Covariance and Correlation

Two alternate expressions for Covariance

$\mathrm{Cov}(X,Y) = E[(X - E[X])(Y - E[Y])]$	
$\mathrm{Cov}(X,Y) = E[XY] - E[Y]E[X]$	This one is used in Excel file

The Dance of the Covariance

Say X and Y are arbitrary random variables

Covariance of X and Y:

$$\mathrm{Cov}(X,Y) = E[(X - E[X])(Y - E[Y])]$$

Equivalently:

$$\begin{aligned}\mathrm{Cov}(X,Y) &= E[XY - E[X]Y - XE[Y] + E[Y]E[X]] \\ &= E[XY] - E[X]E[Y] - E[X]E[Y] + E[X]E[Y] \\ &= E[XY] - E[X]E[Y]\end{aligned}$$

In the case where two discrete random variables X and Y have a joint probability distribution, represented by elements $p_{i,j}$ corresponding to the joint probabilities of $P(X = x_i, Y = y_j)$, the covariance is calculated using a double summation over the indices of the matrix:

$$\mathrm{cov}(X,Y) = \sum_{i=1}^{n}\sum_{j=1}^{n} p_{i,j}(x_i - E[X])(y_j - E[Y]).$$

Suppose that X and Y have the following joint probability mass function,[6] in which the six central cells give the discrete joint probabilities $f(x,y)$ of the six hypothetical realizations $(x,y) \in S = \{(5,8),(6,8),(7,8),(5,9),(6,9),(7,9)\}$:

$f(x,y)$		x			$f_Y(y)$
		5	**6**	**7**	
y	**8**	0	0.4	0.1	0.5
	9	0.3	0	0.2	0.5
$f_X(x)$		0.3	0.4	0.3	1

X can take on three values (5, 6 and 7) while Y can take on two (8 and 9). Their means are
$\mu_X = 5(0.3) + 6(0.4) + 7(0.1 + 0.2) = 6$ and
$\mu_Y = 8(0.4 + 0.1) + 9(0.3 + 0.2) = 8.5$. Then,

$$\begin{aligned}\operatorname{cov}(X, Y) = \sigma_{XY} = \sum_{(x,y) \in S} f(x, y)\,(x - \mu_X)\,(y - \mu_Y) \\ &= (0)(5-6)(8-8.5) + (0.4)(6-6)(8-8.5) + (0.1)(7-6)(8-8.5) + \\ &\quad (0.3)(5-6)(9-8.5) + (0)(6-6)(9-8.5) + (0.2)(7-6)(9-8.5) \\ &= -0.1\ .\end{aligned}$$

<u>Using 2nd Formula:</u>

$$\operatorname{Cov}(X, Y) = E[XY] - E[Y]E[X]$$

E[XY] = [0*40 + 0.4*48 + 0.1*56 + 0.3*45+ 0*54 + 0.2*63]

= [19.2+5.6+13.5+12.6]

= 50.9

E[x] E[Y] = 6 * 8.5 = 51

Cov (X,Y) = 50.9 – 51 = -0.1 **<u>Same Result</u>**

And Now Correlation

Say X and Y are arbitrary random variables

- Correlation of X and Y, denoted ρ(X, Y):

$$\rho(X, Y) = \frac{\operatorname{Cov}(X, Y)}{\sqrt{\operatorname{Var(X)Var(Y)}}}$$

- Note: -1 ≤ ρ(X, Y) ≤ 1