into N scales $s_{\min} = s_1, s_2, \ldots, s_N = s_{\max}$. The s_i are equally spaced on a $\log_2$-scale (not on a linear scale!).
  - The analyzing wavelet. This can be either 'Gaussian', 'Gaussian_first_derivative' or 'Ricker', which are already pre-defined. Optionally, any other wavelet $\psi(t; a)$ may be passed. The wavelet takes two arguments: the position x and the 'width parameter' a.
  - Finally, one can also pass a 'finite size cutoff' (> 0). To understand the usefulness of this parameter, consider a signal consisting of n data points $X_1, \ldots, X_n$, sampled at n different times, $t_1, \ldots, t_n$. We want to calculate the wavelet coefficient at time b and at scale a. Recall that $W_\psi(a, b)$ has the interpretation of averaging X over the interval $\sim [b - a, b + a]$. In order to avoid finite size effects, it is recommended to ignore wavelet coefficients at a time b with a (temporal) distanceless than some multiple of a from the 'boundaries' t_1 and t_n. This multiple can be specified with the 'finite size cutoff'-parameter. Its default is set to 2. Setting it to zero one can expect significant finite size effects close to the boundaries, as long as the signal is not vanishing at t_1 and t_n.
- Calling `O.heatmap()` returns a heatmap of the wavelet transform in the $(\log_2(a), b)$ plane. Several additional parameters may be passed to the method. See documentation inside the code for details.
- `O.skeleton()` returns just the skeleton of the heatmap, i.e. the structure of local extrema (or only the maxima/minima, if requested) as a function of b. Again, several parameters may be passed, see documentation inside the code for details.
- `O.get_WT(a,b)` returns the wavelet transform at scale a and position b.
- `O.get_WT_matrix()` returns the entire wavelet transform at all scales and positions in form of an $N \times \mathrm{size}(X)$-matrix.
- `O.test()` reproduces figure 1(a) in [4], see also section 4.
- The integration (1) uses straight forward Simpson's integration method. Performance could be improved using the built-in Python convolution operator. This is, however, not necessary as the code is executed reasonably fast.

3 Proper normalization of the wavelet transform

In order to interpret (1) as an average over the interval $[b-a, b+a]$, the wavelet must be properly normalized. A typical choice of the mother wavelet is the Gaussian $\psi(t) = \mathcal{N} \exp\left(-t^2/2\right)$. By $\mathcal{N}$ we denote the normalization of the wavelet (and hence of the entire wavelet transform!). If one works with the Gaussian mother wavelet, it is straight forward to see that normalization is given by $\mathcal{N} = \left(\sqrt{2\pi}a\right)^{-1}$.

Assume we are interested in examining the structure of slopes of the signal $X(t)$ at different scales a and times b. According to the integration by part argument from above, we must then analyze the signal $X(t)$ with the first derivative $\psi^{(1)}(t)$ of the mother wavelet $\psi(t)$. Here, we derive the normalization $\mathcal{N}$ of the first derivative of the Gaussian mother wavelet $\psi(t) = \mathcal{N} t \exp\left(-t^2/2\right)$. To this end, we consider the signal $X(t) = pt$, which has a constant slope p at all scales. Due to the above interpretation of the wavelet transform of $X(t)$ with $\psi^{(1)}$, we expect $W_{\psi^{(1)}}(a,b) = p \equiv \mathit{const}$ at all positions and scales. This condition allows us to fix the normalization:

$$\begin{aligned} p &\stackrel{!}{=} \int_{-\infty}^{\infty} \mathrm{d}t\ X(t)\ \psi^{(1)}\left(\frac{t-b}{a}\right) \\ &= \frac{1}{a^2}\mathcal{N} \int_{-\infty}^{\infty} \mathrm{d}t\ X(t)\ (t-b)\ \exp\left(-\frac{1}{2}\left(\frac{t-b}{a}\right)^2\right) \\ &= p\frac{1}{a^2}\mathcal{N} \int_{-\infty}^{\infty} \mathrm{d}t\ t\ (t-b)\ \exp\left(-\frac{1}{2}\left(\frac{t-b}{a}\right)^2\right) \\ &= p\sqrt{2\pi}a\mathcal{N} \\ &\Downarrow \\ \mathcal{N} &= \frac{1}{\sqrt{2\pi}a}. \end{aligned} \tag{4}$$

Thus, the first derivative of the Gaussian mother wavelet is properly normalized as

$$\psi^{(1)}(t;a) \equiv \frac{1}{\sqrt{2\pi}a^3} t \exp\left(-\frac{1}{2}\left(\frac{t}{a}\right)^2\right). \tag{5}$$

The wavelet transform (1), with proper normalization, is then given by

$$W_{\psi^{(1)}}(a,b) = \int_{-\infty}^{\infty} \mathrm{d}t\ \psi^{(1)}(t-b;a) X(t). \tag{6}$$

Lets apply a numerical test to check that everything works fine. We pass the signal $X(t) = 2t$ to the class `WT`. We expect to have a wavelet coefficient equal to two on all positions and scales. Figure 1 confirms that theory and implementation are consistent.

4 Testing the class

Let us reproduce a plot from [4]. The original plot is shown in figure 2, depicting the "devil's staircase function", defined as $s(x) = \mu([0,x])$, the uniform measure lying on the triadic Cantor set. This function can be implemented recursively.[1] Analyzing $s(x)$ with wavelet

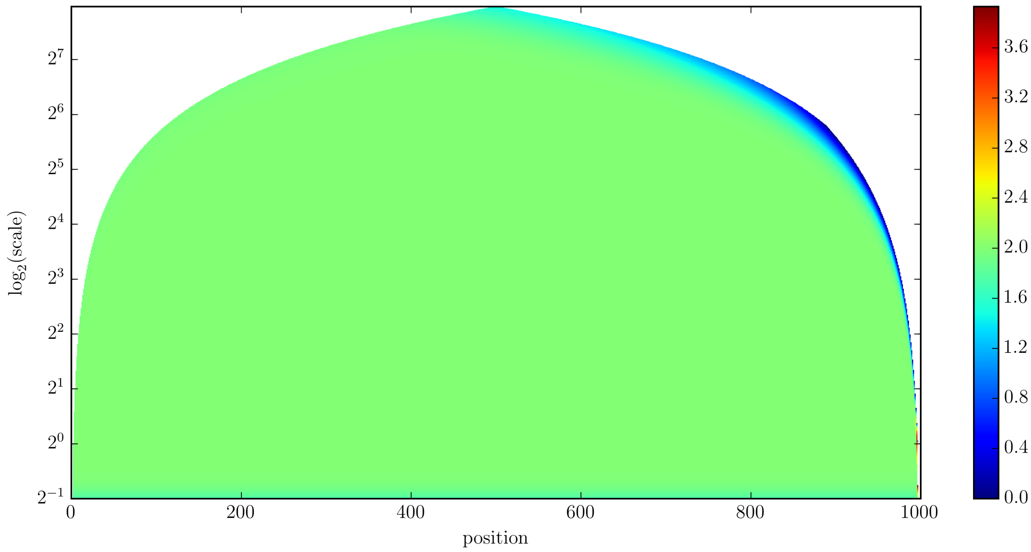

Figure 1: We show the wavelet transform (6) of the discrete signal $X(t) = 2t$ for $t = 1, 2, \ldots, 1000$ with analyzing wavelet (5). As predicted, $W_{\psi^{(1)}}(a, b) = 2 \ \forall a, b$ (apart from finite size effects on the right edge).

(5) we expect to resolve steeper and steeper slopes at small scales. In figure 3 we show the heatmap produced by the class `WT`, and figure 4 depicts the corresponding skeleton structure. The result is as expected. Call `O.test()` to reproduce these figures.

References

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[2] I. Daubechies. *Ten lectures on wavelets.* SIAM, 1992.

[3] A. Arneodo et al. "Beyond classical multifractal analysis using wavelets: Uncovering a multiplicative process hidden in the geometrical complexity of diffusion limited aggregates". In: *Fractals* 01.03 (1993), pp. 629–649.

[4] A. Arneodo, E. Bacry, and J. Muzy. "The thermodynamics of fractals revisited with wavelets". In: *Physica A: Statistical Mechanics and its Applications* 213.1-2 (1995), pp. 232–275.

[1] See for instance https://en.wikipedia.org/wiki/Cantor_function

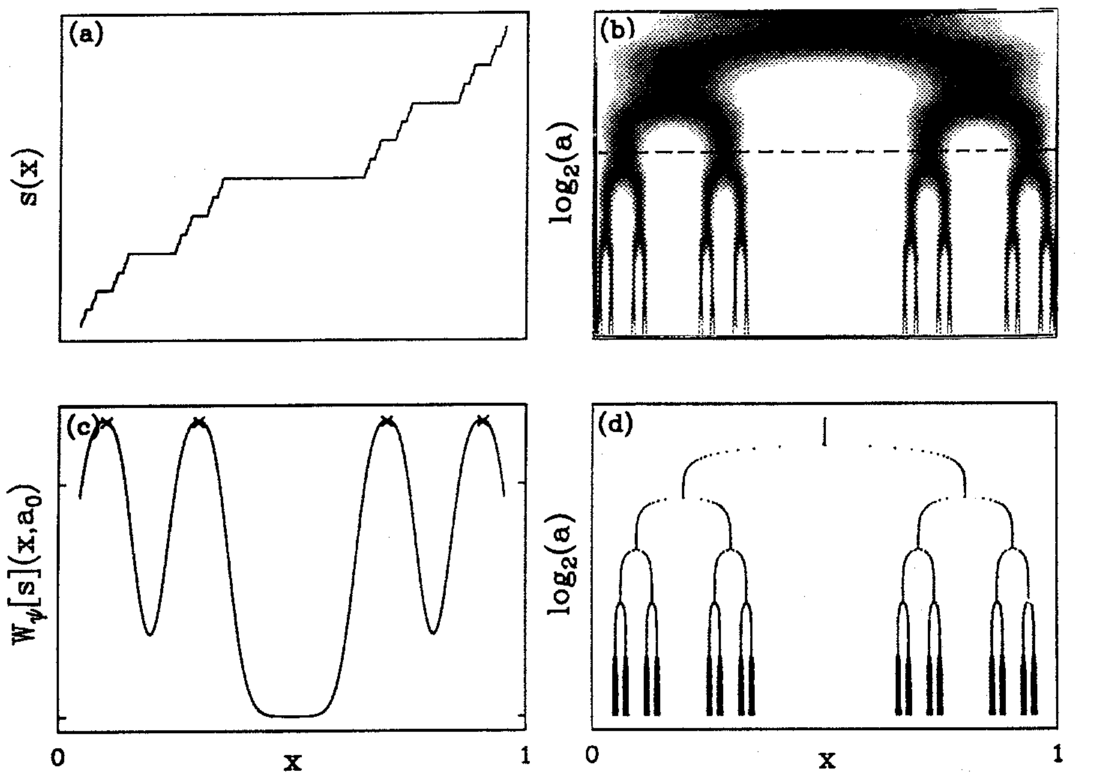

Fig. 1. Continuous wavelet transform of the devil's staircase corresponding to the uniform triadic Cantor set. (a) Graph of the function. (b) Wavelet transform computed with the analyzing wavelet $\psi^{(1)}$; the amplitude is coded, independently at each scale a, using 32 grey levels from white ($W_\psi[s](x,a) < 0$) to black ($\max_x W_\psi[s](x,a)$). (c) Definition of the modulus maxima at a given scale a_0 corresponding to the dashed line in (b). (d) The skeleton of the wavelet transform, i.e., the set of all the maxima lines. In (b) and (d) the large scales are at the top.

Figure 2: Figure copied from [4].

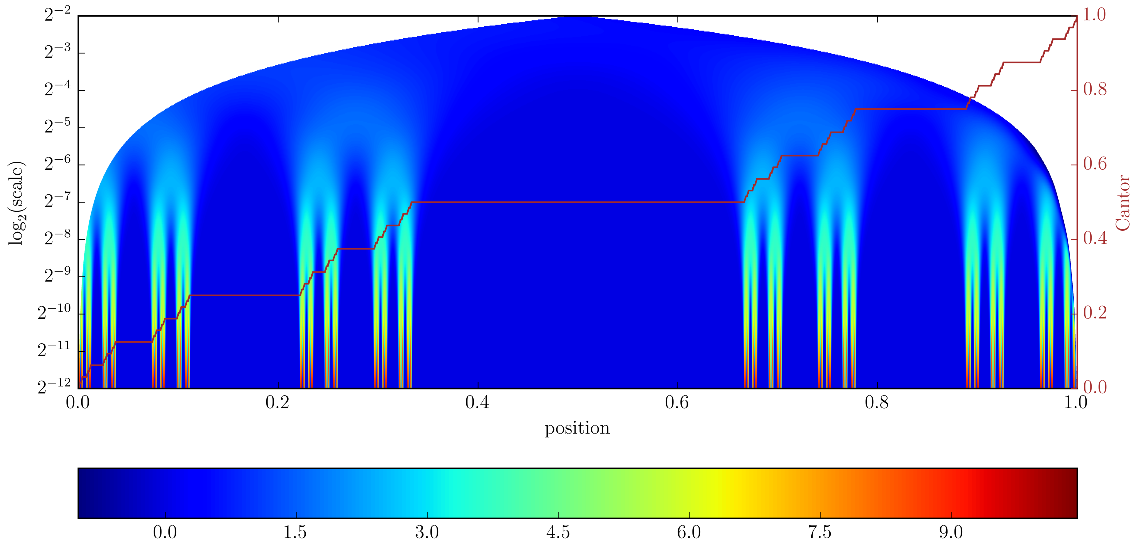

Figure 3: We show the wavelet transform of the devil's staircase function according to definition (1). As analyzing wavelet, we use the first derivative of the Gaussian mother wavelet, (5). The white space represents (scale,position)-pairs that are ignored to omit finite size effects.

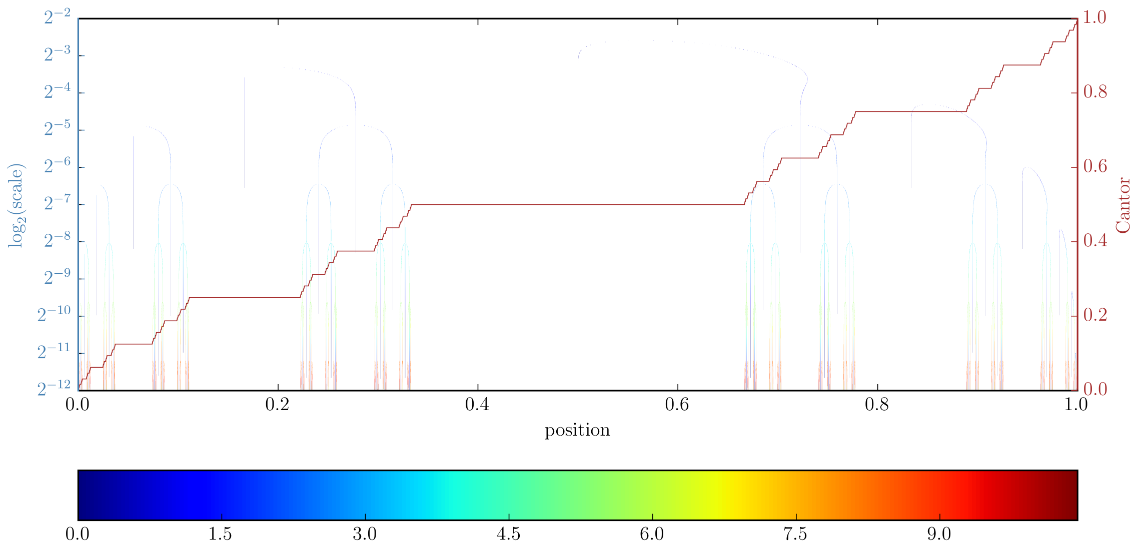

Figure 4: We show the skeleton (local extrema structure) of the wavelet transform of the devil's staircase function according to definition (1). You can also zoom into to plot, and see the fine resolution at small scales. As analyzing wavelet, we use the first derivative of the Gaussian mother wavelet. Zoom into the plot to resolve the details at fine scales.