Moment generating functions (mgf)

Reference: 1. Meyer PL. Introductory probability and statistical applications. Oxford and IBH Publishing; 1965.
2. Johnson, Richard A., Irwin Miller, and John E. Freund. "Probability and statistics for engineers." (2000).

- **Definition:**

The *moment generating function* of a r.v. X is defined by

$$M_X(t) = E(e^{tX}) = \begin{cases} \sum_i e^{tx_i} p_X(x_i) & \text{(discrete case)} \\ \int_{-\infty}^{\infty} e^{tx} f_X(x)\, dx & \text{(continuous case)} \end{cases} \tag{4.40}$$

where t is a real variable. Note that $M_X(t)$ may not exist for all r.v.'s X. In general, $M_X(t)$ will exist only for those values of t for which the sum or integral of Eq. (4.40) converges absolutely. Suppose that $M_X(t)$ exists. If we express e^{tX} formally and take expectation, then

$$\begin{aligned} M_X(t) = E(e^{tX}) &= E\left[1 + tX + \frac{1}{2!}(tX)^2 + \cdots + \frac{1}{k!}(tX)^k + \cdots\right] \\ &= 1 + tE(X) + \frac{t^2}{2!}E(X^2) + \cdots + \frac{t^k}{k!}E(X^k) + \cdots \end{aligned} \tag{4.41}$$

and the kth moment of X is given by

$$m_k = E(X^k) = M_X^{(k)}(0) \qquad k = 1, 2, \ldots \tag{4.42}$$

where

$$M_X^{(k)}(0) = \frac{d^k}{dt^k} M_X(t)\bigg|_{t=0} \tag{4.43}$$

NOTE:

1. $E(X)$ is coefficient of t In $M_X(t)$.
2. $E(X^2)$ is coefficient of $\frac{t^2}{2!}$ In $M_X(t)$.
3. $E(X^n)$ is coefficient of $\frac{t^n}{n!}$ In $M_X(t)$.
4. $E(X^n) = M_X^n(0)$

Example:

1. Suppose that X has pdf $f(x) = \frac{e^{-|x|}}{2}, -\infty < x < \infty$ then find E(X) and V(X) using mgf.
   Solution:

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} f(x)\, dx$$

$= \int_{-\infty}^{\infty} e^{tx} \frac{e^{-|x|}}{2}\, dx$

$= \int_{-\infty}^{0} e^{tx} \frac{e^{x}}{2}\, dx + \int_{0}^{\infty} e^{tx} \frac{e^{-x}}{2}\, dx = \int_{-\infty}^{0} \frac{e^{(1+t)x}}{2}\, dx + \int_{0}^{\infty} \frac{e^{-(1-t)x}}{2}\, dx$

$M_X(t) = \frac{1}{1-t^2}$, -1< t <1

$$M_X^1(0) = 0$$
$$M_X^2(0) = 2$$

E(X)= 0 and V(X)=2.

Or: $M_X(t) = \frac{1}{1-t^2} = 1 + t^2 + t^4 + t^6 + \cdots = 1 + \frac{2t^2}{2} + t^4 + t^6 + \cdots$

Cofficient of "t" E(X)= 0. Cofficient of "$\frac{t^2}{2}$" V(X)= 2-0=2

2. Let X be a random variable taking the values 0,1,2,… and $f(x) = ab^x, a, b > 0$ & a+b=1. Find mgf of X. If $E(X) = m_1, E(X^2) = m_2$ then S.T $m_2 = m_1(2m_1 + 1)$.
Solution:
$M_X(t) = \sum_0^\infty ab^x e^{tx} = a\sum_0^\infty (be^t)^x =$ a $\frac{1}{1-be^t} = \frac{a}{1-be^t}$

$$E(X) = M_X^1(0) = \frac{ab}{(1-b)^2}$$

$$E(X^2) = M_X^2(0) = \frac{(1+b)ab}{(1-b)^3}$$

Given,
$E(X) = m_1, E(X^2) = m_2$
To Prove, $m_2 = m_1(2m_1 + 1)$.
Consider, $m_1(2m_1 + 1) = \frac{ab}{(1-b)^2}\left(2\frac{ab}{(1-b)^2} + 1\right) = \frac{ab}{(1-b)^2}\left(\frac{2ab+1+b^2-2b}{(1-b)^2}\right)$
$= \frac{ab}{(1-b)^4}(2b(a-1) + 1 + b^2) = = \frac{ab}{(1-b)^4}(2b(-b) + 1 + b^2) = m_2.$

Moment generating function for binomial distribution

Let X have the binomial distribution with probability distribution

$$b(x \mid n, p) = \binom{n}{x} p^x (1-p)^{n-x} \qquad \text{for } x = 0, 1, \ldots, n$$

Show that

(a) $M(t) = (1 - p + pe^t)^n$ for all t

(b) $E(X) = np$ and $Var(X) = np(1-p)$

Solution **(a)** By definition of the moment generating function

$$M(t) = \sum_{x=0}^{n} e^{tx} \binom{n}{x} p^x (1-p)^{n-x}$$

$$= \sum_{x=0}^{n} \binom{n}{x} (e^t p)^x (1-p)^{n-x}$$

$$= (pe^t + 1 - p)^n \qquad \text{for all } t$$

where we have used the binomial formula

$$(a+b)^n = \sum_{x=0}^{n} \binom{n}{x} a^x b^{n-x}$$

(b) Differentiating $M(t)$, we find

$$M'(t) = npe^t(pe^t + 1 - p)^{n-1}$$
$$M''(t) = (n-1)np^2 e^{2t}(pe^t + 1 - p)^{n-2} + npe^t(pe^t + 1 - p)^{n-1}$$

Evaluating these derivatives at $t = 0$, we obtain the moments

$$E(X) = np$$
$$E(X^2) = (n-1)np^2 + np$$

Also, the variance is

$$Var(X) = E(X^2) - [E(X)]^2 = np(1-p) \qquad ■$$

Moment generating function for Poisson distribution

Let X have the Poisson distribution with probability distribution

$$f(x) = \frac{\lambda^x}{x!} e^{-\lambda} \qquad \text{for } x = 0, 1, \ldots, \infty$$

Show that

(a) $M(t) = e^{\lambda(e^t - 1)}$ for all t

(b) $E(X) = \lambda$ and $Var(X) = \lambda$

The mean and variance of the Poisson distribution are equal.

Solution **(a)** By definition of the moment generating function

$$M(t) = \sum_{x=0}^{\infty} e^{tx} \frac{\lambda^x}{x!} e^{-\lambda} = \sum_{x=0}^{\infty} \frac{(\lambda e^t)^x}{x!} e^{-\lambda}$$

$$= e^{-\lambda} e^{\lambda e^t} = e^{\lambda(e^t - 1)} \qquad \text{for } -\infty < t < \infty$$

where we have used the series $e^y = \sum_{k=0}^{\infty} \frac{y^k}{k!}$

(b) Differentiating $M(t)$, we find

$$M'(t) = \lambda e^t e^{\lambda(e^t - 1)}$$
$$M''(t) = \lambda e^t e^{\lambda(e^t - 1)} + \lambda^2 e^{2t} e^{\lambda(e^t - 1)}$$

Evaluating these derivatives at $t = 0$, we obtain the moments

$$E(X) = \lambda$$
$$E(X^2) = \lambda + \lambda^2$$

Also, the variance is

$$Var(X) = E(X^2) - [E(X)]^2 = \lambda$$

■

Moment generating function for Gamma Distribution

$$f(x) = \begin{cases} \dfrac{x^{r-1} e^{-\alpha x} \alpha^r}{\Gamma(r)}, & x > 0,\ \alpha, r > 0 \\ 0, & elsewhere \end{cases}$$

$M_X(t) = \int_{-\infty}^{\infty} e^{tx}\ f(x)\, dx = \frac{\alpha^r}{\Gamma(r)} \int_0^{\infty} e^{-(\alpha - t)x}\ \ x^{r-1}\, dx$

Substitute, $x(\alpha - t) = v$ then $dx = \frac{dv}{\alpha - t}$

$M_X(t) = \frac{\alpha^r}{\Gamma(r)} \int_0^{\infty} e^{-v}\ \left(\frac{v}{\alpha - t}\right)^{r-1} \frac{dv}{\alpha - t}$

$= \frac{\alpha^r}{\Gamma(r)(\alpha - t)^r} \int_0^{\infty} e^{-v}\ \ (v)^{r-1}\, dv$

$= \frac{\alpha^r}{\Gamma(r)(\alpha - t)^r} \Gamma(r) = \left(\frac{\alpha}{\alpha - t}\right)^r$

$$E(X) = \frac{r}{\alpha}$$

$$E(x^2) = \frac{r(r+1)}{\alpha^2}$$

$$V(X) = \frac{r}{\alpha^2}$$

Moment generating function for Exponential Distribution

Note: When we sub r=1 in gamma distribution we get exponential distribution.

$$f(x)=\begin{cases}\lambda e^{-\lambda x}, & x>0\\ 0, & otherwise\end{cases}$$

$$M_X(t) = \int_0^{\infty} e^{tx}\, f(x)\, dx = \lambda \int_0^{\infty} e^{-(\lambda-t)x}\, dx$$

$$M_X(t) = \left(\frac{\lambda}{\lambda-t}\right),\ t<\lambda$$

$$E(X) = \frac{1}{\lambda}$$
$$V(X) = \frac{1}{\lambda^2}$$

Or:

EXAMPLE 10.4. Suppose that X has an *exponential distribution* with parameter α. Therefore

$$M_X(t) = \int_0^{\infty} e^{tx}\alpha e^{-\alpha x}\, dx = \alpha \int_0^{\infty} e^{x(t-\alpha)}\, dx.$$

(This integral converges only if $t<\alpha$. Hence the mgf exists only for those values of t. Assuming that this condition is satisfied, we shall proceed.) Thus

$$\begin{aligned} M_X(t) &= \frac{\alpha}{t-\alpha} e^{x(t-\alpha)}\Big|_0^{\infty} \\ &= \frac{\alpha}{\alpha-t}, \quad t<\alpha. \end{aligned} \tag{10.7}$$

Moment generating function for chi square Distribution

Special case of Gamma distribution: $r=\frac{n}{2}$ and $\alpha=\frac{1}{2}$ in Γ function we get χ^2distribution.

A continuous random variable X is said to have a chi-square distribution if its PDF is given by.

$$f(x)=\begin{cases}\dfrac{x^{\frac{n}{2}-1}e^{-\frac{x}{2}}}{\Gamma(n/2)2^{\frac{n}{2}}}, & x>0\\ 0, & elsewhere\end{cases}$$

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx}\, f(x)\, dx$$
$$= \frac{\alpha^r}{\Gamma\left(\frac{n}{2}\right)2^{\frac{n}{2}}} \int_{-\infty}^{\infty} e^{\left(t-\frac{1}{2}\right)x}\ x^{\frac{n}{2}-1}\, dx$$
$$M_X(t) = (1-2t)^{-\frac{n}{2}}$$

$E(X)$ =n and V(X)= 2n

Moment generating function for Uniform Distribution

$$\boldsymbol{f(x)=\begin{cases}\dfrac{1}{(b-a)} & a\le x\le b\\ 0 & else\ where\end{cases}}$$

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx}\, f(x)\, dx$$
$$= \frac{1}{b-a}\int_{-\infty}^{\infty} e^{tx}\ dx$$
$$= \frac{1}{t(b-a)}\left(e^{bt}-e^{at}\right)$$

Expand and get the suitable coefficient in the expanssion to find E(X) and V(X).

Mean E(X) = $\frac{(a+b)}{2}$

E(X^2) = $\frac{(a^2+b^2+ab)}{3}$

Variance V(X) = E(X^2) – $[E(X)]^2$ = $\frac{(b-a)^2}{12}$

Moment generating function for normal distribution

Show that the normal distribution, whose probability density function

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2} \quad \text{has} \quad M(t) = e^{t\mu + \frac{1}{2}t^2\sigma^2}$$

which exists for all t. Also, verify the first two moments.

By substituting, $v = \left[\frac{[x-(t\sigma^2+\mu]^2}{\sigma^2}\right]$ *and* $dx = \sigma^2 dv$ and $\int_{-\infty}^{\infty} e^{-x^2} dx = \Gamma(\frac{1}{2}) = \sqrt{\pi}$

Solution:

$$M_X(t) = \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{+\infty} e^{tx} \exp\left(-\frac{1}{2}\left[\frac{x-\mu}{\sigma}\right]^2\right) dx.$$

Let $(x-\mu)/\sigma = s$; thus $x = \sigma s + \mu$ and $dx = \sigma\, ds$. Therefore

$$M_X(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \exp\left[t(\sigma s + \mu)\right] e^{-s^2/2}\, ds$$

$$= e^{t\mu} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \exp\left(-\tfrac{1}{2}[s^2 - 2\sigma t s]\right) ds$$

$$= e^{t\mu} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \exp\left\{-\tfrac{1}{2}[(s-\sigma t)^2 - \sigma^2 t^2]\right\} ds$$

$$= e^{t\mu + \sigma^2 t^2/2} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \exp\left(-\tfrac{1}{2}[s - \sigma t]^2\right) ds.$$

Let $s - \sigma t = v$; then $ds = dv$ and we obtain

$$M_X(t) = e^{t\mu + \sigma^2 t^2/2} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-v^2/2}\, dv$$

$$= e^{(t\mu + \sigma^2 t^2/2)}.$$

To obtain the moments of the normal, we differentiate once to obtain

$$M'(t) = e^{t\mu + \frac{1}{2}t^2\sigma^2}(\mu + t\sigma^2)$$

and a second time to get

$$M''(t) = e^{t\mu + \frac{1}{2}t^2\sigma^2}[(\mu + t\sigma^2)^2 + \sigma^2].$$

Setting $t = 0$,

$$E[X] = M'(0) = \mu \quad \text{and} \quad E(X^2) = M''(0) = \sigma^2 + \mu^2$$

so $Var(X) = \sigma^2$ as the notation suggests. ■

A basic property relates the moment generating function of $a + bX$ to that of X.

MGF of some standard distributions:

1. **Binomial Distributions:** $M_X(t) = M_X(t) = (pe^t + q)^n$
2. **Poisson Distributions:** $M_X(t) = e^{\propto(e^t-1)}$
3. **Normal Distributions:** $M_X(t) = e^{t\mu+\frac{\sigma^2 t^2}{2}}$
4. **Exponential Distributions:** $M_X(t) = \frac{\propto}{\propto - t}$
5. **Gamma Distributions:** $M_X(t) = \frac{\propto^r}{(\propto - t)^r}$
6. **Chi square Distributions:** $M_X(t) = (1-2t)^{-n/2}$

Properties of mgf

Theorem 10.2. Suppose that the random variable X has mgf M_X. Let $Y = \alpha X + \beta$. Then M_Y, the mgf of the random variable Y, is given by

$$M_Y(t) = e^{\beta t} M_X(\alpha t). \tag{10.12}$$

In words: To find the mgf of $Y = \alpha X + \beta$, evaluate the mgf of X at αt (instead of t) and multiply by $e^{\beta t}$.

Proof

$$\begin{aligned} M_Y(t) &= E(e^{Yt}) = E[e^{(\alpha X+\beta)t}] \\ &= e^{\beta t} E(e^{\alpha t X}) = e^{\beta t} M_X(\alpha t). \end{aligned}$$

Theorem 10.3. Let X and Y be two random variables with mgf's, $M_X(t)$ and $M_Y(t)$, respectively. If $M_X(t) = M_Y(t)$ for all values of t, then X and Y have the same probability distribution.

Theorem 10.4. Suppose that X and Y are independent random variables. Let $Z = X + Y$. Let $M_X(t)$, $M_Y(t)$, and $M_Z(t)$ be the mgf's of the random variables X, Y, and Z, respectively. Then

$$M_Z(t) = M_X(t) M_Y(t). \tag{10.13}$$

Proof

$$\begin{aligned} M_Z(t) &= E(e^{Zt}) = E[e^{(X+Y)t}] = E(e^{Xt}e^{Yt}) \\ &= E(e^{Xt})E(e^{Yt}) = M_X(t)M_Y(t). \end{aligned}$$

Note: This theorem may be generalized as follows: If $X_1, \ldots, X_n$ are independent random variables with mgf's M_{X_i}, $i = 1, 2, \ldots, n$, then M_Z, the mgf of

$$Z = X_1 + \cdots + X_n,$$

is given by

$$M_Z(t) = M_{X_1}(t) \cdots M_{X_n}(t). \tag{10.14}$$

EXAMPLE 10.10. Suppose that X has distribution $N(\mu, \sigma^2)$. Let $Y = \alpha X + \beta$. Then Y is again normally distributed. From Theorem 10.2, the mgf of Y is $M_Y(t) = e^{\beta t} M_X(\alpha t)$. However, from Example 10.8 we have that

$$M_X(t) = e^{\mu t + \sigma^2 t^2/2}.$$

Hence

$$\begin{aligned} M_Y(t) &= e^{\beta t}[e^{\alpha\mu t + (\alpha\sigma)^2 t^2/2}] \\ &= e^{(\beta + \alpha\mu)t} e^{(\alpha\sigma)^2 t^2/2}. \end{aligned}$$

Reproductive Properties of mgf

EXAMPLE 10.11. Suppose that X and Y are independent random variables with distributions $N(\mu_1, \sigma_1^2)$ and $N(\mu_2, \sigma_2^2)$, respectively. Let $Z = X + Y$. Hence

$$\begin{aligned} M_Z(t) = M_X(t)M_Y(t) &= \exp(\mu_1 t + \sigma_1^2 t^2/2) \exp(\mu_2 t + \sigma_2^2 t^2/2) \\ &= \exp[(\mu_1 + \mu_2)t + (\sigma_1^2 + \sigma_2^2)t^2/2]. \end{aligned}$$

Theorem 10.5 (*the reproductive property of the normal distribution*). Let $X_1, X_2, \ldots, X_n$ be n independent random variables with distribution $N(\mu_i, \sigma_i^2)$, $i = 1, 2, \ldots, n$. Let $Z = X_1 + \cdots + X_n$. Then Z has distribution $N(\sum_{i=1}^n \mu_i, \sum_{i=1}^n \sigma_i^2)$.

The Poisson distribution also possesses a reproductive property.

Theorem 10.6. Let $X_1, \ldots, X_n$ be independent random variables. Suppose that X_i has a Poisson distribution with parameter α_i, $i = 1, 2, \ldots, n$. Let $Z = X_1 + \cdots + X_n$. Then Z has a Poisson distribution with parameter

$$\alpha = \alpha_1 + \cdots + \alpha_n.$$

Theorem 10.7. Suppose that the distribution of X_i is $\chi^2_{n_i}$, $i = 1, 2, \ldots, k$, where the X_i's are independent random variables. Let $Z = X_1 + \cdots + X_k$. Then Z has distribution χ^2_n, where $n = n_1 + \cdots + n_k$.

Proof: From Eq. (10.10) we have $M_{X_i}(t) = (1 - 2t)^{-n_i/2}$, $i = 1, 2, \ldots, k$. Hence

$$M_Z(t) = M_{X_1}(t) \cdots M_{X_k}(t) = (1 - 2t)^{-(n_1 + \cdots + n_k)/2}.$$

Problems:

1. Find the mgf of random variable X which is uniformly distributed with an interval (-a, a) and hence find $E(X^{2n})$.

Solution: $X\sim U(-a,a) = \frac{e^{at}-e^{-at}}{(a+a)t}$ $\quad$ *we know,* $M_X(t) = \frac{1}{t(b-a)}(e^{bt}-e^{at})$

$= \frac{e^{at}-e^{-at}}{(2a)t}$, by expanding

$E(X^{2n}) = coefficient\ of\ \frac{t^{2n}}{(2n)!} = \frac{a^{2n}}{(2n+1)}$

2. If X is normally distributed with mean μ and variance, σ^2 then show that $E(X-\mu)^{2n} = 1.3.5 \dots .(2n-1)\sigma^{2n}$.

Solution: Given $X\sim N(\mu,\sigma^2), M_X(t) = e^{\mu t+\frac{\sigma^2 t^2}{2}}$

Let Y= $(X-\mu)$

To get E(Y^{2n})= The coefficient of $\frac{t^{2n}}{(2n)!}$ in $M_Y(t)$.

$M_Y(t) = E\,(e^{ty})= E\left(e^{t(x-\mu)}\right)$

$= E(e^{tx})E(e^{-\mu t})$

$= M_X(t)\,E(e^{-\mu t})$

$M_Y(t) = e^{\mu t+\frac{\sigma^2 t^2}{2}}\ e^{-\mu t}= e^{\frac{\sigma^2 t^2}{2}} = 1 + \left(\frac{\sigma^2 t^2}{2}\right) + \frac{1}{2!}\left(\frac{\sigma^2 t^2}{2}\right)^2 + \frac{1}{3!}\left(\frac{\sigma^2 t^2}{2}\right)^3 + \cdots$

E(Y^{2n})= The coefficient of $\frac{t^{2n}}{(2n)!}$ in $M_Y(t)$= $\frac{\sigma^{2n}(2n)!}{2^n n!} = \frac{(2n)!}{2^n n!} * \sigma^{2n}$

$$= \frac{(2n)(2n-1)(2n-2)\ldots(3)(2)(1)}{(2n)(2n-2)(2n-4)\ldots(6)(4)(2)} * \sigma^{2n}$$

$= 1.3.5 \dots .(2n-1)\sigma^{2n}$.

3. Let $X_1\sim\chi^2(3), X_2\sim\chi^2(5)$ and Z= $X_1 + X_2$ where X_1 *and* X_2 *independent* random variables. Find $M_Z(t)$ and $V(Z)$ *and pdf of* Z.

Solution:

$X_1\sim\chi^2(3)$ gives $M_{X_1}(t) = (1-2t)^{-\frac{3}{2}}$

$X_2\sim\chi^2(5)$ gives $M_{X_2}(t) = (1-2t)^{-\frac{5}{2}}$

$$M_Z(t) = M_{X_1}(t)M_{X_2}(t) = (1-2t)^{-\frac{8}{2}}\sim\chi^2(8)$$

Therefore, $V(Z) = 2n = 2\,(8) = 16$. Where $n = 8$.

Chi-square Distribution $X\sim\chi^2(n)$	$f(x)=\begin{cases} \frac{x^{\frac{n}{2}-1}e^{-\frac{x}{2}}}{\Gamma(n/2)2^{\frac{n}{2}}}, & x>0 \\ 0, & elsewhere \end{cases}$

$$f(Z)=\begin{cases} \frac{Z^{\frac{n}{2}-1}e^{-\frac{Z}{2}}}{2^{\frac{n}{2}}\Gamma(n/2)}, & Z>0 \\ 0, & Otherwise \end{cases} = \begin{cases} \frac{Z^3 e^{-\frac{Z}{2}}}{2^4\Gamma(4)}, & Z>0 \\ 0, & Otherwise \end{cases}$$

4. Let X_1, X_2 *and* X_3 *are* 3 independent random variable having normal distributions with parameter (4, 1), (5,2), (7, 3) respectively. Let $Y = 2X_1 + 2X_2 + X_3$. Find the pdf of $V = \left(\frac{Y-\mu}{\sigma}\right)^2$, where μ *and* σ are the mean and standard deviation of Y.

Solution:

$$M_X(t) = e^{\mu t + \frac{\sigma^2 t^2}{2}}$$

$$Y \sim N(2 * 4 + 2 * 5 + 1 * 7, \quad 2^2 * 1 + 2^2 * 2 + 1 * 3)$$

$$Y \sim N(25, 15)$$

$$M_Y(t) = e^{25t + \frac{15t^2}{2}}$$

We have,

If $X \sim N(\mu, \sigma^2)$, then show that $Z = \frac{X-\mu}{\sigma} \sim N(0,1)$ and $Y = Z^2 \sim \chi^2(1)$.

Chi-square Distribution $X \sim \chi^2(n)$	$f(x) = \begin{cases} \dfrac{x^{\frac{n}{2}-1} e^{-\frac{x}{2}}}{\Gamma(n/2) 2^{\frac{n}{2}}}, & x > 0 \\ 0, & elsewhere \end{cases}$

Therefore, $V = \left(\frac{Y-25}{\sqrt{15}}\right)^2 = Z^2 \sim \chi^2(1)$

$$f(V) = \begin{cases} \dfrac{e^{-\frac{v}{2}} v^{-\frac{1}{2}}}{2^{\frac{1}{2}} \Gamma(\frac{1}{2})}, & v > 0 \\ 0, & Otherwise \end{cases} = \begin{cases} \dfrac{e^{-\frac{v}{2}} v^{-\frac{1}{2}}}{\sqrt{\pi}\, 2^{\frac{1}{2}}}, & v > 0 \\ 0, & Otherwise \end{cases}$$

5. If a random variable X has mgf $M_X(t) = \frac{3}{3-t}$ then find the standard deviation of the random variable X.

Solution:

$$M_X(t) = \frac{3}{3-t} = \frac{3}{3\left(1-\frac{t}{3}\right)} = \frac{1}{\left(1-\frac{t}{3}\right)} = \left(1 - \frac{t}{3}\right)^{-1} = 1 + \frac{t}{3} + \left(\frac{t}{3}\right)^2 + \cdots$$

$$E(X) = coef\ of\ t\ in\ M_X(t) = \frac{1}{3}$$

$$E(X^2) = coef\ of \frac{t^2}{2!}\ in\ M_X(t) = \frac{1}{9}\ 2! = \frac{2}{9}$$

$$V(X) = \frac{1}{9}\ .Hence, \sigma = \frac{1}{3}$$

6. Let X be random variable having probability mass function p(x=k) = $p(1-p)^{k-1}$, $k = 1,2,3 \ldots n$. Find $M_X(t)$ and V(X).

Solution:

$$M_X(t) = \sum_1^n e^{tx} p(X = x)$$

$$= \sum_1^n e^{tx} p(1-p)^{x-1}$$

$$= \frac{p}{p-1} \sum_1^n e^{tx} (1-p)^x$$

$= \frac{p}{1-p} \sum_1^n (e^t(1-p))^x$, Expanding

$$= \frac{p}{1-p} \{(e^t(1-p)) + (e^t(1-p))^2 + \cdots\}$$

$$= \frac{p}{1-p} (e^t(1-p))\{1 + (e^t(1-p)) + (e^t(1-p))^2 + \cdots\}$$

$$M_X(t) = \frac{p}{1-p} * (e^t(1-p)) * \frac{1}{1-e^t(1-p)} = \frac{pe^t}{1-e^t(1-p)}$$

$M_X^1(t) = \frac{pe^t}{[1-e^t(1-p)]^2}$,

at t=o, E(X)= $\frac{1}{p}$

and

$$E(X^2) = \frac{1}{p} + \frac{2(1-p)}{p^2}$$

$$V(X) = \frac{1-p}{p^2}$$

7. If X has pdf f(x)= $\lambda e^{-\lambda(x-a)}, X \geq a$, find the mgf of X and hence find V(X).
   Ans: Given is an exponential distribution.

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} f(x)\, dx$$

$$= \int_a^{\infty} e^{tx} \lambda e^{-\lambda(x-a)}\, dx$$

$$= \lambda \int_a^{\infty} e^{tx} e^{-\lambda(x-a)}\, dx$$

$$= \lambda e^{\lambda a} \int_a^{\infty} e^{-(\lambda-t)x}\, dx$$

$$= \lambda e^{\lambda a} \frac{e^{-(\lambda-t)x}}{-(\lambda-t)}$$

$$= \lambda e^{\lambda a} \frac{e^{-(\lambda-t)a}}{(\lambda-t)} = \frac{\lambda e^{at}}{\lambda-t}, \lambda > t$$

   E(X)= $a + \frac{1}{\lambda}$ and V(X)= $\frac{1}{\lambda^2}$

8. If the mgf of discrete random variable is $e^{4(e^t-1)}$ then find P($X = \mu + \sigma$) where μ and σ are the mean and S.D of X.
   Solution: $M_X(t) = e^{4(e^t-1)}$
   $X \sim p(\alpha)$ where $\alpha = 4$
   Therefore E(X)= V(X)= 4,
   Which gives $\mu = 4$ *and* $\sigma = 2$.
   P(X=k)= $\frac{e^{-\alpha}\alpha^k}{k!}$, k= 0,1,2,….,n
   For, $\alpha = 4$,
   P(X=k)= $\frac{e^{-4}4^k}{k!}$
   To find P($X = \mu + \sigma$)= P($X = 6$)= $\frac{e^{-4}4^6}{6!}$ = 0.1042.