CMSC 250, Fall 2017 **Sections: all**

Final Exam

Friday, 12-15-2017

Student's first and last name: ______________________________ *Grade (grader only):*______

Student's Section Number : _____________

Student's UID: __________________

P1	**P2**	**P3**	**P4**	**P5**	**P6**	**P7**

University Honor Pledge:

I pledge on my honor that I have not given or received
any unauthorized assistance on this assignment/examination.

Print the text of the University Honor Pledge below:

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Signature: ____________________________________

Exam Guidelines / Rules / Assumed Facts

- **Turn off all unapproved electronic devices** (e.g phones, tablets, laptops, but not calculators if you feel you need them). Setting a device on "silent" or "sleep" mode does not constitute it being turned **off**: Your device is turned off if and only if it requires pushing a power button to begin the execution of a bootloader. **Proctors reserve the right to confiscate an electronic device if it is not turned off according to the definition above.**
- Write **neatly**. If we can't read your response, you will receive no credit for it. Use the scrap paper provided at the end of the exam for note-taking. You can ask for extra scrap paper if you need it.
- There are **6 (six)** problems in this exam, with a total grade value that adds to **100 (one hundred)**, as well as an extra credit problem, worth **1 (one)** point.
- The exam has been printed **two-sided**, **stapled on the top-left corner** and spans **18** (**eighteen**) pages across **9** (**nine**) sheets. The last **2 (two)** pages are **scrap paper** for your notes.
- The total time allocated for this exam is **115 (one hundred and fifteen)** minutes.
- During the **last 5 (five) minutes** of the exam, you may **not** leave the classroom (e.g to go to the bathroom, or because you're done).

Provided materials

Table 1 contains a number of logical equivalences that we have discussed in class.

Commutativity of binary operators	$p \wedge q \equiv q \wedge p$	$p \vee q \equiv q \vee p$
Associativity of binary operators	$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	$(p \vee q) \vee r \equiv p \vee (q \vee r)$
Distributivity of binary operators	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
Identity laws	$p \wedge t \equiv p$	$p \vee c \equiv p$
Negation laws	$p \vee \sim p \equiv t$	$p \wedge \sim p \equiv c$
Double negation	$\sim(\sim p) \equiv p$	
Idempotence	$p \wedge p \equiv p$	$p \vee p \equiv p$
De Morgan's axioms	$\sim(p \wedge q) \equiv \sim p \vee \sim q$	$\sim(p \vee q) \equiv \sim p \wedge \sim q$
Universal bound laws	$p \vee t \equiv t$	$p \wedge c \equiv c$
Absorption laws	$p \vee (p \wedge q) \equiv p$	$p \wedge (p \vee q) \equiv p$
Negations of contradictions / tautologies	$\sim c \equiv t$	$\sim t \equiv c$
Equivalence between biconditional and implication	$a \Leftrightarrow b \equiv (a \Rightarrow b) \wedge (b \Rightarrow a)$	
Equivalence between implication and disjunction	$a \Rightarrow b \equiv \sim a \vee b$	

Table 1: A number of propositional logic axioms you can refer to.

Table 2 contains a number of rules of natural deduction / inference that we have discussed in class.

Modus Ponens	**Modus Tollens**	**Disjunctive addition**	**Conjunctive addition**	**Division into Cases**
p $p \Rightarrow s$ $\therefore s$	$\sim s$ $p \Rightarrow s$ $\therefore \sim p$	p $\therefore p \vee s$	p, s $\therefore p \wedge s$	$p \vee s$ $p \Rightarrow r$ $s \Rightarrow r$ $\therefore r$
Conjunctive Simplification	**Disjunctive syllogism**	**Hypothetical syllogism**	**Resolution**	**A bunny with a pancake on its head**
$p \wedge s$ $\therefore p, s$	$p \vee s$ $\sim p$ $\therefore s$	$p \Rightarrow s$ $s \Rightarrow r$ $\therefore p \Rightarrow r$	$p \vee s$ $(\sim s) \vee z$ $\therefore p \vee z$	

Table 2: Propositional Logic rules of inference / natural deduction you may refer to.

You may use the following mathematical facts **without proof**:

- $\mathbb{Z}$ is closed under addition, subtraction and multiplication.
- $\mathbb{N}$ is closed under addition and multiplication.
- $0 \in \mathbb{N}$.
- $\sqrt{2}$ is an irrational number.

Finally, table 3 contains various Set Theory axioms that you can refer to.

Commutativity	$A \cup B = B \cup A$	$A \cap B = B \cap A$
Associativity of union & intersection	$(A \cap B) \cap C = A \cap (B \cap C)$	$(A \cup B) \cup C = A \cup (B \cup C)$
Distributivity of union & intersection	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
Identity laws	$A \cap U = A$	$A \cup \emptyset = A$
Inverse laws	$A \cup A^c = U$	$A \cap A^c = \emptyset$
Double complementation	$(A^c)^c = A$	
Idempotence	$A \cap A = A$	$A \cup A = A$
De Morgan's axioms	$(A \cap B)^c = A^c \cup B^c$	$(A \cup B)^c = A^c \cap B^c$
Universal bound (Domination) laws	$A \cup U = U$	$A \cap \emptyset = \emptyset$
Absorption laws	$A \cup (A \cap B) = A$	$A \cap (A \cup B) = A$
Absolute Complements of empty set / domain	$\emptyset^c = U$	$U^c = \emptyset$
Relationship between relative and absolute complement	$A - B = A \cap B^c$	

Table 3: Various Set Theory axioms you may refer to.

(The exam problems begin on the next page.)

Problem 1: Logic *(15 pts)*

Question (a): Truth tables (7 pts)

Complete the following **truth table** for the logical expression $\big(p \vee q\big) \wedge \big(\sim\!(z \vee q)\big)$. To start you off, we are giving you the first three columns. You can draw and fill as **many columns as you need**, but please be careful to **not run out of space**. Write **neatly**; if we can't make the difference between a **T** and an **F**, we will be forced to take off points!

p	q	z	
F	**F**	**F**	
F	**F**	**T**	
F	**T**	**F**	
F	**T**	**T**	
T	**F**	**F**	
T	**F**	**T**	
T	**T**	**F**	
T	**T**	**T**	

Question (b): Negating quantifiers (8 pts)

In the following quantified expressions, push the negation operator ($\sim$) **as far inside the expression as possible.** You do **not** need to show us your steps; if you prefer working in your scrap paper and just showing us your final answer, that is fine; if you prefer to show us all the steps below, that is **also** fine, but **be careful not to run out of space**. In addition, you do **not** need to care about the truth values of the statements; that is, it is **not important** whether the statements themselves or their negations are **True** or **False**; just push the negation operator as discussed.

(i) $\sim\big(\exists q \in \mathbb{Q}\big)\big[q^2 \in \mathbb{Q}\big]$ *(1 pt)*

(ii) $\sim\big(\forall a \in \mathbb{Z})(\exists b \in \mathbb{Z}\big)\big[(a+b) \notin \mathbb{Z}\big]$ *(3 pts)*

(iii) $\sim\big(\forall r \in \mathbb{R}\big)\Big[\big((r > 0) \land (\sqrt{r} \in \mathbb{N})\big) \Rightarrow (r \in \mathbb{N})\Big]$ *(4 pts)*

BEGIN YOUR ANSWER TO QUESTION (b) BELOW THIS LINE

Problem 2: Various *(2 pts each for 20 pts total)*

For all of the following statements, **circle True** or **False** depending on whether you believe the statement to be true or false respectively. You do **not** need to justify your answers. Recall that S^c is the *universal complement* of set S. Note that for the questions that involve sets A, B, we are asking you to tell us if you believe that the statement is **True** or **False** for **all possible sets** A, B.

Statement	**True or False?**
$A \cup \emptyset = A$	**TRUE / FALSE**
$A \cap A^c \subseteq \emptyset$	**TRUE / FALSE**
$A - B = A \cup B^c$	**TRUE / FALSE**
$\{\emptyset\} \cup \{\} = \{\}$	**TRUE / FALSE**
$\{\{\emptyset\}\} \subseteq \{\{\emptyset, \{\emptyset\}\}\}$	**TRUE / FALSE**
The function $f : \mathbb{R} \mapsto \mathbb{R}$ defined as: $f(x) = x^3$ is a **bijection**.	**TRUE / FALSE**
The function $f : \mathbb{Z} \mapsto \mathbb{R}$ defined as: $f(x) = \begin{cases} x^2, & x < 0 \\ -x^2, & x \geq 0 \end{cases}$ is a **bijection**.	**TRUE / FALSE**

The relation $$R = \{(x,y) \in \mathbb{R} \times \mathbb{R} \mid (x-y)^2 \leq 4\}$$ is *reflexive*, *symmetric* **and** *transitive*.	**TRUE / FALSE**
The set of positive rationals $\mathbb{Q}^{>0}$ has the same **cardinality** as the set of integers $\mathbb{Z}$.	**TRUE / FALSE**
The set $S = \Big\{f \mid f : \mathbb{N} \mapsto \{0,1\}\Big\}$ (set of **all possible functions** from $\mathbb{N}$ **only to the two integers 0**, **1**) is **un**countable.	**TRUE / FALSE**

SOME EMPTY SPACE FOR YOU IF YOU NEED IT

Problem 3: Formal proofs / Number Theory *(15 pts)*

Using **either** a **direct or** an **indirect** proof methodology, prove that the following statements are **True**.

(a) For any integers a, b, c with $a \neq 0$, if $(a|b)$ **and** $(a|c)$, then $(\forall s, r \in \mathbb{Z})[\ a \mid (s \cdot b + r \cdot c)\ ]$. *(7 pts)*

(b) $\sqrt{2} + 1$ is **not** divisible by 3 . You may use the fact that $\sqrt{2} \notin \mathbb{Q}$ **without proof.** *(8 pts)*

BEGIN YOUR ANSWER TO PROBLEM 3 BELOW THIS LINE

CONTINUE YOUR ANSWER TO PROBLEM 3 BELOW THIS LINE

Problem 4: Induction *(20 pts)*

Let a_n be a sequence recursively defined as follows:

$$a_n = \begin{cases} 3, & n = 0 \\ 4, & n = 1 \\ a_{n-1} + a_{n-2} \quad - \; n, & n \geq 2 \end{cases}$$

Note that, in the recursive case of a_n, the quantity $-n$ is **not** an index into a sequence term, like $n-1$ and $n-2$ are! Another way to write the recursive case would be:

$$-n + a_{n-1} + a_{n-2}$$

Using **whichever** mathematical induction principle you consider appropriate, so long as you **sharply delineate** the **Inductive Base**, **Hypothesis** and **Step**, prove that

$$(\forall n \in \mathbb{N})[a_n = n + 3]$$

Please write **neatly** in the Inductive Step. If we cannot understand what you are writing, we will **not** be able to give you **any credit** for the Step!

BEGIN YOUR ANSWER TO PROBLEM 4 BELOW THIS LINE

CONTINUE YOUR ANSWER TO PROBLEM 4 BELOW THIS LINE

Problem 5: Combinatorics *(25 pts)*

Answer the following questions on the small line available to you after each and every one of them. You do **not** need to justify your answers. Remember that in combinatorics problems, we do **not** want you to perform messy calculations: Just leave your result in terms of permutation or combination symbols, factorials, powers, sums or products of all of those, etc.

(a) How many different strings of length **5**, **6**, or **7** (**five, six** or **seven**) can we create with the English alphabet? Note that the English alphabet has **26** (**twenty-six**) characters and, of course, characters can be **repeated**. For example, there are two 'e's in the word "repeat". ____________ *(1 pt)*

(b) Calculate the (distinguishable) permutations of the following strings:

   (i) *iowa* ____________ *(1 pt)*

   (ii) *oregon* ____________ *(2 pts)*

   (iii) *mississippi* ____________ *(3 pts)*

(c) A bit-string is a string that contains only **1**s and **0**s (**one**s and **zero**es). How many bit-strings of length **12** (**twelve**) have...

   (i) **Exactly** 5 (five) 1s? ____________ *(1 pt)*

   (ii) **At most** 5 (five) 1s? ____________ *(2 pts)*

   (iii) **At least** 5 (five) 1s? ____________ *(2 pts)*

   (iv) An **equal number** of 1s and 0s? ____________ *(2 pts)*

(d) Suppose that we have 50 (fifty) students in a classroom.

   (i) We want to take a photo of the students. In how many ways can we order them in a single row? ____________ *(1 pt)*

   (ii) If we want 11 (eleven) of these students to sit in a row of 11 (eleven) seats, in how many ways can we do that? Recall that when 2 (two) or more people are sitting in a row, their **order** matters. ____________ *(2 pts)*

(iii) Suppose that **20** (**twenty**) of those students have a first name that begins with a 'K'. **10** (**ten**) students have a last name that begins with an 'M'. **5** (**five**) students have a first name that starts with a 'K' **and** a last name that begins with an 'M'.

(i) How many students have **neither** a first name that begins with a 'K' **nor** a last name that begins with an 'M'? ____________ *(2 pts)*

(ii) How many different **pairs** of students have **neither** first name that begins with a 'K' **nor** a last name that begins with an 'M'? ____________ *(1 pt)*

(iv) Suppose that a university department contains **10** (**ten**) men and **15** (**fifteen**) women. How many ways are there to form a committee with **6** (**six**) members if:

(i) It must have *the same* number of men and women? ____________ (*1 pt*)

(ii) It must have *more* women than men? Please note that **0 (zero)** men and a **non-zero** number of women **still** means that there are **more women than men** in the committee! ____________ *(2 pts)*

(v) Suppose that we have a basket with 4 (four) different kinds of apples: 8 (eight) Braeburn, 2 (two) Fuji, 4 (four) Elstar and 6 (six) Gala. We **randomly** pick 4 (four) apples from the basket. What is the **probability** that the apples picked will all be of **a different kind**? ____________ (*2 pts*)

Problem 6: Show me what you got *(5 pts)*

Suppose that n and m are natural numbers. Using **whichever** mathematical induction principle you consider appropriate, so long as you **sharply delineate** the **Inductive Base**, **Hypothesis** and **Step**, prove that

$$\sum_{i=0}^{n} \binom{m+i-1}{i} = \binom{m+n}{n}$$

Hints:

- Your induction should be on **n**.
- When solving the inductive step:
  - **Keep in mind the answer that you want**. This will give you a clue on how to manipulate the expression to reach it.
  - If you want to use combinatorial identities that we have talked about in class, **you are absolutely allowed to do so**. However, please make sure you **explicitly state which identity you are using** so that we can understand what you are doing during grading.

Please write **neatly** in the Inductive Step. If we cannot understand what you are writing, we will **not** be able to give you **any credit** for the Step!

BEGIN YOUR ANSWER TO PROBLEM 6 BELOW THIS LINE

CONTINUE YOUR ANSWER TO PROBLEM 6 BELOW THIS LINE

Problem 7: General Trivia *(1 pt extra credit)*

In the legendary real-time-strategy video game *Age of Empires II: The Age of Kings* (as well as its various expansion packs), what does the cheat code `howdoyouturnthison` accomplish? ___________

____

SCRAP PAPER

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