

# 1 Vector Spaces

## 1.1 Fields and Vector Spaces

### Definition 1.1.1: Definition of a field

A field  $F$  is a set with two functions

- Addition:  $+: F \times F \rightarrow F, (\lambda, \mu) \mapsto \lambda + \mu$
- Multiplication:  $\cdot: F \times F, (\lambda, \mu) \mapsto \lambda\mu$

which satisfy the following axioms:

1.  $(F, +)$  is an abelian group  $F^+$ , with identity  $0_F$
2.  $(F \setminus \{0_F\}, \cdot)$  is an abelian group  $F^\times$ , with identity  $1_F$
3. **Distributive law:** For all  $a, b$ , and  $c$  in  $F$ , we have

$$a(b + c) = ab + ac \in F$$

and the following lemmas:

1. The elements  $0_F$  and  $1_F$  of  $F$  are distinct
2. For all  $a \in F$ ,  $a \cdot 0_F = 0_F$  and  $0_F \cdot a = 0_F$
3. Multiplication in  $F$  is associative, and  $1_F$  is an identity element

A **vector space**  $V$  over a field  $F$  is a pair consisting of an abelian group  $V = (V, +)$  and a mapping

$$F \times V \rightarrow V : (\lambda, \vec{v}) \mapsto \lambda\vec{v}$$

s.t. for all  $\lambda, \mu \in F$  and  $\vec{v}, \vec{w} \in V$  the following identities hold:

- **Distributivity 1:**  $\lambda(\vec{v} + \vec{w}) = \lambda\vec{v} + \lambda\vec{w}$
- **Distributivity 2:**  $(\lambda + \mu)\vec{v} = \lambda\vec{v} + \mu\vec{v}$
- **Associativity:**  $\lambda(\mu\vec{v}) = (\lambda\mu)\vec{v}$
- **Identity:**  $1\vec{v} = \vec{v}$

and so do the following lemmas:

1. If  $V$  is a vector space and  $\vec{v} \in V$ , then  $0\vec{v} = \vec{0}$
2. If  $V$  is a vector space and  $\vec{v} \in V$ , then  $(-1)\vec{v} = -\vec{v}$
3. If  $V$  is a vector space over a field  $F$ , then  $\lambda\vec{0} = \vec{0}$  for all  $\lambda \in F$ . Furthermore, if  $\lambda\vec{v} = \vec{0}$  then either  $\lambda = 0$  or  $\vec{v} = \vec{0}$

## 1.2 Working with Vector Spaces

### Definition 1.2.1: Cartesian Product of $n$ sets

$$X_1 \times \cdots \times X_n := \{(x_1, \dots, x_n) : x_i \in X_i \text{ for } 1 \leq i \leq n\}$$

The elements of a product are called  **$n$ -tuples**. An individual entry  $x_i = (x_1, \dots, x_n)$  is called a **component**. There are special mappings called **projections** for a cartesian product:

$$\begin{aligned} \text{pr}_i : X_1 \times \cdots \times X_n &\rightarrow X_i \\ (x_1, \dots, x_n) &\mapsto x_i \end{aligned}$$

The cartesian product of  $n$  copies of a set  $X$  is written in short as:  $X^n$

### Definition 1.2.2: Vector Subspace

A subset  $U$  of a vector space  $V$  is called a **vector subspace** or **subspace** if  $U$  contains the zero vector, and whenever  $\vec{u}, \vec{v} \in U$  and  $\lambda \in F$  we have  $\vec{u} + \vec{v} \in U$  and  $\lambda\vec{u} \in U$

### Definition 1.2.3: Spans and Linear Independence

Let  $T \subset V$  for some vector space  $V$  over a field  $F$ . Then amongus all subspaces of  $V$  that include  $T$  there is a smallest subspace

$$\langle T \rangle = \langle T \rangle_F \subseteq V$$

“the set of all vectors  $\alpha_1\vec{v}_1 + \cdots + \alpha_r\vec{v}_r$  with  $\alpha_1, \dots, \alpha_r \in F$  and  $\vec{v}_1, \dots, \vec{v}_r \in T$ , together with the zero vector in the case  $T = \emptyset$ ”

### Terminology Dump

- An expression of the form  $\alpha_1\vec{v}_1 + \cdots + \alpha_r\vec{v}_r$  is called a **linear combination** of vectors  $\vec{v}_1, \dots, \vec{v}_r$
- The smallest vector subspace  $\langle T \rangle \subseteq V$  containing  $T$  is called the **vector subspace generated by  $T$**  or the vector subspace **spanned by  $T$**  or even the **span of  $T$**
- If we allow the zero vector to be the “empty linear combination of  $r = 0$  vectors”, then the span of  $T$  is exactly the set of all linear combinations of vectors from  $T$
- A subset of a vector space that spans the entire space is called a **generating** or **spanning set**. A vector space that has a finite generating set is said to be **finitely generated**

### Linear Independence

A subset  $L$  of a vector space  $V$  is called **linearly independent** if for all pairwise different vectors  $\vec{v}_1, \dots, \vec{v}_r \in L$  and arbitrary scalars  $\alpha, \dots, \alpha_r \in F$ ,

$$\alpha_1\vec{v}_1 + \cdots + \alpha_r\vec{v}_r = \vec{0} \implies \alpha_1 = \cdots = \alpha_r = 0$$

A subset  $L$  of a vector space  $V$  is called **linearly dependent** if it is not linearly independent (duh..). This means there exists pairwise different vectors  $\vec{v}_1, \dots, \vec{v}_r \in L$  and scalars  $\alpha_1, \dots, \alpha_r \in F$ , not all zero, such that  $\alpha_1\vec{v}_1 + \cdots + \alpha_r\vec{v}_r = \vec{0}$

## 1.3 Linear Independence and Bases

### Definition 1.3.1: Basis of a Vector Space

A **basis of a vector space**  $V$  is a linearly independent generating set in  $V$

### Example 1.3.2: Standard Basis

Let  $F$  be a field and  $n \in \mathbb{N}$ . We consider the following vectors in  $F^n$

$$\vec{e}_i = (0, \dots, 0, 1, 0, \dots, 0)$$

with one 1 in the  $i$ -th place and zero everywhere else. Then  $\vec{e}_1, \dots, \vec{e}_n$  form an ordered basis of  $F^n$ , the so-called **standard basis of  $F^n$**

### Theorem 1.3.3: Linear combinations of basis elements

Let  $F$  be a field,  $V$  a vector space over  $F$  and  $\vec{v}_1, \dots, \vec{v}_r \in V$  vectors. The family  $(\vec{v}_i)_{1 \leq i \leq r}$  is a basis of  $V$  if and only if the following “evaluation” mapping

$$\begin{aligned} \psi : F^r &\rightarrow V \\ (\alpha_1, \dots, \alpha_r) &\mapsto \alpha_1\vec{v}_1 + \cdots + \alpha_r\vec{v}_r \end{aligned}$$

is a bijection

If we label our ordered family by  $\mathcal{A} = (\vec{v}_1, \dots, \vec{v}_r)$ , then we done the above mapping by

$$\psi = \psi_{\mathcal{A}} : F^r \rightarrow V$$

### Theorem 1.3.4: Characterisations of Bases

The following are equivalent for a subset  $E$  of a vector space  $V$ :

1.  $E$  is a basis, i.e. a linearly independent generating set
2.  $E$  is minimal among all generating sets, meaning that  $E \setminus \{\vec{v}\}$  does not generate  $V$ , for any  $\vec{v} \in E$
3.  $E$  is maximal among all linearly independent subsets, meaning that  $E \cup \{\vec{v}\}$  is linearly dependent for any  $\vec{v} \in V$

**Corollary:** Let  $V$  be a finitely generated vector space over a field  $F$ . Then  $V$  has a finite basis

### Basis Characterisation Variant

1. If  $L \subset V$  is a linearly independent subset and  $E$  is minimal amongst all generating sets of  $V$  with the property that  $L \subseteq E$ , then  $E$  is a basis.
2. If  $E \subseteq V$  is a generating set and if  $L$  is maximal amongst all linearly independent sets of  $V$  with the property  $L \subseteq E$ , then  $L$  is a basis.

### Definition 1.3.5: Free Vector Space

Let  $X$  be a set and  $F$  a field. The set  $\text{Maps}(X, F)$  of all mappings  $f : X \rightarrow F$  becomes an  $F$ -vector space with the operations of pointwise addition and multiplication by a scalar. The subset of all mappings which send almost all elements of  $X$  to zero is a vector subspace

$$F\langle X \rangle \subseteq \text{Maps}(X, F)$$

This subspace is called the **free vector space on the set  $X$**

### Theorem 1.3.6: Variant of Linear Combinations

Let  $F$  be a field,  $V$  be an  $F$ -vector space and  $(\vec{v}_i)_{i \in I}$  a family of vectors from the vector space  $V$ . The following are equivalent:

1. The family  $(\vec{v}_i)_{i \in I}$  is a basis for  $V$
2. For each  $\vec{v} \in V$  there is precisely one family  $(a_i)_{i \in I}$  of elements of  $F$ , almost all which are zero and such that

$$\vec{v} = \sum_{i \in I} a_i \vec{v}_i$$

## 1.4 Dimension of a Vector Space

### Theorem 1.4.1: Fundamental Estimate of LinAlg

No linearly independent subset of a given vector has more elements than a generating set. Thus if  $V$  is a vector space,  $L \subset V$  a linearly independent subset and  $E \subseteq V$  a generating set, then

$$|L| \leq |E|$$

### Theorem 1.4.2: Steinitz Exchange Theorem

Let  $V$  be a vector space,  $L \subset V$  a finite linearly independent subset and  $E \subseteq V$  a generating set. Then there is an injection  $\phi : L \hookrightarrow E$  such that  $(E \setminus \phi(L)) \cup L$  is also a generating set for  $V$

Let  $V$  be a vector space,  $M \subseteq V$  a linearly independent subset, and  $E \subseteq V$  a generating subset, such that  $M \subseteq E$ . If  $\vec{w} \in V \setminus M$  is a vector  $\notin M$  such that  $M \cup \{\vec{w}\}$  is linearly independent, then there exists  $\vec{e} \in E \setminus M$  such that  $(E \setminus \{\vec{e}\}) \cup \{\vec{w}\}$  is a generating set

### Theorem 1.4.3: Cardinality of Bases

Let  $V$  be a finitely generated vector space.

1.  $V$  has a finite basis
2.  $V$  cannot have an infinite basis
3. Any two bases of  $V$  have the same number of elements

### Definition 1.4.4: Dimension of a Vector Space

The cardinality of a basis of a finitely generated vector space  $V$  is called the **dimension** of  $V$ , written  $\dim V$ . If  $F$  is a field, and we want to denote that we mean dimension as an  $F$ -vector space, then we write  $\dim_F V$ . If the vector space is not finitely generated, then we say  $\dim V = \infty$  and call  $V$  **infinite dimensional**.

### Theorem 1.4.5: Dimension Theorems

#### Cardinality Criterion for Bases

1. Each linearly independent subset  $L \subset V$  has at most  $\dim V$  elements, and if  $|L| = \dim V$  then  $L$  is a basis
2. Each generating set  $E \subseteq V$  has at least  $\dim V$  elements, and if  $|E| = \dim V$  then  $E$  is a basis

**Dimension Estimate for Vector Subspaces:** A proper vector subspace of a finite dimensional vector space has itself a strictly smaller dimension

If  $U \subseteq V$  is a vector subspace of an arbitrary vector space, then we have  $\dim U \leq \dim V$  and if we have  $\dim U = \dim V < \infty$  then it follows that  $U = V$

## 1.5 Linear Mappings

### Definition 1.5.1: Linear Mappings

Let  $V, W$  be vector spaces over a field  $F$ . A mapping  $f : V \rightarrow W$  is called **linear**, or  **$F$ -linear**, or even a **homomorphism of  $F$ -vector spaces** if for all  $\vec{v}_1, \vec{v}_2 \in V$  and  $\lambda \in F$  we have

$$f(\vec{v}_1 + \vec{v}_2) = f(\vec{v}_1) + f(\vec{v}_2)$$

$$f(\lambda \vec{v}_1) = \lambda f(\vec{v}_1)$$

A bijective linear mapping is called an **isomorphism** of vector spaces. If there is an isomorphism between two vector spaces, we call them **isomorphic**. A homomorphism  $V \rightarrow V$  is called an **endomorphism** of  $V$ . An isomorphism  $V \rightarrow V$  is called an **automorphism** of  $V$

Two vector subspaces  $V_1, V_2$  of a vector space  $V$  are called **complementary** if addition defines a bijection

$$V_1 \times V_2 \xrightarrow{\sim} V$$

something about direct sums

### Theorem 1.5.2: Classifying VecSpaces by Dimension

Let  $n$  be a natural number. Then a vector space over a field  $F$  is isomorphic to  $F^n$  iff it has dimension  $n$

### Theorem 1.5.3: Linear Mapping and Bases

Let  $V, W$  be vector spaces over a field  $F$ . The set of all homomorphisms from  $V$  to  $W$  is denoted by

$$\text{Hom}_F(V, W) = \text{Hom}(V, W) \subseteq \text{Maps}(V, W)$$

Let  $B \subset V$  be a basis. Then restriction of a mapping gives a bijection

$$\text{Hom}_F(V, W) \xrightarrow{\sim} \text{Maps}(B, W)$$

$$f \mapsto f|_B$$

### Theorem 1.5.4: Inverse Mappings

1. Every injective linear mapping  $f : V \hookrightarrow W$  has a **left inverse**, or a linear mapping  $g : W \rightarrow V$  s.t.  $g \circ f = \text{id}_V$
2. Every surjective linear mapping  $f : V \twoheadrightarrow W$  has a **right inverse**, or a linear mapping  $G : W \rightarrow V$  s.t.  $f \circ g = \text{id}_W$

### Definition 1.5.5: Image and Kernel of a map

The **image** of a linear mapping  $f : V \rightarrow W$  is the subset  $\text{im}(f) = f(V) \subseteq W$ . It is a vector subspace of  $W$ . The preimage of the zero vector of a linear mapping  $f : V \rightarrow W$  is denoted by:

$$\ker(f) := f^{-1}(0) = \{v \in V : f(v) = 0\}$$

and is called the **kernel** of the linear mapping  $f$ . The kernel is a subspace of  $V$

**Mini lemma:** A linear mapping is injective iff its kernel is zero

### Theorem 1.5.6: Rank-Nullity / Dimension Theorem

Let  $f : V \rightarrow W$  be a linear mapping between vector spaces. Then:

$$\dim V = \dim(\ker f) + \dim(\text{im } f)$$

Dimension of  $\text{im } f =$  **rank** of  $f$ , dimension of  $\ker f =$  **nullity** of  $f$

Let  $V$  be a vector space, and  $U, W \subseteq V$  vector subspaces. Then

$$\dim(U + W) + \dim(U \cap W) = \dim U + \dim W$$

## 2 Linear Mappings and Matrices

### 2.1 Linear Mappings $F^m \rightarrow F^n$ and Matrices

#### Theorem 2.1.1: Linear Maps $F^m \rightarrow F^n$ and Matrices

Let  $F$  be a field and let  $m, n \in \mathbb{N}$ . There is a bijection between the space of linear mappings  $F^m \rightarrow F^n$  and the set of matrices with  $n$  rows,  $m$  columns, and entries in  $F$ :

$$M : \text{Hom}_F(F^m, F^n) \xrightarrow{\sim} \text{Mat}(n \times m; F)$$

$$f \mapsto [f]$$

This attaches to each linear mapping  $f$  its **representing matrix**  $M(f) := [f]$ . The columns of this matrix are the images under  $f$  of the standard basis elements of  $F^m$

$$[f] := (f(\vec{e}_1) | f(\vec{e}_2) | \cdots | f(\vec{e}_m))$$

#### Definition 2.1.2: Matrix Multiplication

Let  $n, m, \ell \in \mathbb{N}$ ,  $F$  a field, and let  $A \in \text{Mat}(n \times m; F)$  and  $B \in \text{Mat}(m \times \ell; F)$  be matrices. The **product**  $A \circ B = AB \in \text{Mat}(n \times \ell; F)$  is the matrix defined by

$$(AB)_{ik} = \sum_{j=1}^m A_{ij} B_{jk}$$

#### Theorem 2.1.3: Composition of maps to products

Let  $g : F^\ell \rightarrow F^m$  and  $f : F^m \rightarrow F^n$  be linear mappings. The representing matrix of their composition is the product of their representing matrices:

$$[f \circ g] = [f] \circ [g]$$

#### Theorem 2.1.4: Calculating with Matrices

- $(A + A')B = AB + A'B$
- $AI = A$
- $A(B + B') = AB + AB'$
- $(AB)C = A(BC)$
- $IB = B$

2.2 Matrix Definitions

Definition 2.2.1: Big def-thm pairs

**Def:** A matrix  $A$  is called **invertible** if there exists matrices  $B$  and  $C$  such that  $BA = I$  and  $AC = I$

Thm: Invertible Equivalence

- 1. There exists a square matrix  $B$  such that  $BA = I$
- 2. There exists a square matrix  $C$  such that  $AC = I$
- 3. The square matrix  $A$  is invertible

**Def:** An **elementary matrix** is any square matrix that differs from the identity matrix in at least one entry

**Thm:** Every square matrix with entries in a field can be written as a product of elementary matrices

**Def:** Any matrix whose only non-zero entries lie on the diagonal, and which has first 1's along the diagonal and then 0's, is said to be in **Smith Normal Form**

**Thm:** For each matrix  $A \in \text{Mat}(n \times m; F)$  there exist invertible matrices  $P$  and  $Q$  such that  $PAQ$  is a matrix in Smith Normal Form

**Thm:** Let  $f : V \rightarrow W$  be a linear map between finite dim.  $F$ -vector spaces. There exists two ordered bases  $\mathcal{A}$  of  $V$ , and  $\mathcal{B}$  of  $W$  s.t. the representing matrix  $_{\mathcal{B}}[f]_{\mathcal{A}}$  has zero entries everywhere except possibly on the diagonal, and along the diagonal there are 1's first, followed by 0's

**Def:** The **column rank** of a matrix  $A \in \text{Mat}(n \times m; F)$  is the dimension of the subspace of  $F^n$  generated by the columns of  $A$ . Similarly, the **row rank** of  $A$  is the dimension of the subspace of  $F^m$  generated by the rows of  $A$ .

**Thm:** The column and row rank of any matrix are equal

**Def:** Since they are both the same, "column" and "row" can be omitted for the **rank of a matrix**, written as  $\text{rk } A$ . If the rank is equal to the no. of rows/columns, then the matrix has **full rank**

**Def:** The **trace** of a square matrix is defined to be the sum of its diagonal entries, denoted by  $\text{tr}(A)$

2.3 Abstract Linear Mappings and Matrices

Theorem 2.3.1: Representing Matrices

Let  $F$  be a field,  $V$  and  $W$  vector spaces over  $F$  with ordered bases  $\mathcal{A} = (\vec{v}_1, \dots, \vec{v}_m)$  and  $\mathcal{B} = (\vec{w}_1, \dots, \vec{w}_n)$ . Then to each linear mapping  $f : V \rightarrow W$  we associate a **representing matrix**  $_{\mathcal{B}}[f]_{\mathcal{A}}$  whose entries  $a_{ij}$  are defined by the identity

$$f(\vec{v}_j) = a_{1j}\vec{w}_1 + \dots + a_{nj}\vec{w}_n \in W$$

This makes a bijection, which is an isomorphism of vector spaces:

$$M_{\mathcal{B}}^{\mathcal{A}} : \text{Hom}_F(V, W) \xrightarrow{\sim} \text{Mat}(n \times m; F)$$
$$f \mapsto {}_{\mathcal{B}}[f]_{\mathcal{A}}$$

Theorem 2.3.2: Repr. Mat of Compositions

Let  $F$  be a field and  $U, V, W$  finite dimensional vector spaces over  $kF$  with ordered bases  $\mathcal{A}, \mathcal{B}, \mathcal{C}$ . If  $f : U \rightarrow V$  and  $g : V \rightarrow W$  are linear mappings, then the representing matrix of the composition  $g \circ f : U \rightarrow W$  is the matrix product of the representing matrices of  $f$  and  $g$ :

$$_{\mathcal{C}}[g \circ f]_{\mathcal{A}} = {}_{\mathcal{C}}[g]_{\mathcal{B}} \circ {}_{\mathcal{B}}[f]_{\mathcal{A}}$$

Definition 2.3.3: Representation of a vector

Let  $V$  be a finite dimensional vector space with an ordered basis  $\mathcal{A} = (\vec{v}_1, \dots, \vec{v}_m)$ . We'll denote the inverse to the bijection in 1.3.3 " $\Phi_{\mathcal{A}} : F^m \xrightarrow{\sim} V, (\alpha_1, \dots, \alpha_m)^T \mapsto \alpha_1\vec{v}_1 + \dots + \alpha_m\vec{v}_m$ " by

$$\vec{v} \mapsto {}_{\mathcal{A}}[\vec{v}]$$

The column vector  $_{\mathcal{A}}[\vec{v}]$  is called the **representation of the vector  $\vec{v}$  with respect to the basis  $\mathcal{A}$**

**Thm: Representation of the Image of a Vector:** Let  $V, W$  be finite dim. vector spaces over  $F$  with ordered bases  $\mathcal{A}, \mathcal{B}$  and let  $f : V \rightarrow W$  be a linear mapping. The following holds for  $\vec{v} \in V$ :

$$_{\mathcal{B}}[f(\vec{v})] = {}_{\mathcal{B}}[f]_{\mathcal{A}} \circ {}_{\mathcal{A}}[\vec{v}]$$

2.4 Change of a Matrix by Change of Basis

Definition 2.4.1: Change of Basis Matrix

Let  $\mathcal{A} = (\vec{v}_1, \dots, \vec{v}_n)$  and  $\mathcal{B} = (\vec{w}_1, \dots, \vec{w}_n)$  be ordered bases of the same  $F$ -vector space  $V$ . Then the matrix representing the identity mapping w.r.t. these bases

$$_{\mathcal{B}}[\text{id}_V]_{\mathcal{A}}$$

is called a **change of basis matrix**. By definition, its entries are given by the equalities  $\vec{v}_j = \sum_{i=1}^n a_{ij}\vec{w}_i$

Theorem 2.4.2: Change of Basis

Let  $V$  and  $W$  be finite dimensional vector spaces over  $F$  and let  $f : V \rightarrow W$  be a linear mapping. Suppose that  $\mathcal{A}, \mathcal{A}'$  are ordered bases of  $V$  and  $\mathcal{B}, \mathcal{B}'$  are ordered bases of  $W$ . Then

$$_{\mathcal{B}'}[f]_{\mathcal{A}'} = {}_{\mathcal{B}'}[\text{id}_W]_{\mathcal{B}} \circ {}_{\mathcal{B}}[f]_{\mathcal{A}} \circ {}_{\mathcal{A}}[\text{id}_V]_{\mathcal{A}'}$$

Let  $V$  be a finite dimensional vector space and let  $f : V \rightarrow V$  be an endomorphism of  $V$ . Suppose that  $\mathcal{A}, \mathcal{A}'$  are ordered bases of  $V$ . Then

$$_{\mathcal{A}'}[f]_{\mathcal{A}'} = {}_{\mathcal{A}}[\text{id}_V]_{\mathcal{A}'}^{-1} \circ {}_{\mathcal{A}}[f]_{\mathcal{A}} \circ {}_{\mathcal{A}}[\text{id}_V]_{\mathcal{A}'}$$

3 Rings and Modules

3.1 Ring basics

Definition 3.1.1: Definition of a Ring

A **ring** is a set with two operations  $(\mathbb{R}, +, \cdot)$  that satisfy:

- 1.  $(R, +)$  is an abelian group
- 2.  $(R, \cdot)$  is a **monoid**, meaning that it is a set with **Associativity** and **Identity**, or in other words, a monoid is a group without the necessity of having the **Inverse** axiom
- 3. The distributive laws hold, meaning that for all  $a, b, c \in R$ ,

$$a \cdot (b + c) = (a \cdot b) + (a \cdot c)$$
$$(a + b) \cdot c = (a \cdot c) + (b \cdot c)$$

The two operations are called **addition** and **multiplication** in our ring. A ring in which multiplication, that is  $a \cdot b = b \cdot a$  for all  $a, b \in R$ , is a **commutative ring**

**Note:** We denote the identity of the monoid  $(R, \cdot)$  as 1, and the additive identity of  $(R, +)$  as  $0_R$  or 0

**Note:** We define the **null ring** or **zero ring** as a ring where  $R$  is a single element set, i.e.  $\{0\}$  where  $0 + 0 = 0$  and  $0 \times 0 = 0$

Example 3.1.2: Modulo Rings

Let  $m \in \mathbb{Z}$ . Then the set of **integers modulo  $m$** , written

$$\mathbb{Z}/m\mathbb{Z}$$

is a ring. The elements of  $\mathbb{Z}/m\mathbb{Z}$  consist of **congruence classes** of integers modulo  $m$  - that is, the elements are the subsets  $T$  of  $\mathbb{Z}$  of the form  $T = a + m\mathbb{Z}$  with  $a \in \mathbb{Z}$ . Think of these as the set of integers that have the same remainder when you divide them by  $m$ . I denote the above congruence class by  $\bar{a}$ . Obviously  $\bar{a} = \bar{b}$  is the same as  $a - b \in m\mathbb{Z}$ , and often I'll write

$$a \equiv b \pmod{m}$$

3.2 Linking Rings to Fields and Further Properties

Definition 3.2.1: Ring definition of a field

A **field** is a non-zero commutative ring  $F$  in which every non-zero element  $a \in F$  has an inverse  $a^{-1} \in F$ , that is an element  $a^{-1}$  with the property that  $a \cdot a^{-1} = a^{-1} \cdot a = 1$

Definition 3.2.2: Multiples of an abelian group

Let  $m \in \mathbb{Z}$ . The  **$m$ -th multiple  $ma$  of an element  $a$**  in an abelian group  $R$  is:

$$ma = \underbrace{a + a + \dots + a}_{m \text{ terms}} \quad \text{if } m > 0$$

$0a = 0$  and negative multiples are defined by  $(-m)a = -(ma)$

### Theorem 3.2.3: Properties of Rings

**Lemma set 1:** Let  $R$  be a ring and let  $a, b \in R$ . Then:

1.  $0a = 0 = a0$
2.  $(-a)b = -(ab) = a(-b)$
3.  $(-a)(-b) = ab$

**Lemma set 2:** Let  $R$  be a ring,  $a, b \in R$  and  $m, n \in \mathbb{Z}$ . Then:

1.  $m(a + b) = ma + mb$
2.  $(m + n)a = ma + na$
3.  $m(na) = (mn)a$
4.  $m(ab) = (ma)b = a(mb)$
5.  $(ma)(nb) = (mn)(ab)$

**Prime Property for Fields:** Let  $m$  be a natural number. The commutative ring  $\mathbb{Z}/m\mathbb{Z}$  is a field if and only if  $m$  is prime

### Definition 3.2.4: Unit of a ring

Let  $R$  be a ring. An element  $a \in R$  is called a **unit** if it is *invertible* in  $R$  or in other words *has a multiplicative inverse* in  $R$ , meaning that there exists  $a^{-1} \in R$  such that

$$aa^{-1} = 1 = a^{-1}a$$

**Thm:** The set  $R^\times$  of units in a ring  $R$  forms a group under multiplication

### Definition 3.2.5: zero-divisors of a ring

In a ring  $R$ , a non-zero element  $a$  is called a **zero-divisor** or **divisor of zero** if there exists a non-zero element  $b$  such that either  $ab = 0$  or  $ba = 0$ .

### Definition 3.2.6: Integral Domain

An **integral domain** is a non-zero commutative ring that has no zero-divisors. The following two laws hold:

1.  $ab = 0 \implies a = 0$  or  $b = 0$
2.  $a \neq 0$  and  $b \neq 0 \implies ab \neq 0$

### Theorem 3.2.7: Integral Domain Properties

- **Cancellation Law:** Let  $R$  be an integral domain and let  $a, b, c \in R$ . If  $ab = ac$  and  $a \neq 0$  then  $b = c$
- Let  $m$  be a natural number. Then  $\mathbb{Z}/m\mathbb{Z}$  is an integral domain if and only if  $m$  is prime.
- Every **finite** integral domain is a field.

## 3.3 Polynomials

### Definition 3.3.1: Polynomial

Let  $R$  be a ring. A **polynomial over  $R$**  is an expression of the form

$$P = a_0 + a_1X + a_2X^2 + \cdots + a_mX^m$$

for some non-negative  $m \in \mathbb{Z}$  and elements  $a_i \in R$  for  $0 \leq i \leq m$ .

- The set of all polynomials over  $R$  is denoted by  $R[X]$ .
- In the case where  $a_m$  is non-zero, the polynomial  $P$  has **degree  $m$** , (written  $\deg(P)$ ), and  $a_m$  is its **leading coefficient**
- When the leading coefficient is 1 the polynomial is a **monic polynomial**.
- A polynomial of degree one is called **linear**, degree two is called **quadratic**, and degree three is called **cubic**.

### Definition 3.3.2: Ring of Polynomials

The set  $R[X]$  becomes a ring called the **ring of polynomials with coefficients in  $R$ , or over  $R$** . The zero and the identity of  $R[X]$  are the zero and identity of  $R$ , respectively.

### Theorem 3.3.3: Properties of a Polynomial Ring

- If  $R$  is a ring with no zero-divisors, then  $R[X]$  has no zero-divisors and  $\deg(PQ) = \deg(P) + \deg(Q)$  for non-zero  $P, Q \in R[X]$ .
- If  $R$  is an integral domain, then so is  $R[X]$
- Let  $R$  be an integral domain and let  $P, Q \in R[X]$  with  $Q$  monic. Then there exists unique  $A, B \in R[X]$  such that  $P = AQ + B$  and  $\deg(B) < \deg(Q)$  or  $B = 0$

### Definition 3.3.4: Evaluating a Function

Let  $R$  be a commutative ring and  $P \in R[X]$  a polynomial. Then  $P$  can be **evaluated** at the element  $\lambda \in R$  to produce  $P(\lambda)$  by replacing the powers of  $X$  in  $P$  by the corresponding powers of  $\lambda$ . In this way we have a mapping

$$R[X] \rightarrow \text{Maps}(R, R)$$

This is the precise definition of thinking of a polynomial as a function. An element  $\lambda \in R$  is a **root** of  $P$  if  $P(\lambda) = 0$

**Thm:** Let  $R$  be a commutative ring, let  $\lambda \in R$  and  $P(X) \in R[X]$ . Then  $\lambda$  is a root of  $P(X)$  if and only if  $(X - \lambda)$  divides  $P(X)$

### Theorem 3.3.5: Degrees of Polynomial Roots

Let  $R$  be a field, or more generally an integral domain. Then a non-zero polynomial  $P \in R[X] \setminus \{0\}$  has at most  $\deg(P)$  roots in  $R$

### Definition 3.3.6: Algebraically closed fields

A field  $F$  is **algebraically closed** if each non-constant polynomial  $P \in F[X] \setminus F$  with coefficients in our field has a root in our field  $F$

### Theorem 3.3.7: The Fundamental Theorem of Algebra

The field of complex numbers  $\mathbb{C}$  is algebraically closed.

### Theorem 3.3.8: Linear Factors of Closed Fields

If  $F$  is an algebraically closed field, then every non-zero polynomial  $P \in F[X] \setminus \{0\}$  **decomposes into linear factors**

$$P = c(X - \lambda_1) \cdots (X - \lambda_n)$$

with  $n \geq 0$ ,  $c \in F^\times$  and  $\lambda_1, \dots, \lambda_n \in F$ . This decomposition is unique up to reordering the factors

## 3.4 Homomorphisms, Ideals, and Substrings

### Definition 3.4.1: Ring Homomorphisms

Let  $R$  and  $S$  be rings. A mapping  $f : R \rightarrow S$  is a **ring homomorphism** if the following hold for all  $x, y \in R$ :

$$f(x + y) = f(x) + f(y)$$

$$f(xy) = f(x)f(y)$$

### Theorem 3.4.2: Properties of Ring Homomorphisms

Let  $R$  and  $S$  be rings and  $f : R \rightarrow S$  a ring homomorphism. Then for all  $x, y \in R$  and  $m \in \mathbb{Z}$ :

1.  $f(0_R) = 0_S$ , where  $0_R$  and  $0_S$  are the zeros of  $R$  and  $S$
2.  $f(-x) = -f(x)$
3.  $f(x - y) = f(x) - f(y)$
4.  $f(mx) = mf(x)$
5.  $f(x^n) = (f(x))^n$  for all  $x \in R$  and  $n \in \mathbb{N}$

### Definition 3.4.3: Ideal

A subset  $I$  of a ring  $R$  is an **ideal**,  $I \trianglelefteq R$ , if the following hold:

1.  $I \neq \emptyset$
2.  $I$  is closed under subtraction
3. for all  $i \in I$  and  $r \in R$  we have  $ri, ir \in I$



#### Definition 3.4.4: Generated Ideals

Let  $R$  be a commutative ring and let  $T \subseteq R$ . Then the **ideal of  $R$  generated by  $T$**  is the set

$${}_R\langle T \rangle = \{r_1 t_1 + \cdots + r_m t_m : t_1, \dots, t_m \in T, r_1, \dots, r_m \in R\}$$

#### Theorem 3.4.5

Let  $R$  be a commutative ring and let  $T \subseteq R$ . Then  ${}_R\langle T \rangle$  is the smallest ideal of  $R$  that contains  $T$

#### Definition 3.4.6: Principal Ideal

Let  $R$  be a commutative ring. An ideal  $I$  of  $R$  is called a **principal ideal** if  $I = \langle t \rangle$  for some  $t \in R$

#### Theorem 3.4.7: Kernels as Ideals

- Let  $R$  and  $S$  be rings and  $f : R \rightarrow S$  a ring homomorphism. Then  $\ker f$  is an ideal of  $R$ .
- $f$  is injective if and only if  $\ker f = \{0\}$
- The intersection of any collection of ideals of a ring  $R$  is an ideal of  $R$
- Let  $I$  and  $J$  be ideals of a ring  $R$ . Then

$$I + J = \{a + b : a \in I, b \in J\}$$

is an ideal of  $R$

#### Definition 3.4.8: Subrings

Let  $R$  be a ring.  $R' \subset R$  is a **subring** of  $R$  if  $R'$  is itself a ring under the operations of addition and multiplication defined in  $R$ .

**Thm: Test for subring:** Let  $R$  be a subset of a ring  $R$ . Then  $R'$  is a subring iff:

- $R'$  has a multiplicative identity
- $R'$  is closed under subtraction:  $a, b \in R' \rightarrow a - b \in R'$
- $R'$  is closed under multiplication

**Thm:** Let  $R$  and  $S$  be rings and  $f : R \rightarrow S$  a ring homomorphism.

- If  $R'$  is a subring of  $R$  then  $f(R')$  is a subring of  $S$ . In particular,  $\text{im } f$  is a subring of  $S$ .
- Assume that  $f(1_R) = 1_S$ . Then if  $x$  is a unit in  $R$ ,  $f(x)$  is a unit in  $S$  and  $(f(x))^{-1} = f(x^{-1})$ . In this case,  $f$  restricts to a group homomorphism  $f|_{R^\times} : R^\times \rightarrow S^\times$

#### Definition 3.5.1: Equivalence Relations

A **relation**  $R$  on a set  $X$  is a subset  $R \subseteq X \times X$ . In the context of relations, it's written  $xRy$  instead of  $(x, y) \in R$ .  $R$  is an **equivalence relation on  $X$**  when for all elements  $x, y, z \in X$  the following hold:

- Reflexivity:**  $xRx$
- Symmetry:**  $xRy \iff yRx$
- Transitivity:**  $xRy$  and  $yRz \implies xRz$

#### Definition 3.5.2: Equivalence Classes

Suppose that  $\sim$  is an equivalence relation on a set  $X$ . For  $x \in X$  the set  $E(x) := \{z \in X : z \sim x\}$  is called the **equivalence class of  $x$** . A subset  $E \subseteq X$  is called an **equivalence class** for our equivalence relation if there is an  $x \in X$  for which  $E = E(x)$ . An element of an equivalence class is called a **representative** of the class. A subset  $Z \subseteq X$  containing precisely one element from each equivalence class is called a **system of representatives** for the equivalence relation

#### Definition 3.5.3: Set of Equivalence Classes

Given an equivalence relation  $\sim$  on the set  $X$  I will denote the **set of equivalence classes**, which is a subset of the power set  $\mathcal{P}(X)$ , by

$$(X/\sim) := \{E(x) : x \in X\}$$

There is a canonical mapping  $\text{can} : X \rightarrow (X/\sim)$ ,  $x \mapsto E(x)$  (surjection)

### 3.6 Factor Rings and First Isomorphism Theorem

#### Definition 3.6.1: Coset

Let  $I \trianglelefteq R$  be an ideal in a ring  $R$ . The set

$$x + I := \{x + i : i \in I\} \subseteq R$$

is a **coset of  $I$  in  $R$**  or the **coset of  $x$  w.r.t  $I$  in  $R$**

#### Definition 3.6.2: Factor Ring

Let  $R$  be a ring,  $I \trianglelefteq R$  be an ideal, and  $\sim$  the equivalence relation defined by  $x \sim y \iff x - y \in I$ . Then  $R/I$ , the **factor ring of  $R$  by  $I$**  or the **quotient of  $R$  by  $I$** , is the set  $(R/\sim)$  of cosets of  $I$  in  $R$

#### Theorem 3.6.3

Let  $R$  be a ring and  $I \trianglelefteq R$  an ideal. Then  $R/I$  is a ring, where the operation of addition is defined by

$$(x + I) \dot{+} (y + I) = (x + y) + I \quad \text{for all } x, y \in R$$

and multiplication is defined by

$$(x + I) \cdot (y + I) = xy + I \quad \text{for all } x, y \in R$$

#### Theorem 3.6.4: Universal Property of Factor Rings

Let  $R$  be a ring and  $I$  an ideal of  $R$

- The mapping  $\text{can} : R \rightarrow R/I$  sending  $r$  to  $r + I$  for all  $r \in R$  is a surjective ring homomorphism with kernel  $I$
- If  $f : R \rightarrow S$  is a ring homomorphism with  $f(I) = \{0_S\}$ , so that  $I \subseteq \ker f$  then there is a unique ring homomorphism  $\bar{f} : R/I \rightarrow S$  such that  $f = \bar{f} \circ \text{can}$

#### Theorem 3.6.5: First Isomorphism Theorem for Rings

Let  $R$  and  $S$  be rings. Then every ring homomorphism  $f : R \rightarrow S$  induces a ring isomorphism

$$\bar{f} : R/\ker f \xrightarrow{\sim} \text{im } f$$

### 3.7 Modules

#### Definition 3.7.1: Module

A (left) **module  $M$  over a ring  $R$**  (or an  **$R$ -module**) is a pair consisting of an abelian group  $M = (M, +)$  a mapping

$$R \times M \rightarrow M$$

$$(r, a) \mapsto ra$$

s.t. for all  $r, s \in R$  and  $a, b \in M$ , we have:

- Distributivity 1:**  $r(a+b) = (ra) \dot{+} (rb)$
- Distributivity 2:**  $(r+s)a = (ra) \dot{+} (sa)$
- Associativity:**  $r(sa) = (rs)a$
- Identity:**  $1_R a = a$

#### Theorem 3.7.2: Module Lemmas

Let  $R$  be a ring and  $M$  an  $R$ -module

- $0_R a = 0_M$  for all  $a \in M$
- $r 0_M = 0_M$  for all  $r \in R$
- $(-r)a = r(-a) = -(ra)$  for all  $r \in R, a \in M$

### 3.5 Equivalence Relations

### Definition 3.7.3: Module Homomorphisms

Let  $R$  be a ring and let  $M, N$  be  $R$ -modules. A mapping  $f : M \rightarrow N$  is an  $R$ -**homomorphism** or *homomorphism* if the following hold for all  $a, b \in M$  and  $r \in R$

$$f(a + b) = f(a) + f(b)$$

$$f(ra) = rf(a)$$

- The **kernel** of  $f$  is  $\ker f = \{a \in M : f(a) = 0_N\} \subseteq M$
- The **image** of  $f$  is  $\operatorname{im} f = \{f(a) : a \in M\} \subseteq N$
- If  $f$  is a bijection then it is an  $R$ -**module isomorphism** or **isomorphism**, written  $M \cong N$ , and say  $M$  and  $N$  are **isomorphic**

### Definition 3.7.4: Submodules

A non-empty subset  $M'$  of an  $R$ -module  $M$  is a **submodule** if  $M'$  is an  $R$ -module with respect to the operations of the  $R$ -module  $M$  **restricted** to  $M'$

**Thm:** Let  $R$  be a ring and let  $M$  be an  $R$ -module. A subset  $M'$  of  $M$  is a submodule if and only if

1.  $0_M \in M'$
2.  $a, b \in M' \implies a - b \in M'$
3.  $r \in R, a \in M' \implies ra \in M'$

### Theorem 3.7.5: Submodule lemmas

- Let  $f : M \rightarrow N$  be an  $R$ -homomorphism. Then  $\ker f$  is a submodule of  $M$  and  $\operatorname{im} f$  is a submodule of  $N$
- Let  $R$  be a ring,  $M$  an  $R$ -homomorphism. Then  $f$  is injective if and only if  $\ker f = \{0_M\}$

### Definition 3.7.6: Generated Submodules

Let  $R$  be a ring,  $M$  an  $R$ -module and let  $T \subseteq M$ . Then the **submodule of  $M$  generated by  $T$**  is the set

$${}_R\langle T \rangle = \{r_1 t_1 + \dots + r_m t_m : t_1, \dots, t_m \in T, r_1, \dots, r_m \in R\}$$

together with the zero element in the case  $T = \emptyset$ . If  $T = \{t_1, \dots, t_n\}$ , a finite set, we write  ${}_R\langle t_1, \dots, t_n \rangle$  instead of  ${}_R\langle \{t_1, \dots, t_n\} \rangle$ . The module  $M$  is **finitely generated** if it is generated by a finite set:  $M = {}_R\langle t_1, \dots, t_n \rangle$ . It is called **cyclic** if it is generated by a singleton  $M = {}_R\langle T \rangle$

### Definition 3.7.7: Generated Submodule lemmas

- Let  $T \subseteq M$ . Then  ${}_R\langle T \rangle$  is the smallest submodule of  $M$  that contains  $T$
- The intersection of any collection of submodules of  $M$  is a submodule of  $M$ .
- Let  $M_1$  and  $M_2$  be submodules of a  $M$ . Then
 
$$M_1 + M_2 = \{a + b : a \in M_1, b \in M_2\}$$
 is a submodule of  $M$

### Definition 3.7.8: Submodule Cosets

Let  $R$  be a ring,  $M$  an  $R$ -module, and  $N$  a submodule of  $M$ . For each  $a \in M$  the **coset of  $a$  with respect to  $N$  in  $M$**  is

$$a + N = \{a + b : b \in N\}$$

It is a coset of  $N$  in the abelian group  $M$  and so is an equivalence class for the equivalence relation  $a \sim b \iff a - b \in N$ .

Let  $M/N$ , the **factor of  $N$  by  $N$**  or the **quotient of  $M$  by  $N$**  to be the set  $(M/\sim)$  of all cosets of  $N$  in  $M$ . This becomes an  $R$ -module by introducing the operations of addition and multiplication as follows:

$$(a + N) + (b + N) = (a + b) + N$$

$$r(a + N) = ra + N$$

for all  $a, b \in M, r \in R$ .

The zero of  $M/N$  is the coset  $0_{M/N} = 0_M + N$ . The negative of  $a + N \in M/N$  is the coset  $-(a + N) = (-a) + N$ . The  $R$ -module  $M/N$  is the **factor module** of  $M$  by the submodule  $N$

### Theorem 3.7.9: Universal Property of Factor Modules

Let  $R$  be a ring, let  $L$  and  $M$  be  $R$ -modules, and  $N$  a submodule of  $M$ .

1. The mapping  $\text{can} : M \rightarrow M/N$  sending  $a$  to  $a + N$  for all  $a \in M$  is a surjective  $R$ -homomorphism with kernel  $N$
2. If  $f : M \rightarrow L$  is an  $R$ -homomorphism with  $f(N) = \{0_L\}$ , so that  $N \subseteq \ker f$ , then there is a unique homomorphism  $\bar{f} : M/N \rightarrow L$  such that  $f = \bar{f} \circ \text{can}$

### Theorem 3.7.10: First Isomorphism Thm for Modules

Let  $R$  be a ring and let  $M$  and  $N$  be  $R$ -modules. Then every  $R$ -homomorphism  $f : M \rightarrow N$  induces an  $R$ -isomorphism

$$\bar{f} : M/\ker f \xrightarrow{\sim} \operatorname{im} f$$

### Definition 4.1.1: Symmetric Groups

The group of all permutations of the set  $\{1, 2, \dots, n\}$ , also known as bijections from  $\{1, 2, \dots, n\}$  to itself is denoted by  $\mathfrak{S}_n$  (but i will just write  $S_n$  because icba) and called the  **$n$ -th symmetric group**. It is a group under composition and has  $n!$  elements.

A **transposition** is a permutation that swaps two elements of the set and leaves all the others unchanged.

### Definition 4.1.2: Inversions of a permutation

An **inversion** of a permutation  $\sigma \in S_n$  is a pair  $(i, j)$  such that  $1 \leq i < j \leq n$  and  $\sigma(i) > \sigma(j)$ . The number of inversions of the permutation  $\sigma$  is called the **length of  $\sigma$**  and written  $\ell(\sigma)$ . In formulas:

$$\ell(\sigma) = |\{(i, j) : i < j \text{ but } \sigma(i) > \sigma(j)\}|$$

The **sign of  $\sigma$**  is defined to be the parity of the number of inversions of  $\sigma$ . In formulas:

$$\operatorname{sgn}(\sigma) = (-1)^{\ell(\sigma)}$$

### Theorem 4.1.3: Multiplicativity of the sign

For each  $n \in \mathbb{N}$  the sign of a permutation produces a group homomorphism  $\operatorname{sgn} : S_n \rightarrow \{+1, -1\}$  from the symmetric group to the two-element group of signs. In formulas:

$$\operatorname{sgn}(\sigma\tau) = \operatorname{sgn}(\sigma)\operatorname{sgn}(\tau) \quad \forall \sigma, \tau \in S_n$$

### Definition 4.1.4: Alternating Group of a Permutation

For  $n \in \mathbb{N}$ , the set of even permutations in  $S_n$  forms a subgroup of  $S_n$  because it is the kernel of the group homomorphism  $\operatorname{sgn} : S_n \rightarrow \{+1, -1\}$ . This group is the **alternating group** and is denoted  $A_n$

## 4.2 Determinants

### Definition 4.2.1: Determinants - the Leibniz Formula

Let  $R$  be a commutative ring and  $n \in \mathbb{N}$ . The **determinant** is a mapping  $\det : \operatorname{Mat}(n; R) \rightarrow R$  from square matrices with coefficients in  $R$  to the ring  $R$  that is given by the following formula

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \mapsto \det(A) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) a_{1\sigma(1)} \dots a_{n\sigma(n)}$$

The sum is over all permutations of  $n$ , and the coefficient  $\operatorname{sgn}(\sigma)$  is the sign of the permutation  $\sigma$  defined above. When  $n = 0$ , the determinant is 1

## 4 Determinants and Eigenvalues Redux

### 4.1 Symmetric Groups

## 4.3 Characterising the Determinant

### Definition 4.3.1: Bilinear Forms

Let  $U, V, W$  be  $F$ -vector spaces. A **bilinear form on  $U \times V$  with values in  $W$**  is a mapping  $H : U \times V \rightarrow W$  which is a linear mapping in both of its entries. This means that it must satisfy the following properties for all  $u_1, u_2 \in U$  and  $v_1, v_2 \in V$  and all  $\lambda \in F$ :

$$\begin{aligned} H(u_1 + u_2, v_2) &= H(u_1, v_2) + H(u_2, v_2) \\ H(\lambda u_1, v_1) &= \lambda H(u_1, v_1) \\ H(u_1, v_2 + v_1) &= H(u_1, v_2) + H(u_1, v_1) \\ H(u_1, \lambda v_1) &= \lambda H(u_1, v_1) \end{aligned}$$

A bilinear form  $H$  is **symmetric** if  $U = V$  and

$$H(u, v) = H(v, u) \quad \text{for all } u, v \in U$$

while it is **antisymmetric** or **alternating** if  $U = V$  and

$$H(u, u) = 0 \quad \text{for all } u \in U$$

- antisymmetric  $\implies H(u, v) = -H(v, u)$
- $H(u, v) = -H(v, u) \implies$  antisymmetric iff  $1_F + 1_F \neq 0_F$

### Definition 4.3.2: Multilinear Forms

Let  $V_1, \dots, V_n, W$  be  $F$ -vector spaces. A mapping  $H : V_1 \times V_2 \times \dots \times V_n \rightarrow W$  is a **multilinear form** or just **multilinear** if for each  $j$ , the mapping  $V_j \rightarrow W$  defined by  $v_j \mapsto H(v_1, \dots, v_j, \dots, v_n)$ , with the  $v_i \in V_i$  arbitrary fixed vectors of  $V_i$  for  $i \neq j$  is linear.

Let  $V$  and  $W$  be  $F$ -vector spaces. A multilinear form  $H : V \times \dots \times V \rightarrow W$  is **alternating** if it vanishes on every  $n$ -tuple of elements of  $V$  that has at least two entries equal, in other words if:

$$(\exists i \neq j \text{ with } v_i = v_j) \rightarrow H(v_1, \dots, v_i, \dots, v_j, \dots, v_n) = 0$$

### Theorem 4.3.3: Characterisation of the Determinant

Let  $F$  be a field. The mapping

$$\det : \text{Mat}(n; F) \rightarrow F$$

is the unique alternating multilinear form on  $n$ -tuples of column vectors with values in  $F$  that takes the value  $1_F$  on the identity matrix

## 4.4 Rules for Calculating with Determinants

### Theorem 4.4: Determinant Theorem Bank

**4.4.1:** Let  $R$  be a commutative ring,  $A, B \in \text{Mat}(n; R)$ . Then

$$\det(AB) = \det(A) \det(B)$$

**4.4.2:** The determinant of a square matrix with entries in a field  $F$  is non-zero if and only if the matrix is invertible

- 4.4.3:**
- If  $A$  is invertible then  $\det(A^{-1}) = \det(A)^{-1}$
  - If  $B$  is a square matrix then  $\det(A^{-1}BA) = \det(B)$

**4.4.4:** For all  $A \in \text{Mat}(n; R)$  with  $R$  a commutative ring,

$$\det(A^T) = \det(A)$$

### Definition 4.4.6: Cofactors of a Matrix

Let  $A \in \text{Mat}(n; R)$  for some commutative ring  $R$  and  $n \in \mathbb{N}$ . Let  $i, j \in \mathbb{Z}$  between 1 and  $n$ . Then the  $(i, j)$  **cofactor** of  $A$  is  $C_{ij} = (-1)^{i+j} \det(A_{\langle i, j \rangle})$  where  $A_{\langle i, j \rangle}$  is the matrix obtained from  $A$  by deleting the  $i$ -th row and  $j$ -th column.

$$C_{23} = (-1)^{2+3} \det \begin{pmatrix} a_{11} & a_{12} & \textcolor{red}{a_{13}} \\ \textcolor{red}{a_{21}} & \textcolor{red}{a_{22}} & \textcolor{red}{a_{23}} \\ a_{31} & a_{32} & \textcolor{red}{a_{33}} \end{pmatrix} = -a_{11}a_{32} + a_{31}a_{12}$$

### Theorem 4.4.7: Laplace's Expansion

Let  $A = (a_{ij})$  be an  $(n \times n)$ -matrix with entries from a commutative ring  $R$ . For a fixed  $i$ , the  **$i$ -th row expansion of the determinant** is

$$\det(A) = \sum_{j=1}^n a_{ij} C_{ij}$$

and for a fixed  $j$ , the  **$j$ -th column expansion of the determinant** is

$$\det(A) = \sum_{i=1}^n a_{ij} C_{ij}$$

### Definition 4.4.8: Adjugate Matrix

Let  $A$  be a  $(n \times n)$ -matrix with entries in a commutative ring  $R$ . The **adjugate matrix**  $\text{adj}(A)$  is the  $(n \times n)$ -matrix whose entries are  $\text{adj}(A)_{ij} = C_{ji}$  where  $C_{ji}$  is the  $(j, i)$ -cofactor

### Theorem 4.4.9: Cramer's Rule

Let  $A$  be a  $(n \times n)$ -matrix with entries in a commutative ring  $R$ . Then

$$A \cdot \text{adj}(A) = (\det A) I_n$$

### Theorem 4.4.11: Invertibility of Matrices

A square matrix with entries in a commutative ring  $R$  is invertible if and only if its determinant is a unit in  $R$ . That is,  $A \in \text{Mat}(n; R)$  is invertible if and only if  $\det(A) \in R^\times$

### Theorem 4.4.14: Jacobi's Formula

Let  $A = (a_{ij})$  where the coefficients  $a_{ij} = a_{ij}(t)$  are functions of  $t$ . Then

$$\frac{d}{dt} \det A = \text{Tr} \text{Adj} A \frac{dA}{dt}$$

## 4.5 Eigenvalues and Eigenvectors

### Definition 4.5.1: Eigenvalues and Eigenvectors

Let  $f : V \rightarrow V$  be an endomorphism of an  $F$ -vector space  $V$ . A scalar  $\lambda \in F$  is an **eigenvalue** of  $f$  if and only if there exists a non-zero vector  $\vec{v} \in V$  such that  $f(\vec{v}) = \lambda \vec{v}$ . Each such vector is called an **eigenvector of  $f$  with eigenvalue  $\lambda$** . For any  $\lambda \in F$ , the **eigenspace of  $f$  with eigenvalue  $\lambda$**  is

$$E(\lambda, f) = \{\vec{v} \in V : f(\vec{v}) = \lambda \vec{v}\}$$

### Theorem 4.5.4: Existence of Eigenvalues

Each endomorphism of a non-zero finite dimensional vector space over an algebraically closed field has an eigenvalue

### Definition 4.5.6: Characteristic Polynomial

Let  $R$  be a commutative ring and let  $A \in \text{Mat}(n; R)$  be a square matrix with entries in  $R$ . The polynomial  $\det(xI_n - A) \in R[x]$  is called the **characteristic polynomial of the matrix  $A$** . It is denoted by

$$\chi_A(x) := \det(xI_n - A)$$

(where  $\chi$  stands for  $\chi$ aracteristic, lol)

### Theorem 4.5.8: EVs and Characteristic Polynomials

Let  $F$  be a field and  $A \in \text{Mat}(n; F)$  a square matrix with entries in  $F$ . The eigenvalues of the linear mapping  $A : F^n \rightarrow F^n$  are exactly the roots of the characteristic polynomial  $\chi_A$

### Theorem 4.5.9: Eigenvalue Remarks

- Square matrices  $A, B \in \text{Mat}(n; R)$  of same size are **conjugate** if

$$B = P^{-1}AP \in \text{Mat}(n; R)$$

for an invertible  $P \in GL(n; R)$

- Conjugacy is an equivalence relation on  $\text{Mat}(n; R)$
- The char. polynomials for two conjugate matrices are the same
- We can define the char. polynomials of an endomorphism  $f : V \rightarrow V$  of an  $n$ -dim vector space over a field  $F$  to be

$$\chi_f(x) = \chi_A(x) \in F[x]$$

with  $A = \mathcal{A}[f]_{\mathcal{A}} \in \text{Mat}(n; R)$  the matrix of  $f$  w.r.t *any* basis  $\mathcal{A}$  for  $V$ . The E.V.s of  $f$  are exactly the roots of  $\chi_f$

### Theorem 4.5.10: Extending Bases

Let  $f : V \rightarrow V$  be an endomorphism of an  $n$ -dimensional vector space  $V$  over a field  $F$ . Suppose given an  $m$ -dimensional subspace  $W \subseteq V$  such that  $f(W) \subseteq W$ , so that there are defined endomorphisms of the subspace and the quotient space:

$$g : W \rightarrow W; \vec{w} \mapsto f(\vec{w})$$

$$h : V/W \rightarrow V/W; W + \vec{v} \mapsto W + f(\vec{v})$$

The characteristic polynomial of  $f$  is the product of the characteristic polynomials of  $g$  and  $h$

## 4.6 Triangularisable, Diagonalisable, and Cayley-Hamilton

### Definition 4.6.1: Triangularisability

Let  $f : V \rightarrow V$  be an endomorphism of a finite dimensional  $F$ -vector space  $V$ .  $f$  is **triangularisable** if the vector space  $V$  has an ordered basis  $\mathcal{B} = (\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n)$  such that

$$f(\vec{v}_1) = a_{11}\vec{v}_1,$$

$$f(\vec{v}_2) = a_{12}\vec{v}_1 + a_{22}\vec{v}_2,$$

$\vdots$

$$f(\vec{v}_n) = a_{1n}\vec{v}_1 + a_{2n}\vec{v}_2 + \dots + a_{nn}\vec{v}_n \in V$$

(so that the first basis vector  $\vec{v}_1$  is an eigenvector, with eigenvalue  $a_{11}$ ) or equivalently such that the  $n \times n$  matrix  $\mathcal{B}[f]_{\mathcal{B}} = (a_{ij})$  representing  $f$  with respect to  $\mathcal{B}$  is upper triangular (or any other triangular)

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{pmatrix}$$

### Theorem 4.6.1 - 4.6.3

Let  $f : V \rightarrow V$  be an endomorphism of a finite dimensional  $F$ -vector space  $V$ . Then  $f$  is triangularisable iff the characteristic polynomial  $\chi_f$  decomposes into linear factors in  $F[x]$

Finding ordered bases - Choose from the following subspaces

$$1. W = \{\mu\vec{v}_1 \mid \mu \in F\} \subseteq V$$

$$2. W' = \ker(f - \lambda 1_V). \text{ This has a basis of E.V.s } \{\vec{v}_1, \dots, \vec{v}_r\}$$

$$3. W'' = \text{im}(\lambda 1_V - f)$$

Then extend the basis to another ordered basis  $\mathcal{B}$  for  $V$  (the full space) where  $\text{can}(\vec{v}_j) = \vec{u}_j$  forms a basis for  $V/W$ .  $\mathcal{B}[f]_{\mathcal{B}}$  is upper triangular.

An endomorphism  $A : F^n \rightarrow F^n$  is triangularisable iff  $A = (a_{ij})$  is conjugate to  $B = (b_{ij})$  ( $b_{ij} = 0$  for  $i > j$ ), an upper triangular matrix, with  $P^{-1}AP = B$  for an invertible matrix  $P$

### Definition 4.6.6: Diagonalisability

An endomorphism  $f : V \rightarrow V$  of an  $F$ -vector space  $V$  is **diagonalisable** iff there exists a basis of  $V$  consisting of eigenvectors of  $f$ . If  $V$  is finite dimensional then this is the same as saying that there exists an ordered basis  $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_n\}$  where  $\mathcal{B}[f]_{\mathcal{B}} = \text{diag}(\lambda_1, \dots, \lambda_n)$ . In this case, of course,  $f(\vec{v}_i) = \lambda_i\vec{v}_i$ .

A square matrix  $A \in \text{Mat}(n; F)$  is **diagonalisable** iff  $A$  is conjugate to a diagonal matrix, i.e. there exists  $P \in GL(n; F)$  such that  $P^{-1}AP = \text{diag}(\lambda_1, \dots, \lambda_n)$ . In this case the columns  $P$  are the vectors of a basis of  $F^n$  consisting of eigenvectors of  $A$  with eigenvalues  $\lambda_1, \dots, \lambda_n$

### Theorem 4.6.9: Linear Independence of Eigenvectors

Let  $f : V \rightarrow V$  be an endomorphism of a vector space  $V$  and let  $\vec{v}_1, \dots, \vec{v}_n$  be eigenvectors of  $f$  with pairwise different eigenvalues  $\lambda_1, \dots, \lambda_n$ . Then the vectors  $\vec{v}_1, \dots, \vec{v}_n$  are linearly independent

### Theorem 4.6.10: Cayley-Hamilton Theorem

Let  $A \in \text{Mat}(n; R)$  be a square matrix with entries in a commutative ring  $R$ . Then evaluating its characteristic polynomial  $\chi_A(x) \in R[x]$  at the matrix  $A$  gives zero.

## 4.7 Markov Matrices

### Definition 4.7.5: Markov Matrix

A matrix  $M$  whose entries are non-negative and s.t. the sum of the entries of each column equals 1 is a **Markov matrix** or a **stochastic matrix**

**4.7.6:** Suppose  $M \in \text{Mat}(n; \mathbb{R})$  is a M.M. Then  $\lambda = 1$  is an e.v.

### Theorem 4.7.10: Perron-Frobenius Theorem

If  $M \in \text{Mat}(n; \mathbb{R})$  is a Markov matrix with positive values, then the eigenspace  $E(1, M)$  is one-dimensional. There exists a unique basis vector  $\vec{v} \in E(1, M)$  with positive real entries s.t. the sum of its entries is 1

## 5 Inner Product Spaces

### 5.1 Inner Product Spaces Intro

#### Definition 5.1.1: Inner Product

Let  $V$  be a vector space over  $\mathbb{R}$ . An **inner product** on  $V$  is a mapping

$$(-, -) : V \times V \rightarrow \mathbb{R}$$

that satisfies the following for all  $\vec{x}, \vec{y}, \vec{z} \in V$  and  $\lambda, \mu \in \mathbb{R}$ :

1.  $\lambda\vec{x} + \mu\vec{y}, \vec{z} = \lambda(\vec{x}, \vec{z}) + \mu(\vec{y}, \vec{z})$
2.  $(\vec{x}, \vec{y}) = (\vec{y}, \vec{x})$
3.  $(\vec{x}, \vec{x}) \geq 0$ , with equality iff  $\vec{x} = \vec{0}$

A **real inner product space** is a real vector space equipped with an inner product. **Note:** basically a generalisation of dot prod.

A **complex inner product space** is a complex vector space equipped with an inner product. This is the exact same, but condition 2 uses  $(\vec{x}, \vec{y}) = (\vec{y}, \vec{x})$  where  $\bar{z}$  is the complex conjugate

#### Definition 5.1.5: Norm

In a real or complex inner product space, the **length** or **inner product norm** or **norm**  $\|\vec{v}\| \in \mathbb{R}$  of a vector  $\vec{v}$  is defined as the non-negative square root

$$\|\vec{v}\| = \sqrt{(\vec{v}, \vec{v})}$$

Vectors whose length are 1 are called **units**. Two vectors  $\vec{v}, \vec{w}$  are **orthogonal**, written  $\vec{v} \perp \vec{w}$ , iff  $(\vec{v}, \vec{w}) = 0$

The norm  $\|\cdot\|$  on an inner product space  $V$  satisfies, for any  $\vec{v}, \vec{w} \in V$  and scalar  $\lambda$ :

1.  $\|\vec{v}\| \geq 0$  with equality iff  $\vec{v} = \vec{0}$
2.  $\|\lambda\vec{v}\| = |\lambda|\|\vec{v}\|$
3.  $|\vec{v} + \vec{w}| \leq \|\vec{v}\| + \|\vec{w}\|$  (triangle inequality)



### Definition 5.1.7: Orthonormal Family

A family  $(\vec{v}_i)_{i \in I}$  for vectors from an inner product space is an **orthonormal family** if all the vectors  $\vec{v}_i$  have length 1 and if they are pairwise orthogonal to each other, which, if  $\delta_{i,j}$  is the **Kronecker delta** defined by

$$\delta_{i,j} = \begin{cases} 1 & i = j \\ 0 & \text{otherwise} \end{cases}$$

means that  $(\vec{v}_i, \vec{v}_j) = \delta_{ij}$ .

An orthonormal family that has a basis is an **orthonormal basis**

**Thm 5.1.10:** Every finite dimensional inner product space has an orthonormal basis

## 5.2 Orthogonal Complements and Projections

### Definition 5.2.1: Orthogonals to a Subset

Let  $V$  be an inner product space and let  $T \subseteq V$  be an arbitrary subset. Define

$$T^\perp = \{\vec{v} \in V : \vec{v} \perp \vec{t} \forall \vec{t} \in T\}$$

calling this set the **orthogonal** to  $T$

### Theorem 5.2.2: Complementary Orthogonals

Let  $V$  be an inner product space and let  $U$  be a finite dimensional subspace of  $V$ . Then  $U$  and  $U^\perp$  are complementary in the sense of 1.5.1. i.e.  $V = U \oplus U^\perp$

### Definition 5.2.3: Orthogonal Projection

Let  $U$  be a finite dimensional subspace of an inner product space  $V$ . The space  $U^\perp$  is the **orthogonal complement** to  $U$ . The **orthogonal projection from  $V$  onto  $U$**  is the map

$$\pi_U : V \rightarrow V$$

that sends  $\vec{v} = \vec{p} + \vec{r}$  to  $\vec{p}$

**Prop 5.2.4:** Let  $U$  be a finite dimensional subspace of an inner product space  $V$  and let  $\pi_U$  be the orthogonal projection from  $V$  onto  $U$

- $\pi_U$  is a linear mapping with  $\text{im}(\pi_U) = U$  and  $\ker(\pi_U) = U^\perp$
- If  $\{\vec{v}_1, \dots, \vec{v}_n\}$  is an orthonormal basis of  $U$ , then  $\pi_U$  is given by the following formula for all  $\vec{v} \in V$

$$\pi_U(\vec{v}) = \sum_{i=1}^n (\vec{v}, \vec{v}_i) \vec{v}_i$$

- $\pi_U^2 = \pi_U$ , that is,  $\pi_U$  is an idempotent

### Theorem 5.2.5: Cauchy-Shwarz Inequality

Let  $\vec{v}, \vec{w}$  be vectors in an inner product space. Then

$$|(\vec{v}, \vec{w})| \leq \|\vec{v}\| \|\vec{w}\|$$

with equality if and only if  $\vec{v}$  and  $\vec{w}$  are linearly dependent

### Theorem 5.2.7: Gram-Schmidt Process

Let  $\vec{v}_1, \dots, \vec{v}_k$  be linearly independent vectors in an inner product space  $V$ . Then there exists an orthonormal family  $\vec{w}_1, \dots, \vec{w}_k$  with the property that for all  $1 \leq i \leq k$ ,

$$\vec{w}_i \in \mathbb{R}_{>0} \vec{v}_i + \langle \vec{v}_{i-1}, \dots, \vec{v}_1 \rangle$$

TODO: write how to actually do the gram-schmidt process

## 5.3 Adjoints and Self-Adjoint

### Definition 5.3.1: Adjoints

Let  $V$  be an inner product space. Then two endomorphisms  $T, S : V \rightarrow V$  are called **adjoint** to one another if the following holds for all  $\vec{v}, \vec{w} \in V$ :

$$(T\vec{v}, \vec{w}) = (\vec{v}, S\vec{w})$$

In this case I will write  $S = T^*$  and call  $S$  the **adjoint** of  $T$

**Remark 5.3.2:** Any endomorphism has at most one adjoint.

### Theorem 5.3.4

Let  $V$  be a finite dimensional inner product space. Let  $T : V \rightarrow V$  be an endomorphism. Then  $T^*$  exists. That is, there is a unique linear mapping  $T^* : V \rightarrow V$  such that for all  $\vec{v}, \vec{w} \in V$ :

$$(T\vec{v}, \vec{w}) = (\vec{v}, T^*\vec{w})$$

### Definition 5.3.5: Self Adjoints

An endomorphism of an inner product space  $T : V \rightarrow V$  is **self-adjoint** if it equals its own adjoint, i.e. if  $T^* = T$

### Theorem 5.3.7: Self-Adjoint Theorem bank

Let  $T : V \rightarrow V$  be a self-adjoint linear mapping on an inner product space  $V$

- Every eigenvalue of  $T$  is real
- If  $\lambda$  and  $\mu$  are distinct eigenvalues of  $T$  with corresponding eigenvectors  $\vec{v}$  and  $\vec{w}$ , then  $(\vec{v}, \vec{w}) = 0$
- $T$  has an eigenvalue

### Definition 5.3.11: Orthogonal Matrices

An **Orthogonal matrix** is an  $(n \times n)$ -matrix  $P$  with real entries such that  $P^T P = I_n$ , or in other words such that  $P^{-1} = P^T$

### Definition 4.3.14: Complex Matrices

A **hermitian matrix** is one that is self-adjoint in  $\mathbb{C}$ , or in other words one where  $A = \overline{A}^T$  holds

An **unitary matrix** is an  $(n \times n)$ -matrix  $P$  with complex entries such that  $\overline{P}^T P = I_n$ , or such that  $P^{-1} = \overline{P}^T$

### Theorem 5.3.9: Spectral Theorems

#### 5.3.9: The Spectral Theorem for Self-Adjoint Endomorphisms

Let  $V$  be a finite dimensional inner product space and let  $T : V \rightarrow V$  be a self-adjoint linear mapping. Then  $V$  has an orthonormal basis consisting of eigenvalues of  $T$ .

#### 5.3.11: The Spectral Theorem for Real Symmetric Matrices

Let  $A$  be a real  $(n \times n)$ -symmetric matrix. Then there is an  $(n \times n)$ -orthogonal matrix  $P$  such that

$$P^T A P = P^{-1} A P = \text{diag}(\lambda_1, \dots, \lambda_n)$$

where  $\lambda_1, \dots, \lambda_n$  are the (necessarily real) eigenvalues of  $A$ , repeated according to their multiplicity as roots of  $\chi_A$

#### 5.3.15: The Spectral Theorem for Hermitian Matrices

Let  $A$  be a  $(n \times n)$ -hermitian matrix. Then there is an  $(n \times n)$ -unitary matrix  $P$  such that

$$\overline{P}^T A P = P^{-1} A P = \text{diag}(\lambda_1, \dots, \lambda_n)$$

where  $\lambda_1, \dots, \lambda_n$  are the (necessarily real) eigenvalues of  $A$ , repeated according to their multiplicity as roots of  $\chi_A$

## 6 Jordan Normal Form

### 6.1 Motivation

no time for motivation over here

### 6.2 The Jordan Normal Form

#### Definition 6.2.1: Jordan Blocks

Given an integer  $r \geq 1$  define an  $(r \times r)$ -matrix  $J(r)$  called the **nilpotent Jordan block of size  $r$** , by the rule  $J(r)_{ij} = 1$  for  $j = i + 1$  AND  $J(r)_{ij} = 0$  otherwise  
In particular,  $J(1)$  is a  $(1 \times 1)$ -matrix whose only entry is zero.

Given an integer  $r \geq 1$  and a scalar  $\lambda \in F$ , define an  $(r \times r)$ -matrix  $J(r, \lambda)$  called the **Jordan block of size  $r$  and eigenvalue  $\lambda$**  by the rule

$$J(r, \lambda) = \lambda I_r + J(r) = D + N$$

with  $\lambda I_r = \text{diag}(\lambda, \lambda, \dots, \lambda) = D$  diagonal and  $J(r) = N$  nilpotent such that  $DN = ND$

### Theorem 6.2.2: Jordan Normal Form

Let  $F$  be an algebraically closed field. Let  $V$  be a finite dimensional vector space and let  $\phi : V \rightarrow V$  be an endomorphism of  $V$  with characteristic polynomial

$$\chi_\phi(x) = (x - \lambda_1)^{a_1} (x - \lambda_2)^{a_2} \dots (x - \lambda_s)^{a_s} \in F[x], a_i \geq 1, \sum_{i=1}^s a_i = n$$

For distinct  $\lambda_1, \lambda_2, \dots, \lambda_s \in F$ . Then there exists an ordered basis  $\mathcal{B}$  of  $V$  such that the matrix of  $\phi$  with respect to the block  $\mathcal{B}$  is block diagonal with Jordan blocks on the diagonal,  $\mathcal{B}[\phi]_{\mathcal{B}}$

$$= \text{diag}(J(r_{11}, \lambda_1), \dots, J(r_{1m_1}, \lambda_1), J(r_{21}, \lambda_2), \dots, J(r_{sm_s}, \lambda_s))$$

with  $r_{11}, \dots, r_{1m_1}, r_{21}, \dots, r_{sm_s} \geq 1$  such that

$$a_i = r_{i1} + r_{i2} + \dots + r_{im_i} \quad (1 \leq i \leq s)$$

### Theorem 6.3.1: Bézout's identity for polynomials

For a characteristic polynomial

$$\chi_\phi(x) = \prod_{i=1}^s (x - \lambda_i)^{a_i} \in F[x]$$

where each  $a_i$  is a positive integer,  $\lambda_i \neq \lambda_j$  for  $i \neq j$ , and  $\lambda_i$  are e.v.s of  $\phi$ . For each  $1 \leq j \leq s$  define

$$P_j(x) = \prod_{\substack{i=1 \\ i \neq j}}^s (x - \lambda_i)^{a_i}$$

There exists polynomials  $Q_j(x) \in F[x]$  such that

$$\sum_{j=1}^s P_j(x) Q_j(x) = 1$$

### Definition 6.3.2: Generalised Eigenspace

The **generalised eigenspace** of  $\phi$  with eigenvalue  $\lambda_i$ ,  $E^{\text{gen}}(\lambda_i, \phi)$  is the following subspace of  $V$ :

$$E^{\text{gen}}(\lambda_i, \phi) = \{\vec{v} \in V \mid (\phi - \lambda_i \text{id}_V)^{a_i}(\vec{v}) = \vec{0}\}$$

The dimension of  $E^{\text{gen}}(\lambda_i, \phi)$  is called the **algebraic multiplicity of  $\phi$  with eigenvalue  $\lambda_i$**  while the dimension of the eigenspace  $E(\lambda_i, \phi)$  is called the **geometric multiplicity of  $\phi$  with eigenvalue  $\lambda$**

**Remark 6.3.4:** The actual eigenspace is defined by

$$E(\lambda_i, \phi) = \{\vec{v} \in V \mid (\phi - \lambda_i \text{id}_V)(\vec{v}) = \vec{0}\}$$

$E^{\text{gen}}(\lambda_i, \phi) \subseteq E^{\text{gen}}(\lambda_i, \phi)$ , or the algebraic multiplicity of any e.v. must be greater or equal to the corresponding geometric multiplicity

### Definition 6.3.4: Stable subsets

Let  $f : X \rightarrow X$  be a mapping from a set  $X$  to itself. A subset  $Y \subseteq X$  is **stable under  $f$**  precisely when  $f(Y) \subseteq Y$ , that is if  $y \in Y$  then  $f(y) \in Y$ .

### Theorem 6.3.5: Direct Sum Composition

For each  $1 \leq i \leq s$ , let

$$\mathcal{B}_i = \{\vec{v}_{ij} \in V \mid 1 \leq j \leq a_i\}$$

be a basis of  $E^{\text{gen}}(\lambda_i, \phi)$ , where  $a_i$  is the algebraic multiplicity of  $\phi$  with eigenvalue  $\lambda_i$  s.t.  $\sum_{i=1}^s a_i = n$  is the dimension of  $V$ .

1. Each  $E^{\text{gen}}(\lambda_i, \phi)$  is stable under  $\phi$
2. For each  $\vec{v} \in V$  there exist unique  $\vec{v}_i \in E^{\text{gen}}(\lambda_i, \phi)$  such that  $\vec{v} = \sum_{i=1}^s \vec{v}_i$ . In other words, there is a direct sum decomposition

$$V = \bigoplus_{i=1}^s E^{\text{gen}}(\lambda_i, \phi)$$

with  $\phi$  restricting to endomorphisms of the summands

$$\phi_i = \phi| : E^{\text{gen}}(\lambda_i, \phi) \rightarrow E^{\text{gen}}(\lambda_i, \phi)$$

3. Then

$$\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \dots \cup \mathcal{B}_s = \{\vec{v}_{ij} \mid 1 \leq i \leq s, 1 \leq j \leq a_i\}$$

is a basis of  $V$ . The matrix of the endomorphism  $\phi$  w.r.t. this basis is given by the block diagonal matrix

$$\mathcal{B}[\phi]_{\mathcal{B}} = \left( \begin{array}{c|c|c|c} B_1 & 0 & 0 & 0 \\ \hline 0 & B_2 & 0 & 0 \\ \hline 0 & 0 & \ddots & 0 \\ \hline 0 & 0 & 0 & B_s \end{array} \right) \in \text{Mat}(n; F)$$

with  $B_i = \mathcal{B}_i[\phi_i]_{\mathcal{B}_i} \in \text{Mat}(a_i; F)$

### Theorem 6.3: JNF Theorem Bank

**6.3.6:** For each  $i$ , define a linear mapping

$$\psi_i : \frac{W_i}{W_{i-1}} \rightarrow \frac{W_{i-1}}{W_{i-2}}$$

by  $\psi_i(\vec{w} + W_{i-1}) = \psi(\vec{w}) + W_{i-2}$  for  $\vec{w} \in W_i$ . Then  $\psi_i$  is well-defined and injective

**6.3.7:** Let  $f : X \rightarrow Y$  be an injective linear mapping between the  $F$ -vector spaces  $X$  and  $Y$ . If  $\{\vec{x}_1, \dots, \vec{x}_t\}$  is a linearly independent set in  $X$ , then  $\{f(\vec{x}_1), \dots, f(\vec{x}_t)\}$  is a linearly independent set in  $Y$

**6.3.8:** The set of elements  $\{\vec{v}_{j,k} : 1 \leq j \leq m, 1 \leq k \leq d_j\}$  constructed in the next algorithm is a basis for  $W$

**6.3.9:** Let  $\mathcal{B}$  be the ordered basis of  $W$  -  $\{\vec{v}_{j,k} : 1 \leq j \leq m, 1 \leq k \leq d_j\}$ . Then  $\mathcal{B}[\psi]_{\mathcal{B}}$  =

$$\text{diag} \underbrace{J(m), \dots, J(m), J(m-1), \dots, J(m-1)}_{d_m \text{ times}}, \dots, \underbrace{J(1), \dots, J(1)}_{d_1 - d_2 \text{ times}}$$

where  $J(r)$  denotes the nilpotent Jordan block of size  $r$

### Theorem 6.3: JNF Basis Algorithm

Algorithm to construct a basis for each  $W_i/W_{i-1}$ :

- Choose an arbitrary basis for  $W_m/W_{m-1}$ , say  $\{v_{m,1} + W_{m-1}, \vec{v}_{m,2} + W_{m-1}, \dots, \vec{v}_m, d_m + W_{m-1}\}$
- Since  $\psi_m : W_m/W_{m-1} \rightarrow W_{m-1}/W_{m-2}$  is injective by 6.3.6, 6.3.7 proves that  $\{\psi(\vec{v}_{m,1}) + W_{m-2}, \psi(\vec{v}_{m,2}) + W_{m-2}, \dots, \psi(\vec{v}_m, d_m + W_{m-2})\}$  is a linearly independent set in  $W_{m-1}/W_{m-2}$ . Set  $\vec{v}_{m-1,i} = \psi(\vec{v}_{m,i})$  for  $1 \leq i \leq d_m$
- Choose vectors  $\{\vec{v}_{m-1,i} : d_m + 1 \leq i \leq d_{m-1}\}$  so that  $\{\vec{v}_{m-1,i} + W_{m-i-1} : 1 \leq k \leq d_{m-i}\}$  is a basis of  $W_{m-1}/W_{m-2}$
- Repeat!

## 6.3 PageRank, again

### Theorem 6.5.1

If  $M \in \text{Mat}(n; \mathbb{R})$  is a Markov matrix with all positive entries, consider  $M$  as a complex matrix whose entries just happen to be real. If  $\lambda \in \mathbb{C}$  is an eigenvalue of  $M$  then either  $\lambda = 1$  or  $|\lambda| < 1$

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