Dynamics and Vector Calculus Notes

Leon Lee

February 26, 2024

Contents

1 Couple Oscillations and normal modus

some diagram idk

---

where x_1 and x_2 are displacements from equilibrium
For mass 1

- Force to the left: $-k_1 x_1$
- Force to the right: $-k_2(x_2 - x_1)$

$$m\frac{d^2 x_1}{dt^2} = -k_1 x_1 + k_2(x_2 - x_1) - k_3 x_2$$

Write this in matrix form

$$m\frac{d^2}{dt^2}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \implies m\frac{d^2 x}{dt^2} = -Kx$$

Definition 1.0.1: Normal Mode Solution

Normal Mode Solution: All co-ordinates (here x_1, x_2) oscillate with the same frequency

$x(t) = \cos(\omega t - \phi)\underline{b}$
$\underline{b}$ is constant vector, ω to be determined
sub in matrixeq??

$$\begin{aligned} &-m\omega^2 \cos(\omega t - \phi)\underline{b} + K\cos(\omega t - \varnothing)\underline{b} = 0 \\ &-m\omega^2 \underline{b} + K\underline{b} = 0 \to K\underline{b} = \lambda \underline{b} \quad \lambda = m\omega^2 \end{aligned}$$

where λ is eigenvalue, and b is eigenvector
For simplicity, take $k_1 = k_2 = k_3 = k$
Then,

$$K = \begin{pmatrix} 2k & -k \\ -k & 2k \end{pmatrix} \qquad (K - \lambda\mathbb{1})\underline{b} = 0 \implies |k - \lambda\mathbb{1}| = 0$$

$$\begin{vmatrix} 2k - \lambda & -k \\ -k & 2k - x \end{vmatrix} = 0 \implies (2k - \lambda)^2 - k^2 = 0$$

This is called the "Characteristic Equation"
$(2k - \lambda) = \pm k \quad \lambda = 2k \mp k$
Therefore, $\lambda = k, 3k$
Mode A: $\lambda_A = k \quad (K - k\mathbb{1})\underline{b} = 0$

$$\begin{pmatrix} k & -k \\ -k & k \end{pmatrix}\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = 0 \quad \underline{b}_A = Ct\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Usually, choose a constant s.t. $\underline{b} \cdot \underline{b} = 1$
Mode B: $\lambda_A = 3k \quad (K - 3k\mathbb{1})\underline{b} = \begin{pmatrix} -k & -k \\ -k & -k \end{pmatrix}\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = 0$
and some stuff more i forgor to write

[diagram thing]
Normal mode $\underline{x}(t) = \underline{b}\cos(\omega t - \phi) \to (K - \kappa\mathbb{1})\underline{b} = 0 \quad \lambda = m\omega^2$

$$\lambda_A = k,\, \underline{b}_A = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \lambda_B = 3k,\, \underline{b}_B = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

So we have 2 independent solutions
General solution: $\underline{x}(t) = A\underline{b}_A \cos(\omega_A t - \phi_A) + B\underline{b}_B \cos(\omega_B t - \varnothing_B)$
So there are 4 constants A, B, ϕ_A, $\varnothing_B$ to be fixed

1.1 Motion in Normal modes

$$\text{Mode A} x_1 = x_2 \quad \text{"unphase"} \quad \omega_A = \left(\frac{k}{m}\right)^2$$
$$\text{Mode B} x_1 = -x_2 \quad \text{"antiphase"} \quad \omega_A > \omega B$$

Normal Co-ordinates Take scalar product

$$(1,1)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_1 + x_2 = 2A\cos(\omega_A - \phi_A)$$
$$(1,-1)\begin{pmatrix} x_1 \\ -x_2 \end{pmatrix} = x_1 + x_2 = 2B\cos(\omega_B t - \phi_B)$$

Define

$$z_1 = \frac{1}{\sqrt{2}}(x_1 + x_2) = \alpha^1 \cos(w_A t - \phi) \quad z_1 + \omega_A^2 z_1 = 0 \quad \text{(SHO)}$$
$$z_2 = \frac{1}{\sqrt{2}}(x_1 - x_2) = \beta^1 \cos(w_B t - \phi) \quad z_2 + \omega_B^2 z_2 = 0 \quad \text{(SHO)}$$

z_1 and z_2 are each independent simple harmonic motions, and energy is preserved in each one

$$E_A = \frac{1}{2}m(z_1)^2 + \frac{1}{2}kz_1^2 = \text{constant in time}$$

1.2 Summmary: properties of Normal Modes

- $\underline{x}_\alpha = A_\alpha \underline{b}_A \cos(\omega_\alpha t - \phi_\alpha))$
- All coordinates oscillate at the same frequency
- constants A_α, ϕ_α are fixed by ic (???)
- General motion is superposition of normal modes
- Normal coordinates $z_\alpha = \underline{b}_\alpha \cdot \underline{x}$
- Transforming to the noraml coordinates $\rightarrow$ diagonalise k (see notes i.e. ask alice or fiona for them)
- Energy in each normal mode conserved, mode with lowest ω is the most symmetric