Projection onto a line

Wednesday, 12 March 2025 1:25 pm

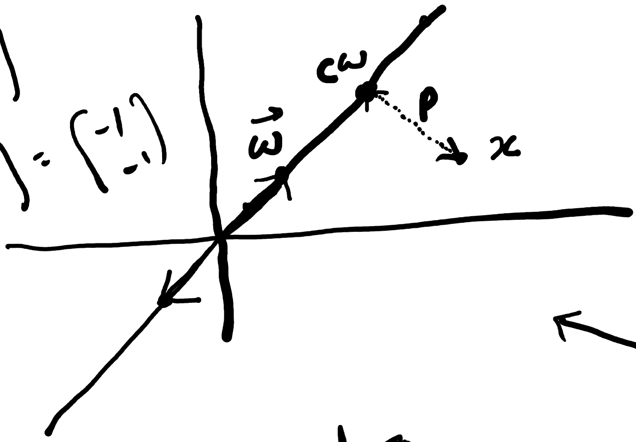
$$\boxed{\mathrm{Proj}_{\vec{w}}\vec{x} = \left(\frac{\vec{x}\cdot\vec{w}}{\vec{w}\cdot\vec{w}}\right)\underline{\underline{\vec{w}}}}$$

c

$c\vec{w}$

$\vec{w} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$-1\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$



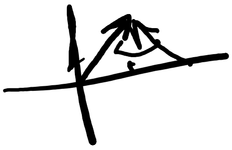
$p = x - cw$

$p \cdot w = 0$

$(x - cw)\cdot w = 0$

$x\cdot w - c\, w\cdot w = 0$

$x \cdot w = c\, w \cdot w$



Example 1

Wednesday, 12 March 2025 1:28 pm

Find the closest point to $\vec{x} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$ on the line L through the origin with direction vector $\vec{w} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

$$cw = \begin{bmatrix} c \\ 2c \end{bmatrix} \quad x, \quad \boxed{y = 2x}$$

$$cw = \begin{bmatrix} 7/5 \\ 14/5 \end{bmatrix}$$

$$c = \frac{x \cdot w}{w \cdot w} = \frac{5 + 2}{1 + 4} = \frac{7}{5}$$

$$cw = \frac{7}{5}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 7/5 \\ 14/5 \end{bmatrix}$$

Example 2

Wednesday, 12 March 2025 1:28 pm

Find the closest point to $\vec{x} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$ on the line $L = \text{span}(\vec{w})$ where $\vec{w} = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}$.

$c = \frac{5}{11}$ dist $= 4.3$ units

$cw = \begin{bmatrix} 15/11 \\ -5/11 \\ 5/11 \end{bmatrix}$ $x - cw = \begin{bmatrix} 1 - \frac{15}{11} \\ 2 + 5/11 \\ 4 - 5/11 \end{bmatrix} = \begin{bmatrix} -4/11 \\ 27/11 \\ 39/11 \end{bmatrix}$

$$\text{dist} = \sqrt{\frac{16 + 729 + 1521}{11^2}} = \frac{\sqrt{2266}}{11} = 4.3$$

Projection onto a General Subspace

Wednesday, 12 March 2025 1:30 pm

If V is spanned by an orthogonal basis $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$, then:

$$\boxed{\mathrm{Proj}_V(\vec{x}) = \mathrm{Proj}_{\vec{v}_1}(\vec{x}) + \mathrm{Proj}_{\vec{v}_2}(\vec{x}) + \cdots + \mathrm{Proj}_{\vec{v}_k}(\vec{x})}$$

Example 3

Wednesday, 12 March 2025 1:33 pm

Find the projection of $\vec{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ onto the plane $V = \mathrm{span}(\vec{v}_1, \vec{v}_2)$ where $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

plane equation:

$$\mathrm{span}(V_1, V_2) = c_1 V_1 + c_2 V_2$$

$$= \begin{bmatrix} c_1 + 0 \\ 0 + c_2 \\ 0 + 0 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ 0 \end{bmatrix} \quad z = 0$$

$$V_1 \cdot V_2 = 0$$

$$\mathrm{proj}_{V_1} x = \left(\frac{\overset{1}{x \cdot V_1}}{\underset{1}{V_1 \cdot V_1}} \right) V_1 \Rightarrow \begin{matrix} x \cdot V_1 = 1 \\ V_1 \cdot V_1 = 1 \end{matrix}$$

$$= V_1$$

$$\mathrm{proj}_{V_2} x = \left(\frac{x \cdot V_2}{V_2 \cdot V_2} \right) V_2 \quad \begin{matrix} x \cdot V_2 = 2 \\ V_2 \cdot V_2 = 1 \end{matrix}$$

$$= 2V_2 = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$$

$$\mathrm{proj}_V x = V_1 + 2V_2$$

$$= \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

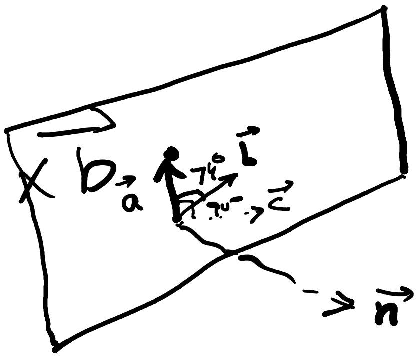
Problem 1

Wednesday, 12 March 2025 1:35 pm

$$\cos\theta = \frac{0+3+0}{\sqrt{4+1+0}\cdot\sqrt{0+9+16}} = \frac{3}{\sqrt{125}} = 0.268 \Rightarrow \theta = 74^\circ$$

Find an orthogonal basis for the plane $V = \operatorname{span}(\vec{a}, \vec{b})$ where $\vec{a} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix}$.

$$\vec{n} = \bar{a} \times b$$



$$\vec{n} = \begin{vmatrix} x_2 & y_1 & z_0 \\ 0 & 3 & 4 \end{vmatrix}$$

$$\vec{n} = (4, -8, 6)$$

$$a = (2, 1, 0)$$
$$v = (x, y, z)$$

$a \cdot v = 0$ $\quad$ $2x + y = 0$

$n \cdot v = 0$ $\quad$ $4x - 8y + 6z = 0$

$a \cdot v = 0$
$n \cdot v = 0$

$4x - 8y + 6z = 0$

$y = -2x$

$4x + 16x + 6z = 0$

$z = -\frac{20}{6} x$

$x = 1 \quad y = -2 \quad z = -\frac{20}{6}, \frac{-10}{3}$

$\vec{n} = (4, -8, 6) \cdot (1, -2, -\frac{10}{3})$

$4 + 16 - 20 = 0$

Example 4: Projection with Non-Orthogonal Basis

Wednesday, 12 March 2025 1:34 pm

Find the projection of $\vec{x} = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$ onto the plane $V = \text{span}(\vec{v}_1, \vec{v}_2)$ where $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$.

$$\text{proj}_V x = \text{proj}_{V_1} x + \text{proj}_{V_2} x \leftarrow \quad V_1 \cdot V_2 = 0$$

$$= \underbrace{\left(\frac{x \cdot V_1}{V_1 \cdot V_1}\right)}_{c_1} V_1 + \underbrace{\left(\frac{x \cdot V_2}{V_2 \cdot V_2}\right)}_{c_2} V_2$$

$x \cdot V_1 = 4$

$x \cdot V_2 = 4$

$V_1 \cdot V_1 = 3$

$V_2 \cdot V_2 = 2$

$$= \frac{4}{3} V_1 + \frac{4}{2} V_2$$

$$= \frac{4}{3}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + 2\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{bmatrix} \frac{4}{3} + 2 \\ \frac{4}{3} - 2 \\ 4/3 + 0 \end{bmatrix} = \begin{bmatrix} 10/3 \\ -2/3 \\ 4/3 \end{bmatrix}$$

plane eq.

$$\text{span}(V_1, V_2) = c_1 V_1 + c_2 V_2 = \begin{bmatrix} c_1 + c_2 \\ c_1 - c_2 \\ c_1 + 0 \end{bmatrix} \begin{matrix} \rightarrow x \\ \rightarrow y \\ \rightarrow z \end{matrix}$$

$$x + y = c_1 + \cancel{c_2} + c_1 - \cancel{c_2}$$

$$2c_1 = 2z$$

$$x + y = 2z$$

$$x + y - 2z = 0$$

$$\vec{a} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix}$$

$$\text{span}(a, b) = c_1 a + c_2 b = \begin{bmatrix} 2c_1 + 0 \\ c_1 + 3c_2 \\ 0 + 4c_2 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$x = 2c_1 \Rightarrow c_1 = \frac{x}{2}$$

$$y = c_1 + 3c_2 \longrightarrow \boxed{y = \frac{x}{2} + \frac{3}{4} z}$$

$$z = 4c_2 \Rightarrow c_2 = \frac{z}{4}$$

$$\frac{x}{2} - y + \frac{3}{4} z = 0$$

$$2x - 4y + 3z = 0$$

$$\vec{n} = \begin{bmatrix} 2 \\ -4 \\ 3 \end{bmatrix}$$

$$v = \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ -\frac{10}{3} \end{bmatrix}$$

$$v \cdot \vec{n} = 0$$

$$v \cdot \vec{a} = 0$$

$$v \cdot \vec{b} = 0$$