# Dynamics and Vector Calculus Notes

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# Contents

# 1 Couple Oscillations and normal modus

some diagram idk

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where $x_1$ and $x_2$ are displacements from equilibrium
**For mass 1**

- Force to the left: $-k_1 x_1$
- Force to the right: $-k_2(x_2 - x_1)$

$$m\frac{d^2x_1}{dt^2} = -k_1x_1 + k_2(x_2 - x_1) - k_3x_2$$

Write this in matrix form

$$m\frac{d^2}{dt^2}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \implies m\frac{d^2x}{dt^2} = -Kx$$

**Definition 1.0.1: Normal Mode Solution**

**Normal Mode Solution**: All co-ordinates (here $x_1$, $x_2$) oscillate with the same frequency

$x(t) = \cos{(\omega t - \phi)}\underline{b}$
$\underline{b}$ is constant vector, $\omega$ to be determined
sub in matrixeq??

$$\begin{aligned} &- m\omega^2\cos(\omega t - \phi)\underline{b} + K\cos(\omega t - \varnothing)\underline{b} = 0 \\ &- m\omega^2\underline{b} + K\underline{b} = 0 \rightarrow K\underline{b} = \lambda\underline{b} \quad \lambda = m\omega^2 \end{aligned}$$

where $\lambda$ is eigenvalue, and $b$ is eigenvector
For simplicity, take $k_1 = k_2 = k_3 = k$
Then,

$$K = \begin{pmatrix} 2k & -k \\ -k & 2k \end{pmatrix} \qquad (K - \lambda\mathbb{1})\underline{b} = 0 \implies |k - \lambda\mathbb{1}| = 0$$

$$\begin{vmatrix} 2k - \lambda & -k \\ -k & 2k - x \end{vmatrix} = 0 \implies (2k - \lambda)^2 - k^2 = 0$$

This is called the "Characteristic Equation"
$(2k - \lambda) = \pm k \quad \lambda = 2k \mp k$
Therefore, $\lambda = k, 3k$
Mode A: $\lambda_A = k \quad (K - k\mathbb{1})\underline{b} = 0$

$$\begin{pmatrix} k & -k \\ -k & k \end{pmatrix}\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = 0 \quad \underline{b}_A = Ct\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Usually, choose a constant s.t. $\underline{b} \cdot \underline{b} = 1$
Mode B: $\lambda_A = 3k \quad (K - 3k\mathbb{1})\underline{b} = \begin{pmatrix} -k & -k \\ -k & -k \end{pmatrix}\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = 0$
and some stuff more i forgor to write

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[diagram thing]
Normal mode $\underline{x}(t) = \underline{b}\cos(\omega t - \phi) \rightarrow (K - \kappa\mathbb{1})\underline{b} = 0 \quad \lambda = m\omega^2$

$$\lambda_A = k,\, \underline{b}_A = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \lambda_B = 3k,\, \underline{b}_B = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

So we have 2 independent solutions
General solution: $\underline{x}(t) = A\underline{b}_A \cos(\omega_A t - \phi_A) + B\underline{b}_B \cos(\omega_B t - \varnothing_B)$
So there are 4 constants $A$, $B$, $\phi_A$, $\varnothing_B$ to be fixed

## 1.1 Motion in Normal modes

$$\begin{aligned} &\text{Mode A} x_1 = x_2 \quad \text{"unphase"} \quad \omega_A = \left(\frac{k}{m}\right)^2 \\ &\text{Mode B} x_1 = -x_2 \quad \text{"antiphase"} \quad \omega_A > \omega B \end{aligned}$$

**Normal Co-ordinates** Take scalar product

$$\begin{aligned} (1,1)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= x_1 + x_2 = 2A\cos(\omega_A - \phi_A) \\ (1,-1)\begin{pmatrix} x_1 \\ -x_2 \end{pmatrix} &= x_1 + x_2 = 2B\cos(\omega_B t - \phi_B) \end{aligned}$$

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**Define**

$$\begin{aligned} z_1 &= \frac{1}{\sqrt{2}}(x_1 + x_2) = \alpha^1 \cos(w_A t - \phi) \quad z_1 + \omega_A^2 z_1 = 0 \quad \text{(SHO)} \\ z_2 &= \frac{1}{\sqrt{2}}(x_1 - x_2) = \beta^1 \cos(w_B t - \phi) \quad z_2 + \omega_B^2 z_2 = 0 \quad \text{(SHO)} \end{aligned}$$

$z_1$ and $z_2$ are each independent simple harmonic motions, and energy is preserved in each one

$$E_A = \frac{1}{2}m(z_1)^2 + \frac{1}{2}kz_1^2 = \text{constant in time}$$

## 1.2 Summmary: properties of Normal Modes

- $\underline{x}_\alpha = A_\alpha \underline{b}_A \cos(\omega_\alpha t - \phi_\alpha))$
- All coordinates oscillate at the same frequency
- constants $A_\alpha, \phi_\alpha$ are fixed by ic (???)
- General motion is superposition of normal modes
- Normal coordinates $z_\alpha = \underline{b}_\alpha \cdot \underline{x}$
- Transforming to the noraml coordinates $\rightarrow$ diagonalise $k$ (see notes i.e. ask alice or fiona for them)
- Energy in each normal mode conserved, mode with lowest $\omega$ is the most symmetric

## 1.3 Coupled Pendulum

[a diagram]
pendulum thing

$$ml\frac{d^2\theta}{dt^2} = -ml\omega_?^2\theta \quad \omega_\theta = \left(\frac{g}{l}\right)^{1/2}$$

Add in the force from the spring exension:

$$x_2 - x_1 = l(\sin\theta_2 - \sin\theta_3) \approx l(\theta_2 - \theta)$$

$$m\frac{d^2\theta_1}{dt^2} = -m\omega_0^2\theta_1 + k(\theta_2 - \theta_1)$$
$$m\frac{d^2\theta_2}{dt^2} = -m\omega_0^2\theta_2 + k(\theta_2 - \theta_1)$$

Putting it in vector form thing

$$m\frac{d^2}{dt^2}\begin{pmatrix}\theta_1\\ \theta_2\end{pmatrix} = -\begin{pmatrix}m\omega_0^2 + k & -k\\ -k & m\omega_0^2 + k\end{pmatrix}\begin{pmatrix}\theta_1\\ \theta_2\end{pmatrix}$$

Normal mode $\underline{\theta} = \underline{b}\cos(\omega t) \quad - m\omega_2\underline{b} + K\underline{b} = 0$
eigenvalue problem $K\underline{b} = \lambda\underline{b} \quad \lambda = m\omega^2$

$$\det(K - \lambda\mathbb{K}) = 0 \quad \begin{vmatrix}m\omega_0^2 + k - \lambda & -k\\ -k & m\omega_0^2 - k - \lambda\end{vmatrix} = 0$$

$$(mw\omega_0^2 + k - \lambda)^2 - k^2 = 0 \quad \lambda_\Delta = m\omega_0^2 \quad \lambda_B = m\omega_0^2 + 2k$$

Eigenvctors

$$\lambda_A = m\omega_0^2 = \begin{pmatrix}k & -k\\ -k & k\end{pmatrix}\underline{b}_A = 0 \quad \underline{b}_A = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ 1\end{pmatrix} \quad \text{inphase} \quad \omega = \omega_0$$
$$\lambda_B = m\omega_0^2 = \begin{pmatrix}k & -k\\ -k & -k\end{pmatrix}\underline{b}_B = 0 \quad \underline{b}_B = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -1\end{pmatrix} \quad \text{antiphase} \quad \omega_B^2 = w_0^2 + \frac{2k}{m}$$

### 1.3.1 Mass Matrix

$$m_1x_1 = (k_1 + k_2)x_1 + k_2x_2$$
$$m_2x_2 = k_2x_1 - (k_3 + k_2)x_2$$

NOTE: have defo missed some double dot $x$ at some points
Write this as $M\ddot{\underline{x}} = -K\underline{x} \quad M = \begin{pmatrix}m_1 & 0\\ 0 & m_2\end{pmatrix}$
Normal mode $\underline{x}(t) = \underline{b}\cos(\omega t - \pi) \quad -\omega^2M\underline{b} = -K\underline{b} \quad (K - \lambda M)\underline{b} = 0$
$(K - \lambda M)\underline{b} = 0$ for nontrivial sol $^n$(??) $\det(K - \lambda M) = 0$

$$\lambda = \omega^2 \quad \begin{vmatrix}k_1 + k_2 - \lambda m_1 & -k_2\\ -k_2 & k_1k_2 - \lambda m_2\end{vmatrix} = 0$$

$\implies$ quadratic for $\lambda$. For equal $k$,

$$(2k - \lambda m_1)(2k - \lambda m_2) - k^2 = 0 \quad \text{(quadratic)}$$

## 1.4 Line integral :)

idk what's happening but line integral

$$\Gamma(x) = \int_a^{a+\frac{\pi}{2}} \sin^2 \lambda d\lambda = \int_a^{a+\frac{\pi}{2}} \cos^2 \lambda d\lambda = \frac{1}{2}\int_a^{a+\frac{\pi}{2}} \sin^2 \lambda + \cos^2 \lambda d\lambda = \frac{\pi}{4}$$

$$= \frac{\pi}{4} - \frac{1}{3}\sin^3 \lambda \cos \lambda \Big|_0^{\frac{\pi}{2}} - \frac{1}{3}\underbrace{\int_0^{\frac{\pi 2}{\sin}} 4\lambda d\lambda}_{I} = \frac{\pi}{4} - \frac{I}{3} = I$$

random facts

- $\underline{\Delta} \times \underline{\Delta}\phi = 0$
- $\underline{\Delta} \cdot (\underline{\Delta \times \underline{a}}) = 0$

## 1.5 Surface Integrals

(shoutout to the generalised stoke's theorem, he got me fr fr)

**Definition 1.5.1: Paramatric form of the surface integral**

$$\begin{aligned}\underline{\Lambda} &= \underline{\Lambda}(u, v)\\ &= x_1(u, v)\underline{e}_1 + x_2(u, v)\underline{e}_2 + x_3(u, v)\underline{e}_3\end{aligned}$$

**Example:** Sphere (in spherical coordinates) idk how to draw diagrams

$$\begin{aligned}x_1(\theta, \phi) &= \sin\theta\cos\phi\\ x_2(\theta, \phi) &= \sin\theta\sin\phi\\ x_3(\theta, \phi) &= \cos\theta\end{aligned}$$

$$d\underline{r} = \underbrace{\partial_X \underline{r} du}_{d\underline{r}_u} + \underbrace{\partial_X \underline{r} dv}_{d\underline{r}_v}$$

$$d\underline{S} = d\underline{R}_u \times d\underline{r}_v = \text{"area of infinitessimal parallelogram"}$$

Actual line integral equation

**Definition 1.5.2: Line Integral Equation**

$$\int_S \underline{a} \cdot d\underline{S} = \iint \underline{a} \cdot (\partial_u \underline{r} \times \partial_v \underline{r}) du dv$$

**Remarksz**

- $\vec{\underline{n}} \propto \partial u\underline{r} \times \partial_v \underline{r}$
- ambiguity in orientation ($u \leftrightarrow v$ interchase)
  - (circle): closed surface, choose $\vec{\underline{u}}$ outwards
  - open surface: can do either - choose one. In Stoke's theorem, there will be a double ambiguity
    - Orientation of $S$

* Direction of line integral

- For applications, $\partial_u \underline{r} \delta_v \underline{r}$. e.g. spherical coords: $\partial_\theta \underline{r} \times \partial_\phi \underline{r} \propto \partial_r \underline{r}$

**Definition 1.5.3: somethinglinear coordinates $I$, flux & surface**

$$(x_1,\, x_2,\, x_3) \to (x, y, z)$$

**Definition 1.5.4: Plan polar coordinates**

$$\begin{aligned} x_1(\rho,\phi) &= \rho\cos\phi & \rho &= \sqrt{x_1^2 + y_2^2} \in [0,\infty) \\ x_2(\rho,\phi) &= \rho\sin\phi & tg\phi &= x_{\frac{2}{x_1}} \in [0,\infty) \end{aligned}$$

$$\underline{\Gamma}(\rho,\phi) = x_1(\rho,\phi)\underline{e}_1 + \underline{x}_2(\rho,\phi)\underline{e_2}$$

$$\underline{e}_\rho = \frac{\partial_\rho \underline{r}}{|\partial_\rho \underline{r}|} = \cos\phi \underline{e_1} + \sin\phi \underline{e_2} = \frac{1}{\rho}\underline{r}(\text{special case final line})$$

**Remarks**

- $\underline{r} \neq \rho \underline{e}_\rho + \phi \underline{e}_\rho$
- generally, $\underline{a} = \underline{a}_\rho \underline{\epsilon}_\rho + a_\phi \underline{e}_\phi$, $\alpha_{\rho,\phi} = \underline{a} \cdot \underline{e}_{\rho,\phi}$
- new aspect: $\{\underline{e}_\rho, \underline{e}_\rho\}$ is position dependent

**Definition 1.5.5: Cylindrical Coordinates**

$$\begin{aligned} x_1 &= \rho\cos\phi \\ x_2 &= \rho\sin\phi \\ x_3 &= z \end{aligned}$$

$$\begin{aligned} \underline{\Gamma}(\rho,\phi,z) &= \rho\cos\phi\underline{e}_1 + \rho\sin\phi\underline{e}_1 + z\underline{e}_3 \\ &= \rho\underline{e}_\rho + z\underline{e}_z \text{ (special case)} \end{aligned}$$

**Note:** $\{\underline{e}_\rho, \underline{e}_\phi, \underline{e}_\zeta\}$ forms a right-hand orthogonal basis

# 2 filler

# 3 more filler

## 3.1 filler

## 3.2 The Vector Potential

Results ahead...

$$\underline{\nabla} \times \underline{a} = 0 \iff \exists \phi \text{ s.t. } \underline{a} = \underline{\nabla}\phi;\ \phi = \int_0^1 d\lambda (\underline{a}(\lambda \underline{r}) \cdot \underline{r})$$

$$\underline{\nabla} \times \underline{B} = 0 \iff \exists \underline{A} \text{ s.t. } \underline{B} = \underline{\nabla A};\ \underline{A} = \int_0^1 d\lambda (\underline{B}(\lambda \underline{r}) \cdot \underline{r}\lambda)$$

**Theorem 3.2.1: Helmholtz Theorem**

Smooth $\underline{Q}$ decomposes (not unique)

$$\underline{Q} = \underline{\nabla} g + \underline{\nabla} \times \underline{G}$$

Conservative curl-free div-free vector free

**Note:** related - Hodge decomposition (valid more generally) whatever that is

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**Example**:

$$\begin{aligned} \underline{B} = \underline{c} \times \underline{r}, \quad \underline{\nabla} \cdot \underline{B} &= \partial_{x_i} \epsilon_{ijk} c_j \times k \\ &= \delta_k \epsilon_{ijk} c_j = 0 \end{aligned}$$

note: what?

$$\begin{aligned} \underline{A} = -\underline{r} \times \int_0^1 \underline{B}(\lambda \underline{r}) d\lambda &= -\underline{r}\lambda(\underline{c} \times \underline{r}) \underbrace{\int_0^1 d\lambda \lambda^2}_{1/3} \\ &= \frac{1}{3}((\underline{r} \cdot \underline{c})\underline{r} - r^2 \underline{c}) \end{aligned}$$

$$\underline{a} \times (\underline{b} \cdot \underline{c})$$

can't keep up with the guy lol

**Formal Check of 2** (second formula, TODO: acstually figure out how to mark number lol)

$$\underline{\nabla} \times \underline{\Delta} = \underline{\nabla} \times \int_0^1 \underline{B}(\lambda \underline{r}) \times \underline{r} \lambda d\lambda = \int_0^1 f(\lambda, \underline{r}) \lambda d\lambda$$

$$\begin{aligned}
f(\lambda, \underline{r}) &=^{(\mathrm{F})} [\underline{\nabla} \cdot \underline{r} \quad \underline{B}(\lambda \underline{r}) + \underline{r} \cdot \underline{\nabla} \quad \underline{B}(\lambda \underline{r})] - [\underbrace{(\underline{\nabla} \cdot \underline{B})}_{0} \underline{r} + \underbrace{(\underline{B}(\lambda \underline{r}) \cdot \underline{\nabla}) \underline{r}}_{\underline{B}}] \\
&= 2\underline{B}(\lambda \underline{r}) + \underbrace{x_i \partial_{x_i} \quad \underline{B}(\lambda x_1, \lambda x_2, \lambda x_3)_{\lambda \partial \lambda \underline{B}(\lambda x_1, \lambda x_2, \lambda x_3)}} \\
&= \int_0^1 [\underbrace{2\underline{B}(\lambda \underline{r}) \lambda}_{I_n} + \underbrace{\lambda^2 \partial_\lambda \underline{B}(\lambda \underline{r})}_{I_2}] d\lambda \\
I_2 &= \underbrace{\lambda^2 \underline{B}(\lambda \underline{r})}_{\underline{B}(\underline{r})} \Big|_0^1 - \underbrace{\int_0^1 2\lambda \underline{B}(\lambda \underline{r}) d\lambda}_{\underline{I}_n} = \underline{B} - \underline{I_n} \\
\underline{\nabla} \times \underline{A} &= \underline{I_n} + \underline{I_2} = \underline{I_n} + (\underline{B} - \underline{I_n}) = \underline{B}
\end{aligned}$$

dunno what that equation means :P

## 3.3 Orthogonal Curvilinear Co-ordinates (OCC)

e.g. spherical co-ords

$$\begin{aligned}
x &= r \sin\theta \cos\phi \\
y &= r \sin\theta \sin\pi \\
z &= r \cos\theta
\end{aligned}$$

In general:
$u_i = u_i(x_1, x_2, x_3)$, $i = 1 \dots 3$ (3 single valued invertible functions of 3 variables)
$x_i = x_i(u_1, u_2, u_3)$

$$\begin{aligned}
r &= \sqrt{x^2 + y^2 + z^2} & r &= \text{ const } \implies \text{ Sphere} \\
\theta &= \cos^{-1}(z/r) & \theta &= \text{ const } \implies \text{ Cones} \\
\phi &= tg^{-1}(y\ x) & \phi &= \text{ const } \implies \text{ Planes}
\end{aligned}$$

### 3.3.1 OCC

- $\partial u_i \underline{r} \cdot \partial_{u_j} \underline{r} = 0 \quad i \neq j \quad$ orthogonality
- Scale Factor: $h_i = |\partial_{u_i} \underline{r}|$ norm
- $\underline{e}_{u_i} = \frac{1}{h_i} \partial_{u_i} \underline{r} \qquad \underline{e}_{u_i} \cdot \underline{e}_{u_j} = \delta_{ij}$

**Examples**:

1. Cartesian Coordinates: $\underline{r} = x_i \underline{e}_i \quad h_i = |\partial_{x_i} \underline{r}| = 1$
2. Spherical: $\underline{r} = r[\sin\theta \cos\phi \underline{e}_1 + \sin\theta \sin\phi \underline{e}_2 + \cos\theta \underline{e}_3]$

$$\begin{aligned}
&\partial_r \underline{r} = [] \implies h_r = |\partial_r \underline{r}| = 1k \\
&\partial_\theta \underline{r} = r\sin\theta(\underbrace{-\sin\phi\underline{c}_1 + \cos\phi\underline{e}_2}_{\underline{e}_f}) \\
&h_\phi = |\partial_\phi \underline{r}| = r\sin\theta
\end{aligned}$$

3. Cylindrical Coordinates: $\underline{r} = \rho\cos\phi\underline{e}_1 + \rho\sin\phi\underline{e}_2 + z\underline{e}_3$

$$\partial_\rho \underline{r} = \underbrace{\cos\phi\underline{e}_1 + \sin\phi\underline{e}_2}_{\underline{e}_\rho} \qquad h_\rho = 1$$

$$\partial_\phi \underline{r} = \rho(\underbrace{-\sin\phi\underline{e}_1 + \cos\phi\underline{e}_2}_{\underline{e}_\phi}) \qquad h_\phi = \rho$$

$$\partial_z \underline{r} = \underline{e}_3 = \underline{e}_z \qquad h_z = 1$$

i cba