

1 Leon's ITCS Exam Notes

Basically adapted from Chris Dalziel's notes :)

Finite Automata

Definition: Finite Automata

A finite automaton takes a string as input and replies "yes" or "no". If an automaton A replies "yes" on a string S we say that A "accepts" S .

Definition: Deterministic Finite Automata

A deterministic finite automaton (DFA) is a quintuple $(Q, \Sigma, q_0, \delta, F)$ where

- Q is a finite set of states
- Σ is an alphabet
- $q_0 \in Q$ is the initial state
- $\delta : Q \times \Sigma \rightarrow Q$ is the transition function
- $F \subseteq Q$ is the set of final states

A DFA accepts a string $w \in \Sigma^*$ iff $\delta^*(q_0, w) \in F$, where δ^* is δ applied successively for each symbol in w .

The language of a DFA A is the set of all strings accepted by A , $\mathcal{L} \subseteq \Sigma^*$ is the set of all strings accepted by A .

The transition function is a total function which gives exactly one next state for each input symbol, i.e. it is deterministic

Definition: Nondeterministic Finite Automata

Non-determinism would mean that δ can return more than one successor state, it instead returns a set of possible states - no states is an empty set. A NFA is a quintuple $(Q, \Sigma, q_0, \delta, F)$ where:

- Q is a finite set of states
- Σ is an alphabet
- $q_0 \in Q$ is the initial state
- $\delta : Q \times \Sigma \rightarrow \mathcal{P}(Q)$ is the transition function
- $F \subseteq Q$ is the set of final states

The only difference between the definition of a DFA and that of an NFA is that in an NFA δ returns an element from the powerset of Q , $\mathcal{P}(Q)$

Adding non-determinism doesn't change "expressivity". Given an NFA A there is an equivalent DFA D such that $\mathcal{L}(D) = \mathcal{L}(A)$ and vice versa.

Definition: ϵ -NFA

If we allow non-deterministic state changes that don't consume any input symbols, we can label silent moves using ϵ - meaning the empty string. We define the ϵ closure $E(q)$ of a state q as the set of all states reachable from q by silent moves. That is, $E(q)$ is the least set satisfying:

- $q \in E(q)$
- For any $s \in E(q)$ we also have $\delta(s, \epsilon) \subseteq E(q)$

DFA, NFA, ϵ -NFA are all equal in expressive power

Regular Languages

Definition: Regular Languages

Any language which can be accepted by a finite automaton is called a regular language.

Regular languages are also those recognised by Regular Expressions

Definition: Regular Language Closure Properties

For two languages L_1 and L_2 , the following operations satisfy the closure property, i.e. for a member $x \in X$, and an operation ϕ we have that $\phi(x) \in \mathbb{R}$ for all x .

- **Union:** $L_1 \cup L_2$ is the language that includes all strings of L_1 and all strings of L_2 .
- **Intersection:** $L_1 \cap L_2$ is the language that includes all strings of L_1 that are not in L_2 , and vice versa
- **Sequential Composition:** $L_1 L_2$ is the language of strings that consist of strings in L_1 followed by a string in L_2 .
- **Kleene closure:** L^* is the language of strings that consist wholly of zero or more strings in L .

$$L^* = \bigcup_{i \in \mathbb{N}} L^i$$

- **Complement:** \bar{L} is the language of every string not in L .

Definition: Regular Expressions

Regular characterise the regular languages, just like finite automata do. The following table shows the syntax and semantics of a regex.

Syntax	Semantics
a	$\llbracket a \rrbracket = \{a\}$
\emptyset	$\llbracket \emptyset \rrbracket = \emptyset$
ϵ	$\llbracket \epsilon \rrbracket = \{\epsilon\}$
$R_1 \cup R_2$	$\llbracket R_1 \cup R_2 \rrbracket = \llbracket R_1 \rrbracket \cup \llbracket R_2 \rrbracket$
$R_1 \circ R_2$	$\llbracket R_1 \circ R_2 \rrbracket = \llbracket R_1 \rrbracket \llbracket R_2 \rrbracket$
R^*	$\llbracket R^* \rrbracket = \llbracket R \rrbracket^*$

Definition: Generalised NFAs

A **generalised NFA**, or GNFA is an NFA where:

- Transitions have **regular expressions** on them instead of symbols
- There is only one unique final state
- The transition relation is **full**, except that the initial state has no incoming transitions, and the final state has no outgoing transitions

Theorem : Pumping Lemma

If $L \subseteq \Sigma^*$ is regular, then there is a **pumping length** $p \in \mathbb{N}$ such that for any $w \in L$ where $|w| \geq p$, we may split w into three pieces $w = xyz$ satisfying three conditions:

- $xy^i z, \forall i \in \mathbb{N}$
- $|y| > 0$
- $|xy| \leq p$

Note that if the pumping lemma fails then the language is not regular, but the inverse is not necessarily true.

Theorem : Myhill-Nerode Theorem

Let $L \subseteq \Sigma^*$ and $x, y \in \Sigma^*$. If there exists a suffix string z such that $xz \in L$, but $yz \notin L$ or vice versa, then x and y are **distinguishable** by L . If x and y are not distinguishable by L , then we say that $x \equiv_L y$ - this is an equivalence relation. A regular language satisfies the following

- The number of equivalence classes \equiv_L is finite.
- The number of equivalence classes is equal to the number of states in the minimal DFA accepting L (not as important)

Therefore, to show a language is non-regular, show that it has infinite equivalence classes - that is, we find an infinite sequence $u_0 u_1 \dots$ of strings such that for any i, j where $i \neq j$, there is a string w_{ij} such that $u_i w_{ij} \in L$ but $u_j w_{ij} \notin L$ or vice-versa

Context-Free Languages

Definition: Context-free Languages

By adding recursion to regexes we can begin to recognise some non-regular languages. All regular languages are also context free.

Definition: Context-free Grammars

A language is context-free iff it is recognised by a Context-free Grammar (CFG), which is a 4-tuple (N, Σ, P, S) where:

- N is a finite set of variables or non-terminals
- Σ is a finite set of terminals
- $P \subseteq N \times (N \cup \Sigma)^*$ is a finite set of rules or productions
 - Typically productions are written $A \rightarrow aBc$
 - Productions with common heads can be combined, $A \rightarrow a$ and $A \rightarrow Aa$ can be combined into $A \rightarrow a \mid Aa$
- $S \in N$ is the starting variable

We use α, β, γ to refer to sequences of terminals

We make a derivation step $\alpha A \beta \Rightarrow_G \alpha \gamma \beta$ whenever $(A \rightarrow \gamma) \in P$; The language of a CFG G is:

$$\mathcal{L}(G) = \{w \in \Sigma^* \mid S \Rightarrow_G^* w\}$$

Where \Rightarrow_G^* is the reflexive, transitive, closure of \Rightarrow_G .

Context-free grammars are ambiguous. They are closed under union, concatenation, and Kleene star, but not under intersection or complementation

Definition: Eliminating Ambiguity

We want to eliminate ambiguity in CFGs while still accepting all the same strings. This can be done for our language of regular expressions:

- First defining atomic expressions: $A \rightarrow (S) \mid \emptyset \mid \epsilon \mid a \mid b$
- Then ones which use Kleene Star: $K \rightarrow A \mid A^*$
- Then ones which may use left-associative composition: $C \rightarrow K \mid C \circ K$
- Finally expressions which use unions: $S \rightarrow C \mid S \cup C$

The order of operations here is therefore bottom to top; unions come before compositions, which come before Kleene etc

Definition: Push-down Automata

Push-down automata are to CFGs what finite automata are to regular expressions. They are implementationally identical to ϵ -NFAs with the addition of a stack. The recursive element of CFGs is implemented using a standard last-in-first-out stack.

Transitions in a push-down automata take the form $x, y \rightarrow z$ which is read as "consume the input x , popping y off the stack, and push z onto the stack". We can allow actions that don't consume, pop, or push by setting variables to ϵ .

Definition: Formal Def. of PDAs

A **push-down automaton** is a 6-tuple $(Q, \Sigma, \Gamma, \delta, q_0, F)$ where Q, Σ, Γ are all finite sets. Γ is the stack alphabet, and δ now may take a stack symbol as input or return one as output:

$$\delta : Q \times \Sigma_\epsilon \times \Gamma_\epsilon \rightarrow \mathcal{P}(Q \times \Gamma_\epsilon)$$

All other components are as with ϵ -NFAs

A string w is accepted by a PDA if it ends in a final state, i.e. $\delta^*(q_0, w, \epsilon)$ gives a state q and a stack γ such that $q \in F$.

Theorem : CFG to PDA

A language is context-free if and only if it is recognised by a push-down automaton. The proof is left as an exercise to the reader.

Theorem : Pumping CFLs intro

Suppose a CFG has n non-terminals, and we have a parse tree of height $k > n$. Then the same non-terminal V must have appeared as its own descendant in the tree

- **Pumping down:** Cut the tree at the higher occurrence of V and replace it with the subtree at the lower occurrence of V
- **Pumping up:** Cut at the lower occurrence and replace it with a fresh copy of the higher occurrence

Theorem : Pumping Lemma for CFLs

If L is context-free then there exists a pumping length $p \in \mathbb{N}$ such that if $w \in L$ with $|w| \geq p$ then w may be split into **five** pieces $w = uvxyz$ such that

- $uw^i xy^i z \in L$ for all $i \in \mathbb{N}$
- $|vy| > 0$
- $|vxy| \leq p$

Definition: Chomsky Grammars

Context-free grammars are a special case of Chomsky Grammars. Chomsky grammars are similar to CFGs, except that the left-hand side of a production may be any string that includes at least one non-terminal. An example is shown below

$$\begin{aligned} S &\rightarrow abc \mid aAbc \\ Ab &\rightarrow bA \\ Ac &\rightarrow Bbcc \\ bB &\rightarrow Bb \\ aB &\rightarrow aaA \mid aa \end{aligned}$$

Such a grammar is called **context-sensitive**

Definition: The Chomsky Hierarchy

A grammar $G = (N, \Sigma, P, S)$ is of type:

0. (or **computably enumerable**) in the general case
1. (or **context sensitive**) if $|\alpha| \leq |\beta|$ for all productions $\alpha \rightarrow \beta$, except we also allow $S \rightarrow \epsilon$ if S does not occur on the RHS of any rule
2. (or **context free**) if all productions are of the form $A \rightarrow \alpha$ (i.e. a CFG)
3. (or **right-linear/regular**) if all productions are of the form $A \rightarrow w$ or $A \rightarrow wB$, where $w \in \Sigma$ and $B \in N$

Algorithms for Languages

Theorem : Emptiness for Regular Languages

Can we write a program to determine if a given regular language is empty?

Given a finite-automaton this is an instance of graph reachability, so we can use a depth-first search.

Theorem : Emptiness for Context-free languages

Can we write a program to determine if a given context-free language is empty?

Given a CFG for our language, we can perform the following process:

1. Mark the terminals and ϵ as generating
2. Mark all non-terminals which have a production with only generating symbols in their right hand side as generating
3. Repeat until nothing new is marked
4. Check if S is marked as generating or not

Theorem : Equivalence of DFA

Is it possible to write a program to determine if two discrete finite automata are equivalent?

Given two DFA for L_1 and L_2 , we can use our standard constructions to produce a DFA of the symmetric set difference:

$$(L_1 \cap \bar{L}_2) \cup (L_2 \cap \bar{L}_1)$$

Register Machines

Definition: Register Machines

A **register machine**, or RM, consists of:

- A fixed number m of registers $R_0 \dots R_{m-1}$, which each holds a natural number
- A fixed program P which is a sequence of n instructions $I_0 \dots I_{n-1}$

Each instruction is one of the following:

- **INC(i)**: which increments the register R_i by one
- **DECJZ(i, j)**: which decrements register R_i unless $R_i = 0$ in which case it jumps to instruction I_j

RMs can compute anything any other computer can

Definition: Pairing Functions for RMs

A **pairing function** is an injective function $\mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$. An example is $f(x, y) = 2^x 3^y$. We write $\langle x, y \rangle_2$ for $f(x, y)$. If $z = \langle x, y \rangle_2$, let $z_0 = x$ and $z_1 = y$. This lets us encode multiple values into a single value, and a 2-tuple pairing function is enough to cram an arbitrary sequence of natural numbers into one $\mathbb{N}^* \rightarrow \mathbb{N}$

Theorem : Church-Turing Thesis

The Church-Turing thesis states that any problem is computable by any model of computation iff it is computable by a **Turing machine**. For our purposes this matters for RMs, TMs, and λ -calculus. Other examples are combinator calculus, general recursive functions, pointer machines, counter machines, cellular automata, queue automata, enzyme-based DNA computers, Minecraft, Magic the Gathering, and others.

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Example : Regular Expressions

At least one 0:

$$(0 \cup 1)^* 0 (0 \cup 1)^*$$

At least one 1 and at least one 0:

$$((0 \cup 1)^* 0 1 (0 \cup 1)^*) \cup ((0 \cup 1)^* 1 0 (0 \cup 1)^*)$$