# 1 Leon's ITCS Exam Notes

Basically adapted from Chris Dalziel's notes :)

## Finite Automata

### Definition: Finite Automata

A finite automaton takes a string as input and replies "yes" or "no". If an automaton $A$ replies "yes" on a string $S$ we say that $A$ "accepts" $S$.

### Definition: Deterministic Finite Automata

A deterministic finite automaton (DFA) is a quintuple $(Q, \Sigma, q_0, \delta, F)$ where

- $Q$ is a finite set of states
- $\Sigma$ is an alphabet
- $q_0 \in Q$ is the initial state
- $\delta : Q \times \Sigma \to Q$ is the transition function
- $F \subseteq Q$ is the set of final states

A DFA accepts a string $w \in \Sigma^*$ iff $\delta^*(q_0, w) \in F$, where $\delta^*$ is $\delta$ applied successively for each symbol in $w$.
The language of a DFA $A$ is the set of all strings accepted by $a$, $\mathcal{L} \subseteq \Sigma^*$ is the set of all strings accepted by $A$.
The transition function is a total function which gives exactly one next state for each input symbol, i.e. it is deterministic

### Definition: Nondeterministic Finite Automata

Non-determinism would mean that $\delta$ can return more than one successor state, it instead returns a set of possible states - no states is an empty set. A NFA is a quintuple $(Q, \Sigma, q_0, \delta, F)$ where:

- $Q$ is a finite set of states
- $\Sigma$ is an alphabet
- $q_0 \in Q$ is the initial state
- $\delta : Q \times \Sigma \to \mathcal{P}(Q)$ is the transition function
- $F \subseteq Q$ is the set of final states

The only difference between the definition of a DFA and that of an NFA is that in an NFA $\delta$ returns an element from the powerset of $Q$, $\mathcal{P}(Q)$

Adding non-determinism doesn't change "expressivity". Given an NFA $A$ there is an equivalent DFA $D$ such that $\mathcal{L}(D) = \mathcal{L}(A)$ and vice versa.

### Definition: $\epsilon$-NFA

If we allow non-deterministic state chnages that don't consume any input symbols, we can label silent moves using $\epsilon$ - meaning the empty string We define the $\epsilon$ closure $E(q)$ of a state $q$ as the set of all states reachable from $q$ by silent moves. That is, $E(q)$ is the least set satisfying:

- $q \in E(q)$
- For any $s \in E(q)$ we also have $\delta(s, \epsilon) \subseteq E(q)$

DFA, NFA, $\epsilon$-NFA are all equal in expressive power

## Regular Languages

### Definition: Regular Languages

Any language which can be accepted by a finite automaton is called a regular language.
Regular languages are also those recognised by Regular Expressions

### Definition: Regular Language Closure Properties

For two languages $L_1$ and $L_2$, the following operations satisfy the closure property, i.e. for a member $x \in X$, and an operation $\phi$ we have that $\phi(x) \in \mathbb{R}$ for all $x$.

- **Union**: $L_1 \cup L_2$ is the language that includes all strings of $L_1$ and all strings of $L_2$.
- **Intersection**: $L_1 \cap L_2$ is the language that includes all strings of $L_1$ that are not in $L_2$, and vice versa
- **Sequential Composition**: $L_1 L_2$ is the language of strings that consist of strings in $L_1$ followed by a string in $L_2$.
- **Kleene closure**: $L^*$ is the language of strings that consist wholly of zero or more strings in $L$.
$$L^* = \bigcup_{i \in \mathbb{N}} L^i$$
- **Complement**: $\bar{L}$ is the language of every string not in $L$.

### Definition: Regular Expressions

Regular characterise the regular languages, just like finite automata do. The following table shows the syntax and semantics of a regex.

| Syntax | Semantics | |
|---|---|---|
| $a$ | $[\![a]\!] = \{a\}$ | $(a \in \Sigma)$ |
| $\emptyset$ | $[\![\emptyset]\!] = \emptyset$ | |
| $\epsilon$ | $[\![\epsilon]\!] = \{\epsilon\}$ | |
| $R_1 \cup R_2$ | $[\![R_1 \cup R_2]\!] = [\![R_1]\!] \cup [\![R_2]\!]$ | |
| $R_1 \circ R_2$ | $[\![R_1 \circ R_2]\!] = [\![R_1]\!][\![R_2]\!]$ | |
| $R^*$ | $[\![R^*]\!] = [\![R]\!]^*$ | |

### Definition: Generalised NFAs

A **generalised NFA**, or GNFA is an NFA where:

- Transitions have **regular expressions** on them instead of symbols
- There is only one unique final state
- The transition relation if **full**, except that the initial state has no incoming transitions, and the final state has no outgoing transitions

### Theorem 1: Pumping Lemma

If $L \subseteq \Sigma^*$ is regular, then there is a **pumping length** $p \in \mathbb{N}$ such that for any $w \in L$ where $|w| \geq p$, we may split $w$ into three piexes $w = xyz$ satisfying three conditions:

- $xy^iz, \quad \forall i \in \mathbb{N}$
- $|y| > 0$
- $|xy| \leq p$

Note that if the pumping lemma fails then the language is not regular, but the inverse is not necessarily true.

### Theorem 2: Myhill-Nerode Theorem

Let $L \subseteq \Sigma^*$ and $x, y \in \Sigma^*$. If there exists a suffix string $z$ such that $xz \in L$, but $yz \notin L$ or vice versa, then $x$ and $y$ are **distinguishable** by $L$. If $x$ and $y$ are not distinguishable by $L$, then we say that $x \equiv_L y$ - this is an equivalence relation. A regular language satisfies the following

- The number of equivalence classes $\equiv_L$ is finite.
- The number of equivalence classes is equal to the number of states in the minimal DFA accepting $L$ (not as important)

Therefore, to show a language is non-regular, show that it has infinite equivalence classes - that is, we find an infinite sequence $u_0 u_1 \ldots$ of strings such that for any $i, j$ where $i \neq j$, there is a string $w_{ij}$ such that $u_i w_{ij} \in L$ but $u_j w_{ij} \notin L$ or vice-versa

## Context-Free Languages

### Definition: Context-free Languages

By adding recursion to regexs we can begin to recognise some non-regular languages. All regular languages are also context free.

### Definition: Context-free Grammars

A language is context-free iff it is recognised by a Context-free Grammar (CFG), which is a 4-tuple $(N, \Sigma, P, S)$ where:

- $N$ is a finite set of variables or non-terminals
- $\Sigma$ is a finite set of terminals
- $P \subseteq N \times (N \cup \Sigma)^*$ is a finite set of rules or productions
  - Typically productions are written $A \to aBc$
  - Productions with common heads can be combined, $A \to a$ and $A \to Aa$ can be combined into $A \to a \mid Aa$
- $S \in N$ is the starting variable

We use $\alpha$, $\beta$, $\gamma$ to refer to sequences of terminals
We make a derivation step $\alpha A \beta \Rightarrow_G \alpha\gamma\beta$ whenever $(A \to \gamma) \in P$;
The language of a CFG $G$ is:

$$\mathcal{L}(G) = \{w \in \Sigma^* \mid S \Rightarrow_G^* w^*\}$$

Where $\Rightarrow_G^*$ is the reflexive, transitive, closure of $\Rightarrow_G$.
Context-free grammars are ambiguous. They are closed under union, concatenation, and kleene star, but not under intersection or complementation

### Definition: Eliminating Ambiguity

We want to eliminate ambiguity in CFGs while still accepting all the same strings. This can be done for our language of regular expressions:

- First defining atomic expressions: $A \to (S)|\emptyset|\epsilon|a|b$
- Then ones which use Kleene Star: $K \to A|A^*$
- Then ones which may use left-associative composition: $C \to K|C \circ K$
- Finally expressions which use unions: $S \to C|S \cup C$

The order of operations here is therefore bottom to top; unions come before compositions, which come before Kleene etc

### Definition: Push-down Automata

Push-down automata are to CFGs what finite automatas are to regular expressions. They are implementationally identical to $\epsilon$-NFAs with the addition of a stack. The recursive element of CFGs is implemented using a standard last-in-first-out stack.

Transitions in a push-down automata take the form $x, y \to z$ which is read as "consume the input $x$, popping $y$ off the stack, and push $z$ onto the stack". We can allow actions that don't consume, pop, or push by setting variables to $\epsilon$.

### Definition: Formal Def. of PDAs

A **push-down automaton** is a 6-tuple $(Q, \Sigma, \Gamma, \delta, q_0, F)$ where $Q$, $\Sigma$, $\Gamma$ are all finite sets. $\Gamma$ is the stack alphabet, and $\delta$ now may take a stack symbol as input or return one as output:

$$\delta : Q \times \Sigma_\epsilon \times \Gamma_\epsilon \to \mathcal{P}(Q \times \Gamma_\epsilon)$$

All other compoments are as with $\epsilon$-NFAs

A string $w$ is accepted by a PDA if it ends in a final state, i.e. $\delta^*(q_0, w, \epsilon)$ gives a state $q$ and a stack $\gamma$ wuch that $q \in F$.

### Theorem 3: CFG to PDA

A language is context-free if and only if it is recognised by a push-down automaton. The proof is left as an exercise to the reader.

### Theorem 4: Pumping CFLs intro

Suppose a CFG has $n$ non-terminals, and we have a parse tree of height $k > n$. Then the same non-terminal $V$ must have appeared as its own descendant in the tree

- **Pumping down**: Cut the tree at the higher occurance of $V$ and replace it with the subtree at the lower occurance of $V$
- **Pumping up**: Cut at the lower occurance and replace it with a fresh copy of the higher occurance

### Theorem 5: Pumping Lemma for CFLs

If $L$ is context-free then there exists a pumping length $p \in \mathbb{N}$ such that if $w \in L$ with $|w| \geq p$ then $w$ may be split into **five** pieces $w = uvxyz$ such that

- $uv^ixy^iz \in L$ for all $i \in \mathbb{N}$
- $|vy| > 0$
- $|vxy| \leq p$

### Definition: Chomsky Grammars

Context-free grammars are a special case of Chomsky Grammars. Chomsky grammars are similar to CFGs, except that the left-hand side of a production may be any string that includes at least one non-terminal. An example is shown below

$$\begin{aligned} S &\to abc \mid aAbc \\ Ab &\to bA \\ Ac &\to Bbcc \\ bB &\to Bb \\ aB &\to aaA \mid aa \end{aligned}$$

Such a grammar is called **context-sensitive**

### Definition: The Chomsky Heirarchy

A grammar $G = (N, \Sigma, P, S)$ is of type:

0. (or **computably enumerable**) in the general case
1. (or **context sensitive**) if $|\alpha| \leq |\beta|$ for all productions $\alpha \to \beta$, except we also allow $S \to \epsilon$ if $S$ foes not occur on the RHS of any rule
2. (or **context free**) if all productions are of the form $A \to \alpha$ (i.e. a CFG)
3. (or **right-linear/regular**) if all productions are of the form $A \to w$ or $A \to wB$, where $w \in \Sigma$ and $B \in N$

## Algorithms for Languages

### Theorem 6: Emptiness for Regular Languages

Can we write a program to determine if a given regular language is empty?
Given a finite-automaton this is an instance of graph reach-ability, so we can use a depth-first search.

### Theorem 7: Emptiness for Context-free languages

Can we write a program to determine if a given context-free language is empty?
Given a CFG for our language, we can perform the following process:

1. Mark the terminals and $\epsilon$ as generating
2. Mark all non-terminals which have a production with only generating symbols in their right hand side as generating
3. Repeat until nothing new is marked
4. Check if $S$ is marked as generating or not

### Theorem 8: Equivalence of DFA

Is it possible to write a program to determine if two discrete finite automata are equivalent?

Given two DFA for $L_1$ and $L_2$, we can use our standard constructions to produce a DFA of the symmetric set difference:

$$(L_1 \cap \overline{L}_2) \cup (L_2 \cap \overline{L}_1)$$

## Register Machines

**Definition: Register Machines**

A **register machine**, or RM, consists of:

- A fixed number $m$ of registers $R_0 \dots R_{m-1}$, which each holds a natural number
- A fixed program $P$ which is a sequence of $n$ instructions $I_0 \dots I_{n-1}$

Each instruction is one of the following:

- `INC(i)`: which increments the register $R_i$ by one
- `DECJZ(i, j)`: which decrements register $R_i$ unless $R_i = 0$ in which case it jumps to instruction $I_j$

RMs can compute anything any other computer can

**Definition: Pairing Functions for RMs**

A **pairing function** is an injective function $\mathbb{N} \times \mathbb{N} \to \mathbb{N}$. An example is $f(x, y) = 2^x 3^y$.
We write $\langle x, y \rangle_2$ for $f(x, y)$. If $z = \langle x, y \rangle_2$, let $z_0 = x$ and $z_1 = y$.
This lets us encode multiple values into a single value, and a 2-tuple pairing function is enough to cram an arbitrary sequence of natural numbers into one $\mathbb{N}^* \to \mathbb{N}$

**Theorem 9: Church-Turing Thesis**

The Church-Turing thesis states that any problem is computable by any model of computation iff it is computable by a **Turing machine**.
For our purposes this matters for RMs, TMs, and $\lambda$-calculus.
Other examples are combinator calculus, general recursive functions, pointer machines, counter machines, cellular automata, queue automata, enzyme-based DNA computers, Minecraft, Magic the Gathering, and others.

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**Example : Regular Expressions**

At least one 0:

$$(0 \cup 1)^* 0 (0 \cup 1) *$$

At least one 1 and at least one 0:

$$((0 \cup 1)^* 01 (0 \cup 1)^*) \cup ((0 \cup 1)^* 10 (0 \cup 1)^*)$$