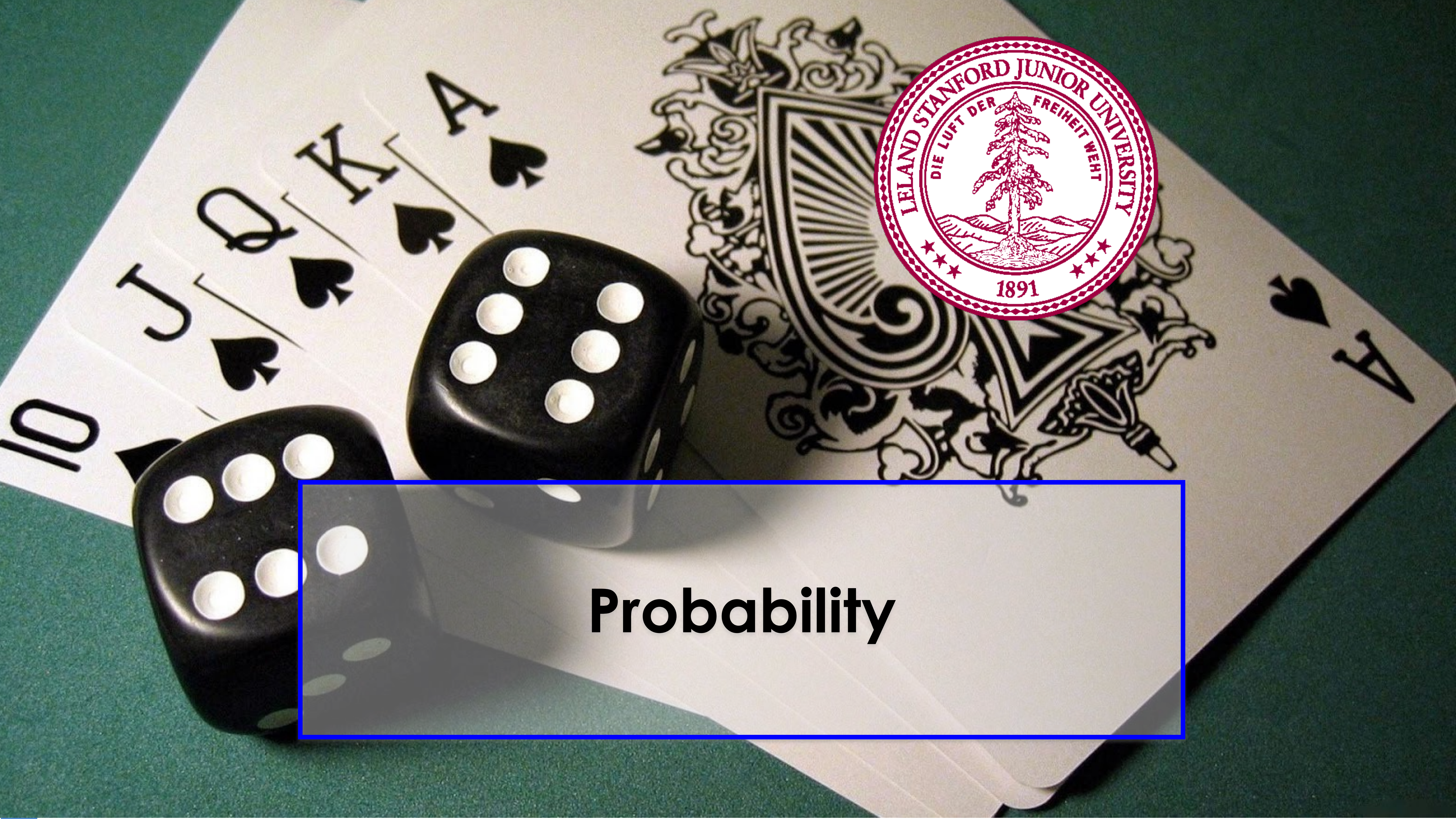


Probability

Review

Counting Rules

Counting operations on *n* objects

- Sort, order matters {perms}
  - Distinct: $n!$
  - Some Distinct: $\dfrac{n!}{n_1!n_2!\ldots}$
- Choose *k* {combinations}
  - Distinct: $\dbinom{n}{k} = \dfrac{n!}{k!(n-k)!}$
- Put in *r* buckets
  - Distinct: r^n
  - None Distinct: $\dfrac{(n+r-1)!}{n!(r-1)!}$

End Review

Sample Space

- **<u>Sample space</u>**, S, is set of all possible outcomes of an experiment
  - Coin flip: S = {Head, Tails}
  - Flipping two coins: S = {{H, H}, {H, T}, {T, H}, {T, T}}
  - Roll of 6-sided die: S = {1, 2, 3, 4, 5, 6}
  - # emails in a day: $S = \{x \mid x \in \mathbf{Z}, x \geq 0\}$ {non-neg. ints}
  - YouTube hrs. in day: $S = \{x \mid x \in \mathbf{R}, 0 \leq x \leq 24\}$

Event Space

- **Event**, E, is some subset of S $\{E \subseteq S\}$
  - Coin flip is heads: $E = \{\text{Head}\}$
  - $\geq$ 1 head on 2 coin flips: $E = \{\{H, H\}, \{H, T\}, \{T, H\}\}$
  - Roll of die is 3 or less: $E = \{1, 2, 3\}$
  - # emails in a day $\leq$ 20: $E = \{x \mid x \in \mathbf{Z}, 0 \leq x \leq 20\}$
  - Wasted day {$\geq$ 5 YT hrs.}: $E = \{x \mid x \in \mathbf{R}, 5 \leq x \leq 24\}$

Note: When Ross uses: $\subset$, he really means: $\subseteq$

Event Space

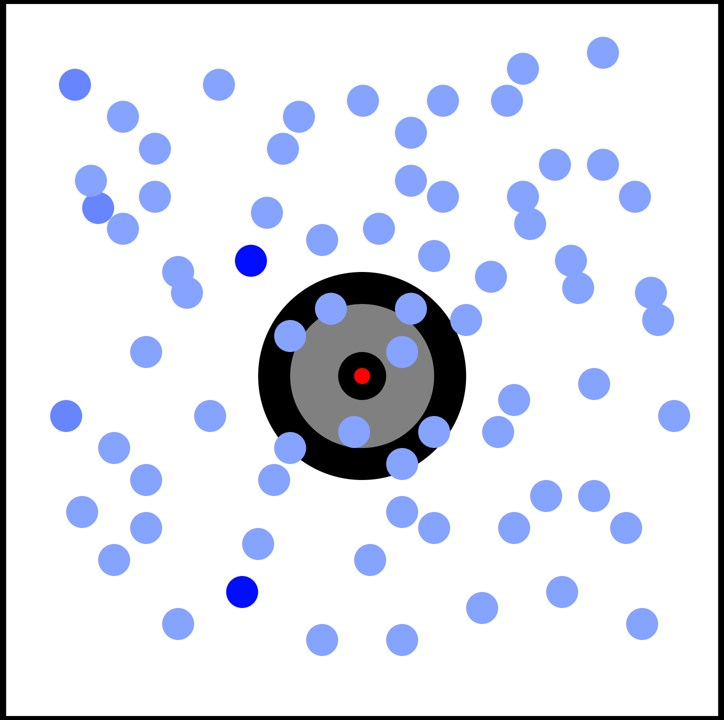
Sample Space, S

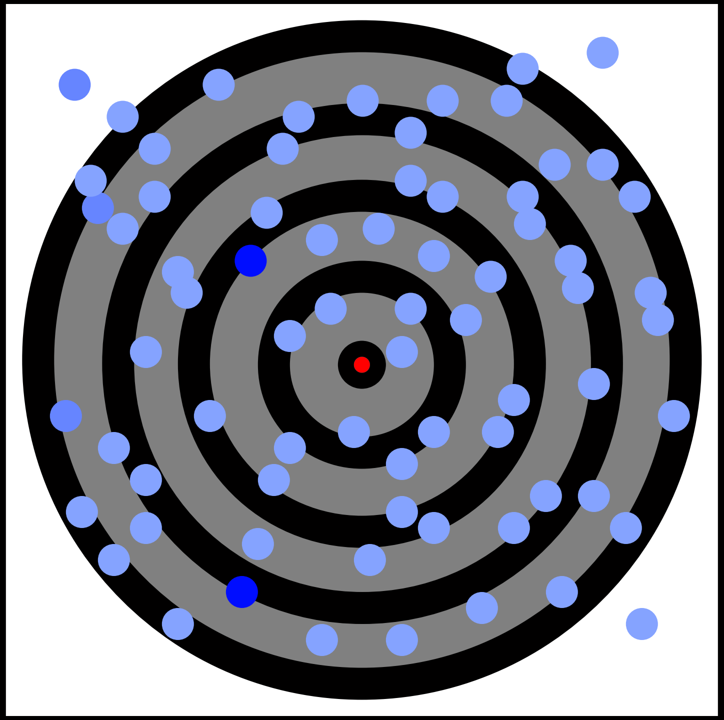
- Coin flip
  $S = \{\text{Heads, Tails}\}$
- Flipping two coins
  $S = \{(\text{H,H}), (\text{H,T}), (\text{T,H}), (\text{T,T})\}$
- Roll of 6-sided die
  $S = \{1, 2, 3, 4, 5, 6\}$
- # emails in a day
  $S = \{x \mid x \in \mathbb{Z}, \; x \geq 0\}$
- TikTok hours in a day
  $S = \{x \mid x \in \mathbb{R}, 0 \leq x \leq 24\}$

Event, E

- Flip lands heads
  $E = \{\text{Heads}\}$
- ≥ 1 head on 2 coin flips
  $E = \{(\text{H,H}), (\text{H,T}), (\text{T,H})\}$
- Roll is 3 or less:
  $E = \{1, 2, 3\}$
- Low email day (≤ 20 emails)
  $E = \{x \mid x \in \mathbb{Z}, \; 0 \leq x \leq 20\}$
- Wasted day (≥ 5 TT hours):
  $E = \{x \mid x \in \mathbb{R}, 5 \leq x \leq 24\}$

What is a probability?





Number between 0 and 1

A number to which we ascribe meaning

$$P(E)$$

* Our belief that an event E occurs

A number to which we ascribe meaning

$$\Pr(E)$$

* Our belief that an event E occurs

Axioms of Probability

Recall: S = all possible outcomes. E = the event.

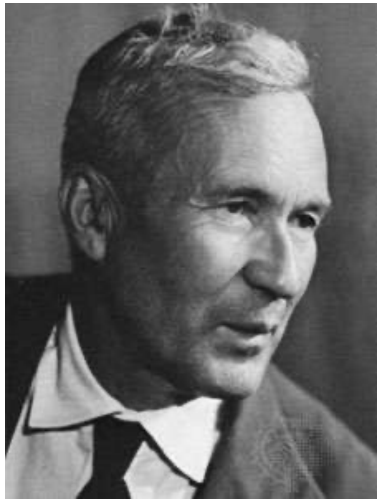
- Axiom 1: $0 \leq P(E) \leq 1$
- Axiom 2: $P(S) = 1$
- Axiom 3: If events E and F are mutually exclusive:

$$P(E \cup F) = P(E) + P(F)$$

Axioms of Probability

Recall: S = all possible outcomes. E = the event.

Kolmogorov, 1933



- Axiom 1: $0 \leq P(E) \leq 1$
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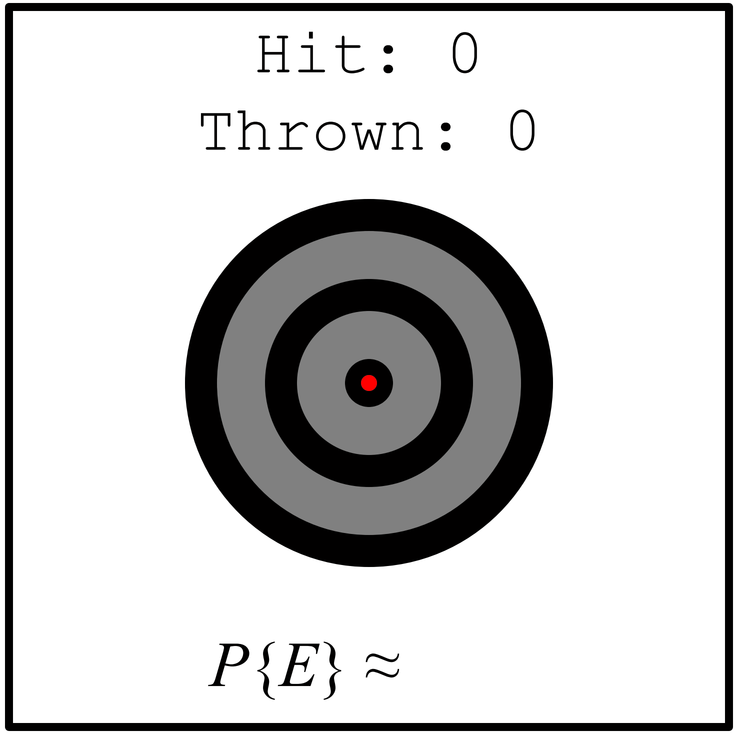
Real-life meaning of Probability

$$P(E) = \lim_{n \to \infty} \frac{n(E)}{n}$$

Real-life meaning of Probability

$$P(E) = \lim_{n \to \infty} \frac{n(E)}{n}$$

n is the number of trails



The "event" *E* is that you hit the target

Real-life meaning of Probability

$$P(E) = \lim_{n \to \infty} \frac{n(E)}{n}$$

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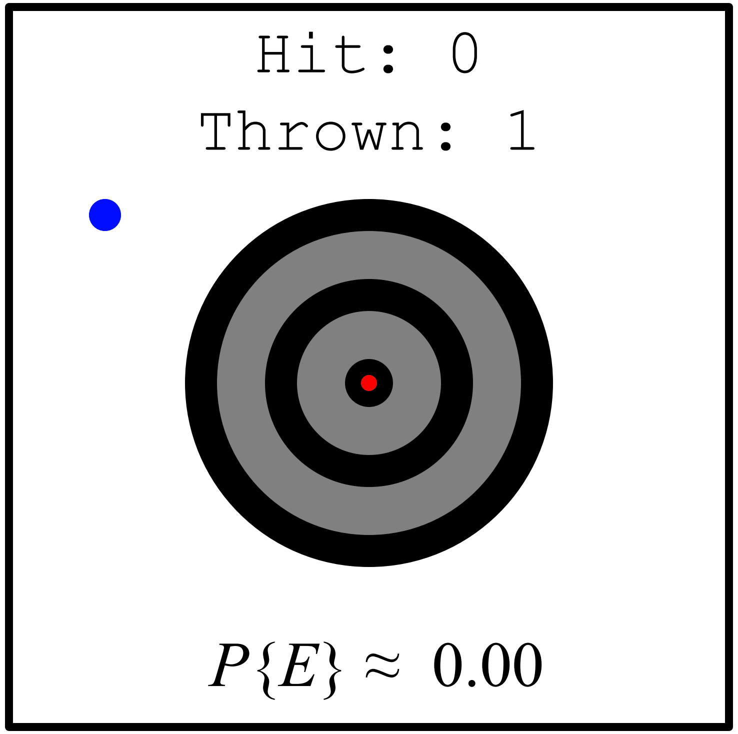

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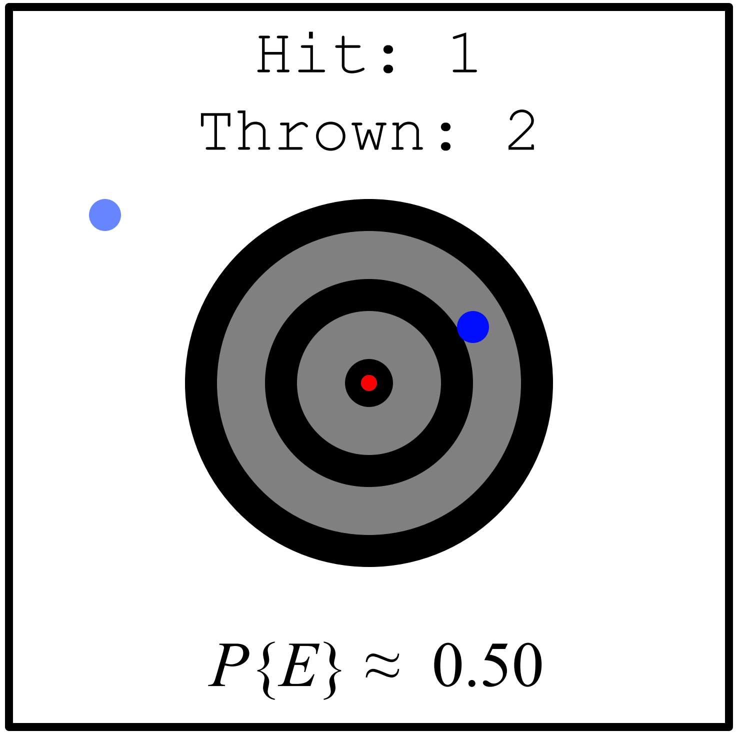

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Real-life meaning of Probability

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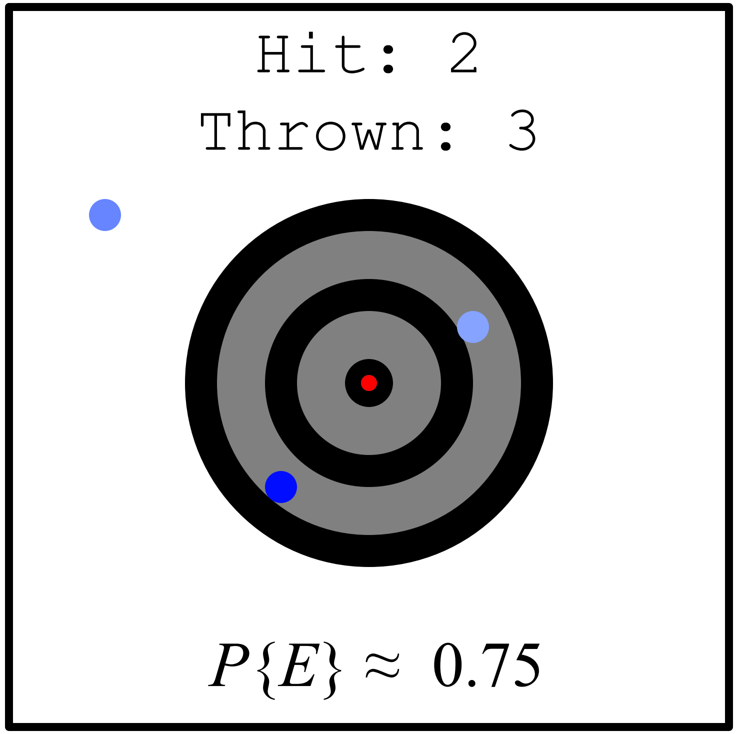
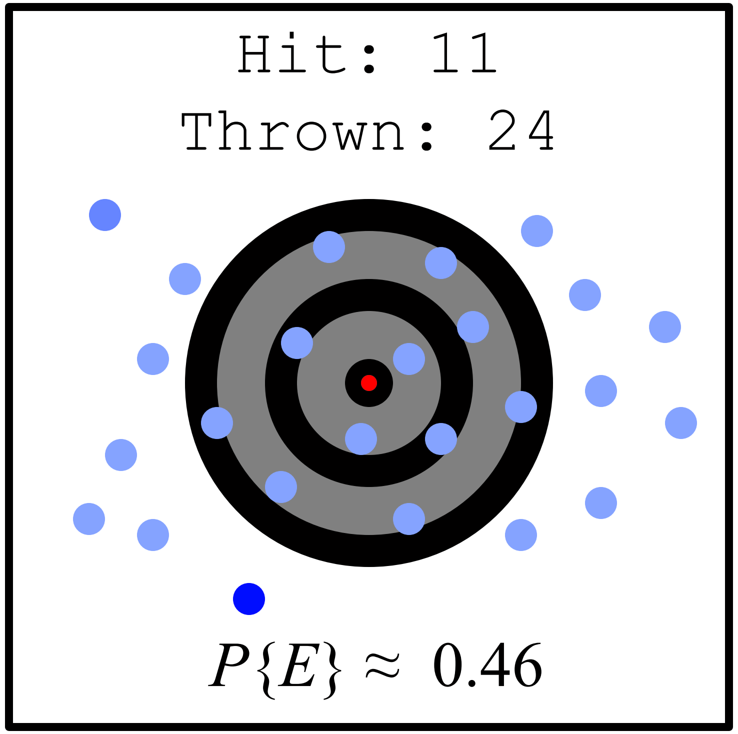


The "event" *E* is that you hit the target

Real-life meaning of Probability

$$P(E) = \lim_{n \to \infty} \frac{n(E)}{n}$$

n is the number of trails



The "event" *E* is that you hit the target

Special Case of Probability

Equally Likely Outcomes

Equally Likely Outcomes

Some sample spaces have equally likely outcomes.

- Coin flip: S = {Head, Tails}
- Flipping two coins: S = {{H, H}, {H, T}, {T, H}, {T, T}}
- Roll of 6-sided die: S = {1, 2, 3, 4, 5, 6}

If we have equally likely outcomes, then P{Each outcome} $= \dfrac{1}{|S|}$

Therefore $$P(E) = \frac{\text{\# outcomes in E}}{\text{\# outcomes in S}} = \frac{|E|}{|S|}$$ {by Axiom 3}

Not Everything is Equally Likely

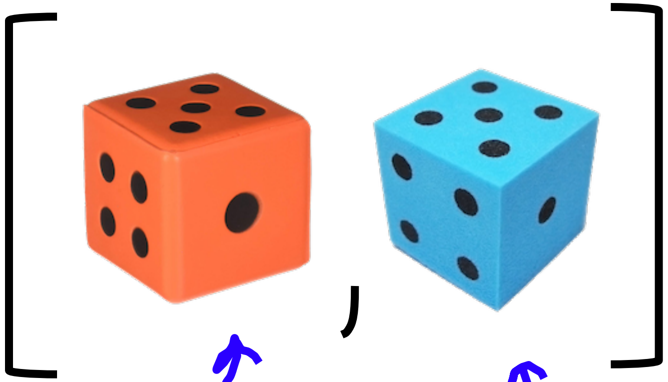
- Play lottery.
  - What is P{Win}?

- S = {Lose, Win}
- E = {Win}
- P{Win} = |E|/|S| = 1/2 = 50%

Sum of Two Die = 7?

Roll two 6-sided dice. What is P[sum = 7]?

Each outcome



Value dice 1

Value dice 2

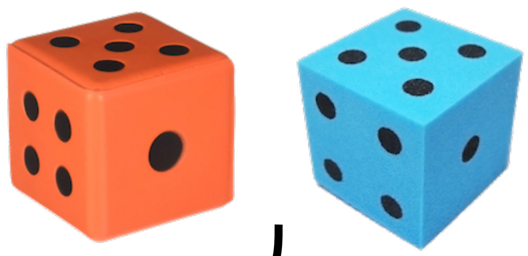
S = { [1,1] [1,2] [1,3] [1,4] [1,5] [1,6]
[2,1] [2,2] [2,3] [2,4] [2,5] [2,6]
[3,1] [3,2] [3,3] [3,4] [3,5] [3,6]
[4,1] [4,2] [4,3] [4,4] [4,5] [4,6]
[5,1] [5,2] [5,3] [5,4] [5,5] [5,6]
[6,1] [6,2] [6,3] [6,4] [6,5] [6,6] }

1. Choose a sample space S (hopefully one that's equally likely)
2. Define the event set E that is of interest

Sum of Two Die = 7?

Roll two 6-sidex dice. What is probability the sum = 7?
Let E be the event that the sum is 7

Each outcome



Value dice 1

Value dice 2

S = {

[1,1]	[1,2]	[1,3]	[1,4]	[1,5]	**[1,6]**
[2,1]	[2,2]	[2,3]	[2,4]	**[2,5]**	[2,6]
[3,1]	[3,2]	[3,3]	**[3,4]**	[3,5]	[3,6]
[4,1]	[4,2]	**[4,3]**	[4,4]	[4,5]	[4,6]
[5,1]	**[5,2]**	[5,3]	[5,4]	[5,5]	[5,6]
[6,1]	[6,2]	[6,3]	[6,4]	[6,5]	[6,6] }

E = *in blue*

$$P(E) = \frac{|E|}{|S|} = \frac{6}{36} = 0.16\bar{6}$$

Is it correct?

$$P(E) = \frac{|E|}{|S|} = \frac{6}{36} = 0.16\bar{6}$$

Sum of Two Die = 7?

```python
import random
from tqdm import tqdm

N_TRIALS = 10000000  # getting close to infinity
TARGET_SUM = 7       # do the two dice sum to 6?

def main():
    n_events = 0
    for i in tqdm(range(N_TRIALS)):
        dice_total = run_experiment()
        if dice_total == TARGET_SUM:
            n_events += 1
    pr_e = n_events / N_TRIALS
    print(f'after {N_TRIALS} trials')
    print('P(E) ≈ ', pr_e)

def run_experiment():
    d_1 = roll_dice()
    d_2 = roll_dice()
    return d_1 + d_2

def roll_dice():
    # give me a random dice roll
    # alternatively random.randint(1, 7)
    return random.choice([1,2,3,4,5,6])

if __name__ == '__main__':
    # this starts the program in main
    main()
```

$$P(E) = \frac{|E|}{|S|} = \frac{6}{36} = 0.16\bar{6}$$

PROBLEMS OUTPUT DEBUG CONSOLE TERMINAL

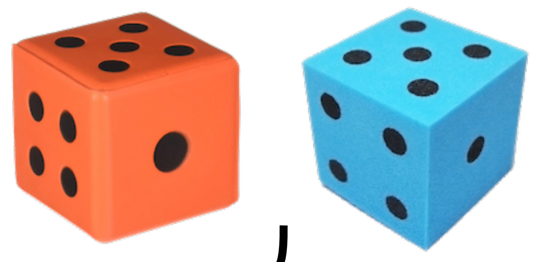
```
piech@Chriss-MBP-2 3 % python dice_soln.py
after 10000000 trials
P(E) =  0.1666913
```

Sum of Two Die = 2?

Roll two 6-sidex dice. What is probability the sum = ~~7~~ 2?
Let E be the event that the sum is ~~7~~ 2

$$S = \{\begin{array}{cccccc} [1,1] & [1,2] & [1,3] & [1,4] & [1,5] & [1,6] \\ [2,1] & [2,2] & [2,3] & [2,4] & [2,5] & [2,6] \\ [3,1] & [3,2] & [3,3] & [3,4] & [3,5] & [3,6] \\ [4,1] & [4,2] & [4,3] & [4,4] & [4,5] & [4,6] \\ [5,1] & [5,2] & [5,3] & [5,4] & [5,5] & [5,6] \\ [6,1] & [6,2] & [6,3] & [6,4] & [6,5] & [6,6] \end{array}\}$$

Each outcome



Value dice 1

Value dice 2

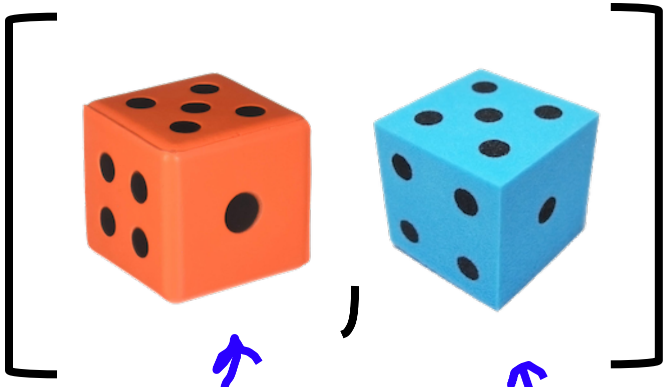
E =

Sum of Two Die = 2?

Roll two 6-sidex dice. What is probability the sum = 2?
Let E be the event that the sum is 2

S = {	**[1,1]**	[1,2]	[1,3]	[1,4]	[1,5]	[1,6]
	[2,1]	[2,2]	[2,3]	[2,4]	[2,5]	[2,6]
	[3,1]	[3,2]	[3,3]	[3,4]	[3,5]	[3,6]
	[4,1]	[4,2]	[4,3]	[4,4]	[4,5]	[4,6]
	[5,1]	[5,2]	[5,3]	[5,4]	[5,5]	[5,6]
	[6,1]	[6,2]	[6,3]	[6,4]	[6,5]	[6,6] }

Each outcome



Value dice 1

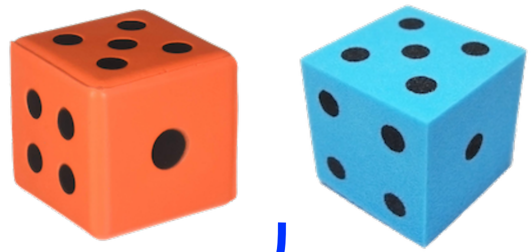
Value dice 2

E = *in red*

$$P(E) = \frac{|E|}{|S|} = \frac{1}{36} = 0.02\bar{7}$$

Other ways to make a Sample Space?

Each outcome

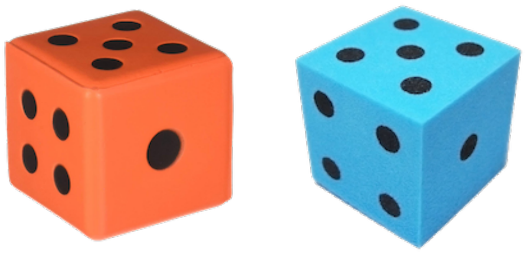


Value dice 1

Value dice 2

Sum of Two Die: Three options for the sample space

Value dice 1



Value dice 2

Think of the die as **distinct**

[5 , 5]

Value dice 1

Value dice 2

Think of the die as **indistinct**

{ 5 , 5 }

Value of a dice

Value of a dice

Just look at the sum

10

Sum of Two Die = 7? Bug: Die are Indistinct

Each outcome

Just look at the sum

Roll two 6-sidex dice. What is probability the sum = 7?
Let E be the event that the sum is 7

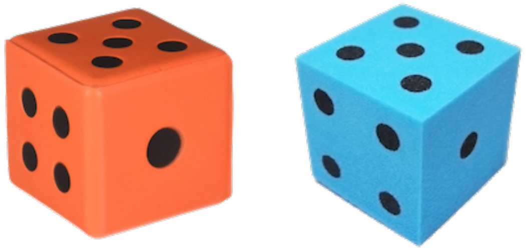
S = { ~~1~~ 7 2 3 4 5 6
7 8 9 10 11 12 }

E = *in red*

$$P(E) = \frac{|E|}{|S|} = \frac{1}{1\cancel{2}} = 0.0\overline{909}$$

Sum of Two Die: Three options for the sample space

Value dice 1



Value dice 2

Think of the die as **distinct**

[5 , 5]

Value dice 1

Value dice 2

Think of the die as **indistinct**

{ 5 , 5 }

Value of a dice

Value of a dice

Just look at the sum

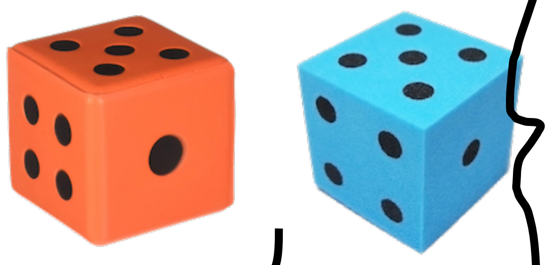
10

Sum of Two Die = 7? Bug: Die are Indistinct

Roll two 6-sidex dice. What is P(sum = 7)?

S = { {1,1} {1,2} {1,3} {1,4} {1,5} **{1,6}**
{2,2} {2,3} {2,4} **{2,5}** {2,6}
{3,3} **{3,4}** {3,5} {3,6}
{4,4} {4,5} {4,6}
{5,5} {5,6}
{6,6} }

Each outcome



Value dice 1

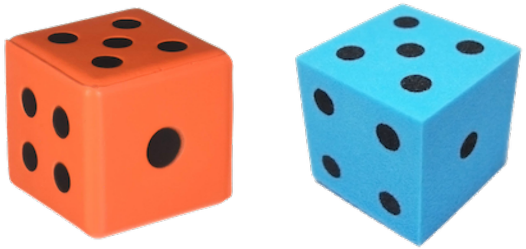
Value dice 2

E = *in blue*

$$P(E) = \frac{|E|}{|S|} = \frac{3}{20} = 0.15?$$

Sum of Two Die: Three options for the sample space

Value dice 1



Value dice 2

Think of the die as **distinct**

$[\,5\;,\;5\,]$

Value dice 1

Value dice 2

Think of the die as **indistinct**

$\{\,5\;,\;5\,\}$

Value of a dice

Value of a dice

Just look at the sum

10

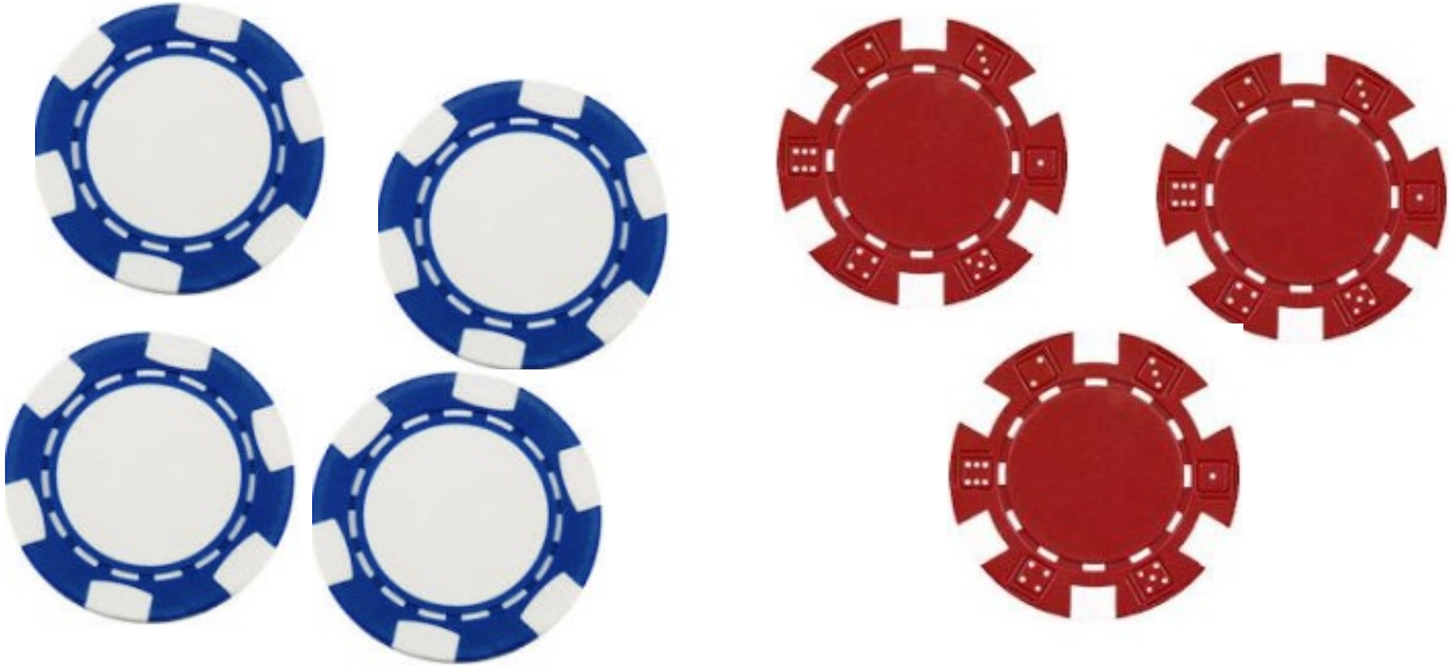
Sum of Two Die: Three options for the sample space

To get equally likely outcomes, it often helps to think of <u>items as distinct</u>, rather than indistinct.

Casino Chips

- 4 blue chips ($10) and 3 red chips ($50). 3 chips are drawn.
  - What is P(3 chips are worth $110)? =P(1 blue chip and 2 red chips)

Equally likely sample space? Thought experiment



4 blue

3 red

The Choice of Sample Space is Yours!

	Distinct	Indistinct
Unordered	$\{B_1, R_2, R_3\}$ $\{B_1, B_2, B_3\}$	{2 red, 1 blue} {3 blues}
Ordered	$[B_1, R_2, R_3]$ $[B_1, B_2, B_3]$	[blue, red, red] [blue, blue, blue]

Which choice will lead to equally likely outcomes?

pigs and cows

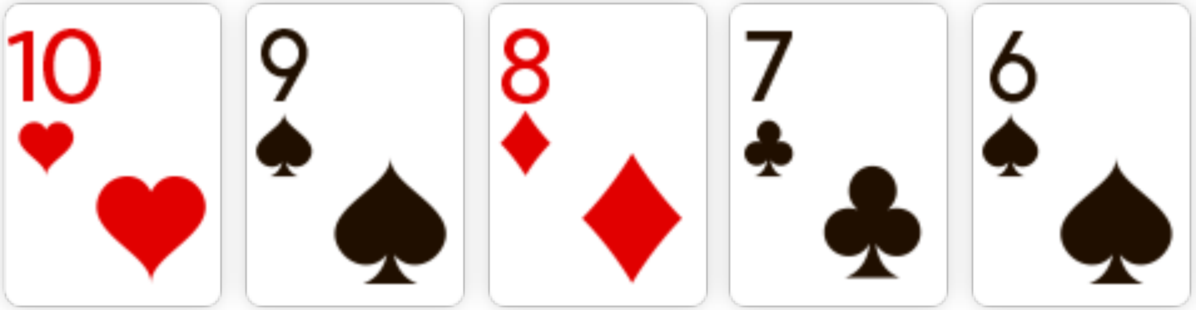
- 4 blues and 3 reds in a Bag. 3 drawn.
  - What is P(1 blue and 2 red drawn)?
- Ordered and Distinct:
  - Pick 3 ordered items: |S| = 7 * 6 * 5 = 210
  - Pick blue as either 1st, 2nd, or 3rd item:
    |E| = {4 * 3 * 2} + {3 * 4 * 2} + {3 * 2 * 4} = 72
  - P(1 blue, 2 red) = 72/210 = 12/35
- Unordered:
  - $|S| = \binom{7}{3} = 35$
  - $|E| = \binom{4}{1}\binom{3}{2} = 12$
  - P(1 blue, 2 red) = 12/35

Make indistinct items **distinct** to get equally likely sample space outcomes

*You will need to use this “trick” with high probability

Straight Poker Hand

- Consider 5 card poker hands.
  - "straight" is 5 consecutive rank cards of any suit
  - What is P(straight)?



Straight Poker Hand

- Consider 5 card poker hands.
  - "straight" is 5 consecutive rank cards of any suit
  - What is P(straight)?

$$|S| = \binom{52}{5}$$

$$|E| = 10 \cdot \binom{4}{1}^5$$

What is an example of one outcome?

Is each outcome equally likely?

$$P(\text{straight}) = \frac{|E|}{|S|} = \frac{10 \cdot \binom{4}{1}^5}{\binom{52}{5}} \approx 0.00394$$

Straight Poker Hand

- Consider 5 card poker hands.
  - "straight" is 5 consecutive rank cards of any suit
  - "straight flush" is 5 consecutive rank cards of same suit
  - What is P(straight, but not straight flush)?

$$|S| = \binom{52}{5}$$

$$|E| = 10\binom{4}{1}^5 - 10\binom{4}{1}$$

$$P(\text{straight}) = \frac{|E|}{|S|} = \frac{10\binom{4}{1}^5 - 10\binom{4}{1}}{\binom{52}{5}} \approx 0.00392$$

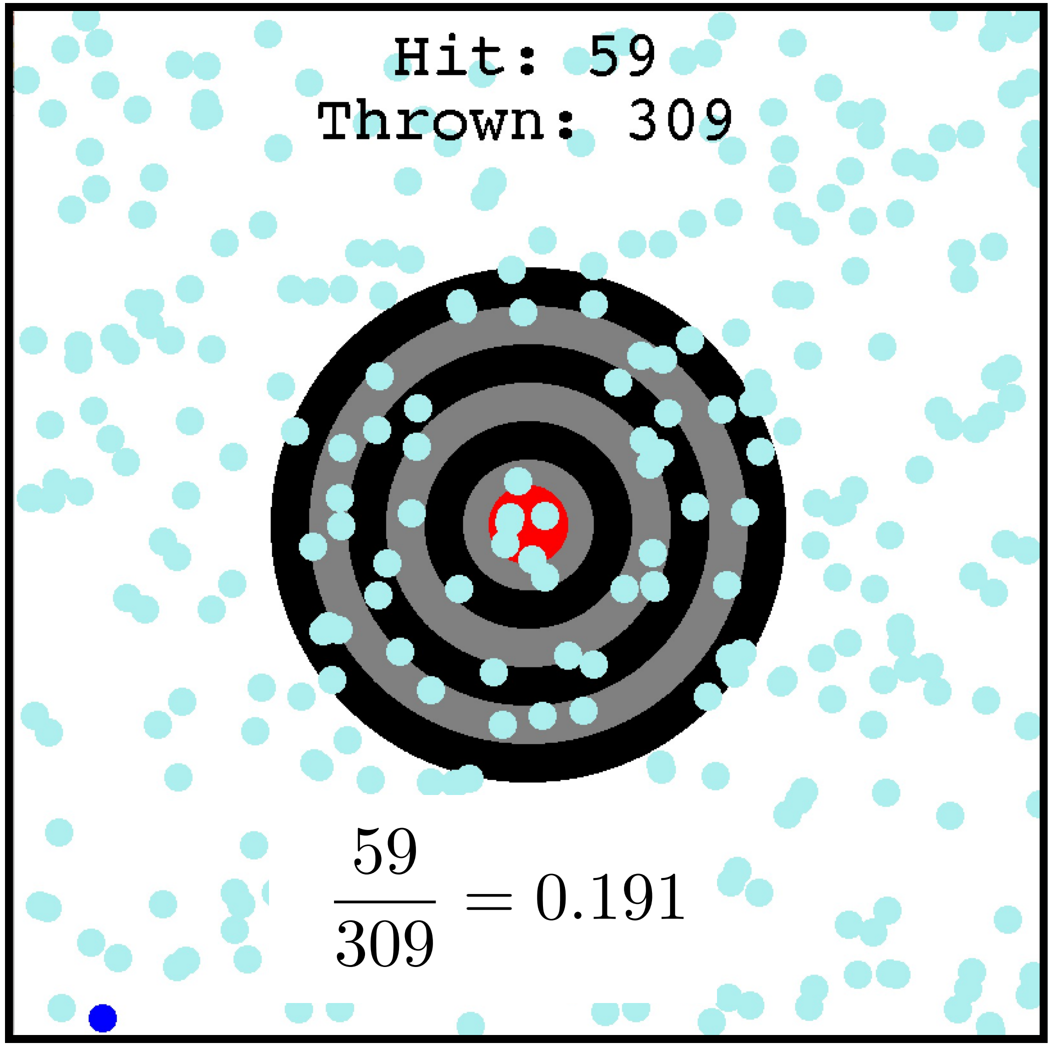
When approaching an "**equally likely probability**" problem, start by defining **sample spaces** and **event spaces**.

Chip Defect Detection

- *n* chips manufactured, 1 of which is defective.
- *k* chips randomly selected from *n* for testing.
  - What is P{defective chip is in *k* selected chips}?
- $|S| = \binom{n}{k}$
- $|E| = \binom{1}{1}\binom{n-1}{k-1}$
- P(defective chip is in *k* selected chips)

$$= \frac{\binom{1}{1}\binom{n-1}{k-1}}{\binom{n}{k}} = \frac{\frac{(n-1)!}{(k-1)!(n-k)!}}{\frac{n!}{k!(n-k)!}} = \frac{k}{n}$$

Target Revisited



Screen size = 800x800
Radius of target = 200

The dart is equally likely to land anywhere on the screen.

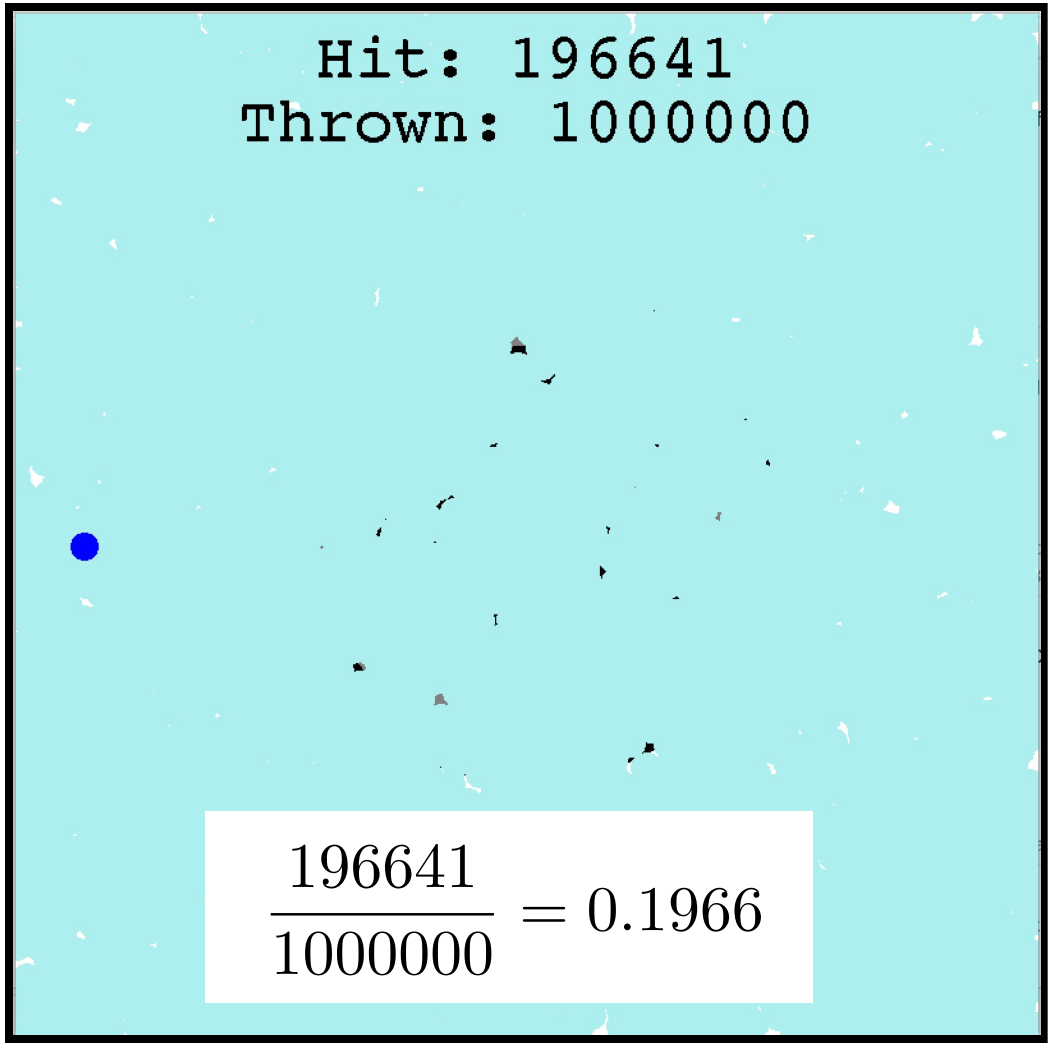
What is the probability of hitting the target?

$$|S| = 800^2$$

$$|E| = \pi 200^2$$

$$p(E) = \frac{\pi \cdot 200^2}{800^2} \approx 0.1963$$

Target Revisited



$$\frac{196641}{1000000} = 0.1966$$

Screen size = 800x800
Radius of target = 200

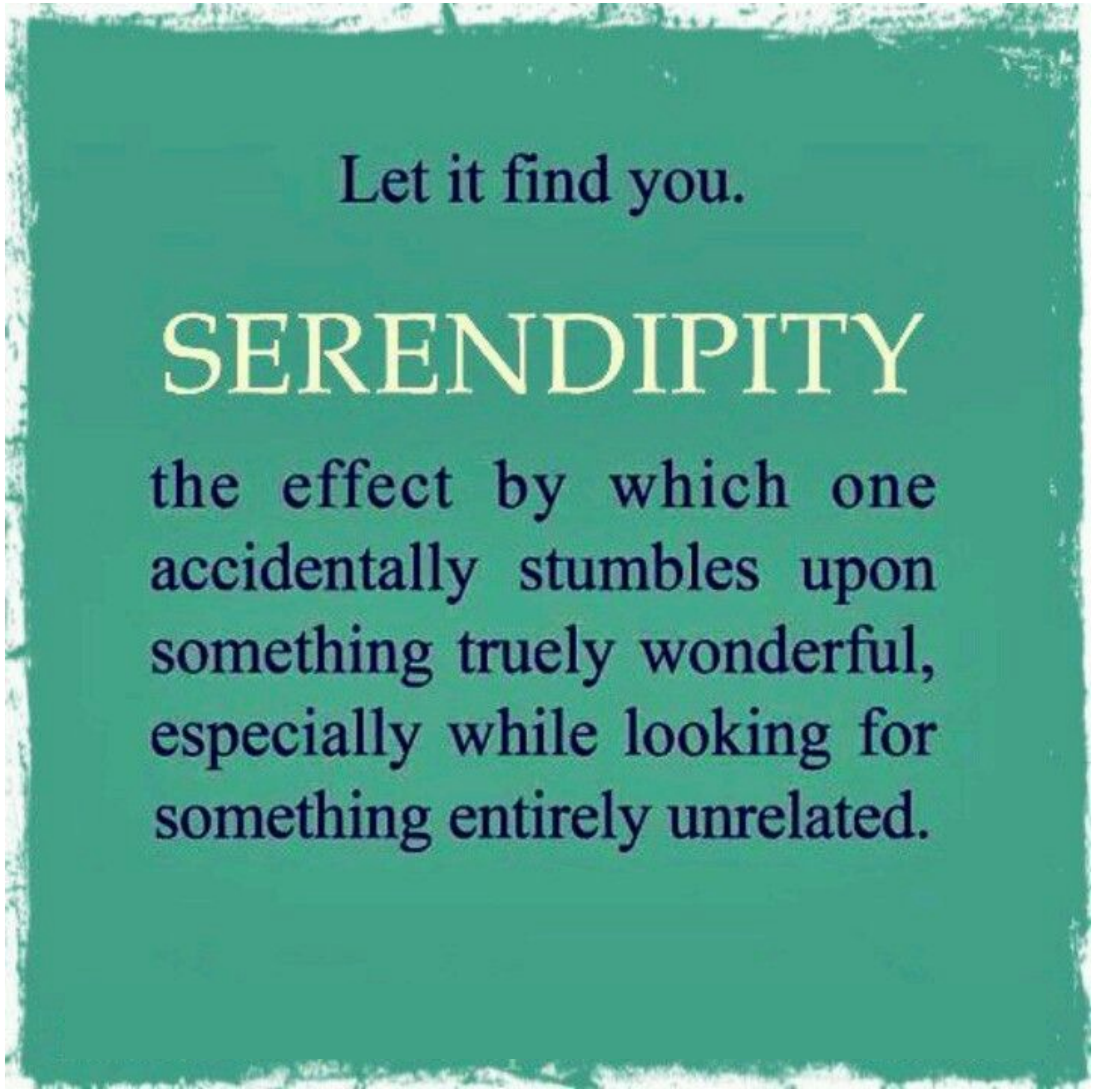
The dart is equally likely to land anywhere on the screen.

What is the probability of hitting the target?

$$|S| = 800^2$$

$$|E| = \pi 200^2$$

$$p(E) = \frac{\pi \cdot 200^2}{800^2} \approx 0.1963$$


Let it find you.
SERENDIPITY
the effect by which one accidentally stumbles upon something truely wonderful, especially while looking for something entirely unrelated.

WHEN YOU MEET YOUR BEST FRIEND

Somewhere you didn't expect to.

Serendipity

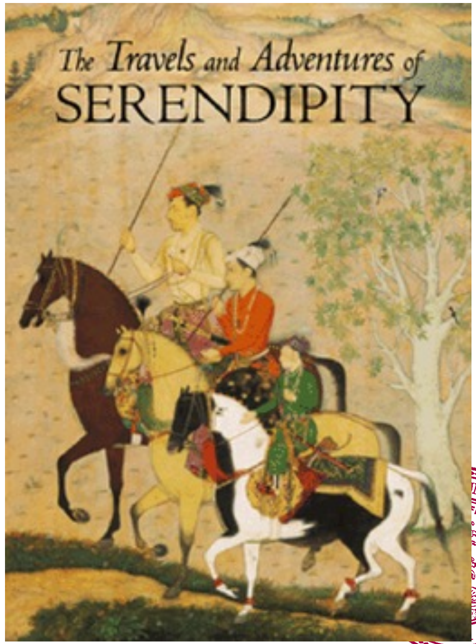
- Say the population of Stanford is 17,000 people
  - You are friends with 80 people?
  - Walk into a room, see 62 random people.
  - What is the probability that you see someone you know?
  - Assume you are equally likely to see each person at Stanford

$$P(\text{see someone you know})$$
$$= P(\text{see 1 or more friends})$$
$$= 1 - P(\text{don't see anyone you know})$$

$$|S| = \binom{17{,}000}{62}$$

$$|E^C| = \binom{17{,}000 - 80}{62}$$

$$P(E) = 1 - P(E^C) = 1 - \frac{|E^C|}{|S|} \approx 0.1914$$



Many times it is easier to calculate $P(E^C)$.

$$P(E^C) = 1 - P(E)$$

(We'll prove this in just a bit)

Back to 3 Axioms

Axioms of Probability

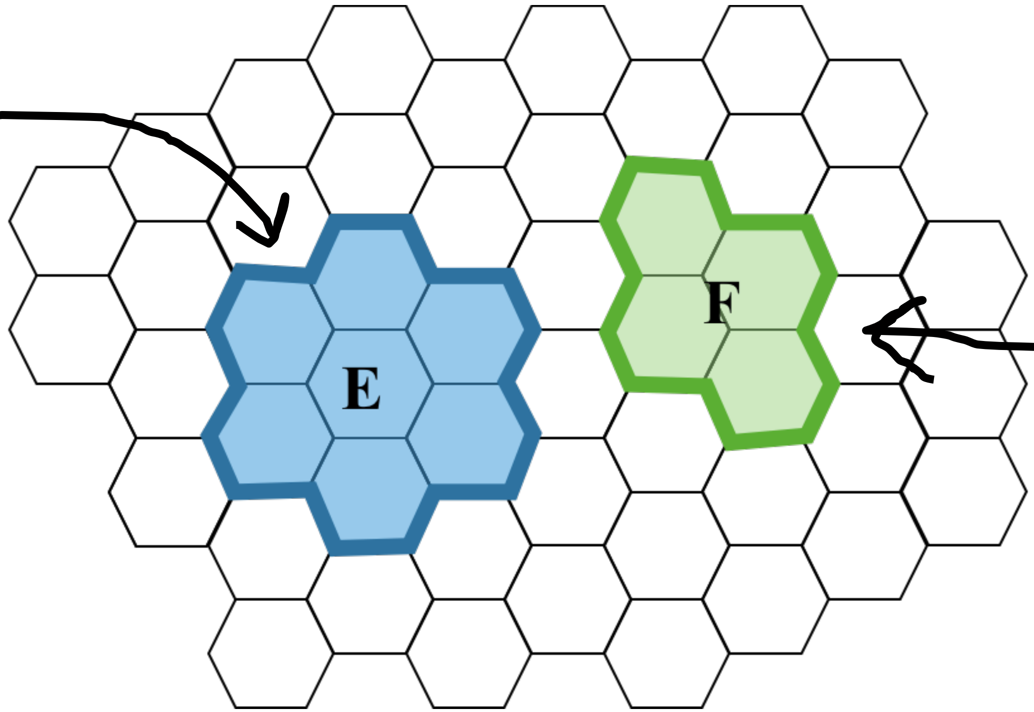
Recall: S = all possible outcomes. E = the event.

- Axiom 1: $0 \leq P(E) \leq 1$
- Axiom 2: $P(S) = 1$
- Axiom 3: If events E and F are mutually exclusive:

$$P(E \cup F) = P(E) + P(F)$$

Mutually Exclusive Events

7/50



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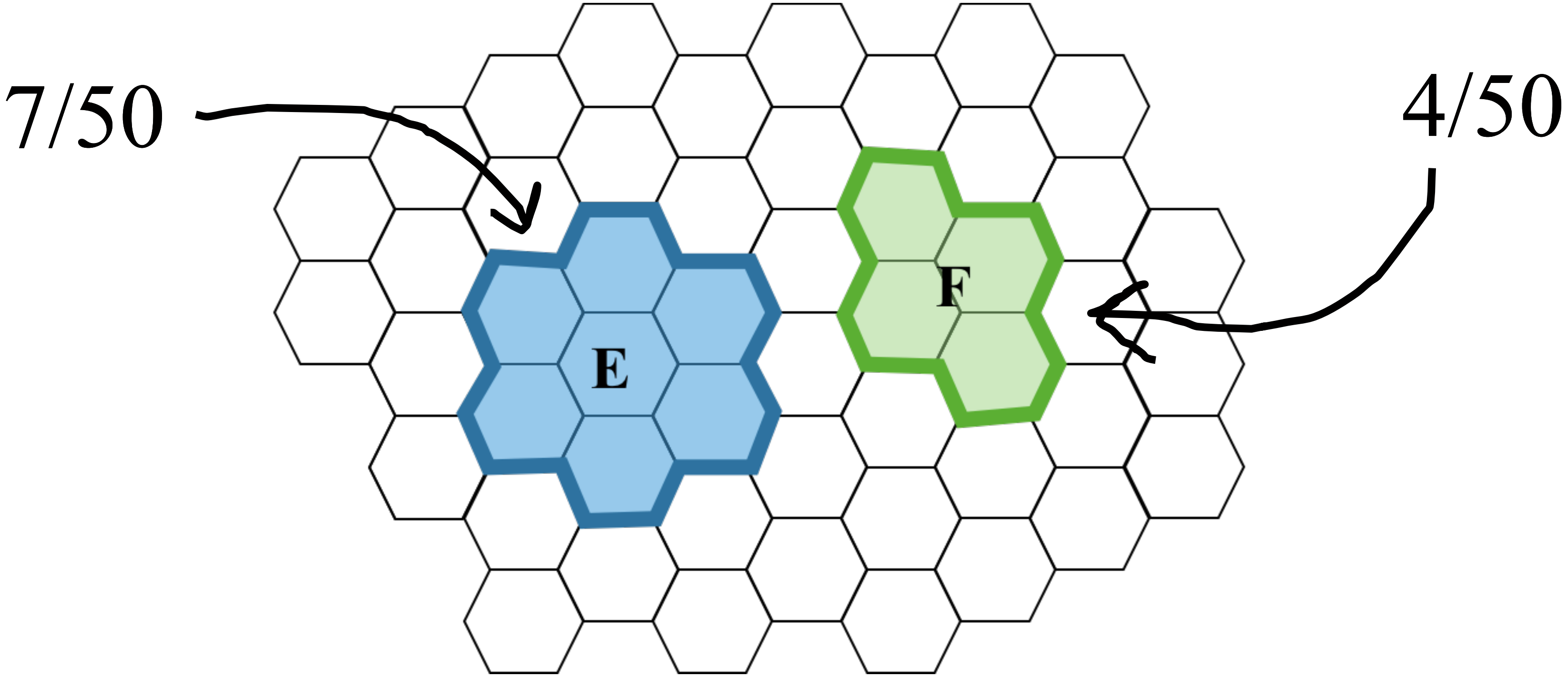
If events are mutually exclusive, probability of OR is simple:

$$P(E \cup F) = P(E) + P(F)$$

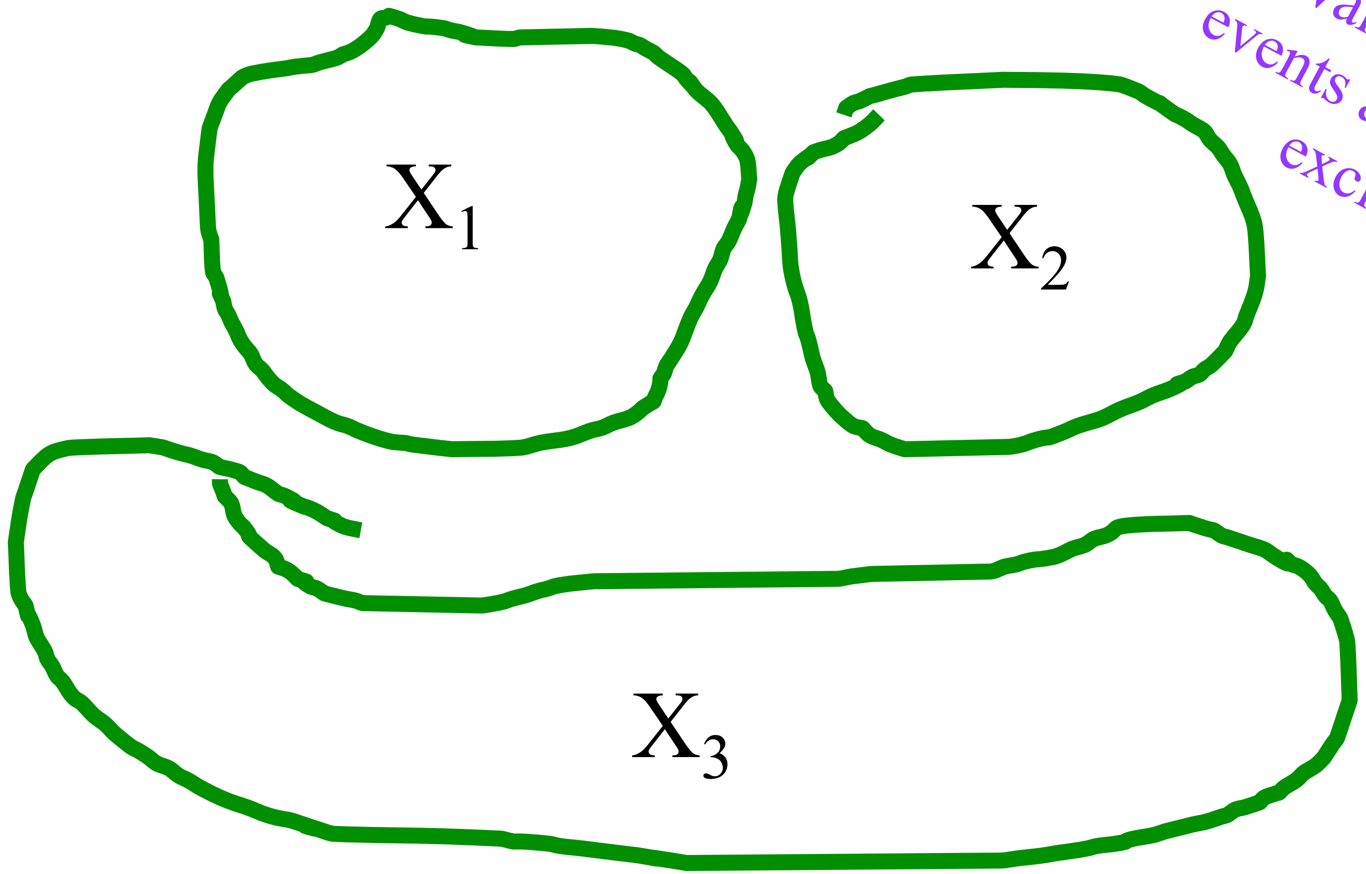
Mutually Exclusive Events



If events are mutually exclusive, probability of OR is simple:

$$P(E \cup F) = \frac{7}{50} + \frac{4}{5} = \frac{11}{50}$$

Probability of "or"



Wahoo! All my events are mutually exclusive

$$P(X_1 \cup X_2 \cup \cdots \cup X_n) = \sum_{i=1}^{n} P(X_i)$$

If events are *mutually exclusive* probability of OR is easy!

$$P(E^c) = 1 - P(E)?$$

$$P(E \cup E^c) = P(E) + P(E^c)$$

Axiom 3. Since E and E^c are mutually exclusive

$$P(S) = P(E) + P(E^c)$$

Since everything must either be in E or E^c

$$1 = P(E) + P(E^c)$$

Axiom 2

$$P(E^c) = 1 - P(E)$$

Rearrange

Probability of "or"

- What is the probability that in the *n* shuffles seen since the start of time, yours is unique?
  - $|S| = \{52!\}^n$
  - $|E| = \{52! - 1\}^n$
  - P{no deck matching yours} = $\{52!-1\}^n/\{52!\}^n$

- For $n = 10^{20}$,
  - P{deck matching yours} < 0.000000001

* Assume 7 billion people have been shuffling cards once a second since cards were invented