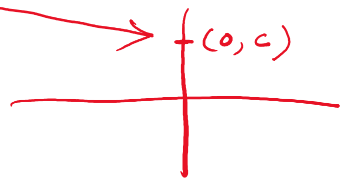
Span

Thursday, 6 March 2025 10:33 am

The span of a collection of vectors is the set of all possible linear combinations of those vectors. Given vectors $v_1, v_2, \ldots, v_k$ in $\mathbb{R}^n$, the span is:

$$\text{span}(v_1, v_2, \ldots, v_k) = \{c_1v_1 + c_2v_2 + \cdots + c_kv_k : c_1, c_2, \ldots, c_k \in \mathbb{R}\}$$

$V = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \in R^2$



R

R^2

R^3

$\text{span}(V) = cV = c\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ c \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ c \end{bmatrix}$ $\quad x = 0$, $y = c$

Example 1

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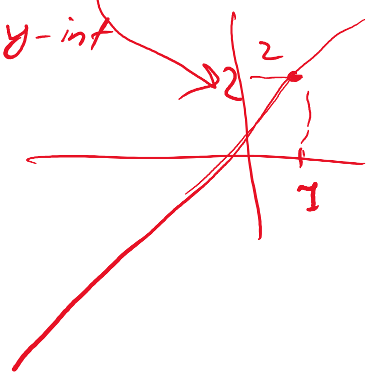
Find the span of $v = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ in R^2

$\text{span}(v) = cv \qquad c \in R$

$= c\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} c \\ 2c \end{bmatrix} \begin{matrix} \to x \\ \to \boxed{y = \underbrace{2}_{m} x} + \underbrace{0}_{y\text{-int}} \end{matrix}$



Example 2

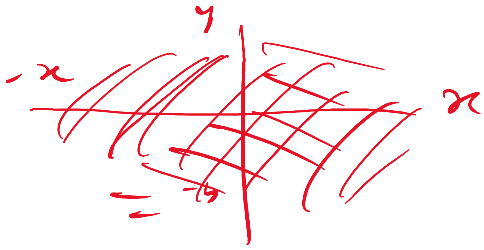
Thursday, 6 March 2025 10:38 am

Find the span of $v_1 = \begin{bmatrix}1\\0\end{bmatrix}$ and $v_2 = \begin{bmatrix}0\\1\end{bmatrix}$ in R^2

Sol

$$\text{span}(v_1, v_2) = c_1 v_1 + c_2 v_2$$

$$= c_1 \begin{bmatrix}1\\0\end{bmatrix} + c_2 \begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}c_1\\0\end{bmatrix} + \begin{bmatrix}0\\c_2\end{bmatrix} = \begin{bmatrix}c_1\\c_2\end{bmatrix} \begin{matrix}\to x\\ \to y\end{matrix}$$



$v_1 = \begin{bmatrix}1\\0\end{bmatrix}$, $v_2 = \begin{bmatrix}2\\0\end{bmatrix}$ $\quad v_2 = 2v_1$

Linearly dependent vectors

$$\text{span}(v_1, v_2) = c_1 v_1 + c_2 v_2$$

$$= c_1 \begin{bmatrix}1\\0\end{bmatrix} + c_2 \begin{bmatrix}2\\0\end{bmatrix}$$

$$= c_1 \begin{bmatrix}1\\0\end{bmatrix} + 2c_2 \begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix} \overbrace{(c_1 + 2c_2)}^{k}$$

$$= k v_1 = \begin{bmatrix}k\\0\end{bmatrix} \begin{matrix}\to x\\ \to y\end{matrix} \quad (k, 0)$$

Problem 1

Thursday, 6 March 2025 10:39 am

Find the span of $v_1 = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ in R^2

$y = mx + c$

$V_1 = 2V_2$ or $V_2 = \frac{1}{2}V_1$

$\text{span}(V_1, V_2) = c_1 V_1 + c_2 V_2$

$= c_1(2V_2) + c_2 V_2$

$= 2c_1 V_2 + c_2 V_2 = \overbrace{(2c_1 + c_2)}^{k} V_2$

$= kV_2 = k\begin{bmatrix} 1 \\ 2 \end{bmatrix} : \begin{Bmatrix} k \\ 2k \end{Bmatrix} \begin{matrix} x \\ y \end{matrix}$

$y = 2k$

$\boxed{y = 2x}$

Determine the span of $v_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}$, $v_2 = \begin{pmatrix}0\\1\\0\end{pmatrix}$, and $v_3 = \begin{pmatrix}1\\1\\0\end{pmatrix}$.

$$V_3 = V_1 + V_2$$

$$\overbrace{c_1 V_1 + c_2 V_2}$$
$$(V_1 + V_2)$$

$$\text{Span}\underbrace{(V_1, V_2, V_3)} = c_1V_1 + c_2V_2 + c_3V_3$$
$$= c_1V_1 + c_2V_2 + c_3(V_1 + V_2)$$
$$= \overline{c_1V_1} + c_2V_2 + \overline{c_3V_1} + c_3V_2$$
$$= (c_1 + c_3)V_1 + (c_2 + c_3)V_2$$
$$= \underbrace{k_1V_1 + k_2V_2} = \text{Span}(V_1, V_2)$$

$$= k_1\begin{bmatrix}1\\0\\0\end{bmatrix} + k_2\begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}k_1\\k_2\\0\end{bmatrix} \Rightarrow (\overset{\overset{x}{d}}{k_1}, \overset{\overset{y}{d}}{k_2}, \overset{\overset{z}{d}}{0})$$

V_1, V_2, V_3 span xy plane

$$V_3 = V_1 + V_2 \Rightarrow V_1 = V_3 - V_2$$

$$\text{Span}(V_1, V_2, V_3, V_4, V_5) = \text{Span}(V_1, V_2)$$

$$\underbrace{V_1 = \begin{bmatrix}1\\0\end{bmatrix}, V_2 = \begin{bmatrix}0\\1\end{bmatrix}} \quad \underbrace{V_3 = \begin{bmatrix}1\\1\end{bmatrix}, V_4 = \begin{bmatrix}0\\0\end{bmatrix}, V_5 = \begin{bmatrix}-2\\-2\end{bmatrix}}$$

$$V_3 = V_1 + V_2$$

$$v_3 = v_1 + v_2$$
$$v_4 = 0v_1 = 0v_2 = 0(v_1 + v_2)$$
$$v_5 : -2(v_1 + v_2) = -2v_1 - 2v_2$$

$$\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 + \cdots = 0$$

$$\frac{\alpha_1 v_1}{\alpha_1} = -\frac{\alpha_2 v_2}{\alpha_1} - \frac{\alpha_3 v_3}{\alpha_1} + \cdots$$

$$v_1 = $$

Geometric Interpretation of Span

Thursday, 6 March 2025 10:42 am

- Span of a single nonzero vector: a line through the origin
- Span of two linearly independent vectors: a plane through the origin
- Span of three linearly independent vectors in $\mathbb{R}^3$: all of $\mathbb{R}^3$

Linear Subspace

Thursday, 6 March 2025 10:45 am

A **linear subspace** of R^n is a subset that satisfies three key properties:

1. **Closure under addition:** If $u, v \in V$, then their sum $u + v$ must also be in V.

2. **Closure under scalar multiplication:** If $v \in V$ and $c \in R$, then the scaled vector cv must also be in V

3. **Contains the zero vector:** The **zero vector** 0 is always in V. This follows from property (2) by setting $c = 0$, which forces $0v = 0$.

Example 3

Thursday, 6 March 2025 10:50 am

Is $W = \{(x, y, z) \in R^3 : z = 2x - y\}$ a subspace of R^3?

(i) $u = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$, $v = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$

$z_1 = 2x_1 - y_1$

$z_2 = 2x_2 - y_2$

$z_3 = 2x_3 - y_3$ (True)

$\begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{bmatrix} \rightarrow z_1 + z_2 = (2x_1 - y_1) + (2x_2 - y_2)$

if $u \in W$

$v \in W$

$u + v \in W$

$z_3 = 2x_1 + 2x_2 - y_1 - y_2$

$= 2(x_1 + x_2) - (y_1 + y_2)$

$= 2x_3 - y_3 \in W$

(ii) $cu = \begin{bmatrix} cx_1 \\ cy_1 \\ cz_1 \end{bmatrix} \rightarrow$

$z_1 = 2x_1 - y_1$

$cz_1 = c(2x_1 - y_1)$

$= 2(cx_1) - (cy_1) \in W$

(iii) $\vec{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \in W$ $\quad \overset{0}{z} = 2\overset{0}{x} - \overset{0}{y}$

Problem 3

Thursday, 6 March 2025 10:52 am

Is $U = \{(x, y, z) \in R^3 : z = 2x - y + 1\}$ a subspace of R^3?

(ii) $u = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$, $v = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$ $u + v \in U$

$\begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{bmatrix}$

$z_1 = 2x_1 - y_1 + 1$

$z_2 = 2x_2 - y_2 + 1$ +

$z_1 + z_2 = 2(x_1 + x_2) - (y_1 + y_2) + 2$

$z_3 = 2x_3 - y_3 + 2$

S_3: $xy = 0$ R^2

(i) $u = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$, $v = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$ $x_1 y_1 = 0$ $x_2 y_2 = 0$

$\begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{bmatrix}$ $\notin S_3$ $(x_1 + x_2)(y_1 + y_2) = 0$

$x_3 y_3 = 0$

$\underbrace{x_1 y_1}_{0} + \underbrace{x_1 y_2} + \underbrace{x_2 y_1}_{0} + \underbrace{x_2 y_2}_{=0}$

$\begin{array}{cc} x_1 & y_1 \\ (0, & 1) \end{array}$, $\begin{array}{cc} x_2 & y_2 \\ (1, & 0) \end{array}$

$cu = \begin{bmatrix} cx_1 \\ cy_1 \\ cz_1 \end{bmatrix}$ $(cx_1)(cy_1) = 0$

$c^2\underbrace{(x_1 y_1)}_{0} = 0$ $x_3 y_3 = 0$

$(x_1 + x_2)$ $(y_1 + y_2)$

$$x^2 y = 1 \qquad R^3$$

$$xyz = 0 \qquad R^3$$

$$x_3 y_3 z_3 \overset{\text{is}}{=} 0 \qquad (x_1 + x_2)(y_1 + y_2)(z_1 + z_3) = 0$$

$$x_1 y_1 + \overbrace{x_2 y_0} + \cdot$$

(i) $\quad x = 2y \qquad \vec{0} \in S_4$

(ii) $$\begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{bmatrix}$$

$$x_1 = 2y_1$$
$$x_2 = 2y_2$$

$$x_1 + x_2 = 2(y_1 + y_2)$$
$$x_3 = 2y_3$$

$$c \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} cx_1 \\ cy_1 \\ cz_1 \end{bmatrix} \qquad cx_1 = c(2y_1) = 2(cy_1)$$

Example 4: Subspace from Orthogonality

Thursday, 6 March 2025 10:53 am

Find a description of the set of all vectors in $\mathbb{R}^4$ that are perpendicular to $v = \begin{pmatrix} 2 \\ 3 \\ 1 \\ 7 \end{pmatrix}$.

$G = \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix}$ $\qquad G \cdot V = 0$

$$\underbrace{2w + 3x + y + 7z = 0}$$

$S: \quad y = -2w - 3x - 7z$

(i) $\vec{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \in S$

$y_1 = -2w_1 - 3x_1 - 7z_1$

(ii) Let $u = \begin{bmatrix} w_1 \\ x_1 \\ y_1 \\ z_1 \end{bmatrix}$, $V = \begin{bmatrix} w_2 \\ x_2 \\ y_2 \\ z_2 \end{bmatrix}$ $\quad u + v = \begin{bmatrix} w_1 + w_2 \\ x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{bmatrix}$

$y_2 = -2w_2 - 3x_1 - 7z_2$

$y_1 + y_2 = -2(w_1 + w_2) - 3(x_1 + x_2) - 7(z_1 + z_2)$

Dimension and Basis

Thursday, 6 March 2025 10:55 am

The dimension of a nonzero subspace V, denoted $\dim(V)$, is the minimum number of vectors needed to span V.

By convention, $\dim(\{0\}) = 0$.

For $\mathbb{R}^n$, $\dim(\mathbb{R}^n) = n$.

If V and W are subspaces with $W \subset V$ (i.e., W is contained in V), then $\dim(W) \leq \dim(V)$, with equality if and only if $W = V$.

A basis for a nonzero subspace V in $\mathbb{R}^n$ is a spanning set for V consisting of exactly $\dim(V)$ vectors.

Example 5

Thursday, 6 March 2025 10:56 am

What is the dimension of the subspace $W = \{(x, y, z) \in \mathbb{R}^3 : z = 2x - y\}$?

$z = 2x - y$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ 2x - y \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

Dimensio = 2

Basis = $\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right.$

Problem 4

Thursday, 6 March 2025 10:57 am

Find the dimension of the subspace $V = \text{span}\left(\begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}2\\4\\6\end{pmatrix}, \begin{pmatrix}3\\6\\9\end{pmatrix}\right)$.

v_1 v_2 v_3

$v_2 = 2v_1$

$v_3 = 3v_1$

Dim = 1

Basis = $\begin{bmatrix}1\\2\\3\end{bmatrix}$

Problem 5

Thursday, 6 March 2025 10:58 am

Find the dimension of span $\left(\begin{pmatrix} 3/2 \\ -2 \end{pmatrix}, \begin{pmatrix} -2 \\ 8/3 \end{pmatrix}\right)$.

v_1 v_2

$v_1 = c v_2$

$\frac{3}{2} = -2c \Rightarrow c = -3/4$

$\begin{bmatrix} 3/2 \\ -2 \end{bmatrix} = \begin{bmatrix} -2c \\ 8/3\, c \end{bmatrix}$

$-2 = \frac{8}{3} c \Rightarrow c = -3/4$

$v_2 = c v_1$

Problem 6

Thursday, 6 March 2025 10:59 am

Find the dimension of span $\left(\begin{pmatrix} 2 \\ -4 \\ 5 \end{pmatrix}, \begin{pmatrix} 3 \\ -6 \\ 15 \end{pmatrix} \right)$.

v_1 v_2

Let $v_2 = c v_1$

$$\begin{bmatrix} 3 \\ -6 \\ 15 \end{bmatrix} = \begin{bmatrix} 2c \\ -4c \\ 5c \end{bmatrix}$$

$3 = 2c$

$c = 3/2$

$-6 = -4c$

$c = 3/2$

$15 = 5c$

$c = 3$

Problem 7

Thursday, 6 March 2025 10:59 am

Find the dimension of span $\left(\begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ -4 \\ 3 \end{pmatrix} \right)$.

$v_1 \quad v_2 \quad v_3$

$$v_3 = a v_1 + b v_2$$

$$\begin{bmatrix} 2 \\ -4 \\ 3 \end{bmatrix} = a \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} + b \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 3a \\ -a+2b \\ 2a-b \end{bmatrix}$$

$$2 = 3a \Rightarrow a = 2/3$$

$$-a + 2b = -4$$

$$-\frac{2}{3} + 2b = -4 \Rightarrow b = -5/3$$

$$2a - b = 3 \qquad 2\left(\frac{2}{3}\right) - b = 3$$

$$\frac{4}{3} - 3 = b$$

$$-5/3 = b$$

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Standard Basis

The standard basis for $\mathbb{R}^n$ consists of the vectors:

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \ldots, e_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

For example, the standard basis for $\mathbb{R}^3$ is:

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Any vector $v = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$ in $\mathbb{R}^n$ can be written as $v = v_1 e_1 + v_2 e_2 + \cdots + v_n e_n$.

$$R^3 = \begin{array}{ccc} e_1 & e_2 & e_3 \end{array}$$
$$R^3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$v = \begin{bmatrix} -5 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -5 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}$$

$$= -5 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= -5e_1 + 2e_2 + 3e_3$$