2

Realistic Projectile Motion

In this chapter we consider several problems involving the motion of objects through the atmosphere. The problems are all described by ordinary differential equations in which initial values are given and all can be solved using the Euler method, which was introduced in the last chapter. These problems are good examples of interesting physics involving the mechanics of macroscopic objects, which can't be solved analytically, but can be easily tackled with a computer.

We begin with a discussion of the motion of a bicycle traveling on flat terrain. We will find that air resistance must be included if we are to obtain a realistic description of the problem, and this leads us to a simple but fairly accurate model of the drag force due to the atmosphere. Next we treat projectile motion in two dimensions as we consider the trajectory of a shell fired by a large cannon. Again, the effect of air resistance is important, but now its variation with altitude plays a key role. We proceed to the national pastime, baseball, and consider the trajectory of a batted ball and the motion of thrown balls (curve balls and knuckleballs). To model a batted ball we are led to consider air resistance a little more realistically than in the earlier problems, while for a thrown ball we must also include spin dependent forces. These themes are developed further in our discussion of the motion of golf balls, where we answer the eternal question: "Why do golf balls have dimples?"

2.1 Bicycle Racing: The Effect of Air Resistance

The bicycle is an extremely efficient form of transportation, a fact that is well known to anyone who rides one. Our goal in this section is to understand the factors that determine the ultimate speed of a bicycle and to estimate this speed for a realistic case. We will begin by ignoring friction; we'll have to add it eventually, of course, but let us first understand how to deal with the simpler case without friction.

Our equation of motion is Newton's second law, which can be written in the form

$$\frac{dv}{dt} = \frac{F}{m}, \tag{2.1}$$

where v is the velocity, m is the mass of the bicycle-rider combination, t is the time, and F is the force on the bicycle that comes from the effort of the rider (here we will assume that the bicycle is moving on flat terrain). Dealing properly with F is complicated by the mechanics of a bicycle, since the force exerted by the rider is transmitted to the wheels by way of the chainring, gears, etc. This makes it very difficult to derive an accurate expression for F. However, there is another way to attack this problem that avoids the need to know the force. This alternative approach involves formulating the problem in terms of the power generated by the rider. Physiological studies of elite racing bicyclists have shown that these athletes are able to produce a power output of approximately 400 watts over extended periods of time ($\sim$ 1 h). Using work-energy ideas we can rewrite (2.1) as

$$\frac{dE}{dt} = P, \tag{2.2}$$

where E is the total energy of the bicycle-rider combination, and P is the power output of the rider. (This assumes implicitly that very little energy is lost to friction in the bicycle itself; we'll include other sources of friction in a moment.) For a flat course the energy is all kinetic, so $E = \frac{1}{2}mv^2$, and $dE/dt = mv(dv/dt)$. Inserting this into (2.2) yields

$$\frac{dv}{dt} = \frac{P}{mv}. \tag{2.3}$$

If P is a constant, (2.3) can be solved analytically. Rearranging gives

$$\int_{v_0}^{v} v'\, dv' = \int_0^t \frac{P}{m}\, dt', \tag{2.4}$$

where v_0 is the velocity of the bicycle at $t = 0$. Integrating both sides and solving for v then leads to

$$v = \sqrt{v_0^2 + 2\,P\,t/m}\,. \tag{2.5}$$

While this is the correct solution of the equation of motion (2.2), it cannot be the whole story, since it predicts that the velocity will increase without bound at long times. We will correct this "unphysical" result in a moment, when we generalize our model to include the effect of air resistance. The new term we will add to the equation of motion will require us to develop a numerical solution, so with that in mind we consider a numerical treatment of (2.3). We begin with the finite difference form for the derivative of the velocity

$$\frac{dv}{dt} \approx \frac{v_{i+1} - v_i}{\Delta t}, \tag{2.6}$$

where we have assumed small, discrete time steps of size Δt and taken v_i to be the velocity at time $t_i \equiv i\,\Delta t$, where i is an integer (this should be familiar from Chapter 1). Inserting this into (2.3) we obtain, after a little rearranging

$$v_{i+1} = v_i + \frac{P}{mv_i}\,\Delta t\,. \tag{2.7}$$

The approximation made in (2.6) leads to correction terms here that are proportional to $(\Delta t)^2$ (and higher powers).

Given the velocity at time step i (i.e., v_i), we can use (2.7) to calculate an *approximate* value of the velocity at the next step, v_{i+1}. Hence, if we know the initial velocity, v_0, we can obtain v_1, then v_2, and so on, and thereby estimate the velocity at all future times. This is just the Euler method, which we encountered in Chapter 1. There are more sophisticated methods for numerically solving differential equations of this form, and a few of them are described in Appendix 1. In this book we will only rarely encounter problems for which the Euler method, or simple variants, are not adequate. However, you should be aware that: (1) methods that are more powerful[1] than the Euler method exist; (2) these more powerful methods can certainly be used to solve all of the problems in this book where we use the Euler method; and (3) if you are going to do this sort of thing for a living it is worth your while to learn about these other methods. In taking what some purists might consider a quick-and-dirty approach (our use of the Euler method), we are *not* trying to minimize the importance of other algorithms. Our intent is merely to use the simplest numerical method (which will give the correct solution, of course) appropriate for the job, so that we can emphasize the *physics* of our problems and not let the numerical techniques get in the way. Students and instructors are encouraged to follow their own tastes here, and with all of the other problems discussed in this book, and to use the methods with which they feel most comfortable.

A program that performs this calculation is listed below. The overall structure is similar to the one we used in the nuclear decay problem in Chapter 1. The `main` program references the graphics libraries and declares the arrays `t()` and `velocity()`, which hold values for the time and velocity, respectively. The `initialize` subroutine initializes the necessary variables, which all have names that fit the problem. `dt` is the time step, `power` is the power output of the bicyclist P, and `mass` is the mass of the bicycle-rider combination. The routine `calculate` then uses the Euler method to calculate the velocity as a function of time. After the calculation is finished, the `mat redim` statement is used to resize the arrays to make them exactly the length used in the calculation (`nmax`). This is handy when using the *True Basic* graphics routines, but may not be needed if you use a different language package. The `display` routine is not listed here. It is very similar to that used in the nuclear decay problem in Chapter 1 and is included in the complete listing of this program, which is given in Appendix 4.

```
! Simulation of velocity vs. time for a bicyclist
! 10-1-94   N. Giordano
! assume no air resistance
program bike
   option nolet
   library "sglib*","sgfunc*"
   dim t(5000),velocity(5000)
   call initialize(t,velocity,dt,power,mass,nmax)
   call calculate(t,velocity,dt,power,mass,nmax)
   call display(t,velocity)
end
```

[1] By "more powerful" we mean that they yield more accurate solutions for a given amount of computer time, or that they are more stable in a numerical sense.

```
! t() = time    v() = velocity
! dt = time step    power = rider power
! mass = mass of rider + bicycle
sub initialize(t(),v(),dt,power,mass,nmax)
   t(1) = 0
   v(1) = 4             ! m/s - all units are SI
   dt = 1               ! second
   power = 400          ! watts
   mass = 70            ! kg
   tmax = 200           ! seconds
   nmax = tmax / dt     ! total number of time steps required
end sub
sub calculate(t(),v(),dt,pmax,mass,nmax)
   for i = 2 to nmax
      t(i) = t(i-1) + dt
      v(i) = v(i-1) + pmax * dt / (mass * v(i-1))     ! Euler method
   next i
   mat redim t(nmax),v(nmax)     ! trim arrays to the size actually used
end sub
```

We are now ready to compute a numerical solution. We assume that the bicycle starts with a velocity of $v_0 = 4$ m/s (about 10 mph) and take $P = 400$ W, the value obtained from physiological measurements of well-trained athletes, as mentioned above. The last point to consider is the choice of Δt. Roughly speaking, Δt should be sufficiently small that the velocity changes only a little during such an interval. What it means to be "sufficiently" small is hard to say, in general. A useful rule of thumb is to begin with a time step that is about 1 percent of any time scales in the problem, and then repeat the calculation with several smaller values.[2] Smaller time steps will give smaller correction terms [e.g., in (2.7)] and thus more accurate results, but the calculation will take a computational time proportional to $(\Delta t)^{-1}$, so there is a tradeoff here. For our bicycle problem it will turn out that time steps smaller than about 1 s are adequate. It is usually instructive to repeat a calculation using different values of the time step so as to observe how the solution converges to the correct one as Δt is reduced.

The results for our frictionless bicycle are shown in Figure 2.1, where we have used a mass of 70 kg for the bicycle-rider combination (elite bicycle racers tend to be rather slender). We see immediately that, as anticipated above, our model has a serious problem; the velocity reaches 45 m/s (about 100 mph) in less than 3 min. While this performance would not be particularly impressive for a car, it is certainly not within reach of any known bicycles (or bicyclists). We also see that v appears to grow indefinitely, and this makes the origin of our problem clear. We have not included any sources of dissipation, so given our assumption of constant power exerted by the rider, the kinetic energy will increase without limit. If we want to have a realistic model, we must add some mechanism for energy loss.

For a well-tuned bicycle traveling at more than about 5 or 10 mph, the energy lost to friction in the hubs and tires is negligible compared to that caused by air resistance, that is, atmospheric drag. Thus, a reasonably realistic model of the motion of a bicycle need only consider this one source of friction. As we will come to appreciate in later sections, the physics of air resistance is a very complicated problem.

[2]While the "characteristic" time scale is often not a unique or precise quantity, in some cases it is easy to estimate. For example, in the radioactive decay problem in Chapter 1 it was the time constant τ of the decay. In the present problem, one natural choice is the time it takes to attain a velocity of the order of terminal velocity (see Figure 2.2).

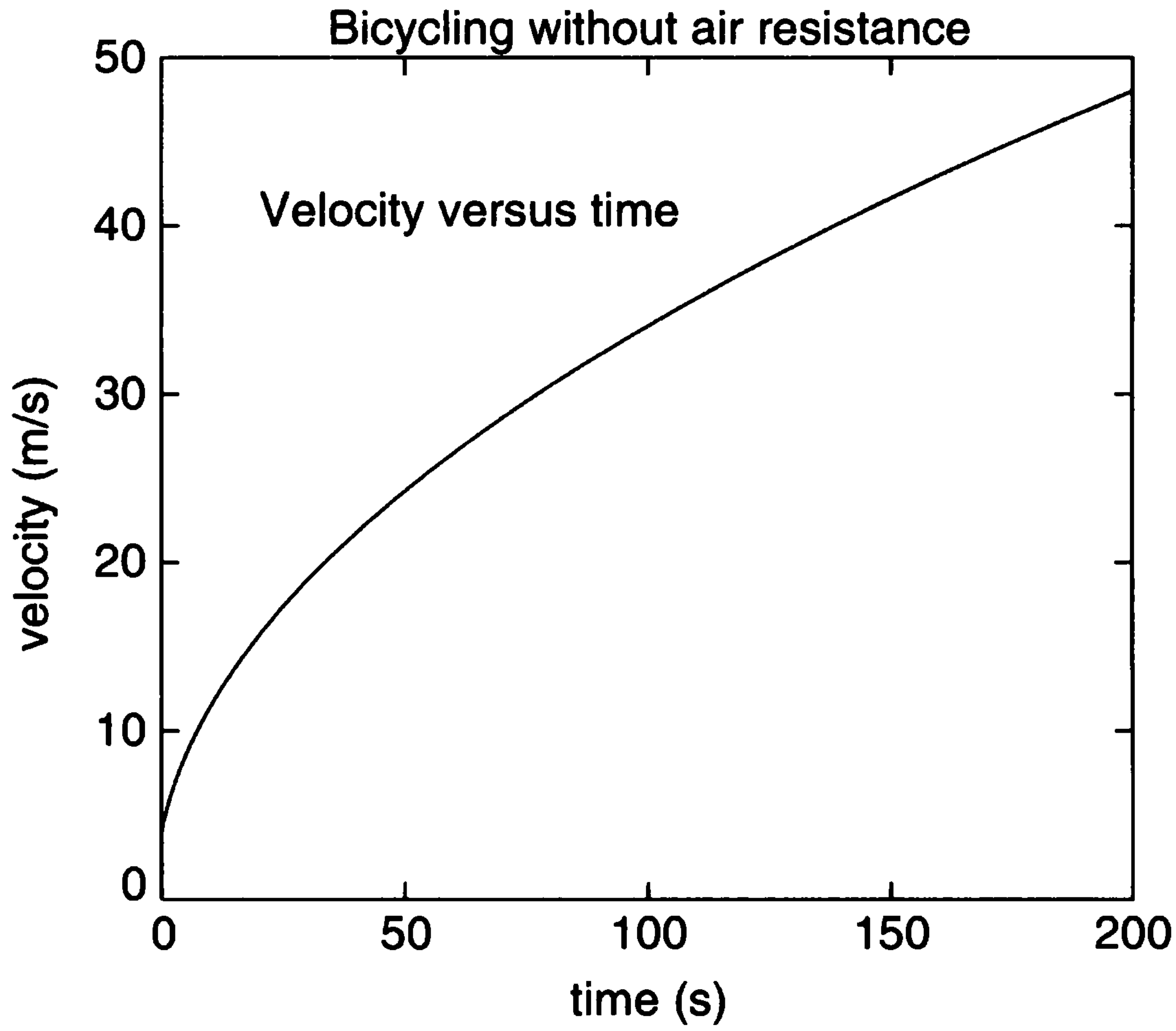

Figure 2.1: Velocity as a function of time for our bicycle problem, assuming no air resistance. The mass of the bicycle-rider combination was 70 kg, the initial velocity was 4 m/s, and the time step was 0.1 s. Here and in the other figures in this chapter, the results at each time step are connected to obtain an essentially smooth curve.

In general, this force can be written in the fairly innocent form

$$F_{\mathrm{drag}} \approx -B_1 v - B_2 v^2 . \tag{2.8}$$

You will no doubt notice that (2.8) bears a strong resemblance to a Taylor expansion. At extremely low velocities the first term dominates, and its coefficient B_1 can be calculated for objects with simple shapes. This is known as Stokes' law and is considered in most elementary mechanics texts. However, at any reasonable velocity the v^2 term in (2.8) dominates for most objects. Moreover, B_2 cannot be calculated exactly for objects as simple as a baseball, and certainly not for a complicated object like a bicycle. We can, however, make an *approximate* estimate of B_2, as follows. As an object moves through the atmosphere, it must push the air in front of it out of the way. The mass of air moved in time dt is $m_{\mathrm{air}} \sim \rho A v dt$, where ρ is the density of air and A the frontal area of the object. This air is given a velocity of order v, and hence its kinetic energy is $E_{\mathrm{air}} \sim m_{\mathrm{air}} v^2/2$. This is also the work done by the drag force (the force on the object due to air resistance) in time dt, so $F_{\mathrm{drag}} v dt = E_{\mathrm{air}}$. Putting this all together we find

$$F_{\mathrm{drag}} \approx -C\rho A v^2 . \tag{2.9}$$

C is known as the drag coefficient, and our simple argument predicts that it is equal to $\frac{1}{2}$. However, we caution that our calculation was only approximate; while we expect it to give the correct functional dependence of F_{drag} on A and v, we certainly do not expect the precise value it predicts for C to be correct. The best way to determine the drag coefficient of any particular object is via wind tunnel measurements, or similar experiments.

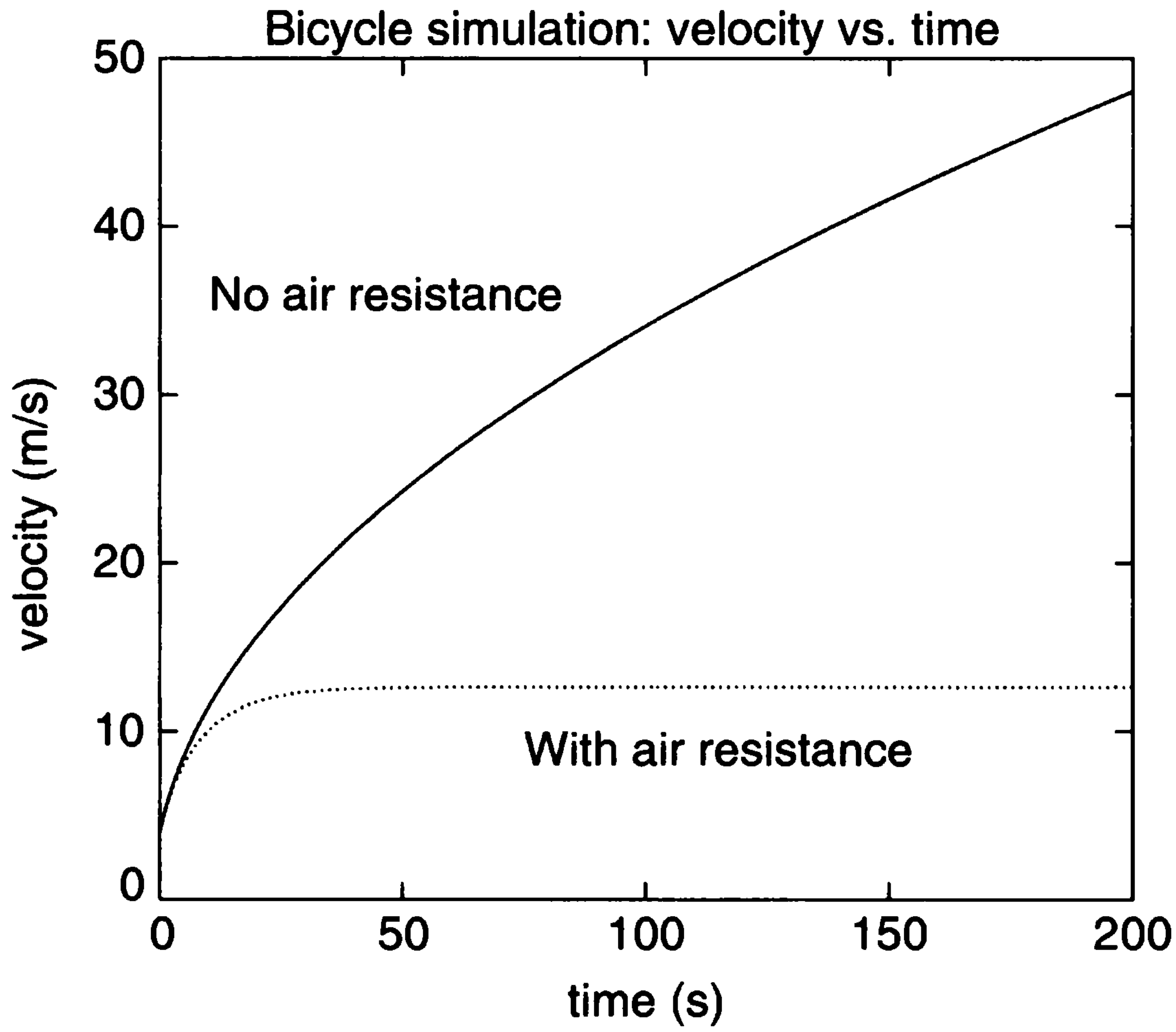

Figure 2.2: Velocity as a function of time for our bicycle problem, with and without air resistance. The drag coefficient was 0.5, the frontal area 0.33 m^2, the mass of the bicycle-rider combination 70 kg, the initial velocity 4 m/s, and the time step 0.1 s.

It is easy to include this drag force in our calculation; Equation (2.9) contributes another term to the right-hand side of (2.7), which becomes

$$v_{i+1} = v_i + \frac{P}{m v_i} \Delta t - \frac{C \rho A v_i^2}{m} \Delta t \,, \tag{2.10}$$

and we can use the Euler method to obtain v as a function of t as before. The program given above can again be used, provided that we add the drag term to the Euler equation in the `calculate` subroutine (a task we will leave to the reader).

The result obtained from solving (2.10) is shown in Figure 2.2, where we have assumed a frontal area of $A = 0.33$ m^2, which is a typical estimate for a rider in a racing crouch. We obtain an ultimate (i.e., terminal) velocity of approximately 13 m/s (about 30 mph), which is in reasonable accord with the performance of an elite racer.[3]

Besides giving a good estimate of the speed of a bicycle racer, our model provides some insight into racing strategy. It is clear from Figure 2.2 that reducing the air resistance is the key to increasing the speed for a given power exertion or decreasing the power required to maintain a given speed. This is, of course, why bicycle racers often ride in a pack. The idea is to let the riders at the front of the pack break the wind, and thereby lower the effective frontal area for those in the pack who "draft" off those

[3]It is interesting to note that the 1995 record for the distance traveled by bicycle in 1 h is a little more than 55 km. This corresponds to an average velocity of approximately 15 m/s, which is within about 15 percent of our result for the terminal velocity.

in front. For this reason, it is extremely difficult for a single rider to escape from a large group; the energy required of the single rider is much greater than that exerted by the riders in the pack (assuming they share time at the front).[4] This also explains why the relatively recent development of "aerobars," handle bars that allow the rider to assume an extremely narrow profile and thereby reduce his frontal area, enables an isolated rider to go faster than with conventional handlebars. This is useful in individual events such as time trials and triathalons, where following closely behind other riders is not allowed.

With a little work to include air resistance, we have developed a simple but reasonably accurate model of bicycle performance. Both our model and the numerical approach can be easily generalized to include other factors, which will be our job in the following sections.

Exercises

1. Compare the exact solution for the velocity as a function of time without air resistance with the numerical results in Figure 2.1 and show that they agree.

2. Investigate the effect of varying both the rider's power and frontal area on the ultimate velocity. In particular, for a rider in the middle of a pack, the effective frontal area is about 30 percent less than for a rider at the front. How much less energy does a rider in the pack expend than does one at the front, assuming they both move at a velocity of 13 m/s?

*3. Generalize the model to treat motion through mountainous terrain. A steep hill is one with a 10 percent grade (that is, $\tan\theta = 0.1$, where θ is the angle the hill makes with the horizontal). Calculate how fast our bicyclist can travel up and down such a slope. Does racing strategy change in these situations? Determine what conditions (the steepness of the grade and the rider's frontal area) would be required for a bicycle to reach a velocity of 70 mph. This is reportedly the speed that professional riders sometimes attain on steep descents.[5]

*4. You might wonder why we did not let our bicycle begin from rest, but instead gave it a nonzero initial velocity. The reason for this is that (2.7) breaks down when $v_i = 0$, since then the term involving P is infinite.[6] If $v_i = 0$, then for a nonzero P the derivative dv/dt is, according to (2.3), infinite. This is difficult to handle in a numerical approach, and it also doesn't make sense from a physical point of view. The problem arises from our assumption that the bicyclist maintains a constant power output. This assumption must break down when the bicycle has a very small velocity, since it would then require that the rider exert extremely large forces (recall that the instantaneous power is the product of the force and the velocity). At low velocities it is more realistic to assume that the rider is able to exert a constant force. To account for this we can modify our bicycle model so that for small v there is a constant force on the bicycle, F_0, which leads to the equation of motion

$$\frac{dv}{dt} = \frac{F_0}{m} . \tag{2.11}$$

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[4]Estimates are that the effective frontal area is reduced by approximately 30 percent for a rider in the middle of a pack, as compared with a rider at the front.

[5]It is interesting to also include the effect of a tail wind in this calculation, as discussed in the section on the motion of a batted baseball.

[6]The problem is also evident from (2.3), and thus is not a product of the Euler method.

The corresponding Euler equation is

$$v_{i+1} = v_i + \frac{F_0}{m} \Delta t , \tag{2.12}$$

and the difficulty that occurs when $v = 0$ is eliminated.

Rewrite your bicycle program to incorporate (2.12). That is, use the Euler method with (2.12) when the velocity is small, and (2.7) when v is large. Let the crossover from small to large v occur when the power ($= F_0 v$) reaches P. Use the same parameters as in Figure 2.2, and take $F_0 = P/v^*$ where $v^* = 7$ m/s (this corresponds to a force approximately twice that found when the bicycle is traveling at its maximum velocity, which seems like a reasonable approximation).

2.2 Projectile Motion: The Trajectory of a Cannon Shell

The Euler method we used to treat the bicycle problem can easily be generalized to deal with motion in two spatial dimensions. To be specific, we consider a projectile such as a shell shot by a cannon. We have a very large cannon in mind, and the large size will determine some of the important physics. If we ignore air resistance, the equations of motion, which are again obtained from Newton's second law, can be written as

$$\begin{aligned} \frac{d^2x}{dt^2} &= 0 \\ \frac{d^2y}{dt^2} &= -g , \end{aligned} \tag{2.13}$$

where x and y are the horizontal and vertical coordinates of the projectile, and g is the acceleration due to gravity. These are second-order differential equations, as opposed to the first-order equations we have encountered so far, so we must generalize our approach a bit. We have seen that with a first-order equation, such as (2.1), it is possible to use a finite difference approximation for the derivative (2.6) to obtain a simple algebraic equation containing the variable of interest at two adjacent time steps. However, if we were to take the same approach with one of the equations in (2.13) and write a finite difference approximation to the second derivative, we would obtain a more complicated algebraic equation involving our variable at three time steps. We will see how this works out in Chapter 6 (and later), as sometimes this is the best approach. However, in the present case it is possible to avoid this complication by recasting the differential equations in the following way.

Let us write each of these second-order equations as two first-order differential equations

$$\begin{aligned} \frac{dx}{dt} &= v_x \\ \frac{dv_x}{dt} &= 0 \\ \frac{dy}{dt} &= v_y \\ \frac{dv_y}{dt} &= -g , \end{aligned} \tag{2.14}$$

where v_x and v_y are the x and y components of the velocity. While we now have twice as many equations to deal with (four altogether), we can use our standard Euler approach to solve each one. To use the Euler method, we write each derivative in finite difference form, as in (2.6), which leads to

$$\begin{aligned} x_{i+1} &= x_i + v_{x,i}\,\Delta t \\ v_{x,i+1} &= v_{x,i} \\ y_{i+1} &= y_i + v_{y,i}\,\Delta t \\ v_{y,i+1} &= v_{y,i} - g\,\Delta t\,. \end{aligned} \tag{2.15}$$

Given the initial values of x, y, v_x, and v_y, we can use (2.15) to estimate their values at later times. These estimates are *approximate*, since there are correction terms to (2.15) that are of order $(\Delta t)^2$ and higher. By choosing Δt to be sufficiently small, we will usually be able to make these corrections negligible. However, their existence should not be forgotten.

In our treatment of the bicycle problem we found that air resistance was very important, so we now add that to the model. As was the case with a bicycle, we will assume that the magnitude of the drag force on our cannon shell is given by

$$F_{\text{drag}} = -B_2\,v^2\,, \tag{2.16}$$

where $v = \sqrt{v_x^2 + v_y^2}$ is the speed of the shell. This force is always directed opposite to the velocity, so we must consider the vector components as illustrated in Figure 2.3. We find

$$F_{\text{drag},x} = F_{\text{drag}}\cos\theta = F_{\text{drag}}\,v_x\,/\,v\,, \tag{2.17}$$

with a similar expression for $F_{\text{drag},y}$.

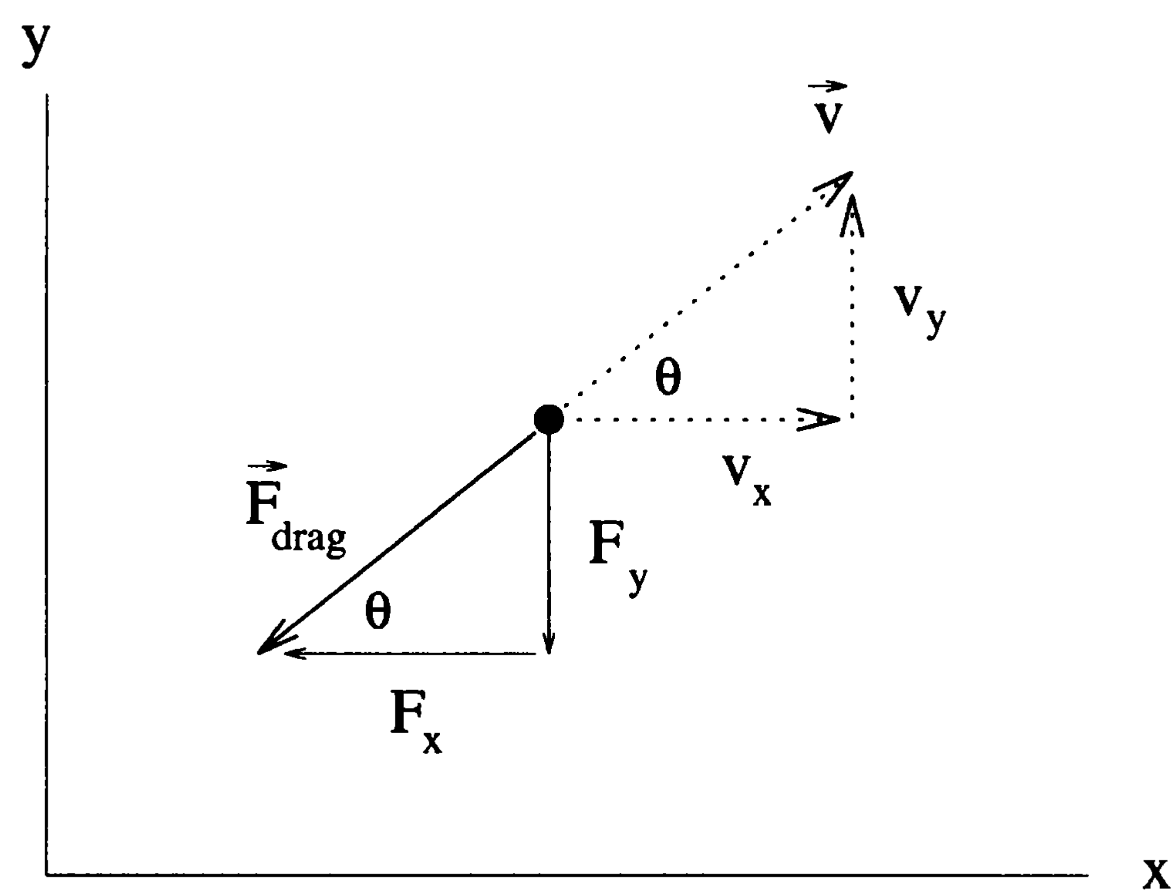

Figure 2.3: Components of the force due to air resistance on an object moving with a velocity $\vec{v}$. The direction of the drag force is opposite to $\vec{v}$.

The components of the drag force are thus

$$F_{\text{drag},x} = -B_2\, v\, v_x \qquad F_{\text{drag},y} = -B_2\, v\, v_y\,. \tag{2.18}$$

Adding this force to the equations of motion leads to

$$\begin{aligned} x_{i+1} &= x_i + v_{x,i}\,\Delta t \\ v_{x,i+1} &= v_{x,i} - \frac{B_2\, v\, v_{x,i}}{m}\,\Delta t \\ y_{i+1} &= y_i + v_{y,i}\,\Delta t \\ v_{y,i+1} &= v_{y,i} - g\,\Delta t - \frac{B_2\, v\, v_{y,i}}{m}\,\Delta t\,. \end{aligned} \tag{2.19}$$

The complete listing of a program that computes the trajectory of our cannon shell is given in Appendix 4; below we list the subroutine that performs the actual calculation. The arrays `x()` and `y()` are used to store the trajectory of the shell, while `vx` and `vy` hold the velocity components (since we only plan to plot the trajectory, we don't need to save the velocity results in arrays). Each time through the `for` loop we calculate the next values of position and velocity using (2.19). This process is continued until `y` becomes negative, which means the shell struck the ground somewhere during the previous time step. We then exit the loop and interpolate between the last two calculated positions to estimate where the shell struck the ground. The arrays are also redimensioned to the size actually used, in preparation for plotting.[7]

```
! x() and y() are the position of the projectile
! dt = time step   v_init = initial speed   theta = launch angle
! B_m = proportional to drag force = B2/m
sub calculate(x(),y(),dt,v_init,theta,B_m)
   option angle degrees          ! use degrees rather than radians
   x(1) = 0
   y(1) = 0
   vx = v_init * cos(theta)      ! initial velocity components
   vy = v_init * sin(theta)
   nmax = size(x)     ! this is the number of elements in the array x()
   for i = 2 to nmax
      x(i) = x(i-1) + vx * dt             ! Euler method equations
      y(i) = y(i-1) + vy * dt
      f = B_m * sqr(vx^2 + vy^2)          ! drag force from air resistance
      vy = vy - 9.8 * dt - f * vy * dt    ! Euler method for velocity
      vx = vx - f * vx * dt
      if y(i) <= 0 then exit for          ! shell has hit the ground
   next i
   a = -y(i) / y(i-1)                     ! interpolate to find landing point
   x(i) = (x(i) + a*x(i-1)) / (1+a)
   y(i) = 0
   mat redim x(i),y(i)
end sub
```

[7] The *True Basic* graphics routines plot all of the points in the input arrays, so resizing the arrays in this way avoids plotting any unused points at the ends of the original arrays.

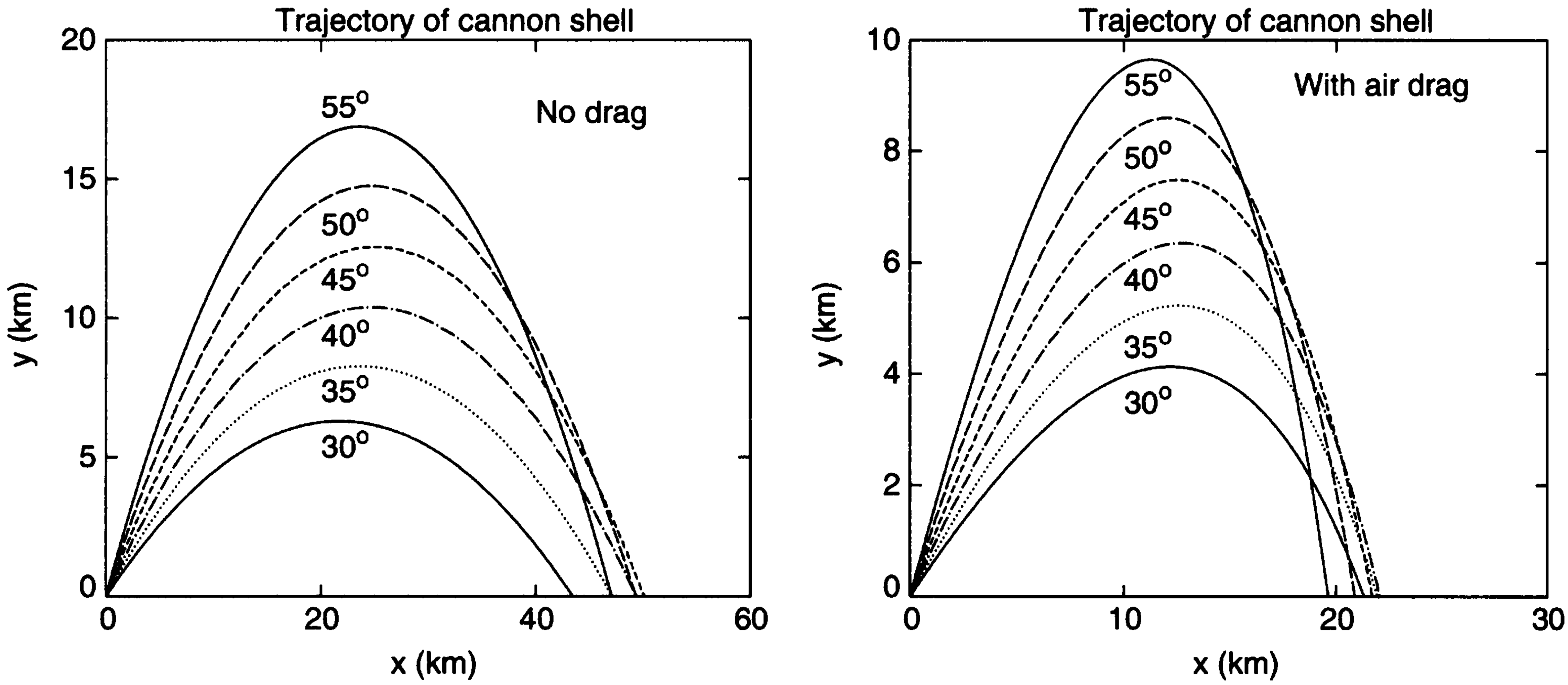

Figure 2.4: Left: trajectory of a cannon shell without air resistance. We assumed an initial speed of 700 m/s, and the firing angles are indicated. Right: trajectory of the same cannon shell, but now with air resistance included, with $B_2/m = 4 \times 10^{-5}$ m^{-1}.

Results for the trajectory are shown in Figure 2.4, where we have assumed an initial velocity of 700 m/s, which is appropriate for a large cannon shell. The plot on the left shows the results without air resistance,[8] while on the right we show results with air resistance for $B_2/m = 4 \times 10^{-5}$ (m^{-1}), which is a value appropriate for a very large cannon shell.[9] There are two important points to note from these results. First, air resistance decreases the range *a lot*, approximately by a factor of 2. The second point concerns the behavior as a function of firing angle. You probably have learned in your introductory mechanics course that without air resistance the maximum range occurs for $\theta = 45°$, and this is indeed observed in Figure 2.4. However, when air resistance is included, the maximum range occurs at a lower firing angle. For the parameters we have used here, the largest range with atmospheric drag is obtained with a firing angle of approximately 37°. This is in accord with our intuition that the longest range is obtained by shooting low into the "wind."

Our calculations clearly show that air resistance plays a major role in the problem. However, there is another important piece of physics that we have not yet accounted for. From Figure 2.4 we see that the shell travels to a very high altitude, where the air density will be lower than at sea level. We saw in our discussion of the drag on a bicycle that the force from air resistance is proportional to the density of the air, so the drag force at high altitudes will be less than that at sea level. To investigate the

[8]For this case the trajectory can also be calculated analytically. We will leave it to the reader to verify that the trajectories we have obtained numerically agree with the analytic results.

[9]See Chapter 5 of Bennett (1976) for a description of the cannon for which this value of B_2 was measured. The shells were approximately 10 cm in diameter. We emphasize that this value of B_2 was measured via "experimental" observations, and not estimated using the value $C = \frac{1}{2}$ in (2.9). We will see in the next section that the drag force for rapidly moving objects, such as cannon shells and baseballs, is more complicated than implied by (2.9).

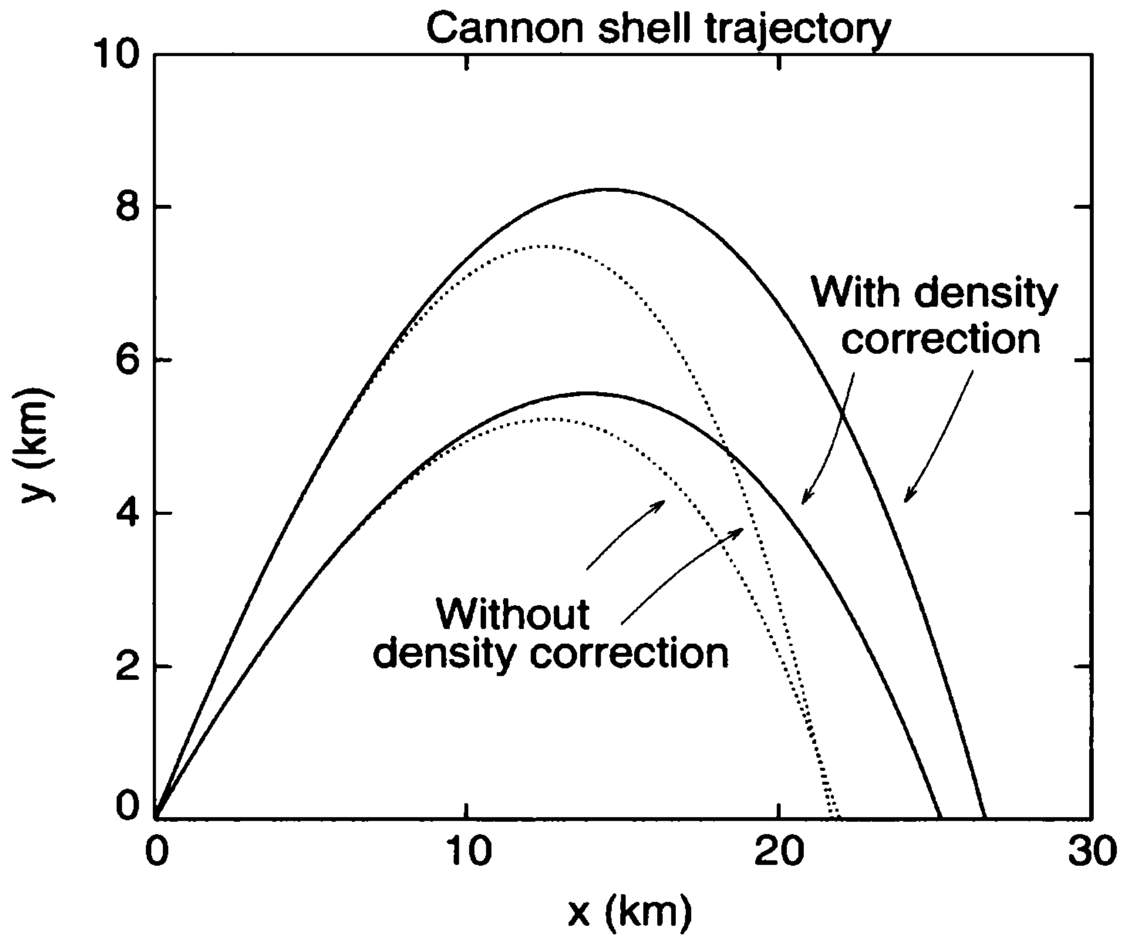

Figure 2.5: Trajectory of a cannon shell with (solid curves) and without (dotted curves) the effect of the lower air density at high altitudes taken into account. In all cases the air resistance was included with $B_2/m = 4 \times 10^{-5}$ m^{-1}, and the initial speed was 700 m/s. The lower two curves were for a firing angle of 35°, while for the top curves it was 45°.

magnitude of this effect we note that the density of the atmosphere varies as

$$\rho = \rho_0 \exp(-y/y_0) , \tag{2.20}$$

where y is the altitude, $y_0 = 1.0 \times 10^4$ m, and ρ_0 is density at sea level ($y = 0$). The drag force due to air resistance is proportional to the density, so

$$F^*_{\text{drag}} = \frac{\rho}{\rho_0} F_{\text{drag}}(y = 0) , \tag{2.21}$$

where $F_{\text{drag}}(y = 0)$ is the force at sea level [given by (2.16)], and F^*_{drag} is the drag force at altitude. Putting this into the calculation is straightforward; we replace B_2 in (2.19) with $B_2\rho/\rho_0$. We will leave it to you to modify the program given above. Some results are shown in Figure 2.5 where we compare trajectories with and without the variation of ρ with altitude taken into account. We see that for this cannon, the decrease of ρ at high altitudes increases the range by about 20 percent. In addition, the maximum range now occurs for a firing angle slightly *larger* than 45°. Hence, to get the longest range for a projectile such as this, it is best to take advantage of the reduced drag at high altitudes.

Exercises

1. Use the Euler method to calculate cannon shell trajectories ignoring both air drag and the effect of air density (actually, ignoring the former automatically rules out the latter). Compare your results with those in Figure 2.4, and with the exact solution.

2. In our model of the cannon shell trajectory we have assumed that the acceleration due to gravity, g, is a constant. It will, of course, depend on altitude. Add this to the model and calculate how much it affects the range.

3. Calculate the trajectory of our cannon shell including both air drag and the reduced air density at high altitudes so that you can reproduce the results in Figure 2.5. Perform your calculation for different firing angles and determine the value of the angle that gives the maximum range.

*4. Generalize the program developed for the previous problem so that it can deal with situations in which the target is at a different altitude than the cannon. Consider cases in which the target is higher and lower than the cannon. Also investigate how the minimum firing velocity required to hit the target varies as the altitude of the target is varied.

*5. In warfare you generally want to hit a particular target (as opposed to having the cannon shells land indiscriminately). However, this is not easy, since very small changes in any of the parameters can lead to large changes in the landing site of the shell. Investigate this by calculating how much the range of the cannon shell considered in this section would change if the initial speed is increased by 1 percent. Also compute the change in the range if there is a slight (10 km/h) wind (this effect can be added using the methods developed in the next section). You should find that even these relatively small changes alter the landing site by quite significant amounts.

2.3 Baseball: Motion of a Batted Ball

The game of baseball has fascinated fans of the game, and also physicists (some of us are both), for many years. There have probably been more books and papers written on the physics of baseball than any other sport, which is perhaps justified considering the wide variety of physical principles that are important in different aspects of the game. In this section we will consider the trajectory of a batted ball. Our goal is to understand how far a *real* baseball should be expected to travel when hit by a "typical" power hitter. The most important ingredient in our model will be the effect of atmospheric drag, and we will employ a somewhat more realistic model of the drag force than was used in previous sections. We will also see that wind and atmospheric density have smaller, but quite significant effects on the game. In the next few sections we will depart from our usual exclusive use of SI units and use also the units of feet and miles per hour (mph), which are so closely tied with how many of us (or at least the author) usually think about the sport.

The basic equations of motion for a baseball are the same as those of the cannon shell, (2.19), and we will again use the Euler method in our simulations. Before we go too far, however, it is interesting to estimate the distance a baseball would travel in a vacuum, that is, without atmospheric drag. This range can be calculated analytically, or numerically, by using (2.19) with $B_2 = 0$. A good power hitter can give the ball an initial speed of about 110 mph (49 m/s), and if it is hit at an initial angle of 45° starting from a height of 1 m, the range in a vacuum would be 248 m, or approximately 815 ft. So far as I know, no one has ever hit a baseball that far (except, perhaps, in the presence of a tornado). A typical outfield fence is 350–400 ft from home plate, and from practical experience we know that a 500-ft home run is an exceptionally long one.[10] Thus, it is clear that baseball in a vacuum would be a very different game than the one we are used to watching.

The force on a baseball due to air resistance is given by a form similar to (2.9). However, it turns out that the drag coefficient C for a baseball is not independent of v. Wind tunnel measurements with real baseballs show that C is actually a strong function of v, as illustrated in Figure 2.6, which shows

[10] Chapter 4 of Adair (1990) has a nice discussion of the distances traveled by the longest known home runs.

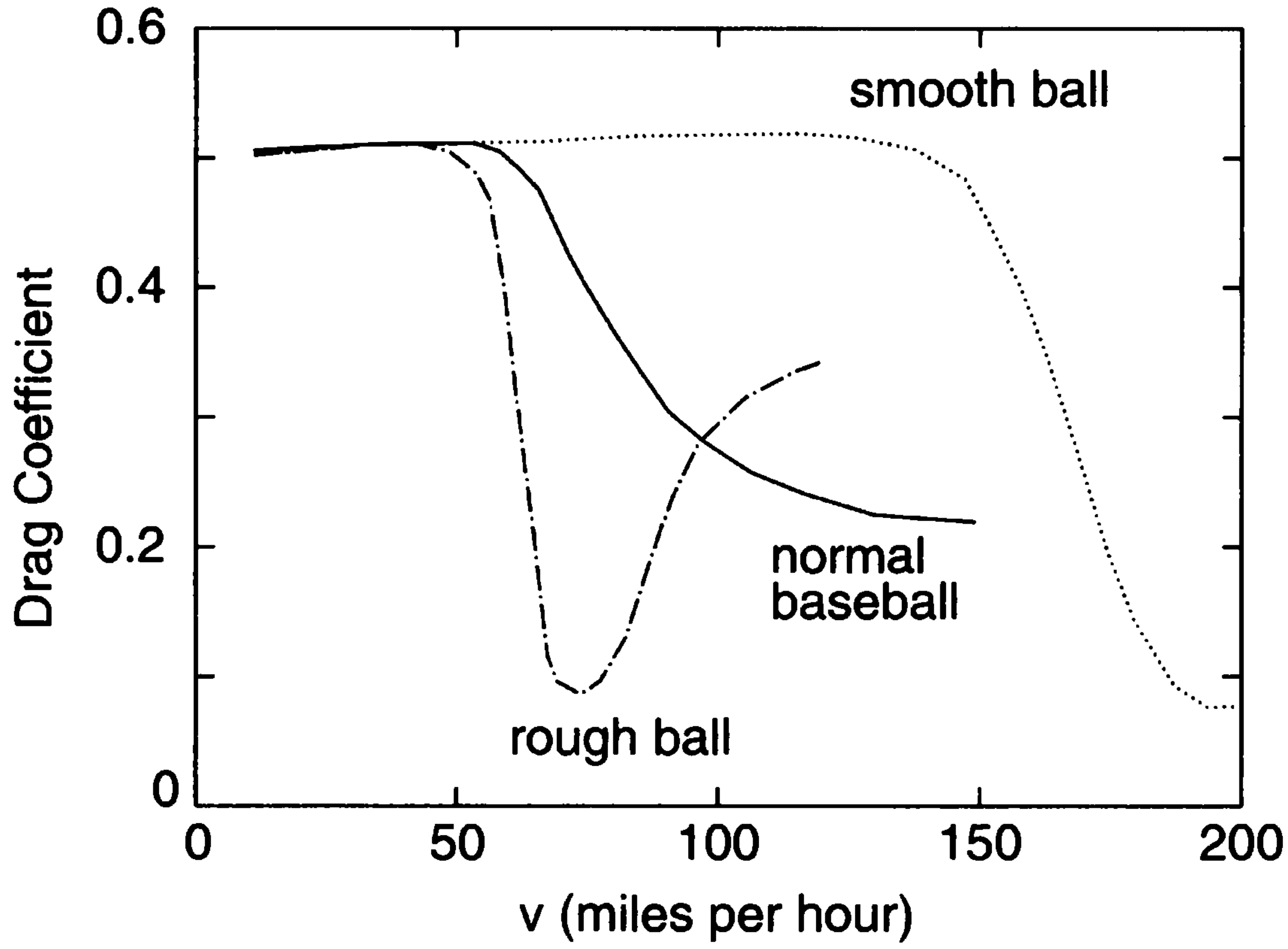

Figure 2.6: Variation of the drag coefficient, C, with velocity for normal, rough, and smooth baseballs. Note that for a normal ball the drag force is equal to the weight of the ball at a velocity of approximately 95 mph. From R.K. Adair, *The Physics of Baseball.*

a plot of the drag coefficient as a function of the speed of the ball. At low speeds, C is close to the value of $\frac{1}{2}$ predicted by the simple argument we gave in the derivation of (2.9). However, as the speed increases, C drops substantially, and it is more than a factor of 2 smaller at high speeds. This does *not* mean that the drag force is lower at high speeds; F_{drag} is proportional to the product Cv^2, and this quantity[11] is a monotonically increasing function of v.

The behavior of C can be understood qualitatively as follows. At low speeds the air flow around the ball is "well behaved" and orderly. However, at high speeds the flow becomes turbulent, and it turns out that an object is able to "slip" through the air more easily in this case than when the flow is nonturbulent. This makes the drag coefficient smaller at high speeds when the flow pattern is turbulent, than at low speeds when it is nonturbulent. Such behavior is quite general and is important for other objects as well (we will consider the interesting case of a golf ball later in this chapter). We see from Figure 2.6 that this transition occurs at a speed of about 80 mph for an ordinary baseball. This is a common speed for both a batted and a pitched ball, so the velocity dependence of C must be considered in any realistic modeling. The figure also shows the variation of C for balls that are rougher or smoother than typical baseballs, and the behavior is quite different in these cases. For a rough ball the transition to turbulent flow occurs at a lower velocity than for a regular ball, so the drop in C takes place sooner, while the opposite trend is observed for a smooth ball (a ball without stitches would qualify as smooth). This will turn out to have important implications for the motion of the pitch known as a knuckleball, since the stitches on the ball are equivalent to a sort of roughness.

[11] A plot of F_{drag} as a function of velocity is given in Chapter 2 of Adair (1990) (it could also be derived from the data given in Figure 2.6). While it does not appear to happen in the case of a baseball, it is possible that for certain objects the drag force might become smaller as the speed is increased. Some of the consequences of such behavior are discussed by Adair.

In order to calculate the trajectory of a baseball, we need to solve the Euler equations (2.19), including the velocity dependence of the drag force. The variation of C observed in Figure 2.6 is clearly a complicated function. For our numerical work it is convenient to have an (admittedly approximate) analytic representation of this function. Taking the results for the drag coefficient from Figure 2.6 and adding in the appropriate factors of air density, and so on, it turns out that the drag factor for a normal baseball is described reasonably well by the function (now using SI units)

$$\frac{B_2}{m} = 0.0039 + \frac{0.0058}{1 + \exp\left[(v - v_d)/\Delta\right]}, \tag{2.22}$$

with $v_d = 35$ m/s and $\Delta = 5$ m/s. Using this parameterization of the drag we can construct a program to calculate the trajectory of our batted baseball. The program is very similar to the one given earlier for the cannon shell problem, so we will leave the details to the exercises.

A typical result for the trajectory is shown as the solid curve in Figure 2.7. Recalling that the range in a vacuum would be more than 800 ft, we see that air resistance has an *enormous* effect. The range of our power hitter is now approximately 120 m, or about 395 ft, which is in much better accord with the size of typical major league ballparks.[12] It turns out that with air resistance described by (2.22), the maximum range is largest for angles near 35°. This is much lower than the 45° angle that gives the maximum range without air resistance. The trajectory is seen to be noticeably nonparabolic[13] (in a vacuum it would be an exact parabola), which is also in agreement with our everyday experience.

Let us next add the effect of the wind. We will assume that it is blowing in a horizontal (x) direction and has a constant magnitude and direction during the flight of the ball.[14] In this case the components of the drag force become

$$\begin{aligned} F_{\text{drag},x} &= -B_2 \mid \vec{v} - \vec{v}_{\text{wind}} \mid (v_x - v_{\text{wind}}) \\ F_{\text{drag},y} &= -B_2 \mid \vec{v} - \vec{v}_{\text{wind}} \mid v_y , \end{aligned} \tag{2.23}$$

where v_{wind} is the velocity of the wind, and a positive value corresponds to a tailwind. The derivation of (2.23) is understood most easily if you consider the drag force (2.18) in the reference frame at rest with respect to the wind. In that frame the drag force is just (2.18). Transforming into the batter's reference frame then requires that we subtract the velocity of the wind from $\vec{v}$, which yields (2.23).

We can include the wind in our program using (2.23), an exercise we will leave to the reader. Some results from such a calculation are shown in Figure 2.7, where we compare the trajectories calculated with a tail wind and a head wind (both 10 mph) with the result for still air. This relatively modest breeze alters the range by approximately ±25 ft, which is more than enough to turn a routine fly ball out into a home run, or a home run into a long out. The wind can thus have a pronounced effect on the game,

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[12] In writing a program to determine the range, it is advisable to locate the place where the ball strikes the ground by using an interpolation procedure, similar to that used in our work on the cannon shell problem. The program "knows" that the ball has landed when $y_n < 0$. You could just take x_n to be the range, but this would overestimate the landing point by as much as $v_{x,n}\,\Delta t$, which can be significant when you want to consider small differences in the range. A better method is to interpolate between (x_n, y_n) and (x_{n-1}, y_{n-1}), assuming the trajectory to be a straight line, to get a much improved estimate of where $y = 0$.

[13] This can be seen by noting that the ball reaches its maximum height at about $x = 70$ m while it strikes the ground at $x = 120$ m in the absence of any wind.

[14] It is not difficult to treat the case of a swirling wind like that found in most stadiums, but we'll leave that for the interested student. Such an effect would certainly be important in developing a model of judging and catching a fly ball.

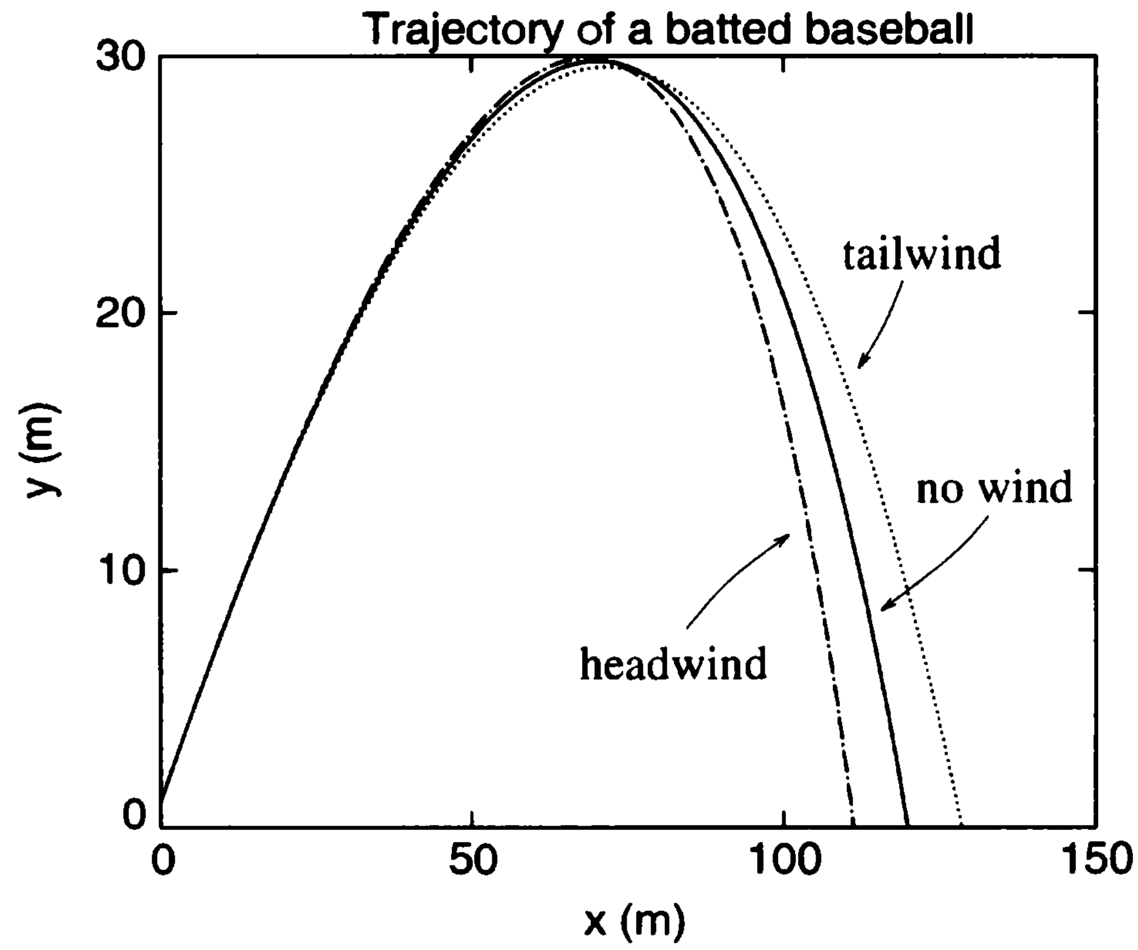

Figure 2.7: Calculated trajectory of a baseball hit at an initial velocity of 110 mph, with the effects of atmospheric drag included. Solid curve: no wind; dotted and dot-dashed curves: a tailwind and headwind of 10 mph. In all cases we assume that the initial velocity makes an angle of 35° with the horizontal.

as any fan who has been to Wrigley Field or Candlestick Park can verify.[15] It is interesting that major league parks are oriented so that the generally prevailing westerly breeze, which averages about 5 mph in the United States, is at the batter's back and thus favors a home run. As in most sports, offense is favored over defense; that is, the hitter over the pitcher.

It is also interesting to consider the effect of altitude. This can be added to the model just as we did in the case of the cannon shell, with the only difference being that we don't have to worry about density changes during the flight of the ball. The results for the range of our home-run hitter at several altitudes are shown in Table 2.1. The highest major league park is currently in Denver (altitude $\approx$ 5280 ft). Our long fly ball will travel about 31 ft farther there than at sea level. This is a substantial difference and suggests that home-run hitters should enjoy playing in Denver. The next highest major league park is in

Table 2.1: Calculated range of a batted ball at various altitudes. In all cases we have assumed an initial velocity of 110 mph, that there is no wind, and that the ball is hit at an initial angle of 35° with respect to the horizontal. (Pike's Peak is a mountain in Colorado.)

Location	*Altitude (feet)*	*Range (feet)*
sea level	0	390
Atlanta	1,000	396
Denver	5,280	421
Pike's Peak	14,110	473

[15]These two baseball parks are notorious for being very windy. Note: Candlestick Park is now named 3Com Park.

Atlanta (altitude $\approx$ 1000 ft), and here the effect of the altitude is to add about 6 ft to the range. We also note that if a ballpark is ever built on Pike's Peak, home runs there will be very interesting to watch.

One of the conclusions that can be drawn from our results for the range of a batted baseball is that atmospheric drag simply must be taken into account to get an accurate picture of the range and trajectory of a fly ball. Air resistance reduces the range of a home-run hitter by more than a factor of 2. For the statisticians, it seems clear that playing in Denver will significantly enhance a hitter's record and be unfair to a pitcher's earned-run average.[16] There are a number of other effects that can be added to our model, and a nice qualitative discussion of many of them is given by Adair (1990). One more comment concerning the "philosophy" of numerical simulations is appropriate here. It is not realistic to expect our model to be accurate in a truly quantitative sense. There are just too many complications. Besides the approximation involved in our parameterization of the drag factor, we must expect that this force will vary somewhat from ball to ball, and that it will also be a function of the humidity and other similar factors. Factors such as these will almost certainly add (or subtract) a few feet or perhaps more to the range. The usefulness of our model is *not* that we can use it to calculate the range of a home run with great precision. Rather, it should give us insight into the relative importance of different effects, and enable us to estimate *changes* in the range as a result of the wind, or other factors. We will have more to say concerning the philosophy of model building in physics during the course of this book. That said, the predictions of our model are certainly consistent with what is observed on the ballfield.

Exercises

1. Construct a program to calculate the trajectory of a baseball, as in Figure 2.7. Use it to investigate the following questions:
   (a) Calculate the range at sea level, with no wind, of a ball hit at 110 mph for different initial angles. Determine to within 1° the angle that gives the maximum range.
   (b) Determine the initial angles that give the maximum range for a ball hit at 110 mph into a 25-mph head wind, and with a 25-mph tail wind. What are the maximum ranges in the two cases?
   (c) In all of our calculations we have assumed that the initial velocity is 110 mph, a value typical for a good power hitter. How much does the range depend on this value? Calculate the range for initial velocities of 120 and 100 mph. How does the range scale with the initial kinetic energy of the ball and why?
   (d) Calculate how much a fastball slows down on its way to home plate. Assume a pitch that leaves the pitchers hand at 100 mph and find its speed when it crosses home plate, which is 60.5 ft away.

*2. Consider the effect of a crosswind on the trajectory of a fly ball. How much will a wind of 10 mph directed at right angles to the initial velocity alter the place where one of the fly balls in Figure 2.7 lands?

*3. Calculate the range for the smooth and rough balls described by the drag functions in Figure 2.6.

4. Estimate the initial speed required to hit a home run 550 ft at sea level in the absence of any wind. How far would this ball travel on Pike's Peak?

[16] Although the reduced density at altitude will add slightly to the velocity of a fastball.

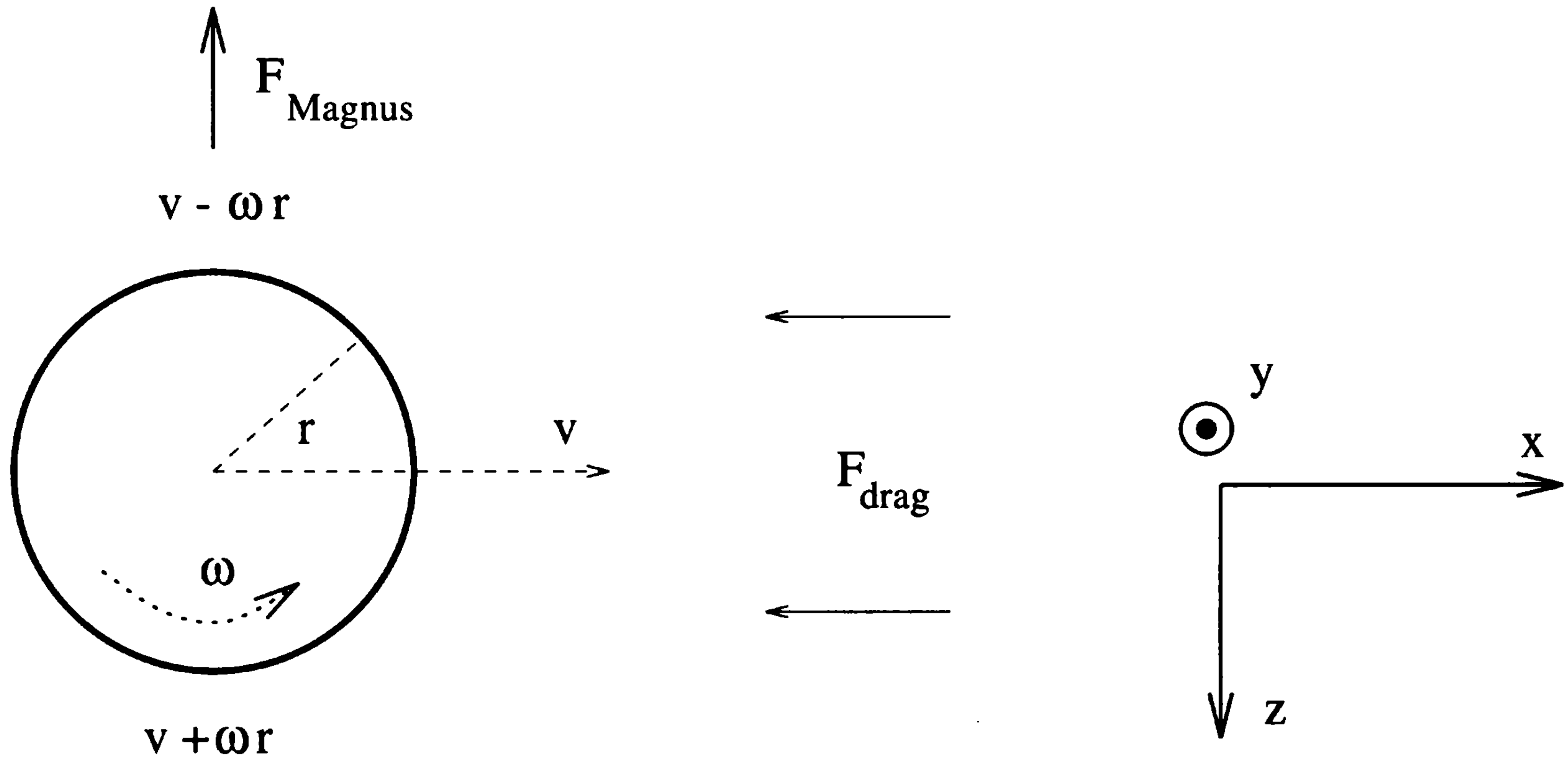

Figure 2.8: Forces acting on a spinning ball. The ball is moving from left to right, and the rotation axis is directed along y, out of the page. The lower edge of the ball has a speed relative to the air of $v + \omega r$, which is larger than that of the the top edge, $v - \omega r$, so the drag force on the bottom part of the ball is larger. This figure corresponds to a sidearm curve ball (see Figure 2.9) as viewed from above. After Adair (1990).

2.4 Throwing a Baseball: The Effects of Spin

To treat the problem of a thrown or pitched ball we must deal with two different effects. One effect is the spin of the ball; this will turn out to dominate the motion of a curve ball. The other effect concerns the difference in the drag coefficients for rough and smooth balls, which we encountered in Figure 2.6, and this will be crucial for the behavior of a knuckleball. We begin with the curve ball.

The origin of the force that makes a spinning ball curve can be appreciated if we recall that the drag force has the form $F_{\text{drag}} \sim v^2$, where v is the speed of the object relative to the air. For a ball spinning about an axis perpendicular to the direction of travel, this speed will be different on opposite edges of the ball, as illustrated in Figure 2.8.

Because of the spin, the lower edge of the ball in Figure 2.8 will have a larger velocity relative to the air than will the upper edge. This will result in a larger drag force on the lower part of the ball than on the top edge. When the forces on the bottom and top parts of the ball are then added, there will be a component of the force in the $-z$ direction (upward in Figure 2.8), perpendicular to the (center of mass) velocity. This is known as the Magnus force and is believed to be the dominant spin-dependent force for objects such as baseballs.[17] We expect this force to be proportional to B_2 in (2.16), but the precise value of the coefficient of proportionality is hard to estimate since it is determined from averaging the appropriate component of the drag force over the curved face of the ball. Rather than trying to perform

[17]The Bernoulli effect is sometimes mentioned as the origin of the spin-dependent force on baseballs. See the discussion by Adair (1990) for more on both this and the Magnus force.

this averaging, we will simply observe that the spin-dependent force in this simple geometry will have the functional form[18]

$$F_M = S_0\,\omega\,v_x\,, \tag{2.24}$$

where ω is the angular speed of the ball and the numerical coefficient S_0 takes care of averaging the drag force over the face of the ball, as just discussed. We will determine the value of S_0 through experimental measurements.[19] Since the drag coefficient of a baseball depends on its speed (see Figure 2.6), we must also expect that S_0 will depend on v. However, for simplicity we will assume that S_0 is a constant in the following. Over the velocity range that is usually of interest to a pitcher, 50–110 mph, we estimate from the data given in Adair (1990) that $S_0/m \approx 4.1 \times 10^{-4}$, where $m = 149$ g is the mass of the ball (note that S_0/m is unitless). The magnitude of the Magnus force (2.24) is about one-third the weight of the ball for a typical curve ball.

To calculate the trajectory of a curve ball, we have to consider motion in three dimensions. We will let x be the axis running from home plate to the pitcher, z be the horizontal direction perpendicular to x, and y be the height above the ground, as in Figure 2.8. The equations of motion for a sidearm curve ball are then

$$\begin{aligned}
\frac{dx}{dt} &= v_x \\
\frac{dv_x}{dt} &= -\frac{B_2}{m}\,v\,v_x \\
\frac{dy}{dt} &= v_y \\
\frac{dv_y}{dt} &= -g \\
\frac{dz}{dt} &= v_z \\
\frac{dv_z}{dt} &= -\frac{S_0\,v_x\,\omega}{m}\,.
\end{aligned} \tag{2.25}$$

Here we assume that the axis of rotation is parallel to y, that is, perpendicular to the ground. These equations of motion include the effects of atmospheric drag on the largest component of the velocity (v_x), with a velocity dependent coefficient (2.22), but we haven't included it for v_y or v_z, since the forces are much smaller in these cases.

We can again use the Euler method to solve (2.25), and some typical results are shown in Figure 2.9. Here and below we assume that ω is a constant; that is, the ball's rotation rate does not decrease significantly during the course of the pitch. The ball begins on a trajectory that would carry

[18] The general vector form is $\vec{F}_M = S_0\vec{\omega} \times \vec{v}$.

[19] In a sense we have swept all of our ignorance, that is, the complicated problem of averaging $F_{\rm drag}$ over the curved face of the ball, into the constant S_0. By then obtaining S_0 from experiment, you might conclude that we are giving in to the problem, and in a sense we are. However, even in taking this approach it still seems fair to say that we "understand" the physics contained in S_0; it is only that the mathematics required to calculate its value is more complicated than we wish to tackle at this time.

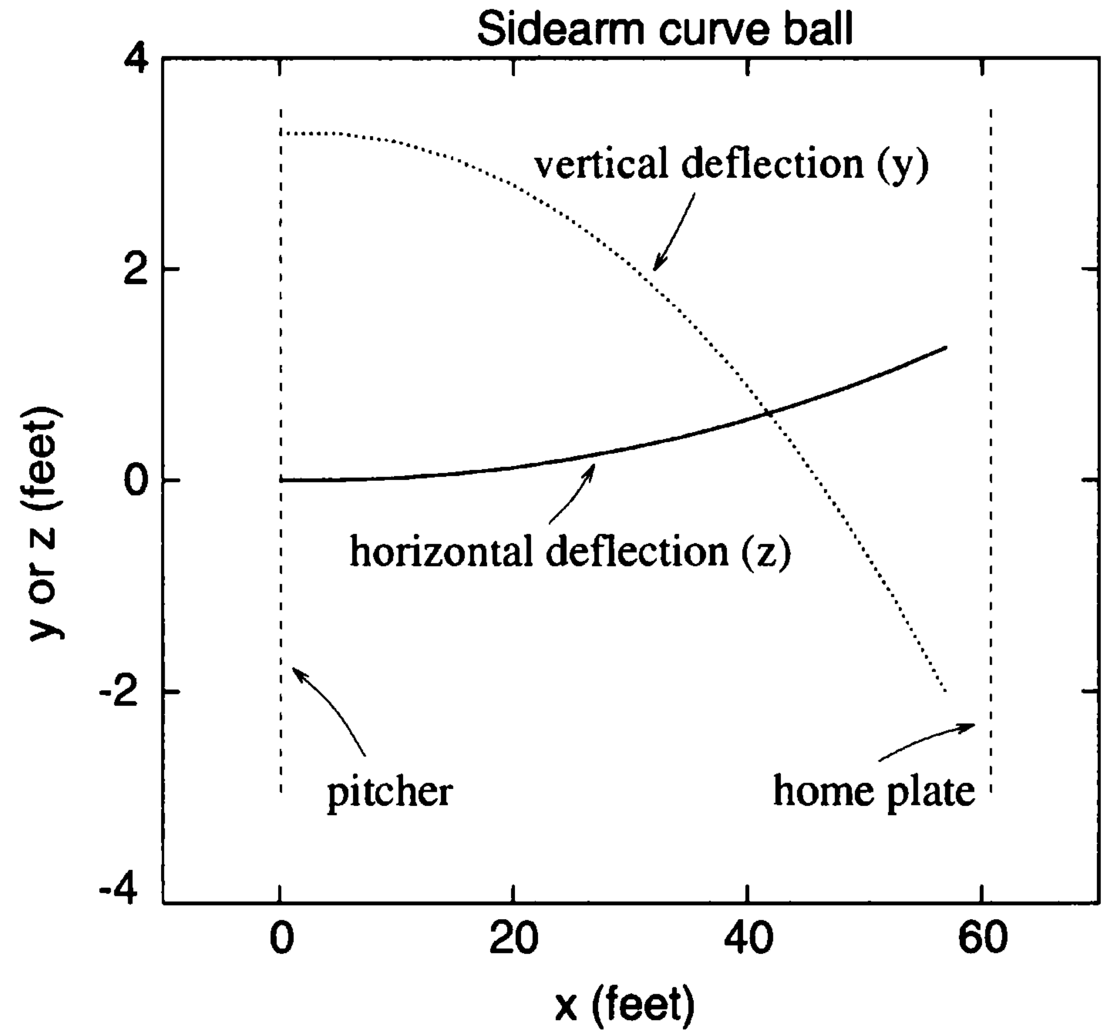

Figure 2.9: Trajectory of a sidearm curve ball. The pitch traveled from left to right, with the y axis vertical. The z axis is horizontal, and perpendicular to the line between the pitcher and home plate. The solid curve shows the horizontal (z) displacement, which is due to the Magnus force, while the dashed curve shows the vertical (y) displacement, which is due solely to gravity. The ball was thrown with an initial velocity of 70 mph in the x direction, and was spinning at 30 revolutions/s, with $\vec{\omega}$ parallel to the y axis.

it over the center of the plate ($z = 0$), but the Magnus force deflects it by nearly a foot, which would take it well outside. The dotted curve shows how much the pitch falls during its trip to the plate. We have taken the initial velocity to be purely horizontal, and it turns out that the vertical deflection due to gravity is so great that it would cause this pitch to bounce in front of the plate. One conclusion here is that gravity has a significant effect on the trajectory, which in this case is larger than the effect of the spin. The fact that a curve ball is so much harder to hit than a fastball thrown at the same velocity implies that a hitter is able to estimate very rapidly and accurately the deflection due to gravity. This is probably because the trajectories of all fastballs are basically similar.

In Figure 2.10 we show the trajectories of several other pitches. The solid line shows an overhand curve. The axis of rotation is now horizontal, so that the Magnus force is (to a good approximation) vertical. The dotted curve shows the trajectory of a similar pitch, the only difference being this one is *not* spinning, so the deflection due to the Magnus force alone is now evident. The additional break due to the Magnus force is seen to be a little less than 1 ft, which is similar to that found for the sidearm curve ball. Note that we have given these two pitches a small, upward, initial velocity so that unlike the example in Figure 2.9, they cross the plate in or near the strike zone. Also shown as the dot-dashed curve in Figure 2.10 is the trajectory of a 95-mph fastball. This has much less downward deflection due to gravity, since it travels to the plate in about 70 percent of the time taken in the other cases.

We next consider one of most entertaining pitches, the knuckleball. Unlike other pitches, this one is thrown with very little spin. Even so, its trajectory typically exhibits one or two rapid and large displacements in directions perpendicular to the line connecting the pitcher with home plate (the x direction). These displacements are different from pitch to pitch, making it extremely difficult for the

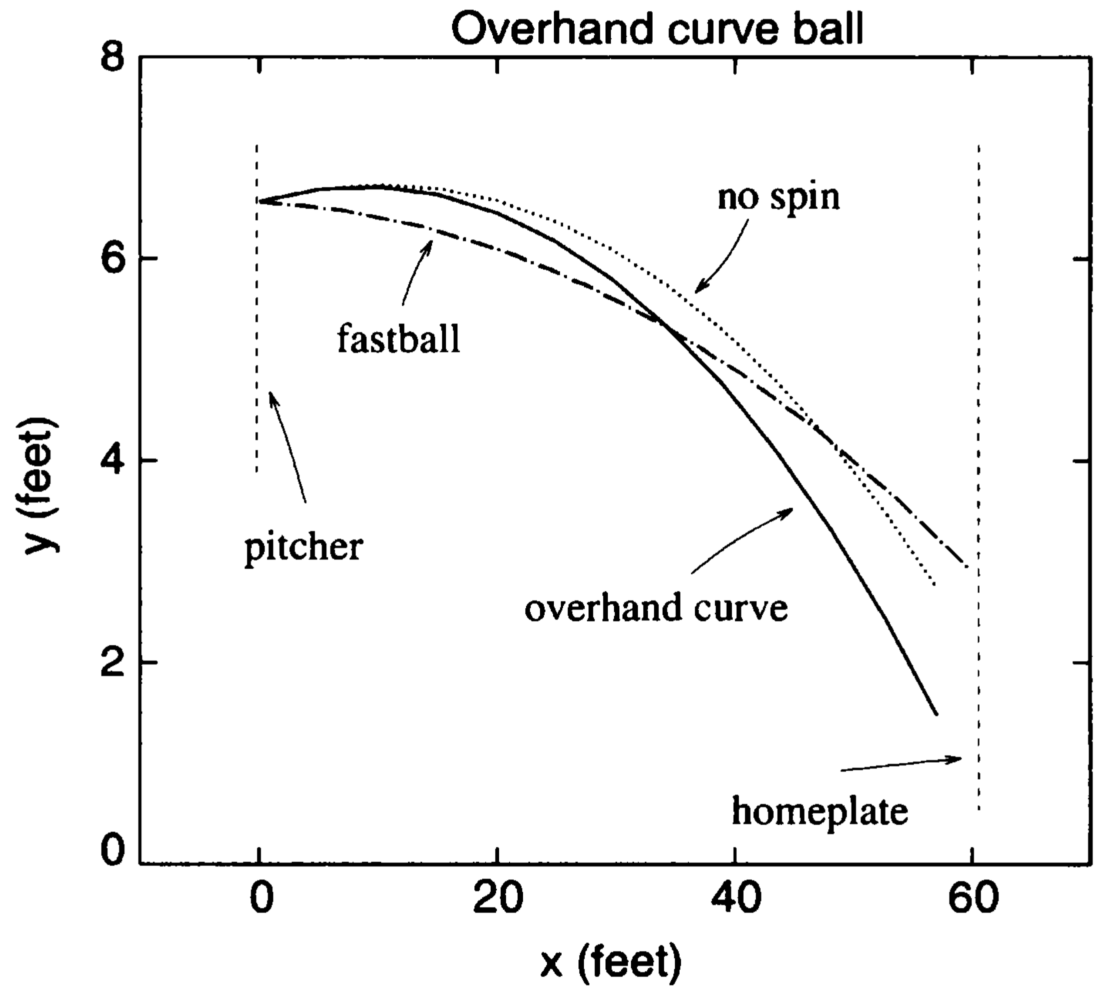

Figure 2.10: Trajectories of several pitches. The pitches travel from left to right, and for these pitches the deflection of the ball is purely in the vertical (y) direction. The solid curve shows the break of an overhand curve ball ($\vec{\omega}$ is along $-z$), which is due to both gravity and the Magnus force. We have assumed the same velocity and the same angular velocity as with the sidearm curve in Figure 2.9. The dotted curve shows the break due to gravity alone; that is, we have assumed a pitch thrown at the same velocity, but with no spin. The dot-dashed curve shows the trajectory of a 95-mph fastball for comparison.

hitter to judge where to swing in order to make contact with the ball. In fact, the motion is so irregular that even the pitcher cannot predict exactly how the pitch will move or curve.

The origin of the lateral force on a knuckleball can be appreciated from Figure 2.6. Suppose that a moving ball is not spinning at all and is oriented such that a stitch is exposed on one side while the other side is smooth (no stitches exposed). Since the drag force is greater for a smooth ball than for a rough one, there will be an imbalance of forces on the two sides of the ball, giving a net force in the direction of the rough side. It turns out that this force is large enough to make the pitch curve as much or more than a typical curve ball. Now, if the ball rotates just a little as it moves toward home plate, so that the exposed stitch moves to the opposite side of the ball, the lateral force will reverse direction and the ball will then curve in the *opposite* direction. This is a knuckleball.

A curve ball is difficult to throw, since it requires the pitcher to impart a large amount of spin on the ball. At the other extreme, a knuckleball is difficult to throw since it requires the pitcher to impart *very little* spin on the ball. A well-thrown knuckleball might complete only half of a revolution on its way to the plate, so a stitch (i.e., rough surface) will be exposed for a significant amount of time.[20]

This force due to the effective roughness of the ball could in principle be estimated from the drag coefficients in Figure 2.6, but it is clearly preferable to obtain it from specially designed experiments. Fortunately Watts and Sawyer (1975) have performed wind tunnel measurements of this force, and a schematic of their results is shown in Figure 2.11. As anticipated, the lateral force is a function of the

[20] If the ball completes too many (several) rotations on the way to the plate, this lateral force will oscillate rapidly in direction, and cancel out the effect.

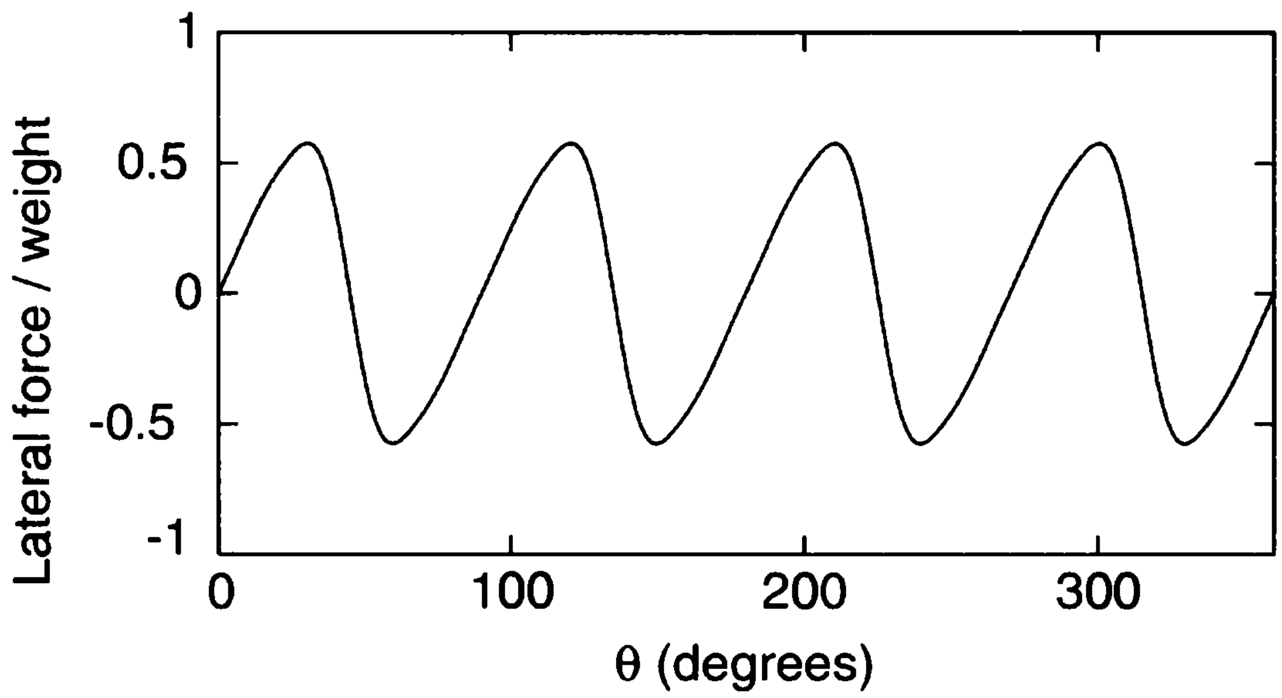

Figure 2.11: Approximate horizontal force on a baseball traveling without spinning at a speed of 65 mph as a function of the angular orientation of the ball. The angle θ is measured about an axis that is vertical, and perpendicular to the center of mass velocity of the ball, which is horizontal. The four oscillations of the force for each complete revolution of the ball correspond to the four times the stitches pass by an observation point. These results are estimated from the data given by Watts and Sawyer (1975). Here we have expressed the force in terms of the weight of the ball, and scaled it as appropriate for a pitch thrown at 65 mph (the measurements were performed at a lower speed).

angular orientation of the ball. In this case the angle is measured about an axis such that stitches pass by any one point four times for each complete revolution of the ball. Hence there are four maxima and four minima in the lateral force for each revolution ($\Delta\theta = 360°$). The magnitude of this force is approximately one-half the weight of the ball at a speed of 65 mph. Besides the magnitude of the force, the nonsinusoidal shape of the curve is also worth noting (its not a plotting error!). The rapid variation of the force at certain angles no doubt contributes to the apparently irregular trajectory of this pitch. To calculate the trajectory of a knuckleball, we treat it as a projectile moving horizontally with a drag coefficient given by (2.22), and assume that it spins very slowly. We now have to keep track of the angular orientation of the ball, and we do this with the equation of motion $d\theta/dt = \omega$. For simplicity we will assume that the ball spins about a vertical axis so that the deflection due to the force in Figure 2.11 will be purely horizontal. Thus, the rotation angle θ is measured with respect to this vertical axis. This leads to another Euler equation

$$\theta_{i+1} = \theta_i + \omega \, \Delta t \,, \tag{2.26}$$

with ω taken as constant during the course of the pitch (but adjustable from one pitch to the next). As the ball travels to home plate, θ varies with time, thereby altering the lateral force. The force in Figure 2.11 is given approximately by the function

$$\frac{F_{\text{lateral}}}{mg} = 0.5\,[\sin(4\theta) - 0.25\sin(8\theta) + 0.08\sin(12\theta) - 0.025\sin(16\theta)] \,. \tag{2.27}$$

This expression is simply an empirical choice[21] that has a form close to the wind tunnel results of Watts and Sawyer (1975). Putting this all together, the trajectory can now be calculated. Two parameters

[21] Our choice of this particular function to describe the data of Watts and Sawyer (1975) was motivated by the form of a Fourier series that has a period of $\pi/2$. For more on this topic see Appendix 2.

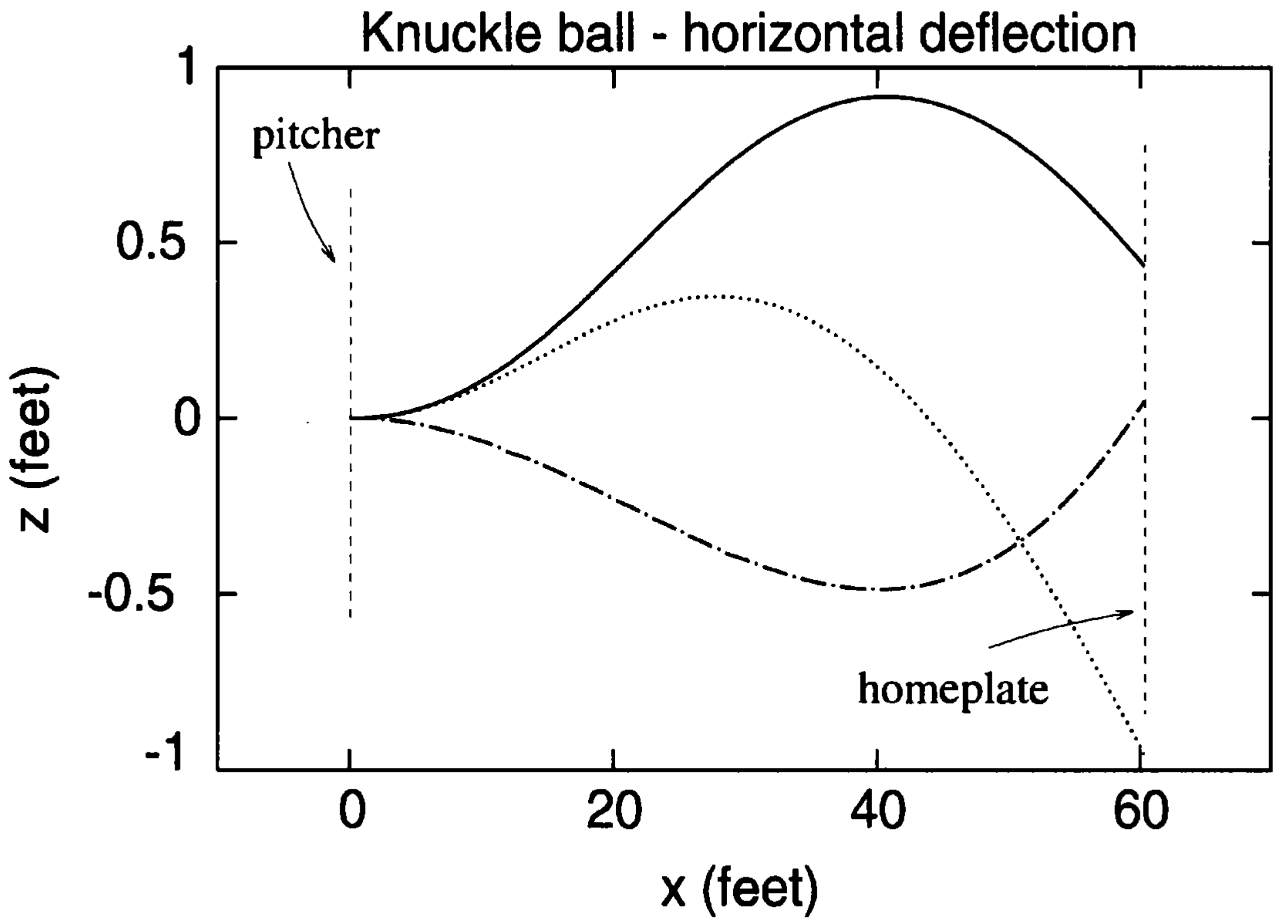

Figure 2.12: Calculated horizontal deflections of several knuckleballs, traveling from left to right. All had a velocity of 65 mph, with an angular velocity of 0.2 revolutions/s (with $\vec{\omega}$ parallel to the vertical direction, y). The different pitches had different initial orientations of the stitches.

control the behavior, the angular velocity of the ball, ω in (2.26), and its initial angular orientation. Some typical trajectories calculated for $\omega = 0.2$ rev/s are shown in Figure 2.12, where we have considered three different initial orientations. Our model reproduces the expected behavior very well. Typical deflections are a foot or more and are comparable to or even larger than those found for curve balls. The multiple deflections in the trajectory make it no surprise that catchers do not enjoy trying to stay in front of these pitches. The trajectories become more complicated if the rotation rate is increased, but the deflections are smaller since the direction of lateral force then varies more rapidly.

Exercises

1. Investigate the trajectories of knuckleballs as a function of the angular velocity ω, the initial angular orientation, and the (center of mass) velocity.

2. Calculate the effect of backspin on a fastball. How much does an angular velocity of 1000 rpm (typical for a fastball) affect the trajectory?

3. Model the effect of backspin on the range of a batted ball. Assume an angular velocity of 2000 rpm.

4. Calculate the effect of the knuckleball force on a batted ball. Assume that the ball is a line drive hit at an initial speed of 90 mph and an angle of 20° and that the ball does not spin at all (clearly a gross assumption). Let the rough side of the ball always face one side, which is perpendicular to the direction the ball is hit, and use Figure 2.11 to estimate the magnitude of the lateral force. With all of these assumptions, calculate the lateral deflection of the ball. Such balls hit with very little spin are observed by outfielders to effectively "flutter" from side to side as they move through the air.

*5. Calculate the trajectory of a batted ball hit with side spin. That is, let the rotation axis be vertical, corresponding to a ball that is hit so that it "hooks" or "slices." This is commonly encountered in a ball hit near one of the foul lines. Calculate how much a spin angular velocity of 2000 rpm would cause a line drive to curve. Is this consistent with your experiences? If not, calculate what angular velocity would be required.

**6. Estimate the value of S_0 in (2.24) by averaging $F_{\rm drag}$ as given in Figure 2.6 over the face of the ball. Don't be afraid to make some approximations, such as assuming that one side of the ball is completely rough while the other is smooth, in the sense of Figure 2.6. You might also want to perform your averaging numerically, using the integration methods discussed in Chapters 5 and 7.

*7. Estimate the velocity dependence of S_0 from the data given in Adair (1990). Use your results to calculate what effect this has on the trajectory of a typical curve ball.

2.5 Golf

We conclude this chapter with a treatment of the motion of a golf ball. While all of the ingredients we will need for our simulations have already been touched on in our discussions of cannon shells and baseballs, there are still some interesting lessons to be learned.

We again use a coordinate system in which y is the height of the ball above the ground, x is the horizontal direction in the same plane as the ball's initial velocity, and with z horizontal and perpendicular to the velocity. The equations of motion for the golf ball are then

$$\begin{aligned}\frac{dv_x}{dt} &= -\frac{F_{{\rm drag},x}}{m} - \frac{S_0\,\omega\,v_y}{m} \\ \frac{dv_y}{dt} &= -\frac{F_{{\rm drag},y}}{m} + \frac{S_0\,\omega\,v_x}{m} - g\,,\end{aligned} \tag{2.28}$$

with the usual equations for dx/dt and dy/dt. Here we have assumed that the ball is hit with backspin with an angular velocity of ω, so that the spin axis is along z and thus perpendicular to both x and y. We have encountered all of these variables before, including the atmospheric drag, $F_{\rm drag}$, and the Magnus force, which is proportional to the product of ω and the velocity. The magnitudes of these forces have been estimated from measurements with real golf balls. The drag force is given by an expression of the form (2.9), and it turns out that the drag coefficient C varies with velocity in a manner that is qualitatively similar to what we observed for a baseball in Figure 2.6. At low speeds $C \approx \frac{1}{2}$, but this coefficient drops sharply with increasing speed, with C measured to be $\approx 7.0/v$ (with v in m/s) at high speeds (see Erlichson [1983]). We will include the behavior in these two limits in our modeling by assuming that

$$F_{\rm drag} = -\,C\,\rho\,A\,v^2\,, \tag{2.29}$$

with $C = \frac{1}{2}$ for speeds up to $v = 14$ m/s, and $C = 7.0/v$ at higher velocities. Our drag force will thus be a continuous function of v, although it will have a discontinuous derivative ($dF_{\rm drag}/dv$) at $v = 14$ m/s. This behavior is qualitatively similar to that of a baseball, but the transition to a reduced drag coefficient, that is, to turbulent flow, occurs at a much smaller velocity.

A golf ball is hit with a significant amount of backspin, and the associated Magnus force gives a large upward (vertical) force. In fact, the initial Magnus force can actually be larger than the force of gravity. This is apparent to any (physics) student of golf, since the initial trajectory of a well-hit drive has upward curvature. Using the measurements discussed by Erlichson, we estimate $S_0\omega/m \approx 0.25\ \mathrm{s}^{-1}$ for a typical case. It is believed to be a good approximation to assume that ω does not change significantly over the course of the trajectory. In our simulations we will assume that $\vec{\omega}$ is constant (in both magnitude and direction), so the Magnus force then varies only as $\vec{v}$ changes.

The equations of motion (2.28) can (yet again) be solved with the Euler method. The programming is similar to what we have encountered previously in this chapter, so we will leave it to the exercises. Some results are shown in Figure 2.13, where we have assumed an initial velocity of 70 m/s (about 230 feet/s), which is a typical value for an average player. The maximum range is approximately 215 m (approximately 235 yards), in reasonable accord with our expectations, so the model seems to capture the essential physics. It turns out that the maximum range occurs with an initial angle of only 9°, much reduced from the 45° that would be found for a simple projectile in a vacuum. This shows the importance of the Magnus force.

We also consider several hypothetical cases in Figure 2.13. If we increase the spin on the ball by 50 percent, as might occur with an iron shot, for example, the trajectory is much higher even for our 9° initial angle, although for the parameters considered here the range is almost unchanged. If we put no spin on the ball, the range drops dramatically to about 112 m, again assuming an initial angle of 9°. Thus, as any golfer knows, the spin is indeed extremely important.

It is especially interesting to consider the case of a smooth ball, that is, a ball *without* dimples. This can be modeled with a constant drag coefficient, $C = \frac{1}{2}$, independent of the speed.[22] This case has the shortest range of all, only about 94 m. (Note: We have still assumed that the ball is spinning, with the same Magnus force, the same value of S_0, as above.) Now we can understand why a golf ball has dimples. They cause the transition to turbulent flow to occur at very low velocities, with a corresponding drop in the drag coefficient. This, in turn, allows a golf ball to be hit much farther than would otherwise be possible (thereby inflating the ego of the golfer). While it is probably too late to change such a familiar aspect of the game, the use of a smooth ball would have some advantages. For example, golf courses could be made much smaller and the players wouldn't have to walk as far.

Exercises

1. Model the behavior of a ping-pong ball. The drag force can be estimated from (2.9) with $C = \frac{1}{2}$ (since the ball is smooth). The Magnus acceleration of a ping-pong ball is $\approx S_0\,\vec{\omega} \times \vec{v}/m$, where $S_0/m = 0.040$ (in SI units, with ω in rad/s and v in m/s; see the discussion in Bennett [1976]). Assume $v = 3$ m/s, and angular velocities in the range 1–10 revolutions per second (better yet, try to measure ω in your own experiment).

2. Calculate the trajectory for a hooking or slicing golf ball by giving the ball sidespin, with the same angular velocity used in the calculations in Figure 2.13.

3. Investigate the trajectory of the topspin lob in tennis. Topspin is also important for ground strokes and serves. See Štěpánek (1988) for a discussion of the relevant parameters.

[22] This amounts to assuming that the transition to turbulent flow occurs at a velocity above that used in our calculation, 70 m/s. As with baseballs, we would expect that even for a smooth ball there must eventually be a transition to turbulent flow, but we do not know of any accurate studies of where this transition occurs. Thus, our modeling of a "smooth" golf ball is less quantitative than our simulations for a ball with dimples. Even so, these uncertainties do not affect our main conclusion, which is that the dimples substantially increase the range of the ball.

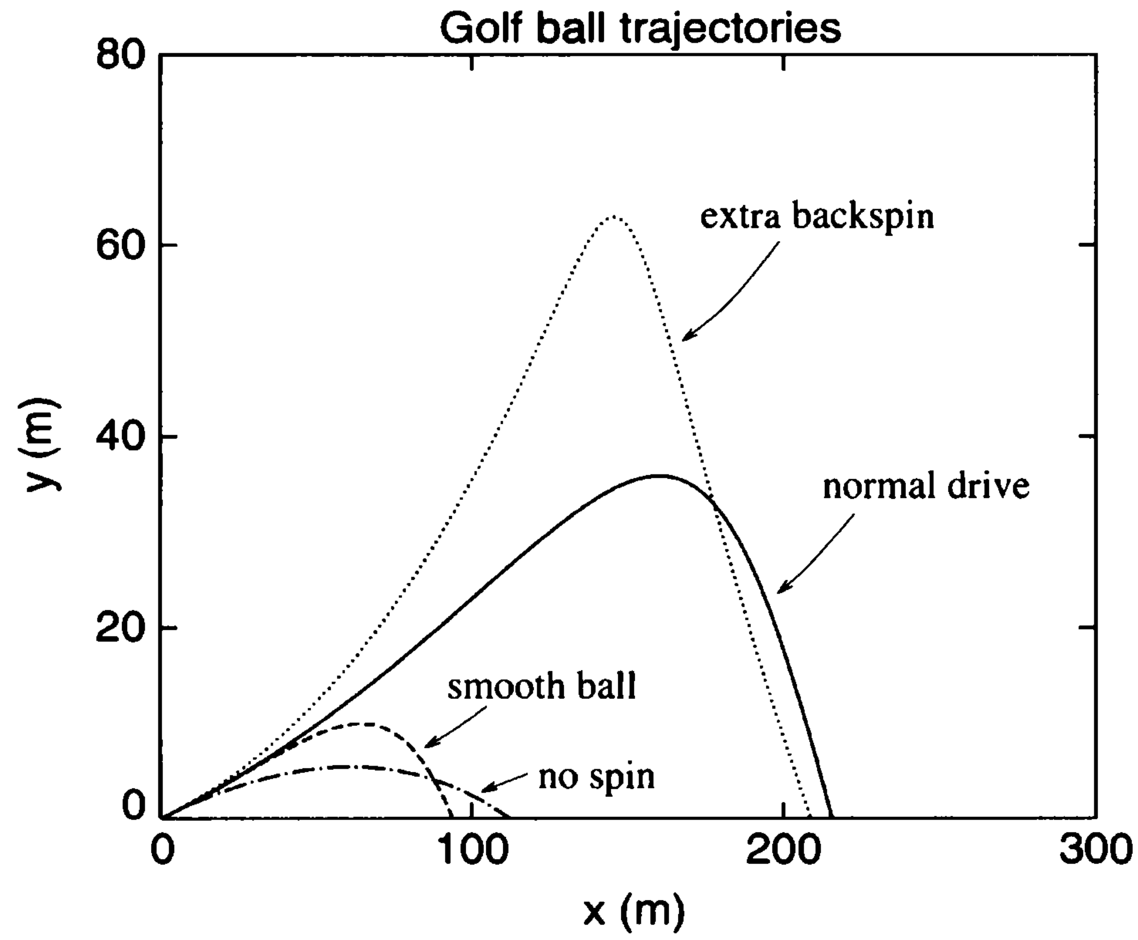

Figure 2.13: Several calculated trajectories of golf balls. In all cases the initial velocity was 70 m/s, at an angle 9° from the horizontal. Solid curve: result for a normal drive (the calculation includes the effect of air resistance and backspin); dotted curve: same as a normal drive, but with 50 percent more backspin to provide more lift; dot-dashed curve: without any backspin; dashed curve: normal amount of backspin, but for a smooth ball.

4. Calculate the effects of air drag, wind, or spin (or all three) for your favorite sports projectile. De Mestre (1990) gives a handy table of the relevant parameters for a wide variety of objects.

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