1 Algebra

Functions and Symmetries

Definition 0.1.1 Functions

A function $f : X \to Y$ is called

- **injective** if $f(x_1) = f(x_2) \implies x_1 = x_2$
- **surjective** if for every $y \in Y$, $\exists x \in X$ s.t. $f(x) = y$
- **bijective** if it is both injective and surjective

Definition 1.1.3 Graph Isomorphisms

An **isomorphism** between two graphs is a *bijection* between them that preserves all edges. More precisely, if Γ_1 and Γ_2 are graphs, with sets of vertices V_1 and V_2 respectively, then an isomorphism from Γ_1 and Γ_2 is a bijection

$$f : V_1 \to V_2$$

such that $f(v_1)$ and $f(v_2)$ are joined by an edge if and only if v_1 and v_2 are also joined by an edge. We say that Γ_1 and Γ_2 are *isomorphic* if there exists an isomorphism $f : \Gamma_1 \to \Gamma_2$

Definition 1.1.9 Symmetry

A **symmetry** of a graph is an *isomorphism* from the graph to itself, i.e. if the set of vertices is V, then the symmetry is a bijection $f : V \to V$ that preserves edges. That is, a symmetry is a bijection $f : V \to V$ such that $f(v_1)$ and $f(v_2)$ are joined by an edge if and only if v_1 and v_2 are joined by an edge.

Groups

Definition 1.2.3 Groups

For an operation $*$, We say a non-empty set G is a **group** under $*$ if the following four axioms hold:

- **G1 - Closure:** $*$ is a binary operation on G, that is $a*b \in G$ for all $a, b \in G$.
- **G2 - Associativity:** $(a*b)*c = a*(b*c)$ for all $a, b, c \in G$
- **G3 - Identity:** There exists an *identity* element of G such that $e * g = g * e = e$ for all $g \in G$.
- **G4 - Inverse:** Every element $g \in G$ has an *inverse* g^{-1} such that $g * g^{-1} = g^{-1} * g = e$

Definition 1.2.6 Abelian Group

The definition of a group doesn't require that $a * b = b * a$. We say that a group is **abelian** or **commutative** if $a * b = b * a$ for every $a, b \in G$. We say that a *commutes* with b, or that a and b *commute*

Subgroups

Definition 2.1.1 Subgroups

Let G be a group. We say that a non-empty subset H of G is a **subgroup** of G if H itself is a group (under the operation from G). We write $H \leq G$ if H is a subgroup of G. If $H \neq G$, we write $H < G$ and say H is a proper subgroup

Theorem 2.1.3: Subgroup Test

$H \subseteq G$ is a subgroup of G if and only if:

- **S1:** H is not empty
- **S2:** If $h, k \in H$ then $h * k \in H$
- **S3:** If $h \in H$ then $h^{-1} \in H$

Alternative test for subgroups:

- $\widetilde{S1}$: H is not empty.
- $\widetilde{S2}$: If $h, k \in H$ then $h * k^{-1} \in H$

Definition 2.2.4 Order of an Element

Let G be a group and $g \in G$. Then the **order** $o(g)$ of g is the *least* natural number n such that

$$g^n = e$$

If no such n exists, we say that g has infinite order

Definition 2.2.3 Order of a Group

The **order** of a finite group, written $|G|$, is the number of elements in G. If G is infinite we say that $|G| = \infty$, or the order of G is infinite.

Theorem 2.2.6: Order of a Finite Group

In a finite group, every element has finite order.
If g is an element of a finite group G, then there exists $k \in \mathbb{N}$ such that $g^k = g^{-1}$

Definition 2.2.8 Generating Subset

Let G be a group and let $g \in G$ be an element. We define the subset

$$\langle g \rangle := \{g^k \mid k \in \mathbb{Z}\} = \{\ldots, g^{-2}, g^{-1}, e, g, g^2, \ldots\}$$

Note that if G is finite, then by 2.2.6 $\langle g \rangle$ is finite, and we can think of $\langle g \rangle$ as

$$\langle g \rangle = \{e, g, \ldots, g^{o(g)-1}\}$$

Definition 2.2.10 Cyclic Subgroup

A subgroup $H \leq G$ is **cyclic** if $H = \langle h \rangle$ for some $h \in H$. In this case, we say that H is the *cyclic subgroup generated by* h. If $G = \langle g \rangle$ for some $g \in G$, then we say that the group G is *cyclic*, and that g is a *generator*.

Remark 2.2.12 - 16: Consequences of Cyclic groups

- **2.2.12** If $g \in G$, then $o(g) = |\langle g \rangle|$
- **2.2.13:** If G is cyclic, then G is abelian.
- **2.2.14:** Let G be a finite group. Then
  $$G \text{ is cyclic} \iff G \text{ has an element of order} |G|$$
- **2.2.15:** Let G be a cyclic group and let H be a subgroup of G. Then H is cyclic.
- **2.2.16:** Let $m, n \in \mathbb{N}$, let $G = \langle g \rangle$ be a cyclic group of order m and $H = \langle h \rangle$ be a cyclic group of order n. Then
  $$G \times H \text{ cyclic} \iff m \text{ and } n \text{ are coprime } (\gcd(\text{m,n}) = 1)$$

Cosets and Lagrange

Definition 2.3.2 Relation

Let X be a set, and R a subset of $X \times X$; thus R consists of some ordered pairs (s, t) with $s, t \in X$. If $(s, t) \in R$ we write $s \sim t$ and say "s is related to t". We call $\sim$ a **relation** on X.

Definition 2.3.2 Equivalence Relation

- **Reflexive:** $x \sim x$ for all $x \in X$
- **Symmetric:** $x \sim y$ implies that $y \sim x$ for all $x, y \in X$
- **Transitive:** $x \sim y$ and $y \sim z$ implies that $x \sim z$ for all $x, y, z \in X$

A relation $\sim$ is called an **equivalence relation** on X if it satisfies the following three axioms:

Definition 2.3.4 Coset

Let $H \leq G$ and let $g \in G$. Then a *left coset* of H in G is a subset of G of the form gH, for some $g \in G$. We denote the set of left cosets of H in G by G/H

Theorem 2.4.2: Lagrange's Theorem

Suppose that G is a finite group.

- If $H \leq G$, then $|H|$ divides $|G|$
- Let $g \in G$. Then $o(g)$ divides $|G|$
- For all $g \in G$, we have that $g^{|G|} = e$

Theorem 2.3.8: Coset Rules

Let $H \leq G$

- For all $h \in H$, $hH = H$. In particular $eH = H$
- For $g_1, g_2 \in G$, the following are equivalent
  - $g_1 H = g_2 H$
  - there exists $h \in H$ such that $g_2 = g_1 H$
  - $g_2 \in g_1 H$
- For g_1, $g_2 \in G$, define $g_1 \sim g_2$ if and only if $g_1 H = g_2 H$. Then $\sim$ defines an equivalence relation on G.

Theorem 2.4.4: Index of a Subgroup

The **index** of $H \leq G$ is defined as the number of *distinct* left cosets of H in G, which by Lagrange's is $|G/H| = \frac{|G|}{|H|}$

Remark 2.4.6 - 8: Consequences of Lagrange

- **2.4.6:** Suppose that G is a group with $|G| = p$, where p is prime. Then G is a cyclic group
- **2.4.7:** Suppose that G is a group with $|G| < 6$. Then G is abelian
- **2.4.8:** If p is a prime and $a \in \mathbb{Z}$, then $a^p \equiv a \mod p$

Homomorphisms and Isomorphisms

Definition 3.1.1 Group Homomorphism

Let $(G, *), (H, \circ)$ be groups. A map $\phi : G \to H$ is called a **homomorphism** if

$$\phi(x * y) = \phi(x) \circ \phi(y) \quad \text{for all } x, y \in G$$

Note that the product on the left is formed using $*$, while the product on the right is formed using $\circ$

Definition 3.1.2 Group Isomorphism

A group homomorphism $\phi : G \to H$ that is also a bijection is called an **isomorphism** of groups. In this case we say that G and H are *isomorphic* and we write $G \cong H$. An isomorphism $G \to G$ is called an **automorphism** of G.

Theorem 3.1.L: Cyclic Isomorphisms

All finite cyclic groups of the same order are *isomorphic* to each other. Therefore, cyclic groups of order n are isomorphic to $(\mathbb{Z}_n, +)$
All infinite cyclic groups are *isomorphic* to each other. Therefore, each cyclic group of infinite order is isomorphic to $(\mathbb{Z}, +)$

Remark 3.1.5: Consequences of Homomorphisms

Let $\phi : G \to H$ be a group homomorphism. Then

- $\phi(e_G) = e_H$
- $\phi(g^k) = (\phi(g))^k$ and $\phi(g^{-1}) = (\phi(g))^{-1}$ for all $g \in G$
- If ϕ is injective, the order of $g \in G$ equals the order of $\phi(g) \in H$.

Definition 3.1.7 Normal Subgroup

A subgroup $N \leq G$ is **normal** if the left and right cosets of N are equal, i.e. $gN = Ng$ for all $g \in G$. If N is a normal subgroup of G, we write $N \triangleleft G$. Kernels of homomorphisms are always normal subgroups

Definition 3.1.6 Image and Kernel of a Group

Let $\phi : G \to H$ be a group homomorphism.

- The **image** of ϕ is defined to be
$$\operatorname{im} \phi := \{h \in H \mid h = \phi(g) \text{ for some } g \in G\}$$
- The **kernel** of ϕ is defined to be
$$\ker \phi := \{g \in G \mid \phi(g) = e_H\}$$

Note: $\operatorname{im} \phi$ is a subgroup of H and $\ker \phi$ is a subgroup of G

Theorem 3.2.1: Product Isomorphisms

Let $H, K \leq G$ be subgroups with $H \cup K = \{e\}$.

- The map $\phi : H \times K \to HK$ given by $\phi : (h, k) \to hk$ is bijective
- If every element of H commutes with every element of K when multiplied in G (i.e. $hk = kh \quad \forall h \in H, k \in K$), then HK is a subgroup of G, and it is isomorphic to $H \times K$ via ϕ

Theorem 3.2.3: Size of Product Group

Let $H, K \leq G$ be finite subgroups of a group G such that $H \cup K = \{e\}$ Then $|HK| = |H| \times |K|$.

Group Actions

Definition 4.1.1 Group Action

Let $(G, *)$ be a group, and let X be a nonempty set. Then a (left) **action** of G on X is a map

$$G \times X \to X$$

written $(g, x) \mapsto g \cdot x$, such that

$$g_1 \cdot (g_2 \cdot x) = (g * h) \cdot x \qquad \text{and} \qquad e \cdot x = x$$

for all $g_1, g_2 \in G$ and all $x \in X$.

Definition 4.1.4 Kernel of an Action, Faithful Action

Suppsoe that G acts on X. Then the set

$$N := \{g \in G \mid g \cdot x = x forall x \in X\}$$

is a subgroup of G, and is called the **kernel** of the action. If $N = \{e\}$, then we say the action is **faithful**

Definition 4.2.1 Orbit, Stabilizer, and Fix

For every x in X, the **orbit** of x is defined by

$$\operatorname{Orb}_G(x) = \{g \cdot x \mid g \in G\}$$

This is a subset of X
For every x in X, the **stabilizer** of x is defined by

$$\operatorname{Stab}_G(x) = \{g \in G : g \cdot x = x\}$$

This is a subgroup of G
For every g in G, the **fix** of g is defined by

$$\operatorname{Fix}(g) = \{x \in X \mid g \cdot x = x\}$$

Let G act on X, let $x \in X$ and set $H := \operatorname{Stab}_G(x)$. If $y = g \cdot x$ for some $g \in G$, then

$$\operatorname{send}_x(y) = gH$$

Theorem 4.2.5: Orbit Equivalence

Let G act on X. Then

$$x \sim y \iff y = g \cdot x \text{ for some } g \in G$$

defines an equivalence relation on X. The equivalence classes are the orbits of G. Thus when G acts on X, we obtain a partition of X into orbits

Theorem 4.3.1: Orbit-Stabilizer Theorem

Suppose G is a finite group acting on a set$X$, and let $x \in X$. Then $|\operatorname{Orb}_G(x)| \times |\operatorname{Stab}_G(x)| = |G|$, or in words:

$$\text{size of orbit } \times \text{ size of stabilizer } = \text{order of group}$$

Theorem 4.3.4: Orbit Send Theorem

Let G act on X, let $x \in X$, and let set $H := \operatorname{Stab}_G(x)$. Then the map

$$\operatorname{send}_x : \operatorname{Orb}_G(x) \to G/H \text{ which sends } y \mapsto \operatorname{send}_x(y)$$

Theorem 4.4.2: Cauchy's Theorem

Let G be a group, p be prime. If p divides $|G|$, then G contains an element of order p

2 Analysis

Real Numbers and Bounds

Definition 1.1 The Real Numbers

$\mathbb{R}$ is defined as the set of real numbers. It has two operations $+$ and $*$, and it is a field, i.e. satisfies group axioms for both, in addition the Distributive law:

$$a \cdot (b + c) = a \cdot b + a \cdot c$$

The set of real numbers is also ordered, i.e. there is a relation $<$ which satisfies pretty much what you think it does
Finally, the set of real numbers is complete, i.e. there are no gaps between any numbers.

Definition 1.3.2 Suprema and Bounds

Let $E \subset \mathbb{R}$ be nonempty

- The set E is said to be bounded above if there is $M \in R$ such that $a \leq M$ for all $a \in E$
- A real number M is called an upper bound of the set$E$ if $a \leq M$ for all $a \in E$
- A real number s is called the **supremum** of the set E if
  - s is an upper bound of E
  - $s \leq M$ for all upper bounds M of the set E

  If a number s exists, we shall say that E has a supremum and write $s = \sup E$

If the supremum s exists, then s is the least upper bound of the set E. The supremum is also unique if it exists.

Definition 1.3.10 Infimum

If the same properties as a supremum apply but in the other direction, a number s is instead called the **infimum** of the set E. Infimum and Supremum are related via the reflection principle:

- Set E has a supremum if and only if the set $-E$ has an infimum. Also $\inf(-E) = -\sup(E)$
- Set E has an infimum if and only if the set $-E$ has a supremum. Also $\sup(-E) = -\inf(E)$

Theorem 1.3.5: Suprema Approximation Property

If the set $E \subset \mathbb{R}$ has a supremum then for any positive number $\epsilon > 0$ there exists $a \in E$ such that

$$\sup E - \epsilon < a \leq \sup E$$

Theorem 1.3.7: Archimedean Principle

Given positive real numbers $a, b \in \mathbb{R}$ there is an integer $n \in N$ such that $b < na$

Definition 1.5.1 Injection/Surjection Terminology

Let f be a function from a set X into a set Y.

- f is said to be **one-to-one** on X if and only if f is injective
- f is said to take X **onto** Y if f is surjective

Definition 1.5.2 Countability

Let E be a set

- E is said to be **finite** if either $E = \varnothing$, or there is an integer $n \in \mathbb{N}$ and a bijection $f : \{1, 2, 3, \ldots, n\} \to E$. We say that the set E has n elements
- E is said to be **countable** if there is a bijective function $f : \mathbb{N} \to E$
- E is said to be **at most countable** if E is finite or countable
- E is said to be **uncountable** if E is neither finite nor countable

Additionally, a nonempty set E is at most countable if and only if there is a surjective function $f : \mathbb{N} \to E$

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