1 Algebra

Note: Any reference numbers are to the lecture notes

Functions and Symmetries

Definition 0.1.1: Functions

A function $f : X \to Y$ is called

- **injective** if $f(x_1) = f(x_2) \implies x_1 = x_2$. f is said to be **one-to-one** on X
- **surjective** if for every $y \in Y$, $\exists x \in X$ s.t. $f(x) = y$. f is said to take X **onto** Y
- **bijective** if it is both injective and surjective

Definition 1.1.3: Graph Isomorphisms

An **isomorphism** between two graphs is a *bijection* between them that preserves all edges. More precisely, if Γ_1 and Γ_2 are graphs, with sets of vertices V_1 and V_2 respectively, then an isomorphism from Γ_1 and Γ_2 is a bijection

$$f : V_1 \to V_2$$

such that $f(v_1)$ and $f(v_2)$ are joined by an edge if and only if v_1 and v_2 are also joined by an edge. We say that Γ_1 and Γ_2 are *isomorphic* if there exists an isomorphism $f : \Gamma_1 \to \Gamma_2$

Definition 1.1.9: Symmetry

A **symmetry** of a graph is an *isomorphism* from the graph to itself, i.e. if the set of vertices is V, then the symmetry is a bijection $f : V \to V$ that preserves edges. That is, a symmetry is a bijection $f : V \to V$ such that $f(v_1)$ and $f(v_2)$ are joined by an edge if and only if v_1 and v_2 are joined by an edge.

Groups

Definition 1.2.3: Groups

For an operation $*$, We say a non-empty set G is a **group** under $*$ if the following four axioms hold:

- **G1 - Closure:** $*$ is a binary operation on G, that is $a*b \in G$ for all $a, b \in G$.
- **G2 - Associativity:** $(a*b)*c = a*(b*c)$ for all $a, b, c \in G$
- **G3 - Identity:** There exists an *identity* element of G such that $e*g = g*e = g$ for all $g \in G$.
- **G4 - Inverse:** Every element $g \in G$ has an *inverse* g^{-1} such that $g*g^{-1} = g^{-1}*g = e$

Definition 1.2.6: Abelian Group

The definition of a group doesn't require that $a*b = b*a$. We say that a group is **abelian** or **commutative** if $a*b = b*a$ for every $a, b \in G$. We say that a *commutes* with b, or that a and b *commute*

Subgroups

Definition 2.1.1: Subgroups

Let G be a group. We say that a non-empty subset H of G is a **subgroup** of G if H itself is a group (under the operation from G). We write $H \leq G$ if H is a subgroup of G. If $H \neq G$, we write $H < G$ and say H is a proper subgroup

Theorem 2.1.3: Subgroup Test

$H \subseteq G$ is a subgroup of G if and only if:

- **S1:** H is not empty
- **S2:** If $h, k \in H$ then $h*k \in H$
- **S3:** If $h \in H$ then $h^{-1} \in H$

Alternative test for subgroups:

- $\widetilde{S1}$: H is not empty.
- $\widetilde{S2}$: If $h, k \in H$ then $h*k^{-1} \in H$

Definition 2.2.4: Order of an Element

Let G be a group and $g \in G$. Then the **order** $o(g)$ of g is the *least* natural number n such that

$$g^n = e$$

If no such n exists, we say that g has infinite order

Definition 2.2.3: Order of a Group

The **order** of a finite group, written $|G|$, is the number of elements in G. If G is infinite we say that $|G| = \infty$, or the order of G is infinite.

Theorem 2.2.6: Order of a Finite Group

In a finite group, every element has finite order.
If g is an element of a finite group G, then there exists $k \in \mathbb{N}$ such that $g^k = g^{-1}$

Definition 2.2.8: Generating Subset

Let G be a group and let $g \in G$ be an element. We define the subset

$$\langle g \rangle := \{g^k \mid k \in \mathbb{Z}\} = \{\ldots, g^{-2}, g^{-1}, e, g, g^2, \ldots\}$$

Note that if G is finite, then by 2.2.6 $\langle g \rangle$ is finite, and we can think of $\langle g \rangle$ as

$$\langle g \rangle = \{e, g, \ldots, g^{o(g)-1}\}$$

Definition 2.2.10: Cyclic Subgroup

A subgroup $H \leq G$ is **cyclic** if $H = \langle h \rangle$ for some $h \in H$. In this case, we say that H is the *cyclic subgroup generated by* h. If $G = \langle g \rangle$ for some $g \in G$, then we say that the group G is *cyclic*, and that g is a *generator*.

Remark 2.2.12 - 16: Consequences of Cyclic groups

- **2.2.12** If $g \in G$, then $o(g) = |\langle g \rangle|$
- **2.2.13:** If G is cyclic, then G is abelian.
- **2.2.14:** Let G be a finite group. Then

  $$G \text{ is cyclic} \iff G \text{ has an element of order} |G|$$

- **2.2.15:** Let G be a cyclic group and let H be a subgroup of G. Then H is cyclic.
- **2.2.16:** Let $m, n \in \mathbb{N}$, let $G = \langle g \rangle$ be a cyclic group of order m and $H = \langle h \rangle$ be a cyclic group of order n. Then

  $$G \times H \text{ cyclic} \iff m \text{ and } n \text{ are coprime } (\gcd(\text{m,n}) = 1)$$

Cosets and Lagrange

Definition 2.3.2: Relation

Let X be a set, and R a subset of $X \times X$; thus R consists of some ordered pairs (s, t) with $s, t \in X$. If $(s, t) \in R$ we write $s \sim t$ and say "s is related to t". We call $\sim$ a **relation** on X.

Definition 2.3.2: Equivalence Relation

- **Reflexive:** $x \sim x$ for all $x \in X$
- **Symmetric:** $x \sim y$ implies that $y \sim x$ for all $x, y \in X$
- **Transitive:** $x \sim y$ and $y \sim z$ implies that $x \sim z$ for all $x, y, z \in X$

A relation $\sim$ is called an **equivalence relation** on X if it satisfies the following three axioms:

Definition 2.3.4: Coset

Let $H \leq G$ and let $g \in G$. Then a *left coset* of H in G is a subset of G of the form gH, for some $g \in G$. We denote the set of left cosets of H in G by G/H
(Notation) Let A, B be subgroups of a group G and let $g \in G$. Then

$$AB := \{ab \mid a \in A, b \in B\}, \quad gA := \{gA \mid a \in A\}$$

Theorem 2.4.2: Lagrange's Theorem

Suppose that G is a finite group.

- If $H \leq G$, then $|H|$ divides $|G|$
- Let $g \in G$. Then $o(g)$ divides $|G|$
- For all $g \in G$, we have that $g^{|G|} = e$

Theorem 2.3.8: Coset Rules

Let $H \leq G$

- For all $h \in H$, $hH = H$. In particular $eH = H$
- For $g_1, g_2 \in G$, the following are equivalent
  - $g_1 H = g_2 H$
  - there exists $h \in H$ such that $g_2 = g_1 h$
  - $g_2 \in g_1 H$
- For g_1, $g_2 \in G$, define $g_1 \sim g_2$ if and only if $g_1 H = g_2 H$. Then $\sim$ defines an equivalence relation on G.

Theorem 2.4.4: Index of a Subgroup

The **index** of $H \leq G$ is defined as the number of *distinct* left cosets of H in G, which by Lagrange's is $|G/H| = \frac{|G|}{|H|}$

Remark 2.4.6 - 8: Consequences of Lagrange

- **2.4.6:** Suppose that G is a group with $|G| = p$, where p is prime. Then G is a cyclic group
- **2.4.7:** Suppose that G is a group with $|G| < 6$. Then G is abelian
- **2.4.8:** If p is a prime and $a \in \mathbb{Z}$, then $a^p \equiv a \mod p$

Homomorphisms and Isomorphisms

Definition 3.1.1: Group Homomorphism

Let $(G, *), (H, \circ)$ be groups. A map $\phi : G \to H$ is called a **homomorphism** if

$$\phi(x * y) = \phi(x) \circ \phi(y) \quad \text{for all } x, y \in G$$

Note that the product on the left is formed using $*$, while the product on the right is formed using $\circ$

Definition 3.1.2: Group Isomorphism

A group homomorphism $\phi : G \to H$ that is also a bijection is called an **isomorphism** of groups. In this case we say that G and H are *isomorphic* and we write $G \cong H$. An isomorphism $G \to G$ is called an **automorphism** of G.

Theorem J.J 9.5 Thm 6: Cyclic Isomorphisms

All finite cyclic groups with the same order are *isomorphic* to each other. Therefore, cyclic groups of order n are isomorphic to $(\mathbb{Z}_n, +)$
All infinite cyclic groups are *isomorphic* to each other. Therefore, each cyclic group of infinite order is isomorphic to $(\mathbb{Z}, +)$

Remark 3.1.5: Consequences of Homomorphisms

Let $\phi : G \to H$ be a group homomorphism. Then

- $\phi(e_G) = e_H$
- $\phi(g^k) = (\phi(g))^k$ and $\phi(g^{-1}) = (\phi(g))^{-1}$ for all $g \in G$
- If ϕ is injective, the order of $g \in G$ equals the order of $\phi(g) \in H$.

Definition 3.1.7: Normal Subgroup

A subgroup $N \leq G$ is **normal** if the left and right cosets of N are equal, i.e. $gN = Ng$ for all $g \in G$. If N is a normal subgroup of G, we write $N \triangleleft G$. Kernels of homomorphisms are always normal subgroups

Definition 3.1.6: Image and Kernel of a Group

Let $\phi : G \to H$ be a group homomorphism.

- The **image** of ϕ is defined to be
$$\operatorname{im} \phi := \{h \in H \,|\, h = \phi(g) \text{ for some } g \in G\}$$
- The **kernel** of ϕ is defined to be
$$\ker \phi := \{g \in G \,|\, \phi(g) = e_H\}$$

Note: $\operatorname{im} \phi$ is a subgroup of H and $\ker \phi$ is a subgroup of G

Theorem 3.2.1: Product Isomorphisms

Let $H, K \leq G$ be subgroups with $H \cap K = \{e\}$.

- The map $\phi : H \times K \to HK$ given by $\phi : (h, k) \to hk$ is bijective
- If every element of H commutes with every element of K when multiplied in G (i.e. $hk = kh \quad \forall h \in H, k \in K$), then HK is a subgroup of G, and it is isomorphic to $H \times K$ via ϕ

Theorem 3.2.3: Size of Product Group

Let $H, K \leq G$ be finite subgroups of a group G such that $H \cap K = \{e\}$ Then $|HK| = |H| \times |K|$.

Group Actions

Definition 4.1.1: Group Action

Let $(G, *)$ be a group, and let X be a nonempty set. Then a (left) **action** of G on X is a map

$$G \times X \to X$$

written $(g, x) \mapsto g \cdot x$, such that

$$g_1 \cdot (g_2 \cdot x) = (g_1 * g_2) \cdot x \qquad \text{and} \qquad e \cdot x = x$$

for all $g_1, g_2 \in G$ and all $x \in X$.

Definition 4.2.1: Orbit, Stabilizer, and Fix

- Suppose that G acts on X. Then the set
$$N := \{g \in G \mid g \cdot x = x \quad \forall x \in X\}$$
is a subgroup of G, and is called the **kernel** of the action. If $N = \{e\}$, then we say the action is **faithful**
- For every x in X, the **orbit** of x is a subset of X defined by
$$\mathrm{Orb}_G(x) = \{g \cdot x \mid g \in G\}$$
- For every x in X, the **stabilizer** of x is a subgroup of G defined by
$$\mathrm{Stab}_G(x) = \{g \in G \mid g \cdot x = x\}$$
- For every g in G, the **fix** of g is a subset of X defined by
$$\mathrm{Fix}(g) = \{x \in X \mid g \cdot x = x\}$$
- Let G act on X, let $x \in X$ and set $H := \mathrm{Stab}_G(x)$. If $y = g \cdot x$ for some $g \in G$, then
$$\mathrm{send}_x(y) = gH$$
- Let $h \in H$ and $g \in G := X$. The **conjugate action** is:
$$h \cdot g := hgh^{-1}$$
- An action of G on X is **transitive** if for all $x, y \in X$ there exists $g \in G$ such that $y = g \cdot x$. Equivalently, X is a single orbit under G
- We define the **centre** of a group G to be
$$C(G) := \{g \in G \mid hg = gh \text{ for all } h \in G\}$$
The **centralizer** of g in G is defined as
$$G(g) := \{h \in G \mid gh = hg\}$$

Theorem 4.2.5: Orbit Equivalence

Let G act on X. Then

$$x \sim y \iff y = g \cdot x \text{ for some } g \in G$$

defines an equivalence relation on X. The equivalence classes are the orbits of G. Thus when G acts on X, we obtain a partition of X into orbits

Theorem 4.3.1: Orbit-Stabilizer Theorem

Suppose G is a finite group acting on a set X, and let $x \in X$. Then $|\mathrm{Orb}_G(x)| \times |\mathrm{Stab}_G(x)| = |G|$, or in words:

$$\text{size of orbit} \times \text{size of stabilizer} = \text{order of group}$$

Theorem 4.3.4: Orbit Send Theorem

Let G act on X, let $x \in X$, and let set $H := \mathrm{Stab}_G(x)$. Then

$$\mathrm{send}_x : \mathrm{Orb}_G(x) \to G/H \text{ which sends } y \mapsto \mathrm{send}_x(y)$$

Theorem 4.4.2: Cauchy's Theorem

Let G be a group, p be prime. If p divides $|G|$, then G contains an element of order p

2 Analysis

Note: Any reference numbers are to the lecture notes

Real Numbers and Bounds

Definition 1.1: The Real Numbers

$\mathbb{R}$ is defined as the set of real numbers. It has two operations $+$ and $*$, and it is a field, i.e. satisfies group axioms for both operations, in addition to the Distributive law:

$$a \cdot (b + c) = a \cdot b + a \cdot c$$

The set of real numbers is also ordered, i.e. there is a relation $<$ which satisfies pretty much what you think it does
Finally, the set of real numbers is complete, i.e. there are no gaps between any numbers.

Definition 1.2.3: Triangle Inequality

The most important property of the absolute value $|a|$:

$$|a + b| \leq |a| + |b| \quad \text{and} \quad ||a| - |b|| \leq |a - b|$$

Definition 1.3.2: Suprema and Bounds

Let $E \subset \mathbb{R}$ be nonempty

- The set E is said to be bounded above if there is $M \in R$ such that $a \leq M$ for all $a \in E$
- A real number M is called an upper bound of the set$E$ if $a \leq M$ for all $a \in E$
- A real number s is called the **supremum** of the set E if
  - s is an upper bound of E
  - $s \leq M$ for all upper bounds M of the set E

  If a number s exists, we shall say that E has a supremum and write $s = \sup E$

If the supremum s exists, then s is the least upper bound of the set E. The supremum is also unique if it exists.

If the same properties as a supremum apply but in the other direction, a number s is instead called the **infimum** of the set E. Infimum and Supremum are related via the reflection principle:

- Set E has a supremum if and only if the set $-E$ has an infimum. Also $\inf(-E) = -\sup(E)$
- Set E has an infimum if and only if the set $-E$ has a supremum. Also $\sup(-E) = -\inf(E)$

Theorem 1.3.5: Suprema Approximation Property

If the set $E \subset \mathbb{R}$ has a supremum then for any positive number $\epsilon > 0$ there exists $a \in E$ such that

$$\sup E - \epsilon < a \leq \sup E$$

Theorem 1.3.7: Archimedean Principle

Given positive real numbers $a, b \in \mathbb{R}$ there is an integer $n \in N$ such that $b < na$

Definition 1.5.2: Countability

Let E be a set. E is said to be:

- **Finite** if either $E = \varnothing$, or there is an integer $n \in \mathbb{N}$ and a bijection $f : \{1, 2, 3, \ldots, n\} \to E$. We say that the set E has n elements
- **Countable** if there is a bijective function $f : \mathbb{N} \to E$
- **At most countable** if E is finite or countable
- **Uncountable** if E is neither finite nor countable

Additionally, a nonempty set E is at most countable if and only if there is a surjective function $f : \mathbb{N} \to E$

Sequences and Series

Definition 2.1.1: Convergence of a Sequence

A sequence of real numbers (x_n) is said to converge to a real number a if for every $\epsilon > 0$ there is a $N \in \mathbb{N}$ where for every $n \geq N$ we have that $|x_n - a| < \epsilon$

For a sequence (x_n), we write $\lim x_n = +\infty$ if for each $M > 0$ there is a number N such that $n > N$ implies $x_n > M$. Reverse every inequality for $-\infty$ case.

Definition 2.1.9: Bounds of Sequences

Let (x_n) be a sequence of real numbers.

- $(x_n)_{n \in \mathbb{N}}$ is said to be **bounded above** if $x_n \leq M$ for some $M \in \mathbb{R}$ and all $n \in \mathbb{N}$
- $(x_n)_{n \in \mathbb{N}}$ is said to be **bounded below** if $x_n \geq m$ for some $m \in \mathbb{R}$ and all $n \in \mathbb{N}$
- $(x_n)_{n \in N}$ is said to be **bounded** if it is both bounded above and below

Remark 2.2.1 - ?: Limit Theorems

- Let $E \subset \mathbb{R}$. If E has a finite supremum then there is a sequence (x_n) with each $x_n \in E$ such that $x_n \to \sup E$ as $n \to \infty$. The same goes for a finite infimum
- **Comparison Theorem for sequences:** Suppose that (x_n), (y_n) are real sequences. If both $\lim_{n\to\infty} x_n$, $\lim_{y\to\infty} y_n$ exist and belong to $\mathbb{R}*$, and if $x_n \leq y_n$ for all $n \geq N$ for some $N \in \mathbb{N}$, then $\lim_{n\to\infty} x_n \leq \lim_{n\to\infty} y_n$

Definition 2.3.1: Monotone Sequences

Let (s_n) be a sequence of real numbers.

- (s_n) is said to be increasing if $s_1 \leq s_2 \leq s_3 \leq \cdots$, and strictly increasing if $s_1 < s_2 < s_3 < \cdots$
- (s_n) is said to be decreasing if $s_1 \geq s_2 \geq s_3 \geq \cdots$, and strictly decreasing if $s_1 > s_2 > s_3 > \cdots$
- (s_n) is said to be monotone if it is either increasing or decreasing

Theorem lots: Top 10 Limit Theorems

- **Squeeze Theorem (for sequences):** Suppose that (x_n), (y_n), and (w_n) are real sequences
  - If both $x_n \to a$ and $y_n \to a$ as $n \to \infty$, and if $x_n \leq w_n \leq y_n$ for all $n \geq N_0$, then $w_n \to a$ as $n \to \infty$
  - If $x_n \to 0$ and (y_n) is bounded, $x_n y_n \to_0$ as $n \to \infty$
- **Divergence Test:**
  - If $\sum_{n=1}^{\infty} a_n$ converges, then $a_n \to 0$.
  - If $(a_n)_{n \in \mathbb{N}}$ doesn't converge to 0, then $\sum_{n=1}^{\infty} a_n$ diverges. Be careful that the converse isn't true.
- **Comparison Test:** Let $(a_n)_{n \in \mathbb{N}}$ and $(b_n)_{n \in \mathbb{N}}$ be two sequences such that $0 \leq a_n \leq b_n$ for all n.
  - If $\sum_n b_n$ converges, then $\sum_n a_n$ converges as well.
  - If $\sum_n a_n$ diverges, then $\sum_n b_n$ diverges as well.
- **Limit Comparison Test:** Let $(a_n)_{n \in \mathbb{N}}$ and $(b_n)_{n \in \mathbb{N}}$ be two real sequences with $a_n \geq 0$ and $b_n > 0$ for all n. Assume that $a_n / b_n \to L$ for some $L \in (0, \infty)$. Then, $\sum_{n=1}^{\infty} a_n$ converges iff $\sum_{n=1}^{\infty} b_n$ converges.
  - If $L = 0$ and $\sum_{n=1}^{\infty} b_n$ converges then $\sum_{n=1}^{\infty} a_n$ converges
  - If $L = \infty$ and $\sum_{n=1}^{\infty} a_n$ diverges then $\sum_{n=1}^{\infty} b_n$ diverges
- **Root Test:** Let $\sum_{n=0}^{\infty} a_n$ be a series with non-negative terms such that $\sqrt[n]{a_n} \to L$ where $0 \leq L \leq +\infty$.
  - If $0 \leq L < 1$ then the series $\sum_{n=0}^{\infty} a_n$ converges.
  - If $L > 1$ then the series $\sum_{n=0}^{\infty} a_n$ diverges.
  - If $L = 1$, the series may or may not converge
- **Ratio Test:** Let $\sum_{n=0}^{\infty} a_n$ be a series with positive terms such that $(a_{n+1})/(a_n) \to L$, where $0 \leq L \leq +\infty$.
  - If $0 \leq L < 1$ then the series $\sum_{n=0}^{\infty} a_n$ converges.
  - If $L > 1$ then the series $\sum_{n=0}^{\infty} a_n$ diverges.
  - If $L = 1$ then compare to p series
- **Cauchy's Condensation Test:** Let $(a_n)_{n \in \mathbb{N}}$ be a decreasing sequence with non-negative terms. Then the following are equivalent:
  - The series $\sum_{n=1}^{\infty} a_n$ converges
  - The series $\sum_{n=0}^{\infty} 2^n a_{2^n}$ converges.
- **Alternating Series Test:** Let $(b_n)_{n \in \mathbb{N}}$ be a *decreasing* sequence of non-negative real numbers that converges to zero. Then the series $\sum_{n=1}^{\infty} (-1)^{n-1} b_n$ converges.
- **Monotone Convergence Theorem:** If a sequence of real numbers (s_n) is increasing and bounded above, or decreasing and bounded below, then (s_n) is convergent (and converges to the sup/inf of the set $\{x_n \,|\, n \in \mathbb{N}\}$ respectively).
- **Geometric Series Test:** Assume $a, r \in \mathbb{R}, a, r \neq 0$. Then

$$\sum_{n=1}^{\infty} a \cdot r^n = \begin{cases} \dfrac{a}{1 - r} & \text{if } |r| < 1 \\ \text{diverges} & \text{if } |r| \geq 1 \end{cases}$$

  Notice that a is always the first term in the series, and r is the *common ratio*

Continuity and Functional Limits

Definition 4.1.1: Continuity

Let f be a real-valued function whose domain is a subset of $\mathbb{R}$. The function f is **continuous** at x_0 in $\operatorname{dom}(f)$ if, for every sequence (x_n) in $\operatorname{dom}(f)$ converging to x_0, we have

$$\lim_{n\to\infty} f(x_n) = f(x_0)$$

If f is continuous at each $a \in S \subseteq \operatorname{dom}(f)$ and then we say that f is continuous on S. If f is continuous on $\operatorname{dom}(f)$ then we say that f is continuous

Theorem 4.1.6: $\epsilon - \delta$ Definition of Continuity

A function $f : A \to \mathbb{R}$ is continuous if for all $\epsilon > 0$, there exists some $\delta > 0$ s.t. for all $x \in A$ for which $0 < |x - c| < \delta$, we have

$$|f(x) - f(c)| < \epsilon$$

Theorem 6.1.4: Evil $\epsilon - \delta$ definition of continuity

A function $f : A \to \mathbb{R}$ is not continuous if there exists $\epsilon > 0$ such that for all $\delta > 0$ there exists some $x \in A$ satisfying $0 < |x - c| < \delta$ for which $|f(x) - f(c)| \geq \epsilon$

Definition 4.2.1: Bounds of a Function

Let $E \subseteq \mathbb{R}$ be nonempty. A function $f : E \to \mathbb{R}$ is said to be bounded on E if

$$|f(x)| \leq M, \qquad \text{for all } x \in E$$

where M is some (large) real number.

Theorem 4.2.2: Extreme Value Theorem

Let $I \subseteq \mathbb{R}$ be a closed and bounded interval. Let $f : I \to \mathbb{R}$ be continuous on I. Then f is bounded on the interval I, denoted by

$$m = \inf\{f(x) \mid x \in I\}, \qquad M = \sup\{f(x) \mid x \in I\}$$

Then there exist points $x_m, x_M \in I$ such that

$$f(x_m) = m \quad \text{and} \quad f(x_M) = M$$

Theorem 4.2.4: $\epsilon - \delta$ Limit jr.

Let $f : I \to \mathbb{R}$ where I is an open nonempty interval. If f is continuous at a point $a \in I$ and $f(a) > 0$ then for some δ, $\epsilon > 0$ we have that

$$f(x) > \epsilon, \qquad \text{for all } x \in (a - \delta, a + \delta)$$

Theorem 4.2.5: Intermediate Value Theorem

Let I be a non-degenerate interval and let $f : I \to \mathbb{R}$ be a continuous function. If $a, b \in I$, $a < b$, then on the interval (a, b), f attains all values between $f(a)$ and $f(b)$. i.e. given y_0 between $f(a)$ and $f(b)$, there exists $x_0 \in (a, b)$ such that $f(x_0) = y_0$

Theorem 4.2.6: Bolzano's Theorem

Let $f(x)$ be continuous on $[a, b]$ such that $f(a)f(b) < 0$, then there exists $c \in (a, b)$ such that $f(c) = 0$

Theorem 4.3.1: $\epsilon - \delta$ definition of a limit

Let $f : A \to \mathbb{R}$ and let c be a limit point of A. Then we say that

$$\lim_{x\to c} f(x) = L$$

if for all $\epsilon > 0$ there exists some $\delta > 0$ such that for every $x \in A$ for which $0 < |x - c| < \delta$, we have

$$|f(x) - L| < \epsilon$$

We also say $\lim_{x\to c} f(x)$ **converges** to L in such a situation

Differentiation

Definition 5.1.1: First Principle Differentiation

A real function f is said to be differentiable at a point $x \in \mathbb{R}$ if f is defined at some open interval containing x, and

$$f'(x) = \lim_{h\to 0} \frac{f(x + h) - f(x)}{h}$$

exists. $f'(x)$ is called the derivative of f at the point x

Theorem 5.1.3: Differentiable implies Continuous

Let I be an open interval, $x_0 \in I$ and $f : I \to \mathbb{R}$ be differentiable at x_0. Then f is continuous at x_0. The converse is not true, an example is $f(x) = |x|$ which isn't differentiable at 0.

Theorem 5.1.4: Differentiable Intervals

Let $f : I \to \mathbb{R}$ be a given fucntion, where I is an open interval. We say that f is differentiable in I iff it is differentiable at every point in I. At endpoints, derivatives only have to be one-sided

Theorem 5.2.1: Differentiation Rules

Let $f, g : (a, b) \to \mathbb{R}$ be differentiable on (a, b). Then $f + g$ and $f \cdot g$ are differentiable on (a, b). If $g(x) \neq 0$ for all $x \in (a, b)$, then f/g is differentiable. Moreover,

- **Sum rule:** $(f + g)' = f' + g'$
- **Product Rule:** $(fg)' = f'g + fg'$
- **Quotient Rule:** $(f/g)' = (f'g - fg')/g^2$

Theorem 5.4.6: Inverse Function Theorem

Let f be injective and continuous on an open interval I. If $a \in f(I)$ and f' at the point $f^{-1}(a) \neq 0$ exists and is nonzero, then f^{-1} is differentiable at a and

$$(f^{-1})'(a) = \frac{1}{f'(f^{-1}(a))}$$

Theorem 5.2.2: Chain Rule

Let f, g be real functions. If f is differentiable at a and g is differentiable at $f(a)$ then $g \circ f$ is differentiable at a and

$$(g \circ f)'(a) = g'(f(a))f'(a)$$

Theorem 5.3.1 - 3: Differentiation Theorem ladder

- **Rolle's Theorem:** Let $a, b \in \mathbb{R}$, $a < b$. If $f : [a, b] \to \mathbb{R}$ is continuous in $[a, b]$, differentiable in (a, b) and $f(a) = f(b)$, then there exists a point c in (a, b) such that $f'(c) = 0$
- **Mean Value Theorem:** If $f : [a, b] \to \mathbb{R}$, $a < b$ is continuous in $[a, b]$, differentiable in (a, b) then $\exists c \in (a, b)$ s.t.

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

- **Generalised MVT:** If $f, g : [a, b] \to \mathbb{R}$ is continuous in $[a, b]$ and differentiable in (a, b), then $\exists c$ in (a, b) such that

$$(f(b) - f(a))g'(c) = (g(b) - g(a))f'(c)$$

If $g(b) - g(a)$, $g'(c) \neq 0$ then this can be written as

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}$$

Theorem 5.4.2: Monotone Functions

Let $a < b$ be real and f be continuous on $[a, b]$ and differentiable on (a, b).

- If $f'(x) > 0$ $\forall x \in (a, b)$, then f is strictly increasing on $[a, b]$
- If $f'(x) < 0$ $\forall x \in (a, b)$, then f is strictly decreasing on $[a, b]$
- If $f'(x) = 0$ $\forall x \in (a, b)$, then f is constant on $[a, b]$

Additionally, if f is injective and continuous on an interval I, Then f and f^{-1} is strictly monotone on I and $f(I)$ respectively

Theorem 5.5.1: Taylor Series

Let $f : I \to \mathbb{R}$ be $n+1$ times differentiable and $x_0 \in I$, for an open interval I. For each $x \in I$, there is a c between x_0 and x such that

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k + \frac{f^{(n+1)}(c)}{(n + 1)!}(x - x_0)^{n+1}$$

(c is a value between x and x_0 (maybe))
Now suppose that $f : (a, b) \to \mathbb{R}$ is infinitely differentiable and let $x_0 \in (a, b)$. Fix x in (a, b). For every positive integer N we have

$$f(x) = \sum_{k=0}^{N} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k + R_N(x)$$

If $R_N(x) \to 0$ as $n \to \infty$, we have

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k$$

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3 Examples Catalogue

Examples of a Group

Example 1.2.4: Dihedral Group D_n

The set of symmetries of an n-gon forms a group under composition. We call this group the **Dihedral Group** D_n
The Dihedral group of n has precisely $|D_n| = 2n$ elements, namely

- The identity e
- $n-1$ anticlockwise rotations of $\frac{2pi}{n}$. We denote this operation with g
- n reflections. If n is odd, then there are n reflections from a point to the opposite edge. If n is even, there are $\frac{n}{2}$ reflections from point to point, and $\frac{n}{2}$ from edge to edge. We denote a vertical reflection with h, and rotated reflections as compositions of h and g

From this, we see that

$$D_n = \{e,\ g,\ g^2,\ \ldots,\ g^{n-1},\ h,\ gh,\ g^2h,\ \ldots,\ g^{n-1}h\}$$

Example 1.3.2: Symmetric Group

The set of all symmetries of $\{1,\ 2,\ \ldots,\ n\}$ is called the **symmetric group** S_n. It is a group under composition with order $|S_n| = n!$ The symmetric group can be thought of as every permutation of the set $\{1,\ 2,\ \ldots,\ n\}$, or can also be thought of as an n-gon where every edge is connected to each other.

Example $\mathbb{Z}_3 \times \mathbb{Z}_4$: Group Properties pick'n'mix

- Any group $\mathbb{Z}_n$ is **Abelian** and **Cyclic**
- Any cross product of $\mathbb{Z}_n \times \mathbb{Z}_m$ where n and m are coprime is **Abelian** and **Cyclic**.
- Any cross product of $\mathbb{Z}_n \times \mathbb{Z}_m$ where n and m are not coprime is **Abelian** but **Not Cyclic**.
- Any dihedral group D_n is **Not Abelian**, and **Not Cyclic**
- The trivial action $g \cdot x = x$ of any group is **Not Faithful** and **Not Transitive**
- The trivial action of any group acting on $\{1\}$ is **Not Faithful** and **Transitive**
- $\mathbb{Z}_{an}$ acting on $\mathbb{Z}_n$, where $a \in \mathbb{Z}$ where $g \cdot x = g + x$ is **Not Faithful** and **Not Transitive**
- A symmetry group of a graph with a middle point has at least two orbits (**Not Transitive**) e.g.

Example 0: Example of a Coset

Consider $\mathbb{Z}_4$ under addition, and let $H = \{0, 2\}$ $(e = 0.)$ The cosets of H in G are:

$$eH = e * H = \{e * h \,|\, h \in H\} = \{0 + h \,|\, h \in H\} = \{0, 2\}$$
$$1H = 1 * H = \{1 * h \,|\, h \in H\} = \{1 + h \,|\, h \in H\} = \{1, 3\}$$
$$2H = 2 * H = \{2 * h \,|\, h \in H\} = \{2 + h \,|\, h \in H\} = \{0, 2\}$$
$$3H = 3 * H = \{3 * h \,|\, h \in H\} = \{3 + h \,|\, h \in H\} = \{1, 3\}$$

Hence there are two cosets, namely

$$0 * H = 2 * H = \{0, 2\} \quad \text{and} \quad 1 * H = 3 * H = \{1, 3\}$$

$$G/H = \{eH = 2H, 1H = 3H\} = \{\{0, 2\}, \{1, 3\}\}$$

Example 3.3.5: p-series

The series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$, and it diverges if $p \leq 1$. At $p = 1$, this series is called the **Harmonic Series**.
To show divergence/convergence of a series, we can compare it to the p-series

Example $\sum_{n=0}^{\infty}$: Deciphering Taylor Series

- **Showing convergence of a Taylor Series:** An infinite Taylor series will converge to $f(x)$ iff we have $R_N(x) \to 0$ as $N \to \infty$ in the finite Taylor series

$$f(x) = \lim_{N\to\infty} \sum_{k=0}^{N} \frac{f^k(x_0)}{k!}(x - x_0)^k + R_N(x)$$

  Therefore, showing that $(R_N(x))_{n\in\mathbb{N}}$ converges to 0 is enough to show that the infinite Taylor Series converges to $f(x)$

- **Simplifying series-like terms:** If you have a Taylor Series / function that is in the same equation as a bunch of series-like terms, then a good idea is to try to expand the Taylor series at N for the N amount of elements and then try and figure something out using the remainder term $R_N(x)$
  Example: extract of 2018 May A3
  At some point we end up with an equation

$$0 \leq \ln(x+1) - x + \frac{x^2}{2} - \frac{x^3}{3} + \frac{x^4}{4}$$

  and a Taylor Series

$$\ln(x+1) = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n} x^n$$

  Expand Taylor series with $N = 4$

$$\begin{aligned} 0 &\leq \ln(x+1) - x + \frac{x^2}{2} - \frac{x^3}{3} + \frac{x^4}{4} \\ &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + R_4(x) - x + \frac{x^2}{2} - \frac{x^3}{3} + \frac{x^4}{4} \\ &= R_4(x) \end{aligned}$$

The museum of $\epsilon - \delta$ limits

Example of rigour: $\epsilon - \delta$ and $\epsilon - N$ Template

Proof: Let $\epsilon > 0$ be given. Set $\delta =$ __ (If there is a constant, then set as "$<$" e.g. $\delta < \min\{1, \epsilon\}$). Then for all $x \in \mathbb{R}$ such that $|x -$ __$| < \delta$ we have
Optional: preliminary step to determine an upper bound
Therefore, / Therefore since "x term" < "constant",
"Same steps as rough working"
$\cdots =$ __ $\cdot |x -$ __$| <$ __$\delta = \epsilon$ (same rule applies about constants)

Proof: Let $\epsilon > 0$ be given. Choose $N \in \mathbb{N}$ such that $N >$ __, (optional(?) for example $N =$ __). Then we have __ $< \epsilon$. Then for all $n \geq N$ we have
"Same steps as rough working"
$\cdots =$ "equation in terms of n" $\leq$ "same thing in terms of N" $< \epsilon$

Example 10000: ϵ-N Convergence

Show that the sequence $\left(\frac{2n+1}{3n+2}\right)_{n\in\mathbb{N}}$ *converges to* $\frac{2}{3}$
We start with the rough work. Start with an arbitrary $\epsilon > 0$ and find an N_ϵ s.t. $\left|\frac{2n+1}{3n+2} - \frac{2}{3}\right| < \epsilon$ for all $n > N_\epsilon$. Let's explore this.

$$\left|\frac{2n+1}{3n+2} - \frac{2}{3}\right| = \frac{11}{3(3n+2)} < \epsilon \quad \Longrightarrow \quad n > \frac{1}{3}\left(\frac{11}{3\epsilon} - 2\right)$$

Proof: Let $\epsilon > 0$. Pick a positive integer N such that

$$N > \frac{1}{3}\left(\frac{11}{3\epsilon} - 2\right)$$

Then,

$$\frac{11}{3(3N+2)} < \epsilon$$

For all n with $n \geq N$ we have

$$|a_n - L| = \left|\frac{2n+5}{3n+2} - \frac{2}{3}\right| = \frac{11}{3(3n+2)} \leq \frac{11}{3(3N+2)} < \epsilon$$

Another method of finding a limit is,

$$\left|\frac{2n+1}{3n+2} - \frac{2}{3}\right| = \frac{11}{3(3n+2)} = \frac{11}{9n+6} < \frac{11}{9n} < \epsilon$$

Since $9n + 6 > 9n$, this means that the right fraction is larger than the left fraction in all cases. This means if we can find a right fraction that is smaller than ϵ then the left fraction must also.

Proof: let $\epsilon > 0$. Pick a positive integer N such that $N > \frac{11}{9\epsilon}$. Then $\frac{11}{9N} < \epsilon$. For all n with $n \geq N$, we have

$$|a_n - L| = \left|\frac{2n+5}{3n+2} - \frac{2}{3}\right| = \frac{11}{9n+6} < \frac{11}{9n} \leq \frac{11}{9N} < \epsilon$$

Example (: $\epsilon - \delta$ Continuity

Using the definition of continuity, prove that the function $f : \mathbb{R}\backslash\{\frac{9}{5}\} \to \mathbb{R}$ *defined by* $f(x) = \frac{x^2}{5x-9}$ *is continuous at* $x_0 = 2$
Since $x_0 = 2$, our delta should end up as $|x-2| < \delta$. Start with $|f(x) - f(a)| < \epsilon$

$$\begin{aligned} |f(x) - f(a)| &= \left| \frac{x^2}{5x-9} - \frac{4}{10-9} \right| \\ &= \left| \frac{x^2}{5x-9} - 4 \right| \\ &= \left| \frac{x^2 - 20x + 36}{5x-9} \right| \\ &= \left| \frac{(x-18)(x-2)}{5x-9} \right| \\ &= |x-2| \left| \frac{x-18}{5x-9} \right| \end{aligned}$$

We have $|x-2|$, so we want to turn the RH fraction into a constant. If we let the neighbourhood around δ to be no less than $\frac{1}{10}$ (i.e. $x \in (1.9, 2.1)$) (this number can be anything, but smaller than $\frac{1}{5}$ since there is an asymptote at $\frac{9}{5}$), using the number with the largest value in that range we can get an upper bound for δ.

$$\left| \frac{x-18}{5x-9} \right| < \left| \frac{1.9-18}{9.5-9} \right| = \left| \frac{-16.1}{0.5} \right| = |-32.2| \implies \left| \frac{x-18}{5x-9} \right| < 32.2$$

Therefore

$$|x-2| \left| \frac{x-18}{5x-9} \right| < |x-2| \cdot 32.2 < \epsilon$$

Therefore, we can take $\delta = \max\{1/10, \epsilon/32.2\}$

Proof: Let $\epsilon > 0$ be given. set $\delta = \min\{\frac{1}{10}, \frac{\epsilon}{32.2}\}$. Then for all $x \in \mathbb{R}$ such that $|x-2| < \delta$ we have

$$\left| \frac{x-18}{5x-9} \right| < \left| \frac{1.9-18}{9.5-9} \right| = \left| \frac{-16.1}{0.5} \right| = |-32.2| \implies \left| \frac{x-18}{5x-9} \right| < 32.2$$

Therefore, since $\left|\frac{x-18}{5x-9}\right| < 32.2$,

$$|f(x) - f(a)| = \left| \frac{x^2}{5x-9} - \frac{4}{10-9} \right| = \left| \frac{x^2 - 20x + 36}{5x-9} \right|$$

$$= \left| \frac{(x-18)(x-2)}{5x-9} \right| = |x-2| \left| \frac{x-18}{5x-9} \right| \leq 32.2 \cdot |x-2| < 32.2 \cdot \delta = \epsilon$$

Example Dumb Assumptions: $\epsilon - \delta$ using other limits

If $f, g : \mathbb{R}\backslash\{1\} \to \mathbb{R}$ *are two functions such that* $\lim_{x\to 1} f(x) = 2$, $\lim_{x\to 1} g(x) = 3$, *show that* $\lim_{x\to 1}(4f(x) + g(x)^2) = 17$ We want to try and turn the limit into compositions of other limits. From the assumptions, we know that

- There exists a δ_1 s.t. $\forall x$ where $0 < |x-1| < \delta_1$, we have
$$|f(x) - 2| < \epsilon \tag{1}$$
- There exists a δ_2 s.t. $\forall x$ where $0 < |x-1| < \delta_2$, we have
$$|g(x) - 3| < \epsilon \tag{2}$$

So, start with the main function. We want to show

$$|4f(x) + g(x)^2 - 17| < \epsilon$$

We want to turn this into a composition of (1) and (2). By "Trusting our professors won't be too mean" this should be possible

$$\begin{aligned} |4f(x) + g(x)^2 - 17| &= |4(f(x) - 2) + g(x)^2 - 9| \\ &= |4(f(x) - 2) + (g(x) - 3)(g(x) + 3)| \\ \textit{(via triangle ineq)} &\leq 4|f(x) - 2| + |g(x) - 3||g(x) + 3| \end{aligned}$$

To find an upper bound for $|g(x)+3|$ we want to manipulate again

$$|g(x) + 3| = |g(x) - 3 + 6| \quad \leq |g(x) - 3| + 6 \quad < \epsilon + 6$$

Therefore now we can substitute equations (1) and (2) into everything

$$4|f(x) - 2| + |g(x) - 3||g(x)+3| \quad < 4\epsilon + \epsilon|g(x)+3| \quad < 4\epsilon + \epsilon(\epsilon + 6)$$

Let the epsilon boundary be less than 1. Then $\epsilon + 6 < 7$, therefore

$$4\epsilon + \epsilon(\epsilon + 6) < 4\epsilon + \epsilon(7) = 11\epsilon$$

We want to finish with ϵ but since (1) and (2) work for any ϵ by definition, set those inequalities to $\frac{\epsilon}{11}$ instead and the final result will be ϵ on its own

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Proof: Let $\epsilon > 0$ be given. First assume $\epsilon \leq 1$. By our assumptions, there exists a δ_1 where $\forall x$ s.t. $0 < |x-1| < \delta_1$, we have $|f(x) - 2| < \epsilon/11$, and a δ_2 where $\forall x$ s.t. $0 < |x-1| < \delta_2$, we have $|g(x) - 3| < \epsilon/11$. For all x s.t. $0 < |x-1| < \delta_2$, we have

$$|g(x) + 3| = |g(x) - 3 + 6| \leq |g(x) - 3| + 6 \leq \frac{\epsilon}{11} + 6 \leq \frac{1}{11} + 6 \leq 7$$

Let $\delta = \min\{\delta_1, \delta_2\}$ Therefore, since $|g(x) + 3| \leq 7$,

$$\begin{aligned} |4f(x) + g(x)^2 - 17| &= |4(f(x) - 2) + g(x)^2 - 9| \\ &= |4(f(x) - 2) + (g(x) - 3)(g(x) + 3)| \\ \textit{(via triangle ineq)} &\leq 4|f(x) - 2| + |g(x) - 3||g(x) + 3| \\ &< 4\frac{\epsilon}{11} + 7\frac{\epsilon}{11} = \epsilon \end{aligned}$$

Assume now that $\epsilon > 1$. By what we have shown above there exists a $\epsilon > 0$ such that for all x such that $0 < |x-1| < \epsilon$,

$$|4f(x) + g(x)^2 - 17| < 1$$

Therefore,

$$|4f(x) + g(x)^2 - 17| < \epsilon$$

Example): $\epsilon - \delta$ Discontinuity

From negation of $\epsilon - \delta$ continuity - A function $f : A \to \mathbb{R}$ is not continuous if there exists $\epsilon > 0$ such that for all $\delta > 0$ there exists some $x \in A$ satisfying $0 < |x - c| < \delta$ for which $|f(x) - f(c)| \geq \epsilon$

$$|f(x) - f(a)| < \epsilon \implies \left| sin\left(\frac{1}{x}\right) - 0 \right| < \epsilon \quad \implies \left| sin\left(\frac{1}{x}\right) \right| < \epsilon$$

So we want to show that we can find an ϵ such that for every $\delta > 0$, we can find an x where $|x| < \delta$ and also $|sin(\frac{1}{x})| \geq \epsilon$.
Since $sin(x)$ repeats, if we can find an x such that $sin(\frac{1}{x})$ is an exact value then we can define ϵ as something lower than that. If we want a value where $sin(\frac{1}{x}) = 1$, this will be true if $x = 1/(\frac{\pi}{2} + 2N\pi)$, where N is a positive integer.
Since x has to be bounded by δ, go from δ

$$\begin{aligned} |x| &< \delta \\ \left| \frac{1}{\frac{\pi}{2} + 2N\pi} \right| &< \delta \\ \frac{1}{\frac{\pi}{2} + 2N\pi} &< \delta \quad \text{(will always be positive since N positive int)} \\ \frac{\pi}{2} + 2N\pi &> \frac{1}{\delta} \\ N &> \frac{1}{2\pi}\left(\frac{1}{\delta} - \frac{\pi}{2}\right) \end{aligned}$$

Proof: Let $\epsilon = \frac{1}{2}$. Let $\delta > 0$ be given. Pick a positive integer N such that $N > \frac{1}{2\pi}\left(\frac{1}{\delta} - \frac{\pi}{2}\right)$ and set $x = \frac{1}{\frac{\pi}{2} + 2N\pi}$. Then for all $x \in \mathbb{R}$ such that $0 < x < \delta$, we have

$$|f(x)| = \left| sin\left(\frac{1}{x}\right) \right| = \left| sin\left(\frac{\pi}{2} + 2N\pi\right) \right| = 1 \geq \frac{1}{2} = \epsilon$$

Example Number: Roots of a Function

Prove that the equation

$$f(x) = x^5 + x^3 + x + 1$$

has exactly one real root
Part 1: The function has *at least* one real root. This is because as it is composed of three odd-power polynomials, so as $x \to \infty$, $f(x) > 0$ and as $x \to -\infty$, $f(x) < 0$ Therefore by the Intermediate Value Theorem, there exists at least one point where $f(x) = 0$.
Part 2: Differentiating the function, we get

$$f(x) = x^5 + x^3 + x + 1 \quad \implies f'(x) = 5x^4 + 3x^2 + 1$$

However, $x^4, x^2 \geq 0$ for every $x \in \mathbb{R}$. Therefore $5x^4 + 3x^2 + 1$ is always positive, meaning there is no roots (critical points).
Part 3: Therefore, by Rolle's Theorem, since there exists no critical points, it means there is no points where $f(a) = f(b)$ which implies that $f(x)$ is strictly increasing/decreasing.
This means there cannot be more than 1 real root. Therefore, there is exactly one real root