Discrete-Time Fourier Analysis

EE 453 / CE 352
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Response of a Linear Time-Invariant System

$$\delta[n] \longrightarrow \boxed{\text{LTI System}} \longrightarrow h[n]$$

$$x[n] \longrightarrow \boxed{\text{LTI System}} \longrightarrow y(n) = x[n] * h[n]$$

- Any signal can be represented by a linear combination of scaled and delayed unit samples.
- We can also represent any arbitrary discrete signal as a linear combination of basic signals.
- When the system is linear and time-invariant, only one representation stands out as the most useful. It is based on the complex exponential signal set $e^{j\omega n}$ and called the discrete-time Fourier transform.

Discrete-Time Fourier Transform

- If $x(n)$ is absolutely summable, that is, $\sum_{-\infty}^{\infty}|x(n)| < \infty$, then its discrete-time Fourier transform is given by:

$$X\left(e^{j\omega}\right) \triangleq \mathcal{F}[x(n)] = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}$$

- The inverse discrete-time Fourier transform (IDTFT) is given by:

$$x(n) \triangleq \mathcal{F}^{-1}\left[X\left(e^{j\omega}\right)\right] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X\left(e^{j\omega}\right)e^{j\omega n}d\omega$$

DT Fourier Transform: Practice

❑Determine the DTFT of $x(n) = \{1,\underset{\uparrow}{2},3,4,5\}$.

- Solution:

$$X(e^{j\omega}) = \sum_{0}^{\infty} x(n)e^{-j\omega n}$$

$$X(e^{j\omega}) = e^{j\omega} + 2 + 3e^{-j\omega} + 4e^{-j2\omega} + 5e^{-j3\omega}$$

DT Fourier Transform: Practice

❑ Determine the DTFT of $x(n) = (0.5)^n u(n)$.

- Solution:

$$X(e^{j\omega}) = \sum_{0}^{\infty} (0.5)^n e^{-j\omega n} = \sum_{0}^{\infty} (0.5e^{-j\omega})^n$$

$$X(e^{j\omega}) = \frac{1}{1 - 0.5e^{-j\omega}} = \frac{e^{j\omega}}{e^{j\omega} - 0.5}$$

DT Fourier Transform: Example

- Example: $x[n] = a^{|n|}, \quad |a| < 1$
- DTFT:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} a^{|n|} e^{-j\omega n}$$

$$X(e^{j\omega}) = \sum_{n=0}^{\infty} a^{n} e^{-j\omega n} + \sum_{n=-\infty}^{-1} a^{-n} e^{-j\omega n}$$

- Substituting $m = -n$ for the second summation:

$$X(e^{j\omega}) = \sum_{n=0}^{\infty} (ae^{-j\omega})^{n} + \sum_{n=-\infty}^{-1} (ae^{-j\omega})^{m}$$

- Both summations are infinite geometric series:

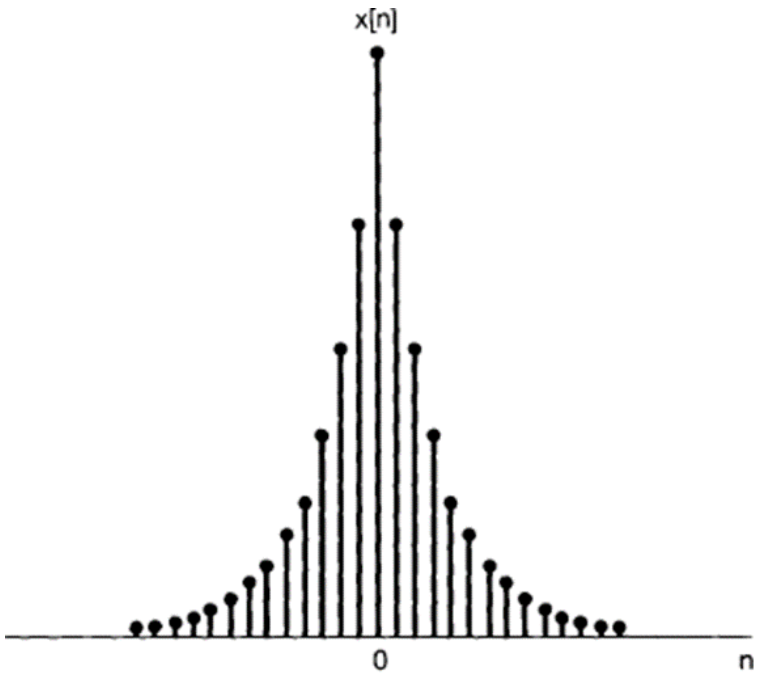
$$X(e^{j\omega}) = \frac{1}{1 - ae^{-j\omega}} + \frac{ae^{-j\omega}}{1 - ae^{-j\omega}}$$

$$X(e^{j\omega}) = \frac{1 - a^2}{1 - 2a\cos\omega + a^2}$$

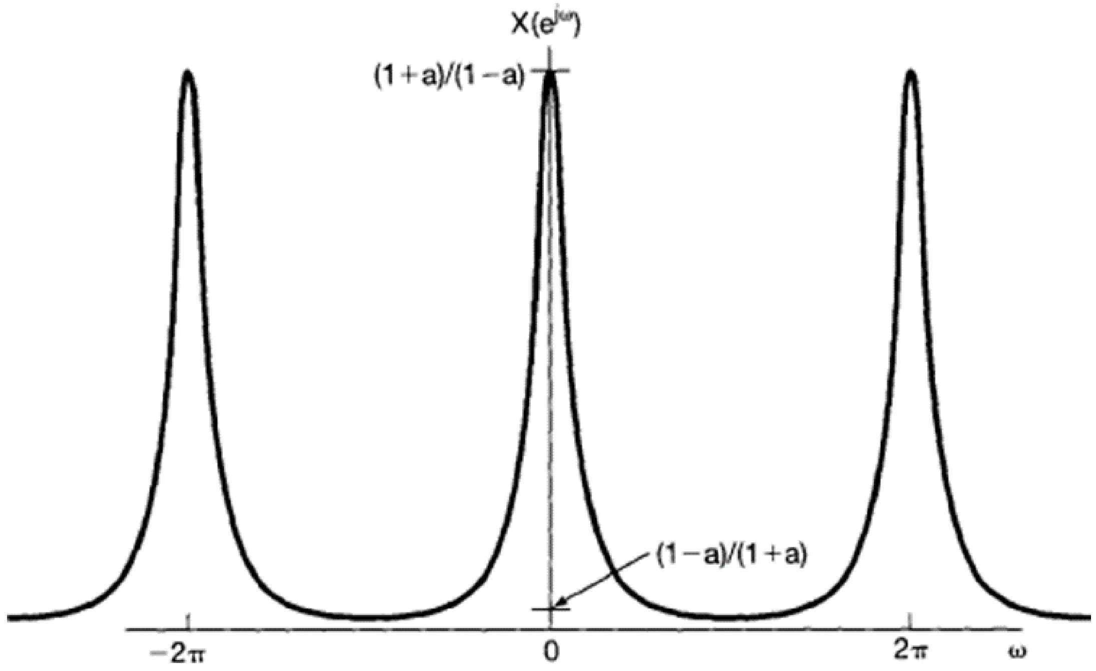
DT Fourier Transform: Example

$$x[n] = a^{|n|}, \qquad 0 < a < 1$$

$$X(e^{j\omega}) = \frac{1 - a^2}{1 - 2a\cos\omega + a^2}$$



(a)



(b)

Common DTFT Pairs

Signal Type	*Sequence* $x(n)$	DTFT $X\left(e^{j\omega}\right)$, $-\pi \leq \omega \leq \pi$
Unit impulse	$\delta(n)$	1
Constant	1	$2\pi\delta(\omega)$
Unit step	$u(n)$	$\dfrac{1}{1-e^{-j\omega}} + \pi\delta(\omega)$
Causal exponential	$\alpha^n u(n)$	$\dfrac{1}{1-\alpha e^{-j\omega}}$
Complex exponential	$e^{j\omega_0 n}$	$2\pi\delta(\omega-\omega_0)$
Cosine	$\cos(\omega_0 n)$	$\pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]$
Sine	$\sin(\omega_0 n)$	$j\pi[\delta(\omega+\omega_0)-\delta(\omega-\omega_0)]$
Double exponential	$\alpha^{\lvert n\rvert}$	$\dfrac{1-\alpha^2}{1-2\alpha\cos(\omega)+\alpha^2}$

Note: Since $X\left(e^{j\omega}\right)$ is periodic with period 2π, expressions over only the primary period of $-\pi \leq \omega \leq \pi$ are given.

Properties of DTFT: Periodicity

Periodicity: The discrete-time Fourier transform $X(e^{j\omega})$ is periodic in ω with period 2π.

$$X(e^{j\omega}) = X(e^{j[\omega+2\pi]})$$

Implication: We need only one period of $X(e^{j\omega})$ (i.e., $\omega \in [0, 2\pi]$, or $[-\pi, \pi]$, etc.) for analysis and not the whole domain $-\infty < \omega < \infty$.

Properties of DTFT: Complex Conjugate

- If $x[n] \overset{\mathcal{F}}{\leftrightarrow} X(e^{j\omega})$, then:

$$x^*[n] \overset{\mathcal{F}}{\leftrightarrow} X^*(e^{-j\omega})$$

- Interesting Consequences:
  - If $x[n]$ is real, then Fourier Transform $X(e^{j\omega})$ has conjugate symmetry.

$$X\left(e^{j\omega}\right) = X^*(e^{-j\omega})$$

  - If $x[n]$ is real, the real part of the Fourier transform is an even function of frequency (ω), and the imaginary part is an odd function of frequency (ω).
  - If $x[n]$ is real, the magnitude of the Fourier transform is an even function of frequency (ω), and the phase part is an odd function of frequency (ω).

Properties of DTFT: Convolution

- If:

$$x[n] \overset{\mathcal{F}}{\leftrightarrow} X(e^{j\omega})$$
$$h[n] \overset{\mathcal{F}}{\leftrightarrow} H(e^{j\omega})$$
$$y[n] \overset{\mathcal{F}}{\leftrightarrow} Y(e^{j\omega})$$

- Then: $y[n] = x[n] * h[n] \overset{\mathcal{F}}{\leftrightarrow} Y\left(e^{j\omega}\right) = X(e^{j\omega})\, H(e^{j\omega})$
- Interpretation:
  - The Fourier Transform maps convolution of two signals in time domain into product of their Fourier transforms.

Other Properties of DTFT

Linearity: The discrete-time Fourier transform is a linear transformation; that is,

$$\mathcal{F}\left[\alpha x_1(n) + \beta x_2(n)\right] = \alpha \mathcal{F}\left[x_1(n)\right] + \beta \mathcal{F}\left[x_2(n)\right]$$

for every α, β, $x_1(n)$, and $x_2(n)$.

Time shifting: A shift in the time domain corresponds to the phase shifting.

$$\mathcal{F}\left[x(n-k)\right] = X(e^{j\omega})e^{-j\omega k}$$

Frequency shifting: Multiplication by a complex exponential corresponds to a shift in the frequency domain.

$$\mathcal{F}\left[x(n)e^{j\omega_0 n}\right] = X(e^{j(\omega-\omega_0)})$$

Other Properties of DTFT

Folding: Folding in the time domain corresponds to the folding in the frequency domain.

$$\mathcal{F}\left[x(-n)\right] = X(e^{-j\omega})$$

Symmetries in real sequences: We have already studied the conjugate symmetry of real sequences. These real sequences can be decomposed into their even and odd parts, as discussed in Chapter 2.

$$x(n) = x_{\mathrm{e}}(n) + x_{\mathrm{o}}(n)$$

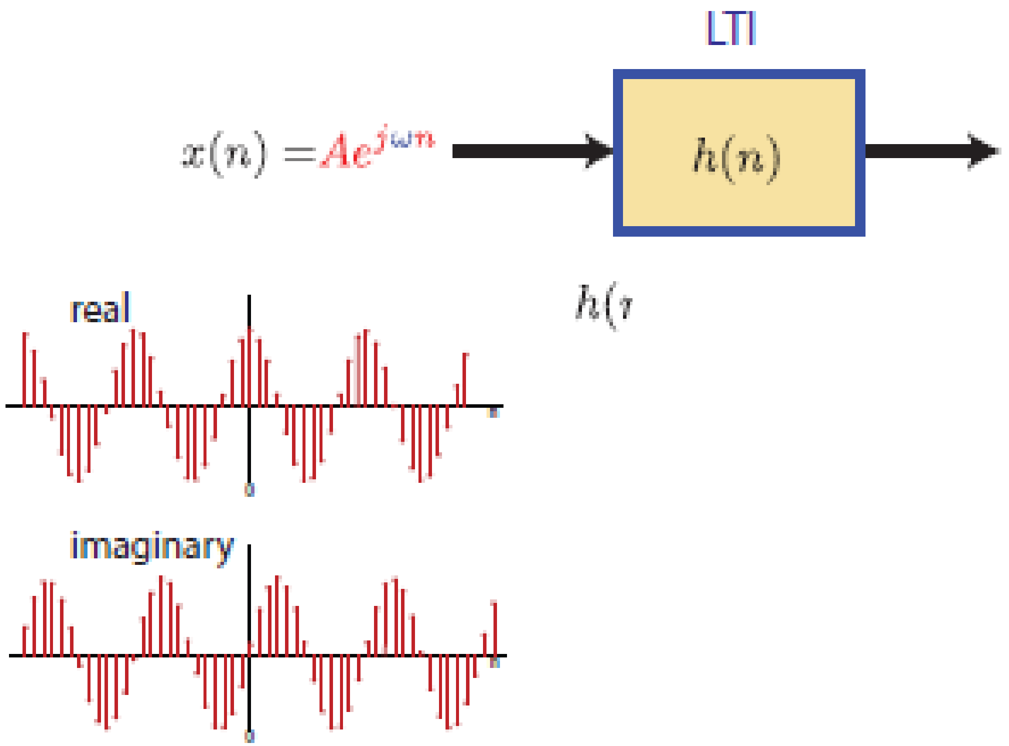
Then

$$\mathcal{F}\left[x_{\mathrm{e}}(n)\right] = \mathrm{Re}\left[X(e^{j\omega})\right]$$

$$\mathcal{F}\left[x_{\mathrm{o}}(n)\right] = j\,\mathrm{Im}\left[X(e^{j\omega})\right]$$

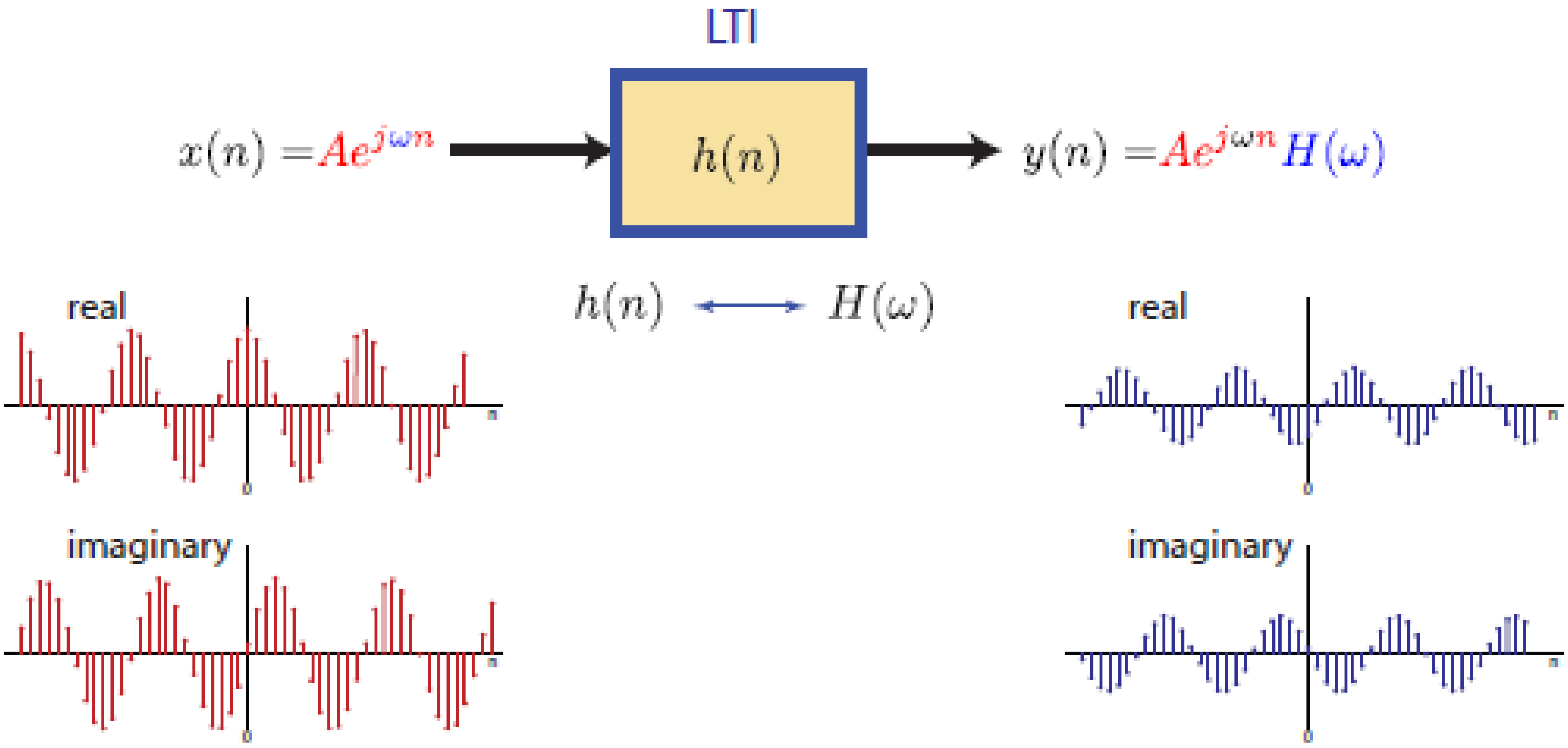
Implication: If the sequence $x(n)$ is real and even, then $X(e^{j\omega})$ is also real and even. Hence only one plot over $[0, \pi]$ is necessary for its complete representation.

DT LTI System with Complex Exponential as Input



$$
\begin{aligned}
\therefore\ y(n) &= \sum_{k=-\infty}^{\infty} h(k) A e^{j\omega(n-k)} \\
&= \sum_{k=-\infty}^{\infty} h(k) A e^{j\omega n} \cdot e^{-j\omega k} \\
&= A e^{j\omega n} \cdot \underbrace{\left[\sum_{k=-\infty}^{\infty} h(k) e^{-j\omega k} \right]}_{\equiv H(\omega) = DTFT\{h(n)\}} \\
&= A e^{j\omega n} H(\omega)
\end{aligned}
$$

DT LTI System with Complex Exponential as Input



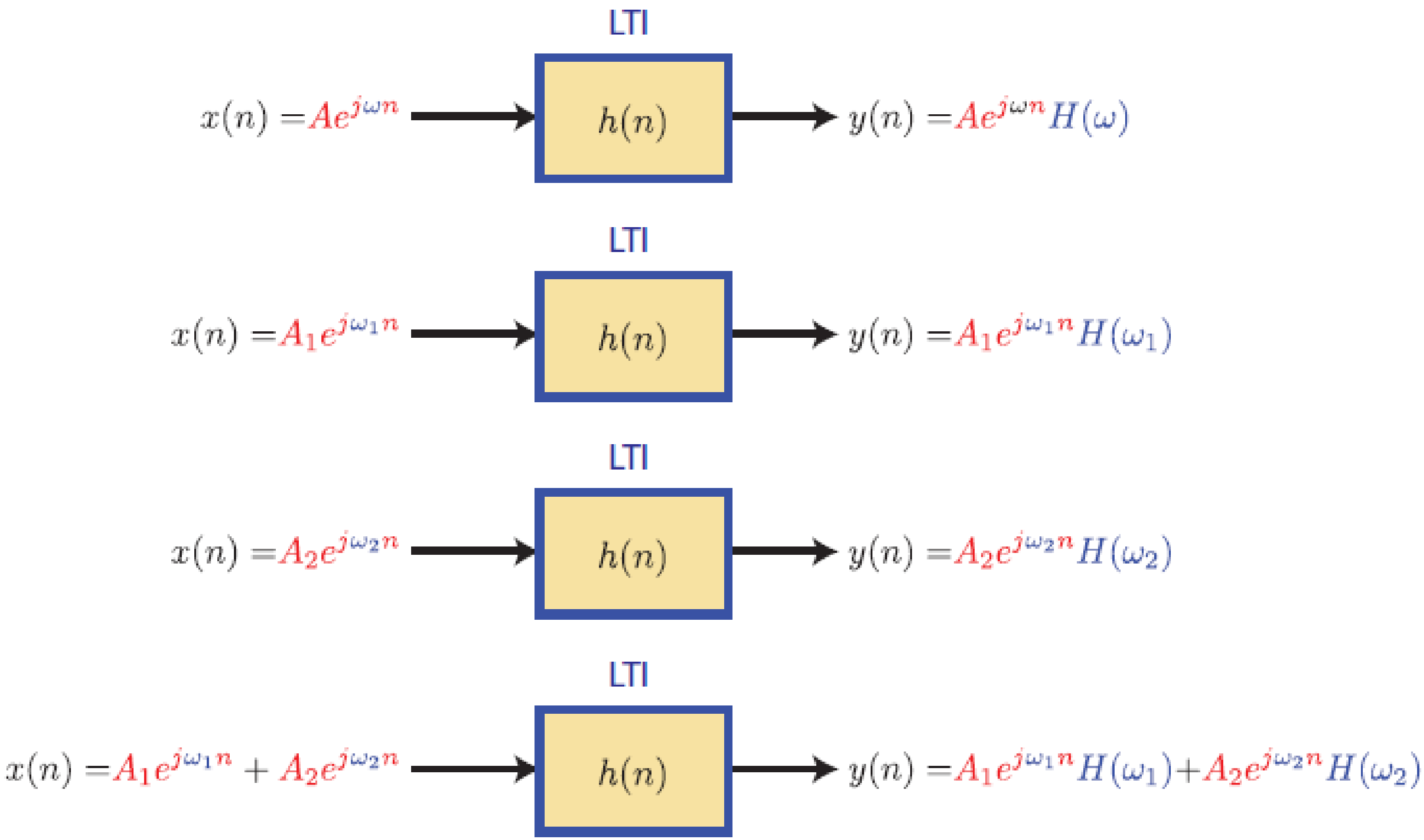
Complex Exponential Signals as “Basic Signals”

- An appropriate set of basic signals must have the following two properties:
  - The set of basic signals can be used to construct a broad and useful class of signals.
  - The response of an LTI system to each basic signal should be “simple”.
    - Provide us with a convenient representation for the system response to any signal constructed as a linear combination of the basic signals.
- The response of an LTI system to a complex exponential input is the same complex exponential with only a change in amplitude: $e^{j\omega n} \rightarrow H(e^{j\omega})e^{j\omega n}$
- Where:

$$H\left(e^{j\omega}\right) = \sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n}$$

is described as the frequency response of the system.

DT LTI System with Complex Exponential as Input



LTI System Response to a Complex Exponential

$$\boxed{x(n) = e^{j\omega_0 n} \longrightarrow \boxed{H(e^{j\omega})} \longrightarrow y(n) = H(e^{j\omega_0}) \times e^{j\omega_0 n}}$$

In general, the frequency response $H(e^{j\omega})$ is a complex function of ω. The magnitude $|H(e^{j\omega})|$ of $H(e^{j\omega})$ is called the *magnitude (or gain) response* function, and the angle $\angle H(e^{j\omega})$ is called the *phase response* function.

LTI System Response to Sinusoidal Sequences

Let $x(n) = A\cos(\omega_0 n + \theta_0)$ be an input to an LTI system $h(n)$. Then we can show that the response $y(n)$ is another sinusoid of the same frequency ω_0, with amplitude *gained* by $|H(e^{j\omega_0})|$ and phase *shifted* by $\angle H(e^{j\omega_0})$, that is,

$$y(n) = A|H(e^{j\omega_0})|\cos(\omega_0 n + \theta_0 + \angle H(e^{j\omega_0}))$$

This response is called the *steady-state response*, denoted by $y_{ss}(n)$. It can be extended to a linear combination of sinusoidal sequences.

$$\boxed{\sum_k A_k \cos(\omega_k n + \theta_k) \longrightarrow \boxed{H(e^{j\omega})} \longrightarrow \sum_k A_k |H(e^{j\omega_k})| \cos(\omega_k n + \theta_k + \angle H(e^{j\omega_k}))}$$

Response Function from Difference Equations

When an LTI system is represented by the difference equation

$$y(n) + \sum_{\ell=1}^{N} a_\ell y(n-\ell) = \sum_{m=0}^{M} b_m x(n-m)$$

We know that when $x(n) = e^{j\omega n}$, then $y(n)$ must be $H(e^{j\omega})e^{j\omega n}$.

$$H(e^{j\omega})e^{j\omega n} + \sum_{\ell=1}^{N} a_\ell H(e^{j\omega})e^{j\omega(n-\ell)} = \sum_{m=0}^{M} b_m\ e^{j\omega(n-m)}$$

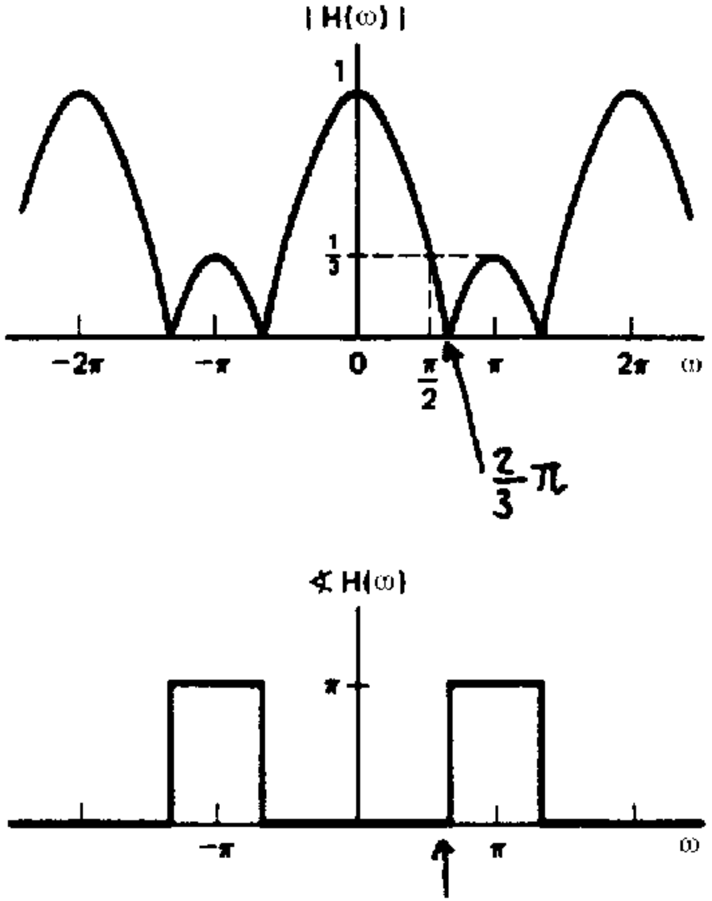
or

$$\boxed{H(e^{j\omega}) = \frac{\sum_{m=0}^{M} b_m\ e^{-j\omega m}}{1 + \sum_{\ell=1}^{N} a_\ell\ e^{-j\omega\ell}}}$$

Frequency Response: Example

- Determine the magnitude and phase of $H(\omega)$ for the three-point moving average system:

$$y(n) = \frac{1}{3}[x(n+1) + x(n) + x(n-1)]$$



Frequency Response: Practice

- Determine the frequency response $H(e^{j\omega})$ of a system characterized by $h(n) = (0.9)^n u(n)$. Plot the magnitude and phase responses.

$$X(e^{j\omega}) = \sum_{0}^{\infty} (0.9)^n e^{-j\omega n} = \frac{1}{1 - 0.9e^{-j\omega}}$$

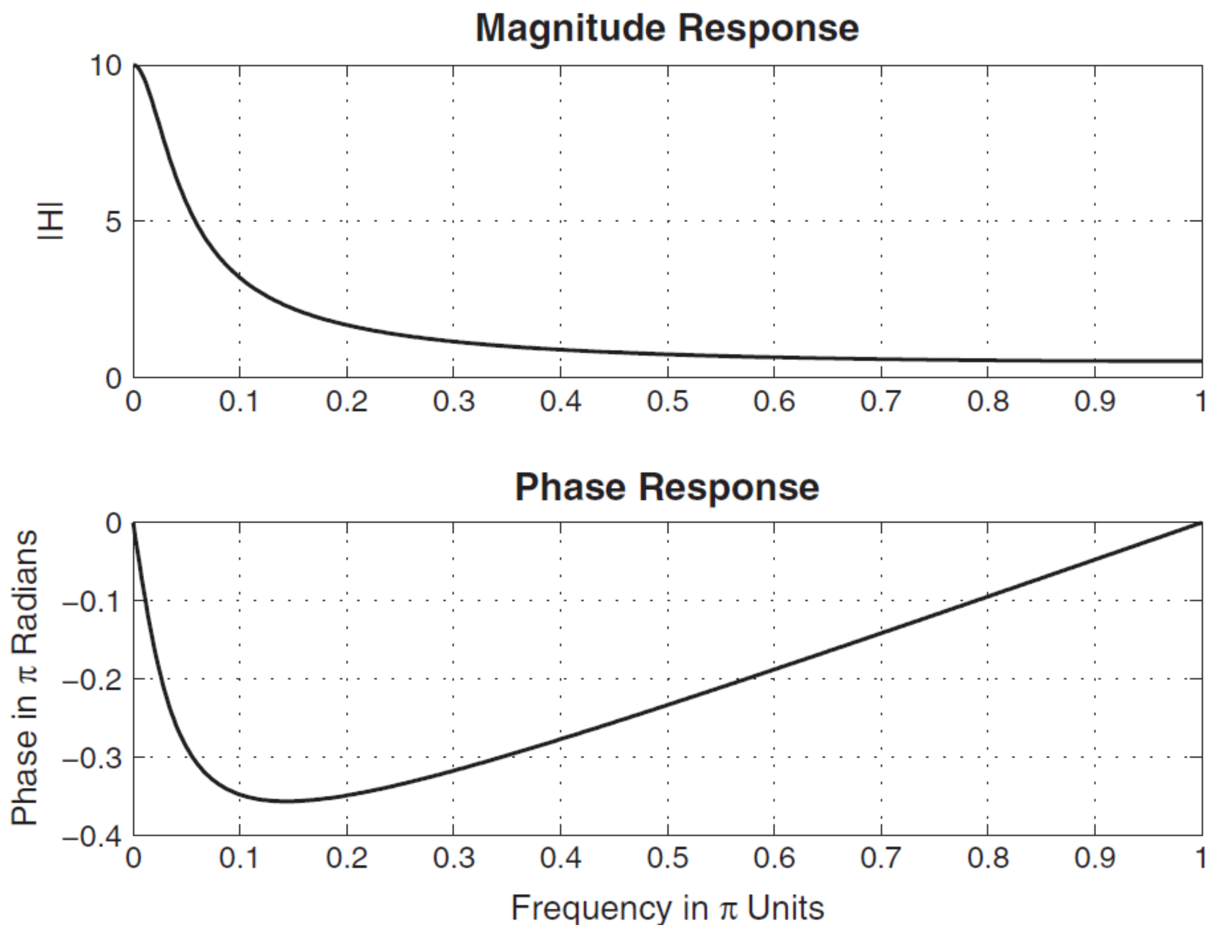
$$|H(e^{j\omega})| = \sqrt{\frac{1}{(1 - 0.9\cos\omega)^2 + (0.9\sin\omega)^2}} = \frac{1}{\sqrt{1.81 - 1.8\cos\omega}}$$

$$\angle H(e^{j\omega}) = -\tan^{-1}\left(\frac{0.9\sin\omega}{1 - 0.9\cos\omega}\right)$$

Frequency Response: Practice

$$\left|H\left(e^{j\omega}\right)\right| = \frac{1}{\sqrt{1.81 - 1.8\cos\omega}}$$

$$\angle H\left(e^{j\omega}\right) = -\tan^{-1}\left(\frac{0.9\sin\omega}{1 - 0.9\cos\omega}\right)$$



An Interpretation of the Convolution Property

$$\boxed{y[n] = x[n] * h[n] \overset{\mathcal{F}}{\leftrightarrow} Y(e^{j\omega}) = X(e^{j\omega})\, H(e^{j\omega})}$$

- An LTI system can only change the amplitude and phase of a sinusoidal signal. It cannot change the frequency.
- Fourier Transform (frequency components) of the output of an LTI system is the Fourier Transform (frequency components) of the input, multiplied by the frequency response of the system
- This creates the possibility to change the relative amplitude of the frequency components in a signal.
- This process is referred to as filtering.

LTI System as "Frequency Selective Filter"

- A filter is a device that discriminates, according to some attribute of the input, what passes through it.
- For LTI systems, given $Y(\omega) = X(\omega)H(\omega)$, $H(\omega)$ acts as a kind of weighting function or spectral shaping function of the different frequency components of the signal.