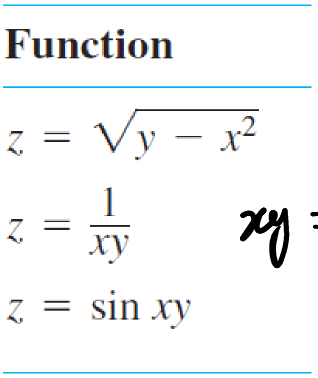
Multivariate Functions: Domain and Range

Monday, 17 February 2025 11:30 am

$P($area, location, #storeys, #rooms, ...$)$

depend ↓ $y = f(x)$ ← independent

↓ Transformation



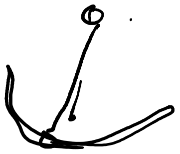
Function

$z = \sqrt{y - x^2}$

$z = \dfrac{1}{xy}$ $\quad xy \neq 0$

$z = \sin xy$

$y = \boxed{\sin}(x)$ —



$\overset{x}{(0}, \overset{y}{5})$

$f(x, y, z) \qquad f(w, x, y, z)$

$$\underset{\text{Target}}{y} = f(\underbrace{x_1, x_2, x_3 \ldots \ldots x_n}_{\text{features}})$$

$z = \sqrt{y - x^2}$

$y - x^2 \geq 0$

$\boxed{y \geq x^2}$ ← domain

$(2, -7)$

$x = 5$

$y \geq 25$

Example

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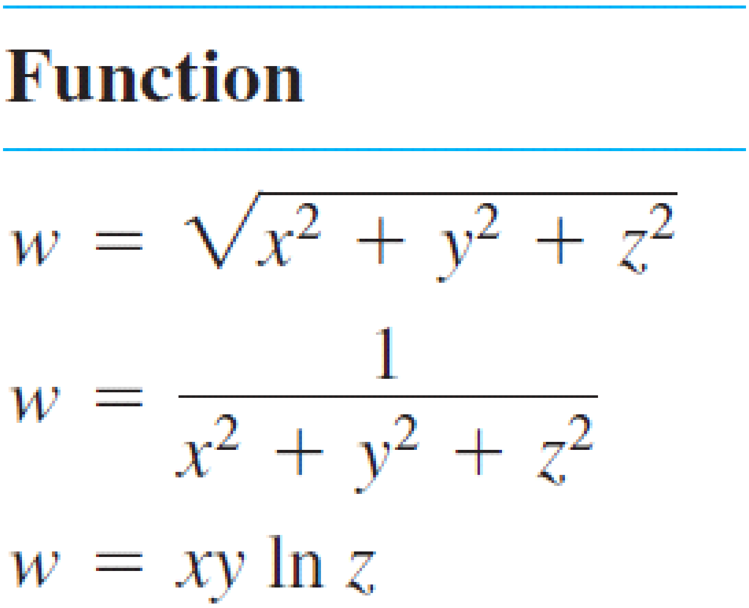

Function
$w = \sqrt{x^2 + y^2 + z^2}$
$w = \dfrac{1}{x^2 + y^2 + z^2}$
$w = xy \ln z$

Graphs of functions of 2 variables

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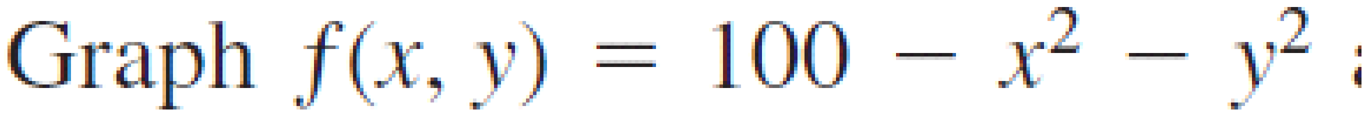

Graph $f(x, y) = 100 - x^2 - y^2$

$y =$ milege m_1 model_year m_2

$\frac{dy}{dm_1} \checkmark \rightarrow \frac{\partial y}{\partial m_1}$ $y = 10m_1 + 20m_2$

$\frac{dy}{dm_2} \rightarrow \frac{\partial y}{\partial m_2}$

Problem

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(a) $z = \sin x + 2 \sin y$

(b) $z = (4x^2 + y^2)e^{-x^2-y^2}$

(c) $z = xye^{-y^2}$

$-\boxed{y^2} \neq (\overset{\downarrow}{-}y)^2$

$= -\left(\overset{2}{y}\right)^2$

Partial Derivatives

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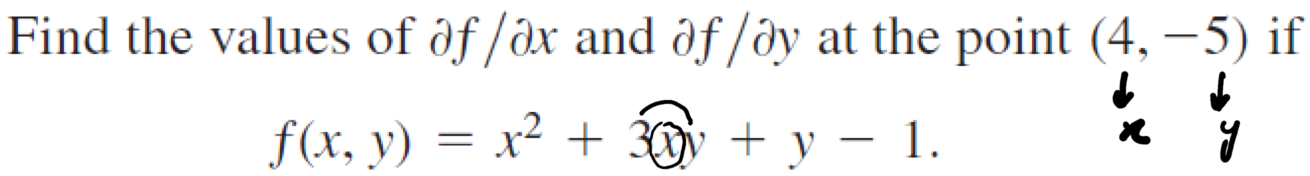

Find the values of $\partial f/\partial x$ and $\partial f/\partial y$ at the point $(4, -5)$ if

$$f(x, y) = x^2 + 3xy + y - 1.$$

$\downarrow x \quad \downarrow y$

$$\frac{\partial f}{\partial x} = 2x + 3y \qquad \left.\frac{\partial f}{\partial x}\right|_{(4,-5)} = 2(4)+3(-5) = -7$$

$$\frac{\partial f}{\partial y} = 3x + 1 \qquad \left.\frac{\partial f}{\partial y}\right|_{(4,-5)} = 3(4)+1 = 13$$

Problem 1

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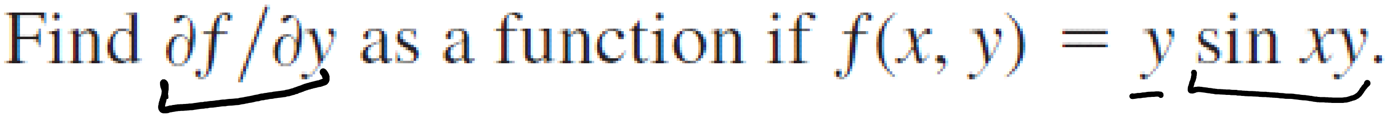

Find $\underbrace{\partial f / \partial y}_{f_y}$ as a function if $f(x, y) = \underline{y}\,\underbrace{\sin xy}$.

$$f_y = (1)\sin xy + y \cdot \cos \overset{\frown}{xy} \cdot x$$

$$= \sin xy + xy \cos xy$$

Problem 2

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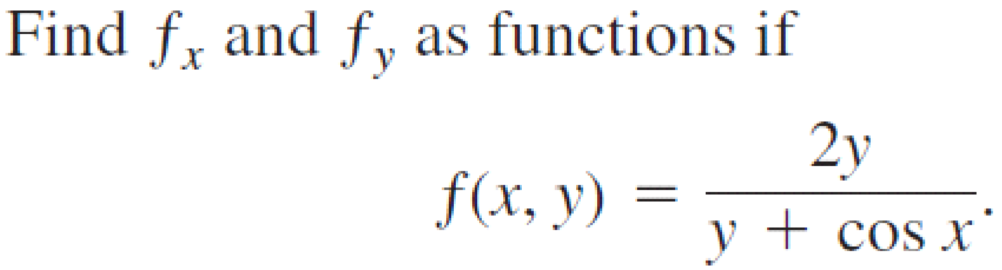

Find f_x and f_y as functions if

$$f(x, y) = \frac{2y}{y + \cos x}.$$

$$f_x = \frac{\partial}{\partial x} 2y (y + \cos x)^{-1} = 2y(-1)\underbrace{(y + \cos x)}^{-2} \cdot (0 - \sin x)$$

$$= \frac{2y \sin x}{(y + \cos x)^2}$$

$$f_y = \frac{(y + \cos x)(2) - 2y(1)}{(y + \cos x)^2} = \frac{\cancel{2y} + 2\cos x - \cancel{2y}}{(y + \cos x)^2}$$

Problem 3

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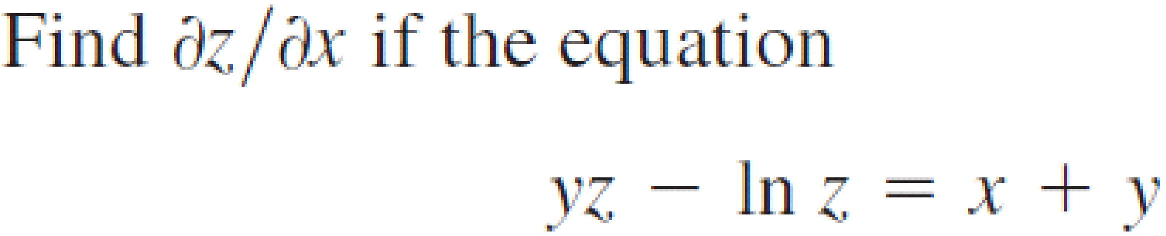

Find $\partial z/\partial x$ if the equation

$$yz - \ln z = x + y$$

$$\frac{\partial}{\partial x}(yz) - \frac{\partial}{\partial x}(\ln z) = \frac{\partial}{\partial x} x + \frac{\partial}{\partial x} y$$

$$y \frac{\partial z}{\partial x} - \frac{1}{z} \cdot \frac{\partial z}{\partial x} = 1 + 0$$

$$\frac{\partial z}{\partial x}\left(y - \frac{1}{z}\right) = 1 \Rightarrow \frac{\partial z}{\partial x} = \frac{1}{y - z^{-1}}$$

Second Order Partial Derivatives

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$$\frac{\partial^2 f}{\partial x^2} \text{ or } f_{xx}, \qquad \frac{\partial^2 f}{\partial y^2} \text{ or } f_{yy},$$

$$\frac{\partial^2 f}{\partial x \partial y} \text{ or } f_{yx}, \quad \text{and} \quad \frac{\partial^2 f}{\partial y \partial x} \text{ or } f_{xy}.$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right), \qquad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right),$$

Problem 4

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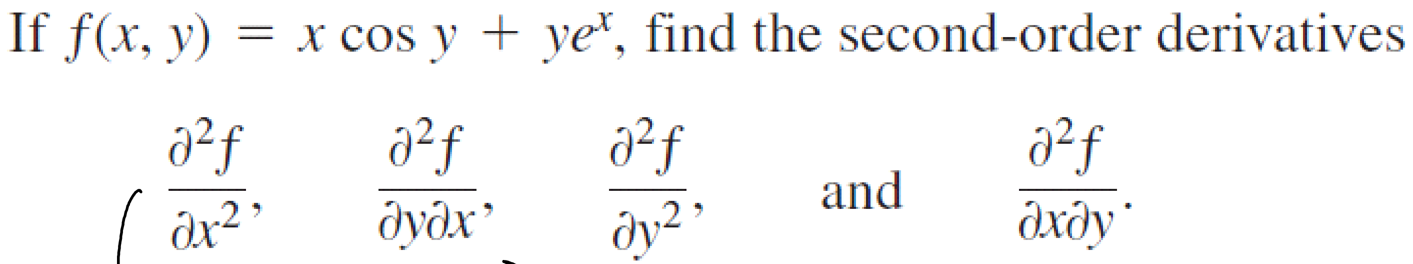

If $f(x, y) = x \cos y + ye^x$, find the second-order derivatives

$$\frac{\partial^2 f}{\partial x^2}, \quad \frac{\partial^2 f}{\partial y \partial x}, \quad \frac{\partial^2 f}{\partial y^2}, \quad \text{and} \quad \frac{\partial^2 f}{\partial x \partial y}.$$

$f_x = \cos y + ye^x$ $\qquad$ $f_{xy} = -\sin y + e^x$

$$\Updownarrow$$

$f_{xx} = 0 + ye^x = ye^x$

$f_y = -x \sin y + e^x$, $f_{yx} = -\sin y + e^x$

$$f_{xy} = f_{yx}$$

Problem 5

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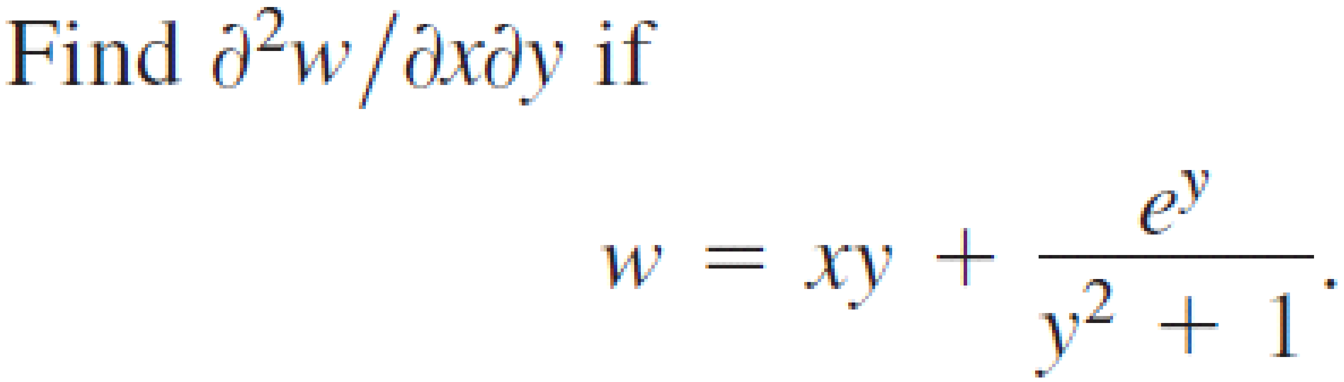

Find $\partial^2 w/\partial x \partial y$ if

$$w = xy + \frac{e^y}{y^2 + 1}.$$

Problem 6

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Find f_{yxyz} if $f(x, y, z) = 1 - 2xy^2z + x^2y$.

Problem 7

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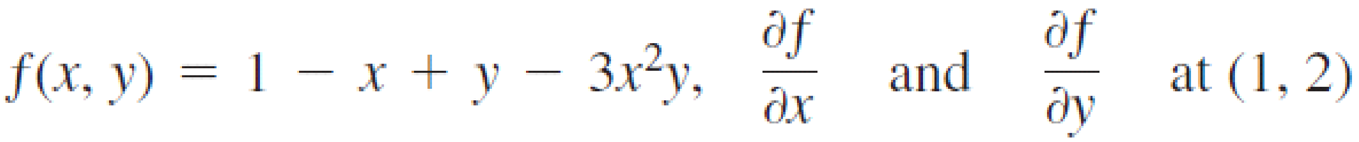
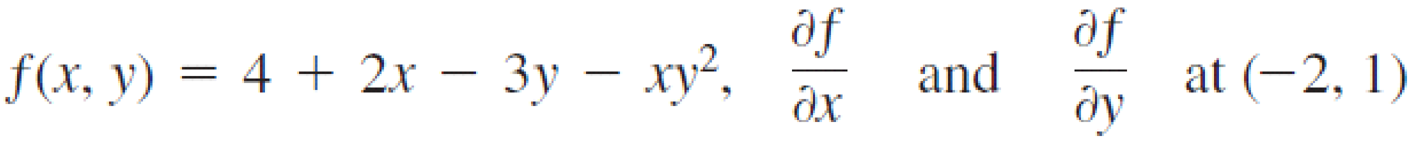


$$f(x, y) = 1 - x + y - 3x^2y, \quad \frac{\partial f}{\partial x} \quad \text{and} \quad \frac{\partial f}{\partial y} \quad \text{at } (1, 2)$$



$$f(x, y) = 4 + 2x - 3y - xy^2, \quad \frac{\partial f}{\partial x} \quad \text{and} \quad \frac{\partial f}{\partial y} \quad \text{at } (-2, 1)$$

Chain Rule

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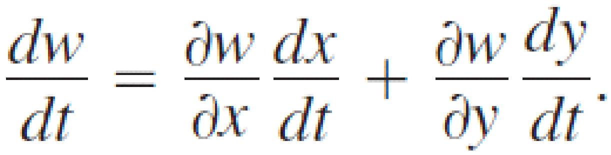
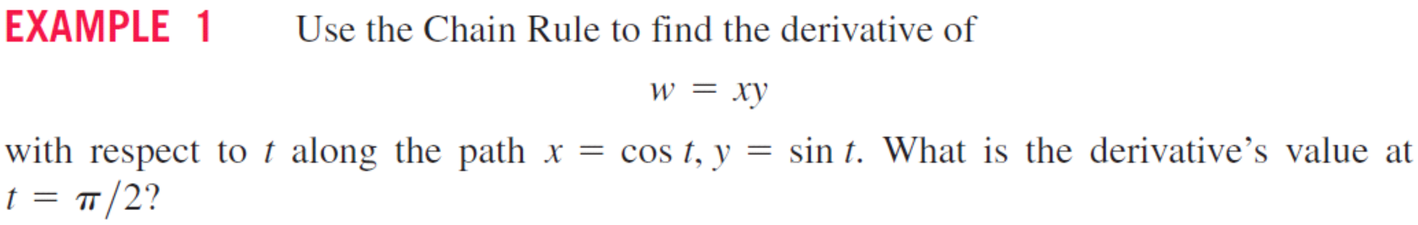


$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}.$$



EXAMPLE 1 Use the Chain Rule to find the derivative of

$$w = xy$$

with respect to t along the path $x = \cos t, y = \sin t$. What is the derivative's value at $t = \pi/2$?

Problem 8

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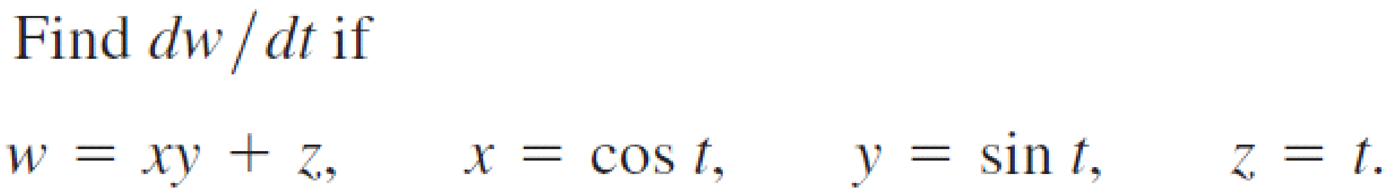
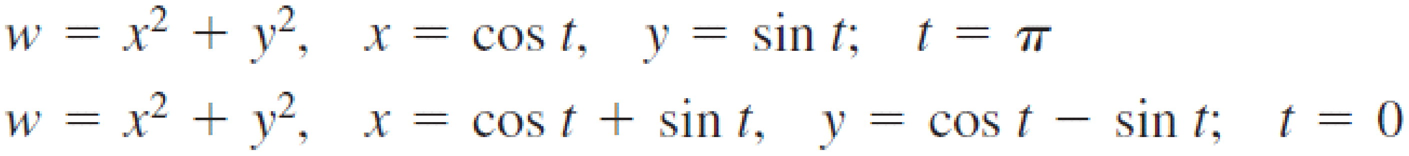


Find dw/dt if

$$w = xy + z, \qquad x = \cos t, \qquad y = \sin t, \qquad z = t.$$

Problem 9

Monday, 17 February 2025 2:12 pm



$$w = x^2 + y^2, \quad x = \cos t, \quad y = \sin t; \quad t = \pi$$

$$w = x^2 + y^2, \quad x = \cos t + \sin t, \quad y = \cos t - \sin t; \quad t = 0$$