1 Algebra

Note: Any reference numbers are to the lecture notes

Functions and Symmetries

Definition 0.1.1 Functions

A function $f : X \to Y$ is called

- **injective** if $f(x_1) = f(x_2) \implies x_1 = x_2$. f is said to be **one-to-one** on X
- **surjective** if for every $y \in Y$, $\exists x \in X$ s.t. $f(x) = y$. f is said to take X **onto** Y
- **bijective** if it is both injective and surjective

Definition 1.1.3 Graph Isomorphisms

An **isomorphism** between two graphs is a *bijection* between them that preserves all edges. More precisely, if Γ_1 and Γ_2 are graphs, with sets of vertices V_1 and V_2 respectively, then an isomorphism from Γ_1 and Γ_2 is a bijection

$$f : V_1 \to V_2$$

such that $f(v_1)$ and $f(v_2)$ are joined by an edge if and only if v_1 and v_2 are also joined by an edge. We say that Γ_1 and Γ_2 are *isomorphic* if there exists an isomorphism $f : \Gamma_1 \to \Gamma_2$

Definition 1.1.9 Symmetry

A **symmetry** of a graph is an *isomorphism* from the graph to itself, i.e. if the set of vertices is V, then the symmetry is a bijection $f : V \to V$ that preserves edges. That is, a symmetry is a bijection $f : V \to V$ such that $f(v_1)$ and $f(v_2)$ are joined by an edge if and only if v_1 and v_2 are joined by an edge.

Groups

Definition 1.2.3 Groups

For an operation $*$, We say a non-empty set G is a **group** under $*$ if the following four axioms hold:

- **G1 - Closure:** $*$ is a binary operation on G, that is $a*b \in G$ for all $a, b \in G$.
- **G2 - Associativity:** $(a*b)*c = a*(b*c)$ for all $a, b, c \in G$
- **G3 - Identity:** There exists an *identity* element of G such that $e * g = g * e = g$ for all $g \in G$.
- **G4 - Inverse:** Every element $g \in G$ has an *inverse* g^{-1} such that $g * g^{-1} = g^{-1} * g = e$

Definition 1.2.6 Abelian Group

The definition of a group doesn't require that $a * b = b * a$. We say that a group is **abelian** or **commutative** if $a * b = b * a$ for every $a, b \in G$. We say that a *commutes* with b, or that a and b *commute*

Subgroups

Definition 2.1.1 Subgroups

Let G be a group. We say that a non-empty subset H of G is a **subgroup** of G if H itself is a group (under the operation from G). We write $H \leq G$ if H is a subgroup of G. If $H \neq G$, we write $H < G$ and say H is a proper subgroup

Theorem 2.1.3: Subgroup Test

$H \subseteq G$ is a subgroup of G if and only if:

- **S1:** H is not empty
- **S2:** If $h, k \in H$ then $h * k \in H$
- **S3:** If $h \in H$ then $h^{-1} \in H$

Alternative test for subgroups:

- $\widetilde{S1}$: H is not empty.
- $\widetilde{S2}$: If $h, k \in H$ then $h * k^{-1} \in H$

Definition 2.2.4 Order of an Element

Let G be a group and $g \in G$. Then the **order** $o(g)$ of g is the *least* natural number n such that

$$g^n = e$$

If no such n exists, we say that g has infinite order

Definition 2.2.3 Order of a Group

The **order** of a finite group, written $|G|$, is the number of elements in G. If G is infinite we say that $|G| = \infty$, or the order of G is infinite.

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Theorem 2.2.6: Order of a Finite Group

In a finite group, every element has finite order.
If g is an element of a finite group G, then there exists $k \in \mathbb{N}$ such that $g^k = g^{-1}$

Definition 2.2.8 Generating Subset

Let G be a group and let $g \in G$ be an element. We define the subset

$$\langle g \rangle := \{g^k \mid k \in \mathbb{Z}\} = \{\ldots, g^{-2}, g^{-1}, e, g, g^2, \ldots\}$$

Note that if G is finite, then by 2.2.6 $\langle g \rangle$ is finite, and we can think of $\langle g \rangle$ as

$$\langle g \rangle = \{e, g, \ldots, g^{o(g)-1}\}$$

Definition 2.2.10 Cyclic Subgroup

A subgroup $H \leq G$ is **cyclic** if $H = \langle h \rangle$ for some $h \in H$. In this case, we say that H is the *cyclic subgroup generated by h*. If $G = \langle g \rangle$ for some $g \in G$, then we say that the group G is *cyclic*, and that g is a *generator*.

Remark 2.2.12 - 16: Consequences of Cyclic groups

- **2.2.12** If $g \in G$, then $o(g) = |\langle g \rangle|$
- **2.2.13:** If G is cyclic, then G is abelian.
- **2.2.14:** Let G be a finite group. Then
$$G \text{ is cyclic} \iff G \text{ has an element of order} |G|$$
- **2.2.15:** Let G be a cyclic group and let H be a subgroup of G. Then H is cyclic.
- **2.2.16:** Let $m, n \in \mathbb{N}$, let $G = \langle g \rangle$ be a cyclic group of order m and $H = \langle h \rangle$ be a cyclic group of order n. Then
$$G \times H \text{ cyclic} \iff m \text{ and } n \text{ are coprime } (\gcd(m,n) = 1)$$

Cosets and Lagrange

Definition 2.3.2 Relation

Let X be a set, and R a subset of $X \times X$; thus R consists of some ordered pairs (s, t) with $s, t \in X$. If $(s, t) \in R$ we write $s \sim t$ and say "s is related to t". We call $\sim$ a **relation** on X.

Definition 2.3.2 Equivalence Relation

- **Reflexive:** $x \sim x$ for all $x \in X$
- **Symmetric:** $x \sim y$ implies that $y \sim x$ for all $x, y \in X$
- **Transitive:** $x \sim y$ and $y \sim z$ implies that $x \sim z$ for all $x, y, z \in X$

A relation $\sim$ is called an **equivalence relation** on X if it satisfies the following three axioms:

Definition 2.3.4 Coset

Let $H \leq G$ and let $g \in G$. Then a *left coset* of H in G is a subset of G of the form gH, for some $g \in G$. We denote the set of left cosets of H in G by G/H
(Notation) Let A, B be subgroups of a group G and let $g \in G$. Then

$$AB := \{ab \mid a \in A, b \in B\}, \quad gA := \{gA \mid a \in A\}$$

Theorem 2.4.2: Lagrange's Theorem

Suppose that G is a finite group.

- If $H \leq G$, then $|H|$ divides $|G|$
- Let $g \in G$. Then $o(g)$ divides $|G|$
- For all $g \in G$, we have that $g^{|G|} = e$

Theorem 2.3.8: Coset Rules

Let $H \leq G$

- For all $h \in H$, $hH = H$. In particular $eH = H$
- For $g_1, g_2 \in G$, the following are equivalent
  - $g_1H = g_2H$
  - there exists $h \in H$ such that $g_2 = g_1h$
  - $g_2 \in g_1H$
- For g_1, $g_2 \in G$, define $g_1 \sim g_2$ if and only if $g_1H = g_2H$. Then $\sim$ defines an equivalence relation on G.

Theorem 2.4.4: Index of a Subgroup

The **index** of $H \leq G$ is defined as the number of *distinct* left cosets of H in G, which by Lagrange's is $|G/H| = \frac{|G|}{|H|}$

Remark 2.4.6 - 8: Consequences of Lagrange

- **2.4.6:** Suppose that G is a group with $|G| = p$, where p is prime. Then G is a cyclic group
- **2.4.7:** Suppose that G is a group with $|G| < 6$. Then G is abelian
- **2.4.8:** If p is a prime and $a \in \mathbb{Z}$, then $a^p \equiv a \mod p$

Homomorphisms and Isomorphisms

Definition 3.1.1 Group Homomorphism

Let $(G, *), (H, \circ)$ be groups. A map $\phi : G \to H$ is called a **homomorphism** if

$$\phi(x * y) = \phi(x) \circ \phi(y) \quad \text{for all } x, y \in G$$

Note that the product on the left is formed using $*$, while the product on the right is formed using $\circ$

Definition 3.1.2 Group Isomorphism

A group homomorphism $\phi : G \to H$ that is also a bijection is called an **isomorphism** of groups. In this case we say that G and H are *isomorphic* and we write $G \cong H$. An isomorphism $G \to G$ is called an **automorphism** of G.

Theorem 3.1.L: Cyclic Isomorphisms

All finite cyclic groups with the same order are *isomorphic* to each other. Therefore, cyclic groups of order n are isomorphic to $(\mathbb{Z}_n, +)$
All infinite cyclic groups are *isomorphic* to each other. Therefore, each cyclic group of infinite order is isomorphic to $(\mathbb{Z}, +)$

Remark 3.1.5: Consequences of Homomorphisms

Let $\phi : G \to H$ be a group homomorphism. Then

- $\phi(e_G) = e_H$
- $\phi(g^k) = (\phi(g))^k$ and $\phi(g^{-1}) = (\phi(g))^{-1}$ for all $g \in G$
- If ϕ is injective, the order of $g \in G$ equals the order of $\phi(g) \in H$.

Definition 3.1.7 Normal Subgroup

A subgroup $N \leq G$ is **normal** if the left and right cosets of N are equal, i.e. $gN = Ng$ for all $g \in G$. If N is a normal subgroup of G, we write $N \triangleleft G$. Kernels of homomorphisms are always normal subgroups

Definition 3.1.6 Image and Kernel of a Group

Let $\phi : G \to H$ be a group homomorphism.

- The **image** of ϕ is defined to be
$$\operatorname{im}\phi := \{h \in H \,|\, h = \phi(g) \text{ for some } g \in G\}$$
- The **kernel** of ϕ is defined to be
$$\ker\phi := \{g \in G \,|\, \phi(g) = e_H\}$$

Note: $\operatorname{im}\phi$ is a subgroup of H and $\ker\phi$ is a subgroup of G

Theorem 3.2.1: Product Isomorphisms

Let $H, K \leq G$ be subgroups with $H \cap K = \{e\}$.

- The map $\phi : H \times K \to HK$ given by $\phi : (h, k) \to hk$ is bijective
- If every element of H commutes with every element of K when multiplied in G (i.e. $hk = kh \quad \forall h \in H, k \in K$), then HK is a subgroup of G, and it is isomorphic to $H \times K$ via ϕ

Theorem 3.2.3: Size of Product Group

Let $H, K \leq G$ be finite subgroups of a group G such that $H \cap K = \{e\}$ Then $|HK| = |H| \times |K|$.

Group Actions

Definition 4.1.1 Group Action

Let $(G, *)$ be a group, and let X be a nonempty set. Then a (left) **action** of G on X is a map

$$G \times X \to X$$

written $(g, x) \mapsto g \cdot x$, such that

$$g_1 \cdot (g_2 \cdot x) = (g_1 * g_2) \cdot x \qquad \text{and} \qquad e \cdot x = x$$

for all $g_1, g_2 \in G$ and all $x \in X$.

Definition 4.2.1 Orbit, Stabilizer, and Fix

- Suppose that G acts on X. Then the set
$$N := \{g \in G \mid g \cdot x = x \quad \forall x \in X\}$$
is a subgroup of G, and is called the **kernel** of the action. If $N = \{e\}$, then we say the action is **faithful**
- For every x in X, the **orbit** of x is a subset of X defined by
$$\operatorname{Orb}_G(x) = \{g \cdot x \mid g \in G\}$$
- For every x in X, the **stabilizer** of x is a subgroup of G defined by
$$\operatorname{Stab}_G(x) = \{g \in G \mid g \cdot x = x\}$$
- For every g in G, the **fix** of g is a subset of X defined by
$$\operatorname{Fix}(g) = \{x \in X \mid g \cdot x = x\}$$
- Let G act on X, let $x \in X$ and set $H := \operatorname{Stab}_G(x)$. If $y = g \cdot x$ for some $g \in G$, then
$$\operatorname{send}_x(y) = gH$$
- Let $h \in H$ and $g \in G := X$. The **conjugate action** is:
$$h \cdot g := hgh^{-1}$$
- An action of G on X is **transitive** if for all $x, y \in X$ there exists $g \in G$ such that $y = g \cdot x$. Equivalently, X is a single orbit under G
- We define the **centre** of a group G to be
$$C(G) := \{g \in G \mid hg = gh \text{ for all } h \in G\}$$
The **centralizer** of g in G is defined as
$$G(g) := \{h \in G \mid gh = hg\}$$

Theorem 4.2.5: Orbit Equivalence

Let G act on X. Then

$$x \sim y \iff y = g \cdot x \text{ for some } g \in G$$

defines an equivalence relation on X. The equivalence classes are the orbits of G. Thus when G acts on X, we obtain a partition of X into orbits

Theorem 4.3.1: Orbit-Stabilizer Theorem

Suppose G is a finite group acting on a set X, and let $x \in X$. Then $|\operatorname{Orb}_G(x)| \times |\operatorname{Stab}_G(x)| = |G|$, or in words:

$$\text{size of orbit} \ \times \ \text{size of stabilizer} \ = \text{order of group}$$

Theorem 4.3.4: Orbit Send Theorem

Let G act on X, let $x \in X$, and let set $H := \operatorname{Stab}_G(x)$. Then

$$\operatorname{send}_x : \operatorname{Orb}_G(x) \to G/H \text{ which sends } y \mapsto \operatorname{send}_x(y)$$

Theorem 4.4.2: Cauchy's Theorem

Let G be a group, p be prime. If p divides $|G|$, then G contains an element of order p

2 Analysis

Note: Any reference numbers are to the lecture notes

Real Numbers and Bounds

Definition 1.1 The Real Numbers

$\mathbb{R}$ is defined as the set of real numbers. It has two operations $+$ and $*$, and it is a field, i.e. satisfies group axioms for both operations, in addition to the Distributive law:

$$a \cdot (b + c) = a \cdot b + a \cdot c$$

The set of real numbers is also ordered, i.e. there is a relation $<$ which satisfies pretty much what you think it does
Finally, the set of real numbers is complete, i.e. there are no gaps between any numbers.

Definition 1.2.3 Triangle Inequality

The most important property of the absolute value $|a|$:

$$|a + b| \leq |a| + |b| \quad \text{and} \quad ||a| - |b|| \leq |a - b|$$

Definition 1.3.2 Suprema and Bounds

Let $E \subset \mathbb{R}$ be nonempty

- The set E is said to be bounded above if there is $M \in R$ such that $a \leq M$ for all $a \in E$
- A real number M is called an upper bound of the set$E$ if $a \leq M$ for all $a \in E$
- A real number s is called the **supremum** of the set E if
  - s is an upper bound of E
  - $s \leq M$ for all upper bounds M of the set E

  If a number s exists, we shall say that E has a supremum and write $s = \sup E$

If the supremum s exists, then s is the least upper bound of the set E. The supremum is also unique if it exists.

If the same properties as a supremum apply but in the other direction, a number s is instead called the **infimum** of the set E. Infimum and Supremum are related via the reflection principle:

- Set E has a supremum if and only if the set $-E$ has an infimum. Also $\inf(-E) = -\sup(E)$
- Set E has an infimum if and only if the set $-E$ has a supremum. Also $\sup(-E) = -\inf(E)$

Theorem 1.3.5: Suprema Approximation Property

If the set $E \subset \mathbb{R}$ has a supremum then for any positive number $\epsilon > 0$ there exists $a \in E$ such that

$$\sup E - \epsilon < a \leq \sup E$$

Theorem 1.3.7: Archimedean Principle

Given positive real numbers $a, b \in \mathbb{R}$ there is an integer $n \in N$ such that $b < na$

Definition 1.5.2 Countability

Let E be a set. E is said to be:

- **Finite** if either $E = \varnothing$, or there is an integer $n \in \mathbb{N}$ and a bijection $f : \{1, 2, 3, \ldots, n\} \to E$. We say that the set E has n elements
- **Countable** if there is a bijective function $f : \mathbb{N} \to E$
- **At most countable** if E is finite or countable
- **Uncountable** if E is neither finite nor countable

Additionally, a nonempty set E is at most countable if and only if there is a surjective function $f : \mathbb{N} \to E$

Sequences and Series

Definition 2.1.1 Convergence of a Sequence

A sequence of real numbers (x_n) is said to converge to a real number a if for every $\epsilon > 0$ there is a $N \in \mathbb{N}$ where for every $n \geq N$ we have that $|x_n - a| < \epsilon$

For a sequence (x_n), we write $\lim x_n = +\infty$ if for each $M > 0$ there is a number N such that $n > N$ implies $x_n > M$. Reverse every inequality for $-\infty$ case.

Definition 2.1.9 Bounds of Sequences

Let (x_n) be a sequence of real numbers.

- $(x_n)_{n \in \mathbb{N}}$ is said to be **bounded above** if $x_n \leq M$ for some $M \in \mathbb{R}$ and all $n \in \mathbb{N}$
- $(x_n)_{n \in \mathbb{N}}$ is said to be **bounded below** if $x_n \geq m$ for some $m \in \mathbb{R}$ and all $n \in \mathbb{N}$
- $(x_n)_{n \in N}$ is said to be **bounded** if it is both bounded above and below

Remark 2.2.1 - ?: Limit Theorems

- Let $E \subset \mathbb{R}$. If E has a finite supremum then there is a sequence (x_n) with each $x_n \in E$ such that $x_n \to \sup E$ as $n \to \infty$. The same goes for a finite infimum
- **Comparison Theorem for sequences:** Suppose that (x_n), (y_n) are real sequences. If both $\lim_{n\to\infty} x_n$, $\lim_{y\to\infty} y_n$ exist and belong to $\mathbb{R}*$, and if $x_n \leq y_n$ for all $n \geq N$ for some $N \in \mathbb{N}$, then $\lim_{n\to\infty} x_n \leq \lim_{n\to\infty} y_n$

Definition 2.3.1 Monotone Sequences

Let (s_n) be a sequence of real numbers.

- (s_n) is said to be increasing if $s_1 \leq s_2 \leq s_3 \leq \cdots$, and strictly increasing if $s_1 < s_2 < s_3 < \cdots$
- (s_n) is said to be decreasing if $s_1 \geq s_2 \geq s_3 \geq \cdots$, and strictly decreasing if $s_1 > s_2 > s_3 > \cdots$
- (s_n) is said to be monotone if it is either increasing or decreasing

Theorem lots: Top 10 Limit Theorems

- **Squeeze Theorem (for sequences):** Suppose that (x_n), (y_n), and (w_n) are real sequences
  - If both $x_n \to a$ and $y_n \to a$ as $n \to \infty$, and if $x_n \leq w_n \leq y_n$ for all $n \geq N_0$, then $w_n \to a$ as $n \to \infty$
  - If $x_n \to 0$ and (y_n) is bounded, $x_n y_n \to_0$ as $n \to \infty$
- **Divergence Test:**
  - If $\sum_{n=1}^{\infty} a_n$ converges, then $a_n \to 0$.
  - If $(a_n)_{n\in\mathbb{N}}$ doesn't converge to 0, then $\sum_{n=1}^{\infty} a_n$ diverges. Be careful that the converse isn't true.
- **Comparison Test:** Let $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be two sequences such that $0 \leq a_n \leq b_n$ for all n.
  - If $\sum_n b_n$ converges, then $\sum_n a_n$ converges as well.
  - If $\sum_n a_n$ diverges, then $\sum_n b_n$ diverges as well.
- **Limit Comparison Test:** Let $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be two real sequences with $a_n \geq 0$ and $b_n > 0$ for all n. Assume that $a_n/b_n \to L$ for some $L \in (0, \infty)$. Then, $\sum_{n=1}^{\infty} a_n$ converges iff $\sum_{n=1}^{\infty} b_n$ converges.
  - If $L = 0$ and $\sum_{n=1}^{\infty} b_n$ converges then $\sum_{n=1}^{\infty} a_n$ converges
  - If $L = \infty$ and $\sum_{n=1}^{\infty} a_n$ diverges then $\sum_{n=1}^{\infty} b_n$ diverges
- **Root Test:** Let $\sum_{n=0}^{\infty} a_n$ be a series with non-negative terms such that $\sqrt[n]{a_n} \to L$ where $0 \leq L \leq +\infty$.
  - If $0 \leq L < 1$ then the series $\sum_{n=0}^{\infty} a_n$ converges.
  - If $L > 1$ then the series $\sum_{n=0}^{\infty} a_n$ diverges.
  - If $L = 1$, the series may or may not converge
- **Ratio Test:** Let $\sum_{n=0}^{\infty} a_n$ be a series with positive terms such that $(a_{n+1})/(a_n) \to L$, where $0 \leq L \leq +\infty$.
  - If $0 \leq L < 1$ then the series $\sum_{n=0}^{\infty} a_n$ converges.
  - If $L > 1$ then the series $\sum_{n=0}^{\infty} a_n$ diverges.
  - If $L = 1$ then compare to p series
- **Cauchy's Condensation Test:** Let $(a_n)_{n\in\mathbb{N}}$ be a decreasing sequence with non-negative terms. Then the following are equivalent:
  - The series $\sum_{n=1}^{\infty} a_n$ converges
  - The series $\sum_{n=0}^{\infty} 2^n a_{2^n}$ converges.
- **Alternating Series Test:** Let $(b_n)_{n\in\mathbb{N}}$ be a *decreasing* sequence of non-negative real numbers that converges to zero. Then the series $\sum_{n=1}^{\infty} (-1)^{n-1} b_n$ converges.
- **Monotone Convergence Theorem:** If a sequence of real numbers (s_n) is increasing and bounded above, or decreasing and bounded below, then (s_n) is convergent (and converges to the sup/inf of the set $\{x_n \mid n \in \mathbb{N}\}$ respectively).
- **Geometric Series Test:** Assume $a, r \in \mathbb{R}, a, r \neq 0$. Then

$$\sum_{n=1}^{\infty} a \cdot r^n = \begin{cases} \dfrac{a}{1-r} & \text{if } |r| < 1 \\ \text{diverges} & \text{if } |r| \geq 1 \end{cases}$$

Notice that a is always the first term in the series, and r is the *common ratio*

Continuity and Functional Limits

Definition 4.1.1 Continuity

Let f be a real-valued function whose domain is a subset of $\mathbb{R}$. The function f is **continuous** at x_0 in $\operatorname{dom}(f)$ if, for every sequence (x_n) in $\operatorname{dom}(f)$ converging to x_0, we have

$$\lim_{n\to\infty} f(x_n) = f(x_0)$$

If f is continuous at each $a \in S \subseteq \operatorname{dom}(f)$ and then we say that f is continuous on S. If f is continuous on $\operatorname{dom}(f)$ then we say that f is continuous

Theorem 4.1.6: $\epsilon - \delta$ Definition of Continuity

A function $f : A \to \mathbb{R}$ is continuous if for all $\epsilon > 0$, there exists some $\delta > 0$ s.t. for all $x \in A$ for which $0 < |x - c| < \delta$, we have

$$|f(x) - f(c)| < \epsilon$$

Theorem 6.1.4: Evil $\epsilon - \delta$ definition of continuity

A function $f : A \to \mathbb{R}$ is not continuous if there exists $\epsilon > 0$ such that for all $\delta > 0$ there exists some $x \in A$ satisfying $0 < |x - c| < \delta$ for which $|f(x) - f(c)| \geq \epsilon$

Definition 4.2.1 Bounds of a Function

Let $E \subseteq \mathbb{R}$ be nonempty. A function $f : E \to \mathbb{R}$ is said to be bounded on E if

$$|f(x)| \leq M, \qquad \text{for all } x \in E$$

where M is some (large) real number.

Theorem 4.2.2: Extreme Value Theorem

Let $I \subseteq \mathbb{R}$ be a closed and bounded interval. Let $f : I \to \mathbb{R}$ be continuous on I. Then f is bounded on the interval I, denoted by

$$m = \inf\{f(x) \mid x \in I\}, \qquad M = \sup\{f(x) \mid x \in I\}$$

Then there exist points $x_m, x_M \in I$ such that

$$f(x_m) = m \quad \text{and} \quad f(x_M) = M$$

Theorem 4.2.4: $\epsilon - \delta$ Limit jr.

Let $f : I \to \mathbb{R}$ where I is an open nonempty interval. If f is continuous at a point $a \in I$ and $f(a) > 0$ then for some δ, $\epsilon > 0$ we have that

$$f(x) > \epsilon, \qquad \text{for all } x \in (a - \delta, a + \delta)$$

Theorem 4.2.5: Intermediate Value Theorem

Let I be a non-degenerate interval and let $f : I \to \mathbb{R}$ be a continuous function. If $a, b \in I$, $a < b$, then on the interval (a, b), f attains all values between $f(a)$ and $f(b)$. i.e. given y_0 between $f(a)$ and $f(b)$, there exists $x_0 \in (a, b)$ such that $f(x_0) = y_0$

Theorem 4.2.6: Bolzano's Theorem

Let $f(x)$ be continuous on $[a, b]$ such that $f(a)f(b) < 0$, then there exists $c \in (a, b)$ such that $f(c) = 0$

Theorem 4.3.1: $\epsilon - \delta$ definition of a limit

Let $f : A \to \mathbb{R}$ and let c be a limit point of A. Then we say that

$$\lim_{x\to c} f(x) = L$$

if for all $\epsilon > 0$ there exists some $\delta > 0$ such that for every $x \in A$ for which $0 < |x - c| < \delta$, we have

$$|f(x) - L| < \epsilon$$

We also say $\lim_{x\to c} f(x)$ **converges** to L in such a situation

Differentiation

Definition 5.1.1 First Principle Differentiation

A real function f is said to be differentiable at a point $x \in \mathbb{R}$ if f is defined at some open interval containing x, and

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

exists. $f'(x)$ is called the derivative of f at the point x

Theorem 5.1.3: Differentiable implies Continuous

Let I be an open interval, $x_0 \in I$ and $f : I \to \mathbb{R}$ be differentiable at x_0. Then f is continuous at x_0. The converse is not true, an example is $f(x) = |x|$ which isn't differentiable at 0.

Theorem 5.1.4: Differentiable Intervals

Let $f : I \to \mathbb{R}$ be a given fucntion, where I is an open interval. We say that f is differentiable in I iff it is differentiable at every point in I. At endpoints, derivatives only have to be one-sided

Theorem 5.2.1: Differentiation Rules

Let $f, g : (a, b) \to \mathbb{R}$ be differentiable on (a, b). Then $f + g$ and $f \cdot g$ are differentiable on (a, b). If $g(x) \neq 0$ for all $x \in (a, b)$, then f/g is differentiable. Moreover,

- **Sum rule:** $(f + g)' = f' + g'$
- **Product Rule:** $(fg)' = f'g + fg'$
- **Quotient Rule:** $(f/g)' = (f'g - fg')/g^2$

Theorem 5.4.6: Inverse Function Theorem

Let f be injective and continuous on an open interval I. If $a \in f(I)$ and f' at the point $f^{-1}(a) \neq 0$ exists and is nonzero, then f^{-1} is differentiable at a and

$$(f^{-1})'(a) = \frac{1}{f'(f^{-1}(a))}$$

Theorem 5.2.2: Chain Rule

Let f, g be real functions. If f is differentiable at a and g is differentiable at $f(a)$ then $g \circ f$ is differentiable at a and

$$(g \circ f)'(a) = g'(f(a))f'(a)$$

Theorem 5.3.1 - 3: Differentiation Theorem ladder

- **Rolle's Theorem:** Let $a, b \in \mathbb{R}$, $a < b$. If $f : [a, b] \to \mathbb{R}$ is continuous in $[a, b]$, differentiable in (a, b) and $f(a) = f(b)$, then there exists a point c in (a, b) such that $f'(c) = 0$
- **Mean Value Theorem:** If $f : [a, b] \to \mathbb{R}$, $a < b$ is continuous in $[a, b]$, differentiable in (a, b) then $\exists c \in (a, b)$ s.t.

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

- **Generalised MVT:** If $f, g : [a, b] \to \mathbb{R}$ is continuous in $[a, b]$ and differentiable in (a, b), then $\exists c$ in (a, b) such that

$$(f(b) - f(a))g'(c) = (g(b) - g(a))f'(c)$$

If $g(b) - g(a)$, $g'(c) \neq 0$ then this can be written as

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}$$

Theorem 5.4.2: Monotone Functions

Let $a < b$ be real and f be continuous on $[a, b]$ and differentiable on (a, b).

- If $f'(x) > 0\ \forall x \in (a, b)$, then f is strictly increasing on $[a, b]$
- If $f'(x) < 0\ \forall x \in (a, b)$, then f is strictly decreasing on $[a, b]$
- If $f'(x) = 0\ \forall x \in (a, b)$, then f is constant on $[a, b]$

Additionally, if f is injective and continuous on an interval I, Then f and f^{-1} is strictly monotone on I and $f(I)$ respectively

Theorem 5.5.1: Taylor Series

Let $f : I \to \mathbb{R}$ be $n+1$ times differentiable and $x_0 \in I$, for an open interval I. For each $x \in I$, there is a c between x_0 and x such that

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k + \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}$$

(c depends on x and n)
Now suppose that $f : (a, b) \to \mathbb{R}$ is infinitely differentiable and let $x_0 \in (a, b)$. Fix x in (a, b). For every positive integer N we have

$$f(x) = \sum_{k=0}^{N} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k + R_N(x)$$

If $R_N(x) \to 0$ as $n \to \infty$, we have

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k$$

Made by Leon

3 Examples Catalogue

Examples of a Group

Example 1.2.4: Dihedral Group D_n

The set of symmetries of an n-gon forms a group under composition. We call this group the **Dihedral Group** D_n
The Dihedral group of n has precisely $|D_n| = 2n$ elements, namely

- The identity e
- $n-1$ anticlockwise rotations of $\frac{2pi}{n}$. We denote this operation with g
- n reflections. If n is odd, then there are n reflections from a point to the opposite edge. If n is even, there are $\frac{n}{2}$ reflections from point to point, and $\frac{n}{2}$ from edge to edge. We denote a vertical reflection with h, and rotated reflections as compositions of h and g

From this, we see that

$$D_n = \{e,\, g,\, g^2,\, \ldots,\, g^{n-1},\, h,\, gh,\, g^2h,\, \ldots,\, g^{n-1}h\}$$

Example 1.3.2: Symmetric Group

The set of all symmetries of $\{1,\, 2,\, \ldots,\, n\}$ is called the **symmetric group** S_n. It is a group under composition with order $|S_n| = n!$
The symmetric group can be thought of as every permutation of the set $\{1,\, 2,\, \ldots,\, n\}$, or can also be thought of as an n-gon where every edge is connected to each other.

Example $\mathbb{Z}_3 x \mathbb{Z}_4$: Group Properties pick'n'mix

- Any group $\mathbb{Z}_n$ is **Abelian** and **Cyclic**
- Any cross product of $\mathbb{Z}_n \times \mathbb{Z}_m$ where n and m are coprime is **Abelian** and **Cyclic**.
- Any cross product of $\mathbb{Z}_n \times \mathbb{Z}_m$ where n and m are not coprime is **Abelian** but **Not Cyclic**.
- Any dihedral group D_n is **Not Abelian**, and **Not Cyclic**
- The trivial action $g \cdot x = x$ of any group is **Not Faithful** and **Not Transitive**
- The trivial action of any group acting on $\{1\}$ is **Not Faithful** and **Transitive**
- $\mathbb{Z}_{an}$ acting on $\mathbb{Z}_n$, where $a \in \mathbb{Z}$ where $g \cdot x = g + x$ is **Not Faithful** and **Not Transitive**
- A symmetry group of a graph with a middle point has at least two orbits (**Not Transitive**)

Example 0: Example of a Coset

Consider $\mathbb{Z}_4$ under addition, and let $H = \{0, 2\}$ $(e = 0.)$ The cosets of H in G are:

$$eH = e * H = \{e * h \,|\, h \in H\} = \{0 + h \,|\, h \in H\} = \{0, 2\}$$
$$1H = 1 * H = \{1 * h \,|\, h \in H\} = \{1 + h \,|\, h \in H\} = \{1, 3\}$$
$$2H = 2 * H = \{2 * h \,|\, h \in H\} = \{2 + h \,|\, h \in H\} = \{0, 2\}$$
$$3H = 3 * H = \{3 * h \,|\, h \in H\} = \{3 + h \,|\, h \in H\} = \{1, 3\}$$

Hence there are two cosets, namely

$$0 * H = 2 * H = \{0, 2\} \quad \text{and} \quad 1 * H = 3 * H = \{1, 3\}$$

$$G/H = \{eH = 2H, 1H = 3H\} = \{\{0, 2\}, \{1, 3\}\}$$

Example 3.3.5: p-series

The series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{k^p}$ converges if $p > 1$, and it diverges if $p \leq 1$. At $p = 1$, this series is called the **Harmonic Series**.
To show divergence/convergence of a series, we can compare it to the p-series

Example $\sum_{n=0}^{\infty}$: Deciphering Taylor Series

- **Showing convergence of a Taylor Series:** An infinite Taylor series will converge to $f(x)$ iff we have $R_N(x) \to 0$ as $N \to \infty$ in the finite Taylor series

$$f(x) = \lim_{N\to\infty} \sum_{k=0}^{N} \frac{f^k(x_0)}{k!}(x - x_0)^k + R_N(x)$$

  Therefore, showing that $(R_N(x))_{n\in\mathbb{N}}$ converges to 0 is enough to show that the infinite Taylor Series converges to $f(x)$

- **Simplifying series-like terms:** If you have a Taylor Series / function that is in the same equation as a bunch of series-like terms, then a good idea is to try to expand the Taylor series at N for the N amount of elements and then try and figure something out using the remainder term $R_N(x)$
  Example: extract of 2018 May A3
  At some point we end up with an equation

$$0 \leq \ln(x+1) - x + \frac{x^2}{2} - \frac{x^3}{3} + \frac{x^4}{4}$$

  and a Taylor Series

$$\ln(x+1) = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n} x^n$$

  Expand Taylor series with $N = 4$

$$\begin{aligned} 0 &\leq \ln(x+1) - x + \frac{x^2}{2} - \frac{x^3}{3} + \frac{x^4}{4} \\ &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + R_4(x) - x + \frac{x^2}{2} - \frac{x^3}{3} + \frac{x^4}{4} \\ &= R_4(x) \end{aligned}$$

The museum of $\epsilon - \delta$ limits

Example of rigour: $\epsilon - \delta$ and $\epsilon - N$ Template

Proof: Let $\epsilon > 0$ be given. Set δ=__ (If there is a constant, then set as "<" e.g. $\delta < \min\{1, \epsilon\}$). Then for all $x \in \mathbb{R}$ such that $|x - \_\_| < \delta$ we have
Optional: preliminary step to determine an upper bound
Therefore, / Therefore since "x term" < "constant",
"Same steps as rough working"
$\cdots = \_\_ \cdot |x - \_\_| < \_\_\delta = \epsilon$ (same rule applies about constants)

Proof: Let $\epsilon > 0$ be given. Choose $N \in \mathbb{N}$ such that $N >$ __, for example $N =$ __. Then for all $n \geq N$ we have
"Same steps as rough working"
$\cdots =$ "equation in terms of n" $\leq$ "same thing in terms of N" $< \epsilon$

Example 10000: ϵ-N Convergence

Show that the sequence $\left(\frac{2n+1}{3n+2}\right)_{n\in\mathbb{N}}$ *converges to* $\frac{2}{3}$
We start with the rough work. Start with an arbitrary $\epsilon > 0$ and find an N_ϵ s.t. $\left|\frac{2n+1}{3n+2} - \frac{2}{3}\right| < \epsilon$ for all $n > N_\epsilon$. Let's explore this.

$$\left|\frac{2n+1}{3n+2} - \frac{2}{3}\right| = \frac{11}{3(3n+2)} < \epsilon \quad\implies\quad n > \frac{1}{3}\left(\frac{11}{3\epsilon} - 2\right)$$

Proof: Let $\epsilon > 0$. Pick a positive integer N such that

$$N > \frac{1}{3}\left(\frac{11}{3\epsilon} - 2\right)$$

Then,

$$\frac{11}{3(3N+2)} < \epsilon$$

For all n with $n \geq N$ we have

$$|a_n - L| = \left|\frac{2n+5}{3n+2} - \frac{2}{3}\right| = \frac{11}{3(3n+2)} \leq \frac{11}{3(3N+2)} < \epsilon$$

Another method of finding a limit is,

$$\left|\frac{2n+1}{3n+2} - \frac{2}{3}\right| = \frac{11}{3(3n+2)} = \frac{11}{9n+6} < \frac{11}{9n} < \epsilon$$

Since $9n + 6 > 9n$, this means that the right fraction is larger than the left fraction in all cases. This means if we can find a right fraction that is smaller than ϵ then the left fraction must also.

Proof: let $\epsilon > 0$. Pick a positive integer N such that $N > \frac{11}{9\epsilon}$. Then $\frac{11}{9N} < \epsilon$. For all n with $n \geq N$, we have

$$|a_n - L| = \left|\frac{2n+5}{3n+2} - \frac{2}{3}\right| = \frac{11}{9n+6} \leq \frac{11}{9n} \leq \frac{11}{9N} < \epsilon$$

Example (: $\epsilon - \delta$ Continuity

Using the definition of continuity, prove that the function $f : \mathbb{R}\backslash\{\frac{9}{5}\} \to \mathbb{R}$ *defined by* $f(x) = \frac{x^2}{5x-9}$ *is continuous at* $x_0 = 2$
Since $x_0 = 2$, our delta should end up as $|x-2| < \delta$. Start with $|f(x) - f(a)| < \epsilon$

$$\begin{aligned} |f(x) - f(a)| &= \left|\frac{x^2}{5x-9} - \frac{4}{10-9}\right| \\ &= \left|\frac{x^2}{5x-9} - 4\right| \\ &= \left|\frac{x^2 - 20x + 36}{5x-9}\right| \\ &= \left|\frac{(x-18)(x-2)}{5x-9}\right| \\ &= |x-2|\left|\frac{x-18}{5x-9}\right| \end{aligned}$$

We have $|x-2|$, so we want to turn the RH fraction into a constant. If we let the neighbourhood around δ to be no less than $\frac{1}{10}$ (i.e. $x \in (1.9, 2.1)$) (this number can be anything, but smaller than $\frac{1}{5}$ since there is an asymptote at $\frac{9}{5}$), using the number with the largest value in that range we can get an upper bound for δ.

$$\left|\frac{x-18}{5x-9}\right| < \left|\frac{1.9-18}{9.5-9}\right| = \left|\frac{-16.1}{0.5}\right| = |-32.2| \implies \left|\frac{x-18}{5x-9}\right| < 32.2$$

Therefore

$$|x-2|\left|\frac{x-18}{5x-9}\right| < |x-2| \cdot 32.2 < \epsilon$$

Therefore, we can take $\delta = \max\{1/10, \epsilon/32.2\}$

Proof: Let $\epsilon > 0$ be given. set $\delta = \min\{\frac{1}{10}, \frac{\epsilon}{32.2}\}$. Then for all $x \in \mathbb{R}$ such that $|x-2| < \delta$ we have

$$\left|\frac{x-18}{5x-9}\right| < \left|\frac{1.9-18}{9.5-9}\right| = \left|\frac{-16.1}{0.5}\right| = |-32.2| \implies \left|\frac{x-18}{5x-9}\right| < 32.2$$

Therefore, since $\left|\frac{x-18}{5x-9}\right| < 32.2$,

$$|f(x) - f(a)| = \left|\frac{x^2}{5x-9} - \frac{4}{10-9}\right| = \left|\frac{x^2-20x+36}{5x-9}\right|$$

$$= \left|\frac{(x-18)(x-2)}{5x-9}\right| = |x-2|\left|\frac{x-18}{5x-9}\right| \leq 32.2 \cdot |x-2| < 32.2 \cdot \delta = \epsilon$$

Example): $\epsilon - \delta$ Discontinuity

From negation of $\epsilon - \delta$ continuity - A function $f : A \to \mathbb{R}$ is not continuous if there exists $\epsilon > 0$ such that for all $\delta > 0$ there exists some $x \in A$ satisfying $0 < |x - c| < \delta$ for which $|f(x) - f(c)| \geq \epsilon$

$$|f(x) - f(a)| < \epsilon \implies \left|sin\left(\frac{1}{x}\right) - 0\right| < \epsilon \quad \implies \left|sin\left(\frac{1}{x}\right)\right| < \epsilon$$

So we want to show that we can find an ϵ such that for every $\delta > 0$, we can find an x where $|x| < \delta$ and also $|sin(\frac{1}{x})| \geq \epsilon$.
Since $sin(x)$ repeats, if we can find an x such that $sin(\frac{1}{x})$ is an exact value then we can define ϵ as something lower than that. If we want a value where $sin(\frac{1}{x}) = 1$, this will be true if $x = 1/(\frac{\pi}{2} + 2N\pi)$, where N is a positive integer.
Since x has to be bounded by δ, go from δ

$$\begin{aligned} &|x| < \delta \\ &\left|\frac{1}{\frac{\pi}{2} + 2N\pi}\right| < \delta \\ &\frac{1}{\frac{\pi}{2} + 2N\pi} < \delta \quad \text{(will always be positive since N positive int)} \\ &\frac{\pi}{2} + 2N\pi > \frac{1}{\delta} \\ &N > \frac{1}{2\pi}\left(\frac{1}{\delta} - \frac{\pi}{2}\right) \end{aligned}$$

Proof: Let $\epsilon = \frac{1}{2}$. Let $\delta > 0$ be given. Pick a positive integer N such that $N > \frac{1}{2\pi}\left(\frac{1}{\delta} - \frac{\pi}{2}\right)$ and set $x = \frac{1}{\frac{\pi}{2} + 2N\pi}$. Then for all $x \in \mathbb{R}$ such that $0 < x < \delta$, we have

$$|f(x)| = \left|sin\left(\frac{1}{x}\right)\right| = \left|sin\left(\frac{\pi}{2} + 2N\pi\right)\right| = 1 \geq \frac{1}{2} = \epsilon$$

Example Dumb Assumptions: $\epsilon - \delta$ using other limits

If $f, g : \mathbb{R}\backslash\{1\} \to \mathbb{R}$ *are two functions such that* $\lim_{x\to 1} f(x) = 2$, $\lim_{x\to 1} g(x) = 3$, *show that* $\lim_{x\to 1}(4f(x) + g(x)^2) = 17$ We want to try and turn the limit into compositions of other limits. From the assumptions, we know that

- There exists a δ_1 s.t. $\forall x$ where $0 < |x-1| < \delta_1$, we have
$$|f(x) - 2| < \epsilon \tag{1}$$
- There exists a δ_2 s.t. $\forall x$ where $0 < |x-1| < \delta_2$, we have
$$|g(x) - 3| < \epsilon \tag{2}$$

So, start with the main function. We want to show

$$|4f(x) + g(x)^2 - 17| < \epsilon$$

We want to turn this into a composition of (1) and (2). By "Trusting our professors won't be too mean" this should be possible

$$\begin{aligned} |4f(x) + g(x)^2 - 17| &= |4(f(x) - 2) + g(x)^2 - 9| \\ &= |4(f(x) - 2) + (g(x) - 3)(g(x) + 3)| \\ \textit{(via triangle ineq)} &\leq 4|f(x) - 2| + |g(x) - 3||g(x) + 3| \end{aligned}$$

To find an upper bound for $|g(x)+3|$ we want to manipulate again

$$|g(x) + 3| = |g(x) - 3 + 6| \quad \leq |g(x) - 3| + 6 \quad < \epsilon + 6$$

Therefore now we can substitute equations (1) and (2) into everything

$$4|f(x) - 2| + |g(x) - 3||g(x) + 3| \quad < 4\epsilon + \epsilon|g(x) + 3| \quad < 4\epsilon + \epsilon(\epsilon + 6)$$

Let the epsilon boundary be less than 1. Then $\epsilon + 6 < 7$, therefore

$$4\epsilon + \epsilon(\epsilon + 6) < 4\epsilon + \epsilon(7) = 11\epsilon$$

We want to finish with ϵ but since (1) and (2) work for any ϵ by definition, set those inequalities to $\frac{\epsilon}{11}$ instead and the final result will be ϵ on its own

Proof: Let $\epsilon > 0$ be given. First assume $\epsilon \leq 1$. By our assumptions, there exists a δ_1 where $\forall x$ s.t. $0 < |x-1| < \delta_1$, we have $|f(x) - 2| < \epsilon/11$, and a δ_2 where $\forall x$ s.t. $0 < |x-1| < \delta_2$, we have $|g(x) - 3| < \epsilon/11$. For all x s.t. $0 < |x-1| < \delta_2$, we have

$$|g(x) + 3| = |g(x) - 3 + 6| \leq |g(x) - 3| + 6 \leq \frac{\epsilon}{11} + 6 \leq \frac{1}{11} + 6 \leq 7$$

Let $\delta = \min\{\delta_1, \delta_2\}$ Therefore, since $|g(x) + 3| \leq 7$,

$$\begin{aligned} |4f(x) + g(x)^2 - 17| &= |4(f(x) - 2) + g(x)^2 - 9| \\ &= |4(f(x) - 2) + (g(x) - 3)(g(x) + 3)| \\ \textit{(via triangle ineq)} &\leq 4|f(x) - 2| + |g(x) - 3||g(x) + 3| \\ &< 4\frac{\epsilon}{11} + 7\frac{\epsilon}{11} = \epsilon \end{aligned}$$

Assume now that $\epsilon > 1$. By what we have shown above there exists a $\epsilon > 0$ such that for all x such that $0 < |x-1| < \epsilon$,

$$|4f(x) + g(x)^2 - 17| < 1$$

Therefore,

$$|4f(x) + g(x)^2 - 17| < \epsilon$$