The balance on the rate-limiting substrate without maintenance energy is

$$\frac{dS^t}{dt} = FS_0 - \frac{\mu_{net} X^t}{Y_{X/S}^M} \tag{9.35}$$

where S^t is the total amount of the rate-limiting substrate in the culture and S_0 is the concentration of substrate in the feed stream.

At quasi-steady state, $X^t = VX_m$ and essentially all the substrate is consumed, so no significant level of substrate can accumulate. Therefore,

$$FS_0 = \frac{\mu_{\text{net}} X^t}{Y_{X/S}^M} \tag{9.36}$$

Equation 9.31 at quasi-steady state with $S \approx 0$ yields

$$\frac{dX^t}{dt} = X_m \left(\frac{dV}{dt}\right) = X_m F = FY_{X/S}^M S_0 \tag{9.37}$$

Integration of eq. 9.37 from $t = 0$ to t with the initial amount of biomass in the reactor being X_0^t yields

$$X^t = X_o^t + FY_{X/S}^M S_0 t \tag{9.38}$$

That is, the total amount of cell in the culture increases linearly with time (which is experimentally observed) in a fed-batch culture. Dilution rate and therefore μ_{net} decrease with time in a fed-batch culture. Since $\mu_{\text{net}} = D$ at quasi-steady state, the growth rate is controlled by the dilution rate. The use of unstructured models is an approximation, since μ_{net} is a function of time.

Product profiles in a fed-batch culture can be obtained by using the definitions of $Y_{P/S}$ or q_P. When the product yield coefficient $Y_{P/S}$ is constant, at quasi-steady state with $S << S_0$

$$P \cong Y_{P/S}\, S_0 \tag{9.39}$$

or the potential product output is

$$FP \approx Y_{P/S} S_0 F \tag{9.40}$$

When the specific rate of product formation q_P is constant,

$$\frac{dP^t}{dt} = q_P X^t \tag{9.41}$$

where P^t is the total amount of product in culture.

Substituting $X^t = (V_0 + Ft)X_m$ into eq. 9.41 yields

$$\frac{dP^t}{dt} = q_P X_m (V_0 + Ft) \tag{9.42}$$

Integration of eq. 9.42 yields

$$P^t = P_0^t + q_P X_m \left(V_0 + \frac{Ft}{2}\right) t \tag{9.43}$$