

1 Algebra

Note: Any reference numbers are to the lecture notes

Functions and Symmetries

Definition 0.1.1 Functions

A function  $f : X \rightarrow Y$  is called

- **injective** if  $f(x_1) = f(x_2) \implies x_1 = x_2$ .  $f$  is said to be **one-to-one** on  $X$
- **surjective** if for every  $y \in Y, \exists x \in X$  s.t.  $f(x) = y$ .  $f$  is said to take  $X$  **onto**  $Y$
- **bijective** if it is both injective and surjective

Definition 1.1.3 Graph Isomorphisms

An **isomorphism** between two graphs is a *bijection* between them that preserves all edges. More precisely, if  $\Gamma_1$  and  $\Gamma_2$  are graphs, with sets of vertices  $V_1$  and  $V_2$  respectively, then an isomorphism from  $\Gamma_1$  and  $\Gamma_2$  is a bijection

$$f : V_1 \rightarrow V_2$$

such that  $f(v_1)$  and  $f(v_2)$  are joined by an edge if and only if  $v_1$  and  $v_2$  are also joined by an edge. We say that  $\Gamma_1$  and  $\Gamma_2$  are *isomorphic* if there exists an isomorphism  $f : \Gamma_1 \rightarrow \Gamma_2$

Definition 1.1.9 Symmetry

A **symmetry** of a graph is an *isomorphism* from the graph to itself, i.e. if the set of vertices is  $V$ , then the symmetry is a bijection  $f : V \rightarrow V$  that preserves edges. That is, a symmetry is a bijection  $f : V \rightarrow V$  such that  $f(v_1)$  and  $f(v_2)$  are joined by an edge if and only if  $v_1$  and  $v_2$  are joined by an edge.

Groups

Definition 1.2.3 Groups

For an operation  $*$ , We say a non-empty set  $G$  is a **group** under  $*$  if the following four axioms hold:

- **G1 - Closure:**  $*$  is a binary operation on  $G$ , that is  $a*b \in G$  for all  $a, b \in G$ .
- **G2 - Associativity:**  $(a*b)*c = a*(b*c)$  for all  $a, b, c \in G$
- **G3 - Identity:** There exists an *identity* element of  $G$  such that  $e*g = g*e = g$  for all  $g \in G$ .
- **G4 - Inverse:** Every element  $g \in G$  has an *inverse*  $g^{-1}$  such that  $g*g^{-1} = g^{-1}*g = e$

Definition 1.2.6 Abelian Group

The definition of a group doesn't require that  $a*b = b*a$ . We say that a group is **abelian** or **commutative** if  $a*b = b*a$  for every  $a, b \in G$ . We say that  $a$  *commutes* with  $b$ , or that  $a$  and  $b$  *commute*

Subgroups

Definition 2.1.1 Subgroups

Let  $G$  be a group. We say that a non-empty subset  $H$  of  $G$  is a **subgroup** of  $G$  if  $H$  itself is a group (under the operation from  $G$ ). We write  $H \leq G$  if  $H$  is a subgroup of  $G$ . If  $H \neq G$ , we write  $H < G$  and say  $H$  is a proper subgroup

Theorem 2.1.3: Subgroup Test

$H \subseteq G$  is a subgroup of  $G$  if and only if:

- **S1:**  $H$  is not empty
- **S2:** If  $h, k \in H$  then  $h*k \in H$
- **S3:** If  $h \in H$  then  $h^{-1} \in H$

Alternative test for subgroups:

- $\widetilde{S1}$ :  $H$  is not empty.
- $\widetilde{S2}$ : If  $h, k \in H$  then  $h*k^{-1} \in H$

Definition 2.2.4 Order of an Element

Let  $G$  be a group and  $g \in G$ . Then the **order**  $o(g)$  of  $g$  is the *least* natural number  $n$  such that

$$g^n = e$$

If no such  $n$  exists, we say that  $g$  has infinite order

Definition 2.2.3 Order of a Group

The **order** of a finite group, written  $|G|$ , is the number of elements in  $G$ . If  $G$  is infinite we say that  $|G| = \infty$ , or the order of  $G$  is infinite.

Theorem 2.2.6: Order of a Finite Group

In a finite group, every element has finite order. If  $g$  is an element of a finite group  $G$ , then there exists  $k \in \mathbb{N}$  such that  $g^k = g^{-1}$

Definition 2.2.8 Generating Subset

Let  $G$  be a group and let  $g \in G$  be an element. We define the subset

$$\langle g \rangle := \{g^k \mid k \in \mathbb{Z}\} = \{\dots, g^{-2}, g^{-1}, e, g, g^2, \dots\}$$

Note that if  $G$  is finite, then by 2.2.6  $\langle g \rangle$  is finite, and we can think of  $\langle g \rangle$  as

$$\langle g \rangle = \{e, g, \dots, g^{o(g)-1}\}$$

Definition 2.2.10 Cyclic Subgroup

A subgroup  $H \leq G$  is **cyclic** if  $H = \langle h \rangle$  for some  $h \in H$ . In this case, we say that  $H$  is the *cyclic subgroup generated by  $h$* . If  $G = \langle g \rangle$  for some  $g \in G$ , then we say that the group  $G$  is *cyclic*, and that  $g$  is a *generator*.

Remark 2.2.12 - 16: Consequences of Cyclic groups

- **2.2.12** If  $g \in G$ , then  $o(g) = |\langle g \rangle|$
- **2.2.13:** If  $G$  is cyclic, then  $G$  is abelian.
- **2.2.14:** Let  $G$  be a finite group. Then  $G$  is cyclic  $\iff G$  has an element of order  $|G|$
- **2.2.15:** Let  $G$  be a cyclic group and let  $H$  be a subgroup of  $G$ . Then  $H$  is cyclic.
- **2.2.16:** Let  $m, n \in \mathbb{N}$ , let  $G = \langle g \rangle$  be a cyclic group of order  $m$  and  $H = \langle h \rangle$  be a cyclic group of order  $n$ . Then  $G \times H$  cyclic  $\iff m$  and  $n$  are coprime ( $\gcd(m,n) = 1$ )

Cosets and Lagrange

Definition 2.3.2 Relation

Let  $X$  be a set, and  $R$  a subset of  $X \times X$ ; thus  $R$  consists of some ordered pairs  $(s, t)$  with  $s, t \in X$ . If  $(s, t) \in R$  we write  $s \sim t$  and say "s is related to t". We call  $\sim$  a **relation** on  $X$ .

Definition 2.3.2 Equivalence Relation

- **Reflexive:**  $x \sim x$  for all  $x \in X$
  - **Symmetric:**  $x \sim y$  implies that  $y \sim x$  for all  $x, y \in X$
  - **Transitive:**  $x \sim y$  and  $y \sim z$  implies that  $x \sim z$  for all  $x, y, z \in X$
- A relation  $\sim$  is called an **equivalence relation** on  $X$  if it satisfies the following three axioms:

Definition 2.3.4 Coset

Let  $H \leq G$  and let  $g \in G$ . Then a *left coset* of  $H$  in  $G$  is a subset of  $G$  of the form  $gH$ , for some  $g \in G$ . We denote the set of left cosets of  $H$  in  $G$  by  $G/H$  (Notation) Let  $A, B$  be subgroups of a group  $G$  and let  $g \in G$ . Then

$$AB := \{ab \mid a \in A, b \in B\}, \quad gA := \{gA \mid a \in A\}$$

Theorem 2.4.2: Lagrange's Theorem

Suppose that  $G$  is a finite group.

- If  $H \leq G$ , then  $|H|$  divides  $|G|$
- Let  $g \in G$ . Then  $o(g)$  divides  $|G|$
- For all  $g \in G$ , we have that  $g^{|G|} = e$

### Theorem 2.3.8: Coset Rules

Let  $H \leq G$

- For all  $h \in H$ ,  $hH = H$ . In particular  $eH = H$
- For  $g_1, g_2 \in G$ , the following are equivalent
  - $g_1H = g_2H$
  - there exists  $h \in H$  such that  $g_2 = g_1h$
  - $g_2 \in g_1H$
- For  $g_1, g_2 \in G$ , define  $g_1 \sim g_2$  if and only if  $g_1H = g_2H$ . Then  $\sim$  defines an equivalence relation on  $G$ .

### Theorem 2.4.4: Index of a Subgroup

The **index** of  $H \leq G$  is defined as the number of *distinct* left cosets of  $H$  in  $G$ , which by Lagrange's is  $|G/H| = \frac{|G|}{|H|}$

### Remark 2.4.6 - 8: Consequences of Lagrange

- **2.4.6:** Suppose that  $G$  is a group with  $|G| = p$ , where  $p$  is prime. Then  $G$  is a cyclic group
- **2.4.7:** Suppose that  $G$  is a group with  $|G| < 6$ . Then  $G$  is abelian
- **2.4.8:** If  $p$  is a prime and  $a \in \mathbb{Z}$ , then  $a^p \equiv a \pmod{p}$

### Remark 3.1.5: Consequences of Homomorphisms

Let  $\phi : G \rightarrow H$  be a group homomorphism. Then

- $\phi(e_G) = e_H$
- $\phi(g^k) = (\phi(g))^k$  and  $\phi(g^{-1}) = (\phi(g))^{-1}$  for all  $g \in G$
- If  $\phi$  is injective, the order of  $g \in G$  equals the order of  $\phi(g) \in H$ .

### Definition 3.1.7 Normal Subgroup

A subgroup  $N \leq G$  is **normal** if the left and right cosets of  $N$  are equal, i.e.  $gN = Ng$  for all  $g \in G$ . If  $N$  is a normal subgroup of  $G$ , we write  $N \triangleleft G$ . Kernels of homomorphisms are always normal subgroups

### Definition 3.1.6 Image and Kernel of a Group

Let  $\phi : G \rightarrow H$  be a group homomorphism.

- The **image** of  $\phi$  is defined to be
$$\text{im } \phi := \{h \in H \mid h = \phi(g) \text{ for some } g \in G\}$$
- The **kernel** of  $\phi$  is defined to be
$$\text{ker } \phi := \{g \in G \mid \phi(g) = e_H\}$$

Note:  $\text{im } \phi$  is a subgroup of  $H$  and  $\text{ker } \phi$  is a subgroup of  $G$

### Theorem 3.2.1: Product Isomorphisms

Let  $H, K \leq G$  be subgroups with  $H \cap K = \{e\}$ .

- The map  $\phi : H \times K \rightarrow HK$  given by  $\phi : (h, k) \rightarrow hk$  is bijective
- If every element of  $H$  commutes with every element of  $K$  when multiplied in  $G$  (i.e.  $hk = kh \ \forall h \in H, k \in K$ ), then  $HK$  is a subgroup of  $G$ , and it is isomorphic to  $H \times K$  via  $\phi$

### Theorem 3.2.3: Size of Product Group

Let  $H, K \leq G$  be finite subgroups of a group  $G$  such that  $H \cap K = \{e\}$ . Then  $|HK| = |H| \times |K|$ .

## Group Actions

### Definition 4.1.1 Group Action

Let  $(G, *)$  be a group, and let  $X$  be a nonempty set. Then a (left) **action** of  $G$  on  $X$  is a map

$$G \times X \rightarrow X$$

written  $(g, x) \mapsto g \cdot x$ , such that

$$g_1 \cdot (g_2 \cdot x) = (g_1 * g_2) \cdot x \quad \text{and} \quad e \cdot x = x$$

for all  $g_1, g_2 \in G$  and all  $x \in X$ .

### Definition 4.2.1 Orbit, Stabilizer, and Fix

- Suppose that  $G$  acts on  $X$ . Then the set

$$N := \{g \in G \mid g \cdot x = x \ \forall x \in X\}$$

is a subgroup of  $G$ , and is called the **kernel** of the action. If  $N = \{e\}$ , then we say the action is **faithful**

- For every  $x$  in  $X$ , the **orbit** of  $x$  is a subset of  $X$  defined by

$$\text{Orb}_G(x) = \{g \cdot x \mid g \in G\}$$

- For every  $x$  in  $X$ , the **stabilizer** of  $x$  is a subgroup of  $G$  defined by

$$\text{Stab}_G(x) = \{g \in G \mid g \cdot x = x\}$$

- For every  $g$  in  $G$ , the **fix** of  $g$  is a subset of  $X$  defined by

$$\text{Fix}(g) = \{x \in X \mid g \cdot x = x\}$$

- Let  $G$  act on  $X$ , let  $x \in X$  and set  $H := \text{Stab}_G(x)$ . If  $y = g \cdot x$  for some  $g \in G$ , then

$$\text{send}_x(y) = gH$$

- Let  $h \in H$  and  $g \in G := X$ . The **conjugate action** is:

$$h \cdot g := hgh^{-1}$$

- An action of  $G$  on  $X$  is **transitive** if for all  $x, y \in X$  there exists  $g \in G$  such that  $y = g \cdot x$ . Equivalently,  $X$  is a single orbit under  $G$

- We define the **centre** of a group  $G$  to be

$$C(G) := \{g \in G \mid hg = gh \text{ for all } h \in G\}$$

The **centralizer** of  $g$  in  $G$  is defined as

$$G(g) := \{h \in G \mid gh = hg\}$$

### Theorem 4.2.5: Orbit Equivalence

Let  $G$  act on  $X$ . Then

$$x \sim y \iff y = g \cdot x \text{ for some } g \in G$$

defines an equivalence relation on  $X$ . The equivalence classes are the orbits of  $G$ . Thus when  $G$  acts on  $X$ , we obtain a partition of  $X$  into orbits

### Theorem 4.3.1: Orbit-Stabilizer Theorem

Suppose  $G$  is a finite group acting on a set  $X$ , and let  $x \in X$ . Then  $|\text{Orb}_G(x)| \times |\text{Stab}_G(x)| = |G|$ , or in words:

$$\text{size of orbit} \times \text{size of stabilizer} = \text{order of group}$$

### Theorem 4.3.4: Orbit Send Theorem

Let  $G$  act on  $X$ , let  $x \in X$ , and let set  $H := \text{Stab}_G(x)$ . Then

$$\text{send}_x : \text{Orb}_G(x) \rightarrow G/H \text{ which sends } y \mapsto \text{send}_x(y)$$

### Theorem 4.4.2: Cauchy's Theorem

Let  $G$  be a group,  $p$  be prime. If  $p$  divides  $|G|$ , then  $G$  contains an element of order  $p$

## Homomorphisms and Isomorphisms

### Definition 3.1.1 Group Homomorphism

Let  $(G, *), (H, \circ)$  be groups. A map  $\phi : G \rightarrow H$  is called a **homomorphism** if

$$\phi(x * y) = \phi(x) \circ \phi(y) \quad \text{for all } x, y \in G$$

Note that the product on the left is formed using  $*$ , while the product on the right is formed using  $\circ$

### Definition 3.1.2 Group Isomorphism

A group homomorphism  $\phi : G \rightarrow H$  that is also a bijection is called an **isomorphism** of groups. In this case we say that  $G$  and  $H$  are *isomorphic* and we write  $G \cong H$ . An isomorphism  $G \rightarrow G$  is called an **automorphism** of  $G$ .

### Theorem J.J 9.5 Thm 6: Cyclic Isomorphisms

All finite cyclic groups with the same order are *isomorphic* to each other. Therefore, cyclic groups of order  $n$  are isomorphic to  $(\mathbb{Z}_n, +)$

All infinite cyclic groups are *isomorphic* to each other. Therefore, each cyclic group of infinite order is isomorphic to  $(\mathbb{Z}, +)$

## 2 Analysis

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### Real Numbers and Bounds

#### Definition 1.1 The Real Numbers

$\mathbb{R}$  is defined as the set of real numbers. It has two operations  $+$  and  $*$ , and it is a field, i.e. satisfies group axioms for both operations, in addition to the Distributive law:

$$a \cdot (b + c) = a \cdot b + a \cdot c$$

The set of real numbers is also ordered, i.e. there is a relation  $<$  which satisfies pretty much what you think it does  
Finally, the set of real numbers is complete, i.e. there are no gaps between any numbers.

#### Definition 1.2.3 Triangle Inequality

The most important property of the absolute value  $|a|$ :

$$|a + b| \leq |a| + |b| \quad \text{and} \quad ||a| - |b|| \leq |a - b|$$

#### Definition 1.3.2 Suprema and Bounds

Let  $E \subset \mathbb{R}$  be nonempty

- The set  $E$  is said to be bounded above if there is  $M \in \mathbb{R}$  such that  $a \leq M$  for all  $a \in E$
- A real number  $M$  is called an upper bound of the set  $E$  if  $a \leq M$  for all  $a \in E$
- A real number  $s$  is called the **supremum** of the set  $E$  if
  - $s$  is an upper bound of  $E$
  - $s \leq M$  for all upper bounds  $M$  of the set  $E$

If a number  $s$  exists, we shall say that  $E$  has a supremum and write  $s = \sup E$

If the supremum  $s$  exists, then  $s$  is the least upper bound of the set  $E$ . The supremum is also unique if it exists.

If the same properties as a supremum apply but in the other direction, a number  $s$  is instead called the **infimum** of the set  $E$ . Infimum and Supremum are related via the reflection principle:

- Set  $E$  has a supremum if and only if the set  $-E$  has an infimum. Also  $\inf(-E) = -\sup(E)$
- Set  $E$  has an infimum if and only if the set  $-E$  has a supremum. Also  $\sup(-E) = -\inf(E)$

#### Theorem 1.3.5: Suprema Approximation Property

If the set  $E \subset \mathbb{R}$  has a supremum then for any positive number  $\epsilon > 0$  there exists  $a \in E$  such that

$$\sup E - \epsilon < a \leq \sup E$$

#### Theorem 1.3.7: Archimedean Principle

Given positive real numbers  $a, b \in \mathbb{R}$  there is an integer  $n \in \mathbb{N}$  such that  $b < na$

#### Definition 1.5.2 Countability

Let  $E$  be a set.  $E$  is said to be:

- **Finite** if either  $E = \emptyset$ , or there is an integer  $n \in \mathbb{N}$  and a bijection  $f : \{1, 2, 3, \dots, n\} \rightarrow E$ . We say that the set  $E$  has  $n$  elements
- **Countable** if there is a bijective function  $f : \mathbb{N} \rightarrow E$
- **At most countable** if  $E$  is finite or countable
- **Uncountable** if  $E$  is neither finite nor countable

Additionally, a nonempty set  $E$  is at most countable if and only if there is a surjective function  $f : \mathbb{N} \rightarrow E$

### Sequences and Series

#### Definition 2.1.1 Convergence of a Sequence

A sequence of real numbers  $(x_n)$  is said to converge to a real number  $a$  if for every  $\epsilon > 0$  there is a  $N \in \mathbb{N}$  where for every  $n \geq N$  we have that  $|x_n - a| < \epsilon$

For a sequence  $(x_n)$ , we write  $\lim x_n = +\infty$  if for each  $M > 0$  there is a number  $N$  such that  $n > N$  implies  $x_n > M$ . Reverse every inequality for  $-\infty$  case.

#### Definition 2.1.9 Bounds of Sequences

Let  $(x_n)$  be a sequence of real numbers.

- $(x_n)_{n \in \mathbb{N}}$  is said to be **bounded above** if  $x_n \leq M$  for some  $M \in \mathbb{R}$  and all  $n \in \mathbb{N}$
- $(x_n)_{n \in \mathbb{N}}$  is said to be **bounded below** if  $x_n \geq m$  for some  $m \in \mathbb{R}$  and all  $n \in \mathbb{N}$
- $(x_n)_{n \in \mathbb{N}}$  is said to be **bounded** if it is both bounded above and below

#### Remark 2.2.1 - ?: Limit Theorems

- Let  $E \subset \mathbb{R}$ . If  $E$  has a finite supremum then there is a sequence  $(x_n)$  with each  $x_n \in E$  such that  $x_n \rightarrow \sup E$  as  $n \rightarrow \infty$ . The same goes for a finite infimum
- **Comparison Theorem for sequences:** Suppose that  $(x_n)$ ,  $(y_n)$  are real sequences. If both  $\lim_{n \rightarrow \infty} x_n$ ,  $\lim_{n \rightarrow \infty} y_n$  exist and belong to  $\mathbb{R}^*$ , and if  $x_n \leq y_n$  for all  $n \geq N$  for some  $N \in \mathbb{N}$ , then  $\lim_{n \rightarrow \infty} x_n \leq \lim_{n \rightarrow \infty} y_n$

#### Definition 2.3.1 Monotone Sequences

Let  $(s_n)$  be a sequence of real numbers.

- $(s_n)$  is said to be increasing if  $s_1 \leq s_2 \leq s_3 \leq \dots$ , and strictly increasing if  $s_1 < s_2 < s_3 < \dots$
- $(s_n)$  is said to be decreasing if  $s_1 \geq s_2 \geq s_3 \geq \dots$ , and strictly decreasing if  $s_1 > s_2 > s_3 > \dots$
- $(s_n)$  is said to be monotone if it is either increasing or decreasing

#### Theorem lots: Top 10 Limit Theorems

- **Squeeze Theorem (for sequences):** Suppose that  $(x_n)$ ,  $(y_n)$ , and  $(w_n)$  are real sequences
  - If both  $x_n \rightarrow a$  and  $y_n \rightarrow a$  as  $n \rightarrow \infty$ , and if  $x_n \leq w_n \leq y_n$  for all  $n \geq N_0$ , then  $w_n \rightarrow a$  as  $n \rightarrow \infty$
  - If  $x_n \rightarrow 0$  and  $(y_n)$  is bounded,  $x_n y_n \rightarrow 0$  as  $n \rightarrow \infty$
- **Divergence Test:**
  - If  $\sum_{n=1}^{\infty} a_n$  converges, then  $a_n \rightarrow 0$ .
  - If  $(a_n)_{n \in \mathbb{N}}$  doesn't converge to 0, then  $\sum_{n=1}^{\infty} a_n$  diverges. Be careful that the converse isn't true.
- **Comparison Test:** Let  $(a_n)_{n \in \mathbb{N}}$  and  $(b_n)_{n \in \mathbb{N}}$  be two sequences such that  $0 \leq a_n \leq b_n$  for all  $n$ .
  - If  $\sum_n b_n$  converges, then  $\sum_n a_n$  converges as well.
  - If  $\sum_n a_n$  diverges, then  $\sum_n b_n$  diverges as well.
- **Limit Comparison Test:** Let  $(a_n)_{n \in \mathbb{N}}$  and  $(b_n)_{n \in \mathbb{N}}$  be two real sequences with  $a_n \geq 0$  and  $b_n > 0$  for all  $n$ . Assume that  $a_n/b_n \rightarrow L$  for some  $L \in (0, \infty)$ . Then,  $\sum_{n=1}^{\infty} a_n$  converges iff  $\sum_{n=1}^{\infty} b_n$  converges.
  - If  $L = 0$  and  $\sum_{n=1}^{\infty} b_n$  converges then  $\sum_{n=1}^{\infty} a_n$  converges
  - If  $L = \infty$  and  $\sum_{n=1}^{\infty} a_n$  diverges then  $\sum_{n=1}^{\infty} b_n$  diverges
- **Root Test:** Let  $\sum_{n=0}^{\infty} a_n$  be a series with non-negative terms such that  $\sqrt[n]{a_n} \rightarrow L$  where  $0 \leq L \leq +\infty$ .
  - If  $0 \leq L < 1$  then the series  $\sum_{n=0}^{\infty} a_n$  converges.
  - If  $L > 1$  then the series  $\sum_{n=0}^{\infty} a_n$  diverges.
  - If  $L = 1$ , the series may or may not converge
- **Ratio Test:** Let  $\sum_{n=0}^{\infty} a_n$  be a series with positive terms such that  $(a_{n+1})/(a_n) \rightarrow L$ , where  $0 \leq L \leq +\infty$ .
  - If  $0 \leq L < 1$  then the series  $\sum_{n=0}^{\infty} a_n$  converges.
  - If  $L > 1$  then the series  $\sum_{n=0}^{\infty} a_n$  diverges.
  - If  $L = 1$  then compare to  $p$  series
- **Cauchy's Condensation Test:** Let  $(a_n)_{n \in \mathbb{N}}$  be a decreasing sequence with non-negative terms. Then the following are equivalent:
  - The series  $\sum_{n=1}^{\infty} a_n$  converges
  - The series  $\sum_{n=0}^{\infty} 2^n a_{2^n}$  converges.
- **Alternating Series Test:** Let  $(b_n)_{n \in \mathbb{N}}$  be a decreasing sequence of non-negative real numbers that converges to zero. Then the series  $\sum_{n=1}^{\infty} (-1)^{n-1} b_n$  converges.
- **Monotone Convergence Theorem:** If a sequence of real numbers  $(s_n)$  is increasing and bounded above, or decreasing and bounded below, then  $(s_n)$  is convergent (and converges to the sup/inf of the set  $\{x_n \mid n \in \mathbb{N}\}$  respectively).
- **Geometric Series Test:** Assume  $a, r \in \mathbb{R}, a, r \neq 0$ . Then

$$\sum_{n=1}^{\infty} a \cdot r^n = \begin{cases} \frac{a}{1-r} & \text{if } |r| < 1 \\ \text{diverges} & \text{if } |r| \geq 1 \end{cases}$$

Notice that  $a$  is always the first term in the series, and  $r$  is the *common ratio*

Continuity and Functional Limits

Definition 4.1.1 Continuity

Let  $f$  be a real-valued function whose domain is a subset of  $\mathbb{R}$ . The function  $f$  is **continuous** at  $x_0$  in  $\text{dom}(f)$  if, for every sequence  $(x_n)$  in  $\text{dom}(f)$  converging to  $x_0$ , we have

$$\lim_{n \rightarrow \infty} f(x_n) = f(x_0)$$

If  $f$  is continuous at each  $a \in S \subseteq \text{dom}(f)$  and then we say that  $f$  is continuous on  $S$ . If  $f$  is continuous on  $\text{dom}(f)$  then we say that  $f$  is continuous

Theorem 4.1.6:  $\epsilon - \delta$  Definition of Continuity

A function  $f : A \rightarrow \mathbb{R}$  is continuous if for all  $\epsilon > 0$ , there exists some  $\delta > 0$  s.t. for all  $x \in A$  for which  $0 < |x - c| < \delta$ , we have

$$|f(x) - f(c)| < \epsilon$$

Theorem 6.1.4: Evil  $\epsilon - \delta$  definition of continuity

A function  $f : A \rightarrow \mathbb{R}$  is not continuous if there exists  $\epsilon > 0$  such that for all  $\delta > 0$  there exists some  $x \in A$  satisfying  $0 < |x - c| < \delta$  for which  $|f(x) - f(c)| \geq \epsilon$

Definition 4.2.1 Bounds of a Function

Let  $E \subseteq \mathbb{R}$  be nonempty. A function  $f : E \rightarrow \mathbb{R}$  is said to be bounded on  $E$  if

$$|f(x)| \leq M, \quad \text{for all } x \in E$$

where  $M$  is some (large) real number.

Theorem 4.2.2: Extreme Value Theorem

Let  $I \subseteq \mathbb{R}$  be a closed and bounded interval. Let  $f : I \rightarrow \mathbb{R}$  be continuous on  $I$ . Then  $f$  is bounded on the interval  $I$ , denoted by

$$m = \inf\{f(x) \mid x \in I\}, \quad M = \sup\{f(x) \mid x \in I\}$$

Then there exist points  $x_m, x_M \in I$  such that

$$f(x_m) = m \quad \text{and} \quad f(x_M) = M$$

Theorem 4.2.4:  $\epsilon - \delta$  Limit jr.

Let  $f : I \rightarrow \mathbb{R}$  where  $I$  is an open nonempty interval. If  $f$  is continuous at a point  $a \in I$  and  $f(a) > 0$  then for some  $\delta, \epsilon > 0$  we have that

$$f(x) > \epsilon, \quad \text{for all } x \in (a - \delta, a + \delta)$$

Theorem 4.2.5: Intermediate Value Theorem

Let  $I$  be a non-degenerate interval and let  $f : I \rightarrow \mathbb{R}$  be a continuous function. If  $a, b \in I$ ,  $a < b$ , then on the interval  $(a, b)$ ,  $f$  attains all values between  $f(a)$  and  $f(b)$ . i.e. given  $y_0$  between  $f(a)$  and  $f(b)$ , there exists  $x_0 \in (a, b)$  such that  $f(x_0) = y_0$

Theorem 4.2.6: Bolzano's Theorem

Let  $f(x)$  be continuous on  $[a, b]$  such that  $f(a)f(b) < 0$ , then there exists  $c \in (a, b)$  such that  $f(c) = 0$

Theorem 4.3.1:  $\epsilon - \delta$  definition of a limit

Let  $f : A \rightarrow \mathbb{R}$  and let  $c$  be a limit point of  $A$ . Then we say that

$$\lim_{x \rightarrow c} f(x) = L$$

if for all  $\epsilon > 0$  there exists some  $\delta > 0$  such that for every  $x \in A$  for which  $0 < |x - c| < \delta$ , we have

$$|f(x) - L| < \epsilon$$

We also say  $\lim_{x \rightarrow c} f(x)$  **converges** to  $L$  in such a situation

Differentiation

Definition 5.1.1 First Principle Differentiation

A real function  $f$  is said to be differentiable at a point  $x \in \mathbb{R}$  if  $f$  is defined at some open interval containing  $x$ , and

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

exists.  $f'(x)$  is called the derivative of  $f$  at the point  $x$

Theorem 5.1.3: Differentiable implies Continuous

Let  $I$  be an open interval,  $x_0 \in I$  and  $f : I \rightarrow \mathbb{R}$  be differentiable at  $x_0$ . Then  $f$  is continuous at  $x_0$ . The converse is not true, an example is  $f(x) = |x|$  which isn't differentiable at 0.

Theorem 5.1.4: Differentiable Intervals

Let  $f : I \rightarrow \mathbb{R}$  be a given function, where  $I$  is an open interval. We say that  $f$  is differentiable in  $I$  iff it is differentiable at every point in  $I$ . At endpoints, derivatives only have to be one-sided

Theorem 5.2.1: Differentiation Rules

Let  $f, g : (a, b) \rightarrow \mathbb{R}$  be differentiable on  $(a, b)$ . Then  $f + g$  and  $f \cdot g$  are differentiable on  $(a, b)$ . If  $g(x) \neq 0$  for all  $x \in (a, b)$ , then  $f/g$  is differentiable. Moreover,

- **Sum rule:**  $(f + g)' = f' + g'$
- **Product Rule:**  $(fg)' = f'g + fg'$
- **Quotient Rule:**  $(f/g)' = (f'g - fg')/g^2$

Theorem 5.4.6: Inverse Function Theorem

Let  $f$  be injective and continuous on an open interval  $I$ . If  $a \in f(I)$  and  $f'$  at the point  $f^{-1}(a) \neq 0$  exists and is nonzero, then  $f^{-1}$  is differentiable at  $a$  and

$$(f^{-1})'(a) = \frac{1}{f'(f^{-1}(a))}$$

Theorem 5.2.2: Chain Rule

Let  $f, g$  be real functions. If  $f$  is differentiable at  $a$  and  $g$  is differentiable at  $f(a)$  then  $g \circ f$  is differentiable at  $a$  and

$$(g \circ f)'(a) = g'(f(a))f'(a)$$

Theorem 5.3.1 - 3: Differentiation Theorem ladder

- **Rolle's Theorem:** Let  $a, b \in \mathbb{R}$ ,  $a < b$ . If  $f : [a, b] \rightarrow \mathbb{R}$  is continuous in  $[a, b]$ , differentiable in  $(a, b)$  and  $f(a) = f(b)$ , then there exists a point  $c$  in  $(a, b)$  such that  $f'(c) = 0$
- **Mean Value Theorem:** If  $f : [a, b] \rightarrow \mathbb{R}$ ,  $a < b$  is continuous in  $[a, b]$ , differentiable in  $(a, b)$  then  $\exists c \in (a, b)$  s.t.

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

- **Generalised MVT:** If  $f, g : [a, b] \rightarrow \mathbb{R}$  is continuous in  $[a, b]$  and differentiable in  $(a, b)$ , then  $\exists c$  in  $(a, b)$  such that

$$(f(b) - f(a))g'(c) = (g(b) - g(a))f'(c)$$

If  $g(b) - g(a), g'(c) \neq 0$  then this can be written as

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}$$

Theorem 5.4.2: Monotone Functions

Let  $a < b$  be real and  $f$  be continuous on  $[a, b]$  and differentiable on  $(a, b)$ .

- If  $f'(x) > 0 \forall x \in (a, b)$ , then  $f$  is strictly increasing on  $[a, b]$
- If  $f'(x) < 0 \forall x \in (a, b)$ , then  $f$  is strictly decreasing on  $[a, b]$
- If  $f'(x) = 0 \forall x \in (a, b)$ , then  $f$  is constant on  $[a, b]$

Additionally, if  $f$  is injective and continuous on an interval  $I$ , Then  $f$  and  $f^{-1}$  is strictly monotone on  $I$  and  $f(I)$  respectively

Theorem 5.5.1: Taylor Series

Let  $f : I \rightarrow \mathbb{R}$  be  $n + 1$  times differentiable and  $x_0 \in I$ , for an open interval  $I$ . For each  $x \in I$ , there is a  $c$  between  $x_0$  and  $x$  such that

$$f(x) = \sum_{k=0}^n \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k + \frac{f^{(n+1)}(c)}{(n+1)!} (x - x_0)^{n+1}$$

( $c$  is a value between  $x$  and  $x_0$  (maybe))  
Now suppose that  $f : (a, b) \rightarrow \mathbb{R}$  is infinitely differentiable and let  $x_0 \in (a, b)$ . Fix  $x$  in  $(a, b)$ . For every positive integer  $N$  we have

$$f(x) = \sum_{k=0}^N \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k + R_N(x)$$

If  $R_N(x) \rightarrow 0$  as  $n \rightarrow \infty$ , we have

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k$$



3 Examples Catalogue

Examples of a Group

Example 1.2.4: Dihedral Group  $D_n$

The set of symmetries of an  $n$ -gon forms a group under composition. We call this group the **Dihedral Group  $D_n$** . The Dihedral group of  $n$  has precisely  $|D_n| = 2n$  elements, namely

- The identity  $e$
- $n - 1$  anticlockwise rotations of  $\frac{2\pi}{n}$ . We denote this operation with  $g$
- $n$  reflections. If  $n$  is odd, then there are  $n$  reflections from a point to the opposite edge. If  $n$  is even, there are  $\frac{n}{2}$  reflections from point to point, and  $\frac{n}{2}$  from edge to edge. We denote a vertical reflection with  $h$ , and rotated reflections as compositions of  $h$  and  $g$

From this, we see that

$$D_n = \{e, g, g^2, \dots, g^{n-1}, h, gh, g^2h, \dots, g^{n-1}h\}$$

Example 1.3.2: Symmetric Group

The set of all symmetries of  $\{1, 2, \dots, n\}$  is called the **symmetric group  $S_n$** . It is a group under composition with order  $|S_n| = n!$ . The symmetric group can be thought of as every permutation of the set  $\{1, 2, \dots, n\}$ , or can also be thought of as an  $n$ -gon where every edge is connected to each other.

Example  $\mathbb{Z}_3 \times \mathbb{Z}_4$ : Group Properties pick'n'mix

- Any group  $\mathbb{Z}_n$  is **Abelian** and **Cyclic**
- Any cross product of  $\mathbb{Z}_n \times \mathbb{Z}_m$  where  $n$  and  $m$  are coprime is **Abelian** and **Cyclic**.
- Any cross product of  $\mathbb{Z}_n \times \mathbb{Z}_m$  where  $n$  and  $m$  are not coprime is **Abelian** but **Not Cyclic**.
- Any dihedral group  $D_n$  is **Not Abelian**, and **Not Cyclic**
- The trivial action  $g \cdot x = x$  of any group is **Not Faithful** and **Not Transitive**
- The trivial action of any group acting on  $\{1\}$  is **Not Faithful** and **Transitive**
- $\mathbb{Z}_{an}$  acting on  $\mathbb{Z}_n$ , where  $a \in \mathbb{Z}$  where  $g \cdot x = g + x$  is **Not Faithful** and **Not Transitive**
- A symmetry group of a graph with a middle point has at least two orbits (**Not Transitive**) e.g.

• ————— • ————— •

Example 0: Example of a Coset

Consider  $\mathbb{Z}_4$  under addition, and let  $H = \{0, 2\}$  ( $e = 0$ ). The cosets of  $H$  in  $G$  are:

$$\begin{aligned} eH &= e * H = \{e * h \mid h \in H\} = \{0 + h \mid h \in H\} = \{0, 2\} \\ 1H &= 1 * H = \{1 * h \mid h \in H\} = \{1 + h \mid h \in H\} = \{1, 3\} \\ 2H &= 2 * H = \{2 * h \mid h \in H\} = \{2 + h \mid h \in H\} = \{0, 2\} \\ 3H &= 3 * H = \{3 * h \mid h \in H\} = \{3 + h \mid h \in H\} = \{1, 3\} \end{aligned}$$

Hence there are two cosets, namely

$$0 * H = 2 * H = \{0, 2\} \quad \text{and} \quad 1 * H = 3 * H = \{1, 3\}$$

$$G/H = \{eH = 2H, 1H = 3H\} = \{\{0, 2\}, \{1, 3\}\}$$

Example 3.3.5: p-series

The series  $\sum_{n=1}^{\infty} \frac{1}{n^p}$  converges if  $p > 1$ , and it diverges if  $p \leq 1$ . At  $p = 1$ , this series is called the **Harmonic Series**.

To show divergence/convergence of a series, we can compare it to the p-series

Example  $\sum_{n=0}^{\infty}$ : Deciphering Taylor Series

- **Showing convergence of a Taylor Series:** An infinite Taylor series will converge to  $f(x)$  iff we have  $R_N(x) \rightarrow 0$  as  $N \rightarrow \infty$  in the finite Taylor series

$$f(x) = \lim_{N \rightarrow \infty} \sum_{k=0}^N \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k + R_N(x)$$

Therefore, showing that  $(R_N(x))_{n \in \mathbb{N}}$  converges to 0 is enough to show that the infinite Taylor Series converges to  $f(x)$

- **Simplifying series-like terms:** If you have a Taylor Series / function that is in the same equation as a bunch of series-like terms, then a good idea is to try to expand the Taylor series at  $N$  for the  $N$  amount of elements and then try and figure something out using the remainder term  $R_N(x)$

Example: extract of 2018 May A3

At some point we end up with an equation

$$0 \leq \ln(x + 1) - x + \frac{x^2}{2} - \frac{x^3}{3} + \frac{x^4}{4}$$

and a Taylor Series

$$\ln(x + 1) = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n} x^n$$

Expand Taylor series with  $N = 4$

$$\begin{aligned} 0 &\leq \ln(x + 1) - x + \frac{x^2}{2} - \frac{x^3}{3} + \frac{x^4}{4} \\ &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + R_4(x) - x + \frac{x^2}{2} - \frac{x^3}{3} + \frac{x^4}{4} \\ &= R_4(x) \end{aligned}$$

The museum of  $\epsilon - \delta$  limits

Example of rigour:  $\epsilon - \delta$  and  $\epsilon - N$  Template

**Proof:** Let  $\epsilon > 0$  be given. Set  $\delta = \underline{\hspace{1cm}}$  (If there is a constant, then set as " $<$ " e.g.  $\delta < \min\{1, \epsilon\}$ ). Then for all  $x \in \mathbb{R}$  such that  $|x - \underline{\hspace{1cm}}| < \delta$  we have

Optional: preliminary step to determine an upper bound

Therefore, / Therefore since "x term" < "constant", "Same steps as rough working"

$$\dots = \underline{\hspace{1cm}} \cdot |x - \underline{\hspace{1cm}}| < \underline{\hspace{1cm}} \delta = \epsilon \quad (\text{same rule applies about constants})$$

**Proof:** Let  $\epsilon > 0$  be given. Choose  $N \in \mathbb{N}$  such that  $N > \underline{\hspace{1cm}}$ , (optional(?) for example  $N = \underline{\hspace{1cm}}$ ). Then we have  $\underline{\hspace{1cm}} < \epsilon$ . Then for all  $n \geq N$  we have

"Same steps as rough working"

$$\dots = \text{"equation in terms of } n" \leq \text{"same thing in terms of } N" < \epsilon$$

Example 10000:  $\epsilon$ -N Convergence

Show that the sequence  $\left(\frac{2n+1}{3n+2}\right)_{n \in \mathbb{N}}$  converges to  $\frac{2}{3}$

We start with the rough work. Start with an arbitrary  $\epsilon > 0$  and find an  $N_\epsilon$  s.t.  $\left|\frac{2n+1}{3n+2} - \frac{2}{3}\right| < \epsilon$  for all  $n > N_\epsilon$ . Let's explore this.

$$\left|\frac{2n+1}{3n+2} - \frac{2}{3}\right| = \frac{11}{3(3n+2)} < \epsilon \implies n > \frac{1}{3} \left(\frac{11}{3\epsilon} - 2\right)$$

**Proof:** Let  $\epsilon > 0$ . Pick a positive integer  $N$  such that

$$N > \frac{1}{3} \left(\frac{11}{3\epsilon} - 2\right)$$

Then,

$$\frac{11}{3(3N+2)} < \epsilon$$

For all  $n$  with  $n \geq N$  we have

$$|a_n - L| = \left|\frac{2n+1}{3n+2} - \frac{2}{3}\right| = \frac{11}{3(3n+2)} \leq \frac{11}{3(3N+2)} < \epsilon$$

Another method of finding a limit is,

$$\left|\frac{2n+1}{3n+2} - \frac{2}{3}\right| = \frac{11}{3(3n+2)} = \frac{11}{9n+6} < \frac{11}{9n} < \epsilon$$

Since  $9n + 6 > 9n$ , this means that the right fraction is larger than the left fraction in all cases. This means if we can find a right fraction that is smaller than  $\epsilon$  then the left fraction must also.

**Proof:** let  $\epsilon > 0$ . Pick a positive integer  $N$  such that  $N > \frac{11}{9\epsilon}$ . Then  $\frac{11}{9N} < \epsilon$ . For all  $n$  with  $n \geq N$ , we have

$$|a_n - L| = \left|\frac{2n+1}{3n+2} - \frac{2}{3}\right| = \frac{11}{9n+6} < \frac{11}{9n} \leq \frac{11}{9N} < \epsilon$$

### Example (: $\epsilon - \delta$ Continuity)

Using the definition of continuity, prove that the function  $f : \mathbb{R} \setminus \{\frac{9}{5}\} \rightarrow \mathbb{R}$  defined by  $f(x) = \frac{x^2}{5x-9}$  is continuous at  $x_0 = 2$ . Since  $x_0 = 2$ , our delta should end up as  $|x - 2| < \delta$ . Start with  $|f(x) - f(a)| < \epsilon$

$$\begin{aligned} |f(x) - f(a)| &= \left| \frac{x^2}{5x-9} - \frac{4}{10-9} \right| \\ &= \left| \frac{x^2}{5x-9} - 4 \right| \\ &= \left| \frac{x^2 - 20x + 36}{5x-9} \right| \\ &= \left| \frac{(x-18)(x-2)}{5x-9} \right| \\ &= |x-2| \left| \frac{x-18}{5x-9} \right| \end{aligned}$$

We have  $|x - 2|$ , so we want to turn the RH fraction into a constant. If we let the neighbourhood around  $\delta$  to be no less than  $\frac{1}{10}$  (i.e.  $x \in (1.9, 2.1)$ ) (this number can be anything, but smaller than  $\frac{1}{5}$  since there is an asymptote at  $\frac{9}{5}$ ), using the number with the largest value in that range we can get an upper bound for  $\delta$ .

$$\left| \frac{x-18}{5x-9} \right| < \left| \frac{1.9-18}{9.5-9} \right| = \left| \frac{-16.1}{0.5} \right| = |-32.2| \implies \left| \frac{x-18}{5x-9} \right| < 32.2$$

Therefore

$$|x-2| \left| \frac{x-18}{5x-9} \right| < |x-2| \cdot 32.2 < \epsilon$$

Therefore, we can take  $\delta = \max\{1/10, \epsilon/32.2\}$

**Proof:** Let  $\epsilon > 0$  be given. set  $\delta = \min\{\frac{1}{10}, \frac{\epsilon}{32.2}\}$ . Then for all  $x \in \mathbb{R}$  such that  $|x - 2| < \delta$  we have

$$\left| \frac{x-18}{5x-9} \right| < \left| \frac{1.9-18}{9.5-9} \right| = \left| \frac{-16.1}{0.5} \right| = |-32.2| \implies \left| \frac{x-18}{5x-9} \right| < 32.2$$

Therefore, since  $\left| \frac{x-18}{5x-9} \right| < 32.2$ ,

$$\begin{aligned} |f(x) - f(a)| &= \left| \frac{x^2}{5x-9} - \frac{4}{10-9} \right| = \left| \frac{x^2 - 20x + 36}{5x-9} \right| \\ &= \left| \frac{(x-18)(x-2)}{5x-9} \right| = |x-2| \left| \frac{x-18}{5x-9} \right| \leq 32.2 \cdot |x-2| < 32.2 \cdot \delta = \epsilon \end{aligned}$$

### Example Dumb Assumptions: $\epsilon - \delta$ using other limits

If  $f, g : \mathbb{R} \setminus \{1\} \rightarrow \mathbb{R}$  are two functions such that  $\lim_{x \rightarrow 1} f(x) = 2$ ,  $\lim_{x \rightarrow 1} g(x) = 3$ , show that  $\lim_{x \rightarrow 1} (4f(x) + g(x)^2) = 17$ . We want to try and turn the limit into compositions of other limits. From the assumptions, we know that

$$\bullet \text{ There exists a } \delta_1 \text{ s.t. } \forall x \text{ where } 0 < |x - 1| < \delta_1, \text{ we have } |f(x) - 2| < \epsilon \quad (1)$$

$$\bullet \text{ There exists a } \delta_2 \text{ s.t. } \forall x \text{ where } 0 < |x - 1| < \delta_2, \text{ we have } |g(x) - 3| < \epsilon \quad (2)$$

So, start with the main function. We want to show

$$|4f(x) + g(x)^2 - 17| < \epsilon$$

We want to turn this into a composition of (1) and (2). By “Trusting our professors won’t be too mean” this should be possible

$$\begin{aligned} |4f(x) + g(x)^2 - 17| &= |4(f(x) - 2) + g(x)^2 - 9| \\ &= |4(f(x) - 2) + (g(x) - 3)(g(x) + 3)| \\ &\quad (\text{via triangle ineq}) \leq 4|f(x) - 2| + |g(x) - 3||g(x) + 3| \end{aligned}$$

To find an upper bound for  $|g(x) + 3|$  we want to manipulate again

$$|g(x) + 3| = |g(x) - 3 + 6| \leq |g(x) - 3| + 6 < \epsilon + 6$$

Therefore now we can substitute equations (1) and (2) into everything

$$4|f(x) - 2| + |g(x) - 3||g(x) + 3| < 4\epsilon + \epsilon|g(x) + 3| < 4\epsilon + \epsilon(\epsilon + 6)$$

Let the epsilon boundary be less than 1. Then  $\epsilon + 6 < 7$ , therefore

$$4\epsilon + \epsilon(\epsilon + 6) < 4\epsilon + \epsilon(7) = 11\epsilon$$

We want to finish with  $\epsilon$  but since (1) and (2) work for any  $\epsilon$  by definition, set those inequalities to  $\frac{\epsilon}{11}$  instead and the final result will be  $\epsilon$  on its own

**Proof:** Let  $\epsilon > 0$  be given. First assume  $\epsilon \leq 1$ . By our assumptions, there exists a  $\delta_1$  where  $\forall x$  s.t.  $0 < |x - 1| < \delta_1$ , we have  $|f(x) - 2| < \epsilon/11$ , and a  $\delta_2$  where  $\forall x$  s.t.  $0 < |x - 1| < \delta_2$ , we have  $|g(x) - 3| < \epsilon/11$ . For all  $x$  s.t.  $0 < |x - 1| < \delta_2$ , we have

$$|g(x) + 3| = |g(x) - 3 + 6| \leq |g(x) - 3| + 6 \leq \frac{\epsilon}{11} + 6 \leq \frac{1}{11} + 6 \leq 7$$

Let  $\delta = \min\{\delta_1, \delta_2\}$ . Therefore, since  $|g(x) + 3| \leq 7$ ,

$$\begin{aligned} |4f(x) + g(x)^2 - 17| &= |4(f(x) - 2) + g(x)^2 - 9| \\ &= |4(f(x) - 2) + (g(x) - 3)(g(x) + 3)| \\ &\quad (\text{via triangle ineq}) \leq 4|f(x) - 2| + |g(x) - 3||g(x) + 3| \end{aligned}$$

$$< 4 \frac{\epsilon}{11} + 7 \frac{\epsilon}{11} = \epsilon$$

Assume now that  $\epsilon > 1$ . By what we have shown above there exists a  $\epsilon > 0$  such that for all  $x$  such that  $0 < |x - 1| < \epsilon$ ,

$$|4f(x) + g(x)^2 - 17| < 1$$

Therefore,

$$|4f(x) + g(x)^2 - 17| < \epsilon$$

### Example ): $\epsilon - \delta$ Discontinuity

From negation of  $\epsilon - \delta$  continuity - A function  $f : A \rightarrow \mathbb{R}$  is not continuous if there exists  $\epsilon > 0$  such that for all  $\delta > 0$  there exists some  $x \in A$  satisfying  $0 < |x - c| < \delta$  for which  $|f(x) - f(c)| \geq \epsilon$

$$|f(x) - f(a)| < \epsilon \implies \left| \sin\left(\frac{1}{x}\right) - 0 \right| < \epsilon \implies \left| \sin\left(\frac{1}{x}\right) \right| < \epsilon$$

So we want to show that we can find an  $\epsilon$  such that for every  $\delta > 0$ , we can find an  $x$  where  $|x| < \delta$  and also  $|\sin(\frac{1}{x})| \geq \epsilon$ .

Since  $\sin(x)$  repeats, if we can find an  $x$  such that  $\sin(\frac{1}{x})$  is an exact value then we can define  $\epsilon$  as something lower than that. If we want a value where  $\sin(\frac{1}{x}) = 1$ , this will be true if  $x = 1/(\frac{\pi}{2} + 2N\pi)$ , where  $N$  is a positive integer.

Since  $x$  has to be bounded by  $\delta$ , go from  $\delta$

$$\begin{aligned} |x| &< \delta \\ \left| \frac{1}{\frac{\pi}{2} + 2N\pi} \right| &< \delta \\ \frac{1}{\frac{\pi}{2} + 2N\pi} &< \delta \quad (\text{will always be positive since } N \text{ positive int}) \\ \frac{\pi}{2} + 2N\pi &> \frac{1}{\delta} \\ N &> \frac{1}{2\pi} \left( \frac{1}{\delta} - \frac{\pi}{2} \right) \end{aligned}$$

**Proof:** Let  $\epsilon = \frac{1}{2}$ . Let  $\delta > 0$  be given. Pick a positive integer  $N$  such that  $N > \frac{1}{2\pi} (\frac{1}{\delta} - \frac{\pi}{2})$  and set  $x = \frac{1}{\frac{\pi}{2} + 2N\pi}$ . Then for all  $x \in \mathbb{R}$  such that  $0 < x < \delta$ , we have

$$|f(x)| = \left| \sin\left(\frac{1}{x}\right) \right| = \left| \sin\left(\frac{\pi}{2} + 2N\pi\right) \right| = 1 \geq \frac{1}{2} = \epsilon$$

### Example Number: Roots of a Function

Prove that the equation

$$f(x) = x^5 + x^3 + x + 1$$

has exactly one real root

**Part 1:** The function has *at least* one real root. This is because as it is composed of three odd-power polynomials, so as  $x \rightarrow \infty$ ,  $f(x) > 0$  and as  $x \rightarrow -\infty$ ,  $f(x) < 0$ . Therefore by the Intermediate Value Theorem, there exists at least one point where  $f(x) = 0$ .

**Part 2:** Differentiating the function, we get

$$f(x) = x^5 + x^3 + x + 1 \implies f'(x) = 5x^4 + 3x^2 + 1$$

However,  $x^4, x^2 \geq 0$  for every  $x \in \mathbb{R}$ . Therefore  $5x^4 + 3x^2 + 1$  is always positive, meaning there is no roots (critical points).

**Part 3:** Therefore, by Rolle's Theorem, since there exists no critical points, it means there is no points where  $f(a) = f(b)$  which implies that  $f(x)$  is strictly increasing/decreasing.

This means there cannot be more than 1 real root. Therefore, there is exactly one real root