E.

1. Vypočítejte integrál $\displaystyle\int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \frac{dx}{x+\sqrt{1-x^2}}$ pomocí substituce $x=\sin t$

$$\int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \frac{dx}{x+\sqrt{1-x^2}} = \left| \begin{array}{ll} x=\sin t & x=\frac{\sqrt{3}}{2} \Rightarrow t=\frac{\pi}{3} \\ dx=\cos t\,dt & x=\frac{1}{2} \quad\Rightarrow t=\frac{\pi}{6} \end{array} \right| = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\cos t}{\sin t+\sqrt{1-\sin^2 t}}\,dt = \left| \begin{array}{c} t\in\left\langle \frac{\pi}{6},\frac{\pi}{3}\right\rangle \Rightarrow \\ \cos t\geq 0 \end{array} \right| = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\cos t}{\sin t+\cos t}\,dt =$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{\operatorname{tg} t+1}\,dt = \left| \begin{array}{lll} \operatorname{tg} t = z & & t=\frac{\pi}{3} \Rightarrow z=\sqrt{3} \\ t=\operatorname{arctg} z & dt=\frac{1}{1+z^2}dz & t=\frac{\pi}{6} \Rightarrow z=\frac{\sqrt{3}}{3} \end{array} \right| = \int_{\frac{\sqrt{3}}{3}}^{\sqrt{3}} \frac{1}{z+1}\cdot\frac{1}{z^2+1}\,dz = \frac{1}{2}\int_{\frac{\sqrt{3}}{3}}^{\sqrt{3}} \left(\frac{1}{z+1}-\frac{z}{z^2+1}+\frac{1}{z^2+1}\right)dz =$$

$$\frac{1}{2}\left[\ln(z+1)-\tfrac{1}{2}\ln(z^2+1)+\operatorname{arctg} z\right]_{\frac{\sqrt{3}}{3}}^{\sqrt{3}} = \frac{1}{4}\left[\ln\frac{(z+1)^2}{z^2+1}+2\operatorname{arctg} z\right]_{\frac{\sqrt{3}}{3}}^{\sqrt{3}} =$$

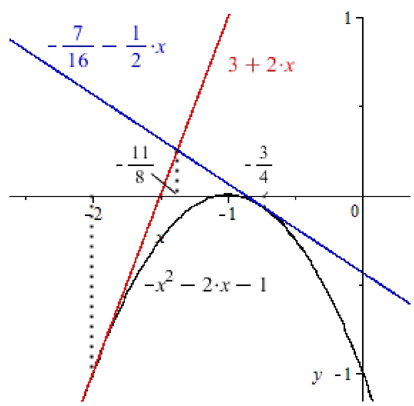
$$= \frac{1}{4}\left(\ln\frac{(\sqrt{3}+1)^2}{4}-\ln\frac{\left(\frac{1}{\sqrt{3}}+1\right)^2}{\frac{4}{3}}+2\operatorname{arctg}\sqrt{3}-2\operatorname{arctg}\frac{1}{\sqrt{3}}\right) = \frac{1}{4}\left(\ln\frac{(\sqrt{3}+1)^2}{4}\cdot\frac{4}{3\left(\frac{\sqrt{3}+1}{\sqrt{3}}\right)^2}+2\frac{\pi}{3}-2\frac{\pi}{6}\right) = \frac{1}{4}\left(\ln 1+2\frac{\pi}{6}\right) = \frac{\pi}{12}$$

*jiný postup* :

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\cos t}{\sin t+\cos t}\,dt = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\cos t+\cos t+\sin t-\sin t}{2(\sin t+\cos t)}\,dt = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin t+\cos t+\cos t-\sin t}{2(\sin t+\cos t)}\,dt =$$

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \left(\frac{\sin t+\cos t}{2(\sin t+\cos t)}+\frac{\cos t-\sin t}{2(\sin t+\cos t)}\right)dt = \left[\frac{1}{2}t+\frac{1}{2}\ln\left|\cos t+\sin t\right|\right]_{\frac{\pi}{6}}^{\frac{\pi}{3}} = \frac{\pi}{6}+\frac{1}{2}\ln\left(\frac{1}{2}+\frac{\sqrt{3}}{2}\right)-\frac{\pi}{12}+\frac{1}{2}\ln\left(\frac{\sqrt{3}}{2}+\frac{1}{2}\right) = \frac{\pi}{12}$$

2. Najděte rovnice dvou tečen k parabole $y=-\left(x^2+2x+1\right)$; jedna je rovnoběžná s přímkou $2x-y-1=0$, druhá je na ni kolmá.
Vypočítejte obsah části roviny omezené danou parabolou a nalezenými tečnami.
(Pro numerické výpočty můžete použít kalkulačku.)




$2x-y-1=0 \quad\Leftrightarrow\quad y=2x-1$ - hledáme tečny k parabole

se směrnicemi $k_1=2,\ k_2=-\frac{1}{2}$.

$y'=-2x-2 \quad -2x-2=2 \Leftrightarrow x=-2,\ f(2)=-1,\quad T_1=[-2,-1]$

$-2x-2=-\frac{1}{2} \Leftrightarrow x=-\frac{3}{4},\ f\left(-\frac{3}{4}\right)=-\frac{7}{16},\ T_2=\left[-\frac{3}{4},-\frac{7}{16}\right]$

$t_1:\quad y=-1+2\left(x+2\right)=2x+3,$

$t_2:\quad y=-\frac{7}{16}-\frac{1}{2}\left(x+\frac{3}{4}\right)=-\frac{7}{16}-\frac{1}{2}x$

průsečík tečen: $-\frac{7}{16}-\frac{1}{2}x=2x+3 \Rightarrow x=-\frac{11}{8}$.

$$S=\int_{-2}^{-\frac{11}{8}}(2x+3)\,dx+\int_{-\frac{11}{8}}^{-\frac{3}{4}}\left(-\frac{7}{16}-\frac{1}{2}x\right)dx-\int_{-\frac{3}{4}}^{2}\left(-x^2-2x-1\right)dx=\cdots=\frac{125}{768}\doteq 0{,}16275$$