CS109: Probability for Computer Scientists

CS109: From Counting to Machine Learning

Counting Theory

Core Probability

Random Variables

Probabilistic Models

Uncertainty Theory

Machine Learning

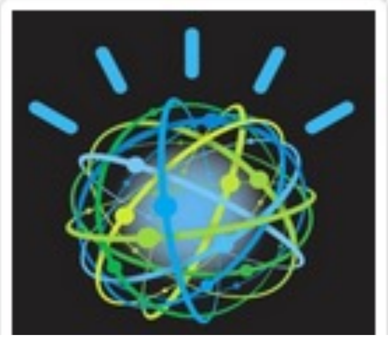
Early Optimism 1950s

"Machines will be capable, within twenty years, of doing any work a man can do."
–Herbert Simon, 1952

Big Milestones Part 1

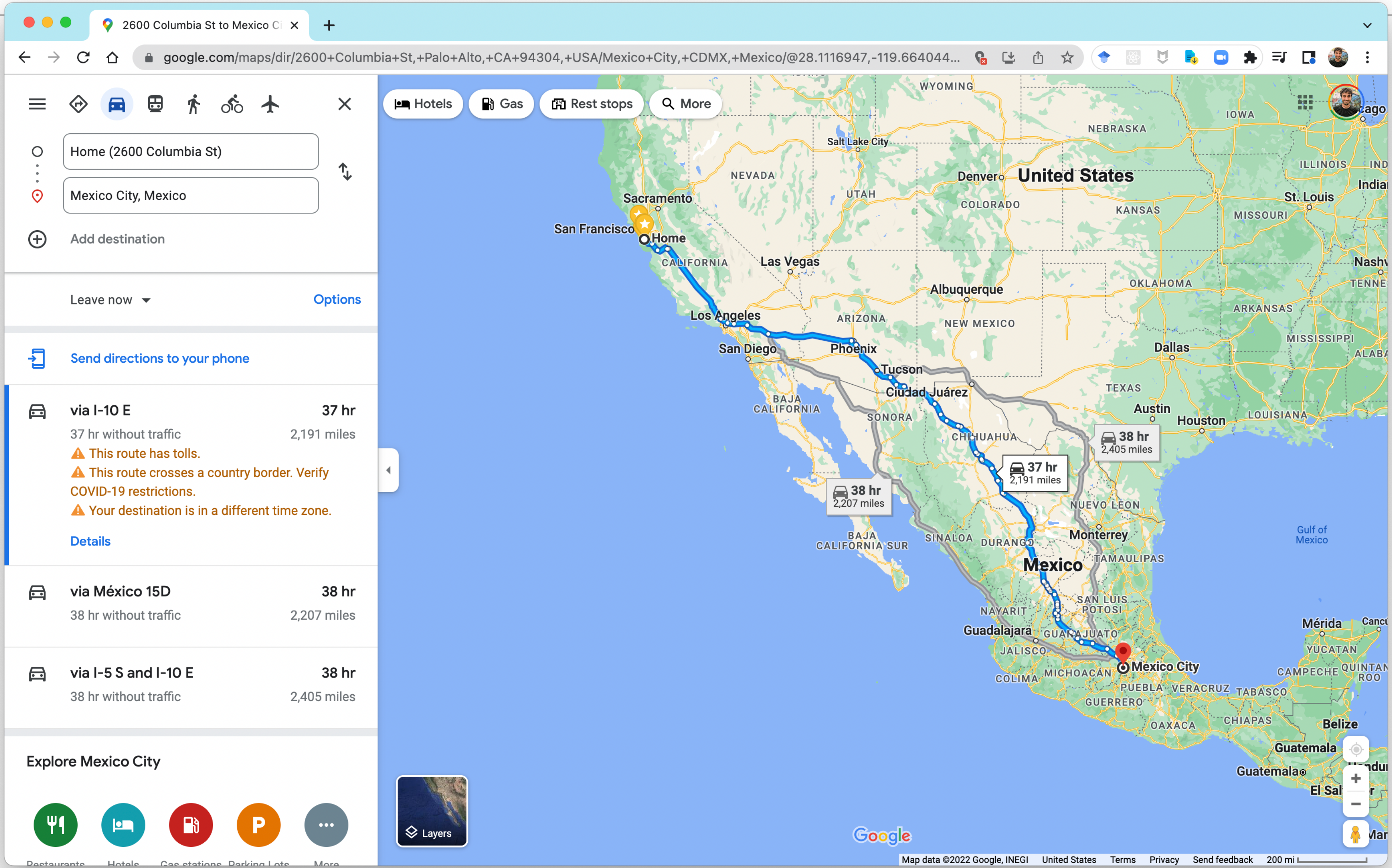
1997 Deep Blue

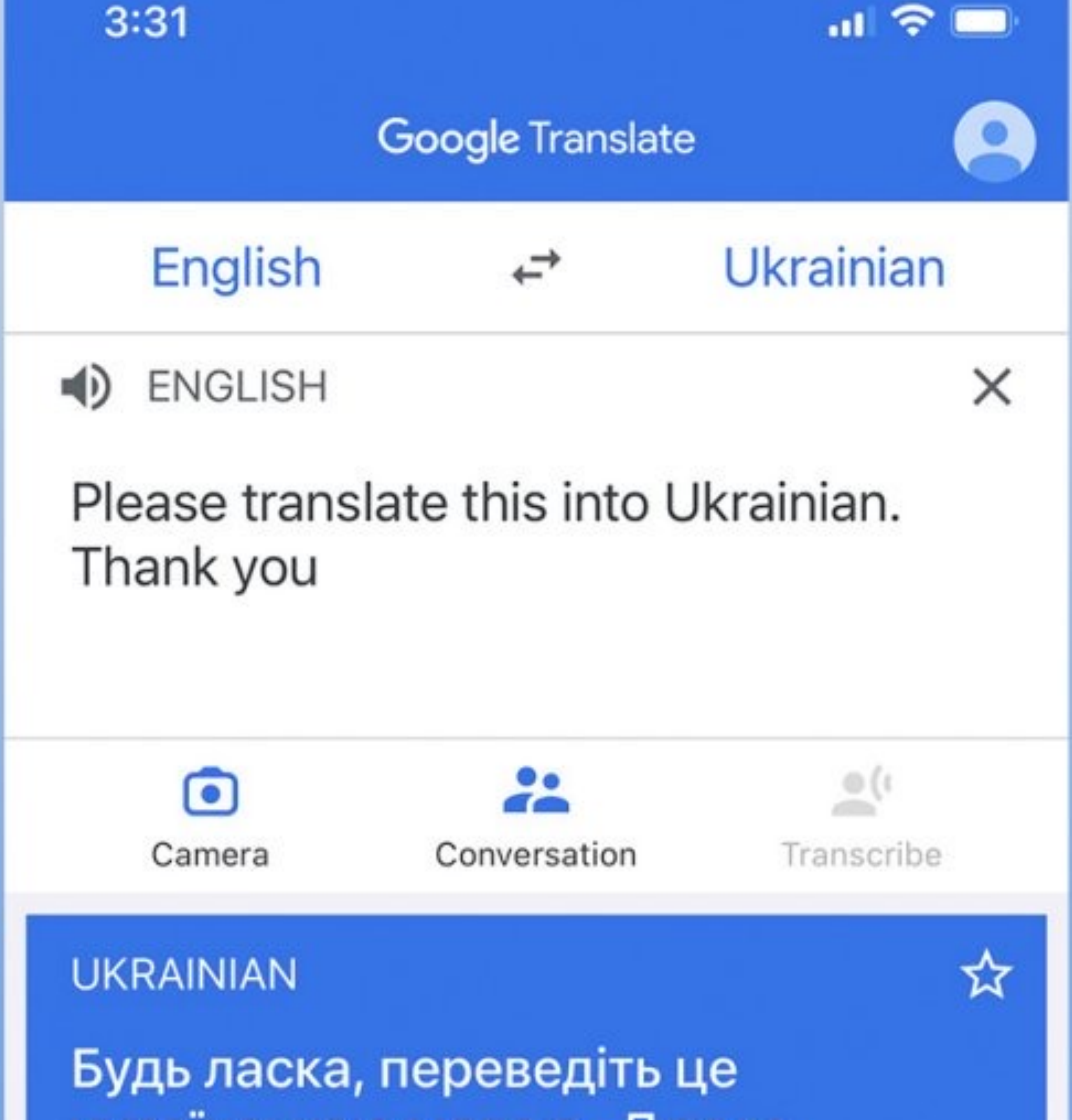
2005 Stanley



2011 Watson

Directions From A to B

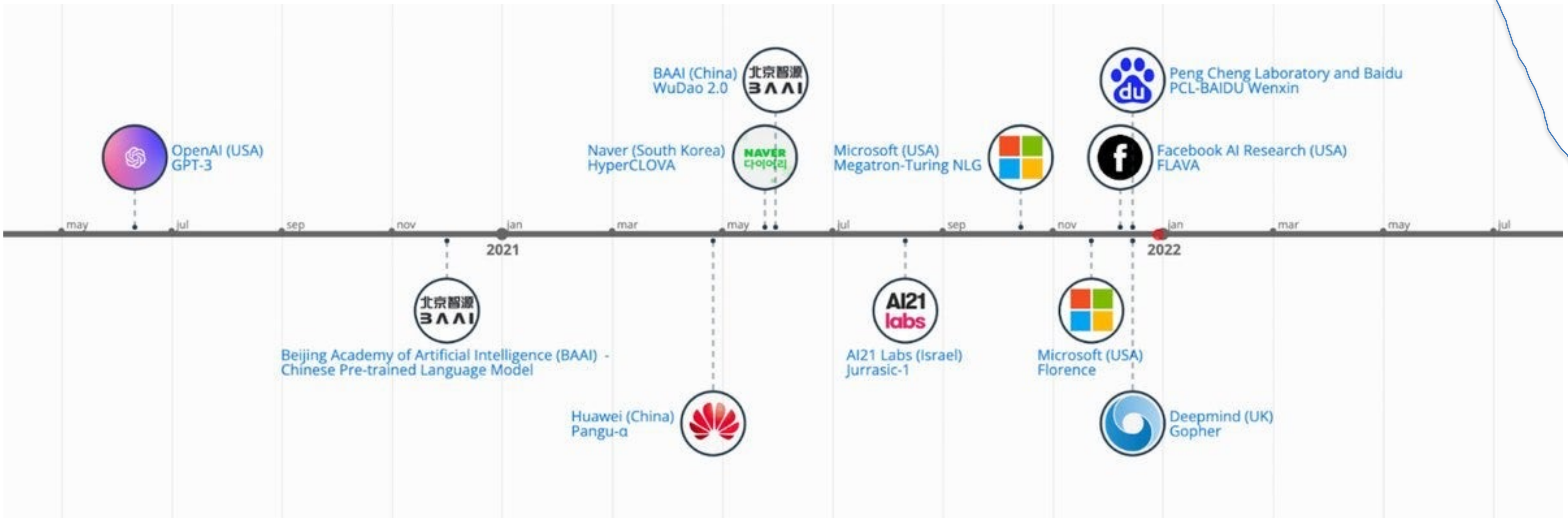



3:31
Google Translate
English
Ukrainian
ENGLISH
Please translate this into Ukrainian.
Thank you
Camera
Conversation
Transcribe
UKRAINIAN
Будь ласка, переведіть це

Self Driving Cars

AI that (seems) to understand language

ChatGPT 11/2022



Focus on one problem

Computer Vision

Chihuahua or muffin?

Can you do it?

Chihuahua or Muffin?

Chihuahua or Muffin?

How about now?

What a computer sees

0	0	1	0	1	0	1	0	0	0	1	1	1	0	1
1	0	0	1	0	1	1	1	0	1	0	0	0	0	0
1	1	1	0	1	0	0	1	1	0	0	1	0	1	0
1	1	1	1	1	0	0	0	0	0	1	1	0	1	1
0	0	0	1	1	0	0	1	0	0	0	1	1	1	0
1	0	0	1	1	0	0	0	1	0					
1	1	0	1	1	0	0	1	1	0					
1	0	1	0	0	1	0	0	1	0					
0	0	0	0	1	0	1	0	1	1					
0	1	1	0	0	0	0	0	1	1					
0	0	1	0	1	1	1	0	0	0					
0	1	1	1	0	1	0	0	1	0					
1	1	0	0	0	0	0	0	0	0					
0	0	0	0	0	0	0	0	1	1					
0	0	1	1	1	0	1	0	1	1					

What a human sees

Why is it easy for Humans?

About 30% of your cortex is used from vision
3% is used to process hearing

1

Make a Harry Potter Sorting Hat

Classification

Most Desired Skill in Industry

Forbes Billionaires Innovation Leadership Money Consumer

30,575 views | Jan 29, 2018, 02:47pm

Data Scientist Is the Best Job In America According Glassdoor's 2018 Rankings

TWEET THIS

- Data Scientist has been named the best job in America for three years running, with a median base salary of $110,000 and 4,524 job openings.
- DevOps Engineer is the second-best job in 2018, paying a median base salary of $105,000 and 3,369 job openings.

Job Score is based on:

- Earning potential
- Number of jobs
- Job satisfaction rating

"Data science and machine learning are generating more jobs than candidates right now, making these two areas the *fastest growing employment areas*."

9.8 times more jobs than five years ago.

[LinkedIn's 2017 U.S. Emerging Jobs Report]

Most Desired Skill in Academia

Most CS PhD students list their highest desiderata upon graduation as:

"Better understanding of probability"

But Its NOT Always Intuitive

A patient has a positive Zika test.

What is the probability they have zika?

- *0.8% of people have zika*
- *Test has 90% positive rate for people with zika*
- *Test has 7% positive rate for people without zika*

The right answer is 9%

CS109

AI

Uncertainty Theory

Single Random Variables

Probabilistic Models

Counting

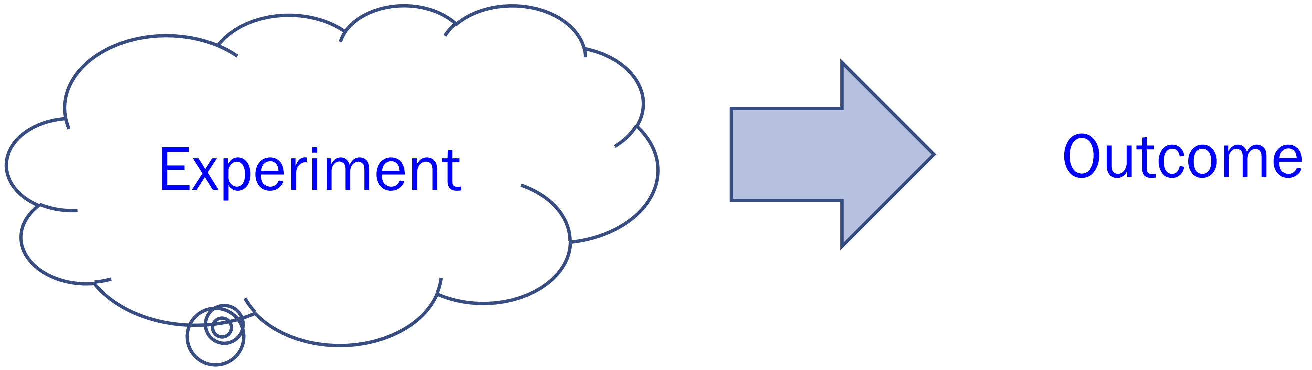
Probability Fundamentals

Lets dive in…

Counting I

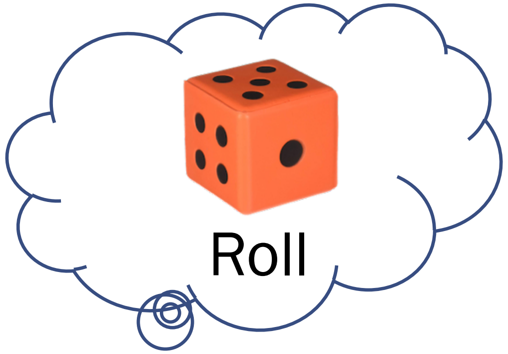
What is Counting?

An experiment
in probability:



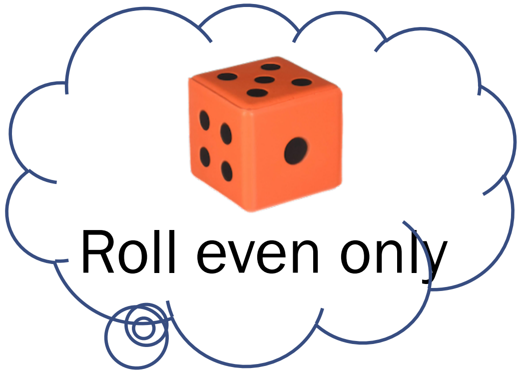
Counting: How many possible outcomes satisfy some event?

What is Counting?



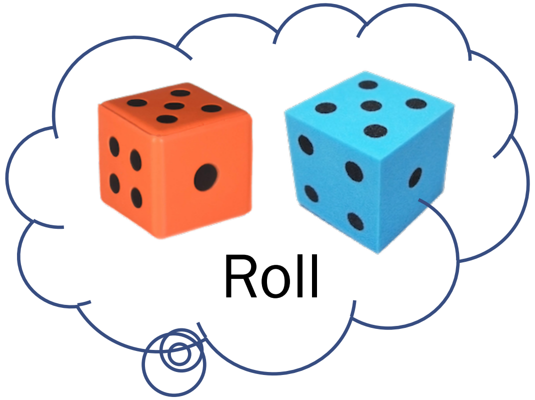
6

{1, 2, 3,
4, 5, 6}



3

{2, 4, 6}



36

{(1, 1) , (1, 2), (1, 3), (1, 4), (1, 5), (1, 6),
(2, 1) , (2, 2), (2, 3), (2, 4), (2, 5), (2, 6),
(3, 1) , (3, 2), (3, 3), (3, 4), (3, 5), (3, 6),
(4, 1) , (4, 2), (4, 3), (4, 4), (4, 5), (4, 6),
(5, 1) , (5, 2), (5, 3), (5, 4), (5, 5), (5, 6),
(6, 1) , (6, 2), (6, 3), (6, 4), (6, 5), (6, 6)}

Step Rule of Counting (aka Product Rule of Counting)

If an experiment has two steps, where

The first step (step A) can have one of m different outcomes,
and the second step (step B) can have one of n different outcomes,

and n is unaffected by outcome of first step.

Then the number of outcomes of the experiment is mn.

Two-step experiment

$\rightarrow$ A $\rightarrow$ B

[On board]

How Many Different Lock Patterns?

4-digit numeric combination lock

1st Digit

↓

2nd Digit

↓

3rd Digit

↓

4th Digit

10 Choices per digit

10^4=10,000 total patterns

How Many Different Lock Patterns (2)?

4-digit numeric combination lock

BUT each digit used exactly once?

1st Digit

↓

2nd Digit

↓

3rd Digit

↓

4th Digit

Choices for 2nd digit depend on the choice of 1st digit…

How Many Different Lock Patterns (2)?

4-digit numeric combination lock

BUT each digit used exactly once?

1^{st} Digit

2^{nd} Digit

3^{rd} Digit

4^{th} Digit

$10 \times 9 \times 8 \times 7 = 5040$
Possible patterns

For the step rule, only the number of possible next choices matters!

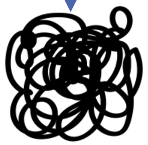
How Many Different Lock Patterns (3)?

3,5,3,_

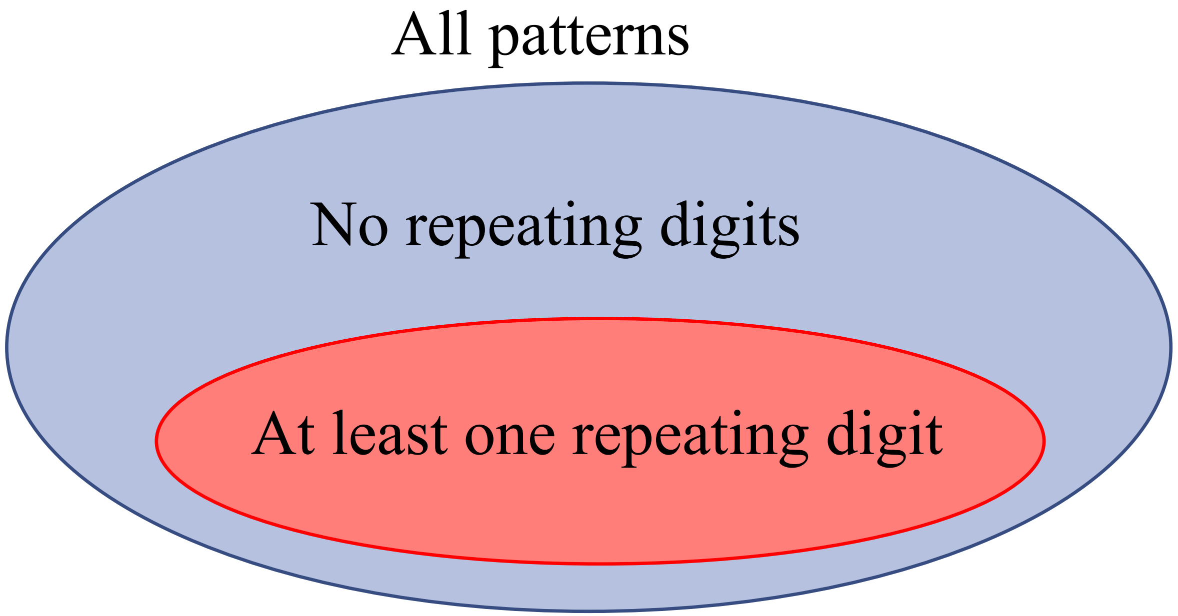
3,4,6,_

4-digit numeric combination lock

BUT has at least one digit repeated?

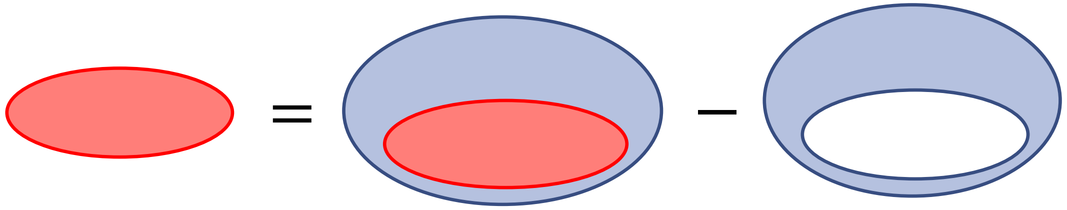
1^{st} Digit	10 Choices
↓	↓
2^{nd} Digit	10 Choices
↓	↓
3^{rd} Digit	10 Choices
↓	↓
4^{th} Digit	

How Many Different Lock Patterns (3)?



4-digit numeric combination lock

BUT has at least one digit repeated?



(At least one repeating digit)
= # (All patterns) - # (No repeating digit)
= 10,000 – 5,040
= 4,960

<u>Complementary Counting</u>

Sometimes it helps to "count away" complementary events

Sum Rule of Counting

If the outcome of an experiment can be either from

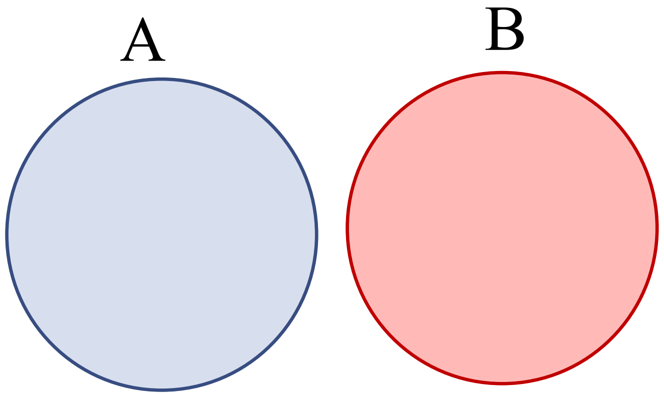
Set A, where $|A| = m$,

<u>or</u> Set B, where $|B| = n$,

<u>**where $A \cap B = \emptyset$,**</u>

Then the number of outcomes of the experiment is

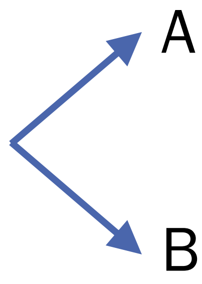
$$|A| + |B| = m + n.$$

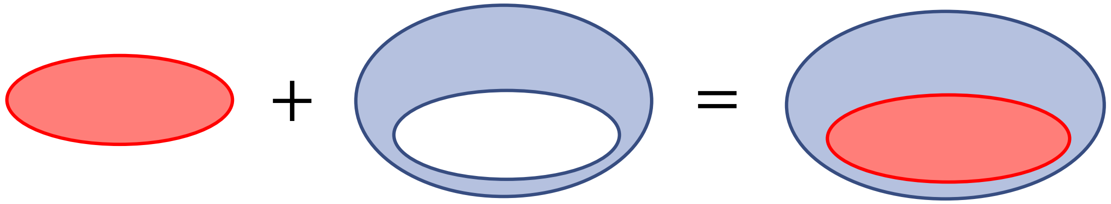


If $|A \cap B| = 0$,

$|A \cup B| = |A| + |B|$

One experiment





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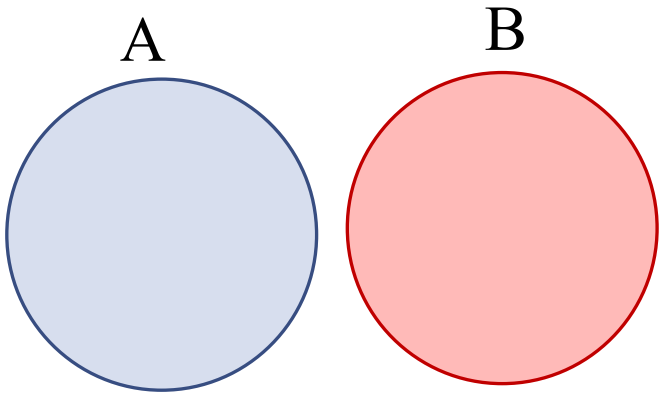

If $|A \cap B| = 0$,

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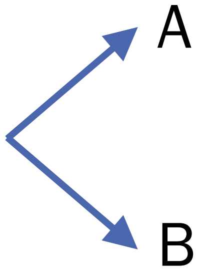
One experiment



Can you derive the Step Rule from the Sum Rule?

How Many Bit Stings?

Problem: A 6-bit string is sent over a network. The valid set of strings recognized by the receiver must either start with "01" **OR** start with "10". How many such strings are there?

Answer

$$N = |A| + |B|$$
$$= 2^4 + 2^4$$
$$= 32$$

2^4 start with 01		2^4 start with 10
010000		100000
010001		100001
010010		100010
010011	3rd Digit	100011
010100	↓	100100
010101		100101
010110		100110
010111	4th Digit	100111
011000	↓	101000
011001		101001
011010	5th Digit	101010
011011	↓	101011
011100		101100
011101		101101
011110	6th Digit	101110
011111		101111
Set A		Set B

$$|A \cap B| = 0$$

How Many Bit Stings (2)?

Problem: A 6-bit string is sent over a network. The valid set of strings recognized by the receiver must either start with "01" **OR** **end** with "10". How many such strings are there?

Answer

2^4 start with 01

010000
010001
010010
010011
010100
010101
010110
010111
011000
011001
011010
011011
011100
011101
011110
011111

Set A

2^4 end with 10

000010
000110
001010
001110
010010
010110
011010
011110
100010
100110
101010
101110
110010
110110
111010
111110

Set B

How Many Bit Stings (2)?

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Sum Rule of Counting

If the outcome of an experiment can be either from

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Then the number of outcomes of the experiment is

$|A| + |B| = m + n$.

2^4 start with 01	2^4 end with 10
010000	000010
010001	000110
010010	001010
010011	001110
010100	010010
010101	010110
010110	011010
010111	011110
011000	100010
011001	100110
011010	101010
011011	101110
011100	110010
011101	110110
011110	111010
011111	111110
Set A	Set B

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010001	000110
010010	001010
010011	001110
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011000	100010
011001	100110
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Set A	Set B

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Answer

$$N = |A| + |B| - |A \text{ and } B|$$
$$= 16 + 16 - 4$$
$$= 28$$

2^4 start with 01	2^4 end with 10
010000	000010
010001	000110
010010	001010
010011	001110
010100	010010
010101	010110
010110	011010
010111	011110
011000	100010
011001	100110
011010	101010
011011	101110
011100	110010
011101	110110
011110	111010
011111	111110
Set A	Set B

Or Rule of Counting (aka Inclusion/ Exclusion)

If the outcome of an experiment can be either from

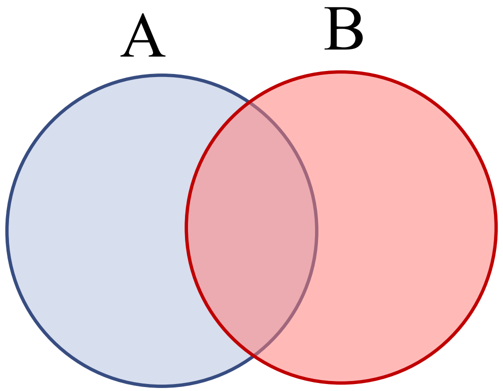
Set A, where $|A| = m$,

<u>or</u> Set B, where $|B| = n$,

where $\boldsymbol{A \cap B}$ ***may not be empty***,

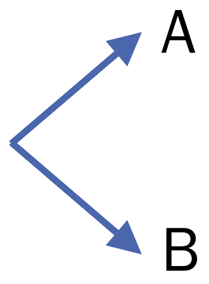
Then the number of outcomes of the experiment is

N = $|A| + |B| - |A \cap B|$.



$|A \cup B| = |A| + |B| - |A \cap B|$

One experiment



Core Counting

Counting with steps

Definition: Step Rule of Counting (aka Product Rule of Counting)

If an experiment has two parts, where the first part can result in one of m outcomes and the second part can result in one of n outcomes regardless of the outcome of the first part, then the total number of outcomes for the experiment is $m \cdot n$.

Can you derive the Step Rule from the Sum Rule (or the OR rule)?

Counting with "or"

Definition: Inclusion Exclusion Counting

If the outcome of an experiment can either be drawn from set A or set B, and sets A and B may potentially overlap (i.e., it is not the case that A and B are mutually exclusive), then the number of outcomes of the experiment is $|A \text{ or } B| = |A| + |B| - |A \text{ and } B|$.

Challenge Problem

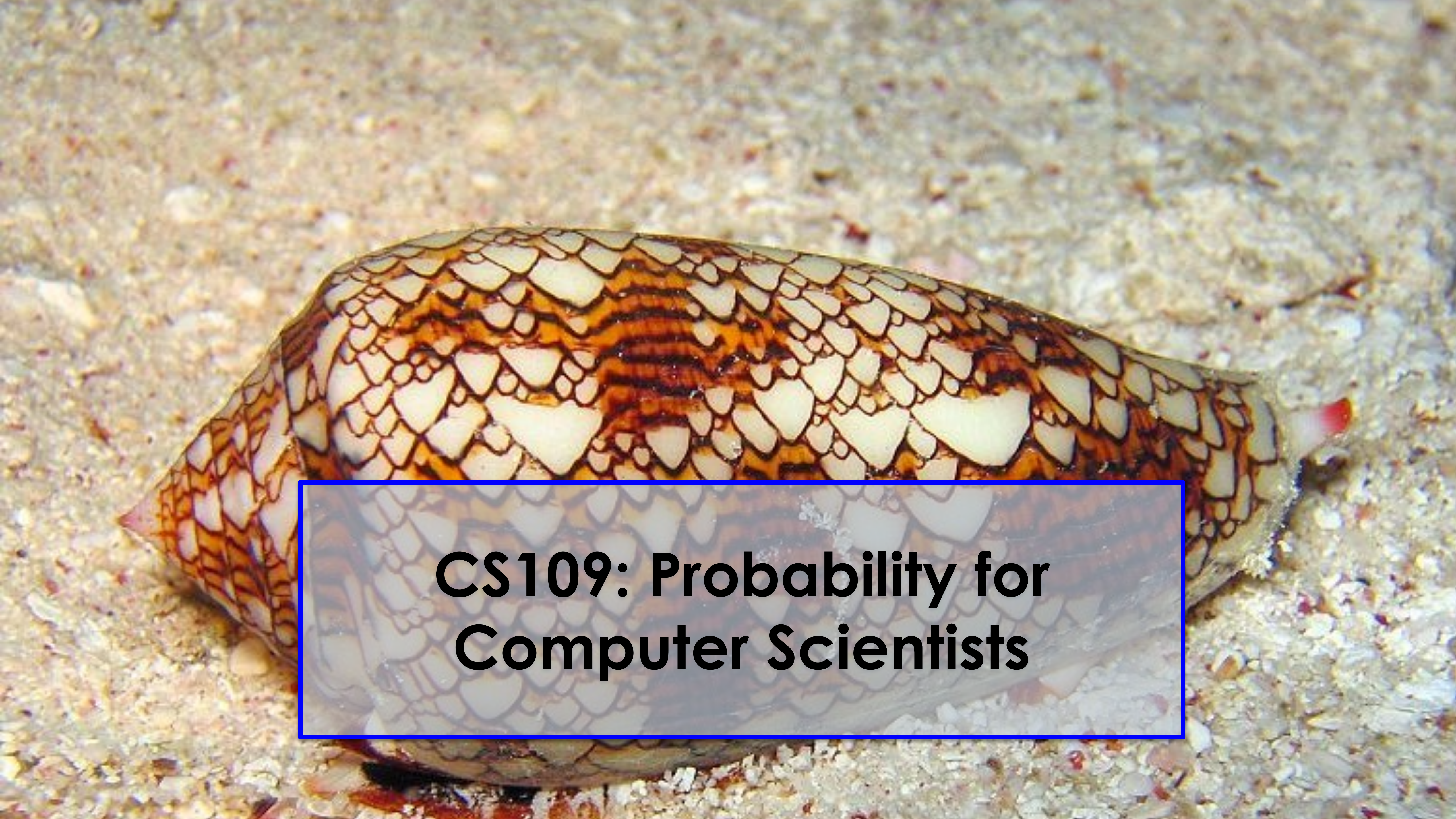
1. Strings
   - How many *different* orderings of letters are possible for the string BOBA?

`BOBA, ABOB, OBBA...`

🤔

Incredible time. Incredible school at which to study probability!

Exciting.



CS109: Probability for Computer Scientists

Core Counting

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$$= 28$$

2^4 start with 01	2^4 end with 10
010000	000010
010001	000110
010010	001010
010011	001110
010100	**010010**
010101	**010110**
010110	**011010**
010111	**011110**
011000	100010
011001	100110
011010	101010
011011	101110
011100	110010
011101	110110
011110	111010
011111	111110
Set A	Set B

Challenge Problem

1. Strings
   – How many *different* orderings of letters are possible for the string BOBA?

BOBA, ABOB, OBBA…

🤔

End Review

Permutations I

Summary of Combinatorics

Counting tasks on n objects

- Sort objects (permutations)
  - Distinct (distinguishable)
- Choose k objects (combinations)
- Put objects in r buckets

Orderings of Letters

How many letter orderings are possible for the following strings?

1. GATES

Orderings of Letters

gates	getsa	atges	asteg	tegas	egtsa	esgat	sateg
gatse	gesat	atgse	asegt	tegsa	egsat	esgta	saegt
gaets	gesta	ategs	asetg	teags	egsta	esagt	saetg
gaest	gsate	atesg	tgaes	teasg	eagts	esatg	stgae
gaste	gsaet	atsge	tgase	tesga	eagst	estga	stgea
gaset	gstae	atseg	tgeas	tesag	eatgs	estag	stage
gtaes	gstea	aegts	tgesa	tsgae	eatsg	sgate	staeg
gtase	gseat	aegst	tgsae	tsgea	easgt	sgaet	stega
gteas	gseta	aetgs	tgsea	tsage	eastg	sgtae	steag
gtesa	agtes	aetsg	tages	tsaeg	etgas	sgtea	segat
gtsae	agtse	aesgt	tagse	tsega	etgsa	sgeat	segta
gtsea	agets	aestg	taegs	tseag	etags	sgeta	seagt
geats	agest	asgte	taesg	egats	etasg	sagte	seatg
geast	agste	asget	tasge	egast	etsga	saget	setga
getas	agset	astge	taseg	egtas	etsag	satge	setag

Orderings of letters

Step 1:
Chose first letter

Step 2:
Chose 2nd letter

Step 3:
Chose 3rd letter

Step 4:
Chose 4th letter

Step 5:
Chose 5th letter

Orderings of letters

S A T G E

Step 1:
Chose first letter
(5 options)

Step 2:
Chose 2nd letter
(4 options)

Step 3:
Chose 3rd letter
(3 options)

Step 4:
Chose 4th letter
(2 options)

Step 5:
Chose 5th letter
(1 option)

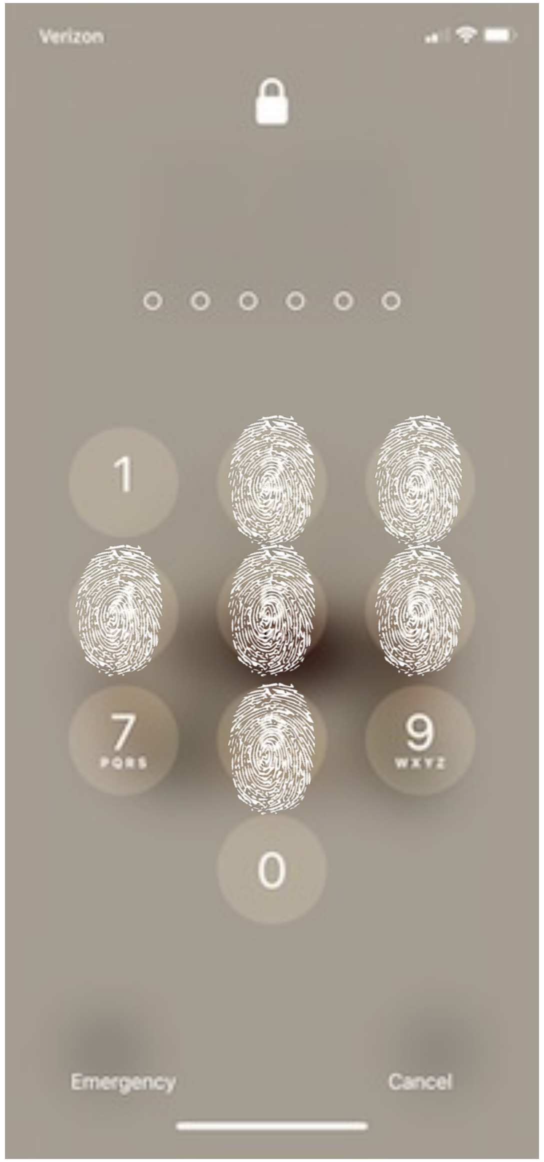
Permutations

A permutation is an ordered arrangement of objects.

The number of unique orderings (permutations) of n distinct objects is

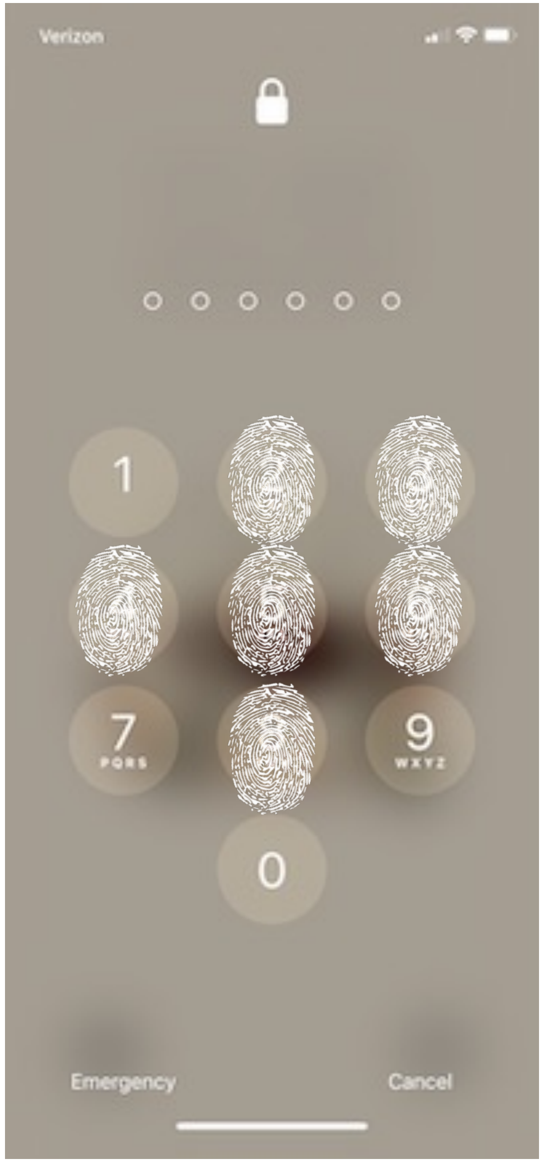
$$n! = n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1.$$

Unique 6-digit passcodes with six smudges



How many unique 6-digit passcodes are possible if a phone password uses each of **six** distinct numbers?

Unique 6-digit passcodes with six smudges



How many unique 6-digit passcodes are possible if a phone password uses each of **six** distinct numbers?

$$\text{Total} = 6! = 720 \text{ passcodes}$$

How many unique passcodes are possible if a phone password is some ordered subset of any 6 unique digits?

$$\text{Total} = 10 \times 9 \times 8 \times 7 \times 6 \times 5$$
$$= \frac{10!}{4!} = 151200 \text{ passcodes}$$

Unique Bit Strings

1, 0, 1, 0, 0

Sort indistinct objects?

Sort n distinct objects

Ayesha Tim Irina Joey Waddie

Sort n distinct objects

Ayesha Tim Irina Joey Waddie

1^{st} 2^{nd} 3^{rd} 4^{th} 5^{th}

<u>Steps</u>:

1. Choose 1^{st} can 5 options
2. Choose 2^{nd} can 4 options

…

5. Choose 5^{th} can 1 option

$$\text{Total} = 5 \times 4 \times 3 \times 2 \times 1$$
$$= 120$$

Permutations II

Summary of Combinatorics

Counting tasks on n objects

Sort objects (permutations)

Choose k objects (combinations)

Put objects in r buckets

Distinct (distinguishable)

Semi-distinct

$n!$

How many ways can we sort coke cans!

Sort n distinct objects

Ayesha Tim Irina Joey Waddie

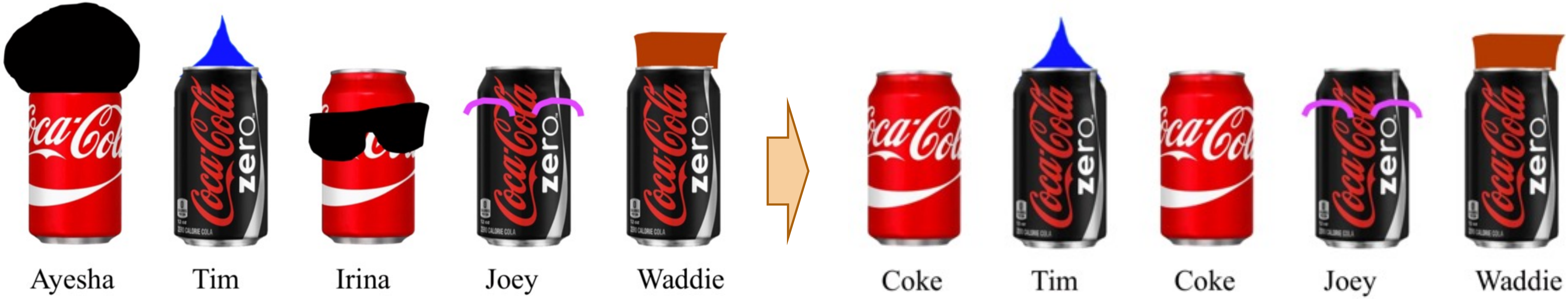
# of permutations =

Sort semi-distinct objects

Order n distinct objects	$n!$

All distinct

Some indistinct



Sort semi-distinct objects

How do we find the number of permutations considering some objects are indistinct?

By the product rule, permutations of <u>distinct</u> objects is a two-step process:

permutations of distinct objects	=	permutations considering some objects are indistinct	×	Permutations of just the indistinct objects

Sort semi-distinct objects

How do we find the number of permutations considering some objects are indistinct?

By the product rule, permutations of <u>distinct</u> objects is a two-step process:

$$\frac{\text{permutations of distinct objects}}{\text{Permutations of just the indistinct objects}} = \text{permutations considering some objects are indistinct}$$

In combinatorics it often helps to overcount permutations and fix for the overcounting

General approach to counting permutations

When there are n objects such that

n_1 are the same (indistinguishable or indistinct), and

n_2 are the same, and

…

n_r are the same,

The number of unique orderings (permutations) is

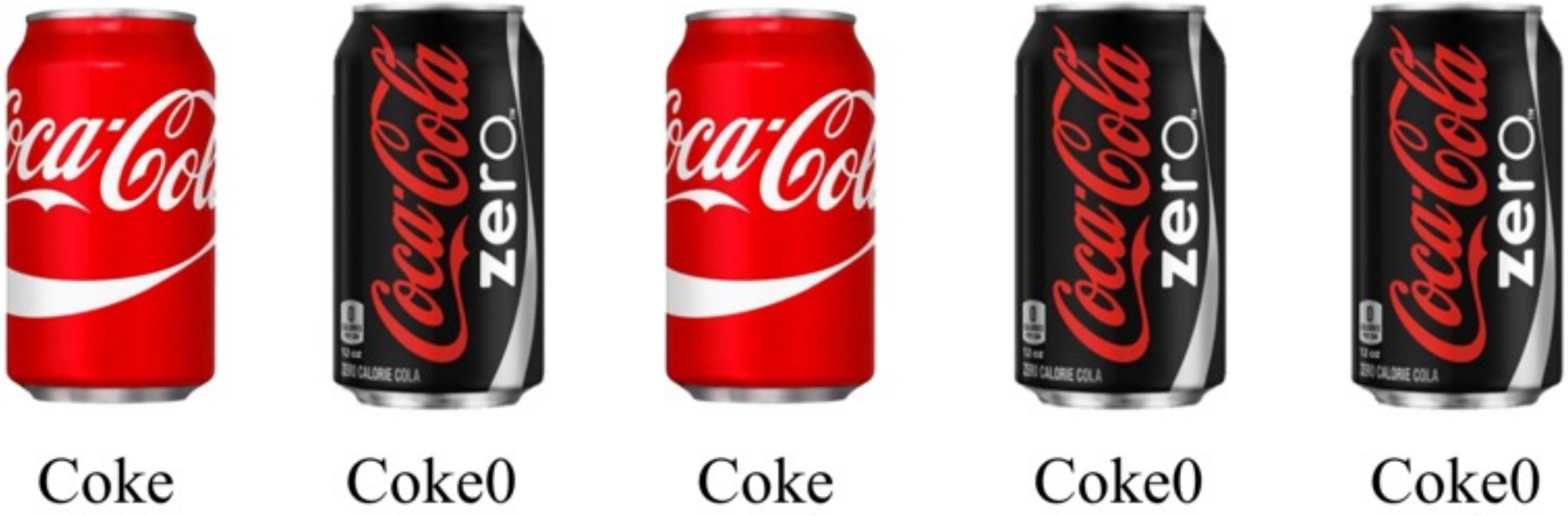
$$\frac{n!}{n_1!\,n_2!\cdots n_r!}.$$

For each group of indistinct objects,
Divide by the overcounted permutations.

Sort semi-distinct objects

Order n semi-distinct objects $\frac{n!}{n_1!\, n_2! \cdots n_r!}$

How many permutations?



Strings

Order n semi-distinct objects	$\frac{n!}{n_1!\,n_2!\cdots n_r!}$

How many letter orderings are possible for the following strings?

1. BOBA

2. MISSISSIPPI

Strings

Order n semi-distinct objects $\dfrac{n!}{n_1!\,n_2!\cdots n_r!}$

How many letter orderings are possible for the following strings?

1. BOBA $= \dfrac{4!}{2!} = 12$

2. MISSISSIPPI $= \dfrac{11!}{1!4!4!2!} = 34{,}650$

To the Code!

baob
bbao
obba
oabb
boab
babo
abbo
aobb
boba
abob
bboa
obab

```python
import itertools

def main():
   letters = ['b','o','b','a']
   perms = set(itertools.permutations(letters))
   for perm in perms:
      pretty_perm = "".join(perm)
      print(pretty_perm)
```

```python
import math

def main():
   n = math.factorial(4)
   d = math.factorial(2)
   print(n / d)
```

Summary of Combinatorics

Counting tasks on n objects

- Sort objects (permutations)
  - Distinct (distinguishable): $n!$
  - Semi-distinct: $\dfrac{n!}{n_1!\, n_2! \cdots n_r!}$
- Choose k objects (combinations)
- Put objects in r buckets

Combinations I

Summary of Combinatorics

Counting tasks on n objects

Sort objects (permutations)

- Distinct (distinguishable): $n!$
- Some distinct: $\dfrac{n!}{n_1!\,n_2!\cdots n_r!}$

Choose k objects (combinations)

- Distinct

Put objects in r buckets

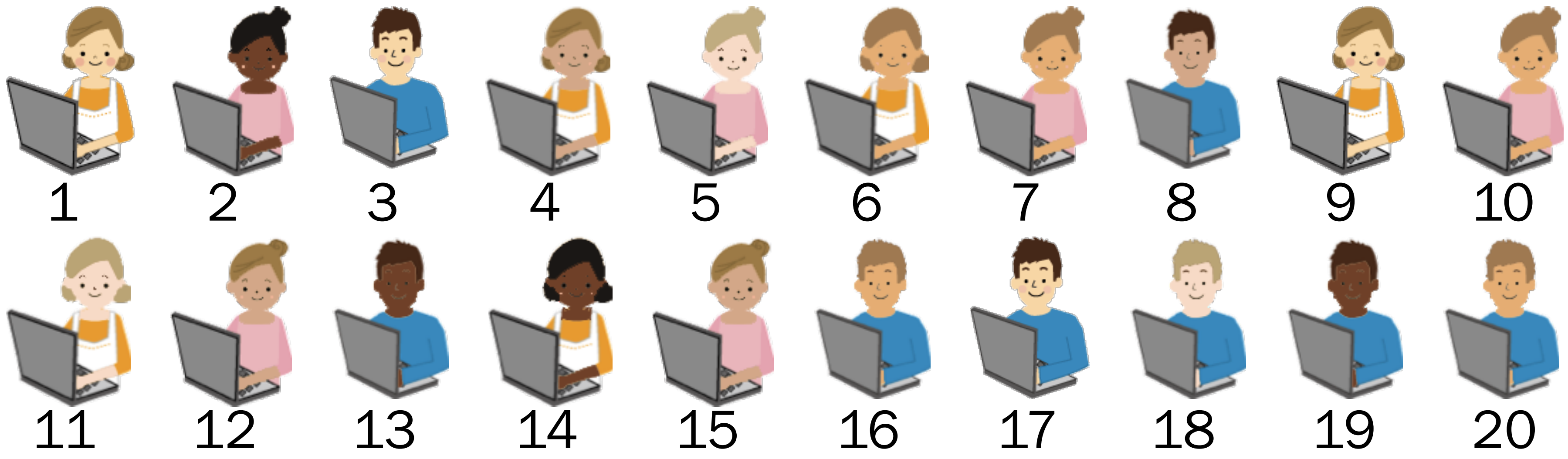
Combinations with cake

There are $n = 20$ people.
How many ways can we choose $k = 5$ people to get cake?

Consider the following generative process...

Combinations with cake

There are $n = 20$ people.
How many ways can we choose $k = 5$ people to get cake?

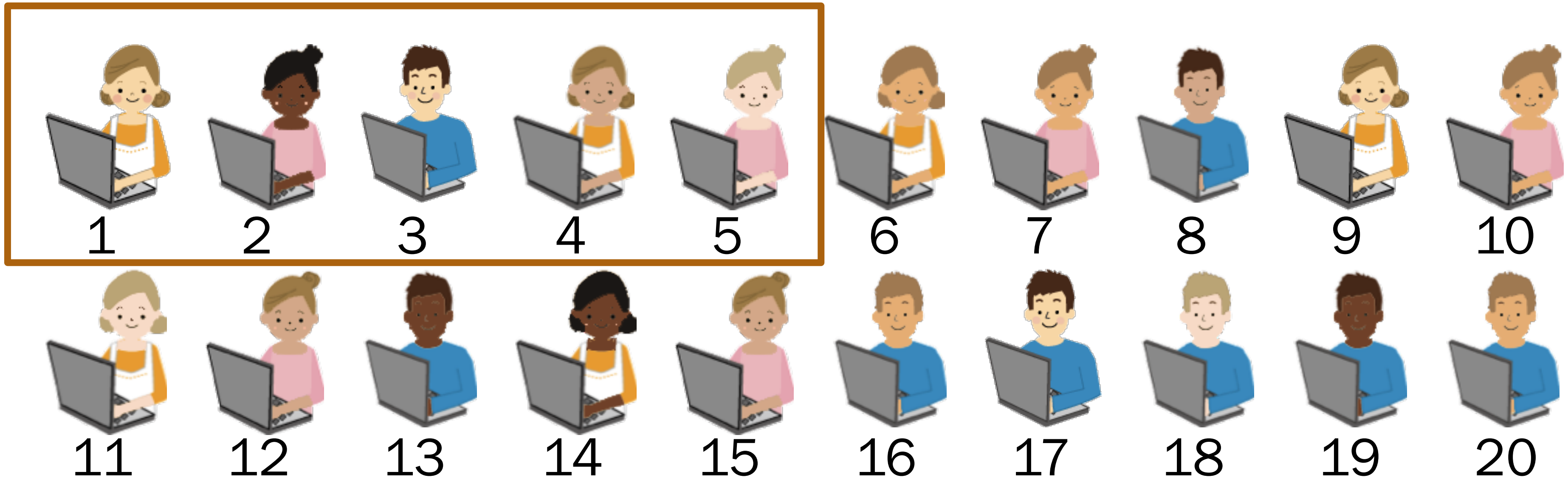


1. n people get in line

$n!$ ways

Combinations with cake

There are $n = 20$ people.
How many ways can we choose $k = 5$ people to get cake?



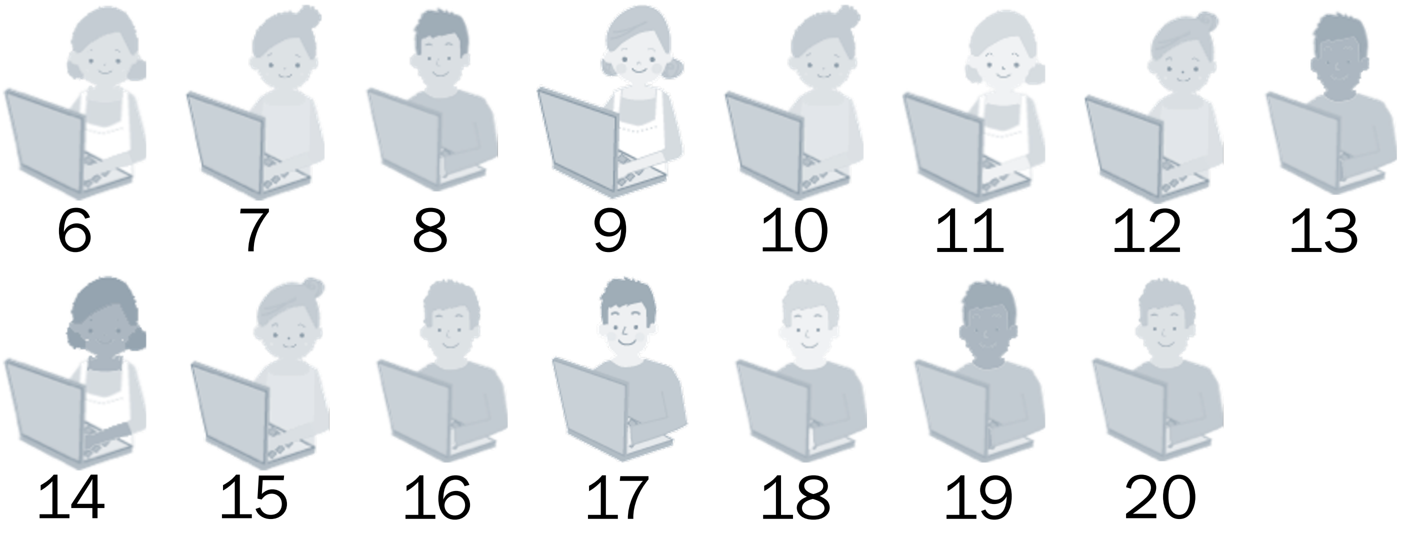
1. n people get in line
2. Put first k in cake room

$n!$ ways

1 way

Combinations with cake

There are $n = 20$ people.
How many ways can we choose $k = 5$ people to get cake?



1. n people get in line

   $n!$ ways

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   1 way

Combinations with cake

There are $n = 20$ people.
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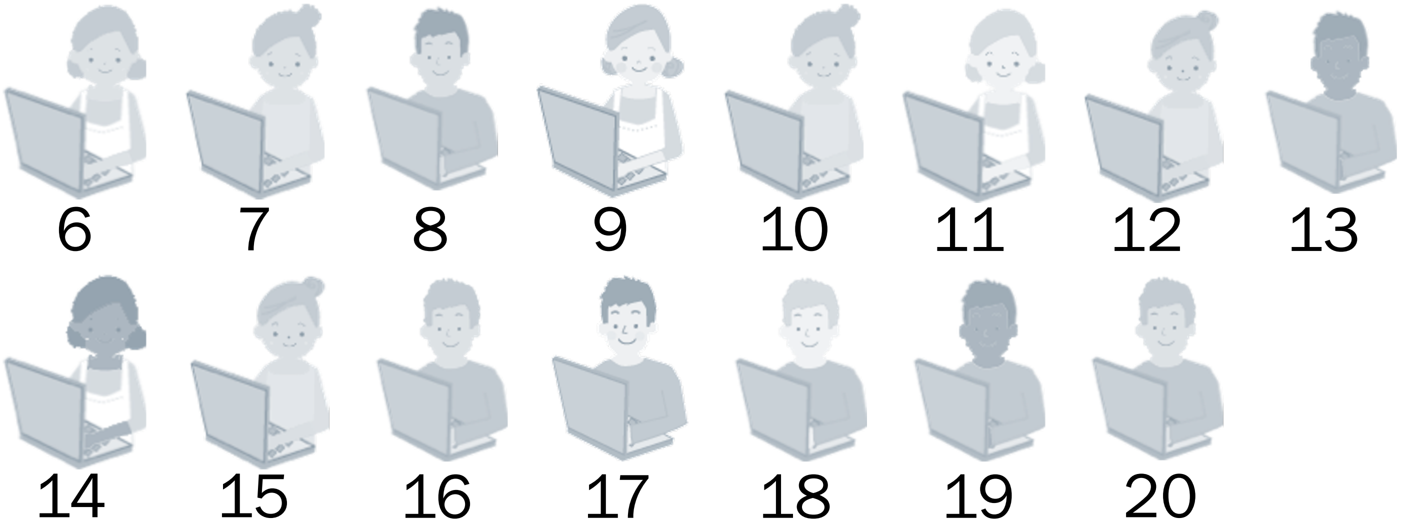

1. n people get in line

   $n!$ ways

2. Put first k in cake room

   1 way

3. Allow cake group to mingle

   $k!$ different permutations lead to the same mingle

Combinations with cake

There are $n = 20$ people.
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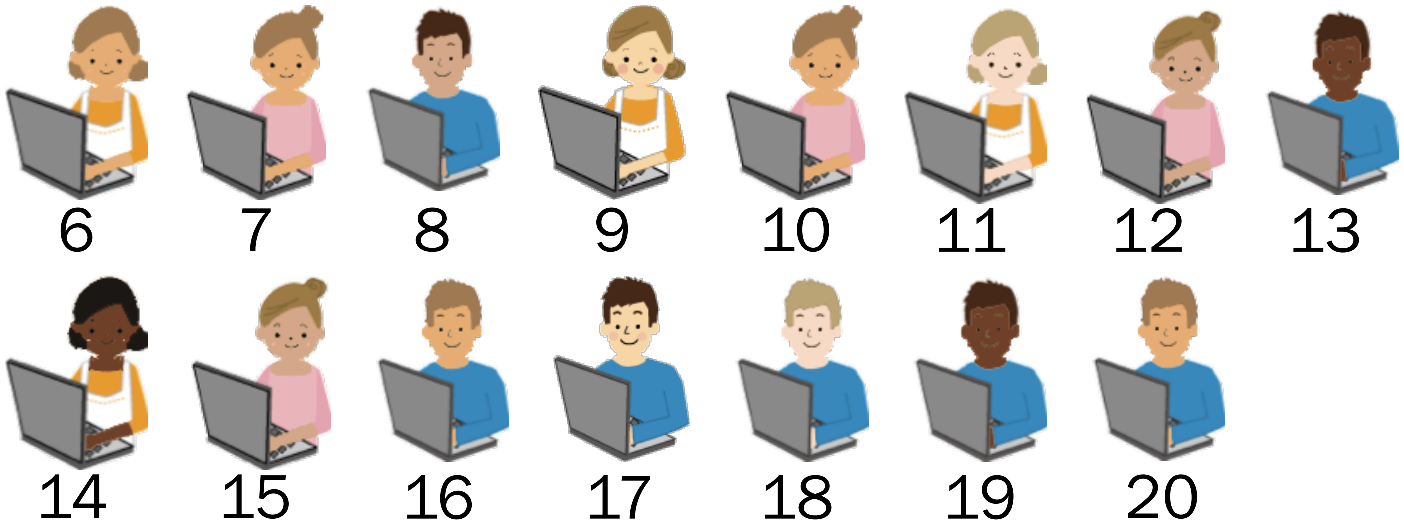


1. n people get in line
2. Put first k in cake room
3. Allow cake group to mingle
4. Allow non-cake group to mingle

$n!$ ways

1 way

$k!$ different permutations lead to the same mingle

Combinations with cake

There are $n = 20$ people.
How many ways can we choose $k = 5$ people to get cake?

1. n people get in line

   $n!$ ways

2. Put first k in cake room

   1 way

3. Allow cake group to mingle

   $k!$ different permutations lead to the same mingle

4. Allow non-cake group to mingle

   $(n-k)!$ different permutations lead to the same mingle

Combinations

A combination is an <u>unordered</u> selection of k objects from a set of n distinct objects.

The number of ways of making this selection is

$$\frac{n!}{k!\,(n-k)!} = n! \times 1 \times \frac{1}{k!} \times \frac{1}{(n-k)!}$$

1. Order n distinct objects
2. Take first k as chosen
3. Overcounted: any ordering of chosen group is same choice
4. Overcounted: any ordering of unchosen group is same choice

Combinations

A combination is an unordered selection of k objects from a set of n distinct objects.

The number of ways of making this selection is

$$\frac{n!}{k!\,(n-k)!} = n! \times 1 \times \frac{1}{k!} \times \frac{1}{(n-k)!} = \binom{n}{k}$$

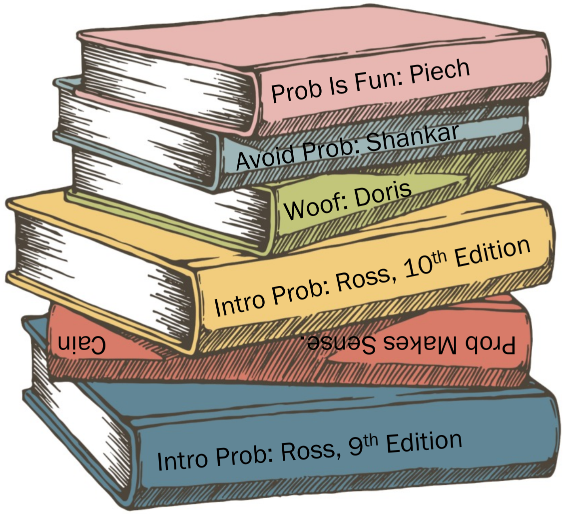
Binomial coefficient

Fun Fact: $\binom{n}{k} = \binom{n}{n-k}$

$(x+y)^n$

Probability textbooks

Choose k of n distinct objects $\binom{n}{k}$



How many ways are there to choose 3 books from a set of 6 distinct books?

$$\binom{6}{3} = \frac{6!}{3!\,3!} = 20 \text{ ways}$$

To the code!

How many unique hands of 5 cards are there in a 52 card deck?

```
def main():
    cards = make_deck()
    all_hands = itertools.combinations(cards, 5)
    for hand in all_hands:
        print(hand)
```

```
def main():
    total = math.comb(52, 5)
    print(total)
```

Buckets and The Divider Method

Summary of Combinatorics

Counting tasks on n objects

Sort objects (permutations)

- Distinct (distinguishable): $n!$
- Some distinct: $$\frac{n!}{n_1!\,n_2!\cdots n_r!}$$

Choose k objects (combinations)

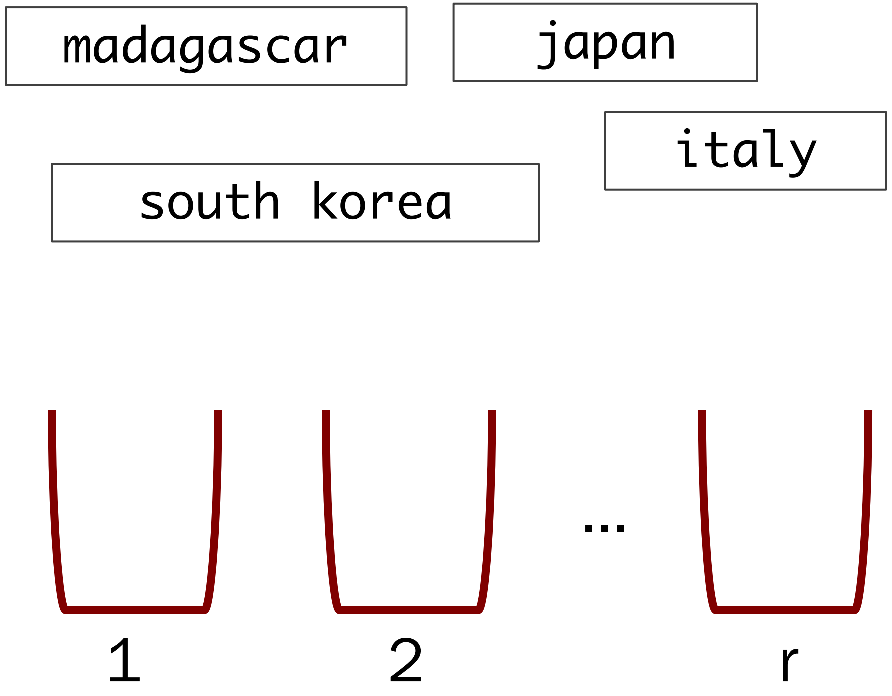
- Distinct: $\binom{n}{k}$

Put objects in r buckets

- Distinct
- Indistinct

~~Balls and urns~~ Hash tables and distinct strings

How many ways are there to hash n distinct strings to r buckets?



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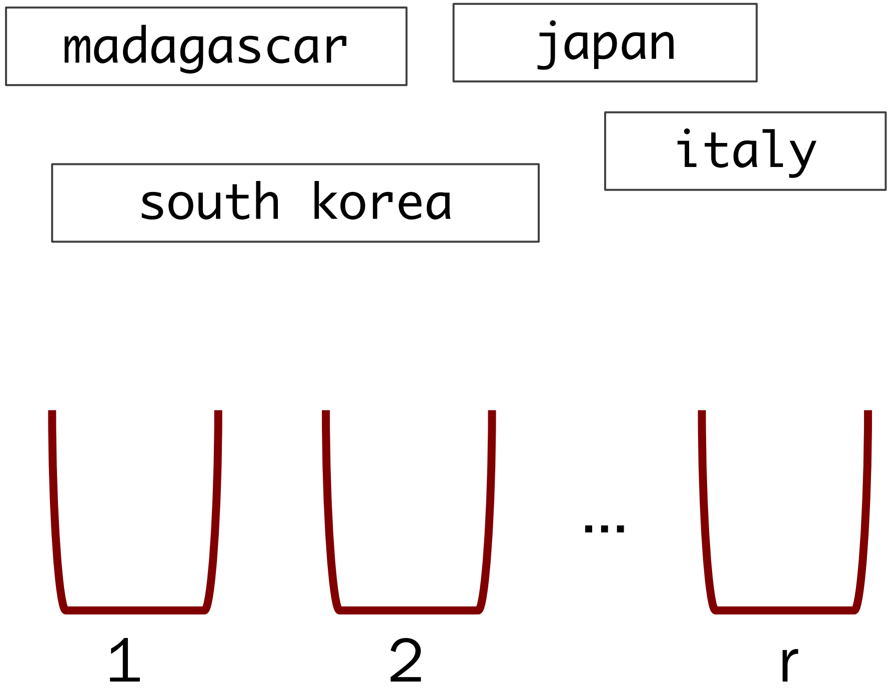


Steps:

1. Bucket 1st string
2. Bucket 2nd string

…

n. Bucket n^{th} string

r^n outcomes

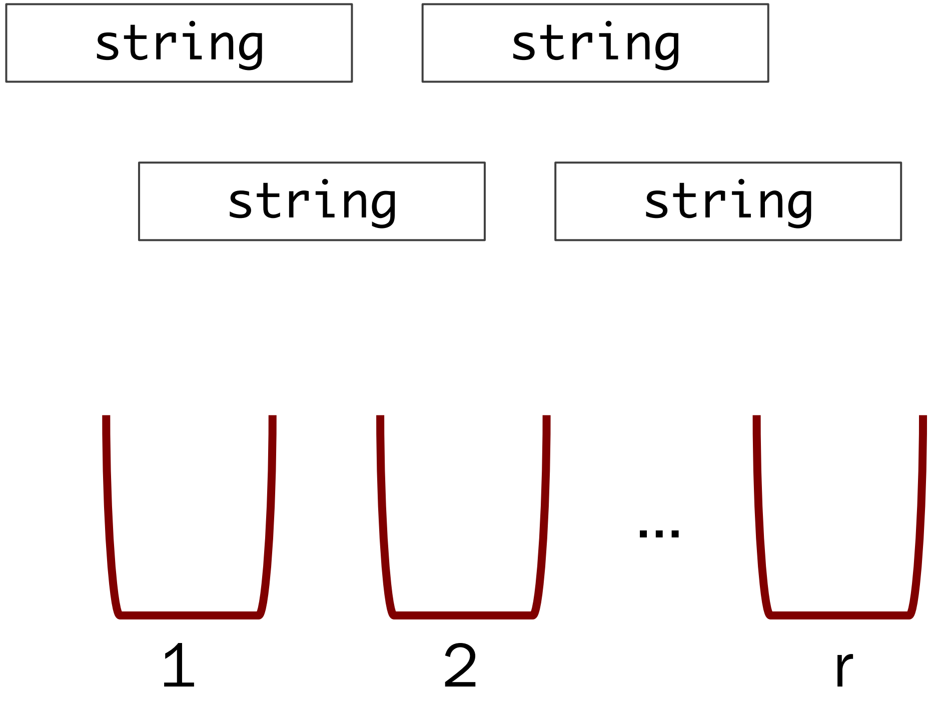
Summary of Combinatorics

Counting tasks on n objects

- **Sort objects (permutations)**
  - Distinct (distinguishable): $n!$
  - Some distinct: $\dfrac{n!}{n_1!\,n_2!\cdots n_r!}$
- **Choose k objects (combinations)**
  - Distinct: $\binom{n}{k}$
- **Put objects in r buckets**
  - Distinct: r^n
  - Indistinct

Servers and indistinct requests

How many ways are there to distribute n indistinct strings to r buckets?



Goal

Bucket 1 has x_1 `string`,
Bucket 2 has x_2 `string`,
…
Bucket r has x_r `string`

constraint: $\sum_{i=1}^{r} x_i = n$

Servers and indistinct requests

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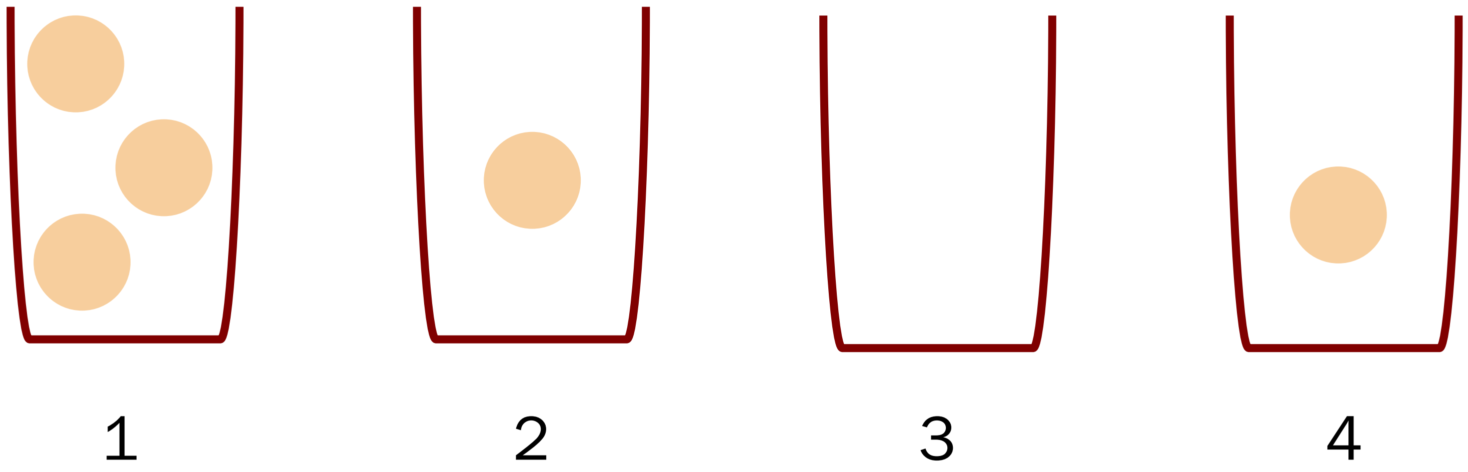

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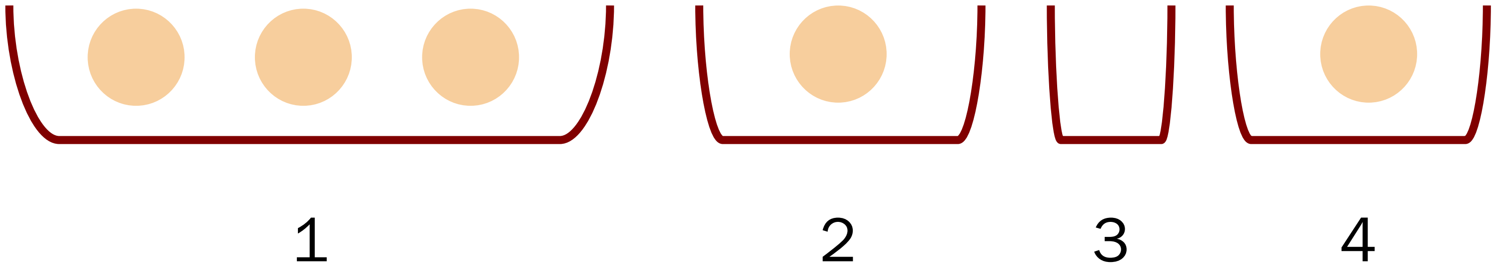

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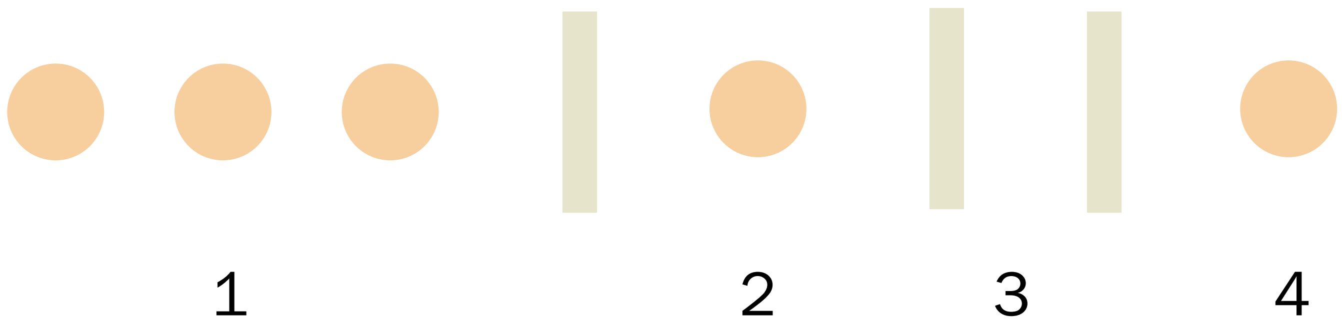


n objects

$(r-1)$ dividers

Servers and indistinct requests

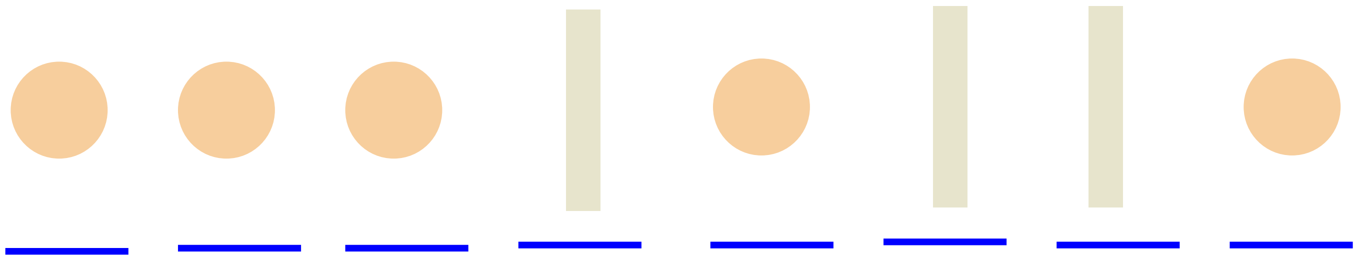
How many ways are there to distribute n indistinct strings to r buckets?

: Slots where either a divider or an object can go into

$(n + r - 1)$ slots

n objects

$(r - 1)$ dividers

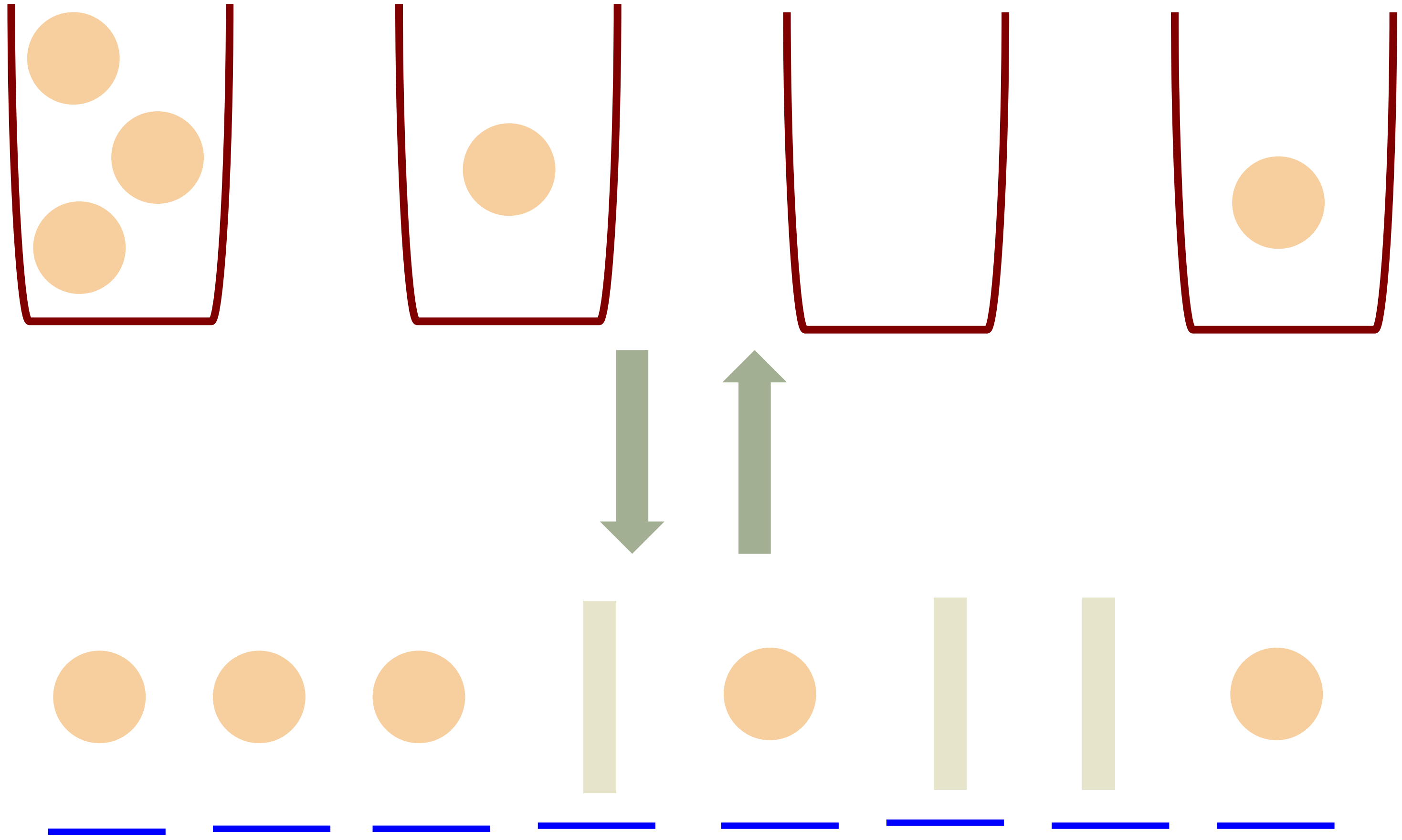


Servers and indistinct requests

How many ways are there to distribute n indistinct strings to r buckets?

n objects

r buckets



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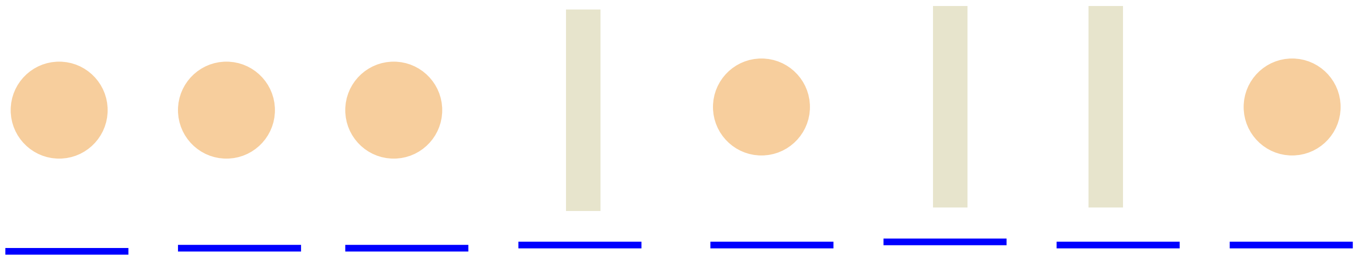

Answer

<u>1. (Combination) Choose $(\boldsymbol{r} - \mathbf{1})$ divider slots out of $(\boldsymbol{n} + \boldsymbol{r} - \mathbf{1})$ distinct slots</u>

<u>2. (Permutation) Arrange $(\boldsymbol{r} - \mathbf{1})$ indistinct dividers and $\boldsymbol{n}$ indistinct objects</u>

The divider method

The number of ways to distribute n indistinct objects into r buckets is equivalent to the number of ways of:

1. Choosing (r-1) slots among (n+r-1) distinct slots to place the divider
2. Arranging (r-1) indistinct dividers and n indistinct objects.

$$\text{Total } = \boxed{\binom{n+r-1}{r-1}} = \frac{(n+r-1)!}{n!\,(r-1)!} \text{ outcomes}$$

Integer solutions to equations

Divider method $\binom{n+r-1}{r-1}$
(n indistinct objects, r buckets)

How many integer solutions are there to the following equation:

$$x_1 + x_2 + \cdots + x_r = n,$$

where for all i, x_i is an integer such that $0 \leq x_i \leq n$?

Positive integer equations can be solved with the divider method.

Summary of Combinatorics

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  - Distinct (distinguishable)
  - Some distinct
- Choose k objects (combinations)
  - Distinct
    - 1 group
- Put objects in r buckets
  - Distinct
  - Indistinct

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```