The Z-Transform

EE 453 / CE 352
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Z-Transform

- Analysis and design of DSP systems require the capability to analyze and design Discrete-Time systems.
- We focus on LTI Discrete-Time systems due to their mathematical tractability.
- Analysis of LTI DT techniques:
  - Time-domain techniques (such as convolution)
  - Transform-domain techniques (such as Fourier Transform)

Z-Transform

- Why transform-domain analysis?
- Analysis simplification.
  - E.g. convolution of two time-domain signals are equivalent to multiplication of their corresponding transforms.
- Z-transform provides us additional means of characterizing an LTI system.

Z-Transform

- Direct z-Transform:

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

- Notation:

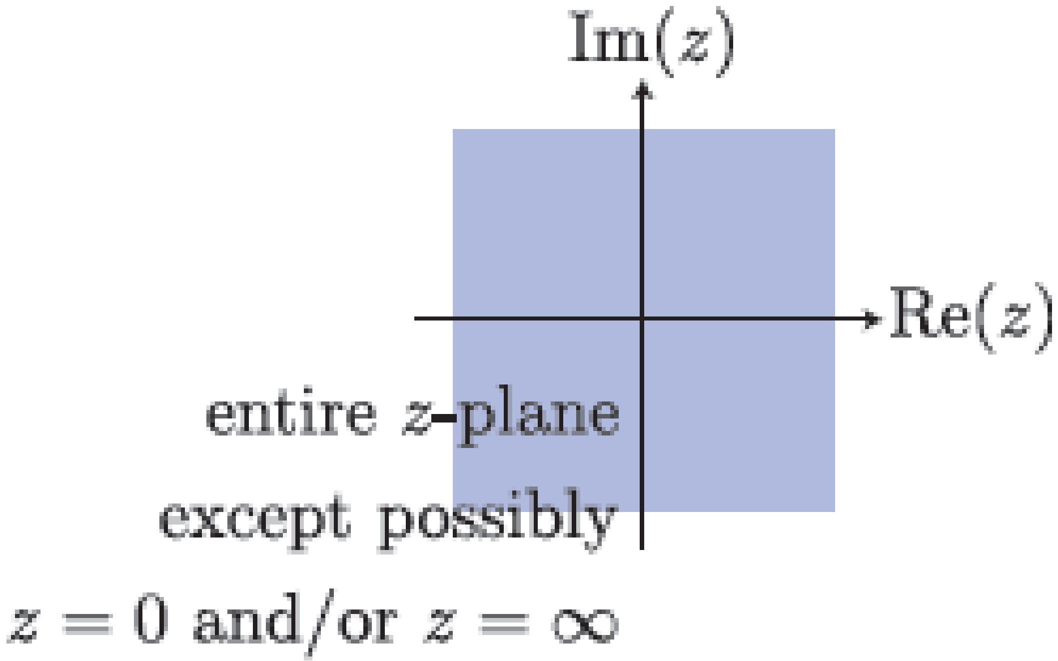
$$X(z) \equiv \mathcal{Z}\{x(n)\}$$

$$x(n) \overset{\mathcal{Z}}{\leftrightarrow} X(z)$$

Region of Convergence

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

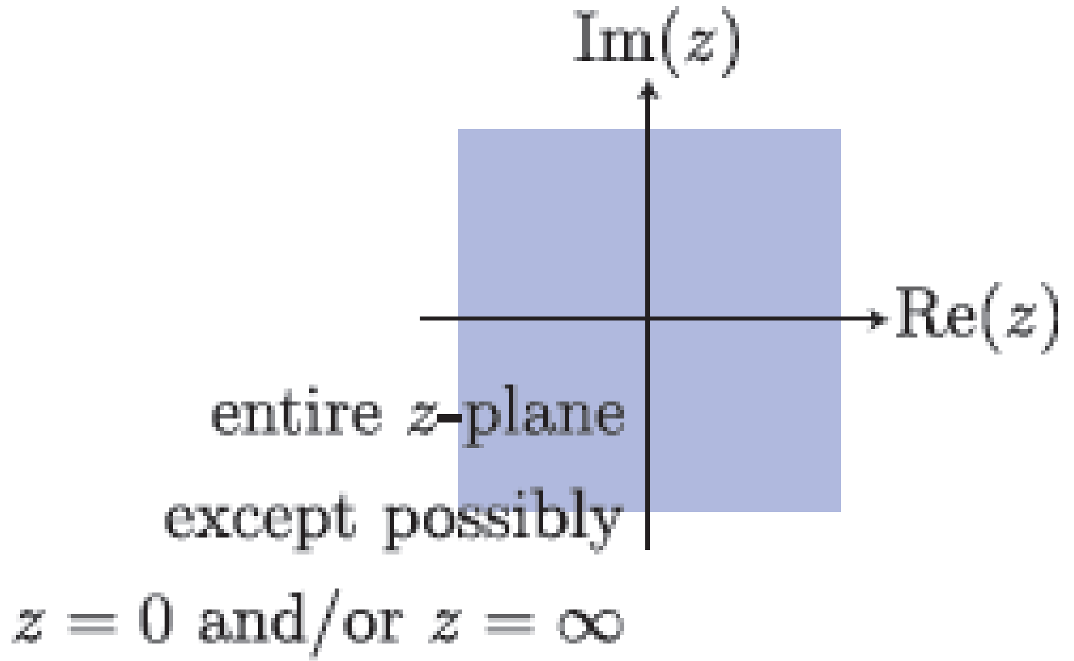
- The ROC of $X(z)$ is the set of all values of z for which $X(z)$ attains a finite value.
- The z-transform is uniquely characterized by the expression of $X(z)$ as well as its ROC.



Region of Convergence

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

- The ROC plays an important role when we're looking at inverting z-transforms and understanding qualities like causality and stability.

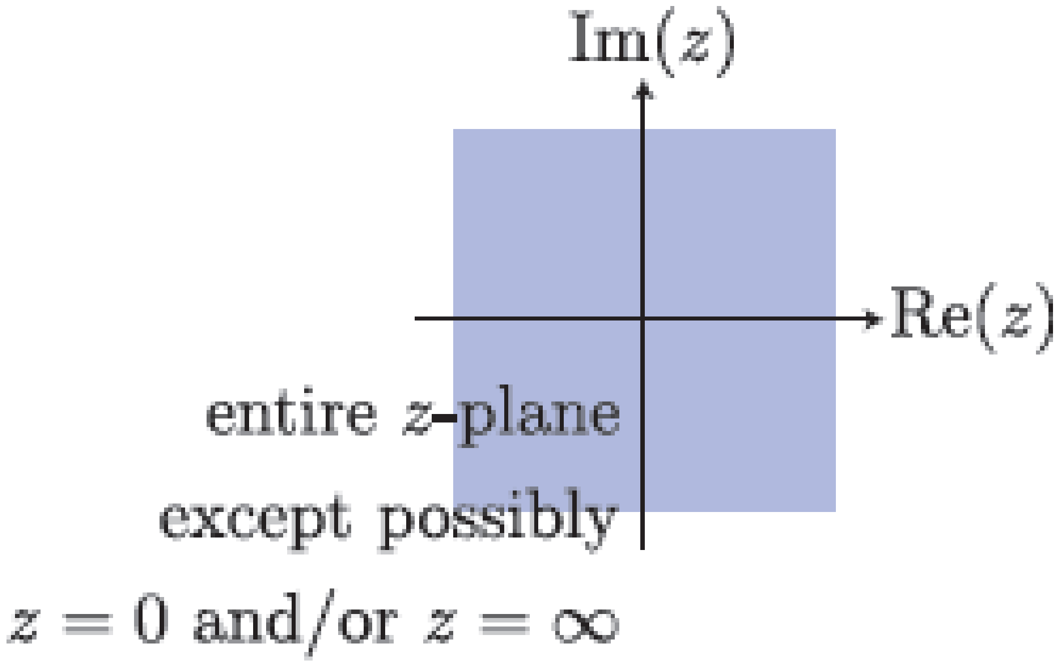


Region of Convergence: Finite-Duration Signals

- Example 1: $x_1(n) = \{1,2,5,7,0,1\}$
$$\uparrow$$

$$X_1(z) = z^2 + 2z + 5 + 7z^{-1} + z^{-3}$$

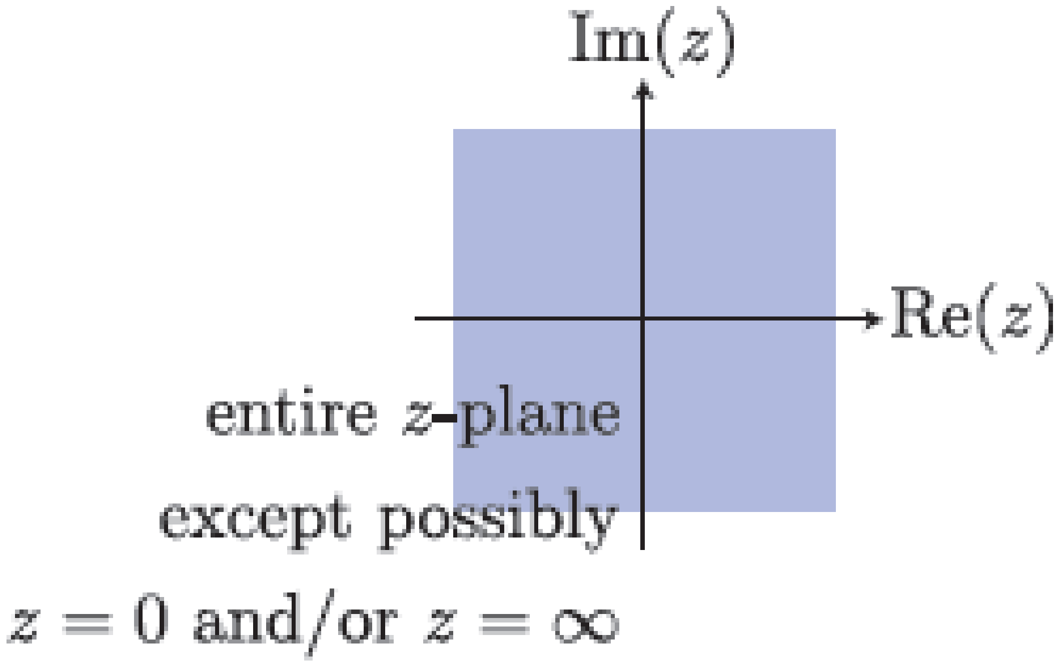
ROC: Entire plane except $z = 0$ and $z = \infty$.



Region of Convergence: Finite-Duration Signals

- Example 2: $x_2(n) = \delta(n-2)$
  - $X_2(z) = z^{-2}$
  - ROC: Entire z-plane except $z = 0$.

- Example 3: $x_3(n) = \delta(n+2)$
  - $X_3(z) = z^2$
  - ROC: Entire z-plane except $z = \infty$.



Region of Convergence: Infinite-Duration Signal

- **Practice:** $x[n] = \alpha^n u[n]$

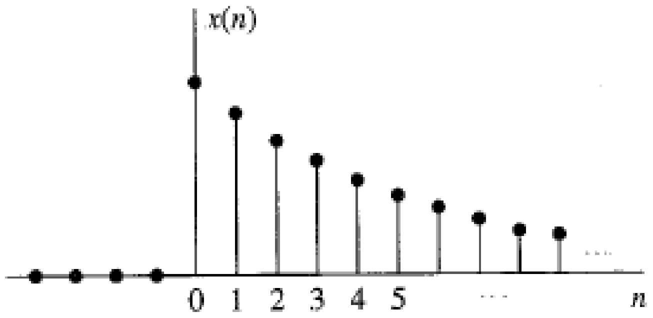
$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

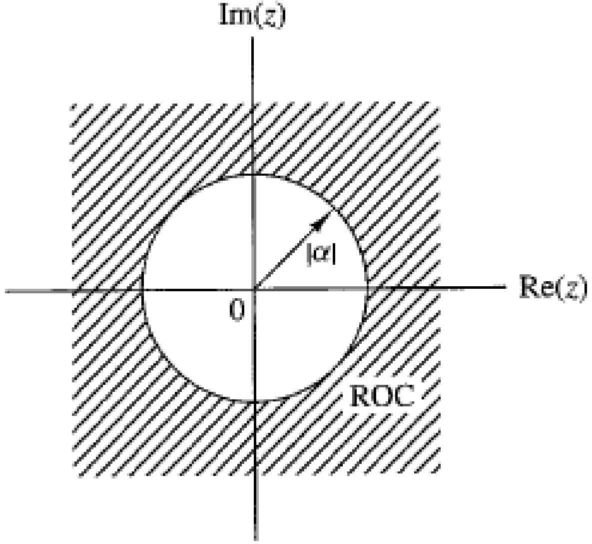
$$X(z) = \sum_{n=0}^{\infty} \alpha^n z^{-n}$$

$$X(z) = \sum_{n=0}^{\infty} (\alpha z^{-1})^n$$

$$X(z) = \frac{1}{1 - \alpha z^{-1}} = \frac{z}{z - \alpha}$$

ROC: $|\alpha z^{-1}| < 1 \rightarrow |z| > |\alpha|$





Region of Convergence: Infinite-Duration Signal

- **Example:** $x[n] = -\alpha^n u[-n-1]$

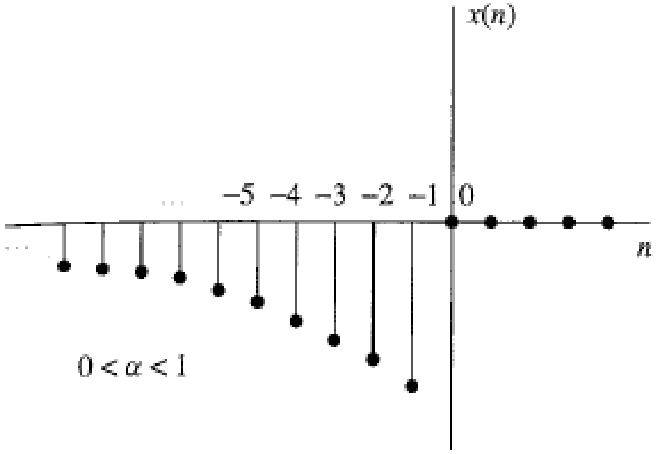
$$X(z) = \sum_{n=-\infty}^{-1} (-\alpha^n) z^{-n}$$

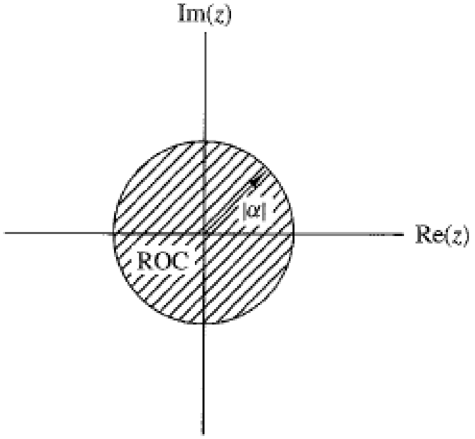
Let $t = -n$:

$$X(z) = -\sum_{t=1}^{\infty} (\alpha^{-1} z)^t$$

$$X(z) = -\frac{\alpha^{-1} z}{1 - \alpha^{-1} z} = \frac{z}{z - \alpha}$$

ROC: $|\alpha^{-1} z| < 1 \rightarrow |z| < |\alpha|$





Region of Convergence: Infinite-Duration Signal

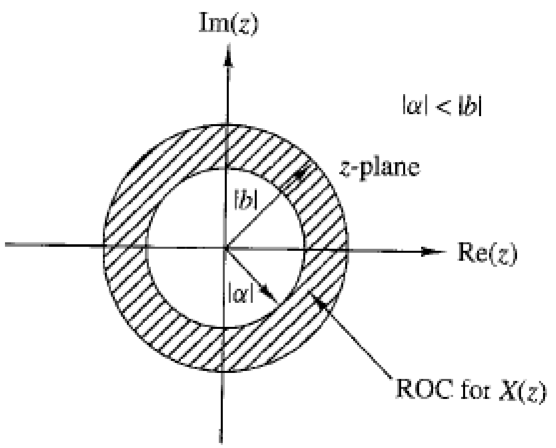
- **Practice:** $g[n] = \left(\frac{1}{4}\right)^n u[n] - \left(\frac{1}{2}\right)^n u[-n-1]$

$$\left(\frac{1}{4}\right)^n u[n] \overset{\mathcal{Z}}{\leftrightarrow} \frac{z}{z-1/4}, |z| > 1/4$$

$$-\left(\frac{1}{2}\right)^n u[-n-1] \overset{\mathcal{Z}}{\leftrightarrow} \frac{z}{z-1/2}, |z| < 1/2$$

$$G(z) = \frac{z}{z-1/4} + \frac{z}{z-1/2}$$

ROC: $1/4 < |z| < 1/2$



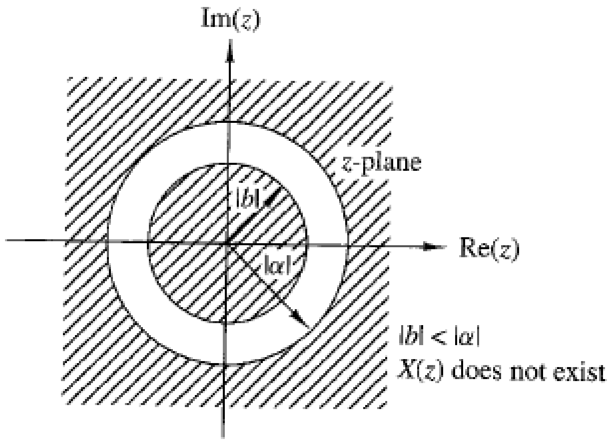
(b)

Region of Convergence: Infinite-Duration Signal

- **Practice:** $x[n] = u[n] + \left(-\frac{3}{4}\right)^n u[-n]$

$$|z| < 3/4\,, \qquad |z| > 1$$

$X(z)$ does not exist!



(a)

Region of Convergence: Infinite-Duration Signal

- For a power series:

$$f(z) = \sum_{n=0}^{\infty} a_n (z-c)^n = a_0 + a_1(z-c) + a_2(z-c)^2 + \cdots$$

- There exists a number $0 \le r \le \infty$ such that the series:
  1. Converges for $|z-c| < r$
  2. Diverges for $|z-c| > r$
  3. May or may not converge for values on $|z-c| = r$

- For a power series:

$$f(z) = \sum_{n=0}^{\infty} a_n (z-c)^{-n} = a_0 + \frac{a_1}{(z-c)} + \frac{a_2}{(z-c)^2} + \cdots$$

- There exists a number $0 \le r \le \infty$ such that the series:
  1. Converges for $|z-c| > r$
  2. Diverges for $|z-c| < r$
  3. May or may not converge for values on $|z-c| = r$

In short, the convergence of a power series will only occur if it is absolutely summable.

Region of Convergence: Infinite-Duration Signal

For a power series,

$$f(z) = \sum_{n=0}^{\infty} a_n (z-c)^n = a_0 + a_1(z-c) + a_2(z-c)^2 + \cdots$$

there exists a number $0 \leq r \leq \infty$ such that the series

- convergences for $|z-c| < r$, and
- diverges for $|z-c| > r$
- may or may not converge for values on $|z-c| = r$.

For a power series,

$$f(z) = \sum_{n=0}^{\infty} a_n (z-c)^{-n} = a_0 + \frac{a_1}{(z-c)} + \frac{a_2}{(z-c)^2} + \cdots$$

there exists a number $0 \leq r \leq \infty$ such that the series

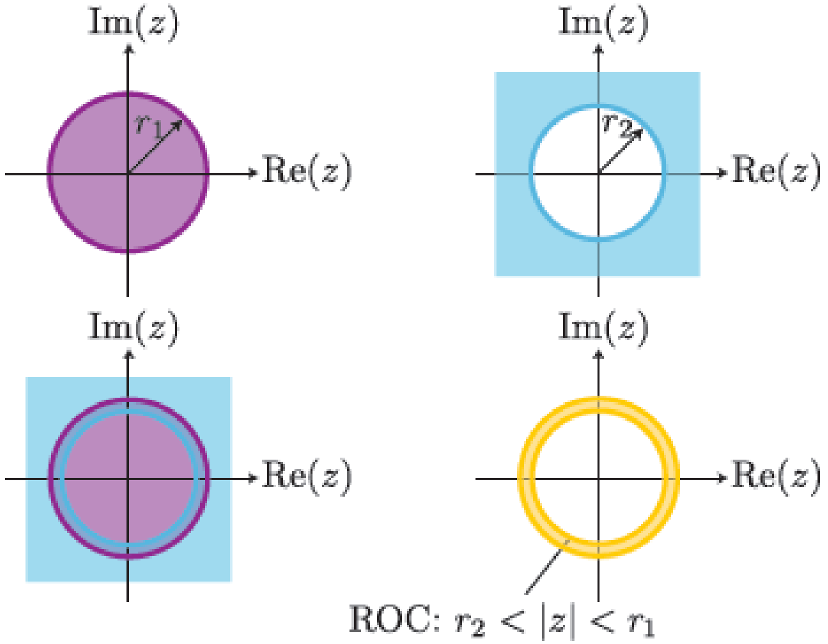
- convergences for $|z-c| > r$, and
- diverges for $|z-c| < r$
- may or may not converge for values on $|z-c| = r$.

$$\begin{aligned}
X(z) &= \sum_{n=-\infty}^{\infty} x(n) z^{-n} \\
&= \sum_{n=-\infty}^{-1} x(n) z^{-n} + \sum_{n=0}^{\infty} x(n) z^{-n} \\
&= \underbrace{\sum_{n'=0}^{\infty} x(-n') z^{n'}}_{\text{ROC: } |z| < r_1} + \underbrace{\sum_{n=0}^{\infty} x(n) z^{-n}}_{\text{ROC: } |z| > r_2} - \underbrace{x(0)}_{\text{ROC: all } z}
\end{aligned}$$

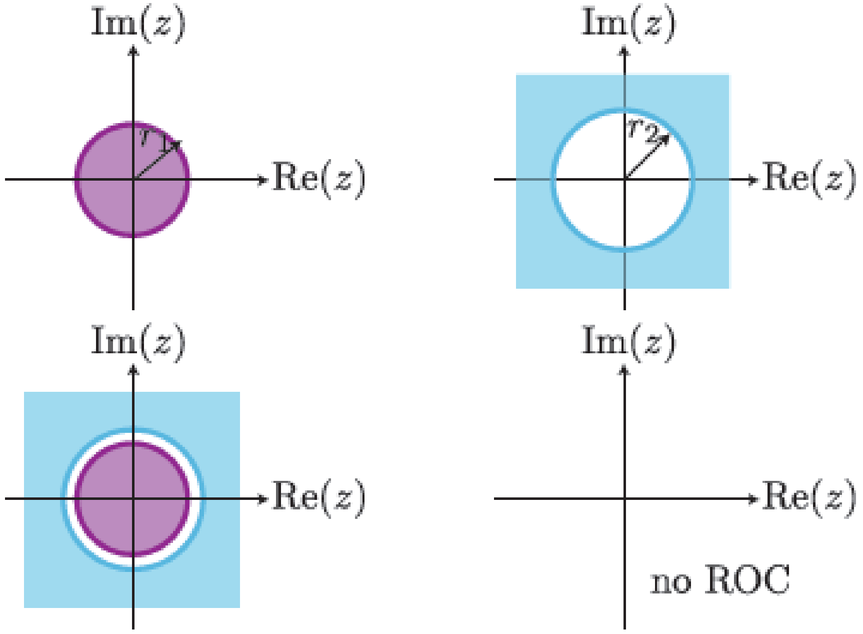
Region of Convergence: Infinite-Duration Signal

$$\underbrace{\sum_{n'=0}^{\infty} x(-n')z^{n'}}_{\text{ROC: } |z| < r_1} + \underbrace{\sum_{n=0}^{\infty} x(n)z^{-n}}_{\text{ROC: } |z| > r_2} - \underbrace{x(0)}_{\text{ROC: all } z}$$

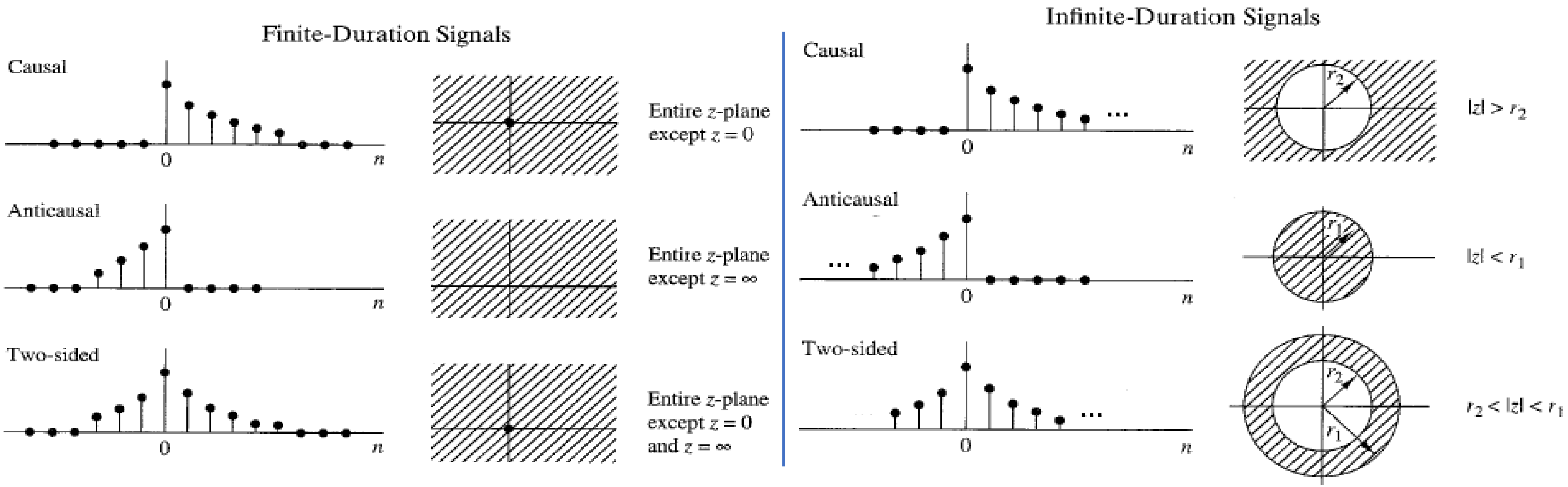
$r_1 > r_2$



$r_1 < r_2$



Region of Convergence: Summary



z-Transform: Properties

Property	Time Domain	z-Domain	ROC
Notation:	$x(n)$	$X(z)$	ROC: $r_2 < \|z\| < r_1$
	$x_1(n)$	$X_1(z)$	ROC_1
	$x_2(n)$	$X_1(z)$	ROC_2
Linearity:	$a_1 x_1(n) + a_2 x_2(n)$	$a_1 X_1(z) + a_2 X_2(z)$	At least $\mathrm{ROC}_1 \cap \mathrm{ROC}_2$
Time shifting:	$x(n-k)$	$z^{-k} X(z)$	ROC, except $z = 0$ (if $k > 0$) and $z = \infty$ (if $k < 0$)
z-Scaling:	$a^n x(n)$	$X(a^{-1} z)$	$\|a\| r_2 < \|z\| < \|a\| r_1$
Time reversal	$x(-n)$	$X(z^{-1})$	$\frac{1}{r_1} < \|z\| < \frac{1}{r_2}$
Conjugation:	$x^*(n)$	$X^*(z^*)$	ROC
z-Differentiation:	$n\,x(n)$	$-z\frac{dX(z)}{dz}$	$r_2 < \|z\| < r_1$
Convolution:	$x_1(n) * x_2(n)$	$X_1(z) X_2(z)$	At least $\mathrm{ROC}_1 \cap \mathrm{ROC}_2$

among others . . .

Linearity Property

$$x_1(n) \xleftrightarrow{z} X_1(z), \mathrm{ROC}_1$$
$$x_2(n) \xleftrightarrow{z} X_2(z), \mathrm{ROC}_2$$

$$a_1x_1(n) + a_2x_2(n) \xleftrightarrow{z} a_1X_1(z) + a_2X_2(z),$$
$$\text{At least } \mathrm{ROC}_1 \cap \mathrm{ROC}_2$$

Time Shifting Property

$$x(n) \stackrel{z}{\longleftrightarrow} X(z), \qquad \text{ROC}$$

$$x(n-k) \stackrel{z}{\longleftrightarrow} z^{-k}X(z), \qquad \text{At least ROC}$$

$$\text{except } z = 0 \ (k > 0) \text{ or}$$

$$z = \infty \ (k < 0)$$

Time Shifting Property: ROC

$$x(n) = \delta(n) \quad \stackrel{z}{\longleftrightarrow} \quad X(z) = 1, \text{ROC: } \underline{\text{entire } z\text{-plane}}$$

- Example: For $k = -1$

$$y(n) = x(n - (-1)) = x(n+1) = \delta(n+1)$$

$$y(n) = \delta(n+1) \quad \stackrel{z}{\longleftrightarrow} \quad Y(z) = z, \text{ROC: } \underline{\text{entire } z\text{-plane}}$$

except $z = \infty$

- Example: For $k = 1$

$$y(n) = x(n - (1)) = x(n-1) = \delta(n-1)$$

$$y(n) = \delta(n-1) \quad \stackrel{z}{\longleftrightarrow} \quad Y(z) = z^{-1}, \text{ROC: } \underline{\text{entire } z\text{-plane}}$$

except $z = 0$

Scaling in the z-Domain

$$x(n) \overset{z}{\longleftrightarrow} X(z), \qquad \text{ROC: } r_1 < |z| < r_2$$

$$a^n x(n) \overset{z}{\longleftrightarrow} X(a^{-1}z), \qquad \text{ROC: } |a|r_1 < |z| < |a|r_2$$

Convolution Property

$$x(n) = x_1(n) * x_2(n) \iff X(z) = X_1(z) \cdot X_2(z)$$

1. Compute z-Transform of each of the signals to convolve (time domain $\rightarrow$ z-domain):

$$\begin{aligned} X_1(z) &= \mathcal{Z}\{x_1(n)\} \\ X_2(z) &= \mathcal{Z}\{x_2(n)\} \end{aligned}$$

2. Multiply the two z-Transforms (in z-domain):

$$X(z) = X_1(z)X_2(z)$$

3. Find the inverse z-Transformof the product (z-domain $\rightarrow$ time domain):

$$x(n) = \mathcal{Z}^{-1}\{X(z)\}$$

Rational z-Transforms

- $X(z)$ is a rational function if it can be represented as the ratio of two polynomials in z^{-1} (or z):

$$X(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_M z^{-M}}{a_0 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_N z^{-N}}$$

- For LTI systems represented by LCCDEs, the z-transform of the unit sample response $h(n)$, denoted by $H(z) = \mathcal{Z}\{h(n)\}$, is rational.

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

Rational z-Transforms

- Many signals of practical interest have a rational z-Transform.
- For LTI systems represented by LCCDEs, the z-transform of the unit sample response $h(n)$, denoted by $H(z) = \mathcal{Z}\{h(n)\}$, is rational.

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

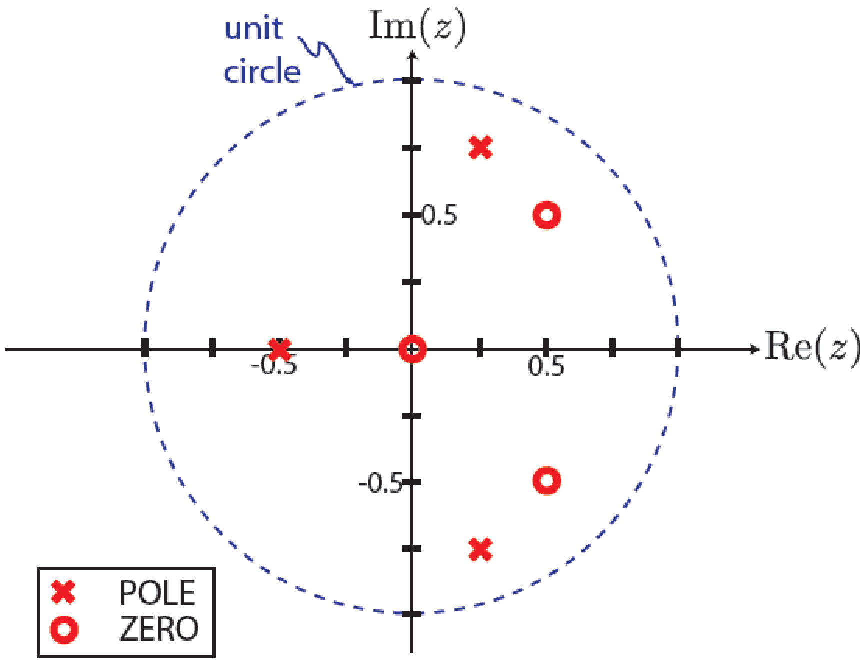
Rational z-Transforms: Poles and Zeros

- Zeroes of $X(z)$:

  Values of z for which $X(z) = 0$.

- Poles of $X(z)$:

  Values of z for which $X(z) = \infty$.



Rational z-Transforms: Poles and Zeros

- Let $a_0, b_0 \neq 0$:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_M z^{-M}}{a_0 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_N z^{-N}}$$

$$H(z) = \left(\frac{b_0 z^{-M}}{a_0 z^{-N}}\right) \frac{z^M + (b_1/b_0) z^{M-1} + \cdots + b_M/b_0}{z^N + (a_1/a_0) z^{N-1} + \cdots + a_N/b_0}$$

$$= \left(\frac{b_0}{a_0} z^{-M+N}\right) \frac{(z - r_1)(z - r_2) \ldots (z - r_M)}{(z - p_1)(z - p_2) \ldots (z - p_N)}$$

$$= G z^{N-M} \frac{\prod_{k=1}^{M} (z - r_k)}{\prod_{k=1}^{N} (z - p_k)}$$

Poles and Zeros of Rational z-Transform

$$H(z) = Gz^{N-M}\frac{\prod_{k=1}^{M}(z - r_k)}{\prod_{k=1}^{N}(z - p_k)}, \text{where } G \equiv \frac{b_o}{a_o}$$

- $X(z)$ has M finite zeroes at $r = r_1, r_2, \dots, r_M$
- $X(z)$ has N finite poles at $p = p_1, p_2, \dots, p_M$

- **Poles and zeroes at $z = 0$:**
  - If $N > M$, there are $|N - M|$ zeroes at $z = 0$.
  - If $N < M$, there are $|N - M|$ poles at $z = 0$.
- **Poles and zeroes at $z = \infty$:**
  - A zero exists at $z = \infty$ if $X(\infty) = 0$.
  - A pole exists at $z = \infty$ if $X(\infty) = \infty$.

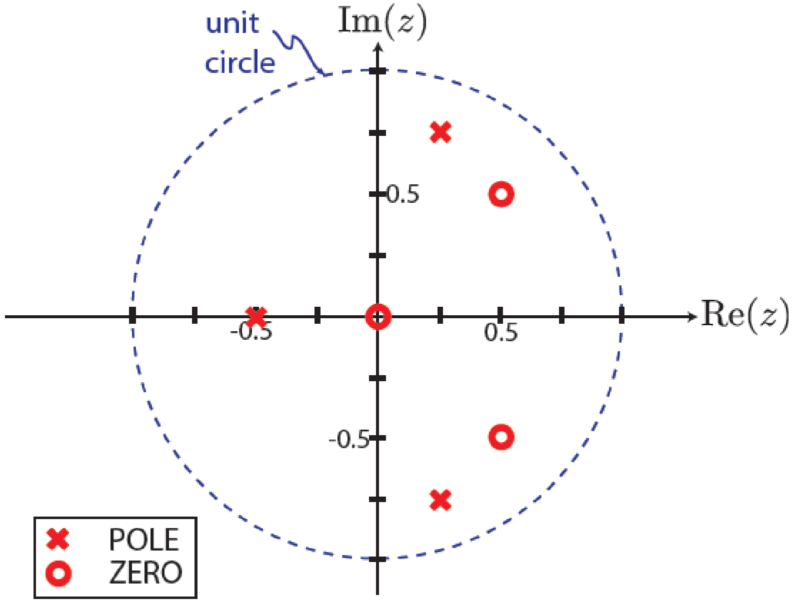
If we count the poles and zeros at zero and infinity:
total number of zeroes = total number of poles

Rational z-Transforms: Poles and Zeros

- **Practice:** $X(z) = z\frac{2z^2-2z+1}{16z^3+6z+5}$

$$X(z) = (z-0)\frac{\left(z-\left(\frac{1}{2}+j\frac{1}{2}\right)\right)\left(z-\left(\frac{1}{2}-j\frac{1}{2}\right)\right)}{\left(z-\left(\frac{1}{4}+j\frac{3}{4}\right)\right)\left(z-\left(\frac{1}{4}+j\frac{3}{4}\right)\right)\left(z-\left(-\frac{1}{2}\right)\right)}$$

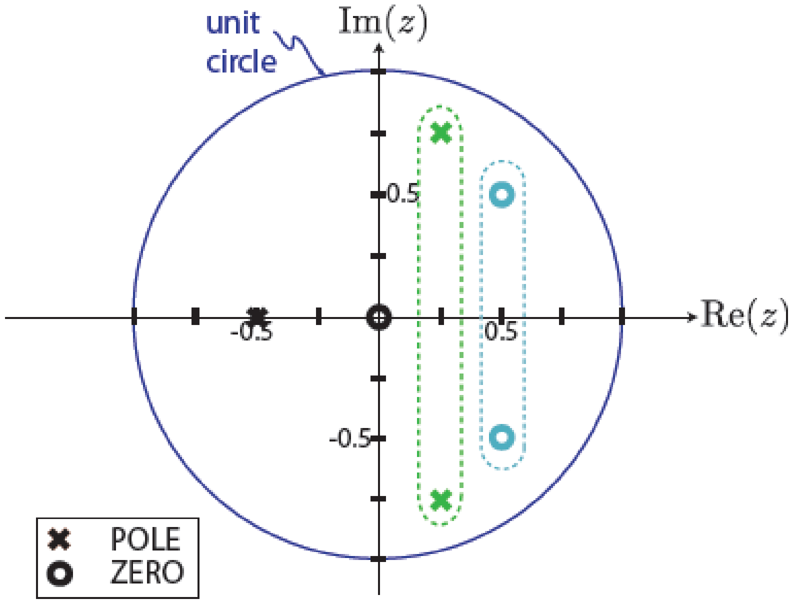
  - Zeroes: $0, \frac{1}{2} \pm j\frac{1}{2}$
  - Poles: $\frac{1}{4} \pm j\frac{3}{4}, -\frac{1}{2}$



Pole-Zero Plot and Conjugate Pairs

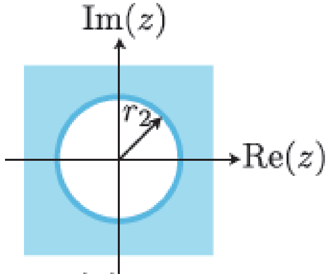
$$X(z) = z\,\frac{2z^2-2z+1}{16z^3+6z+5}$$

- For real time-domain signals, the coefficients of $X(z)$ are necessarily real.
- Complex poles and zeros must occur in conjugate pairs.
  - Real poles and zeros do not have to be paired up.



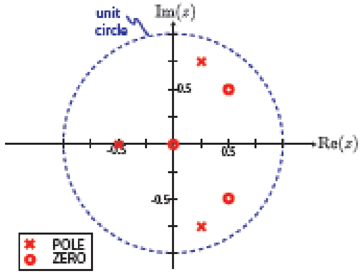
Pole-Zero Plot, Causality and Stability

- Recall, for causal signals, the ROC will be the outer region of a disk

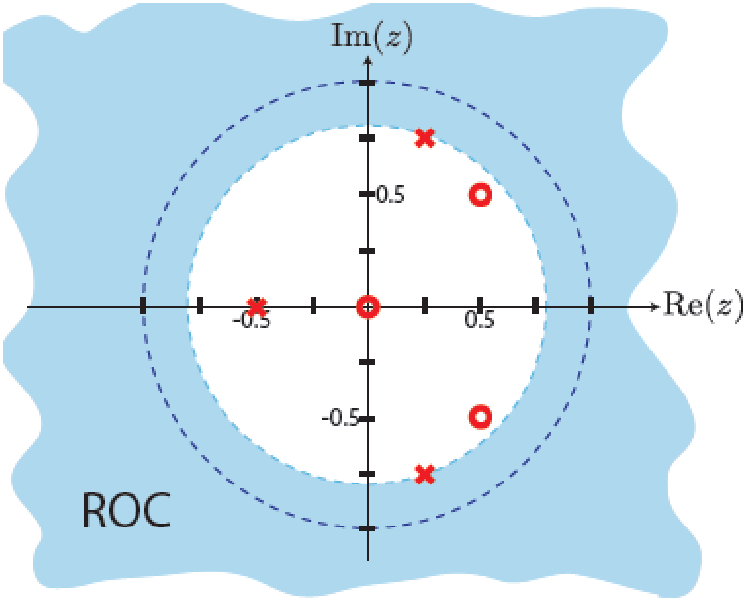


Causal: $x(n) = 0 \;\; n < 0$

- ROC cannot necessarily include poles ($\because X(p_k) = \infty$)

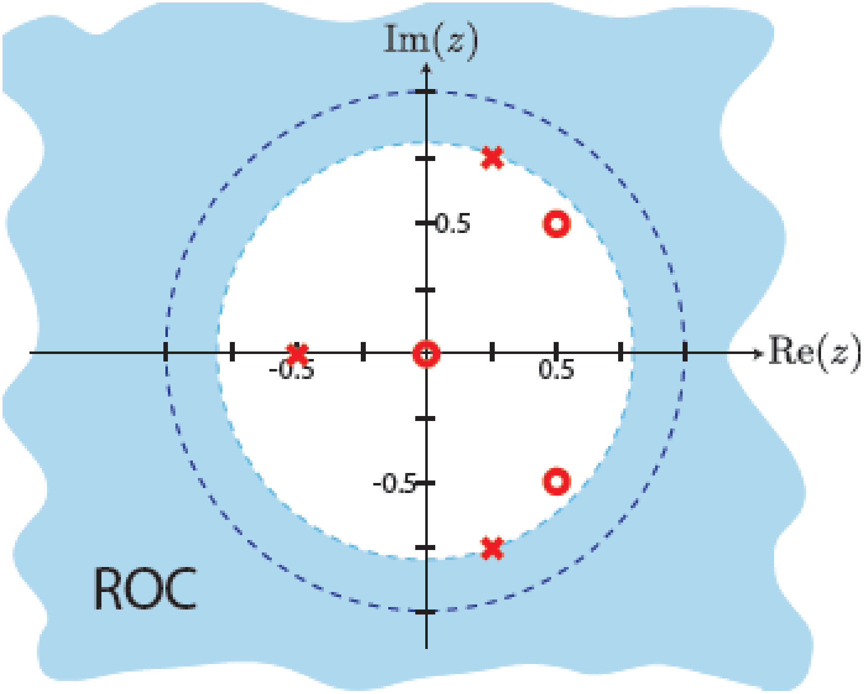


- Therefore, for a causal signal the ROC is the smallest (origin-centered) circle encompassing all the poles.



Pole-Zero Plot, Causality and Stability

- For stable systems, the ROC will include the unit circle.



- For stability of a causal system, the poles must lie inside the unit circle.

Poles/Zeroes and Time Behavior of Signals

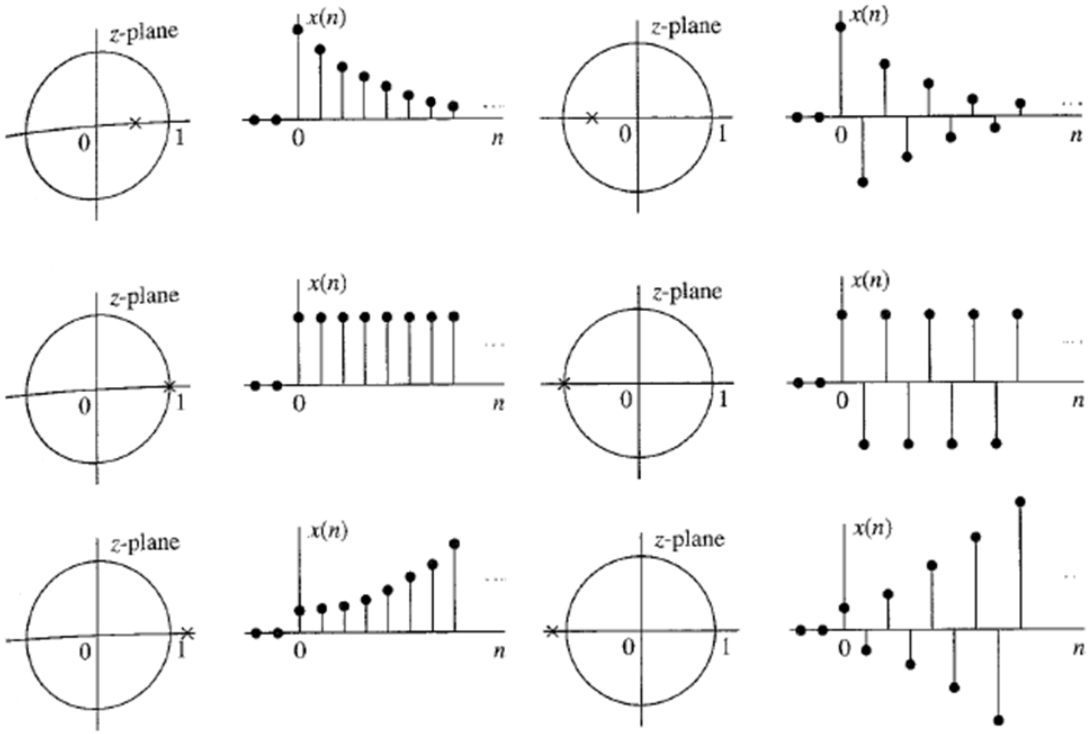


Figure 3.3.5 Time-domain behavior of a single-real-pole causal signal as a function of the location of the pole with respect to the unit circle.

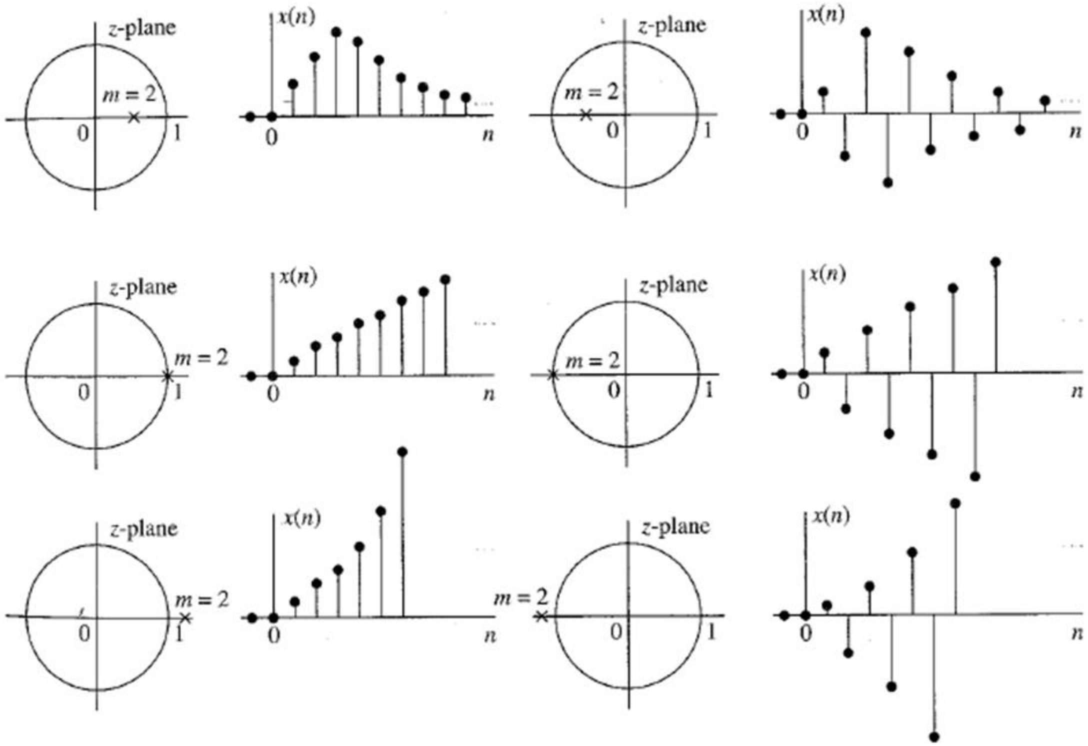


Figure 3.3.6 Time-domain behavior of causal signals corresponding to a double ($m = 2$) real pole, as a function of the pole location.

Practice

- A LTI System is characterized by the system function:

$$H(z) = \frac{z^2}{(z-0.5)(z+2)}$$

- What are the possibilities of its ROC? What are the implications of these ROC choices on the stability and causality of the LTI system?

Practice

- Given a causal system:

$$y(n) = 0.9y(n-1) + x(n)$$

- Determine $H(z)$ and sketch its pole-zero plot.

Practice

- Given that the following is a causal system:

$$H(z) = \frac{z+1}{z^2 - 0.9z + 0.81}$$

- Sketch its pole-zero plot and find its difference equation representation.