

1 Leon's (WIP) ITCS Exam Notes

Basically adapted from Chris Dalziel's notes :)
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Finite Automata

Definition: Finite Automata

A finite automaton takes a string as input and replies "yes" or "no". If an automaton A replies "yes" on a string S we say that A "accepts" S .

Definition: Deterministic Finite Automata

A deterministic finite automaton (DFA) is a quintuple $(Q, \Sigma, q_0, \delta, F)$ where

- Q is a finite set of states
- Σ is an alphabet
- $q_0 \in Q$ is the initial state
- $\delta : Q \times \Sigma \rightarrow Q$ is the transition function
- $F \subseteq Q$ is the set of final states

A DFA accepts a string $w \in \Sigma^*$ iff $\delta^*(q_0, w) \in F$, where δ^* is δ applied successively for each symbol in w .
The language of a DFA A is the set of all strings accepted by A , $\mathcal{L} \subseteq \Sigma^*$ is the set of all strings accepted by A .
The transition function is a total function which gives exactly one next state for each input symbol, i.e. it is deterministic

Definition: Nondeterministic Finite Automata

Non-determinism would mean that δ can return more than one successor state, it instead returns a set of possible states - no states is an empty set. A NFA is a quintuple $(Q, \Sigma, q_0, \delta, F)$ where:

- Q is a finite set of states
- Σ is an alphabet
- $q_0 \in Q$ is the initial state
- $\delta : Q \times \Sigma \rightarrow \mathcal{P}(Q)$ is the transition function
- $F \subseteq Q$ is the set of final states

The only difference between the definition of a DFA and that of an NFA is that in an NFA δ returns an element from the powerset of Q , $\mathcal{P}(Q)$

Adding non-determinism doesn't change "expressivity". Given an NFA A there is an equivalent DFA D such that $\mathcal{L}(D) = \mathcal{L}(A)$ and vice versa.

Definition: ϵ -NFA

If we allow non-deterministic state changes that don't consume any input symbols, we can label silent moves using ϵ - meaning the empty string. We define the ϵ closure $E(q)$ of a state q as the set of all states reachable from q by silent moves. That is, $E(q)$ is the least set satisfying:

- $q \in E(q)$
- For any $s \in E(q)$ we also have $\delta(s, \epsilon) \subseteq E(q)$

DFA, NFA, ϵ -NFA are all equal in expressive power

Regular Languages

Definition: Regular Languages

Any language which can be accepted by a finite automaton is called a regular language.
Regular languages are also those recognised by Regular Expressions

Definition: Regular Language Closure Properties

For two languages L_1 and L_2 , the following operations satisfy the closure property, i.e. for a member $x \in X$, and an operation ϕ we have that $\phi(x) \in \mathbb{R}$ for all x .

- **Union:** $L_1 \cup L_2$ is the language that includes all strings of L_1 and all strings of L_2 .
- **Intersection:** $L_1 \cap L_2$ is the language that includes all strings of L_1 that are not in L_2 , and vice versa
- **Sequential Composition:** $L_1 L_2$ is the language of strings that consist of strings in L_1 followed by a string in L_2 .
- **Kleene closure:** L^* is the language of strings that consist wholly of zero or more strings in L .

$$L^* = \bigcup_{i \in \mathbb{N}} L^i$$

- **Complement:** \bar{L} is the language of every string not in L .

Definition: Regular Expressions

Regular characterise the regular languages, just like finite automata do. The following table shows the syntax and semantics of a regex.

Syntax	Semantics
a	$\llbracket a \rrbracket = \{a\}$
\emptyset	$\llbracket \emptyset \rrbracket = \emptyset$
ϵ	$\llbracket \epsilon \rrbracket = \{\epsilon\}$
$R_1 \cup R_2$	$\llbracket R_1 \cup R_2 \rrbracket = \llbracket R_1 \rrbracket \cup \llbracket R_2 \rrbracket$
$R_1 \circ R_2$	$\llbracket R_1 \circ R_2 \rrbracket = \llbracket R_1 \rrbracket \llbracket R_2 \rrbracket$
R^*	$\llbracket R^* \rrbracket = \llbracket R \rrbracket^*$

Definition: Generalised NFAs

A **generalised NFA**, or GNFA is an NFA where:

- Transitions have **regular expressions** on them instead of symbols
- There is only one unique final state
- The transition relation is **full**, except that the initial state has no incoming transitions, and the final state has no outgoing transitions

Theorem 1: Pumping Lemma

If $L \subseteq \Sigma^*$ is regular, then there is a **pumping length** $p \in \mathbb{N}$ such that for any $w \in L$ where $|w| \geq p$, we may split w into three pieces $w = xyz$ satisfying three conditions:

- $xy^i z, \quad \forall i \in \mathbb{N}$
- $|y| > 0$
- $|xy| \leq p$

Note that if the pumping lemma fails then the language is not regular, but the inverse is not necessarily true.

Theorem 2: Myhill-Nerode Theorem

Let $L \subseteq \Sigma^*$ and $x, y \in \Sigma^*$. If there exists a suffix string z such that $xz \in L$, but $yz \notin L$ or vice versa, then x and y are **distinguishable** by L . If x and y are not distinguishable by L , then we say that $x \equiv_L y$ - this is an equivalence relation. A regular language satisfies the following

- The number of equivalence classes \equiv_L is finite.
- The number of equivalence classes is equal to the number of states in the minimal DFA accepting L (not as important)

Therefore, to show a language is non-regular, show that it has infinite equivalence classes - that is, we find an infinite sequence $u_0 u_1 \dots$ of strings such that for any i, j where $i \neq j$, there is a string w_{ij} such that $u_i w_{ij} \in L$ but $u_j w_{ij} \notin L$ or vice-versa

Context-Free Languages

Definition: Context-free Languages

By adding recursion to regexes we can begin to recognise some non-regular languages. All regular languages are also context free.

Definition: Context-free Grammars

A language is context-free iff it is recognised by a Context-free Grammar (CFG), which is a 4-tuple (N, Σ, P, S) where:

- N is a finite set of variables or non-terminals
- Σ is a finite set of terminals
- $P \subseteq N \times (N \cup \Sigma)^*$ is a finite set of rules or productions
 - Typically productions are written $A \rightarrow aBc$
 - Productions with common heads can be combined, $A \rightarrow a$ and $A \rightarrow Aa$ can be combined into $A \rightarrow a \mid Aa$
- $S \in N$ is the starting variable

We use α, β, γ to refer to sequences of terminals

We make a derivation step $\alpha A \beta \Rightarrow_G \alpha \gamma \beta$ whenever $(A \rightarrow \gamma) \in P$; The language of a CFG G is:

$$\mathcal{L}(G) = \{w \in \Sigma^* \mid S \Rightarrow_G^* w\}$$

Where \Rightarrow_G^* is the reflexive, transitive, closure of \Rightarrow_G .

Context-free grammars are ambiguous. They are closed under union, concatenation, and Kleene star, but not under intersection or complementation

Definition: Eliminating Ambiguity

We want to eliminate ambiguity in CFGs while still accepting all the same strings. This can be done for our language of regular expressions:

- First defining atomic expressions: $A \rightarrow (S) \mid \emptyset \mid \epsilon \mid a \mid b$
- Then ones which use Kleene Star: $K \rightarrow A \mid A^*$
- Then ones which may use left-associative composition: $C \rightarrow K \mid C \circ K$
- Finally expressions which use unions: $S \rightarrow C \mid S \cup C$

The order of operations here is therefore bottom to top; unions come before compositions, which come before Kleene etc

Definition: Push-down Automata

Push-down automata are to CFGs what finite automata are to regular expressions. They are implementationally identical to ϵ -NFAs with the addition of a stack. The recursive element of CFGs is implemented using a standard last-in-first-out stack.

Transitions in a push-down automata take the form $x, y \rightarrow z$ which is read as "consume the input x , popping y off the stack, and push z onto the stack". We can allow actions that don't consume, pop, or push by setting variables to ϵ .

Definition: Formal Def. of PDAs

A **push-down automaton** is a 6-tuple $(Q, \Sigma, \Gamma, \delta, q_0, F)$ where Q, Σ, Γ are all finite sets. Γ is the stack alphabet, and δ now may take a stack symbol as input or return one as output:

$$\delta : Q \times \Sigma_\epsilon \times \Gamma_\epsilon \rightarrow \mathcal{P}(Q \times \Gamma_\epsilon)$$

All other components are as with ϵ -NFAs

A string w is accepted by a PDA if it ends in a final state, i.e. $\delta^*(q_0, w, \epsilon)$ gives a state q and a stack γ such that $q \in F$.

Theorem 3: CFG to PDA

A language is context-free if and only if it is recognised by a push-down automaton. The proof is left as an exercise to the reader.

Theorem 4: Pumping CFLs intro

Suppose a CFG has n non-terminals, and we have a parse tree of height $k > n$. Then the same non-terminal V must have appeared as its own descendant in the tree

- **Pumping down:** Cut the tree at the higher occurrence of V and replace it with the subtree at the lower occurrence of V
- **Pumping up:** Cut at the lower occurrence and replace it with a fresh copy of the higher occurrence

Theorem 5: Pumping Lemma for CFLs

If L is context-free then there exists a pumping length $p \in \mathbb{N}$ such that if $w \in L$ with $|w| \geq p$ then w may be split into **five** pieces $w = uvxyz$ such that

- $uw^i xy^i z \in L$ for all $i \in \mathbb{N}$
- $|vy| > 0$
- $|vxy| \leq p$

Definition: Chomsky Grammars

Context-free grammars are a special case of Chomsky Grammars. Chomsky grammars are similar to CFGs, except that the left-hand side of a production may be any string that includes at least one non-terminal. An example is shown below

$$\begin{aligned} S &\rightarrow abc \mid aAbc \\ Ab &\rightarrow bA \\ Ac &\rightarrow Bbcc \\ bB &\rightarrow Bb \\ aB &\rightarrow aaA \mid aa \end{aligned}$$

Such a grammar is called **context-sensitive**

Definition: The Chomsky Hierarchy

A grammar $G = (N, \Sigma, P, S)$ is of type:

0. (or **computably enumerable**) in the general case
1. (or **context sensitive**) if $|\alpha| \leq |\beta|$ for all productions $\alpha \rightarrow \beta$, except we also allow $S \rightarrow \epsilon$ if S does not occur on the RHS of any rule
2. (or **context free**) if all productions are of the form $A \rightarrow \alpha$ (i.e. a CFG)
3. (or **right-linear/regular**) if all productions are of the form $A \rightarrow w$ or $A \rightarrow wB$, where $w \in \Sigma$ and $B \in N$

Algorithms for Languages

Theorem 6: Emptiness for Regular Languages

Can we write a program to determine if a given regular language is empty?

Given a finite-automaton this is an instance of graph reachability, so we can use a depth-first search.

Theorem 7: Emptiness for Context-free languages

Can we write a program to determine if a given context-free language is empty?

Given a CFG for our language, we can perform the following process:

1. Mark the terminals and ϵ as generating
2. Mark all non-terminals which have a production with only generating symbols in their right hand side as generating
3. Repeat until nothing new is marked
4. Check if S is marked as generating or not

Theorem 8: Equivalence of DFA

Is it possible to write a program to determine if two discrete finite automata are equivalent?

Given two DFA for L_1 and L_2 , we can use our standard constructions to produce a DFA of the symmetric set difference:

$$(L_1 \cap \bar{L}_2) \cup (L_2 \cap \bar{L}_1)$$

Register Machines

Definition: Register Machines

A **register machine**, or RM, consists of:

- A fixed number m of registers $R_0 \dots R_{m-1}$, which each holds a natural number
- A fixed program P which is a sequence of n instructions $I_0 \dots I_{n-1}$

Each instruction is one of the following:

- **INC(i)**: which increments the register R_i by one
- **DECJZ(i, j)**: which decrements register R_i unless $R_i = 0$ in which case it jumps to instruction I_j

RMs can compute anything any other computer can

Definition: Pairing Functions for RMs

A **pairing function** is an injective function $\mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$. An example is $f(x, y) = 2^x 3^y$. We write $\langle x, y \rangle_2$ for $f(x, y)$. If $z = \langle x, y \rangle_2$, let $z_0 = x$ and $z_1 = y$. This lets us encode multiple values into a single value, and a 2-tuple pairing function is enough to cram an arbitrary sequence of natural numbers into one $\mathbb{N}^* \rightarrow \mathbb{N}$

Definition: Turing Machine

A **turing machine** is a 7-tuple $(Q, \Sigma, \Gamma, \delta, q_0, q_{\text{accept}}, q_{\text{reject}})$

- Q : states
- Σ : input symbols
- $\Gamma \subseteq \Sigma$: **tape** symbols, including a **blank** symbol \sqcup
- $\delta : Q \times \Gamma \rightarrow Q \times \Gamma \times \{-1, 1\}$
- $q_0, q_{\text{accept}}, q_{\text{reject}} \in Q$: start, accept, reject states

Theorem 9: Church-Turing Thesis

The Church-Turing thesis states that any problem is computable by any model of computation iff it is computable by a **Turing machine**. For our purposes this matters for RMs, TMs, and λ -calculus. Other examples are combinator calculus, general recursive functions, pointer machines, counter machines, cellular automata, queue automata, enzyme-based DNA computers, Minecraft, Magic the Gathering, and others.

Decidability

Definition: Problems with no Algorithm

While the above problems are all computable, this is not always the case. The questions asked, i.e. "can we determine x ?", are not always something we can answer using a program - if this is the case then that problem is considered undecidable. Many such problems exist for Context-free Languages

- Are two CFG equivalent?
- Is a given CFG ambiguous?
- Is there a way to make a given CFG unambiguous?
- Is the intersection of two CFLs empty?
- Does a CFG generate all strings Σ^* ?
- ...

Theorem 10: The Halting Problem

Given a register machine encoding, can we write a program to determine if the simulated machine will halt or not? If we suppose H is such a register machine, which takes a machine encoding $[M]$ in R_0 , halts with 1 if M halts, and halts with 0 if M doesn't halt. We can construct a new machine $L = (P_L, R_0, \dots)$ which, given a program $[P]$ runs H on the program with itself as input, the machine $(P, [P])$ and loops if it halts. If we run L on P_L itself we get a problem. If L halts on $[P_L]$ that means H says $(P_L, [P_L])$. If L loops on $[P_L]$ that means that H says $(P_L, [P_L])$ halts. This is a contradiction! The halting problem proves that there are some programs which cannot be decided by register machines. What about other machines?

Definition: Computability

A (total) function $\mathbb{N} \rightarrow \mathbb{N}$ is **computable** if there is an RM/TM which computes f , i.e., given an x in R_0 , leaves $f(x)$ in R_0 . A **decision problem** is a set D and a query subset $Q \subseteq D$. A problem is **decidable** or **computable** if $d \in Q$ is characterised by a computable function $f : D \rightarrow \{0, 1\}$.

Definition: Reductions

A **reduction** is a transformation from one problem to another. To prove that a problem P_2 is hard, show that there is an easy reduction from a known hard problem P_1 to P_2 . To show a problem P_2 is undecidable, show that there is a computable reduction from a known undecidable P_1 to P_2 . The direction here matters - it tells us nothing to know that there's an easy way to make an easy problem difficult!

Example : Reduction analogy

If it is a well known fact that Hyunwoo cannot lift a car, we can prove that Hyunwoo cannot lift a loaded truck by making the following reduction: If we suppose he could lift the loaded truck, then we could have him lift the car by putting it in the loaded truck - but we know he cannot lift a car.

Definition: Mapping Reductions

A **Turing transducer** is a RM (or TM) which takes an instance d of a problem $P_1 = (D_1, Q_1)$ in R_0 and halts with an instance $d' = f(d)$ of $P_2 = (D_2, Q_2)$ in R_0 . Thus, f is a computable function $D_1 \rightarrow D_2$. A **mapping reduction** (or a many-to-one reduction) from P_1 to P_2 is a turing transducer f such that $d \in Q_1$ iff $f(d) \in Q_2$. If A is mapping reducible to B , and A is undecidable, then B is undecidable.

Definition: Oracles

Given a decision problem (D, Q) , an **oracle** for Q is a 'magic' RM instruction $\text{ORACLE}_Q(i)$ which, given an encoding of $d \in D$ in R_i , sets R_i to contain 1 iff $d \in Q$

Definition: Turing Reductions

A **Turing reduction** from P_1 to P_2 is an RM/TM equipped with an oracle for P_2 which solves P_1 . Decidability results carry across during Turing reductions as with mapping reductions, but mapping reductions make finer distinctions of computing power.

Theorem 11: Rice's Theorem

- A **property** is a set of RM (or TM) descriptions
- A property is **non-trivial** if it contains some but not all descriptions
- A property P is **semantic** if

$$\mathcal{L}(M_1) = \mathcal{L}(M_2) \Rightarrow ([M_1] \in P \Leftrightarrow [M_2] \in P)$$

In other words, it concerns the **language** and not the particular implementation of the language

Rice's Theorem - All non-trivial semantic properties are undecidable.

Theorem 12: Rice's Theorem part 2

Rice's theorem is useful for deciding properties like whether a language is empty, non-empty, regular, context-free etc. It cannot be applied to questions like whether a TM has fewer than 7 states, a final state, a start state, etc. These properties are of machines and not languages. It also doesn't apply to questions like is a language a subset of Σ^* or whether a language of a RM is a language of a TM - these are trivial properties! The consequences of this is that we cannot write programs which answer non-trivial questions about the black-box behaviours of programs.

Computability in depth

Definition: Semi-decidability

A problem (D, Q) is **semi-decidable** if there is a TM/RM that returns "yes" for any $d \in Q$, but may return "no" or loop forever when $d \notin Q$.

A problem (D, Q) is **co-semi-decidable** if there is a TM/RM that returns "no" for any $d \notin Q$, but may return "yes" or loop forever when $d \in Q$.

Definition: Enumerable Computability

A set S is **enumerable** if there is a bijection between S and \mathbb{N} . A set S is called **computably enumerable** (or c.e.) if the enumeration function $f: \mathbb{N} \rightarrow S$ is computable. In terms of RM and TM we can think of enumeration as outputting an infinite list as it executes forever.

Theorem 13: Decidability theorems

Any problem that is both semi-decidable and co-semi-decidable is decidable.

If a problem P is semi-decidable then its complement \bar{P} is co-semi-decidable, and vice versa.

All semi-decidable problems are computably enumerable, and any computably enumerable problem is semi-decidable - being semi-decidable is the same as being computably enumerable.

Theorem 14: Decidability Reductions

To prove that a problem P_2 is not c.e. we show that there is a mapping reduction from a known not-c.e. problem P_1 to P_2 . We must use mapping reductions, as H is c.e. but L is not - but L is Turing reducible to H by flipping the answer.

Definition: Time Complexity

The **time complexity** of a (deterministic) machine M that halts on all inputs is a function $f: \mathbb{N} \rightarrow \mathbb{N}$ where $f(n)$ is the maximum number of steps that M uses on any input of size n .

Definition: Complexity Measures

When performing addition in a TM we have a time complexity of $\mathcal{O}(\log n)$, where it is $\mathcal{O}(n)$ for RMs - there's an exponential penalty for using a register machine! Addition can be $\mathcal{O}(1)$ if we add dedicated $\text{ADD}(i, j)$ and $\text{SUB}(i, j)$ instructions - but this doesn't remove the inaccuracy it just makes it smaller.

Complexity is useful, but the measures are slightly bogus:

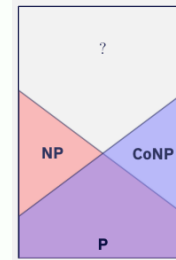
- $\mathcal{O}(n)$ is not always easy - if n is massive for example
- $\Omega(2^n)$ isn't always hard - there are problems much worse than this that are still solvable for real examples
- $\mathcal{O}(n^{10})$ and $\Omega(n^{10})$ seem extremely difficult, but a new model or algorithm could reduce that significantly
- Since we also ignore coefficients, we could have something like $f(n) \geq 10^{100} \log n$ which is slow but still only logarithmic - this isn't common enough to worry generally however.

Definition: Complexity Classes

Let $t: \mathbb{N} \rightarrow \mathbb{R}_{\geq 0}$. A time complexity class $\text{TIME}(t(n))$ is the collection of all problems that are decidable by a deterministic machine RM, TM etc.) in $\mathcal{O}(t(n))$ time.

Given $A = \{0^i 1^i \mid i \in \mathbb{N}\}$, a TM can decide this in $\mathcal{O}(n^2)$, which means that $A \in \text{TIME}(n^2)$.

We also define $\text{NTIME}(t(n))$ to be the collection of all problems decidable by a nondeterministic machine NRM, NTM, etc.) in $\mathcal{O}(t(n))$.



Definition: NP / Nondeterministic-polynomial time

The polynomial complexity class **NP** is the class of problems decidable with some nondeterministic polynomial time complexity.

$$\text{NP} = \bigcup_{k \in \mathbb{N}} \text{NTIME}(n^k)$$

We don't know if every exponentially bounded problem is in **NP**, we think that it's probably not the case.

Definition: P / Polynomial Time

The polynomial complexity class **P** is the class of problems decidable with some deterministic polynomial time complexity.

$$\text{P} = \bigcup_{k \in \mathbb{N}} \text{TIME}(n^k)$$

Problems in **P** are called tractable. Any problem not in **P** is $\Omega(n^k)$ for every k .

The class itself is robust, reasonable changes to model don't change it and reasonable translations between problems preserves membership in **P**.

A polynomially-bounded RM together with a polynomial (n^k for some k , without loss of generality), such that given an input w it will always halt after executing $|w|^k$ instructions. A problem Q is in **P** iff it is computed by such a machine.

Definition: CoNP

We don't know if the class **NP** is closed under complement. We cannot flip the result because the result of flipping in a nondeterministic machine involves turning it from angelic nondeterminism to demonic nondeterminism (or co-nondeterminism) or vice versa.

Definition: AP / Alternating Poly time

The class **AP** is the class of all problems decidable by an alternating machine in polynomial time with no restriction on swapping quantifiers.

AP is known to be equal to PSPACE.

Definition: PSPACE / Polynomial Space

An RM/TM is $f(n)$ -space-bounded if it may use only $f(\text{inputsize})$ space. For TMs this is the number of cells on the tape, where register machines use bits in registers.

Theorem 15: Polynomial Reductions

A polynomial reduction from $P_1 = (D_1, Q_1)$ to $P_2 = (D_2, Q_2)$ is a **P**-computable function $f: D_1 \rightarrow D_2$ such that $d \in Q_1$ iff $d \in Q_2$.

If P_2 is in **P**, then P_1 is in **P** straightforwardly. Therefore to prove a problem is not in **P** we can show that there is a polynomial reduction from a problem P_2 which isn't in **P** to our problem P_1 .

- A problem P_1 is **polynomially reducible** to P_2 , written $P_1 \leq_P P_2$ if there is a polynomially-bounded reduction from P_1 to P_2 .
- A problem P is **NP-Hard** if for every $A \in \text{NP}$, $A \leq_P P$.

That is, if a problem P_1 is **NP-Hard** and P_1 is polynomially reducible to P_2 , then P_2 is also **NP-Hard**.

We can use this fact to prove other problems are **NP-Hard** by showing a reduction from a known **NP-hard** problem.

- A problem is **NP-Complete** if it is both **NP-Hard** and in **NP**.

Time Complexity

Theorem 16: Cook-Levin Theorem

The Cook-Levin theorem states that the NP problem SAT is NP-Complete.

The proof of this involves showing it is NP by nondeterministically guessing assignments and checking them in polynomial time, and then showing it's NP- Hard by reducing any NP problem to SAT

The Polynomial Heirarchy

Definition: Sigma Notation

The set Σ_1^P describes all problems that can be phrased as

$$\{y \mid \exists^P x \in \mathbb{N}. R(x, y)\}$$

where R is a **P**-decidable predicate and $\exists^P x \dots$ indicates that x is of size polynomial in the size of y .

We can say that x is a certificate showing which guesses can be made by our NRM giving an accepting run.

If a problem $Q \in \Sigma_1^P$ then Q is **NP**, because it is a problem for which we can verify the answer in polynomial time. If a problem Q is in **NP** then $Q \in \Sigma_1^P$.

So, **NP** = Σ_1^P

Definition: Pi notation

The set Π_1^P describes all problems that can be phrased as

$$\{y \mid \forall^P x \in \mathbb{N}. R(x, y)\}$$

where R is a **P**-decidable predicate, and $\forall^P x \dots$ indicates that x is of size polynomial in the size of y .

We have that

$$\Pi_1^P = \overline{\Sigma_1^P}, \quad \text{and} \quad \Pi_1^P = \mathbf{CoNP}$$

Definition: Delta Notation

There are two conflicting definitions of Δ_1^P "For reasons that are unknown to me" - lecturer

- The set Δ_1^P describes the intersection of Σ_1^P and Π_1^P
- The set Δ_1^P describes the set **P**

From our characterisations of Σ_1^P and Π_1^P , we have that $\Delta_1^P \subseteq \mathbf{P}$, but we don't know if these definitions are equal

Definition: Moving higher

The next layer of the hierarchy goes as follows:

- Σ_2^P is all problems of the form $\{x \mid \exists^P y. \forall^P z. R(x, y, z)\}$
- Π_2^P is all problems of the form $\{x \mid \forall^P y. \exists^P z. R(x, y, z)\}$
- $\Delta_2^P = \Sigma_2^P \cap \Pi_2^P$

We can also use oracles to get an alternate definition:

- Δ_2^P is all problems that are decidable in polynomial time by some deterministic RM/TM with an $\mathcal{O}(1)$ oracle for some problem in Σ_1^P (it is **P** with an $\mathcal{O}(1)$ oracle for **NP**)
- Σ_2^P allows the TM/RM to be nondeterministic (it is **NP** with an $\mathcal{O}(1)$ oracle for **NP**)
- Π_2^P is **CoNP** with an oracle for **NP**

In general for any $n > 1$:

- Δ_n^P is all problems decidable by a deterministic polynomially bounded TM/RM with an $\mathcal{O}(1)$ oracle for some problem in Σ_{n-1}^P .
- Σ_n^P is all problems decidable by some nondeterministic polynomially bounded TM/RM with an $\mathcal{O}(1)$ oracle for some problem in Σ_{n-1}^P .
- Π_n^P is all problems decidable by some co-nondeterministic polynomially bounded TM/RM with an $\mathcal{O}(1)$ oracle for some problem in Σ_{n-1}^P .

Note: Co-nondeterminism could also be called **demonic** nondeterminism, like regular (angelic) nondeterminism but only accepts if **all** paths accept

Definition: Alternation

Equivalently Σ_n^P are all problems that can be phrased as some **alternation** of (**P**-bounded) quantifiers, starting with \exists^P

$$\{w \mid \exists^P x_1. \exists^P x_2. \exists^P x_3. \exists^P x_4. \dots x_n. R(w, x_1, \dots, x_n)\}$$

Π_n^P has a similar definition, starting instead with \forall^P

$$\{w \mid \forall^P x_1. \exists^P x_2. \exists^P x_3. \exists^P x_4. \dots x_n. R(w, x_1, \dots, x_n)\}$$

Alternating machines combine the acceptance modes of both angelic and demonic non-deterministic machines.

Alternating register machines would replace the NRM's **MAYBE** instruction with the **MAYBE**[∃] and **MAYBE**[∀] instructions, which are nondeterministic branching choices where acceptance depends on if one branch accepts (∃) or both branches accept (∀).

Alternating Turing Machines are defined by labelling states with either ∀ or ∃.

The class Σ_n^P can therefore be described as the class of problems decided in polynomial time by an alternating machine that initially uses ∃-nondeterminism and swaps quantifiers at most $n - 1$ times. This extends to Π_n^P swapping starting with ∃ to starting with ∀.

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Definition: Name

Example : Regular Expressions

At least one 0:

$$(0 \cup 1)^* 0 (0 \cup 1)^*$$

At least one 1 and at least one 0:

$$((0 \cup 1)^* 0 1 (0 \cup 1)^*) \cup ((0 \cup 1)^* 1 0 (0 \cup 1)^*)$$