1 Algebra

1.1 Functions and Symmetries

Definition 0.1.1 Functions

A function $f : X \to Y$ is called

- *injective* if $f(x_1) = f(x_2) \implies x_1 = x_2$
- *surjective* if for every $y \in Y$, $\exists x \in X$ s.t. $f(x) = y$
- *bijective* if it is both injective and surjective

Definition 1.1.3 Graph Isomorphisms

An **isomorphism** between two graphs is a *bijection* between them that preserves all edges. More precisely, if Γ_1 and Γ_2 are graphs, with sets of vertices V_1 and V_2 respectively, then an isomorphism from Γ_1 and Γ_2 is a bijection

$$f : V_1 \to V_2$$

such that $f(v_1)$ and $f(v_2)$ are joined by an edge if and only if v_1 and v_2 are also joined by an edge. We say that Γ_1 and Γ_2 are *isomorphic* if there exists an isomorphism $f : \Gamma_1 \to \Gamma_2$

Definition 1.1.9 Symmetry

A **symmetry** of a graph is an *isomorphism* from the graph to itself, i.e. if the set of vertices is V, then the symmetry is a bijection $f : V \to V$ that preserves edges. That is, a symmetry is a bijection $f : V \to V$ such that $f(v_1)$ and $f(v_2)$ are joined by an edge if and only if v_1 and v_2 are joined by an edge.

1.2 Groups

Definition 1.2.3 Groups

For an operation $*$, We say a non-empty set G is a **group** under $*$ if the following four axioms hold:

- **G1 - Closure:** $*$ is a binary operation on G, that is $a * b \in G$ for all $a, b \in G$.
- **G2 - Associativity:** $(a*b)*c = a*(b*c)$ for all $a, b, c \in G$
- **G3 - Identity:** There exists an *identity* element of G such that $e * g = g * e = e$ for all $g \in G$.
- **G4 - Inverse:** Every element $g \in G$ has an *inverse* g^{-1} such that $g * g^{-1} = g^{-1} * g = e$

Definition 1.2.6 Abelian Group

The definition of a group doesn't require that $a * b = b * a$. We say that a group is **abelian** or **commutative** if $a * b = b * a$ for every $a, b \in G$. We say that a *commutes* with b, or that a and b *commute*

Definition 2.2.3 Order of a Group

The **order** of a finite group, written $|G|$, is the number of elements in G. If G is infinite we say that $|G| = \infty$, or the order of G is infinite.

1.3 Subgroups

Definition 2.1.1 Subgroups

Let G be a group. We say that a non-empty subset H of G is a **subgroup** of G if H itself is a group (under the operation from G). We write $H \leq G$ if H is a subgroup of G. If $H \neq G$, we write $H < G$ and say H is a proper subgroup

Theorem 2.1.3: Subgroup Test

$H \subseteq G$ is a subgroup of G if and only if:

- **S1:** H is not empty
- **S2:** If $h, k \in H$ then $h * k \in H$
- **S3:** If $h \in H$ then $h^{-1} \in H$

Alternative test for subgroups:

- $\widetilde{S1}$: H is not empty.
- $\widetilde{S2}$: If $h, k \in H$ then $h * k^{-1} \in H$

Definition 2.2.4 Order of an Element

Let G be a group and $g \in G$. Then the **order** $o(g)$ of g is the *least* natural number n such that

$$g^n = e$$

If no such n exists, we say that g has infinite order

Theorem 2.2.6: Order of a Finite Group

In a finite group, every element has finite order.
If g is an element of a finite group G, then there exists $k \in \mathbb{N}$ such that $g^k = g^{-1}$

Definition 2.2.8 Generating Subset

Let G be a group and let $g \in G$ be an element. We define the subset

$$\langle g \rangle := \{g^k \mid k \in \mathbb{Z}\} = \{\ldots, g^{-2}, g^{-1}, e, g, g^2, \ldots\}$$

Note that if G is finite, then by 2.2.6 $\langle g \rangle$ is finite, and we can think of $\langle g \rangle$ as

$$\langle g \rangle = \{e, \mathbf{g} \ldots, g^{o(g)-1}\}$$

Definition 2.2.10 Cyclic Subgroup

A subgroup $H \leq G$ is **cyclic** if $H = \langle h \rangle$ for some $h \in H$. In this case, we say that H is the *cyclic subgroup generated by h*. If $G = \langle g \rangle$ for some $g \in G$, then we say that the group G is *cyclic*, and that g is a *generator*.

Remark 2.2.14 - 16: Consequences of Cyclic groups

- **2.2.12** If $g \in G$, then $o(g) = |\langle g \rangle|$
- **2.2.13:** If G is cyclic, then G is abelian.
- **2.2.14:** Let G be a finite group. Then

$$G \text{ is cyclic} \iff G \text{ has an element of order} |G|$$

- **2.2.15:** Let G be a cyclic group and let H be a subgroup of G. Then H is cyclic.
- **2.2.16:** Let $m, n \in \mathbb{N}$, let $G = \langle g \rangle$ be a cyclic group of order m and $H = \langle h \rangle$ be a cyclic group of order n. Then

$$G \times H \text{ cyclic} \iff m \text{ and } n \text{ are coprime } (\gcd(m,n) = 1)$$

1.4 Cosets and Lagrange

Definition 2.3.2 Relation

Let X be a set, and R a subset of $X \times X$; thus R consists of some ordered pairs (s, t) with $s, t \in X$. If $(s, t) \in R$ we write $s \sim t$ and say "s is related to t". We call $\sim$ a **relation** on X.

Definition 2.3.2 Equivalence Relation

- **Reflexive:** $x \sim x$ for all $x \in X$
- **Symmetric:** $x \sim y$ implies that $y \sim x$ for all $x, y \in X$
- **Transitive:** $x \sim y$ and $y \sim z$ implies that $x \sim z$ for all $x, y, z \in X$

A relation $\sim$ is called an **equivalence relation** on X if it satisfies the following three axioms:

Definition 2.3.4 Coset

Let $H \leq G$ and let $g \in G$. Then a *left coset* of H in G is a subset of G of the form gH, for some $g \in G$. We denote the set of left cosets of H in G by G/H

Theorem 2.3.8: Coset Rules

Let $H \leq G$

- For all $h \in H$, $hH = H$. In particular $eH = H$
- For $g_1, g_2 \in G$, the following are equivalent
  - $g_1 H = g_2 H$
  - there exists $h \in H$ such that $g_2 = g_1 H$
  - $g_2 \in g_1 H$
- For g_1, $g_2 \in G$, define $g_1 \sim g_2$ if and only if $g_1 H = g_2 H$. Then $\sim$ defines an equivalence relation on G.

Theorem 2.4.2: Lagrange's Theorem

Suppose that G is a finite group.

- If $H \leq G$, then $|H|$ divides $|G|$
- Let $g \in G$. Then $o(g)$ divides $|G|$
- For all $g \in G$, we have that $g^{|G|} = e$

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