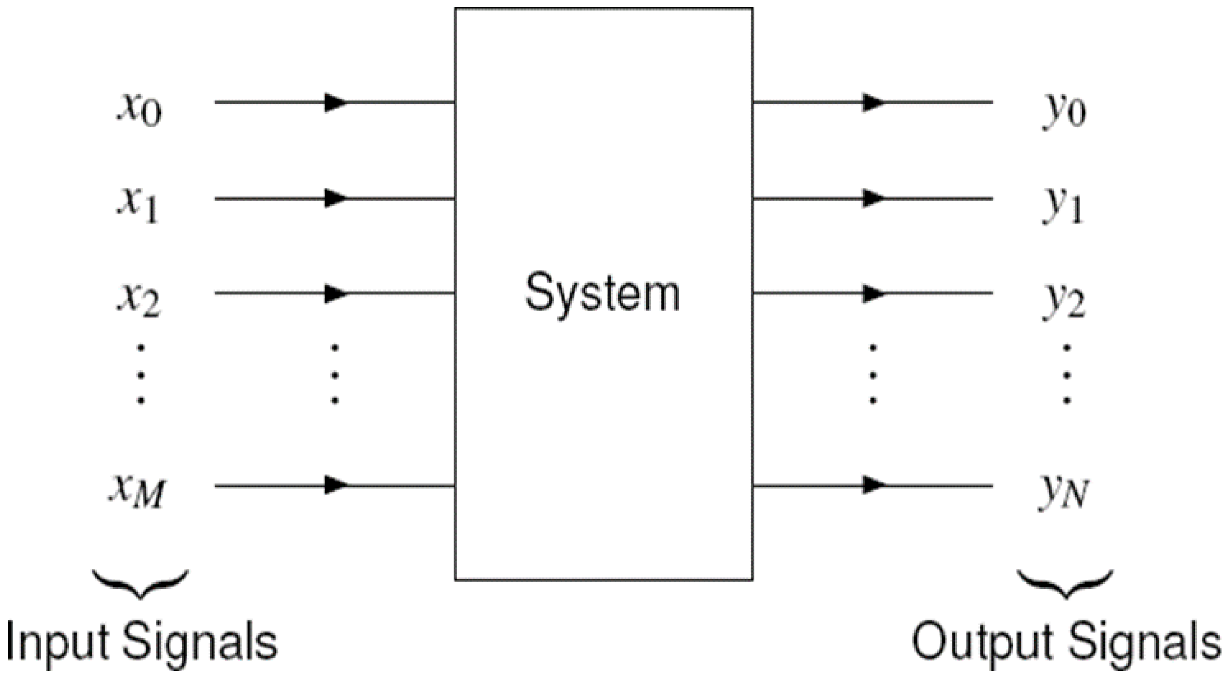
# Digital LTI Systems

EE 453 / CE 352

Saad Baig

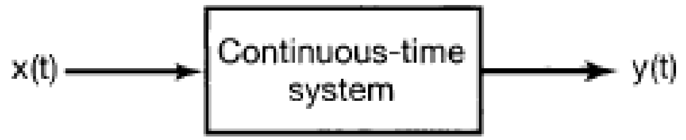
# What is a System?

- An entity that processes one or more input signals in order to produce one or more output signals.



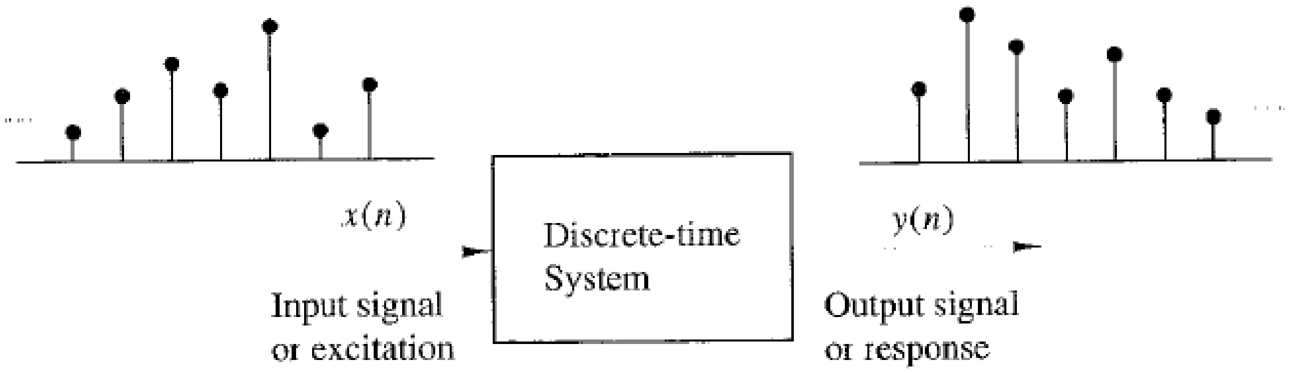
# System: Continuous Time vs Discrete Time

- Continuous-Time (CT) System
  - Input and output are continuous-time signals
  - $x(t) \rightarrow y(t)$




- Discrete-Time (DT) System
  - Input and output are discrete-time signals
  - $x[n] \rightarrow y[n]$

# Discrete-Time System




- We say that the input signal $x(n)$ is transformed by the system into a signal $y(n)$.

$$y(n) \equiv \tau[x(n)]$$

  - Where the symbol $\tau$ (also called an operator) denotes the transformation or processing performed by the system on signal $x(n)$ to produce signal $y(n)$.

# Description of a DT System

- There are various ways to describe the operation a DT system performs on input signal $x(n)$ to produce the output signal $y(n)$.
- The input-output description:
  - Focuses on the behavior at the terminals of the system
  - Ignores the detailed internal construction or realization of the system
  - "Black Box representation"
  - Consists of a mathematical expression or a rule which explicitly defines the relationship between the input and output signals.

$$x(n) \xrightarrow{\tau} y(n)$$

# Input-Output Description of a DT System

(a) $y(n) = x(n)$ (identity system)

(b) $y(n) = x(n-1)$ (unit delay system)

(c) $y(n) = x(n+1)$ (unit advance system)

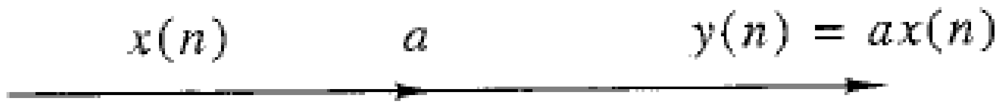
(d) $y(n) = \frac{1}{3}[x(n+1) + x(n) + x(n-1)]$ (moving average filter)

(e) $y(n) = \text{median}\{x(n+1), x(n), x(n-1)\}$ (median filter)

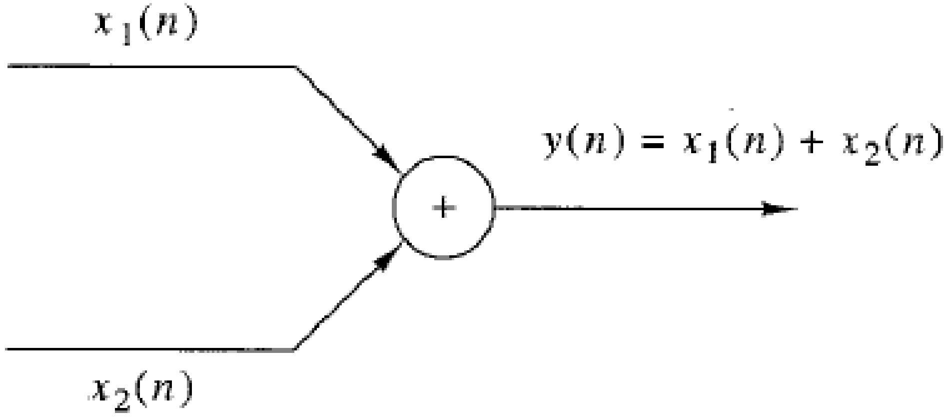
(f) $y(n) = \sum_{k=-\infty}^{n} x(k) = x(n) + x(n-1) + x(n-2) + \cdots$ (accumulator)

# Block Diagram Representation: Basic Building Blocks

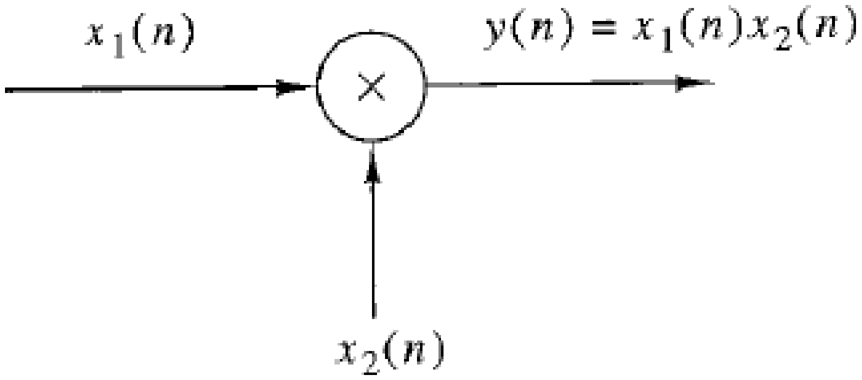
## Constant Multiplier




## Adder

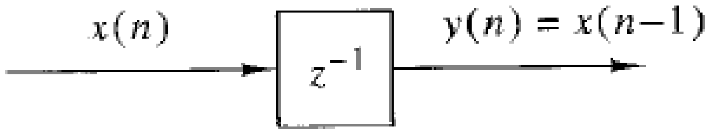



## Signal Multiplier

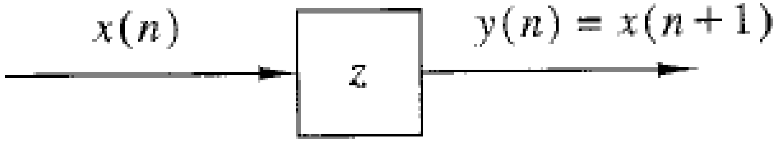


# Block Diagram Representation: Basic Building Blocks

- Unit Delay
  - This basic building block requires memory

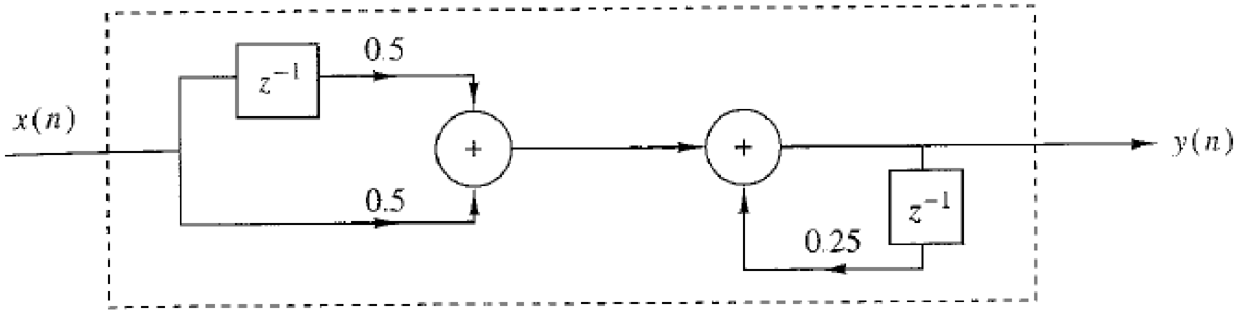



- Unit Advance
  - Any such advance is physically impossible in real-time since it involves looking into the future of the signal
  - On the other hand, If we have the signal stored in computer memory, we can advance the signal in time.



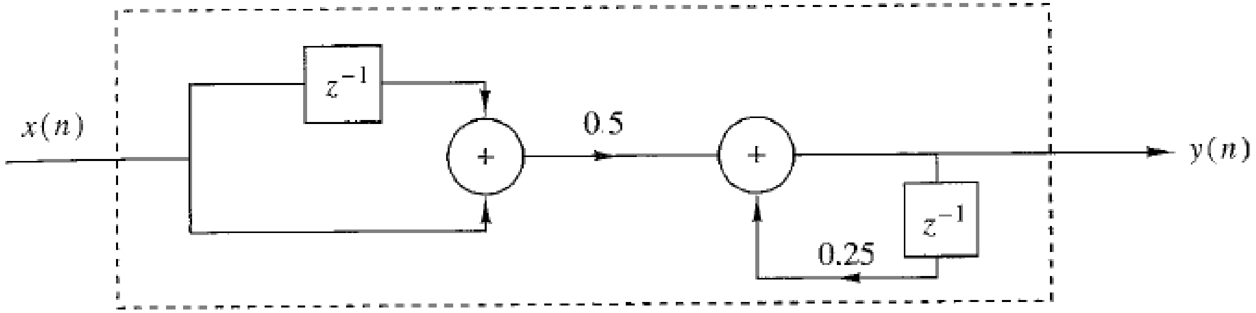
# Block Diagram Representation: Example

$$y(n) = \frac{1}{4}y(n-1) + \frac{1}{2}x(n) + \frac{1}{2}x(n-1)$$




- After rearrangement as:

$$y(n) = \frac{1}{4}y(n-1) + \frac{1}{2}[x(n) + x(n-1)]$$



# Considering Initial Conditions

- Consider the DT system known as an "accumulator".
- Option 1 is not feasible as it requires to keep track of (or remember or store in memory) infinite number of past inputs.
- Option 2 is feasible as the system computes the current value of the output by adding the current value of the input to the previous value of the output.
- Option 2 still has a complexity compared to option 1.

$$y[n] = \sum_{k=-\infty}^{n} x[k] \rightarrow (1)$$

$$y[n] = \sum_{k=-\infty}^{n-1} x[k] + x[n]$$

$$y[n] = y[n-1] + x[n] \rightarrow (2)$$

# Considering Initial Conditions

$$y[n] = y[n-1] + x[n]$$

For an input signal $x[n]$ where $n \leq n_0$:

$$y[n_0] = y[n_0 - 1] + x[n_0]$$

$$y[n_0 + 1] = y[n_0] + x[n_0 + 1]$$

- There will always be additional information required: $y[n-1]$. This is known as the initial condition.
  - The initial condition summarizes the effect on the system from all inputs applied before time $n_0$.
  - The system is said to be initially relaxed if initial condition equals zero.

# Classification of DT Systems (Based on their Properties)

Common System Properties:

| | | |
|---|---|---|
| static | vs. | dynamic |
| time-invariant | vs. | time-variant |
| linear | vs. | nonlinear |
| causal | vs. | non-causal |
| stable | vs. | unstable systems |

# Classification of DT Systems (Based on their Properties)

- Why is this so important?
  - For developing DSP systems, we shall depend heavily on our ability to analyze and design DT systems.
  - The mathematical techniques that we develop for analyzing as well as designing DT systems depend heavily on the characteristics/properties of the systems that are being considered.
  - Therefore, it is important to focus on a number of properties of DT systems that can be used to categorize these systems.
- Important point about properties of systems
  - For a system to possess a given property, the property must hold for every possible input to the system.
  - Therefore, a counter example is sufficient to prove that a system does not possess a property
  - However, to prove that the system has some property, we must prove that this property holds for a generic input to the system.

# Static vs Dynamic Systems

- A discrete-time system is said to be static or memoryless if its output any instant $n$ depends at most on the input sample at the same time, but not on past or future samples of the input.
- In any other case, the system is said to be dynamic or have memory.
- Examples:

$$y(n) = ax(n)$$

$$y(n) = nx(n) + bx^3(n)$$

$$y(n) = x(n) + 3x(n-1)$$

$$y(n) = \sum_{k=0}^{n} x(n-k)$$

$$y(n) = \sum_{k=0}^{\infty} x(n-k)$$

# Invertibility

- A system is said to be invertible if distinct inputs lead to distinct outputs.
- Examples of Invertible Systems

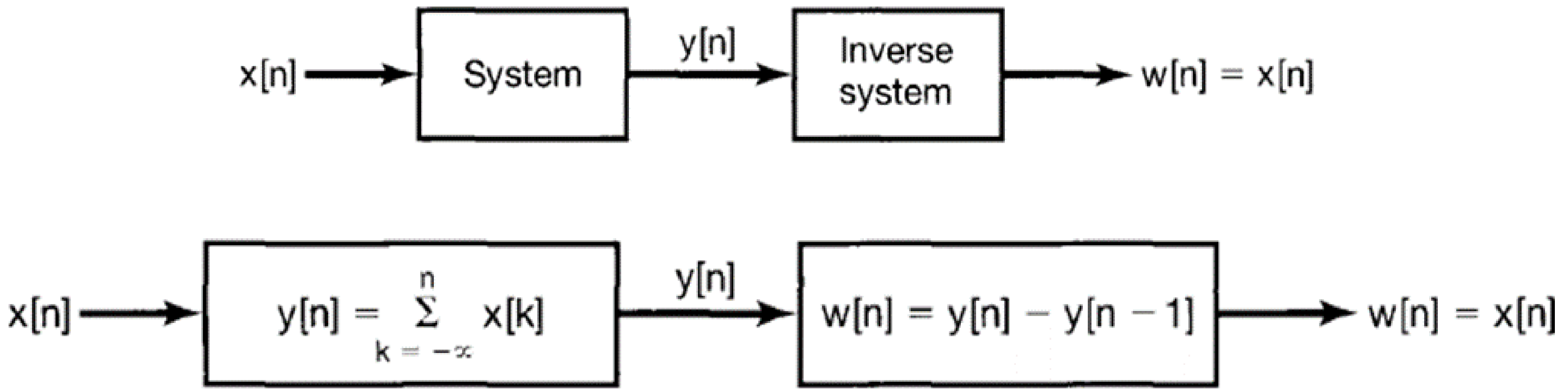
$$y[n] = \sum_{k=-\infty}^{n} x[k], \qquad y[n] = 2x[n]$$

- Examples of Non-Invertible Systems

$$y[n] = 0, \qquad y[n] = x^2[n]$$

# Invertibility

- If a system is invertible, then an inverse system exists that, when cascaded with the original system, yields an output w[n] equal to the input x[n] to the first system.



# Causal vs Non-Causal Systems

- A system is said to be causal if the output at any time depends only on values of the input at the present time and in the past.
- Non-causal systems are dependent on future values of input as well.
- Examples:

$$y[n] = \sum_{k=-\infty}^{n} x[k],$$

$$y[n] = x[n] - x[n+1]$$

$$y[n] = \frac{1}{2M+1} \sum_{k=-M}^{+M} x[n-k].$$

# Causal vs Non-Causal Systems

- All memoryless systems are causal, since the output responds only to the current value of the input.
- if two inputs to a causal system are identical up to some point in time $t_0$ or $n_0$, the corresponding outputs must also be equal up to this same time.

# Causal vs Non-Causal Systems

- In real-time signal processing applications, we cannot observe future values of the signal, and hence a non-causal system is physically unrealizable (i.e. it cannot be implemented).

- On the other hand, if the signal is recorded so that the processing is done offline (nonreal time), it is possible to implement a noncausal system.

# Stable vs Unstable Systems

- A system is said to be stable if a bounded input always leads to a bounded output.
- The condition that the input sequence x(n) and the output sequence y(n) are bounded is translated mathematically to mean that there exists some finite numbers, say Mx and My such that

$$|x(n)| \leq M_x < \infty, \qquad |y(n)| \leq M_y < \infty \qquad \text{for all } n$$

- If for some bounded input sequence $x(n)$, the output is unbounded (infinite), the system is classified as unstable.

# Stable vs Unstable Systems

- Examples of a Stable System

$$y[n] = \frac{1}{2M+1} \sum_{k=-M}^{M} x[n-k]$$

- Example of an Unstable System

$$y[n] = \sum_{k=-\infty}^{n} x[k]$$

# Stability

- Exercise: Are the following two systems stable

$$y(n) = nx(n) \qquad\qquad y(n) = e^{x(n)}$$

# Time Invariance

- Conceptually, a system is time invariant if the behavior and characteristics of the system are fixed over time.
  - The RC circuit is time invariant if the resistance and capacitance values R and C are constant over time: We would expect to get the same results from an experiment with this circuit today as we would if we ran the identical experiment tomorrow.
  - On the other hand, if the values of R and C are changed or fluctuate over time, then we would expect the results of our experiment to depend on the time at which we run it.

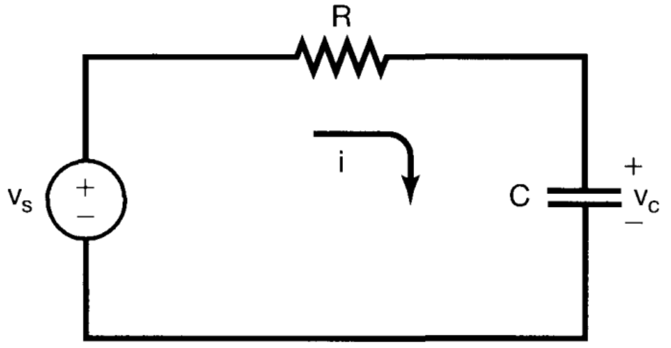



**Figure 1.1** A simple *RC* circuit with source voltage $v_s$ and capacitor voltage $v_c$.

# Time Invariance

- A system is said to be time invariant if a time shift in the input signal results in an identical time shift in the output signal. i.e.

$$x(n) \xrightarrow{\tau} y(n) \Rightarrow x(n-k) \xrightarrow{\tau} y(n-k)$$

for every input $x(n)$ and every time shift $k$.

- Example of a Time Invariant System

$$y(n) = e^{x(n)}$$

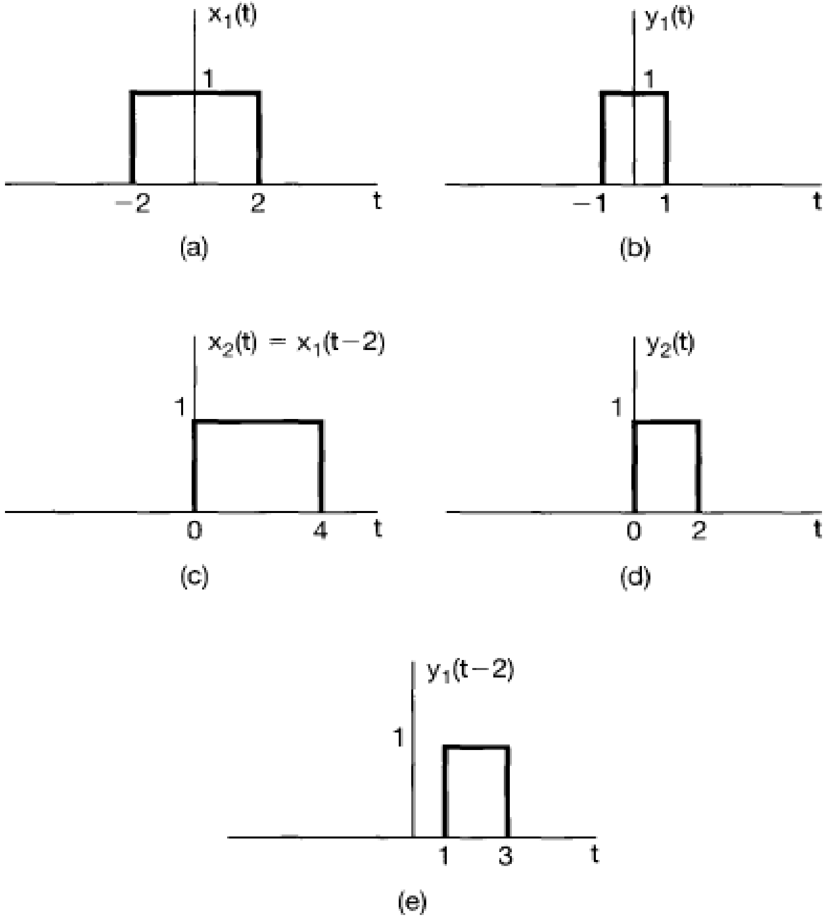
- Example of a Time Variant System

$$y[n] = nx[n]$$

# Time Invariance

- Exercise: Is the following system time-invariant?

$$y(t) = x(2t)$$



# Linearity

- A linear system, in continuous-time or discrete-time, is a system that possesses the important property of superposition.
  - If an input consists of the weighted sum of several signals, then the output is the weighted sum of the responses of the system to each of those signals.
  - If $x_1[n] \to y_1[n]$ and $x_2[n] \to y_2[n]$, then:

$$ax_1[n] + bx_2[n] \to ay_1[n] + by_2[n]$$

  - where $a$ and b are any complex constants

# Linearity

- Exercise: Is the following system linear? $y(t) = tx(t)$

To determine whether or not $S$ is linear, we consider two arbitrary inputs $x_1(t)$ and $x_2(t)$.

$$x_1(t) \rightarrow y_1(t) = tx_1(t)$$
$$x_2(t) \rightarrow y_2(t) = tx_2(t)$$

Let $x_3(t)$ be a linear combination of $x_1(t)$ and $x_2(t)$. That is,

$$x_3(t) = ax_1(t) + bx_2(t)$$

where $a$ and $b$ are arbitrary scalars. If $x_3(t)$ is the input to $S$, then the corresponding output may be expressed as

$$\begin{aligned} y_3(t) &= tx_3(t) \\ &= t(ax_1(t) + bx_2(t)) \\ &= atx_1(t) + btx_2(t) \\ &= ay_1(t) + by_2(t) \end{aligned}$$

We conclude that the system $S$ is linear.

# Linearity

- Exercise: Is the following system linear? $y(t) = x^2(t)$

Defining $x_1(t)$, $x_2(t)$, and $x_3(t)$ as in the previous example, we have

$$x_1(t) \rightarrow y_1(t) = x_1^2(t)$$
$$x_2(t) \rightarrow y_2(t) = x_2^2(t)$$

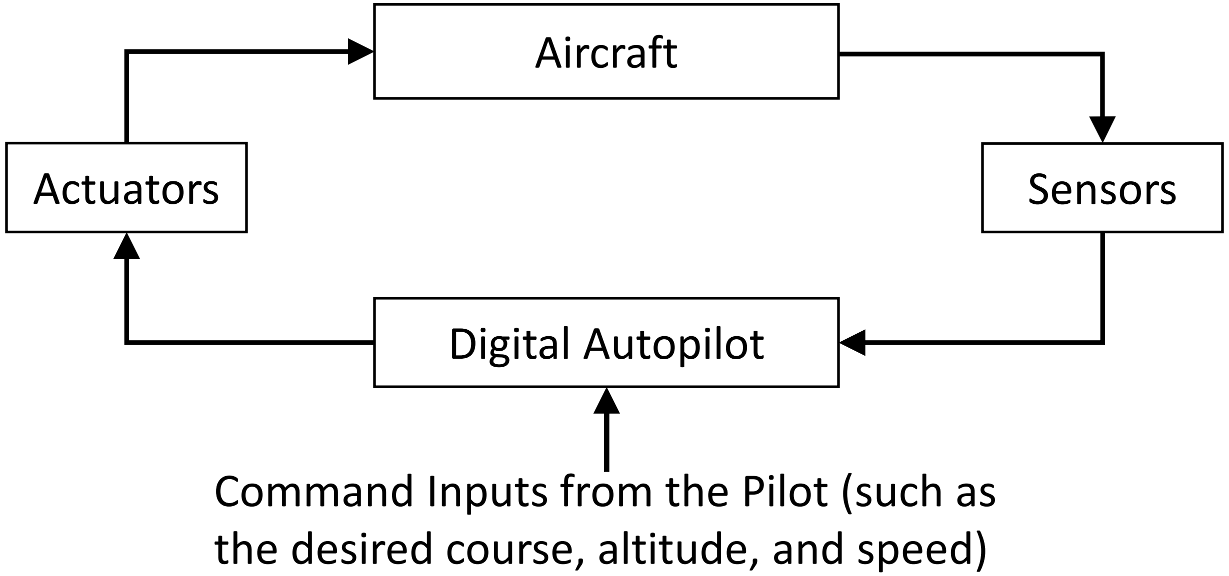
and

$$\begin{aligned} x_3(t) \rightarrow y_3(t) &= x_3^2(t) \\ &= (ax_1(t) + bx_2(t))^2 \\ &= a^2x_1^2(t) + b^2x_2^2(t) + 2abx_1(t)x_2(t) \\ &= a^2y_1(t) + b^2y_2(t) + 2abx_1(t)x_2(t) \end{aligned}$$

Clearly, we can specify $x_1(t)$, $x_2(t)$, $a$, and $b$ such that $y_3(t)$ is not the same as $ay_1(t) + by_2(t)$. For example, if $x_1(t) = 1$, $x_2(t) = 0$, $a = 2$, and $b = 0$, then $y_3(t) = (2x_1(t))^2 = 4$, but $2y_1(t) = 2(x_1(t))^2 = 2$. We conclude that the system $S$ is not linear.

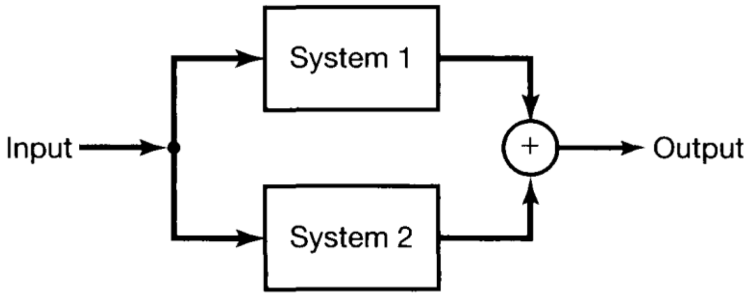
# Interconnection of Systems

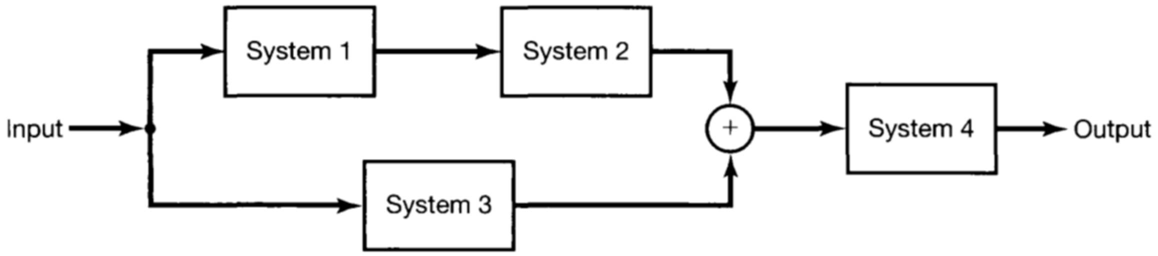
- Many real systems can be considered as interconnection of several subsystems.
- Example: Digitally-controlled Aircraft

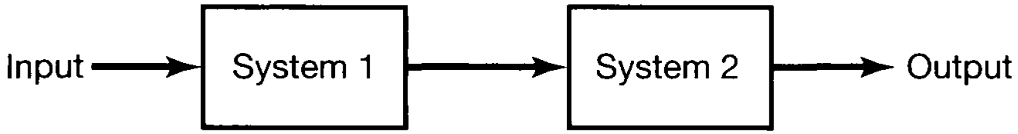


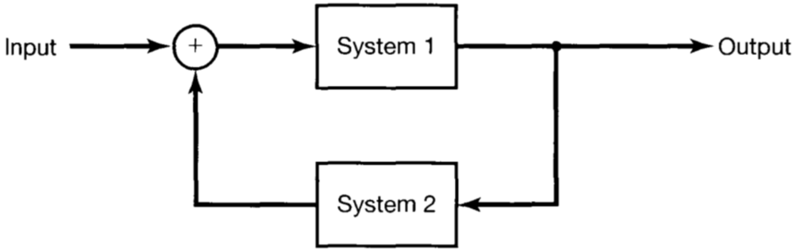
# Interconnection of Systems

- Instead of developing a complicated system model of a complex system, we can use simpler system models of subsystems to analyze the operation and behavior of the overall complex system



# LTI Systems: Big Picture

- Why focus on Linear, Time-Invariant (LTI) Systems:
  - Many physical processes can be modeled as linear, time-invariant (LTI) systems.
  - More powerful mathematical tools are available for analyzing LTI systems (as compared to non-LTI systems).
  - In practice, systems that are not LTI can be well approximated using LTI models. So, even when dealing with systems that are not LTI, LTI systems still play an important role.
- Analysis Strategy for LTI Systems
  - If we can represent the inputs to an LTI system in terms of a linear combination, we can then use superposition to compute the output of the system in terms of its responses to these basic signals.

# Analysis Strategy for LTI System

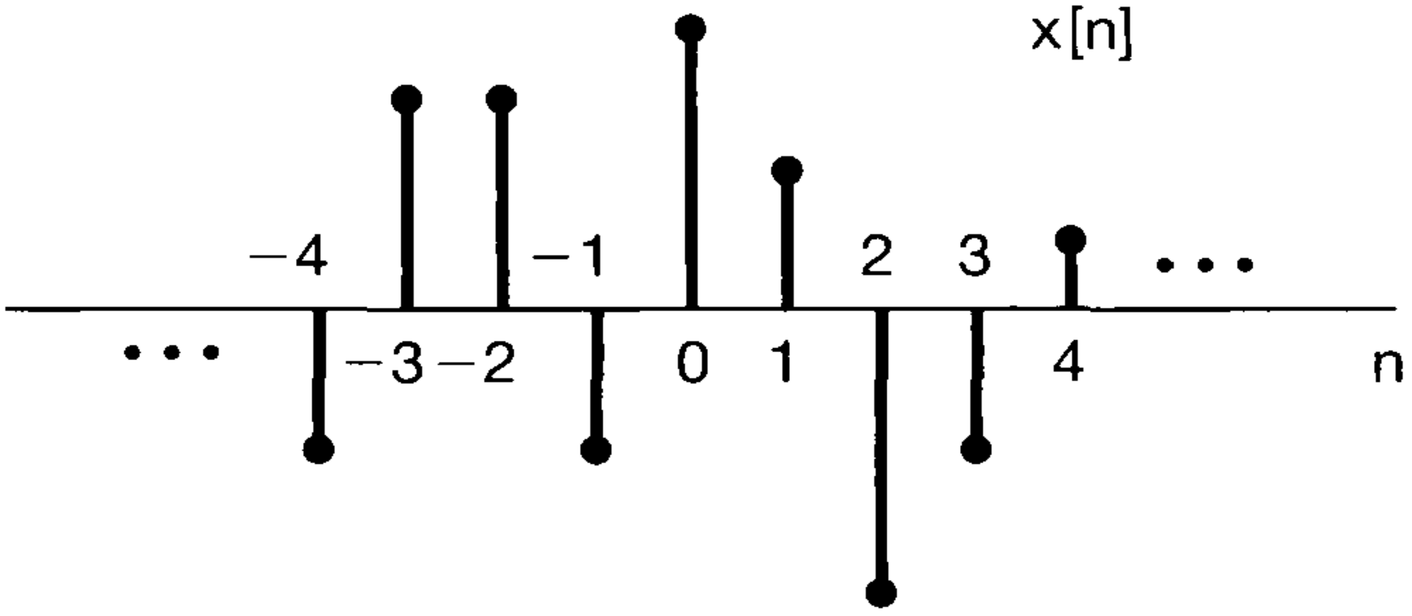
*Basic Signals* → LTI System → *Response to Basic Signals*

*Generic Signal = Linear Combination of Basic Signals* → LTI System → *Response to Generic Signal = Linear Combination of Response to Basic Signals*

# Representation of Signals using Unit Impulses

- Recall the sampling property of discrete-time unit impulse given by

$$x[n]\delta[n-n_0] = x[n_0]\delta[n-n_0]$$



# Representation of Signals using Unit Impulses

- Recall the sampling property of discrete-time unit impulse given by
$$x[n]\delta[n-n_0] = x[n_0]\delta[n-n_0]$$
- Then:

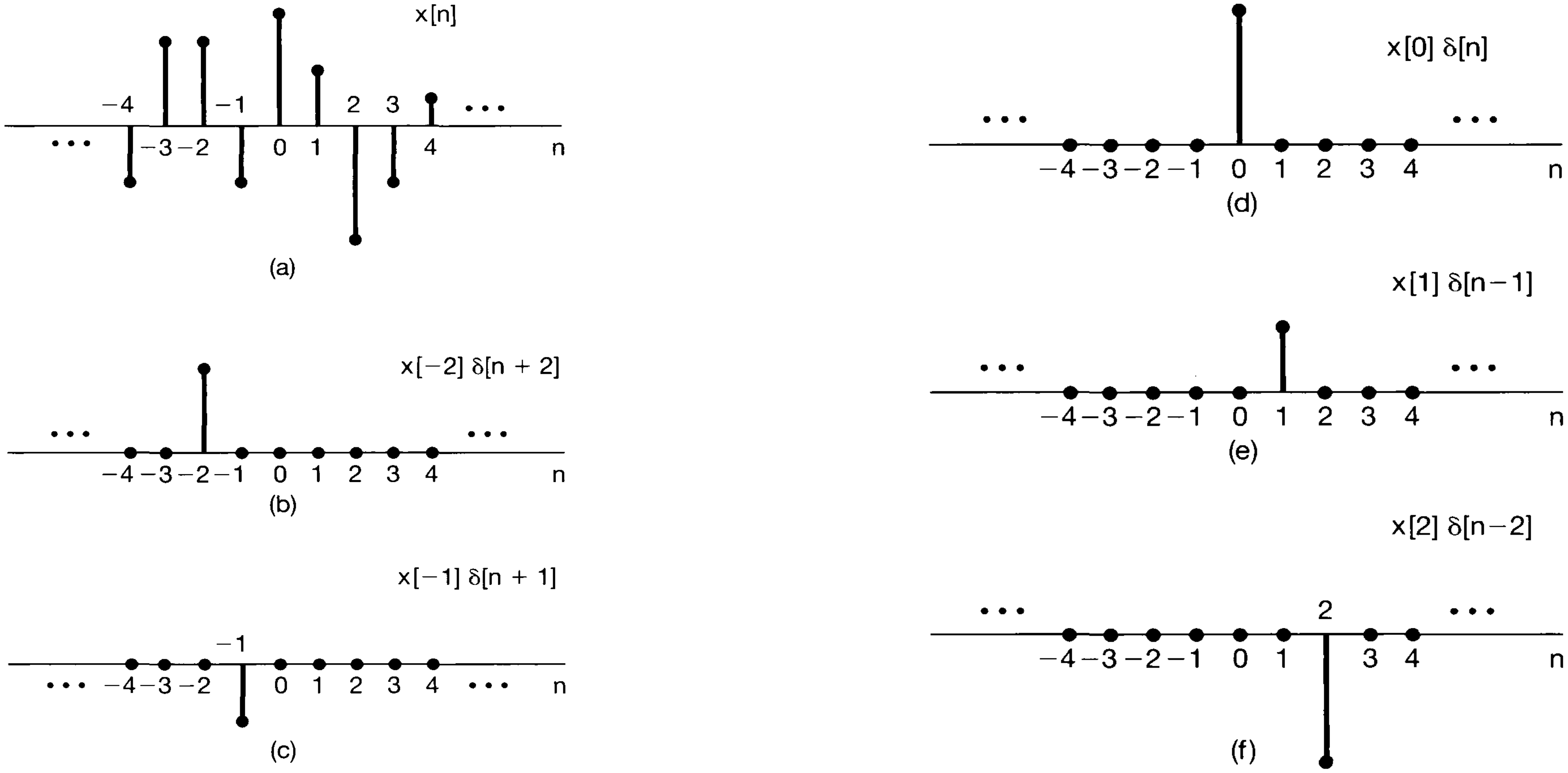
$$x[-1]\delta[n+1] = \begin{cases} x[-1], & n = -1 \\ 0, & n \neq -1 \end{cases}$$

$$x[0]\delta[n] = \begin{cases} x[0], & n = 0 \\ 0, & n \neq 0 \end{cases}$$

$$x[1]\delta[n-1] = \begin{cases} x[1], & n = 1 \\ 0, & n \neq 1 \end{cases}$$

# Representation of Signals using Unit Impulses

$$x[n]\delta[n-n_0] = x[n_0]\delta[n-n_0]$$



# Representation of Signals using Unit Impulses

- Using sampling property, we can decompose $x[n]$:

$$x[n] = \cdots + x[-3]\delta[n+3] + x[-2]\delta[n+2] + x[-1]\delta[n+1] + x[0]\delta[n] \\ + x[1]\delta[n-1] + x[2]\delta[n-2] + x[3]\delta[n-3] + \cdots$$

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

  – This corresponds to the representation of an arbitrary signal as a linear combination of shifted unit impulses $\delta[n-k]$, where the weights in this linear combination are $x[k]$.

# Response of a Linear Time-Invariant System

- What will be the response of $x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$ in a LTI system?

$$\delta[n] \longrightarrow \boxed{\text{LTI System}} \longrightarrow h[n]$$

$$\delta[n-k] \longrightarrow \boxed{\text{LTI System}} \longrightarrow h[n-k]$$

$$x[k]\delta[n-k] \longrightarrow \boxed{\text{LTI System}} \longrightarrow x[k]h[n-k]$$

$$x[n] = \sum_{k=-\infty}^{+\infty} x[k]\,\delta[n-k] \longrightarrow \boxed{\text{LTI System}} \longrightarrow y[n] = \sum_{k=-\infty}^{+\infty} x[k]\,h[n-k]$$

# Response of a Linear Time-Invariant System

- What will be the response of $x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$ in a LTI system?

$\delta[n] \longrightarrow$ LTI System $\longrightarrow$ $\boldsymbol{h[n]}$ **= unit impluse response**

$\delta[n-k] \longrightarrow$ LTI System $\longrightarrow$ $h[n-k]$

$x[k]\delta[n-k] \longrightarrow$ LTI System $\longrightarrow$ $x[k]h[n-k]$

$$x[n] = \sum_{k=-\infty}^{+\infty} x[k]\,\delta[n-k] \longrightarrow \text{LTI System} \longrightarrow y[n] = \sum_{k=-\infty}^{+\infty} x[k]\,h[n-k]$$

# Response of a Linear Time-Invariant System

- If we know the response of an LTI system to the unit impulse, we can construct the response to any arbitrary input.

$$\delta[n] \longrightarrow \boxed{\text{LTI System}} \longrightarrow \boldsymbol{h[n]} = \textbf{unit impluse response}$$

$$x[n] = \sum_{k=-\infty}^{+\infty} x[k]\,\delta[n-k] \longrightarrow \boxed{\text{LTI System}} \longrightarrow \boldsymbol{y[n]} = \sum_{\boldsymbol{k=-\infty}}^{+\infty} \boldsymbol{x[k]\,h[n-k]}$$

# Convolution

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]\, h[n-k]$$

- The operation on the right-hand side of this equation is known as the convolution of $x[n]$ and $h[n]$.
- Convolution takes two signals to produce a third signal.
- Symbolically, we represent the operation of convolution through "*".

$$y[n] = x[n] * h[n]$$

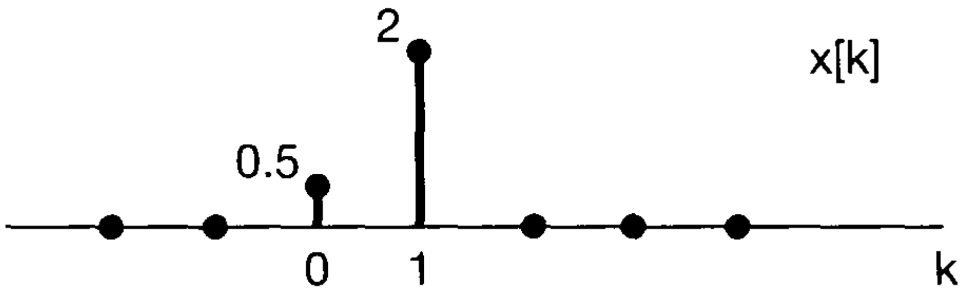
# Convolution (Graphical Method)

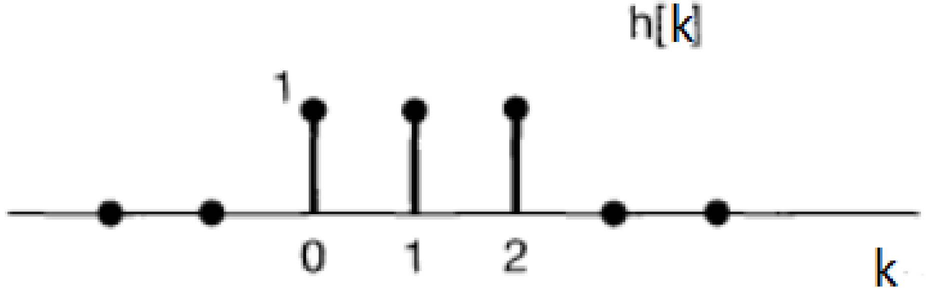
$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]\, h[n-k]$$

- Consider the evaluation of the output value at some specific time $n$.
  - A particularly convenient way of displaying this calculation graphically begins with sketching the two signals $x[k]$ and $h[n-k]$ viewed as functions of $k$.
  - To sketch $h[n-k]$, $h[k]$ is first time reversed (to become $h[-k]$) and then shifted left/right by $n$ samples.
  - Having sketched $x[k]$ and $h[n-k]$ for any particular value of $n$, we multiply these two signals and sum over all values of $k$.

# Convolution (Graphical Method)

- Example: Convolving $x[k]$ with $h[k]$



# Convolution (Graphical Method)

$y[n] = 0 \; for \; n < 0$

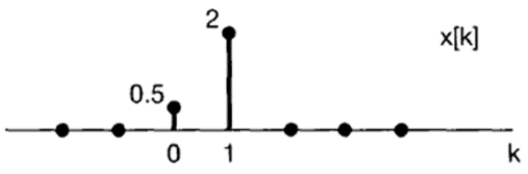
$$y[0] = \sum_{k=-\infty}^{\infty} x[k]h[0-k] = 0.5.$$

$$y[1] = \sum_{k=-\infty}^{\infty} x[k]h[1-k] = 0.5 + 2.0 = 2.5.$$

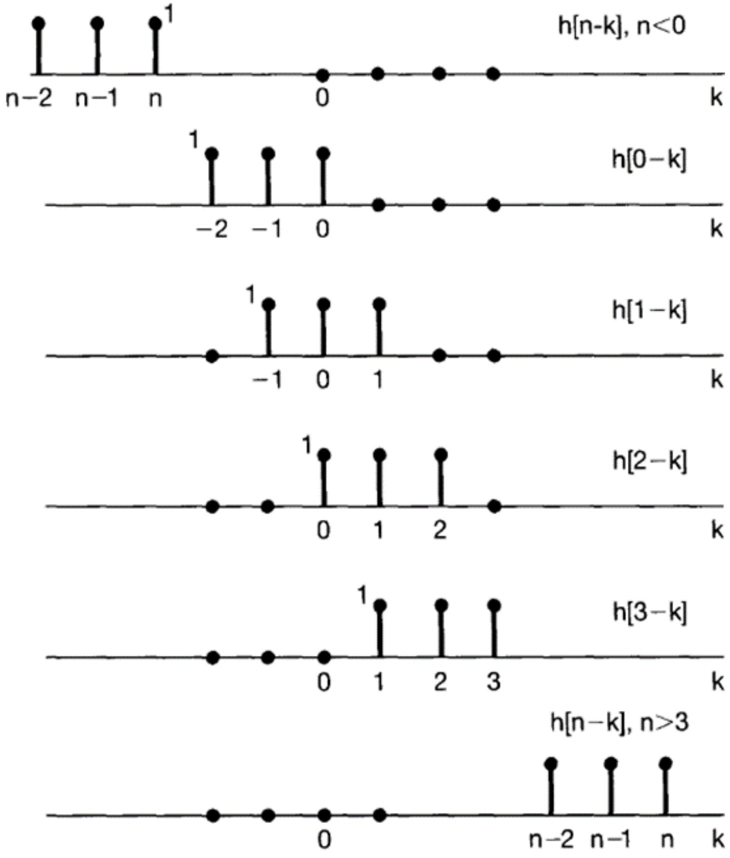
$$y[2] = \sum_{k=-\infty}^{\infty} x[k]h[2-k] = 0.5 + 2.0 = 2.5,$$

$$y[3] = \sum_{k=-\infty}^{\infty} x[k]h[3-k] = 2.0.$$

$y[n] = 0 \; for \; n > 3$

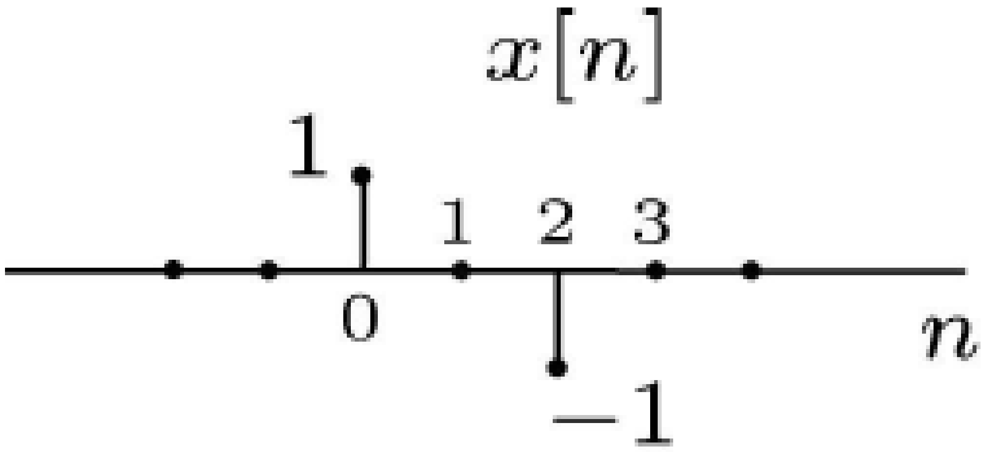


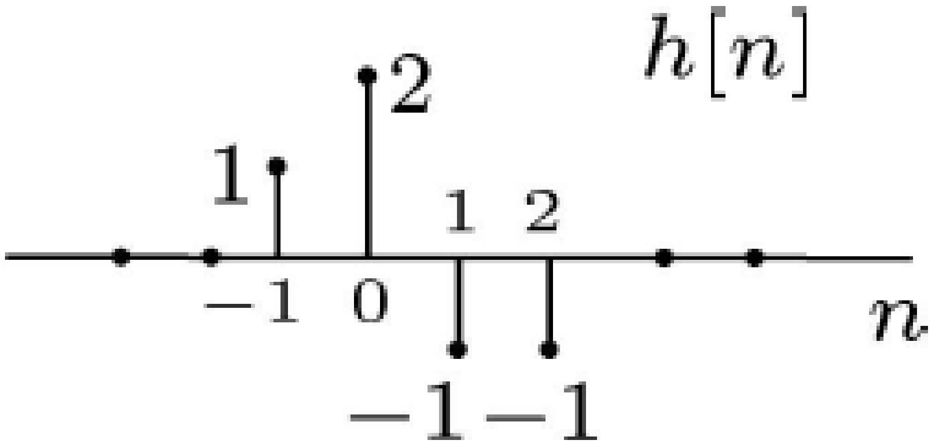

(a)




(b)

# Convolution (Graphical Method): Practice




$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]\, h[n-k]$$

# Exercise

- A LTI system has an impulse response

$$h[n] = \delta[n] + \delta[n-1] + \delta[n-2]$$

- Compute the output y[n] for the following inputs:

a) $x[n] = \delta[n] + \delta[n-1]$

b) $x[n] = \delta[n] + \delta[n-1] + \delta[n-2] + \delta[n-3] + \delta[n-4]$

c) $x[n] = \delta[n] - \delta[n-1]$

d) $x[n] = \delta[n] - \delta[n-6]$

# Summary

***Characteristics of a LTI system are completely determined by its Impulse Response!***

# Properties of LTI System

- The Commutative Property
- The Distributive Property
- The Associative Property
- LTI Systems with and without Memory
- Invertibility of LTI Systems
- Causality for LTI Systems
- Stability of LTI Systems

# LTI Systems: Commutative Property

- A basic  property of convolution in both continuous and discrete time is that it is a commutative operation.

$$x[n] * h[n] = h[n] * x[n] = \sum_{k=-\infty}^{+\infty} h[k]\, x[n-k]$$

- The output of an LTI system with input $x[n]$ and unit impulse response $h[n]$ is identical to the output of an LTI system with input $h[n]$ and unit impulse response $x[n]$.

# LTI Systems: Distributive Property

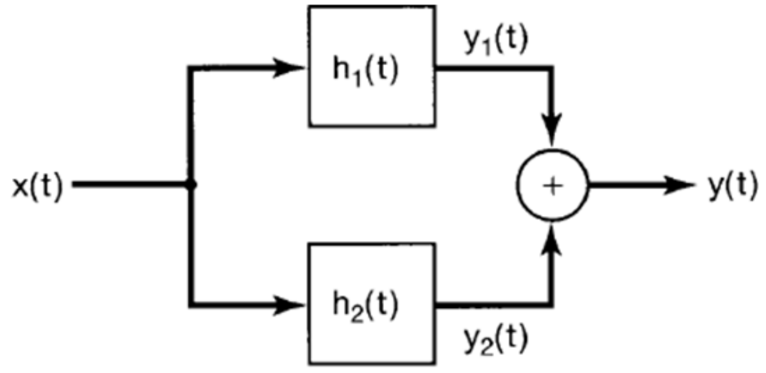
- Convolution distributes over addition.

$$x[n] * (h_1[n] + h_2[n]) = x[n] * h_1[n] + x[n] * h_2[n]$$

$$y_1(t) = x(t) * h_1(t)$$

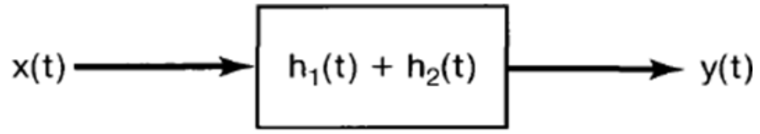
$$y_2(t) = x(t) * h_2(t),$$

$$y(t) = x(t) * h_1(t) + x(t) * h_2(t),$$




(a)

$$y(t) = x(t) * [h_1(t) + h_2(t)],$$




(b)

# LTI Systems: Distributive Property

$$x[n] = \left(\frac{1}{2}\right)^n u[n] + 2^n u[-n], \qquad (2.52)$$

$$h[n] = u[n]. \qquad (2.53)$$

$$\boldsymbol{y}[\boldsymbol{n}] = \boldsymbol{x}[\boldsymbol{n}] * \boldsymbol{h}[\boldsymbol{n}] = ?$$

Note that the sequence $x[n]$ is nonzero along the entire time axis. Direct evaluation of such a convolution is somewhat tedious. Instead, we may use the distributive property to express $y[n]$ as the sum of the results of two simpler convolution problems. In particular, if we let $x_1[n] = (1/2)^n u[n]$ and $x_2[n] = 2^n u[-n]$, it follows that

$$y[n] = (x_1[n] + x_2[n]) * h[n]. \qquad (2.54)$$

Using the distributive property of convolution, we may rewrite eq. (2.54) as

$$y[n] = y_1[n] + y_2[n], \qquad (2.55)$$

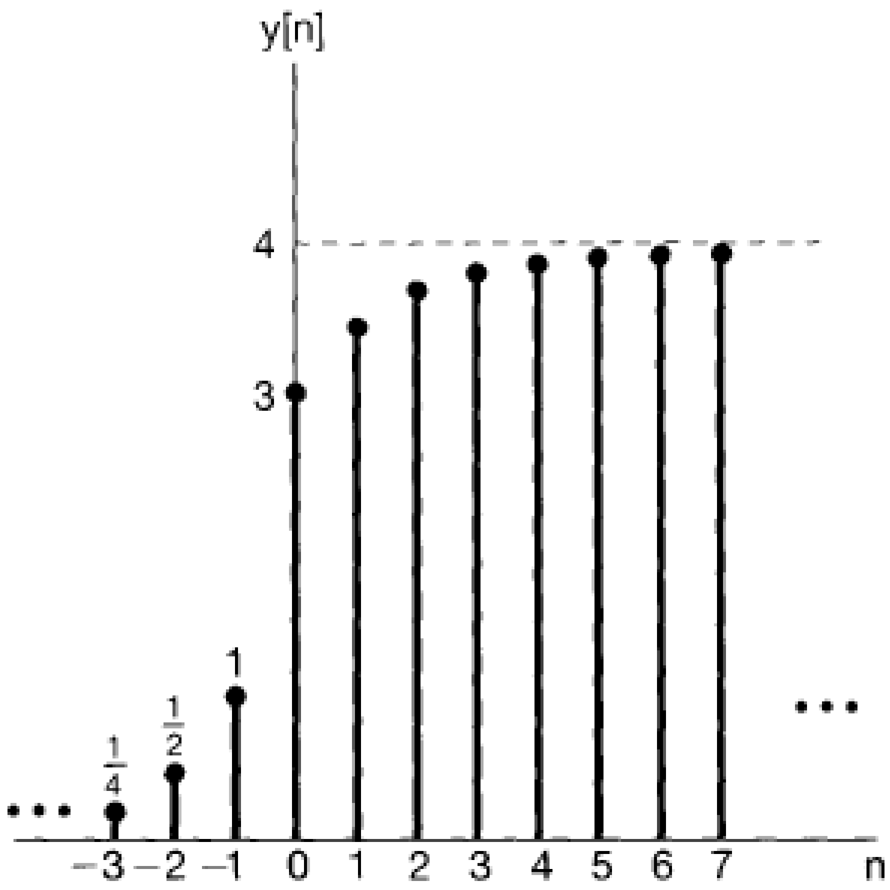
where

$$y_1[n] = x_1[n] * h[n] \qquad (2.56)$$

and

$$y_2[n] = x_2[n] * h[n]. \qquad (2.57)$$

The convolution in eq. (2.56) for $y_1[n]$ can be obtained from Example 2.3 (with $\alpha$ = 1/2), while $y_2[n]$ was evaluated in Example 2.5. Their sum is $y[n]$, which is shown in Figure 2.24.



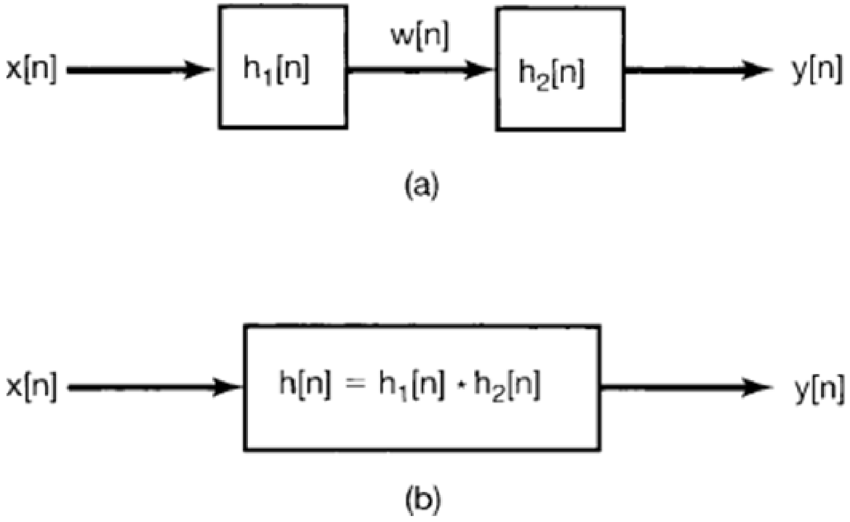
# LTI Systems: Associative Property

- Convolution is associative.

$$x[n] * (h_1[n] * h_2[n]) = (x[n] * h_1[n]) * h_2[n]$$

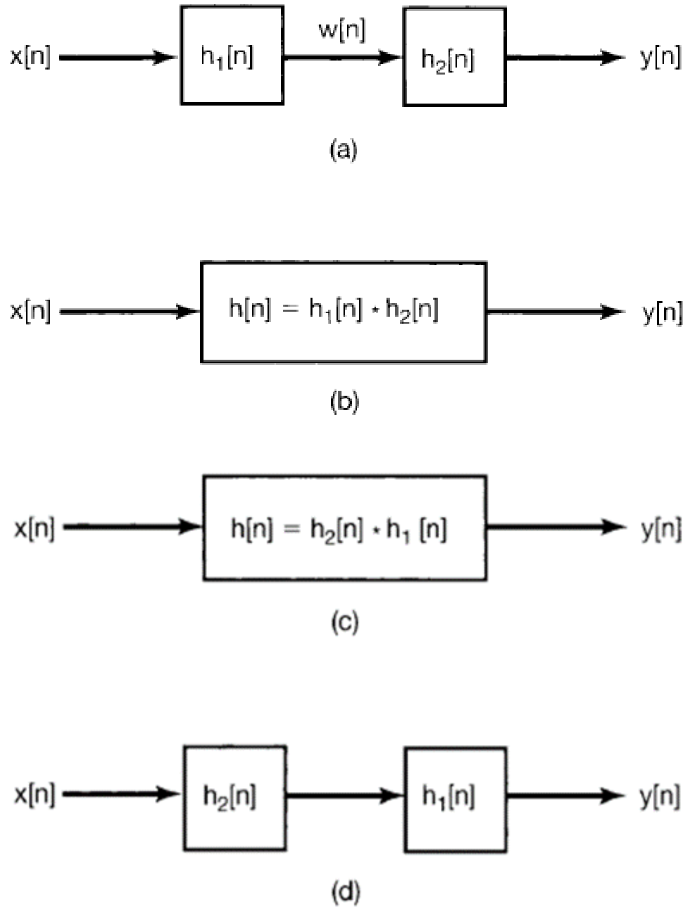
$$\begin{aligned} y[n] &= w[n] * h_2[n] \\ &= (x[n] * h_1[n]) * h_2[n]. \end{aligned}$$

$$\begin{aligned} y[n] &= x[n] * h[n] \\ &= x[n] * (h_1[n] * h_2[n]). \end{aligned}$$



# LTI Systems: Associative Property

- Through the Commutative Property
  - The unit impulse response of a cascade of two LTI systems does not depend on the order in which they are cascaded.



# Memory/Memoryless Systems

- A system is said to have memory if its output at a given time is dependent on the input at that same time as well as past time or future time.
- Discrete-time examples:

Accumulator

$$y[n] = \sum_{k=-\infty}^{n} x[k]$$

Delay

$$y[n] = x[n-1]$$

# LTI Systems without Memory

- If a discrete-time LTI system is memoryless, then:

$$h[n] = 0 \text{ for } n \neq 0$$

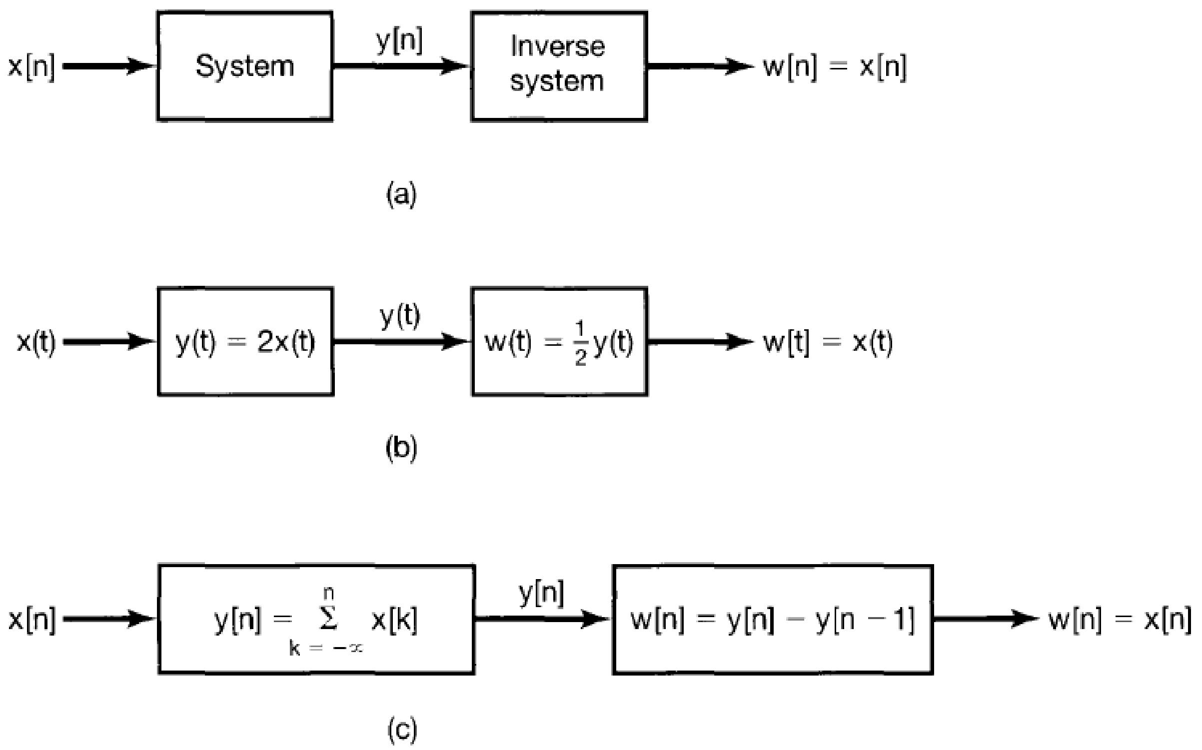
- Why?
  - Hint: $y[n] = \sum_{k=-\infty}^{+\infty} x[k]\, h[n-k]$
- The impulse response has the following form:

$$h[n] = K\delta[n]$$

where $K$ is a constant.

# LTI Systems: Invertibility

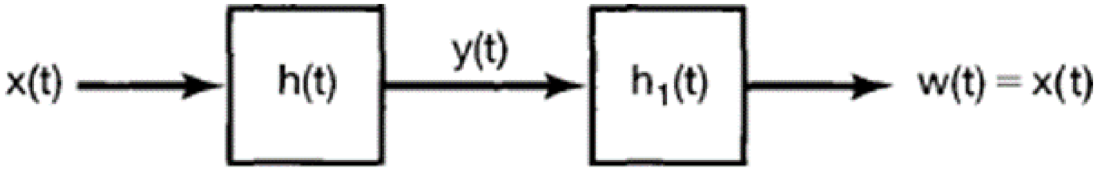
- If a system is invertible, then an inverse system exists that, when cascaded with the original system, yields an output $w[n]$ or $w(t)$ equal to the input $x[n]$ or $x(t)$ to the first system.



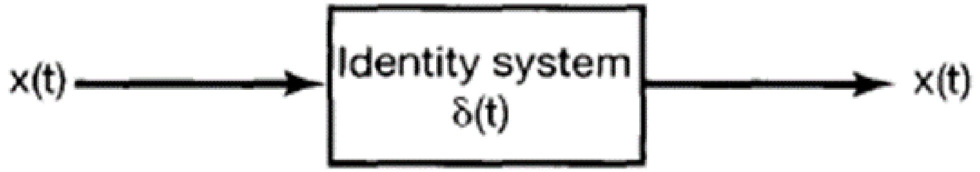
# LTI Systems: Invertibility

- If an LTI system is invertible, then its inverse system is also an LTI system.
- If the inverse system has an impulse response $h_1(t)$, then:

$$h[n] * h_1[n] = \delta[n]$$



(a)



(b)

# LTI Systems: Invertibility

| System | Inverse |
|---|---|
| $y[n] = \sum_{k=-\infty}^{n} x[k]$ | $y[n] = x[n] - x[n-1]$ |

**Impulse Response**

| | |
|---|---|
| $h[n] = u[n]$ | $h_1[n] = \delta[n] - \delta[n-1]$ |

$$
\begin{aligned}
h[n] * h_1[n] &= u[n] * \{\delta[n] - \delta[n-1]\} \\
&= u[n] * \delta[n] - u[n] * \delta[n-1]\} \\
&= u[n] - u[n-1] \\
&= \delta[n]
\end{aligned}
$$

# LTI Systems: Causality

- The output at any time depends only on values of the input at the present time and in the past.
  - Discrete-time Example: $y[n] = \sum_{k=-\infty}^{n} x[k]$
  - Non-Causal Systems E.g.: $y[n] = x[n] - x[n-1]$

$$y[n] = \frac{1}{2M+1} \sum_{k=-M}^{M} x[n-k]$$

# LTI Systems: Causality

- If an LTI system is causal then:

$$h[n] = 0 \text{ for } n < 0$$

- Why?
  - Intuitively, the impulse response of a causal LTI system must be zero before the impulse occurs.
  - Mathematically: $y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$

  $y[n]$ must not depend on $x[k]$ for $k > n$. For this to be true, all of the coefficients $h[n-k]$ that multiply values of $x[k]$ for $k > n$ must be zero.

# Causality and Updated Convolution Formula

$$y(n) = \sum_{k=-\infty}^{+\infty} h(k)\,x(n-k)$$

$$y(n) = \sum_{k=-\infty}^{-1} h(k)\,x(n-k) + \sum_{k=0}^{\infty} h(k)\,x(n-k)$$

$$y(n) = \sum_{k=0}^{\infty} h(k)\,x(n-k)$$

- Here $h[n] = 0$ for $\mathrm{n} < 0$ to ensure stability.

# LTI Systems: Stability

- A discrete-time LTI system is stable if its impulse response is absolutely summable. i.e.

$$\sum_{k=-\infty}^{\infty} |h[k]| < \infty$$

$$|y[n]| = \left| \sum_{k=-\infty}^{+\infty} h[k]x[n-k] \right|.$$

- Why?
  - Consider an input $x[n]$ that is bounded in magnitude: $|x[n]| < B \quad \text{for all } n$
  - Suppose that we apply this input to an LTI system with unit impulse response $h[n]$. Then,

$$|y[n]| = \left| \sum_{k=-\infty}^{\infty} h[k]x[n-k] \right|$$

  - Since the magnitude of the sum of a set of numbers is no larger than the sum of the magnitudes of the numbers, it follows that:

$$|y[n]| \leq \sum_{k=-\infty}^{+\infty} |h[k]||x[n-k]|.$$

# LTI Systems: Stability

- Consider an input $x[n]$ that is bounded in magnitude: $|x[n]| < B \quad \text{for all } n$
- Suppose that we apply this input to an LTI system with unit impulse response $h[n]$. Then,

$$|y[n]| = \left| \sum_{k=-\infty}^{\infty} h[k]x[n-k] \right|$$

- Since the magnitude of the sum of a set of numbers is no larger than the sum of the magnitudes of the numbers, it follows that:

$$|y[n]| = \sum_{k=-\infty}^{\infty} |h[k]||x[n-k]| \qquad\qquad |y[n]| \leq B \sum_{k=-\infty}^{\infty} |h[k]| \qquad \text{for all } n$$

- We can conclude that if the impulse response is absolutely summable, i.e. $\sum_{k=-\infty}^{\infty} |h[k]| < \infty$, then $|y[n]|$ is bounded and thus the LTI system is stable.

# LTI Systems: Stability

- Exercise: Are the following two systems stable?

$$y(t) = x(t-1)$$

$$y[n] = \sum_{k=-\infty}^{n} x[k]$$

# LTI Systems: Unit Step Response

$\delta[n] \rightarrow$ LTI System $\rightarrow h[n]$ = unit impulse response

$u[n] \rightarrow$ LTI System $\rightarrow \boldsymbol{s[n]}$ **= unit step response**

$x[n] \rightarrow$ LTI System $\boldsymbol{h[n]}$ $\rightarrow y[n] = x[n] * h[n]$

$u[n] \rightarrow$ LTI System $\boldsymbol{h[n]}$ $\rightarrow \boldsymbol{s[n] = u[n] * h[n]}$

# Relationship between Unit Step Response and Unit Impulse Response

- The step response of a discrete-time LTI system is the running sum of its impulse response.

$$s[n] = \sum_{k=-\infty}^{n} h[k]$$

- The impulse response of a discrete-time LTI system is the first difference of its step response.

$$h[n] = s[n] - s[n-1]$$