Honours Algebra Notes

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1 Vector Spaces

1.1 Fields and Vector Spaces

Definition 1.1.1: Definition of a field

A **field** F is a set with functions

- Addition: $+ : F \times F \to F$, $(\lambda, \mu) \mapsto \lambda + \mu$
- Multiplication: $\cdot : F \times F$, $(\lambda, \mu) \mapsto \lambda\mu$

and two distinguished members 0_F, 1_F with $0_F \neq 1_F$ s.t. $(F, +)$ and $F \backslash \{0_F, \cdot\}$ are *abelian groups* whose neutral elements are 0_F and 1_F respectively, and which also satisifies

$$\lambda(\mu + \nu) = \lambda\mu + \lambda\nu \in F$$

for any $\lambda, \mu, \nu \in F$. Additional Requirements: For all $\lambda, \mu \in F$,

- $\lambda + \mu = \mu + \lambda$
- $\lambda \cdot \mu = \mu \cdot \lambda$
- $\lambda + 0_F = \lambda$
- $\lambda \cdot 1_F = \lambda \in F$

For every $\lambda \in F$ there exists $-\lambda \in F$ such that

$$\lambda + (-\lambda) = 0_F \in F$$

For every $\lambda \neq 0 \in F$ there exists $\lambda^{-1} \neq 0 \in F$ such that

$$\lambda(\lambda^{-1}) = 1_F \in F$$

NOTE: This is a terrible definition of a field, just think of it as a group with two operations instead of one

Definition 1.1.2: Definition of a Vector Space

A **vector space** V **over a field** F is a pair consisting of an abelian group $V = (V, \dot{+})$ and a mapping

$$F \times V \to V : (\lambda, \vec{v}) \mapsto \lambda\vec{v}$$

such that for all $\lambda, \mu \in F$ and $\vec{v}, \vec{w} \in V$ the following identities hold:

$$\begin{aligned}
\lambda(\vec{v}\dot{+}\vec{w}) &= (\lambda\vec{v})\dot{+}(\lambda\vec{w}) \\
(\lambda + \mu)\vec{v} &= (\lambda\vec{v})\dot{+}(\mu\vec{v}) \\
\lambda(\mu\vec{v}) &= (\lambda\mu)\vec{v} \\
1_F\vec{v} &= \vec{v}
\end{aligned}$$

The first two laws are the **Distributive Laws**, the third law is called the **Associativity Law**. A vector field V over a field F is commonly called an **F-vector space**

1.1.3 Vector Space Terminology

- Elements of a vector space: **vectors**
- Elements of the field F: **scalars**
- The field F itself: **ground field**
- The map $(\lambda, \vec{v}) \mapsto \lambda\vec{v}$: **multiplication by scalars**, or the **action of the field F on V**

Notes:

- This is not the same as the "scalar product", as that produces a scalar from two vectors
- Let the zero element of the abelian group V be written as $\vec{0}$ and called the **zero vector**
- The use of $\dot{+}$ and 1_F is there for mostly pedantic rigorous reasons, and a much less confusing way of defining a vector field is defined below:

Definition 1.1.4: Alternative Vector Space definition

A **vector space V over a field F** is a pair consisting of an abelian group $V = (V, \dot{+})$ and a mapping

$$F \times V \to V : (\lambda, \vec{v}) \mapsto \lambda\vec{v}$$

such that for all $\lambda, \mu \in F$ and $\vec{v}, \vec{w} \in V$ the following identities hold:

$$\begin{aligned}
\lambda(\vec{v}\dot{+}\vec{w}) &= \lambda\vec{v}\dot{+}\lambda\vec{w} \\
(\lambda + \mu)\vec{v} &= \lambda\vec{v}\dot{+}\mu\vec{v} \\
\lambda(\mu\vec{v}) &= (\lambda\mu)\vec{v} \\
1\vec{v} &= \vec{v}
\end{aligned}$$

1.1.5 Vector Space Lemmas

Product with the scalar zero: If V is a *vector space* and $\vec{v} \in V$, then $0\vec{v} = \vec{0}$, or in words "zero times a vector is the zero vector"
Product with the scalar (-1): If V is a *vector space* and $\vec{v} \in V$, then $(-1)\vec{v} = -\vec{v}$
Product with the zero vector: If V is a *vector space* over a field F, then $\lambda\vec{0} = \vec{0}$ for all $\lambda \in F$. Furthermore, if $\lambda\vec{v} = \vec{0}$ then either $\lambda = 0$ or @$\vec{v} = \vec{0}$

1.2 Product of Sets and of Vector Spaces

Definition 1.2.1: Cartesian Product of n sets

Trivially: $X \times Y = \{(x, y) : x \in X,\, y \in Y\}$
Just extend this to n numbers

$$X_1 \times \cdots \times X_n := \{(x_1, \ldots, x_n) : x_i \in X_i \text{ for } 1 \leq i \leq n\}$$

The elements of a product are called **n-tuples**. An individual entry $x_i = (x_1, \ldots, x_n)$ is called a **component**.
There are special mappings called **projections** for a cartesian product:

$$\begin{aligned} \mathrm{pr}_i : X_1 \times \cdots \times X_n &\to X_i \\ (x_1, \ldots, x_n) &\mapsto x_i \end{aligned}$$

The cartesian product of n copies of a set X is written in short as: X^n

The elements of X^n are n-tuples of elements from X. In the special case $n = 0$ we use the general convention that X^0 is "the" one element set, so that for all $n, m \geq 0$, we then have the canonical bijection

$$X^n \times X^m \to X^{n+m}$$
$$((x_1, x_2, \ldots, x_n),\, (x_{n+1}, x_{n+2}, \ldots, x_{n+m})) \mapsto (x_1, x_2, \ldots, x_n, x_{n+1}, x_{n+2}, \ldots, x_{n+m})$$

Note: the $\to$ should have a tilde but idk how to typeset it like that
[Bunch of examples: check LN 1.3]

1.3 Vector Subspaces

Definition 1.3.1: Vector Subspace

A subset U of a vector space V is called a **vector subspace** or **subspace** if U contains the zero vector, and whenever $\vec{u}, \vec{v} \in U$ and $\lambda \in F$ we have $\vec{u} + \vec{v} \in U$ and $\lambda\vec{u} \in U$

Note There is a more generalized definition using concepts we haven't learned yet, it is as follows: Let F be a field. A subset of an F-vector space is called a vector subspace if it can be given the structure of an F-vector space such that the embedding is a "homomorphism of F-vector spaces". This definition is a lot more general since it also applies to subgroups, subfields, sub-"any structure", etc

Definition 1.3.2: Spanning Subspace

Let T be a subset of a vector space V over a field F. Then amongst all vector subspaces of V that include T there is a smallest vector subspace

$$\langle T \rangle = \langle T \rangle_F \subseteq V$$

It can be described as the set of all vectors $\alpha_1\vec{v}_1 + \cdots + \alpha_r\vec{v}_r$ with $\alpha_1, \ldots, \alpha_r \in F$ and $\vec{v}_1, \ldots, \vec{v}_r \in T$, together with the zero vector in the case $T = \emptyset$

1.3.3 Subspace terminology

- An expression of the form $a_1\vec{v}_1 + \cdots + \alpha_r\vec{v}_r$ is called a **linear combination** of vectors $\vec{v}_1, \ldots, \vec{v}_r$.
- The smallest vector subspace $\langle T \rangle \subseteq V$ containing T is called the **vector subspace generated by** T or the vector subspace **spanned by** T or even the **span of** T
- If we allow the zero vector to be the "empty linear combination of $r = 0$ vectors", which is what we will mean from hereon, then the span of T is exactly the set of all linear combinations of vectors from T

Definition Number: Generating Subspace

A subset of a vector space is called a **generating** or **spanning set** of our vector space if its span is all of the vector space. A vector space that has a finite generating set is said to be **finitely generated**.

1.4 Linear Independence and Bases

Definition 1.4.1: Linear Independence

A subset L of a vector space V is called **linearly independent** if for all pairwise different vectors $\vec{v}_1, \ldots, \vec{v}_r \in L$ and arbitrary scalars $\alpha, \ldots, \alpha_r \in F$,

$$a_1\vec{v}_1 + \cdots + \alpha_r\vec{v}_r = \vec{0} \implies a_1 = \cdots = \alpha_r = 0$$

Definition 1.4.2: Linear Dependence

A subset L of a vector space V is called **ilnearly dependent** if it is not linearly independent (duh..). This means there exists pairwise different vectors $\vec{v}j_1, \ldots, \vec{v}_r \in L$ and scalars $\alpha_1, \ldots, \alpha_r \in F$, not all zero, such that $\alpha_1\vec{v}_1 + \cdots \alpha_r\vec{v}_r = \vec{0}$

Definition 1.4.3: Basis of a Vector Space

A **basis of a vector space** V is a linearly independent generating set in V

1.4.4 Family notation

Let A and I be sets. We will refer to a mapping $I \to A$ as a **family of elements of** A **indexed by** I and use the notation

$$(a_i)i \in I$$

This is used mainly when I plays a secondary role to A. In the case $I = \emptyset$, we will talk about the **empty family** of elements of A.
Random facts:

- The family $(\vec{v}_i)_{i\in I}$ would be called a generating set if the set $\{\vec{v}_i : i \in I\}$ is a generating set.
- It would be called **linearly independent** or a **linearly independent family** if, for pairwise distinct indices $i(1), \ldots, i(r) \in I$ and arbitrary scalars $a_1, \ldots, a_r \in F$,

$$a_1\vec{v}_{i(1)} + \cdots + a_r\vec{v}_{i(r)} = \vec{0} \to \alpha_1 = \cdots = a_r = 0$$

A difference between families and subsets is that the same vector can be represented by different indices in a family, in which case linear independence as a family is not possible. A family of vectors that is not linearly independent is called a **linearly dependent family**. A family of vectors that is a generating set and linearly independent is called either a **basis** or a **basis indexed by** $i \in I$

Example 1.4.5: Standard Basis

Let F be a field and $n \in \mathbb{N}$. We consider the following vectors in F^n

$$\vec{e}_i = (0, \dots, 0, 1, 0, \dots, 0)$$

with one 1 in the i-th place and zero everywhere else. Then $\vec{e}_1, \dots, \vec{e}_n$ form an ordered basis of F^n, the so-called **standard basis of** F^n

Theorem 1.4.6: Linear combinations of basis elements

Let F be a field, V a vector space over F and $\vec{v}_1, \dots, \vec{v}_r \in V$ vectors. The family $(\vec{v}_i)_{1 \leq i \leq r}$ is a basis of V if and only if the following "evaluation" mapping

$$\begin{aligned} \psi : F^r &\to V \\ (\alpha_1, \dots, a_r) &\mapsto a_1\vec{v}_1 + \cdots + \alpha_r\vec{v}_r \end{aligned}$$

is a bijection
If we label our ordered family by $\mathcal{A} = (\vec{v}_1, \dots, \vec{v}_r)$, then we done the above mapping by

$$\psi = \psi_{\mathcal{A}} : F^r \to V$$