Exponential Distribution

$$f(x,\lambda) = \begin{cases} \lambda e^{-\lambda x}, & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

continuous

- A random variable X is said to have an exponential distribution with parameter $\theta > 0$, if its p.d.f. is given by

$$f(x,\theta) = \begin{cases} \theta e^{-\theta x}, & x \geq 0 \\ 0, & otherwise \end{cases}$$

- Cumulative distribution function $F(x)$ is given by

$$F(x) = \int_0^x f(u)du = \theta \int_0^x e^{-\theta u} du$$

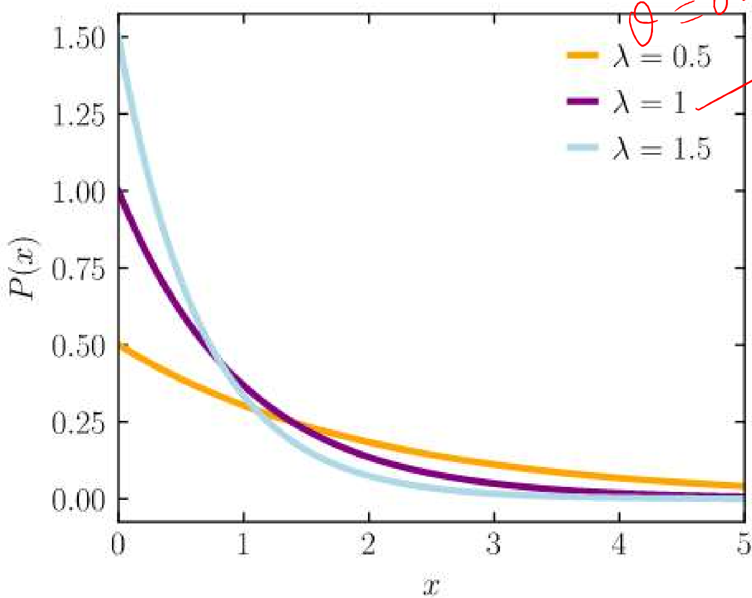
$$\theta \left[\frac{e^{-\theta u}}{-\theta} \right]_0^x$$

$$= -\left[e^{-\theta x} - 1 \right]$$

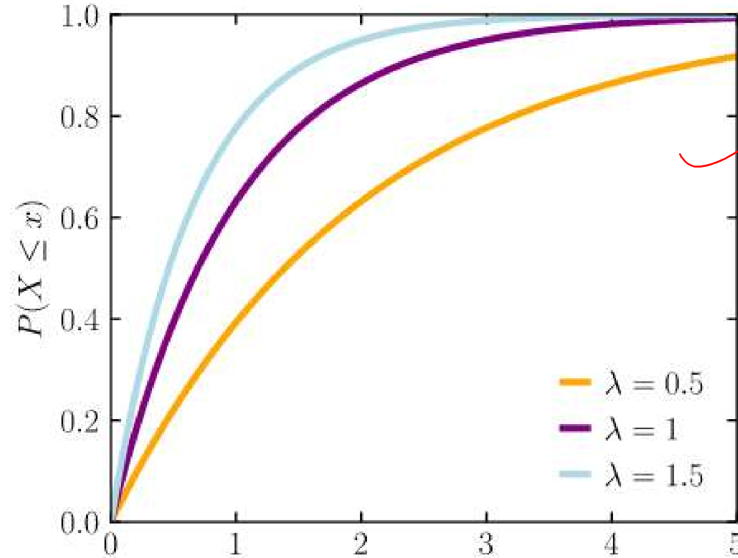
$$= 1 - e^{-\theta x}$$

$$\therefore F(x) = \begin{cases} 1 - e^{-\theta x}, & x \geq 0 \\ 0, & otherwise \end{cases}$$

$f(x)$



$F(x)$



Moment Generating Function of Exponential Distribution

$(1-x)^{-1} = 1+x+x^2+x^3+\ldots\ldots$

$$M_X(t) = E(e^{tX}) = \int_{-\infty}^{\infty} e^{tx} f(x)\,dx = \int_{-\infty}^{0} e^{tx} f(x)\,dx + \int_{0}^{\infty} e^{tx} f(x)\,dx$$

$E(X) = \int_{-\infty}^{\infty} x f(x)\,dx$

$\because\ E(e^{tX}) = \int_{-\infty}^{\infty} e^{tx} f(x)\,dx$

$$= 0 + \int_{0}^{\infty} e^{tx}\,\theta\, e^{-\theta x}\,dx = \theta \int_{0}^{\infty} e^{-(\theta - t)x}\,dx$$

$$= \theta \left(\frac{e^{-(\theta-t)x}}{-(\theta-t)}\right)_0^{\infty} = \theta\left[0 + \frac{1}{\theta - t}\right] = \frac{\theta}{\theta - t} = \frac{1}{\left(1-\frac{t}{\theta}\right)}$$

$$= \left[1-\frac{t}{\theta}\right]^{-1} = 1 + \frac{t}{\theta} + \frac{t^2}{\theta^2} + \frac{t^3}{\theta^3} + \cdots = \sum_{r=0}^{\infty}\left(\frac{t}{\theta}\right)^r \times \frac{r!}{r!}, \quad \theta > t$$

r^{th} moment about origin $= \mu'_r =$ coefficient of $\frac{t^r}{r!} = \frac{r!}{\theta^r}$, $\mu'_1 = \frac{1}{\theta}$, $\mu'_2 = \frac{2}{\theta^2}$ .....

mean $= \mu_1' = \frac{1}{\theta}$

$X \sim N(\mu, \sigma)$

moments about mean :-

$\mu_1 = \mu_1' - \mu_1' = 0$ [Always]

$Var = E(X^2) - [E(X)]^2$
$= \mu_2' - (\mu_1')^2$

Variance $= \mu_2 = \mu_2' - (\mu_1')^2 = \frac{2}{\theta^2} - \left(\frac{1}{\theta}\right)^2 = \frac{1}{\theta^2}$

If the random variable follows exponential distribution i.e., $X \sim \exp(\theta)$,

then mean $= \frac{1}{\theta}$ and variance $= \frac{1}{\theta^2}$

Note: ① variance $= \frac{1}{\theta^2} = \frac{1}{\theta} \cdot \frac{1}{\theta} = \frac{\text{mean}}{\theta}$

② variance > mean, if $0 < \theta < 1$

③ variance = mean, if $\theta = 1$

④ variance < mean, if $\theta > 1$

Hence for the exponential distribution

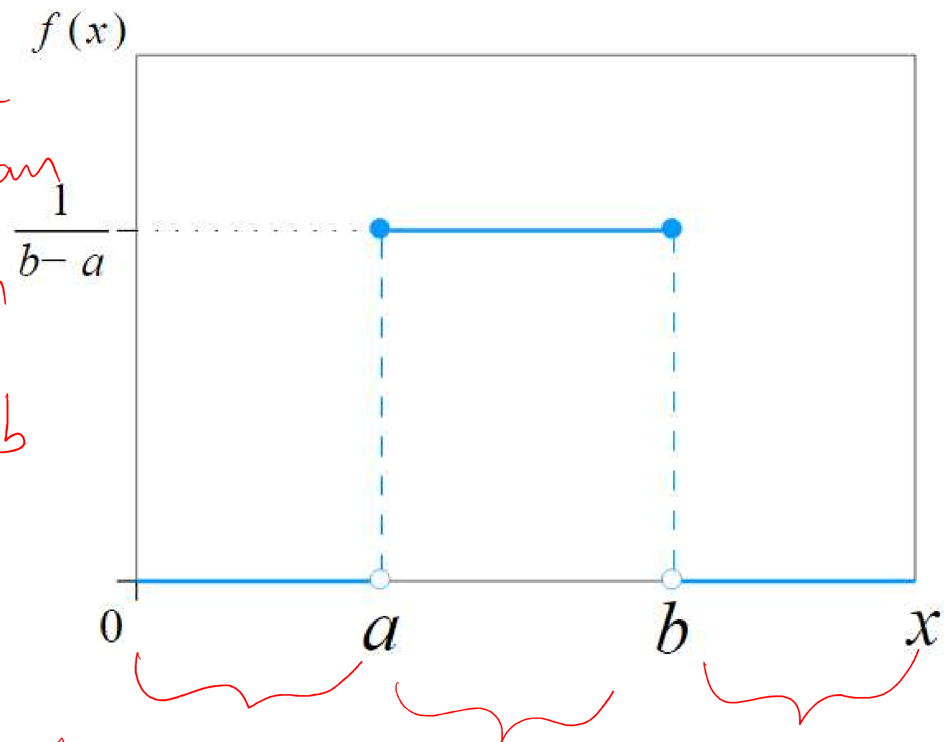
variance >, =, < mean,

for different values of θ

Uniform Distribution (Rectangular Distribution)

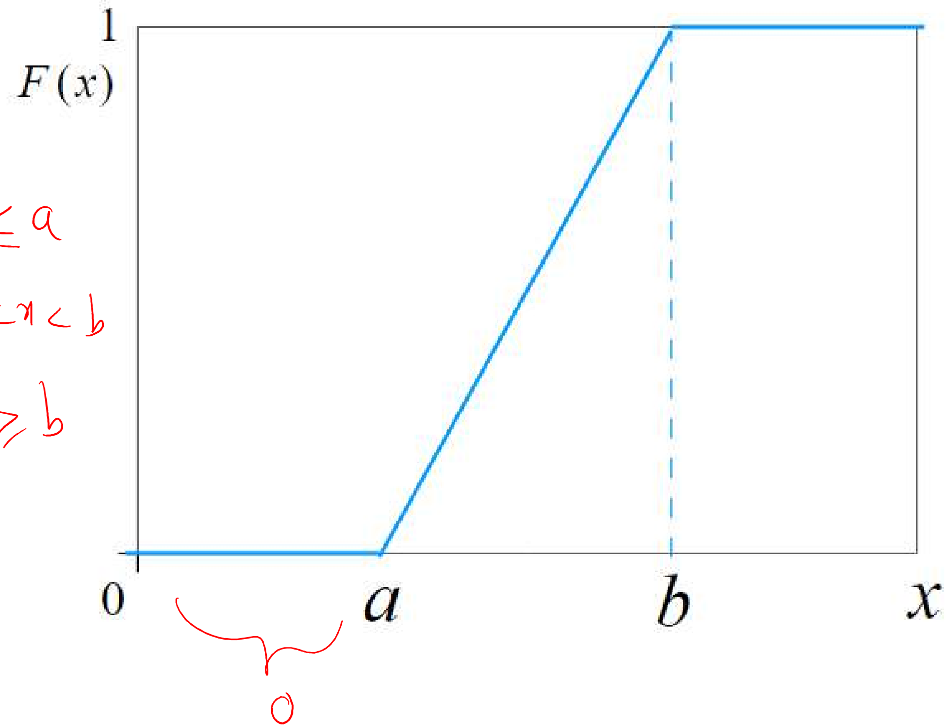
A random variable X is said to have Uniform (Rectangular) distribution over an interval (a,b) if its p.d.f. is given by

$$f(x;a,b) = \begin{cases} \frac{1}{b-a}, & \text{if } a<x<b \\ 0, & \text{otherwise} \end{cases}$$



Notation: $X \sim U[a,b]$ or $X \sim R[a,b]$ denote that the random variable X follows the uniform or rectangular distribution.

Cumulative distribution function



$$F(x) = \begin{cases} 0 & \text{if } x \le a \\ \dfrac{x-a}{b-a} & \text{if } a < x < b \\ 1 & \text{if } x \ge b \end{cases}$$

Note ① If the rectangular distribution from $(-a, a)$ then $f(x) = \begin{cases} \dfrac{1}{2a} & -a < x < a \\ 0 & \text{otherwise} \end{cases}$

Moments of Rectangular Distribution:-

Let $X \sim U[a, b]$,

rth moment about origin $= \mu_r' = \int_a^b x^r f(x)\,dx = \int_a^b x^r \frac{1}{b-a} = \frac{1}{b-a}\left[\frac{x^{r+1}}{r+1}\right]_a^b$

$$= \frac{1}{(b-a)}\left(\frac{b^{r+1} - a^{r+1}}{r+1}\right) \quad \text{——— (1)}$$

mean $= \mu_1' = \frac{1}{b-a} \cdot \frac{b^2 - a^2}{(1+1)} = \frac{(b-a)(b+a)}{2(b-a)} = \frac{a+b}{2}$ $\quad$ [$\because$ put $r = 1$ in (1)]

$\mu_2' = \frac{1}{b-a}\left(\frac{b^3 - a^3}{3}\right) = \frac{(b-a)(b^2+ab+a^2)}{(b-a)3} = \frac{(a^2+ab+b^2)}{3}$ $\quad$ [put $r = 2$ in (1)]

Variance $= \mu_2' - (\mu_1')^2 = \frac{a^2+ab+b^2}{3} - \left(\frac{a+b}{2}\right)^2 = \frac{a^2+ab+b^2}{3} - \frac{a^2+b^2+2ab}{4} = \frac{(b-a)^2}{12}$

Moment Generating Function

$$M_X(t) = E(e^{tX}) = \int_a^b e^{tx} f(x)\,dx = \int_a^b e^{tx} \frac{1}{b-a}\,dx = \frac{1}{b-a}\left[\frac{e^{tx}}{t}\right]_a^b$$

$$= \frac{e^{bt} - e^{at}}{t(b-a)}, \quad t \neq 0$$

$$\int_2^4 |x-3|\,dx = \int_2^3 |x-3|\,dx + \int_3^4 |x-3|\,dx$$

$3-x$ $\quad$ $x-3$

Mean Deviation about mean:-

$$\text{mean} = \frac{a+b}{2} \qquad \sum f_i |x_i - \bar{x}|$$

$$\eta = E|X - \text{mean}| = \int_a^b |x - \text{mean}|\, f(x)\,dx = \int_a^b \frac{|x-\text{mean}|}{b-a}\,dx = \frac{1}{b-a}\int_a^b \left|x - \left(\frac{a+b}{2}\right)\right| dx$$

$$= \frac{1}{b-a}\int_a^{\frac{a+b}{2}} \left(\frac{a+b}{2} - x\right) dx + \frac{1}{b-a}\int_{\frac{a+b}{2}}^{b} \left(x - \left(\frac{a+b}{2}\right)\right) dx$$

$$= \frac{1}{b-a}\left[\left(\frac{a+b}{2}\right)x - \frac{x^2}{2}\right]_a^{\frac{a+b}{2}} + \frac{1}{b-a}\left[\frac{x^2}{2} - \left(\frac{a+b}{2}\right)x\right]_{\frac{a+b}{2}}^{b}$$

$$= \frac{1}{b-a}\left(\left(\frac{a+b}{2}\right)^2 - \frac{(a+b)^2}{8} - \frac{a(a+b)}{2} + \frac{a^2}{2}\right) + \frac{1}{b-a}\left[\frac{b^2}{2} - \frac{(a+b)b}{2} - \frac{(a+b)^2}{8} + \left(\frac{a+b}{2}\right)^2\right]$$

$$= \frac{1}{b-a}\left[(a+b)^2\left(\frac{1}{4} - \frac{1}{8} - \frac{1}{8} + \frac{1}{4}\right) - \frac{(a^2+ab)}{2} + \frac{a^2}{2} + \frac{b^2}{2} - \frac{(ab+b^2)}{2}\right]$$

$$= \frac{1}{b-a}\left[\frac{(a+b)^2}{4} - ab\right] = \frac{1}{b-a}\left[\frac{a^2+2ab+b^2-4ab}{4}\right] = \frac{1}{b-a}\cdot\frac{(b-a)^2}{4}$$

$$= \frac{b-a}{4}$$

Problem: If X is uniformly distributed with mean 1 and variance $\frac{4}{3}$ find $P(X<0)$.

Soln:- If $X \sim U[a,b]$ then mean $= \frac{b+a}{2} = 1 \Rightarrow b = 2-a$ — (1)

variance $= \frac{(b-a)^2}{12} = \frac{4}{3}$ — (2)

$$\frac{(2-a-a)^2}{12} = \frac{4}{3} \Rightarrow (2-2a)^2 = 16$$

$$(1-a)^2 = 4$$

$$1-a = \pm 2 \Rightarrow + \Rightarrow a = -1$$

$$- \Rightarrow a = 3$$

If $a=-1 \Rightarrow b = 3$ ✓

If $a = 3 \Rightarrow b = -1$ ✗ $\because a<b$

Given mean $=1$, variance $= \frac{4}{3}$

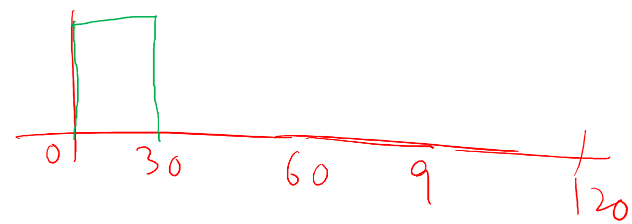
Solving (1) & (2), $a=-1$, $b=3$

$$P(X<0) = \int_{-1}^{0} f(x)\,dx = \int_{-1}^{0} \frac{1}{b-a}\,dx = \frac{1}{4}\left[x\right]_{-1}^{0}$$

$$= \frac{1}{4}\left[0+1\right] = \frac{1}{4}$$

problem: Subway trains on a certain line run every half an hour between midnight and six in the morning. What is the probability that a man entering the station at a random time during this period will have to wait atleast twenty minutes?

Soln :- Let the random variable X represent the waiting time. Under the assumption that man arrives at random, X is distributed uniformly on (0, 30) with p.d.f



$$f(x) = \begin{cases} \frac{1}{30} & 0 < x < 30 \\ 0 & \text{otherwise} \end{cases}$$

Here $a = 0$, $b = 30$

$$P(X \geq 20) = \int_{20}^{30} f(x)\,dx = \int_{20}^{30} \frac{1}{b-a}\,dx = \frac{1}{30}\left[x\right]_{20}^{30} = \frac{10}{30} = \frac{1}{3}.$$

A random variable X has a rectangular distribution over (-3,3). Compute (i) P(X<1) (ii) $P(|X| < 2)$

Rectangular/uniform distribution PDF is $f(x) = \begin{cases} \frac{1}{b-a} & \text{if } a<x<b \\ 0 & \text{otherwise} \end{cases}$

$$P(X<1) = \int_{-3}^{1} f(x)\,dx = \int_{-3}^{1} \frac{1}{3+3}\,dx = \frac{4}{9}$$

$$P(|X|<2) = P(-2<X<2) = \int_{-2}^{2} \frac{1}{3+3}\,dx = \frac{4}{9}$$