Honours Analysis Exam Notes

Made by Leon :) *Note: Any reference numbers are to the lecture notes*

1 Revisiting FPM

Definition 1.1: Nested Sequences and covers

A sequence $(I_n)_{n\in\mathbb{N}}$ of sets is said to be **nested** if

$$I_1 \subset I_2 \subset I_3 \subset \cdots$$

Thm 1.1: If (I_n) is a nested sequence of nonempty closed bounded intervals then

$$E = \bigcap_{n\in\mathbb{N}} I_n = \{x \in \mathbb{R} : x \in I_n,\ \forall n \in \mathbb{N}\}$$

is nonempty (i.e. it contains at least one number). Moreover if $\lambda(I_n) \to 0$, where $\lambda(I_n)$ denotes the length of interval I_n, then E contains exactly one number

Thm 1.2: Let E be a subset of $\mathbb{R}^n$

- A **cover** of E is a collection of sets $\{I_\alpha\}_{\alpha\in A}$ such that

$$E \subseteq \bigcup_{\alpha\in A} I_\alpha$$

- An **open covering** of E is a cover such that each I_α is open, i.e.(a, b) compared to $[a, b]$
- A **finite subcover** of E is a collection of sets $(I_\alpha)_{\alpha\in A_0}$ where there exists a subset $A_0 = \{\alpha_1, \alpha_2, \ldots, a_N\}$ of A such that $(I_\alpha)_{\alpha\in A_0}$ is a finite subset of $(I_\alpha)_{\alpha\in A}$ that is also a cover
- The set E is said to be **compact** iff every open covering of E has a **finite subcovering**; that is

$$E \subseteq \bigcup_{j=1}^{N} I_{aj} \quad \text{or} \quad E \subseteq I_{\alpha_1} \cup I_{a_2} \cup \cdots \cup I_{a_N}$$

Definition 1.2: Convergence of Sequences and Cauchy

A sequence of real numbers (x_n) is said to **converge** to a real number $a \in \mathbb{R}$ iff for every $\epsilon > 0$ there is an $N \in \mathbb{N}$ such that

$$n \geq N \text{ implies } |x_n - a| < \epsilon$$

If (x_n) converges to a, we will write $\lim_{n\to\infty} x_n = a$, or $x_n \to a$. The number a is called the limit of the sequence (x_n). A sequence that does not converge to some real number is said to *diverge

Def 1.3: A sequence (x_n) of numbers $x_n \in \mathbb{R}$ is said to be **Cauchy** if for every $\epsilon > 0$ there is $N \in \mathbb{N}$ such that

$$|x_n - x_m| < \epsilon \quad \forall n, m \geq N$$

Thm 1.3: Let (x_n) be a sequence of real numbers. Then (x_n) is a Cauchy sequence if and only if (x_n) is a convergent sequence.
Thm 1.4: Let (x_n) be a sequence of real numbers. Then (x_n) is a Cauchy sequence iff (x_n) is a convergent sequence

Definition 1.4: Subsequences

Suppose $(x_n)_{n\in\mathbb{N}}$ is a sequence. A subsequence of this sequence is a sequence of the form $(x_{n_k})_{k\in\mathbb{N}}$ where for each k there is a positive integer n_k such that

$$n_1 < n_2 < \cdots < n_k < n_{k+1} < \cdots$$

Thus, $(x_n)_{n\in\mathbb{N}}$ is just a selection of some (possibly all) of the x_n's taken in order
Thm 1.5 (Bolzano-Weierstrass): Every bounded sequence of real numbers has a convergent subsequence

Definition 1.5: Limit Superior and Inferior

If (x_n) is a bounded sequence of real numbers we denote by

$$\limsup_{n\to\infty} x_n = \lim_{n\to\infty}\left(\sup_{k\geq n} x_k\right), \qquad \liminf_{n\to\infty} x_n = \lim_{n\to\infty}\left(\inf_{k\geq n} x_k\right)$$

Note: These are only defined for bounded sequences

- If (x_n) is not bounded from above then we write $\limsup_{n\to\infty} x_n = +\infty$
- If (x_n) is not bounded from below then we write $\liminf_{n\to\infty} x_n = +\infty$

Thm 1.6: A sequence (x_n) of real numbers is convergent if and only if $\limsup_{n\to\infty} x_n$ and $\liminf_{n\to\infty} x_n$ are real numbers and

$$\limsup_{n\to\infty} x_n = \liminf_{n\to\infty} x_n$$

Definition 1.7: Continuity

Let f be a function $f : \text{dom}(f) \to \mathbb{R}$ where $\text{dom}(f) \subset \mathbb{R}$. We say that f is **continuous** at some $a \in \text{dom}(f)$, if for any sequence (x_n) whose terms lie in $\text{dom}(f)$ and which converges to a, we have $\lim_{n\to\infty} f(x_n) = f(a)$.
If f is continuous at each $a \in S \subset \text{dom}(f)$ then we say f is continuous on S. If f is continuous of $\text{dom}(f)$ then we say f is continuous

Thm 1.10: Let $f, g : D \to \mathbb{R}$ be continuous on D, and let $\alpha \in \mathbb{R}$. Then the following functions are continuous on D :

1. α f
2. $f + g$
3. fg

Thm 1.12 ($\epsilon - \delta$ Definition of Continuity): Let f be a function $f : \text{dom}(f) \to \mathbb{R}$ where $\text{dom}(f) \subset \mathbb{R}$. Then f is continuous at $a \in \text{dom}(f)$ iff for any $\epsilon > 0$ there exists $\delta > 0$ s.t. whenever $x \in \text{dom}(f)$ and $|x - a| < \delta$ we have $|f(x) - f(a)| < \epsilon$

Thm 1.13 (Intermediate Value Theorem): Let $a < b$ real numbers and $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$. If $f(a)f(b) < 0$ then there exists at least one $c \in (a, b)$ s.t. $f(c) = 0$

Thm 1.14 (Extreme Value Theorem): Let $a < b$ real numbers and $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$. Then there exists points $c, d \in [a, b]$ s.t.

$$f(c) = \inf\{f(x) : x \in [a, b]\}, \quad f(d) = \sup\{f(x) : x \in [a, b]\}$$

That is, the function f on the interval $[a, b]$ is bounded and attains its minimal value at some point $c \in [a, b]$. Similarly, the maximal value of f is also attained at some point $d \in [a, b]$

Definition 1.6: Convergent Infinite Series

Let $S = \sum_{k=1}^{\infty} a_k$ be an infinite series a_k. For each $n \in \mathbb{N}$, the partial sum of S of order n is defined by

$$s_n = \sum_{k=1}^{n} a_k$$

S is said to **converge** iff its sequence of partial sums (s_n) converges to some $s \in \mathbb{R}$ as $n \to \infty$; that is, iff for every $\epsilon > 0$ there is an $N \in \mathbb{N}$ s.t. for all $n \geq N$ we have $|s_n - s| < \epsilon$. In this case we shall write

$$\sum_{k=1}^{\infty} a_k = s$$

and call s the **sum** or **value** of the series $\sum_{k=1}^{\infty} a_k$

- **Absolutely convergent**: the series $\sum_{k=1}^{\infty} |a_k|$ is convergent
- **Conditionally convergent**: Convergent but not absolutely

Thm 1.7 (Cauchy Criteron): Let $S = \sum_{k=1}^{\infty} a_k$ be a series. Then S is convergent iff for any $\epsilon > 0$ there exists N such that for all $m \geq n \geq N$ we have that

$$\left|\sum_{k=n+1}^{m} a_k\right| < \epsilon$$

Thm 1.8 (Rearranging Absolutely Convergent Series)
Let $S = \sum_{k=1}^{\infty} a_k$ be an absolutely convergent series. Then

- The series S is convergent
- Let $z : \mathbb{N} \to \mathbb{N}$ be a bijection. Then the series $\sum_{k=1}^{\infty} a_{z(k)}$ is convergent and

$$\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} a_{z(k)}$$

The series $\sum_{k=1}^{\infty} a_{z(k)}$ is called a **rearrangement** of the series $\sum_{k=1}^{\infty} a_k$. What we do here is add the terms of the sum in a different order to the original one, for example

$$a_3 + a_7 + a_1 + a_{100} + a_2 + \ldots$$

Since $z : \mathbb{N} \to \mathbb{N}$ is a bijection, we will miss no terms.

Thm 1.9 (Rearranging Conditionally Convergent Series)
Let $S = \sum_{k=1}^{\infty} a_k$ be any conditionally convergent series. Then there exists rearrangements $z : \mathbb{N} \to \mathbb{N}$ (where z is a bijection) such that

- For any $r \in \mathbb{R}$, $\sum_{k=1}^{\infty} a_{z(k)}$ is conditionally convergent with sum r
- The series $\sum_{k=1}^{\infty} a_{z(k)}$ diverges to $+\infty$
- The series $\sum_{k=1}^{\infty} a_{z(k)}$ diverges to $-\infty$
- The partial sums of the series $\sum_{k=1}^{\infty} a_{z(k)}$ oscillate between any two real numbers

Definition 1.8: Composition

Let $A, B \subseteq \mathbb{R}$ be nonempty, let $f : A \to \mathbb{R}$, $g : B \to \mathbb{R}$ and $f(A) \subseteq B$. The composition of g with f is the function $g \circ f : A \to \mathbb{R}$ defined by

$$(g \circ f)(x) = g(f(x)), \quad \text{for all } x \in A$$

Thm 1.11: If f is continuous at $a \in \mathbb{R}$ and g is continuous at $f(a)$ then the composition $g \circ f$ is continuous at a

2 Uniform convergence

Definition 2.1: Pointwise Convergence

Let E be a nonempty subset of $\mathbb{R}$. A sequence of functions $f_n : E \to \mathbb{R}$ is said to **converge pointwise** on E, written $f_n \to f$ pointwise on E as $n \to \infty$, iff $f(x) = \lim_{n\to\infty} f_n(x)$ exists for each $x \in E$

---

f_n converges pointwise on E, as $n \to \infty$, iff for every $\epsilon > 0$ and $x \in E$ there is an $N \in \mathbb{N}$ (which may depend on x as well we ϵ) such that

$$n \geq N \quad \text{implies} \quad |f_n(x) - f(x)| < \epsilon$$

Remarks:

- The pointwise limit of continuous (or differentiable) functions is not necessarily continuous (or differentiable).
- The pointwise limit of integrable functions is not always integrable.
- There exist continuous functions f_n and f such that $f_n \to f$ pointwise on $[0, 1]$ but

$$\lim_{n\to\infty} \int_0^1 f_n(x)\,dx \neq \int_0^1 \left(\lim_{n\to\infty} f_n(x)\right) dx$$

Def 2.2: Let E be a nonempty subset of $\mathbb{R}$. A sequence of functions $f_n : E \to \mathbb{R}$ is said to **converge uniformly** on E to a function f (notation: $f_n \to f$ uniformly on E as $n \to \infty$) if and only if for every $\epsilon > 0$ there is an $N \in \mathbb{N}$ such that for all $x \in E$

$$n \geq N \quad \text{implies} \quad |f_n(x) - f(x)| < \epsilon$$

Remark (The difference between Pointwise and Uniform): For a sequence of functions to be pointwise convergent, it is enough to have an N_n for every x_n, but for it to be uniformly convergent, it has to have **the same** N for every x in the sequence

---

Def 2.2: A sequence of functions f_n is said to be **uniformly bounded** on a set E if there is a $M > 0$ such that $|f_n(x)| \leq M$ for all $x \in E$ and all $n \in N$

Definition 2.3: Convergence of series

Let f_k be a sequence of real functions defined on some set E and set

$$s_n(x) = \sum_{k=1}^{n} f_k(x), \quad x \in E,\, n \in \mathbb{N}$$

- The series $\sum_{k=1}^{\infty} f_k$ **converges pointwise** on E iff the sequence $s_n(x)$ converges pointwise on E as $n \to \infty$
- The series $\sum_{k=1}^{\infty} f_k$ **converges uniformly** on E iff the sequence $s_n(x)$ converges uniformly on E as $n \to \infty$
- The series $\sum_{k=1}^{\infty} f_k$ **converges absolutely** (pointwise) on E iff $\sum_{k=1}^{\infty} |f_k(x)|$ converges for each $x \in E$

Theorem 2.1 - 2.3: Uniform Continuity Theorems

- The following are equivalent concerning a sequence of functions $f_n : E \to \mathbb{R}$ and $f : E \to \mathbb{R}$:
  - $f_n \to f$ uniformly on E
  - $\sup_{x\in E} |f_n(x) - f(x)| \to 0$ as $n \to \infty$
  - there exists a seq $a_n \to 0$ s.t. $|f_n(x) - f(x)| \leq a_n$, $\forall x \in E$

2.1: Let E be a nonempty subset of $\mathbb{R}$ and suppose that $f_n \to f$ uniformly on E as $n \to \infty$. If each f_n is continuous at some $x_0 \in E$, then f is continuous at $x_0 \to E$

2.2: Suppose that $f_n \to f$ uniformly on a closed interval $[a, b]$. If each f_n is integrable on $[a, b]$, then so is f and

$$\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b \left(\lim_{n\to\infty} f_n(x)\right) dx$$

2.3: Let (a, b) be a bounded interval and suppose that f_n is a sequence of functions which converges at some $x_0 \in (a, b)$. If each f_n is differentiable on (a, b), and f_n' converges uniformly on (a, b) as $n \to \infty$, then f_n converges uniformly on (a, b) and

$$\lim_{n\to\infty} f_n'(x) = \left(\lim_{n\to\infty} f_n(x)\right)'$$

Theorem 2.4: Results of Convergent Series

Let E be a nonempty subset of $\mathbb{R}$ and let (f_k) be a sequence of real functions defined on E.

- Suppose that $x_0 \in E$ and that each f_k is continuous at $x_0 \in E$. If $f = \sum_{k=1}^{\infty} f_k$ converges uniformly on E, then f is continuous at $x_0 \in E$.
- Term-by-term integration: Suppose that $E = [a, b]$ and that each f_k is integrable on $[a, b]$. If $f = \sum_{k=1}^{\infty} f_k$ converges uniformly on $[a, b]$, then f is integrable on $[a, b]$ and

$$\int_a^b \sum_{k=1}^{\infty} f_k(x)\,dx = \sum_{k=1}^{\infty} \int_a^b f_k(x)\,dx$$

- Term-by-term differentiation: Suppose that E is a bounded, open interval and that each f_k is differentiable on E. If $\sum_{k=1}^{\infty} f_k(x_0)$ converges at some $x_0 \in E$, and $g = \sum_{k=1}^{\infty} f'(k)$ converges uniformly on E, then $f = \sum_{k=1}^{\infty} f_k$ converges uniformly on E, is differentiable on E, and

$$f'(x) = \left(\sum_{k=1}^{\infty} f_k(x)\right)' = \sum_{k=1}^{\infty} f_k'(x) = g(x)$$

for $x \in E$

Theorem 2.5: Weierstrass M-test

Let E be a nonempty subset of $\mathbb{R}$, let $f_k : E \to \mathbb{R}$, $k \in \mathbb{N}$, and suppose that $M_k > 0$ satisfies $\sum_{k=1}^{\infty} M_k < \infty$. If $|f_k(x)| \leq M_k$ for $k \in \mathbb{N}$ and $x \in E$, then $f = \sum_{k=1}^{\infty} f_k$ converges absolutely and uniformly on E.

Definition 3.1: Power Series and RoC

The **radius of convergence** R of the power series

$$\sum_{n=0}^{\infty} a_n(x-c)^n \qquad (*)$$

is defined by

$$R = \sup\{r \geq 0 : (a_n r^n) \text{ is bounded}\}$$

unless $(a_n r^n)$ is bounded for all $r \geq 0$, where we say that $R = \infty$

Thm 3.1: Suppose the radius of convergence R of $*$ satisfies $0 < R < \infty$. If $|x - c| < R$, the power series $*$ converges absolutely. If $|x - c| > R$, the power series $*$ diverges

Theorem 3.2: Continuity of Power Series

Assume that $R > 0$. Suppose that $0 < r < R$. Then a power series converges uniformly and absolutely on $|x - c| \leq r$ to a continuous function f. Hence

$$f(x) = \sum_{n=0}^{\infty} a_n(x-c)^n$$

defines a continuous function $f : (c - R, c + R) \to \mathbb{R}$

Lemma 3.1: The two power series

$$\sum_{n=1}^{\infty} a_n(x-c)^n \text{ and } \sum_{n=1}^{\infty} n a_n(x-c)^{n-1}$$

have the same radius of convergence

Theorem 3.3: Differentiation of Power Series

Suppose the radius of convergence of a power series is R. Then the function

$$f(x) = \sum_{n=0}^{\infty} a_n(x-c)^n$$

is infinitely differentiable on $|x - c| < R$, and for such x,

$$f'(x) = \sum_{n=0}^{\infty} n a_n(x-c)^{n-1}$$

and the series converges absolutely, and also uniformly on $[c - r, c + r]$ for any $r < R$. Moreover,

$$a_n = \frac{f^{(n)}(c)}{n!}$$

3 Lebesgue Integration

Definition 4.0: Characteristic Function

Let E be a subset of $\mathbb{R}$. We define its **characteristic function** $\chi_E : \mathbb{R} \to \mathbb{R}$ by $\chi_E(x) = 1$ if $x \in E$ and $\chi_E(x) = 0$ if $x \notin E$. In other words,

$$\chi_E = \begin{cases} 1 & x \in E \\ 0 & x \notin E \end{cases}$$

In other words, this is a function that is 1 at all points of a bounded interval, and 0 elsewhere

---

Let I be a bounded interval with endpoints a, b and $a \leq b$. We call the number $b - a$ the **length of the interval** I and we denote it by $\lambda(I)$. This might also be referred to as $|I|$. That is,

$$\lambda((a,b)) = \lambda([a,b]) = \lambda((a,b]) = \lambda([a,b)) = b - a$$

From our definition of a characteristic function and the length of an interval, we have that the area of the characteristic function is a rectangle with width $\lambda(I)$ and height 1, therefore

$$\int \chi_I = 1 \cdot \lambda(I) = \lambda(I)$$

Definition 4.1: Step function

We say that $\phi : \mathbb{R} \to \mathbb{R}$ is a **step function** if there exist real numbers $x_0 < x_1 < x_2 < \cdots < x_n$ (for some $n \in \mathbb{N}$) such that

1. $\phi(x) = 0$ for $x < x_0$ and $x > x_n$
2. ϕ is constant on (x_{j-1}, x_j) for $1 \leq j \leq n$

We shall use the phrase "ϕ is a step function with respect to $\{x_0, x_1, \ldots, x_n\}$" to describe this situation

---

Properties of Step Functions

1. The class of step functions is a vector space - i.e. if ϕ and ψ are step functions and α and β are real numbers, then $\alpha\phi + \beta\psi$ is a step function, and that if ϕ and ψ are step functions, then $\max\{\phi, \psi\}$, $\min\{\phi\psi\}$, $|\phi|$ and $\phi\psi$ are also step functions
2. If ϕ and ψ are step functions, then $\phi + \psi$ is a step function
3. ϕ is a step function if and only if it is of the form
$$\phi = \sum_{j=1}^{n} c_j \chi_{J_j}$$
for some n, c_j, and bounded intervals J_j

Def 4.2: Integral of a Step Function

If ϕ is a step function with respect to $\{x_0, x_1, \ldots, x_n\}$ which takes the value c_j on (x_{j-1}, x_j), then

$$\int \phi := \sum_{j=1}^{n} c_j (x_j - x_{j-1})$$

Therefore, using the characteristic definition of a step function, the integral is

$$\int \phi = \int \sum_{j=1}^{n} c_j \chi_{J_j} = \sum_{j=1}^{n} c_j \int \chi_{I J_j} = \sum_{j=1}^{n} c_j \lambda(J_j)$$

Definition 4.3: Lebesgue Integrals

A function $f : I \to \mathbb{R}$ is said to be **integrable** or more precisely **Lebesgue integrable** on an interval I if there exist numbers c_j and bounded intervals $J_j \subset I$, $j = 1, 2, 3, \ldots$ such that

$$\sum_{j=1}^{\infty} |c_j| \lambda(J_j) < \infty$$

and the equality

$$f(x) = \sum_{j=1}^{\infty} c_j \chi_{J_j}(x)$$

holds for all $x \in I$ at which

$$\sum_{j=1}^{\infty} |c_j| \chi_{J_j}(x) < \infty$$

We denote by $\int_I f$ the number

$$\int_I f = \sum_{j=1}^{\infty} c_j \lambda(J_j)$$

and call it the integral of f over the interval I. If the function f is not integrable on the interval I then we say that the integral of f on I does not exist. Hence if we say that the integral of f on I exists it just means that f is (Lebesgue) integrable on I.

Theorem 4.1: Lebesgue Equality

Suppose that c_j, d_j are real numbers and J_j, K_j are bounded intervals for all $j = 1, 2, 3, \ldots$, and

$$\sum_{j=1}^{\infty} |c_j| \lambda(J_j) < \infty, \quad \sum_{j=1}^{\infty} |d_j| \lambda(K_j) < \infty$$

If

$$\sum_{j=1}^{\infty} c_j \chi_{J_j}(x) = \sum_{j=1}^{\infty} d_j \chi_{K_j}(x)$$

holds for all x such that

$$\sum_{j=1}^{\infty} |c_j| \chi_{J_j}(x) < \infty, \quad \sum_{j=1}^{\infty} |d_j| \chi_{K_j}(x) < \infty$$

Then

$$\sum_{j=1}^{\infty} c_j \lambda(J_j) = \sum_{j=1}^{\infty} d_j \lambda(K_j)$$

Theorem 4.2: Lebesgue Integral Properties

Suppose f and g are integrable on I and α and β are real numbers. Then

1. $\alpha f + \beta g$ is integrable on I and
$$\int_I (\alpha f + \beta g) = \alpha \int_I f + \beta \int_I g$$
2. If $f \geq 0$ on I then $\int_I f \geq 0$; if $f \geq g$ on I then $\int_I f \geq \int_I g$
3. $|f|$ is integrable on I and $\left| \int_f f \right| \leq \int_I |f|$
4. $\max\{f, g\}$ and $\min\{f, g\}$ are integrable on I
5. If one of the functions is bounded then the product fg is integrable on I
6. If $f \geq 0$ with $\int_I f = 0$ then any function h such that $0 \leq h \leq f$ on I is integrable on I

Theorem 4.3: Integrability of Sequences and Series

Suppose that $(f_n)_{n \in \mathbb{N}}$ is a sequence of functions each of which is integrable on I

1. Assume that
$$\sum_{n=1}^{\infty} \int_I |f_n| < \infty$$
Let f be a function on the interval I such that
$$f(x) = \sum_{n=1}^{\infty} f_n(x) \quad \text{for all } x \in I \text{ such that} \quad \sum_{n=1}^{\infty} |f_n(x)| < \infty$$
Then f is integrable on I and its integral on I is equal to
$$\int_I f = \sum_{n=1}^{\infty} \int_I f_n$$
2. Assume that each $f_n \geq 0$ on I and let $f(x) = \sum_{n=1}^{\infty} f_n(x)$ for all $x \in I$ (we allow for the possibility that at some points this sum is infinite). Then f is integrable on I if and only if
$$\sum_{n=1}^{\infty} \int_I f_n < \infty$$

Theorem 4.4: Monotone Convergence Theorem

Suppose that (f_n) is a monotone increasing sequence of integrable functions on an interval I. That is, $f_1(x) \leq f_2(x) \leq f_3(x) \leq \ldots$ for all $x \in I$. For all $x \in I$, let

$$f(x) = \lim_{n \to \infty} f_n(x)$$

where we allow for the possibility that at some points this limit is infinite. Then f is integrable on I iff

$$sup_{n \in \mathbb{N}} \int_I f_n = \lim_{n \to \infty} \int_I f_n < \infty. \quad \text{Also,} \int_I f = \lim_{n \to \infty} \int_I f_n$$

Definition 4.4: Riemann Integrable Functions

Let $f:\mathbb{R}\to\mathbb{R}$. We say that f is **Riemann-integrable** if for every $\epsilon>0$ there exists step functions ϕ and ψ such that

$$\phi \le f \le \psi$$

and

$$\int \psi - \int \phi < \epsilon$$

Thm 4.5: A function $f:\mathbb{R}\to\mathbb{R}$ is Riemann-integrable iff

$$\sup\left\{\int \phi : \phi \text{ is a step function and } \phi \le f\right\}$$
$$= \inf\left\{\int \psi : \psi \text{ is a step function and } \phi \ge f\right\}$$

Def 4.5: If f is Riemann-integrable we define its Riemann integral $(R)\int f$ as the common value

$$(R)\int f := \sup\left\{\int \phi : \phi \text{ is a step function and } \phi \le f\right\}$$
$$= \inf\left\{\int \psi : \psi \text{ is a step function and } \phi \ge f\right\}$$

Theorem 4.1: Riemann lemmas

Let $f:\mathbb{R}\to\mathbb{R}$ be a bounded function with bounded support $[a,b]$. The following are equivalent:

1. f is Riemann-integrable
2. for every $\epsilon>0$ there exists $a=x_0<\cdots<x_n=b$ s.t. if M_j and m_j denote the sup and inf of f on (x_{j-1},x_j) respectively, then
$$\sum_{j=1}^{n}(M_j-m_j)(x_j-x_{j-1})<\epsilon$$
3. for every $\epsilon>0$ there exists $\alpha=x_0<\cdots<x_n=b$ s.t. with $I_j=(x_{j-1},x_j)$ for $j\ge 1$
$$\sum_{j=1}^{n}\sup_{x,y\in I_j}|f(x)-f(y)|\lambda(I_j)<\epsilon$$

Notation to aid these lemmas: For $f:\mathbb{R}\to\mathbb{R}$ a bounded function with bounded support $[a,b]$ and for $a=x_0<\cdots<x_n=b$, we let $I_j=(x_{j-1},x_j)$, $m_j:=\inf_{x\in I_j}f(x)$ and $M_j:=\sup_{x\in I_j}f(x)$. We define the **lower step function of f with respect to** $\{x_0,\ldots,x_n\}$ as

$$\phi_*(x)=\sum_{j=1}^{n}m_j\chi_{I_j}(x)+\sum_{j=0}^{n}f(x_j)\chi_{\{x_j\}}(x)$$

and the **upper step function of f with respect to** $\{x_0,\ldots,x_n\}$ as

$$\phi^*(x)=\sum_{j=1}^{n}M_j\chi_{I_j}(x)+\sum_{j=0}^{n}f(x_j)\chi_{\{x_j\}}(x)$$

$\phi_*(x)$ and $\phi^*(x)$ are step functions, and $\phi_*(x)\le f\le\phi^*(x)$

Suppose that $g:[a,b]\to\mathbb{R}$ and let f be defined by $f(x)=g(x)$ for $x\in[a,b]$ and $f(x)=0$ otherwise.

1. If g is continuous on $[a,b]$, then f is Riemann-integrable
2. If g is a monotone function then f is Riemann-integrable

Theorem 4.6: Connection between Riemann and Lebesgue

Suppose that $f:\mathbb{R}\to\mathbb{R}$ is Riemann-integrable. Then f is also Lebesgue integrable on $\mathbb{R}$ and moreoever

$$(R)\int f=\int f$$

where the number on the lefthand side is the value of the Riemann integral of f, while the righthand side denotes the value of the Lebesgue integral of f on $\mathbb{R}$

Theorem 4.8: Dependence on Intervals for Lebesgue

Let I and J be two intervals such that $J\subset I$.

1. If f is integrable on I then f is also integrable on the subinterval J
2. If f is integrable on J and simultaneously $f(x)=0$ for all $x\in I\backslash J$ then f is integrable on I and
$$\int_J f=\int_I f$$
3. If f is integrable on I and $f(x)\ge 0$ for all $x\in I$ then
$$\int_J f\le\int_I f$$
4. Suppose that I can be written as the union of disjoint intervals I_n, $n=1,2,3,\ldots$ and let f be integrable on each of the intervals I_n. Then f is integrable on I iff
$$\sum_{n=1}^{\infty}\int_{I_n}|f|<\infty$$
If this holds, then
$$\int_I f=\sum_{n=1}^{\infty}\int_{I_n}f$$

Theorem 4.9: Addition of Intervals

If any two of these integrals

$$\int_a^b f,\quad \int_b^c f,\quad \int_a^c f$$

exist then so does the third and

$$\int_a^b f,+\int_b^c f=\int_a^c f$$

Theorem 4.10: Fundamental Theorem of Calculus

Let I be an interval and let $g:I\to\mathbb{R}$ be integrable on I. For all $x\in I$ and some fixed $x_0\in I$ let $G(x)=\int_{x_0}^x g$. Suppose g is continuous at x for some $x\in I$ [if x is an endpoint we mean one-sided continuity.] Then G is differentiable at x and $G'(x)=g(x)$. [if x if an endpoint we mean one-sided differentiable]

Suppose $f:I\to\mathbb{R}$ has continuous derivative f' on the interval I. Then for any $a,b\in I$:

$$\int_a^b f'=f(b)-f(a)$$

Lemma 4.2: Fatoux Lemma

Let (f_n) be a sequence of non-negative integrable functions on an interval I. Let

$$f(x)=\liminf_{n\to\infty}f_n(x),\quad \text{for all } x\in I$$

If $\liminf_{n\to\infty}\int_I f_n<\infty$ then f is integrable on I and

$$\int_I f\le\liminf_{n\to\infty}\int_i f_n$$

Theorem 4.12: Dominated Convergence Theorem

Let (f_n) be a sequence of integrable functions on an interval I and assume that

$$f(x)=\lim_{n\to\infty}f_n(x),\quad \text{for all } x\in I$$

. Assume also that the sequence (f_n) is **dominated** by some integrable function g, that is

$$|f_n(x)|\le g(x),\quad \text{for all } x\in I \text{ and } n=1,2,\ldots,\quad \int_I g<\infty$$

Then the function f is integrable on I and

$$\int_I f=\lim_{n\to\infty}\int_I f_n$$

Theorem 4.13

Let (a,b) be a bounded interval and suppose that $f_n:(a,b)\to\mathbb{R}$ are integrable functions which converges uniformly to a function f. Then f is integrable on (a,b) and

$$\int_a^b f=\lim_{n\to\infty}\int_a^b f_n$$

4 Fourier Series and Orthogonality

Definition 5.1: The Space L^2

Define the space $L^2 = L^2([a,b])$ as the set of measurable functions $f : [a,b] \to \mathbb{C}$ so that the function $x \mapsto |f(x)|^2$ is Lebesgue integrable, i.e.

$$\|f\|_2^2 := \int_a^b |f(x)|^2 dx < \infty$$

The quantity $\|f\|_2$ is called the **L^2-norm** of f. If $\|f\|_2 = 1$, then we say that f is **L^2-normalised**

Definition 5.2: Inner Product

For two functions $f, g \in L^2([a,b])$, we define their **inner product** by

$$\langle f, g \rangle = \int_a^b f(x)\overline{g(x)} dx$$

Theorem 5.1: Cauchy-Shwarz Inequality

Let $f, g \in L^2([a,b])$. then the function $x \mapsto f(x)\overline{g(x)}$ is Lebesgue integrable and we have

$$|\langle f, g \rangle| \leq \|f\|_2 \|g\|_2$$

Minkowski's Inequality: For two functions $f, g \in L^2([a,b])$,

$$\|f + g\|_2 \leq \|f\|_2 + \|g\|_2$$

Definition 5.3: Convergent Sequences in L^2

Let $f, f_1, f_2, \ldots$ be functions in $L^2([a,b])$. We say that the function $(f_n)_n$ converges to f in L^2 if the sequence

$$\|f_n - f\|_2 = \left(\int_a^b |f_n(x) - f(x)|^2 dx \right)^{1/2}$$

converges to zero as $n \to \infty$. We will also write $f_n \to f$ in L^2

Definition 5.4: Orthonormal Systems

A sequence $(\phi_n)_n$ of L^2 functions on $[a,b]$ is called an **orthonormal system on** $[a,b]$ if

$$\langle \phi_n, \phi_m \rangle = \int_a^b \phi_n(x)\overline{\phi_m(x)} dx = \begin{cases} 0, & \text{if } n \neq m \\ 1, & \text{if } n = m \end{cases}$$

(The index n may run over any countable set. We will write $\sum_n$ to denote a sum over all the indices. In proofs we will always adopt the interpretation that n runs over $1, 2, 3, \ldots$ without loss of genereрality)

Theorem 5.2

Let $(\phi_n)_n$ be an orthonormal system on $[a,b]$ and $f \in L^2$. Consider

$$s_N(x) = \sum_{n=1}^N \langle f, \phi_n \rangle \phi_n(x)$$

Denote the linear span of the functions $(\phi_n)_{n=1,\ldots,N}$ by X_N. Then

$$\|f - s_N\|_2 \leq \|f - g\|_2$$

holds for all $g \in X_N$ with equality iff $g = s_N$

Definition 5.3: Bessel's Inequality

If $(\phi_n)_n$ is an orthonormal system on $[a,b]$ and $f \in L^2$, then

$$\sum_n |\langle f, \phi_n \rangle|^2 \leq \|f\|_2^2$$

Corollary - Riemann-Lebesgue lemma in L^2. Let $(\phi_n)_{n=1,2,\ldots}$ be an orthonormal system and $f \in L^2$, then

$$\lim_{n \to \infty} \langle f, \phi_n \rangle = 0$$

Definition 5.5: Complete Orthonormal Systems

An orthonormal system $(\phi_n)_n$ is called **complete** if

$$\sum_n |\langle f, \phi_n \rangle|^2 = \|f\|_2^2$$

for all $f \in L^2$

Thm 5.4: Let $(\phi_n)_n$ be an orthonormal system on $[a,b]$. Let $(s_N)_N$ be as in Theorem 5.2. Then $(\phi_n)_n$ is complete iff $(s_N)_N$ converges to f in the L^2-norm for every $f \in L^2$

Definition 5.6: Trigonometric Polynomials

A **trigonometric polynomial** is a function of the form

$$f(x) = \sum_{n=-N}^N c_n e^{2\pi i n x} \quad (x \in \mathbb{R})$$

where $N \in \mathbb{N}$ and $c_n \in \mathbb{C}$. If c_N or c_{-N} is non-zero, then N is called the **degree** of f
Observe that trigonometric polynomials are continuous functions.
From Euler's identity $e^{ix} = \cos(x) + i\sin(x)$, $(x \in \mathbb{R})$ we see that every trigonometric polynomial can also be written in the form

$$f(x) = a_0 + \sum_{n=0}^N (a_n \cos(2\pi n x) + b_n \sin(2\pi n x))$$

Lemma 5.1: $(e^{2\pi i n x})_{n \in \mathbb{Z}}$ forms an orthonormal system on $[0,1]$. In particular,

1. for all $n \in \mathbb{Z}$,

$$\int_0^1 e^{2\pi i n x} dx = \begin{cases} 0, & \text{if } n \neq 0 \\ 1, & \text{if } n = 0 \end{cases}$$

2. if $f(x) = \sum_{n=-N}^N c_n e^{2\pi i n x}$ is a trigonometric polynomial, then

$$c_n = \langle f, \phi_n \rangle = \int_0^1 f(t) e^{-2\pi i n t} dt$$

Definition 5.7: Fourier Coefficient

For a 1-periodic integrable function f and $n \in \mathbb{Z}$ we define the **nth Fourier coefficieent** by

$$\widehat{f}(n) = \int_0^1 f(t) e^{-2\pi i n t} dt = \langle f, \phi_n \rangle$$

(the integral on the right exists since f is integrable and $|\phi_n| \leq 1$.)
The doubly infinite series

$$\sum_{n=-\infty}^{\infty} \widehat{f}(n) e^{2\pi i n x}$$

is called the **Fourier series** of f

Def 5.8 (Partial Sums): For a 1-periodic integrable function f, we define the **partial sums**

$$S_N f(x) = \sum_{n=-N}^N \widehat{f}(n) e^{2\pi i n x}$$

Note: for all $f \in L^2$ and trigonometric polynomials g of degree $\leq N$, we have

$$\|f - S_N f\|_2 \leq \|f - g\|_2$$

Definition 5.9: Convolution

For two 1-periodic functions $f, g \in L^2$ we define their **convolution** by

$$f * g(x) = \int_0^1 f(t) g(x - t) dt$$

(The integral on the right hand side exists by Cauchy-Shwarz)

Lemma bank

5.2 For 1-periodic functions $f, g \in L^2$,

$$f * g = g * f$$

5.3 Dirichlet Kernel: We have

$$D_N(x) = \sum_{n=-N}^N e^{2\pi i n x} = \frac{\sin(2\pi(N + \frac{1}{2})x)}{\sin(\pi x)}$$

5.4 Fejér Kernel: We have

$$K_N(x) = \frac{1}{N+1} \sum_{n=0}^N D_n(x) = \frac{1}{2(N+1)} \frac{1 - \cos(2\pi(N+1)x)}{\sin(\pi x)^2}$$

$$= \frac{1}{N+1} \left(\frac{\sin(\pi(N+1)x)}{\sin(\pi x)} \right)^2$$

Thm 5.5 (Fejér): For every 1-periodic continuous function f,

$$K_N * f \to f$$

uniformly on $\mathbb{R}$ as $N \to \infty$
Corollary: Every 1-periodic continuous function can be uniformly approximated by trigonometric polynomials. That is, for every 1-periodic continuous f there exists a sequence $(f_n)_n$ of trigonometric polynomials so that $f_n \to f$ uniformly

Definition 5.10: Approximation of Unity

A sequence of 1-periodic integrable functions $(k_n)_n$ is called **approximation of unity** if for all 1-periodic continuous functions f we have that $f * k_n$ converges uniformly to f on $\mathbb{R}$. That is,

$$\sup_{x\in\mathbb{R}}|f * k_n(x) - f(x)| \to 0 \quad \text{as } n \to \infty$$

Thm 5.6: Let $(k_n)_n$ be a sequence of 1-periodic integrable functions such that

1. $k_n(x) \geq 0$ for all $x \in \mathbb{R}$
2. $\int_{-1/2}^{1/2} k_n(t)dt = 1$
3. For all $1/2 \geq \delta > 0$ we have
$$\int_{-\delta}^{\delta} k_n(t)dt \to 1 \quad \text{as } n \to \infty$$

Then $(k_n)_n$ is an approximation of unity

Corollary: The Fejér kernel $(K_N)_N$ is an approximation of unity

Lemma 5.5

Let f be a 1-periodic and continuous function. Then

$$\lim_{N\to\infty} \|S_N f - f\|_2 = 0$$

Theorem 5.7: Completeness of Trigonometric System

The trigonometric system is complete. In view of Theorem 5.4 this means that for every 1-periodic L^2 function f we have

$$\lim_{N\to\infty} \|S_N f - f\|_2 = 0$$

In other words, the Fourier series of f converges to f in the L^2 sense

---

Corollary (Parseval's Theorem): If f, g are 1-periodic L^2 functions then

$$\langle f, g\rangle = \sum_{n=-\infty}^{\infty} \widehat{f}(n)\overline{\widehat{g}(n)}$$

In particular,

$$\|f\|_2^2 = \sum_{n=-\infty}^{\infty} |\widehat{f}(n)|^2$$

Lorem ipsum dolor sit amet, consectetuer adipiscing elit. Ut purus elit, vestibulum ut, placerat ac, adipiscing vitae, felis. Curabitur dictum gravida mauris. Nam arcu libero, nonummy eget, consectetuer id, vulputate a, magna. Donec vehicula augue eu neque. Pellentesque habitant morbi tristique senectus et netus et malesuada fames ac turpis egestas. Mauris ut leo. Cras viverra metus rhoncus sem. Nulla et lectus vestibulum urna fringilla ultrices. Phasellus eu tellus sit amet tortor gravida placerat. Integer sapien est, iaculis in, pretium quis, viverra ac, nunc. Praesent eget sem vel leo ultrices bibendum. Aenean faucibus. Morbi dolor nulla, malesuada eu, pulvinar at, mollis ac, nulla. Curabitur auctor semper nulla. Donec varius orci eget risus. Duis nibh mi, congue eu, accumsan eleifend, sagittis quis, diam. Duis eget orci sit amet orci dignissim rutrum.

Nam dui ligula, fringilla a, euismod sodales, sollicitudin vel, wisi. Morbi auctor lorem non justo. Nam lacus libero, pretium at, lobortis vitae, ultricies et, tellus. Donec aliquet, tortor sed accumsan bibendum, erat ligula aliquet magna, vitae ornare odio metus a mi. Morbi ac orci et nisl hendrerit mollis. Suspendisse ut massa. Cras nec ante. Pellentesque a nulla. Cum sociis natoque penatibus et magnis dis parturient montes, nascetur ridiculus mus. Aliquam tincidunt urna. Nulla ullamcorper vestibulum turpis. Pellentesque cursus luctus mauris.

Nulla malesuada porttitor diam. Donec felis erat, congue non, volutpat at, tincidunt tristique, libero. Vivamus viverra fermentum felis. Donec nonummy pellentesque ante. Phasellus adipiscing semper elit. Proin fermentum massa ac quam. Sed diam turpis, molestie vitae, placerat a, molestie nec, leo. Maecenas lacinia. Nam ipsum ligula, eleifend at, accumsan nec, suscipit a, ipsum. Morbi blandit ligula feugiat magna. Nunc eleifend consequat lorem. Sed lacinia nulla vitae enim. Pellentesque tincidunt purus vel magna. Integer non enim. Praesent euismod nunc eu purus. Donec bibendum quam in tellus. Nullam cursus pulvinar lectus. Donec et mi. Nam vulputate metus eu enim. Vestibulum pellentesque felis eu massa.

Quisque ullamcorper placerat ipsum. Cras nibh. Morbi vel justo vitae lacus tincidunt ultrices. Lorem ipsum dolor sit amet, consectetuer adipiscing elit. In hac habitasse platea dictumst. Integer tempus convallis augue. Etiam facilisis. Nunc elementum fermentum wisi. Aenean placerat. Ut imperdiet, enim sed gravida sollicitudin, felis odio placerat quam, ac pulvinar elit purus eget enim. Nunc vitae tortor. Proin tempus nibh sit amet nisl. Vivamus quis tortor vitae risus porta vehicula.

Fusce mauris. Vestibulum luctus nibh at lectus. Sed bibendum, nulla a faucibus semper, leo velit ultricies tellus, ac venenatis arcu wisi vel nisl. Vestibulum diam. Aliquam pellentesque, augue quis sagittis posuere, turpis lacus congue quam, in hendrerit risus eros eget felis. Maecenas eget erat in sapien mattis porttitor. Vestibulum porttitor. Nulla facilisi. Sed a turpis eu lacus commodo facilisis. Morbi fringilla, wisi in dignissim interdum, justo lectus sagittis dui, et vehicula libero dui cursus dui. Mauris tempor ligula sed lacus. Duis cursus enim ut augue. Cras ac magna. Cras nulla. Nulla egestas. Curabitur a leo. Quisque egestas wisi eget nunc. Nam feugiat lacus vel est. Curabitur consectetuer.

Suspendisse vel felis. Ut lorem lorem, interdum eu, tincidunt sit amet, laoreet vitae, arcu. Aenean faucibus pede eu ante. Praesent enim elit, rutrum at, molestie non, nonummy vel, nisl. Ut lectus eros, malesuada sit amet, fermentum eu, sodales cursus, magna. Donec eu purus. Quisque vehicula, urna sed ultricies auctor, pede lorem egestas dui, et convallis elit erat sed nulla. Donec luctus. Curabitur et nunc. Aliquam dolor odio, commodo pretium, ultricies non, pharetra in, velit. Integer arcu est, nonummy in, fermentum faucibus, egestas vel, odio.

Sed commodo posuere pede. Mauris ut est. Ut quis purus. Sed ac odio. Sed vehicula hendrerit sem. Duis non odio. Morbi ut dui. Sed accumsan risus eget odio. In hac habitasse platea dictumst. Pellentesque non elit. Fusce sed justo eu urna porta tincidunt. Mauris felis odio, sollicitudin sed, volutpat a, ornare ac, erat. Morbi quis dolor. Donec pellentesque, erat ac sagittis semper, nunc dui lobortis purus, quis congue purus metus ultricies tellus. Proin et quam. Class aptent taciti sociosqu ad litora torquent per conubia nostra, per inceptos hymenaeos. Praesent sapien turpis, fermentum vel, eleifend faucibus, vehicula eu, lacus.

Pellentesque habitant morbi tristique senectus et netus et malesuada fames ac turpis egestas. Donec odio elit, dictum in, hendrerit sit amet, egestas sed, leo. Praesent feugiat sapien aliquet odio. Integer vitae justo. Aliquam vestibulum fringilla lorem. Sed neque lectus, consectetuer at, consectetuer sed, eleifend ac, lectus. Nulla facilisi. Pellentesque eget lectus. Proin eu metus. Sed porttitor. In hac habitasse platea dictumst. Suspendisse eu lectus. Ut mi mi, lacinia sit amet, placerat et, mollis vitae, dui. Sed ante tellus, tristique ut, iaculis eu, malesuada ac, dui. Mauris nibh leo, facilisis non, adipiscing quis, ultrices a, dui.

Morbi luctus, wisi viverra faucibus pretium, nibh est placerat odio, nec commodo wisi enim eget quam. Quisque libero justo, consectetuer a, feugiat vitae, porttitor eu, libero. Suspendisse sed mauris vitae elit sollicitudin malesuada. Maecenas ultricies eros sit amet ante. Ut venenatis velit. Maecenas sed mi eget dui varius euismod. Phasellus aliquet volutpat odio. Vestibulum ante ipsum primis in faucibus orci luctus et ultrices posuere cubilia Curae; Pellentesque sit amet pede ac sem eleifend consectetuer. Nullam elementum, urna vel imperdiet sodales, elit ipsum pharetra ligula, ac pretium ante justo a nulla. Curabitur tristique arcu eu metus. Vestibulum lectus. Proin mauris. Proin eu nunc eu urna hendrerit faucibus. Aliquam auctor, pede consequat laoreet varius, eros tellus scelerisque quam, pellentesque hendrerit ipsum dolor sed augue. Nulla nec lacus.

Suspendisse vitae elit. Aliquam arcu neque, ornare in, ullamcorper quis, commodo eu, libero. Fusce sagittis erat at erat tristique mollis. Maecenas sapien libero, molestie et, lobortis in, sodales eget, dui. Morbi ultrices rutrum lorem. Nam elementum ullamcorper leo. Morbi dui. Aliquam sagittis. Nunc placerat. Pellentesque tristique sodales est. Maecenas imperdiet lacinia velit. Cras non urna. Morbi eros pede, suscipit ac, varius vel, egestas non, eros. Praesent malesuada, diam id pretium elementum, eros sem dictum tortor, vel consectetuer odio sem sed wisi.

Sed feugiat. Cum sociis natoque penatibus et magnis dis parturient montes, nascetur ridiculus mus. Ut pellentesque augue sed urna. Vestibulum diam eros, fringilla et, consectetuer eu, nonummy id, sapien. Nullam at lectus. In sagittis ultrices mauris. Curabitur malesuada erat sit amet massa. Fusce blandit. Aliquam erat volutpat. Aliquam euismod. Aenean vel lectus. Nunc imperdiet justo nec dolor.

Etiam euismod. Fusce facilisis lacinia dui. Suspendisse potenti. In mi erat, cursus id, nonummy sed, ullamcorper eget, sapien. Praesent pretium, magna in eleifend egestas, pede pede pretium lorem, quis consectetuer tortor sapien facilisis magna. Mauris quis magna varius nulla scelerisque imperdiet. Aliquam non quam. Aliquam porttitor quam a lacus. Praesent vel arcu ut tortor cursus volutpat. In vitae pede quis diam bibendum placerat. Fusce elementum convallis neque. Sed dolor orci, scelerisque ac, dapibus nec, ultricies ut, mi. Duis nec dui quis leo sagittis commodo.