

# Exam Notes

Made by Leon :) *Note: Any reference numbers are to the lecture notes*

## 1 Revisiting FPM

### Definition 1.1: Nested Sequences and covers

A sequence  $(I_n)_{n \in \mathbb{N}}$  of sets is said to be **nested** if

$$I_1 \subset I_2 \subset I_3 \subset \dots$$

**Thm 1.1:** If  $(I_n)$  is a nested sequence of nonempty closed bounded intervals then

$$E = \bigcap_{n \in \mathbb{N}} I_n = \{x \in \mathbb{R} : x \in I_n, \forall n \in \mathbb{N}\}$$

is nonempty (i.e. it contains at least one number). Moreover if  $\lambda(I_n) \rightarrow 0$ , where  $\lambda(I_n)$  denotes the length of interval  $I_n$ , then  $E$  contains exactly one number

**Thm 1.2:** Let  $E$  be a subset of  $\mathbb{R}^n$

- A **cover** of  $E$  is a collection of sets  $\{I_\alpha\}_{\alpha \in A}$  such that

$$E \subseteq \bigcup_{\alpha \in A} I_\alpha$$

- An **open covering** of  $E$  is a cover such that each  $I_\alpha$  is open, i.e.  $(a, b)$  compared to  $[a, b]$
- A **finite subcover** of  $E$  is a collection of sets  $(I_\alpha)_{\alpha \in A_0}$  where there exists a subset  $A_0 = \{\alpha_1, \alpha_2, \dots, \alpha_N\}$  of  $A$  such that  $(I_\alpha)_{\alpha \in A_0}$  is a finite subset of  $(I_\alpha)_{\alpha \in A}$  that is also a cover
- The set  $E$  is said to be **compact** iff every open covering of  $E$  has a **finite subcovering**; that is

$$E \subseteq \bigcup_{j=1}^N I_{a_j} \quad \text{or} \quad E \subseteq I_{\alpha_1} \cup I_{\alpha_2} \cup \dots \cup I_{\alpha_N}$$

### Definition 1.2: Convergence of Sequences and Cauchy

A sequence of real numbers  $(x_n)$  is said to **converge** to a real number  $a \in \mathbb{R}$  iff for every  $\epsilon > 0$  there is an  $N \in \mathbb{N}$  such that

$$n \geq N \text{ implies } |x_n - a| < \epsilon$$

If  $(x_n)$  converges to  $a$ , we will write  $\lim_{n \rightarrow \infty} x_n = a$ , or  $x_n \rightarrow a$ . The number  $a$  is called the limit of the sequence  $(x_n)$ . A sequence that does not converge to some real number is said to \*diverge

**Def 1.3:** A sequence  $(x_n)$  of numbers  $x_n \in \mathbb{R}$  is said to be **Cauchy** if for every  $\epsilon > 0$  there is  $N \in \mathbb{N}$  such that

$$|x_n - x_m| < \epsilon \quad \forall n, m \geq N$$

**Thm 1.3:** Let  $(x_n)$  be a sequence of real numbers. Then  $(x_n)$  is a Cauchy sequence if and only if  $(x_n)$  is a convergent sequence.

**Thm 1.4:** Let  $(x_n)$  be a sequence of real numbers. Then  $(x_n)$  is a Cauchy sequence iff  $(x_n)$  is a convergent sequence

### Definition 1.4: Subsequences

Suppose  $(x_n)_{n \in \mathbb{N}}$  is a sequence. A subsequence of this sequence is a sequence of the form  $(x_{n_k})_{k \in \mathbb{N}}$  where for each  $k$  there is a positive integer  $n_k$  such that

$$n_1 < n_2 < \dots < n_k < n_{k+1} < \dots$$

Thus,  $(x_n)_{n \in \mathbb{N}}$  is just a selection of some (possibly all) of the  $x_n$ 's taken in order

**Thm 1.5 (Bolzano-Weierstrass):** Every bounded sequence of real numbers has a convergent subsequence

### Definition 1.5: Limit Superior and Inferior

If  $(x_n)$  is a bounded sequence of real numbers we denote by

$$\limsup_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} \left( \sup_{k \geq n} x_k \right), \quad \liminf_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} \left( \inf_{k \geq n} x_k \right)$$

**Note:** These are only defined for bounded sequences

- If  $(x_n)$  is not bounded from above then we write  $\limsup_{n \rightarrow \infty} x_n = +\infty$
- If  $(x_n)$  is not bounded from below then we write  $\liminf_{n \rightarrow \infty} x_n = -\infty$

**Thm 1.6:** A sequence  $(x_n)$  of real numbers is convergent if and only if  $\limsup_{n \rightarrow \infty} x_n$  and  $\liminf_{n \rightarrow \infty} x_n$  are real numbers and

$$\limsup_{n \rightarrow \infty} x_n = \liminf_{n \rightarrow \infty} x_n$$

### Definition 1.7: Continuity

Let  $f$  be a function  $f : \text{dom}(f) \rightarrow \mathbb{R}$  where  $\text{dom}(f) \subset \mathbb{R}$ . We say that  $f$  is **continuous** at some  $a \in \text{dom}(f)$ , if for any sequence  $(x_n)$  whose terms lie in  $\text{dom}(f)$  and which converges to  $a$ , we have  $\lim_{n \rightarrow \infty} f(x_n) = f(a)$ . If  $f$  is continuous at each  $a \in S \subset \text{dom}(f)$  then we say  $f$  is continuous on  $S$ . If  $f$  is continuous of  $\text{dom}(f)$  then we say  $f$  is continuous

**Thm 1.10:** Let  $f, g : D \rightarrow \mathbb{R}$  be continuous on  $D$ , and let  $\alpha \in \mathbb{R}$ . Then the following functions are continuous on  $D$  :

- $\alpha f$
- $f + g$
- $fg$

**Thm 1.12 ( $\epsilon - \delta$  Definition of Continuity):** Let  $f$  be a function  $f : \text{dom}(f) \rightarrow \mathbb{R}$  where  $\text{dom}(f) \subset \mathbb{R}$ . Then  $f$  is continuous at  $a \in \text{dom}(f)$  iff for any  $\epsilon > 0$  there exists  $\delta > 0$  s.t. whenever  $x \in \text{dom}(f)$  and  $|x - a| < \delta$  we have  $|f(x) - f(a)| < \epsilon$

**Thm 1.13 (Intermediate Value Theorem):** Let  $a < b$  real numbers and  $f : [a, b] \rightarrow \mathbb{R}$  be continuous on  $[a, b]$ . If  $f(a)f(b) < 0$  then there exists at least one  $c \in (a, b)$  s.t.  $f(c) = 0$

**Thm 1.14 (Extreme Value Theorem):** Let  $a < b$  real numbers and  $f : [a, b] \rightarrow \mathbb{R}$  be continuous on  $[a, b]$ . Then there exists points  $c, d \in [a, b]$  s.t.

$$f(c) = \inf\{f(x) : x \in [a, b]\}, \quad f(d) = \sup\{f(x) : x \in [a, b]\}$$

That is, the function  $f$  on the interval  $[a, b]$  is bounded and attains its minimal value at some point  $c \in [a, b]$ . Similarly, the maximal value of  $f$  is also attained at some point  $d \in [a, b]$

### Definition 1.6: Convergent Infinite Series

Let  $S = \sum_{k=1}^{\infty} a_k$  be an infinite series  $a_k$ . For each  $n \in \mathbb{N}$ , the partial sum of  $S$  of order  $n$  is defined by

$$s_n = \sum_{k=1}^n a_k$$

$S$  is said to **converge** iff its sequence of partial sums  $(s_n)$  converges to some  $s \in \mathbb{R}$  as  $n \rightarrow \infty$ ; that is, iff for every  $\epsilon > 0$  there is an  $N \in \mathbb{N}$  s.t. for all  $n \geq N$  we have  $|s_n - s| < \epsilon$ . In this case we shall write

$$\sum_{k=1}^{\infty} a_k = s$$

and call  $s$  the **sum** or **value** of the series  $\sum_{k=1}^{\infty} a_k$

- Absolutely convergent:** the series  $\sum_{k=1}^{\infty} |a_k|$  is convergent
- Conditionally convergent:** Convergent but not absolutely

**Thm 1.7 (Cauchy Criterion):** Let  $S = \sum_{k=1}^{\infty} a_k$  be a series. Then  $S$  is convergent iff for any  $\epsilon > 0$  there exists  $N$  such that for all  $m \geq n \geq N$  we have that

$$\left| \sum_{k=n+1}^m a_k \right| < \epsilon$$

### Thm 1.8 (Rearranging Absolutely Convergent Series)

Let  $S = \sum_{k=1}^{\infty} a_k$  be an absolutely convergent series. Then

- The series  $S$  is convergent
- Let  $z : \mathbb{N} \rightarrow \mathbb{N}$  be a bijection. Then the series  $\sum_{k=1}^{\infty} a_{z(k)}$  is convergent and

$$\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} a_{z(k)}$$

The series  $\sum_{k=1}^{\infty} a_{z(k)}$  is called a **rearrangement** of the series  $\sum_{k=1}^{\infty} a_k$ . What we do here is add the terms of the sum in a different order to the original one, for example

$$a_3 + a_7 + a_1 + a_{100} + a_2 + \dots$$

Since  $z : \mathbb{N} \rightarrow \mathbb{N}$  is a bijection, we will miss no terms.

### Thm 1.9 (Rearranging Conditionally Convergent Series)

Let  $S = \sum_{k=1}^{\infty} a_k$  be any conditionally convergent series. Then there exists rearrangements  $z : \mathbb{N} \rightarrow \mathbb{N}$  (where  $z$  is a bijection) such that

- For any  $r \in \mathbb{R}$ ,  $\sum_{k=1}^{\infty} a_{z(k)}$  is conditionally convergent with sum  $r$
- The series  $\sum_{k=1}^{\infty} a_{z(k)}$  diverges to  $+\infty$
- The series  $\sum_{k=1}^{\infty} a_{z(k)}$  diverges to  $-\infty$
- The partial sums of the series  $\sum_{k=1}^{\infty} a_{z(k)}$  oscillate between any two real numbers

### Definition 1.8: Composition

Let  $A, B \subseteq \mathbb{R}$  be nonempty, let  $f : A \rightarrow \mathbb{R}$ ,  $g : B \rightarrow \mathbb{R}$  and  $f(A) \subseteq B$ . The composition of  $g$  with  $f$  is the function  $g \circ f : A \rightarrow \mathbb{R}$  defined by

$$(g \circ f)(x) = g(f(x)), \quad \text{for all } x \in A$$

**Thm 1.11:** If  $f$  is continuous at  $a \in \mathbb{R}$  and  $g$  is continuous at  $f(a)$  then the composition  $g \circ f$  is continuous at  $a$

## 2 Uniform convergence

### Definition 2.1: Pointwise Convergence

Let  $E$  be a nonempty subset of  $\mathbb{R}$ . A sequence of functions  $f_n : E \rightarrow \mathbb{R}$  is said to **converge pointwise** on  $E$ , written  $f_n \rightarrow f$  pointwise on  $E$  as  $n \rightarrow \infty$ , iff  $f(x) = \lim_{n \rightarrow \infty} f_n(x)$  exists for each  $x \in E$

$f_n$  converges pointwise on  $E$ , as  $n \rightarrow \infty$ , iff for every  $\epsilon > 0$  and  $x \in E$  there is an  $N \in \mathbb{N}$  (which may depend on  $x$  as well we  $\epsilon$ ) such that

$$n \geq N \quad \text{implies} \quad |f_n(x) - f(x)| < \epsilon$$

#### Remarks:

- The pointwise limit of continuous (or differentiable) functions is not necessarily continuous (or differentiable).
- The pointwise limit of integrable functions is not always integrable.
- There exist continuous functions  $f_n$  and  $f$  such that  $f_n \rightarrow f$  pointwise on  $[0, 1]$  but

$$\lim_{n \rightarrow \infty} \int_0^1 f_n(x) dx \neq \int_0^1 \left( \lim_{n \rightarrow \infty} f_n(x) \right) dx$$

**Def 2.2:** Let  $E$  be a nonempty subset of  $\mathbb{R}$ . A sequence of functions  $f_n : E \rightarrow \mathbb{R}$  is said to **converge uniformly** on  $E$  to a function  $f$  (notation:  $f_n \rightarrow f$  uniformly on  $E$  as  $n \rightarrow \infty$ ) if and only if for every  $\epsilon > 0$  there is an  $N \in \mathbb{N}$  such that for all  $x \in E$

$$n \geq N \quad \text{implies} \quad |f_n(x) - f(x)| < \epsilon$$

**Remark (The difference between Pointwise and Uniform):** For a sequence of functions to be pointwise convergent, it is enough to have an  $N_n$  for every  $x_n$ , but for it to be uniformly convergent, it has to have **the same**  $N$  for every  $x$  in the sequence

**Def 2.2:** A sequence of functions  $f_n$  is said to be **uniformly bounded** on a set  $E$  if there is a  $M > 0$  such that  $|f_n(x)| \leq M$  for all  $x \in E$  and all  $n \in \mathbb{N}$

### Definition 2.3: Convergence of series

Let  $f_k$  be a sequence of real functions defined on some set  $E$  and set

$$s_n(x) = \sum_{k=1}^n f_k(x), \quad x \in E, n \in \mathbb{N}$$

- The series  $\sum_{k=1}^{\infty} f_k$  **converges pointwise** on  $E$  iff the sequence  $s_n(x)$  converges pointwise on  $E$  as  $n \rightarrow \infty$
- The series  $\sum_{k=1}^{\infty} f_k$  **converges uniformly** on  $E$  iff the sequence  $s_n(x)$  converges uniformly on  $E$  as  $n \rightarrow \infty$
- The series  $\sum_{k=1}^{\infty} f_k$  **converges absolutely** (pointwise) on  $E$  iff  $\sum_{k=1}^{\infty} |f_k(x)|$  converges for each  $x \in E$

### Theorem 2.1 - 2.3: Uniform Continuity Theorems

- The following are equivalent concerning a sequence of functions  $f_n : E \rightarrow \mathbb{R}$  and  $f : E \rightarrow \mathbb{R}$ :
  - $f_n \rightarrow f$  uniformly on  $E$
  - $\sup_{x \in E} |f_n(x) - f(x)| \rightarrow 0$  as  $n \rightarrow \infty$
  - there exists a seq  $a_n \rightarrow 0$  s.t.  $|f_n(x) - f(x)| \leq a_n, \forall x \in E$

**2.1:** Let  $E$  be a nonempty subset of  $\mathbb{R}$  and suppose that  $f_n \rightarrow f$  uniformly on  $E$  as  $n \rightarrow \infty$ . If each  $f_n$  is continuous at some  $x_0 \in E$ , then  $f$  is continuous at  $x_0 \rightarrow E$

**2.2:** Suppose that  $f_n \rightarrow f$  uniformly on a closed interval  $[a, b]$ . If each  $f_n$  is integrable on  $[a, b]$ , then so is  $f$  and

$$\lim_{n \rightarrow \infty} \int_a^b f_n(x) dx = \int_a^b \left( \lim_{n \rightarrow \infty} f_n(x) \right) dx$$

**2.3:** Let  $(a, b)$  be a bounded interval and suppose that  $f_n$  is a sequence of functions which converges at some  $x_0 \in (a, b)$ . If each  $f_n$  is differentiable on  $(a, b)$ , and  $f'_n$  converges uniformly on  $(a, b)$  as  $n \rightarrow \infty$ , then  $f_n$  converges uniformly on  $(a, b)$  and

$$\lim_{n \rightarrow \infty} f'_n(x) = \left( \lim_{n \rightarrow \infty} f_n(x) \right)'$$

### Theorem 2.4: Results of Convergent Series

Let  $E$  be a nonempty subset of  $\mathbb{R}$  and let  $(f_k)$  be a sequence of real functions defined on  $E$ .

- Suppose that  $x_0 \in E$  and that each  $f_k$  is continuous at  $x_0 \in E$ .

If  $f = \sum_{k=1}^{\infty} f_k$  converges uniformly on  $E$ , then  $f$  is continuous at  $x_0 \in E$ .

- Term-by-term integration: Suppose that  $E = [a, b]$  and that each  $f_k$  is integrable on  $[a, b]$ . If  $f = \sum_{k=1}^{\infty} f_k$  converges uniformly on  $[a, b]$ , then  $f$  is integrable on  $[a, b]$  and

$$\int_a^b \sum_{k=1}^{\infty} f_k(x) dx = \sum_{k=1}^{\infty} \int_a^b f_k(x) dx$$

- Term-by-term differentiation: Suppose that  $E$  is a bounded, open interval and that each  $f_k$  is differentiable on  $E$ . If  $\sum_{k=1}^{\infty} f_k(x_0)$  converges at some  $x_0 \in E$ , and  $g = \sum_{k=1}^{\infty} f'_k(k)$  converges uniformly on  $E$ , then  $f = \sum_{k=1}^{\infty} f_k$  converges uniformly on  $E$ , is differentiable on  $E$ , and

$$f'(x) = \left( \sum_{k=1}^{\infty} f_k(x) \right)' = \sum_{k=1}^{\infty} f'_k(x) = g(x)$$

for  $x \in E$

### Theorem 2.5: Weierstrass M-test

Let  $E$  be a nonempty subset of  $\mathbb{R}$ , let  $f_k : E \rightarrow \mathbb{R}, k \in \mathbb{N}$ , and suppose that  $M_k > 0$  satisfies  $\sum_{k=1}^{\infty} M_k < \infty$ . If  $|f_k(x)| \leq M_k$  for  $k \in \mathbb{N}$  and  $x \in E$ , then  $f = \sum_{k=1}^{\infty} f_k$  converges absolutely and uniformly on  $E$ .

### Definition 3.1: Power Series and RoC

The **radius of convergence**  $R$  of the power series

$$\sum_{n=0}^{\infty} a_n(x-c)^n \quad (*)$$

is defined by

$$R = \sup\{r \geq 0 : (a_n r^n) \text{ is bounded}\}$$

unless  $(a_n r^n)$  is bounded for all  $r \geq 0$ , where we say that  $R = \infty$

**Thm 3.1:** Suppose the radius of convergence  $R$  of  $*$  satisfies  $0 < R < \infty$ . If  $|x - c| < R$ , the power series  $*$  converges absolutely. If  $|x - c| > R$ , the power series  $*$  diverges

### Theorem 3.2: Continuity of Power Series

Assume that  $R > 0$ . Suppose that  $0 < r < R$ . Then a power series converges uniformly and absolutely on  $|x - c| \leq r$  to a continuous function  $f$ . Hence

$$f(x) = \sum_{n=0}^{\infty} a_n(x-c)^n$$

defines a continuous function  $f : (c - R, c + R) \rightarrow \mathbb{R}$

**Lemma 3.1:** The two power series

$$\sum_{n=1}^{\infty} a_n(x-c)^n \quad \text{and} \quad \sum_{n=1}^{\infty} n a_n(x-c)^{n-1}$$

have the same radius of convergence

### Theorem 3.3: Differentiation of Power Series

Suppose the radius of convergence of a power series is  $R$ . Then the function

$$f(x) = \sum_{n=0}^{\infty} a_n(x-c)^n$$

is infinitely differentiable on  $|x - c| < R$ , and for such  $x$ ,

$$f'(x) = \sum_{n=0}^{\infty} n a_n(x-c)^{n-1}$$

and the series converges absolutely, and also uniformly on  $[c - r, c + r]$  for any  $r < R$ . Moreover,

$$a_n = \frac{f^{(n)}(c)}{n!}$$

### 3 Lebesgue Integration

#### Definition 4.0: Characteristic Function

Let  $E$  be a subset of  $\mathbb{R}$ . We define its **characteristic function**  $\chi_E : \mathbb{R} \rightarrow \mathbb{R}$  by  $\chi_E(x) = 1$  if  $x \in E$  and  $\chi_E(x) = 0$  if  $x \notin E$ . In other words,

$$\chi_E = \begin{cases} 1 & x \in E \\ 0 & x \notin E \end{cases}$$

In other words, this is a function that is 1 at all points of a bounded interval, and 0 elsewhere

Let  $I$  be a bounded interval with endpoints  $a, b$  and  $a \leq b$ . We call the number  $b - a$  the **length of the interval**  $I$  and we denote it by  $\lambda(I)$ . This might also be referred to as  $|I|$ . That is,

$$\lambda((a, b)) = \lambda([a, b]) = \lambda((a, b]) = \lambda([a, b)) = b - a$$

From our definition of a characteristic function and the length of an interval, we have that the area of the characteristic function is a rectangle with width  $\lambda(I)$  and height 1, therefore

$$\int \chi_I = 1 \cdot \lambda(I) = \lambda(I)$$

#### Definition 4.1: Step function

We say that  $\phi : \mathbb{R} \rightarrow \mathbb{R}$  is a **step function** if there exist real numbers  $x_0 < x_1 < x_2 < \dots < x_n$  (for some  $n \in \mathbb{N}$ ) such that

1.  $\phi(x) = 0$  for  $x < x_0$  and  $x > x_n$
2.  $\phi$  is constant on  $(x_{j-1}, x_j)$  for  $1 \leq j \leq n$

We shall use the phrase " $\phi$  is a step function with respect to  $\{x_0, x_1, \dots, x_n\}$ " to describe this situation

#### Properties of Step Functions

1. The class of step functions is a vector space - i.e. if  $\phi$  and  $\psi$  are step functions and  $\alpha$  and  $\beta$  are real numbers, then  $\alpha\phi + \beta\psi$  is a step function, and that if  $\phi$  and  $\psi$  are step functions, then  $\max\{\phi, \psi\}$ ,  $\min\{\phi, \psi\}$ ,  $|\phi|$  and  $\phi\psi$  are also step functions
2. If  $\phi$  and  $\psi$  are step functions, then  $\phi + \psi$  is a step function
3.  $\phi$  is a step function if and only if it is of the form

$$\phi = \sum_{j=1}^n c_j \chi_{J_j}$$

for some  $n$ ,  $c_j$ , and bounded intervals  $J_j$

#### Def 4.2: Integral of a Step Function

If  $\phi$  is a step function with respect to  $\{x_0, x_1, \dots, x_n\}$  which takes the value  $c_j$  on  $(x_{j-1}, x_j)$ , then

$$\int \phi := \sum_{j=1}^n c_j (x_j - x_{j-1})$$

Therefore, using the characteristic definition of a step function, the integral is

$$\int \phi = \int \sum_{j=1}^n c_j \chi_{J_j} = \sum_{j=1}^n c_j \int \chi_{J_j} = \sum_{j=1}^n c_j \lambda(J_j)$$

#### Definition 4.3: Lebesgue Integrals

A function  $f : I \rightarrow \mathbb{R}$  is said to be **integrable** or more precisely **Lebesgue integrable** on an interval  $I$  if there exist numbers  $c_j$  and bounded intervals  $J_j \subset I$ ,  $j = 1, 2, 3, \dots$  such that

$$\sum_{j=1}^{\infty} |c_j| \lambda(J_j) < \infty$$

and the equality

$$f(x) = \sum_{j=1}^{\infty} c_j \chi_{J_j}(x)$$

holds for all  $x \in I$  at which

$$\sum_{j=1}^{\infty} |c_j| \chi_{J_j}(x) < \infty$$

We denote by  $\int_I f$  the number

$$\int_I f = \sum_{j=1}^{\infty} c_j \lambda(J_j)$$

and call it the integral of  $f$  over the interval  $I$ . If the function  $f$  is not integrable on the interval  $I$  then we say that the integral of  $f$  on  $I$  does not exist. Hence if we say that the integral of  $f$  on  $I$  exists it just means that  $f$  is (Lebesgue) integrable on  $I$ .

#### Theorem 4.1: Lebesgue Equality

Suppose that  $c_j, d_j$  are real numbers and  $J_j, K_j$  are bounded intervals for all  $j = 1, 2, 3, \dots$ , and

$$\sum_{j=1}^{\infty} |c_j| \lambda(J_j) < \infty, \quad \sum_{j=1}^{\infty} |d_j| \lambda(K_j) < \infty$$

If

$$\sum_{j=1}^{\infty} c_j \chi_{J_j}(x) = \sum_{j=1}^{\infty} d_j \chi_{K_j}(x)$$

holds for all  $x$  such that

$$\sum_{j=1}^{\infty} |c_j| \chi_{J_j}(x) < \infty, \quad \sum_{j=1}^{\infty} |d_j| \chi_{K_j}(x) < \infty$$

Then

$$\sum_{j=1}^{\infty} c_j \lambda(J_j) = \sum_{j=1}^{\infty} d_j \lambda(K_j)$$

### Theorem 4.2: Lebesgue Integral Properties

Suppose  $f$  and  $g$  are integrable on  $I$  and  $\alpha$  and  $\beta$  are real numbers. Then

1.  $\alpha f + \beta g$  is integrable on  $I$  and

$$\int_I (\alpha f + \beta g) = \alpha \int_I f + \beta \int_I g$$

2. If  $f \geq 0$  on  $I$  then  $\int_I f \geq 0$ ; if  $f \geq g$  on  $I$  then  $\int_I f \geq \int_I g$
3.  $|f|$  is integrable on  $I$  and  $\left| \int_I f \right| \leq \int_I |f|$
4.  $\max\{f, g\}$  and  $\min\{f, g\}$  are integrable on  $I$
5. If one of the functions is bounded then the product  $fg$  is integrable on  $I$
6. If  $f \geq 0$  with  $\int_I f = 0$  then any function  $h$  such that  $0 \leq h \leq f$  on  $I$  is integrable on  $I$

### Theorem 4.3: Integrability of Sequences and Series

Suppose that  $(f_n)_{n \in \mathbb{N}}$  is a sequence of functions each of which is integrable on  $I$

1. Assume that

$$\sum_{n=1}^{\infty} \int_I |f_n| < \infty$$

Let  $f$  be a function on the interval  $I$  such that

$$f(x) = \sum_{n=1}^{\infty} f_n(x) \quad \text{for all } x \in I \text{ such that } \sum_{n=1}^{\infty} |f_n(x)| < \infty$$

Then  $f$  is integrable on  $I$  and its integral on  $I$  is equal to

$$\int_I f = \sum_{n=1}^{\infty} \int_I f_n$$

2. Assume that each  $f_n \geq 0$  on  $I$  and let  $f(x) = \sum_{n=1}^{\infty} f_n(x)$  for all  $x \in I$  (we allow for the possibility that at some points this sum is infinite). Then  $f$  is integrable on  $I$  if and only if

$$\sum_{n=1}^{\infty} \int_I f_n < \infty$$

### Theorem 4.4: Monotone Convergence Theorem

Suppose that  $(f_n)$  is a monotone increasing sequence of integrable functions on an interval  $I$ . That is,  $f_1(x) \leq f_2(x) \leq f_3(x) \leq \dots$  for all  $x \in I$ . For all  $x \in I$ , let

$$f(x) = \lim_{n \rightarrow \infty} f_n(x)$$

where we allow for the possibility that at some points this limit is infinite. Then  $f$  is integrable on  $I$  iff

$$\sup_{n \in \mathbb{N}} \int_I f_n = \lim_{n \rightarrow \infty} \int_I f_n < \infty. \quad \text{Also, } \int_I f = \lim_{n \rightarrow \infty} \int_I f_n$$

### Definition 4.4: Riemann Integrable Functions

Let  $f : \mathbb{R} \rightarrow \mathbb{R}$ . We say that  $f$  is **Riemann-integrable** if for every  $\epsilon > 0$  there exists step functions  $\phi$  and  $\psi$  such that

$$\phi \leq f \leq \psi$$

and

$$\int \psi - \int \phi < \epsilon$$

---

**Thm 4.5:** A function  $f : \mathbb{R} \rightarrow \mathbb{R}$  is Riemann-integrable iff

$$\sup \left\{ \int \phi : \phi \text{ is a step function and } \phi \leq f \right\} \\ = \inf \left\{ \int \psi : \psi \text{ is a step function and } \phi \geq f \right\}$$


---

**Def 4.5:** If  $f$  is Riemann-integrable we define its Riemann integral  $(R) \int f$  as the common value

$$(R) \int f := \sup \left\{ \int \phi : \phi \text{ is a step function and } \phi \leq f \right\} \\ = \inf \left\{ \int \psi : \psi \text{ is a step function and } \phi \geq f \right\}$$

### Theorem 4.6: Connection between Riemann and Lebesgue

Suppose that  $f : \mathbb{R} \rightarrow \mathbb{R}$  is Riemann-integrable. Then  $f$  is also Lebesgue integrable on  $\mathbb{R}$  and moreover

$$(R) \int f = \int f$$

where the number on the lefthand side is the value of the Riemann integral of  $f$ , while the righthand side denotes the value of the Lebesgue integral of  $f$  on  $\mathbb{R}$

### Theorem 4.1: Riemann lemmas

Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a bounded function with bounded support  $[a, b]$ . The following are equivalent:

1.  $f$  is Riemann-integrable
2. for every  $\epsilon > 0$  there exists  $a = x_0 < \dots < x_n = b$  s.t. if  $M_j$  and  $m_j$  denote the supremum and infimum values of  $f$  on  $(x_{j-1}, x_j)$  respectively, then

$$\sum_{j=1}^n (M_j - m_j)(x_j - x_{j-1}) < \epsilon$$

3. for every  $\epsilon > 0$  there exists  $\alpha = x_0 < \dots < x_n = b$  s.t. with  $I_j = (x_{j-1}, x_j)$  for  $j \geq 1$

$$\sum_{j=1}^n \sup_{x, y \in I_j} |f(x) - f(y)| \lambda(I_j) < \epsilon$$

---

Notation to aid these lemmas: For  $f : \mathbb{R} \rightarrow \mathbb{R}$  a bounded function with bounded support  $[a, b]$  and for  $a = x_0 < \dots < x_n = b$ , we let  $I_j = (x_{j-1}, x_j)$ ,  $m_j := \inf_{x \in I_j} f(x)$  and  $M_j := \sup_{x \in I_j} f(x)$ . We define the **lower step function of  $f$  with respect to  $\{x_0, \dots, x_n\}$**  as

$$\phi_*(x) = \sum_{j=1}^n m_j \chi_{I_j}(x) + \sum_{j=0}^n f(x_j) \chi_{\{x_j\}}(x)$$

and the **upper step function of  $f$  with respect to  $\{x_0, \dots, x_n\}$**  as

$$\phi^*(x) = \sum_{j=1}^n M_j \chi_{I_j}(x) + \sum_{j=0}^n f(x_j) \chi_{\{x_j\}}(x)$$

**Note:**  $\phi_*(x)$  and  $\phi^*(x)$  are step functions, and that  $\phi_*(x) \leq f \leq \phi^*(x)$

---

Suppose that  $g : [a, b] \rightarrow \mathbb{R}$  and let  $f$  be defined by  $f(x) = g(x)$  for  $x \in [a, b]$  and  $f(x) = 0$  otherwise.

1. If  $g$  is continuous on  $[a, b]$ , then  $f$  is Riemann-integrable
2. If  $g$  is a monotone function then  $f$  is Riemann-integrable

#### Theorem 4.8: Dependence on Intervals for Lebesgue

Let  $I$  and  $J$  be two intervals such that  $J \subset I$ .

1. If  $f$  is integrable on  $I$  then  $f$  is also integrable on the subinterval  $J$
2. If  $f$  is integrable on  $J$  and simultaneously  $f(x) = 0$  for all  $x \in I \setminus J$  then  $f$  is integrable on  $I$  and

$$\int_J f = \int_I f$$

3. If  $f$  is integrable on  $I$  and  $f(x) \geq 0$  for all  $x \in I$  then

$$\int_J f \leq \int_I f$$

4. Suppose that  $I$  can be written as the union of disjoint intervals  $I_n$ ,  $n = 1, 2, 3, \dots$  and let  $f$  be integrable on each of the intervals  $I_n$ . Then  $f$  is integrable on  $I$  iff

$$\sum_{n=1}^{\infty} \int_{I_n} |f| < \infty$$

If this holds, then

$$\int_I f = \sum_{n=1}^{\infty} \int_{I_n} f$$

#### Theorem 4.9: Addition of Intervals

If any two of these integrals

$$\int_a^b f, \quad \int_b^c f, \quad \int_a^c f$$

exist then so does the third and

$$\int_a^b f + \int_b^c f = \int_a^c f$$

#### Theorem 4.10: Fundamental Theorem of Calculus

Let  $I$  be an interval and let  $g : I \rightarrow \mathbb{R}$  be integrable on  $I$ . For all  $x \in I$  and some fixed  $x_0 \in I$  let  $G(x) = \int_{x_0}^x g$ . Suppose  $g$  is continuous at  $x$  for some  $x \in I$  [if  $x$  is an endpoint we mean one-sided continuity.] Then  $G$  is differentiable at  $x$  and  $G'(x) = g(x)$ . [if  $x$  is an endpoint we mean one-sided differentiable]

Suppose  $f : I \rightarrow \mathbb{R}$  has continuous derivative  $f'$  on the interval  $I$ . Then for any  $a, b \in I$ :

$$\int_a^b f' = f(b) - f(a)$$

#### Lemma 4.2: Fatoux Lemma

Let  $(f_n)$  be a sequence of non-negative integrable functions on an interval  $I$ . Let

$$f(x) = \liminf_{n \rightarrow \infty} f_n(x), \quad \text{for all } x \in I$$

If  $\liminf_{n \rightarrow \infty} \int_I f_n < \infty$  then  $f$  is integrable on  $I$  and

$$\int_I f \leq \liminf_{n \rightarrow \infty} \int_I f_n$$

#### Theorem 4.12: Dominated Convergence Theorem

Let  $(f_n)$  be a sequence of integrable functions on an interval  $I$  and assume that

$$f(x) = \lim_{n \rightarrow \infty} f_n(x), \quad \text{for all } x \in I$$

. Assume also that the sequence  $(f_n)$  is **dominated** by some integrable function  $g$ , that is

$$|f_n(x)| \leq g(x), \quad \text{for all } x \in I \text{ and } n = 1, 2, \dots, \quad \int_I g < \infty$$

Then the function  $f$  is integrable on  $I$  and

$$\int_I f = \lim_{n \rightarrow \infty} \int_I f_n$$

#### Theorem 4.13

Let  $(a, b)$  be a bounded interval and suppose that  $f_n : (a, b) \rightarrow \mathbb{R}$  are integrable functions which converges uniformly to a function  $f$ . Then  $f$  is integrable on  $(a, b)$  and

$$\int_a^b f = \lim_{n \rightarrow \infty} \int_a^b f_n$$

## 4 Fourier Series and Orthogonality

#### Definition 5.1: The Space $L^2$

Define the space  $L^2 = L^2([a, b])$  as the set of measurable functions  $f : [a, b] \rightarrow \mathbb{C}$  so that the function  $x \mapsto |f(x)|^2$  is Lebesgue integrable, i.e.

$$\|f\|_2^2 := \int_a^b |f(x)|^2 dx < \infty$$

The quantity  $\|f\|_2$  is called the  **$L^2$ -norm** of  $f$ . If  $\|f\|_2 = 1$ , then we say that  $f$  is  **$L^2$ -normalised**

#### Definition 5.2: Inner Product

For two functions  $f, g \in L^2([a, b])$ , we define their **inner product** by

$$\langle f, g \rangle = \int_a^b f(x) \overline{g(x)} dx$$

#### Theorem 5.1: Cauchy-Shwarz Inequality

Let  $f, g \in L^2([a, b])$ . then the function  $x \mapsto f(x) \overline{g(x)}$  is Lebesgue integrable and we have

$$|\langle f, g \rangle| \leq \|f\|_2 \|g\|_2$$

**Minkowski's Inequality:** For two functions  $f, g \in L^2([a, b])$ ,

$$\|f + g\|_2 \leq \|f\|_2 + \|g\|_2$$

#### Definition 5.3: Convergent Sequences in $L^2$

Let  $f, f_1, f_2, \dots$  be functions in  $L^2([a, b])$ . We say that the function  $(f_n)_n$  converges to  $f$  in  $L^2$  if the sequence

$$\|f_n - f\|_2 = \left( \int_a^b |f_n(x) - f(x)|^2 dx \right)^{1/2}$$

converges to zero as  $n \rightarrow \infty$ . We will also write  $f_n \rightarrow f$  in  $L^2$

#### Definition 5.4: Orthonormal Systems

A sequence  $(\phi_n)_n$  of  $L^2$  functions on  $[a, b]$  is called an **orthonormal system** on  $[a, b]$  if

$$\langle \phi_n, \phi_m \rangle = \int_a^b \phi_n(x) \overline{\phi_m(x)} dx = \begin{cases} 0, & \text{if } n \neq m \\ 1, & \text{if } n = m \end{cases}$$

(The index  $n$  may run over any countable set. We will write  $\sum_n$  to denote a sum over all the indices. In proofs we will always adopt the interpretation that  $n$  runs over  $1, 2, 3, \dots$  without loss of generality)

#### Theorem 5.2

Let  $(\phi_n)_n$  be an orthonormal system on  $[a, b]$  and  $f \in L^2$ . Consider

$$s_N(x) = \sum_{n=1}^N \langle f, \phi_n \rangle \phi_n(x)$$

Denote the linear span of the functions  $(\phi_n)_{n=1, \dots, N}$  by  $X_N$ . Then

$$\|f - s_N\|_2 \leq \|f - g\|_2$$

holds for all  $g \in X_N$  with equality iff  $g = s_N$

#### Definition 5.3: Bessel's Inequality

If  $(\phi_n)_n$  is an orthonormal system on  $[a, b]$  and  $f \in L^2$ , then

$$\sum_n |\langle f, \phi_n \rangle|^2 \leq \|f\|_2^2$$

**Corollary - Riemann-Lebesgue lemma in  $L^2$ .** Let  $(\phi_n)_{n=1, 2, \dots}$  be an orthonormal system and  $f \in L^2$ , then

$$\lim_{n \rightarrow \infty} \langle f, \phi_n \rangle = 0$$

### Definition 5.5: Complete Orthonormal Systems

An orthonormal system  $(\phi_n)_n$  is called **complete** if

$$\sum_n |\langle f, \phi_n \rangle|^2 = \|f\|_2^2$$

for all  $f \in L^2$

**Thm 5.4:** Let  $(\phi_n)_n$  be an orthonormal system on  $[a, b]$ . Let  $(s_N)_N$  be as in Theorem 5.2. Then  $(\phi_n)_n$  is complete iff  $(s_N)_N$  converges to  $f$  in the  $L^2$ -norm for every  $f \in L^2$

### Definition 5.6: Trigonometric Polynomials

A **trigonometric polynomial** is a function of the form

$$f(x) = \sum_{n=-N}^N c_n e^{2\pi i n x} \quad (x \in \mathbb{R})$$

where  $N \in \mathbb{N}$  and  $c_n \in \mathbb{C}$ . If  $c_N$  or  $c_{-N}$  is non-zero, then  $N$  is called the **degree** of  $f$ . Observe that trigonometric polynomials are continuous functions. Moreover, from Euler's identity  $e^{ix} = \cos(x) + i \sin(x)$ , ( $x \in \mathbb{R}$ ) we see that every trigonometric polynomial can also be written in the alternate form

$$f(x) = a_0 + \sum_{n=0}^N (a_n \cos(2\pi n x) + b_n \sin(2\pi n x))$$

**Lemma 5.1:**  $(e^{2\pi i n x})_{n \in \mathbb{Z}}$  forms an orthonormal system on  $[0, 1]$ . In particular,

- for all  $n \in \mathbb{Z}$ ,

$$\int_0^1 e^{2\pi i n x} dx = \begin{cases} 0, & \text{if } n \neq 0 \\ 1, & \text{if } n = 0 \end{cases}$$

- if  $f(x) = \sum_{n=-N}^N c_n e^{2\pi i n x}$  is a trigonometric polynomial, then

$$c_n = \langle f, \phi_n \rangle = \int_0^1 f(t) e^{-2\pi i n t} dt$$

### Definition 5.7: Fourier Coefficient

For a 1-periodic integrable function  $f$  and  $n \in \mathbb{Z}$  we define the  **$n$ th Fourier coefficient** by

$$\widehat{f}(n) = \int_0^1 f(t) e^{-2\pi i n t} dt = \langle f, \phi_n \rangle$$

(the integral on the right exists since  $f$  is integrable and  $|\phi_n| \leq 1$ .) The doubly infinite series

$$\sum_{n=-\infty}^{\infty} \widehat{f}(n) e^{2\pi i n x}$$

is called the **Fourier series** of  $f$

**Def 5.8 (Partial Sums):** For a 1-periodic integrable function  $f$ , we define the **partial sums**

$$S_N f(x) = \sum_{n=-N}^N \widehat{f}(n) e^{2\pi i n x}$$

**Note:** for all  $f \in L^2$  and trigonometric polynomials  $g$  of degree  $\leq N$ , we have

$$\|f - S_N f\|_2 \leq \|f - g\|_2$$

### Definition 5.9: Convolution

For two 1-periodic functions  $f, g \in L^2$  we define their **convolution** by

$$f * g(x) = \int_0^1 f(t) g(x - t) dt$$

(The integral on the right hand side exists by Cauchy-Schwarz)

**Lemma bank**

**5.2** For 1-periodic functions  $f, g \in L^2$ ,

$$f * g = g * f$$

**5.3** We have

$$D_N(x) = \frac{\sin(2\pi(N + \frac{1}{2})x)}{\sin(\pi x)}$$

**5.4** We have

$$\begin{aligned} K_N(x) &= \frac{1}{2(N+1)} \frac{1 - \cos(2\pi(N+1)x)}{\sin(\pi x)^2} \\ &= \frac{1}{N+1} \left( \frac{\sin(\pi(N+1)x)}{\sin(\pi x)} \right)^2 \end{aligned}$$

**Thm 5.5 (Fejér):** For every 1-periodic continuous function  $f$ ,

$$K_N * f \rightarrow f$$

uniformly on  $\mathbb{R}$  as  $N \rightarrow \infty$

**Corollary:** Every 1-periodic continuous function can be uniformly approximated by trigonometric polynomials. That is, for every 1-periodic continuous  $f$  there exists a sequence  $(f_n)_n$  of trigonometric polynomials so that  $f_n \rightarrow f$  uniformly

### Definition 5.10: Approximation of Unity

A sequence of 1-periodic integrable functions  $(k_n)_n$  is called **approximation of unity** if for all 1-periodic continuous functions  $f$  we have that  $f * k_n$  converges uniformly to  $f$  on  $\mathbb{R}$ . That is,

$$\sup_{x \in \mathbb{R}} |f * k_n(x) - f(x)| \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

**Thm 5.6:** Let  $(k_n)_n$  be a sequence of 1-periodic integrable functions such that

- $k_n(x) \geq 0$  for all  $x \in \mathbb{R}$
- $\int_{-1/2}^{1/2} k_n(t) dt = 1$
- For all  $1/2 \geq \delta > 0$  we have

$$\int_{-\delta}^{\delta} k_n(t) dt \rightarrow 1 \quad \text{as } n \rightarrow \infty$$

Then  $(k_n)_n$  is an approximation of unity

**Corollary:** The Fejér kernel  $(K_N)_N$  is an approximation of unity

### Lemma 5.5

Let  $f$  be a 1-periodic and continuous function. Then

$$\lim_{N \rightarrow \infty} \|S_N f - f\|_2 = 0$$

### Theorem 5.7: Completeness of Trigonometric System

The trigonometric system is complete. In view of Theorem 5.4 this means that for every 1-periodic  $L^2$  function  $f$  we have

$$\lim_{N \rightarrow \infty} \|S_N f - f\|_2 = 0$$

In other words, the Fourier series of  $f$  converges to  $f$  in the  $L^2$  sense

**Corollary (Parseval's Theorem):** If  $f, g$  are 1-periodic  $L^2$  functions then

$$\langle f, g \rangle = \sum_{n=-\infty}^{\infty} \widehat{f}(n) \overline{\widehat{g}(n)}$$

In particular,

$$\|f\|_2^2 = \sum_{n=-\infty}^{\infty} |\widehat{f}(n)|^2$$

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