Metric Spaces Notes

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Contents

1 Introduction to Metric Spaces

1.1 Defining a Metric

Metric is another name for distance. A **Metric Space** is a set equipped with a metric. A standard example is $\mathbb{R}$ with the standard metric

$$d(x, y) = |x - y|$$

We will now formally define what it means to have a metric

Theorem 1.1.1: Definition of a Metric

Let X be a non-empty set. A function $d : X \times X \to \mathbb{R}$ is called a **metric** iff for all $x,\ y,\ z \in X$,

- $d(x, y) \geq 0$ and $d(x, y) = 0 \iff x = y$
- $d(x, y) = d(y, x)$
- $d(x, y) \leq d(x, z) + d(z, y)$ (Triangle Inequality)

A non-empty set X equipped with a metric d is called a **metric space**

1.2 Examples of Metric Spaces

We can construct a metric space using the **Absolute value** equipped with the standard triangle inequality

Example 1.2.1: The Real Line

Let $X = \mathbb{R}$. Define our metric $x : X \times X \to \mathbb{R}$ by

$$d(x, y) = |x - y|$$

The first two properties are fairly trivial. The third property follows using the regular triangle inequality

$$d(x, y) = |x - y| = |(x - z) + (z - y)| \ \leq \ |x - z| + |z - y| = d(x, z) + d(z, y)$$

Remark: This can be extended not just in $\mathbb{R}^2$, but to all $\mathbb{R}^n$. By induction,

$$|x_1 + \cdots + x_N| \leq |x_1| + \cdots + |x_N|$$

If $\sum_{n=1}^{\infty} x_n$ converges absolutely, let $N \to +\infty$ to see that

$$\left|\sum_{n=1}^{\infty} x_n\right| \leq \sum_{n=1}^{\infty} |x_n|$$

A second example is the **Euclidean Plane**. The metric is defined using the **inner product** and the **norm**.

Definition 1.2.2: Inner Product

The **inner product** is defined as

$$\langle x, y\rangle = x_1 y_1 + x_2 y_2$$

Properties of the inner product: For all vectors x, y, $z \in \mathbb{R}^2$ and all real scalars a, b,

- $\langle x, x\rangle \geq 0$ and $\langle x, x\rangle = 0 \iff x = 0$
- $\langle x, y\rangle = \langle y, x\rangle$
- $\langle ax + by, z\rangle = a\langle x, z\rangle + b\langle y, z\rangle$

Remark: This is basically a formalisation of the dot product

Definition 1.2.3: Norm

The **norm** is defined as:

$$\|x\|_2 = \langle x, x\rangle^{1/2} = \sqrt{x_1^2 + x_2^2}$$

Properties of the norm: For all x, $y \in \mathbb{R}^2$, $a \in \mathbb{R}$

- $\|x\|_2 \geq 0$ and $\|x\|_2 = 0 \iff x = 0$
- $\|ax\|_2 = |a|\|x\|_2$
- $\|x + y\|_2 \leq \|x\|_2 + \|y\|_2$ (triangle inequality)

Remark: This is a formalisation of the "length of a vector"
With these two properties, we can now define the **Euclidean Metric**

Example 1.2.4: Euclidean Metric

For all $x = (x_1, x_2)$, $y = (y_1, y_2) \in \mathbb{R}^2$, define

$$d_2(x, y) = \|x - y\|_2 = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$$

Remark: Derivation of the triangle inequality is basically the same as Example 1.2.1.

$$d_2(x, y) = \|x - y\|_2 = \|(x - z) + (z - y)\|_2 \leq \|x - z\|_2 + \|z - y\|_2 = d_2(x, z) + d_2(z, y)$$

1.2.5 Proof of the euclidean triangle inequality

W.T.S:

$$\|x + y\|_2 \leq \|x\|_2 + \|y\|_2$$

Proof: Square both sides

$$\begin{aligned} \text{LHS}^2 &= \langle x + y, x + y\rangle \\ &= \langle x, x\rangle + 2\langle x, y\rangle + \langle y, y\rangle \\ &= \|x\|_2^2 + 2\langle x, y\rangle + \|y\|_2^2 \end{aligned} \qquad \text{RHS}^2 = \|x\|_2^2 + \|y\|_2^2 + 2\|x\|_2\|y\|_2$$

Discarding the equal terms, we get

$$\|x\|_2^2 + 2\langle x,y\rangle + \|y\|_2^2 \le \|x\|_2^2 + \|y\|_2^2 + 2\|x\|_2\|y\|_2$$
$$\langle x,y\rangle \le \|x\|_2\|y\|_2$$
$$\text{i.e. } x_1y_1 + x_2y_2 \le \sqrt{x_1^2 + x_2^2}\sqrt{y_1^2 + y_2^2}$$

This is the **Cauchy-Schwarz Inequality**. Various ways to prove this (watch lecture 1)

Example 1.2.6: Complex Plane

Let $X = \mathbb{C}$, $d : \mathbb{C} \times \mathbb{C} \to \mathbb{R}$

$$d(z,w) = |z - w|$$

If $z = a + ib, w = c + id$, $a, b, c, d \in \mathbb{R}$, then

$$z - w = (a - c) + i(b - d)$$

therefore,

$$d(z,w) = \sqrt{(a-c)^2 + (b-d)^2}$$

Definition 1.2.7: n-dimensional Euclidean space

Let $X = \mathbb{R}^n = \{(x_1, x_2, \ldots, x_n) : x_1, x_2, \ldots, x_n \in \mathbb{R}\}$
For $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n)$ in $\mathbb{R}^n$, define

$$\langle x,y\rangle = x_1y_1 + x_2y_2 + \cdots + x_ny_n \text{ (inner product)}$$

Properties of n-inner product: For all vectors $x, y, z \in \mathbb{R}^n$ and all real scalars a, b,

- $\langle x,x\rangle \ge 0$ and $\langle x,x\rangle = 0 \iff x = 0$
- $\langle x,y\rangle = \langle y,x\rangle$
- $\langle ax + by, z\rangle = a\langle x,z\rangle + b\langle y,z\rangle$

For $x = (x_1, x_2, \ldots, .x_n) \in \mathbb{R}^n$ define

$$\|x\|_2 = \langle x,x\rangle^{1/2} = \sqrt{x_1^2 + x_2^2 + \cdots + c_n^2}\text{(norm)}$$

Properties of n-norm: For $x, y \in \mathbb{R}^n$, $a \in \mathbb{R}$,

- $\|x\|_2 \ge 0$ and $\|x\|_2 = 0 \iff x = 0$
- $\|ax\|_2 = |a|\|x\|_2$
- $\|x + y\|_2 \le \|x\|_2 + \|y\|_2$ (triangle inequality)

Example 1.2.8: Metric in n-dim euclidean space

For $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n)$ in $\mathbb{R}^n$, define

$$\begin{aligned} d_2(x,y) &= \|x-y\|_2 \\ &= \sqrt{(x_1-y_1)^2 + (x_2-y_2)^2 + \cdots + (x_n-y_n)^2} \end{aligned}$$

Triangle inequality, cauchy schwarz, yadda yadda same as 2-dim case

1.2.9 L space

For two sequences $x = (x_1, \ldots, x_n, \ldots)$, $y = (y_1, \ldots, y_n, \ldots)$ of real numbers we wish to define

$$d_1(x,y) = \sum_{n=0}^{\infty} |x_n - y_n|$$

We need this series to converge - in particular when $y = (0, \ldots, 0, \ldots)$, we need the series $\sum_{n=1}^{\infty} |x_n|$ to converge

Definition 1.2.10: l space

We denote by ℓ^1 the set of real sequences $(x_n)_{n \in \mathbb{N}}$ for which the series $\sum_{n=1}^{\infty} |x_n|$ converges.

If $x, y \in \ell^1$ i.e. if $\sum_{n=1}^{\infty} |x_n|$ and $\sum_{n=1}^{\infty} |y_n|$ converge, then $\sum_{n=1}^{\infty} |x_n - y_n|$ converges, because for all n,

$$|x_n - y_n| \leq |x_n| + y_n$$

For $x = (x_1, \ldots, x_n, \ldots)$ in ℓ^1, we may now define

$$\|x\|_1 = \sum_{n=1}^{\infty} |x_n|$$

For $x = (x_1, \ldots, x_n, \ldots)$, $y = (y_1, \ldots, y_n, \ldots)$ in ℓ^1 we may now define

$$d_1(x,y) = \|x-y\|_1 = \sum_{n=1}^{\infty} |x_n - y_n|$$

1.3 Real Vector Spaces

Definition 1.3.1: Real Vector Spaces

A *real vector space* is a set X with two operations, addition(+) and scalar multiplication $\cdot$, with the following properties: for all $x, y, z \in X$, $a, b \in \mathbb{R}$, we have $x + y, a \cdot x \in X$, and

- $x + y = y + x$
- $x + (y + z) = (x + y) + z$
- There is an element of X denoted by 0 such that, for all x, $0 + x = x + 0 = x$
- For every $x \in X$ there exists an element of X denoted by $-x$ such that $x + (-x) = (-x) + x = 0$
- $a \cdot (x + y) = a \cdot x + a \cdot y$
- $(a + b) \cdot x = a \cdot x + b \cdot x$
- $a \cdot (b \cdot x) = (ab) \cdot X$
- $1 \cdot x = x$

(we usually write ax instead of x)

1.3.2 Normalising l 1

Properties: For all sequences $x, y \in \ell^1$ and all real scalars a,

- $\|x\|_1 \geq 0$ and $\|x\|_1 = 0 \iff x = 0$
- $\|ax\|_1 = |a|\|x\|_1$
- $\|x + y\|_1 \leq \|x\|_1 + \|y\|_1$

1.3.3 Space l-2

We denote by ℓ^2 the set of real sequences $(x_1, \dots, x_n, \dots)$ such that the seriese $\sum\limits_{n=1}^{\infty} |x_n|^2$ converges

For $x = (x_1, \dots, x_n, \dots) \in \ell^2$, $y = (y_1, \dots, y_n, \dots) \in \ell^2$ we define

- $\langle x, y \rangle = \sum_{n=1}^{\infty} x_n y_n$ (inner product)
- $\|x\|_2 = \left(\sum\limits_{n=1}^{\infty} |x_n|^2 \right)^{1/2}$ (norm)
- $d_2(x, y) = \|x - y\|_2 = \left(\sum\limits_{n=1}^{\infty} |x_n - y_n|^2 \right)^{1/2}$ (Metric)

Theorem 1.3.4: 4

ℓ^2 is a real vector space proof icba

more stuff on ℓ^2 - typical properties watch video 1

1.4 Generalising metric space features

Definition 1.4.1: Normed Vector Spaces

A *normed vector space* (or *normed linear space* or *normed space*) is a real vector space X equipped with a *norm*, i.e. a function that assigns to every vector $x \in X$ a real number $\|x\|$ so that, for all vectors x and y in X and all real scalars a,

- $\|x\| \geq 0$ and $\|x\| = 0 \iff x = 0$
- $\|ax\| = |a|\|x\|$
- $\|x + y\| \leq \|x\| + \|y\|$

If $(X, \|\cdot\|)$ is a normed vector space then

$$d(x, y) = \|x - y\|$$

defines a metric in X

Definition 1.4.2: Inner Product Spaces

Let X be a real vector space. An *inner product* on X is a function that assigns to every pair $(x, y) \in X \times X$ a real number denoted by $\langle x, y\rangle$ and has the following properties

- $\langle x, x\rangle \geq 0$ and $\langle x, x\rangle = 0 \iff x = 0$
- $\langle x, y\rangle = \langle y, x\rangle$
- $ax + by, z = a\langle x, z\rangle + b\langle y, z\rangle$

A *real inner product space* is a real vector space equipped with an inner product. If $\|\cdot, \cdot\|$ is an inner product on X, then

$$\|x\| = \sqrt{\langle x, x\rangle}$$

defines a norm and

$$d(x, y) = \|x - y\|$$

defines a metric

Example 1.4.3: Discrete metric

Let X be a non-empty set. Define $d : X \times X \to \mathbb{R}$ by

$$d(x, y) = \begin{cases} 0, & x = y \\ 1, x \neq y \end{cases}$$

Example of metric space without norm or inner prod. Another example is post office metric

theres lots of examples, i kinda cba

1.5 Open Balls

Definition 1.5.1: Open Ball

Let (X, d) be a metric space, c be a point in X, and $r > 0$. The **open ball** with center c and radius r is defined by

$$B(c, r) = \{x \in X : d(c, x) < r\}$$

Note: there are lots of different notations for this, e.g. calling it a sphere
Example: on the real line with the standard metric

$$b(c, r) = \{x \in \mathbb{R} : |x - c| < r\} = (c - r, c + r)$$

Example: on the real plane with the Euclidean metric, $X = \mathbb{R}^2$m

$$d_2(x, y) = \sqrt{|x_1 - y_1|^2 + |x_2 - y_2|}^2$$

$B(c, r)$ is the open disc with center c and radius r

Watch lecture recording for examples of open balls on:

- Discrete metric
- $\mathbb{R}^2$ with the d_1 metric
- $\mathbb{R}^2$ with the d_∞ metric

2 Convergence

2.1 Convergent Sequences in Metric Spaces

On the real line, $x_n \to x$ iff for every positive ϵ, there exists an index N such that for all indices n where $n \geq N$, we have $|x_n - x| < \epsilon$.

Definition 2.1.1: Convergent Sequence

Let (X, d) be a metric space, $(x_n)_{n=1}^{\infty}$ be a sequence in X, and $x \in X$. We say that $(x_n)_{n=1}^{\infty}$ converges to x iff for every positive ϵ, there exists an index N s.t. for all indices n with $n \geq N$a we have $d(x_n, x) < \epsilon$.
Observe that:

- $d(x_n, x) < \epsilon$ is equivalent to $x_n \in B(x, \epsilon)$.
- $x_n \to x$ in (X, d) iff $d(x_n, x) \to 0$ on the real line

Theorem 2.1.2: Uniqueness of metric limit

- Let (X, d) be a metric space, and $x, x' \in X$, $x \neq x'$. Then there exists a positive radius r s.t. $B(x, r) \cap B(x', r) = \emptyset$
- A sequence in a metric space can have at most one limit

Proof of first: $d(x, x') > 0$ because $x \neq x'$. Choose any r with $0 < r \leq \frac{d(x,x')}{2}$. If $y \in B(x, r)$, then $d(y, x) < r$, therefore

$$d(y, x' \geq d(x, x') - d(y, x) > d(x, x') - r)$$

and $d(x, x') - r \geq r$, therefore

$$d(y, x') > r$$

Therefore, $y \notin B(x', r)$
Proof of second: Let $x_n \to x$ and $x_n \to x'$ in a metric space (X, d). We claim that $x = x'$. Assume $x \neq x'$. Let $r > 0$ be s.t.

$$B(x, r) \cap B(x', r) = \emptyset$$

Since $x_n \to x$, there exists N s.t. for all n with $n \geq N$ we have

$$x_n \in B(x, r)$$

Since $x_n \to x$, there exists N' s.t. for all n with $n \geq N'$ we have

$$x_n \in B(x', r)$$

For any n with $n \geq \max\{N, N'\}$, the term x_n belongs to both balls - contradiction

Example 2.1.3: convergence in $(\mathbb{R}^N, d_2)$

A sequence

$$
\begin{aligned}
x_1 &= (x_{11}, \ldots, x_{1j}, \ldots x_{1N}) \\
x_2 &= (x_{21}, \ldots, x_{2j}, \ldots x_{2N}) \\
&\vdots \\
x_n &= (x_{n1}, \ldots, x_{nj}, \ldots x_{nN}) \\
&\vdots \\
&\downarrow \\
x &= (x_1, \ldots, x_j, \ldots, x_N)
\end{aligned}
$$

in $\mathbb{R}^N, d_2$ converges to $x = (x_1, \ldots, x_j, \ldots, x_N)$ iff for each j,

$$x_{nj} \xrightarrow[j \to +\infty]{} x_j$$

Watch lecture recording 23/01 for examples of:

- Convergence in ℓ^2
- Convergecnce in $C([a, b])$

Definition 2.1.4: Bounded Sequence

A sequence in a metric space is said to be **bounded** iff there exists an open ball that contains all of its terms

Note: this is the same definition as "sequence is bounded if there is upper and lower bound", as open ball implies the same thing

Theorem 2.1.5

Every convergence is bounded

Proof: Let $x_n \to x$ in a metric space (X, d). There exists an index N s.t. for all n with $n \geq N$,

$$x_n \in B(x, 1)$$

Let r be any positive number such that

$$r > 1,\ r > d(x, x_1), \ldots, r > d(x, x_{N-1})$$

Then, for all n,

$$d(x_n, x) < r$$

therefore

$$x_n \in B(x, r)$$

2.2 Cauchy Sequences

Convergence: For every ϵ, there is an N such that for $n \geq N$, $d(x_n, x) < \epsilon$

$$x_1 \quad x_2 \quad \cdots \quad x_N \quad \cdots x_n \quad \cdots \quad \to x$$

Replace x by any x_m with $m \geq N$

$$x_1 \quad x_2 \quad \cdots \quad x_N \quad \cdots x_n \quad \cdots \quad x_m \quad \cdots$$

'$d(x_n, x) < \epsilon$' becomes '$\forall m \geq N$, $d(x_n, x_m) < \epsilon$'

Definition 2.2.1: Cauchy Sequence

A sequence $(x_n)_{n=1}^{\infty}$ in a metric space (X, d) is said to be a **Cauchy sequence** iff for every positive ϵ, there exists an index N, s.t. for all indices n, m with $n, m \geq N$,

$$d(x_n, x_m) < \epsilon$$

Theorem 2.2.2

If a sequence in a metric space converges, then it is a Cauchy sequence

Proof: If $x_n \to L$ in a metric space (X, d), then for every positive ϵ, there exists an index N, such that for all indices n with $n \geq N, d(x_n, L) < \frac{\epsilon}{2}$. Therefore for all $n, m \geq N$,

$$d(x_n, x_m) \leq d(x_n, L) + d(x_m, L) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Note: The converse is not true.
Counterexample:

$$X = (0, 1), d(x, y) = |x - y|, \; x_n = \frac{1}{n}, \; (n \geq 2)$$

This sequence is Cauchy but not convergent
Cauchy: Let ϵ be positive. Pick N s.t. $\frac{1}{N} < \frac{\epsilon}{2}$. For $n, m \geq N$ we have

$$d(x_n, x_m) = \left| \frac{1}{n} - \frac{1}{m} \right| \leq \frac{1}{n} + \frac{1}{m} \leq \frac{2}{N} < \epsilon$$

Not convergent: Let $x \in (0, 1)$. Find N s.t. $\frac{1}{N} < x$. For $n \geq N$ we have $x_n = \frac{1}{n} \leq \frac{1}{N}$, so the open interval $(\frac{1}{N}, 1)$ contains x and only finitely many terms of the sequence. Therefore $x_n \not\to x$

Watch Lecture 23/01 for example of counterexample

- Metric spaces $(\mathbb{R}, d_{\mathbb{R}})$ and $(\mathbb{Q}, d_{\mathbb{Q}})$

Definition 2.2.3: Complete Metric Spaces

A metric space is said to be **complete** if and only if every Cauchy Sequence is convergent

Examples:

- $\mathbb{R}$ with the standard metric is complete
- $\mathbb{Q}$ with the standard metric is not complete

- $(0, 1)$ with the standard metric is not complete
- $[0, 1]$ with the standard metric is complete
- $\mathbb{R}^n$, ℓ^p, $C([a, b])$ is complete (proof later)

2.3 Open sets and closed sets

Definition 2.3.1: Open Sets and Closed Sets

Let (X, d) be a metric space.

- A subset G of X is said to be **open** iff for every point x in G there exists a positive radius r such that $B(x, r) \subseteq G$.
- A subset F of X is said to be **closed** iff F^c is open

Example: In any metric space (X, d), the sets $\emptyset$ and X are both open and closed.
$\emptyset$ is open because the following statement is true:

$$\forall x(x \in \emptyset \implies \exists r \dots)$$

X is open because, for every x in X we can take $r = 1234$ to have $B(x, r) \subseteq X$
$\emptyset^c = X$ and $X^c = \emptyset$ are closed

Watch lecture recording 26/01 for details on examples

- Every open ball is an open set
- If d is the discrete metric on a non-empty set X, then every subset of X is both open and closed
- $X = \mathbb{Z}$, $d(x, y) = |x - y|$, all subsets of X are both open and closed

Definition 2.3.2: Discrete Metric Space

A metric space is called **discrete** iff all its subsets are open (equiv. all subsets are closed)

Example: $[0, 1] \cap (2, 3)$

Theorem 2.3.3: Properties of open sets

Let (X, d) be a metric space

1. The union of any family of open sets is an open set
2. The intersection of finitely many open sets is an open set

Proof for 1: Let $(G_i)_{i \in I}$ be a family of open sets and define $G = \bigcup_{i \in I} G_i$. If $x \in G$, then $x \in G_i$ for some i. Since G_i is open, there exists a positive r such that $B(x, r) \subseteq G_i$. Then $B(x, r) \subseteq G$
Proof for 2: Let $G_1, \dots, G_n$ be open sets. Define $G = G_1 \cap \dots \cap G_n$. If $x \in G$, then $x \in G_i$ for all i. Since each G_i is open, there exists a positive r_i such that $B(x, r_i) \subseteq G_i$. Let $r = \min\{r_1, \dots, r_n\}$. For each i,

$$B(x, r) \subseteq B(x, r_i) \subseteq G_i$$

Therefore, $B(x, r) \subseteq G_1 \cap \dots \cap G_n = G$

Theorem 2.3.4: Infinite open sets

The intersection of infinitely many open sets is not always an open set
For example, let $G_n = (-\frac{1}{n}, \frac{1}{n}), n = 1, 2, \ldots$ on the real line with the standard metric.
Each G_n is open but

$$\bigcap_{n=1}^{\infty} G_n = \{0\}$$

Theorem 2.3.5: Relatively open sets

Let (X, d) be a metric space and A be a non-empty subset of X equipped with the induced metric d_A. Let $G \subseteq A$. G is open in (A, d_A) iff there exists a subset O of X, open in (X, d), such that $G = A \cap O$
The open sets of (A, d_A) are sometimes referred to as **relatively open**

Theorem 2.3.6

Let (X, d) be a metric space, $(x_n)_{n=1}^{\infty}$ be a sequence in X and x be a point in X.
$x_n \to x$ iff every open set that contains x contains eventually all terms of the sequence

Proof: Assume $x_n \to x$. Let G be any open set with $x \in G$. There is a positive r such that $B(x, r) \subseteq G$. There is an N such that for all n with $n \geq N$ we have $x_n \in B(x, r)$, hence, $x_n \in G$. Conversely, assume that every open set containing x contains eventually all terms of the sequence. Every open ball centered at x is an open set, therefore it contains eventually all terms of the sequence. It follows that $x_n \to x$.

Definition 2.3.7: Neighbourhoods of points

An **open neighbourhood** of a point x is any open set that contains x. $x_n \to x$ iff every open neighbourhood of x contains eventually all terms of the sequence.

A **neighbourhood** of a point x is a set that contains an open neighbourhood of x. $x_n \to x$ iff every neighbourhood of x contains eventually all terms of the sequence.

Theorem 2.3.8: Properties of Closed sets

Let (X, d) be a metric space.

1. The intersection of any family of closed sets is a closed set
2. The union of finitely many closed sets is a closed set.

Proof for 1: Let $(F_i)_{i \in I}$ be a family of closed sets. Then each F_i^c is open, therefore, $\bigcup_{i \in I} F_i^c$ is open, therefore $\left(\bigcup_{i \in I} F_i^c\right)$ is closed. By De Morgan's rule, $\left(\bigcup_{i \in I} F_i^c\right)^c = \bigcap_{i \in I} F_i$. Therefore, $\bigcap_{i \in I} F_i$ is closed.
Proof for 2: Let $F_1, \ldots, F_n$ be closed sets. Then $F_1^c, .., F_n^c$ are open, therefore $F_1^c \cap \cdots \cap F_n^c$ is open, therefore $(F_1^c \cap \cdots \cap F_n^c)^c$ is closed. By de Morgan's rule, $(F_1^c \cap \cdots \cap F_n^c)^c = F \cup \cdots \cup F_n$. Therefore, $F \cup \cdots \cup F_n$ is closed

Theorem 2.3.9: Infinite closed sets

The union of infinitely many closed sets is not always a closed set.
For example, let $F_n = [\frac{1}{n}, 1], n = 1, 2, \dots,$ on the real line with the standard metric. Each F_n is closed but

$$\bigcup_{n=1}^{\infty} F_n = (0, 1]$$

is not closed.

Watch lecture recording 30/01 for examples

Theorem 2.3.10

A subset F of a metric space is closed iff the limit of every convergent sequence of elements of F belongs to F

Proof $\implies$: Assume F is closed, and let $(x_n)_{n=1}^{\infty}$ be a convergent sequence of elements of F. Let x be its limit. We wish to show that $x \in F$. We argue by contradiction. Suppose $x \notin F$. Then $x \in F^c$, and since F^c is open, there exists a positive r such that $B(x, r) \subseteq F^c$. Then $B(x, r)$ contains no terms of the sequence - contradiction
Proof $\impliedby$: assume that the limit of every convergent sequence of elements of F belongs to F. We wish to show that F is closed.
We show that F^c is open. Let $x \in F^c$. We need to show that there exists a postive r such that $B(x, r) \subseteq F^c$. If not, then for every r there exists a point in $B(x, r)$ that belongs to F.
Using this with $r = \frac{1}{n}, n = 1, 2, 3, \dots,$ we find points x_n with $x_n \in B(x, 1/n)$ and $x_n \in F$. Then $x_n \to x$ but $x \notin F$a. Contradiction

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Watch lecture recording 30/01 for examples

- In any metric space (X, d), singletons $F = \{x\}$ are closed.
- In any metric space, any finite set is closed because

$$\{x_1, \dots, x_n\} = \{x_1\} \cup \dots \cup \{x_n\}$$

2.4 Closure

Definition 2.4.1: Closure

Let (X, d) be a metric space and $A \subseteq X$. The **closure** of A, deonted by $\overline{A}$, is the smallest closed subset of X that contains A
There exists at least one closed subset of X that contains A, namely X itself. The smallest closed subset of X that contains A is

$$\bigcap_{A \subseteq F \subseteq X,\ F \text{closed}} F$$

Theorem 2.4.2: Properties of Closure

Let (X, d) be a metric space and A, $B \subseteq X$.

1. $\overline{\emptyset} = \emptyset$ and $\overline{X} = X$
2. $A \subseteq \overline{A}$ and $\overline{A}$ is closed
3. A is closed iff $A = \overline{A}$
4. $\overline{\overline{A}} = \overline{A}$
5. If $A \subseteq B$, then $\overline{A} \subseteq \overline{B}$
6. $\overline{A \cup B} = \overline{A} \cup \overline{B}$

Lecture 30/01 45m for proofs

Example: $X = \mathbb{R}$, $d(x, y) = |x - y|$, $A = (0, 1)$. We claim that $\overline{A} = [0, 1]$
$A \subseteq [0, 1]$ and $[0, 1]$ is a closed set. The smallest such set is $\overline{A}$. Therefore $\overline{A} \subseteq [0, 1]$.
Next we show that $[0, 1] \subseteq \overline{A}$. clearly, $(0, 1) = A \subseteq \overline{A}$
$(1/2, 1/3, \dots, 1/n \dots) \to 0$, each term belongs to $\overline{A}$, and $\overline{A}$ is closed, therefore $0 \in \overline{A}$. Similarly, $1 \in \overline{A}$

Watch lecture recording 02/02 10m for more in-depth examples of closure things

- On the real line with the standard metric, $\overline{(a, b)} = [a, b]$
- In $\mathbb{R}^n$ with the Euclidean metric d_2, the closure of the open ball $B(c, r)$ is the closed ball $\{x \in \mathbb{R}^n : d_2(x, c) \leq r\}$
- On the complex plane with its standard metric, the closure of an open disc is the corresponding closed disc
- Let X be a non-empty set with the discrete metric, $c \in X$ and $r = 1$. Then $B(c, 1) = \{c\}$, therefore $\overline{B(c, 1)} - \overline{\{c\}} = \{c\}$, while

$$\{x \in X : d(x, c) \leq 1\} = X$$

  The closure of an open ball is not always equal to the corresponding closed ball
- $X = \mathbb{R}$, $d(x, y) = |x - y|$. $\overline{\mathbb{Q}} = \mathbb{R}$

Definition 2.4.3: Dense Subset of a Metric Space

Let (X, d) be a metric space. A subset D of X is said to be **dense** iff $\overline{D} = X$

Random fact: In $\mathbb{R}^n$ with the Euclidean metric d_2, $\mathbb{Q}^n$ is dense.

Theorem 2.4.4: Closure Equivalence

Let (X, d) be a metric space, $A \subseteq X, x \in X$. The following are equivalent

1. $x \in \overline{A}$
2. For every positive r, $B(x, r) \cap A \neq \emptyset$
3. There exists a sequence $(a_n)_{n \in \mathbb{N}}$ with $a_n \in A$ for all n, such that $a_n \to x$

A point x with any of these properties is called an **adherent point** of A. So, $\overline{A}$ is the set of all adherent points of A.

Example: $X = \mathbb{R}$, $d(x, y) = |x - y|, A = (0, 1) \cup \{2\}$, $\overline{A} = [0, 1] \cup \{2\}$
2 is an adherent point of A. 0 is an adherent point of A.
Observe: $2 \in A, 0 \notin A$

Proof: $1 \implies 2$
Assume $x \in \overline{A}$. Fix a positive r. We show: $B(x, r) \cap A \neq \emptyset$.
The set $\overline{A} \backslash B(x, r)$ is closed and $\overline{A} \backslash B(x, r) \subsetneq \overline{A}$
Therefore, $A \nsubseteq \overline{A} \backslash B(x, r)$
Therefore there exists an element $a \in A$ s.t. $a \notin \overline{A} \backslash B(x, r)$. But $a \in \overline{A}$. Therefore $a \in B(x, r)$
Proof: $2 \implies 3$
If A intersects every open ball centered at x, then for every n there is a point a_n that belongs to A and to $B(x, 1/n)$. Then $d(a_n, x) < 1/n$, therefore $a_n \to x$
Proof: $3 \implies 1$ Assume that there is a sequence $(a_n)_{n=1}^{\infty}$ such that $a_n \in A$ for all n, and $a_n \to x$. We show that $x \in \overline{A}$.
For each n we have $a_n \in \overline{A}$. Also, $a_n \to x$ and $\overline{A}$ is closed. Therefore $x \in \overline{A}$

Definition 2.4.5: Limit points of sets

Let (X, d) be a metric space, $A \subseteq X$ and $x \in X$. We say that x is a **limit point** or an **accumulation point** of A iff every open ball centered at x contains an element of A distinct from x, i.e.

$$\forall r > 0 \quad (B(x, r) \backslash \{x\}) \cap A \neq \emptyset$$

The set of all limit points of A is called the **derived set** of A and is denoted by A' or $\tilde{A}$.

Note w/o proof: x is a limit point of A iff there exists a sequence $(a_n)_{n=1}^{\infty}$ such that $a_n \in A, a_n \neq x$ for all n, and $a_n \to x$
Note w/o proof: Let (X, d) be a metric space and $A \subseteq X$. Then $\overline{A} = A \cup A'$

Example: On the real line with the standard metric, let $A = (0, 1) \cup \{2\}$. Then $\overline{A} = [0, 1] \cup \{2\}$, so $0, 2 \in \overline{A}$ 0 is a limit point of A 2 isn't a limit point of A

2.5 Continuous functions between metric spaces

Definition 2.5.1: Continuity at a point

Let (X, d_X), (Y, d_Y) be metric spaces and $f : X \to Y$ be a function. We say that f is **continuous at a point** x_0 in X iff for for every positive ϵ, there exists a positive δ, s.t., for all $x \in X$ with $d_X(x, x_0) < \delta$ we have $d_Y(f(x), f(x_0)) < \epsilon$

Alternatively, f is **continuous at a point** $x_0 \in X$ iff, for every positive ϵ, there exists a positive δ, such that, for all $x \in B_X(x_0, \delta)$ we have $f(x) \in B_Y(f(x_0), \epsilon)$

Definition 2.5.2: Continuity of a function

Let $(X, d_X), (Y, d_Y)$ be metric spaces. A function $f : X \to Y$ is said to be **continuous** iff it is continuous at every point in X

Example: Let (X, d) be a metric space and p be a point in X. Define $f : X \to \mathbb{R}$ by $f(x) = d(x, p)$. f is continuous.
Watch lecture recording 02/02 40m for proof

Theorem 2.5.3

Let $(X, d_X), (Y, d_Y)$ be metric spaces, $f : X \to Y$ be a function and x_0 be a point in X. Then f is continuous at x_0 iff for every open neighbourhood G of $f(x_0)$ there exists an open neighbourhood O of x_0 such that, for all $x \in O$, we have $f(x) \in G$

Proof: Assume f is continuous at x_0. Let G be an open set in Y with $f(x_0) \in G$. There exists a positive ϵ such that $B_Y(f(x_0), \epsilon) \subseteq G$. By continuity, there exists a positive δ such that for all $x \in B_X(x_0, \delta)$ we have $f(x) \in B_Y(f(x_0), \epsilon)$. Let $O = B_X(x_0, \delta)$. For all $x \in O$ we have $f(x) \in G$

Conversely, assume that for every open neighbourhood G of $f(x_0)$ there exists an open neighbourhood O of x_0 s.t. for all $x \in O$, we have $f(x) \in G$. We wish to show that f is continuous at x_0
Let ϵ be positive. Apply our hypothesis with $G = B_Y(f(x_0), \epsilon)$ to see that there exists an open set O in X with $x_0 \in O$, s.t. for all $x \in O$ we have $f(x) \in G$.
Since O is open, there exists a positive δ such that $B_X(x_0, \delta) \subseteq O$.
For all x in $B_X(x_0, \delta)$ we have $f(x) \in B_Y(f(x_0), \epsilon)$

Theorem 2.5.4: Continuity and Convergence

Let (X, d_X), (Y, d_Y) be metric spaces, x_0 be a point in X, and $f : X \to Y$ be a function. The following are equivalent:

1. f is continuous at x_0
2. For every sequence $(x_n)_{n=1}^{\infty}$ in X, if $x_n \xrightarrow[n \to +\infty]{} $ in (X, d_X), then $f(x_n) \xrightarrow[n \to +\infty]{} f(x_0)$ in (Y, d_Y)

Watch lecture recording 02/02 50m for the proof