Digital FIR Filters

EE 453 / CE 352

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FIR vs IIR Filters

- FIR filters: normally used when there is a requirement of linear phase
  - FIR filter with the following symmetry is linear phase:

$$h(n) = \pm h(M-1-n) \qquad n = 0, 1, 2, \ldots, M-1$$

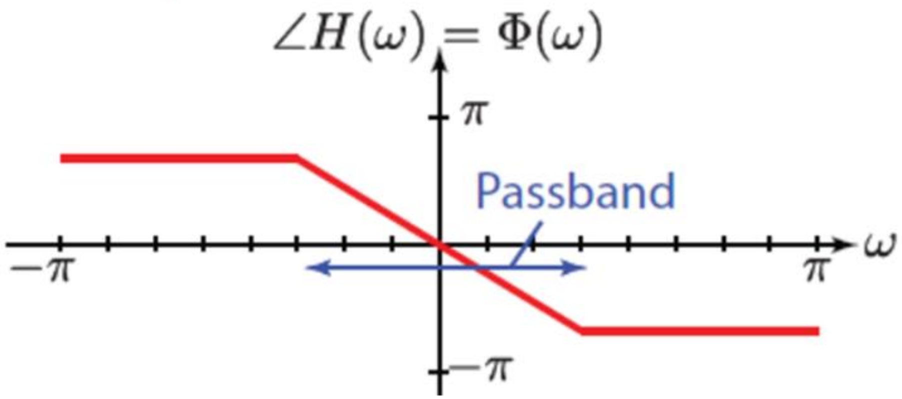
- IIR filters: normally used when linear phase is not required and cost effectiveness is needed
  - IIR filter has lower sidelobes in the stopband than an FIR having the same number of parameters
  - if some phase distortion is tolerable, an IIR filter has an implementation with fewer parameters requiring less memory and lower complexity

Filter Specifications: Phase

- At the specification stage, we need to decide whether we need a linear phase filter or not.
- The linear phase is a straight line in the passband of the system.

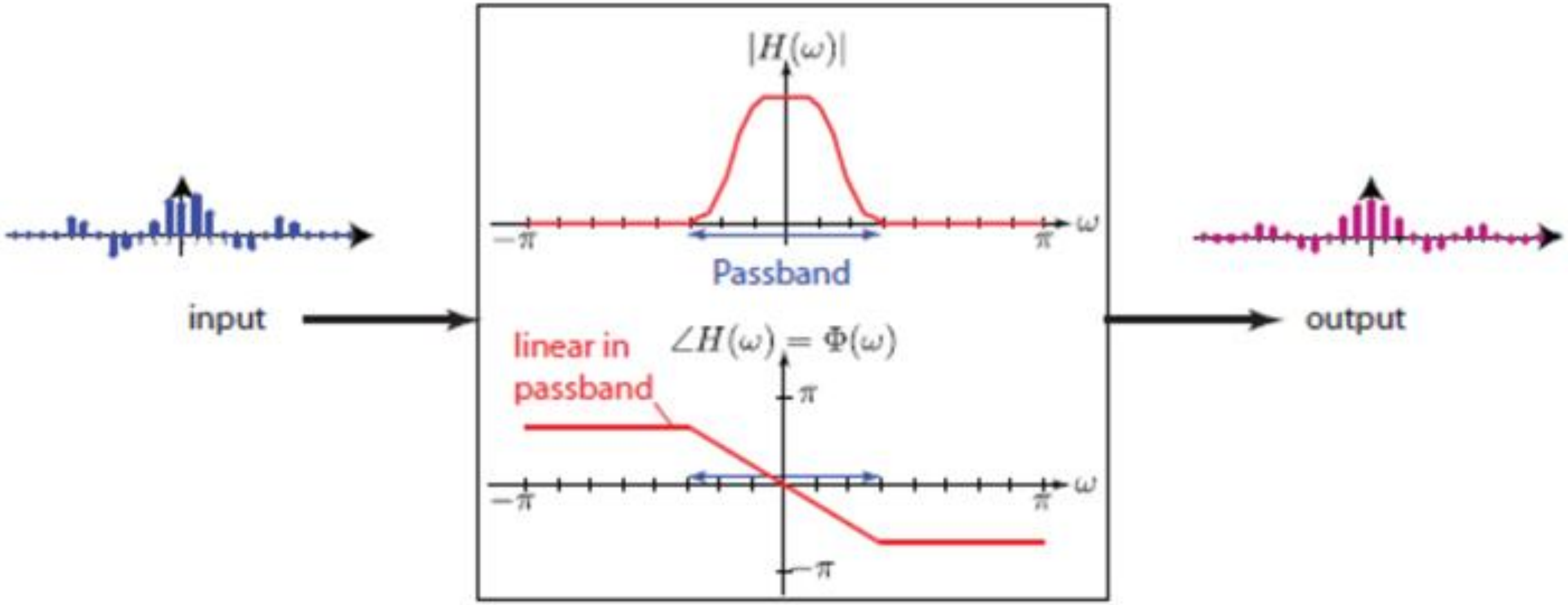
Example: linear phase (low pass system)

- Linear characteristics only need to pertain to the passband frequencies only.



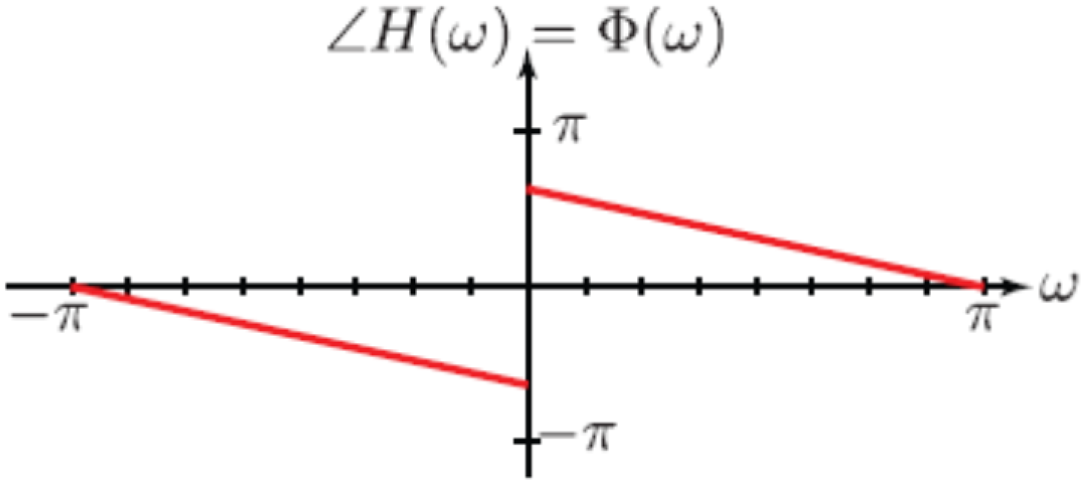
Filter Specifications: Phase

- Linear phase filters maintain the relative positioning of the sinusoids in the filter passband.
- This maintains the structure of the signal while removing unwanted frequency components.

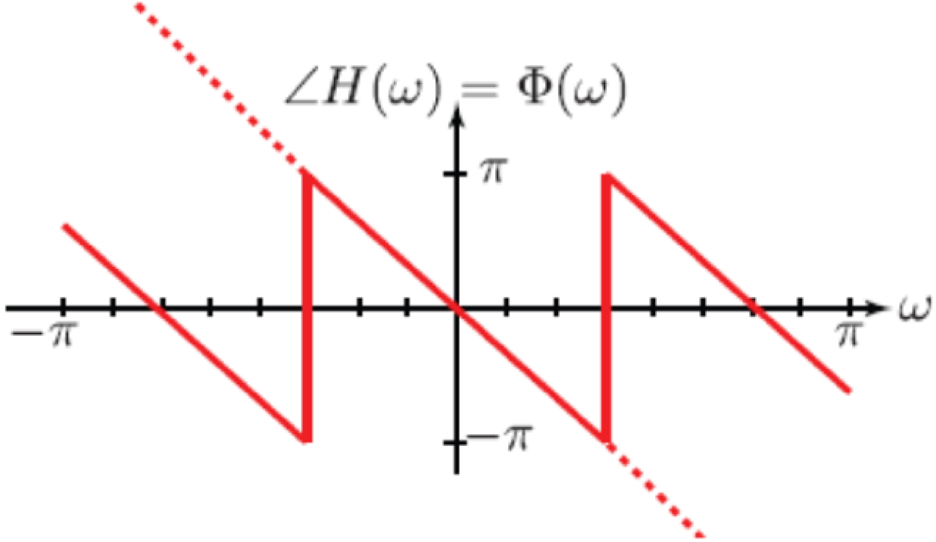


Filter Specifications: Phase

- **Example:** High Pass System with linear phase.
  - Discontinuities at the origin still correspond to a linear phase system.



- **Example:** All Pass System with linear phase.
  - Phase wrapping may occur, but the phase is still considered to be linear.



Symmetric FIR Filters

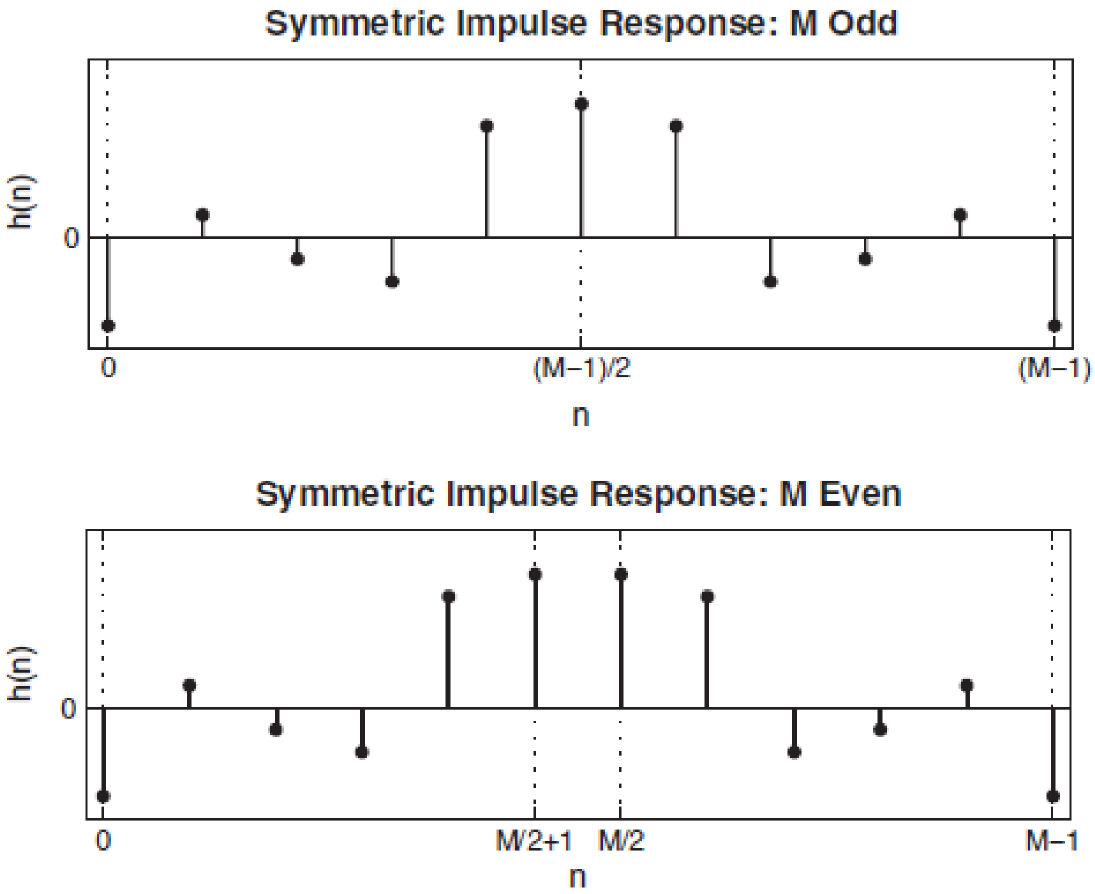
- Symmetric FIR filters are easier to design and provide a linear phase response, which makes them useful in many applications.

$$h[n] = \pm h[M-1-n]$$

- Phase function:

$$\angle H\left(e^{j\omega}\right) = -\alpha\omega$$

  - α is the constant phase delay.

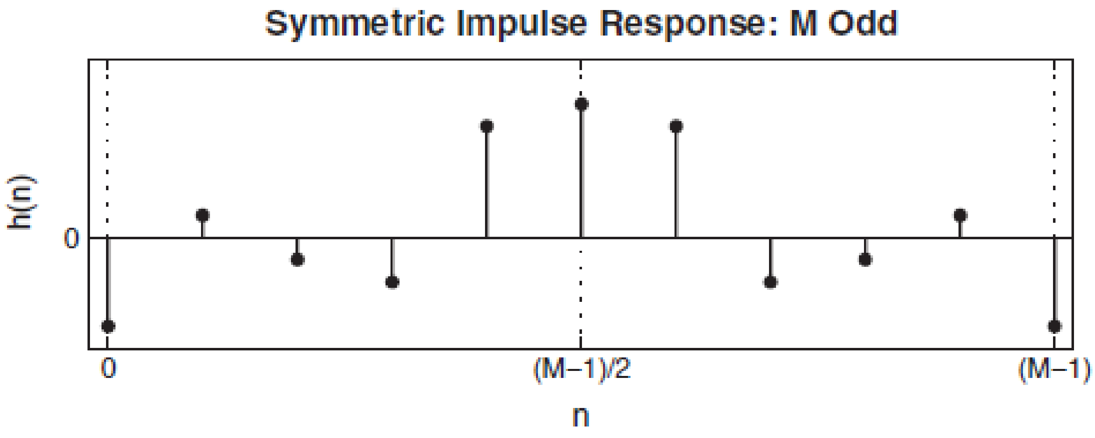


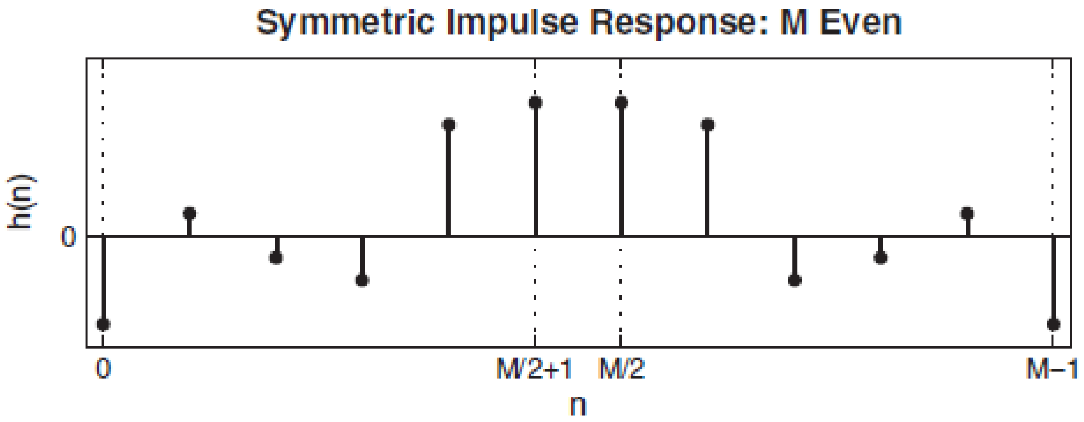
Symmetric FIR Filters

$$h[n] = \pm h[M-1-n]$$

- Index of symmetry: $\alpha = \frac{M-1}{2}$

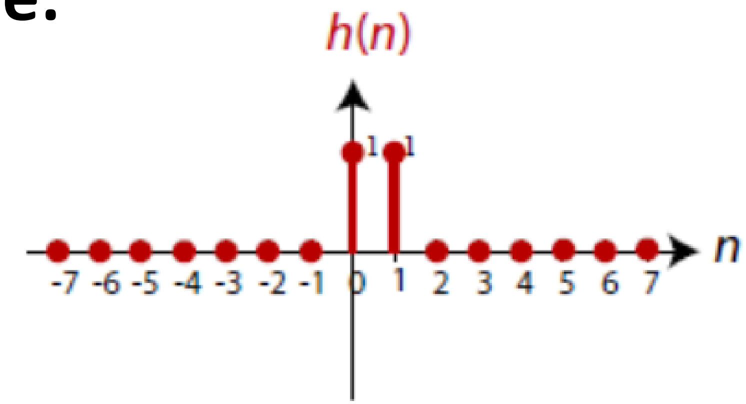
1. Odd-length (Type I)
   - M is odd, length is from 0 to $M-1$, α is an integer.
   - Most versatile.
2. Even-length (Type II)
   - M is even, length is from 0 to $M-1$, α is not an integer.
   - Not suitable for HPF.

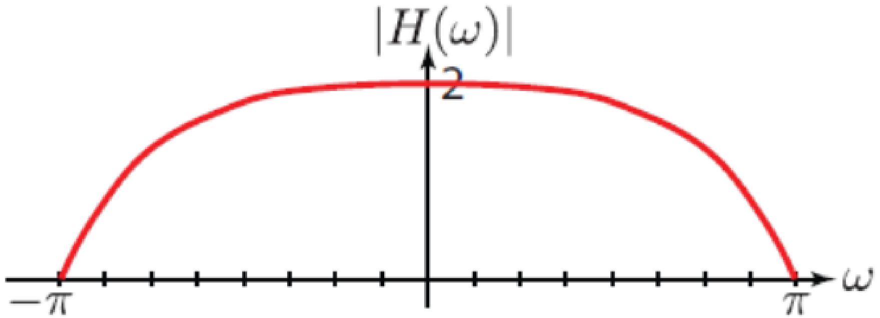




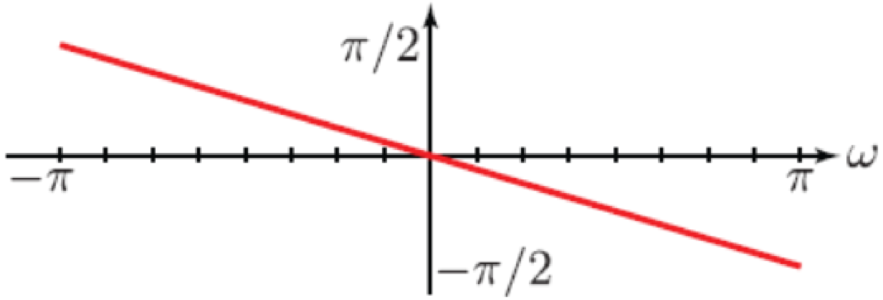
Symmetric FIR Filters

- **Example:**





$M = 2, h(n) = +h(1-n)$ for $n = 0,1.$



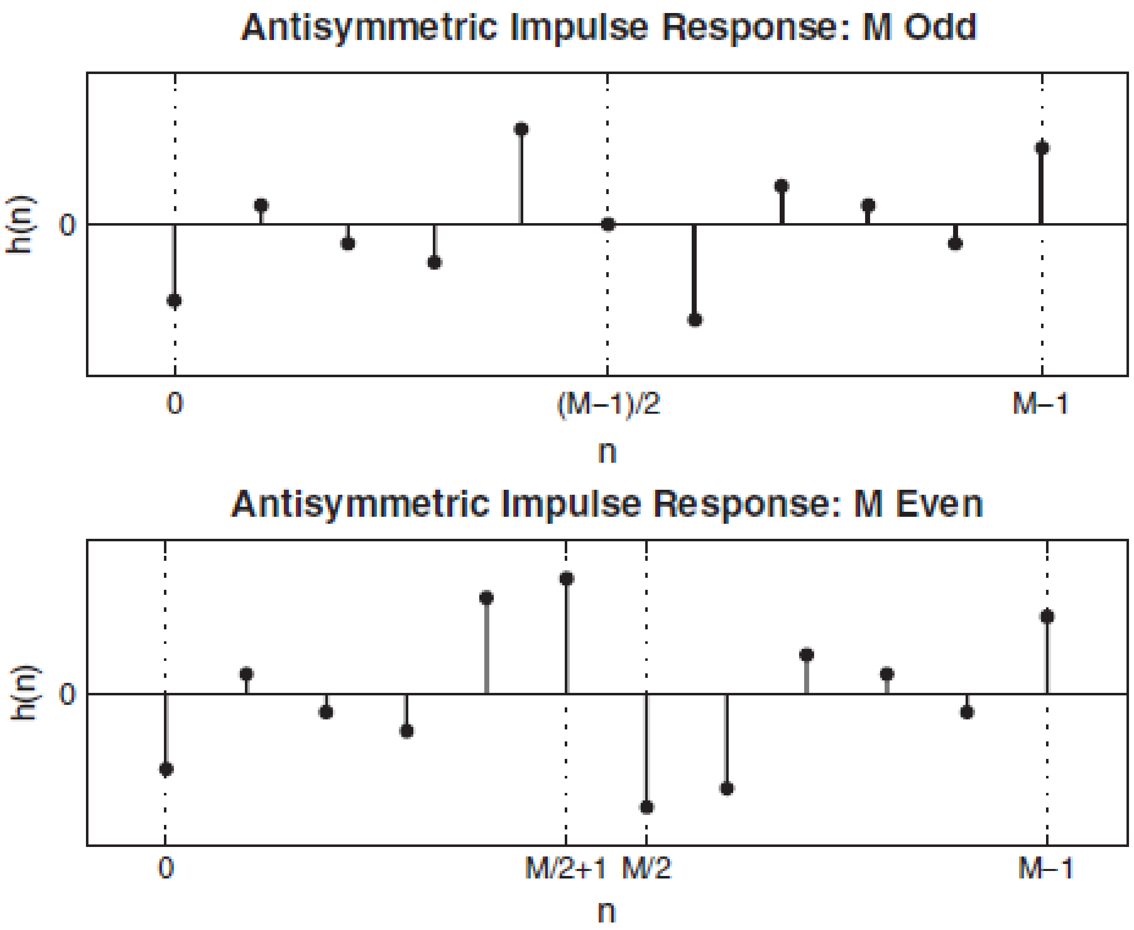
- For $n = 0$: $h(0) = +h(1-0) = 1$
- For $n = 1$, $h(1) = +h(1-1) = 1$

Antisymmetric FIR Filters

- Antisymmetric FIR filters are useful when a specific frequency response needs to be achieved that cannot be obtained with a symmetric filter.
- Phase function:

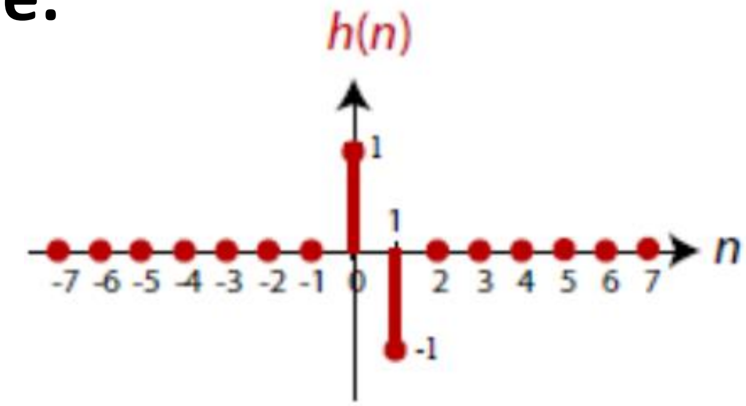
$$\angle H(\omega) = \frac{\pi}{2} - \alpha\omega$$

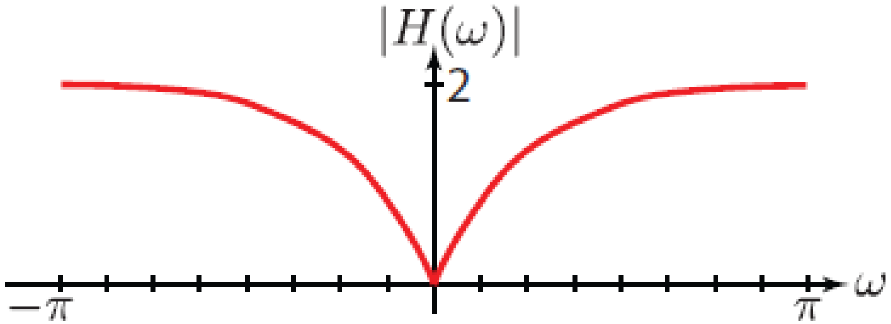
1. Odd-length (Type III)
2. Even-length(Type IV)



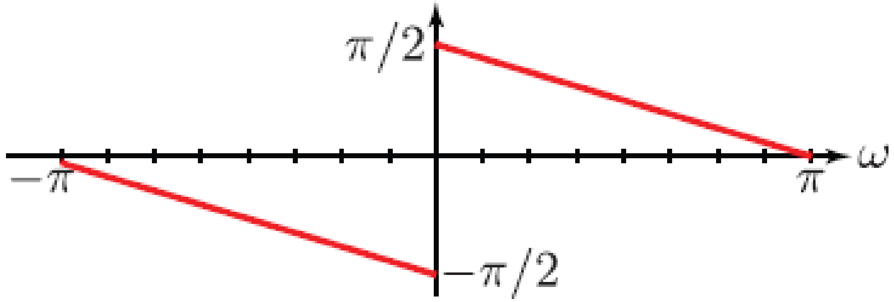
Antisymmetric FIR Filters

- **Example:**





$M = 2, h(n) = -h(1-n)$ for $n = 0,1.$

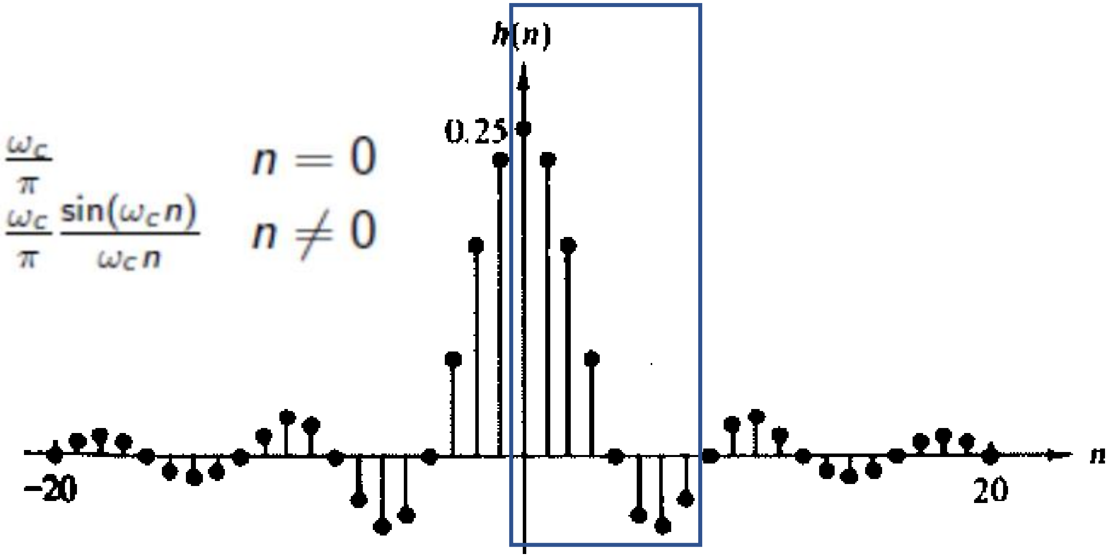


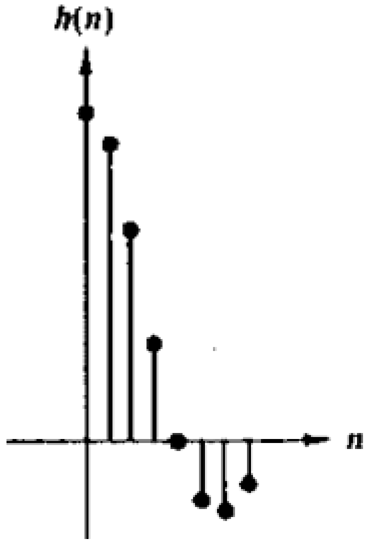
- For $n = 0$: $h(0) = -h(1-0) = 1$
- For $n = 1$, $h(1) = -h(1-1) = -1$

Filter Realization via Truncation

- Ideal filters are non-causal and hence cannot be realized in practice.
- We can clip off the response at some large value of n.
- Choose a proper ideal frequency-selective filter (noncausal and infinite impulse response), then truncate it to obtain a linear-phase and causal FIR filter.

$$h(n) = \begin{cases} \frac{\omega_c}{\pi} & n = 0 \\ \frac{\omega_c}{\pi} \frac{\sin(\omega_c n)}{\omega_c n} & n \neq 0 \end{cases}$$





Rectangular Window Function

- If we consider a rectangular window:

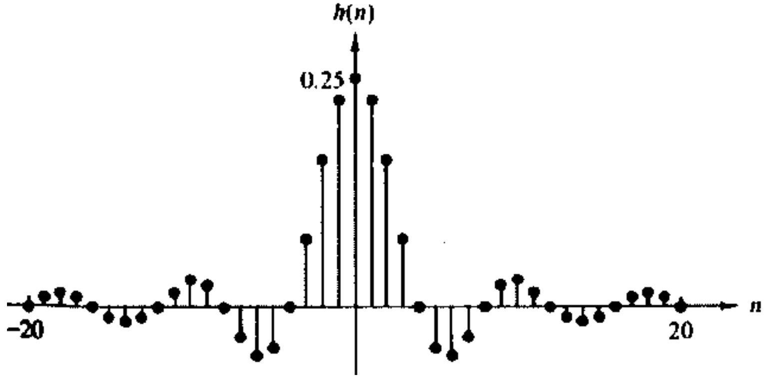
$$w_R[n] = \begin{cases} 1 & 0 \leq n \leq M-1 \\ 0 & \text{else} \end{cases}$$

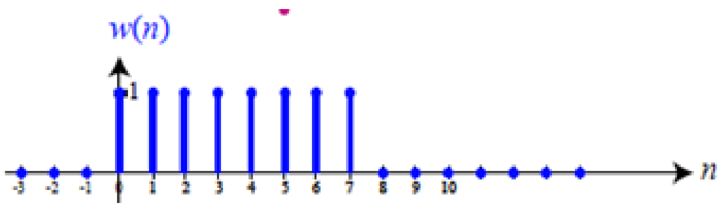
- Then:

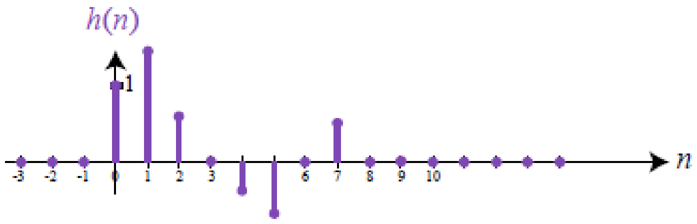
$$W_R(e^{j\omega}) = e^{-j\omega\frac{M-1}{2}} \frac{\sin(\omega M/2)}{\sin(\omega/2)}$$

- We can consider this a digital sinc function.
- W_R has a piecewise linear phase.

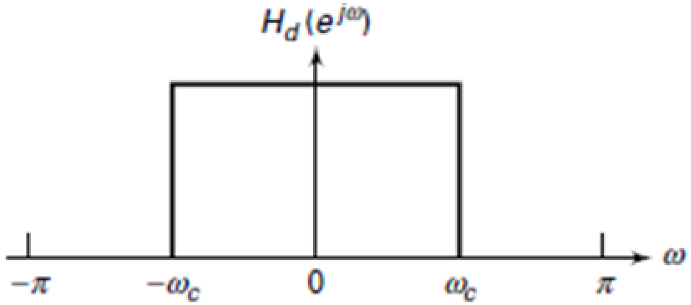
Windowing

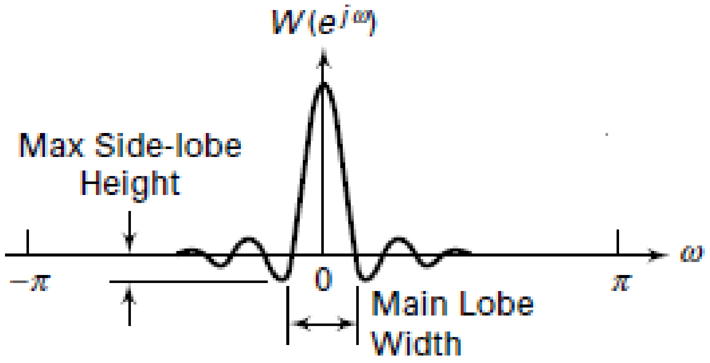


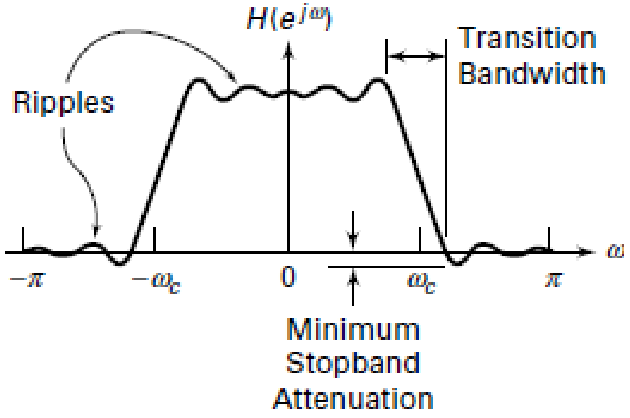




$$h_d(n)w(n) \xleftrightarrow{\mathcal{F}} H_d(\omega) * W(\omega)$$

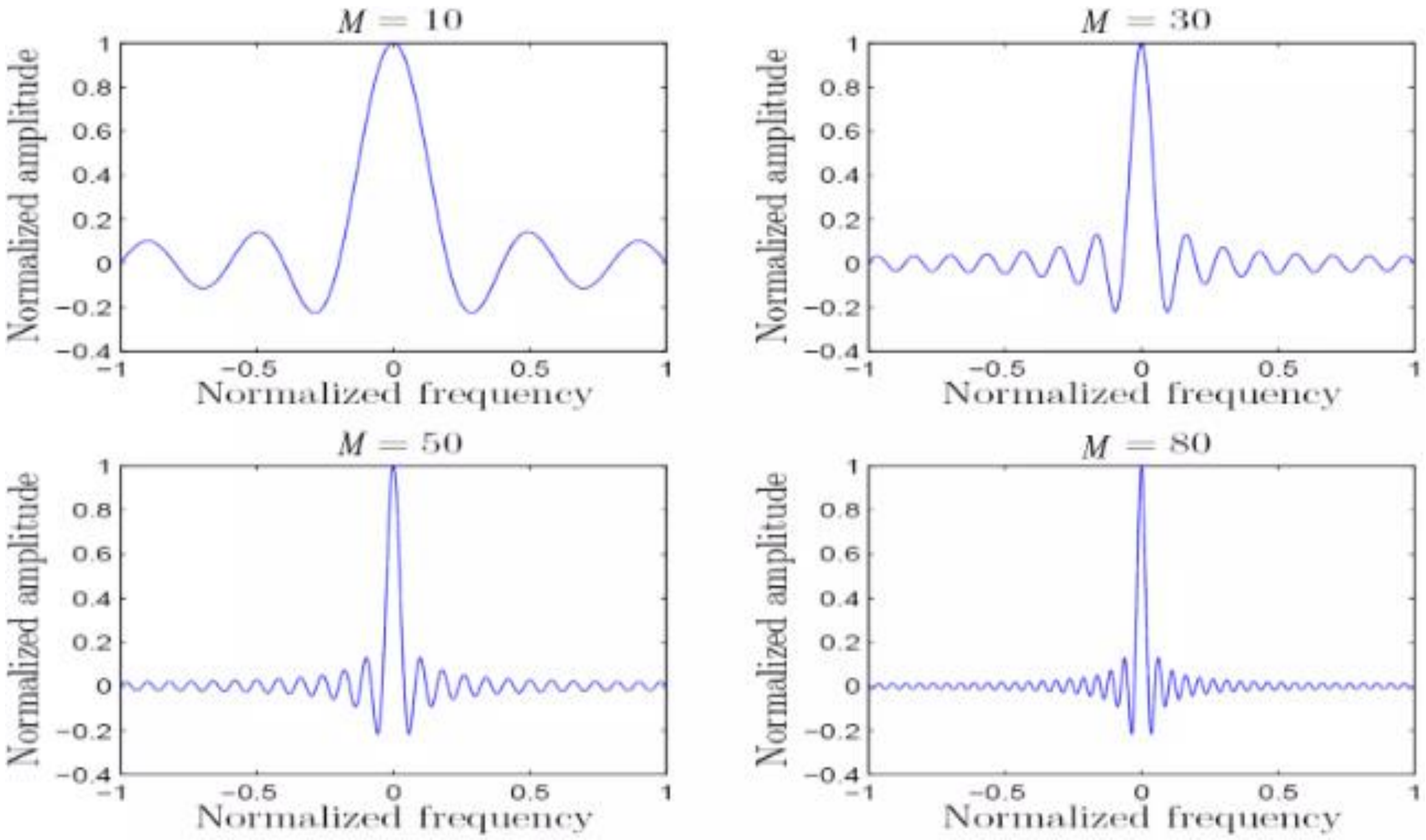






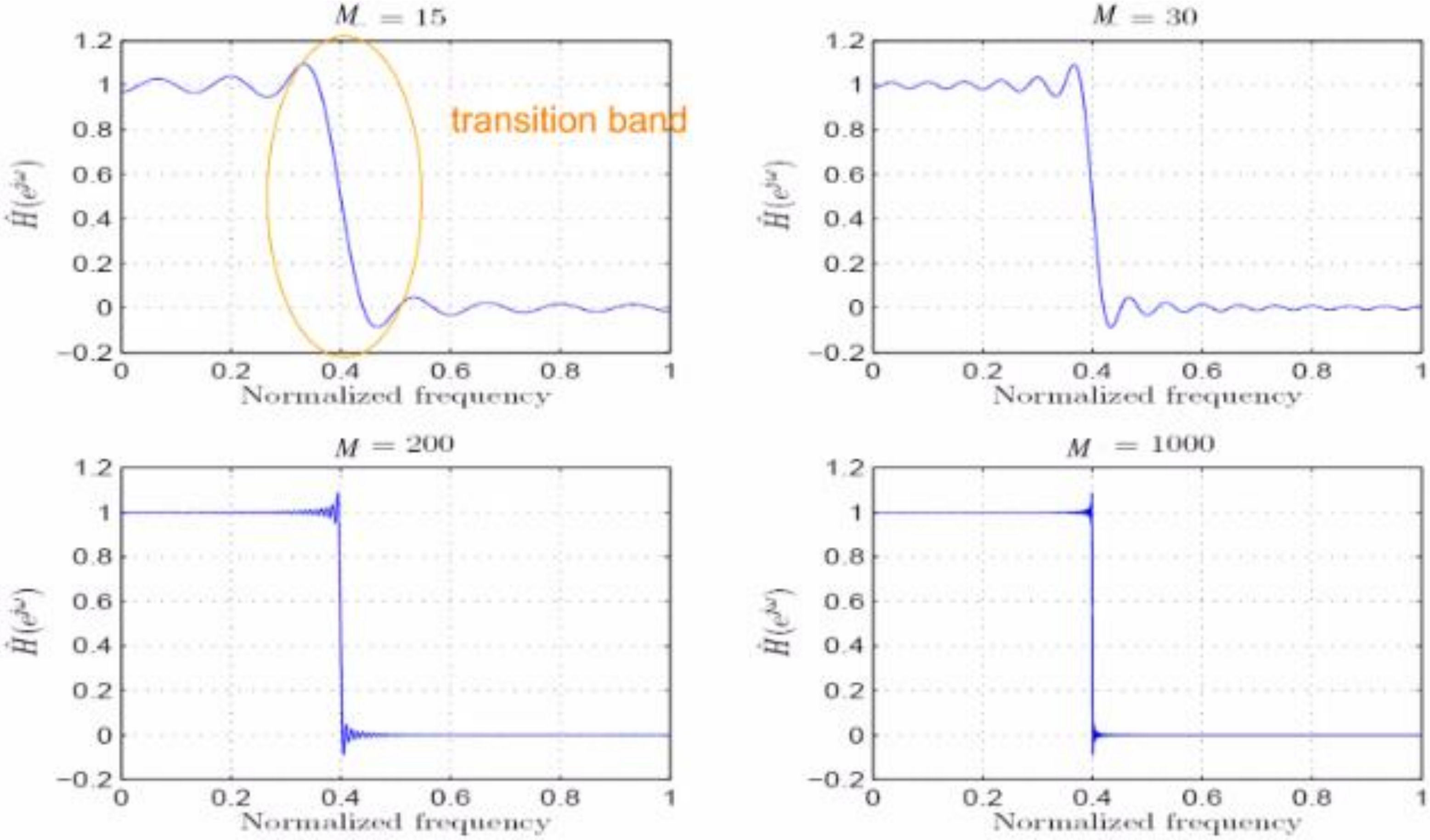
Effects of Rectangular Window Functions

- Ideally M should be small to reduce number of computations.
- But a large M will being us closer to a better filter:
  - Side lobes cause ripples in passband and stop bands.
  - A smaller main lobe creates sharper transition band.



Effects of Rectangular Window Functions

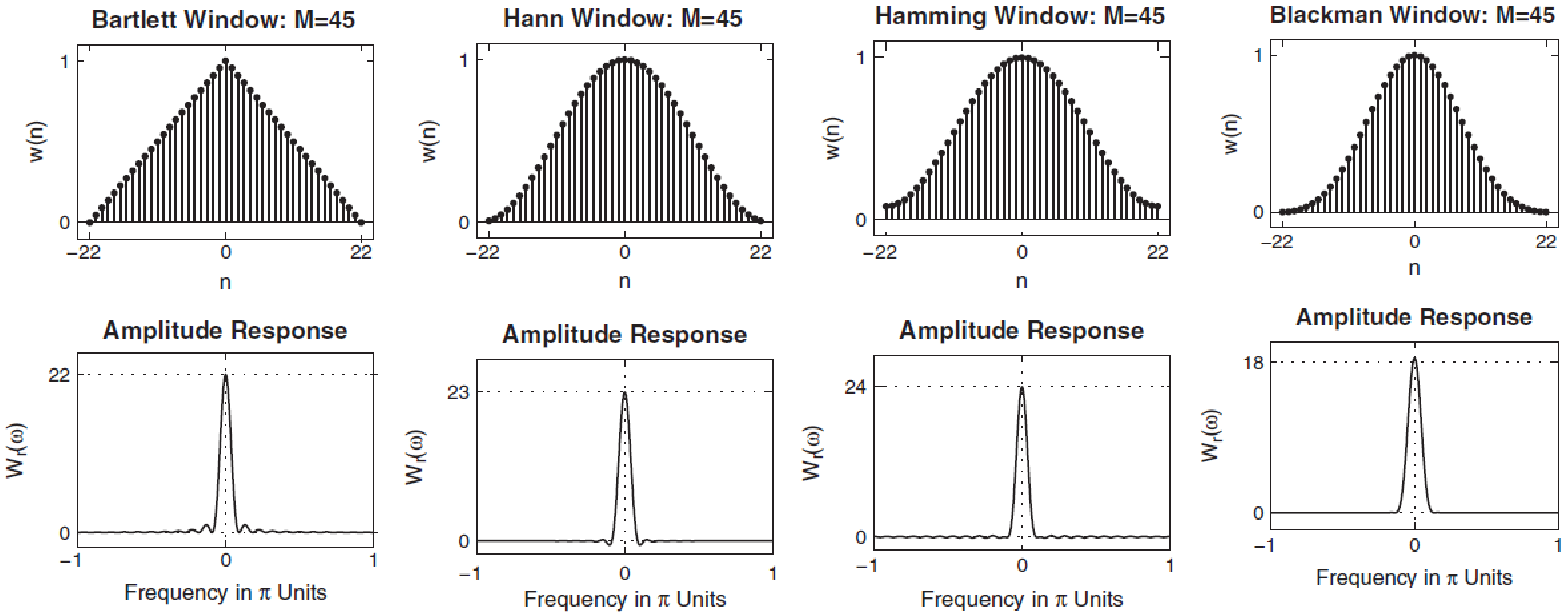
- Even as M tends to infinity, we still see ripples in both the passband and the stopband.
    - This is due to Gibb's phenomenon.
- The actual problem is sharp discontinuities because of the rectangular function.



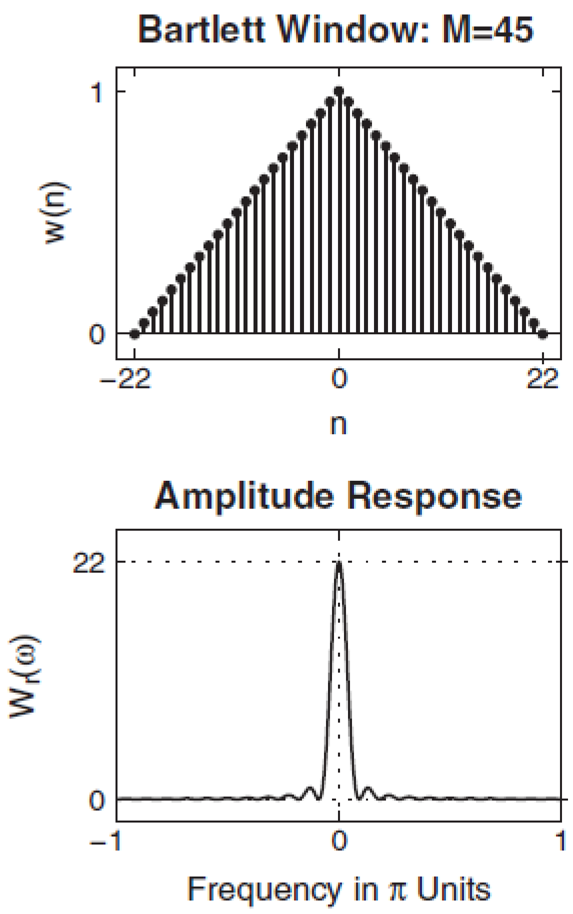
Alternative Window Functions

- **Solution:** Use windows with no abrupt discontinuity in their time-domain response.
- This will cause low side lobes in the frequency response.
- The reduced ripple comes at the expense of a wider transition region.
- But this can be compensated for by increasing the length of the filter.

Alternative Window Functions

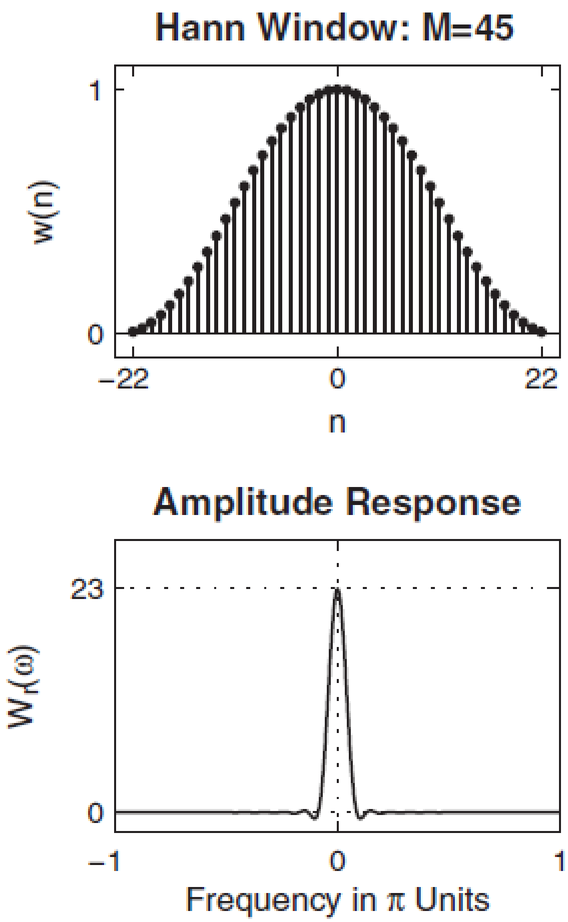


Bartlett Window



$$w(n) = \begin{cases} \dfrac{2n}{M-1}, & 0 \le n \le \dfrac{M-1}{2} \\ 2 - \dfrac{2n}{M-1}, & \dfrac{M-1}{2} \le n \le M-1 \\ 0, & \text{otherwise} \end{cases}$$

Hann Window



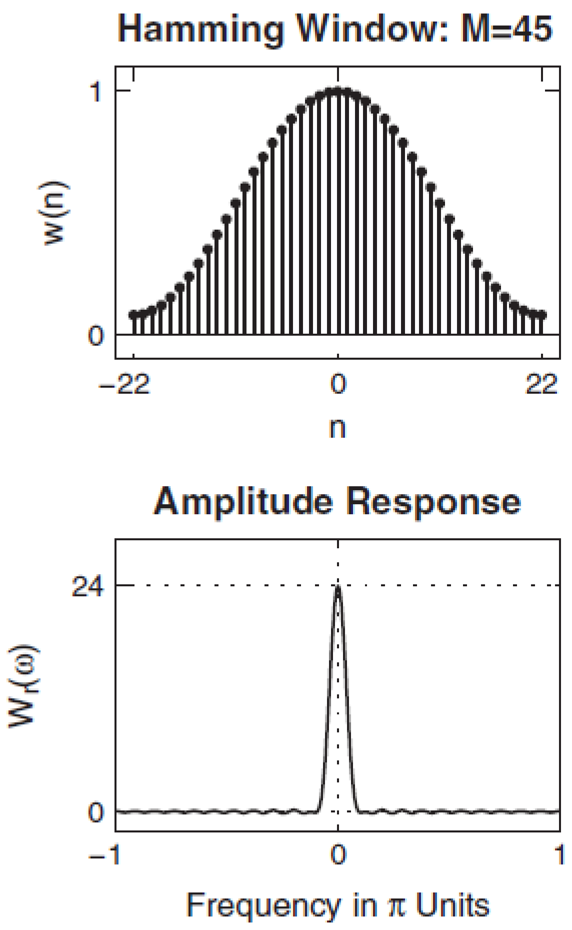
- Also known as a raised cosine window.

$$w(n) = \begin{cases} 0.5\left[1 - \cos\left(\frac{2\pi n}{M-1}\right)\right], & 0 \le n \le M-1 \\ 0, & \text{otherwise} \end{cases}$$

Hamming Window

- Similar to Hann except that it has a small amount of discontinuity.

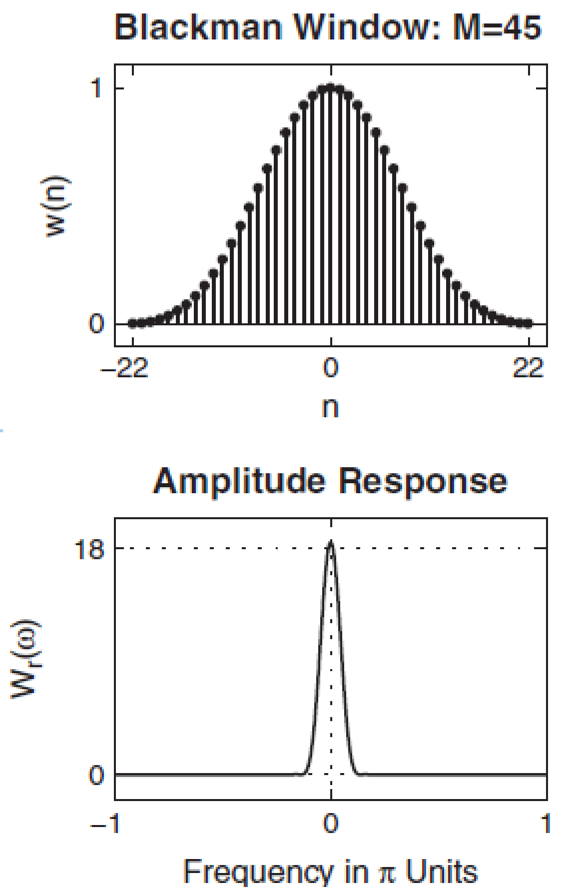
$$w(n) = \begin{cases} 0.54 - 0.46\cos\left(\frac{2\pi n}{M-1}\right), & 0 \le n \le M-1 \\ 0, & \text{otherwise} \end{cases}$$



Blackman Window

- Similar to the previous two but contains a second harmonic.

$$w(n) = \begin{cases} 0.42 - 0.5\cos\left(\frac{2\pi n}{M-1}\right) + 0.08\cos\left(\frac{4\pi n}{M-1}\right), & 0 \le n \le M-1 \\ 0, & \text{otherwise} \end{cases}$$



Summary of Window Function Characteristics

Window Name	*Transition Width $\Delta\omega$ Approximate*	*Transition Width $\Delta\omega$ Exact Values*	*Min. Stopband Attenuation*
Rectangular	$\frac{4\pi}{M}$	$\frac{1.8\pi}{M}$	21 dB
Bartlett	$\frac{8\pi}{M}$	$\frac{6.1\pi}{M}$	25 dB
Hann	$\frac{8\pi}{M}$	$\frac{6.2\pi}{M}$	44 dB
Hamming	$\frac{8\pi}{M}$	$\frac{6.6\pi}{M}$	53 dB
Blackman	$\frac{12\pi}{M}$	$\frac{11\pi}{M}$	74 dB

Summary of Window Function Characteristics

Window type	Main lobe width	Peak sidelobe (dB)
Rectangular	$4\pi/M$	-13
Bartlett	$8\pi/M$	-25
Hanning	$8\pi/M$	-31
Hamming	$8\pi/M$	-41
Blackman	$12\pi/M$	-57

Note:

- the larger the main lobe, the larger the filter transition region
- the larger the peak sidelobe, the higher the degree of ringing in the pass/stopbands

Kaiser's Heuristic Formula

- We have a heuristic formula (attributed to Kaiser) for approximating the filter length M:

$$\widehat{M} = \frac{-20\log_{10}\left(\sqrt{\delta_1\delta_2}\right) - 13}{14.6\Delta f} + 1$$

- We can use this estimate to carry out the design, and the length of the filter can be increased until we obtain a sidelobe level that meets the specification.
- It's important to note that the actual window length required may differ from the estimate due to the specific characteristics of the filter and the chosen window function.

Disadvantage of Window Functions

- Lack of precise control of critical frequencies (ω_s and ω_p).
- The value of these critical frequencies depends on the type of window and filter length.
- We can go for IIR filters in this case.
  - Normally used when linear phase is not required and cost effectiveness is needed.
  - Has lower sidelobes in the stopband than FIR filters.
  - Can be implemented with less memory and lower complexity due to recursiveness.