At steady state, eq. 9.17 becomes

$$\mu_2 = D_2\left(1 - \frac{X_1}{X_2}\right) \tag{9.18}$$

where $X_1/X_2 < 1$ and $\mu_2 < D_2$.

The substrate balance for the limiting substrate in the second stage is

$$FS_1 - FS_2 - \frac{\mu_2 X_2}{Y_{X/S}^M} V_2 = V_2 \frac{dS_2}{dt} \tag{9.19}$$

At steady state, eq. 9.19 becomes

$$S_2 = S_1 - \frac{\mu_2}{D_2}\frac{X_2}{Y_{X/S}^M} \tag{9.20}$$

where

$$D_2 = F/V_2 \quad \text{and} \quad \mu_2 = \frac{\mu_m S_2}{K_s + S_2}$$

Equations 9.18 and 9.20 can be solved simultaneously for X_2 and S_2 by substituting $\mu_2 = \mu_m S_2/(K_s + S_2)$ in both equations or any other functional form that describes μ_2.

When a feed stream is added to the second stage, then the design equations change. The second feed stream may contain additional nutrients, inducers, hormones, or inhibitors. Biomass balance for the second stage in this case is

$$F_1 X_1 + F'X' - (F_1 + F')X_2 + V_2\mu_2 X_2 = V_2 \frac{dX_2}{dt} \tag{9.21}$$

At steady state when $X' = 0$, eq. 9.21 becomes

$$\mu_2 = D_2' - \frac{F_1}{V_2}\frac{X_1}{X_2} \tag{9.22}$$

where

$$D_2' = \frac{F_1 + F'}{V_2} \quad \text{and} \quad \mu_2 = \frac{\mu_m S_2}{K_s + S_2}$$

Substrate balance for the second stage yields

$$F_1 S_1 + F'S_0' - (F_1 + F')S_2 - \frac{V_2\mu_2 X_2}{Y_{X/S}^M} = V_2 \frac{dS_2}{dt} \tag{9.23}$$

Equations 9.22 and 9.23 need to be solved simultaneously for X_2 and S_2.

We can generalize these equations for a system with no additional streams added to second or subsequent units. If we do a balance around the nth stage on biomass, substrate, and product, we find

$$r_{x,n}(X_x, S_n) = D_n(X_n - X_{n-1}) \tag{9.24a}$$