Leon's ITCS Exam Notes

Big thanks to Chris Dalziel, this is mostly adapted from their notes :)
In collaboration with Alex Brodbelt :)

Finite Automata

Definition: Finite Automata

A finite automaton takes a string as input and replies "yes" or "no". If an automaton A replies "yes" on a string S we say that A "accepts" S.

Definition: Deterministic Finite Automata

A deterministic finite automaton (DFA) is a quintuple $(Q, \Sigma, q_0, \delta, F)$ where

- Q is a finite set of states
- Σ is an alphabet
- $q_0 \in Q$ is the initial state
- $\delta : Q \times \Sigma \to Q$ is the transition function
- $F \subseteq Q$ is the set of final states

A DFA accepts a string $w \in \Sigma^*$ iff $\delta^*(q_0, w) \in F$, where δ^* is δ applied successively for each symbol in w.
The language of a DFA A is the set of all strings accepted by a, $\mathcal{L} \subseteq \Sigma^*$ is the set of all strings accepted by A.
The transition function is a total function which gives exactly one next state for each input symbol, i.e. it is deterministic

Definition: Nondeterministic Finite Automata

Non-determinism would mean that δ can return more than one successor state, it instead returns a set of possible states - no states is an empty set. A NFA is a quintuple $(Q, \Sigma, q_0, \delta, F)$ where:

- Q is a finite set of states
- Σ is an alphabet
- $q_0 \in Q$ is the initial state
- $\delta : Q \times \Sigma \to \mathcal{P}(Q)$ is the transition function
- $F \subseteq Q$ is the set of final states

The only difference between the definition of a DFA and that of an NFA is that in an NFA δ returns an element from the powerset of Q, $\mathcal{P}(Q)$

Adding non-determinism doesn't change "expressivity". Given an NFA A there is an equivalent DFA D such that $\mathcal{L}(D) = \mathcal{L}(A)$ and vice versa.

Definition: ϵ-NFA

If we allow non-deterministic state chnages that don't consume any input symbols, we can label silent moves using ϵ - meaning the empty string We define the ϵ closure $E(q)$ of a state q as the set of all states reachable from q by silent moves. That is, $E(q)$ is the least set satisfying:

- $q \in E(q)$
- For any $s \in E(q)$ we also have $\delta(s, \epsilon) \subseteq E(q)$

DFA, NFA, ϵ-NFA are all equal in expressive power

Regular Languages

Definition: Regular Languages

Any language which can be accepted by a finite automaton is called a regular language.
Regular languages are also those recognised by Regular Expressions

Definition: Regular Language Closure Properties

For two languages L_1 and L_2, the following operations satisfy the closure property, i.e. for a member $x \in X$, and an operation ϕ we have that $\phi(x) \in \mathbb{R}$ for all x.

- **Union**: $L_1 \cup L_2$ is the language that includes all strings of L_1 and all strings of L_2.
- **Intersection**: $L_1 \cap L_2$ is the language that includes all strings of L_1 that are not in L_2, and vice versa
- **Sequential Composition**: $L_1 L_2$ is the language of strings that consist of strings in L_1 followed by a string in L_2.
- **Kleene closure**: L^* is the language of strings that consist wholly of zero or more strings in L.
$$L^* = \bigcup_{i \in \mathbb{N}} L^i$$
- **Complement**: $\bar{L}$ is the language of every string not in L.

Definition: Regular Expressions

Regular characterise the regular languages, just like finite automata do. The following table shows the syntax and semantics of a regex.

Syntax	Semantics	
a	$[\![a]\!] = \{a\}$	$(a \in \Sigma)$
$\emptyset$	$[\![\emptyset]\!] = \emptyset$	
ϵ	$[\![\epsilon]\!] = \{\epsilon\}$	
$R_1 \cup R_2$	$[\![R_1 \cup R_2]\!] = [\![R_1]\!] \cup [\![R_2]\!]$	
$R_1 \circ R_2$	$[\![R_1 \circ R_2]\!] = [\![R_1]\!][\![R_2]\!]$	
R^*	$[\![R^*]\!] = [\![R]\!]^*$	

Definition: Generalised NFAs

A **generalised NFA**, or GNFA is an NFA where:

- Transitions have **regular expressions** on them instead of symbols
- There is only one unique final state
- The transition relation if **full**, except that the initial state has no incoming transitions, and the final state has no outgoing transitions

Theorem 1: Pumping Lemma

If $L \subseteq \Sigma^*$ is regular, then there is a **pumping length** $p \in \mathbb{N}$ such that for any $w \in L$ where $|w| \geq p$, we may split w into three piexes $w = xyz$ satisfying three conditions:

- $xy^i z \in L, \quad \forall i \in \mathbb{N}$
- $|y| > 0$
- $|xy| \leq p$

Note that if the pumping lemma fails then the language is not regular, but the inverse is not necessarily true.

Theorem 2: Myhill-Nerode Theorem

Let $L \subseteq \Sigma^*$ and $x, y \in \Sigma^*$. If there exists a suffix string z such that $xz \in L$, but $yz \notin L$ or vice versa, then x and y are **distinguishable** by L. If x and y are not distinguishable by L, then we say that $x \equiv_L y$ - this is an equivalence relation. A regular language satisfies the following

- The number of equivalence classes $\equiv_L$ is finite.
- The number of equivalence classes is equal to the number of states in the minimal DFA accepting L (not as important)

Therefore, to show a language is non-regular, show that it has infinite equivalence classes - that is, we find an infinite sequence $u_0 u_1 \ldots$ of strings such that for any i, j where $i \neq j$, there is a string w_{ij} such that $u_i w_{ij} \in L$ but $u_j w_{ij} \notin L$ or vice-versa

Context-Free Languages

Definition: Context-free Languages

By adding recursion to regexs we can begin to recognise some non-regular languages. All regular languages are also context free.
Context free languages are closed under:
Union, Concatenation, Kleene Star.
But are **not** closed under:
Interesection, Complementation (shown via de Morgan's laws).

Definition: Context-free Grammars

A language is context-free iff it is recognised by a Context-free Grammar (CFG), which is a 4-tuple (N, Σ, P, S) where:

- N is a finite set of variables or non-terminals
- Σ is a finite set of terminals
- $P \subseteq N \times (N \cup \Sigma)^*$ is a finite set of rules or productions
  - Typically productions are written $A \to aBc$
  - Productions with common heads can be combined, $A \to a$ and $A \to Aa$ can be combined into $A \to a \mid Aa$
- $S \in N$ is the starting variable

We use α, β, γ to refer to sequences of terminals
We make a derivation step $\alpha A\beta \Rightarrow_G \alpha\gamma\beta$ whenever $(A \to \gamma) \in P$;
The language of a CFG G is:

$$\mathcal{L}(G) = \{w \in \Sigma^* \mid S \Rightarrow^*_G w^*\}$$

Where $\Rightarrow^*_G$ is the reflexive, transitive, closure of $\Rightarrow_G$.
Context-free grammars are ambiguous. They are closed under union, concatenation, and kleene star, but not under intersection or complementation

Definition: Eliminating Ambiguity

We want to eliminate ambiguity in CFGs while still accepting all the same strings. This can be done for our language of regular expressions:

- First defining atomic expressions: $A \to (S)|\emptyset|\epsilon|a|b$
- Then ones which use Kleene Star: $K \to A|A^*$
- Then ones which may use left-associative composition: $C \to K|C \circ K$
- Finally expressions which use unions: $S \to C|S \cup C$

The order of operations here is therefore bottom to top; unions come before compositions, which come before Kleene etc

Definition: Push-down Automata

Push-down automata are to CFGs what finite automatas are to regular expressions. They are implementationally identical to ϵ-NFAs with the addition of a stack. The recursive element of CFGs is implemented using a standard last-in-first-out stack.

Transitions in a push-down automata take the form $x, y \to z$ which is read as "consume the input x, popping y off the stack, and push z onto the stack". We can allow actions that don't consume, pop, or push by setting variables to ϵ.

Definition: Formal Def. of PDAs

A **push-down automaton** is a 6-tuple $(Q, \Sigma, \Gamma, \delta, q_0, F)$ where Q, Σ, Γ are all finite sets. Γ is the stack alphabet, and δ now may take a stack symbol as input or return one as output:

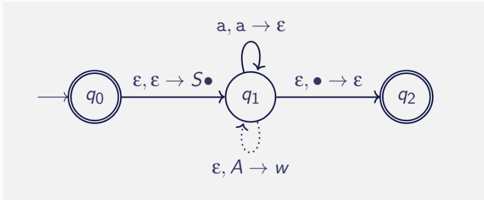
$$\delta : Q \times \Sigma_\epsilon \times \Gamma_\epsilon \to \mathcal{P}(Q \times \Gamma_\epsilon)$$

All other compoments are as with ϵ-NFAs

A string w is accepted by a PDA if it ends in a final state, i.e. $\delta^*(q_0, w, \epsilon)$ gives a state q and a stack γ wuch that $q \in F$.

Theorem 3: CFG to PDA and PDA to CFG

The upper loop on q_1 is added for every terminal a in the CFG. The lower loop on q_1 is shorthand for a looping sequence of states added for each production $A \to w$ that builds up w on the stack one symbol at a time.



Firstly we ensure that the PDA has only one accept state, empties its stack before terminating, and has only transitions that either push or pop a symbol (but not transitions that do both or neither).

Given such a PDA $P = (Q, \Sigma, \Gamma, \delta, q_0, F)$ we provide a CFG (V, Σ, R, S) with V containing a non-terminal A_{pq} for every pair of states $(p, q) \in Q \times Q$.

The non-terminal A_{pq} generates all strings that go from p with an empty stack to q with an empty stack. Then S is just $A_{q_0 q_{accept}}$.

R consists of:

- $A_{pq} \to aA_{rs}b$ if $p \xrightarrow{a,\epsilon\to t} r$ and $s \xrightarrow{b,t,\to\epsilon} q$ (for intermediate states r, s and stack symbol t
- $A_{pq} \to A_{pr}A_{rq}$ for all intermediate states r
- $A_{pp} \to \epsilon$

Theorem 4: Pumping CFLs intro

Suppose a CFG has n non-terminals, and we have a parse tree of height $k > n$. Then the same non-terminal V must have appeared as its own descendant in the tree

- **Pumping down**: Cut the tree at the higher occurance of V and replace it with the subtree at the lower occurance of V
- **Pumping up**: Cut at the lower occurance and replace it with a fresh copy of the higher occurance

Theorem 5: Pumping Lemma for CFLs

If L is context-free then there exists a pumping length $p \in \mathbb{N}$ such that if $w \in L$ with $|w| \geq p$ then w may be split into **five** pieces $w = uvxyz$ such that

- $uv^ixy^iz \in L$ for all $i \in \mathbb{N}$
- $|vy| > 0$
- $|vxy| \leq p$

Definition: Chomsky Grammars

Context-free grammars are a special case of Chomsky Grammars. Chomsky grammars are similar to CFGs, except that the left-hand side of a production may be any string that includes at least one non-terminal. An example is shown below

$$\begin{aligned} S &\to abc \mid aAbc \\ Ab &\to bA \\ Ac &\to Bbcc \\ bB &\to Bb \\ aB &\to aaA \mid aa \end{aligned}$$

Such a grammar is called **context-sensitive**

Definition: The Chomsky Heirarchy

A grammar $G = (N, \Sigma, P, S)$ is of type:

0. (or **computably enumerable**) in the general case
1. (or **context sensitive**) if $|\alpha| \leq |\beta|$ for all productions $\alpha \to \beta$, except we also allow $S \to \epsilon$ if S foes not occur on the RHS of any rule
2. (or **context free**) if all productions are of the form $A \to \alpha$ (i.e. a CFG)
3. (or **right-linear/regular**) if all productions are of the form $A \to w$ or $A \to wB$, where $w \in \Sigma$ and $B \in N$

Algorithms for Languages

Theorem 6: Emptiness for Regular Languages

Can we write a program to determine if a given regular language is empty?
Given a finite-automaton this is an instance of graph reach-ability, so we can use a depth-first search.

Theorem 7: Equivalence of DFA

Is it possible to write a program to determine if two discrete finite automata are equivalent?

Given two DFA for L_1 and L_2, we can use our standard constructions to produce a DFA of the symmetric set difference:

$$(L_1 \cap \overline{L}_2) \cup (L_2 \cap \overline{L}_1)$$

Theorem 8: Emptiness for Context-free languages

Can we write a program to determine if a given context-free language is empty?
Given a CFG for our language, we can perform the following process:

1. Mark the terminals and ϵ as generating
2. Mark all non-terminals which have a production with only generating symbols in their right hand side as generating
3. Repeat until nothing new is marked
4. Check if S is marked as generating or not

Register Machines

Definition: Register Machines

A **register machine**, or RM, consists of:

- A fixed number m of registers $R_0 \dots R_{m-1}$, which each holds a natural number
- A fixed program P which is a sequence of n instructions $I_0 \dots I_{n-1}$

Each instruction is one of the following:

- `INC(i)`: which increments the register R_i by one
- `DECJZ(i, j)`: which decrements register R_i unless $R_i = 0$ in which case it jumps to instruction I_j

RMs can compute anything any other computer can

Definition: Pairing Functions for RMs

A **pairing function** is an injective function $\mathbb{N} \times \mathbb{N} \to \mathbb{N}$. An example is $f(x, y) = 2^x 3^y$.
We write $\langle x, y \rangle_2$ for $f(x, y)$. If $z = \langle x, y \rangle_2$, let $z_0 = x$ and $z_1 = y$.
This lets us encode multiple values into a single value, and a 2-tuple pairing function is enough to cram an arbitrary sequence of natural numbers into one $\mathbb{N}^* \to \mathbb{N}$

Definition: Turing Machine

A **turing machine** is a 7-tuple $(Q, \Sigma, \Gamma, \delta, q_0, q_{\text{accept}}, q_{\text{reject}})$

- Q: states
- Σ: input symbols
- $\Gamma \subseteq \Sigma$: **tape** symbols, including a **blank** symbol $\sqcup$
- $\delta : Q \times \Gamma \to Q \times \Gamma \times \{-1, 1\}$
- $q_0, q_{\text{accept}}, q_{\text{reject}} \in Q$: start, accept, reject states

Theorem 9: Church-Turing Thesis

The Church-Turing thesis states that any problem is computable by any model of computation iff it is computable by a **Turing machine**.
For our purposes this matters for RMs, TMs, and λ-calculus.
Other examples are combinator calculus, general recursive functions, pointer machines, counter machines, cellular automata, queue automata, enzyme-based DNA computers, Minecraft, Magic the Gathering, and others.

Theorem 10: The Halting Problem

Given a register machine encoding, can we write a program to determine if the simulated machine will halt or not?
If we suppose H is such a register machine, which takes a machine encoding $\lceil M \rceil$ in R_0, halts with 1 if M halts, and halts with 0 if M doesn't halt.
We can construct a new machine $L = (P_L, R_0, \dots)$ which, given a program $\lceil P \rceil$ runs H on the program with itself as input, the machine $(P, \lceil P \rceil)$ and loops if it halts.
If we run L on P_L itself we get a problem. If L halts on $\lceil P_L \rceil$ that means H says $(P_L, \lceil P_L \rceil)$. If L loops on $\lceil P_L \rceil$ that means that H says $(P_L, \lceil P_L \rceil)$ halts.
This is a contradiction!
The halting problem proves that there are some programs which cannot be decided by register machines. What about other machines?

Decidability

Definition: Computability

A (total) function $\mathbb{N} \to \mathbb{N}$ is **computable** if there is an RM/TM which computes f, i.e., given an x in R_0, leaves $f(x)$ in R_0
A **decision problem** is a set D and a quiery subset $Q \subseteq D$. A problem is **decidable** or **computable** if $d \in Q$ is characterised by a computable function $f : D \to \{0, 1\}$.

Definition: Problems with no Algorithm

While the above problems are all computable, this is not always the case. The questions asked, i.e. "can we determine x?", are not always something we can answer using a program - if this is the case then that problem is considered undecidable.
Many such problems exist for Context-free Languages

- Are two CFG equivalent?
- Is a given CFG ambiguous?
- Is there a way to make a given CFG unambiguous?
- Is the intersection of two CFLs empty?
- Does a CFG generate all strings Σ^*?
- ...

Definition: Reductions

A **reduction** is a transformation from one problem to another.
To prove that a problem P_2 is hard, show that there is an easy reduction from a known hard problem P_1 to P_2.
To show a problem P_2 is undecidable, show that there is a computable reduction from a known undecidable P_1 to P_2. The direction here matters - it tells us nothing to know that there's an easy way to make an easy problem difficult!

Example : Reduction analogy

If it is a well known fact that Hyunwoo cannot lift a car, we can prove that Hyunwoo cannot lift a loaded truck by making the following reduction: If we suppose he could lift the loaded truck, then we could have him lift the car by putting it in the loaded truck - but we know he cannot lift a car.

Definition: Mapping Reductions

A **Turing transducer** is a RM (or TM) which takes an instance d of a problem $P_1 = (D_1, Q_1)$ in R_0 and halts with an instance $d' = f(d)$ of $P_2 = (D_2, Q_2)$ in R_0. Thus, f is a computable function $D_1 \to D_2$
A **mapping reduction** (or a many-to-one reduction) from P_1 to P_2 is a turing transducer f such that $d \in Q_1$ iff $f(d) \in Q_2$
If A is mapping reducible to B, and A is undecidable, then B is undecidable.

Definition: Oracles

Given a decision problem (D, Q), an **oracle** for Q is a 'magic' RM instruction $\texttt{ORACLE}_Q(i)$ which, given an encoding of $d \in D$ in R_i, sets R_i to contain 1 iff $d \in Q$

Definition: Turing Reductions

A **Turing reduction** from P_1 to P_2 is an RM/TM equipped with an oracle for P_2 which solves P_1.
Decidability results carry across during Turing reductions as with mapping reductions, but mapping reductions make finer distinctions of computing power.

Theorem 11: Rice's Theorem

- A **property** is a set of RM (or TM) descriptions
- A property is **non-trivial** if it contains some but not all descriptions
- A property P is **semantic** if
$$\mathcal{L}(M_1) = \mathcal{L}(M_2) \Rightarrow (\lceil M_1 \rceil \in P \Leftrightarrow \lceil M_2 \rceil \in P)$$
In other words, it concerns the **language** and not the particular implentation of the language

---

Rice's Theorem - All non-trivial semantic properties are undecidable.

Theorem 12: Rice's Theorem part 2

Rice's theorem is useful for deciding propertiese like whether a language is empty, non-empty, regular, context-free etc
It cannot be applied to questions like whether a TM has fewer than 7 states, a final state, a start state, etc. These properties are of machines and not languages
It also doesn't apply to questions like is a language a subset of Σ^* or whether a language of a RM is a language of a TM - these are trivial properties! The consequences of this is that we cannot write programs which answer non-trivial questions about the black-box behaviours of programs

Computability in depth

Definition: Semi-decidability

A problem (D, Q) is **semi-decidable** if there is a TM/RM that returns "yes" for any $d \in Q$, but may return "no" or loop forever when $d \notin Q$

A problem (D, Q) is **co-semi-decidable** if ther eis a TM/RM that returns "no" for any $d \notin Q$, but may return "yes" or loop forever when $d \in Q$

Definition: Enumerable Computability

A set S is **enumerable** if there is a bijection between S and $\mathbb{N}$
A set S is called **computably enumerable** (or c.e.) if the enumeration function $f : \mathbb{N} \to S$ is computable
In terms of RM and TM we can think of enumeration as outputting an infinite list as it executes forever.

Theorem 13: Decidability Reductions

To prove that a problem P_2 is not c.e. we show that there is a mapping reduction from a known not-c.e. problem P_1 to P_2. We must use mapping reductions, as H is c.e. but L is not - but L is Turing reducible to H by flipping the answer.

Theorem 14: Decidability theorems

Any problem that is both semi-decidable and co-semi-decidable is decidable

---

If a problem P is semi-decidable then its complement $\overline{P}$ is co-semi-decidable, and vice versa

---

All semi-decidable problems are computably enumerable, and any computably enumerable problem is semi-decidable - being semi-decidable is the same as being computably enumerable.

Time Complexity

Definition: Time Complexity

The **time complexity** of a (deterministic) machine M that halts on all inputs is a functino $f : \mathbb{N} \to \mathbb{N}$ where $f(n)$ is the maximum number of steps that M uses on any input of size n.

Definition: Complexity Measures

When performing addition in a TM we have a time complexity of $\mathcal{O}(\log n)$, where it is $\mathcal{O}(n)$ for RMs - there's an exponential penalty for using a register machine!
Addition can be $\mathcal{O}(1)$ if we add dedicated `ADD(i,j)` and `SUB(i,j)` instructions - but this doesn't remove the inaccuracy it just makes it smaller.

Complexity is useful, but the measures are slightly bogus:

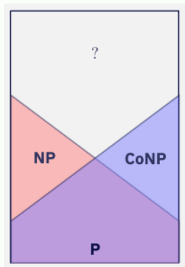
- $\mathcal{O}(n)$ is not always easy - if n is massive for example
- $\Omega(2^n)$ isn't always hard - there are problems much worse than this that are still solvable for real examples
- $\mathcal{O}(n^{10})$ and $\Omega(n^{10})$ seem extremely difficult, but a new model or algorithm could reduce that significantly
- Since we also ignore coefficients, we could have something like $f(n) \geq 10^{100} \log n$ which is slow but still only logarithmic - this isn't common enough to worry generally however.

Definition: Complexity Classes

Let $t : \mathbb{N} \to \mathbb{R}_{\geq 0}$. A time complexity class $\mathbf{TIME}(t(n))$ is the collection of all problems that are decidable by a deterministic machine RM, TM etc.) in $\mathcal{O}(t(n))$ time.

Given $A = \{0^i 1^i \mid i \in \mathbb{N}\}$, a TM can decide this in $\mathcal{O}(n^2)$, which means that $A \in \mathbf{TIME}(n^2)$.

We also define $\mathbf{NTIME}(t(n))$ to be the collection of all problems decidable by a nondeterministic machine NRM, NTM, etc.) in $\mathcal{O}(t(n))$.



Definition: NP / Nondeterministic-polynomial time

The polynomial complexity class **NP** is the class of problems decidable with some nondeterministic polynomial time complexity.

$$\mathbf{NP} = \bigcup_{k \in \mathbb{N}} \mathbf{NTIME}(n^k)$$

We don't know if every exponentially bounded problem is in **NP**, we think that it's probably not the case.

Definition: P / Polynomial Time

The polynomial complexity class **P** is the class of problems decidable with some deterministic polynomial time complexity.

$$\mathbf{P} = \bigcup_{k \in \mathbb{N}} \mathbf{TIME}(n^k)$$

Problems in **P** are called tractable. Any problem not in **P** is $\Omega(n^k)$ for every k.

The class itself is robust, reasonable changes to model don't change it and reasonable translations between problems preserves membership in **P**.

A polynomially-bounded RM together with a polynomial (n^k for some k, without loss of generality), such that given an input w it will always halt after executing $|w|^k$ instructions. A problem Q is in **P** iff it is computed by such a machine.

Definition: CoNP

We don't know if the class NP is closed under complement. We cannot flip the result because the result of flipping in a nondeterministic machine involves turning it from angelic nondeterminism to demonic nondeterminism (or co-nondeterminism) or vice versa.

Definition: AP / Alternating Poly time

The class **AP** is the class of all problems decidable by an alternating machine in polynomial time with no restriction on swapping quantifiers.

AP is known to be equal to PSPACE.

Definition: PSPACE / Polynomial Space

An RM/TM is $f(n)$-space-bounded if it may use only $f(inputsize)$ space. For TMs this is the number of cells on the tape, where register machines uses bits in registers.

Theorem 15: Polynomial Reductions

A polynomial reduction from $P_1 = (D_1, Q_1)$ to $P_2 = (D_2, Q_2)$ is a **P**-computable function $f : D_1 \to D_2$ such that $d \in Q_1$ iff $d \in Q_2$.

If P_2 is in **P**, then P_1 is in **P** straightforwardly. Therefore to prove a problem is not in **P** we can show that there is a polynomial reduction from a problem P_2 which isn't in **P** to our problem P_1

- A problem P_1 is **polynomially reducible** to P_2, written $P_1 \leq_P P_2$ if there is a polynomially-bounded reduction from P_1 to P_2.
- A problem P is **NP**-Hard if for every $A \in \mathbf{NP}$, $A \leq_P P$.

  That is, if a problem P_1 is **NP**-Hard and P_1 is polynomially reducible to P_2, then P_2 is also **NP**-Hard.

  We can use this fact to prove other problems are **NP**-Hard by showing a reduction from a known **NP**-hard problem.
- A problem is **NP**-Complete if it is both **NP**-Hard and in **NP**.

Theorem 16: Cook-Levin Theorem

The Cook-Levin theorem states that the NP problem SAT is NP-Complete.
The proof of this involves showing it is NP by nondeterministically guessing assignments and checking them in polynomial time, and then showing it's NP- Hard by reducing any NP problem to SAT

The Polynomial Heirarchy

Definition: Sigma Notation

The set Σ_1^P describes all problems that can be phrased as

$$\{y \mid \exists^p x \in \mathbb{N}. R(x, y)\}$$

where R is a **P**-decidable predicate and $\exists^p x \ldots$ indicates that x is of size polynomial in the size of y.
We can say that x is a certificate showing which guesses can be made by our NRM giving an accepting run.
If a problem $Q \in \Sigma_1^P$ then Q is **NP**, because it is a problem for which we can verify the answer in polynomial time. If a problem Q is in **NP** then $Q \in \Sigma_1^P$.
So, $\mathbf{NP} = \Sigma_1^P$

Definition: Pi notation

The set Π_1^P describes all problems that can be phrased as

$$\{y \mid \forall^P x \in \mathbb{N}. R(x, y)\}$$

where R is a **P**-decidable predicate, and $\forall^P x \ldots$ indicates that x is of size polynomial in the size of y.
We have that

$$\Pi_1^P = \overline{\Sigma_1^P}, \quad \text{and} \quad \Pi_1^P = \mathbf{CoNP}$$

Definition: Delta Notation

There are two conflicting definitions of Δ_1^P "For reasons that are unknown to me" - lecturer

- The set Δ_1^P describes the intersection of Σ_1^P and Π_1^P
- The set Δ_1^P describes the set **P**

From our characterisations of Σ_1^P and Π_1^P, we have that $\Delta_1^P \subseteq \mathbf{P}$, but we don't konw if these definitions are equal

Definition: Moving higher

The next layer of the heirarchy goes as follows:

- Σ_2^P is all problems of the form $\{x \mid \exists^P y. \forall^P z. R(x, y, z)\}$
- Π_2^P is all problems of the form $\{x \mid \forall^P y. \exists^P z. R(x, y, z)\}$
- $\Delta_2^P = \Sigma_2^P \cap \Pi_2^P$

We can also use oracles to get an alternate definition:

- Δ_2^P is all problems that are decidable in polynomial time by some deterministic RM/TM with an $\mathcal{O}(1)$ oracle for some problem in Σ_1^P (it is **P** with an $\mathcal{O}(1)$ oracle for **NP**)
- Σ_2^P allows the TM/RM to be nondeterministic (it is **NP** with an $\mathcal{O}(1)$ oracle for **NP**)
- Π_2^P is **CoNP** with an oracle for **NP**

In general for any $n > 1$:

- Δ_n^P is all problems decidable by a deterministic polynomially bounded TM/RM with an $\mathcal{O}(1)$ oracle for some problem in Σ_{n-1}^P.
- Σ_n^P is all problems decidable by some nondeterministic polynomially bounded TM/RM with an $\mathcal{O}(1)$ oracle for some problem in Σ_{n-1}^P.
- Π_n^P is all problems decidable by some co-nondeterministic polynomially bounded TM/RM with an $\mathcal{O}(1)$ oracle for some problem in Σ_{n-1}^P.

Note: Co-nondeterminism could also be called **demonic** nondeterminism, like regular (angelic) nondeterminism but only accepts if **all** paths accept

Definition: Alternation

Equivalently Σ_n^P are all problems that can be phrased as some **alternation** of (**P**-bounded) quantifiers, starting with $\exists^P$

$$\{w \mid \exists^P x_1 . \exists^P x_2 . \exists^P x_3 . \exists^P x_4 . \ldots x_n . R(w, x_1, \ldots, x_n)\}$$

Π_n^P has a similar definition, starting instead with $\forall^P$

$$\{w \mid \forall^P x_1 . \exists^P x_2 . \exists^P x_3 . \exists^P x_4 . \ldots x_n . R(w, x_1, \ldots, x_n)\}$$

Alternating machines combine the acceptance modes of both angelic and demonic non-deterministic machines.

Alternating register machines would replace the NRM's `MAYBE` instruction with the $\texttt{MAYBE}^\exists$ and $\texttt{MAYBE}^\forall$ instructions, which are nondeterministic branching choices where acceptance depends on if one branch accepts ($\exists$) or both branches accept ($\forall$).

Alternating Turing Machines are defined by labelling states with either $\forall$ or $\exists$.

The class Σ_n^P can therefore be described as the class of problems decided in polynomial time by an alternating machine that initially uses $\exists$-nondeterminism and swaps quantifiers at most $n - 1$ times. This extends to Π_n^P swapping starting with $\exists$ to starting with $\forall$.

λ-calculus

Definition: Syntax

λ-calculus computations are expressed as λ-terms:

$$\begin{array}{rcl} t & ::== & x \text{ (variables)} \\ & | & t_1\ t_2 \text{ (application)} \\ & | & \lambda x.\ t \text{ (λ-abstraction)} \end{array}$$

A λ-term $(\lambda x.\ y)$ can be thought of as a function that given an input bound to the variable x, returns the term y.

- Function application is left associative:
  $$f\ a\ b\ c = ((f\ a)\ b)\ c$$
- λ-abstraction extends as far as possible:
  $$\lambda a.\ f\ a\ b = \lambda a.\ (f\ a\ b)$$
- All functions are unary, multiple argument functions are modeled via nested λ-abstractions:
  $$\lambda x.\ \lambda y.\ f\ y\ x$$
- λ-calculus is higher-order, in that functions may be arguments to functions themselves:
  $$\lambda f.\ \lambda g.\ \lambda x.\ f\ (g\ x)$$

Definition: α-equivalence

α-equivalence is a way of saying that two statements are semantically identical but differ in the choice of bound variable names.

$$e_1 = (\lambda x.\ \lambda x.\ x + x)$$

$$e_2 = (\lambda a.\ \lambda y.\ y + y)$$

These two statements are α-equivalent, we say that $e_1 \equiv_\alpha e_2$, the relation $\equiv_\alpha$ is an equivalence-relation. The process of consistently renaming variables that preserves α-equivalence is called α-renaming or α-conversion.

Definition: β-reduction

The rule to evaluate function applications is called β-reduction:

$$(\lambda x.\ t)\ u \ \mapsto_\beta\ t[^u/_x]$$

β-reduction is a congruence:

$$\frac{}{(\lambda x\ t)\ u \mapsto_\beta t[^u/_x]}$$

$$\frac{t \mapsto_\beta t'}{s\ t \mapsto_\beta s\ t'} \quad \frac{s \mapsto_\beta s'}{s\ t \mapsto_\beta s'\ t} \quad \frac{t \mapsto_\beta t'}{\lambda x.\ t \mapsto_\beta \lambda x.\ t'}$$

This means we can pick any reducible subexpression (known as a redex) and perform β-reduction.

Definition: η-reduction

$$(\lambda x.\, f\, x) \mapsto_\eta f$$

Definition: Substitution

A variable x is free in a term e if x occurs in e but is not bound by a λ-abstraction in $e\cdot$ The variable x is free in $\lambda y.\, x + y$, but not in $\lambda x.\, \lambda y.\, x + y$.
A substitution, written $e[^t/_x]$ is the replacement of all free occurrences of x in e with t.

Definition: Variable capture

$e[^t/_x]$ whenever there is a bound variable in the term e with the same name as a free variable occurring in t.
Fortunately, it is always possible to avoid capture by α-rename the offending bound variable to an unused name.

Definition: Normal Forms

A λ-term which cannot be beta-reduction further is called a normal form.
Not every term has a normal form. Thankfully, all terms that do have a normal form are unique due to the Church-Rosser theorem.

Theorem 17: Church-Rosser Theorem (Confluence)

If a term t β-reduces to two terms a and b, then there is a common term t' to which both a and b are β-reducible.

Definition: Church encodings

In order to demonstrate that λ-calculus is a usable programming language we need to show how to encode certain useful structures and primitives like booleans and natural numbers with their operations as λ-terms.
The general idea is to turn a data type into the type of its eliminator, in other words we make a function which serves the same purpose as the data type when used.

Definition: Booleans

We use booleans to choose between results, so the encoding of a boolean is a function that given two arguments returns the first if true and the second if false.

$$\text{True} \equiv \lambda a.\, \lambda b.\, a$$
$$\text{False} \equiv \lambda a.\, \lambda b.\, b$$

An If statement becomes

$$\text{If} \equiv \lambda c.\, \lambda t.\, \lambda e.\, cte$$

Definition: Church Numerals

We use natural numbers to repeat processes n times, so the encoding of a natural number is a function that takes a function f and a value x and will apply f to x that number of times.

$$\begin{aligned}\text{Zero} &\equiv \lambda f.\, \lambda x.\, x\\ \text{One} &\equiv \lambda f.\, \lambda x.\, f\, x\\ \text{Two} &\equiv \lambda f.\, \lambda x.\, f\, (f\, x)\\ &\vdots\end{aligned}$$

Then operations like the successor and addition are relatively simple:

$$\text{Suc} \equiv \lambda n.\, \lambda f.\, \lambda x.\, f\, (n\, f\, x)$$
$$\text{Add} \equiv \lambda m.\, \lambda n.\, \lambda f.\, \lambda x.\, m\, f\, (n\, f\, x)$$

Definition: $\mathcal{Y}$ Combinator

The $\mathcal{Y}$ combinator is a "fixed point combinator", and it's the way we achieve recursion in λ-calculus. One $\mathcal{Y}$ combinator is the following:

$$\mathcal{Y} \equiv (\lambda f.\, (\lambda x.\, f\, (x\, x))\, (\lambda x.\, f\, (x\, x)))$$

Simply Typed λ-Calculus

Definition: Higher Order Logic

Originally the λ-calculus was intended for use as a term language for a logic, called higher-order logic.

The existence of the $\mathcal{Y}$ combinator and recursion in general causes a problem for this because we can cause statements like:

$$\mathcal{Y}\, \neg \equiv_\beta \neg\, (\mathcal{Y}\, \neg)$$

A logic which can have statements equivalent to their own negation is not ideal at all. To solve this church introduced types.

Definition: Types

Typed λ-calculus has a set of base types like `nat` and `bool`.

Given two types σ and τ, $\sigma \to \tau$ is the type signature of a function from σ to τ, type signatures are right associative:

$$\sigma \to \tau \to \rho = \sigma \to (\tau \to \rho)$$

λ-abstraction also specifies the type of the parameter: $\lambda x : \tau.\, t$

Things like the $\mathcal{Y}$ combinator or other recursive structures (things like the term which β-reduces to itself) cannot be typed.

Definition: Natural Deduction

We can specify a logical system as a deductive system by providing a set of rules and axioms that describe how to prove various connectives. The same can be done for typing.

$$\frac{x : \tau \in \Gamma}{\Gamma \vdash t\, u : \tau}A \qquad \frac{x : \sigma, \Gamma \vdash t : \tau}{\Gamma \vdash (\lambda x : \sigma.\, t) : \sigma \to \tau}I$$

$$\frac{\Gamma \vdash t : \sigma \to \tau \quad \Gamma \vdash u : \sigma}{\Gamma \vdash t\, u : \tau}E$$

This is the full rule-set for simply typed λ-calculus. The structure is that from the bottom you can derive the top, and that the right of the $\vdash$ symbol can be assumed true as a result of the left of it.

A is assignment, I is introduction, and E is elimination.

Definition: The Results of Simply Typed λ-Calculus

This simply typed λ-calculus has the following properties:

- Uniqueness of types: In a given context (types for free variables), any simply typed λ-terms has at most one type. Deciding this is in **P**.
- Subject reduction (or type safety): Typing respects $\equiv_{\alpha\beta\eta}$, i.e. reduction does not affect a term's type.
- Strong normalisation: Any well-typed term evaluates in finitely many reductions to a unique irreducible term. If the type is a base type, this term is a constant.

The $\mathcal{Y}$ combinator cannot be typed, and strong normalisation means that such a term cannot exist.

If we want to do general computation in λ-calculus we need recursion back. The way this is done is by extending it to include a new built in feature called **fix**.

We extend β-reduction to unroll recursion in a single step (which will keep strong normalisation).

Some type-theoretic languages avoid adding general recursion to the underlying λ-calculus - for reasons which are not examinable.