🔍 Overview

In this lesson, you'll explore the **foundations of linear systems** — sets of linear equations — and understand their **geometric meaning** in two and three dimensions. You'll learn how solutions to these systems correspond to **points of intersection** between lines or planes, and gain insight into when solutions exist, are unique, or don't exist at all.

By the end of this lesson, you will:

- Define a system of linear equations.
- Solve simple 2×2 systems algebraically and geometrically.
- Interpret solutions geometrically (intersecting, parallel, coincident lines).
- Understand the three possible outcomes for linear systems.
- Extend intuition to 3D (planes in space).

📐 1. What Is a Linear System?

A **system of linear equations** is a collection of two or more linear equations involving the same set of variables.

Example (2×2 System):

$$\begin{aligned} &(1)\quad 2x + y = 5 \\ &(2)\quad x - y = 1 \end{aligned}$$

We seek values of x and y that satisfy **both equations simultaneously**.

> ✅ A **solution** to a linear system is an assignment of values to the variables that makes **all equations true** at once.

🧮 2. Solving a System Algebraically

Let's solve the above system using **substitution** or **elimination**.

Using Elimination:

Add equations (1) and (2):

$$(2x + y) + (x - y) = 5 + 1 \Rightarrow 3x = 6 \Rightarrow x = 2$$

Plug $x = 2$ into equation (2):

$$2 - y = 1 \Rightarrow y = 1$$

✅ Solution: $(x, y) = (2, 1)$

This is the **only pair** that satisfies both equations.

🖼️ 3. Geometric Interpretation in 2D

Each linear equation in two variables (x, y) represents a **line** in the 2D coordinate plane.

So, solving a 2×2 system means:

> 🔎 **Finding the point(s) where the two lines intersect.**

Let's plot our example:

- Line 1: $2x + y = 5$ → Passes through (0,5) and (2.5,0)
- Line 2: $x - y = 1$ → Passes through (0,-1) and (1,0)

They intersect at $(2, 1)$ — exactly our algebraic solution!

🎯 Three Possible Cases in 2D:

Case	Geometry	Number of Solutions	Description
1	✅ Intersecting lines	**One unique solution**	Lines cross at one point
2	❌ Parallel lines	**No solution**	Same slope, different intercepts
3	🔁 Coincident lines	**Infinitely many solutions**	Same line, just different forms

Example: No Solution (Parallel)

$$\begin{aligned} x + y &= 3 \\ x + y &= 5 \end{aligned}$$

→ Parallel lines → ❌ No intersection

Example: Infinite Solutions (Same Line)

$$\begin{aligned} 2x + 3y &= 6 \\ 4x + 6y &= 12 \quad \text{(just doubled)} \end{aligned}$$

→ Both represent the same line → 🔁 Every point on the line is a solution

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🧊 4. Extending to 3D: Planes in Space

Now consider three variables: x, y, z. Each linear equation represents a **plane** in 3D space.

A system of three equations defines **three planes**. The solution is the point (or set of points) where **all three planes intersect**.

Example (3×3 System):

$$\begin{aligned} x + y + z &= 6 \\ 2x - y + z &= 3 \\ x + z &= 4 \end{aligned}$$

Solving this (via substitution or matrix methods) gives:
$x = 1,\ y = 2,\ z = 3$ → A single point: $(1, 2, 3)$

Geometrically: The three planes meet at a **single point**.

🎯 Possible Cases in 3D:

- ✅ **One solution**: Planes intersect at a single point.
- ❌ **No solution**: Planes are parallel or form a "triangular prism" with no common point.
- 🔁 **Infinite solutions**: Planes intersect along a line or are identical.

> 💡 Just like in 2D, consistency and independence determine the nature of the solution.

🧠 Key Concepts Summary

Concept	Meaning
Linear Equation	An equation like $ax + by = c$, graphed as a straight line (or plane)
System of Equations	Multiple equations to solve together
Solution Set	All variable assignments that satisfy every equation
Consistent System	Has **at least one** solution
Inconsistent System	Has **no solution**
Independent Equations	Each adds new information (not redundant)

Concept	Meaning
Dependent Equations	One can be derived from others → infinite solutions

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🧩 Real-World Connection

Linear systems model real situations:

- Balancing chemical equations
- Finding break-even points in business
- Computer graphics (intersection of surfaces)
- GPS triangulation (solving for position)

Understanding the **geometry** behind them helps you **visualize and verify** solutions.

📝 Check Your Understanding

Q1: What is the geometric meaning of the solution to a 2×2 linear system?

> A: The point where the two lines intersect.

Q2: Can two lines in a plane have exactly two intersection points? Why or why not?

> A: No. Two distinct lines can only intersect 0 times (parallel), 1 time (crossing), or infinitely many (same line). Lines are straight — they can't "curve back".

Q3: If three planes in 3D space are parallel, how many solutions does the system have?

> A: Zero — no common point of intersection.

🚀 What's Next?

In the next lesson, we'll formalize solving linear systems using:

- **Row operations**
- **Augmented matrices**
- **Gaussian elimination**

This will give you a powerful algorithmic tool to solve larger systems (4×4, 10×10, or more!) efficiently.

📚 Additional Resources

- Khan Academy: Systems of Equations
- Interactive 3D graphing: GeoGebra 3D Calculator