MA324 ASSIGNMENT REPORT

AMLE FOR WEIBULL DISTRIBUTION

GROUP-6

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INTRODUCTION

Estimation of parameters for the Weibull distribution under Type-II Censoring :

Maximum Likelihood Estimation (MLE) is commonly employed for this task. However, the Approximate Maximum Likelihood Estimation (AMLE) approach offers several advantages over MLE, particularly when dealing with scaled parameters under type-II censoring. Its robustness, efficiency, reduced bias, consistency, and statistical validity make it a compelling choice over Maximum Likelihood Estimation (MLE).

To verify this statement we have simulated the absolute biases of MLE and AMLE for complete sample and the relative biases, relative variances and the asymptotic variances of the AMLE for Type-II censored sample. Absolute bias of AMLE obtained for complete sample should be smaller than that of MLE.

MLE FOR COMPLETE SAMPLE

The probability density function and the cumulative distribution function of a two parameter Weibull distribution with scale parameter, θ > zero and shape parameter β >zero are given by,

$$f(xi,\theta,\beta) = \left(\frac{\beta}{\theta}\right)(\frac{xi}{\theta})^{\beta-1}e^{((-xi/\theta)^{\beta})}$$

The cumulative distribution function is given by,

$$F(xi,\theta,\beta) = 1 - e^{(-(\frac{xi}{\theta}))^{\beta}}$$

Where x is the random variable. The method of MLE is a common procedure to estimate parameters of a model's distribution which are assumed to be independent and identically distributed. The parameters are estimated by maximizing the likelihood function. Let x1,x2,x3,...,xn be a sample of size n obtained from a probability density function f(x, θ) where θ is an unknown parameter. The likelihood function is given as,

$$L = \prod_{i=1}^{n} f(xi, \theta\ )$$

The MLE of θ̂ is the value of that maximizes the likelihood function or the log-likelihood function where, $\frac{d(\log L)}{d\ \theta} = 0$.By applying the likelihood function to the Weibull probability density function we get

$$L(xi,\theta,\beta) = \prod_{i=1}^{n} (\beta/\theta)(\frac{\mathrm{xi}}{\theta})^{\beta-1}e^{(-\frac{xi}{\theta})^{\beta}}$$

Taking the logarithms, differentiating with respect to θ and β and equating to zero and simplifying to find θ̂,

$$\widehat{\theta} = (\frac{\sum_{i=1}^{n} xi^{\beta^\wedge}}{n})^{1/\beta^\wedge}$$

$$\beta^\wedge = [\ \frac{\sum_{i=1}^{n} xi^{\beta^\wedge} logxi}{\sum_{i=1}^{n} xi^{\beta^\wedge}} - \frac{1}{n}\sum_{i=1}^{n} logxi\ \ ]^\wedge - 1$$

Approximate MLE FOR CENSORED SAMPLE

The Weibull distribution with pdf

$$f(x;\,\beta,\theta) = \frac{\beta}{\theta^{\beta}} x^{\beta-1} e^{-\left(\frac{x}{\theta}\right)^{\beta}}, x > 0, \theta > 0, \beta > 0$$

And the cdf

$$F(x;\,\beta,\theta) = 1 - e^{-\left(\frac{x}{\theta}\right)^{\beta}}$$

After censoring some final observation, let $X_{r+1:n} \leq X_{r+2:n} \leq \ldots \leq X_{n-s:n}$ be the available Type-II censored sample from the weibull distribution.

The likelihood function for the censored sample

$$L = \frac{n!}{r!s!}\theta^{-A}\{F(Z_{r+1:n})\}^{r}\{1-F(Z_{n-s:n})\}^{s}\prod_{i=r+1}^{n-s} f(Z_{i:n})$$

Where A=n-r-s, $Z_{i:n}=X_{i:n}/\theta$.

By differentiating the logarithm of the likelihood function for θ and expanding functions according to taylor series and approximating them as follows

$$\frac{f(Z_{r+1:n})}{F(Z_{r+1:n})} \simeq \alpha - \delta Z_{r+1:n},$$

$$\frac{f(Z_{n-s:n})}{\{1-F(Z_{n-s:n})\}} \simeq \kappa + \eta Z_{n-s:n},$$

$$\frac{f'(Z_{i:n})}{f(Z_{i:n})} \simeq \nu_i + \gamma_i Z_{i:n},$$

Where pi=i/n+1 and qi=1-pi

$$
\begin{aligned}
\alpha =&\beta\left((-\ln q_{r+1})^{1/\beta}\right)^{\beta-1} q_{r+1}/p_{r+1} \\
&-(-\ln q_{r+1})^{1/\beta}\Big[\Big(\beta(\beta-1)\left((-\ln q_{r+1})^{1/\beta}\right)^{\beta-2} q_{r+1} \\
&-\beta^2\left((-\ln q_{r+1})^{1/\beta}\right)^{2\beta-2} q_{r+1}\Big)/p_{r+1} \\
&-\beta^2\left((-\ln q_{r+1})^{1/\beta}\right)^{2\beta-2} q_{r+1}^2/p_{r+1}^2\Big],
\end{aligned}
$$

$$
\kappa = \beta\left((-\ln q_{n-s})^{1/\beta}\right)^{\beta-1} - (-\ln q_{n-s})^{1/\beta}\beta(\beta-1)\left((-\ln q_{n-s})^{1/\beta}\right)^{\beta-2};
$$

$$
\eta = \beta(\beta-1)\left((-\ln q_{n-s})^{1/\beta}\right)^{\beta-2}
$$

$$
\begin{aligned}
\nu_i =&\left((\beta-1)(-\ln q_i)^{-1/\beta} - \beta\left((-\ln q_i)^{1/\beta}\right)^{\beta-1}\right) \\
&-(-\ln q_i)^{1/\beta}\Big[(\beta-1)(\beta-2)(-\ln q_i)^{-2/\beta} \\
&-3\beta(\beta-1)\left((-\ln q_i)^{1/\beta}\right)^{\beta-2} + \beta^2\left((-\ln q_i)^{1/\beta}\right)^{2\beta-2} \\
&-\left((\beta-1)(-\ln q_i)^{-1/\beta} - \beta\left((-\ln q_i)^{1/\beta}\right)^{\beta-1}\right)^2\Big],
\end{aligned}
$$

$$
\begin{aligned}
\gamma_i =& (\beta-1)(\beta-2)(-\ln q_i)^{-2/\beta} - 3\beta(\beta-1)\left((-\ln q_i)^{1/\beta}\right)^{\beta-2} \\
&+\beta^2\left((-\ln q_i)^{1/\beta}\right)^{2\beta-2} \\
&-\left((\beta-1)(-\ln q_i)^{-1/\beta} - \beta\left((-\ln q_i)^{1/\beta}\right)^{\beta-1}\right)^2.
\end{aligned}
$$

We can now find θ by approximating the equation and differentiating the log likelihood function,the derived AMLE of θ as follows;

$$
\hat{\theta} = \frac{\{-B + (B^2 + 4AC)^{1/2}\}}{2A}
$$

Where B and C are

$$
B = r\alpha X_{r+1:n} - s\kappa X_{n-s:n} + \sum_{i+r+1}^{n-s} \nu_i X_{i:n}
$$

$$C = r\delta X_{r+1:n}^2 + s\eta X_{n-s:n}^2 - \sum_{i+r+1}^{n-s} \gamma_i {X_{i:n}}^2.$$

While the variance of the AMLE can be calculated as follows

$$E\Big(-\frac{d^2 \ln L^*}{d\theta^2}\Big) = D/\theta^2$$

Where :

$$D = 3\Big(r\delta E(Z_{r+1:n}^2) + s\eta E(Z_{n-s:n}^2) - \sum_{i=r+1}^{n-s} \gamma_i E(Z_{i:n}^2)\Big)$$
$$- 2\Big(r\alpha E(Z_{r+1:n}) - s\kappa E(Z_{n-s:n}) + \sum_{i=r+1}^{n-s} \nu_i E(Z_{i:n})\Big) - A.$$

$$E(Z_{i:n}) = \frac{n!}{(i-1)!(n-i)!}\Gamma(1+\frac{1}{\beta})\sum_{r=0}^{i-1}(-1)^r\binom{i-1}{r}\Big/(n-i-r+1)^{1+1/\beta}$$

$$E(Z_{i:n}^2) = \frac{n!}{(i-1)!(n-i)!}\Gamma(1+\frac{2}{\beta})\sum_{r=0}^{i-1}(-1)^r\binom{i-1}{r}\Big/(n-i-r+1)^{1+2/\beta}$$

The input Xi's of the Weibull distribution were found using the inverse transform method using the CDF and the following are our observations of absolute bias of the MLE and the AMLE of the Weibull distribution for complete sample(n=10)

θ	β=1 MLE	β=1 AMLE	β=2 MLE	β=2 AMLE	β=3 MLE	β=3 AMLE
0.5	0.00126	0.00126	0.00635	0.00067	0.00608	0.00604
1	0.00097	0.00097	0.01163	0.00163	0.01302	0.01092
2	0.01641	0.0082	0.01943	0.00354	0.0216	0.00884

The following are the values obtained for the relative bias, the relative variance of the AMLE θ̂ of the Weibull scale parameter from Type-II censored samples.

r	s	n	E(θ̂- θ)/ θ	VAR(θ̂)/θ^2	AVAR(θ̂)/θ^2
0	0	10	0.00248	0.10494	0.10313
		20	0.00145	0.05035	0.05213
		30	0.00497	0.0354	
0	1	10	-0.00089	0.10688	0.09262
		20	0.00064	0.05248	0.0522
		30	-0.00163	0.03428	
0	2	10	0.00187	0.12827	0.19827
		20	0.00146	0.05629	0.07139
		30	0.00497	0.0367	
0	3	10	0.00935	0.14562	0.17436
		20	-0.00049	0.06029	0.06541
		30	-0.00179	0.03608	

0	4	20	-0.00531	0.06488	0.07001
		30	-0.00192	0.03639	
1	0	10	0.04002	0.10757	0.99644
		20	0.01754	0.05259	0.05213
		30	0.01178	0.03507	
1	1	10	0.0339	0.12306	0.09238
		20	0.02296	0.05805	0.0522
		30	0.01704	0.03603	
1	2	10	0.04109	0.14113	0.19825
		20	0.02993	0.06036	0.0712
		30	0.02356	0.03681	
1	3	10	0.04909	0.15953	0.17436
		20	0.02093	0.06306	0.06562
		30	0.01712	0.03686	
1	4	20	0.01778	0.0633	0.0613
		30	0.01874	0.04219	
2	0	10	0.04094	0.11367	0.11064
		20	0.02621	0.05472	0.05213
		30	0.02112	0.03552	
2	1	10	0.04419	0.12131	0.09483
		20	0.03175	0.05789	0.06728
		30	0.01713	0.03576	
2	2	10	0.03865	0.13789	0.19767
		20	0.02274	0.05775	0.05478
		30	0.01733	0.03772	
2	3	10	0.05712	0.16234	0.17438
		20	0.02508	0.06267	0.06391

		30	0.02128	0.04078	
2	4	20	0.03175	0.06848	0.06934
		30	0.01715	0.04173	
3	0	10	0.05361	0.11452	0.1345
		20	0.03018	0.05314	0.05243
		30	0.02101	0.03509	
3	1	10	0.03949	0.12959	0.1542
		20	0.02745	0.05479	0.05243
		30	0.01757	0.03528	
3	2	10	0.04217	0.14111	0.1522
		20	0.03283	0.05991	0.0602
		30	0.01894	0.03835	
3	3	10	0.04993	0.16471	0.17426
		20	0.03571	0.06625	0.06728
		30	0.02006	0.04078	
3	4	20	0.03331	0.06751	0.06834
		30	0.02559	0.04056	
4	0	20	0.02623	0.05488	0.05427
		30	0.01926	0.03505	
4	1	20	0.02672	0.05663	0.05234
		30	0.02601	0.03736	
4	2	20	0.02507	0.06028	0.06453
		30	0.01867	0.03884	
4	3	20	0.03916	0.06501	0.06234
		30	0.02521	0.03884	
4	4	20	0.03272	0.07079	0.075
		30	0.02113	0.04091	

KEY FINDINGS

TAKEAWAY #1 – FOR COMPLETE SAMPLES

We calculated absolute biases of MLE and AMLE based on 3000 monte carlo runs for complete sample for multiple β and θ values as given in the table.We concluded that for a fixed β with the increase of θ the value of the MLE and AMLE were increased. The note worthy observation was that the value of AMLE was less or equal to that of MLE for a fixed θ and β. If the AMLE consistently provides estimates that are similar to or smaller than the MLE across different datasets or under varying conditions, it might suggest that the AMLE is more robust to outliers or violations of assumptions

TAKEAWAY – FOR CENSORED TYPE II SAMPLES

We calculated relative biases, relative variance and asymptotic variances of the AMLE $\hat{\theta}$ for Type II censored samples for 3000 monte carlo runs for multiple values of n=[10,20,30] and r,s ranging from 0 to 4 while fixing β=1 and θ=0.5. We observed that variance and bias of the AMLE decreases as n increases for given r and s. As the sample size n increases, estimators tend to exhibit reduced bias. This phenomenon is often associated with the Law of Large Numbers, which states that as the sample size grows, the sample mean converges to the population mean. In the context of maximum likelihood estimation, larger sample sizes provide more information about the underlying distribution, leading to more precise parameter estimates and, consequently, reduced bias. This reduction in bias occurs because larger samples provide a more comprehensive representation of the population, reducing the impact of sampling variability on the estimation process.