The Inverse Z-Transform

EE 453 / CE 352
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Inversion of the z-Transform

Three popular methods:

1. Contour integration:

$$x(n) = \frac{1}{2\pi j} \oint_C X(z) z^{n-1} dz$$

2. Expansion into a power series in z or z^{-1}:

$$X(z) = \sum_{k=-\infty}^{\infty} x(k) z^{-k}$$

   and obtaining $x(k)$ for all k by inspection.

3. Partial-fraction expansion and table look-up.

Inversion of Rational z-Transform

$$X(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_M z^{-M}}{a_0 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_N z^{-N}}$$

In the table lookup method, we attempt to express the function X(z) as a linear combination:

$$X(z) = \alpha_1 X_1(z) + \alpha_2 X_2(z) + \cdots + \alpha_k X_k(z)$$

Each of these terms have an inverse transform available in a table. Then due to linearity:

$$x(n) = \alpha_1 x_1(n) + \alpha_2 x_2(n) + \cdots + \alpha_k x_k(n)$$

Example:

$$X(z) = \frac{1}{1 - 1.5z^{-1} + 0.5z^{-2}} \xrightarrow{\text{Partial Fractions}} X(z) = \frac{2}{1 - z^{-1}} - \frac{1}{1 - 0.5z^{-1}} \xrightarrow{\text{Table Lookup}} x(n) = 2(1)^n u(n) - (0.5)^n u(n)$$

Rational z-Transforms: Proper vs Improper

$$X(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_M z^{-M}}{a_0 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_N z^{-N}}$$

- A rational function of the above form is called proper, if $a_N \neq 0$ and $M < N$.
- An improper rational function $(M > N)$ can always be written as the sum of a polynomial and a proper rational function.

Proper Rational z-Transforms

- Let $X(z)$ be a proper rational function with $a_N \neq 0$ and $M < N$:

$$X(z) = \frac{B(z)}{A(z)} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_M z^{-M}}{1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_N z^{-N}}$$

- To simplify, we eliminate negative powers of z by multiplying both the numerator and denominator by z^N:

$$X(z) = \frac{b_0 z^N + b_1 z^{N-1} + \cdots + b_M z^{N-M}}{z^N + a_1 z^{N-1} + \cdots + a_N}$$

which contains only positive powers of z and is always proper.

Proper Rational z-Transforms

$$X(z) = \frac{b_0 z^N + b_1 z^{N-1} + \cdots + b_M z^{N-M}}{z^N + a_1 z^{N-1} + \cdots + a_N}$$

- Our task in partial fraction expansion is to express this as a sum of simple fractions.

$$\frac{X(z)}{z} = \frac{A_1}{z - p_1} + \frac{A_2}{z - p_2} + \cdots + \frac{A_N}{z - p_N}$$

- where $p_1, p_2, \ldots, p_N$ are the roots of the denominator polynomial $z^N + a_1 z^{N-1} + \cdots + a_N$.
- If X(z) has a pole of multiplicity m, i.e. it contains in its denominator the factor $(z - p_k)^m$, then the partial fraction expansion must contain the terms:

$$\frac{X(z)}{z} = \frac{A_{1k}}{z - p_k} + \frac{A_{2k}}{(z - p_k)^2} + \cdots + \frac{A_{mk}}{(z - p_k)^m}$$

Common Transform Pairs

Sequence	*Transform*	*ROC*
$\delta(n)$	1	$\forall z$
$u(n)$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
$-u(-n-1)$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert < 1$
$a^n u(n)$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
$-b^n u(-n-1)$	$\frac{1}{1-bz^{-1}}$	$\lvert z\rvert < \lvert b\rvert$
$[a^n \sin \omega_0 n]\, u(n)$	$\frac{(a \sin \omega_0) z^{-1}}{1-(2a\cos\omega_0)z^{-1}+a^2z^{-2}}$	$\lvert z\rvert > \lvert a\rvert$
$[a^n \cos \omega_0 n]\, u(n)$	$\frac{1-(a \cos \omega_0) z^{-1}}{1-(2a\cos\omega_0)z^{-1}+a^2z^{-2}}$	$\lvert z\rvert > \lvert a\rvert$
$na^n u(n)$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert > \lvert a\rvert$
$-nb^n u(-n-1)$	$\frac{bz^{-1}}{(1-bz^{-1})^2}$	$\lvert z\rvert < \lvert b\rvert$

Partial-Fraction Expansion

- **Example:** Find $x(n)$, given:

$$X(z) = \frac{1}{(1+2z^{-1})(1-z^{-1})^2}$$

$$\frac{X(z)}{z} = \frac{z^2}{(z+2)(z-1)^2}$$

$$\frac{z^2}{(z+2)(z-1)^2} = \frac{A_1}{z+2} + \frac{A_2}{z-1} + \frac{A_3}{(z-1)^2}$$

Partial-Fraction Expansion

$$\frac{z^2}{(z+2)(z-1)^2} = \frac{A_1}{z+2} + \frac{A_2}{z-1} + \frac{A_3}{(z-1)^2}$$

For A_1:

$$\frac{z^2\cancel{(z+2)}}{\cancel{(z+2)}(z-1)^2} = \frac{A_1\cancel{(z+2)}}{\cancel{z+2}} + \frac{A_2(z+2)}{z-1} + \frac{A_3(z+2)}{(z-1)^2}$$

$$\frac{z^2}{(z-1)^2} = A_1 + \frac{A_2(z+2)}{z-1} + \frac{A_3(z+2)}{(z-1)^2}$$

Setting $z = -2$, we get: $A_1 = \frac{4}{9}$

Partial-Fraction Expansion

$$\frac{z^2}{(z+2)(z-1)^2} = \frac{A_1}{z+2} + \frac{A_2}{z-1} + \frac{A_3}{(z-1)^2}$$

For A_3:

$$\frac{z^2\cancel{(z-1)^2}}{(z+2)\cancel{(z-1)^2}} = \frac{A_1(z-1)^2}{z+2} + \frac{A_2(z-1)^{\cancel{2}}}{\cancel{z-1}} + \frac{A_3\cancel{(z-1)^2}}{\cancel{(z-1)^2}}$$

$$\frac{z^2}{(z+2)} = \frac{A_1(z-1)^2}{z+2} + A_2(z-1) + A_3$$

Setting $z = 1$, we get: $A_3 = \frac{1}{3}$

Partial-Fraction Expansion

For A_2:

$$\frac{z^2}{(z+2)} = \frac{A_1(z-1)^2}{z+2} + A_2(z-1) + A_3$$

$$\frac{d}{dz}\frac{z^2}{(z+2)} = \frac{d}{dz}\left[\frac{A_1(z-1)^2}{z+2} + A_2(z-1) + A_3\right]$$

Setting $z = 1$, we get: $A_2 = \frac{5}{9}$

$$\frac{X(z)}{z} = \frac{4}{9}\frac{1}{z+2} + \frac{5}{9}\frac{1}{z-1} + \frac{1}{3}\frac{1}{(z-1)^2}$$

Partial-Fraction Expansion

$$\frac{X(z)}{z} = \frac{4}{9}\frac{1}{(z+2)} + \frac{5}{9}\frac{1}{(z-1)} + \frac{1}{3}\frac{1}{(z-1)^2}$$

$$X(z) = \frac{4}{9}\frac{z}{(z+2)} + \frac{5}{9}\frac{z}{(z-1)} + \frac{1}{3}\frac{z}{(z-1)^2}$$

$$X(z) = \frac{4}{9}\frac{1}{(1+2z^{-1})} + \frac{5}{9}\frac{1}{(1-z^{-1})} + \frac{1}{3}\frac{1}{(1-z^{-1})^2}$$

Using Table Lookup

$$X(z) = \frac{4}{9}\frac{1}{(1+2z^{-1})} + \frac{5}{9}\frac{1}{(1-z^{-1})} + \frac{1}{3}\frac{1}{(1-z^{-1})^2}$$

Let’s assume we are dealing with a causal system:

$$X(z) = \frac{4}{9}(-2)^n u(n) + \frac{5}{9}u(n) + \frac{1}{3}nu(n)$$

$$X(z) = \left[\frac{4}{9}(-2)^n + \frac{5}{9} + \frac{n}{3}\right]u(n)$$

Sequence	*Transform*	*ROC*
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$a^n u(n)$	$\frac{1}{1-az^{-1}}$	$\|z\| > \|a\|$
$-b^n u(-n-1)$	$\frac{1}{1-bz^{-1}}$	$\|z\| < \|b\|$
$[a^n \sin\omega_0 n]\, u(n)$	$\frac{(a\sin\omega_0)z^{-1}}{1-(2a\cos\omega_0)z^{-1}+a^2z^{-2}}$	$\|z\| > \|a\|$
$[a^n \cos\omega_0 n]\, u(n)$	$\frac{1-(a\cos\omega_0)z^{-1}}{1-(2a\cos\omega_0)z^{-1}+a^2z^{-2}}$	$\|z\| > \|a\|$
$na^n u(n)$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\|z\| > \|a\|$
$-nb^n u(-n-1)$	$\frac{bz^{-1}}{(1-bz^{-1})^2}$	$\|z\| < \|b\|$

Improper Rational z-Transforms

- **Practice:** Express the improper rational transform in terms of a polynomial and a proper function.

$$X(z) = \frac{1 + 3z^{-1} + \frac{11}{6}z^{-2} + \frac{1}{3}z^{-3}}{1 + \frac{5}{6}z^{-1} + \frac{1}{6}z^{-2}}$$

- **Answer:** $X(z) = 1 + 2z^{-1} + \frac{\frac{1}{6}z^{-1}}{1+\frac{5}{6}z^{-1}+\frac{1}{6}z^{-2}}$

Improper Rational z-Transforms

$$X(z) = \frac{B(z)}{A(z)} = \frac{b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}}{1 + a_1 z^{-1} + \cdots + a_N z^{-N}}$$

- In general, any improper rational function ($M > N$) can be expressed as:

$$X(z) = \frac{B(z)}{A(z)} = c_0 + c_1 z^{-1} + \cdots + c_{M-N} z^{-(M-N)} + \frac{B_1(z)}{A(z)}$$

where $\frac{B_1(z)}{A(z)}$ is a proper rational function.

- Example (from last slide)

$$X(z) = \frac{1 + 3z^{-1} + \frac{11}{6}z^{-2} + \frac{1}{3}z^{-3}}{1 + \frac{5}{6}z^{-1} + \frac{1}{6}z^{-2}} \longrightarrow X(z) = 1 + 2z^{-1} + \frac{\frac{1}{6}z^{-1}}{1 + \frac{5}{6}z^{-1} + \frac{1}{6}z^{-2}}$$

- The inversion strategy
  - The inverse z-transform of the polynomial $c_0 + c_1 z^{-1} + \cdots + c_{M-N} z^{-(M-N)}$ can easily be found by inspection and comparison with z-transform of finite-duration sequences.
  - For the inversion of proper rational function, partial fraction expansion and table lookup approach is applied.

Examples from MATLAB

- **Example:** $X(z) = \dfrac{1+3z^{-1}+\frac{11}{6}z^{-2}+\frac{1}{3}z^{-3}}{1+\frac{5}{6}z^{-1}+\frac{1}{6}z^{-2}}$

- $X(z) = 1 + 2z^{-1} + \dfrac{\frac{1}{6}z^{-1}}{1+\frac{5}{6}z^{-1}+\frac{1}{6}z^{-2}}$

- Comparison with MATLAB's *residuez()* function:

```
b = [1 3 11/6 1/3]; a = [1 5/6 1/6];
[R, p, c] = residuez(b,a)
```

```
R =

   -1.0000
    1.0000


p =

   -0.5000
   -0.3333


c =

    1.0000    2.0000
```

Examples from MATLAB

- **Example:** $X(z) = \dfrac{1}{1-0.9\,z^{-1}-0.81z^{-2}+0.729z^{-3}}$

- b = 1   a = [1 -0.9 -0.81 0.729]

```
>> [R,p,c]=residuez(b,a)

R =

    0.2500
    0.5000
    0.2500


p =

    0.9000
    0.9000
   -0.9000


c =

     []
```

$$X(z) = \frac{0.25}{1-0.9z^{-1}} + \frac{0.5}{(1-0.9z^{-1})^2} + \frac{0.25}{1+0.9z^{-1}},$$