Digital Filters Implementation

EE 453 / CE 352

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Some Implementation/Realization Options

$$\sum_{k=0}^{N} a_k y(n-k) = \sum_{k=0}^{M} b_k x(n-k) \qquad a_0 \triangleq 1$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\left[1 + \sum_{k=1}^{N} a_k z^{-k}\right]}$$

- **Direct Form**
  - In this form, the difference equation is implemented directly.
  - Two versions: Direct-Form I and Direct-Form II.
- **Cascade Form**
  - In this form, $H(z)$ is factored into smaller second-order sections, called biquads.
  - The system function is then represented as a product of these biquads.
  - Each biquad is implemented in a direct form.
  - The entire system function is then implemented as a cascade of biquad sections.

FIR System Realization: General

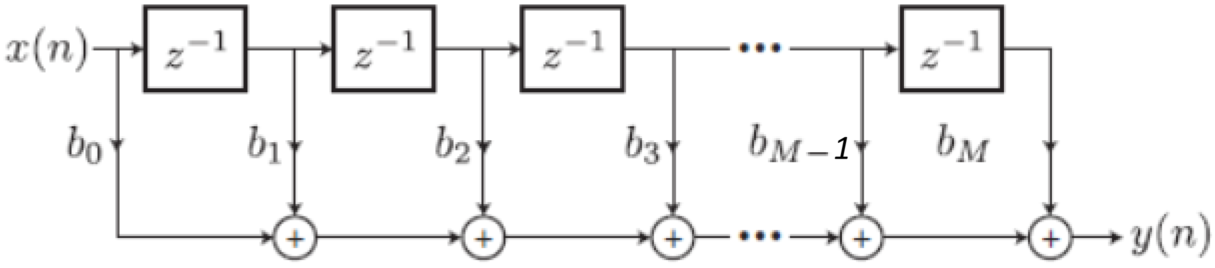
- General expression for Nth-order LCCDE:

$$\sum_{k=0}^{N} a_k y(n-k) = \sum_{k=0}^{M} b_k x(n-k) \qquad a_0 \triangleq 1$$

Initial conditions: $y(-1), y(-2), y(-3), \ldots, y(-N)$

- Requires:
  - $M+1$ multiplications
  - M additions
  - M memory elements

$$y(n) = \sum_{k=0}^{M} b_k x(n-k)$$

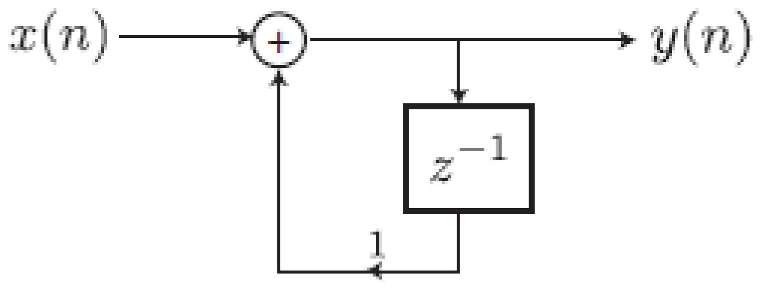


IIR System Realization: Example

- Consider an accumulator:

$$y(n) = \sum_{k=0}^{n} x(k) \quad n = 0, 1, 2, \ldots \qquad \text{for } y(-1) = 0.$$

$$\begin{aligned} y(n) &= \sum_{k=0}^{n} x(k) \\ &= \sum_{k=0}^{n-1} x(k) + x(n) \\ &= y(n-1) + x(n) \\ \therefore\ y(n) &= y(n-1) + x(n) \end{aligned}$$



<u>recursive</u> implementation

Realization: Direct Form I vs Direct Form II

$$y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k)$$

is equivalent to the cascade of the following systems:

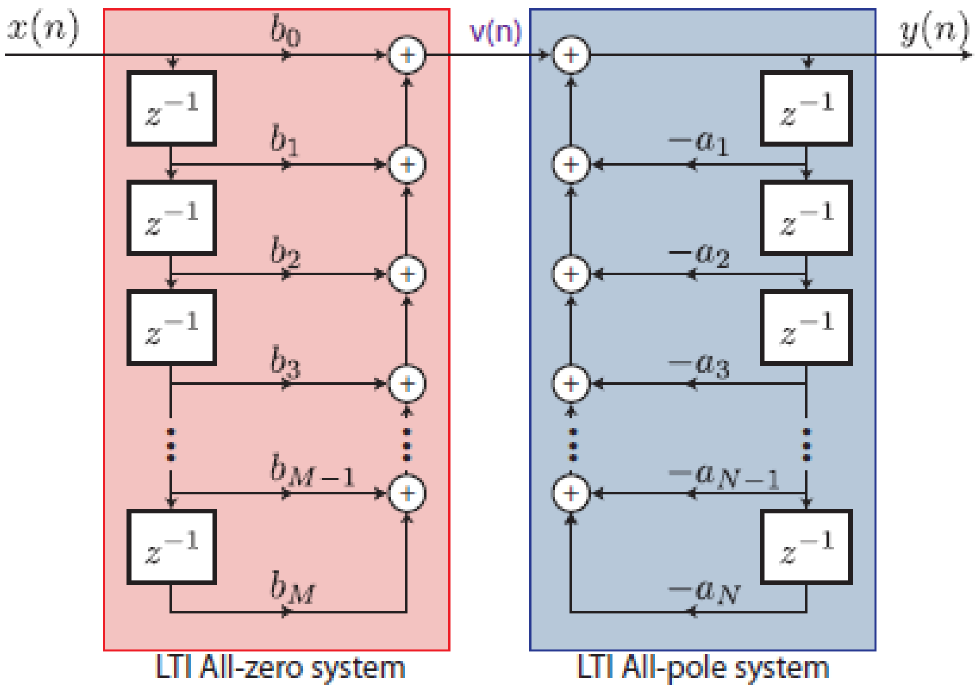
$$\underbrace{v(n)}_{\text{output 1}} = \sum_{k=0}^{M} b_k \underbrace{x(n-k)}_{\text{input 1}} \qquad \underline{\text{nonrecursive}}$$

$$\underbrace{y(n)}_{\text{output 2}} = -\sum_{k=1}^{N} a_k y(n-k) + \underbrace{v(n)}_{\text{input 2}} \qquad \underline{\text{recursive}}$$

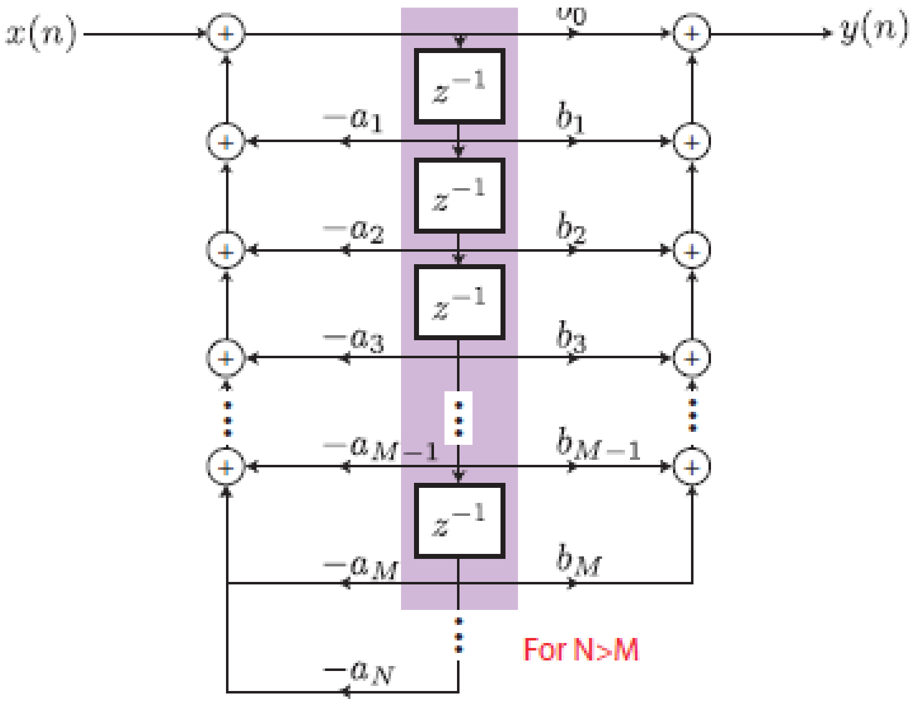
Comparison of Direct Form I vs Direct Form II

$$y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k)$$

Direct Form I



Direct Form II



Realization: Direct Form I vs Direct Form II

- General expression for Nth-order LCCDE:

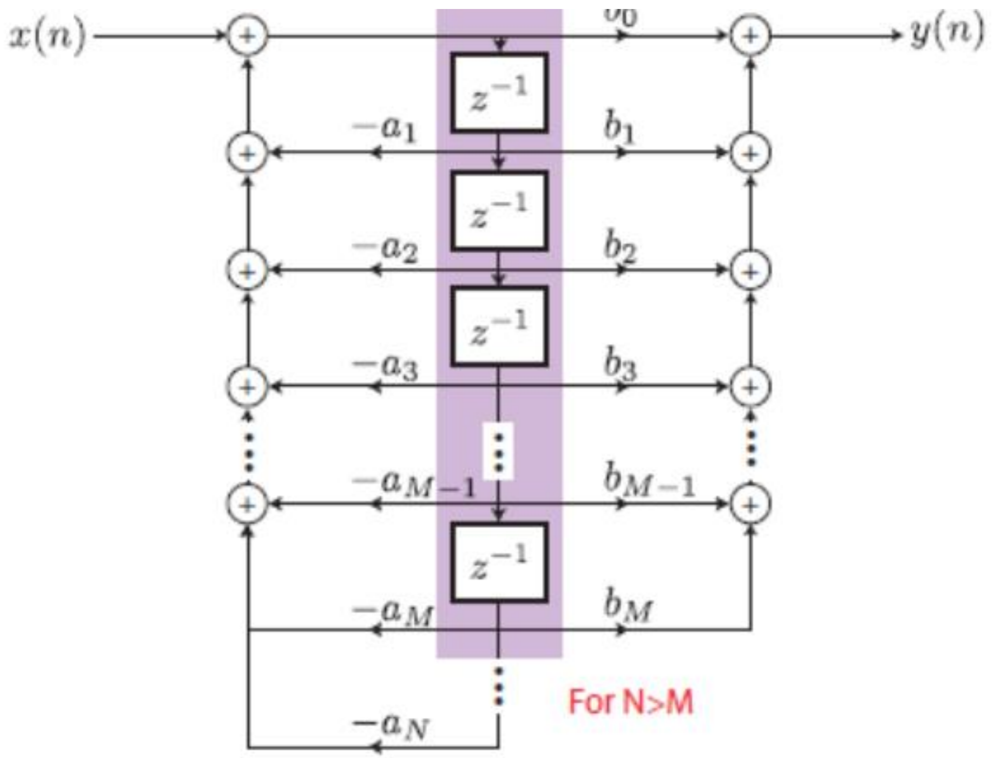
$$\sum_{k=0}^{N} a_k y(n-k) = \sum_{k=0}^{M} b_k x(n-k) \qquad a_0 \triangleq 1$$

Initial conditions: $y(-1), y(-2), y(-3), \ldots, y(-N)$

$$y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k)$$

- Requires:
  - $M + N + 1$ multiplications
  - $M + N$ additions
  - $\mathrm{Max}(M, N)$ memory locations

Direct Form II



Linear Phase FIR Filters

- Recall that an FIR filter has linear phase if its unit sample response satisfied the condition:

$$h(n) = \pm h(N-1-n), \qquad n = 0,1,2,\ldots,N-1$$

- Note that this means that:
  - Symmetry condition:

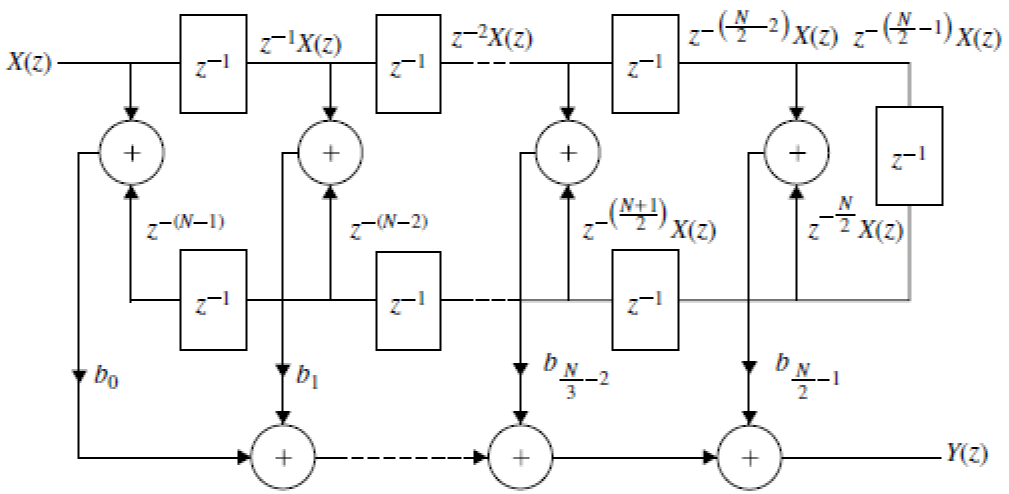
$$h(n) = +h(N-1-n), \qquad n = 0,1,2,\ldots,N-1$$

  - Antisymmetry condition:

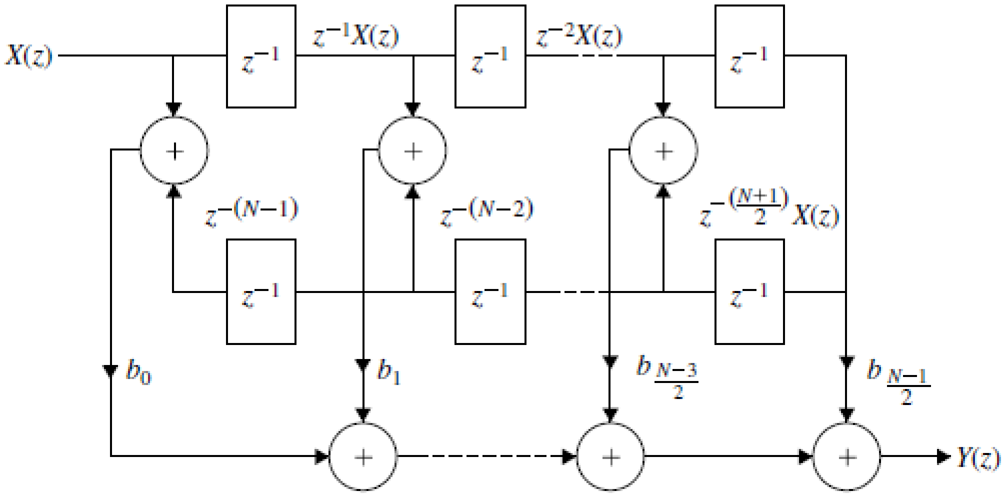
$$h(n) = -h(N-1-n), \qquad n = 0,1,2,\ldots,N-1$$

Symmetric FIR Filters: Direct Form

- <u>N is even:</u>



- <u>N is odd:</u>



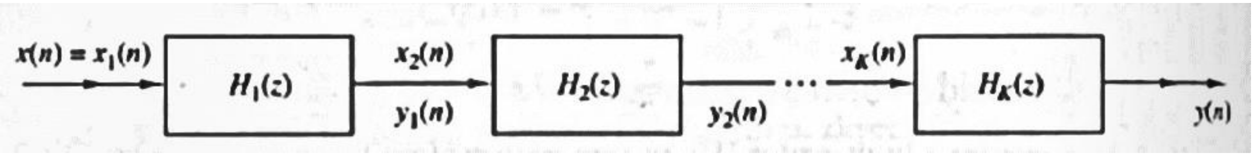
- Multiplications reduced from N to $N/2$.
- Multiplications reduced from N to $(N-1)/2$.

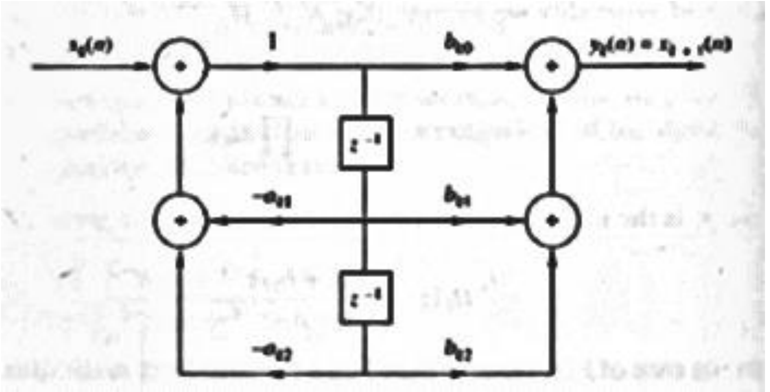
Cascade Form

- The system function $H(z)$ is written as a product of second-order sections with real coefficients.

$$
\begin{aligned}
H(z) &= \frac{b_0 + b_1 z^{-1} + \cdots + b_N z^{-N}}{1 + a_1 z^{-1} + \cdots + a_N z^{-N}} \\
&= b_0 \frac{1 + \frac{b_1}{b_0} z^{-1} + \cdots + \frac{b_N}{b_0} z^{-N}}{1 + a_1 z^{-1} + \cdots + a_N z^{-N}} \\
&= b_0 \prod_{k=1}^{K} \frac{1 + B_{k,1} z^{-1} + B_{k,2} z^{-2}}{1 + A_{k,1} z^{-1} + A_{k,2} z^{-2}}
\end{aligned}
$$

- Each of the second order subsystem can be realized in previously-seen forms:





Development of Cascade Form

- The system function $H(z)$ needs to be written as a product of second-order sections with real coefficients.
  - Factor the numerator and denominator polynomials into their respective roots.
  - Combine either a complex conjugate root pair or any two real roots into second-order polynomials.

$$H(z) = b_0 \prod_{k=1}^{K} \frac{1 + B_{k,1} z^{-1} + B_{k,2} z^{-2}}{1 + A_{k,1} z^{-1} + A_{k,2} z^{-2}}$$

  - Where K is equal to $N/2$ and $B_{k,1}$, $B_{k,2}$, $A_{k,1}$, and $A_{k,2}$ are real numbers representing the coefficients of each 2nd order section, known as a biquad.

Why Focus on Different Realization Options

- Accuracy with which filter coefficients can be specified is limited by the word length of the computer or register. If coefficients are not exact, the poles and zeroes will be different from the desired poles and zeroes.
- Consequently, quantization of filter coefficients (during software and hardware implementation) changes the frequency response of the implemented filter from the frequency response of the designed filter.
- Sensitivity of the filter frequency response to quantization of filter coefficients is minimized by realizing a high-order filters as interconnection of second-order filter sections.