1 Geometry Sheet - WIP

Definitions and stuff

Definition A: Standard Hyperbolic Derivatives

$$y = \sinh(x) \implies y' = \cosh(x)$$
$$y = \cosh(x) \implies y' = \sinh(x)$$
$$y = \tanh(x) \implies y' = \operatorname{sech}^2(x)$$
$$y = \operatorname{csch}(x) \implies y' = -\operatorname{csch}(x)\coth(x)$$
$$y = \operatorname{sech}(x) \implies y' = -\operatorname{sech}(x)\tanh(x)$$
$$y = \coth(x) \implies y' = -\operatorname{csch}^2(x)$$

Definition B: Standard Hyperbolic Identities

$$\tanh(x) = \frac{\sinh(X)}{\cosh(X)} \qquad \coth(x) = \frac{\cosh(x)}{\sinh(x)}$$

$$\operatorname{sech}(x) = \frac{1}{\cosh(x)} \qquad \operatorname{csch}(x) = \frac{1}{\sinh(x)}$$

Definition C: Cross Product

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} (a_2 \cdot b_3) - (a_3 \cdot b_2) \\ (a_3 \cdot b_1) - (a_1 \cdot b_3) \\ (a_1 \cdot b_2) - (a_2 \cdot b_1) \end{pmatrix}$$

Definition D: Taylor Series Expansion

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \cdots$$

Bonus: quick unit form parameterised derivations of x

- $x'(s) = T(s)$

- $x''(s) = T'(s) = \kappa(s)N(s)$

- $x'''(s) = \frac{d\kappa}{ds}(s)N(s) - \kappa^2(s)T(s) + \kappa(s)\tau(s)B(s)$

Definition E: Gauss Curvature on a Graph

Random equation that was in 2022 PP sols, can't find it anywhere in the book. Presumably only works on a graph, i.e.

$$x : (u,v) \mapsto \begin{pmatrix} u \\ v \\ f(u,v) \end{pmatrix}$$
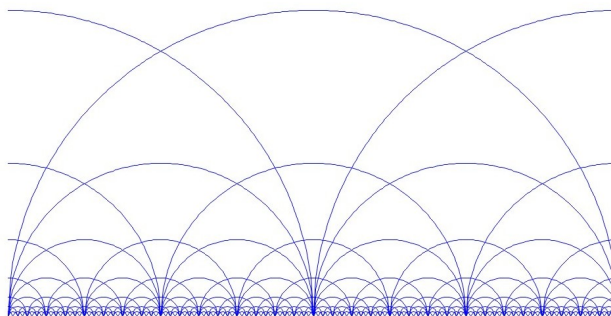
Equation is as follows

$$K = \frac{f_{uu}f_{vv} - f_{uv}^2}{(1 + f_u^2 + f_v^2)^2}$$
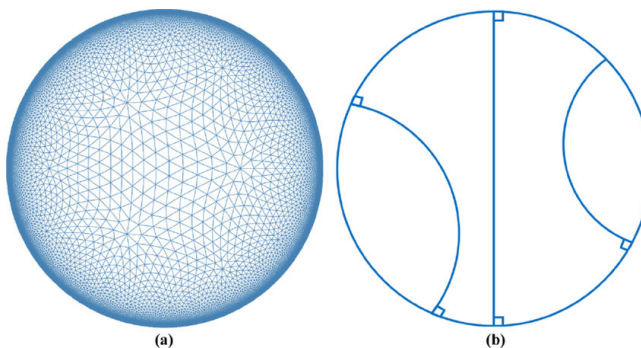
Definition F: Closed vs Exact Forms

A form $\alpha \in \Omega^K(D)$ is said to be **closed** if $d\alpha = 0$ and is said to be **exact** if $\alpha = d\beta$ for some $\beta \in \Omega^{k-1}(D)$

Every exact form is closed, since $d^2 = 0$. The converse is not neccesarily true

If α is exact, and β is closed, then $\alpha \wedge \beta$ is also exact

Definition G: Hyperbolic plane representations

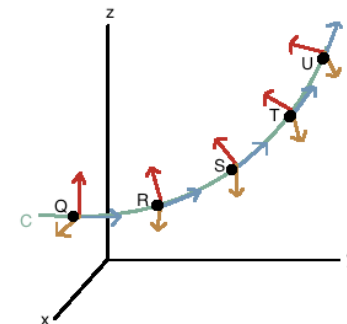Poincare Upper half plane model

Poincare Hyperbolic disk model

(a) (b)

Definition H: Standard Orientation

The **standard orientation** (which we always assume) is defined by

$$dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n$$

Coordinates $(y^1, \ldots, y^n)$ (an ordered set) are said to be **oriented** on D iff $dy^1 \wedge \cdots \wedge dy^n$ is a positive multiple of $dx^1 \wedge \cdots \wedge dx^n$ for all $x \in D \subseteq \mathbb{R}^n$

Definition I: Unit Frame Vector Diagram

The **blue vectors** represent the **unit tangent vectors** at the points Q, R, S, T, and U on **C**.
The **red vectors** represent the **unit normal vectors** at the points Q, R, S, T, and U on **C**.
The **yellow vectors** represent the **unit binormal vectors** at the points Q, R, S, T, and U on **C**.

Notice that the **unit binormal vectors** are perpendicular to both the **unit tangent vectors** and the **unit normal vectors**.

Theorem nah i'd win: nah i'd win

Examples

Example 1: Wedge Product Exmaple

Find the wedge product of

$$(x^1 dx^2 - dx^3) \wedge ((x^1)^2 dx^1 \wedge dx^2 + x^3 dx^1 \wedge dx^3)$$

$$(x^1 dx^2 - dx^3) \wedge ((x^1)^2 dx^1 \wedge dx^2 + x^3 dx^1 \wedge dx^3)$$

$$= 0 + x^1 x^3 dx^2 \wedge dx^1 \wedge dx^3 - (x^1)^2 dx^3 \wedge dx^1 \wedge dx^2 - 0$$

$$= x^1 x^3 dx^2 \wedge dx^1 \wedge dx^3 - (x^1)^2 dx^3 \wedge dx^1 \wedge dx^2$$

$$= -x^1 x^3 dx^1 \wedge dx^2 \wedge dx^3 + (x^1)^2 dx^1 \wedge dx^3 \wedge dx^2$$

$$= -x^1 x^3 dx^1 \wedge dx^2 \wedge dx^3 - (x^1)^2 dx^1 \wedge dx^2 \wedge dx^3$$

$$= -x^1(x^3 + x^1) dx^1 \wedge dx^2 \wedge dx^3$$

Example 2: Exterior Derivative Example

Find the Exterior Derivative of the 1-form $\alpha = x^1 x^2 dx^1 + x^3 dx^2 - dx^3$. (This should turn from a 1-form to a 2-form, i.e. $\Omega^1(D) \to \Omega^2(D)$)

$$\alpha = d(x^1 x^2 dx^1 + x^3 dx^2 - dx^3)$$

$$= d(x^1 x^2) \wedge dx^1 + dx^3 \wedge dx^2 + d(-1) \wedge dx^3$$

$$= (x^2 dx^1 + x^1 dx^2) \wedge dx^1 - dx^2 \wedge dx^3$$

$$= x^2 dx^1 \wedge dx^1 - x^1 dx^1 \wedge dx^2 - dx^2 \wedge dx^3$$

$$= -x^1 dx^1 \wedge dx^2 - dx^2 \wedge dx^3$$

Example 3: Pullback Example

Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be given by

$$f(u^1, u^2) = ((u^1)^2, (u^2)^2, u^1 u^2))$$

where we use the Cartesian coordinates (u^1, u^2) on $\mathbb{R}^2$ and (x^1, x^2, x^3) on $\mathbb{R}^3$. Calculate the pullback $f^* \rho$ for the form

$$\rho = x^1 dx^2 \wedge dx^3$$

$$f^* \rho = (u^1)^2 d((u^2)^2) \wedge d(u^1 u^2)$$

$$= (u^1)^2 2u^2 du^2 \wedge (u^2 du^1 + u^1 du^2)$$

$$= -2(u^1 u^2)^2 du^1 \wedge du^2$$

Example 4: Theorema Egregium

Example - Finding Gauss Curvature on a sphere defined with the equation

$$x : (\alpha, \phi) \mapsto \begin{pmatrix} a \sin \alpha \cos \phi \\ a \sin \alpha \sin \phi \\ a \cos \alpha \end{pmatrix}$$

with the first fundamental form

$$\mathrm{I} = a^2 d\alpha^2 + a^2 \sin^2 \alpha d\phi^2$$

Pick θ^1 and θ^2 such that $\mathrm{I} = (\theta^1)^2 + (\theta^1)^2$. i.e.

$$\theta^1 = a d\alpha, \quad \theta^2 = a \sin \alpha d\phi$$

Find exterior derivatives

$$d\theta^1 = 0, \quad d\theta^2 = a \cos \alpha d\alpha \wedge d\phi$$

Substitute into the equations $d\theta^1 + \omega_2^1 \wedge \theta^2 = 0$ and $d\theta^2 + \omega_1^2 \wedge \theta^1 = 0$
Substituting into the first equation, we get

$$\theta^1 + \omega_2^1 \wedge \theta^2 = 0 \implies 0 + \omega_2^1 \wedge a \sin \alpha d\phi = 0$$

$$\implies (a \sin \alpha) w_2^1 \wedge d\phi = 0$$

This implies that ω_2^1 must be proportional to $d\phi$ only, so that the wedge product can evaluate to $d\phi \wedge d\phi = 0$. Therefore, $w_2^1 = \psi d\phi$ for some function ψ. Substituting into the second equation, we get

$$\theta^2 + \omega_1^2 \wedge \theta^1 = 0 \implies a \cos \alpha d\alpha \wedge d\phi + \omega_1^2 \wedge a d\alpha = 0$$

$$\implies a \cos \alpha d\alpha \wedge d\phi = -\omega_1^2 \wedge a d\alpha$$

$$\implies a \cos \alpha d\alpha \wedge d\phi = a \omega_2^1 \wedge d\alpha$$

$$\implies \cos \alpha d\alpha \wedge d\phi = \omega_2^1 \wedge d\alpha$$

This can then be solved by having $\omega_2^1 = -\cos\alpha d\phi$ (the minus sign coz the wedge needs flipped) Now we can find the Gauss Curvature with the equation

$$d\omega_2^1 = K \theta^1 \wedge \theta^2$$

by substituting values for θ^1 and θ^2, and finding the exterior derivative of ω_2^1

$$\omega_2^1 = -\cos \alpha d\phi$$

$$\implies d\omega_2^1 = \sin \alpha d\alpha \wedge d\phi$$

Compare to wedge

$$\sin \alpha d\alpha \wedge d\phi = K(a d\alpha) \wedge (a \sin \alpha d\phi)$$

$$\sin \alpha d\alpha \wedge d\phi = K a^2 \sin \alpha d\alpha \wedge d\phi$$

Therefore, $K = \dfrac{1}{a^2}$

Example 5: random q

I was just doing this on latex to be neater cos the calculations were really tedious, might remove later idk
Find the exterior derivative of

$$\beta = \frac{x^1 dx^2 - x^2 dx^1}{(x^1)^2 + (x^2)^2}$$

Let f be the function

$$f = \frac{1}{(x^1)^2 + (x^2)^2} = ((x^1)^2 + (x^2)^2)^{-1}$$

Then, we have

$$df = \frac{\partial f}{\partial x^1} dx^1 + \frac{\partial f}{\partial x^2} dx^2$$

$$= -\frac{2x^1}{((x^1)^2 + (x^2)^2)^2} dx^1 - \frac{2x^2}{((x^1)^2 + (x^2)^2)^2} dx^2$$

Returning to the original equation, rewrite as follows

$$\beta = f(x^1 dx^2 - x^2 dx^1)$$

$$= f x^1 dx^2 - f x^2 dx^1$$

$$d\beta = d(f x^1) \wedge dx^2 - d(f x^2) \wedge dx^1$$

$$= (x_1 df + f dx^1) \wedge dx^2 - (x^2 df + f dx^2) \wedge dx^1$$

$$= x^1 df \wedge dx^2 + \frac{dx^1 \wedge dx^2}{(x^1)^2 + (x^2)^2} - x^2 df \wedge dx^1 - \frac{dx^2 \wedge dx^1}{(x^1)^2 + (x^2)^2}$$

$$= \frac{2 dx^1 \wedge dx^2}{(x^1)^2 + (x^2)^2} + x^1 df \wedge dx^2 - x^2 df \wedge dx^1$$

$$= \frac{2 dx^1 \wedge dx^2}{(x^1)^2 + (x^2)^2} + x^1 \left(-\frac{2x^1}{((x^1)^2 + (x^2)^2)^2} \right) dx^1 \wedge dx^2$$

$$+ x^2 \left(-\frac{2x^2}{((x^1)^2 + (x^2)^2)^2} \right) dx^2 \wedge dx^1$$

$$= \frac{2 dx^1 \wedge dx^2}{(x^1)^2 + (x^2)^2} - \left(\frac{2(x^1)^2}{((x^1)^2 + (x^2)^2)^2} \right) dx^1 \wedge dx^2$$

$$+ \left(\frac{2(x^2)^2}{((x^1)^2 + (x^2)^2)^2} \right) dx^1 \wedge dx^2$$

$$= \frac{2 dx^1 \wedge dx^2}{(x^1)^2 + (x^2)^2} - \frac{2((x^1)^2 + 2(x^2)^2) dx^1 \wedge dx^2}{((x^1)^2 + (x^2)^2)^2}$$