

1 Vector Spaces

1.1 Fields and Vector Spaces

Definition 1.1.1: Definition of a field

A field F is a set with two functions

- Addition: $+: F \times F \rightarrow F, (\lambda, \mu) \mapsto \lambda + \mu$
- Multiplication: $\cdot: F \times F, (\lambda, \mu) \mapsto \lambda\mu$

which satisfy the following axioms:

1. $(F, +)$ is an abelian group F^+ , with identity 0_F
2. $(F \setminus \{0_F\}, \cdot)$ is an abelian group F^\times , with identity 1_F
3. **Distributive law:** For all a, b , and c in F , we have

$$a(b + c) = ab + ac \in F$$

and the following lemmas:

1. The elements 0_F and 1_F of F are distinct
2. For all $a \in F$, $a \cdot 0_F = 0_F$ and $0_F \cdot a = 0_F$
3. Multiplication in F is associative, and 1_F is an identity element

A **vector space** V over a field F is a pair consisting of an abelian group $V = (V, +)$ and a mapping

$$F \times V \rightarrow V : (\lambda, \vec{v}) \mapsto \lambda\vec{v}$$

s.t. for all $\lambda, \mu \in F$ and $\vec{v}, \vec{w} \in V$ the following identities hold:

- **Distributivity 1:** $\lambda(\vec{v} + \vec{w}) = \lambda\vec{v} + \lambda\vec{w}$
- **Distributivity 2:** $(\lambda + \mu)\vec{v} = \lambda\vec{v} + \mu\vec{v}$
- **Associativity:** $\lambda(\mu\vec{v}) = (\lambda\mu)\vec{v}$
- **Identity:** $1\vec{v} = \vec{v}$

and so do the following lemmas:

1. If V is a vector space and $\vec{v} \in V$, then $0\vec{v} = \vec{0}$
2. If V is a vector space and $\vec{v} \in V$, then $(-1)\vec{v} = -\vec{v}$
3. If V is a vector space over a field F , then $\lambda\vec{0} = \vec{0}$ for all $\lambda \in F$. Furthermore, if $\lambda\vec{v} = \vec{0}$ then either $\lambda = 0$ or $\vec{v} = \vec{0}$

1.2 Working with Vector Spaces

Definition 1.2.1: Cartesian Product of n sets

$$X_1 \times \cdots \times X_n := \{(x_1, \dots, x_n) : x_i \in X_i \text{ for } 1 \leq i \leq n\}$$

The elements of a product are called **n -tuples**. An individual entry $x_i = (x_1, \dots, x_n)$ is called a **component**. There are special mappings called **projections** for a cartesian product:

$$\begin{aligned} \text{pr}_i : X_1 \times \cdots \times X_n &\rightarrow X_i \\ (x_1, \dots, x_n) &\mapsto x_i \end{aligned}$$

The cartesian product of n copies of a set X is written in short as: X^n

Definition 1.2.2: Vector Subspace

A subset U of a vector space V is called a **vector subspace** or **subspace** if U contains the zero vector, and whenever $\vec{u}, \vec{v} \in U$ and $\lambda \in F$ we have $\vec{u} + \vec{v} \in U$ and $\lambda\vec{u} \in U$

Definition 1.2.3: Spans and Linear Independence

Let $T \subset V$ for some vector space V over a field F . Then amongus all subspaces of V that include T there is a smallest subspace

$$\langle T \rangle = \langle T \rangle_F \subseteq V$$

“the set of all vectors $\alpha_1\vec{v}_1 + \cdots + \alpha_r\vec{v}_r$ with $\alpha_1, \dots, \alpha_r \in F$ and $\vec{v}_1, \dots, \vec{v}_r \in T$, together with the zero vector in the case $T = \emptyset$ ”

Terminology Dump

- An expression of the form $\alpha_1\vec{v}_1 + \cdots + \alpha_r\vec{v}_r$ is called a **linear combination** of vectors $\vec{v}_1, \dots, \vec{v}_r$
- The smallest vector subspace $\langle T \rangle \subseteq V$ containing T is called the **vector subspace generated by T** or the vector subspace **spanned by T** or even the **span of T**
- If we allow the zero vector to be the “empty linear combination of $r = 0$ vectors”, then the span of T is exactly the set of all linear combinations of vectors from T
- A subset of a vector space that spans the entire space is called a **generating** or **spanning set**. A vector space that has a finite generating set is said to be **finitely generated**

Linear Independence

A subset L of a vector space V is called **linearly independent** if for all pairwise different vectors $\vec{v}_1, \dots, \vec{v}_r \in L$ and arbitrary scalars $\alpha, \dots, \alpha_r \in F$,

$$\alpha_1\vec{v}_1 + \cdots + \alpha_r\vec{v}_r = \vec{0} \implies \alpha_1 = \cdots = \alpha_r = 0$$

A subset L of a vector space V is called **linearly dependent** if it is not linearly independent (duh.). This means there exists pairwise different vectors $\vec{v}_{j_1}, \dots, \vec{v}_r \in L$ and scalars $\alpha_1, \dots, \alpha_r \in F$, not all zero, such that $\alpha_1\vec{v}_1 + \cdots + \alpha_r\vec{v}_r = \vec{0}$

1.3 Linear Independence and Bases

Definition 1.3.1: Basis of a Vector Space

A **basis of a vector space** V is a linearly independent generating set in V

Example 1.3.2: Standard Basis

Let F be a field and $n \in \mathbb{N}$. We consider the following vectors in F^n

$$\vec{e}_i = (0, \dots, 0, 1, 0, \dots, 0)$$

with one 1 in the i -th place and zero everywhere else. Then $\vec{e}_1, \dots, \vec{e}_n$ form an ordered basis of F^n , the so-called **standard basis of F^n**

Theorem 1.3.3: Linear combinations of basis elements

Let F be a field, V a vector space over F and $\vec{v}_1, \dots, \vec{v}_r \in V$ vectors. The family $(\vec{v}_i)_{1 \leq i \leq r}$ is a basis of V if and only if the following “evaluation” mapping

$$\begin{aligned} \psi : F^r &\rightarrow V \\ (\alpha_1, \dots, \alpha_r) &\mapsto \alpha_1\vec{v}_1 + \cdots + \alpha_r\vec{v}_r \end{aligned}$$

is a bijection

If we label our ordered family by $\mathcal{A} = (\vec{v}_1, \dots, \vec{v}_r)$, then we done the above mapping by

$$\psi = \psi_{\mathcal{A}} : F^r \rightarrow V$$

Theorem 1.3.4: Characterisations of Bases

The following are equivalent for a subset E of a vector space V :

1. E is a basis, i.e. a linearly independent generating set
2. E is minimal among all generating sets, meaning that $E \setminus \{\vec{v}\}$ does not generate V , for any $\vec{v} \in E$
3. E is maximal among all linearly independent subsets, meaning that $E \cup \{\vec{v}\}$ is linearly dependent for any $\vec{v} \in V$

Corollary: Let V be a finitely generated vector space over a field F . Then V has a finite basis

Basis Characterisation Variant

1. If $L \subset V$ is a linearly independent subset and E is minimal amongst all generating sets of V with the property that $L \subseteq E$, then E is a basis.
2. If $E \subseteq V$ is a generating set and if L is maximal amongst all linearly independent sets of V with the property $L \subseteq E$, then L is a basis.

Definition 1.3.5: Free Vector Space

Let X be a set and F a field. The set $\text{Maps}(X, F)$ of all mappings $f : X \rightarrow F$ becomes an F -vector space with the operations of pointwise addition and multiplication by a scalar. The subset of all mappings which send almost all elements of X to zero is a vector subspace

$$F\langle X \rangle \subseteq \text{Maps}(X, F)$$

This subspace is called the **free vector space on the set X**

Theorem 1.3.6: Variant of Linear Combinations

Let F be a field, V be an F -vector space and $(\vec{v}_i)_{i \in I}$ a family of vectors from the vector space V . The following are equivalent:

1. The family $(\vec{v}_i)_{i \in I}$ is a basis for V
2. For each $\vec{v} \in V$ there is precisely one family $(a_i)_{i \in I}$ of elements of F , almost all which are zero and such that

$$\vec{v} = \sum_{i \in I} a_i \vec{v}_i$$

1.4 Dimension of a Vector Space

Theorem 1.4.1: Fundamental Estimate of LinAlg

No linearly independent subset of a given vector has more elements than a generating set. Thus if V is a vector space, $L \subset V$ a linearly independent subset and $E \subseteq V$ a generating set, then

$$|L| \leq |E|$$

Theorem 1.4.2: Steinitz Exchange Theorem

Let V be a vector space, $L \subset V$ a finite linearly independent subset and $E \subseteq V$ a generating set. Then there is an injection $\phi : L \hookrightarrow E$ such that $(E \setminus \phi(L)) \cup L$ is also a generating set for V

Let V be a vector space, $M \subseteq V$ a linearly independent subset, and $E \subseteq V$ a generating subset, such that $M \subseteq E$. If $\vec{w} \in V \setminus M$ is a vector $\notin M$ such that $M \cup \{\vec{w}\}$ is linearly independent, then there exists $\vec{e} \in E \setminus M$ such that $(E \setminus \{\vec{e}\}) \cup \{\vec{w}\}$ is a generating set

Theorem 1.4.3: Cardinality of Bases

Let V be a finitely generated vector space.

1. V has a finite basis
2. V cannot have an infinite basis
3. Any two bases of V have the same number of elements

Definition 1.4.4: Dimension of a Vector Space

The cardinality of a basis of a finitely generated vector space V is called the **dimension** of V , written $\dim V$. If F is a field, and we want to denote that we mean dimension as an F -vector space, then we write $\dim_F V$. If the vector space is not finitely generated, then we say $\dim V = \infty$ and call V **infinite dimensional**.

Theorem 1.4.5: Dimension Theorems

Cardinality Criterion for Bases

1. Each linearly independent subset $L \subset V$ has at most $\dim V$ elements, and if $|L| = \dim V$ then L is a basis
2. Each generating set $E \subseteq V$ has at least $\dim V$ elements, and if $|E| = \dim V$ then E is a basis

Dimension Estimate for Vector Subspaces: A proper vector subspace of a finite dimensional vector space has itself a strictly smaller dimension

If $U \subseteq V$ is a vector subspace of an arbitrary vector space, then we have $\dim U \leq \dim V$ and if we have $\dim U = \dim V < \infty$ then it follows that $U = V$

1.5 Linear Mappings

Definition 1.5.1: Linear Mappings

Let V, W be vector spaces over a field F . A mapping $f : V \rightarrow W$ is called **linear**, or **F -linear**, or even a **homomorphism of F -vector spaces** if for all $\vec{v}_1, \vec{v}_2 \in V$ and $\lambda \in F$ we have

$$f(\vec{v}_1 + \vec{v}_2) = f(\vec{v}_1) + f(\vec{v}_2)$$

$$f(\lambda \vec{v}_1) = \lambda f(\vec{v}_1)$$

A bijective linear mapping is called an **isomorphism** of vector spaces. If there is an isomorphism between two vector spaces, we call them **isomorphic**. A homomorphism $V \rightarrow V$ is called an **endomorphism** of V . An isomorphism $V \rightarrow V$ is called an **automorphism** of V

Two vector subspaces V_1, V_2 of a vector space V are called **complementary** if addition defines a bijection

$$V_1 \times V_2 \xrightarrow{\sim} V$$

something about direct sums

Theorem 1.5.2: Classifying VecSpaces by Dimension

Let n be a natural number. Then a vector space over a field F is isomorphic to F^n iff it has dimension n

Theorem 1.5.3: Linear Mapping and Bases

Let V, W be vector spaces over a field F . The set of all homomorphisms from V to W is denoted by

$$\text{Hom}_F(V, W) = \text{Hom}(V, W) \subseteq \text{Maps}(V, W)$$

Let $B \subset V$ be a basis. Then restriction of a mapping gives a bijection

$$\begin{aligned} \text{Hom}_F(V, W) &\xrightarrow{\sim} \text{Maps}(B, W) \\ f &\mapsto f|_B \end{aligned}$$

Theorem 1.5.4: Inverse Mappings

1. Every injective linear mapping $f : V \hookrightarrow W$ has a **left inverse**, or a linear mapping $g : W \rightarrow V$ s.t. $g \circ f = \text{id}_V$
2. Every surjective linear mapping $f : V \twoheadrightarrow W$ has a **right inverse**, or a linear mapping $G : W \rightarrow V$ s.t. $f \circ g = \text{id}_W$

Definition 1.5.5: Image and Kernel of a map

The **image** of a linear mapping $f : V \rightarrow W$ is the subset $\text{im}(f) = f(V) \subseteq W$. It is a vector subspace of W . The preimage of the zero vector of a linear mapping $f : V \rightarrow W$ is denoted by:

$$\ker(f) := f^{-1}(0) = \{v \in V : f(v) = 0\}$$

and is called the **kernel** of the linear mapping f . The kernel is a subspace of V

Mini lemma: A linear mapping is injective iff its kernel is zero

Theorem 1.5.6: Rank-Nullity / Dimension Theorem

Let $f : V \rightarrow W$ be a linear mapping between vector spaces. Then:

$$\dim V = \dim(\ker f) + \dim(\text{im } f)$$

Dimension of $\text{im } f =$ **rank** of f , dimension of $\ker f =$ **nullity** of f

Let V be a vector space, and $U, W \subseteq V$ vector subspaces. Then

$$\dim(U + W) + \dim(U \cap W) = \dim U + \dim W$$

2 Linear Mappings and Matrices

2.1 Linear Mappings $F^m \rightarrow F^n$ and Matrices

Theorem 2.1.1: Linear Maps $F^m \rightarrow F^n$ and Matrices

Let F be a field and let $m, n \in \mathbb{N}$. There is a bijection between the space of linear mappings $F^m \rightarrow F^n$ and the set of matrices with n rows, m columns, and entries in F :

$$M : \text{Hom}_F(F^m, F^n) \xrightarrow{\sim} \text{Mat}(n \times m; F)$$

$$f \mapsto [f]$$

This attaches to each linear mapping f its **representing matrix** $M(f) := [f]$. The columns of this matrix are the images under f of the standard basis elements of F^m

$$[f] := (f(\vec{e}_1) | f(\vec{e}_2) | \dots | f(\vec{e}_m))$$

Definition 2.1.2: Matrix Multiplication

Let $n, m, \ell \in \mathbb{N}$, F a field, and let $A \in \text{Mat}(n \times m; F)$ and $B \in \text{Mat}(m \times \ell; F)$ be matrices. The **product** $A \circ B = AB \in \text{Mat}(n \times \ell; F)$ is the matrix defined by

$$(AB)_{ik} = \sum_{j=1}^m A_{ij} B_{jk}$$

Theorem 2.1.3: Composition of maps to products

Let $g : F^\ell \rightarrow F^m$ and $f : F^m \rightarrow F^n$ be linear mappings. The representing matrix of their composition is the product of their representing matrices:

$$[f \circ g] = [f] \circ [g]$$

Theorem 2.1.4: Calculating with Matrices

- $(A + A')B = AB + A'B$
- $AI = A$
- $A(B + B') = AB + AB'$
- $(AB)C = A(BC)$
- $IB = B$

3 Rings

I can't be bothered doing changes of basis and stuff, time for something more interesting :D

3.1 Ring basics

Definition 3.1.1: Definition of a Ring

A **ring** is a set with two operations $(\mathbb{R}, +, \cdot)$ that satisfy:

- 1. $(R, +)$ is an abelian group
- 2. (R, \cdot) is a **monoid** - this means that the second operation $\cdot : R \times R \rightarrow R$ is associative and that there is an **identity element** $1 = 1_R \in R$, often just called the identity, with the property that $1 \cdot a = a \cdot 1 = a$ for all $a \in R$.
- 3. The distributive laws hold, meaning that for all $a, b, c \in R$,
$$a \cdot (b + c) = (a \cdot b) + (a \cdot c)$$
$$(a + b) \cdot c = (a \cdot c) + (b \cdot c)$$

The two operations are called **addition** and **multiplication** in our ring. A ring in which multiplication, that is $a \cdot b = b \cdot a$ for all $a, b \in R$, is a **commutative ring**

Note: We'll call the element $1 \in R$ as the identity element of the monoid (R, \cdot) , and we call the additive identity of $(R, +)$ zero, written as 0_R or 0

Example: We can define the **null ring** or **zero ring** as a ring where R is a single element set, e.g. $\{0\}$, with the operations $0 + 0 = 0$ and $0 \times 0 = 0$. We will call any ring that isn't the zero ring a **non-zero ring**

Example 3.1.2: Modulo Rings

Let $m \in \mathbb{Z}$ be an integer. Then the set of **integers modulo m** , written

$$\mathbb{Z}/m\mathbb{Z}$$

is a ring. The elements of $\mathbb{Z}/m\mathbb{Z}$ consist of **congruence classes** of integers modulo m - that is the elements are the subsets T of \mathbb{Z} of the form $T = a + m\mathbb{Z}$ with $a \in \mathbb{Z}$. Think of these as the set of integers that have the same remainder when you divide them by m . I denote the above congruence class by \bar{a} . Obviously $\bar{a} = \bar{b}$ is the same as $a - b \in m\mathbb{Z}$, and often I'll write

$$a \equiv b \pmod{m}$$

3.2 Linking Rings to Fields and Further Properties

Definition 3.2.1: Ring definition of a field

A **field** is a non-zero commutative ring F in which every non-zero element $a \in F$ has an inverse $a^{-1} \in F$, that is an element a^{-1} with the property that $a \cdot a^{-1} = a^{-1} \cdot a = 1$

Theorem 3.2.2: Prime property of fields

Let m be a positive integer. The commutative ring $\mathbb{Z}/m\mathbb{Z}$ is a field if and only if m is prime.

Theorem 3.2.3: Lemmas for multiplying

Let R be a ring and let $a, b \in R$. Then

- 1. $0a = 0 = a0$
- 2. $(-a)b = -(ab) = a(-b)$
- 3. $(-a)(-b) = ab$

Definition 3.2.4: Multiples of an abelian group

Let $m \in \mathbb{Z}$. The m -th **multiple** ma of an element a in an abelian group R is:

$$ma = \underbrace{a + a + \dots + a}_{m \text{ terms}} \quad \text{if } m > 0$$

$0a = 0$ and negative multiples are defined by $(-m)a = -(ma)$

Theorem 3.2.5: Lemmas for multiples

Let R be a ring, let $a, b \in R$ and let $m, n \in \mathbb{Z}$. Then:

- 1. $m(a + b) = ma + mb$
- 2. $(m + n)a = ma + na$
- 3. $m(na) = (mn)a$
- 4. $m(ab) = (ma)b = a(mb)$
- 5. $(ma)(nb) = (mn)(ab)$

Definition 3.2.6: Unit of a ring

Let R be a ring. An element $a \in R$ is called a **unit** if it is *invertible* in R or in other words *has a multiplicative inverse in R* , meaning that there exists $a^{-1} \in R$ such that

$$aa^{-1} = 1 = a^{-1}a$$

Theorem 3.2.7

The set R^\times of units in a ring R forms a group under multiplication

Definition 3.2.8: zero-divisors of a ring

In a ring R , a non-zero element a is called a **zero-divisor** or **divisor of zero** if there exists a non-zero element b such that either $ab = 0$ or $ba = 0$.

Theorem 3.2.9: Cancellation Law

Let R be an *integral domain* and let $a, b, c \in R$. If $ab = ac$ and $a \neq 0$ then $b = c$

Theorem 3.2.10: Prime Property for Integral Domains

Let m be a natural number. Then $\mathbb{Z}/m\mathbb{Z}$ is an integral domain if and only if m is prime.

Theorem 3.2.11

Every **finite** integral domain is a field.

3.3 Polynomials

Definition 3.3.1: Polynomial

Let R be a ring. A **polynomial over R** is an expression of the form

$$P = a_0 + a_1X + a_2X^2 + \dots + a_mX^m$$

for some non-negative integer m and elements $a_i \in R$ for $0 \leq i \leq m$. The set of all polynomials over R is denoted by $R[X]$. In the case where a_m is non-zero, the polynomial P has **degree m** , (written $\deg(P)$), and a_m is its **leading coefficient**. When the leading coefficient is 1 the polynomial is a **monic polynomial**. A polynomial of degree one is called **linear**, a polynomial of degree two is called **quadratic**, and a polynomial of degree three is called **cubic**.

Definition 3.3.2: Ring of Polynomials

The set $R[X]$ becomes a ring called the **ring of polynomials with coefficients in R , or over R** . The zero and the identity of $R[X]$ are the zero and identity of R , respectively.

Theorem 3.3.3: Zero-Divisors of a Polynomial Ring

If R is a ring with no zero-divisors, then $R[X]$ has no zero-divisors and $\deg(PQ) = \deg(P) + \deg(Q)$ for non-zero $P, Q \in R[X]$.

If R is an integral domain, then so is $R[X]$

Theorem 3.3.4: Division and Remainder

Let R be an integral domain and let $P, Q \in R[X]$ with Q monic. Then there exists unique $A, B \in R[X]$ such that $P = AQ + B$ and $\deg(B) < \deg(Q)$ or $B = 0$

Definition 3.3.5: Formal definition of a function

Let R be a commutative ring and $P \in R[X]$ a polynomial. Then the polynomial P can be **evaluated** at the element $\lambda \in R$ to produce $P(\lambda)$ by replacing the powers of X in the polynomial P by the corresponding powers of λ . In this way we have a mapping

$$R[X] \rightarrow \text{Maps}(R, R)$$

This is the precise mathematical description of thinking of a polynomial as a function. An element $\lambda \in R$ is a **root** of P is $P(\lambda) = 0$

Theorem 3.3.6: Roots of a Polynomial

Let R be a commutative ring, let $\lambda \in R$ and $P(X) \in R[X]$. Then λ is a root of $P(X)$ if and only if $(X - \lambda)$ divides $P(X)$

Theorem 3.3.7: Degrees of Polynomial Roots

Let R be a field, or more generally an integral domain. Then a non-zero polynomial $P \in R[X] \setminus \{0\}$ has at most $\deg(P)$ roots in R

Definition 3.3.8: Algebraically closed fields

A field F is **algebraically closed** if each non-constant polynomial $P \in F[X] \setminus F$ with coefficients in our field has a root in our field F

Theorem 3.3.9: Fundamental Theorem of Algebra

The field of complex numbers \mathbb{C} is algebraically closed.

Theorem 3.3.10: Linear Factors of Closed Fields

If F is an algebraically closed field, then every non-zero polynomial $P \in F[X] \setminus \{0\}$ **decomposes into linear factors**

$$P = c(X - \lambda_1) \cdots (X - \lambda_n)$$

with $n \geq 0$, $c \in F^\times$ and $\lambda_1, \dots, \lambda_n \in F$. This decomposition is unique up to reordering the factors

Definition 4.1.1: Symmetric Groups

The group of all permutations of the set $\{1, 2, \dots, n\}$, also known as bijections from $\{1, 2, \dots, n\}$ to itself is denoted by \mathfrak{S}_n (but i will just write S_n because icba) and called the **n -th symmetric group**. It is a group under composition and has $n!$ elements.

A **transposition** is a permutation that swaps two elements of the set and leaves all the others unchanged.

Definition 4.1.2: Inversions of a permutation

An **inversion** of a permutation $\sigma \in S_n$ is a pair (i, j) such that $1 \leq i < j \leq n$ and $\sigma(i) > \sigma(j)$. The number of inversions of the permutation σ is called the **length** of σ and written $\ell(\sigma)$. In formulas:

$$\ell(\sigma) = |\{(i, j) : i < j \text{ but } \sigma(i) > \sigma(j)\}|$$

The **sign** of σ is defined to be the parity of the number of inversions of σ . In formulas:

$$\text{sgn}(\sigma) = (-1)^{\ell(\sigma)}$$

Theorem 4.1.3: Multiplicativity of the sign

For each $n \in \mathbb{N}$ the sign of a permutation produces a group homomorphism $\text{sgn} : S_n \rightarrow \{+1, -1\}$ from the symmetric group to the two-element group of signs. In formulas:

$$\text{sgn}(\sigma\tau) = \text{sgn}(\sigma)\text{sgn}(\tau) \quad \forall \sigma, \tau \in S_n$$

Definition 4.1.4: Alternating Group of a Permutation

For $n \in \mathbb{N}$, the set of even permutations in S_n forms a subgroup of S_n because it is the kernel of the group homomorphism $\text{sgn} : S_n \rightarrow \{+1, -1\}$. This group is the **alternating group** and is denoted A_n

4.2 Determinants

Definition 4.2.1: Determinants

Let R be a commutative ring and $n \in \mathbb{N}$. The **determinant** is a mapping $\det : \text{Mat}(n; R) \rightarrow R$ from square matrices with coefficients in R to the ring R that is given by the following formula

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \mapsto \det(A) = \sum_{\sigma \in S_n} \text{sgn}(\sigma) a_{1\sigma(1)} \cdots a_{n\sigma(n)}$$

4.3 Characterising the Determinant

Definition 4.3.1: Bilinear Forms

Let U, V, W be F -vector spaces. A **bilinear form on $U \times V$ with values in W** is a mapping $H : U \times V \rightarrow W$ which is a linear mapping in both of its entries. This means that it must satisfy the following properties for all $u_1, u_2 \in U$ and $v_1, v_2 \in V$ and all $\lambda \in F$:

$$H(u_1 + u_2, v_2) = H(u_1, v_1) + H(u_2, v_1)$$

$$H(\lambda u_1, v_1) = \lambda H(u_1, v_1)$$

$$H(u_1, v_2 + u_2) = H(u_1, v_1) + H(u_2, v_1)$$

$$H(u_1, \lambda v_1) = \lambda H(u_1, v_1)$$

Definition 4.3.2: Multilinear Forms

Let V_1, \dots, V_n, W be F -vector spaces. A mapping $H : V_1 \times V_2 \times \cdots \times V_n \rightarrow W$ is a **multilinear form** or just **multilinear** if for each j , the mapping $V_j \rightarrow W$ defined by $v_j \mapsto H(v_1, \dots, v_j, \dots, v_n)$, with the $v_i \in V_i$ arbitrary fixed vectors of V_i for $i \neq j$ is linear.

Definition 4.3.3: Alternating Multilinear Forms

Let V and W be F -vector spaces. A multilinear form $H : V \times \cdots \times V \rightarrow W$ is **alternating** if it vanishes on every n -tuple of elements of V that has at least two entries equal, in other words if:

$$(\exists i \neq j \text{ with } v_i = v_j) \rightarrow H(v_1, \dots, v_i, \dots, v_j, \dots, v_n) = 0$$

Theorem 4.3.4: Characterisation of the Determinant

Let F be a field. The mapping

$$\det : \text{Mat}(n; F) \rightarrow F$$

is the unique alternating multilinear form on n -tuples of column vectors with values in F that takes the value 1_F on the identity matrix

4.4 Rules for Calculating with Determinants

Theorem 4.4.1: Multiplicativity of the Determinant

Let R be a commutative ring and let $A, B \in \text{Mat}(n; R)$. Then

$$\det(AB) = \det(A)\det(B)$$

Theorem 4.4.2

The determinant of a square matrix with entries in a field F is non-zero if and only if the matrix is invertible

4 Determinants and Eigenvalues Redux

4.1 Symmetric Groups

4.4.3 Consequences of determinant rules

- If A is invertible then $\det(A^{-1}) = \det(A)^{-1}$
- If B is a square matrix then $\det(A^{-1}BA) = \det(B)$

Theorem 4.4.4: Determinants of a Transpose Matrix

The determinant of a square matrix and of the transpose of the square matrix are equal, that is for all $A \in \text{Mat}(n; R)$ with R a commutative ring,

$$\det(A^T) = \det(A)$$

Definition 4.4.5: Cofactors of a Matrix

Let $A \in \text{Mat}(n; R)$ for some commutative ring R and natural number n . Let i and j be integers between 1 and n . Then the (i, j) **cofactor** of A is $C_{ij} = (-1)^{i+j} \det(A\langle i, j \rangle)$ where $A\langle i, j \rangle$ is the matrix obtained from A by deleting the i -th row and j -th column.

$$C_{23} = (-1)^{2+3} \det \begin{pmatrix} a_{11} & a_{12} & \textcolor{red}{a_{13}} \\ \textcolor{red}{a_{21}} & \textcolor{red}{a_{22}} & \textcolor{red}{a_{23}} \\ a_{31} & a_{32} & \textcolor{red}{a_{33}} \end{pmatrix} = -a_{11}a_{32} + a_{31}a_{12}$$

Theorem 4.4.6: Laplace's Expansion

Let $A = (a_{ij})$ be an $(n \times n)$ -matrix with entries from a commutative ring R . For a fixed i , the **i -th row expansion of the determinant** is

$$\det(A) = \sum_{j=1}^n a_{ij} C_{ij}$$

and for a fixed j , the **j -th column expansion of the determinant** is

$$\det(A) = \sum_{i=1}^n a_{ij} C_{ij}$$

Definition 4.4.7: Adjugate Matrix

Let A be a $(n \times n)$ -matrix with entries in a commutative ring R . The **adjugate matrix** $\text{adj}(A)$ is the $(n \times n)$ -matrix whose entries are $\text{adj}(A)_{ij} = C_{ji}$ where C_{ji} is the (j, i) -cofactor

Theorem 4.4.8: Cramer's Rule

Let A be a $(n \times n)$ -matrix with entries in a commutative ring R . Then

$$A \cdot \text{adj}(A) = (\det A) I_n$$

4.4.9 Alternative Definition of Cramer's

In many sources, such as Wikipedia, Cramer's Rule means the formula

$$x_i = \frac{\det(a_{*1} \mid \cdots \mid b_* \mid \cdots \mid a_{*n})}{\det(a_{*1} \mid \cdots \mid a_{*i} \mid \cdots \mid a_{*n})}$$

for solving a field F the system $A\vec{x} = \vec{b}$ of n linear equations in n unknowns, provided that a unique solution exists. A unique solution exists if and only if A is invertible. So, instead of applying the Gaussian algorithm, you can calculate lots of determinants, replacing the i -th column of A by the given solution vector \vec{b} . It turns out that if you implement this rule on a computer, it has the same efficiency as the Gaussian algorithm. The relationship between this version of Cramer's rule and the above theorem is got by successively taking the vector \vec{b} in the system of linear equations to be the standard basis elements \vec{e}_i with $1 \leq i \leq n$.

Theorem 4.4.10: Invertibility of Matrices

A square matrix with entries in a commutative ring R is invertible if and only if its determinant is a unit in R . That is, $A \in \text{Mat}(n; R)$ is invertible if and only if $\det(A) \in R^\times$

So for instance, an integral matrix $A \in \text{Mat}(n; \mathbb{Z})$ is invertible if and only if $\det(A)$ is 1 or -1 , since $\mathbb{Z}^\times = \{\pm 1\}$. On the other hand, a matrix $A \in \text{Mat}(n; F)$ with entries in a field F is invertible if and only if $\det(A) \neq 0$ since F^\times consists of the non-zero elements of F .

Theorem 4.4.11: Jacobi's Formula

Let $A = (a_{ij})$ where the coefficients $a_{ij} = a_{ij}(t)$ are functions of t . Then

$$\frac{d}{dt} \det A = \text{Tr} \text{Adj} A \frac{dA}{dt}$$

4.5 Eigenvalues and Eigenvectors

Definition 4.5.1: Eigenvalues and Eigenvectors

Let $f : V \rightarrow V$ be an endomorphism of an F -vector space V . A scalar $\lambda \in F$ is an **eigenvalue** of f if and only if there exists a non-zero vector $\vec{v} \in V$ such that $f(\vec{v}) = \lambda \vec{v}$. Each such vector is called an **eigenvector of f with eigenvalue λ** . For any $\lambda \in F$, the **eigenspace of f with eigenvalue λ** is

$$E(\lambda, f) = \{\vec{v} \in V : f(\vec{v}) = \lambda \vec{v}\}$$

Theorem 4.5.2: Existence of Eigenvalues

Each endomorphism of a non-zero finite dimensional vector space over an algebraically closed field has an eigenvalue

Definition 4.5.3: Characteristic Polynomial

Let R be a commutative ring and let $A \in \text{Mat}(n; R)$ be a square matrix with entries in R . The polynomial $\det(xI_n - A) \in R[x]$ is called the **characteristic polynomial of the matrix A** . It is denoted by

$$\chi_A(x) := \det(xI_n - A)$$

(where χ stands for χ aracteristic, lol)

Theorem 4.5.4: EVs and Characteristic Polynomials

Let F be a field and $A \in \text{Mat}(n; F)$ a square matrix with entries in F . The eigenvalues of the linear mapping $A : F^n \rightarrow F^n$ are exactly the roots of the characteristic polynomial χ_A

4.5.5 Eigenvalue remarks

1. Square matrices $A, B \in \text{Mat}(n; R)$ of the same size are *conjugate* if

$$B = P^{-1}AP \in \text{Mat}(n; R)$$

for an invertible $P \in \text{GL}(n; R)$. Conjugacy is an equivalence relation on $\text{Mat}(n; R)$. (The definition makes sense for any commutative ring R , although we will mainly be concerned with the case of a field)

2. The motivation for conjugacy comes from the various matrix representations for an endomorphism $f : V \rightarrow V$ of an n -dimensional vector space V over a field F . Let

$$A = (a_{ij}) = {}_{\mathcal{A}}[f]_{\mathcal{A}}, B = (b_{ij}) = {}_{\mathcal{B}}[f]_{\mathcal{B}} \in \text{Mat}(n; f)$$

be the matrices of f with respect to bases $\mathcal{A} = (\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n)$, $\mathcal{B} = (\vec{w}_1, \vec{w}_2, \dots, \vec{w}_n)$ for V

$$f(\vec{v}_j) = \sum_{i=1}^n a_{ij} \vec{v}_i, f(\vec{w}_j) = \sum_{i=1}^n b_{ij} \vec{w}_i \in V$$

The change of basis matrix $P = (p_{ij}) = {}_{\mathcal{A}}[\text{id}_V]_{\mathcal{B}} \in \text{Mat}(n; F)$ is invertible, with

$$\vec{w}_j = \sum_{i=1}^n p_{ij} \vec{v}_i \in V$$

We have the identity

$$B = P^{-1}AP \in \text{Mat}(n; F)$$

so A, B are conjugate

3. **Key observation:** the characteristic polynomials of conjugate $A, B \in \text{Mat}(n, R)$ are the same

$$\begin{aligned} \chi_B(x) &= \det(xI_n - B) = \det(xI_n - P^{-1}AP) \\ &= \det(P^{-1}(xI_n - A)P) = \det(P)^{-1} \det(xI_n - A) \det(P) \\ &= \det(xI_n - A) = \chi_A(x) \in R[x] \end{aligned}$$

4. In view of 2 and 3 we can define the characteristic polynomial of an endomorphism $f : V \rightarrow V$ of an n -dimensional vector space over a field F to be

$$\chi_f(x) = \chi_A(x) \in F[x]$$

with $A = \mathcal{A}[f]_{\mathcal{A}} \in \text{Mat}(n; R)$ the matrix of f with respect to *any* basis \mathcal{A} for V . Thanks to 4.5.4, the eigenvalues of f are exactly the roots of χ_f , the characteristic polynomial of f

Remark: Let $f : V \rightarrow V$ be an endomorphism of an n -dimensional vector space V over a field F . Suppose given an m -dimensional subspace $W \subseteq V$ such that $f(W) \subseteq W$, so that there are defined endomorphisms of the subspace and the quotient space

$$g : W \rightarrow W; \vec{w} \mapsto f(\vec{w})$$

$$h : V/W \rightarrow V/W; W + \vec{v} \mapsto W + f(\vec{v})$$

Any ordered basis $\mathcal{A} = (\vec{w}_1, \vec{w}_2, \dots, \vec{w}_m)$ for W can be extended to an ordered basis for V

$$\mathcal{B} = (\vec{w}_1, \vec{w}_2, \dots, \vec{w}_m, \vec{v}_{m+1}, \vec{v}_{m+2}, \dots, \vec{v}_n)$$

The images of the \vec{v}_j 's under the canonical projection $\text{can} : V \rightarrow V/W$ are then an ordered basis for V/W

$$\mathcal{C} = (\text{can}(\vec{v}_{m+1}), \text{can}(\vec{v}_{m+2}), \dots, \text{can}(\vec{v}_n))$$

Let $a_{ij}, b_{jk}, c_{ik} \in F$ be the coefficients in the linear combinations

$$f(\vec{w}_j) = \sum_{i=1}^m a_{ij} \vec{w}_i \in W, \quad f(\vec{v}_k) = \sum_{j=m+1}^n b_{jk} \vec{v}_j + \sum_{i=1}^m c_{ik} \vec{w}_i \in V$$

[WIP SO MUCH WRITING OMG]

4.6 Triangularisable, Diagonalisable, and Cayley-Hamilton

Definition 4.6.1: Triangularisability

Let $f : V \rightarrow V$ be an endomorphism of a finite dimensional F -vector space V . f is **triangularisable** if the vector space V has an ordered basis $\mathcal{B} = (\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n)$ such that

$$f(\vec{v}_1) = a_{11} \vec{v}_1,$$

$$f(\vec{v}_2) = a_{12} \vec{v}_1 + a_{22} \vec{v}_2,$$

\vdots

$$f(\vec{v}_n) = a_{1n} \vec{v}_1 + a_{2n} \vec{v}_2 + \dots + a_{nn} \vec{v}_n \in V$$

(so that the first basis vector \vec{v}_1 is an eigenvector, with eigenvalue a_{11}) or equivalently such that the $n \times n$ matrix $_{\mathcal{B}}[f]_{\mathcal{B}} = (a_{ij})$ representing f with respect to \mathcal{B} is upper triangular (or any other triangular)

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{pmatrix}$$

Theorem 4.6.2

Let $f : V \rightarrow V$ be an endomorphism of a finite dimensional F -vector space V . Then f is triangularisable iff the characteristic polynomial χ_f decomposes into linear factors in $F[x]$

Theorem 4.6.3: Triangularisability and Conjugacy

An endomorphism $A : F^n \rightarrow F^n$ is triangularisable if and only if $A = (a_{ij})$ is conjugate to an upper triangular matrix $B = (b_{ij})$ ($b_{ij} = 0$ for $i > j$), with $P^{-1}AP = B$ for an invertible matrix P

Definition 4.6.4: Diagonalisability

An endomorphism $f : V \rightarrow V$ of an F -vector space V is **diagonalisable** if and only if there exists a basis of V consisting of eigenvectors of f . If V is finite dimensional then this is the same as saying that there exists an ordered basis $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_n\}$ such that corresponding matrix representing f is diagonal, that is $_{\mathcal{B}}[f]_{\mathcal{B}} = \text{diag}(\lambda_1, \dots, \lambda_n)$. In this case, of course, $f(\vec{v}_i) = \lambda_i \vec{v}_i$. A square matrix $A \in \text{Mat}(n; F)$ is **diagonalisable** if and only if the corresponding linear mapping $F^n \rightarrow F^n$ given by left multiplication by A is diagonalisable. Thanks to [something] this just means that A is conjugate to a diagonal matrix, there exists an invertible matrix $P \in \text{GL}(n; F)$ such that $P^{-1}AP = \text{diag}(\lambda_1, \dots, \lambda_n)$. In this case the columns P are the vectors of a basis of F^n consisting of eigenvectors of A with eigenvalues $\lambda_1, \dots, \lambda_n$

Theorem 4.6.5: Linear Independence of Eigenvectors

Let $f : V \rightarrow V$ be an endomorphism of a vector space V and let $\vec{v}_1, \dots, \vec{v}_n$ be eigenvectors of f with pairwise different eigenvalues $\lambda_1, \dots, \lambda_n$. Then the vectors $\vec{v}_1, \dots, \vec{v}_n$ are linearly independent

Theorem 4.6.6: Cayley-Hamilton Theorem

Let $A \in \text{Mat}(n; R)$ be a square matrix with entries in a commutative ring R . Then evaluating its characteristic polynomial $\chi_A(x) \in R[x]$ at the matrix A gives zero.

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