CAI 2.0, Linear Algebra
Worksheet 5: Basis and Orthogonality

Problem 1: Basis of Vector Spaces

(a) Determine whether the vectors $v_1 = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$, $v_2 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$, and $v_3 = \begin{bmatrix} 5 \\ 2 \\ 8 \end{bmatrix}$ form a basis for $\mathbb{R}^3$.

(b) Find a basis for the subspace $W = \{(x, y, z, w) \in \mathbb{R}^4 : x + y = 0, z + w = 0\}$.

(c) Determine the dimension of the subspace $V = \text{span}\left(\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} \right)$.

Problem 2: Orthogonal and Orthonormal Vectors

(a) Determine whether the following collection of vectors is orthogonal: $u_1 = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$, $u_2 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$, $u_3 = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}$

(b) Convert the orthogonal collection $\left\{ \begin{bmatrix} 3 \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 5 \\ 0 \end{bmatrix} \right\}$ into an orthonormal collection.

(c) Verify that $\left\{ \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix} \right\}$ is an orthonormal basis for $\mathbb{R}^3$.

Problem 3: Orthogonal Basis for Planes

(a) Find an orthogonal basis for the plane in $\mathbb{R}^3$ given by the equation $3x - 2y + z = 0$.

(b) Find an orthogonal basis for the subspace $U = \{(x, y, z) \in \mathbb{R}^3 : x + y + z = 0\}$.

(c) Find an orthonormal basis for the plane in $\mathbb{R}^3$ passing through the origin with normal vector $\begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}$.