1 Geometry Sheet - WIP

Definitions and stuff

Definition A: Standard Hyperbolic Derivatives

$$\begin{aligned}
y = \sinh(x) &\implies & y' &= \cosh(x) \\
y = \cosh(x) &\implies & y' &= \sinh(x) \\
y = \tanh(x) &\implies & y' &= \operatorname{sech}^2(x) \\
y = \operatorname{csch}(x) &\implies & y' &= -\operatorname{csch}(x)\coth(x) \\
y = \operatorname{sech}(x) &\implies & y' &= -\operatorname{sech}(x)\tanh(x) \\
y = \coth(x) &\implies & y' &= -\operatorname{csch}^2(x)
\end{aligned}$$

Definition B: Standard Hyperbolic Identities

$$\tanh(x) = \frac{\sinh(X)}{\cosh(X)} \qquad \coth(x) = \frac{\cosh(x)}{\sinh(x)}$$

$$\operatorname{sech}(x) = \frac{1}{\cosh(x)} \qquad \operatorname{csch}(x) = \frac{1}{\sinh(x)}$$

Definition C: Cross Product

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} (a_2 \cdot b_3) - (a_3 \cdot b_2) \\ (a_3 \cdot b_1) - (a_1 \cdot b_3) \\ (a_1 \cdot b_2) - (a_2 \cdot b_1) \end{pmatrix}$$

Definition D: Taylor Series Expansion

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \cdots$$

Bonus: quick unit form parameterised derivations of x

- $x'(s) = T(s)$
- $x''(s) = T'(s) = \kappa(s)N(s)$
- $x'''(s) = \frac{d\kappa}{ds}(s)N(s) - \kappa^2(s)T(s) + \kappa(s)\tau(s)B(s)$

Definition E: Gauss Curvature on a Graph

Random equation that was in 2022 PP sols, can't find it anywhere in the book. Presumably only works on a graph, i.e.

$$x : (u, v) \mapsto \begin{pmatrix} u \\ v \\ f(u,v) \end{pmatrix}$$

Equation is as follows

$$K = \frac{f_{uu}f_{vv} - f_{uv}^2}{(1 + f_u^2 + f_v^2)^2}$$

Definition F: Closed vs Exact Forms

A form $\alpha \in \Omega^K(D)$ is said to be **closed** if $d\alpha = 0$ and is said to be **exact** if $\alpha = d\beta$ for some $\beta \in \Omega^{k-1}(D)$
Every exact form is closed, since $d^2 = 0$. The converse is not neccesarily true
If α is exact, and β is closed, then $\alpha \wedge \beta$ is also exact

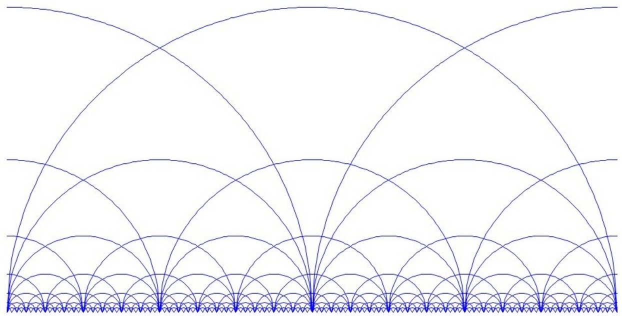
Definition G: Hyperbolic plane representations

Poincare Upper half plane model

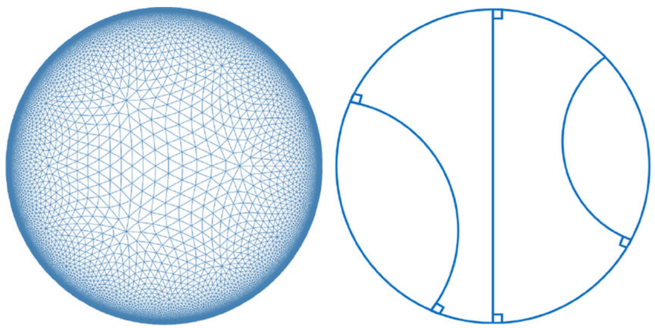
$$H = \{(x, y) \mid y > 0\} \subset \mathbb{R}^2$$

with the first fundamental form

$$I = \frac{(dx)^2 + (dy)^2}{y^2}$$



Poincare Hyperbolic disk model



(a) (b)

Definition H: Standard Orientation

The **standard orientation** (which we always assume) is defined by

$$dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n$$

Coordinates $(y^1, \ldots, y^n)$(an ordered set) are said to be **oriented** on D iff $dy^1 \wedge \cdots \wedge dy^n$ is a positive multiple of $dx^1 \wedge \cdots \wedge dx^n$ for all $x \in D \subseteq \mathbb{R}^n$

Examples

Example 1: Wedge Product Exmaple

Find the wedge product of

$$(x^1dx^2 - dx^3) \wedge ((x^1)^2dx^1 \wedge dx^2 + x^3dx^1 \wedge dx^3)$$

$$\begin{aligned}
&(x^1dx^2 - dx^3) \wedge ((x^1)^2dx^1 \wedge dx^2 + x^3dx^1 \wedge dx^3)\\
&= 0 + x^1x^3dx^2 \wedge dx^1 \wedge dx^3 - (x^1)^2dx^3 \wedge dx^1 \wedge dx^2 - 0\\
&= x^1x^3dx^2 \wedge dx^1 \wedge dx^3 - (x^1)^2dx^3 \wedge dx^1 \wedge dx^2\\
&= -x^1x^3dx^1 \wedge dx^2 \wedge dx^3 + (x^1)^2dx^1 \wedge dx^3 \wedge dx^2\\
&= -x^1x^3dx^1 \wedge dx^2 \wedge dx^3 - (x^1)^2dx^1 \wedge dx^2 \wedge dx^3\\
&= -x^1(x^3 + x^1)dx^1 \wedge dx^2 \wedge dx^3
\end{aligned}$$

Example 2: Exterior Derivative Example

Find the Exterior Derivative of the 1-form $\alpha = x^1x^2dx^1 + x^3dx^2 - dx^3$. (This should turn from a 1-form to a 2-form, i.e. $\Omega^1(D) \to \Omega^2(D)$)

$$\begin{aligned}
\alpha &= d(x^1x^2dx^1 + x^3dx^2 - dx^3)\\
&= d(x^1x^2) \wedge dx^1 + dx^3 \wedge dx^2 + d(-1) \wedge dx^3\\
&= (x^2dx^1 + x^1dx^2) \wedge dx^1 - dx^2 \wedge dx^3\\
&= x^2dx^1 \wedge dx^1 - x^1dx^1 \wedge dx^2 - dx^2 \wedge dx^3\\
&= -x^1dx^1 \wedge dx^2 - dx^2 \wedge dx^3
\end{aligned}$$

Example 3: Pullback Example

Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be given by

$$f(u^1, u^2) = ((u^1)^2, (u^2)^2, u^1u^2))$$

where we use the Cartesian coordinates (u^1, u^2) on $\mathbb{R}^2$ and (x^1, x^2, x^3) on $\mathbb{R}^3$. Calculate the pullback $f^*\rho$ for the form

$$\rho = x^1dx^2 \wedge dx^3$$

$$\begin{aligned}
f^*\rho &= (u^1)^2d((u^2)^2) \wedge d(u^1u^2)\\
&= (u^1)^22u^2du^2 \wedge (u^2du^1 + u^1du^2)\\
&= -2(u^1u^2)^2du^1 \wedge du^2
\end{aligned}$$

Example 4: Theorema Egregium

Example - Finding Gauss Curvature on a sphere defined with the equation

$$x : (\alpha, \phi) \mapsto \begin{pmatrix} a\sin\alpha\cos\phi \\ a\sin\alpha\sin\phi \\ a\cos\alpha \end{pmatrix}$$

with the first fundamental form

$$\mathrm{I} = a^2d\alpha^2 + a^2\sin^2\alpha d\phi^2$$

Pick θ^1 and θ^2 such that $\mathrm{I} = (\theta^1)^2 + (\theta^1)^2$. i.e.

$$\theta^1 = ad\alpha, \quad \theta^2 = a\sin\alpha d\phi$$

Find exterior derivatives

$$d\theta^1 = 0, \quad d\theta^2 = a\cos\alpha d\alpha \wedge d\phi$$

Substitute into the equations $d\theta^1+\omega^1_2\wedge\theta^2 = 0$ and $d\theta^2+\omega^2_1\wedge\theta^1 = 0$ Substituting into the first equation, we get

$$\begin{aligned}
\theta^1 + \omega^1_2 \wedge \theta^2 = 0 &\implies 0 + \omega^1_2 \wedge a\sin\alpha d\phi = 0\\
&\implies (a\sin\alpha)w^1_2 \wedge d\phi = 0
\end{aligned}$$

This implies that ω^1_2 must be proportional to $d\phi$ only, so that the wedge product can evaluate to $d\phi \wedge d\phi = 0$. Therefore, $w^1_2 = \psi d\phi$ for some function ψ. Substituting into the second equation, we get

$$\begin{aligned}
\theta^2 + \omega^2_1 \wedge \theta^1 = 0 &\implies a\cos\alpha d\alpha \wedge d\phi + \omega^2_1 \wedge ad\alpha = 0\\
&\implies a\cos\alpha d\alpha \wedge d\phi = -\omega^2_1 \wedge ad\alpha\\
&\implies a\cos\alpha d\alpha \wedge d\phi = a\omega^1_2 \wedge d\alpha\\
&\implies \cos\alpha d\alpha \wedge d\phi = \omega^1_2 \wedge d\alpha
\end{aligned}$$

This can then be solved by having $\omega^1_2 = -cos\alpha d\phi$ (the minus sign coz the wedge needs flipped) Now we can find the Gauss Curvature with the equation

$$d\omega^1_2 = K\theta^1 \wedge \theta^2$$

by substituting values for θ^1 and θ^2, and finding the exterior derivative of ω^1_2

$$\begin{aligned}
\omega^1_2 &= -\cos\alpha d\phi\\
\implies d\omega^1_2 &= \sin\alpha d\alpha \wedge d\phi
\end{aligned}$$

Compare to wedge

$$\begin{aligned}
\sin\alpha d\alpha \wedge d\phi &= K(ad\alpha) \wedge (a\sin\alpha d\phi)\\
\sin\alpha d\alpha \wedge d\phi &= Ka^2\sin\alpha d\alpha \wedge d\phi
\end{aligned}$$

Therefore, $K = \dfrac{1}{a^2}$

Example 5: random q

I was just doing this on latex to be neater cos the calculations were really tedious, might remove later idk

Find the exterior derivative of

$$\beta = \frac{x^1dx^2 - x^2dx^1}{(x^1)^2 + (x^2)^2}$$

Let f be the function

$$f = \frac{1}{(x^1)^2 + (x^2)^2} = ((x^1)^2 + (x^2)^2)^{-1}$$

Then, we have

$$\begin{aligned}
df &= \frac{\partial f}{\partial x^1}dx^1 + \frac{\partial f}{\partial x^2}dx^2\\
&= -\frac{2x^1}{((x^1)^2 + (x^2)^2)^2}dx^1 - \frac{2x^2}{((x^1)^2 + (x^2)^2)^2}dx^2
\end{aligned}$$

Returning to the original equation, rewrite as follows

$$\begin{aligned}
\beta &= f(x^1dx^2 - x^2dx^1)\\
&= fx^1dx^2 - fx^2dx^1\\
d\beta &= d(fx^1) \wedge dx^2 - d(fx^2) \wedge dx^1\\
&= (x_1df + fdx^1) \wedge dx^2 - (x^2df + fdx^2) \wedge dx^1\\
&= x^1df \wedge dx^2 + \frac{dx^1 \wedge dx^2}{(x^1)^2 + (x^2)^2} - x^2df \wedge dx^1 - \frac{dx^2 \wedge dx^1}{(x^1)^2 + (x^2)^2}\\
&= \frac{2dx^1 \wedge dx^2}{(x^1)^2 + (x^2)^2} + x^1df \wedge dx^2 - x^2df \wedge dx^1\\
&= \frac{2dx^1 \wedge dx^2}{(x^1)^2 + (x^2)^2} + x^1\left(-\frac{2x^1}{((x^1)^2 + (x^2)^2)^2}\right)dx^1 \wedge dx^2\\
&+ x^2\left(-\frac{2x^2}{((x^1)^2 + (x^2)^2)^2}\right)dx^2 \wedge dx^1\\
&= \frac{2dx^1 \wedge dx^2}{(x^1)^2 + (x^2)^2} - \left(\frac{2(x^1)^2}{((x^1)^2 + (x^2)^2)^2}\right)dx^1 \wedge dx^2\\
&+ \left(\frac{2(x^2)^2}{((x^1)^2 + (x^2)^2)^2}\right)dx^1 \wedge dx^2\\
&= \frac{2dx^1 \wedge dx^2}{(x^1)^2 + (x^2)^2} - \frac{2((x^1)^2 + 2(x^2)^2)dx^1 \wedge dx^2}{((x^1)^2 + (x^2)^2)^2}
\end{aligned}$$