

Honours Algebra Exam Notes

Made by Leon :) *Note: Any reference numbers are to the lecture notes*

1 Abstractions upon Abstractions

see you guys in UG4 category theory!

Definition A: Rings and Fields

A **ring** (left) is a set with two operations $(\mathbb{R}, +, \cdot)$ that satisfies the following lemmas.

A **field** (right) is an extension of a ring where (\cdot) is a group

- | | |
|--|---|
| <ol style="list-style-type: none">1. $(R, +)$ is an abelian group with identity 02. (R, \cdot) is a monoid, i.e. it is a set with Associativity and Identity (written as 1)3. Distributive law: For all a, b, and c in F, we have | <ol style="list-style-type: none">1. $(F, +)$ is an abelian group F^+, with identity 0_F2. $(F \setminus \{0_F\}, \cdot)$ is an abelian group F^\times, with identity 1_F3. Distributive law: For all a, b, and c in F, we have |
|--|---|

$$a(b + c) = ab + ac \in F$$

$$\begin{aligned} a \cdot (b + c) &= (a \cdot b) + (a \cdot c) \\ (a + b) \cdot c &= (a \cdot c) + (b \cdot c) \end{aligned}$$

and they satisfy the following lemmas (for both):

1. $0a = 0 = a0$
2. The elements 0 and 1 are distinct (only ring case is zero ring)

Field Specific Lemmas:

1. (\cdot) in F is associative, 1_F is an identity (it's an abelian group only in $(F \setminus \{0_F\}, \cdot)$)

Ring Specific Lemmas and Definitions:

1. The **null ring** or **zero ring** is defined as a ring where R is a single element - i.e. $\{0\}$ where $0 + 0 = 0$ and $0 \times 0 = 0$
2. A **commutative ring** is one where $a \cdot b = b \cdot a$ for all $a, b \in R$
 - $(-a)(b) = -(ab) = a(-b)$
 - $(-a)(-b) = ab$
 - $m(a + b) = ma + mb$
 - $(m + n)a = ma + na$
 - $m(na) = (mn)a$
 - $m(ab) = (ma)b = a(mb)$
 - $(ma)(nb) = (mn)(ab)$

Definition B: Modules and Vector Spaces

A **left module** M over a ring R (or an **R -module**) (left) is a pair consisting of an abelian group $M = (M, +)$ and a mapping

A **vector space** V over a field F (right) is an extension of a module but over a field instead, and using vectors - $V = (V, +)$

$R \times M \rightarrow M : (r, a) \mapsto ra$	$F \times V \rightarrow V : (\lambda, \vec{v}) \mapsto \lambda \vec{v}$
s.t. $\forall r, s \in R$ and $a, b \in M$, the following axioms apply:	s.t. $\forall \lambda, \mu \in F$ and $\vec{v}, \vec{w} \in v$, the following axioms apply:

$\begin{aligned} r(a \dot{+} b) &= (ra) \dot{+} (rb) \\ (r + s)a &= (ra) \dot{+} (sa) \\ r(sa) &= (rs)a \\ 1_R a &= a \end{aligned}$	<div>Distributivity 1</div> <div>Distributivity 2</div> <div>Associativity</div> <div>Identity</div>	$\begin{aligned} \lambda(\vec{v} \dot{+} \vec{w}) &= \lambda \vec{v} \dot{+} \lambda \vec{w} \\ (\lambda + \mu)\vec{v} &= \lambda \vec{v} \dot{+} \mu \vec{v} \\ \lambda(\mu \vec{v}) &= (\lambda \mu)\vec{v} \\ 1 \vec{v} &= \vec{v} \end{aligned}$
--	--	--

and they satisfy the following lemmas (for both):

1. $0_R a = 0_M$ for all $a \in M$ or $0 \vec{v} = \vec{0}$ for all $\vec{v} \in V$
2. $r 0_M = 0_M$ for all $r \in R$ or $\lambda \vec{0} = \vec{0}$ for all $\lambda \in F$
3.
 - $(-r)a = r(-a) = -(ra)$ for all $r \in R, a \in M$
 - $(-1)\vec{v} = -\vec{v}$ for all $\vec{v} \in V$

Definition C: Sub-things

A sub-thing is basically something that is a smaller but self-contained version of a thing

- **Vector Subspace** (left): A subset U of a vector space V
- **Subring** (centre): A subset R' of a ring R under the same operations of addition and multiplication defined in R
- **Submodule** (right): A subset M' of a module M under the same operations of the R -module M **restricted** to M

Subspace Criterion $\forall \vec{u}, \vec{v} \in U, \lambda \in F$	Subring Criterion $\forall a, b \in R'$	Submod. Criterion $\forall a, b \in M', r \in R$
1. $\vec{0} \in U$	1. R' has a multiplicative identity	1. $0_M \in M'$
2. $\vec{u} + \vec{v} \in U$	2. $a - b \in R'$	2. $a - b \in M'$
3. $\lambda \vec{u} \in U$	3. $a \cdot b \in R'$	3. $ra \in M'$

Definition D: Homo no homo

Everything has its own homomorphism and they are all the exact same thing

- **Linear Mapping** (left): Homomorphism on a Vector Space
- **Ring Homomorphism** (centre): Homomorphism on a ring
- **R -homomorphism** (right): Homomorphism on a module

V. Space Criterion $\forall \vec{u}, \vec{v} \in U, \lambda \in F$	Ring Criterion $\forall x, y \in R'$	Module Criterion $\forall a, b \in M', r \in R$
• $f(\vec{v}_1 + \vec{v}_2) = f(\vec{v}_1) + f(\vec{v}_2)$	• $f(x + y) = f(x) + f(y)$	• $f(a + b) = f(a) + f(b)$
• $f(\lambda \vec{v}_1) = \lambda f(\vec{v}_1)$	• $f(xy) = f(x)f(y)$	• $f(ra) = rf(a)$

- A bijective homomorphism is called a **isomorphism**
- Two objects with an iso. are called **isomorphic**, written $A \cong B$
- A homomorphism $V \rightarrow V$ is called an **endomorphism** of V
- An isomorphism $V \rightarrow V$ is called an **automorphism** of V

Image and Kernel

The image and kernel of a mapping $f : M \rightarrow N$ are as follows:

- **Image**: $\text{im } f = \{f(a) : a \in M\} \subseteq N$
- **Kernel**: $\ker f = \{a \in M : f(a) = 0_N\} \subseteq M$

Theorem E: Universal Properties and First Iso Thm

Thm: Universal Properties

Let A be an object of type σ , and I be an ideal-ish σ object

- The mapping $\text{can} : A \rightarrow A/I$ sending a to $a + I$ for all $a \in A$ is a surjective σ -homomorphism with kernel I
- If $f : A \rightarrow B$ is an σ -homomorphism with $f(I) = \{0_B\}$, so that $I \subseteq \ker f$, then there is a unique σ -homomorphism $\bar{f} : A/I \rightarrow B$ such that $f = \bar{f} \circ \text{can}$

Thm: First Isomorphism Theorem

Every σ homomorphism $f : A \rightarrow B$ induces an σ -homomorphism

$$\bar{f} : A / \ker f \xrightarrow{\sim} \text{im } f$$

This can be applied to pretty much everything!

- **Factor Rings**: σ are rings (so A is a ring), and I is an ideal
- **Factor Modules**: σ are R -modules, and I is a submodule
- **Groups**: σ are groups, and I is a normal subgroup

2 Rings and Modules

Example 3.1.4: Modulo Rings

Let $m \in \mathbb{Z}$. Then the set of **integers modulo** m is a ring, written

$$\mathbb{Z}/m\mathbb{Z}$$

The elements of $\mathbb{Z}/m\mathbb{Z}$ consist of **congruence classes** of integers modulo m , written \bar{a} , - i.e. “the subsets T of \mathbb{Z} of the form $T = a + m\mathbb{Z}$ with $a \in \mathbb{Z}$ ”, or “set of integers that have the same remainder when you divide them by m ”. $\bar{a} = \bar{b}$ is the same as $a - b \in m\mathbb{Z}$, and often I'll write

$$a \equiv b \pmod{m}$$

Thm 3.1.11 - Prime Property for Fields: Let $m \in \mathbb{N}$. The commutative ring $\mathbb{Z}/m\mathbb{Z}$ is a field if and only if m is prime

Definition 3.2.3: Multiples of an abelian group

Let $m \in \mathbb{Z}$. The **m -th multiple** ma of an element a in an abelian group R is:

$$ma = \underbrace{a + a + \dots + a}_{m \text{ terms}} \quad \text{if } m > 0$$

$0a = 0$ and negative multiples are defined by $(-m)a = -(ma)$

Definition 3.2: Units and Field Construction

Def 3.2.6: Let R be a ring. An element $a \in R$ is called a **unit** if it is invertible in R , i.e. there exists $r^{-1} \in R$ such that

$$aa^{-1} = 1 = a^{-1}a$$

Prop 3.2.9: The set of R^\times units in a ring R forms a group under multiplication

Definition 3.1.8: A **field** is a non-zero commutative ring F in which every non-zero element $a \in F$ is a unit.

Definition 3.2.11: zero-divisors of a ring

In a ring R , a non-zero element a is called a **zero-divisor** or **divisor of zero** if there exists a non-zero element b such that either $ab = 0$ or $ba = 0$.

Definition 3.2.12: Integral Domain

An **integral domain** is a non-zero commutative ring that has no zero-divisors. The following two laws hold:

1. $ab = 0 \implies a = 0$ or $b = 0$
2. $a \neq 0$ and $b \neq 0 \implies ab \neq 0$

Theorem 3.2: Integral Domain Properties

3.2.15 (Cancellation Law): Let R be an integral domain and let $a, b, c \in R$. If $ab = ac$ and $a \neq 0$ then $b = c$

3.2.16 Let m be a natural number. Then $\mathbb{Z}/m\mathbb{Z}$ is an integral domain if and only if m is prime.

3.2.17 Every **finite** integral domain is a field.

Definition 3.1.1: Polynomial

Let R be a ring. A **polynomial over R** is an expression of the form

$$P = a_0 + a_1X + a_2X^2 + \cdots + a_mX^m$$

for some non-negative $m \in \mathbb{Z}$ and elements $a_i \in R$ for $0 \leq i \leq m$.

- The set of all polynomials over R is denoted by $R[X]$.
- In the case where a_m is non-zero, the polynomial P has **degree m** , (written $\deg(P)$), and a_m is its **leading coefficient**
- When the leading coefficient is 1 the polynomial is a **monic polynomial**.
- A polynomial of degree one is called **linear**, degree two is called **quadratic**, and degree three is called **cubic**.

Thm 3.3.2: The set $R[X]$ becomes a ring called the **ring of polynomials with coefficients in R , or over R** . The zero and the identity of $R[X]$ are the zero and identity of R , respectively.

Theorem 3.3: Properties of a Polynomial Ring

- 3.3.3:** If R is a ring with no zero-divisors, then $R[X]$ has no zero-divisors and $\deg(PQ) = \deg(P) + \deg(Q)$ for non-zero $P, Q \in R[X]$.
- If R is an integral domain, then so is $R[X]$
- 3.3.4:** Let R be an integral domain and let $P, Q \in R[X]$ with Q monic. Then there exists unique $A, B \in R[X]$ such that $P = AQ + B$ and $\deg(B) < \deg(Q)$ or $B = 0$

Definition 3.3.6: Evaluating a Function

Let R be a commutative ring and $P \in R[X]$ a polynomial. P can be **evaluated** at $\lambda \in R$ to make $P(\lambda)$ by replacing the powers of X in P by the corresponding powers of λ . In this way we have a mapping

$$R[X] \rightarrow \text{Maps}(R, R)$$

This is the precise definition of thinking of a polynomial as a function. An element $\lambda \in R$ is a **root** of P if $P(\lambda) = 0$

Thm 3.3.9: Let R be a commutative ring, let $\lambda \in R$ and $P(X) \in R[X]$. Then λ is a root of $P(X)$ iff $(X - \lambda)$ divides $P(X)$

Theorem 3.3.10: Degrees of Polynomial Roots

Let R be a field, or more generally an integral domain. Then a non-zero polynomial $P \in R[X] \setminus \{0\}$ has at most $\deg(P)$ roots in R

Definition 3.3.11: Algebraically closed fields

A field F is **algebraically closed** if each non-constant polynomial $P \in F[X] \setminus F$ with coefficients in our field has a root in our field F

Thm 3.3.13 (Fundamental Thm of Algebra): The field of complex numbers \mathbb{C} is algebraically closed.

Thm 3.3.14 (Linear factors of closed fields): If F is an algebraically closed field, then every non-zero polynomial $P \in F[X] \setminus \{0\}$ **decomposes into linear factors**

$$P = c(X - \lambda_1) \cdots (X - \lambda_n)$$

with $n \geq 0$, $c \in F^\times$ and $\lambda_1, \dots, \lambda_n \in F$. This decomposition is unique up to reordering the factors

Theorem 3.4.5: Properties of Ring Homomorphisms

Let R and S be rings and $f : R \rightarrow S$ a ring homomorphism. Then for all $x, y \in R$ and $m \in \mathbb{Z}$ (where 0_R and 0_S are the zeros of R and S):

1. $f(0_R) = 0_S$
2. $f(-x) = -f(x)$
3. $f(x - y) = f(x) - f(y)$
4. $f(mx) = mf(x)$
5. $f(x^n) = (f(x))^n$ for all $x \in R$ and $n \in \mathbb{N}$

Definition 3.4: All about Ideals

Def 3.4.7: $I \subseteq R$ is an **ideal**, $I \trianglelefteq R$, if the following hold:

1. $I \neq \emptyset$
2. I is closed under subtraction
3. for all $i \in I$ and $r \in R$ we have $ri, ir \in I$

Def 3.4.11: R be a commutative ring and let $T \subset R$. Then the **ideal of R generated by T** is the set

$$_R\langle T \rangle = \{r_1t_1 + \cdots + r_mt_m : t_1, \dots, t_m \in T, r_1, \dots, r_m \in R\}$$

Thm 3.4.14: Let R be a commutative ring and let $T \subseteq R$. Then $_R\langle T \rangle$ is the smallest ideal of R that contains T

Def 3.4.15: Let R be a commutative ring. An ideal I of R is called a **principal ideal** if $I = \langle t \rangle$ for some $t \in R$

Theorem 3.4: Kernels as Ideals

- 3.4.18** Let R and S be rings and $f : R \rightarrow S$ a ring homomorphism. Then $\ker f$ is an ideal of R .
- 3.4.20** f is injective if and only if $\ker f = \{0\}$
- 3.4.21** The intersection of any collection of ideals of a ring R is an ideal of R
- 3.4.22** Let I and J be ideals of a ring R . Then
- $$I + J = \{a + b : a \in I, b \in J\}$$
- is an ideal of R

Definition 3.5.1: Equivalence Relations

A **relation R** on a set X is a subset $R \subseteq X \times X$. In the context of relations, it's written xRy instead of $(x, y) \in R$. R is an **equivalence relation on X** when for all elements $x, y, z \in X$ the following hold:

1. **Reflexivity:** xRx
2. **Symmetry:** $xRy \iff yRx$
3. **Transitivity:** xRy and $yRz \implies xRz$

Suppose that is an equivalence relation on a set X .

- **Equivalence class of x :** $E(x) := \{z \in X : z \sim x \text{ for } x \in X\}$
- **Equivalence class for \sim :** $E \subseteq X$, if $\exists x \in X$ s.t. $E = E(x)$
- **Representative:** Element of an equivalence class
- **System of representatives for \sim :** A subset $Z \subseteq X$ containing precisely one element from each equivalence class

Given an equivalence relation \sim on the set X I will denote the **set of equivalence classes**, which is a subset of the power set $\mathcal{P}(X)$, by

$$(X/\sim) := \{E(x) : x \in X\}$$

There is a canonical mapping $\text{can} : X \rightarrow (X/\sim)$, $x \mapsto E(x)$ (surjection)

Definition 3.6.1: Coset

Let $I \trianglelefteq R$ be an ideal in a ring R . The set

$$x + I := \{x + i : i \in I\} \subseteq R$$

is a **coset of I in R** or the **coset of x w.r.t I in R**

Let R be a ring, $I \trianglelefteq R$ be an ideal, and \sim the equivalence relation defined by $x \sim y \iff x - y \in I$. Then R/I , the **factor ring of R by I** or **the quotient of R by I** , is the set (R/\sim) of cosets of I in R

Thm 3.6.4: Let R be a ring and $I \trianglelefteq R$ an ideal. Then R/I is a ring, where the operation of addition and multiplication is defined by

$$(x + I) + (y + I) = (x + y) + I, \quad (x + I) \cdot (y + I) = xy + I \quad \forall x, y \in R$$

Theorem 3.7: Submodule lemmas

- 3.7.21** Let $f : M \rightarrow N$ be an R -homomorphism. Then $\ker f$ is a submodule of M and $\text{im } f$ is a submodule of N
- 2.7.22** Let R be a ring, M an R -homomorphism. Then f is injective if and only if $\ker f = \{0_M\}$

Definition 3.7.23: Generated Submodules

Let R be a ring, M an R -module and let $T \subseteq M$. Then the **submodule of M generated by T** is the set

$$_R\langle T \rangle = \{r_1t_1 + \cdots + r_mt_m : t_1, \dots, t_m \in T, r_1, \dots, r_m \in R\}$$

together with the zero element in the case $T = \emptyset$. If $T = \{t_1, \dots, t_n\}$, a finite set, we write $_R\langle t_1, \dots, t_n \rangle$ instead of $_R\langle \{t_1, \dots, t_n\} \rangle$. M is **finitely generated** if it's generated by a finite set $M = _R\langle t_1, \dots, t_n \rangle$. M is **cyclic** if it's generated by a singleton $M = _R\langle T \rangle$

- 3.7.28** Let $T \subseteq M$. Then $_R\langle T \rangle$ is the smallest submodule of M that contains T
- 3.7.29** The intersection of any collection of submodules of M is a submodule of M .
- 3.7.30** Let M_1 and M_2 be submodules of a M . Then

$$M_1 + M_2 = \{a + b : a \in M_1, b \in M_2\}$$

is a submodule of M

Definition 3.7.31: Submodule Cosets

Let R be a ring, M an R -module, and N a submodule of M . For each $a \in M$ the **coset of a with respect to N in M** is

$$a + N = \{a + b : b \in N\}$$

It is a coset of N in the abelian group M and so is an equivalence class for the equivalence relation $a \sim b \iff a - b \in N$.

Let M/N , the **factor of N by N** or the **quotient of M by N** to be the set (M/\sim) of all cosets of N in M . This becomes an R -module by introducing the operations of addition and multiplication:

$$(a + N) + (b + N) = (a + b) + N$$
$$r(a + N) = ra + N$$

for all $a, b \in M$, $r \in R$.

The zero of M/N is the coset $0_{M/N} = 0_M + N$. The negative of $a + N \in M/N$ is the coset $-(a + N) = (-a) + N$
The R -module M/N is the **factor module of M by the submod. N**

3 Linear algebra (ew)

Definition 1.4.5: Spans and Linear Independence

Let $T \subset V$ for some vector space V over a field F . Then amongus all subspaces of V that include T there is a smallest subspace

$\langle T \rangle = \langle T \rangle_F \subseteq V$

“the set of all vectors $\alpha_1 \vec{v}_1 + \dots + \alpha_r \vec{v}_r$ with $\alpha_1, \dots, \alpha_r \in F$ and $\vec{v}_1, \dots, \vec{v}_r \in T$, together with the zero vector in the case $T = \emptyset$ ”

Terminology Dump

- **Linear Combination** of vectors $\vec{v}_1, \dots, \vec{v}_r$: An expression of the form $\alpha_1 \vec{v}_1 + \dots + \alpha_r \vec{v}_r$
- **Vec. Subspace generated(or spanned) by T / span of T** : The smallest vector subspace $\langle T \rangle \subseteq V$ containing T
- If we allow the zero vector to be the “empty linear combination of $r = 0$ vectors”, then the span of T is exactly the set of all linear combinations of vectors from T

1.4.7: Generating / Spanning set: A subset of a vector space that spans the entire space. A vector space that has a finite generating set is said to be **finitely generated**

1.5.8: Basis of a vector space V : a linearly independent generating set in V

1.5.9: Let A and I be sets. A **family of elements of A indexed by I** , written $(a_i)_{i \in I}$ is a mapping $I \rightarrow A$

Theorem 1.5.11: Basis Theorems

Thm 1.5.11 (Linear combinations of basis elements): Let F be a field, V a vector space over F and $\vec{v}_1, \dots, \vec{v}_r \in V$ vectors. The family $(\vec{v}_i)_{1 \leq i \leq r}$ is a basis of V iff the following “evaluation” mapping, or if we label the family as \mathcal{A} , written $\psi = \psi_{\mathcal{A}} : F^r \rightarrow V$,

$$\psi : F^r \rightarrow V$$
$$(\alpha_1, \dots, \alpha_r) \mapsto \alpha_1 \vec{v}_1 + \dots + \alpha_r \vec{v}_r$$

is a bijection

Thm 1.5.12 (Characterisation of Bases): The following are equivalent for a subset E of a vector space V :

1. E is a basis, i.e. a linearly independent generating set
2. E is minimal among all generating sets, meaning that $E \setminus \{\vec{v}\}$ does not generate V , for any $\vec{v} \in E$
3. E is maximal among all linearly independent subsets, meaning that $E \cup \{\vec{v}\}$ is linearly dependent for any $\vec{v} \in V$

Thm 1.5.14 (Basis Characterisation Variant)

1. If $L \subset V$ is a linearly indep. subset and E is minimal over all generating sets of V where $L \subseteq E$, then E is a basis.
2. If $E \subseteq V$ is a generating set and if L is maximal amongst all linearly indep. sets of V where $L \subseteq E$, then L is a basis.

Thm 1.5.16 (Variant of Linear Combis of basis elements): Let F be a field, V be an F -vector space and $(\vec{v}_i)_{i \in I}$ a family of vectors from the vector space V . The following are equivalent:

1. The family $(\vec{v}_i)_{i \in I}$ is a basis for V
2. For each $\vec{v} \in V$ there is precisely one family $(a_i)_{i \in I}$ of elements of F , almost all which are zero and such that

$$\vec{v} = \sum_{i \in I} a_i \vec{v}_i$$

Definition 1.4 - 1.5: Random sets

Def 1.4.9: The set of all subsets $\mathcal{P}(X) = \{U : U \subseteq X\}$ of X is the **power set** of X , $\mathcal{P}(X)$ is referred to as a **system of subsets of X** . We can now define 2 new subsets - the **union** and **intersection**

$$\bigcup_{U \in \mathcal{U}} U = \{x \in X : \text{there is } U \in \mathcal{U} \text{ with } x \in U\}$$
$$\bigcap_{U \in \mathcal{U}} U = \{x \in X : x \in U \text{ for all } U \in \mathcal{U}\}$$

Def 1.5.15: Let X be a set and F a field. The set $\text{Maps}(X, F)$ of all mappings $f : X \rightarrow F$ becomes an F -vector space with the operations of pointwise addition and multiplication by a scalar. The subset of all mappings which send almost all elements of X to zero is a vector subspace called the **free vector space on the set X**

$$F\langle X \rangle \subseteq \text{Maps}(X, F)$$

Theorem 1.6.1: Fundamental Estimate of LinAlg

No linearly independent subset of a given vector has more elements than a generating set. Thus if V is a vector space, $L \subset V$ a linearly independent subset and $E \subseteq V$ a generating set, then

$$|L| \leq |E|$$

Theorem 1.6: Steinitz Exchange Theorem

1.6.2: Let V be a vector space, $L \subset V$ a finite linearly indep. subset and $E \subseteq V$ a generating set. Then there is an injection $\phi : L \hookrightarrow E$ such that $(E \setminus \phi(L)) \cup L$ is also a generating set for V

1.6.3: Let V be a vector space, $M \subseteq V$ a linearly indep. subset, and $E \subseteq V$ a generating subset, such that $M \subseteq E$. If $\vec{w} \in V \setminus M$ is a vector $\notin M$ such that $M \cup \{\vec{w}\}$ is linearly independent, then there exists $\vec{e} \in E \setminus M$ such that $(E \setminus \{\vec{e}\}) \cup \{\vec{w}\}$ is a generating set

Theorem 1.6: Cardinality of Bases and Dimension

Def 1.6.4: Let V be a finitely generated vector space. V has a finite basis, and any two bases of V also have the same number of elements

Def 1.6.5: The cardinality of a basis of a finitely generated vector space V is called the **dimension** of V , written $\dim V$.

Theorems

1.6.7 (Cardinality Criterion for Bases)

1. Each linearly independent subset $L \subset V$ has at most $\dim V$ elements, and if $|L| = \dim V$ then L is a basis
2. Each generating set $E \subseteq V$ has at least $\dim V$ elements, and if $|E| = \dim V$ then E is a basis

1.6.8 (Dimension Estimate for Vector Subspaces): A proper vector subspace of a finite dimensional vector space has itself a strictly smaller dimension

1.6.9 If $U \subseteq V$ is a subspace of an arbitrary vector space, then we have $\dim U \leq \dim V$, and if $\dim U = \dim V = \infty$ then $U = V$

1.6.10 (The Dimension Theorem): Let V be a vector space containing vector subspaces $U, W \subseteq V$. Then

$$\dim(U + W) + \dim(U \cap W) = \dim U + \dim W$$

Definition 1.7.1: Linear Mappings

Def 1.7.6: Two vector subspaces V_1, V_2 of a vector space V are called **complementary** if addition defines a bijection

$$V_1 \times V_2 \xrightarrow{\sim} V$$

something about direct sums

Theorem 1.7: Vector Spaces and Linear Maps

1.7.7 Let n be a natural number. Then a vector space over a field F is isomorphic to F^n iff it has dimension n

1.7.8 (Linear Mapping and Bases): Let V, W be vector spaces over a field F . The set of all homoms $V \rightarrow W$ is denoted by

$$\text{Hom}_F(V, W) = \text{Hom}(V, W) \subseteq \text{Maps}(V, W)$$

Let $B \subset V$ be a basis. Then restriction of a mapping gives a bijection

$$\text{Hom}_F(V, W) \xrightarrow{\sim} \text{Maps}(B, W) : f \mapsto f|_B$$

1.7.9: (Inverse Mappings)

1. Every injective linear map $f : V \hookrightarrow W$ has a **left inverse**, or a linear mapping $g : W \rightarrow V$ s.t. $g \circ f = \text{id}_V$
2. Every surjective linear map $f : V \twoheadrightarrow W$ has a **right inverse**, or a linear mapping $G : W \rightarrow V$ s.t. $f \circ g = \text{id}_W$

1.8.2 A linear mapping is injective iff its kernel is zero

1.8.4 (Rank-Nullity Theorem): Let $f : V \rightarrow W$ be a linear mapping between vector spaces. Then:

$$\dim V = \dim(\ker f) + \dim(\text{im } f)$$

Dim. of $\text{im } f$ = **rank** of f , and the dim. of $\ker f$ = **nullity** of f

Theorem 2.1.1: Linear Maps $F^m \rightarrow F^n$ and Matrices

Let F be a field and let $m, n \in \mathbb{N}$. There is a bijection between the space of linear mappings $F^m \rightarrow F^n$ and the set of matrices with n rows, m columns, and entries in F :

$$M : \text{Hom}_F(F^m, F^n) \xrightarrow{\sim} \text{Mat}(n \times m; F)$$
$$f \mapsto [f]$$

This attaches to each linear mapping f its **representing matrix** $M(f) := [f]$. The columns of this matrix are the images under f of the standard basis elements of F^m

$$[f] := ((f(\vec{e}_1)|f(\vec{e}_2)| \dots |f(\vec{e}_m)))$$

Theorem 2.1.8: Composition of maps to products

Let $g : F^\ell \rightarrow F^m$ and $f : F^m \rightarrow F^n$ be linear mappings. The representing matrix of their composition is the product of their representing matrices:

$$[f \circ g] = [f] \circ [g]$$

Definition 2.2: Big def-thm pairs

Thm 2.2.3: Every square matrix with entries in a field can be written as a product of elementary matrices

Def 2.2.4: Smith Normal Form: A matrix that is fully zero, except for 1's on the diagonal followed by 0's

Thm 2.2.5: For each matrix $A \in \text{Mat}(n \times m; F)$ there exist invertible matrices P and Q such that PAQ is a matrix in Smith NF

Thm 2.4.5: Let $f: V \rightarrow W$ be a linear map between finite dim. F -vector spaces. There exists two ordered bases \mathcal{A} of V , and \mathcal{B} of W s.t. the representing matrix $_{\mathcal{B}}[f]_{\mathcal{A}}$ is in Smith Normal Form

Def 2.2.9: Rank of a matrix $A \in \text{Mat}(n \times m; F)$, written $\text{rk } A$: The dim. of the subspace of F^n generated by the columns of A , or same with the row (The row/column rank are the same). If the rank is equal to the no. of rows/columns, then the matrix has **full rank**

Def 2.4.6: Trace, written $\text{tr}(A)$ is the sum of diagonal entries

Theorem 2.3: Representing Matrices

Thm 2.3.1: Let F be a field, V and W vector spaces over F with ordered bases $\mathcal{A} = (\vec{v}_1, \dots, \vec{v}_m)$ and $\mathcal{B} = (\vec{w}_1, \dots, \vec{w}_n)$. Then to each linear mapping $f: V \rightarrow W$ we associate a **representing matrix** $_{\mathcal{B}}[f]_{\mathcal{A}}$ whose entries a_{ij} are defined by the identity

$$f(\vec{v}_j) = a_{1j}\vec{w}_1 + \dots + a_{nj}\vec{w}_n \in W$$

This makes a bijection, which is an isomorphism of vector spaces:

$$M_{\mathcal{B}}^{\mathcal{A}}: \text{Hom}_F(V, W) \xrightarrow{\sim} \text{Mat}(n \times m; F) \quad f \mapsto _{\mathcal{B}}[f]_{\mathcal{A}}$$

Thm 2.3.2: Let F field and U, V, W finite dim. vector spaces over kF with ordered bases $\mathcal{A}, \mathcal{B}, \mathcal{C}$. If $f: U \rightarrow V$, $g: V \rightarrow W$ are linear maps, then the representing matrix of the composition $g \circ f: U \rightarrow W$ is the matrix product of the representing matrices of f and g :

$$_{\mathcal{C}}[g \circ f]_{\mathcal{A}} = _{\mathcal{C}}[g]_{\mathcal{B}} \circ _{\mathcal{B}}[f]_{\mathcal{A}}$$

Def 2.3.4: Let V be a finite dimensional vector space with an ordered basis $\mathcal{A} = (\vec{v}_1, \dots, \vec{v}_m)$. We'll denote the inverse to the bijection in 3 “ $\Phi_{\mathcal{A}}: F^m \xrightarrow{\sim} V, (\alpha_1, \dots, \alpha_m)^T \mapsto \alpha_1\vec{v}_1 + \dots + \alpha_m\vec{v}_m$ ” by

$$\vec{v} \mapsto _{\mathcal{A}}[\vec{v}]$$

The column vector $_{\mathcal{A}}[\vec{v}]$ is called the **representation of the vector \vec{v} with respect to the basis \mathcal{A}**

Thm 2.3.4: Representation of the Image of a Vector: Let V, W be finite dim. vector spaces over F with ordered bases \mathcal{A}, \mathcal{B} and let $f: V \rightarrow W$ be a linear mapping. The following holds for $\vec{v} \in V$:

$$_{\mathcal{B}}[f(\vec{v})] = _{\mathcal{B}}[f]_{\mathcal{A}} \circ _{\mathcal{A}}[\vec{v}]$$

Definition 2.4.1: Change of Basis Matrix

Let $\mathcal{A} = (\vec{v}_1, \dots, \vec{v}_n)$, $\mathcal{B} = (\vec{w}_1, \dots, \vec{w}_n)$ be ordered bases of the same F -vector space V . Then the matrix representing the identity mapping w.r.t. these bases

$$_{\mathcal{B}}[\text{id}_V]_{\mathcal{A}}$$

is called a **change of basis matrix**. Its entries are $\vec{v}_j = \sum_{i=1}^n a_{ij}\vec{w}_i$

Thm 2.4.3: Let V and W be finite dimensional vector spaces over F and let $f: V \rightarrow W$ be a linear mapping. Suppose that $\mathcal{A}, \mathcal{A}'$ are ordered bases of V and $\mathcal{B}, \mathcal{B}'$ are ordered bases of W . Then

$$_{\mathcal{B}'}[f]_{\mathcal{A}'} = _{\mathcal{B}'}[\text{id}_W]_{\mathcal{B}} \circ _{\mathcal{B}}[f]_{\mathcal{A}} \circ _{\mathcal{A}}[\text{id}_V]_{\mathcal{A}'}$$

Crl 2.4.4: Let V be a finite dimensional vector space and let $f: V \rightarrow V$ be an endomorphism of V . Suppose that $\mathcal{A}, \mathcal{A}'$ are ordered bases of V . Then

$$_{\mathcal{A}'}[f]_{\mathcal{A}'} = _{\mathcal{A}}[\text{id}_V]_{\mathcal{A}'}^{-1} \circ _{\mathcal{A}}[f]_{\mathcal{A}} \circ _{\mathcal{A}}[\text{id}_V]_{\mathcal{A}'}$$

Definition 4.1.1: Symmetric Groups

The group of all permutations of the set $\{1, 2, \dots, n\}$, also known as bijections from $\{1, 2, \dots, n\}$ to itself is denoted by \mathfrak{S}_n (but i will just write S_n because icba) and called the **n -th symmetric group**. It is a group under composition and has $n!$ elements.

- **Tranposition:** A permutation that swaps two elements of the set and leaves all the others unchanged.
- **Inversion** of a permutation $\sigma \in S_n$: A pair (i, j) such that $1 \leq i < j \leq n$ and $\sigma(i) > \sigma(j)$.
- **Length of σ :** Num. of inversions of the perm. σ , written $\ell(\sigma)$. i.e.
$$\ell(\sigma) = |\{(i, j) : i < j \text{ but } \sigma(i) > \sigma(j)\}|$$
- **Sign of σ :** The parity of the number of inversions of σ . i.e.:

$$\text{sgn}(\sigma) = (-1)^{\ell(\sigma)}$$

Theorem 4.1: Multiplicativity of the sign

Thm 4.1.5: For each $n \in \mathbb{N}$, the sign of a permutation produces a group homomorphism $\text{sgn}: S_n \rightarrow \{+1, -1\}$ from the symmetric group to the two-element group of signs. In formulas:

$$\text{sgn}(\sigma\tau) = \text{sgn}(\sigma)\text{sgn}(\tau) \quad \forall \sigma, \tau \in S_n$$

Def 4.1.6 (Alternating Group): For $n \in \mathbb{N}$, the set of even permutations in S_n forms a subgroup of S_n because it's the kernel of the group homomorphism $\text{sgn}: S_n \rightarrow \{+1, -1\}$, written A_n

Definition 4.3.1: Bilinear Forms

Let U, V, W be F -vector spaces. A **bilinear form on $U \times V$ with values in W** is a mapping $H: U \times V \rightarrow W$ which is a linear mapping in both of its entries. This means that it must satisfy the following properties for all $u_1, u_2 \in U$ and $v_1, v_2 \in V$ and all $\lambda \in F$:

$$\begin{aligned} H(u_1 + u_2, v_2) &= H(u_1, v_2) + H(u_2, v_2), & H(\lambda u_1, v_1) &= \lambda H(u_1, v_1) \\ H(u_1, v_2 + v_2) &= H(u_1, v_1) + H(u_1, v_2), & H(u_1, \lambda v_1) &= \lambda H(u_1, v_1) \end{aligned}$$

A bilinear form H is **symmetric** if $U = V$ and

$$H(u, v) = H(v, u) \quad \text{for all } u, v \in U$$

while it is **antisymmetric** or **alternating** if $U = V$ and

$$H(u, u) = 0 \quad \text{for all } u \in U$$

- antisymmetric $\implies H(u, v) = -H(v, u)$
- $H(u, v) = -H(v, u) \implies$ antisymmetric iff $1_F + 1_F \neq 0_F$

Definition 4.3.3: Multilinear Forms

Let V_1, \dots, V_n, W be F -vector spaces. A mapping $H: V_1 \times V_2 \times \dots \times V_n \rightarrow W$ is a **multilinear form** or just **multilinear** if for each j , the mapping $V_j \rightarrow W$ defined by $v_j \mapsto H(v_1, \dots, v_j, \dots, v_n)$, with the $v_i \in V_i$ arbitrary fixed vectors of V_i for $i \neq j$ is linear.

Let V and W be F -vector spaces. A multilinear form $H: V \times \dots \times V \rightarrow W$ is **alternating** if it vanishes on every n -tuple of elements of V that has at least two entries equal, in other words if:

$$(\exists i \neq j \text{ with } v_i = v_j) \rightarrow H(v_1, \dots, v_i, \dots, v_j, \dots, v_n) = 0$$

Theorem 4.3.6: Characterisation of the Determinant

Let F be a field. The mapping

$$\det: \text{Mat}(n; F) \rightarrow F$$

is the unique alternating multilinear form on n -tuples of column vectors with values in F that takes the value 1_F on the identity matrix

Definition 4.4.6: Cofactors of a Matrix

Let $A \in \text{Mat}(n; R)$ for some commutative ring R and $n \in \mathbb{N}$. Let $i, j \in \mathbb{Z}$ between 1 and n . Then the (i, j) **cofactor** of A is $C_{ij} = (-1)^{i+j} \det(A(i, j))$ where $A(i, j)$ is the matrix obtained from A by deleting the i -th row and j -th column.

$$C_{23} = (-1)^{2+3} \det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = -a_{11}a_{32} + a_{31}a_{12}$$

Theorem 4.4.7: Laplace's Expansion

Let $A = (a_{ij})$ be an $(n \times n)$ -matrix with entries in a commutative ring R . For a fixed i , the **i -th row expansion of the determinant** (left) and similarly, the **j -th column expansion of the determinant** (right) is

$$\det(A) = \sum_{j=1}^n a_{ij} C_{ij} \quad \left| \quad \det(A) = \sum_{i=1}^n a_{ij} C_{ij} \right.$$

Definition 4.4.8: Adjugate Matrix

Let A be a $(n \times n)$ -matrix with entries in a commutative ring R . The **adjugate matrix** $\text{adj}(A)$ is the $(n \times n)$ -matrix whose entries are $\text{adj}(A)_{ij} = C_{ji}$ where C_{ji} is the (j, i) -cofactor

Theorem 4.4: Determinant Theorem Bank

4.4.1: Let R be a commutative ring, $A, B \in \text{Mat}(n; R)$. Then

$$\det(AB) = \det(A)\det(B)$$

4.4.2: The determinant of a square matrix with entries in a field F is non-zero if and only if the matrix is invertible

- 4.4.3:**
- If A is invertible then $\det(A^{-1}) = \det(A)^{-1}$
 - If B is a square matrix then $\det(A^{-1}BA) = \det(B)$

4.4.4: For all $A \in \text{Mat}(n; R)$ with R a commutative ring,

$$\det(A^T) = \det(A)$$

4.4.9 (Cramer's Rule): Let A be a $(n \times n)$ -matrix with entries in a commutative ring R . Then

$$A \cdot \text{adj}(A) = (\det A)I_n$$

4.4.11 A square matrix with entries in a commutative ring R is invertible if and only if its determinant is a unit in R . That is, $A \in \text{Mat}(n; R)$ is invertible if and only if $\det(A) \in R^\times$

4.4.14 (Jacobi's Formula): Let $A = (a_{ij})$ where the coefficients $a_{ij} = a_{ij}(t)$ are functions of t . Then

$$\frac{d}{dt} \det A = \text{Tr} \text{Adj} A \frac{dA}{dt}$$

Definition 4.5.6: Characteristic Polynomial

Let R be a commutative ring and let $A \in \text{Mat}(n; R)$ be a square matrix with entries in R . The polynomial $\det(xI_n - A) \in R[x]$ is called the **characteristic polynomial of the matrix** A . It is denoted by

$$\chi_A(x) := \det(xI_n - A)$$

Thm: 4.5.8: Let F be a field and $A \in \text{Mat}(n; F)$ a square matrix with entries in F . The eigenvalues of the linear mapping $A : F^n \rightarrow F^n$ are exactly the roots of the characteristic polynomial χ_A

Theorem 4.5.9: Eigenvalue Remarks

- **Thm 4.5.4 (Existence of Eigenvalues)** Each endomorphism of a non-zero finite dimensional vector space over an algebraically closed field has an eigenvalue
- Square matrices $A, B \in \text{Mat}(n; R)$ of same size are **conjugate** if

$$B = P^{-1}AP \in \text{Mat}(n; R)$$

for an invertible $P \in GL(n; R)$

- Conjugacy is an equivalence relation on $\text{Mat}(n; R)$
- The char. polynomials for two conjugate matrices are the same
- We can define the char. polynomials of an endomorphism $f : V \rightarrow V$ of an n -dim vector space over a field F to be

$$\chi_f(x) = \chi_{\mathcal{A}}(x) \in F[x]$$

with $A = {}_{\mathcal{A}}[f]_{\mathcal{A}} \in \text{Mat}(n; R)$ the matrix of f w.r.t *any* basis \mathcal{A} for V . The E.V.s of f are exactly the roots of χ_f

Theorem 4.5.10: Extending Bases

Let $f : V \rightarrow V$ be an endomorphism of an n -dimensional vector space V over a field F . Suppose given an m -dimensional subspace $W \subseteq V$ such that $f(W) \subseteq W$, so that there are defined endomorphisms of the subspace and the quotient space:

$$g : W \rightarrow W; \vec{w} \mapsto f(\vec{w})$$

$$h : V/W \rightarrow V/W; W + \vec{v} \mapsto W + f(\vec{v})$$

The char. poly. of f is the product of the char. poly.s of g and h

Definition 4.6.1: Triangularisability

Let $f : V \rightarrow V$ be an endomorphism of a finite dimensional F -vector space V . f is **triangularisable** if the vector space V has an ordered basis $\mathcal{B} = (\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n)$ such that

$$f(\vec{v}_1) = a_{11}\vec{v}_1,$$

$$f(\vec{v}_2) = a_{12}\vec{v}_1 + a_{22}\vec{v}_2,$$

$$\vdots$$

$$f(\vec{v}_n) = a_{1n}\vec{v}_1 + a_{2n}\vec{v}_2 + \dots + a_{nn}\vec{v}_n \in V$$

(so that the first basis vector \vec{v}_1 is an eigenvector, with eigenvalue a_{11}) or equivalently such that the $n \times n$ matrix ${}_{\mathcal{B}}[f]_{\mathcal{B}} = (a_{ij})$ representing f with respect to \mathcal{B} is upper triangular (or any other triangular)

Theorem 4.6.1 - 4.6.3

Let $f : V \rightarrow V$ be an endomorphism of a finite dimensional F -vector space V . Then f is triangularisable iff the characteristic polynomial χ_f decomposes into linear factors in $F[x]$

Finding ordered bases - Choose from the following subspaces

1. $W = \{\mu\vec{v}_1 \mid \mu \in F\} \subseteq V$
2. $W' = \ker(f - \lambda 1_V)$. This has a basis of E.Vs $\{\vec{v}_1, \dots, \vec{v}_r\}$
3. $W'' = \text{im}(\lambda 1_V - f)$

Then extend the basis to another ordered basis \mathcal{B} for V (the full space) where $\text{can}(\vec{v}_j) = \vec{u}_j$ forms a basis for V/W . ${}_{\mathcal{B}}[f]_{\mathcal{B}}$ is upper triangular.

An endomorphism $A : F^n \rightarrow F^n$ is triangularisable iff $A = (a_{ij})$ is conjugate to $B = (b_{ij})$ ($b_{ij} = 0$ for $i > j$), an upper triangular matrix, with $P^{-1}AP = B$ for an invertible matrix P

Definition 4.6.6: Diagonalisability

An endomorphism $f : V \rightarrow V$ of an F -vector space V is **diagonalisable** iff there exists a basis of V consisting of eigenvectors of f . If V is finite dimensional then this is the same as saying that there exists an ordered basis $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_n\}$ where ${}_{\mathcal{B}}[f]_{\mathcal{B}} = \text{diag}(\lambda_1, \dots, \lambda_n)$. In this case, of course, $f(\vec{v}_i) = \lambda_i\vec{v}_i$.

A square matrix $A \in \text{Mat}(n; F)$ is **diagonalisable** iff A is conjugate to a diagonal matrix, i.e. there exists $P \in GL(n; F)$ such that $P^{-1}AP = \text{diag}(\lambda_1, \dots, \lambda_n)$. In this case the columns P are the vectors of a basis of F^n consisting of eigenvectors of A with eigenvalues $\lambda_1, \dots, \lambda_n$

Theorem 4.6.9: Linear Independence of Eigenvectors

Let $f : V \rightarrow V$ be an endomorphism of a vector space V and let $\vec{v}_1, \dots, \vec{v}_n$ be eigenvectors of f with pairwise different eigenvalues $\lambda_1, \dots, \lambda_n$. Then the vectors $\vec{v}_1, \dots, \vec{v}_n$ are linearly independent

Theorem 4.6.10: Cayley-Hamilton Theorem

Let $A \in \text{Mat}(n; R)$ be a square matrix with entries in a commutative ring R . Then evaluating its characteristic polynomial $\chi_A(x) \in R[x]$ at the matrix A gives zero.

4 Inner Product Spaces

Definition 5.1.1: Inner Product

Let V be a vector space over \mathbb{R} . An **inner product** on V is a mapping

$$(_, _) : V \times V \rightarrow \mathbb{R}$$

that satisfies the following for all $\vec{x}, \vec{y}, \vec{z} \in V$ and $\lambda, \mu \in \mathbb{R}$:

1. $\lambda\vec{x} + \mu\vec{y}, \vec{z} = \lambda(\vec{x}, \vec{z}) + \mu(\vec{y}, \vec{z})$
2. $(\vec{x}, \vec{y}) = (\vec{y}, \vec{x})$
3. $(\vec{x}, \vec{x}) \geq 0$, with equality iff $\vec{x} = \vec{0}$

A **real inner product space** is a real vector space equipped with an inner product. **Note:** basically a generalisation of dot prod.

A **complex inner product space** is a complex vector space equipped with an inner product. This is the exact same, but condition 2 uses $(\vec{x}, \vec{y}) = \overline{(\vec{y}, \vec{x})}$ where \bar{z} is the complex conjugate

Definition 5.1.5: Norm

In a real/complex inner product space, the **length** or **inner product norm** or **norm** $\|\vec{v}\| \in \mathbb{R}$ of a vector \vec{v} is the non-negative square root

$$\|\vec{v}\| = \sqrt{(\vec{v}, \vec{v})}$$

Vectors whose length are 1 are called **units**. Two vectors \vec{v}, \vec{w} are **orthogonal**, written $\vec{v} \perp \vec{w}$, iff $(\vec{v}, \vec{w}) = 0$

The norm $\|\cdot\|$ on an inner product space V satisfies, for any $\vec{v}, \vec{w} \in V$ and scalar λ :

1. $\|\vec{v}\| \geq 0$ with equality iff $\vec{v} = \vec{0}$
2. $\|\lambda\vec{v}\| = |\lambda|\|\vec{v}\|$
3. $|\vec{v} + \vec{w}| \leq \|\vec{v}\| + \|\vec{w}\|$ (triangle inequality)

Definition 5.1.7: Orthonormal Family

A family $(\vec{v}_i)_{i \in I}$ for vectors from an inner product space is an **orthonormal family** if all the vectors \vec{v}_i have length 1 and if they are pairwise orthogonal to each other. If $\delta_{i,j}$ is the **Kronecker delta** defined by "1 if $i = j$, and 0 otherwise", this means that $(\vec{v}_i, \vec{v}_j) = \delta_{ij}$. An orthonormal family that has a basis is an **orthonormal basis**

Thm 5.1.10: Every finite dimensional inner product space has an orthonormal basis

Definition 5.2.1: Orthogonals to a Subset

Let V be an inner product space and let $T \subseteq V$ be an arbitrary subset. Define

$$T^\perp = \{\vec{v} \in V : \vec{v} \perp \vec{t} \forall \vec{t} \in T\}$$

calling this set the **orthogonal** to T

Theorem 5.2.2: Complementary Orthogonals

Let V be an inner product space and let U be a finite dimensional subspace of V . Then U and U^\perp are complementary, i.e. $V = U \oplus U^\perp$

Definition 5.2.3: Orthogonal Projection

Let U be a finite dimensional subspace of an inner product space V . The space U^\perp is the **orthogonal complement to** U . The **orthogonal projection from** V **onto** U is the map

$$\pi_U : V \rightarrow V$$

that sends $\vec{v} = \vec{p} + \vec{r}$ to \vec{p}

Prop 5.2.4: Let U be a finite dimensional subspace of an inner product space V and let π_U be the orthogonal projection from V onto U

1. π_U is a linear mapping with $\text{im}(\pi_U) = U$ and $\ker(\pi_U) = U^\perp$
2. If $\{\vec{v}_1, \dots, \vec{v}_n\}$ is an orthonormal basis of U , then π_U is given by the following formula for all $\vec{v} \in V$

$$\pi_U(\vec{v}) = \sum_{i=1}^n (\vec{v}, \vec{v}_i) \vec{v}_i$$

3. $\pi_U^2 = \pi_U$, that is, π_U is an idempotent

Theorem 5.2.5: Cauchy-Shwarz Inequality

Let \vec{v}, \vec{w} be vectors in an inner product space. Then

$$|(\vec{v}, \vec{w})| \leq \|\vec{v}\| \|\vec{w}\|$$

with equality if and only if \vec{v} and \vec{w} are linearly dependent

Theorem 5.2.7: Gram-Shmidt Process

Let $\vec{v}_1, \dots, \vec{v}_k$ be linearly independent vectors in an inner product space V . Then there exists an orthonormal family $\vec{w}_1, \dots, \vec{w}_k$ with the property that for all $1 \leq i \leq k$,

$$\vec{w}_i \in \mathbb{R}_{>0} \vec{v}_i + \langle \vec{v}_{i-1}, \dots, \vec{v}_1 \rangle$$

TODO: write how to actually do the gram-shmidt process

Definition 5.3.1: Adjoints

Let V be an inner product space. Then two endomorphisms $T, S: V \rightarrow V$ are called **adjoint** to one another if the following holds for all $\vec{v}, \vec{w} \in V$:

$$(T\vec{v}, \vec{w}) = (\vec{v}, S\vec{w})$$

In this case I will write $S = T^*$ and call S the **adjoint** of T

Remark 5.3.2: Any endomorphism has at most one adjoint.

Theorem 5.3.4

Let V be a finite dimensional inner product space. Let $T: V \rightarrow V$ be an endomorphism. Then T^* exists. That is, there is a unique linear mapping $T^*: V \rightarrow V$ such that for all $\vec{v}, \vec{w} \in V$:

$$(T\vec{v}, \vec{w}) = (\vec{v}, T^*\vec{w})$$

Definition 5.3.5: Self Adjoints

An endomorphism of an inner product space $T: V \rightarrow V$ is **self-adjoint** if it equals its own adjoint, i.e. if $T^* = T$

Thm 5.3.7: Let $T: V \rightarrow V$ be a self-adjoint linear mapping on an inner product space V

1. Every eigenvalue of T is real
2. If λ and μ are distinct eigenvalues of T with corresponding eigenvectors \vec{v} and \vec{w} , then $(\vec{v}, \vec{w}) = 0$
3. T has an eigenvalue

Definition 5.3.11: Orthogonal Matrices

An **Orthogonal matrix** is an $(n \times n)$ -matrix P with real entries such that $P^T P = I_n$, or in other words such that $P^{-1} = P^T$

A **hermitian matrix** is one that is self-adjoint in \mathbb{C} , or in other words one where $A = \overline{A}^T$ holds

An **unitary matrix** is an $(n \times n)$ -matrix P with complex entries such that $\overline{P}^T P = I_n$, or such that $P^{-1} = \overline{P}^T$

Theorem 5.3.9: Spectral Theorems

5.3.9: The Spectral Theorem for Self-Adjoint Endomorphisms

Let V be a finite dimensional inner product space and let $T: V \rightarrow V$ be a self-adjoint linear mapping. Then V has an orthonormal basis consisting of eigenvalues of T .

5.3.11: The Spectral Theorem for Real Symmetric Matrices

Let A be a real $(n \times n)$ -symmetric matrix. Then there is an $(n \times n)$ -orthogonal matrix P such that

$$P^T A P = P^{-1} A P = \text{diag}(\lambda_1, \dots, \lambda_n)$$

where $\lambda_1, \dots, \lambda_n$ are the (necessarily real) eigenvalues of A , repeated according to their multiplicity as roots of χ_A

5.3.15: The Spectral Theorem for Hermitian Matrices

Let A be a $(n \times n)$ -hermitian matrix. Then there is an $(n \times n)$ -unitary matrix P such that

$$\overline{P}^T A P = P^{-1} A P = \text{diag}(\lambda_1, \dots, \lambda_n)$$

where $\lambda_1, \dots, \lambda_n$ are the (necessarily real) eigenvalues of A , repeated according to their multiplicity as roots of χ_A

5 Jordan Normal Form

Definition 6.2.1: Jordan Blocks

Given an integer $r \geq 1$ define an $(r \times r)$ -matrix $J(r)$ called the **nilpotent Jordan block of size r** , by the rule $J(r)_{ij} = 1$ for $j = i + 1$ AND $J(r)_{ij} = 0$ otherwise
In particular, $J(1)$ is a (1×1) -matrix whose only entry is zero.

Given an integer $r \geq 1$ and a scalar $\lambda \in F$, define an $(r \times r)$ -matrix $J(r, \lambda)$ called the **Jordan block of size r and eigenvalue λ** by the rule

$$J(r, \lambda) = \lambda I_r + J(r) = D + N$$

with $\lambda I_r = \text{diag}(\lambda, \lambda, \dots, \lambda) = D$ diagonal and $J(r) = N$ nilpotent such that $DN = ND$

Theorem 6.2.2: Jordan Normal Form

Let F be an algebraically closed field. Let V be a finite dimensional vector space and let $\phi: V \rightarrow V$ be an endomorphism of V with characteristic polynomial

$$\chi_\phi(x) = (x - \lambda_1)^{a_1} (x - \lambda_2)^{a_2} \dots (x - \lambda_s)^{a_s} \in F[x], a_i \geq 1, \sum_{i=1}^s a_i = n$$

For distinct $\lambda_1, \lambda_2, \dots, \lambda_s \in F$. Then there exists an ordered basis \mathcal{B} of V such that the matrix of ϕ with respect to the block \mathcal{B} is block diagonal with Jordan blocks on the diagonal, $_{\mathcal{B}}[\phi]_{\mathcal{B}}$

$$= \text{diag}(J(r_{11}, \lambda_1), \dots, J(r_{1m_1}, \lambda_1), J(r_{21}, \lambda_2), \dots, J(r_{sm_s}, \lambda_s))$$

with $r_{11}, \dots, r_{1m_1}, r_{21}, \dots, r_{sm_s} \geq 1$ such that

$$a_i = r_{i1} + r_{i2} + \dots + r_{im_i} \quad (1 \leq i \leq s)$$

Theorem 6.3.1: Bézout's identity for polynomials

For a characteristic polynomial

$$\chi_\phi(x) = \prod_{i=1}^s (x - \lambda_i)^{a_i} \in F[x]$$

where each a_i is a positive integer, $\lambda_i \neq \lambda_j$ for $i \neq j$, and λ_i are e.v.s of ϕ . For each $1 \leq j \leq s$ define

$$P_j(x) = \prod_{\substack{i=1 \\ i \neq j}}^s (x - \lambda_i)^{a_i}$$

There exists polynomials $Q_j(x) \in F[x]$ such that

$$\sum_{j=1}^s P_j(x) Q_j(x) = 1$$

Definition 6.3.2: Generalised Eigenspace

The **generalised eigenspace** of ϕ with eigenvalue λ_i , $E^{\text{gen}}(\lambda_i, \phi)$ is the following subspace of V :

$$E^{\text{gen}}(\lambda_i, \phi) = \{\vec{v} \in V \mid (\phi - \lambda_i \text{id}_V)^{a_i}(\vec{v}) = \vec{0}\}$$

The dimension of $E^{\text{gen}}(\lambda_i, \phi)$ is called the **algebraic multiplicity of ϕ with eigenvalue λ_i** while the dimension of the eigenspace $E(\lambda_i, \phi)$ is called the **geometric multiplicity of ϕ with eigenvalue λ**

Remark 6.3.4: The actual eigenspace is defined by

$$E(\lambda_i, \phi) = \{\vec{v} \in V \mid (\phi - \lambda_i \text{id}_V)(\vec{v}) = \vec{0}\}$$

$E^{\text{gen}}(\lambda_i, \phi) \subseteq E^{\text{gen}}(\lambda_i, \phi)$, or the algebraic multiplicity of any e.v. must be greater or equal to the corresponding geometric multiplicity

Definition 6.3.4: Stable subsets

Let $f: X \rightarrow X$ be a mapping from a set X to itself. A subset $Y \subseteq X$ is **stable under f** precisely when $f(Y) \subseteq Y$, that is if $y \in Y$ then $f(y) \in Y$.

Theorem 6.3.5: Direct Sum Composition

For each $1 \leq i \leq s$, let

$$\mathcal{B}_i = \{\vec{v}_{ij} \in V \mid 1 \leq j \leq a_i\}$$

be a basis of $E^{\text{gen}}(\lambda_i, \phi)$, where a_i is the algebraic multiplicity of ϕ with eigenvalue λ_i s.t. $\sum_{i=1}^s a_i = n$ is the dimension of V .

- Each $E^{\text{gen}}(\lambda_i, \phi)$ is stable under ϕ
- For each $\vec{v} \in V$ there exist unique $\vec{v}_i \in E^{\text{gen}}(\lambda_i, \phi)$ such that $\vec{v} = \sum_{i=1}^s \vec{v}_i$. In other words, there is a direct sum decomposition

$$V = \bigoplus_{i=1}^s E^{\text{gen}}(\lambda_i, \phi)$$

with ϕ restricting to endomorphisms of the summands

$$\phi_i = \phi| : E^{\text{gen}}(\lambda_i, \phi) \rightarrow E^{\text{gen}}(\lambda_i, \phi)$$

- Then

$$\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \dots \cup \mathcal{B}_s = \{\vec{v}_{ij} \mid 1 \leq i \leq s, 1 \leq j \leq a_i\}$$

is a basis of V . The matrix of the endomorphism ϕ w.r.t. this basis is given by the block diagonal matrix

$$_{\mathcal{B}}[\phi]_{\mathcal{B}} = \begin{pmatrix} B_1 & 0 & 0 & 0 \\ 0 & B_2 & & 0 \\ & & \ddots & \\ 0 & 0 & 0 & B_s \end{pmatrix} \in \text{Mat}(n; F)$$

with $B_i = _{\mathcal{B}_i}[\phi_i]_{\mathcal{B}_i} \in \text{Mat}(a_i; F)$

Theorem 6.3: JNF Theorem Bank

- 6.3.6:** For each i , define a linear mapping

$$\psi_i : \frac{W_i}{W_{i-1}} \rightarrow \frac{W_{i-1}}{W_{i-2}}$$

by $\psi_i(\vec{w} + W_{i-1}) = \psi(\vec{w}) + W_{i-2}$ for $\vec{w} \in W_i$. Then ψ_i is well-defined and injective

- 6.3.7:** Let $f : X \rightarrow Y$ be an injective linear mapping between the F -vector spaces X and Y . If $\{\vec{x}_1, \dots, \vec{x}_t\}$ is a linearly independent set in X , then $\{f(\vec{x}_1), \dots, f(\vec{x}_t)\}$ is a linearly independent set in Y

- 6.3.8:** The set of elements $\{\vec{v}_{j,k} : 1 \leq j \leq m, 1 \leq k \leq d_j\}$ constructed in the next algorithm is a basis for W

- 6.3.9:** Let \mathcal{B} be the ordered basis of W - $\{\vec{v}_{j,k} : 1 \leq j \leq m, 1 \leq k \leq d_j\}$. Then $_{\mathcal{B}}[\psi]_{\mathcal{B}} =$
- $$\text{diag} \underbrace{J(m), \dots, J(m)}_{d_m \text{ times}}, \underbrace{J(m-1), \dots, J(m-1)}_{d_{m-1} - d_m \text{ times}}, \dots, \underbrace{J(1), \dots, J(1)}_{d_1 - d_2 \text{ times}}$$
- where $J(r)$ denotes the nilpotent Jordan block of size r

Theorem 6.3: JNF Basis Algorithm

Algorithm to construct a basis for each W_i/W_{i-1} :

- Choose an arbitrary basis for W_m/W_{m-1} , say $\{v_{m,1} + W_{m-1}, \vec{v}_{m,2} + W_{m-1}, \dots, \vec{v}_m, d_m + W_{m-1}\}$
- Since $\psi_m : W_m/W_{m-1} \rightarrow W_{m-1}/W_{m-2}$ is injective by 6.3.6, 6.3.7 proves that $\{\psi(\vec{v}_{m,1}) + W_{m-2}, \psi(\vec{v}_{m,2}) + W_{m-2}, \dots, \psi(\vec{v}_m, d_m + W_{m-2})\}$ is a linearly independent set in W_{m-1}/W_{m-2} . Set $\vec{v}_{m-1,i} = \psi(\vec{v}_{m,i})$ for $1 \leq i \leq d_m$
- Choose vectors $\{\vec{v}_{m-1,i} : d_m + 1 \leq i \leq d_{m-1}\}$ so that $\{\vec{v}_{m-1,i} + W_{m-i-1} : 1 \leq k \leq d_{m-i}\}$ is a basis of W_{m-1}/W_{m-2}
- Repeat!

Lorem ipsum dolor sit amet, consectetur adipiscing elit. Ut purus elit, vestibulum ut, placerat ac, adipiscing vitae, felis. Curabitur dictum gravida mauris. Nam arcu libero, nonummy eget, consectetur id, vulputate a, magna. Donec vehicula augue eu neque. Pellentesque habitant morbi tristique senectus et netus et malesuada fames ac turpis egestas. Mauris ut leo. Cras viverra metus rhoncus sem. Nulla et lectus vestibulum urna fringilla ultrices. Phasellus eu tellus sit amet tortor gravida placerat. Integer sapien est, iaculis in, pretium quis, viverra ac, nunc. Praesent eget sem vel leo ultrices bibendum. Aenean faucibus. Morbi dolor nulla, malesuada eu, pulvinar at, mollis ac, nulla. Curabitur auctor semper nulla. Donec varius orci eget risus.

Duis nibh mi, congue eu, accumsan eleifend, sagittis quis, diam. Duis eget orci sit amet orci dignissim rutrum.

Nam dui ligula, fringilla a, euismod sodales, sollicitudin vel, wisi. Morbi auctor lorem non justo. Nam lacus libero, pretium at, lobortis vitae, ultricies et, tellus. Donec aliquet, tortor sed accumsan bibendum, erat ligula aliquet magna, vitae ornare odio metus a mi. Morbi ac orci et nisl hendrerit mollis. Suspendisse ut massa. Cras nec ante. Pellentesque a nulla. Cum sociis natoque penatibus et magnis dis parturient montes, nascetur ridiculus mus. Aliquam tincidunt urna. Nulla ullamcorper vestibulum turpis. Pellentesque cursus luctus mauris.

Nulla malesuada porttitor diam. Donec felis erat, congue non, volutpat at, tincidunt tristique, libero. Vivamus viverra fermentum felis. Donec nonummy pellentesque ante. Phasellus adipiscing semper elit. Proin fermentum massa ac quam. Sed diam turpis, molestie vitae, placerat a, molestie nec, leo. Maecenas lacinia. Nam ipsum ligula, eleifend at, accumsan nec, suscipit a, ipsum. Morbi blandit ligula feugiat magna. Nunc eleifend consequat lorem. Sed lacinia nulla vitae enim. Pellentesque tincidunt purus vel magna. Integer non enim. Praesent euismod nunc eu purus. Donec bibendum quam in tellus. Nullam cursus pulvinar lectus. Donec et mi. Nam vulputate metus eu enim. Vestibulum pellentesque felis eu massa.

Quisque ullamcorper placerat ipsum. Cras nibh. Morbi vel justo vitae lacus tincidunt ultrices. Lorem ipsum dolor sit amet, consectetur adipiscing elit. In hac habitasse platea dictumst. Integer tempus convallis augue. Etiam facilisis. Nunc elementum fermentum wisi. Aenean placerat. Ut imperdiet, enim sed gravida sollicitudin, felis odio placerat quam, ac pulvinar elit purus eget enim. Nunc vitae tortor. Proin tempus nibh sit amet nisl. Vivamus quis tortor vitae risus porta vehicula.

Fusce mauris. Vestibulum luctus nibh at lectus. Sed bibendum, nulla a faucibus semper, leo velit ultricies tellus, ac venenatis arcu wisi vel nisl. Vestibulum diam. Aliquam pellentesque, augue quis sagittis posuere, turpis lacus congue quam, in hendrerit risus eros eget felis. Maecenas eget erat in sapien mattis porttitor. Vestibulum porttitor. Nulla facilisi. Sed a turpis eu lacus commodo facilisis. Morbi fringilla, wisi in dignissim interdum, justo lectus sagittis dui, et vehicula libero dui cursus dui. Mauris tempor ligula sed lacus. Duis cursus enim ut augue. Cras ac magna. Cras nulla. Nulla egestas. Curabitur a leo. Quisque egestas wisi eget nunc. Nam feugiat lacus vel est. Curabitur consectetur.

Suspendisse vel felis. Ut lorem lorem, interdum eu, tincidunt sit amet, laoreet vitae, arcu. Aenean faucibus pede eu ante. Praesent enim elit, rutrum at, molestie non, nonummy vel, nisl. Ut lectus eros, malesuada sit amet, fermentum eu, sodales cursus, magna. Donec eu purus. Quisque vehicula, urna sed ultricies auctor, pede lorem egestas dui, et convallis elit erat sed nulla. Donec luctus. Curabitur et nunc. Aliquam dolor odio, commodo pretium, ultricies non, pharetra in, velit. Integer arcu est, nonummy in, fermentum faucibus, egestas vel, odio.

Sed commodo posuere pede. Mauris ut est. Ut quis purus. Sed ac odio. Sed vehicula hendrerit sem. Duis non odio. Morbi ut dui. Sed accumsan risus eget odio. In hac habitasse platea dictumst. Pellentesque non elit. Fusce sed justo eu urna porta tincidunt. Mauris felis odio, sollicitudin sed, volutpat a, ornare ac, erat. Morbi quis dolor. Donec pellentesque, erat ac sagittis semper, nunc dui lobortis purus, quis congue purus metus ultricies tellus. Proin et quam. Class aptent taciti sociosqu ad litora torquent per conubia nostra, per inceptos hymenaeos. Praesent sapien turpis, fermentum vel, eleifend faucibus, vehicula eu, lacus.