CAI 2.0: Linear Algebra
Assignment 02

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Problem 1: Subspaces in Feature Extraction for Computer Vision

Feature extraction is a critical component in computer vision systems that reduces high-dimensional image data to a set of meaningful features. In this problem, you'll explore the concept of subspaces in a feature extraction context.

Problem: An AI vision system extracts three features from facial images: symmetry, texture, and edge clarity. Researchers have collected the following feature vectors from sample images:

Vector 1: $(2, 1, 3)$
Vector 2: $(4, 2, 6)$
Vector 3: $(1, 0, 2)$

a) Find the span of vectors 1, 2, and 3. What is the dimension of this span?

b) Identify a basis for the span calculated in part (a).

c) Provide a geometric description of the span of vectors 1 and 2 in $\mathbb{R}^3$. What does this tell you about the relationship between these feature vectors?

Problem 2: Vector Subspaces in Natural Language Processing

Modern NLP systems often represent words and sentences as vectors in high-dimensional spaces. Understanding the subspaces formed by these vectors can help in analyzing semantic relationships.

Problem: A natural language processing system represents words as 4D vectors. The system is analyzing a corpus of text about technology and has identified the following word vectors:

"Computer": $(1, 0, 1, 0)$
"Software": $(0, 1, 0, 1)$
"Programming": $(1, 1, 1, 1)$

a) Find the dimension of the span of these word vectors.

b) Identify a basis for the span.

c) The system defines a subspace $W = \{(x, y, z, w) \in \mathbb{R}^4 : x + y = 0, z + w = 0\}$. Show that W is indeed a subspace of $\mathbb{R}^4$ by verifying the subspace conditions.

Problem 3: Orthogonal Projections in Image Processing

Image processing systems often use orthogonal projections to decompose images into components that highlight specific features.

Problem: An image processing algorithm decomposes grayscale images into orthogonal components to enhance specific features. The algorithm works in a 3D space where each dimension represents a different spatial frequency (low, medium, high).

a) A feature vector $\vec{v} = (3, 1, 2)$ needs to be projected onto the plane P with equation $x+y+z=0$. Find the projection of $\vec{v}$ onto P.

b) Find an orthogonal basis for the plane P.

c) For noise filtering, the system needs to find the distance from a point $\vec{p} = (5, -2, 3)$ to the plane R with equation $2x - y + 3z = 4$. Calculate this distance.

Problem 4: Orthonormal Bases in 3D Object Recognition

3D object recognition systems often use orthonormal bases to efficiently represent and analyze object shapes.

Problem: A 3D object recognition system analyzes objects by decomposing their shapes into components along orthonormal bases. The system works with vectors in $\mathbb{R}^3$.

a) Determine whether the following collection of vectors is orthogonal:

$$\vec{u}_1 = (2, -1, 2), \vec{u}_2 = (1, 2, 0), \vec{u}_3 = (0, 2, -1)$$

b) The system uses an orthogonal collection of vectors:

$$\vec{v}_1 = (3, 0, 4), \vec{v}_2 = (0, 5, 0)$$

Convert this orthogonal collection into an orthonormal collection.

c) A feature vector $\vec{x} = (4, 2, -3)$ needs to be projected onto the line spanned by $\vec{w} = (2, 1, 2)$. Find this projection.

Problem 5: Dimensionality Reduction in Machine Learning

Dimensionality reduction is a crucial technique in machine learning to handle high-dimensional data. Linear algebra concepts like basis, span, and orthogonality play a key role in these methods.

Problem: A machine learning algorithm is analyzing a dataset where each data point has 3 features. The algorithm seeks to find a lower-dimensional representation of the data.

The dataset contains the following feature vectors:

$$\vec{d}_1 = (1, 2, 3), \vec{d}_2 = (4, 5, 6), \vec{d}_3 = (7, 8, 9)$$

a) Find the dimension of the span of these feature vectors and identify a basis for it.

b) The algorithm identifies a subspace $U = \{(x, y, z) \in \mathbb{R}^3 : x + y + z = 0\}$ as potentially useful for representing the data. Find an orthogonal basis for U.

c) Two vectors $\vec{c} = (2, 0, 2)$ and $\vec{d} = (1, 1, -1)$ represent key features in the dataset. Determine if they are orthogonal, and if not, construct an orthogonal basis for the plane they span.