Critical Points

Tuesday, 18 March 2025 9:43 am

For a function $f : \mathbb{R}^n \to \mathbb{R}$, a point $\mathbf{a} \in \mathbb{R}^n$ is called a **critical point** if all partial derivatives of f vanish at $\mathbf{a}$:

$$\frac{\partial f}{\partial x_i}(\mathbf{a}) = 0 \text{ for all } i = 1, 2, \ldots, n$$

Example 1

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Let's find the critical points of $f(x, y) = x^2 + y^2$.

$$\frac{\partial f}{\partial x} = 2x = 0 \Rightarrow x = 0$$

$$\frac{\partial f}{\partial y} = 2y = 0 \Rightarrow y = 0$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Problem 1

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Find the critical points of $f(x, y) = 3x^2y + 2y^3 - xy$.

$\frac{\partial f}{\partial x} = 6xy - y = 0 \Rightarrow y(6x-1) = 0 \Rightarrow y = 0,\ 6x - 1 = 0$

$x = 1/6$

$\frac{\partial f}{\partial y} = 3x^2 + 6y^2 - x = 0$

$y = 0$: $3x^2 - x = 0$

$x(3x-1) = 0$

$x = 0,\ x = 1/3$

$(0,0)$

$(\frac{1}{3}, 0)$

$x = 1/6$

$3(\frac{1}{6})^2 + 6y^2 - \frac{1}{6} = 0$

$\frac{1}{12} - \frac{1}{6} + 6y^2 = 0$

$6y^2 = 1/12 \Rightarrow y^2 = \frac{1}{72}$

$y = \pm\sqrt{\frac{1}{72}} = \pm \frac{1}{6\sqrt{2}}$

$(\frac{1}{6}, \frac{1}{6\sqrt{2}}),\ (\frac{1}{6}, -\frac{1}{6\sqrt{2}})$

Problem 2

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Find the critical points of $f(x, y) = 4x^2 - 2xy + y^2 + 8x - 2y + 5$.

$$\frac{\partial f}{\partial x} = 8x - 2y + 8 = 0 \quad \text{— (1)} \quad \Rightarrow \quad 8x + 8 = 2y$$

$$\frac{\partial f}{\partial y} = -2x + 2y - 2 = 0 \quad \text{— (2)} \qquad 2x + 2 = 2y$$

$$\Rightarrow 8x + 8 = 2x + 2 - 8$$

$$6x$$

$$(-1, 0)$$

(1) + (2)

$$6x + 6 = 0$$

$$x = -1$$

$$-8 - 2y + 8 = 0 \Rightarrow -2y = 0 \Rightarrow y = 0$$

Saddle Points

Tuesday, 18 March 2025 9:59 am

Definition: A critical point $\mathbf{a}$ of a function $f : \mathbb{R}^n \to \mathbb{R}$ is a **saddle point** if:

- Moving from $\mathbf{a}$ in some direction causes f to increase (so $\mathbf{a}$ looks like a local minimum in that direction)
- Moving from $\mathbf{a}$ in some other direction causes f to decrease (so $\mathbf{a}$ appears to be a local maximum in that direction)

Example 2

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Find Saddle Points of $f(x, y) = x^2 - y^2$

$$H = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$$

$$f(x, y) = 3x^2y + 2y^3 - xy$$

$f_x = 6xy - y$

$f_y = 3x^2 + 6y^2 - x$

$$D = f_{xx}f_{yy} - f_{xy}f_{yx}$$

Find the Hessian Matrix and its determinant D.

$f_{xx} = 6y \qquad f_{xy} = 6x - 1$

$f_{yy} = 12y \qquad f_{yx} = f_{xy} = 6x - 1$

$$\begin{vmatrix} 6y & 6x-1 \\ 6x-1 & 12y \end{vmatrix}$$

$D = 72y^2 - 36x^2 + 12x - 1$ ✓

$(6x-1)^2$
$= 36x^2 + 1 + 12x$

$P_1 = (0, 0)$, $P_2 = \left(\frac{1}{3}, 0\right)$, $P_3 = \left(\frac{1}{6}, \frac{1}{6\sqrt{2}}\right)$, $P_4 = \left(\frac{1}{6}, -\frac{1}{6\sqrt{2}}\right)$

$D|_{P_1} = -1$,

$D|_{P_2} = -36\left(\frac{1}{3}\right)^2 + 12\left(\frac{1}{3}\right) - 1$
$= -4 + 4 - 1$
$= -1$

$D|_{P_3} =$

$D|_{P_3} = 72\left(\frac{1}{6\sqrt{2}}\right)^2 - 36\left(\frac{1}{6}\right)^2 + 12\left(\frac{1}{6}\right) - 1$
$= \cancel{72}\left(\frac{1}{\cancel{72}}\right) - \cancel{36}\left(\frac{1}{\cancel{36}}\right) + 2 - 1 = 1$

$D|_{P_4} = 1$

$f_{xx}|_{P_3} = 6y = 6\left(\frac{1}{6\sqrt{2}}\right) = \frac{1}{\sqrt{2}}$ +ive min

$f_{xx}|_{P_4} = 6y = 6\left(-\frac{1}{6\sqrt{2}}\right) = -\frac{1}{\sqrt{2}}$ -ive max

Deciding the nature of Critical Points

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D	f_{xx}	Point
> 0	> 0	Local Minimum
> 0	< 0	Local Maximum
< 0	-	Saddle Point
$= 0$	-	Cannot be decided

$$f\left(x, y\right) = 3x^2y + 2y^3 - xy$$

Problem 3

Tuesday, 18 March 2025 10:36 am

Find the critical points of $f(x,y) = 2x^2 - 3xy + 2y^2$ and decide their nature.

c.p

$f_x = 4x - 3y = 0$

$f_y = -3x + 4y = 0$

$\begin{bmatrix} 4 & -3 \\ -3 & 4 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$x = \dfrac{\begin{vmatrix} 0 & -3 \\ 0 & 4 \end{vmatrix}}{\begin{vmatrix} 4 & -3 \\ -3 & 4 \end{vmatrix}} = \dfrac{0}{-} = 0$

$y = 0$

$(0, 0)$

$f_{xx} = 4$ $\quad f_{xy} = -3$

$f_{yy} = 4$ $\quad f_{yx} = -3$

$H = \begin{pmatrix} 4 & -3 \\ -3 & 4 \end{pmatrix}$

$D = 16 - 9 = 7$ (+ive)

Application: Least Squares Regression

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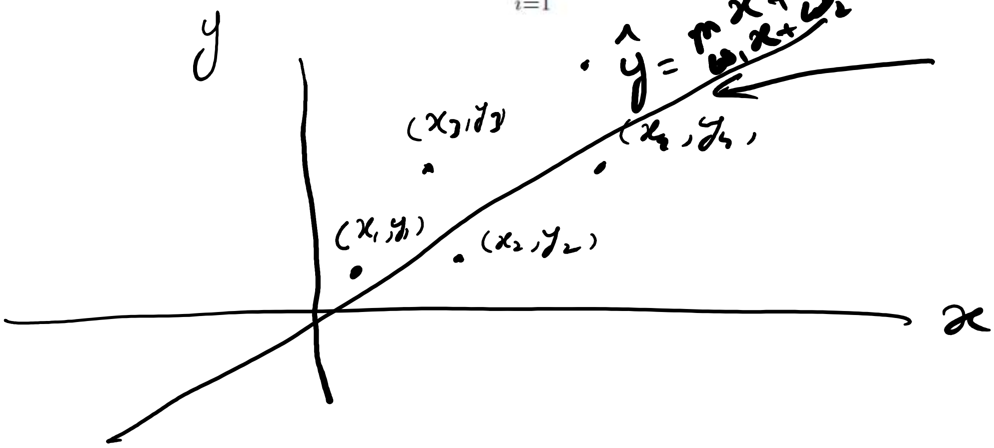
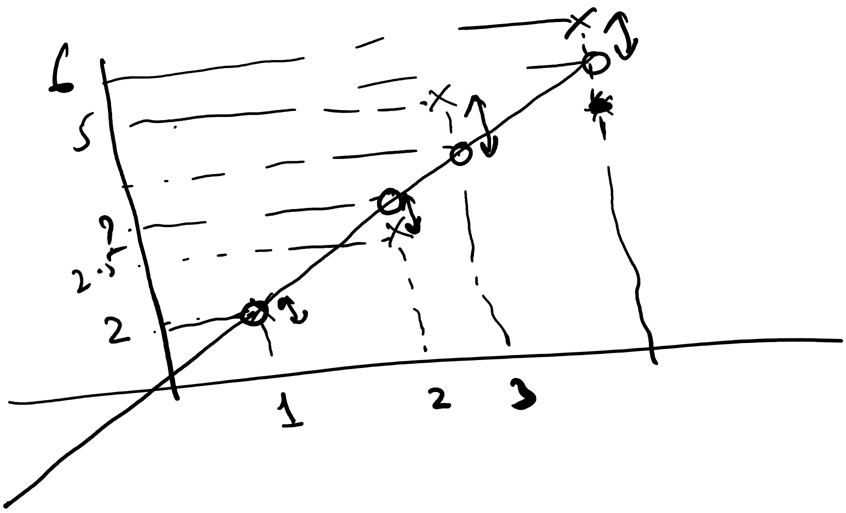
Suppose we have N data points (x_i, y_i) and want to find the line $y = mx + b$ that best fits this data in the sense of minimizing the sum of squared errors:

$$E(m,b) = \sum_{i=1}^{N} (y_i - (mx_i + b))^2$$



t	L
1	2
2	2.5
3	5
4	6
6	? 14.75

$$E(m,b) = \sum_{i=1}^{N} (y_i - \hat{y}_i)^2$$



$$E = \sum_{i=1}^{N} \left[y_i - (mx_i + b) \right]^2$$

$$\frac{\partial E}{\partial m} = \sum_{i=1}^{N} 2\left[y_i - mx_i - b \right](-x_i) = -2 \underbrace{\sum_{i=1}^{N} x_i (y_i - mx_i - b) = 0}_{} \quad (1)$$

$i=1$

$$\sum_{i=1}^{N} x_i (y_i - m x_i - b) = 0 \quad \text{— (A)}$$

$$\frac{\partial E}{\partial b} = \sum_{i=1}^{N} 2 (y_i - m x_i - b)(-1) = 0 \Rightarrow \sum_{i=1}^{N} (y_i - m x_i - b) = 0 \quad \text{— (B)}$$

$$\text{(A)} \quad \underbrace{\sum x_i y_i}_{S_1} - m \underbrace{\sum x_i^2}_{S_2} - b \underbrace{\sum x_i}_{S_3} = 0$$

$$\underbrace{\sum y_i}_{S_4} - m \underbrace{\sum x_i}_{S_3} - \underbrace{b \boxed{\sum_{i=1}^{N} 1}}_{bN} = 0$$

$$S_1 - m S_2 - b S_3 = 0$$
$$S_4 - m S_3 - b N = 0$$

x_i	y_i	$x_i y_i$	x_i^2
1	2	2	1
2	2.5	5	4
3	5	15	9
4	6	24	16
$+\Sigma$ 10	15.5	46	30
S_3	S_4	S_1	S_2

$$46 - 30m - 10b = 0$$
$$15.5 - 10m - 4b = 0$$

$$30m + 10b = 46$$
$$10m + 4b = 15.5$$

$$\boxed{\boxed{m = 1.45} \quad b = 0.25}$$

$$\hat{y} = 1.45x + 0.25$$

Example 3

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Let's find the best-fit line for the data points: $(1, 1)$, $(2, 2)$, $(3, 2)$, $(4, 3)$, $(5, 5)$.

$$\hat{y} = mx + b$$

$m = 0.9$

$b = -0.1$

$48 - 55m - 15b = 0$

$13 - 15m - 5b = 0$

	x_i	y_i	$x_i \cdot y_i$	x_i^2
	1	1	1	1
	2	2	4	4
	3	2	6	9
	4	3	12	16
	5	5	25	25
Σ	15	13	48	55

Problem 4

Tuesday, 18 March 2025 10:52 am

Use the method of least squares to find the line of best fit $y = mx + b$ for the data points: $(0, 1)$, $(1, 3)$, $(2, 2)$, $(3, 5)$, $(4, 4)$.