Z-Transform and DTFT

EE 453 / CE 352
Saad Baig

Big Picture Till Now

$\{a_k, b_k\} \xrightarrow{\textbf{Analysis}} H(\omega)$

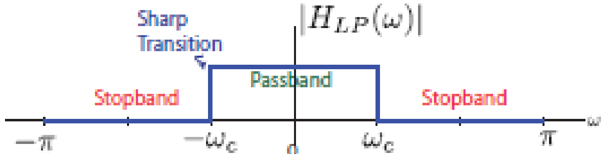
$\{a_k, b_k\} \xleftarrow{\textbf{Design}} H(\omega)$

DT LTI Systems
(Described by LCCDEs)

$$\sum_{k=0}^{N} a_k y(n-k) = \sum_{k=0}^{M} b_k x(n-k) \qquad a_0 \triangleq 1$$

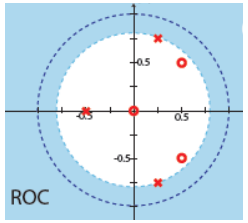
DT LTI Systems as Frequency Selective Filters
(Frequency Response)

$$h(n) \xleftrightarrow{\text{DTFT}} H(\omega)$$



Rational z-Transforms
(Pole-Zero Plots)

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\left[1 + \sum_{k=1}^{N} a_k z^{-k}\right]}$$

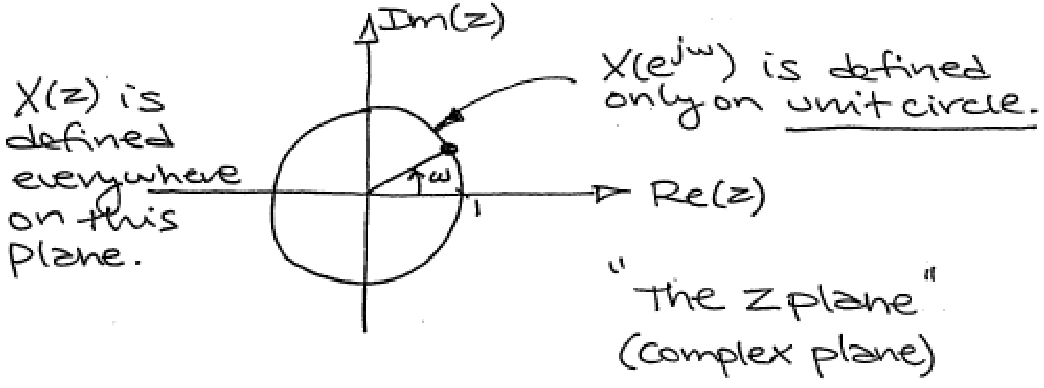


Z-Transform and DTFT

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}$$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n}$$

$$X(e^{j\omega}) = X(z) \text{ for all } z = e^{j\omega}$$

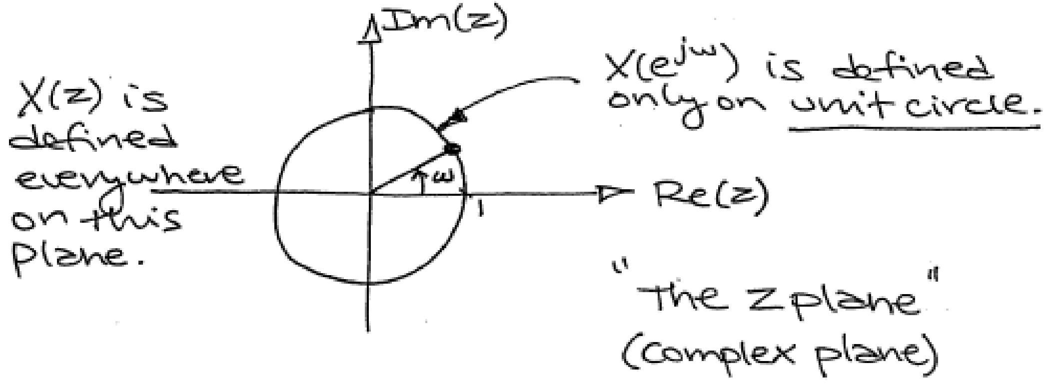


Source: [12.1: Z-Transform - Engineering LibreTexts]

- With the DTFT, we have a complex-valued function of a real-valued variable ω (and 2π periodic).
- The z-transform is a complex-valued function of a complex valued variable z.

Z-Transform and DTFT

- The Fourier Transform can be viewed as the z-transform of the sequence evaluated on the unit circle.
- If $X(z)$ does not converge in the region $|z| = 1$, the Fourier transform $X(\omega)$ does not exist.
  - I.e. if the unit circle is not contained in the ROC of $X(z)$.

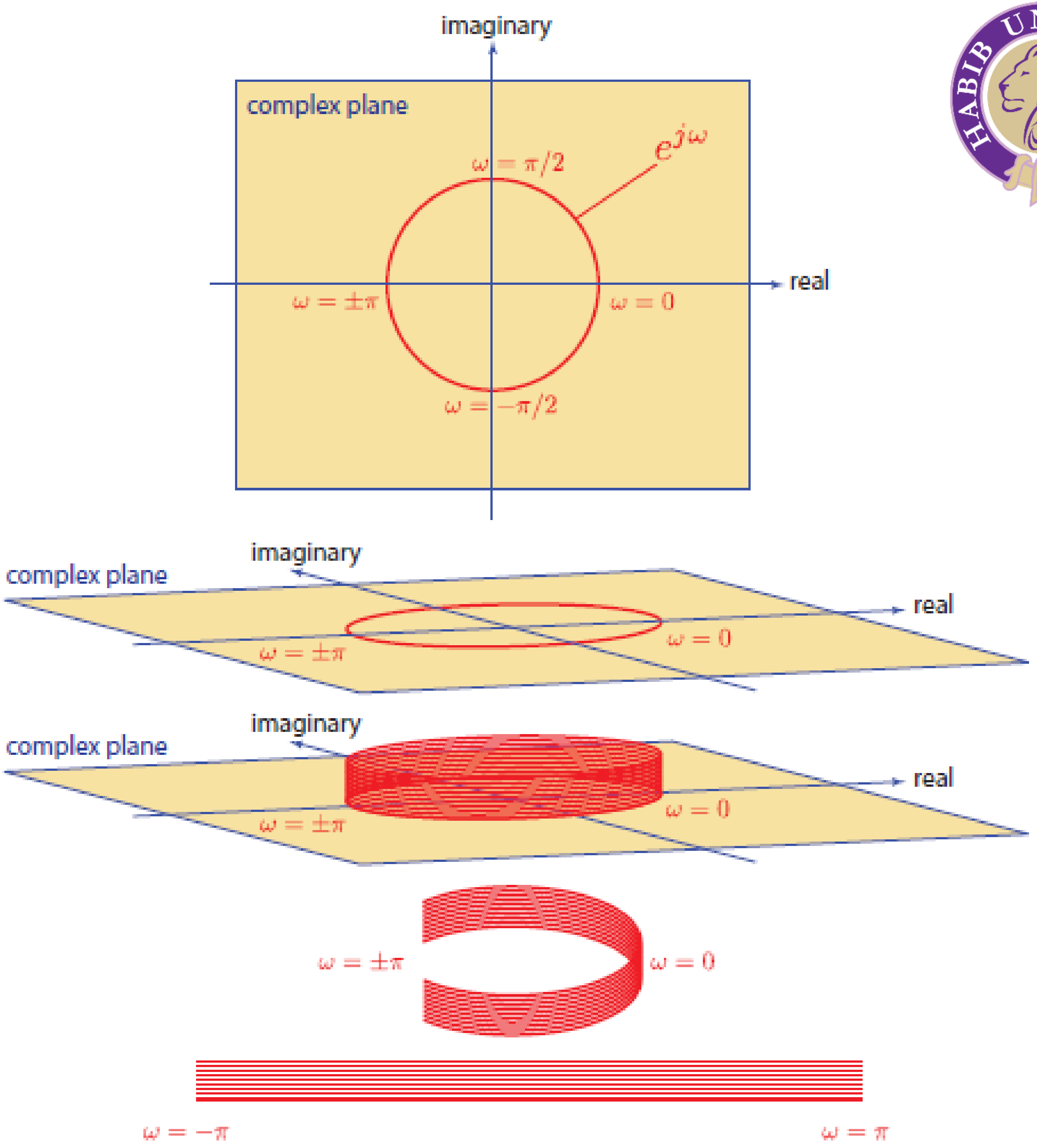


Z-Transform and DTFT

z-Domain		ω-Domain
$H(z)$	$\overset{z=e^{j\omega}}{\Longrightarrow}$	$H(\omega)$
system function	$\overset{z=e^{j\omega}}{\Longrightarrow}$	frequency response
$Y(z) = X(z)H(z)$	$\overset{z=e^{j\omega}}{\Longrightarrow}$	$Y(\omega) = X(\omega)H(\omega)$

- If H(z) converges on the unit circle, then we can obtain the frequency response by letting $z = e^{j\omega}$:

$$H(w) = H(z)\Big|_{z\,=\,e^{j\omega n}} = \sum_{n=-\infty}^{\infty} h(n)\, e^{-j\omega n}$$



Z-Transform and DTFT

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}}$$

$$H(z) = \frac{b_o z^{-M}\left(z^M + \cdots + \frac{b_M}{b_o}\right)}{z^{-N}(z^N + \cdots + a_N)}$$

$$H(z) = \ b_0 z^{N-M} \frac{(z - r_1)(z - r_2) \ldots (z - r_M)}{(z - p_1)(z - p_2) \ldots (z - p_N)}$$

$$H(z) \overset{z=e^{j\omega}}{\Longrightarrow} H(\omega) \qquad H(\omega) = \ b_0 e^{j(N-M)\omega} \frac{(e^{j\omega} - r_1)(e^{j\omega} - r_2) \ldots (e^{j\omega} - r_M)}{(e^{j\omega} - p_1)(e^{j\omega} - p_2) \ldots (e^{j\omega} - p_N)}$$

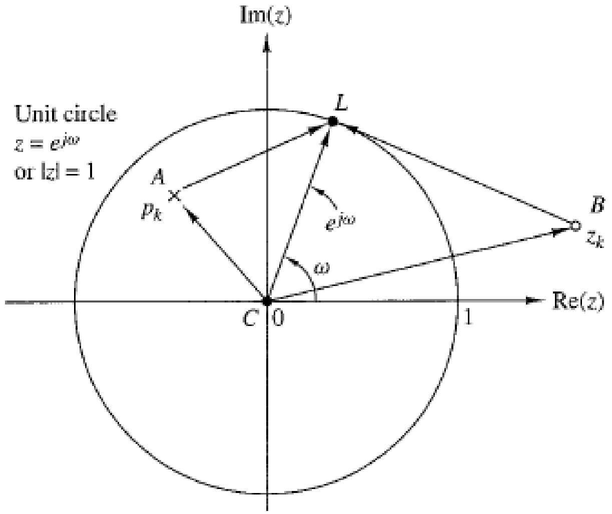
Z-Transform and DTFT

$$H(z) = b_0 z^{N-M} \frac{(z - r_1)(z - r_2) \dots (z - r_M)}{(z - p_1)(z - p_2) \dots (z - p_N)}$$

$$H(z) \overset{z=e^{j\omega}}{\Longrightarrow} H(\omega)$$

$$H(\omega) = b_0 e^{j(N-M)\omega} \frac{(e^{j\omega} - r_1)(e^{j\omega} - r_2) \dots (e^{j\omega} - r_M)}{(e^{j\omega} - p_1)(e^{j\omega} - p_2) \dots (e^{j\omega} - p_N)}$$

- Geometric Interpretation:
  - The factor $(e^{j\omega} - z_k)$ can be interpreted as a vector in the complex z-plane from a zero z_k to the unit circle at $z = e^{j\omega}$.
  - The factor $(e^{j\omega} - p_k)$ can be interpreted as a vector in the complex z-plane from a pole p_k to the unit circle at $z = e^{j\omega}$.

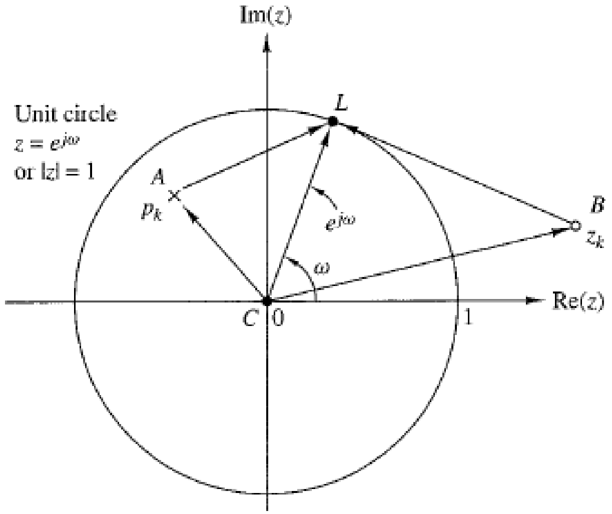


Z-Transform and DTFT

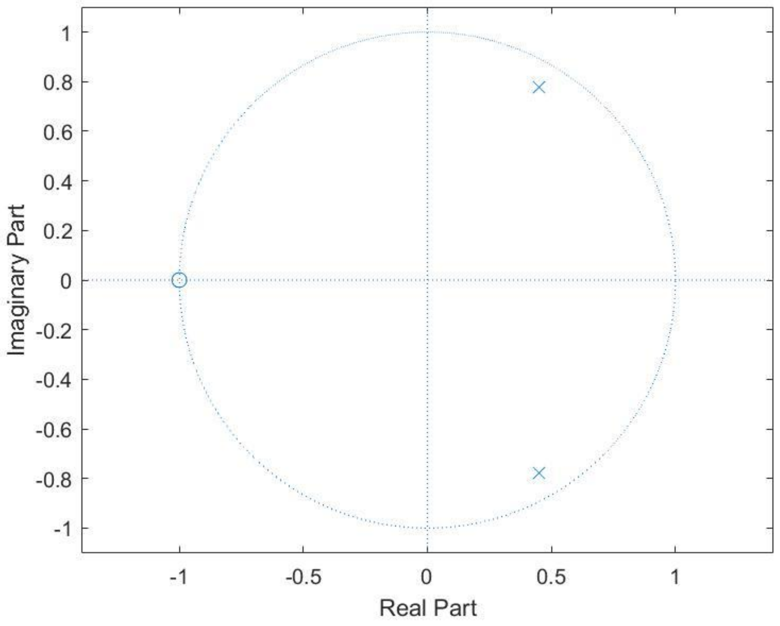
$$H(\omega) = b_0 e^{j(N-M)\omega} \frac{(e^{j\omega} - r_1)(e^{j\omega} - r_2) \dots (e^{j\omega} - r_M)}{(e^{j\omega} - p_1)(e^{j\omega} - p_2) \dots (e^{j\omega} - p_N)}$$

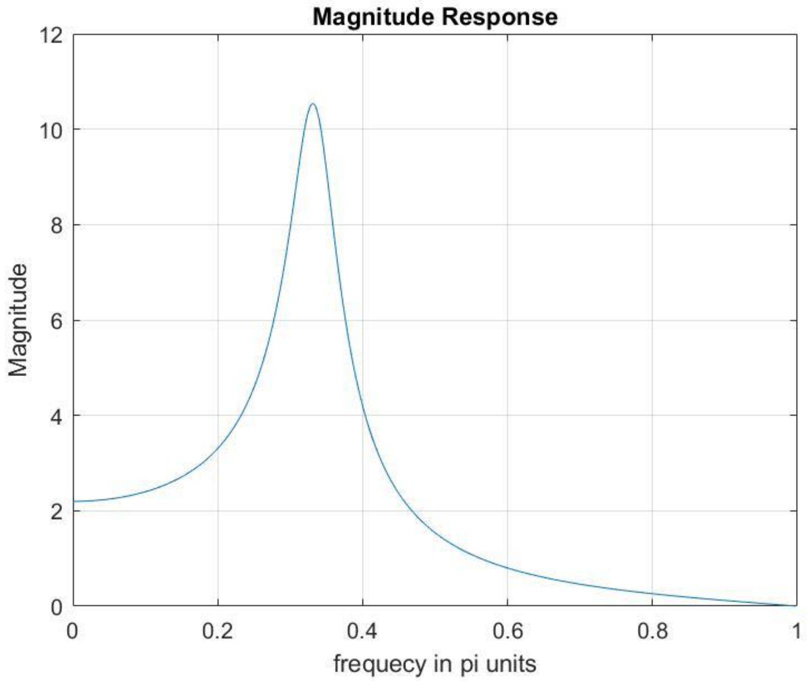
$$|H(\omega)| = b_0 \frac{|e^{j\omega} - r_1||e^{j\omega} - r_2| \dots |e^{j\omega} - r_M|}{|e^{j\omega} - p_1||e^{j\omega} - p_2| \dots |e^{j\omega} - p_N|}$$

$$|H(\omega)| = b_0 \frac{\text{Product of length of vectors from Zeros to the point on Unit Circle for } e^{j\omega}}{\text{Product of length of vectors from Poles to the point on Unit Circle for } e^{j\omega}}$$



System Function vs. Frequency Response

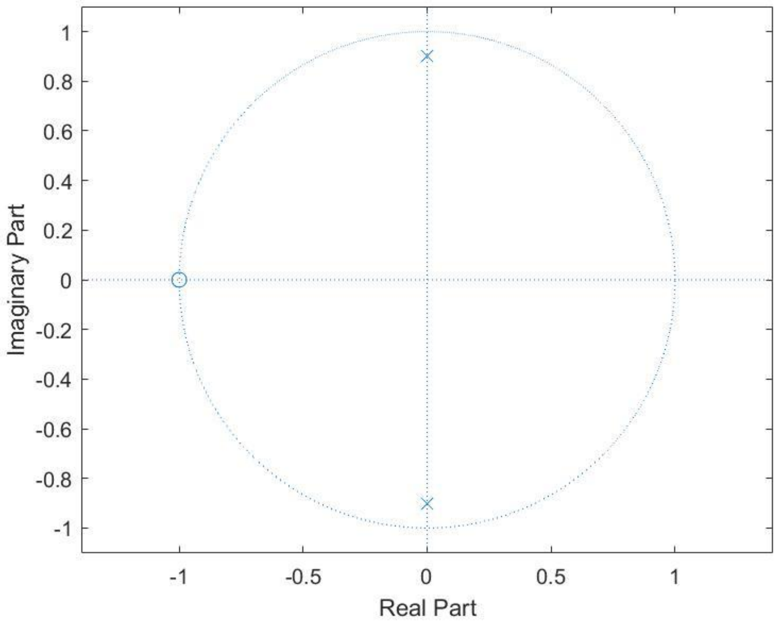


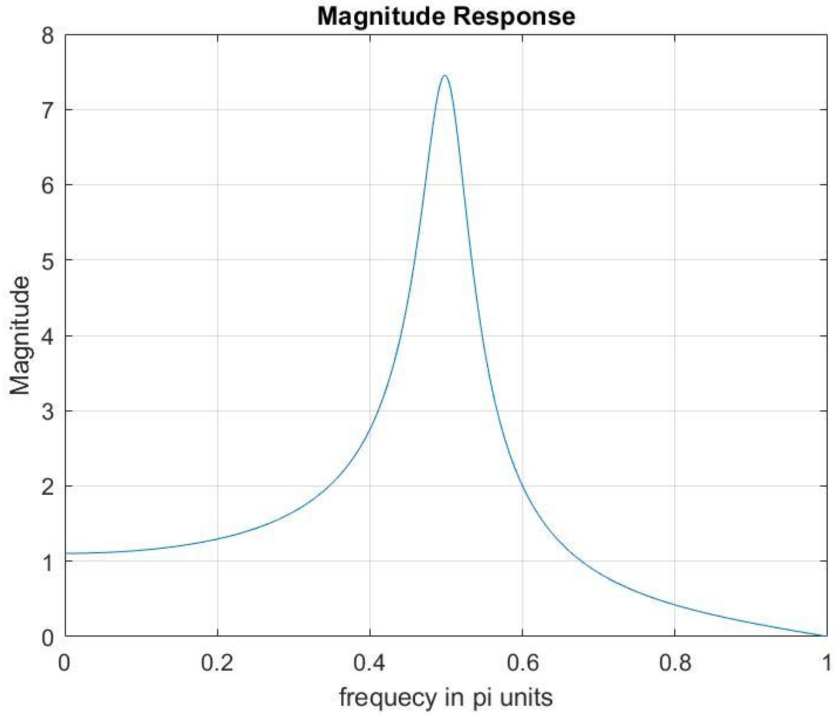


$$|H(\omega)| = b_0 \frac{Product\ of\ length\ of\ vectors\ from\ \boldsymbol{Zeros}\ to\ the\ point\ on\ Unit\ Cicrle\ for\ e^{j\omega}}{Product\ of\ length\ of\ vectors\ from\ \boldsymbol{Poles}\ to\ the\ point\ on\ Unit\ Cicrle\ for\ e^{j\omega}}$$

System Function vs. Frequency Response

Moving the pole locations along the unit circle:





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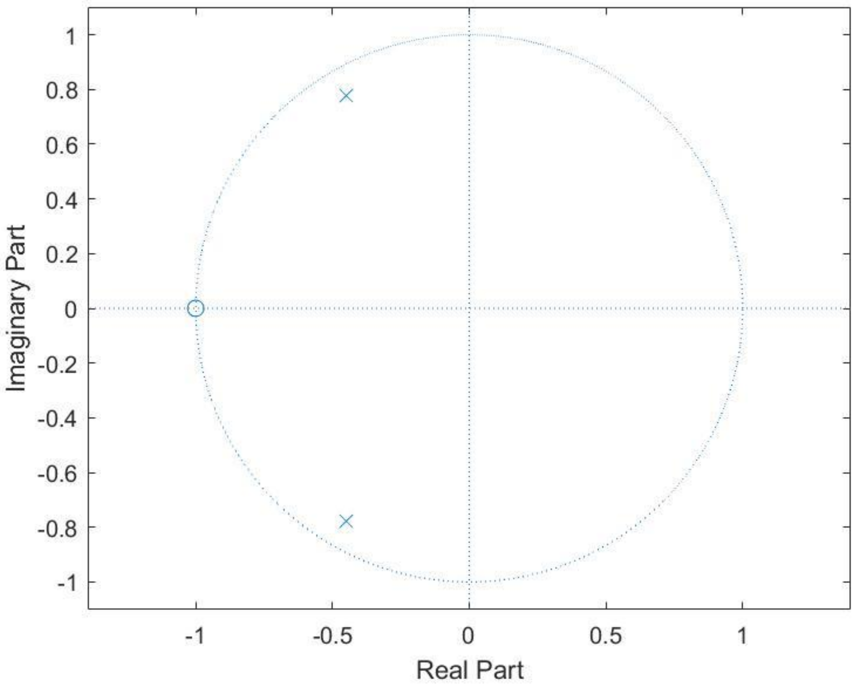

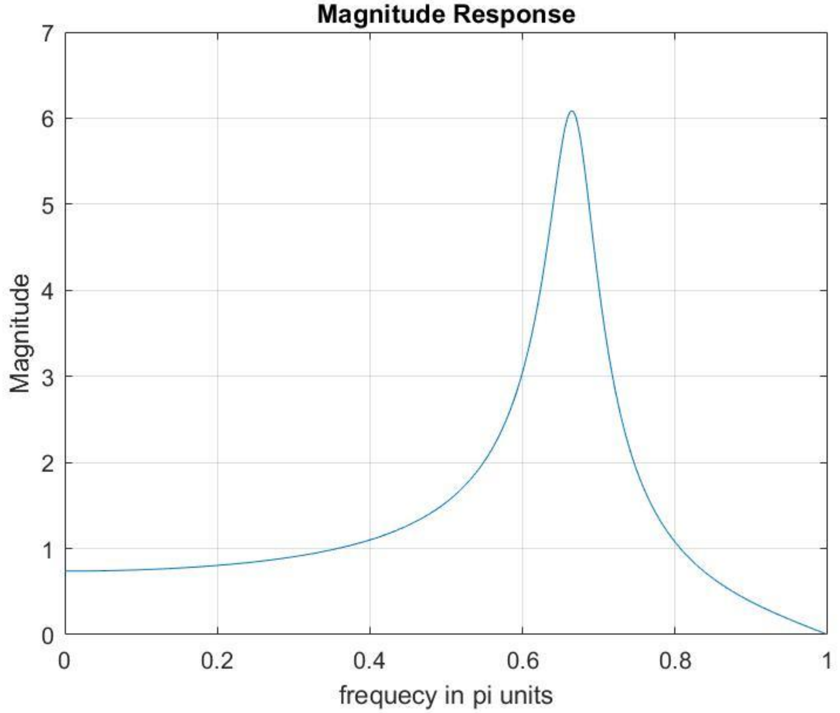

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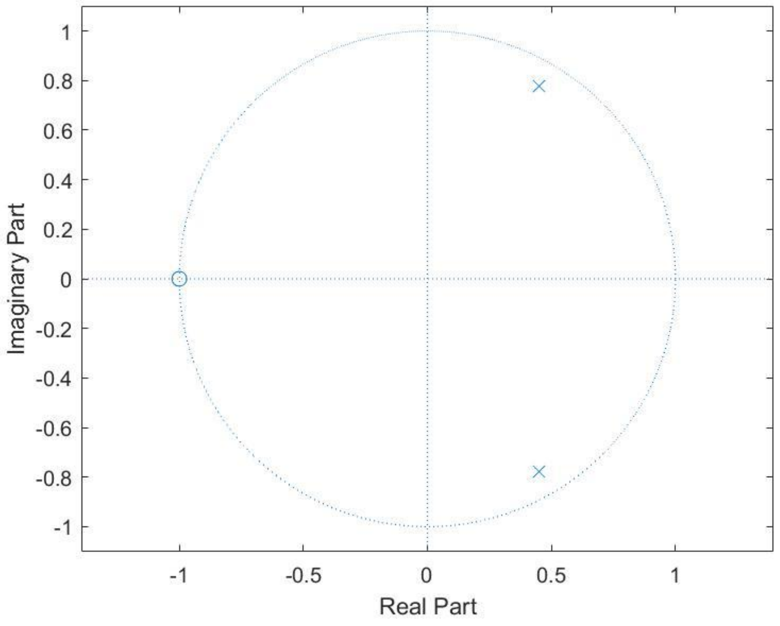

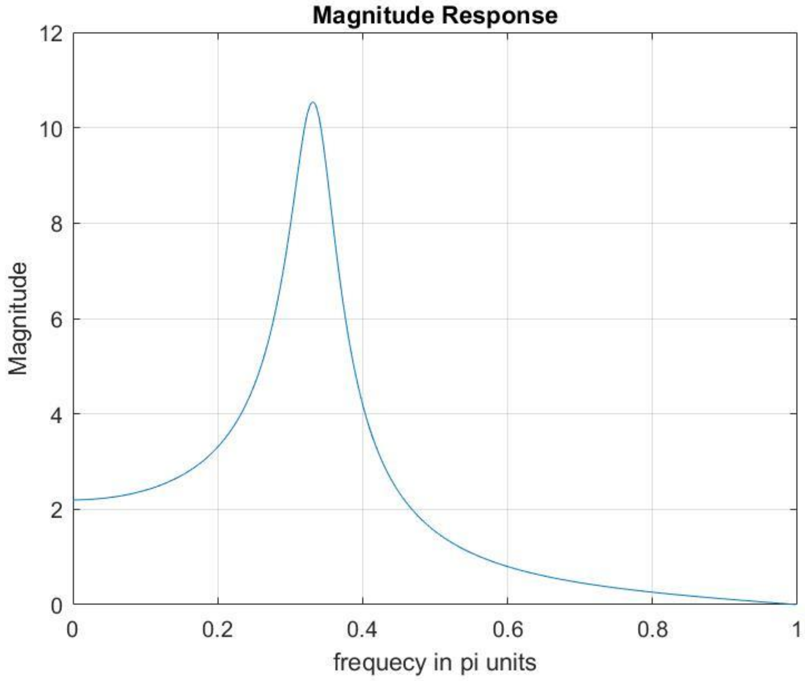

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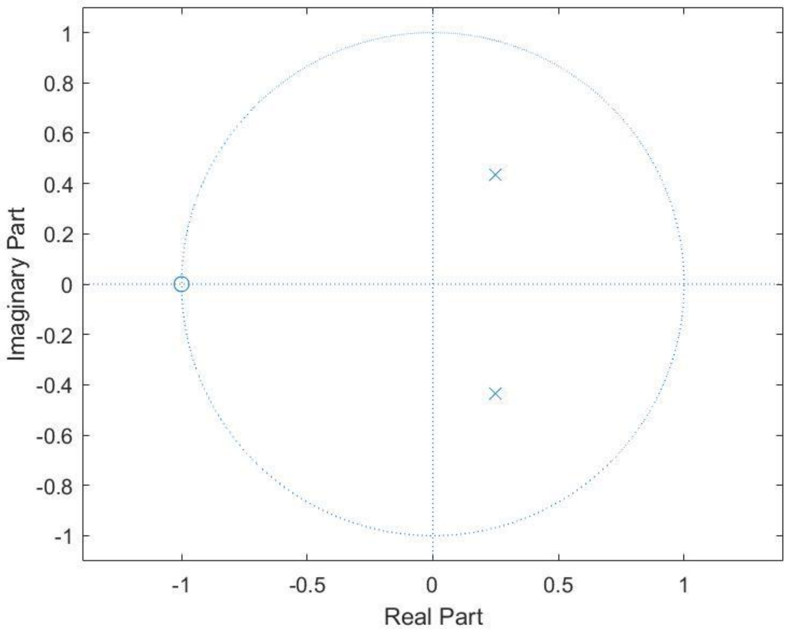
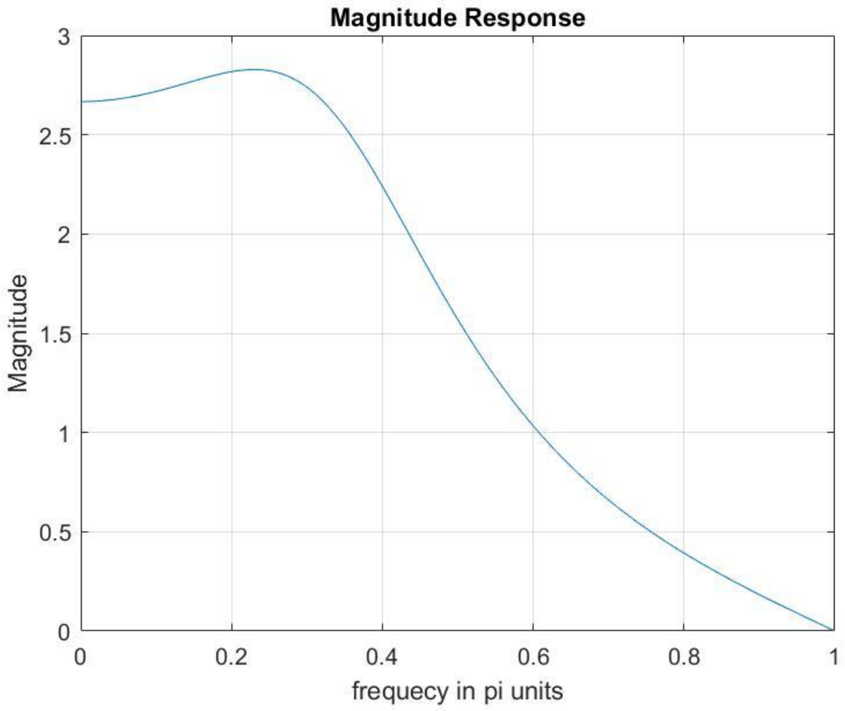
System Function vs. Frequency Response

Moving the pole locations close to (0,0):





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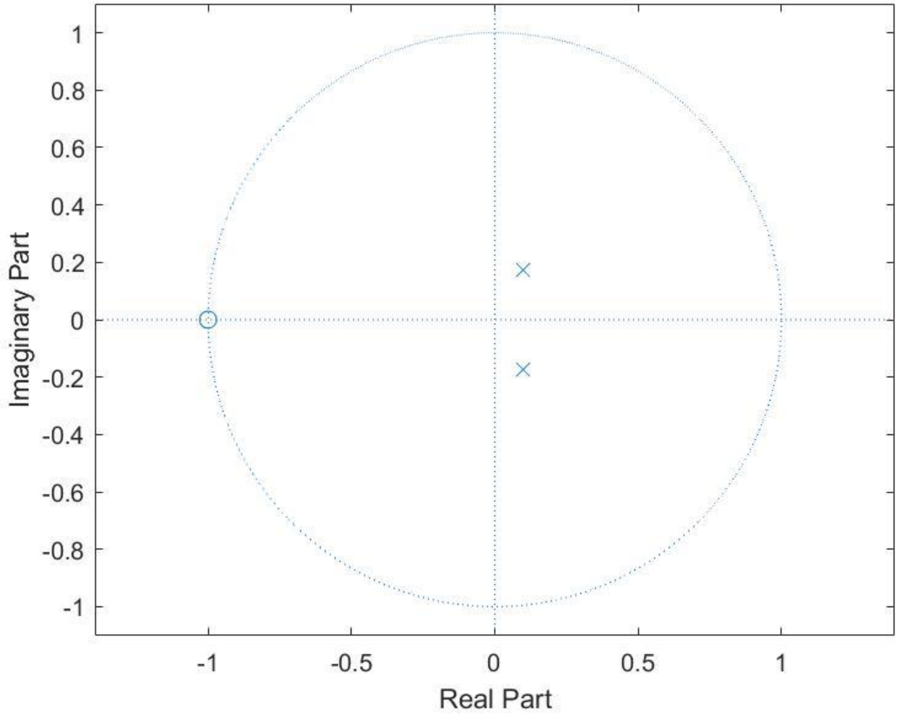

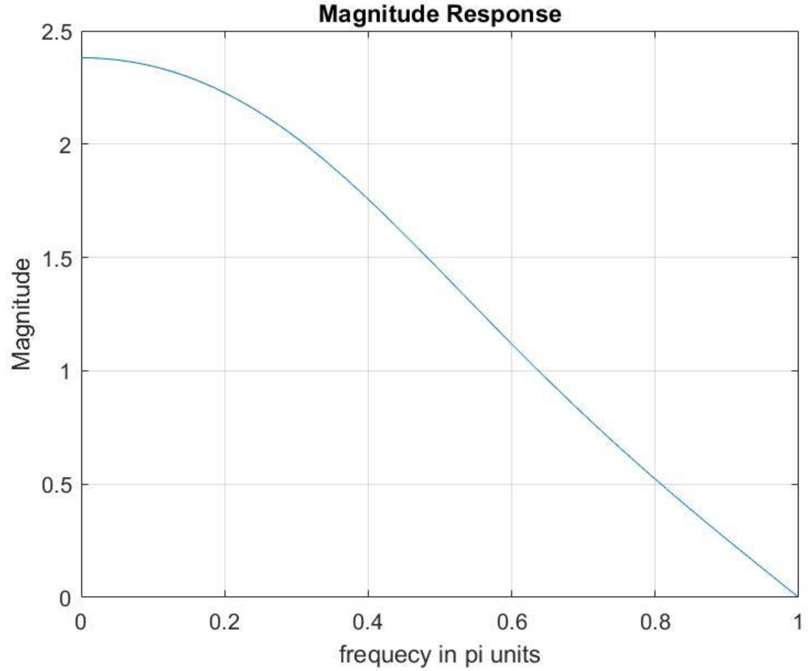

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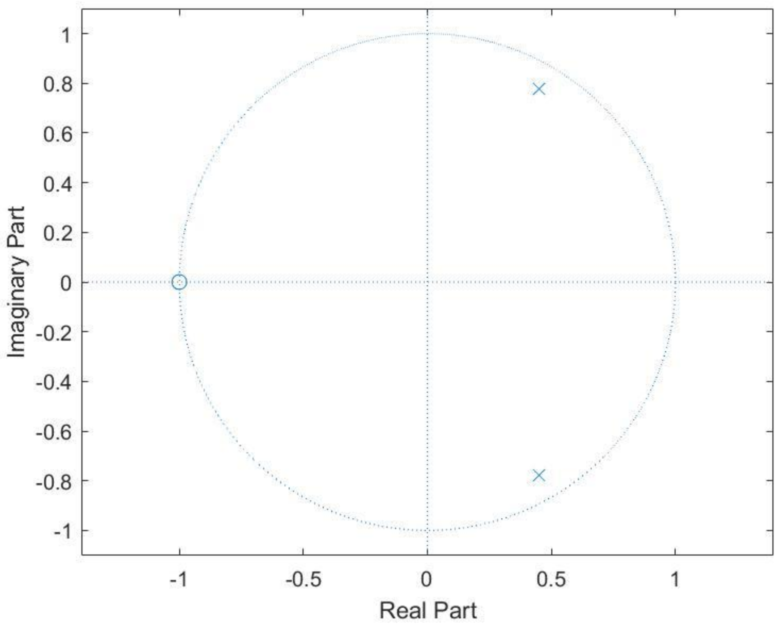
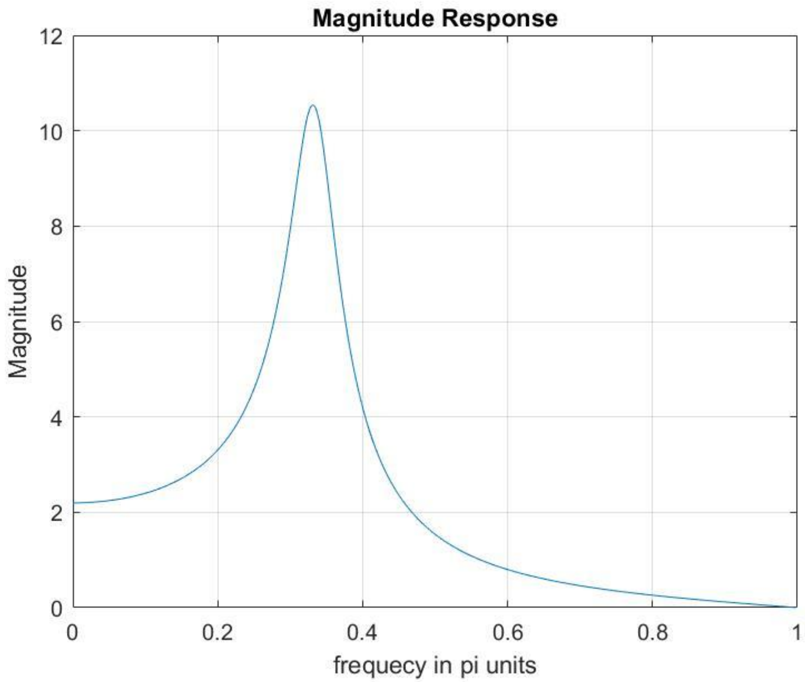
System Function vs. Frequency Response





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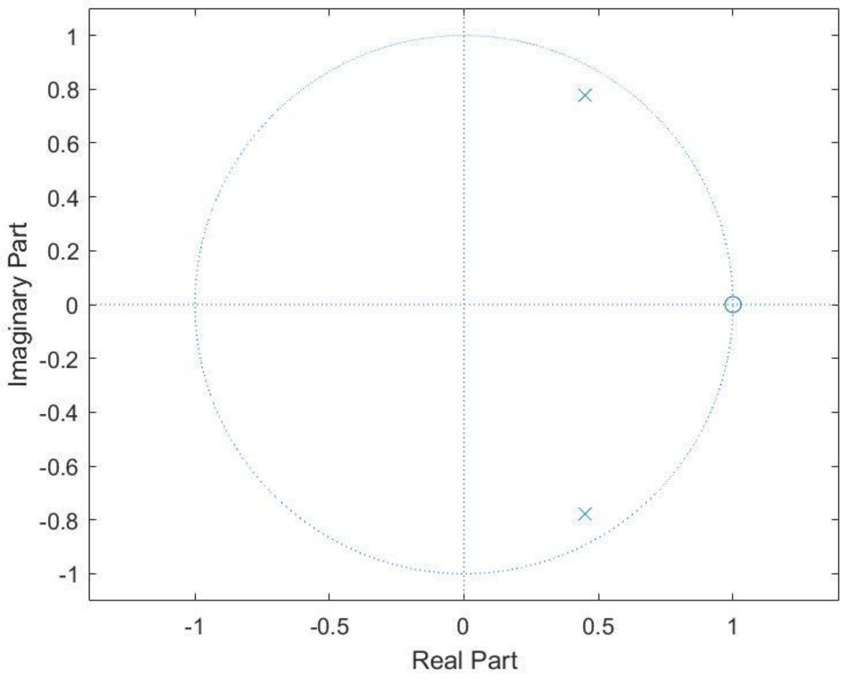
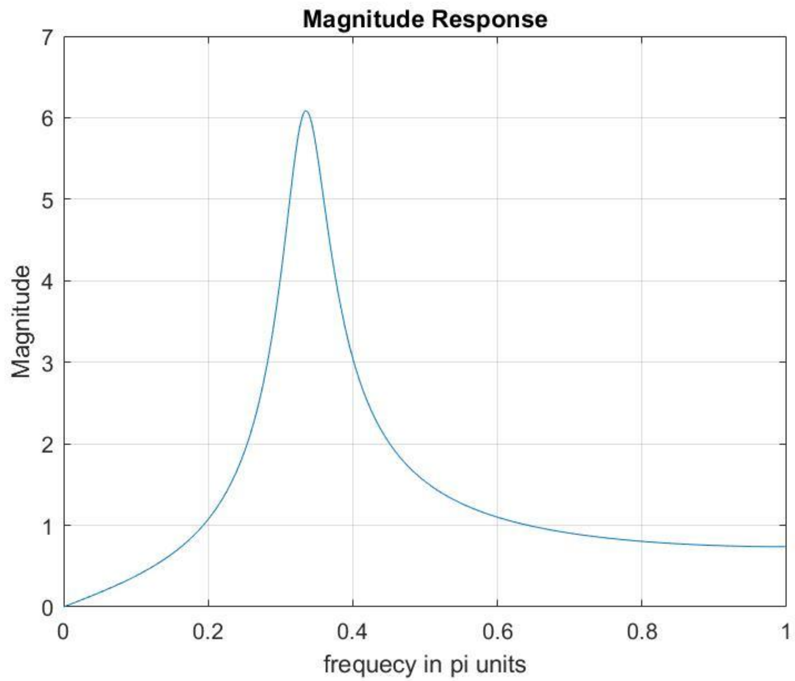
System Function vs. Frequency Response

Moving the zero location:





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Z-Transform and DTFT

$$H(\omega) = \ b_0 e^{j(N-M)\omega} \frac{(e^{j\omega} - r_1)(e^{j\omega} - r_2) \ldots (e^{j\omega} - r_M)}{(e^{j\omega} - p_1)(e^{j\omega} - p_2) \ldots (e^{j\omega} - p_N)}$$

$$\angle H(\omega) = [0 \text{ or } \pi] + [(N-M)\omega] + \sum_{1}^{M} \angle(e^{j\omega} - r_k) - \sum_{1}^{N} \angle(e^{j\omega} - p_k)$$

$$\angle H(\omega) = \text{constant factor} + \text{linear phase factor} + \text{nonlinear phase factor}$$

- **Constant factor:** The phase of the gain term b_0, which is either 0 or π depending on whether b_0 is positive or negative.
- **Non-linear phase factor:**

  Sum of angles from "zero vectors" - sum of angles from "pole vectors"