

CS109: Probability for Computer Scientists

CS109: From Counting to Machine Learning


Counting
Theory


Core
Probability


Random
Variables


Probabilistic
Models


Uncertainty
Theory


Machine
Learning



Early Optimism 1950s

“Machines will be capable,
within twenty years, of doing
any work a man can do.”
—Herbert Simon, 1952



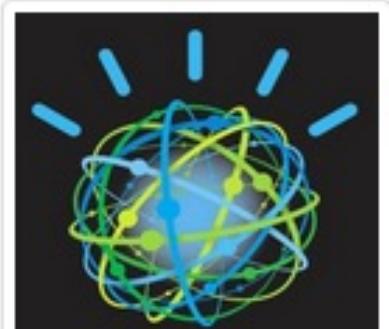
Big Milestones Part 1



1997 Deep Blue

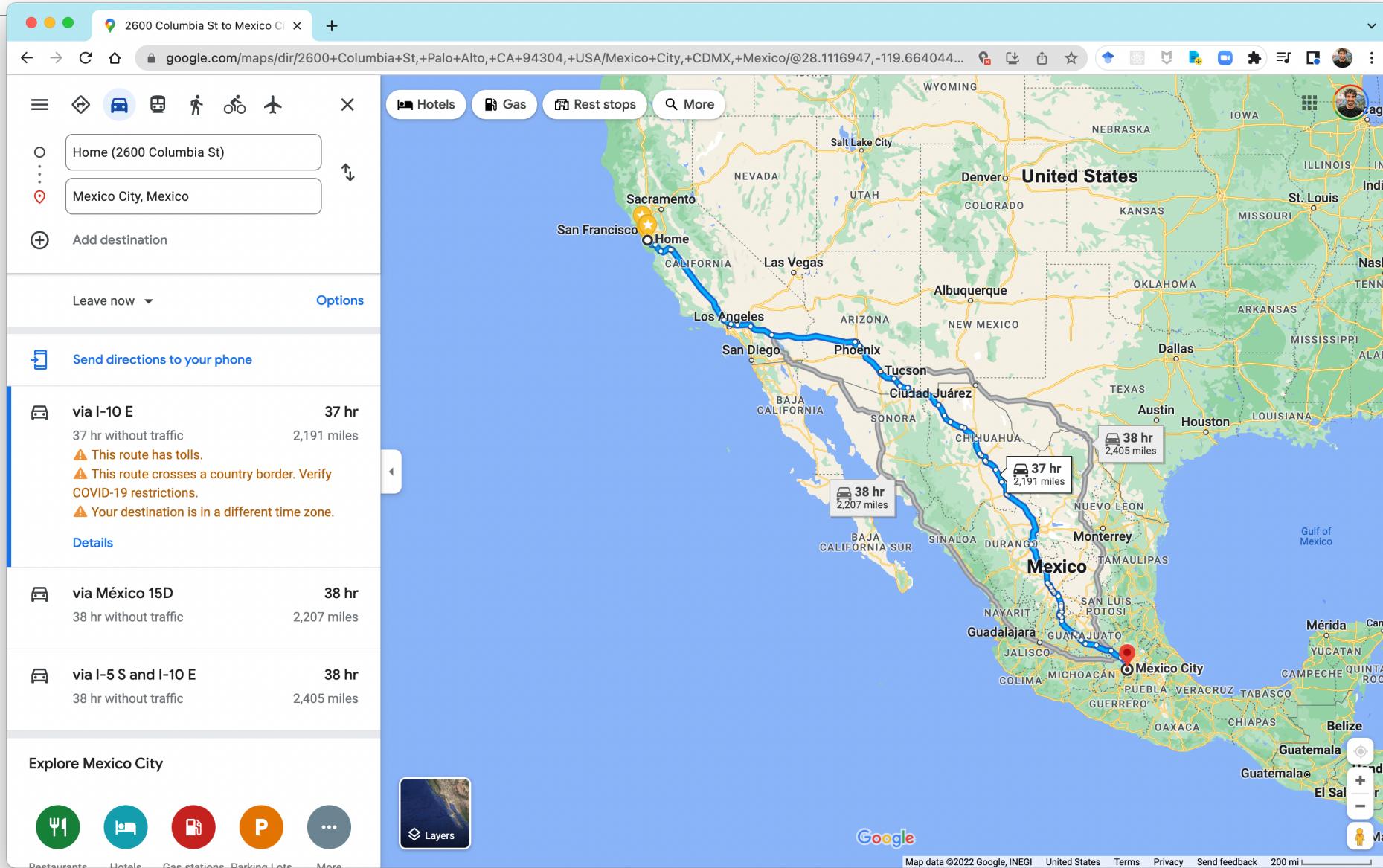


2005 Stanley



2011 Watson

Directions From A to B



3:31



Google Translate



English



Ukrainian



ENGLISH



Please translate this into Ukrainian.
Thank you



Camera



Conversation



Transcribe

UKRAINIAN

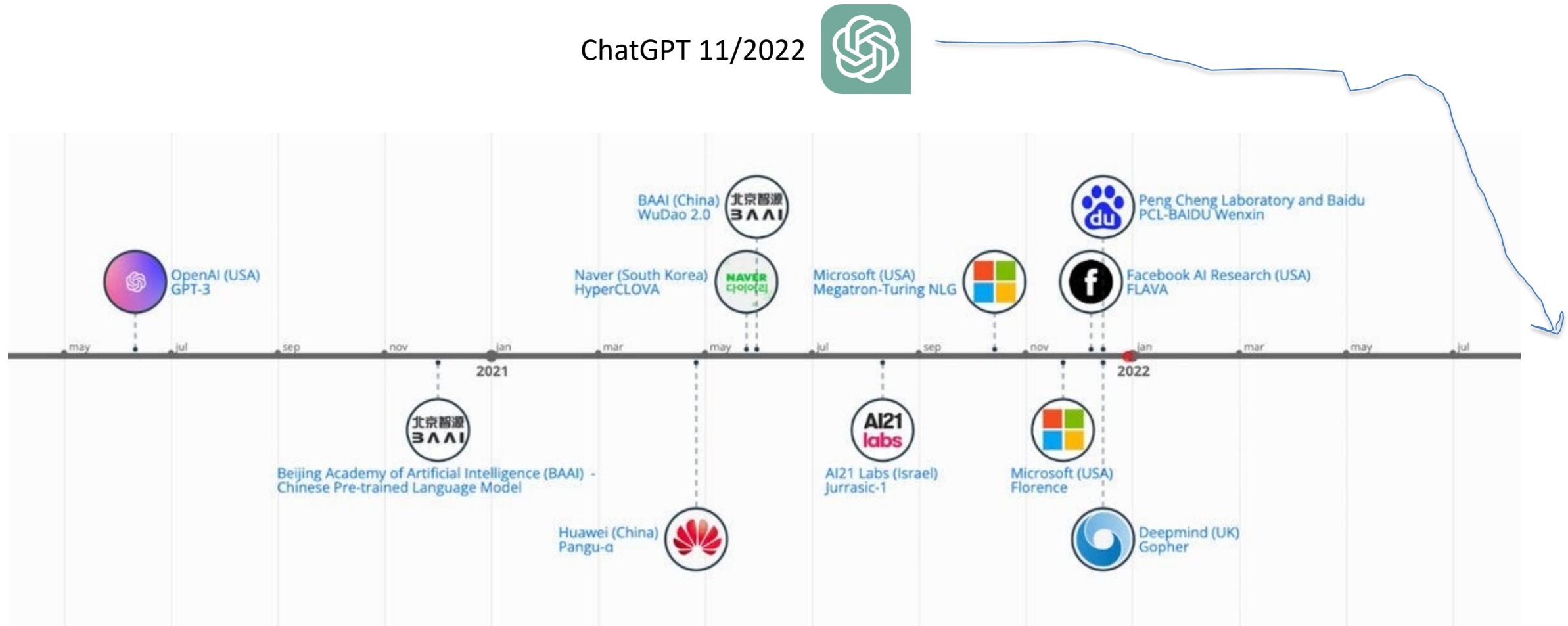


Будь ласка, переведіть це

Self Driving Cars



AI that (seems) to understand language



Most Desired Skill in Industry

Forbes

Billionaires Innovation Leadership Money Consumer

30,575 views | Jan 29, 2018, 02:47pm

Data Scientist Is the Best Job In America According Glassdoor's 2018 Rankings

TWEET THIS

Data Scientist has been named the best job in America for three years running, with a median base salary of \$110,000 and 4,524 job openings.

DevOps Engineer is the second-best job in 2018, paying a median base salary of \$105,000 and 3,369 job openings.

f
t
in



Job Score is based on:

- Earning potential
- Number of jobs
- Job satisfaction rating

“Data science and machine learning are generating more jobs than candidates right now, making these two areas the *fastest growing employment areas*.”

9.8 times more jobs than five years ago.

[LinkedIn's 2017 U.S. Emerging Jobs Report](#)



Most Desired Skill in Academia

Most CS PhD students list their highest desiderata upon graduation as:

“Better understanding of probability”



But Its NOT Always Intuitive



A patient has a positive Zika test.

What is the probability they have zika?

-
- *0.8% of people have zika*
 - *Test has 90% positive rate for people with zika*
 - *Test has 7% positive rate for people without zika*

The right answer is 9%



CS109

AI

Uncertainty Theory

Single Random
Variables

Probabilistic Models

Counting

Probability Fundamentals

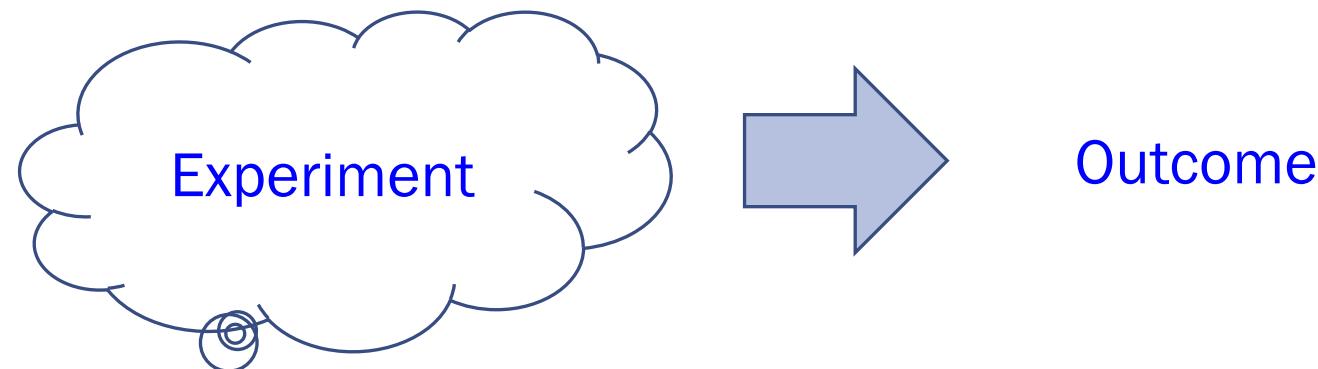
Lets dive in...

Counting I



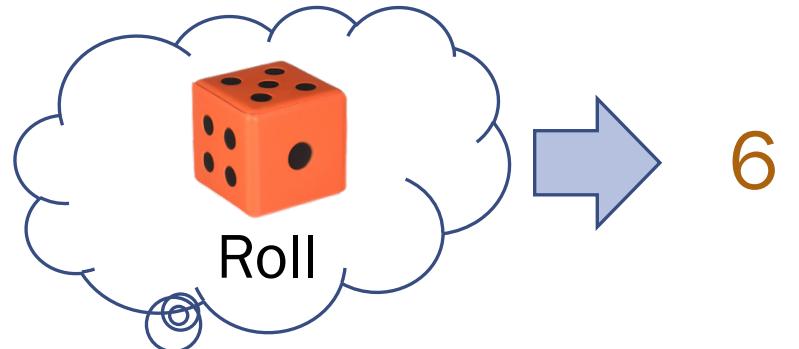
What is Counting?

An experiment
in probability:

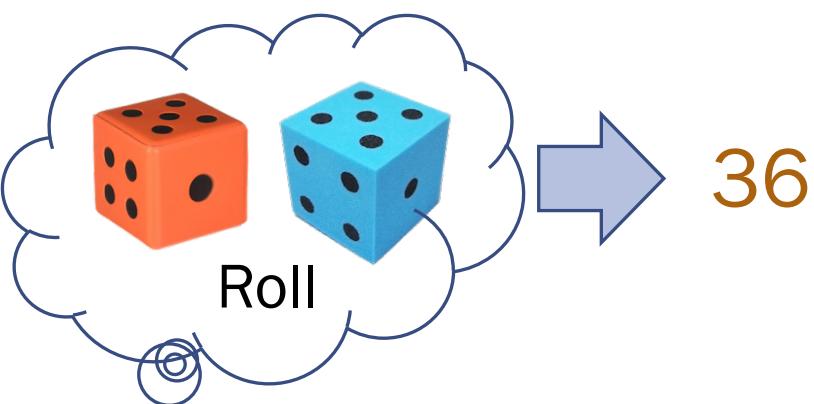
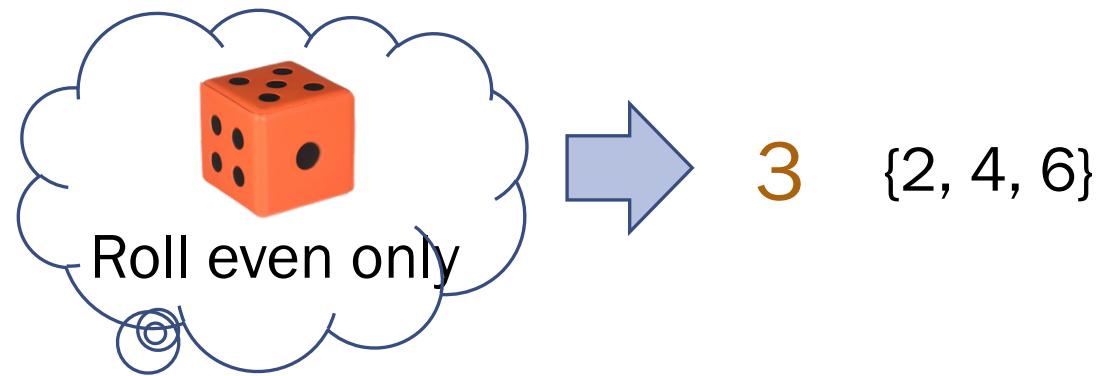


Counting: How many possible outcomes satisfy some event?

What is Counting?



6
 $\{1, 2, 3, 4, 5, 6\}$



36
 $\{(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (2, 1), (2, 2), (2, 3), (2, 4), (2, 5), (2, 6), (3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (3, 6), (4, 1), (4, 2), (4, 3), (4, 4), (4, 5), (4, 6), (5, 1), (5, 2), (5, 3), (5, 4), (5, 5), (5, 6), (6, 1), (6, 2), (6, 3), (6, 4), (6, 5), (6, 6)\}$

Step Rule of Counting (aka Product Rule of Counting)

If an experiment has two steps, where

The first step (step A) can have one of m different outcomes,
and the second step (step B) can have one of n different outcomes,
and n is unaffected by outcome of first step.

Then the number of outcomes of the experiment is mn .

Two-step experiment



How Many Different Lock Patterns?



4-digit numeric combination lock

1st Digit



2nd Digit



3rd Digit



4th Digit

10 Choices per digit

$10^4 = 10,000$ total patterns

How Many Different Lock Patterns (2)?



Choices for 2nd digit depend
on the choice of 1st digit...

4-digit numeric combination lock

BUT each digit used **exactly once**?

1st Digit



2nd Digit



3rd Digit



4th Digit

How Many Different Lock Patterns (2)?



For the step rule, only the number of possible next choices matters!

4-digit numeric combination lock

BUT each digit used **exactly once**?

1st Digit



2nd Digit



3rd Digit



4th Digit

$$10 \times 9 \times 8 \times 7 = 5040$$

Possible patterns

How Many Different Lock Patterns (3)?



3 , 5 , 3 , _

3 , 4 , 6 , _

4-digit numeric combination lock

BUT has **at least one digit repeated?**

1st Digit 10 Choices



2nd Digit 10 Choices



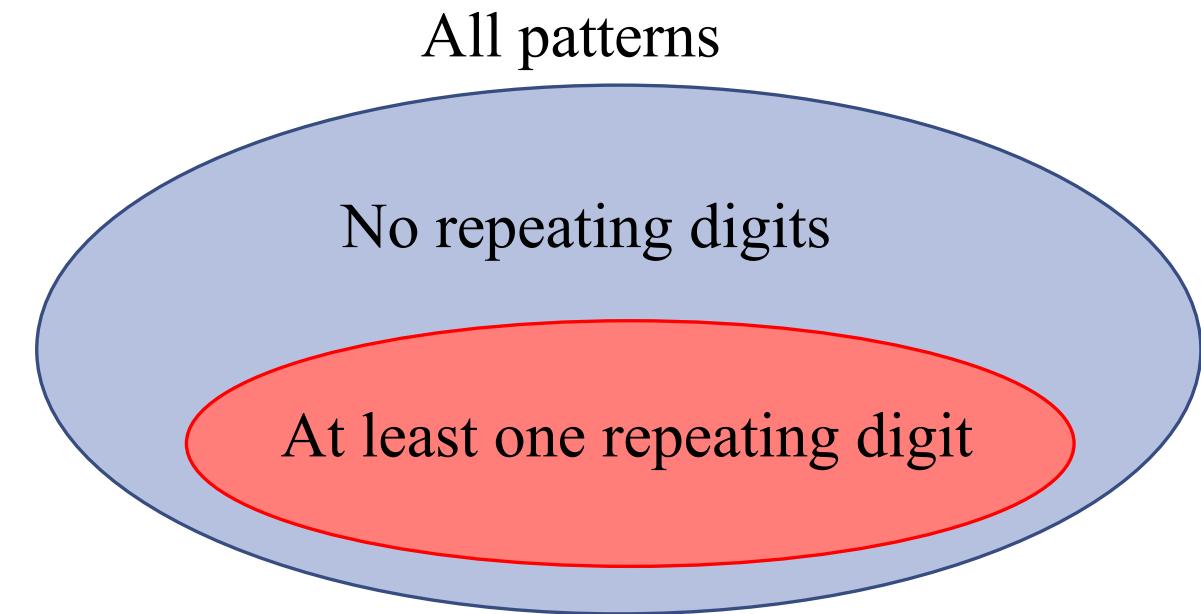
3rd Digit 10 Choices



4th Digit



How Many Different Lock Patterns (3)?

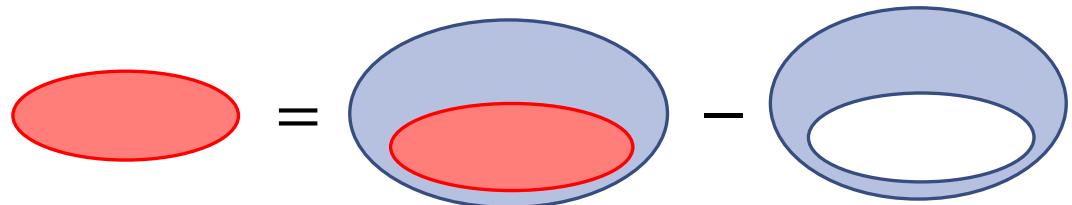


Complementary Counting

Sometimes it helps to “count away” complementary events

4-digit numeric combination lock

BUT has **at least one digit repeated?**



$$\begin{aligned}\#(\text{At least one repeating digit}) &= \#(\text{All patterns}) - \#(\text{No repeating digit}) \\ &= 10,000 - 5,040 \\ &= 4,960\end{aligned}$$

Sum Rule of Counting

If the outcome of an experiment can be either from

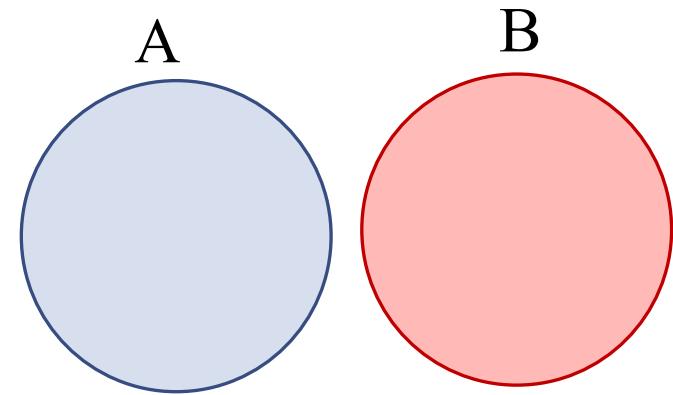
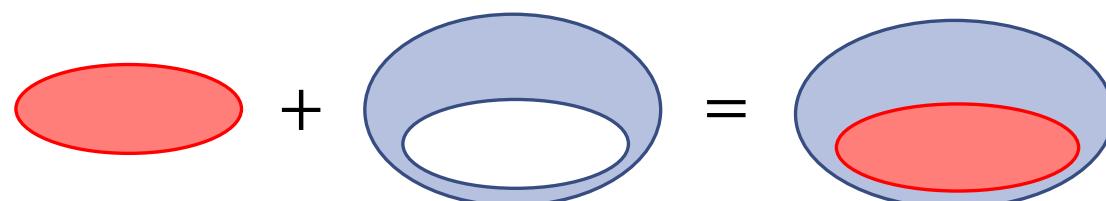
Set A , where $|A| = m$,

or Set B , where $|B| = n$,

where $A \cap B = \emptyset$,

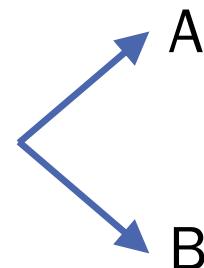
Then the number of outcomes of the experiment is

$$|A| + |B| = m + n.$$



If $|A \cap B| = 0$,
 $|A \cup B| = |A| + |B|$

One experiment



Sum Rule of Counting

If the outcome of an experiment can be either from

Set A , where $|A| = m$,

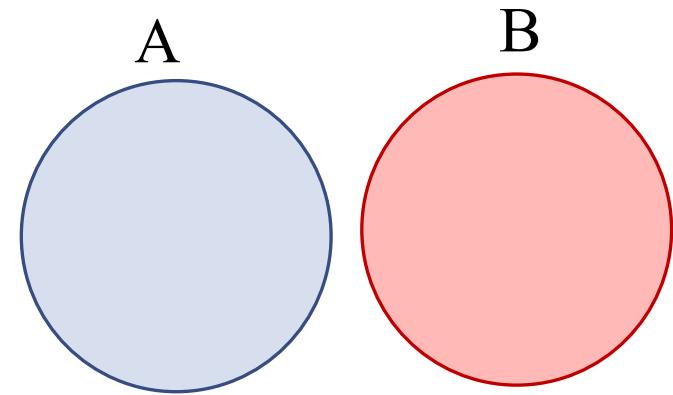
or Set B , where $|B| = n$,

where $A \cap B = \emptyset$,

Then the number of outcomes of the experiment is

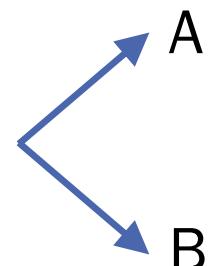
$$|A| + |B| = m + n.$$

Can you derive the Step Rule from the Sum Rule?



If $|A \cap B| = 0$,
 $|A \cup B| = |A| + |B|$

One experiment



How Many Bit Strings?

Problem: A 6-bit string is sent over a network. The valid set of strings recognized by the receiver must either **start with "01"** **OR start with "10"**. How many such strings are there?

Answer

$$\begin{aligned}N &= |A| + |B| \\&= 2^4 + 2^4 \\&= 32\end{aligned}$$

2^4 start with 01

010000
010001
010010
010011
010100
010101
010110
010111
011000
011001
011010
011011
011100
011101
011110
011111

Set *A*

2^4 start with 10

100000
100001
100010
100011
100100
100101
100110
100111
101000
101001
101010
101011
101100
101101
101110
101111

Set *B*

3rd Digit

4th Digit

5th Digit

6th Digit

$$|A \cap B| = 0$$

How Many Bit Strings (2)?

Problem: A 6-bit string is sent over a network. The valid set of strings recognized by the receiver must either **start with "01"** **OR end with "10"**. How many such strings are there?

Answer



2^4 start with 01

010000
010001
010010
010011
010100
010101
010110
010111
011000
011001
011010
011011
011100
011101
011110
011111

Set *A*

2^4 end with 10

000010
000110
001010
001110
010010
010110
011010
011110
100010
100110
101010
101110
110010
110110
111010
111110

Set *B*

How Many Bit Strings (2)?

Problem: A 6-bit string is sent over a network. The valid set of strings recognized by the receiver must either **start with "01"** **OR end with "10"**. How many such strings are there?

Sum Rule of Counting

If the outcome of an experiment can be either from

Set A , where $|A| = m$,

or Set B , where $|B| = n$,

where $A \cap B = \emptyset$,

Then the number of outcomes of the experiment is \square

$$|A| + |B| = m + n.$$

2^4 start with 01

010000
010001
010010
010011
010100
010101
010110
010111
011000
011001
011010
011011
011100
011101
011110
011111

Set A

2^4 end with 10

000010
000110
001010
001110
010010
010110
011010
011110
100010
100110
101010
101110
110010
110110
111010
111110

Set B

How Many Bit Strings (2)?

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Answer

2^4 start with 01

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011010
011011
011100
011101
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Set *A*

2^4 end with 10

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001010
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010010
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100110
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2^4 end with 10

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110010
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111010
111110

Set *B*

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Answer

$$\begin{aligned}N &= |A| + |B| - |A \text{ and } B| \\&= 16 + 16 - 4 \\&= 28\end{aligned}$$

2^4 start with 01

010000
010001
010010
010011
010100
010101
010110
010111
011000
011001
011010
011011
011100
011101
011110
011111

Set *A*

2^4 end with 10

000010
000110
001010
001110
010010
010110
011010
011110
100010
100110
101010
101110
110010
110110
111010
111110

Set *B*

Or Rule of Counting (aka Inclusion/ Exclusion)

If the outcome of an experiment can be either from

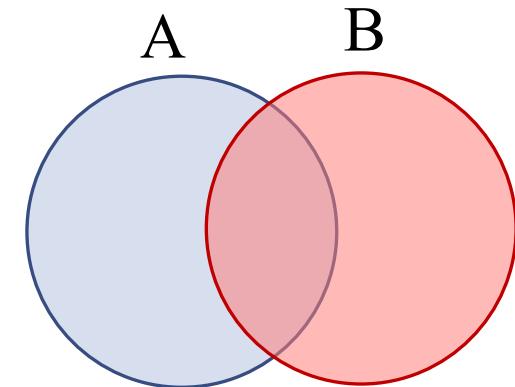
Set A , where $|A| = m$,

or Set B , where $|B| = n$,

where **$A \cap B$ may not be empty**,

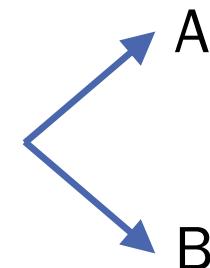
Then the number of outcomes of the experiment is

$$N = |A| + |B| - |A \cap B|.$$



$$|A \cup B| = |A| + |B| - |A \cap B|$$

One experiment



Core Counting

Counting with steps

Definition: Step Rule of Counting (aka Product Rule of Counting)

If an experiment has two parts, where the first part can result in one of m outcomes and the second part can result in one of n outcomes regardless of the outcome of the first part, then the total number of outcomes for the experiment is $m \cdot n$.



Counting with “or”

Definition: Inclusion Exclusion Counting

If the outcome of an experiment can either be drawn from set A or set B , and sets A and B may potentially overlap (i.e., it is not the case that A and B are mutually exclusive), then the number of outcomes of the experiment is $|A \text{ or } B| = |A| + |B| - |A \text{ and } B|$.

Can you derive
the Step Rule
from the Sum
Rule (or the OR
rule)?

Challenge Problem

1. Strings

- How many *different* orderings of letters are possible for the string BOBA?

BOBA, ABOB, OBBA...



Core Counting

Counting with steps

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How Many Bit Strings?

Problem: A 6-bit string is sent over a network. The valid set of strings recognized by the receiver must either start with "01" or end with "10". How many such strings are there?

Answer

$$\begin{aligned}N &= |A| + |B| - |A \text{ and } B| \\&= 16 + 16 - 4 \\&= 28\end{aligned}$$

2^4 start with 01

010000
010001
010010
010011
010100
010101
010110
010111
011000
011001
011010
011011
011100
011101
011110
011111

Set *A*

2^4 end with 10

000010
000110
001010
001110
010010
010110
011010
011110
100010
100110
101010
101110
110010
110110
111010
111110

Set *B*



Challenge Problem

1. Strings

- How many *different* orderings of letters are possible for the string BOBA?

BOBA, ABOB, OBBA...



Permutations I

Summary of Combinatorics

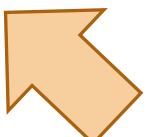
Counting tasks on n objects

Sort objects
(permutations)

Choose k objects
(combinations)

Put objects in r
buckets

Distinct
(distinguishable)



Orderings of Letters

How many letter orderings are possible for the following strings?

1. GATES



Orderings of Letters

gates	getsa	atges	asteg	tegas	egtsa	esgat	sateg
gatse	gesat	atgse	asegt	tegsa	egsat	esgta	saegt
gaets	gesta	ategs	asetg	teags	egsta	esagt	saetg
gaest	gsate	atesg	tgaes	teasg	eagts	esatg	stgae
gaste	gsaet	atsge	tgase	tesga	eagst	estga	stgea
gaset	gstae	atseg	tgeas	tesag	eatgs	estag	stage
gtaes	gstea	aegts	tgesa	tsgae	eatsg	sgate	staeg
gtase	gseat	aegst	tgsae	tsgea	easgt	sgaet	stega
gteas	gseta	aetgs	tgsea	tsage	eastg	sgtae	steag
gtesa	agtes	aetsg	tages	tsaeg	etgas	sgtea	segat
gtsae	agtse	aesgt	tagse	tsega	etgsa	sgeat	segta
gtsea	agets	aestg	taegs	tseag	etags	sgeta	seagt
geats	agest	asgte	taesg	egats	etasg	sagte	seatg
geast	agste	asget	tasge	egast	etsga	saget	setga
getas	agset	astge	taseg	egtas	etsag	satge	setag

Orderings of letters



Step 1:
Chose first letter

Step 2:
Chose 2nd letter

Step 3:
Chose 3rd letter

Step 4:
Chose 4th letter

Step 5:
Chose 5th letter

Orderings of letters



Step 1:
Choose first letter
(5 options)

Step 2:
Choose 2nd letter
(4 options)

Step 3:
Choose 3rd letter
(3 options)

Step 4:
Choose 4th letter
(2 options)

Step 5:
Choose 5th letter
(1 option)

Permutations

A **permutation** is an ordered arrangement of objects.

The number of unique orderings (**permutations**) of n distinct objects is
$$n! = n \times (n - 1) \times (n - 2) \times \cdots \times 2 \times 1.$$

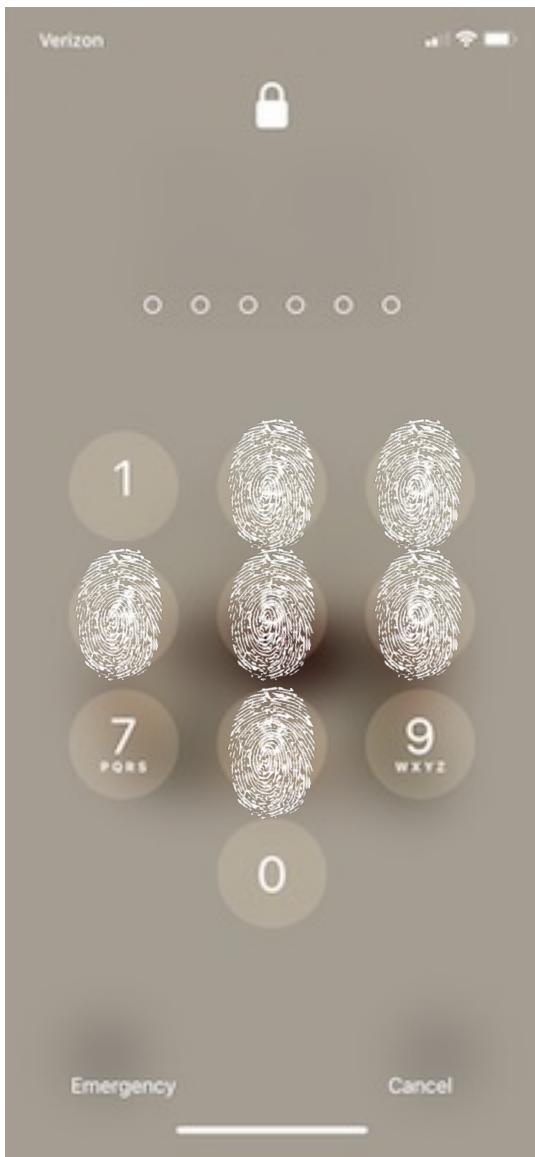
Unique 6-digit passcodes with **six** smudges



How many unique 6-digit passcodes are possible if a phone password uses each of **six** distinct numbers?



Unique 6-digit passcodes with **six** smudges



How many unique 6-digit passcodes are possible if a phone password uses each of **six** distinct numbers?

$$\text{Total} = 6! = 720 \text{ passcodes}$$

How many unique passcodes are possible if a phone password is some ordered subset of any 6 unique digits?

$$\begin{aligned}\text{Total} &= 10 \times 9 \times 8 \times 7 \times 6 \times 5 \\ &= \frac{10!}{4!} = 151200 \text{ passcodes}\end{aligned}$$



Unique Bit Strings

1, 0, 1, 0, 0



Sort indistinct objects?



Sort n distinct objects



Ayesha



Tim



Irina



Joey



Waddie

Sort n distinct objects



Steps:

1. Choose 1st can 5 options
2. Choose 2nd can 4 options
- ...
5. Choose 5th can 1 option

$$\begin{aligned}\text{Total} &= 5 \times 4 \times 3 \times 2 \times 1 \\ &= 120\end{aligned}$$



Permutations II

Summary of Combinatorics

Counting tasks on n objects

Sort objects
(permutations)

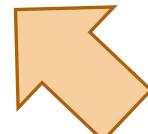
Choose k objects
(combinations)

Put objects in r
buckets

Distinct
(distinguishable)

$n!$

Semi-distinct



How many ways can we sort coke cans!



Coke



Coke0



Coke



Coke0



Coke0

Sort n distinct objects



Ayesha



Tim



Irina



Joey



Waddie

of permutations =

Sort semi-distinct objects

Order n
distinct objects $n!$

All distinct



Some indistinct



Sort semi-distinct objects

How do we find the number of permutations considering some objects are indistinct?

By the product rule, permutations of distinct objects is a two-step process:

$$\text{permutations of distinct objects} = \text{permutations considering some objects are indistinct} \times \text{Permutations of just the indistinct objects}$$

Sort semi-distinct objects

How do we find the number of permutations considering some objects are indistinct?

By the product rule, permutations of distinct objects is a two-step process:

permutations
of distinct objects

=

permutations
considering some
objects are indistinct

Permutations
of just the
indistinct objects



In combinatorics it often helps to overcount permutations and fix for the overcounting

General approach to counting permutations

When there are n objects such that

n_1 are the same (indistinguishable or **indistinct**), and

n_2 are the same, and

...

n_r are the same,

The number of unique orderings (**permutations**) is

$$\frac{n!}{n_1! n_2! \cdots n_r!}.$$

For each group of indistinct objects,
Divide by the overcounted permutations.

Sort semi-distinct objects

Order n semi-
distinct objects $\frac{n!}{n_1! n_2! \cdots n_r!}$

How many permutations?



Coke



Coke0



Coke



Coke0



Coke0

Strings

Order n semi-distinct objects $\frac{n!}{n_1! n_2! \cdots n_r!}$

How many letter orderings are possible for the following strings?

1. BOBA

2. MISSISSIPPI



Strings

Order n semi-distinct objects $\frac{n!}{n_1! n_2! \cdots n_r!}$

How many letter orderings are possible for the following strings?

1. BOBA

$$= \frac{4!}{2!} = 12$$

2. MISSISSIPPI

$$= \frac{11!}{1!4!4!2!} = 34,650$$

To the Code!

baob
bbao
obba
oabb
boab
bab
abbo
aobb
boba
abob
bboa
obab

```
import itertools

def main():
    letters = ['b', 'o', 'b', 'a']
    perms = set(itertools.permutations(letters))
    for perm in perms:
        pretty_perm = "".join(perm)
        print(pretty_perm)

import math

def main():
    n = math.factorial(4)
    d = math.factorial(2)
    print(n / d)
```

Summary of Combinatorics

Counting tasks on n objects

Sort objects
(permutations)

Choose k objects
(combinations)

Put objects in r
buckets

Distinct
(distinguishable)

$$\frac{n!}{n_1! n_2! \cdots n_r!}$$

$n!$

Combinations I

Summary of Combinatorics

Counting tasks on n objects

Sort objects
(permutations)

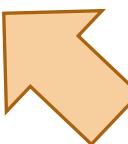
Choose k objects
(combinations)

Put objects in r
buckets

Distinct
(distinguishable)

Some
distinct

Distinct



$n!$

$$\frac{n!}{n_1! n_2! \cdots n_r!}$$

Combinations with cake

There are $n = 20$ people.

How many ways can we choose $k = 5$ people to get cake?

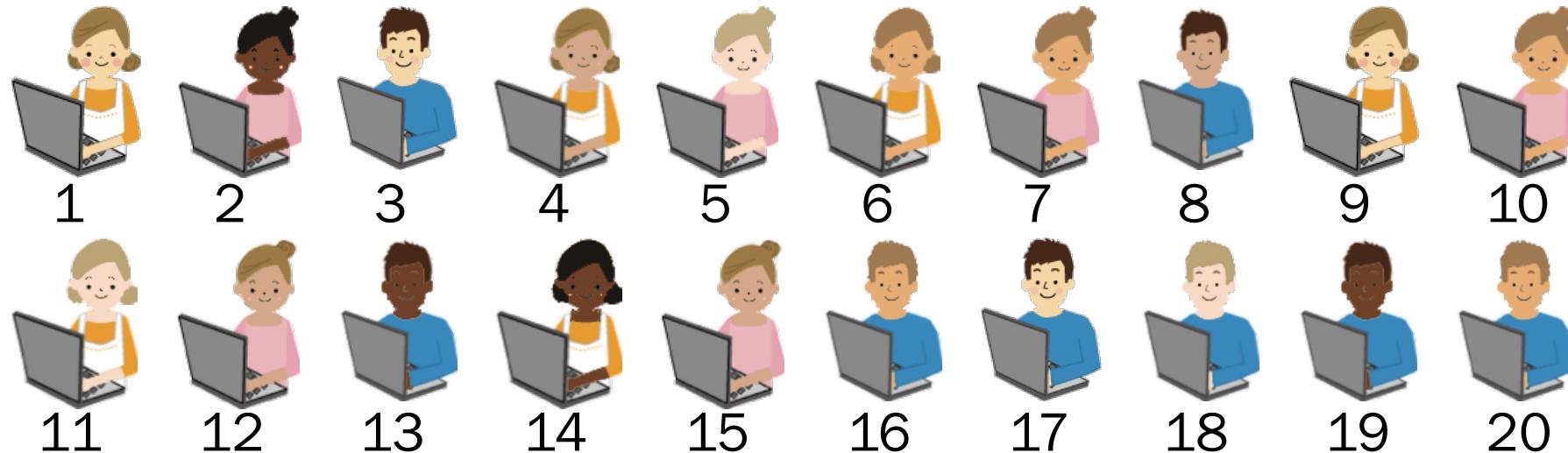


Consider the following
generative process...

Combinations with cake

There are $n = 20$ people.

How many ways can we choose $k = 5$ people to get cake?



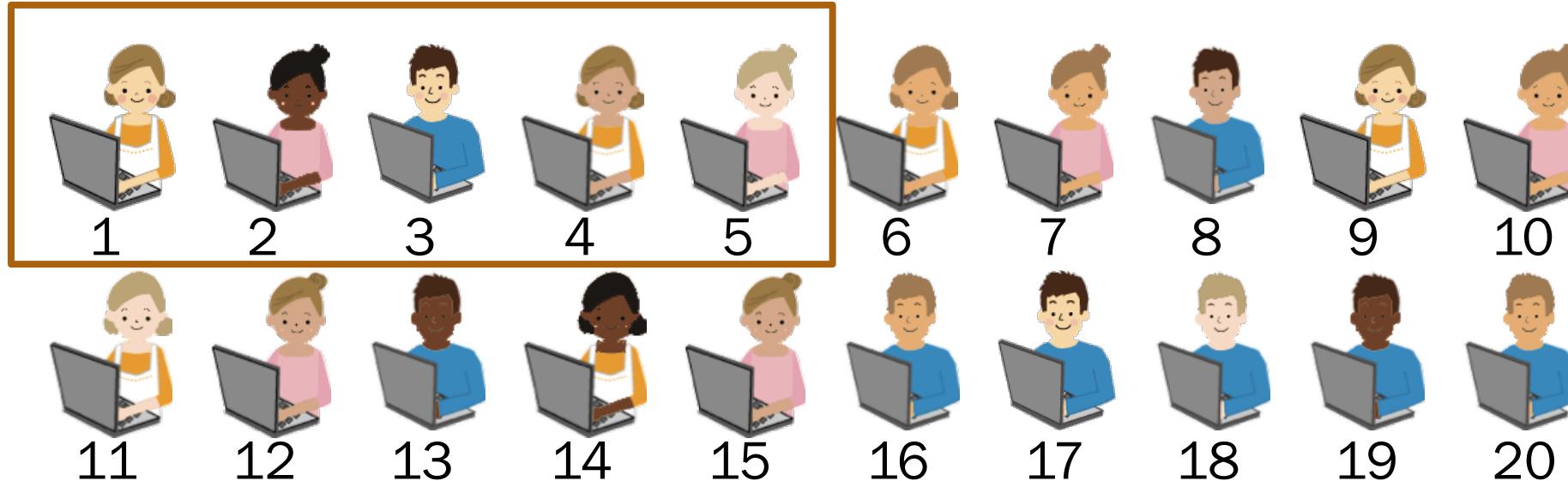
1. n people
get in line

$n!$ ways

Combinations with cake

There are $n = 20$ people.

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1. n people get in line

$n!$ ways

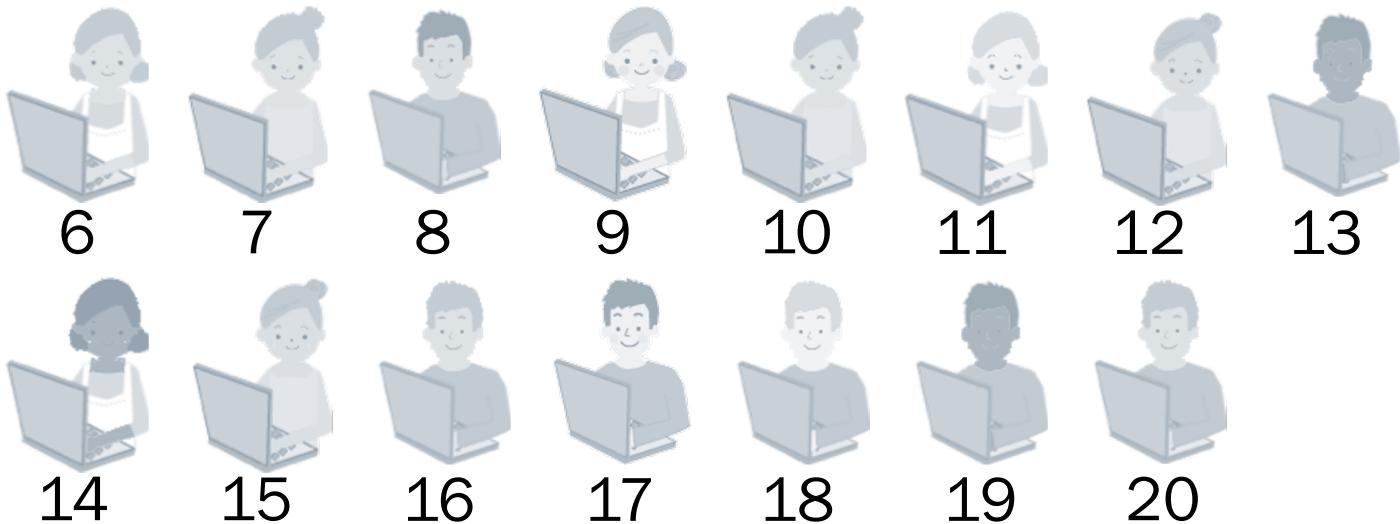
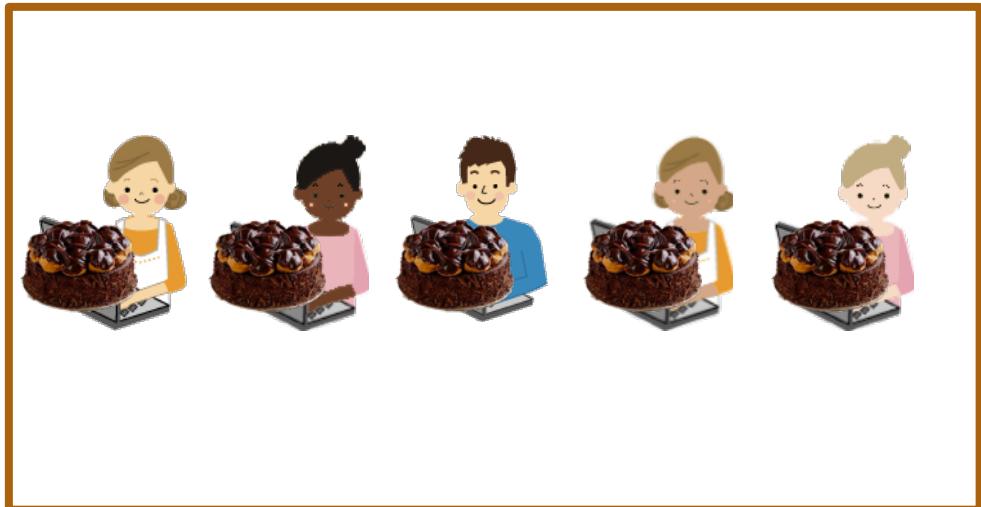
2. Put first k in cake room

1 way

Combinations with cake

There are $n = 20$ people.

How many ways can we choose $k = 5$ people to get cake?



1. n people
get in line

$n!$ ways

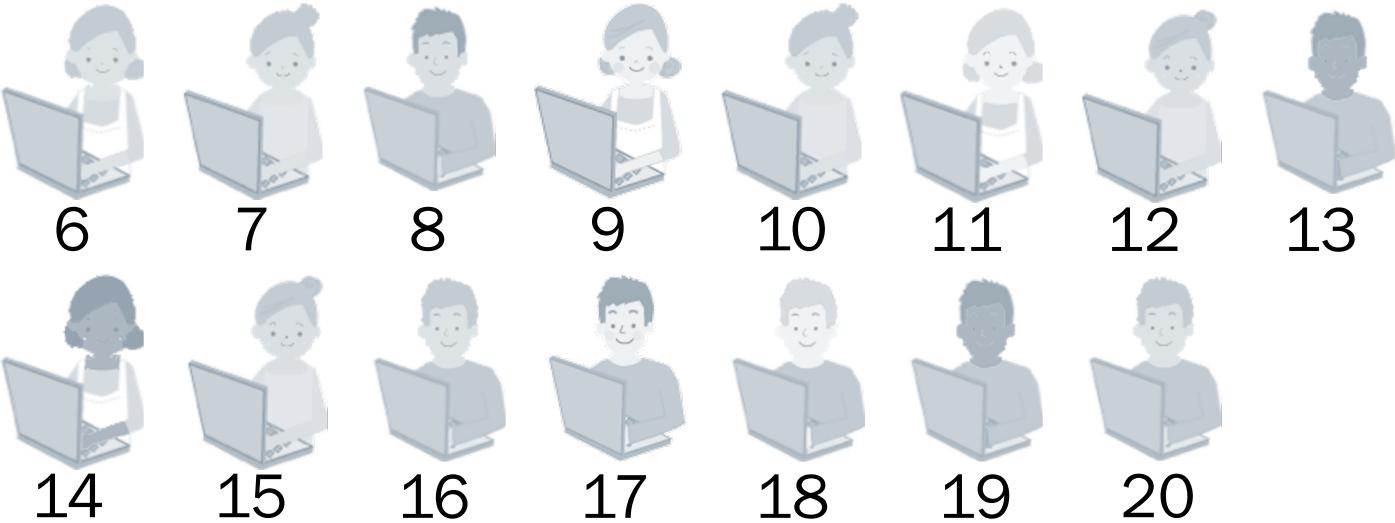
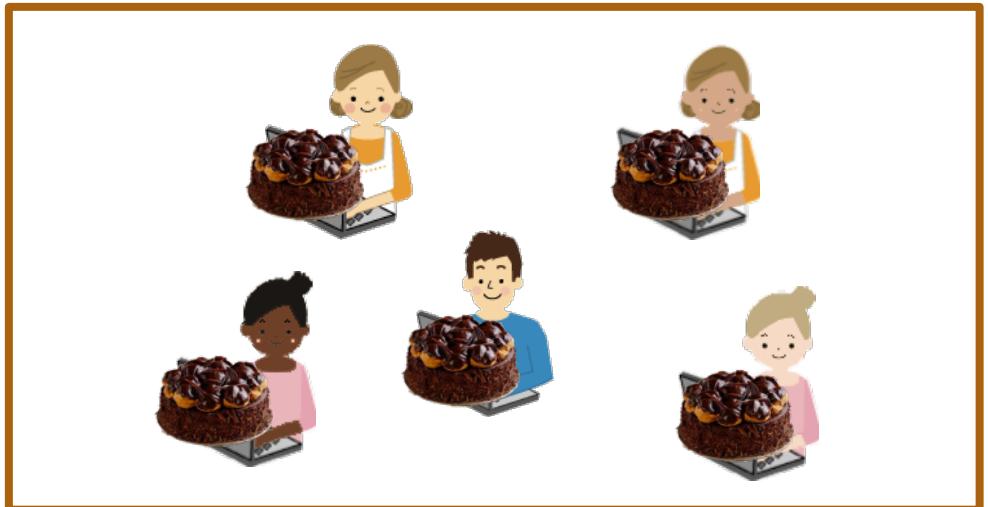
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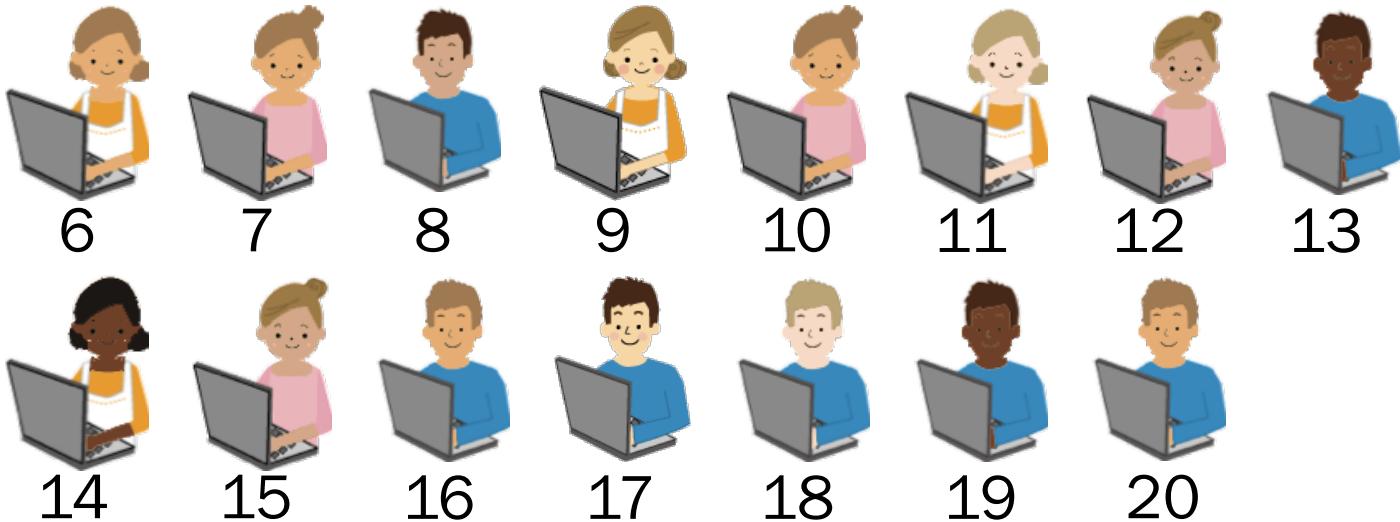
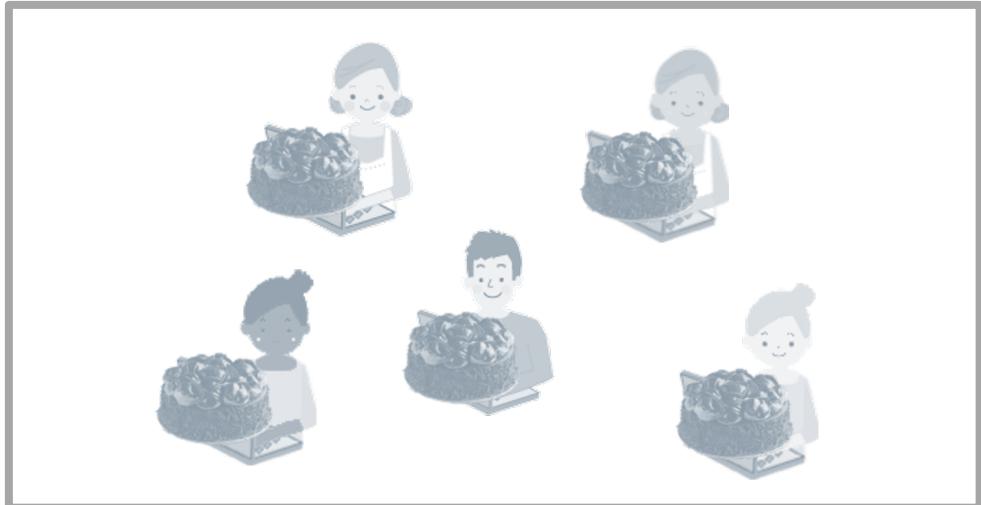
3. Allow cake group to mingle

$k!$ different permutations lead to the same mingle

Combinations with cake

There are $n = 20$ people.

How many ways can we choose $k = 5$ people to get cake?



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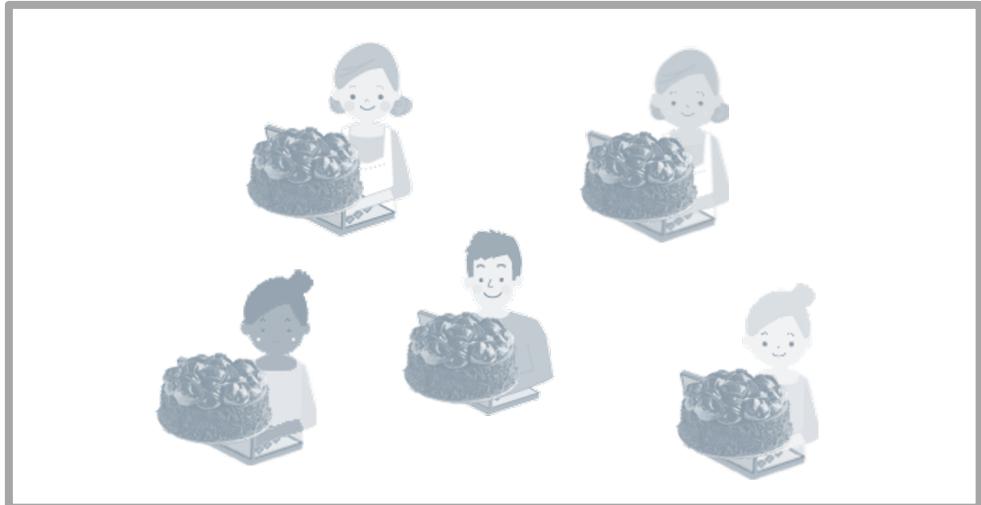
$k!$ different permutations lead to the same mingle

4. Allow non-cake group to mingle

Combinations with cake

There are $n = 20$ people.

How many ways can we choose $k = 5$ people to get cake?



1. n people get in line

$n!$ ways

2. Put first k in cake room

1 way



3. Allow cake group to mingle

$k!$ different permutations lead to the same mingle

4. Allow non-cake group to mingle

$(n - k)!$ different permutations lead to the same mingle

Combinations

A **combination** is an unordered selection of k objects from a set of n **distinct** objects.

The number of ways of making this selection is

$$\frac{n!}{k!(n-k)!} = n! \times 1 \times \frac{1}{k!} \times \frac{1}{(n-k)!}$$

1. Order n distinct objects

2. Take first k as chosen

3. Overcounted:
any ordering of chosen group is same choice

4. Overcounted:
any ordering of unchosen group is same choice

Combinations

A **combination** is an unordered selection of k objects from a set of n **distinct** objects.

The number of ways of making this selection is

$$\frac{n!}{k!(n-k)!} = n! \times 1 \times \frac{1}{k!} \times \frac{1}{(n-k)!} = \binom{n}{k} \text{ Binomial coefficient}$$

Fun Fact: $\binom{n}{k} = \binom{n}{n-k}$ $(x + y)^n$

Probability textbooks

Choose k of
 n distinct objects $\binom{n}{k}$

How many ways are there to choose 3 books
from a set of 6 distinct books?

$$\binom{6}{3} = \frac{6!}{3! 3!} = 20 \text{ ways}$$



To the code!

How many unique hands of 5 cards are there in a 52 card deck?

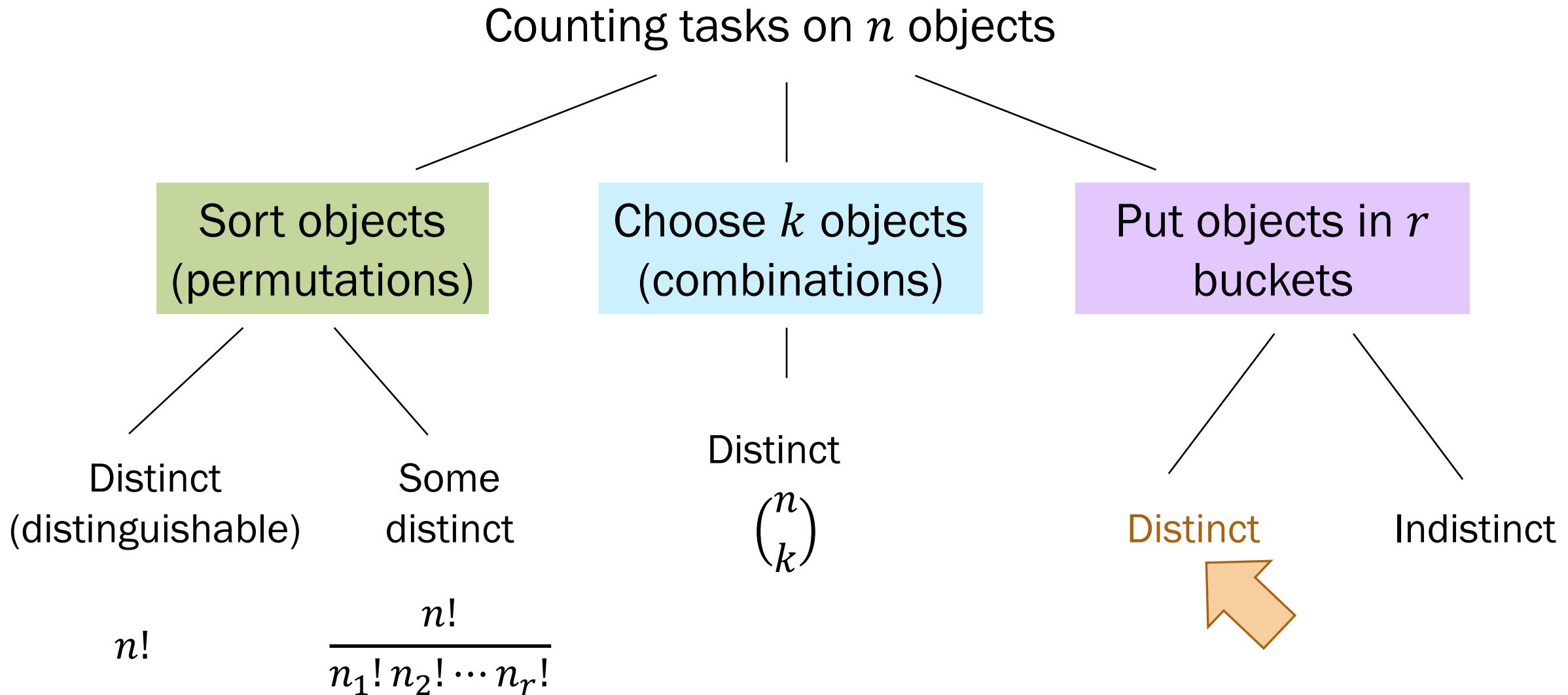


```
def main():
    cards = make_deck()
    all_hands = itertools.combinations(cards, 5)
    for hand in all_hands:
        print(hand)
```

```
def main():
    total = math.comb(52, 5)
    print(total)
```

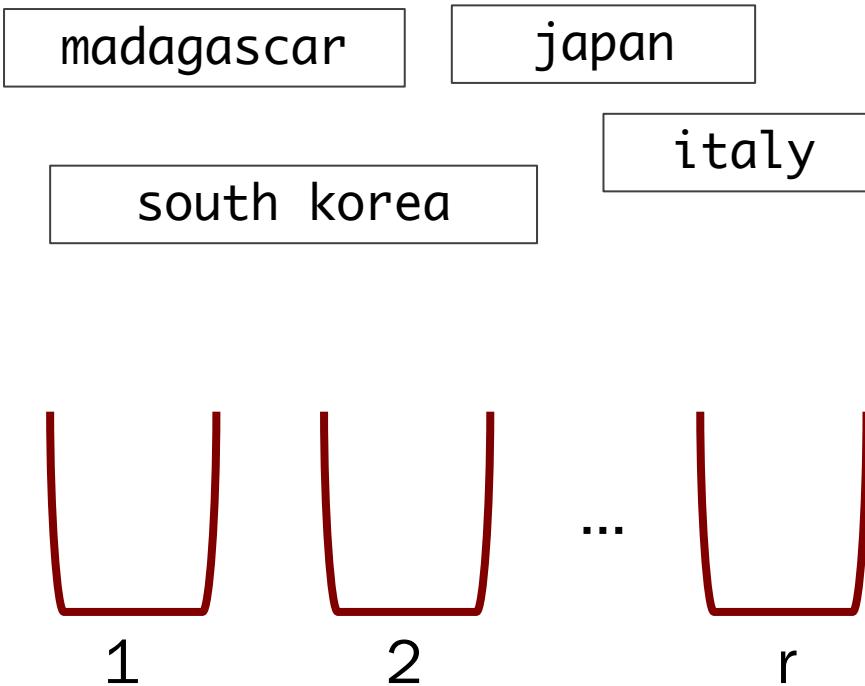
Buckets and The Divider Method

Summary of Combinatorics



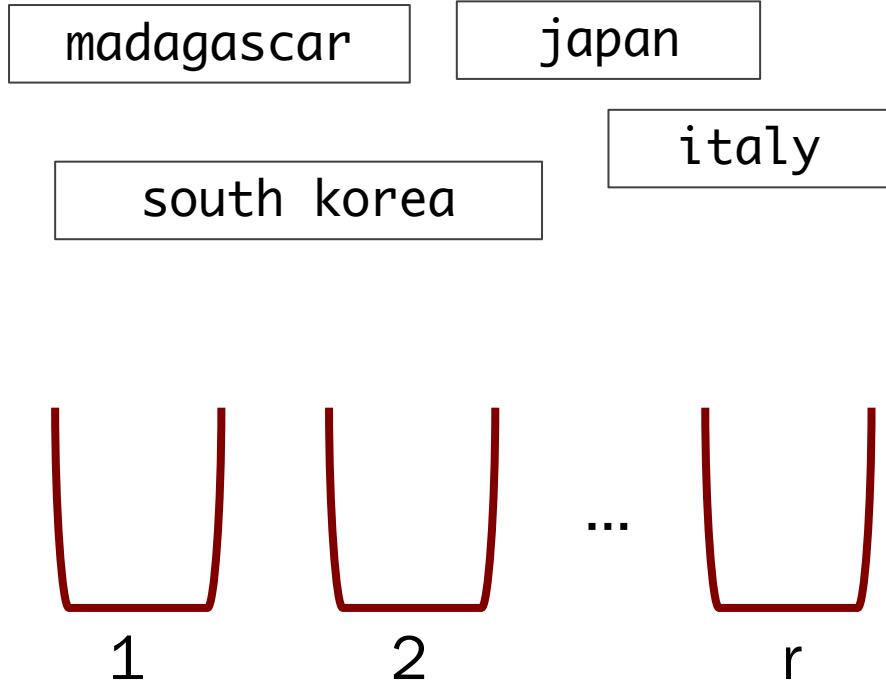
~~Balls and urns~~ Hash tables and **distinct** strings

How many ways are there to hash n **distinct** strings to r buckets?



Balls and urns Hash tables and **distinct** strings

How many ways are there to hash n **distinct** strings to r buckets?

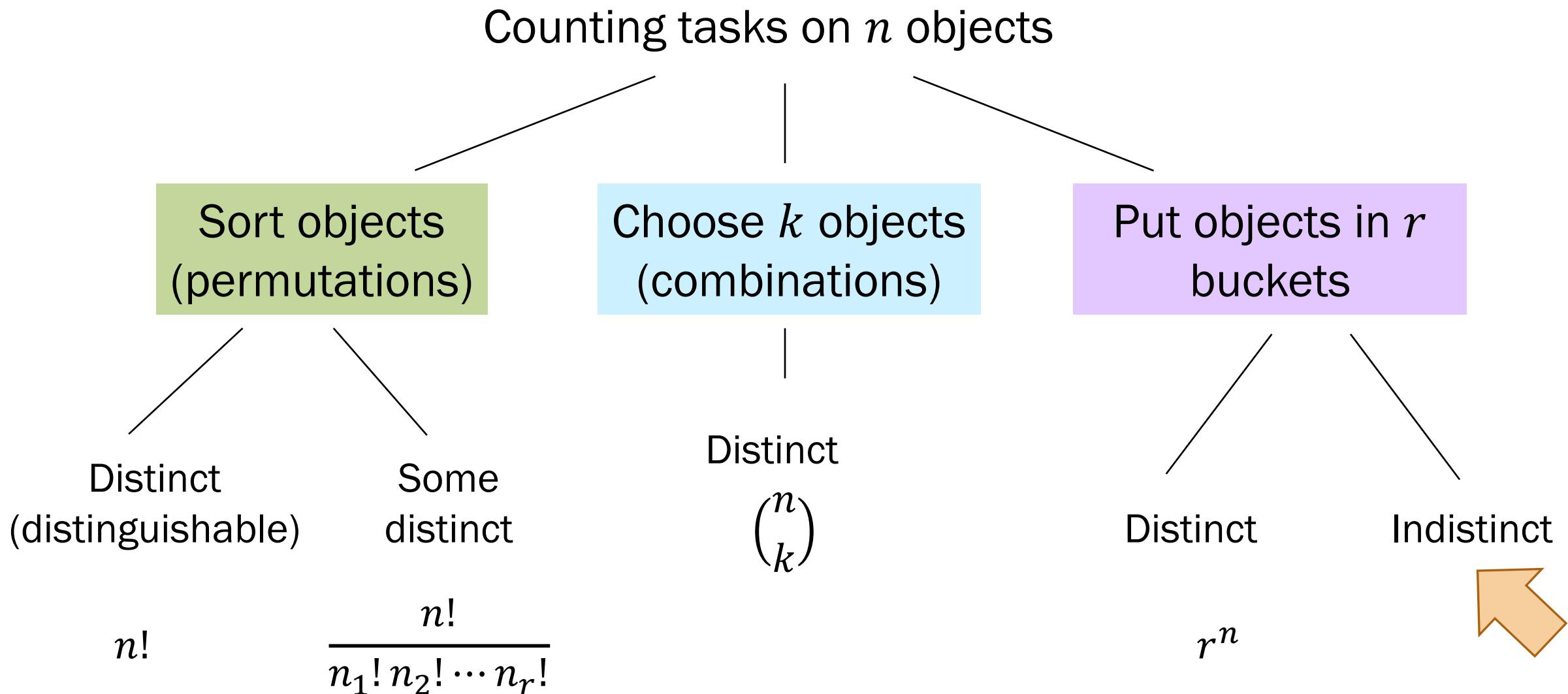


Steps:

1. Bucket 1st string
2. Bucket 2nd string
- ...
- n . Bucket n^{th} string

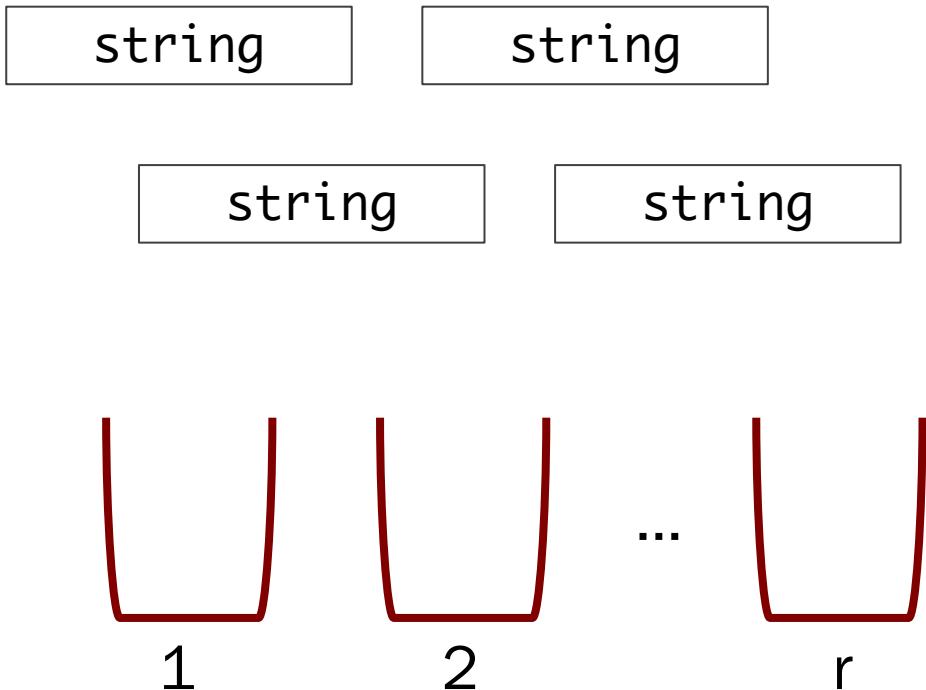
r^n outcomes

Summary of Combinatorics



Servers and **indistinct** requests

How many ways are there to distribute n **indistinct** strings to r buckets?



Goal

Bucket 1 has x_1 string,
Bucket 2 has x_2 string,

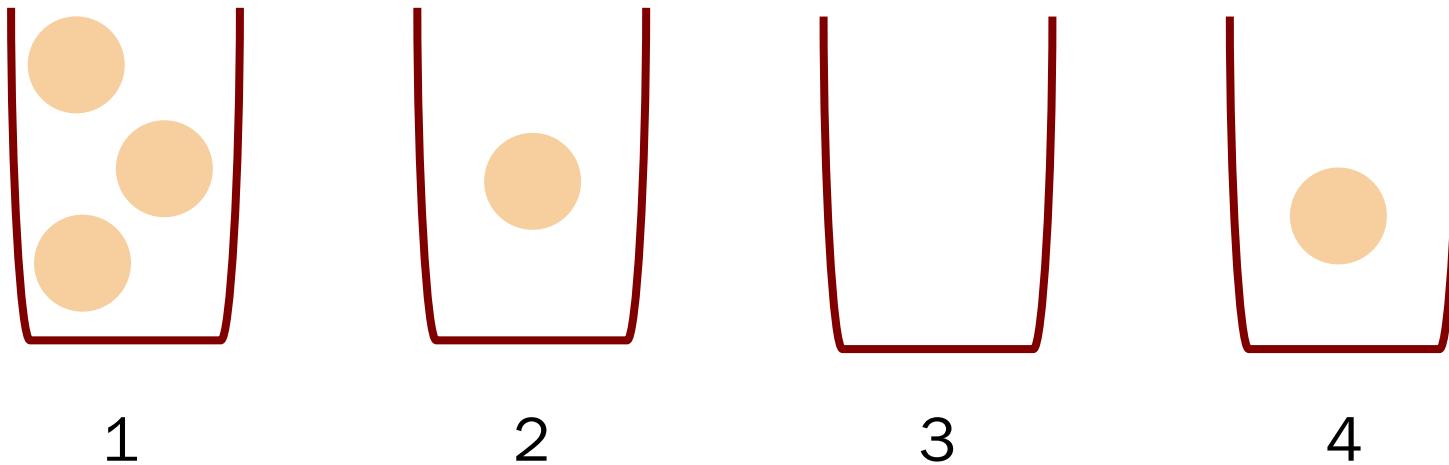
...

Bucket r has x_r string

$$\text{constraint: } \sum_{i=1}^r x_i = n$$

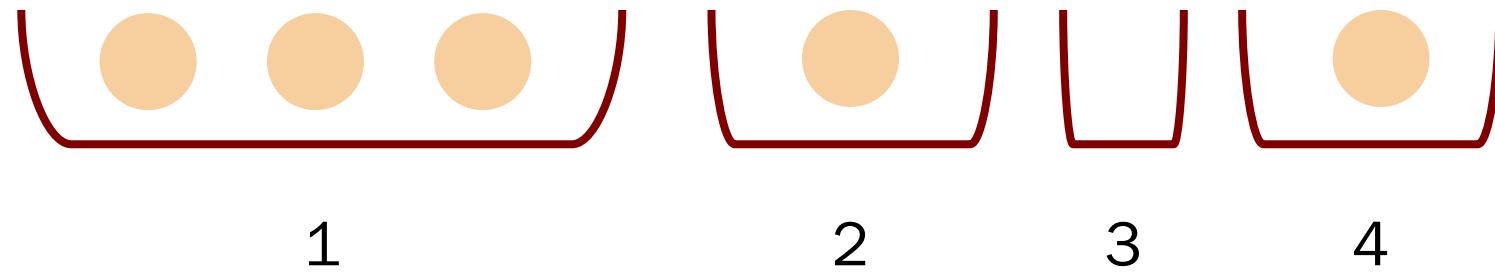
Servers and **indistinct** requests

How many ways are there to distribute n **indistinct** strings to r buckets?



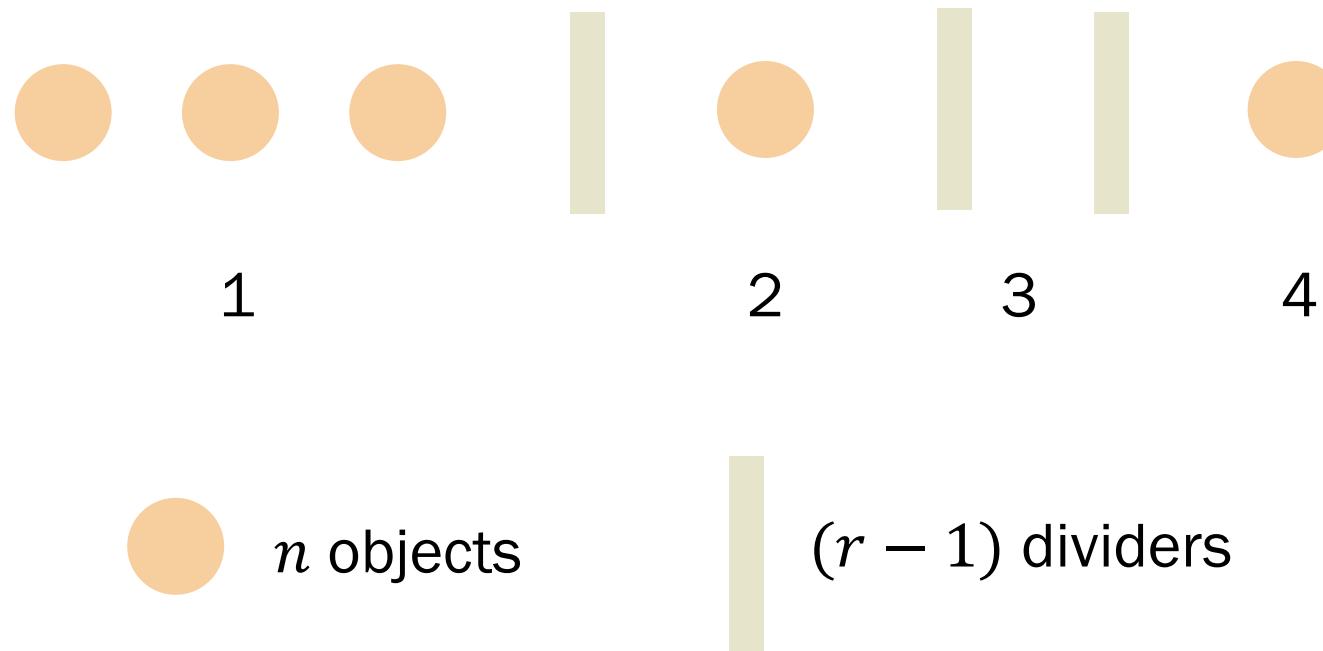
Servers and **indistinct** requests

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Servers and **indistinct** requests

How many ways are there to distribute n **indistinct** strings to r buckets?

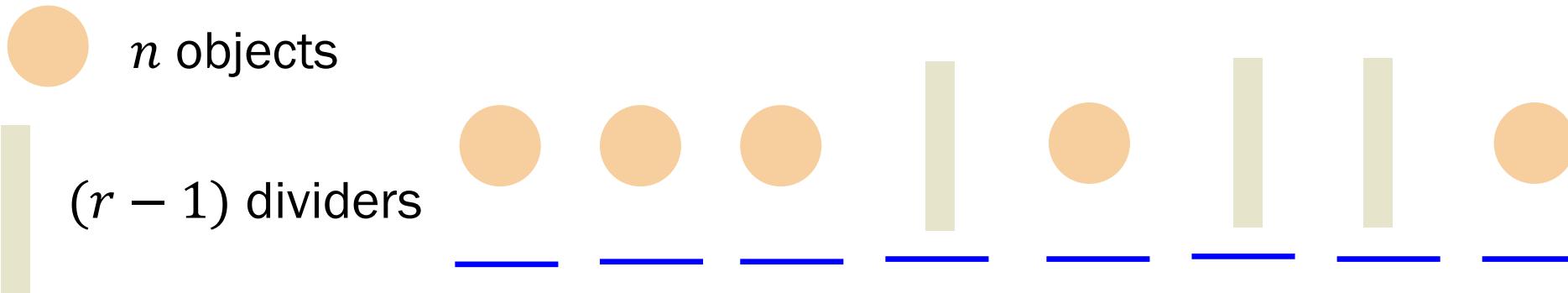


Servers and **indistinct** requests

How many ways are there to distribute n **indistinct** strings to r buckets?

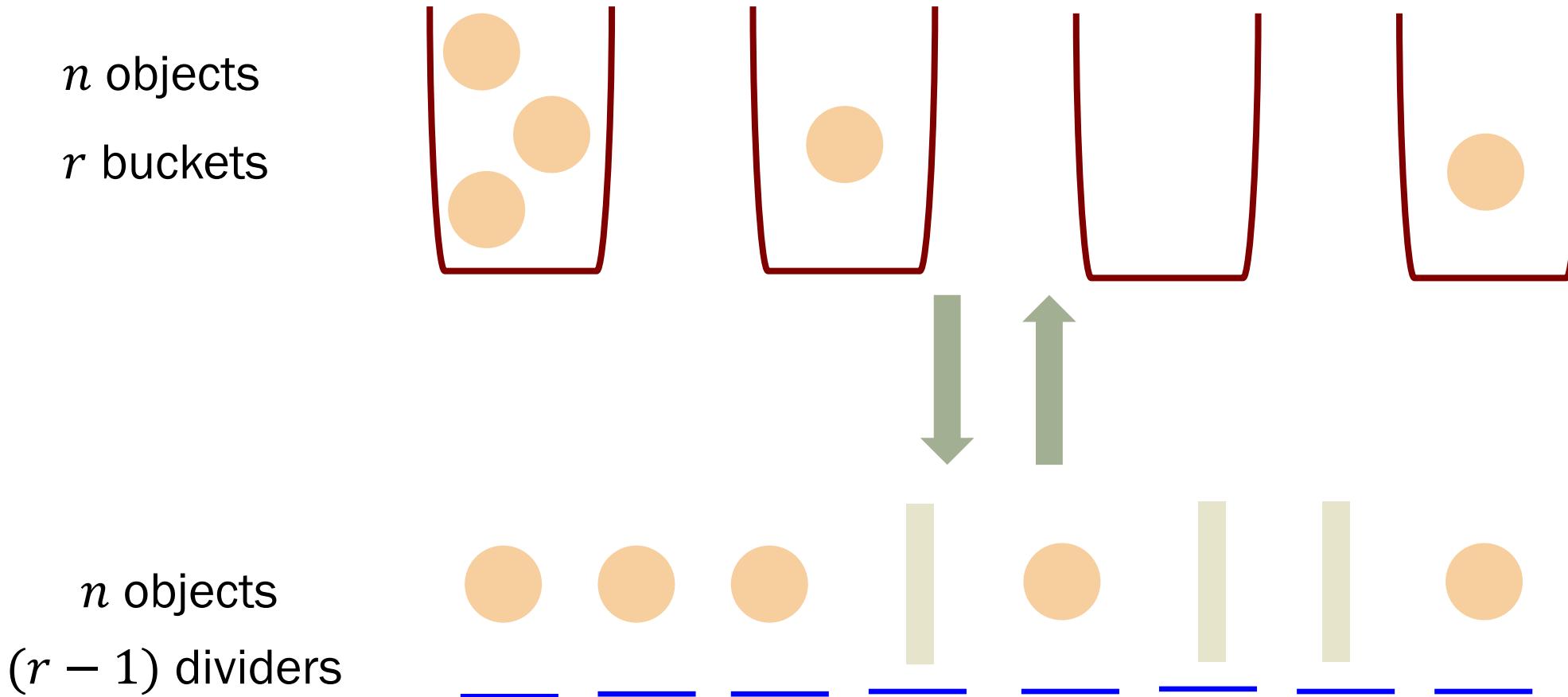
_____ : Slots where either a divider or an object can go into

$(n + r - 1)$ slots



Servers and **indistinct** requests

How many ways are there to distribute n **indistinct** strings to r buckets?

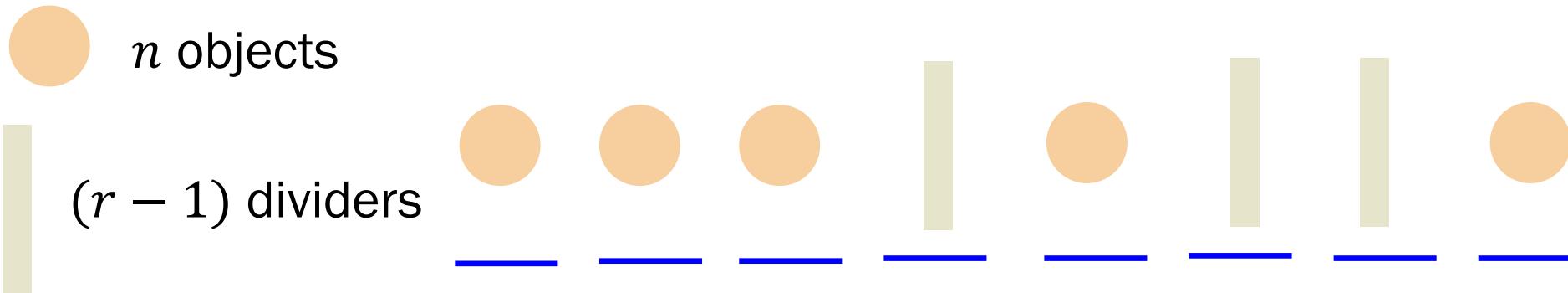


Servers and **indistinct** requests

How many ways are there to distribute n **indistinct** strings to r buckets?

— : Slots where either a divider or an object can go into

$(n + r - 1)$ slots



Answer

1. (Combination) Choose $(r - 1)$ divider slots out of $(n + r - 1)$ distinct slots

2. (Permutation) Arrange $(r - 1)$ indistinct dividers and n indistinct objects

The divider method

The number of ways to distribute n indistinct objects into r buckets is equivalent to the number of ways of:

1. Choosing $(r-1)$ slots among $(n+r-1)$ distinct slots to place the divider
2. Arranging $(r-1)$ indistinct dividers and n indistinct objects.

$$\text{Total} = \binom{n + r - 1}{r - 1} = \frac{(n + r - 1)!}{n! (r - 1)!} \text{ outcomes}$$

Integer solutions to equations

Divider method
(n indistinct objects, r buckets) $\binom{n+r-1}{r-1}$

How many integer solutions are there to the following equation:

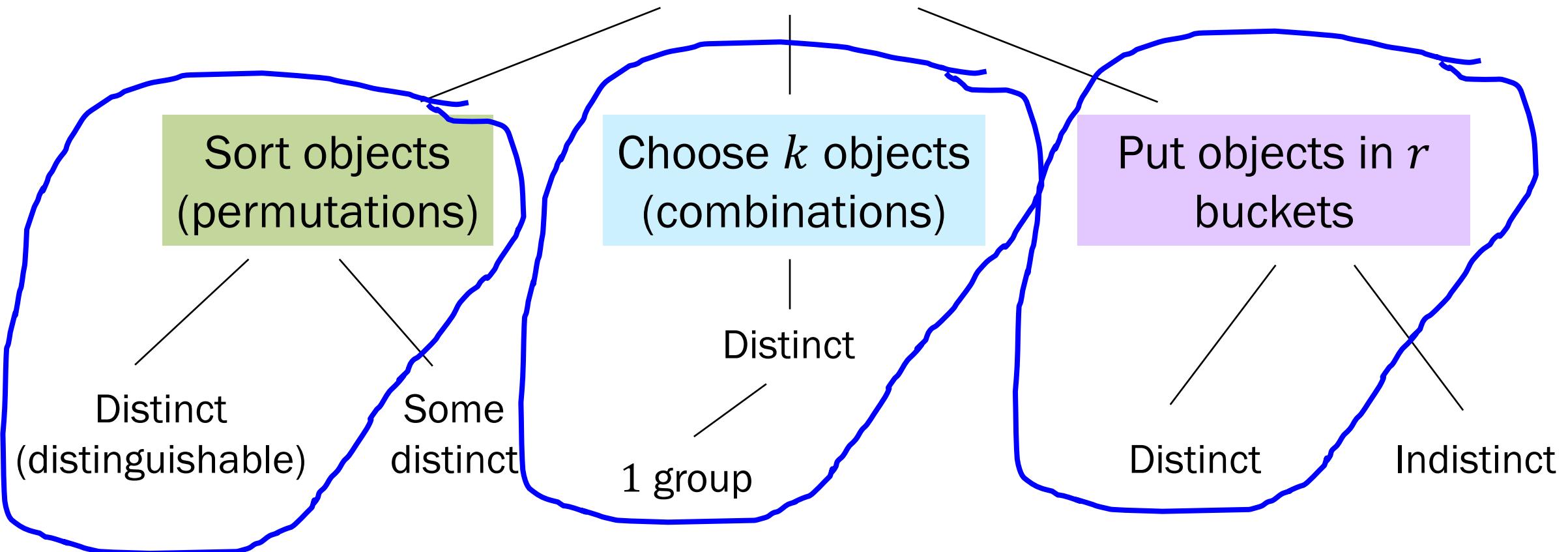
$$x_1 + x_2 + \cdots + x_r = n,$$

where for all i , x_i is an integer such that $0 \leq x_i \leq n$?

Positive integer equations can be solved with the divider method.

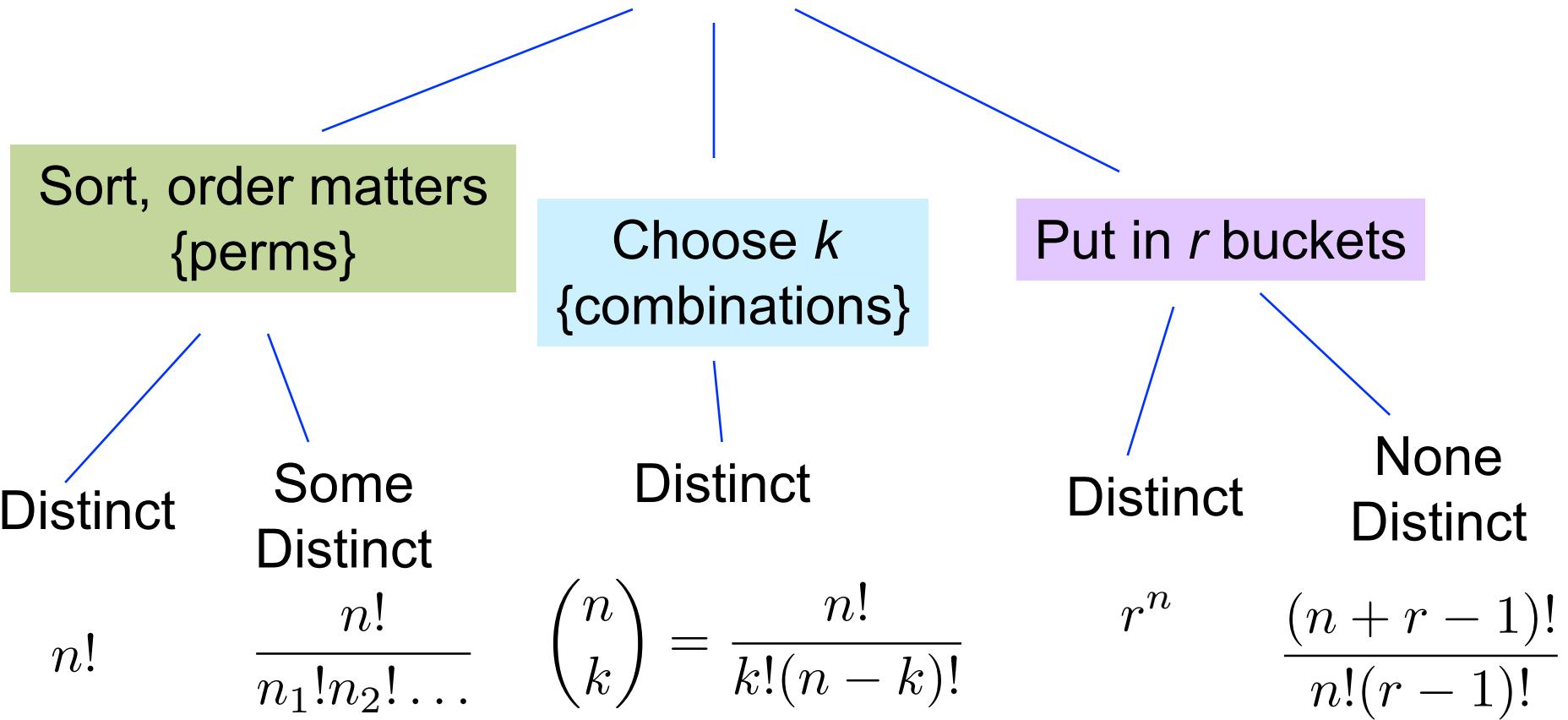
Summary of Combinatorics

Counting tasks on n objects



Counting Rules

Counting operations on n objects



To the Code!

baob
bbao
obba
oabb
boab
bab
abbo
aobb
boba
abob
bboa
obab

```
import itertools

def main():
    letters = ['b', 'o', 'b', 'a']
    perms = set(itertools.permutations(letters))
    for perm in perms:
        pretty_perm = "".join(perm)
        print(pretty_perm)

import math

def main():
    n = math.factorial(4)
    d = math.factorial(2)
    print(n / d)
```