Discrete Fourier Transform

EE 453 / CE 352

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Learning Outcomes

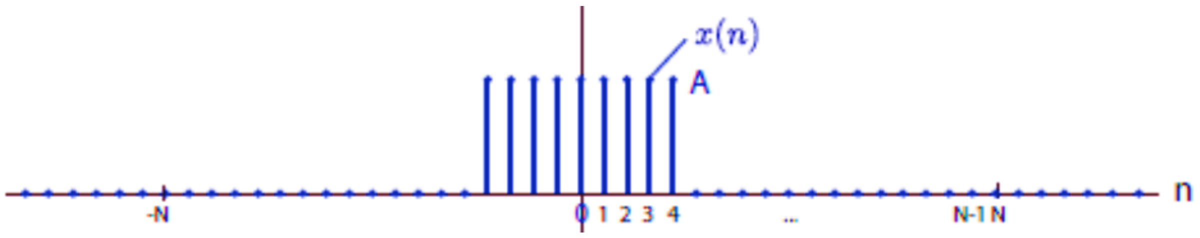
- After completing this chapter, you should be able to:
  - Define Discrete Fourier transform (DFT) and Inverse Discrete Fourier Transform (IDFT).
  - Establish the properties of DFT.
  - Study different methods of circular convolution and solve numerical problems.

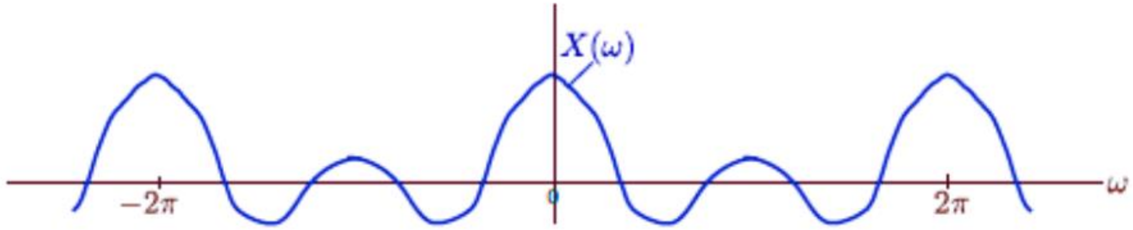
A Lookback at DTFT and Z-Transform

- The DTFT provides frequency domain (ω) representation for absolutely summable sequences
- The Z-transform provides a generalized frequency domain (z) representation of arbitrary sequences.

A Lookback at DTFT and Z-Transform

- Both transforms have the following features:
  - They are defined for infinite-length sequences.
  - They are functions of continuous variables.





Why Use Another Transform

1. From a computational point of view, both features are troublesome, because we have to evaluate infinite sums at uncountably infinite frequencies.
    - In MATLAB sequences have to be truncated and evaluated at finite points.
    - In other words, DTFT and z-transform are not numerically computable transforms.
2. $X(\omega)$ must be computed for a discrete and finite set of real values.

- We now turn our attention to a numerically computable transform: The DFT

Discrete Fourier Transform

- The DFT is the equivalent of the continuous Fourier Transform for signals known only at N instants separated by sample times T.
  - That is, the data is a finite sequence.

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi k \frac{n}{N}}, \qquad k = 0,1,\ldots,N-1$$

- As far as realization on a computer is concerned, DFT is the appropriate representation since it is discrete and of finite length in both the time and frequency domains.

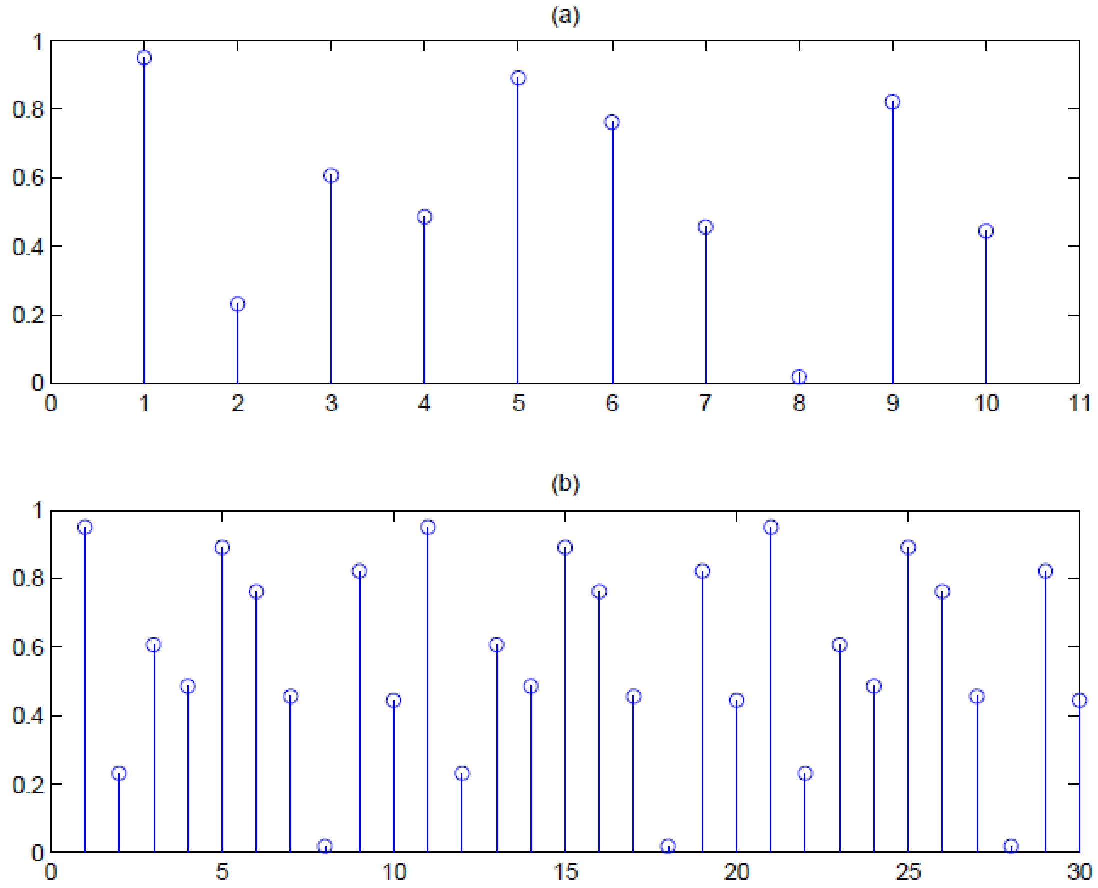
Discrete Fourier Transform

DFT: $$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi k \frac{n}{N}}, \qquad k = 0,1,\ldots,N-1$$

IDFT: $$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{j2\pi k \frac{n}{N}}, \qquad n = 0,1,\ldots,N-1$$

Discrete Fourier Transform

- Since there are only a finite number of input data points, the DFT treats the data as if it were periodic.
  - Figure (a): Sequence of N=10 samples.
  - Figure (b): Implicit periodicity in DFT.



Discrete Fourier Transform

- Since the operation treats the data as if it were periodic, we evaluate the DFT equation for the fundamental frequency as well as its harmonics.

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi k \frac{n}{N}}, \qquad k = 0, 1, \ldots, N-1$$

$$X(k) = \sum_{n=0}^{N-1} x(n) \left(e^{-j\frac{2\pi}{N}}\right)^{kn} = \sum_{n=0}^{N-1} x(n) \left(W_N\right)^{kn}$$

where $e^{-j\frac{2\pi}{N}} = W_N$ is considered as the twiddle factor.

Discrete Fourier Transform

$$X(k) = \sum_{n=0}^{N-1} x(n)\left(e^{-j\frac{2\pi}{N}}\right)^{kn} = \sum_{n=0}^{N-1} x(n)\,(W_N)^{kn}$$

where $e^{-j\frac{2\pi}{N}} = W_N$ is considered as the twiddle factor.

- Harmonics:

$$\omega = 0, \frac{2\pi f}{N}, \frac{4\pi f}{N}, \dots, \frac{2\pi f(N-1)}{N}$$

- In terms of fundamental frequency:

$$W_N^0, W_N^1, W_N^2, \dots, W_N^{kn}$$

Discrete Fourier Transform

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \\ \vdots \\ X(N-1) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & W_N^1 & W_N^2 & W_N^3 & \cdots & W_N^{N-1} \\ 1 & W_N^2 & W_N^4 & W_N^6 & \cdots & W_N^{2(N-1)} \\ 1 & W_N^3 & W_N^6 & W_N^9 & \cdots & W_N^{3(N-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & W_N^{N-1} & W_N^{2(N-1)} & W_N^{3(N-1)} & \cdots & W_N^{(N-1)^2} \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \\ \vdots \\ x(N-1) \end{bmatrix}$$

Discrete Fourier Transform

$$X_N = [W_N]x_N$$
$$x_N = \frac{1}{N}[W_N^*]X_N$$

where $W_N^* = W_N^{-kn}$

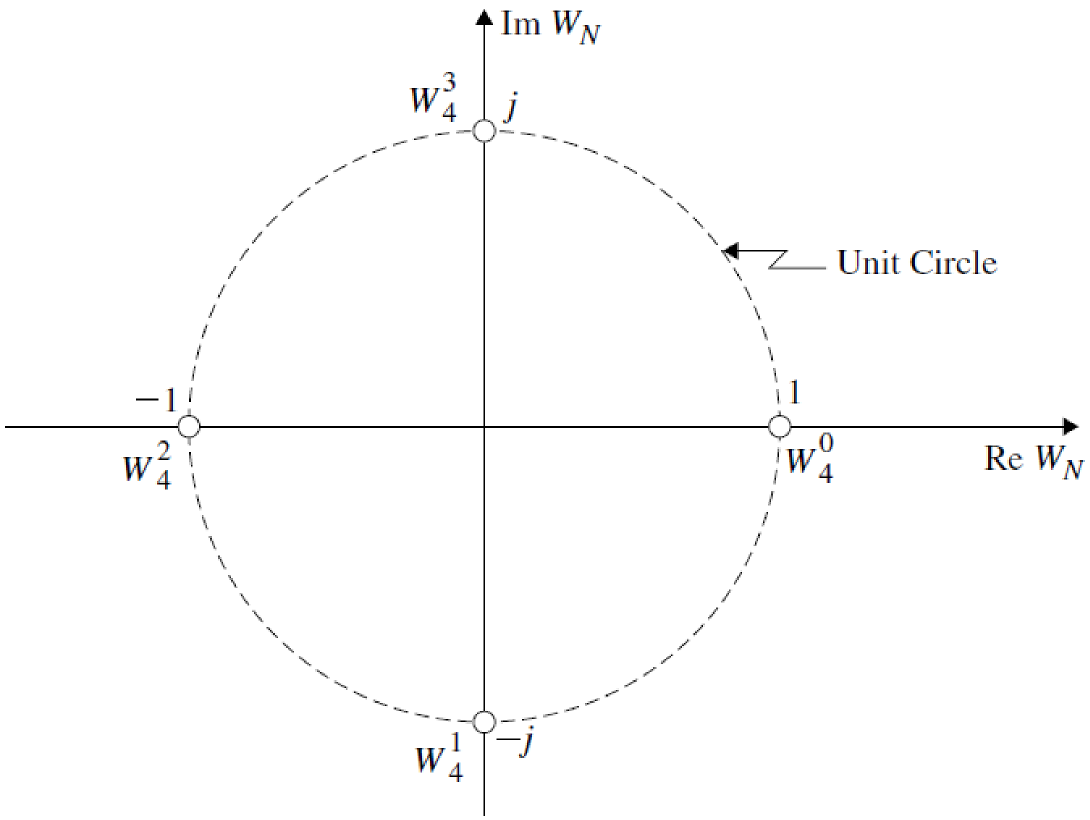
$$W_N = e^{-j\frac{2\pi}{N}}$$
$$= 1\angle{-2\pi/N}$$

the magnitude of the twiddle factor is 1 and the phase angle is $-\frac{2\pi}{N}$. It lies on the unit circle in the complex plane from 0 to 2π angle, and it gets repeated for every cycle.

Four-Point Twiddle Factor

- W_4^{kn}

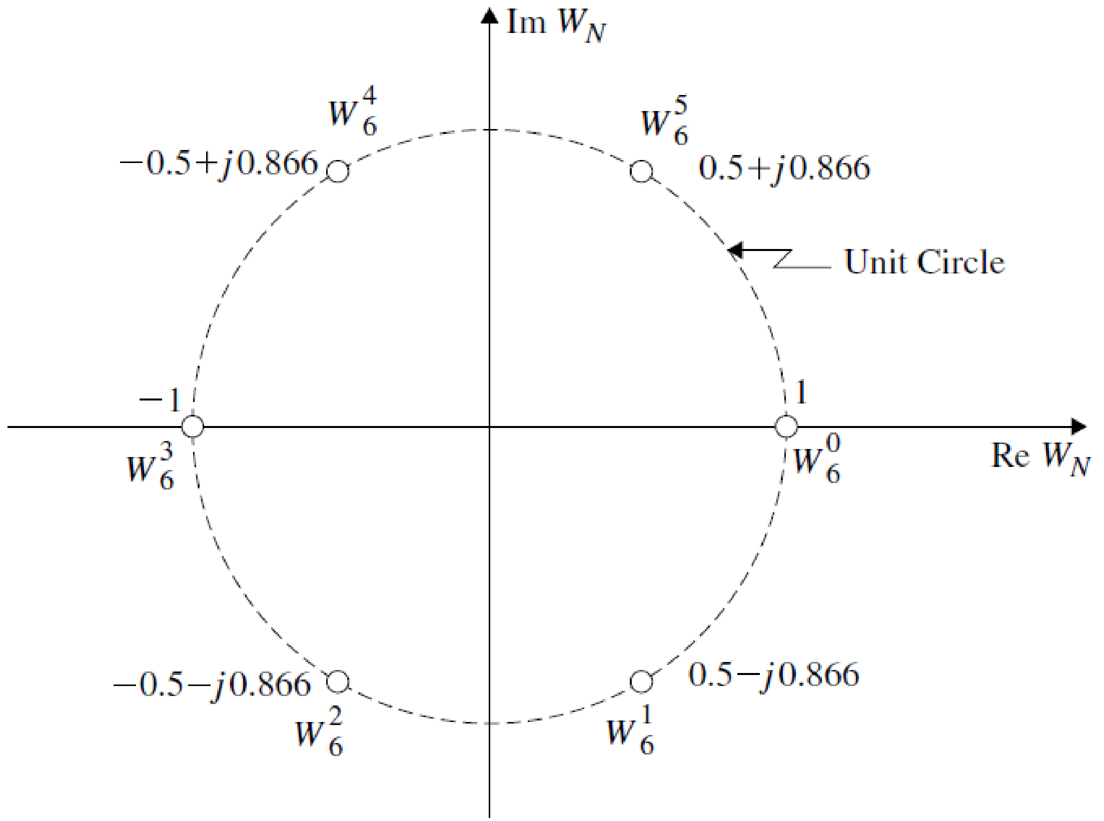
$$W_N = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$$



Six-Point Twiddle Factor

- W_6^{kn}

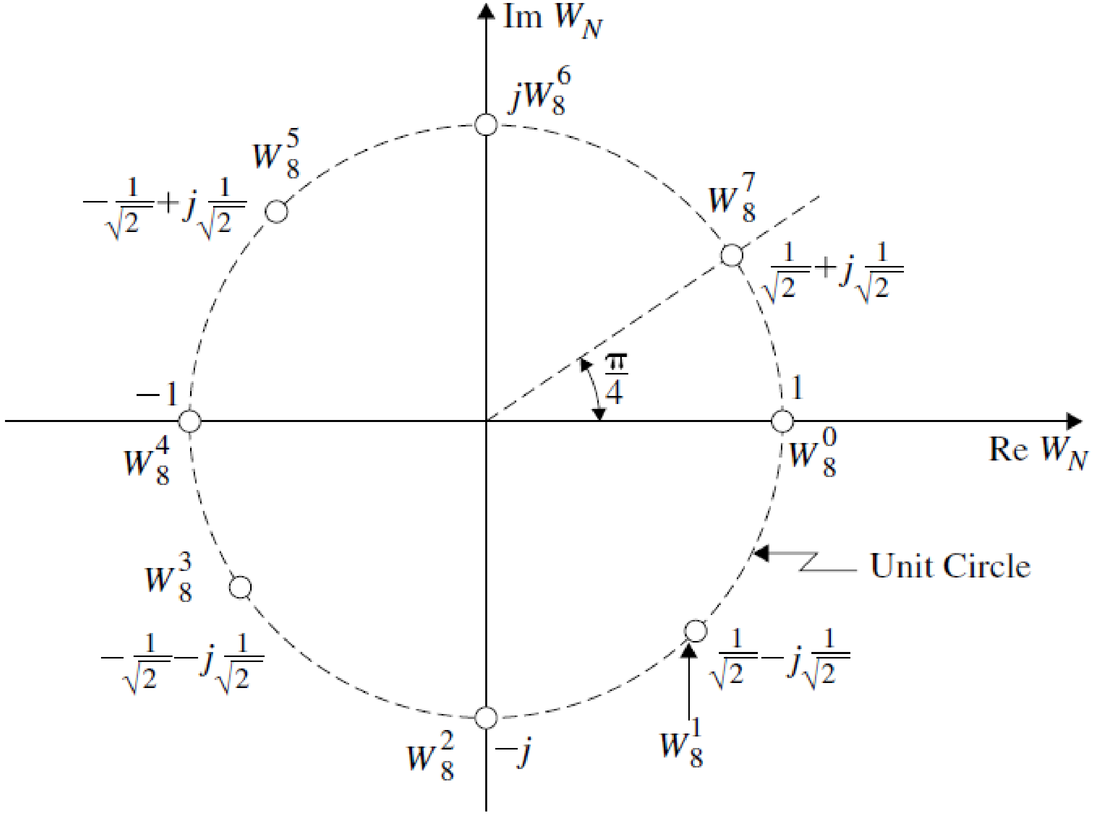
$$W_N = \begin{bmatrix} W_6^0 & W_6^0 & W_6^0 & W_6^0 & W_6^0 & W_6^0 \\ W_6^0 & W_6^1 & W_6^2 & W_6^3 & W_6^4 & W_6^5 \\ W_6^0 & W_6^2 & W_6^4 & W_6^6 & W_6^8 & W_6^{10} \\ W_6^0 & W_6^3 & W_6^6 & W_6^9 & W_6^{12} & W_6^{15} \\ W_6^0 & W_6^4 & W_6^8 & W_6^{12} & W_6^{16} & W_6^{20} \\ W_6^0 & W_6^5 & W_6^{10} & W_6^{15} & W_6^{20} & W_6^{25} \end{bmatrix}$$



Eight-Point Twiddle Factor

- W_8^{kn}

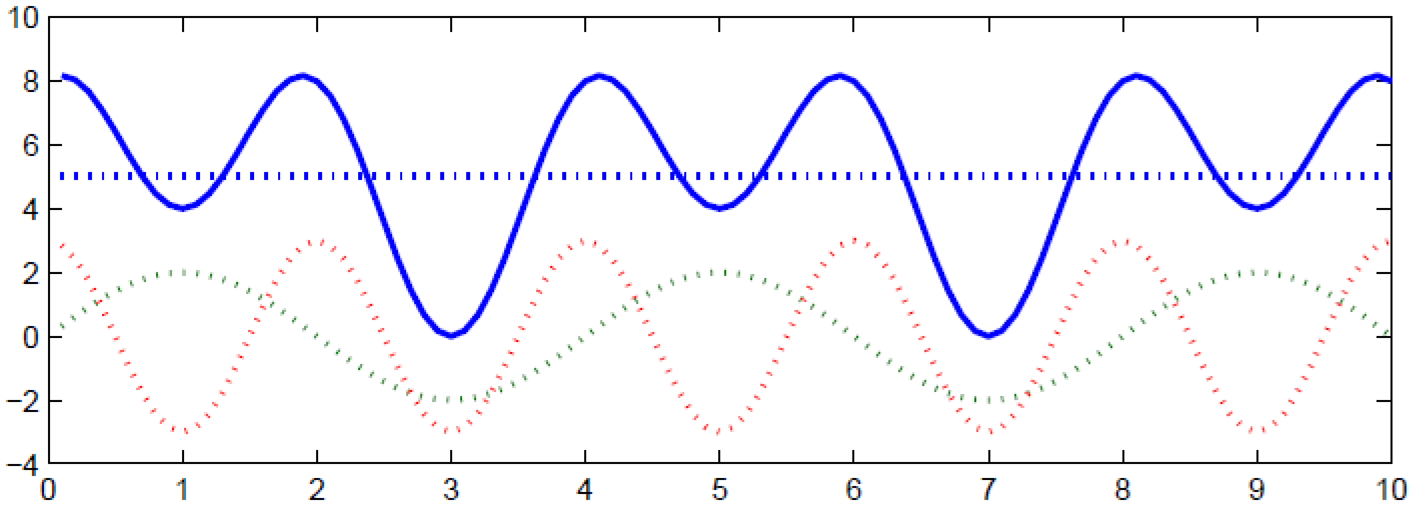
$$W_8 = \begin{bmatrix} W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 \\ W_8^0 & W_8^1 & W_8^2 & W_8^3 & W_8^4 & W_8^5 & W_8^6 & W_8^7 \\ W_8^0 & W_8^2 & W_8^4 & W_8^6 & W_8^8 & W_8^{10} & W_8^{12} & W_8^{14} \\ W_8^0 & W_8^3 & W_8^6 & W_8^9 & W_8^{12} & W_8^{15} & W_8^{18} & W_8^{21} \\ W_8^0 & W_8^4 & W_8^8 & W_8^{12} & W_8^{16} & W_8^{20} & W_8^{24} & W_8^{28} \\ W_8^0 & W_8^5 & W_8^{10} & W_8^{15} & W_8^{20} & W_8^{25} & W_8^{30} & W_8^{35} \\ W_8^0 & W_8^6 & W_8^{12} & W_8^{18} & W_8^{24} & W_8^{30} & W_8^{36} & W_8^{42} \\ W_8^0 & W_8^7 & W_8^{14} & W_8^{21} & W_8^{28} & W_8^{35} & W_8^{42} & W_8^{49} \end{bmatrix}$$



DFT: Example

- Let the continuous signal be:

$$x(t) = \underbrace{5}_{\text{dc}} + \underbrace{2\cos(2\pi t - 90°)}_{\text{1 Hz}} + \underbrace{3\cos 4\pi t}_{\text{2 Hz}}$$



DFT: Example

$$x(t) = 5 + 2\cos(2\pi t - 90°) + 3\cos 4\pi t$$

Let's sample this at 4 times per second, $F_s = 4\text{ Hz}$, from $t = 0$ to $t = 0.75$. Then putting $t = nT_s = n/4$:

$$x(n) = 5 + 2\cos\left(\frac{\pi}{2}n - 90°\right) + 3\cos \pi k$$

$$x(0) = ?$$
$$x(1) = ?$$
$$x(2) = ?$$
$$x(3) = ?$$

DFT: Example

$$x(t) = 5 + 2\cos(2\pi t - 90°) + 3\cos 4\pi t$$

Let's sample this at 4 times per second, $F_s = 4\text{ Hz}$, from $t = 0$ to $t = 0.75$. Then putting $t = nT_s = n/4$:

$$x(n) = 5 + 2\cos\left(\frac{\pi}{2}n - 90°\right) + 3\cos \pi k$$

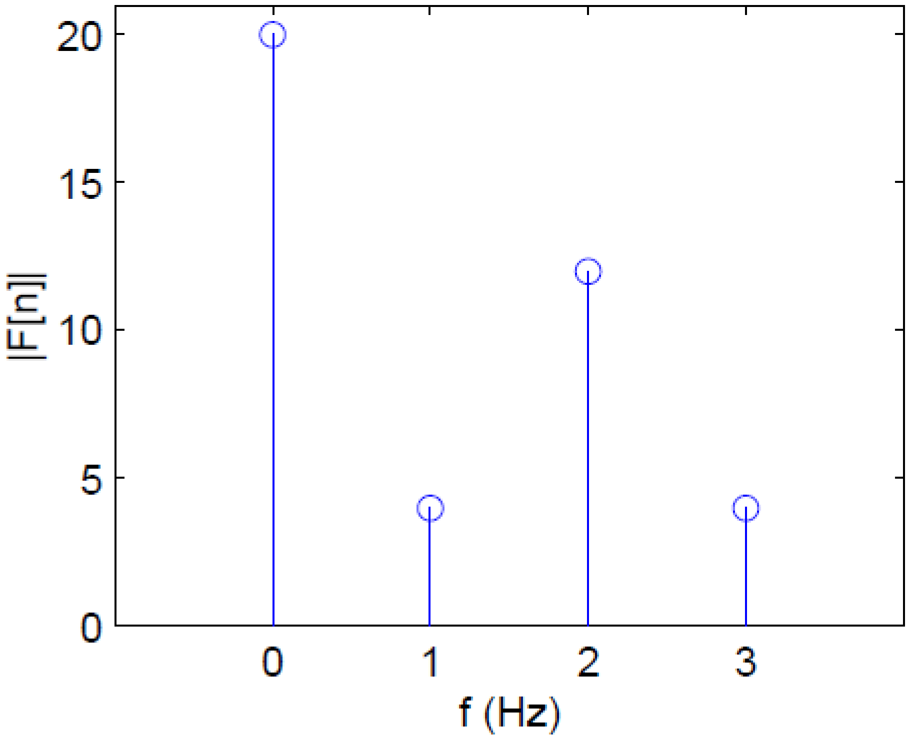
$$x(0) = 8$$
$$x(1) = 4$$
$$x(2) = 8$$
$$x(3) = 0$$

DFT: Example

$$W_N = e^{-j\frac{2\pi}{4}} = e^{-j\frac{\pi}{2}}$$

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \end{bmatrix} = \begin{bmatrix} 20 \\ -j4 \\ 12 \\ j4 \end{bmatrix}$$

The spectrum is symmetrical about $N/2$. In the process of taking the inverse transform, $X(k)$ and $X(N-k)$ are combined, and the lowest of them is valid. The higher frequency is an alias.



DFT: Example

1. $F[0] = 20$ implies a d.c. value of $\frac{1}{N}F[0] = \frac{20}{4} = 5$ (as expected)

2. $F[1] = -j4 = F^*[3]$ implies a fundamental component of peak amplitude $\frac{2}{N}|F[1]| = \frac{2}{4} \times 4 = 2$ with phase given by $\arg F[1] = -90\ ^{\mathrm{o}}$

$$\text{i.e.}\quad 2\cos(\frac{2\pi}{NT}kT - 90\ ^{\mathrm{o}}) = 2\cos(\frac{\pi}{2}k - 90\ ^{\mathrm{o}}) \qquad \text{(as expected)}$$

3. $F[2] = 12 \quad (n = \frac{N}{2}$ – no other $N - n$ component here) and this implies a component

$$f_2[k] = \frac{1}{N}F[2]e^{j\frac{2\pi}{N}\cdot 2k} = \frac{1}{4}F[2]e^{j\pi k} = 3cos\pi k \qquad \text{(as expected)}$$

since $\sin \pi k = 0$ for all k

Practice

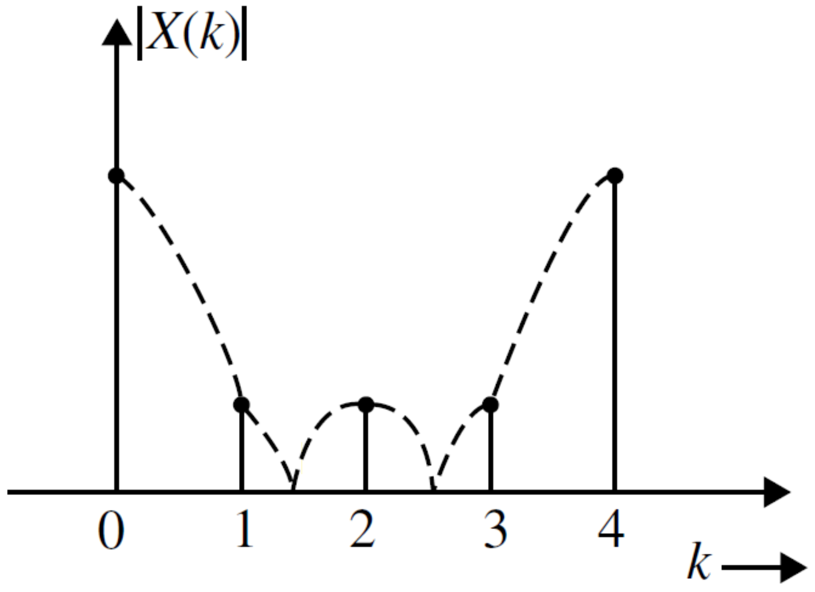
Find the DFS representation of the periodic sequence

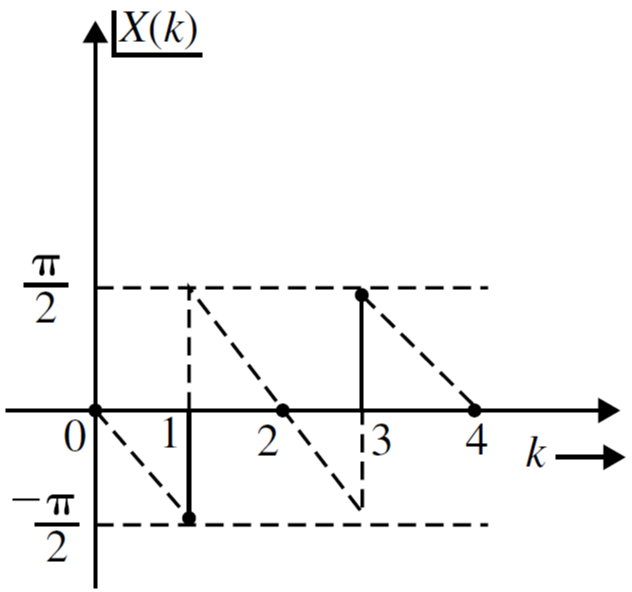
$$\tilde{x}(n) = \{\ldots, 0, 1, 2, 3, \underset{\uparrow}{0}, 1, 2, 3, 0, 1, 2, 3, \ldots\}$$

DFT Magnitude and Phase Plots

- Example: 4-point DFT of:

$$x[n] = \begin{cases} 1/3, & 0 \le n \le 2 \\ 0, & \text{else} \end{cases}$$





Aliasing in DFT

- If the initial samples are not sufficiently closely spaced to represent high-frequency components present in the underlying function, then the DFT values will be corrupted by aliasing.
- As before, the solution is either to increase the sampling rate (if possible) or to pre-filter the signal in order to minimize its high frequency spectral content.

Aliasing in DFT

- $x(n)$ can be recovered from $x_p(n)$ if there is no overlap when taking the periodic repetition.
- If $x(n)$ is finite duration and non-zero in the interval $0 \leq n \leq L-1$, then

$$x(n) = x_p(n), \quad 0 \leq n \leq N-1 \quad \text{when } N \geq L$$

- If $N < L$ then, $x(n)$ cannot be recovered from $x_p(n)$.
  - or equivalently $X(\omega)$ cannot be recovered from its samples $X\left(\frac{2\pi}{N}k\right)$ due to time-domain aliasing

Zero Padding

- Determine the 8-point DFT of:

$$x(n) = \{2,1,2,1\}$$

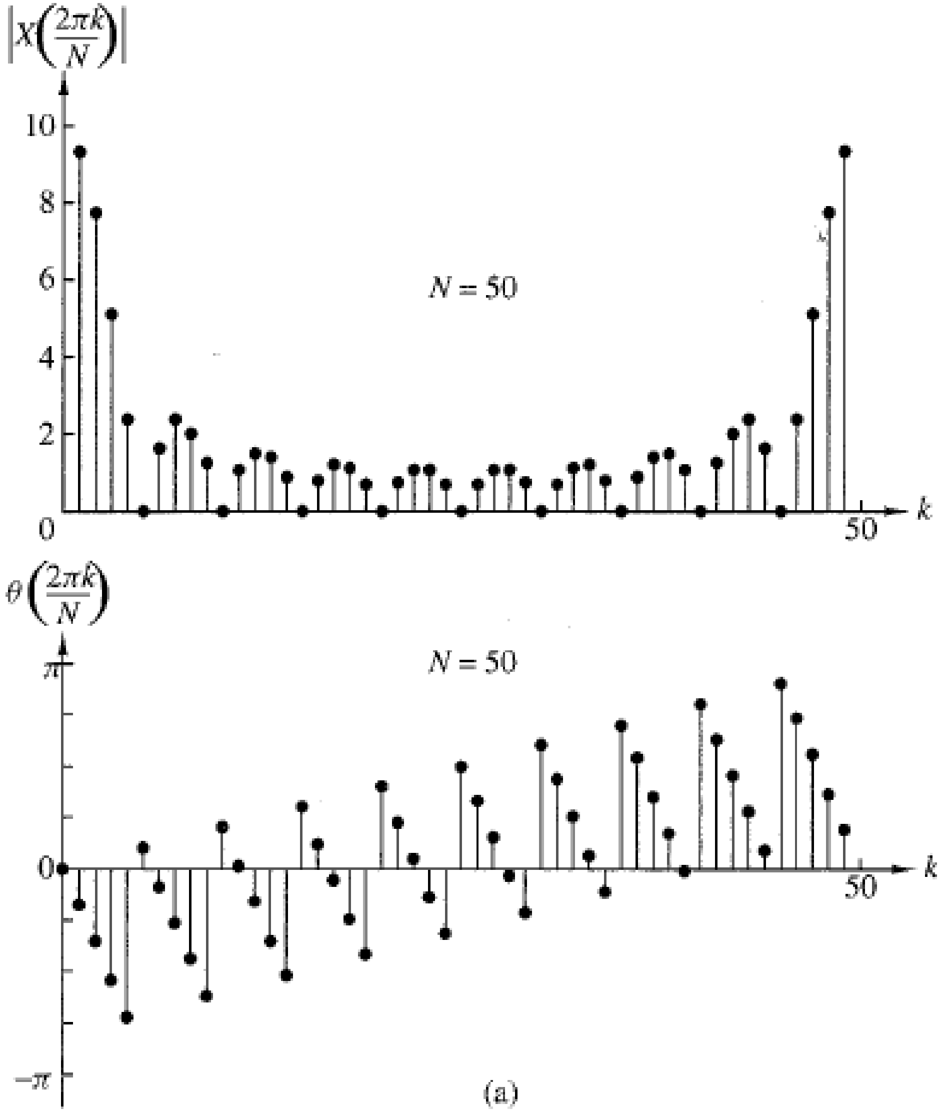
- When we take the N-point DFT of a finite-duration sequence of length L where $N > L$, we pad the original sequence with $(N - L)$ zeros.
- By adding zeros, we effectively increase the number of samples in the signal, which can improve the frequency resolution of the DFT.

DFT: Example

Determine the N-point DFT of the following sequence for $N \geq L$:

$$x(n) = \begin{cases} 1 & 0 \leq n \leq L-1 \\ 0 & \text{otherwise} \end{cases}$$

For L = 10, N = 50



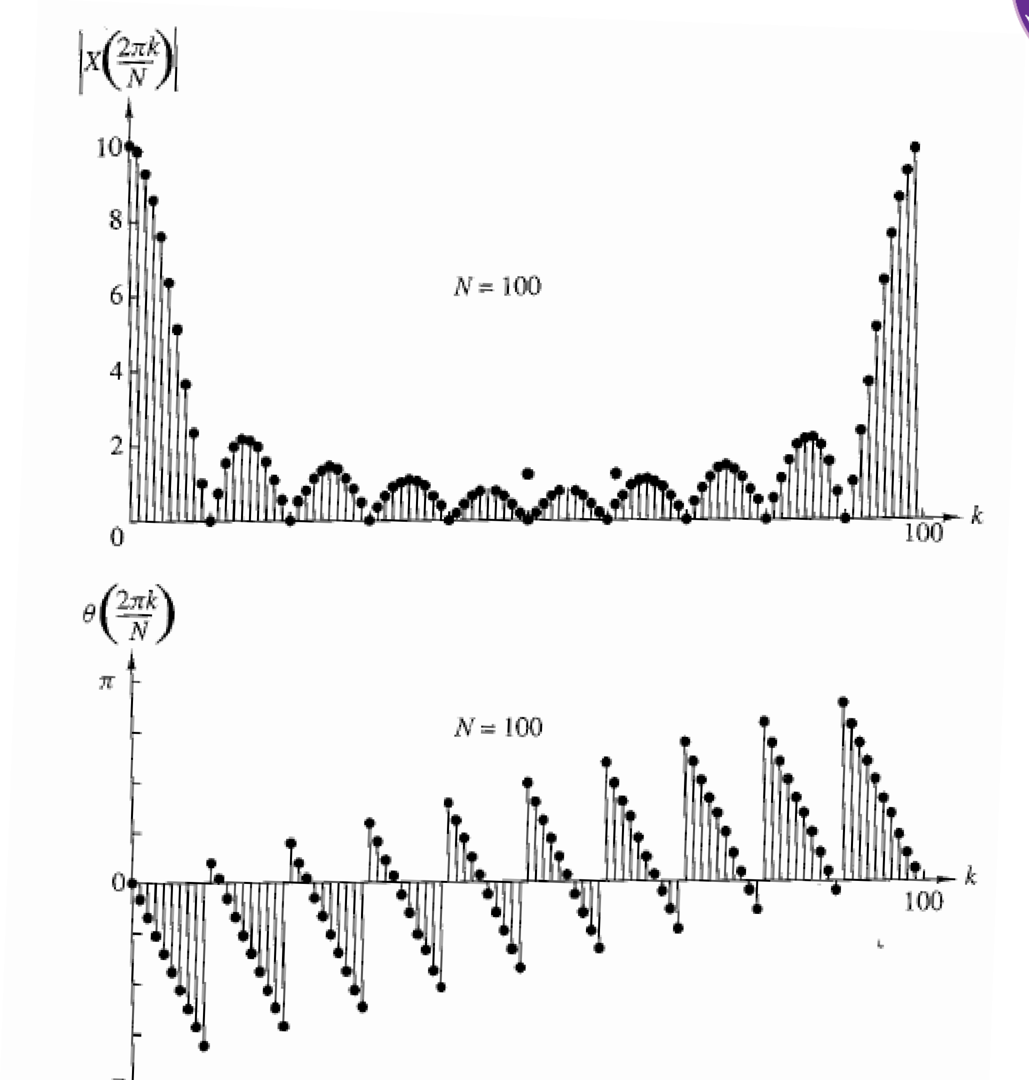
(a)

DFT: Example

Determine the N-point DFT of the following sequence for $N \geq L$:

$$x(n) = \begin{cases} 1 & 0 \leq n \leq L-1 \\ 0 & \text{otherwise} \end{cases}$$

For L = 10, N = 100



(b)

DFT: Properties

- The properties of the DFT are circular in nature (as opposed to the DTFT). That is, they apply to the periodic repetition of the signal.

Property	Time Domain	Frequency Domain
Notation:	$x(n)$	$X(k)$
Periodicity:	$x(n) = x(n+N)$	$X(k) = X(k+N)$
Linearity:	$a_1x_1(n) + a_2x_2(n)$	$a_1X_1(k) + a_2X_2(k)$
Time reversal	$x(N-n)$	$X(N-k)$
Circular time shift:	$x((n-l))_N$	$X(k)e^{-j2\pi kl/N}$
Circular frequency shift:	$x(n)e^{j2\pi ln/N}$	$X((k-l))_N$
Complex conjugate:	$x^*(n)$	$X^*(N-k)$
Circular convolution:	$x_1(n) \otimes x_2(n)$	$X_1(k)X_2(k)$
Multiplication:	$x_1(n)x_2(n)$	$\frac{1}{N}X_1(k) \otimes X_2(k)$
Parseval's theorem:	$\sum_{n=0}^{N-1} x(n)y^*(n)$	$\frac{1}{N}\sum_{k=0}^{N-1} X(k)Y^*(k)$

Circular Operation

- Circular operations: apply the transformation on the periodic repetition of $x(n)$ and then obtain the final result by taking points for $n = 0, 1, \ldots, N-1$

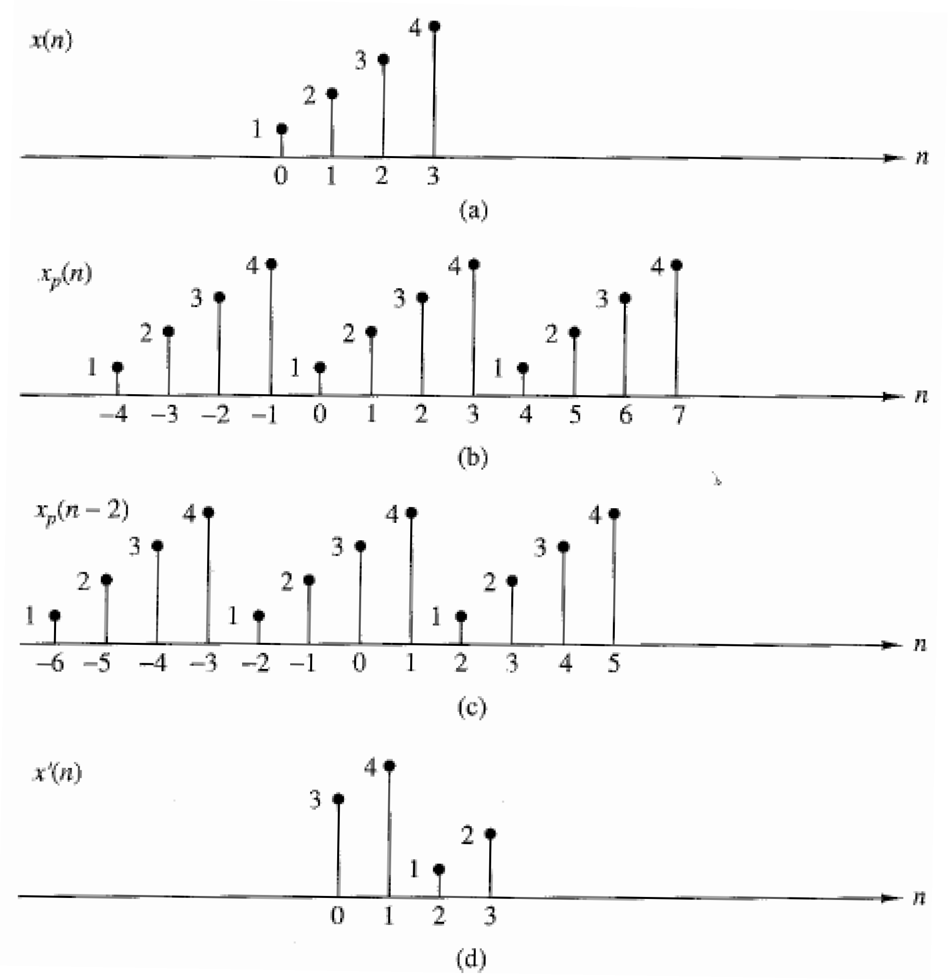
- Often use the modulo notation:

$$(n)_N = n \mod N = \text{remainder of } n/N$$

Example: $N = 4$

n	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8
$(n)_4$	0	1	2	3	0	1	2	3	0	1	2	3	0

Circular Shifting: Two Interpretations

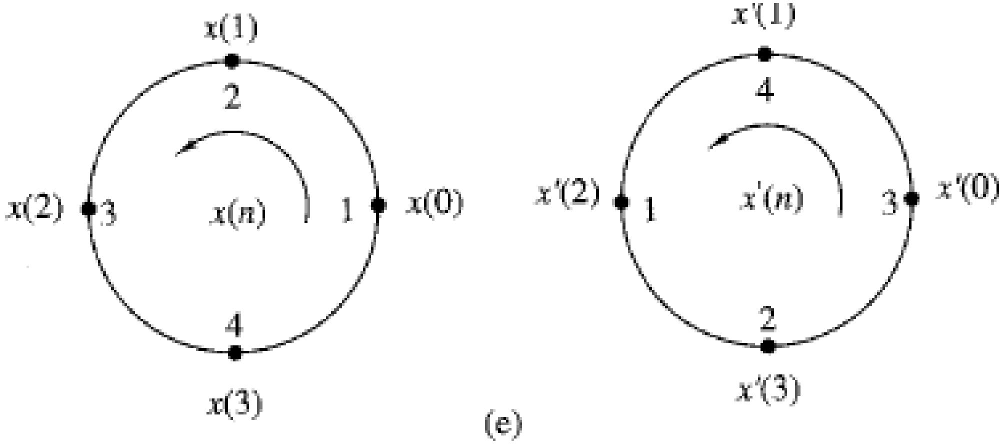


$$x'(n) = x(n-k, \text{modulo } N)$$

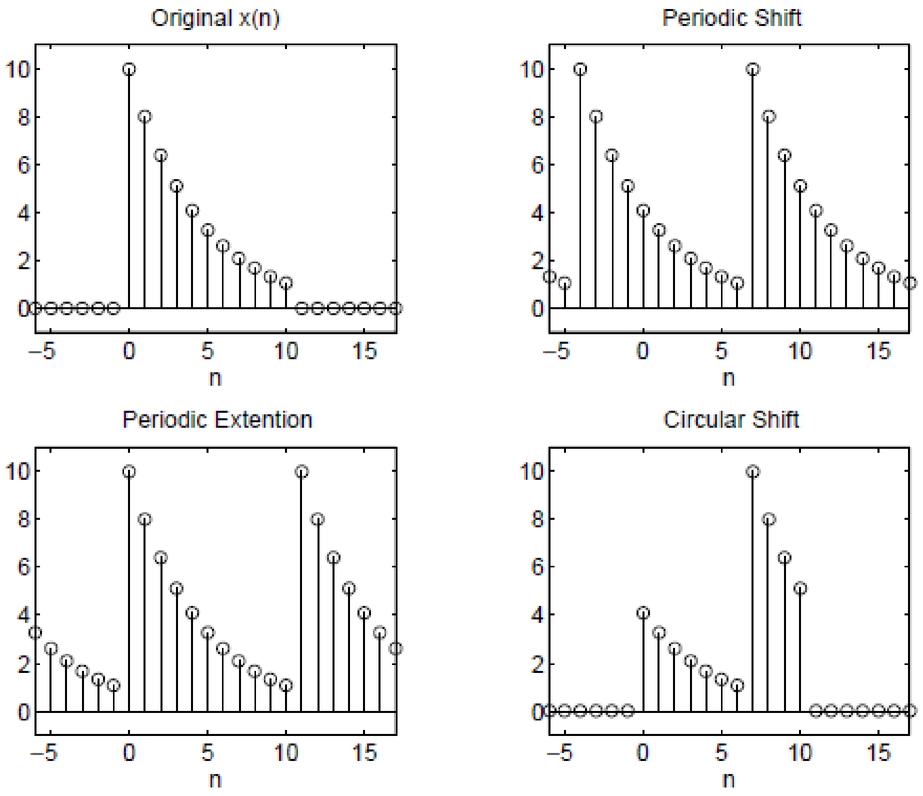
$$\equiv x((n-k))_N$$

if $k = 2$ and $N = 4$, we have

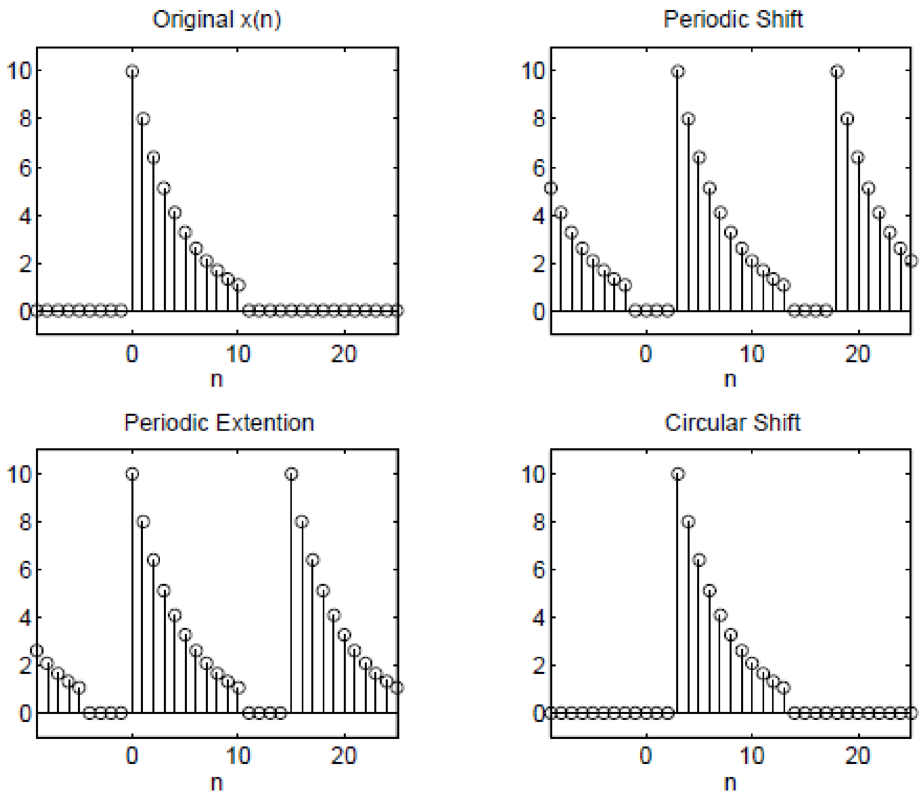
$$x'(n) = x((n-2))_4$$



Circular Shifting



$x((n+4))_{11}$



$x((n-3))_{15}$

Circular Convolution

Circular convolution. If

$$x_1(n) \overset{\text{DFT}}{\underset{N}{\longleftrightarrow}} X_1(k)$$

and

$$x_2(n) \overset{\text{DFT}}{\underset{N}{\longleftrightarrow}} X_2(k)$$

then

$$x_1(n) \, Ⓝ \, x_2(n) \overset{\text{DFT}}{\underset{N}{\longleftrightarrow}} X_1(k)X_2(k)$$

where $x_1(n) \, Ⓝ \, x_2(n)$ denotes the circular convolution of the sequence $x_1(n)$ and $x_2(n)$.

Linear versus Circular Convolution

Linear Convolution

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]\, h[n-k]$$

- Folding (Time Reversal)
- Shifting
- Multiplication
- Summation
- m+n-1 samples

Circular Convolution

$$x_3(m) = \sum_{n=0}^{N-1} x_1(n) x_2((m-n))_N, \quad m = 0, 1, \ldots, N-1$$

- Circular Folding (Time Reversal)
- Circular Shifting
- Multiplication
- Summation
- max(m,n)

Circular Convolution: Example

- Perform the circular convolution of the following two sequences:

$$x_1(n) = \{2,1,2,1\} \qquad x_2(n) = \{1,2,3,4\}$$

DFT and Z-Transform

The z-transform of N-point sequence $x(n)$ is given by,

$$X(z) = \sum_{n=0}^{N-1} x(n) z^{-n}$$

Let us evaluate $X(z)$ at N equally spaced points on unit circle that is at $z = e^{j\frac{2\pi k}{N}}$

$$\begin{aligned} X(z)\Big|_{z=e^{j\frac{2\pi k}{N}}} &= \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi kn}{N}} \\ &= X(k) \\ X(k) &= X(z)\Big|_{z=e^{j\frac{2\pi k}{N}}} \end{aligned}$$

we can conclude that the N-point DFT of a finite duration sequence can be obtained from the z-transform of the sequence at N equally spaced points around the unit circle.

Practice

- Compute the 4-point DFT of the following sequences:
  - $x_1[n] = \{1, j, -1, -j\}$
  - $x_2[n] = 1, \quad 0 \leq n < 2$
  - $x_3[n] = \sin(n\pi/2)\,, \ n = 0,1,2,3 \ldots$
    - Answers: $X_1(k) = \{0,4,0,0\}$, $X_2(k) = \{3, -j, 1, j\}$, $X_3[k] = \{0, -j2, 0, j2\}$

- Find the IDFT of the following functions with $N = 4$:
  - $X_1(k) = \{1,0,1,0\}$
  - $X_2(k) = \{6, -2 + j2, -2, -2 - j2\}$
    - Answers: $x_1[n] = \{0.5,0,0.5,0\}$, $x_2[n] = \{0,1,2,3\}$