Metric Spaces Notes

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Contents

1 Introduction to Metric Spaces

1.1 Defining a Metric

Metric is another name for distance. A **Metric Space** is a set equipped with a metric. A standard example is $\mathbb{R}$ with the standard metric

$$d(x, y) = |x - y|$$

We will now formally define what it means to have a metric

Theorem 1.1.1: Definition of a Metric

Let X be a non-empty set. A function $d : X \times X \to \mathbb{R}$ is called a **metric** iff for all $x, y, z \in X$,

- $d(x, y) \geq 0$ and $d(x, y) = 0 \iff x = y$
- $d(x, y) = d(y, x)$
- $d(x, y) \leq d(x, z) + d(z, y)$ (Triangle Inequality)

A non-empty set X equipped with a metric d is called a **metric space**

1.2 Examples of Metric Spaces

We can construct a metric space using the **Absolute value** equipped with the standard triangle inequality

Example 1.2.1: The Real Line

Let $X = \mathbb{R}$. Define our metric $x : X \times X \to \mathbb{R}$ by

$$d(x, y) = |x - y|$$

The first two properties are fairly trivial. The third property follows using the regular triangle inequality

$$d(x, y) = |x - y| = |(x - z) + (z - y)| \leq |x - z| + |z - y| = d(x, z) + d(z, y)$$

Remark: This can be extended not just in $\mathbb{R}^2$, but to all $\mathbb{R}^n$. By induction,

$$|x_1 + \cdots + x_N| \leq |x_1| + \cdots + |x_N|$$

If $\sum_{n=1}^{\infty} x_n$ converges absolutely, let $N \to +\infty$ to see that

$$\left| \sum_{n=1}^{\infty} x_n \right| \leq \sum_{n=1}^{\infty} |x_n|$$

A second example is the **Euclidean Plane**. The metric is defined using the **inner product** and the **norm**.

Definition 1.2.2: Inner Product

The **inner product** is defined as

$$\langle x, y\rangle = x_1 y_1 + x_2 y_2$$

Properties of the inner product: For all vectors x, y, $z \in \mathbb{R}^2$ and all real scalars a, b,

- $\langle x, x\rangle \geq 0$ and $\langle x, x\rangle = 0 \iff x = 0$
- $\langle x, y\rangle = \langle y, x\rangle$
- $\langle ax + by, z\rangle = a\langle x, z\rangle + b\langle y, z\rangle$

Remark: This is basically a formalisation of the dot product

Definition 1.2.3: Norm

The **norm** is defined as:

$$\|x\|_2 = \langle x, x\rangle^{1/2} = \sqrt{x_1^2 + x_2^2}$$

Properties of the norm: For all x, $y \in \mathbb{R}^2$, $a \in \mathbb{R}$

- $\|x\|_2 \geq 0$ and $\|x\|_2 = 0 \iff x = 0$
- $\|ax\|_2 = |a|\|x\|_2$
- $\|x + y\|_2 \leq \|x\|_2 + \|y\|_2$ (triangle inequality)

Remark: This is a formalisation of the "length of a vector"
With these two properties, we can now define the **Euclidean Metric**

Example 1.2.4: Euclidean Metric

For all $x = (x_1, x_2)$, $y = (y_1, y_2) \in \mathbb{R}^2$, define

$$d_2(x, y) = \|x - y\|_2 = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$$

Remark: Derivation of the triangle inequality is basically the same as Example 1.2.1.

$$d_2(x, y) = \|x - y\|_2 = \|(x - z) + (z - y)\|_2 \leq \|x - z\|_2 + \|z - y\|_2 = d_2(x, z) + d_2(z, y)$$

1.2.5 Proof of the euclidean triangle inequality

W.T.S:

$$\|x + y\|_2 \leq \|x\|_2 + \|y\|_2$$

Proof: Square both sides

$$\begin{aligned} \text{LHS}^2 &= \langle x + y, x + y\rangle \\ &= \langle x, x\rangle + 2\langle x, y\rangle + \langle y, y\rangle \\ &= \|x\|_2^2 + 2\langle x, y\rangle + \|y\|_2^2 \end{aligned}$$

$$\text{RHS}^2 = \|x\|_2^2 + \|y\|_2^2 + 2\|x\|_2\|y\|_2$$

Discarding the equal terms, we get

$$\|x\|_2^2 + 2\langle x, y\rangle + \|y\|_2^2 \leq \|x\|_2^2 + \|y\|_2^2 + 2\|x\|_2\|y\|_2$$
$$\langle x, y\rangle \leq \|x\|_2\|y\|_2$$
$$\text{i.e. } x_1y_1 + x_2y_2 \leq \sqrt{x_1^2 + x_2^2}\sqrt{y_1^2 + y_2^2}$$

This is the **Cauchy-Schwarz Inequality**. Various ways to prove this (watch lecture 1)

Example 1.2.6: Complex Plane

Let $X = \mathbb{C}$, $d : \mathbb{C} \times \mathbb{C} \to \mathbb{R}$

$$d(z, w) = |z - w|$$

If $z = a + ib, w = c + id$, $a, b, c, d \in \mathbb{R}$, then

$$z - w = (a - c) + i(b - d)$$

therefore,

$$d(z, w) = \sqrt{(a - c)^2 + (b - d)^2}$$

Definition 1.2.7: n-dimensional Euclidean space

Let $X = \mathbb{R}^n = \{(x_1, x_2, \ldots, x_n) : x_1, x_2, \ldots, x_n \in \mathbb{R}\}$
For $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n)$ in $\mathbb{R}^n$, define

$$\langle x, y\rangle = x_1y_1 + x_2y_2 + \cdots + x_ny_n \text{ (inner product)}$$

Properties of n-inner product: For all vectors $x, y, z \in \mathbb{R}^n$ and all real scalars a, b,

- $\langle x, x\rangle \geq 0$ and $\langle x, x\rangle = 0 \iff x = 0$
- $\langle x, y\rangle = \langle y, x\rangle$
- $\langle ax + by, z\rangle = a\langle x, z\rangle + b\langle y, z\rangle$

For $x = (x_1, x_2, \ldots, .x_n) \in \mathbb{R}^n$ define

$$\|x\|_2 = \langle x, x\rangle^{1/2} = \sqrt{x_1^2 + x_2^2 + \cdots + c_n^2}\text{(norm)}$$

Properties of n-norm: For $x, y \in \mathbb{R}^n$, $a \in \mathbb{R}$,

- $\|x\|_2 \geq 0$ and $\|x\|_2 = 0 \iff x = 0$
- $\|ax\|_2 = |a|\|x\|_2$
- $\|x + y\|_2 \leq \|x\|_2 + \|y\|_2$ (triangle inequality)

Example 1.2.8: Metric in n-dim euclidean space

For $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n)$ in $\mathbb{R}^n$, define

$$\begin{aligned} d_2(x, y) &= \|x - y\|_2 \\ &= \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_n - y_n)^2} \end{aligned}$$

Triangle inequality, cauchy schwarz, yadda yadda same as 2-dim case

1.2.9 L space

For two sequences $x = (x_1, \dots, x_n, \dots)$, $y = (y_1, \dots, y_n, \dots)$ of real numbers we wish to define

$$d_1(x, y) = \sum_{n=0}^{\infty} |x_n - y_n|$$

We need this series to converge - in particular when $y = (0, \dots, 0, \dots)$, we need the series $\sum_{n=1}^{\infty} |x_n|$ to converge

Definition 1.2.10: l space

We denote by ℓ^1 the set of real sequences $(x_n)_{n \in \mathbb{N}}$ for which the series $\sum_{n=1}^{\infty} |x_n|$ converges.

If $x, y \in \ell^1$ i.e. if $\sum_{n=1}^{\infty} |x_n|$ and $\sum_{n=1}^{\infty} |y_n|$ converge, then $\sum_{n=1}^{\infty} |x_n - y_n|$ converges, because for all n,

$$|x_n - y_n| \leq |x_n| + y_n$$

For $x = (x_1, \dots, x_n, \dots)$ in ℓ^1, we may now define

$$\|x\|_1 = \sum_{n=1}^{\infty} |x_n|$$

For $x = (x_1, \dots, x_n, \dots)$, $y = (y_1, \dots, y_n, \dots)$ in ℓ^1 we may now define

$$d_1(x, y) = \|x - y\|_1 = \sum_{n=1}^{\infty} |x_n - y_n|$$

1.3 Real Vector Spaces

Definition 1.3.1: Real Vector Spaces

A *real vector space* is a set X with two operations, addition(+) and scalar multiplication $\cdot$, with the following properties: for all $x, y, z \in X$, $a, b \in \mathbb{R}$, we have $x + y, a \cdot x \in X$, and

- $x + y = y + x$
- $x + (y + z) = (x + y) + z$
- There is an element of X denoted by 0 such that, for all x, $0 + x = x + 0 = x$
- For every $x \in X$ there exists an element of X denoted by $-x$ such that $x + (-x) = (-x) + x = 0$
- $a \cdot (x + y) = a \cdot x + a \cdot y$
- $(a + b) \cdot x = a \cdot x + b \cdot x$
- $a \cdot (b \cdot x) = (ab) \cdot X$
- $1 \cdot x = x$

(we usually write ax instead of x)

1.3.2 Normalising l 1

Properties: For all sequences $x, y \in \ell^1$ and all real scalars a,

- $\|x\|_1 \geq 0$ and $\|x\|_1 = 0 \iff x = 0$
- $\|ax\|_1 = |a|\|x\|_1$
- $\|x + y\|_1 \leq \|x\|_1 + \|y\|_1$

1.3.3 Space l-2

We denote by ℓ^2 the set of real sequences $(x_1, \ldots, x_n, \ldots)$ such that the seriese $\sum_{n=1}^{\infty} |x_n|^2$ converges

For $x = (x_1, \ldots, x_n, \ldots) \in \ell^2$, $y = (y_1, \ldots, y_n, \ldots) \in \ell^2$ we define

- $\langle x, y \rangle = \sum_{n=1}^{\infty} x_n y_n$ (inner product)
- $\|x\|_2 = \left(\sum_{n=1}^{\infty} |x_n|^2 \right)^{1/2}$ (norm)
- $d_2(x, y) = \|x - y\|_2 = \left(\sum_{n=1}^{\infty} |x_n - y_n|^2 \right)^{1/2}$ (Metric)

Theorem 1.3.4: 4

ℓ^2 is a real vector space proof icba

more stuff on ℓ^2 - typical properties watch video 1

1.4 Generalising metric space features

Definition 1.4.1: Normed Vector Spaces

A *normed vector space* (or *normed linear space* or *normed space*) is a real vector space X equipped with a *norm*, i.e. a function that assigns to every vector $x \in X$ a real number $\|x\|$ so that, for all vectors x and y in X and all real scalars a,

- $\|x\| \geq 0$ and $\|x\| = 0 \iff x = 0$
- $\|ax\| = |a|\|x\|$
- $\|x + y\| \leq \|x\| + \|y\|$

If $(X, \|\cdot\|)$ is a normed vector space then

$$d(x, y) = \|x - y\|$$

defines a metric in X

Definition 1.4.2: Inner Product Spaces

Let Xa be a real vector space. An *inner product* on X is a function that assigns to every pair $(x, y) \in X \times X$ a real number denoted by $\langle x, y\rangle$ and has the following properties

- $\langle x, x\rangle \geq 0$ and $\langle x, x\rangle = 0 \iff x = 0$
- $\langle x, y\rangle = \langle y, x\rangle$
- $ax + by, z = a\langle x, z\rangle + b\langle y, z\rangle$

A *real inner product space* is a real vector space equipped with an inner product. If $\|\cdot, \cdot\|$ is an inner product on X, then

$$\|x\| = \sqrt{\langle x, x\rangle}$$

defines a norm and

$$d(x, y) = \|x - y\|$$

defines a metric

Example 1.4.3: Discrete metric

Let X be a non-empty set. Define $d : X \times X \to \mathbb{R}$ by

$$d(x, y) = \begin{cases} 0, & x = y \\ 1, x \neq y \end{cases}$$

Example of metric space without norm or inner prod