1 Leon's (WIP) ITCS Exam Notes

Basically adapted from Chris Dalziel's notes :)

Finite Automata

Definition: Finite Automata

A finite automaton takes a string as input and replies "yes" or "no". If an automaton A replies "yes" on a string S we say that A "accepts" S.

Definition: Deterministic Finite Automata

A deterministic finite automaton (DFA) is a quintuple $(Q, \Sigma, q_0, \delta, F)$ where

- Q is a finite set of states
- Σ is an alphabet
- $q_0 \in Q$ is the initial state
- $\delta : Q \times \Sigma \to Q$ is the transition function
- $F \subseteq Q$ is the set of final states

A DFA accepts a string $w \in \Sigma^*$ iff $\delta^*(q_0, w) \in F$, where δ^* is δ applied successively for each symbol in w.
The language of a DFA A is the set of all strings accepted by a, $\mathcal{L} \subseteq \Sigma^*$ is the set of all strings accepted by A.
The transition function is a total function which gives exactly one next state for each input symbol, i.e. it is deterministic

Definition: Nondeterministic Finite Automata

Non-determinism would mean that δ can return more than one successor state, it instead returns a set of possible states - no states is an empty set. A NFA is a quintuple $(Q, \Sigma, q_0, \delta, F)$ where:

- Q is a finite set of states
- Σ is an alphabet
- $q_0 \in Q$ is the initial state
- $\delta : Q \times \Sigma \to \mathcal{P}(Q)$ is the transition function
- $F \subseteq Q$ is the set of final states

The only difference between the definition of a DFA and that of an NFA is that in an NFA δ returns an element from the powerset of Q, $\mathcal{P}(Q)$

Adding non-determinism doesn't change "expressivity". Given an NFA A there is an equivalent DFA D such that $\mathcal{L}(D) = \mathcal{L}(A)$ and vice versa.

Definition: ϵ-NFA

If we allow non-deterministic state chnages that don't consume any input symbols, we can label silent moves using ϵ - meaning the empty string We define the ϵ closure $E(q)$ of a state q as the set of all states reachable from q by silent moves. That is, $E(q)$ is the least set satisfying:

- $q \in E(q)$
- For any $s \in E(q)$ we also have $\delta(s, \epsilon) \subseteq E(q)$

DFA, NFA, ϵ-NFA are all equal in expressive power

Regular Languages

Definition: Regular Languages

Any language which can be accepted by a finite automaton is called a regular language.
Regular languages are also those recognised by Regular Expressions

Definition: Regular Language Closure Properties

For two languages L_1 and L_2, the following operations satisfy the closure property, i.e. for a member $x \in X$, and an operation ϕ we have that $\phi(x) \in \mathbb{R}$ for all x.

- **Union**: $L_1 \cup L_2$ is the language that includes all strings of L_1 and all strings of L_2.
- **Intersection**: $L_1 \cap L_2$ is the language that includes all strings of L_1 that are not in L_2, and vice versa
- **Sequential Composition**: $L_1 L_2$ is the language of strings that consist of strings in L_1 followed by a string in L_2.
- **Kleene closure**: L^* is the language of strings that consist wholly of zero or more strings in L.
$$L^* = \bigcup_{i \in \mathbb{N}} L^i$$
- **Complement**: $\bar{L}$ is the language of every string not in L.

Definition: Regular Expressions

Regular characterise the regular languages, just like finite automata do. The following table shows the syntax and semantics of a regex.

Syntax	Semantics	
a	$[\![a]\!] = \{a\}$	$(a \in \Sigma)$
$\emptyset$	$[\![\emptyset]\!] = \emptyset$	
ϵ	$[\![\epsilon]\!] = \{\epsilon\}$	
$R_1 \cup R_2$	$[\![R_1 \cup R_2]\!] = [\![R_1]\!] \cup [\![R_2]\!]$	
$R_1 \circ R_2$	$[\![R_1 \circ R_2]\!] = [\![R_1]\!][\![R_2]\!]$	
R^*	$[\![R^*]\!] = [\![R]\!]^*$	

Definition: Generalised NFAs

A **generalised NFA**, or GNFA is an NFA where:

- Transitions have **regular expressions** on them instead of symbols
- There is only one unique final state
- The transition relation if **full**, except that the initial state has no incoming transitions, and the final state has no outgoing transitions

Theorem 1: Pumping Lemma

If $L \subseteq \Sigma^*$ is regular, then there is a **pumping length** $p \in \mathbb{N}$ such that for any $w \in L$ where $|w| \geq p$, we may split w into three piexes $w = xyz$ satisfying three conditions:

- $xy^iz, \quad \forall i \in \mathbb{N}$
- $|y| > 0$
- $|xy| \leq p$

Note that if the pumping lemma fails then the language is not regular, but the inverse is not necessarily true.

Theorem 2: Myhill-Nerode Theorem

Let $L \subseteq \Sigma^*$ and $x, y \in \Sigma^*$. If there exists a suffix string z such that $xz \in L$, but $yz \notin L$ or vice versa, then x and y are **distinguishable** by L. If x and y are not distinguishable by L, then we say that $x \equiv_L y$ - this is an equivalence relation. A regular language satisfies the following

- The number of equivalence classes $\equiv_L$ is finite.
- The number of equivalence classes is equal to the number of states in the minimal DFA accepting L (not as important)

Therefore, to show a language is non-regular, show that it has infinite equivalence classes - that is, we find an infinite sequence $u_0 u_1 \ldots$ of strings such that for any i, j where $i \neq j$, there is a string w_{ij} such that $u_i w_{ij} \in L$ but $u_j w_{ij} \notin L$ or vice-versa

Context-Free Languages

Definition: Context-free Languages

By adding recursion to regexs we can begin to recognise some non-regular languages. All regular languages are also context free.

Definition: Context-free Grammars

A language is context-free iff it is recognised by a Context-free Grammar (CFG), which is a 4-tuple (N, Σ, P, S) where:

- N is a finite set of variables or non-terminals
- Σ is a finite set of terminals
- $P \subseteq N \times (N \cup \Sigma)^*$ is a finite set of rules or productions
  - Typically productions are written $A \to aBc$
  - Productions with common heads can be combined, $A \to a$ and $A \to Aa$ can be combined into $A \to a \mid Aa$
- $S \in N$ is the starting variable

We use α, β, γ to refer to sequences of terminals
We make a derivation step $\alpha A \beta \Rightarrow_G \alpha\gamma\beta$ whenever $(A \to \gamma) \in P$;
The language of a CFG G is:

$$\mathcal{L}(G) = \{w \in \Sigma^* \mid S \Rightarrow^*_G w^*\}$$

Where $\Rightarrow^*_G$ is the reflexive, transitive, closure of $\Rightarrow_G$.
Context-free grammars are ambiguous. They are closed under union, concatenation, and kleene star, but not under intersection or complementation

Definition: Eliminating Ambiguity

We want to eliminate ambiguity in CFGs while still accepting all the same strings. This can be done for our language of regular expressions:

- First defining atomic expressions: $A \to (S)|\emptyset|\epsilon|a|b$
- Then ones which use Kleene Star: $K \to A|A^*$
- Then ones which may use left-associative composition: $C \to K|C \circ K$
- Finally expressions which use unions: $S \to C|S \cup C$

The order of operations here is therefore bottom to top; unions come before compositions, which come before Kleene etc

Definition: Push-down Automata

Push-down automata are to CFGs what finite automatas are to regular expressions. They are implementationally identical to ϵ-NFAs with the addition of a stack. The recursive element of CFGs is implemented using a standard last-in-first-out stack.

Transitions in a push-down automata take the form $x, y \to z$ which is read as "consume the input x, popping y off the stack, and push z onto the stack". We can allow actions that don't consume, pop, or push by setting variables to ϵ.

Definition: Formal Def. of PDAs

A **push-down automaton** is a 6-tuple $(Q, \Sigma, \Gamma, \delta, q_0, F)$ where Q, Σ, Γ are all finite sets. Γ is the stack alphabet, and δ now may take a stack symbol as input or return one as output:

$$\delta : Q \times \Sigma_\epsilon \times \Gamma_\epsilon \to \mathcal{P}(Q \times \Gamma_\epsilon)$$

All other compoments are as with ϵ-NFAs

A string w is accepted by a PDA if it ends in a final state, i.e. $\delta^*(q_0, w, \epsilon)$ gives a state q and a stack γ wuch that $q \in F$.

Theorem 3: CFG to PDA

A language is context-free if and only if it is recognised by a push-down automaton. The proof is left as an exercise to the reader.

Theorem 4: Pumping CFLs intro

Suppose a CFG has n non-terminals, and we have a parse tree of height $k > n$. Then the same non-terminal V must have appeared as its own descendant in the tree

- **Pumping down**: Cut the tree at the higher occurance of V and replace it with the subtree at the lower occurance of V
- **Pumping up**: Cut at the lower occurance and replace it with a fresh copy of the higher occurance

Theorem 5: Pumping Lemma for CFLs

If L is context-free then there exists a pumping length $p \in \mathbb{N}$ such that if $w \in L$ with $|w| \geq p$ then w may be split into **five** pieces $w = uvxyz$ such that

- $uv^ixy^iz \in L$ for all $i \in \mathbb{N}$
- $|vy| > 0$
- $|vxy| \leq p$

Definition: Chomsky Grammars

Context-free grammars are a special case of Chomsky Grammars. Chomsky grammars are similar to CFGs, except that the left-hand side of a production may be any string that includes at least one non-terminal. An example is shown below

$$S \to abc \mid aAbc$$
$$Ab \to bA$$
$$Ac \to Bbcc$$
$$bB \to Bb$$
$$aB \to aaA \mid aa$$

Such a grammar is called **context-sensitive**

Definition: The Chomsky Heirarchy

A grammar $G = (N, \Sigma, P, S)$ is of type:

0. (or **computably enumerable**) in the general case
1. (or **context sensitive**) if $|\alpha| \leq |\beta|$ for all productions $\alpha \to \beta$, except we also allow $S \to \epsilon$ if S foes not occur on the RHS of any rule
2. (or **context free**) if all productions are of the form $A \to \alpha$ (i.e. a CFG)
3. (or **right-linear/regular**) if all productions are of the form $A \to w$ or $A \to wB$, where $w \in \Sigma$ and $B \in N$

Algorithms for Languages

Theorem 6: Emptiness for Regular Languages

Can we write a program to determine if a given regular language is empty?
Given a finite-automaton this is an instance of graph reach-ability, so we can use a depth-first search.

Theorem 7: Emptiness for Context-free languages

Can we write a program to determine if a given context-free language is empty?
Given a CFG for our language, we can perform the following process:

1. Mark the terminals and ϵ as generating
2. Mark all non-terminals which have a production with only generating symbols in their right hand side as generating
3. Repeat until nothing new is marked
4. Check if S is marked as generating or not

Theorem 8: Equivalence of DFA

Is it possible to write a program to determine if two discrete finite automata are equivalent?

Given two DFA for L_1 and L_2, we can use our standard constructions to produce a DFA of the symmetric set difference:

$$(L_1 \cap \overline{L}_2) \cup (L_2 \cap \overline{L}_1)$$

Register Machines

Definition: Register Machines

A **register machine**, or RM, consists of:

- A fixed number m of registers $R_0 \dots R_{m-1}$, which each holds a natural number
- A fixed program P which is a sequence of n instructions $I_0 \dots I_{n-1}$

Each instruction is one of the following:

- `INC(i)`: which increments the register R_i by one
- `DECJZ(i, j)`: which decrements register R_i unless $R_i = 0$ in which case it jumps to instruction I_j

RMs can compute anything any other computer can

Definition: Pairing Functions for RMs

A **pairing function** is an injective function $\mathbb{N} \times \mathbb{N} \to \mathbb{N}$. An example is $f(x, y) = 2^x 3^y$.
We write $\langle x, y \rangle_2$ for $f(x, y)$. If $z = \langle x, y \rangle_2$, let $z_0 = x$ and $z_1 = y$. This lets us encode multiple values into a single value, and a 2-tuple pairing function is enough to cram an arbitrary sequence of natural numbers into one $\mathbb{N}^* \to \mathbb{N}$

Theorem 9: Church-Turing Thesis

The Church-Turing thesis states that any problem is computable by any model of computation iff it is computable by a **Turing machine**.
For our purposes this matters for RMs, TMs, and λ-calculus. Other examples are combinator calculus, general recursive functions, pointer machines, counter machines, cellular automata, queue automata, enzyme-based DNA computers, Minecraft, Magic the Gathering, and others.

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Example : Regular Expressions

At least one 0:

$$(0 \cup 1)^* 0 (0 \cup 1) *$$

At least one 1 and at least one 0:

$$((0 \cup 1)^* 01 (0 \cup 1)^*) \cup ((0 \cup 1)^* 10 (0 \cup 1)^*)$$