Content

: Addition & Multiplication Rule

: Permutation Basics

: Combination Basics & Properties

: Inverse modulus revision

: $({}^{N}C_{R}) \% p$ — way 1, way 2

: $(a^{r!}) \% p$, given $r < p$

T|F Given 3 T/F questions, every question have to be answered T/F, how many ways we can answer all question?

⇒ 6, 3!, 9, 8,

T T T
T T F
T F T
T F F
F T T
F T F
F F T
F F F

$\Big\}$ 8

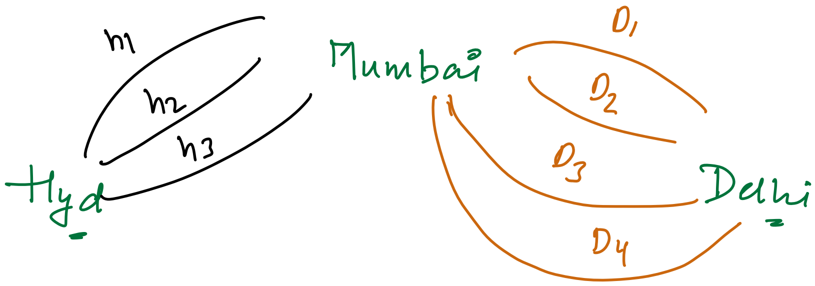
$\underset{T\ F}{\underline{2}} \times \underset{T\ F}{\underline{2}} \times \underset{F\ F}{\underline{2}} = 8$ possibilities

Ex1: Given 10 Girls & 7 Boys, How many different pair are possible

1 Girl and 1 Boy
(10) (7)

10 * 7 = 70 pairs

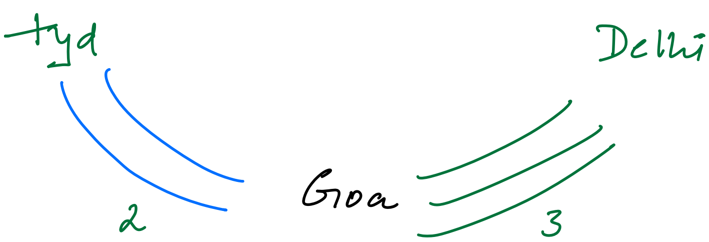
Ex2:



Hyd to Delhi via Mumbai

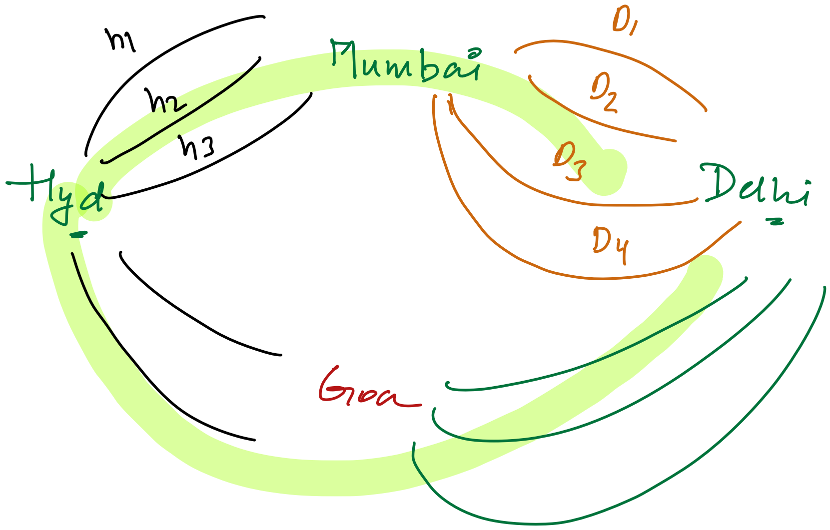
$\underbrace{\text{Hyd} \rightarrow \text{Mumbai}}_{\#\text{ways} = 3}$ and $\underbrace{\text{Mumbai} \rightarrow \text{Delhi}}_{\#\text{ways} = 4}$ $\Rightarrow$ Total = 12

Ex3:



Hyd to Delhi via Goa = 6

Ex4:



// Hyd to Delhi via Mumbai = 12

or → +

Hyd to Delhi via Goa = 6

} 18

// you need to gift

(pen & book) or (flowers & chocolates) & ring

$\underbrace{\text{pen}}_{3}$ $\underbrace{\text{book}}_{5}$ $\underbrace{\text{flowers}}_{7}$ $\underbrace{\text{chocolates}}_{3}$ $\underbrace{\text{ring}}_{3}$

// Total ways = 15 + 21 + 3 ⇒ 39

Permutations : arrangement of Objects

In general : $(i, j) == (j, i)$

$(i, j) \neq (j, i)$

order of arrangement matters.

Q1) Given 3 distinct characters?

how many ways we can arrange them

S = " a c d "

$\underline{3} \times \underline{2} \times \underline{1} \rightarrow 6$ ways

a c d

a d c

c a d

c d a

d a c

d c a

$\Bigg\}$ 6 ways

a → {c, d}
c → {a, d}
d → {a, c}

// If Given N distinct characters how many ways can we arrangements?

$\underline{N}\ \underline{N-1}\ \underline{N-2}\ \ldots\ \underline{\ }\ \underline{\ }\ \underline{\ }\ \underline{\ }\ \underline{1}$

$\rightarrow N * N-1 * N-2 \ldots 1 = N!$

Q2) = a b c d :

$\left.\underline{4} \times \underline{3} \times \underline{2} \times \underline{1}\right\} = 4!$

// Given 5 distinct characters, how many ways we can arrange 2 characters

a b c d e

5 4 → 20 cases

a → {b, c, d, e} :

b → {a, c, d, e} :

c → {a, b, d, e} :

d → {a, b, c, e} :

e → {a, b, c, d} :

5 distinct character, 3 characters

5 4 3 → 12*5 = 60

4 4

N distinct character, 3 character?

N N-1 N-2 → (N)(N-1)(N-2)

// N distinct characters, arrange r characters.

$$(N)(N-1)(N-2)\ldots(N-r+1) * \{(N-r)*(N-r-1)*(N-r-2)\ldots 1^*\}$$

$$(N-r)*(N-r-1)*(N-r-2)\ldots 1^*$$

$$\boxed{{}^{N}P_r = \frac{N!}{(N-r)!}}$$

Combination: Selection of objects. $(i, j) == (j, i)$

$B_1 \quad B_2 \quad B_3 \quad B_4$ } Number of we can select 3 Boys

$B_1 \; B_2 \; B_3 = B_3 \; B_1 \; B_2$ Both are same

$B_1 \; B_2 \; B_4$

$B_1 \; B_3 \; B_4$ = 4 ways

$B_2 \; B_3 \; B_4$

// Number of ways we can arrange 3 boys from 4 boys.

$B_1 \; B_2 \; B_3$	$B_1 \; B_2 \; B_4$	$B_1 \; B_3 \; B_4$	$B_2 \; B_3 \; B_4$
$B_1 \; B_3 \; B_2$	$B_1 \; B_4 \; B_2$	$B_1 \; B_4 \; B_3$	$B_2 \; B_4 \; B_3$
$B_2 \; B_1 \; B_3$	$B_2 \; B_1 \; B_4$	$B_3 \; B_1 \; B_4$	$B_3 \; B_2 \; B_4$
$B_2 \; B_3 \; B_1$	$B_2 \; B_4 \; B_1$	$B_3 \; B_4 \; B_1$	$B_3 \; B_4 \; B_2$
$B_3 \; B_1 \; B_2$	$B_4 \; B_1 \; B_2$	$B_4 \; B_1 \; B_3$	$B_4 \; B_2 \; B_3$
$B_3 \; B_2 \; B_1$	$B_4 \; B_2 \; B_1$	$B_4 \; B_3 \; B_1$	$B_4 \; B_3 \; B_2$
$\{B_1, B_2, B_3\}$	$\{B_1, B_2, B_4\}$	$\{B_1, B_3, B_4\}$	$\{B_2, B_3, B_4\}$

Total arrangement = 24 => $4!/(4-3)!$

Total selections = $24/3!$ = $\underline{4}$

Given N, how many ways arrange r = $\frac{N!}{(N-r)!}$
^NP_r =

Given N, how many ways select r = $\frac{N!}{(N-r)!\,r!}$
^NC_R =

$$\boxed{^NC_R = \frac{^NP_r}{r!}}$$

$$^NC_R = \frac{N!}{(N-R)!\,(R)!} \qquad ^NP_R = \frac{N!}{(N-R)!}$$

// Properties of ^NC_R

$$^NC_1 = \frac{N!}{(N-1)!\,(1)!}$$

$0! = 1$

$$^NC_0 = \frac{N!}{(N-N)!\,(0)!} \Rightarrow \frac{\cancel{N!}}{\cancel{0!}\,\cancel{N!}} \Rightarrow 1$$

↳ Empty Set

$r! \Rightarrow 1$

$$\frac{N!}{(N-r)!}$$

n

→ N items

$$\text{N items} \xrightarrow{0} {}^NC_0,\quad \xrightarrow{1} {}^NC_1,\quad \xrightarrow{2} {}^NC_2,\quad {}^NC_3 \quad \dots \quad {}^NC_N$$

$$ {}^NC_0 + {}^NC_1 + {}^NC_2 + {}^NC_3 + \dots {}^NC_N = 2^N $$

All Selections:

How many ways we can select = 0, 1, 2, 3, .. N

// N Elements:

$$\{\ a_1 \mid a_2 \mid a_3 \mid a_4 \mid \dots \quad a_N \mid \}$$

$$\ \ 2 \quad\ \ 2 \quad\ \ 2 \quad\ \ 2 \qquad\qquad 2$$

Set

$\longrightarrow 2^N$ scenario

a_1 | a_2 | a_3

2 | 2 | 2 $\Rightarrow$ 8 cases

$\Rightarrow \{ \ \} \Rightarrow {}^3C_0$

$\{a_1\}$

$\{a_2\}$ $\Rightarrow {}^3C_1$

$\{a_3\}$

$\{a_1\ a_2\}$

$\{a_2\ a_3\}$ ${}^3C_2$

$\{a_1\ a_3\}$

$\{a_1\ a_2\ a_3\}$ ${}^3C_3$

$${}^3C_0 + {}^3C_1 + {}^3C_2 + {}^3C_3 = 2^3$$

// N, how may ways we can select r items.

$a_1 \quad a_2 \quad a_3 \quad \dots \quad a_{N-1} \quad a_N$

a

Select   Don't Select

$${}^{N}C_r = {}^{N-1}C_{r-1} + {}^{N-1}C_r$$

$$\boxed{{}^{N}C_r = {}^{N-1}C_{r-1} + {}^{N-1}C_r}$$

$r! = r * (r-1)!$

$(N-r)! = (N-r) * (N-r-1)!$

$$= \frac{(N-1)!}{(r-1)!\,(N-r)!} \qquad \frac{(N-1)!}{r!\,(N-r-1)!}$$

$$= \frac{(N-1)!}{(r-1)!\,(N-r)(N-r-1)!} + \frac{(N-1)!}{r * (r-1)!\,(N-r-1)!}$$

$$\frac{(N-1)!}{(r-1)!\,(N-r-1)!}\left[\frac{1}{(N-r)} + \frac{1}{r}\right]$$

$$\frac{(N-1)!}{(r-1)!\,(N-r-1)!}\left[\frac{1}{(N-r)}+\frac{1}{r}\right]$$

$$=\frac{(N-1)!}{(r-1)!\,(N-r-1)!}\left[\frac{r+N-r}{(r)(N-r)}\right]$$

$$=\frac{(N-1)!\times N}{r!\times(N-r)!}=\frac{N!}{r!\times(N-r)!}={}^{N}C_{R}$$

// 5 girls, you need select 2 ? $= {}^{5}C_{2} = \frac{5!}{3!\times 2!} = \frac{5\times 4}{2} = 10$

$G_1 \quad G_2 \quad G_3 \quad G_4 \quad G_5$

$G_1\, G_2 = \{G_3 G_4 G_5\}$

$G_1\, G_3 =$ ___

$G_1\, G_4 =$ ___

$G_1\, G_5 =$ ___

$G_2\, G_3 =$ ___

$G_2\, G_4 =$ ___

$G_2\, G_5 =$ ___

$G_3\, G_5 =$ ___

$G_3\, G_4 =$ ___

$G_4\, G_5 =$ ___

// ${}^{5}C_2 = {}^{5}C_3$

// N girls, how many r girls = ${}^{N}C_R$

// N girls, reject N-r girls = ${}^{N}C_{N-R}$

$$\boxed{{}^{N}C_R = {}^{N}C_{N-R}}$$

// $${}^{N}C_R = \frac{N!}{(N-R)!\,R!}$$

$\begin{cases} N = 50 \\ R = 25 \end{cases} \Rightarrow \dfrac{50!}{25!\,25!}$

Q: Given N r p

calculate

$$\left({}^{N}C_r\right) \% p$$

$$\left(\frac{N!}{(N-r)!\,(r)!}\right) \% p$$

$(a * b) \% p = (a \% p * b \% p) \% p$

$$(a/b) \% p = \left(\frac{(a \% p)}{(b \% p)}\right)$$

// Inverse modulo :

$(1/a) \% p = \underbrace{(a^{-1}) \% p}$ } ↳ Inverse modulo

⇒ If and only if gcd (a, p) = 1

TODO : Extended Euclidian

Special Case :

I) If p is prime && gcd (a, p) = 1

$$\boxed{(a^{-1}) \% p = (a^{p-2}) \% p}$$

} Concept comes from Fermat's little Theorem

// Given N, r, p & p is prime & $\underbrace{N \& r}$ < p

calculate ⇒ $^{N}C_{r}$ ⇒ $\left(\frac{N!}{(N-r)!\, r!}\right) \% p$

$P = \underline{10^9 + 7}$ } In general

// Given N, r, p & p is prime & N & $r < p$

Calculate ⇒

$${}^{N}C_r \,\%\, p \Rightarrow \left(\frac{N!}{(N-r)!\, r!}\right) \% p \quad \begin{cases} p > (N-r) \\ \gcd(p, (N-r)!) = 1 \end{cases}$$

it won't contain p

$$\Rightarrow \left\{ (N!) \% p \ast ((N-r)!)^{-1} \% p \ast ((r!)^{-1}) \% p \right\} \% p$$

b

$a = (N-r)!$

$(a^{-1}) \% p$

⇒ p is prime, & $\gcd(a, p) = 1$

$(N-r)!$

$(a^{p-2}) \% p$

$$\left[\left[(N-r)!\right]^{p-2}\right] \% p$$

$$\left[\left((N-r)! \,\%\, p\right)\right]^{p-2} \% p$$

⇒ calc

⇒ fast power exponentiation

$* \left((r!)^{-1} \right) \% p$

$\left\{ \begin{array}{l} p > r \\ \text{In } r! \text{ p wont be} \\ \text{there} \end{array} \right.$

$b = r!$

$(b^{-1}) \% p = (b^{p-2}) \% p$

→ p is prime ✓ $\gcd(b, p) = 1$

$\gcd(r!, p) = 1$

→ $\left((r!)^{p-2} \right) \% p$

→ $\left((r!) \% p \right)^{p-2} \% p$

✓

fast exponentiation

$\boxed{\Rightarrow \text{pow}(a, N, p) \Rightarrow a^N \% p \text{ using recu} \Rightarrow \log N}$

// If p is not prime ∧ gcd( ) != 1

$$^{N}C_{R} = \left(^{N-1}C_{R} + ^{N-1}C_{R-1} \right) \% p \quad \} \quad ^{3}C_{4} = 0 \quad \} \quad ^{N}C_{R} = 0 \quad R > N$$

$N = 4, \; R = 3$

$^{0}C_{0} =$

	R = 0	1	2	3
N = 0	1	0	0	0
1	1	1	0	0
2	1	2	1	0
3	1	3	3	1
4	1	4	6	4

$$^{2}C_{1} = ^{2-1}C_{1} + ^{2-1}C_{0}$$

$$= ^{1}C_{1} + ^{1}C_{0}$$

$$^{2}C_{2} = ^{2-1}C_{2} + ^{2-1}C_{1}$$

$$^{3}C_{2} = ^{3-1}C_{2} + ^{3-1}C_{1}$$

TC = O(N * R)

SC = O(N * R) $\xrightarrow[\text{optimized}]{\text{Can be}}$ O(R) → At any given point we only need 2 rows data

$$\boxed{mat[N][i] = \left(mat[N-1][i] + mat[N-1][i-1] \right) \% p}$$

Handle Edges

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

⇒ If p is prime

$$\left(a^{p-1}\right) \% p = 1$$

Doubts: