Digital IIR Filters

EE 453 / CE 352

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Digital IIR Filters Design Stages

$$H(z) = \frac{\sum_{k=0}^{N} b_k z^{-k}}{\left[1 + \sum_{k=1}^{M} a_k z^{-k}\right]}$$

1. Filter specification: The function of the filter (lowpass, highpass, bandpass, band rejected filter) and the desired performance are given.
2. Calculation of coefficients *ak* and *bk* .
3. Structure realization.
4. Finite word length effects analysis and solutions.
5. Hardware and software implementation.

IIR Filter Design: Bilinear Transformation

- IIR Filters can be designed by converting a well-known analog filter into a digital filter.
- Rationale:
  - Analog filter design is a mature and well-developed field, with closed-form solutions for some widely-used common filter types.
  - To take advantage of these capabilities, we begin the design of a digital filter in the analog domain and then convert the design into a digital domain.
- The conversion process is known as Bilinear Transformation.

Characterizations of an Analog Filter

- Differential Equation:

$$\sum_{k=0}^{N} \alpha_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} \beta_k \frac{d^k x(t)}{dt^k}$$

- System Function:

$$H_a(s) = \frac{B(s)}{A(s)} = \frac{\sum_{k=0}^{M} \beta_k s^k}{\sum_{k=0}^{N} \alpha_k s^k}$$

$$H_a(s) = \int_{-\infty}^{\infty} h_a(t) e^{-st}\, dt$$

<u>Laplace Transform of the Impulse Response</u>

Definition of Laplace Transform

- Given a function $f(t)$, its Laplace transform, denoted by $F(s)$ or $\mathcal{L}[f(t)]$, is defined by:

$$\mathcal{L}[f(t)] = F(s) = \int_{-0}^{\infty} f(t)e^{-st}dt$$

where s is complex frequency, given by: $s = \sigma + j\Omega$

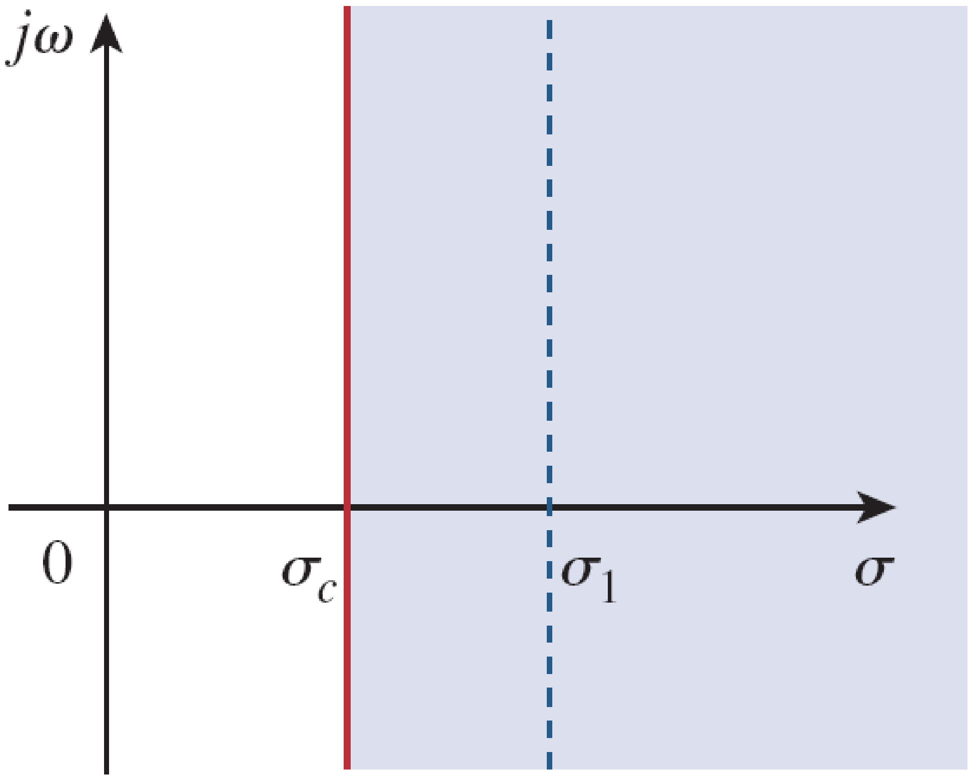
- Laplace transforms cater not only to frequency response, but decay effects as well.

Definition of Laplace Transform

- For some particular value of s, the Laplace transform should converge.

$$\int_{-0}^{\infty} f(t).e^{-\sigma t}.e^{-j\omega t}dt < \infty$$

- The region of convergence of Laplace transformation is solely dependent on $\mathrm{Re(s)} = \sigma$.



Analog Filter: Condition for Stability

- Degree of $N(s)$ should be less than that of $D(s)$.
- All the poles must lie in the left half of the s-plane (i.e. roots must have negative real parts).

$$H(s) = \frac{N(s)}{D(s)} = \frac{N(s)}{(s + p_1)(s + p_2)\cdots(s + p_n)}$$

Properties of the Laplace transform.

Property	$f(t)$	$F(s)$
Linearity	$a_1 f_1(t) + a_2 f_2(t)$	$a_1 F_1(s) + a_2 F_2(s)$
Scaling	$f(at)$	$\frac{1}{a} F\left(\frac{s}{a}\right)$
Time shift	$f(t-a)u(t-a)$	$e^{-as}F(s)$
Frequency shift	$e^{-at}f(t)$	$F(s+a)$
Time differentiation	$\frac{df}{dt}$	$sF(s) - f(0^-)$
	$\frac{d^2f}{dt^2}$	$s^2F(s) - sf(0^-) - f'(0^-)$
	$\frac{d^3f}{dt^3}$	$s^3F(s) - s^2f(0^-) - sf'(0^-) - f''(0^-)$
	$\frac{d^nf}{dt^n}$	$s^nF(s) - s^{n-1}f(0^-) - s^{n-2}f'(0^-) - \cdots - f^{(n-1)}(0^-)$
Time integration	$\int_0^t f(x)dx$	$\frac{1}{s}F(s)$
Frequency differentiation	$tf(t)$	$-\frac{d}{ds}F(s)$
Frequency integration	$\frac{f(t)}{t}$	$\int_s^\infty F(s)ds$
Time periodicity	$f(t) = f(t+nT)$	$\frac{F_1(s)}{1-e^{-sT}}$
Initial value	$f(0)$	$\lim_{s\to\infty} sF(s)$
Final value	$f(\infty)$	$\lim_{s\to 0} sF(s)$
Convolution	$f_1(t) * f_2(t)$	$F_1(s)F_2(s)$

Laplace transform pairs.*

$f(t)$	$F(s)$
$\delta(t)$	1
$u(t)$	$\frac{1}{s}$
e^{-at}	$\frac{1}{s+a}$
t	$\frac{1}{s^2}$
t^n	$\frac{n!}{s^{n+1}}$
te^{-at}	$\frac{1}{(s+a)^2}$
t^ne^{-at}	$\frac{n!}{(s+a)^{n+1}}$
$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
$\sin(\omega t + \theta)$	$\frac{s\sin\theta + \omega\cos\theta}{s^2+\omega^2}$
$\cos(\omega t + \theta)$	$\frac{s\cos\theta - \omega\sin\theta}{s^2+\omega^2}$
$e^{-at}\sin \omega t$	$\frac{\omega}{(s+a)^2+\omega^2}$
$e^{-at}\cos \omega t$	$\frac{s+a}{(s+a)^2+\omega^2}$

*Defined for $t \geq 0$; $f(t) = 0$, for $t < 0$.

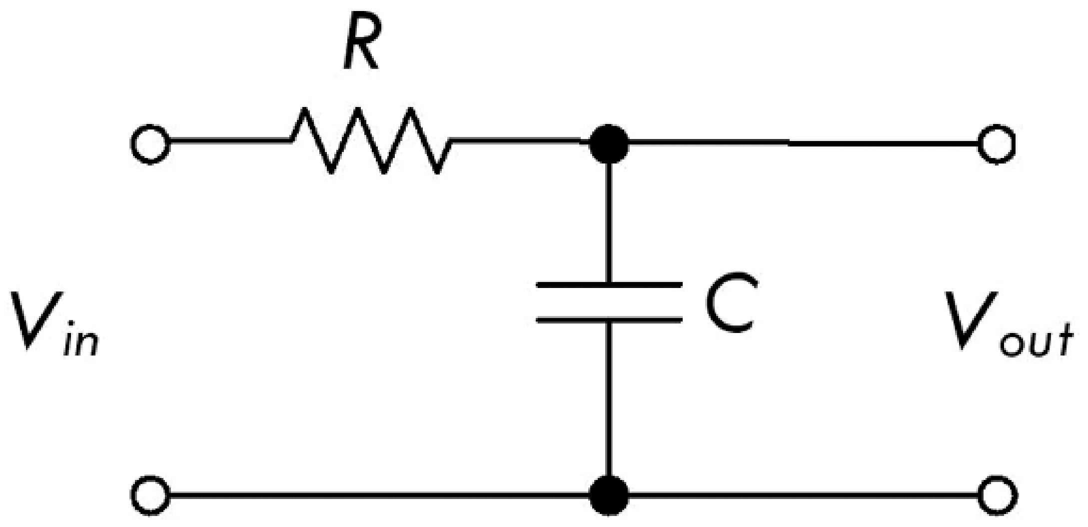
Example: Analog Low Pass Filter (RC)

- Cut-off Frequency: $\omega_c = \frac{1}{RC}$
- Transfer function:

$$H_a(s) = \frac{Y(s)}{X(s)}$$

$$H_a(s) = \frac{Z_C}{Z_C + Z_R} = \frac{1/sC}{1/sC + R}$$

$$H_a(s) = \frac{1}{sRC + 1}$$

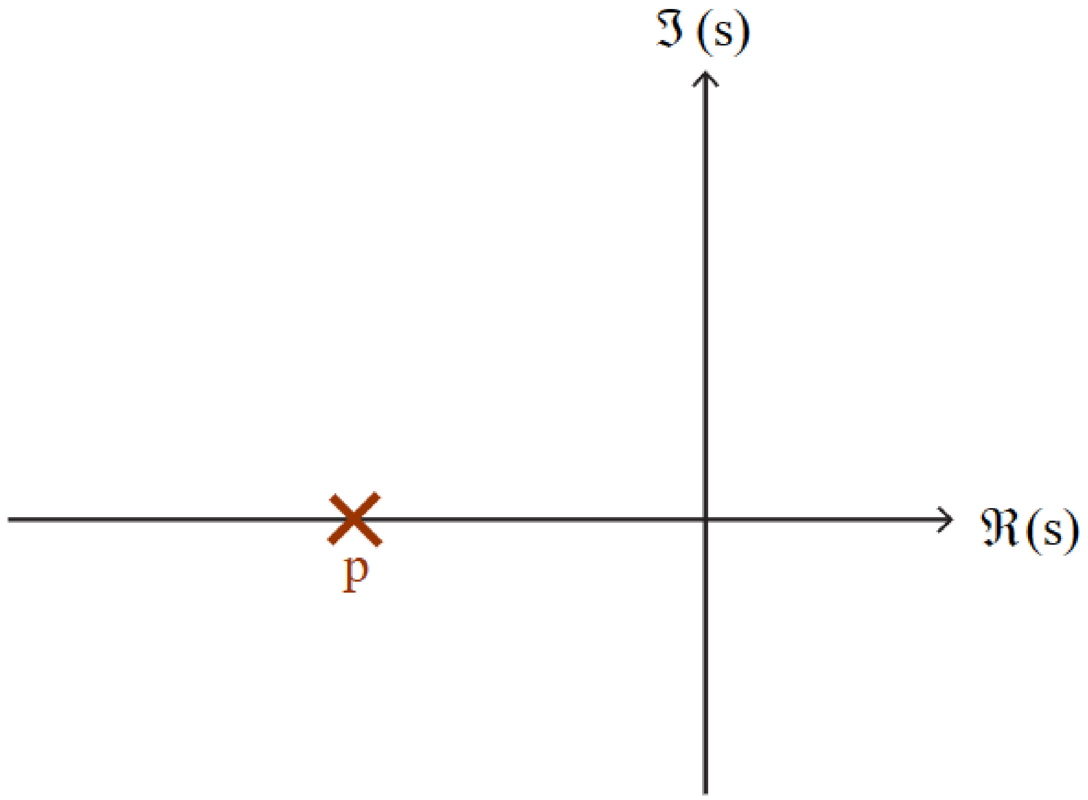


Example: Analog Low Pass Filter (RC)

$$H_a(s) = \frac{1}{sRC + 1}$$

- This first order system has no zeros and one stable pole on the left real axis as visualized in the s-plane.
- Pole located at:

$$s = -1/RC$$



Conversion from s-plane to z-plane

1. The j axis in the s-plane should map into unit circle in the z-plane.
   - This results in a direct relationship between the two frequency variables in s-domain and z-domain.

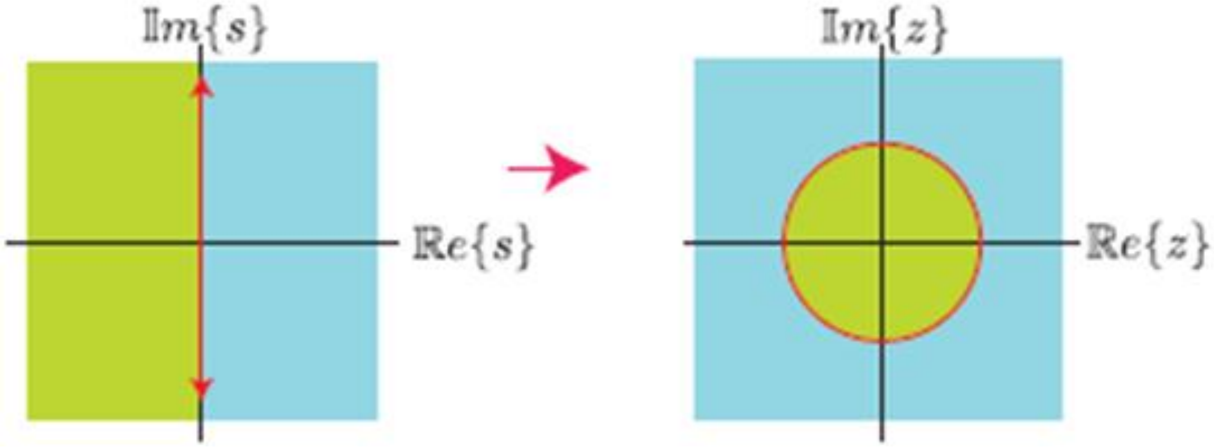
2. The LHP of the s-plane should map into the interior of the unit circle in the z-plane.
   - This guarantees the stability of the digital filter when converted from analog filter.

3. The transfer functions $H_a(s)$ and $H(z)$ should be rational, so that the filter is causal.

Bilinear Transformation

$$s = \frac{2}{T}\left(\frac{z-1}{z+1}\right) \quad \leftrightarrow \quad z = \frac{\frac{2}{T} + s}{\frac{2}{T} - s}$$

$$H(z) = H_a(s)\Big|_{s = \frac{2}{T}\left(\frac{z-1}{z+1}\right)}$$



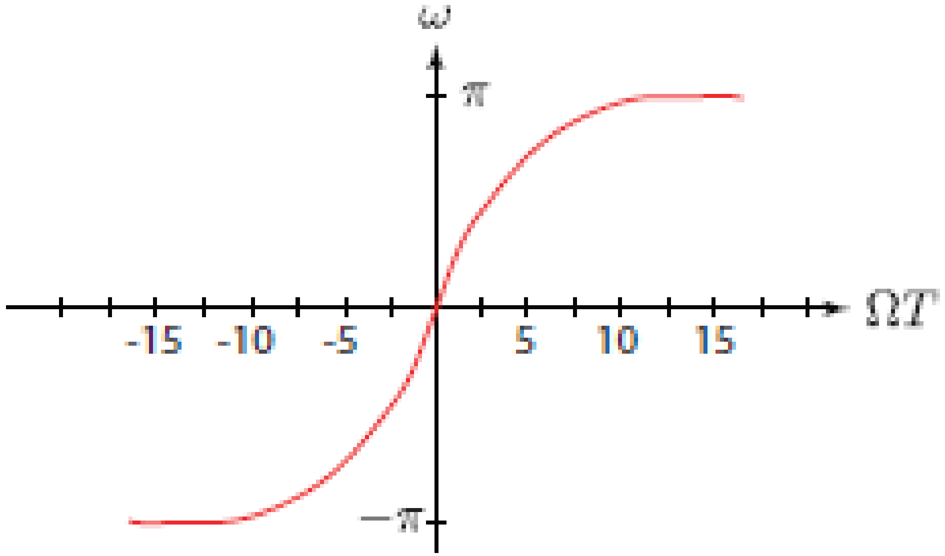
Bilinear Transformation

$$\Omega = \frac{2}{T}\tan\frac{\omega}{2}$$

$$\omega = 2\tan^{-1}\frac{\Omega T}{2}$$

- The entire $-\infty < \Omega < \infty$ axis is mapped to $-\pi < \omega < \pi$.
- There is a huge compression of the frequency response at large values of Ω.

- For $s = j\Omega$ and $z = e^{j\omega}$:



Design using Bilinear Transformation

- Typically, the design of the digital filter begins with specifications in the digital domain that involve the variable ω.
- These specifications (cutoff frequencies) are converted to the analog domain by $\Omega = \frac{2}{T}\tan\frac{\omega}{2}$.
- The analog filter is then designed in terms of $H_a(s)$.
- $H_a(s)$ is converted to $H(z)$ by bilinear transformation, resulting in the digital filter.

Example

- Design a single-pole lowpass filter with a 3-dB bandwidth of 0.2π, using the bilinear transformation applied to the analog filter:

$$H(s) = \frac{\Omega_c}{s + \Omega_c}$$

Where Ω_c is the 3-dB bandwidth of the analog filter.

$$\boxed{\Omega = \frac{2}{T}\tan\frac{\omega}{2}}$$

$$\boxed{G(z) = H(s)\Big|_{s = \frac{2}{T}\left(\frac{z-1}{z+1}\right)}}$$

- **Solution:**

$$\Omega_c = \frac{2}{T}\tan 0.1\pi = \frac{0.65}{T}$$

Thus the analog filter has the system function:

$$H(s) = \frac{0.65/T}{s + 0.65/T}$$

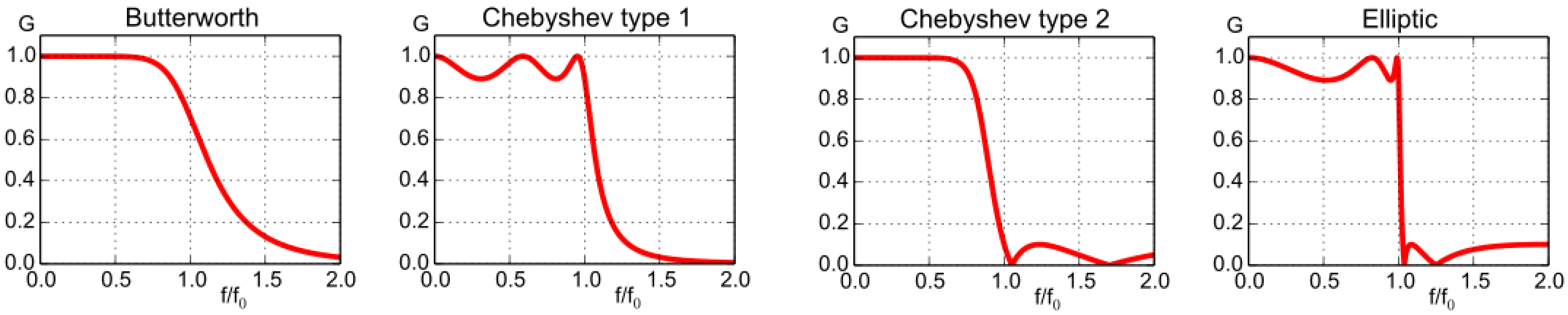
Applying the bilinear transformation:

$$H(z) = \frac{0.245(1 + z^{-1})}{1 - 0.509z^{-1}}$$

Filter Approximations

- On many practical applications, realization of $H(z)$ from analog transfer function $H_a(s)$ may not be available
- Then it has to be determined from the specifications of the desired digital filters, which are R_p, A_s, δ_1, δ_2, ω_p and ω_s. The system function $H_a(s)$ is obtained to satisfy the above specifications.
- For realization, the filter response characteristics have to be approximated. There are several approximation techniques available for the filter design.
- The characteristics of Butterworth and Chebyshev filters are ideally suited to design lowpass filters.

Common Lowpass Analog Filter Types



- Closed Form Solutions exist to design these common filter types.
  - I.e. to figure out the corresponding $H_a(s)$.

Butterworth LP Filter

1. The Butterworth filters are all-pole designs.
   - i.e., the zeros of the filter exist at infinity.
2. At the cutoff frequency Ω_c, the magnitude is $1/\sqrt{2}$ or $-3\ \mathrm{dB}$.
3. The filter order N completely specifies the filter.
4. The magnitude is flat at the origin.
5. The magnitude is a monotonically decreasing function of Ω.
6. The magnitude response approaches the ideal response as the value of N increases.

Butterworth LP Filter

The magnitude response of lowpass filter obtained by the approximation is given by

$$|H_a(\Omega)|^2 = \frac{1}{1 + \left(\frac{\Omega}{\Omega_c}\right)^{2N}}$$

We know that $s = j\Omega$. Substituting Ω by s/j

$$H_a(s)H_a(-s) = \frac{1}{1 + \left(\frac{s/j}{\Omega_c}\right)^{2N}} = \frac{1}{1 + \left(-\frac{s^2}{\Omega_c^2}\right)^{N}}$$

Let $s/\Omega_c = s_n$ is the normalized function. Therefore, the normalized transfer function is

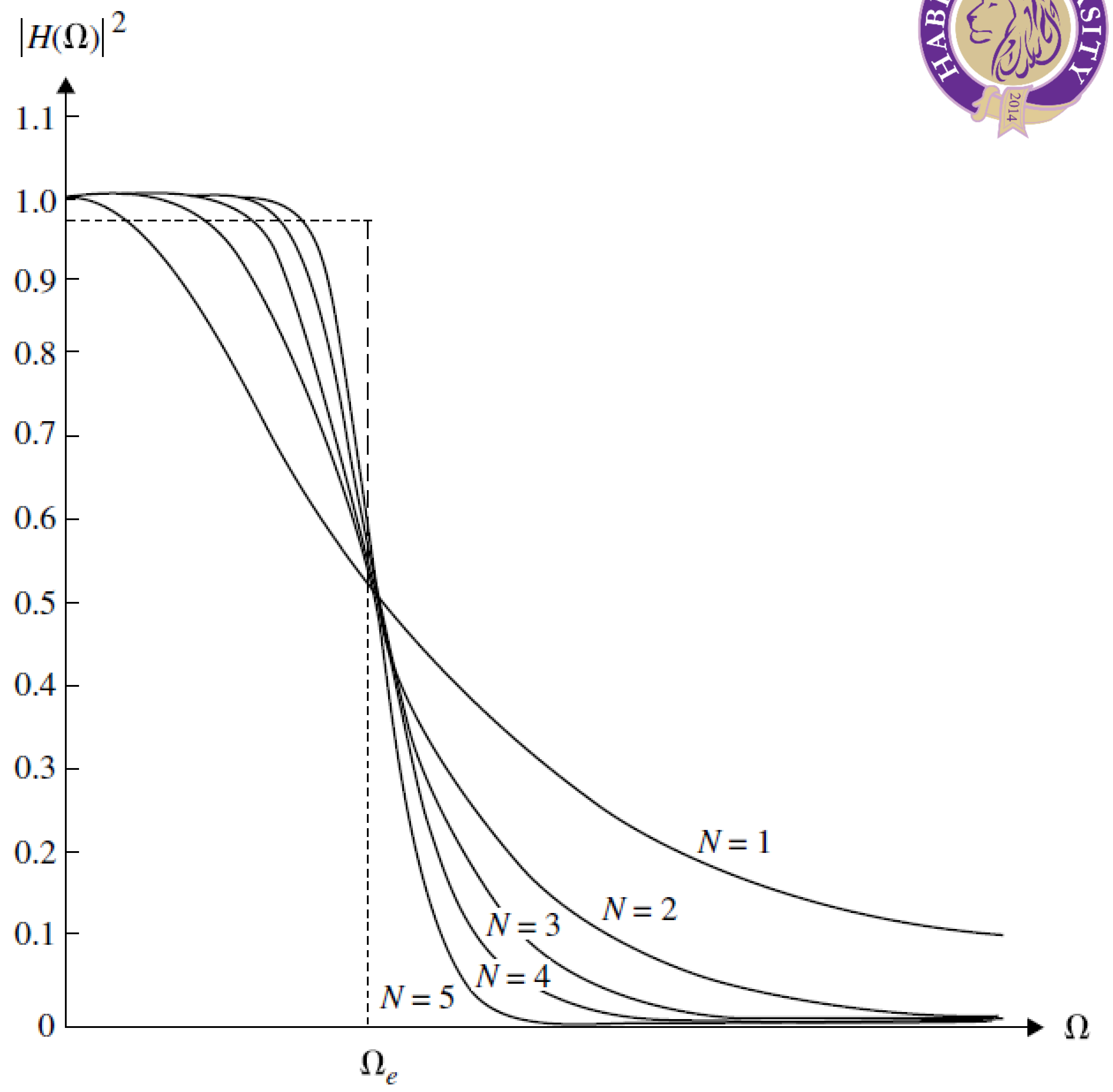
$$H_a(s_n)H_a(-s_n) = \frac{1}{1 + \left(-s_n^2\right)^{N}}$$

Butterworth LP Filter

- The normalized transfer function has $2N$ poles.
  - For odd N: $s^{2N} = 1$
  - For even N:

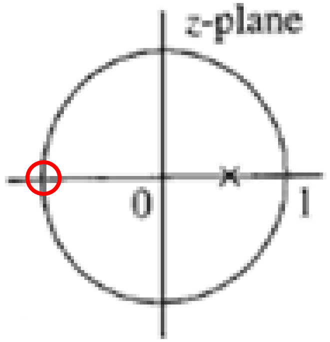
    $s_k = 1 < (2k+1)\pi < 2N,$
    $k = 0,1,\dots,2N$

- For a stable and causal filter, the poles should lie on the left half of s-plane.



Filter Design using Z-Transform

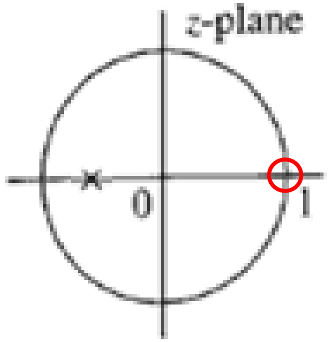
- We can design a filter by placing appropriate poles and zeros. Then we can estimate the frequency response by evaluating the z-transform on the unit circle.



- Example: $H(z) = \frac{1}{1-\alpha z^{-1}} = \frac{z}{z-\alpha}$

- You can improve this LPF by adding a zero at π:

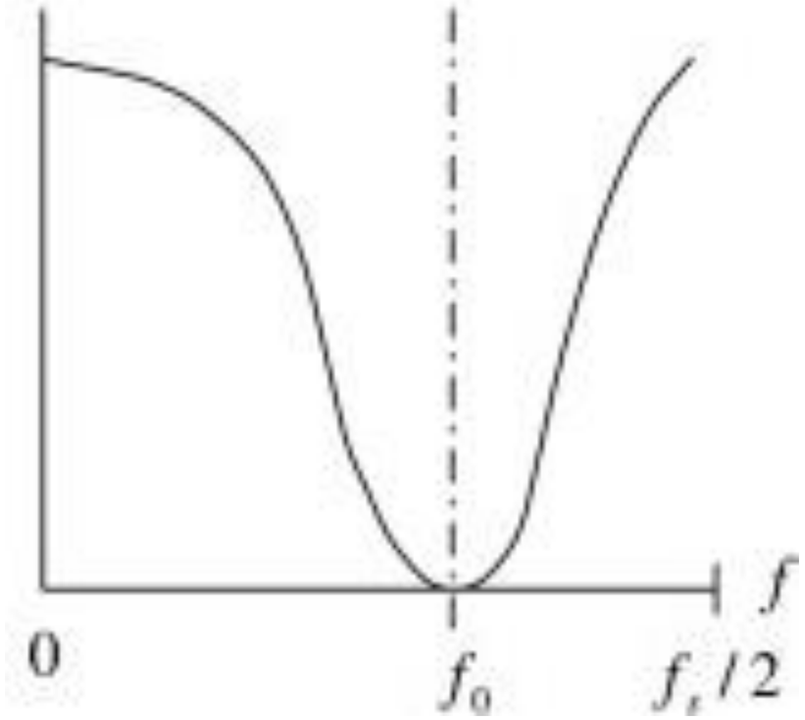
- $H(z) = \frac{z-(-1)}{z-\alpha} = \frac{z+1}{z-\alpha}$

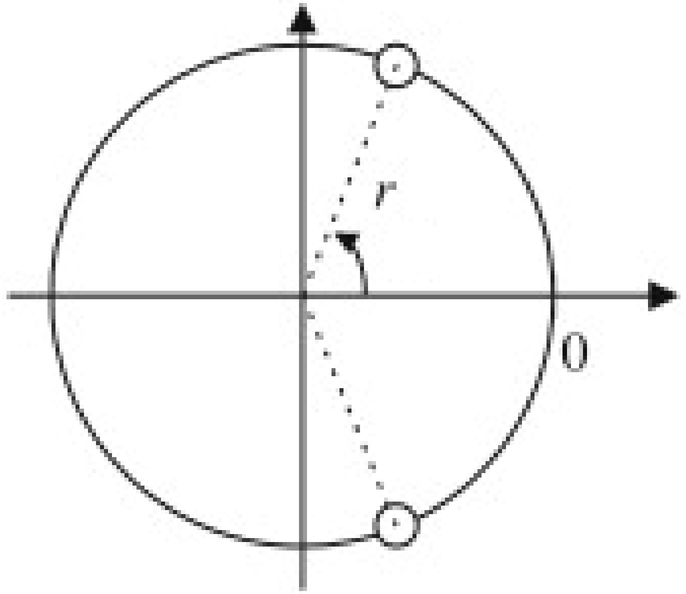


- To convert into a HPF?

Filter Design using Z-Transform

- Example: Build a notch filter:





Filter Design using Z-Transform

- Instability issues can occur with too many poles.
  - There will be too many delayed outputs in time, which increases number of computations.
  - It can cause numerical instability, as the filter coefficients may become more sensitive to quantization errors and round-off noise, causing the filter to produce output samples that are noisy or even diverge to infinity.
- If all zeros are inside the unit circle, we will have a minimum phase filter.
  - A minimum phase filter can be thought of as a filter that produces the minimum amount of distortion of the input signal due to phase shifts.
  - This is useful when we want an invertible system.