1 Geometry Sheet - WIP

Definition A: Standard Hyperbolic Derivatives

$$\begin{aligned}
y = \sinh(x) &\implies & y' &= \cosh(x) \\
y = \cosh(x) &\implies & y' &= \sinh(x) \\
y = \tanh(x) &\implies & y' &= \operatorname{sech}^2(x) \\
y = \operatorname{csch}(x) &\implies & y' &= -\operatorname{csch}(x)\coth(x) \\
y = \operatorname{sech}(x) &\implies & y' &= -\operatorname{sech}(x)\tanh(x) \\
y = \coth(x) &\implies & y' &= -\operatorname{csch}^2(x)
\end{aligned}$$

Definition B: Standard Hyperbolic Identities

$$\tanh(x) = \frac{\sinh(X)}{\cosh(X)} \qquad \coth(x) = \frac{\cosh(x)}{\sinh(x)}$$
$$\operatorname{sech}(x) = \frac{1}{\cosh(x)} \qquad \operatorname{csch}(x) = \frac{1}{\sinh(x)}$$

Definition C: Cross Product

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} (a_2 \cdot b_3) - (a_3 \cdot b_2) \\ (a_3 \cdot b_1) - (a_1 \cdot b_3) \\ (a_1 \cdot b_2) - (a_2 \cdot b_1) \end{pmatrix}$$

Theorem 2.2.2: Gauss Curvature on a Graph

Random equation that was in 2022 PP sols, can't find it anywhere in the book. Presumably only works on a graph, i.e.

$$x : (u, v) \mapsto \begin{pmatrix} u \\ v \\ f(u, v) \end{pmatrix}$$

Equation is as follows

$$K = \frac{f_{uu} f_{vv} - f_{uv}^2}{(1 + f_u^2 + f_v^2)^2}$$

Definition D: Closed vs Exact Forms

A form $\alpha \in \Omega^K(D)$ is said to be **closed** if $d\alpha = 0$ and is said to be **exact** if $\alpha = d\beta$ for some $\beta \in \Omega^{k-1}(D)$
Every exact form is closed, since $d^2 = 0$. The converse is not neccesarily true

Definition E: Standard Orientation

The **standard orientation** (which we always assume) is defined by

$$dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n$$

Coordinates $(y^1, \ldots, y^n)$(an ordered set) are said to be **oriented** on D iff $dy^1 \wedge \cdots \wedge dy^n$ is a positive multiple of $dx^1 \wedge \cdots \wedge dx^n$ for all $x \in D \subseteq \mathbb{R}^n$

Example 0: Wedge Product Exmaple

Find the wedge product of

$$(x^1 dx^2 - dx^3) \wedge ((x^1)^2 dx^1 \wedge dx^2 + x^3 dx^1 \wedge dx^3)$$

$$\begin{aligned}
&(x^1 dx^2 - dx^3) \wedge ((x^1)^2 dx^1 \wedge dx^2 + x^3 dx^1 \wedge dx^3) \\
&= 0 + x^1 x^3 dx^2 \wedge dx^1 \wedge dx^3 - (x^1)^2 dx^3 \wedge dx^1 \wedge dx^2 - 0 \\
&= x^1 x^3 dx^2 \wedge dx^1 \wedge dx^3 - (x^1)^2 dx^3 \wedge dx^1 \wedge dx^2 \\
&= -x^1 x^3 dx^1 \wedge dx^2 \wedge dx^3 + (x^1)^2 dx^1 \wedge dx^3 \wedge dx^2 \\
&= -x^1 x^3 dx^1 \wedge dx^2 \wedge dx^3 - (x^1)^2 dx^1 \wedge dx^2 \wedge dx^3 \\
&= -x^1 (x^3 + x^1) dx^1 \wedge dx^2 \wedge dx^3
\end{aligned}$$

Example 1: Exterior Derivative Example

Find the Exterior Derivative of the 1-form $\alpha = x^1 x^2 dx^1 + x^3 dx^2 - dx^3$. (This should turn from a 1-form to a 2-form, i.e. $\Omega^1(D) \to \Omega^2(D)$)

$$\begin{aligned}
\alpha &= d(x^1 x^2 dx^1 + x^3 dx^2 - dx^3) \\
&= d(x^1 x^2) \wedge dx^1 + dx^3 \wedge dx^2 + d(-1) \wedge dx^3 \\
&= (x^2 dx^1 + x^1 dx^2) \wedge dx^1 - dx^2 \wedge dx^3 \\
&= x^2 dx^1 \wedge dx^1 - x^1 dx^1 \wedge dx^2 - dx^2 \wedge dx^3 \\
&= -x^1 dx^1 \wedge dx^2 - dx^2 \wedge dx^3
\end{aligned}$$

Example 3: Theorema Egregium

Example - Finding Gauss Curvature on a sphere defined with the equation

$$x : (\alpha, \phi) \mapsto \begin{pmatrix} a \sin\alpha \cos\phi \\ a \sin\alpha \sin\phi \\ a \cos\alpha \end{pmatrix}$$

with the first fundamental form

$$\mathrm{I} = a^2 d\alpha^2 + a^2 \sin^2 \alpha d\phi^2$$

Pick θ^1 and θ^2 such that $\mathrm{I} = (\theta^1)^2 + (\theta^1)^2$. i.e.

$$\theta^1 = a d\alpha, \quad \theta^2 = a \sin\alpha d\phi$$

Find exterior derivatives

$$d\theta^1 = 0, \quad d\theta^2 = a \cos\alpha d\alpha \wedge d\phi$$

Substitute into the equations $d\theta^1 + \omega_2^1 \wedge \theta^2 = 0$ and $d\theta^2 + \omega_1^2 \wedge \theta^1 = 0$
Substituting into the first equation, we get

$$\begin{aligned}
\theta^1 + \omega_2^1 \wedge \theta^2 = 0 &\implies 0 + \omega_2^1 \wedge a \sin\alpha d\phi = 0 \\
&\implies (a \sin\alpha) w_2^1 \wedge d\phi = 0
\end{aligned}$$

This implies that ω_2^1 must be proportional to $d\phi$ only, so that the wedge product can evaluate to $d\phi \wedge d\phi = 0$. Therefore, $w_2^1 = \psi d\phi$ for some function ψ. Substituting into the second equation, we get

$$\begin{aligned}
\theta^2 + \omega_1^2 \wedge \theta^1 = 0 &\implies a \cos\alpha d\alpha \wedge d\phi + \omega_1^2 \wedge a d\alpha = 0 \\
&\implies a \cos\alpha d\alpha \wedge d\phi = -\omega_1^2 \wedge a d\alpha \\
&\implies a \cos\alpha d\alpha \wedge d\phi = a \omega_2^1 \wedge d\alpha \\
&\implies \cos\alpha d\alpha \wedge d\phi = \omega_2^1 \wedge d\alpha
\end{aligned}$$

This can then be solved by having $\omega_2^1 = -cos\alpha d\phi$ (the minus sign coz the wedge needs flipped) Now we can find the Gauss Curvature with the equation

$$d\omega_2^1 = K \theta^1 \wedge \theta^2$$

by substituting values for θ^1 and θ^2, and finding the exterior derivative of ω_2^1

$$\begin{aligned}
\omega_2^1 &= -\cos\alpha d\phi \\
\implies d\omega_2^1 &= \sin\alpha d\alpha \wedge d\phi
\end{aligned}$$

Compare to wedge

$$\begin{aligned}
\sin\alpha d\alpha \wedge d\phi &= K(a d\alpha) \wedge (a \sin\alpha d\phi) \\
\sin\alpha d\alpha \wedge d\phi &= K a^2 \sin\alpha d\alpha \wedge d\phi
\end{aligned}$$

Therefore, $K = \dfrac{1}{a^2}$

Theorem 1: random q

I was just doing this on latex to be neater cos the calculations were really tedious, might remove later idk

Find the exterior derivative of

$$\beta = \frac{x^1 dx^2 - x^2 dx^1}{(x^1)^2 + (x^2)^2}$$

Let f be the function

$$f = \frac{1}{(x^1)^2 + (x^2)^2} = ((x^1)^2 + (x^2)^2)^{-1}$$

Then, we have

$$\begin{aligned} df &= \frac{\partial f}{\partial x^1} dx^1 + \frac{\partial f}{\partial x^2} dx^2 \\ &= -\frac{2x^1}{((x^1)^2 + (x^2)^2)^2} dx^1 - \frac{2x^2}{((x^1)^2 + (x^2)^2)^2} dx^2 \end{aligned}$$

Returning to the original equation, rewrite as follows

$$
\begin{aligned}
\beta &= f(x^1 dx^2 - x^2 dx^1) \\
&= f x^1 dx^2 - f x^2 dx^1 \\
d\beta &= d(f x^1) \wedge dx^2 - d(f x^2) \wedge dx^1 \\
&= (x_1 df + f dx^1) \wedge dx^2 - (x^2 df + f dx^2) \wedge dx^1 \\
&= x^1 df \wedge dx^2 + \frac{dx^1 \wedge dx^2}{(x^1)^2 + (x^2)^2} - x^2 df \wedge dx^1 - \frac{dx^2 \wedge dx^1}{(x^1)^2 + (x^2)^2} \\
&= \frac{2dx^1 \wedge dx^2}{(x^1)^2 + (x^2)^2} + x^1 df \wedge dx^2 - x^2 df \wedge dx^1 \\
&= \frac{2dx^1 \wedge dx^2}{(x^1)^2 + (x^2)^2} + x^1 \left(- \frac{2x^1}{((x^1)^2 + (x^2)^2)^2} \right) dx^1 \wedge dx^2 \\
&+ x^2 \left(- \frac{2x^2}{((x^1)^2 + (x^2)^2)^2} \right) dx^2 \wedge dx^1 \\
&= \frac{2dx^1 \wedge dx^2}{(x^1)^2 + (x^2)^2} - \left(\frac{2(x^1)^2}{((x^1)^2 + (x^2)^2)^2} \right) dx^1 \wedge dx^2 \\
&+ \left(\frac{2(x^2)^2}{((x^1)^2 + (x^2)^2)^2} \right) dx^1 \wedge dx^2 \\
&= \frac{2dx^1 \wedge dx^2}{(x^1)^2 + (x^2)^2} - \frac{2((x^1)^2 + 2(x^2)^2) dx^1 \wedge dx^2}{((x^1)^2 + (x^2)^2)^2}
\end{aligned}
$$