The boundary conditions are

$$S = S_{0i} \qquad \text{at } y = 0$$

$$\frac{dS}{dy} = 0 \qquad \text{at } y = L$$

where L is the thickness of biofilm.

If it is also assumed that the liquid nutrient phase is vigorously agitated and the liquid film resistance is negligible, then $S_0 \approx S_{0i}$. By defining a maximum rate of substrate utilization as $r_m = \mu_m X/Y_{X/S}$ (g subs/cm³ h), we rewrite eq. 9.49 as

$$D_e \frac{d^2 S}{dy^2} = \frac{r_m S}{K_s + S} \tag{9.50}$$

In dimensionless form, eq. 9.50 can be written as

$$\frac{d^2 \bar{S}}{d\bar{y}^2} = \frac{\phi^2 \bar{S}}{1 + \beta \bar{S}} \tag{9.51}$$

where

$$\bar{S} = \frac{S}{S_0}, \qquad \bar{y} = \frac{y}{L}, \qquad \beta = \frac{S_0}{K_s}$$

and

$$\phi = L\sqrt{\frac{\mu_m X}{Y_{X/S} D_e K_s}} = L\sqrt{\frac{r_m}{D_e K_s}} \tag{9.52}$$

Equation 9.51 can be solved numerically. An analytical solution can be derived for the limiting cases of zero or first-order reaction rates.

The maximum rate of substrate flux in the absence of diffusion limitations is given by the following equation:

$$N_s A_s = -A_s D_e \frac{dS}{dy}\bigg|_{y=0} = \frac{r_m S_0}{K_s + S_0}(LA_s) \tag{9.53}$$

where A_s is a surface area of biofilm available for substrate flux, N_s is the substrate flux, and L is the thickness of the biofilm.

In the presence of diffusion limitation, the rate of substrate consumption or flux is expressed in terms of the effectiveness factor.

$$N_s = -D_e \frac{dS}{dy}\bigg|_{y=0} = \eta\left(\frac{r_m S_0}{K_s + S_0}\right)L \tag{9.54}$$

where η is the effectiveness factor, defined as the ratio of the rate of substrate consumption in the presence of diffusion limitation to the rate of substrate consumption in the absence of diffusion limitation. In the absence of diffusion limitations, $\eta \cong 1$, and in the presence of diffusion limitations, $\eta < 1$. The effectiveness factor is a function of ϕ and β. Figure 9.13 is a plot of η versus β for various values of ϕ. The ϕ value should be low