1 Vector Spaces

1.1 Fields and Vector Spaces

Definition 1.1.1: Definition of a field

A field F is a set with two functions

- Addition: $+ : F \times F \to F$, $(\lambda, \mu) \mapsto \lambda + \mu$
- Multiplication: $\cdot : F \times F$, $(\lambda, \mu) \mapsto \lambda\mu$

which satisfy the following axioms:

1. $(F, +)$ is an abelian group F^+, with identity 0_F
2. $(F\backslash\{0_F\}, \cdot)$ is an abelian group $F^\times$, with identity 1_F
3. **Distributive law**: For all a, b, and c in F, we have
$$a(b + c) = ab + ac \in F$$

and the following lemmas:

1. The elements 0_F and 1_F of F are distinct
2. For all $a \in F$, $a \cdot 0_F = 0_F$ and $0_F \cdot a = 0_F$
3. Multiplication in F is associative, and 1_F is an identity element

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A **vector space** V **over a field** F is a pair consisting of an abelian group $V = (V, \dot{+})$ and a mapping

$$F \times V \to V : (\lambda, \vec{v}) \mapsto \lambda\vec{v}$$

s.t. for all $\lambda, \mu \in F$ and $\vec{v}, \vec{w} \in V$ the following identities hold:

- **Distributivity 1**: $\lambda(\vec{v}\dot{+}\vec{w}) = \lambda\vec{v}\dot{+}\lambda\vec{w}$
- **Distributivity 2**: $(\lambda + \mu)\vec{v} = \lambda\vec{v}\dot{+}\mu\vec{v}$
- **Associativity**: $\lambda(\mu\vec{v}) = (\lambda\mu)\vec{v}$
- **Identity**: $1\vec{v} = \vec{v}$

and so do the following lemmas:

1. If V is a vector space and $\vec{v} \in V$, then $0\vec{v} = \vec{0}$
2. If V is a vector space and $\vec{v} \in V$, then $(-1)\vec{v} = -\vec{v}$
3. If V is a vector space over a field F, then $\lambda\vec{0} = \vec{0}$ for all $\lambda \in F$. Furthermore, if $\lambda\vec{v} = \vec{0}$ then either $\lambda = 0$ or $\vec{v} = \vec{0}$

1.2 Working with Vector Spaces

Definition 1.2.1: Cartesian Product of n sets

$$X_1 \times \cdots \times X_n := \{(x_1, \ldots, x_n) : x_i \in X_i \text{ for } 1 \leq i \leq n\}$$

The elements of a product are called ***n*-tuples**. An individual entry $x_i = (x_1, \ldots, x_n)$ is called a **component**.
There are special mappings called **projections** for a cartesian product:

$$\mathrm{pr}_i : X_1 \times \cdots \times X_n \to X_i$$
$$(x_1, \ldots, x_n) \mapsto x_i$$

The cartesian product of n copies of a set X is written in short as: X^n

Definition 1.2.2: Vector Subspace

A subset U of a vector space V is called a **vector subspace** or **subspace** if U contains the zero vector, and whenever $\vec{u}, \vec{v} \in U$ and $\lambda \in F$ we have $\vec{u} + \vec{v} \in U$ and $\lambda\vec{u} \in U$

Definition 1.2.3: Spans and Linear Independence

Let $T \subset V$ for some vector space V over a field F. Then amongus all subspaces of V that include T there is a smallest subspace

$$\langle T \rangle = \langle T \rangle_F \subseteq V$$

"the set of all vectors $\alpha_1\vec{v}_1 + \cdots + \alpha_r\vec{v}_r$ with $\alpha_1, \ldots, \alpha_r \in F$ and $\vec{v}_1, \ldots, \vec{v}_r \in T$, together with the zero vector in the case $T = \emptyset$"

Terminology Dump

- An expression of the form $\alpha_1\vec{v}_1 + \cdots + \alpha_r\vec{v}_r$ is called a **linear combination** of vectors $\vec{v}_1, \ldots, \vec{v}_r$
- The smallest vector subspace $\langle T \rangle \subseteq V$ containing T is called the **vector subspace generated by** T or the vector subspace **spanned by** T or even the **span of** T
- If we allow the zero vector to be the "empty linear combination of $r = 0$ vectors", then the span of T is exactly the set of all linear combinations of vectors from T
- A subset of a vector space that spans the entire space is called a **generating** or **spanning set**. A vector space that has a finite generating set is said to be **finitely generated**

Linear Independence

A subset L of a vector space V is called **linearly independent** if for all pairwise different vectors $\vec{v}_1, \ldots, \vec{v}_r \in L$ and arbitrary scalars $\alpha, \ldots, \alpha_r \in F$,

$$a_1\vec{v}_1 + \cdots + \alpha_r\vec{v}_r = \vec{0} \implies a_1 = \cdots = \alpha_r = 0$$

A subset L of a vector space V is called **linearly dependent** if it is not linearly independent (duh..). This means there exists pairwise different vectors $\vec{v}j_1, \ldots, \vec{v}_r \in L$ and scalars $\alpha_1, \ldots, \alpha_r \in F$, not all zero, such that $\alpha_1\vec{v}_1 + \cdots \alpha_r\vec{v}_r = \vec{0}$

1.3 Linear Independence and Bases

Definition 1.3.1: Basis of a Vector Space

A **basis of a vector space** V is a linearly independent generating set in V

Example 1.3.2: Standard Basis

Let F be a field and $n \in \mathbb{N}$. We consider the following vectors in F^n

$$\vec{e}_i = (0, \ldots, 0, 1, 0, \ldots, 0)$$

with one 1 in the i-th place and zero everywhere else. Then $\vec{e}_1, \ldots, \vec{e}_n$ form an ordered basis of F^n, the so-called **standard basis of** F^n

Theorem 1.3.3: Linear combinations of basis elements

Let F be a field, V a vector space over F and $\vec{v}_1, \ldots, \vec{v}_r \in V$ vectors. The family $(\vec{v}_i)_{1 \leq i \leq r}$ is a basis of V if and only if the following "evaluation" mapping

$$\psi : F^r \to V$$
$$(\alpha_1, \ldots, a_r) \mapsto a_1\vec{v}_1 + \cdots + \alpha_r\vec{v}_r$$

is a bijection
If we label our ordered family by $\mathcal{A} = (\vec{v}_1, \ldots, \vec{v}_r)$, then we done the above mapping by

$$\psi = \psi_{\mathcal{A}} : F^r \to V$$

Theorem 1.3.4: Characterisations of Bases

The following are equivalent for a subset E of a vector space V:

1. E is a basis, i.e. a linearly independent generating set
2. E is minimal among all generating sets, meaning that $E\backslash\{\vec{v}\}$ does not generate V, for any $\vec{v} \in E$
3. E is maximal among all linearly independent subsets, meaning that $E \cup \{\vec{v}\}$ is linearly dependent for any $\vec{v} \in V$

Corrollary: Let V be a finitely generated vector space over a field F. Then V has a finite basis

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Basis Characterisation Variant

1. If $L \subset V$ is a linearly independent subset and E is minimal amongst all generating sets of V with the property that $L \subseteq E$, then E is a basis.
2. If $E \subseteq V$ is a generating set and if L is maximal amongst all linearly independent sets of V with the property $L \subseteq E$, then L is a basis.

Definition 1.3.5: Free Vector Space

Let X be a set and F a field. The set $\mathrm{Maps}(X, F)$ of all mappings $f : X \to F$ becomes an F-vector space with the operations of pointwise addition and multiplication by a scalar. The subset of all mappings which send almost all elements of X to zero is a vector subspace

$$F\langle X \rangle \subseteq \mathrm{Maps}(X, F)$$

This subspace is called the **free vector space on the set** X

Theorem 1.3.6: Variant of Linear Combinations

Let F be a field, V be an F-vector space and $(\vec{v}_i)_{i \in I}$ a family of vectors from the vector space V. The following are equivalent:

1. The family $(\vec{v}_i)_{i \in I}$ is a basis for V
2. For each $\vec{v} \in V$ there is precisely one family $(a_i)_{i \in I}$ of elements of F, almost all which are zero and such that
$$\vec{v} = \sum_{i=I} a_i\vec{v}_i$$

1.4 Dimension of a Vector Space

Theorem 1.4.1: Fundamental Estimate of LinAlg

No linearly independent subset of a given vector has more elements than a generating set. Thus if V is a vector space, $L \subset V$ a linearly independent subset and $E \subseteq V$ a generating set, then

$$|L| \leq |E|$$

Theorem 1.4.2: Steinitz Exchange Theorem

Let V be a vector space, $L \subset V$ a finite linearly independent subset and $E \subseteq V$ a generating set. Then there is an injection $\phi : L \hookrightarrow E$ such that $(E \backslash \phi(L)) \cup L$ is also a generating set for V

Let V be a vector space, $M \subseteq V$ a linearly independent subset, and $E \subseteq V$ a generating subset, such that $M \subseteq E$. If $\vec{w} \in V \backslash M$ is a vector $\notin M$ such that $M \cup \{\vec{w}\}$ is linearly independent, then there exists $\vec{e} \in E \backslash M$ such that $(E \backslash \{\vec{e}\}) \cup \{\vec{w}\}$ is a generating set

Theorem 1.4.3: Cardinality of Bases

Let V be a finitely generated vector space.

1. V has a finite basis
2. V cannot have an infinite basis
3. Any two bases of V have the same number of elements

Definition 1.4.4: Dimension of a Vector Space

The cardinality of a basis of a finitely generated vector space V is called the **dimension** of V, written $\dim V$. If F is a field, and we want to denote that we mean dimension as an F-vector space, then we write $\dim_F V$. If the vector space is not finitely generated, then we say $\dim V = \infty$ and call V **infinite dimensional**.

Theorem 1.4.5: Dimension Theorems

Cardinality Criterion for Bases

1. Each linearly independent subset $L \subset V$ has at most $\dim V$ elements, and if $|L| = \dim V$ then L is a basis
2. Each generating set $E \subseteq V$ has at least $\dim V$ elements, and if $|E| = \dim V$ then E is a basis

Dimension Estimate for Vector Subspaces: A proper vector subspace of a finite dimensional vector space has itself a strictly smaller dimension

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If $U \subseteq V$ is a vector subspace of an arbitrary vector space, then we have $\dim U \leq \dim V$ and if we have $\dim U = \dim V < \infty$ then it follows that $U = V$

1.5 Linear Mappings

Definition 1.5.1: Linear Mappings

Let V, W be vector spaces over a field F. A mapping $f : V \to W$ is called **linear**, or **F-linear**, or even a **homomorphism of F-vector spaces** if for all $\vec{v}_1, \vec{v}_2 \in V$ and $\lambda \in F$ we have

$$f(\vec{v}_1 + \vec{v}_2) = f(\vec{v}_1) + f(\vec{v}_2)$$
$$f(\lambda \vec{v}_1) = \lambda f(\vec{v}_1)$$

A bijective linear mapping is called an **isomorphism** of vector spaces. If there is an isomorphism between two vector spaces, we call them **isomorphic**. A homomorphism $V \to V$ is called an **endomorphism** of V. An isomorphism $V \to V$ is called an **automorphism** of V

Two vector subspaces V_1, V_2 of a vector space V are called **complementary** if addition defines a bijection

$$V_1 \times V_2 \xrightarrow{\sim} V$$

something about direct sums

Theorem 1.5.2: Classifying VecSpaces by Dimension

Let n be a natural number. Then a vector space over a field F is isomorphic to F^n iff it has dimension n

Theorem 1.5.3: Linear Mapping and Bases

Let V, W be vector spaces over a field F. The set of all homomorphisms from V to W is denoted by

$$\mathrm{Hom}_F(V, W) = \mathrm{Hom}(V, W) \subseteq \mathrm{Maps}(V, W)$$

Let $B \subset V$ be a basis. Then restriction of a mapping gives a bijection

$$\mathrm{Hom}_F(V, W) \xrightarrow{\sim} \mathrm{Maps}(B, W)$$
$$f \mapsto f|_B$$

Theorem 1.5.4: Inverse Mappings

1. Every injective linear mapping $f : V \hookrightarrow W$ has a **left inverse**, or a linear mapping $g : W \to V$ s.t. $g \circ f = \mathrm{id}_V$
2. Every surjective linear mapping $f : V \twoheadrightarrow W$ has a **right inverse**, or a linear mapping $G : W \to V$ s.t. $f \circ g = \mathrm{id}_W$

Definition 1.5.5: Image and Kernel of a map

The **image** of a linear mapping $f : V \to W$ is the subset $\mathrm{im}(f) = f(V) \subseteq W$. It is a vector subspace of W. The preimage of the zero vector of a linear mapping $f : V \to W$ is denoted by:

$$\ker(f) := f^{-1}(0) = \{v \in V : f(v) = 0\}$$

and is called the **kernel** of the linear mapping f. The kernel is a subspace of V

Mini lemma: A linear mapping is injective iff its kernel is zero

Theorem 1.5.6: Rank-Nullity / Dimension Theorem

Let $f : V \to W$ be a linear mapping between vector spaces. Then:

$$\dim V = \dim(\ker f) + \dim(\mathrm{im}\, f)$$

Dimension of $\mathrm{im}\, f$= **rank** of f, dimension of $\ker f$ = **nullity** of f

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Let V be a vector space, and $U, W \subseteq V$ vector subspaces. Then

$$\dim(U + W) + \dim(U \cap W) = \dim U + \dim W$$

2 Linear Mappings and Matrices

2.1 Linear Mappings $F^m \to F^n$ and Matrices

Theorem 2.1.1: Linear Maps $F^m \to F^n$ and Matrices

Let F be a field and let $m, n \in \mathbb{N}$. There is a bijection between the space of linear mappings $F^m \to F^n$ and the set of matrices with n rows, m columns, and entries in F:

$$M : \mathrm{Hom}_F(F^m, F^n) \xrightarrow{\sim} \mathrm{Mat}(n \times m; F)$$
$$f \mapsto [f]$$

This attaches to each linear mapping f its **representing matrix** $M(f) := [f]$. The columns of this matrix are the images under f of the standard basis elements of F^m

$$[f] := (f(\vec{e}_1) | f(\vec{e}_2) | \mid \cdots \mid f(\vec{e}_m))$$

Definition 2.1.2: Matrix Multiplication

Let $n, m, \ell \in \mathbb{N}$, F a field, and let $A \in \mathrm{Mat}(n \times m; F)$ and $B \in \mathrm{Mat}(m \times \ell; F)$ be matrices. The **product** $A \circ B = AB \in \mathrm{Mat}(n \times \ell; F)$ is the matrix defined by

$$(AB)_{ik} = \sum_{j=1}^{m} A_{ij} B_{jk}$$

Theorem 2.1.3: Composition of maps to products

Let $g : F^\ell \to F^m$ and $f : F^m \to F^n$ be linear mappings. The representing matrix of their composition is the product of their representing matrices:

$$[f \circ g] = [f] \circ [g]$$

Theorem 2.1.4: Calculating with Matrices

- (A + A')B = AB + A'B
- A(B + B') + AB + AB'
- IB = B
- AI = A
- (AB)C = A(BC)

2.2 Matrix Definitions

Definition 2.2.1: Big def-thm pairs

Def: A matrix A is called **invertible** if there exists matrices B and C such that $BA = I$ and $AC = I$

Thm: Invertible Equivalence

1. There exists a square matrix B such that $BA = I$
2. There exists a square matrix C such that $AC = I$
3. The square matrix A is invertible

Def: An **elementary matrix** is any square matrix that differs from the identity matrix in at least one entry

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Thm: Every square matrix with entries in a field can be written as a product of elementary matrices

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Def: Any matrix whose only non-zero entries lie on the diagonal, and which has first 1's along the diagonal and then 0's, is said to be in **Smith Normal Form**

Thm: For each matrix $A \in \text{Mat}(n \times m; F)$ there exist invertible matrices P and Q such that PAQ is a matrix in Smith Normal Form

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Thm: Let $f : V \to W$ be a linear map between finite dim. F-vector spaces. There exists two ordered bases $\mathcal{A}$ of V, and $\mathcal{B}$ of W s.t. the representing matrix ${}_{\mathcal{B}}[f]_{\mathcal{A}}$ has zero entries everywhere except possibly on the diagonal, and along the diagonal there are 1's first, followed by 0's

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Def: The **column rank** of a matrix $A \in \text{Mat}(n \times m; F)$ is the dimension of the subspace of F^n generated by the columns of A. Similarly, the **row rank** of A is the dimension of the subspace of F^m generated by the rows of A.

Thm: The column and row rank of any matrix are equal

Def: Since they are both the same, "column" and "row" can be omitted for the **rank of a matrix**, written as $\text{rk}\, A$. If the rank is equal to the no. of rows/columns, then the matrix has **full rank**

Def: The **trace** of a square matrix is defined to be the sum of its diagonal entries, denoted by $\text{tr}(A)$

2.3 Abstract Linear Mappings and Matrices

Theorem 2.3.1: Representing Matrices

Let F be a field, V and W vector spaces over F with ordered bases $\mathcal{A} = (\vec{v}_1, \dots, \vec{v}_m)$ and $\mathcal{B} = (\vec{w}_1, \dots, \vec{w}_n)$. Then to each linear mapping $f : V \to W$ we associate a **representing matrix** ${}_{\mathcal{B}}[f]\mathcal{A}$ whose entries a_{ij} are defined by the identity

$$f(\vec{v}_j) = a_{1j}\vec{w}_1 + \cdots + a_{nj}\vec{w}_n \in W$$

This makes a bijection, which is an isomorphism of vector spaces:

$$M_{\mathcal{B}}^{\mathcal{A}} : \text{Hom}_F(V, W) \xrightarrow{\sim} \text{Mat}(n \times m; F)$$
$$f \mapsto {}_{\mathcal{B}}[f]_{\mathcal{A}}$$

Theorem 2.3.2: Repr. Mat of Compositions

Let F be a field and U, V, W finite dimensional vector spaces over kF with ordered bases $\mathcal{A}, \mathcal{B}, \mathcal{C}$. If $f : U \to V$ and $g : V \to W$ are linear mappings, then the representing matrix of the composition $g \circ f : U \to W$ is the matrix product of the representing matrices of f and g:

$${}_{\mathcal{C}}[g \circ f]_{\mathcal{A}} = {}_{\mathcal{C}}[g]_{\mathcal{B}} \circ {}_{\mathcal{B}}[f]_{\mathcal{A}}$$

Definition 2.3.3: Representation of a vector

Let V be a finite dimensional vector space with an ordered basis $\mathcal{A} = (\vec{v}_1, \dots, \vec{v}_m)$. We'll denote the inverse to the bijection in 1.3.3 "$\Phi_{\mathcal{A}} : F^m \xrightarrow{\sim} V, (\alpha_1, \dots, \alpha_m)^T \mapsto \alpha_1\vec{v}_1 + \cdots + \alpha_m\vec{v}_m$" by

$$\vec{v} \mapsto {}_{\mathcal{A}}[\vec{v}]$$

The column vector ${}_{\mathcal{A}}[\vec{v}]$ is called the **representation of the vector $\vec{v}$ with respect to the basis $\mathcal{A}$**

Thm: Representation of the Image of a Vector: Let V, W be finite dim. vector spaces over F with ordered bases $\mathcal{A}, \mathcal{B}$ and let $f : V \to W$ be a linear mapping. The following holds for $\vec{v} \in V$:

$${}_{\mathcal{B}}[f(\vec{v})] = {}_{\mathcal{B}}[f]_{\mathcal{A}} \circ {}_{\mathcal{A}}[\vec{v}]$$

2.4 Change of a Matrix by Change of Basis

Definition 2.4.1: Change of Basis Matrix

Let $\mathcal{A} = (\vec{v}_1, \dots, \vec{v}_n)$ and $\mathcal{B} = (\vec{w}_1, \dots, \vec{w}_n)$ be ordered basies of the same F-vector space V. Then the matrix representing the identity mapping w.r.t. these bases

$${}_{\mathcal{B}}[\text{id}_V]_{\mathcal{A}}$$

is called a **change of basis matrix**. By definition, its entries are given by the equalities $\vec{v}_j = \sum_{i=1}^{n} a_{ij}\vec{w}_i$

Theorem 2.4.2: Change of Basis

Let V and W be finite dimensional vector spaces over F and let $f : V \to W$ be a linear mapping. Suppose that $\mathcal{A}, \mathcal{A}'$ are ordered bases of V and $\mathcal{B}, \mathcal{B}'$ are ordered bases of W. Then

$${}_{\mathcal{B}'}[f]_{\mathcal{A}'} = {}_{\mathcal{B}'}[\text{id}_W]_{\mathcal{B}} \circ {}_{\mathcal{B}}[f]_{\mathcal{A}} \circ {}_{\mathcal{A}}[\text{id}_V]_{\mathcal{A}'}$$

Let V be a finite dimensional vector space and let $f : V \to V$ be an endomorphim of V. Suppose that $\mathcal{A}, \mathcal{A}'$ are ordered bases of V. Then

$${}_{\mathcal{A}'}[f]_{\mathcal{A}'} = {}_{\mathcal{A}}[\text{id}_V]_{\mathcal{A}'}^{-1} \circ {}_{\mathcal{A}}[f]_{\mathcal{A}} \circ {}_{\mathcal{A}}[\text{id}_V]_{\mathcal{A}'}$$

3 Rings

I can't be bothered doing changes of basis and stuff, time for something more interesting :D

3.1 Ring basics

Definition 3.1.1: Definition of a Ring

A **ring** is a set with two operations $(\mathbb{R}, +, \cdot)$ that satisfy:

1. $(R, +)$ is an abelian group
2. $(R, \cdot)$ is a **monoid** - this means that the second operation $\cdot : R \times R \to R$ is associative and that there is an **identity element** $1 = 1_R \in R$, often just called the identity, with the property that $1 \cdot a = a \cdot 1 = a$ for all $a \in R$.
3. The distributive laws hold, meaning that for all $a, b, c \in R$,
$$a \cdot (b + c) = (a \cdot b) + (a \cdot c))$$
$$(a + b) \cdot c = (a \cdot c) + (b \cdot c)$$

The two operations are called **addition** and **multiplication** in our ring. A ring in which multiplication, that is $a \cdot b = b \cdot a$ for all $a, b \in R$, is a **commutative ring**

Note: We'll call the element $1 \in R$ as the identity element of the monoid $(R, \cdot)$, and we call the additive identity of $(R, +)$ zero, written as 0_R or 0

Example: We can define the **null ring** or **zero ring** as a ring where R is a single ement set, e.g. $\{0\}$, with the operations $0 + 0 = 0$ and $0 \times 0 = 0$. We will call any ring that isn't the zero ring a **non-zero ring**

Example 3.1.2: Modulo Rings

Let $m \in \mathbb{Z}$ be an integer. Then the set of **integers modulo** m, written
$$\mathbb{Z}/m\mathbb{Z}$$
is a ring. The elements of $\mathbb{Z}/m\mathbb{Z}$ consist of **congruence classes** of integers modulo m - that is the elements are the subsets T of $\mathbb{Z}$ of the form $T = a + m\mathbb{Z}$ with $a \in \mathbb{Z}$. Think of these as the set of integers that have the same remainder when you divide them by m. I denote the above congruence class by $\overline{a}$. Obviously $\overline{a} = \overline{b}$ is the same as $a - b \in m\mathbb{Z}$, and often I'll write
$$a \equiv b \mod m$$

3.2 Linking Rings to Fields and Further Properties

Definition 3.2.1: Ring definition of a field

A **field** is a non-zero commutative ring F in which every non-zero element $a \in F$ has an inverse $a^{-1} \in F$, that is an element a^{-1} with the property that $a \cdot a^{-1} = a^{-1} \cdot a = 1$

Theorem 3.2.2: Prime property of fields

Let m be a positive integer. The commutative ring $\mathbb{Z}/m\mathbb{Z}$ is a field if and only if m is prime.

Theorem 3.2.3: Lemmas for multiplying

Let R be a ring and let $a, b \in R$. Then

1. $0a = 0 = a0$
2. $(-a)b = -(ab) = a(-b)$
3. $(-a)(-b) = ab)$

Definition 3.2.4: Multiples of an abelian group

Let $m \in \mathbb{Z}$. The **m-th multiple ma of an element** $a$in an abelian group R is:
$$ma = \underbrace{a + a + \cdots + a}_{m \text{ terms}} \quad \text{if} m > 0$$
$0a = 0$ and negative multiples are defined by $(-m)a = -(ma)$

Theorem 3.2.5: Lemmas for multiples

Let R be a ring, let $a, b \in R$ and let $m, n \in \mathbb{Z}$. Then:

1. $m(a + b) = ma + mb$
2. $(m + n)a = ma + na$
3. $m(na) = (mn)a$
4. $m(ab) = (ma)b = a(mb)$
5. $(ma)(nb) = (mn)(ab)$

Definition 3.2.6: Unit of a ring

Let R be a ring. An element $a \in R$ is called a **unit** if it is *invertible* in R or in other words *has a multiplicative inverse in* R, meaning that there exists $a^{-1} \in R$ such that
$$aa^{-1} = 1 = a^{-1}a$$

Theorem 3.2.7

The set $R^\times$ of units in a ring R forms a group under multiplication

Definition 3.2.8: zero-divisors of a ring

In a ring R, a non-zero element a is called a **zero-divisor** or **divisor of zero** if there exists a non-zero element b such that either $ab = 0$ or $ba = 0$.

Theorem 3.2.9: Cancellation Law

Let R be an *integral domain* and let $a, b, c \in R$. If $ab = ac$ and $a \neq 0$ then $b = c$

Theorem 3.2.10: Prime Property for Integral Domains

Let m be a natural number. Then $\mathbb{Z}/m\mathbb{Z}$ is an integral domain if and only if m is prime.

Theorem 3.2.11

Every **finite** integral domain is a field.

3.3 Polynomials

Definition 3.3.1: Polynomial

Let R be a ring. A **polynomial over** R is an expression of the form
$$P = a_0 + a_1 X + a_2 X^2 + \cdots + a_m X^m$$
for some non-negative integer m and elements $a_i \in R$ for $0 \leq i \leq m$. The set of all polynomials over R is denoted by $R[X]$. In the case where a_m is non-zero, the polynomial P has **degree** m, (written $\deg(P)$), and a_m is its **leading coefficient**. When the leading coefficient is 1 the polynomial is a **monic polynomial**. A polynomial of degree one is called **linear**, a polynomial od degree two is called **quadractic**, and a polynomial of degree three is called **cubic**.

Definition 3.3.2: Ring of Polynomials

The set $R[X]$ becomes a ring called the **ring of polynomials with coefficients in** R**, or over** R. The zero and the identity of $R[X]$ are the zero and identity of R, respectively.

Theorem 3.3.3: Zero-Divisors of a Polynomial Ring

If R is a ring with no zero-divisors, then $R[X]$ has no zero-divisors and $\deg(PQ) = \deg(P) + \deg(Q)$ for non-zero $P, Q \in R[X]$.

If R is an integral domain, then so is $R[X]$

Theorem 3.3.4: Division and Remainder

Let R be an integral domain and let $P, Q \in R[X]$ with Q monic. Then there exists unique $A, B \in R[X]$ such that $P = AQ + B$ and $\deg(B) < \deg(Q)$ or $B = 0$

Definition 3.3.5: Formal definition of a function

Let R be a commutative ring and $P \in R[X]$ a polynomial. Then the polynomial P can be **evaluated** at the element $\lambda \in R$ to produce $P(\lambda)$ by replacing the powers of X in the polynomial P by the corresponding powers of λ. In this way we have a mapping

$$R[X] \to \mathrm{Maps}(R, R)$$

This is the precise mathematical description of thinking of a polynomial as a function. An element $\lambda \in R$ is a **root** of P is $P(\lambda) = 0$

Theorem 3.3.6: Roots of a Polynomial

Let R be a commutative ring, let $\lambda \in R$ and $P(X) \in R[X]$. Then λ is a root of $P(X)$ if and only if $(X - \lambda)$ divides $P(X)$

Theorem 3.3.7: Degrees of Polynomial Roots

Let R be a field, or more generally an integral domain. Then a non-zero polynomial $P \in R[X] \backslash \{0\}$ has at most $\deg(P)$ roots in R

Definition 3.3.8: Algebraically closed fields

A field F is **algebraically closed** if each non-constant polynomial $P \in F[X] \backslash F$ with coefficients in our field has a root in our field F

Theorem 3.3.9: Fundamental Theorem of Algebra

The field of complex numbers $\mathbb{C}$ is algebraically closed.

Theorem 3.3.10: Linear Factors of Closed Fields

If F is an algebraically closed field, then every non-zero polynomial $P \in F[X] \backslash \{0\}$ **decomposes into linear factors**

$$P = c(X - \lambda_1) \cdots (X - \lambda_n)$$

with $n \geq 0$, $c \in F^\times$ and $\lambda_1, \ldots, \lambda_n \in F$. This decomposition is unique up to reordering the factors

4 Determinants and Eigenvalues Redux

4.1 Symmetric Groups

Definition 4.1.1: Symmetric Groups

The group of all permutations of the set $\{1, 2, \ldots, n\}$, also known as bijections from $\{1, 2, \ldots, n\}$ to itself is denoted by $\mathfrak{S}_n$ (but i will just write S_n because icba) and called the **n-th symmetric group**. It is a group under composition and has $n!$ elements.

A **tranposition** is a permutation that swaps two elements of the set and leaves all the others unchanged.

Definition 4.1.2: Inversions of a permutation

An **inversion** of a permutation $\sigma \in S_n$ is a pair (i, j) such that $1 \leq i < j \leq n$ and $\sigma(i) > \sigma(j)$. The number of inversions of the permutation σ is called the **length of** σ and written $\ell(\sigma)$. In formulas:

$$\ell(\sigma) = |\{(i, j) : i < j \text{ but } \sigma(i) > \sigma(j)\}|$$

The **sign of** σ is defined to be the parity of the number of inversions of σ. In formulas:

$$\mathrm{sgn}(\sigma) = (-1)^{\ell(\sigma)}$$

Theorem 4.1.3: Multiplicativity of the sign

For each $n \in \mathbb{N}$ the sign of a permutation produces a group homomorphism $\mathrm{sgn} : S_n \to \{+1, -1\}$ from the symmetric group to the two-element group of signs. In formulas:

$$\mathrm{sgn}(\sigma\tau) = \mathrm{sgn}(\sigma)\,\mathrm{sgn}(\tau) \quad \forall \sigma, \tau \in S_n$$

Definition 4.1.4: Alternating Group of a Permutation

For $n \in \mathbb{N}$, the set of even permutations in S_n forms a subgroup of S_n because it is the kernel of the group homomorphism $\mathrm{sgn} : S_n \to \{+1, -1\}$. This group is the **alternating group** and is denoted A_n

4.2 Determinants

Definition 4.2.1: Determinants

Let R be a commutative ring and $n \in \mathbb{N}$. The **determinant** is a mapping $\det : \mathrm{Mat}(n; R) \to R$ from square matrices with coefficients in R to the ring R that is given by the following formula

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \mapsto \det(A) = \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) a_{1\sigma(1)} \ldots a_{n\sigma(n)}$$

4.3 Characterising the Determinant

Definition 4.3.1: Bilinear Forms

Let U, V, W be F-vector spaces. A **bilinear form on** $U \times V$ **with values in** W is a mapping $H : U \times V \to W$ which is a linear mapping in both of its entries. This means that it must satisfy the following properties for all $u_1, u_2 \in U$ and $v_1, v_2 \in V$ and all $\lambda \in F$:

$$\begin{aligned} H(u_1 + u_2, v_2) &= H(u_1, v_1) + H(u_2, v_1) \\ H(\lambda u_1, v_1) &= \lambda H(u_1, v_1) \\ H(u_1, v_2 + u_2) &= H(u_1, v_1) + H(u_2, v_1) \\ H(u_1, \lambda v_1) &= \lambda H(u_1, v_1) \end{aligned}$$

Definition 4.3.2: Multilinear Forms

Let $V_1, \ldots, V_n, W$ be F-vector spaces. A mapping $H : V_1 \times V_2 \times \cdots \times V_n \to W$ is a **multilinear form** or just **multilinear** if for each j, the mapping $V_j \to W$ defined by $v_j \mapsto H(v_1, \ldots, v_j, \ldots, v_n)$, with the $v_i \in V_i$ arbitrary fixed vectors of V_i for $i \neq j$ is linear.

Definition 4.3.3: Alternating Multilinear Forms

Let V and W be F-vector spaces. A multilinear form $H : V \times \cdots \times V \to W$ is **alternating** if it vanishes on every n-tuple of elements of V that has at least two entries equal, in other words if:

$$(\exists i \neq j \text{ with } v_i = v_j) \to H(v_1, \ldots, v_i, \ldots, v_j, \ldots, v_n) = 0$$

Theorem 4.3.4: Characterisation of the Determinant

Let F be a field. The mapping

$$\det : \mathrm{Mat}(n; F) \to F$$

is the unique alternating multilinear form on n-tuples of column vectors with values in F that takes the value 1_F on the identity matrix

4.4 Rules for Calculating with Determinants

Theorem 4.4.1: Multiplicativity of the Determinant

Let R be a commutative ring and let $A, B \in \mathrm{Mat}(n; R)$. Then

$$\det(AB) = \det(A)\det(B)$$

Theorem 4.4.2

The determinant of a square matrix with entries in a field F is non-zero if and only if the matrix is invertible

4.4.3 Consequences of determinant rules

- If A is invertible then $\det(A^{-1}) = \det(A)^{-1}$
- If B is a square matrix then $\det(A^{-1}BA) = \det(B)$

Theorem 4.4.4: Determinants of a Transpose Matrix

The determinant of a square matrix and of the transpose of the square matrix are equal, that is for all $A \in \mathrm{Mat}(n; R)$ with R a commutative ring,

$$\det(A^T) = \det(A)$$

Definition 4.4.5: Cofactors of a Matrix

Let $A \in \mathrm{Mat}(n; R)$ for some commutative ring R and natural number n. Let i and j be integers between 1 and n. Then the $(i,)$ **cofactor of** A is $C_{ij} = (-1)^{i+j} \det(A\langle i, j\rangle)$ where $A\langle i, j\rangle$ is the matrix obtained from A by deleting the i-th row and j-th column.

$$C_{23} = (-1)^{2+3} \det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = -a_{11}a_{32} + a_{31}a_{12}$$

Theorem 4.4.6: Laplace's Expansion

Let $A = (a_{ij})$ be an $(n \times n)$-matrix with entries from a commutative ring R. For a fixed i, the **i-th row expansion of the determinant** is

$$\det(A) = \sum_{j=1}^{n} a_{ij} C_{ij}$$

and for a fixed j, the **j-th column expansion of the determinant** is

$$\det(A) = \sum_{i=1}^{n} a_{ij} C_{ij}$$

Definition 4.4.7: Adjugate Matrix

Let A be a $(n \times n)$-matrix with entries in a commutative ring R. The **adjugate matrix** $\mathrm{adj}(A)$ is the $(n \times n)$-matrix whose entries are $adj(A)_{ij} = C_{ji}$ where C_{ji} is the (j, i)-cofactor

Theorem 4.4.8: Cramer's Rule

Let A be a $(n \times n)$-matrix with entries in a commutative ring R. Then

$$A \cdot \mathrm{adj}(A) = (\det A) I_n$$

4.4.9 Alternative Definition of Cramer's

In many sources, such as Wikipedia, Cramer's Rule means the formula

$$x_i = \frac{\det(a_{*1} \mid \cdots \mid b_* \mid \cdots \mid a_{*n})}{\det(a_{*1} \mid \cdots \mid a_{*i} \mid \cdots \mid a_{*n})}$$

for solving a field F the system $A\vec{x} = \vec{b}$ of n linear equations in n unknowns, provided that a unique solution exists. A unique solution exists if and only if A is invertible. So, instead of applying the Gaussian algorithm, you can calculate lots of determinants, replacing the i-th column of A by the given solution vector $\vec{b}$. It turns out that if you implement this rule on a computer, it has the same efficiency as the Gaussian algorithm. The relationship between this version of Cramer's rule and the above theorem is got by successively taking the vector $\vec{b}$ in the system of linear equations to be the standard basis elements $\vec{e_i}$ with $1 \leq i \leq n$.

Theorem 4.4.10: Invertibility of Matrices

A square matrix with entries in a commutative ring R is invertible if and only if its determinant is a unit in R. That is, $A \in \mathrm{Mat}(n; R)$ is invertible if and only if $\det(A) \in R^\times$

So for instance, an integral matrix $A \in \mathrm{Mat}(n; \mathbb{Z})$ is invertible if and only if $\det(A)$ is 1 or -1, since $\mathbb{Z}^\times = \{\pm 1\}$. On the other hand, a matrix $A \in \mathrm{Mat}(n; F)$ with entries in a field F is invertible if and only if $\det(A) \neq 0$ since $F^\times$ consists of the non-zero elements of F.

Theorem 4.4.11: Jacobi's Formula

Let $A = (a_{ij})$ where the coefficients $a_{ij} = a_{ij}(t)$ are functions of t. Then

$$\frac{d}{dt} \det A = \mathrm{Tr} \mathrm{Adj} A \frac{dA}{dt}$$

4.5 Eigenvalues and Eigenvectors

Definition 4.5.1: Eigenvalues and Eigenvectors

Let $f : V \to V$ be an endomorphism of an F-vector space V. A scalar $\lambda \in F$ is an **eigenvalue of f** if and only if there exists a non-zero vector $\vec{v} \in V$ such that $f(\vec{v}) = \lambda\vec{v}$. Each such vector is called an **eigenvector of f with eigenvalue** λ. For any $\lambda \in F$, the **eigenspace of f with eigenvalue** λ is

$$E(\lambda, f) = \{\vec{v} \in V : f(\vec{v}) = \lambda\vec{v}\}$$

Theorem 4.5.2: Existence of Eigenvalues

Each endomorphism of a non-zero finite dimensional vector space over an algebraically closed field has an eigenvalue

Definition 4.5.3: Characteristic Polynomial

Let R be a commutative ring and let $A \in \mathrm{Mat}(n; R)$ be a square matrix with entries in R. The polynomial $\det(xI_n - A) \in R[x]$ is called the **characteristic polynomial of the matrix** A. It is denoted by

$$\chi_A(x) := \det(xI_n - A)$$

(where χ stands for χaracteristic, lol)

Theorem 4.5.4: EVs and Characteristic Polynomials

Let F be a field and $A \in \mathrm{Mat}(n; F)$ a square matrix with entries in F. The eigenvalues of the linear mapping $A : F^n \to F^n$ are exactly the roots of the characteristic polynomial χ_A

4.5.5 Eigenvalue remarks

1. Square matrices $A, B \in \mathrm{Mat}(n; R)$ of the same size are *conjugate* if
$$B = P^{-1}AP \in \mathrm{Mat}(n; R)$$
for an invertible $P \in \mathrm{GL}(n; R)$. Conjugacy is an equivalence relation on $\mathrm{Mat}(n; R)$. (The definition makes sense for any commutative ring R, although we will mainly be concerned with the case of a field)

2. The motivation for conjugacy comes from the various matrix representations for an endomorphism $f : V \to V$ of an n-dimensional vector space V over a field F. Let
$$A = (a_{ij}) = {}_{\mathcal{A}}[f]_{\mathcal{A}},\ B = (b_{ij}) = {}_{\mathcal{B}}[f]_{\mathcal{B}} \in \mathrm{Mat}(n; f)]$$
be the matrices of f with respect to bases $\mathcal{A} = (\vec{v_1}, \vec{v_2}, \ldots, \vec{v_n})$, $\mathcal{B} = (\vec{w_1}, \vec{w_2}, \ldots, \vec{w_n})$ for V
$$f(\vec{v_j}) = \sum_{i=1}^{n} a_{ij}\vec{v_i},\ f(\vec{w_j}) = \sum_{i=1}^{n} b_{ij}\vec{w_i} \in V$$
The change of basis matrix $P = (p_{ij}) = {}_{\mathcal{A}}[\mathrm{id}_V]_{\mathcal{B}} \in \mathrm{Mat}(n; F)$ is invertible, with
$$\vec{w_j} = \sum_{i=1}^{n} p_{ij}\vec{v_i} \in V$$
We have the identity
$$B = P^{-1}AP \in \mathrm{Mat}(n; F)$$
so A, B are conjugate

3. **Key observation**: the characteristic polynomials of conjugate $A, B \in \mathrm{Mat}(n, R)$ are the same
$$\begin{aligned} \chi_B(x) = \det(xI_n - B) &= \det(xI_n = P^{-1}AP) \\ &= \det(P^{-1}(xI_n - A)P) = \det(P)^{-1}\det(xI_n - A)\det(P) \\ &= \det(xI_n - A) = \chi_A(x) \in R[x] \end{aligned}$$

4. In view of 2 and 3 we can define the characteristic polynomial of an endomorphism $f : V \to V$ of an n-dimensional vector space over a field F to be
$$\chi_f(x) = \chi_A(x) \in F[x]$$

with $A = \mathcal{A}[f]_{\mathcal{A}} \in \mathrm{Mat}(n; R)$ the matrix of f with respect to *any* basis $\mathcal{A}$ for V. Thanks to 4.5.4, the eigenvalues of f are exactly the roots of χ_f, the characteristic polynomial of f

Remark: Let $f : V \to V$ be an endomorphism of an n-dimensional vector space V over a field F. Suppose given an m-dimensional subspace $W \subseteq V$ such that $f(W) \subseteq W$, so that there are defined endomorphisms of the subspace and the quotient space

$$g : W \to W;\ \vec{w} \mapsto f(\vec{w})$$
$$h : V/W \to V/W;\ W + \vec{v} \mapsto W + f(\vec{v})$$

Any ordered basis $\mathcal{A} = (\vec{w_1}, \vec{w_2}, \ldots, \vec{w_m})$ for W can be extended to an ordered basis for V

$$\mathcal{B} = (\vec{w_1}, \vec{w_2}, \ldots, \vec{w_m}, \vec{v}_{m+1}, \vec{v}_{m+2} \cdots, \vec{v_n})$$

The images of the $\vec{v}_j$'s under the canonical projection can : $V \to V/W$ are then an ordered basis for V/W

$$\mathcal{C} = (\mathrm{can}(v_{m+1}), \mathrm{can}(v_{m+2}), \ldots, \mathrm{can}(\vec{v}_n))$$

Let $a_{ij}, b_{jk}, c_{ik} \in F$ be the coefficients in the linear combinations

$$f(\vec{w}_j) = \sum_{i=1}^{m} a_{ij}\vec{w}_i \in W,\ f(\vec{v}_k) = \sum_{j=m+1}^{n} b_{jk}\vec{v}_j + \sum_{i=1}^{n} c_{ik}\vec{w}_i \in V$$

[WIP SO MUCH WRITING OMG]

4.6 Triangularisable, Diagonalisable, and Cayley-Hamilton

Definition 4.6.1: Triangularisability

Let $f : V \to V$ be an endomorphism of a finite dimensional F-vector space V. f is **triangularisable** if the vector space V has an ordered basis $\mathcal{B} = (\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n)$ such that

$$\begin{aligned} f(\vec{v}_1) &= a_{11}\vec{v}_1, \\ f(\vec{v_2}) &= a_{12}\vec{v}_1 + a_{22}\vec{v}_2, \\ &\vdots \\ f(\vec{v}_n) &= a_{1n}\vec{v}_1 + a_{2n}\vec{v}_2 + \cdots + a_{nn}\vec{v}_n \in V \end{aligned}$$

(so that the first basis vector $\vec{v}_1$ is an eigenvector, with eigenvalue a_{11}) or equivalently such that the $n \times n$ matrix $_{\mathcal{B}}[f]_{\mathcal{B}} = (a_{ij})$ representing f with respect to $\mathcal{B}$ is upper triangular (or any other triangular)

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{pmatrix}$$

Theorem 4.6.2

Let $f : V \to V$ be an endomorphism of a finite dimensional F-vector space V. Then f is triangularisable iff the characteristic polynomial χ_f decomposes into linear factors in $F[x]$

Theorem 4.6.3: Triangularisability and Conjugacy

An endomorphism $A : F^n \to F^n$ is triangularisable if and only if $A = (a_{ij})$ is conjugate to an upper triangular matrix $B = (b_{ij})(b_{ij} = 0 \text{ for } i > j)$, with $P^{-1}AP = B$ for an invertible matrix P

Definition 4.6.4: Diagonalisability

An endomorphism $f : V \to V$ of an F-vector space V is **diagonalisable** if and only if there exists a basis of V consisting of eigenvectors of f. If V is finite dimensional then this is the same as saying that there exists an ordered basis $\mathcal{B} = \{\vec{v}_1, \ldots, \vec{v}_n\}$ such that corresponding matrix representing f is diagonal, that is $_{\mathcal{B}}[f]_{\mathcal{B}} = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$. In this case, of course, $f(\vec{v}_i) = \lambda_i\vec{v}_i$.
A square matrix $A \in \mathrm{Mat}(n; F)$ is **diagonalisable** if and only if the corresponding linear mapping $F^n \to F^n$ given by left multiplication by A is diagonalisable. Thanks to [something] this just means that A is conjugate to a diagonal matrix, there exists an invertible matrix $P \in \mathrm{GL}j(n; F)$ such that $P^{-1}AP = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$. In this case the columns P are the vectors of a basis of F^n consisting of eigenvectors of A with eigenvalues $\lambda_1, \ldots, \lambda_n$

Theorem 4.6.5: Linear Independence of Eigenvectors

Let $f : V \to V$ be an endomorphism of a vector space V and let $\vec{v}_1, \ldots, \vec{v}_n$ be eigenvectors of f with pairwise different eigenvalues $\lambda_1, \ldots, \lambda_n$. Then the vectors $\vec{v}_1, \ldots, \vec{v}_n$ are linearly independent

Theorem 4.6.6: Cayley-Hamilton Theorem

Let $A \in \mathrm{Mat}(n; R)$ be a square matrix with entries in a commutative ring R. Then evaluating its characteristic polynomial $\chi_A(x) \in R[x]$ at the matrix A gives zero.

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