1 Vector Spaces

1.1 Fields and Vector Spaces

Definition 1.1.1: Definition of a field

A field F is a set with two functions

- Addition: $+ : F \times F \to F$, $(\lambda, \mu) \mapsto \lambda + \mu$
- Multiplication: $\cdot : F \times F$, $(\lambda, \mu) \mapsto \lambda\mu$

which satisfy the following axioms:

1. $(F, +)$ is an abelian group F^+, with identity 0_F
2. $(F \backslash \{0_F\}, \cdot)$ is an abelian group $F^\times$, with identity 1_F
3. **Distributive law**: For all a, b, and c in F, we have
$$a(b + c) = ab + ac \in F$$

and the following lemmas:

1. The elements 0_F and 1_F of F are distinct
2. For all $a \in F$, $a \cdot 0_F = 0_F$ and $0_F \cdot a = 0_F$
3. Multiplication in F is associative, and 1_F is an identity element

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A **vector space** V **over a field** F is a pair consisting of an abelian group $V = (V, \dot{+})$ and a mapping
$$F \times V \to V : (\lambda, \vec{v}) \mapsto \lambda\vec{v}$$
s.t. for all $\lambda, \mu \in F$ and $\vec{v}, \vec{w} \in V$ the following identities hold:

- **Distributivity 1**: $\lambda(\vec{v} \dot{+} \vec{w}) = \lambda\vec{v} \dot{+} \lambda\vec{w}$
- **Distributivity 2**: $(\lambda + \mu)\vec{v} = \lambda\vec{v} \dot{+} \mu\vec{v}$
- **Associativity**: $\lambda(\mu\vec{v}) = (\lambda\mu)\vec{v}$
- **Identity**: $1\vec{v} = \vec{v}$

and so do the following lemmas:

1. If V is a vector space and $\vec{v} \in V$, then $0\vec{v} = \vec{0}$
2. If V is a vector space and $\vec{v} \in V$, then $(-1)\vec{v} = -\vec{v}$
3. If V is a vector space over a field F, then $\lambda\vec{0} = \vec{0}$ for all $\lambda \in F$. Furthermore, if $\lambda\vec{v} = \vec{0}$ then either $\lambda = 0$ or $\vec{v} = \vec{0}$

1.2 Working with Vector Spaces

Definition 1.2.1: Cartesian Product of n sets

$$X_1 \times \cdots \times X_n := \{(x_1, \ldots, x_n) : x_i \in X_i \text{ for } 1 \le i \le n\}$$

The elements of a product are called **n-tuples**. An individual entry $x_i = (x_1, \ldots, x_n)$ is called a **component**.
There are special mappings called **projections** for a cartesian product:
$$\mathrm{pr}_i : X_1 \times \cdots \times X_n \to X_i$$
$$(x_1, \ldots, x_n) \mapsto x_i$$
The cartesian product of n copies of a set X is written in short as: X^n

Definition 1.2.2: Vector Subspace

A subset U of a vector space V is called a **vector subspace** or **subspace** if U contains the zero vector, and whenever $\vec{u}, \vec{v} \in U$ and $\lambda \in F$ we have $\vec{u} + \vec{v} \in U$ and $\lambda\vec{u} \in U$

Definition 1.2.3: Spans and Linear Independence

Let $T \subset V$ for some vector space V over a field F. Then amongus all subspaces of V that include T there is a smallest subspace
$$\langle T \rangle = \langle T \rangle_F \subseteq V$$
"the set of all vectors $\alpha_1\vec{v}_1 + \cdots + \alpha_r\vec{v}_r$ with $\alpha_1, \ldots, \alpha_r \in F$ and $\vec{v}_1, \ldots, \vec{v}_r \in T$, together with the zero vector in the case $T = \emptyset$"

Terminology Dump

- An expression of the form $\alpha_1\vec{v}_1 + \cdots + \alpha_r\vec{v}_r$ is called a **linear combination** of vectors $\vec{v}_1, \ldots, \vec{v}_r$
- The smallest vector subspace $\langle T \rangle \subseteq V$ containing T is called the **vector subspace generated by** T or the vector subspace **spanned by** T or even the **span of** T
- If we allow the zero vector to be the "empty linear combination of $r = 0$ vectors", then the span of T is exactly the set of all linear combinations of vectors from T
- A subset of a vector space that spans the entire space is called a **generating** or **spanning set**. A vector space that has a finite generating set is said to be **finitely generated**

Linear Independence

A subset L of a vector space V is called **linearly independent** if for all pairwise different vectors $\vec{v}_1, \ldots, \vec{v}_r \in L$ and arbitrary scalars $\alpha, \ldots, \alpha_r \in F$,
$$a_1\vec{v}_1 + \cdots + \alpha_r\vec{v}_r = \vec{0} \implies a_1 = \cdots = \alpha_r = 0$$
A subset L of a vector space V is called **linearly dependent** if it is not linearly independent (duh..). This means there exists pairwise different vectors $\vec{v}j_1, \ldots, \vec{v}_r \in L$ and scalars $\alpha_1, \ldots, \alpha_r \in F$, not all zero, such that $\alpha_1\vec{v}_1 + \cdots \alpha_r\vec{v}_r = \vec{0}$

1.3 Linear Independence and Bases

Definition 1.3.1: Basis of a Vector Space

A **basis of a vector space** V is a linearly independent generating set in V

Example 1.3.2: Standard Basis

Let F be a field and $n \in \mathbb{N}$. We consider the following vectors in F^n
$$\vec{e}_i = (0, \ldots, 0, 1, 0, \ldots, 0)$$
with one 1 in the i-th place and zero everywhere else. Then $\vec{e}_1, \ldots, \vec{e}_n$ form an ordered basis of F^n, the so-called **standard basis of** F^n

Theorem 1.3.3: Linear combinations of basis elements

Let F be a field, V a vector space over F and $\vec{v}_1, \ldots, \vec{v}_r \in V$ vectors. The family $(\vec{v}_i)_{1 \le i \le r}$ is a basis of V if and only if the following "evaluation" mapping
$$\psi : F^r \to V$$
$$(\alpha_1, \ldots, a_r) \mapsto a_1\vec{v}_1 + \cdots + \alpha_r\vec{v}_r$$
is a bijection
If we label our ordered family by $\mathcal{A} = (\vec{v}_1, \ldots, \vec{v}_r)$, then we done the above mapping by
$$\psi = \psi_{\mathcal{A}} : F^r \to V$$

Theorem 1.3.4: Characterisations of Bases

The following are equivalent for a subset E of a vector space V:

1. E is a basis, i.e. a linearly independent generating set
2. E is minimal among all generating sets, meaning that $E \backslash \{\vec{v}\}$ does not generate V, for any $\vec{v} \in E$
3. E is maximal among all linearly independent subsets, meaning that $E \cup \{\vec{v}\}$ is linearly dependent for any $\vec{v} \in V$

Corrollary: Let V be a finitely generated vector space over a field F. Then V has a finite basis

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Basis Characterisation Variant

1. If $L \subset V$ is a linearly independent subset and E is minimal amongst all generating sets of V with the property that $L \subseteq E$, then E is a basis.
2. If $E \subseteq V$ is a generating set and if L is maximal amongst all linearly independent sets of V with the property $L \subseteq E$, then L is a basis.

Definition 1.3.5: Free Vector Space

Let X be a set and F a field. The set $\mathrm{Maps}(X, F)$ of all mappings $f : X \to F$ becomes an F-vector space with the operations of pointwise addition and multiplication by a scalar. The subset of all mappings which send almost all elements of X to zero is a vector subspace
$$F\langle X \rangle \subseteq \mathrm{Maps}(X, F)$$
This subspace is called the **free vector space on the set** X

Theorem 1.3.6: Variant of Linear Combinations

Let F be a field, V be an F-vector space and $(\vec{v}_i)_{i \in I}$ a family of vectors from the vector space V. The following are equivalent:

1. The family $(\vec{v}_i)_{i \in I}$ is a basis for V
2. For each $\vec{v} \in V$ there is precisely one family $(a_i)_{i \in I}$ of elements of F, almost all which are zero and such that
$$\vec{v} = \sum_{i = I} a_i\vec{v}_i$$

1.4 Dimension of a Vector Space

Theorem 1.4.1: Fundamental Estimate of LinAlg

No linearly independent subset of a given vector has more elements than a generating set. Thus if V is a vector space, $L \subset V$ a linearly independent subset and $E \subseteq V$ a generating set, then

$$|L| \leq |E|$$

Theorem 1.4.2: Steinitz Exchange Theorem

Let V be a vector space, $L \subset V$ a finite linearly independent subset and $E \subseteq V$ a generating set. Then there is an injection $\phi : L \hookrightarrow E$ such that $(E \backslash \phi(L)) \cup L$ is also a generating set for V

Let V be a vector space, $M \subseteq V$ a linearly independent subset, and $E \subseteq V$ a generating subset, such that $M \subseteq E$. If $\vec{w} \in V \backslash M$ is a vector $\notin M$ such that $M \cup \{\vec{w}\}$ is linearly independent, then there exists $\vec{e} \in E \backslash M$ such that $(E \backslash \{\vec{e}\}) \cup \{\vec{w}\}$ is a generating set

Theorem 1.4.3: Cardinality of Bases

Let V be a finitely generated vector space.

1. V has a finite basis
2. V cannot have an infinite basis
3. Any two bases of V have the same number of elements

Definition 1.4.4: Dimension of a Vector Space

The cardinality of a basis of a finitely generated vector space V is called the **dimension** of V, written $\dim V$. If F is a field, and we want to denote that we mean dimension as an F-vector space, then we write $\dim_F V$. If the vector space is not finitely generated, then we say $\dim V = \infty$ and call V **infinite dimensional**.

Theorem 1.4.5: Dimension Theorems

Cardinality Criterion for Bases

1. Each linearly independent subset $L \subset V$ has at most $\dim V$ elements, and if $|L| = \dim V$ then L is a basis
2. Each generating set $E \subseteq V$ has at least $\dim V$ elements, and if $|E| = \dim V$ then E is a basis

Dimension Estimate for Vector Subspaces: A proper vector subspace of a finite dimensional vector space has itself a strictly smaller dimension

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If $U \subseteq V$ is a vector subspace of an arbitrary vector space, then we have $\dim U \leq \dim V$ and if we have $\dim U = \dim V < \infty$ then it follows that $U = V$

1.5 Linear Mappings

Definition 1.5.1: Linear Mappings

Let V, W be vector spaces over a field F. A mapping $f : V \to W$ is called **linear**, or **F-linear**, or even a **homomorphism of F-vector spaces** if for all $\vec{v}_1, \vec{v}_2 \in V$ and $\lambda \in F$ we have

$$f(\vec{v}_1 + \vec{v}_2) = f(\vec{v}_1) + f(\vec{v}_2)$$
$$f(\lambda \vec{v}_1) = \lambda f(\vec{v}_1)$$

A bijective linear mapping is called an **isomorphism** of vector spaces. If there is an isomorphism between two vector spaces, we call them **isomorphic**. A homomorphism $V \to V$ is called an **endomorphism** of V. An isomorphism $V \to V$ is called an **automorphism** of V

Two vector subspaces V_1, V_2 of a vector space V are called **complementary** if addition defines a bijection

$$V_1 \times V_2 \xrightarrow{\sim} V$$

something about direct sums

Theorem 1.5.2: Classifying VecSpaces by Dimension

Let n be a natural number. Then a vector space over a field F is isomorphic to F^n iff it has dimension n

Theorem 1.5.3: Linear Mapping and Bases

Let V, W be vector spaces over a field F. The set of all homomorphisms from V to W is denoted by

$$\mathrm{Hom}_F(V, W) = \mathrm{Hom}(V, W) \subseteq \mathrm{Maps}(V, W)$$

Let $B \subset V$ be a basis. Then restriction of a mapping gives a bijection

$$\mathrm{Hom}_F(V, W) \xrightarrow{\sim} \mathrm{Maps}(B, W)$$
$$f \mapsto f|_B$$

Theorem 1.5.4: Inverse Mappings

1. Every injective linear mapping $f : V \hookrightarrow W$ has a **left inverse**, or a linear mapping $g : W \to V$ s.t. $g \circ f = \mathrm{id}_V$
2. Every surjective linear mapping $f : V \twoheadrightarrow W$ has a **right inverse**, or a linear mapping $G : W \to V$ s.t. $f \circ g = \mathrm{id}_W$

Definition 1.5.5: Image and Kernel of a map

The **image** of a linear mapping $f : V \to W$ is the subset $\mathrm{im}(f) = f(V) \subseteq W$. It is a vector subspace of W. The preimage of the zero vector of a linear mapping $f : V \to W$ is denoted by:

$$\ker(f) := f^{-1}(0) = \{v \in V : f(v) = 0\}$$

and is called the **kernel** of the linear mapping f. The kernel is a subspace of V

Mini lemma: A linear mapping is injective iff its kernel is zero

Theorem 1.5.6: Rank-Nullity / Dimension Theorem

Let $f : V \to W$ be a linear mapping between vector spaces. Then:

$$\dim V = \dim(\ker f) + \dim(\mathrm{im}\, f)$$

Dimension of $\mathrm{im}\, f$= **rank** of f, dimension of $\ker f$ = **nullity** of f

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Let V be a vector space, and $U, W \subseteq V$ vector subspaces. Then

$$\dim(U + W) + \dim(U \cap W) = \dim U + \dim W$$

2 Linear Mappings and Matrices

2.1 Linear Mappings $F^m \to F^n$ and Matrices

Theorem 2.1.1: Linear Maps $F^m \to F^n$ and Matrices

Let F be a field and let $m, n \in \mathbb{N}$. There is a bijection between the space of linear mappings $F^m \to F^n$ and the set of matrices with n rows, m columns, and entries in F:

$$M : \mathrm{Hom}_F(F^m, F^n) \xrightarrow{\sim} \mathrm{Mat}(n \times m; F)$$
$$f \mapsto [f]$$

This attaches to each linear mapping f its **representing matrix** $M(f) := [f]$. The columns of this matrix are the images under f of the standard basis elements of F^m

$$[f] := (f(\vec{e}_1) | f(\vec{e}_2) | \mid \cdots \mid f(\vec{e}_m))$$

Definition 2.1.2: Matrix Multiplication

Let $n, m, \ell \in \mathbb{N}$, F a field, and let $A \in \mathrm{Mat}(n \times m; F)$ and $B \in \mathrm{Mat}(m \times \ell; F)$ be matrices. The **product** $A \circ B = AB \in \mathrm{Mat}(n \times \ell; F)$ is the matrix defined by

$$(AB)_{ik} = \sum_{j=1}^{m} A_{ij} B_{jk}$$

Theorem 2.1.3: Composition of maps to products

Let $g : F^\ell \to F^m$ and $f : F^m \to F^n$ be linear mappings. The representing matrix of their composition is the product of their representing matrices:

$$[f \circ g] = [f] \circ [g]$$

Theorem 2.1.4: Calculating with Matrices

- (A + A')B = AB + A'B
- A(B + B') + AB + AB'
- IB = B
- AI = A
- (AB)C = A(BC)

2.2 Matrix Definitions

Definition 2.2.1: Big def-thm pairs

Def: A matrix A is called **invertible** if there exists matrices B and C such that $BA = I$ and $AC = I$

Thm: Invertible Equivalence

1. There exists a square matrix B such that $BA = I$
2. There exists a square matrix C such that $AC = I$
3. The square matrix A is invertible

Def: An **elementary matrix** is any square matrix that differs from the identity matrix in at least one entry

Thm: Every square matrix with entries in a field can be written as a product of elementary matrices

Def: Any matrix whose only non-zero entries lie on the diagonal, and which has first 1's along the diagonal and then 0's, is said to be in **Smith Normal Form**

Thm: For each matrix $A \in \text{Mat}(n \times m; F)$ there exist invertible matrices P and Q such that PAQ is a matrix in Smith Normal Form

Thm: Let $f : V \to W$ be a linear map between finite dim. F-vector spaces. There exists two ordered bases $\mathcal{A}$ of V, and $\mathcal{B}$ of W s.t. the representing matrix ${}_{\mathcal{B}}[f]_{\mathcal{A}}$ has zero entries everywhere except possibly on the diagonal, and along the diagonal there are 1's first, followed by 0's

Def: The **column rank** of a matrix $A \in \text{Mat}(n \times m; F)$ is the dimension of the subspace of F^n generated by the columns of A. Similarly, the **row rank** of A is the dimension of the subspace of F^m generated by the rows of A.

Thm: The column and row rank of any matrix are equal

Def: Since they are both the same, "column" and "row" can be omitted for the **rank of a matrix**, written as $\text{rk}\, A$. If the rank is equal to the no. of rows/columns, then the matrix has **full rank**

Def: The **trace** of a square matrix is defined to be the sum of its diagonal entries, denoted by $\text{tr}(A)$

2.3 Abstract Linear Mappings and Matrices

Theorem 2.3.1: Representing Matrices

Let F be a field, V and W vector spaces over F with ordered bases $\mathcal{A} = (\vec{v}_1, \ldots, \vec{v}_m)$ and $\mathcal{B} = (\vec{w}_1, \ldots, \vec{w}_n)$. Then to each linear mapping $f : V \to W$ we associate a **representing matrix** ${}_{\mathcal{B}}[f]\mathcal{A}$ whose entries a_{ij} are defined by the identity

$$f(\vec{v}_j) = a_{1j}\vec{w}_1 + \cdots + a_{nj}\vec{w}_n \in W$$

This makes a bijection, which is an isomorphism of vector spaces:

$$M_{\mathcal{B}}^{\mathcal{A}} : \text{Hom}_F(V, W) \xrightarrow{\sim} \text{Mat}(n \times m; F)$$
$$f \mapsto {}_{\mathcal{B}}[f]_{\mathcal{A}}$$

Theorem 2.3.2: Repr. Mat of Compositions

Let F be a field and U, V, W finite dimensional vector spaces over kF with ordered bases $\mathcal{A}, \mathcal{B}, \mathcal{C}$. If $f : U \to V$ and $g : V \to W$ are linear mappings, then the representing matrix of the composition $g \circ f : U \to W$ is the matrix product of the representing matrices of f and g:

$${}_{\mathcal{C}}[g \circ f]_{\mathcal{A}} = {}_{\mathcal{C}}[g]_{\mathcal{B}} \circ {}_{\mathcal{B}}[f]_{\mathcal{A}}$$

Definition 2.3.3: Representation of a vector

Let V be a finite dimensional vector space with an ordered basis $\mathcal{A} = (\vec{v}_1, \ldots, \vec{v}_m)$. We'll denote the inverse to the bijection in 1.3.3 "$\Phi_{\mathcal{A}} : F^m \xrightarrow{\sim} V, (\alpha_1, \ldots, \alpha_m)^T \mapsto \alpha_1\vec{v}_1 + \cdots + \alpha_m\vec{v}_m$" by

$$\vec{v} \mapsto {}_{\mathcal{A}}[\vec{v}]$$

The column vector ${}_{\mathcal{A}}[\vec{v}]$ is called the **representation of the vector $\vec{v}$ with respect to the basis $\mathcal{A}$**

Thm: Representation of the Image of a Vector: Let V, W be finite dim. vector spaces over F with ordered bases $\mathcal{A}, \mathcal{B}$ and let $f : V \to W$ be a linear mapping. The following holds for $\vec{v} \in V$:

$${}_{\mathcal{B}}[f(\vec{v})] = {}_{\mathcal{B}}[f]_{\mathcal{A}} \circ {}_{\mathcal{A}}[\vec{v}]$$

2.4 Change of a Matrix by Change of Basis

Definition 2.4.1: Change of Basis Matrix

Let $\mathcal{A} = (\vec{v}_1, \ldots, \vec{v}_n)$ and $\mathcal{B} = (\vec{w}_1, \ldots, \vec{w}_n)$ be ordered basies of the same F-vector space V. Then the matrix representing the identity mapping w.r.t. these bases

$${}_{\mathcal{B}}[\text{id}_V]_{\mathcal{A}}$$

is called a **change of basis matrix**. By definition, its entries are given by the equalities $\vec{v}_j = \sum_{i=1}^{n} a_{ij}\vec{w}_i$

Theorem 2.4.2: Change of Basis

Let V and W be finite dimensional vector spaces over F and let $f : V \to W$ be a linear mapping. Suppose that $\mathcal{A}, \mathcal{A}'$ are ordered bases of V and $\mathcal{B}, \mathcal{B}'$ are ordered bases of W. Then

$${}_{\mathcal{B}'}[f]_{\mathcal{A}'} = {}_{\mathcal{B}'}[\text{id}_W]_{\mathcal{B}} \circ {}_{\mathcal{B}}[f]_{\mathcal{A}} \circ {}_{\mathcal{A}}[\text{id}_V]_{\mathcal{A}'}$$

Let V be a finite dimensional vector space and let $f : V \to V$ be an endomorphim of V. Suppose that $\mathcal{A}, \mathcal{A}'$ are ordered bases of V. Then

$${}_{\mathcal{A}'}[f]_{\mathcal{A}'} = {}_{\mathcal{A}}[\text{id}_V]_{\mathcal{A}'}^{-1} \circ {}_{\mathcal{A}}[f]_{\mathcal{A}} \circ {}_{\mathcal{A}}[\text{id}_V]_{\mathcal{A}'}$$

3 Rings and Modules

3.1 Ring basics

Definition 3.1.1: Definition of a Ring

A **ring** is a set with two operations $(\mathbb{R}, +, \cdot)$ that satisfy:

1. $(R, +)$ is an abelian group
2. $(R, \cdot)$ is a **monoid**, meaning that it is a set with **Associativity** and **Identity**, or in other words, a monoid is a group without the neccessity of having the **Inverse** axiom
3. The distributive laws hold, meaning that for all $a, b, c \in R$,

$$a \cdot (b + c) = (a \cdot b) + (a \cdot c))$$
$$(a + b) \cdot c = (a \cdot c) + (b \cdot c)$$

The two operations are called **addition** and **multiplication** in our ring. A ring in which multiplication, that is $a \cdot b = b \cdot a$ for all $a, b \in R$, is a **commutative ring**

Note: We denote the identity of the monoid $(R, \cdot)$ as 1, and the additive identity of $(R, +)$ as 0_R or 0

Note: We define the **null ring** or **zero ring** as a ring where R is a single element set, i.e. $\{0\}$ where $0 + 0 = 0$ and $0 \times 0 = 0$

Example 3.1.2: Modulo Rings

Let $m \in \mathbb{Z}$. Then the set of **integers modulo** m, written

$$\mathbb{Z}/m\mathbb{Z}$$

is a ring. The elements of $\mathbb{Z}/m\mathbb{Z}$ consist of **congruence classes** of integers modulo m - that is, the elements are the subsets T of $\mathbb{Z}$ of the form $T = a + m\mathbb{Z}$ with $a \in \mathbb{Z}$. Think of these as the set of integers that have the same remainder when you divide them by m. I denote the above congruence class by $\overline{a}$. Obviously $\overline{a} = \overline{b}$ is the same as $a - b \in m\mathbb{Z}$, and often I'll write

$$a \equiv b \mod m$$

3.2 Linking Rings to Fields and Further Properties

Definition 3.2.1: Ring definition of a field

A **field** is a non-zero commutative ring F in which every non-zero element $a \in F$ has an inverse $a^{-1} \in F$, that is an element a^{-1} with the property that $a \cdot a^{-1} = a^{-1} \cdot a = 1$

Definition 3.2.2: Multiples of an abelian group

Let $m \in \mathbb{Z}$. The **m-th multiple ma of an element** $a$in an abelian group R is:

$$ma = \underbrace{a + a + \cdots + a}_{m \text{ terms}} \quad \text{if} m > 0$$

$0a = 0$ and negative multiples are defined by $(-m)a = -(ma)$

Theorem 3.2.3: Properties of Rings

Lemma set 1: Let R be a ring and let $a, b \in R$. Then:

1. $0a = 0 = a0$
2. $(-a)b = -(ab) = a(-b)$
3. $(-a)(-b) = ab$

Lemma set 2: Let R be a ring, $a, b \in R$ and $m, n \in \mathbb{Z}$. Then:

1. $m(a+b) = ma + mb$
2. $(m+n)a = ma + na$
3. $m(na) = (mn)a$
4. $m(ab) = (ma)b = a(mb)$
5. $(ma)(nb) = (mn)(ab)$

Prime Property for Fields: Let m be a natural number. The commutative ring $\mathbb{Z}/m\mathbb{Z}$ is a field if and only if m is prime

Definition 3.2.4: Unit of a ring

Let R be a ring. An element $a \in R$ is called a **unit** if it is *invertible* in R or in other words *has a multiplicative inverse in* R, meaning that there exists $a^{-1} \in R$ such that

$$aa^{-1} = 1 = a^{-1}a$$

Thm: The set $R^\times$ of units in a ring R forms a group under multiplication

Definition 3.2.5: zero-divisors of a ring

In a ring R, a non-zero element a is called a **zero-divisor** or **divisor of zero** if there exists a non-zero element b such that either $ab = 0$ or $ba = 0$.

Definition 3.2.6: Integral Domain

An **integral domain** is a non-zero commutative ring that has no zero-divisors. The following two laws hold:

1. $ab = 0 \implies a = 0$ or $b = 0$
2. $a \neq 0$ and $b \neq 0 \implies ab \neq 0$

Theorem 3.2.7: Integral Domain Properties

- **Cancellation Law**: Let R be an integral domain and let $a, b, c \in R$. If $ab = ac$ and $a \neq 0$ then $b = c$
- Let m be a natural number. Then $\mathbb{Z}/m\mathbb{Z}$ is an integral domain if and only if m is prime.
- Every **finite** integral domain is a field.

3.3 Polynomials

Definition 3.3.1: Polynomial

Let R be a ring. A **polynomial over** R is an expression of the form

$$P = a_0 + a_1X + a_2X^2 + \cdots + a_mX^m$$

for some non-negative $m \in \mathbb{Z}$ and elements $a_i \in R$ for $0 \leq i \leq m$.

- The set of all polynomials over R is denoted by $R[X]$.
- In the case where a_m is non-zero, the polynomial P has **degree** m, (written $\deg(P)$), and a_m is its **leading coefficient**
- When the leading coefficient is 1 the polynomial is a **monic polynomial**.
- A polynomial of degree one is called **linear**, degree two is called **quadractic**, and degree three is called **cubic**.

Definition 3.3.2: Ring of Polynomials

The set $R[X]$ becomes a ring called the **ring of polynomials with coefficients in** R**, or over** R. The zero and the identity of $R[X]$ are the zero and identity of R, respectively.

Theorem 3.3.3: Properties of a Polynomial Ring

- If R is a ring with no zero-divisors, then $R[X]$ has no zero-divisors and $\deg(PQ) = \deg(P) + \deg(Q)$ for non-zero $P, Q \in R[X]$.
- If R is an integral domain, then so is $R[X]$
- Let R be an integral domain and let $P, Q \in R[X]$ with Q monic. Then there exists unique $A, B \in R[X]$ such that $P = AQ + B$ and $\deg(B) < \deg(Q)$ or $B = 0$

Definition 3.3.4: Evaluating a Function

Let R be a commutative ring and $P \in R[X]$ a polynomial. Then P can be **evaluated** at the element $\lambda \in R$ to produce $P(\lambda)$ by replacing the powers of X in P by the corresponding powers of λ. In this way we have a mapping

$$R[X] \to \mathrm{Maps}(R, R)$$

This is the precise definition of thinking of a polynomial as a function. An element $\lambda \in R$ is a **root** of P if $P(\lambda) = 0$

Thm: Let R be a commutative ring, let $\lambda \in R$ and $P(X) \in R[X]$. Then λ is a root of $P(X)$ if and only if $(X - \lambda)$ divides $P(X)$

Theorem 3.3.5: Degrees of Polynomial Roots

Let R be a field, or more generally an integral domain. Then a non-zero polynomial $P \in R[X]\backslash\{0\}$ has at most $\deg(P)$ roots in R

Definition 3.3.6: Algebraically closed fields

A field F is **algebraically closed** if each non-constant polynomial $P \in F[X]\backslash F$ with coefficients in our field has a root in our field F

Theorem 3.3.7: The Fundamental Theorem of Algebra

The field of complex numbers $\mathbb{C}$ is algebraically closed.

Theorem 3.3.8: Linear Factors of Closed Fields

If F is an algebraically closed field, then every non-zero polynomial $P \in F[X]\backslash\{0\}$ **decomposes into linear factors**

$$P = c(X - \lambda_1)\cdots(X - \lambda_n)$$

with $n \geq 0$, $c \in F^\times$ and $\lambda_1, \ldots, \lambda_n \in F$. This decomposition is unique up to reordering the factors

3.4 Homomorphisms, Ideals, and Substrings

Definition 3.4.1: Ring Homomorphisms

Let R and S be rings. A mapping $f : R \to S$ is a **ring homomorphism** if the following hold for all $x, y \in R$:

$$f(x+y) = f(x) + f(y)$$
$$f(xy) = f(x)f(y)$$

Theorem 3.4.2: Properties of Ring Homomorphisms

Let R and S be rings and $f : R \to S$ a ring homomorphism. Then for all $x, y \in R$ and $m \in \mathbb{Z}$:

1. $f(0_R) = 0_S$, where 0_R and 0_S are the zeros of R and S
2. $f(-x) = -f(x)$
3. $f(x - y) =$ f(x) - f(y)
4. $f(mx) = mf(x)$
5. $f(x^n) = (f(x))^n$ for all $x \in R$ and $n \in \mathbb{N}$

Definition 3.4.3: Ideal

A subset I of a ring R is an **ideal**, $I \trianglelefteq R$, if the following hold:

1. $I \neq \emptyset$
2. I is closed under subtraction
3. for all $i \in I$ and $r \in R$ we have $ri, ir \in I$

Definition 3.4.4: Generated Ideals

Let R be a commutative ring and let $T \subset R$. Then the **ideal of R generated by T** is the set

$$_R\langle T\rangle = \{r_1t_1 + \cdots + r_mt_m : t_1, \dots, t_m \in T, r_1, \dots, r_m \in R\}$$

Theorem 3.4.5

Let R be a commutative ring and let $T \subseteq R$. Then $_R\langle T\rangle$ is the smallest ideal of R that contains T

Definition 3.4.6: Principal Ideal

Let R be a commutative ring. An ideal I of R is called a **principal ideal** if $I = \langle t\rangle$ for some $t \in R$

Theorem 3.4.7: Kernels as Ideals

- Let R and S be rings and $f : R \to S$ a ring homomorphism. Then $\ker f$ is an ideal of R.
- f is injective if and only if $\ker f = \{0\}$
- The intersection of any collection of ideals of a ring R is an ideal of R
- Let I and J be ideals of a ring R. Then
$$I + J = \{a + b : a \in I, b \in J\}$$
is an ideal of R

Definition 3.4.8: Subrings

Let R be a ring. $R' \subset R$ is a **subring** of R if R' is itself a ring under the operations of addition and multiplication defined in R.

Thm: Test for subring: Let R be a subset of a ring R. Then R' is a subring iff:

1. R' has a multiplicative identity
2. R' is closed under subtraction: $a, b \in R' \to a - b \in R'$
3. R' is closed under multiplication

Thm: Let R and S be rings and $f : R \to S$ a ring homomorphism.

1. If R' is a subring of R then $f(R')$ is a subring of S. In particular, $\operatorname{im} f$ is a subring of S.
2. Assume that $f(1_R) = 1_S$. Then if x is a unit in R, $f(x)$ is a unit in S and $(f(x))^{-1} = f(x^{-1})$. In this case, f restricts to a group homomorphism $f|_{R^X} : R^X \to S^X$

3.5 Equivalence Relations

Definition 3.5.1: Equivalence Relations

A **relation** R on a set X is a subset $R \subseteq X \times X$. In the context of relations, it's written xRy instead of $(x, y) \in R$. R is an **equivalence relation on** X when for all elements $x, y, z \in X$ the following hold:

1. **Reflexivity**: xRx
2. **Symmetry**: $xRy \iff yRx$
3. **Transivity**: xRy and $yRz \implies xRz$

Definition 3.5.2: Equivalence Classes

Suppose that $\sim$ is an equivalence relation on a set X. For $x \in X$ the set $E(x) := \{z \in X : z \sim x\}$ is called the **equivalence class of** x. A subset $E \subseteq X$ is called an **equivalence class** for our equivalence relation if there is an $x \in X$ for which $E = E(x)$. An element of an equivalence class is called a **representive** of the class. A subset $Z \subseteq X$ containing precisely one element from each equivalence class is called a **system of representatives** for the equivalence relation

Definition 3.5.3: Set of Equivalence Classes

Given an equivalence relation $\sim$ on the set X I will denote the **set of equivalence classes**, which is a subset of the power set $\mathcal{P}(X)$, by

$$(X/\sim) := \{E(x) : x \in X\}$$

There is a canonical mapping $\text{can} : X \to (X/\sim)$, $x \mapsto E(x)$ (surjection)

3.6 Factor Rings and First Isomorphism Theorem

Definition 3.6.1: Coset

Let $I \trianglelefteq R$ be an ideal in a ring R. The set

$$x + I := \{x + i : i \in I\} \subseteq R$$

is a **coset of I in** R or the **coset of x w.r.t I in** R

Definition 3.6.2: Factor Ring

Let R be a ring, $I \trianglelefteq R$ be an ideal, and $\sim$ the equivalence relation defined by $x \sim y \iff x - y \in I$. Then R/I, the **factor ring of R by I** or **the quotient of R by** I, is the set $(R/\sim)$ of cosets of I in R

Theorem 3.6.3

Let R be a ring and $I \trianglelefteq R$ an ideal. Then R/I is a ring, where the operation of addition is defined by

$$(x + I)\dot{+}(y + I) = (x + y) + I \qquad \text{for all } x, y \in R$$

and multiplication is defined by

$$(x + I) \cdot (y + I) = xy + I \qquad \text{for all } x, y \in R$$

Theorem 3.6.4: Universal Property of Factor Rings

Let R be a ring and I an ideal of R

1. The mapping $\text{can} : R \to R/I$ sending r to $r + I$ for all $r \in R$ is a surjective ring homomorphism with kernel I
2. If $f : R \to S$ is a ring homomorphism with $f(I) = \{0_S\}$, so that $I \subseteq \ker f$ then there is a unique ring homomorphism $\overline{f} : R/I \to S$ such that $f = \overline{f} \circ \text{can}$

Theorem 3.6.5: First Isomorphism Theorem for Rings

Let R and S be rings. Then every ring homomorphism $f : R \to S$ induces a ring isomorphism

$$\overline{f} : R/\ker f \xrightarrow{\sim} \operatorname{im} f$$

3.7 Modules

Definition 3.7.1: Module

A **(left) module** M **over a ring** R (or an R**-module**) is a pair consisting of an abelian group $M = (M, \dot{+})$ a mapping

$$R \times M \to M$$
$$(r, a) \mapsto ra$$

s.t. for all $r, s \in R$ and $a, b \in M$, we have:

- **Distributivity 1**: $r(a\dot{+}b) = (ra)\dot{+}(rb)$
- **Distributivity 2**: $(r + s)a = (ra)\dot{+}(sa)$
- **Associativity**: $r(sa) = (rs)a$
- **Identity**: $1_Ra = a$

Theorem 3.7.2: Module Lemmas

Let R be a ring and M an R-module

1. $0_Ra = 0_M$ for all $a \in M$
2. $r0_m = 0_M$ for all $r \in R$
3. $(-r)a = r(-a) = -(ra)$ for all $r \in R$, $a \in M$

Definition 3.7.3: Module Homomorphisms

Let R be a ring and let M, N be R-modules. A mapping $f : M \to N$ is an **R-homomorphism** or *homomorphism* if the following hold for all $a, \in M$ and $r \in R$

$$f(a+b) = f(a) + f(b)$$
$$f(ra) = rf(a)$$

- The **kernel** of f is $\ker f = \{a \in M : f(a) = 0_N\} \subseteq M$
- The **image** of f is $\operatorname{im} f = \{f(a) : a \in M\} \subseteq N$
- If f is a bijection then it is an **R-module isomorphim** or **isomorphism**, written $M \cong N$, and say M and N are **isomorphic**

Definition 3.7.4: Submodules

A non-empty subset M' of an R-module M is a **submodule** if M' is an R-module with respect to the operations of the R-module M **restricted** to M'

Thm: Let R be a ring and let M be an R-module. A subset M' of M is a submodule if and only if

1. $0_M \in M'$
2. $a, b \in M' \implies a - b \in M'$
3. $r \in R, a \in M' \implies ra \in M'$

Theorem 3.7.5: Submodule lemmas

- Let $f : M \to N$ be an R-homomorphism. Then $\ker f$ is a submodule of M and $\operatorname{im} f$ is a submodule of N
- Let R be a ring, M an R-homomorphism. Then f is injective if and only if $\ker f = \{0_M\}$

Definition 3.7.6: Generated Submodules

Let R be a ring, M an R-module nad let $T \subseteq M$. Then the **submodule of M generated by T** is the set

$$_R\langle T \rangle = \{r_1 t_1 + \cdots + r_m t_m : t_1, \ldots, t_m \in T, r_1, \ldots, r_m \in R\}$$

together with the zero element in the case $T = \emptyset$. If $T = \{t_1, \ldots, t_n\}$, a finite set, we write $_R\langle t_1, \ldots, t_n \rangle$ instead of $_R\langle \{t_1, \ldots, t_n\} \rangle$. The module M is **finitely generated** if it is generated by a finite set: $M = {_R\langle t_1, \ldots, t_n \rangle}$. It is called **cyclic** if it is generated by a singleton $M = {_R\langle T \rangle}$

Definition 3.7.7: Generated Submodule lemmas

- Let $T \subseteq M$. Then $_R\langle T \rangle$ is the smallest submodule of M that contains T
- The intersection of any collection of submodules of M is a submodule of M.
- Let M_1 and M_2 be submodules of a M. Then
$$M_1 + M_2 = \{a + b : a \in M_1, b \in M_2\}$$
is a submodule of M

Definition 3.7.8: Submodule Cosets

Let R be a ring, M an R-module, and N a submodule of M. For each $a \in M$ the **coset of a with respect to N in M** is

$$a + N = \{a + b : b \in N\}$$

It is a coset of N in the abelian group M and so is an equivalence class for the equivalence relation $a \sim b \iff a - b \in N$.

Let M/N, the **factor of N by N** or the **quotient of M by N** to be the set $(M/\sim)$ of all cosets of N in M. This becomes an R-module by introducing the operations of addition and multiplication as follows:

$$(a+N) \dot{+} (b+N) = (a+b) + N$$
$$r(a+N) = ra + N$$

for all $a, b \in M$, $r \in R$.

The zero of M/N is the coset $0_{M/N} = 0_M + N$. The negative of $a + N \in M/N$ is the coset $-(a+N) = (-a) + N$
The R-module M/N is the **factor module** of M by the submodule N

Theorem 3.7.9: Universal Property of Factor Modules

Let R be a ring, let L and M be R-modules, and N a submodule of M.

1. The mapping $\mathrm{can} : M \to M/N$ sending a to $a + N$ for all $a \in M$ is a surjective R-homomorphism with kernel N
2. If $f : M \to L$ is an R-homomorphism with $f(N) = \{0_L\}$, so that $N \subseteq \ker f$, then there is a unique homomorphism $\overline{f} : M/N \to L$ such that $f = \overline{f} \circ \mathrm{can}$

Theorem 3.7.10: First Isomorphism Thm for Modules

Let R be a ring and let M and N be R-modules. Then every R-homomorphism $f : M \to N$ induces an R-isomorphism

$$\overline{f} : M/\ker f \xrightarrow{\sim} \operatorname{im} f$$

4 Determinants and Eigenvalues Redux

4.1 Symmetric Groups

Definition 4.1.1: Symmetric Groups

The group of all permutations of the set $\{1, 2, \ldots, n\}$, also known as bijections from $\{1, 2, \ldots, n\}$ to itself is denoted by $\mathfrak{S}_n$ (but i will just write S_n because icba) and called the **n-th symmetric group**. It is a group under composition and has $n!$ elements.

A **tranposition** is a permutation that swaps two elements of the set and leaves all the others unchanged.

Definition 4.1.2: Inversions of a permutation

An **inversion** of a permutation $\sigma \in S_n$ is a pair (i, j) such that $1 \le i < j \le n$ and $\sigma(i) > \sigma(j)$. The number of inversions of the permutation σ is called the **length of σ** and written $\ell(\sigma)$. In formulas:

$$\ell(\sigma) = |\{(i,j) : i < j \text{ but } \sigma(i) > \sigma(j)\}|$$

The **sign of σ** is defined to be the parity of the number of inversions of σ. In formulas:

$$\operatorname{sgn}(\sigma) = (-1)^{\ell(\sigma)}$$

Theorem 4.1.3: Multiplicativity of the sign

For each $n \in \mathbb{N}$ the sign of a permutation produces a group homomorphism $\operatorname{sgn} : S_n \to \{+1, -1\}$ from the symmetric group to the two-element group of signs. In formulas:

$$\operatorname{sgn}(\sigma\tau) = \operatorname{sgn}(\sigma)\operatorname{sgn}(\tau) \quad \forall \sigma, \tau \in S_n$$

Definition 4.1.4: Alternating Group of a Permutation

For $n \in \mathbb{N}$, the set of even permutations in S_n forms a subgroup of S_n because it is the kernel of the group homomorphism $\operatorname{sgn} : S_n \to \{+1, -1\}$. This group is the **alternating group** and is denoted A_n

4.2 Determinants

Definition 4.2.1: Determinants - the Leibniz Formula

Let R be a commutative ring and $n \in \mathbb{N}$. The **determinant** is a mapping $\det : \mathrm{Mat}(n; R) \to R$ from square matrices with coefficients in R to the ring R that is given by the following formula

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \mapsto \det(A) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) a_{1\sigma(1)} \ldots a_{n\sigma(n)}$$

The sum is over all permutations of n, and the coefficient $\operatorname{sgn}(\lambda)$ is the sign of the permutation σ defined above. When $n = 0$, the determinant is 1

4.3 Characterising the Determinant

Definition 4.3.1: Bilinear Forms

Let U, V, W be F-vector spaces. A **bilinear form on** $U \times V$ **with values in** W is a mapping $H : U \times V \to W$ which is a linear mapping in both of its entries. This means that it must satisfy the following properties for all $u_1, u_2 \in U$ and $v_1, v_2 \in V$ and all $\lambda \in F$:

$$\begin{aligned} H(u_1 + u_2, v_2) &= H(u_1, v_1) + H(u_2, v_1) \\ H(\lambda u_1, v_1) &= \lambda H(u_1, v_1) \\ H(u_1, v_2 + u_2) &= H(u_1, v_1) + H(u_2, v_1) \\ H(u_1, \lambda v_1) &= \lambda H(u_1, v_1) \end{aligned}$$

A bilinear form H is **symmetric** is $U = V$ and

$$H(u, v) = H(v, u) \quad \text{for all } u, v \in U$$

while it is **antisymmetric** or **alternating** if $U = V$ and

$$H(u, u) = 0 \quad \text{for all } u \in U$$

- antisymmetric $\implies H(u, v) = -H(v, u)$
- $H(u, v) = -H(v, u) \implies$ antisymmetric iff $1_F + 1_F \neq 0_F$

Definition 4.3.2: Multilinear Forms

Let $V_1, \ldots, V_n, W$ be F-vector spaces. A mapping $H : V_1 \times V_2 \times \cdots \times V_n \to W$ is a **multilinear form** or just **multilinear** if for each j, the mapping $V_j \to W$ defined by $v_j \mapsto H(v_1, \ldots, v_j, \ldots, v_n)$, with the $v_i \in V_i$ arbitrary fixed vectors of V_i for $i \neq j$ is linear.

Let V and W be F-vector spaces. A multilinear form $H : V \times \cdots \times V \to W$ is **alternating** if it vanishes on every n-tuple of elements of V that has at least two entries equal, in other words if:

$$(\exists i \neq j \text{ with } v_i = v_j) \to H(v_1, \ldots, v_i, \ldots, v_j, \ldots, v_n) = 0$$

Theorem 4.3.3: Characterisation of the Determinant

Let F be a field. The mapping

$$\det : \mathrm{Mat}(n; F) \to F$$

is the unique alternating multilinear form on n-tuples of column vectors with values in F that takes the value 1_F on the identity matrix

4.4 Rules for Calculating with Determinants

Theorem 4.4: Determinant Theorem Bank

4.4.1: Let R be a commutative ring, $A, B \in \mathrm{Mat}(n; R)$. Then

$$\det(AB) = \det(A)\det(B)$$

4.4.2: The determinant of a square matrix with entries in a field F is non-zero if and only if the matrix is invertible

4.4.3:
- If A is invertible then $\det(A^{-1}) = \det(A)^{-1}$
- If B is a square matrix then $\det(A^{-1}BA) = \det(B)$

4.4.4: For all $A \in \mathrm{Mat}(n; R)$ with R a commutative ring,

$$\det(A^T) = \det(A)$$

Definition 4.4.6: Cofactors of a Matrix

Let $A \in \mathrm{Mat}(n; R)$ for some commutative ring R and $n \in \mathbb{N}$. Let $i, j \in \mathbb{Z}$ between 1 and n. Then the (i, j) **cofactor of** A is $C_{ij} = (-1)^{i+j} \det(A\langle i, j\rangle)$ where $A\langle i, j\rangle$ is the matrix obtained from A by deleting the i-th row and j-th column.

$$C_{23} = (-1)^{2+3} \det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = -a_{11}a_{32} + a_{31}a_{12}$$

Theorem 4.4.7: Laplace's Expansion

Let $A = (a_{ij})$ be an $(n \times n)$-matrix with entries from a commutative ring R. For a fixed i, the ***i*-th row expansion of the determinant** is

$$\det(A) = \sum_{j=1}^{n} a_{ij} C_{ij}$$

and for a fixed j, the ***j*-th column expansion of the determinant** is

$$\det(A) = \sum_{i=1}^{n} a_{ij} C_{ij}$$

Definition 4.4.8: Adjugate Matrix

Let A be a $(n \times n)$-matrix with entries in a commutative ring R. The **adjugate matrix** $\mathrm{adj}(A)$ is the $(n \times n)$-matrix whose entries are $adj(A)_{ij} = C_{ji}$ where C_{ji} is the (j, i)-cofactor

Theorem 4.4.9: Cramer's Rule

Let A be a $(n \times n)$-matrix with entries in a commutative ring R. Then

$$A \cdot \mathrm{adj}(A) = (\det A) I_n$$

Theorem 4.4.11: Invertibility of Matrices

A square matrix with entries in a commutative ring R is invertible if and only if its determinant is a unit in R. That is, $A \in \mathrm{Mat}(n; R)$ is invertible if and only if $\det(A) \in R^\times$

Theorem 4.4.14: Jacobi's Formula

Let $A = (a_{ij})$ where the coefficients $a_{ij} = a_{ij}(t)$ are functions of t. Then

$$\frac{d}{dt} \det A = \mathrm{TrAdj} A \frac{dA}{dt}$$

4.5 Eigenvalues and Eigenvectors

Definition 4.5.1: Eigenvalues and Eigenvectors

Let $f : V \to V$ be an endomorphism of an F-vector space V. A scalar $\lambda \in F$ is an **eigenvalue of** f if and only if there exists a non-zero vector $\vec{v} \in V$ such that $f(\vec{v}) = \lambda\vec{v}$. Each such vector is called an **eigenvector of** f **with eigenvalue** λ. For any $\lambda \in F$, the **eigenspace of** f **with eigenvalue** λ is

$$E(\lambda, f) = \{\vec{v} \in V : f(\vec{v}) = \lambda\vec{v}\}$$

Theorem 4.5.4: Existence of Eigenvalues

Each endomorphism of a non-zero finite dimensional vector space over an algebraically closed field has an eigenvalue

Definition 4.5.6: Characteristic Polynomial

Let R be a commutative ring and let $A \in \mathrm{Mat}(n; R)$ be a square matrix with entries in R. The polynomial $\det(xI_n - A) \in R[x]$ is called the **characteristic polynomial of the matrix** A. It is denoted by

$$\chi_A(x) := \det(xI_n - A)$$

(where χ stands for χaracteristic, lol)

Theorem 4.5.8: EVs and Characteristic Polynomials

Let F be a field and $A \in \mathrm{Mat}(n; F)$ a square matrix with entries in F. The eigenvalues of the linear mapping $A : F^n \to F^n$ are exactly the roots of the characteristic polynomial χ_A

Theorem 4.5.9: Eigenvalue Remarks

- Square matrices $A, B \in \mathrm{Mat}(n; R)$ of same size are **conjugate** if
$$B = P^{-1}AP \in \mathrm{Mat}(n; R)$$
for an invertible $P \in GL(n; R)$
- Conjugacy is an equivalence relation on $\mathrm{Mat}(n; R)$
- The char. polynomials for two conjugate matrices are the same
- We can define the char. polynomials of an endomorphism $f : V \to V$ of an n-dim vector space over a field F to be
$$\chi_f(x) = \chi_{\mathcal{A}}(x) \in F[x]$$
with $A = {}_{\mathcal{A}}[f]_{\mathcal{A}} \in \mathrm{Mat}(n; R)$ the matrix of f w.r.t *any* basis $\mathcal{A}$ for V. The E.V.s of f are exactly the roots of χ_f

Theorem 4.5.10: Extending Bases

Let $f : V \to V$ be an endomorphism of an n-dimensional vector space V over a field F. Suppose given an m-dimensional subspace $W \subseteq V$ such that $f(W) \subseteq W$, so that there are defined endomorphisms of the subspace and the quotient space:
$$g : W \to W;\ \vec{w} \mapsto f(\vec{w})$$
$$h : V/W \to V/W;\ W + \vec{v} \mapsto W + f(\vec{v})$$
The characteristic polynomial of f is the product of the characteristic polynomials of g and h

4.6 Triangularisable, Diagonalisable, and Cayley-Hamilton

Definition 4.6.1: Triangularisability

Let $f : V \to V$ be an endomorphism of a finite dimensional F-vector space V. f is **triangularisable** if the vector space V has an ordered basis $\mathcal{B} = (\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n)$ such that
$$\begin{aligned} f(\vec{v}_1) &= a_{11}\vec{v}_1, \\ f(\vec{v_2}) &= a_{12}\vec{v}_1 + a_{22}\vec{v}_2, \\ &\vdots \\ f(\vec{v}_n) &= a_{1n}\vec{v}_1 + a_{2n}\vec{v}_2 + \cdots + a_{nn}\vec{v}_n \in V \end{aligned}$$
(so that the first basis vector $\vec{v}_1$ is an eigenvector, with eigenvalue a_{11}) or equivalently such that the $n \times n$ matrix ${}_{\mathcal{B}}[f]_{\mathcal{B}} = (a_{ij})$ representing f with respect to $\mathcal{B}$ is upper triangular (or any other triangular)
$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{pmatrix}$$

Theorem 4.6.1 - 4.6.3

Let $f : V \to V$ be an endomorphism of a finite dimensional F-vector space V. Then f is triangularisable iff the characteristic polynomial χ_f decomposes into linear factors in $F[x]$

Finding ordered bases - Choose from the following subspaces

1. $W = \{\mu\vec{v}_1 \mid \mu \in F\} \subseteq V$
2. $W' = \ker(f - \lambda 1_V)$. This has a basis of E.Vs $\{\vec{v}_1, \ldots, \vec{v}_r\}$
3. $W'' = \mathrm{im}(\lambda 1_V - f)$

Then extend the basis to another ordered basis $\mathcal{B}$ for V(the full space) where $\mathrm{can}(\vec{v}_j) = \vec{u}_j$ forms a basis for V/W. ${}_{\mathcal{B}}[f]_{\mathcal{B}}$ is upper triangular.

An endomorphism $A : F^n \to F^n$ is triangularisable iff $A = (a_{ij})$ is conjugate to $B = (b_{ij})(b_{ij} = 0$ for $i > j)$, an upper triangular matrix, with $P^{-1}AP = B$ for an invertible matrix P

Definition 4.6.6: Diagonalisability

An endomorphism $f : V \to V$ of an F-vector space V is **diagonalisable** iff there exists a basis of V consisting of eigenvectors of f. If V is finite dimensional then this is the same as saying that there exists an ordered basis $\mathcal{B} = \{\vec{v}_1, \ldots, \vec{v}_n\}$ where ${}_{\mathcal{B}}[f]_{\mathcal{B}} = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$. In this case, of course, $f(\vec{v}_i) = \lambda_i\vec{v}_i$.

A square matrix $A \in \mathrm{Mat}(n; F)$ is **diagonalisable** iff A is conjugate to a diagonal matrix, i.e. there exists $P \in \mathrm{GL}(n; F)$ such that $P^{-1}AP = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$. In this case the columns P are the vectors of a basis of F^n consisting of eigenvectors of A with eigenvalues $\lambda_1, \ldots, \lambda_n$

Theorem 4.6.9: Linear Independence of Eigenvectors

Let $f : V \to V$ be an endomorphism of a vector space V and let $\vec{v}_1, \ldots, \vec{v}_n$ be eigenvectors of f with pairwise different eigenvalues $\lambda_1, \ldots, \lambda_n$. Then the vectors $\vec{v}_1, \ldots, \vec{v}_n$ are linearly independent

Theorem 4.6.10: Cayley-Hamilton Theorem

Let $A \in \mathrm{Mat}(n; R)$ be a square matrix with entries in a commutative ring R. Then evaluating its characteristic polynomial $\chi_A(x) \in R[x]$ at the matrix A gives zero.

4.7 Markov Matrices

Definition 4.7.5: Markov Matrix

A matrix M whose entires are non-negative and s.t. the sum of the entries of each column equals 1 is a **Markov matrix** or a **stochastic matrix**

4.7.6: Suppose $M \in \mathrm{Mat}(n; \mathbb{R})$ is a M.M. Then $\lambda = 1$ is an e.v.

Theorem 4.7.10: Perron-Frobenius Theorem

If $M \in \mathrm{Mat}(n; \mathbb{R})$ is a Markov matrix with positive values, then the eigenspace $E(1, M)$ is one-dimensional. There exists a unique basis vector $\vec{v} \in E(1, M)$ with positive real entries s.t. the sum of its entries is 1

5 Inner Product Spaces

5.1 Inner Product Spaces Intro

Definition 5.1.1: Inner Product

Let V be a vector space over $\mathbb{R}$. An **inner product** on V is a mapping
$$(-,-) : V \times V \to \mathbb{R}$$
that satisfies the following for all $\vec{x}, \vec{y}, \vec{z} \in V$ and $\lambda, \mu \in \mathbb{R}$:

1. $\lambda\vec{x} + \mu\vec{y}, z = \lambda(\vec{x}, \vec{z} + \mu(\vec{y}, \vec{z}))$
2. $(\vec{x}, \vec{y}) = (\vec{y}, \vec{x})$
3. $(\vec{x}, \vec{x}) \geq 0$, with equality iff $\vec{x} = \vec{0}$

A **real inner product space** is a real vector space equipped with an inner product. **Note**: basically a generalisation of dot prod.

A **complex inner product space** is a complex vector space equipped with an inner product. This is the exact same, but condition 2 uses $(\vec{x}, \vec{y}) = \overline{(\vec{y}, \vec{x})}$ where $\overline{z}$ is the complex conjugate

Definition 5.1.5: Norm

In a real or complex inner product space, the **length** or **inner product norm** or **norm** $\|\vec{v}\| \in \mathbb{R}$ of a vector $\vec{v}$ is defined as the non-negative square root
$$\|\vec{v}\| = \sqrt{(\vec{v}, \vec{v})}$$
Vectors whose length are 1 are called **units**. Two vectors $\vec{v}, \vec{w}$ are **orthogonal**, written $\vec{v} \perp \vec{w}$, iff $(\vec{v}, \vec{w}) = 0$

The norm $\|\cdot\|$ on an inner product spacse V satisfies, for any $\vec{v}, \vec{w} \in V$ and scalar λ:

1. $\|\vec{v}\| \geq 0$ with equality iff $\vec{v} = \vec{0}$
2. $\|\lambda\vec{v}\| = |\lambda|\|\vec{v}\|$
3. $|\vec{v} + \vec{w}| \leq \|\vec{v}\| + \|\vec{w}\|$ (triangle inequality)

Definition 5.1.7: Orthonormal Family

A family $(\vec{v}_i)_{i\in I}$ for vectors from an inner product space is an **orthonormal family** if all the vectors $\vec{v}_i$ have length 1 and if they are pairwise orthogonal to each other, which, if $\delta_{i,j}$ is the **Kronecker delta** defined by

$$\delta_{i,j} = \begin{cases} 1 & i = j \\ 0 & \text{otherwise} \end{cases}$$

means that $(\vec{v}_i, \vec{v}_j) = \delta_{ij}$.
An orthonormal family that has a basis is an **orthonormal basis**

Thm 5.1.10: Every finite dimensional inner product space has an orthonormal basis

5.2 Orthogonal Complements and Projections

Definition 5.2.1: Orthogonals to a Subset

Let V be an inner product space and let $T \subseteq V$ be an arbitrary subset. Define

$$T^\perp = \{\vec{v} \in V : \vec{v} \perp \vec{t} \forall \vec{t} \in T\}$$

calling this set the **orthogonal** to T

Theorem 5.2.2: Complementary Othorgonals

Let V be an inner product space and let U be a finite dimensional subspace of V. Then U and $U^\perp$ are complementary in the sense of 1.5.1. i.e. $V = U \oplus U^\perp$

Definition 5.2.3: Orthogonal Projection

Let U be a finite dimensional subspace of an inner product space V. The space $U^\perp$ is the **orthogonal complement to** U. The **orthogonal projection from** V **onto** U is the map

$$\pi_U : V \to V$$

that sends $\vec{v} = \vec{p} + \vec{r}$ to $\vec{p}$

Prop 5.2.4: Let U be a finite dimensional subspace of an inner product space V and let π_U be the orthogonal projection from V onto U

1. π_U is a linear mapping with $\operatorname{im}(\pi_U) = U$ and $\ker(\pi_U) = U^\perp$
2. If $\{\vec{v}_1, \ldots, \vec{v}_n\}$ is an orthonormal basis of U, then π_U is given by the following formula for all $\vec{v} \in V$
$$\pi_U(\vec{v}) = \sum_{i=1}^{n} (\vec{v}, \vec{v}_i)\vec{v}_i$$
3. $\pi_U^2 = \pi_U$, that is, π_U is an idempotent

Theorem 5.2.5: Cauchy-Shwarz Inequality

Let $\vec{v}$, $\vec{w}$ be vectors in an inner product space. Then

$$|(\vec{v}, \vec{w})| \leq \|\vec{v}\| \|\vec{w}\|$$

with equality if and only if $\vec{v}$ and $\vec{w}$ are linearly dependent

Theorem 5.2.7: Gram-Shmidt Process

Let $\vec{v}_1, \ldots, \vec{v}_k$ be linearly independent vectors in an inner product space V. Then there exists an orthonormal family $\vec{w}_1, \ldots, \vec{w}_k$ with the property that for all $1 \leq i \leq k$,

$$\vec{w}_i \in \mathbb{R}_{>0}\vec{v}_i + \langle \vec{v}_{i-1}, \ldots, \vec{v}_1 \rangle$$

TODO: write how to actually do the gram-shmidt process

5.3 Adjoints and Self-Adjoints

Definition 5.3.1: Adjoints

Let V be an inner product space. Then two endomorphisms $T, S : V \to V$ are called **adjoint** to one another if the following holds for all $\vec{v}, \vec{w} \in V$:

$$(T\vec{v}, \vec{w}) = (\vec{v}, S\vec{w})$$

In this case I will write $S = T^*$ and call S the **adjoint** of T

Remark 5.3.2: Any endomorphism has at most one adjoint.

Theorem 5.3.4

Let V be a finite dimensional inner product space. Let $T : V \to V$ be an endomorphism. Then T^* exists. That is, there is a unique linear mapping $T^* : V \to V$ such that for all $\vec{v}, \vec{w} \in V$:

$$(T\vec{v}, \vec{w}) = (\vec{v}, T^*\vec{w})$$

Definition 5.3.5: Self Adjoints

An endomorphism of an inner product space $T : V \to V$ is **self-adjoint** if it equals its own adjoint, i.e. if $T^* = T$

Theorem 5.3.7: Self-Adjoint Theorem bank

Let $T : V \to V$ be a self-adjoint linear mapping on an inner product space V

1. Every eigenvalue of T is real
2. If λ and μ are distinct eigenvalues of T with corresponding eigenvectors $\vec{v}$ and $\vec{w}$, then $(\vec{v}, \vec{w}) = 0$
3. T has an eigenvalue

Definition 5.3.11: Orthogonal Matrices

An **Orthogonal matrix** is an $(n \times n)$-matrix P with real entries such that $P^T P = I_n$, or in other words such that $P^{-1} = P^T$

Definition 4.3.14: Complex Matrices

A **hermitian matrix** is one that is self-adjoint in $\mathbb{C}$, or in other words one where $A = \overline{A}^T$ holds
An **unitary matrix** is an $(n \times n)$-matrix P with complex entries such that $\overline{P}^T P = I_n$, or such that $P^{-1} = \overline{P}^T$

Theorem 5.3.9: Spectral Theorems

5.3.9: The Spectral Theorem for Self-Adjoint Endomorphisms
Let V be a finite dimensional inner product space and let $T : V \to V$ be a self-adjoint linear mapping. Then V has an orthonormal basis consisting of eigenvalues of T.

5.3.11: The Spectral Theorem for Real Symmetric Matrices
Let A be a real $(n \times n)$-symmetric matrix. Then there is an $(n \times n)$-orthogonal matrix P such that

$$P^T AP = P^{-1}AP = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$$

where $\lambda_1, \ldots, \lambda_n$ are the (necessarily real) eigenvalues of A, repeated according to their multiplicity as roots of χ_A

5.3.15: The Spectral Theorem for Hermitian Matrices
Let A be a $(n \times n)$-hermitian matrix. Then there is an $(n \times n)$-unitary matrix P such that

$$\overline{P}^T AP = P^{-1}AP = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$$

where $\lambda_1, \ldots, \lambda_n$ are the (necessarily real) eigenvalues of A, repeated according to their multiplicity as roots of χ_A

6 Jordan Normal Form

6.1 Motivation

no time for motivation over here

6.2 The Jordan Normal Form

Definition 6.2.1: Jordan Blocks

Given an integer $r \geq 1$ define an $(r \times r)$-matrix $J(r)$ called the **nilpotent Jordan block of size** r, by the rule $J(r)_{ij} = 1$ for $j = i + 1$ AND $J(r)_{ij} = 0$ otherwise
In particular, $J(1)$ is a (1×1)-matrix whose only entry is zero.

Given an integer $r \geq 1$ and a scalar $\lambda \in F$, define an $(r \times r)$-matrix $J(r, \lambda)$ called the **Jordan block of size** r **and eigenvalue** λ by the rule

$$J(r, \lambda) = \lambda I_r + J(r) = D + N$$

with $\lambda I_r = \operatorname{diag}(\lambda, \lambda, \ldots, \lambda) = D$ diagonal and $J(r) = N$ nilpotent such that $DN = ND$

Theorem 6.2.2: Jordan Normal Form

Let F be an algebraically closed field. Let V be a finite dimensional vector space and let $\phi : V \to V$ be an endomorphism of V with characteristic polynomial

$$\chi_\phi(x) = (x-\lambda_1)^{a_1}(x-\lambda_2)^{a_2}..(x-\lambda_s)^{a_s} \in F[x], a_i \geq 1, \sum_{i=1}^{s} a_i = n$$

For distinct $\lambda_1, \lambda_2, \ldots, \lambda_s \in F$. Then there exists an ordered basis $\mathcal{B}$ of V such that the matrix of ϕ with respect to the block $\mathcal{B}$ is block diagonal with Jordan blocks on the diagonal, ${}_\mathcal{B}[\phi]_\mathcal{B}$

$$= \operatorname{diag}(J(r_{11}, \lambda_1), \ldots, J(r_{1m_1}, \lambda_1), J(r_{21}, \lambda_2), \ldots, J(r_{sm_s}, \lambda_s))$$

with $r_{11}, \ldots, r_{1m_1}, r_{21,\ldots,r_{sm_s}} \geq 1$ such that

$$a_i = r_{i_1} + r_{i_2} + \cdots + r_{im_i} \quad (1 \leq i \leq s)$$

Theorem 6.3.1: Bézout's identity for polynomials

For a characteristic polynomial

$$\chi_\phi(x) = \prod_{i=1}^{s}(x - \lambda_i)^{a_i} \in F[x]$$

where each a_i is a positive integer, $\lambda_i \neq \lambda_j$ for $i \neq j$, and λ_i are e.v.s of ϕ. For each $1 \leq j \leq s$ define

$$P_j(x) = \prod_{\substack{i=1 \\ i \neq j}}^{s}(x - \lambda_i)^{a_i}$$

There exists polynomials $Q_j(x) \in F[x]$ such that

$$\sum_{j=1}^{s} P_j(x) Q_j(x) = 1$$

Definition 6.3.2: Generalised Eigenspace

The **generalised eigenspace** of ϕ with eigenvalue λ_i, $E^{\text{gen}}(\lambda_i, \phi)$ is the following subspace of V:

$$E^{\text{gen}}(\lambda_i, \phi) = \{\vec{v} \in V \mid (\phi - \lambda_i \operatorname{id}_V)^{a_i}(\vec{v}) = \vec{0}\}$$

The dimension of $E^{\text{gen}}(\lambda_i, \phi)$ is called the **algebraic multiplicity of ϕ with eigenvalue** λ_i while the dimension of the eigenspace $E(\lambda_i, \phi)$ is called the **geometric multiplicity of ϕ with eigenvalue λ**

Remark 6.3.4: The actual eigenspace is defined by

$$E(\lambda_i, \phi) = \{\vec{v} \in V \mid (\phi - \lambda_i \operatorname{id}_V)(\vec{v}) = \vec{0}\}$$

$E^{\text{gen}}(\lambda_i, \phi) \subseteq E^{\text{gen}}(\lambda_i, \phi)$, or the algebraic multiplicity of any e.v. must be greater or equal to the corresponding geometric multiplicity

Definition 6.3.4: Stable subsets

Let $f : X \to X$ be a mapping from a set X to itself. A subset $Y \subseteq X$ is **stable under** f precisely when $f(Y) \subseteq Y$, that is if $y \in Y$ then $f(y) \in Y$.

Theorem 6.3.5: Direct Sum Composition

For each $1 \leq i \leq s$, let

$$\mathcal{B}_i = \{\vec{v}_{ij} \in V \mid 1 \leq j \leq a_i\}$$

be a basis of $E^{\text{gen}}(\lambda_i, \phi)$, where a_i is the algebraic multiplicity of ϕ with eigenvalue λ_i s.t. $\sum_{i=1}^{s} a_i = n$ is the dimension of V.

1. Each $E^{\text{gen}}(\lambda_i, \phi)$ is stable under ϕ
2. For each $\vec{v} \in V$ there exist unique $\vec{v}_i \in E^{\text{gen}}(\lambda_i, \phi)$ such that $\vec{v} = \sum_{i=1}^{s} \vec{v}_i$. In other words, there is a direct sum decomposition
$$V = \bigoplus_{i=1}^{s} E^{\text{gen}}(\lambda_i, \phi)$$
with ϕ restricting to endomorphisms of the summands
$$\phi_i = \phi| : E^{\text{gen}}(\lambda_i, \phi) \to E^{\text{gen}}(\lambda_i, \phi)$$
3. Then
$$\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \cdots \cup \mathcal{B}_s = \{\vec{v}_{ij} \mid 1 \leq i \leq s, 1 \leq j \leq a_i\}$$
is a basis of V. The matrix of the endomorphism ϕ w.r.t. this basis is given by the block diagonal matrix
$${}_\mathcal{B}[\phi]_\mathcal{B} = \left(\begin{array}{c|c|c|c} B_1 & 0 & 0 & 0 \\ \hline 0 & B_2 & 0 & 0 \\ \hline & & \ddots & \\ 0 & 0 & & 0 \\ \hline 0 & 0 & 0 & B_s \end{array}\right) \in \operatorname{Mat}(n; F)$$
with $B_i = {}_{\mathcal{B}_i}[\phi_i]_{\mathcal{B}_i} \in \operatorname{Mat}(a_i; F)$

Theorem 6.3: JNF Theorem Bank

6.3.6: For each i, define a linear mapping
$$\psi_i : \frac{W_i}{W_{i-1}} \to \frac{W_{i-1}}{W_{i-2}}$$
by $\psi_i(\vec{w} + W_{i-1}) = \psi(\vec{w}) + W_{i-2}$ for $\vec{w} \in W_i$. Then ψ_i is well-defined and injective

6.3.7: Let $f : X \to Y$ be an injective linear mapping between the F-vector spaces X and Y. If $\{\vec{x}_1, \ldots, \vec{x}_t\}$ is a linearly independent set in X, then $\{f(\vec{x}_1, \ldots, \vec{x}_t)\}$ is a linearly independent set in Y

6.3.8: The set of elements $\{\vec{v}_{j,k} : 1 \leq j \leq m, 1 \leq k \leq d_j\}$ constructed in the next algorithm is a basis for W

6.3.9: Let $\mathcal{B}$ be the ordered basis of W - $\{\vec{v}_{j,k} : 1 \leq j \leq m, 1 \leq k \leq d_j\}$. Then ${}_\mathcal{B}[\psi]_\mathcal{B} =$
$$\operatorname{diag}\underbrace{J(m), .., J(m)}_{d_m \text{ times}}, \underbrace{J(m-1), .., J(m-1)}_{d_{m-1} - d_m \text{ times}}, .., \underbrace{J(1), .., J(1)}_{d_1 - d_2 \text{ times}}$$
where $J(r)$ denotes the nilpotent Jordan block of size r

Theorem 6.3: JNF Basis Algorithm

Algorithm to construct a basis for each W_i/W_{i-1}:

- Choose an arbitrary basis for W_m/W_{m-1}, say
$$\{v_{m,1} + W_{m-1}, \vec{v}_{m,2} + W_{m-1}, \ldots, \vec{v}_m, d_m + W_{m-1}\}$$
- Since $\psi_m : W_m/W_{m-1} \to W_{m-1}/W_{m-2}$ is injective by 6.3.6, 6.3.7 proves that
$$\{\psi(\vec{v}_{m,1}) + W_{m-2}, \psi(\vec{v}_m, 2) + W_{m-2}, .., \psi(\vec{v}_m, d_m + W_{m-2})\}$$
is a linearly independent set in W_{m-1}/W_{m-2}. Set $\vec{v}_{m-1,i} = \psi(\vec{v}_{m,i})$ for $1 \leq i \leq d_m$
- Choose vectors $\{\vec{v}_{m-1,i} : d_m + 1 \leq i \leq d_{m-1}\}$ so that $\{\vec{v}_{m-1,i} + W_{m-i-1} : 1 \leq k \leq d_{m-i}\}$ is a basis of W_{m-1}/W_{m-2}
- Repeat!

6.3 PageRank, again

Theorem 6.5.1

If $M \in \operatorname{Mat}(n; \mathbb{R})$ is a Markov matrix with all positive entries, consider M as a complex matrix whose entries just happen to be real. If $\lambda \in \mathbb{C}$ is an eigenvalue of M then either $\lambda = 1$ or $|\lambda| < 1$

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