Honours Analysis Exam Notes

Made by Leon :) *Note: Any reference numbers are to the lecture notes*

1 Revisiting FPM

Definition 1.1: Nested Sequences and covers

A sequence $(I_n)_{n\in\mathbb{N}}$ of sets is said to be **nested** if

$$I_1 \subset I_2 \subset I_3 \subset \cdots$$

Thm 1.1: If (I_n) is a nested sequence of nonempty closed bounded intervals then

$$E = \bigcap_{n\in\mathbb{N}} I_n = \{x \in \mathbb{R} : x \in I_n,\ \forall n \in \mathbb{N}\}$$

is nonempty (i.e. it contains at least one number). Moreover if $\lambda(I_n) \to 0$, where $\lambda(I_n)$ denotes the length of interval I_n, then E contains exactly one number

Thm 1.2: Let E be a subset of $\mathbb{R}^n$

- A **cover** of E is a collection of sets $\{I_\alpha\}_{\alpha\in A}$ such that
$$E \subseteq \bigcup_{\alpha\in A} I_\alpha$$
- An **open covering** of E is a cover such that each I_α is open, i.e.(a, b) compared to $[a, b]$
- A **finite subcover** of E is a collection of sets $(I_\alpha)_{\alpha\in A_0}$ where there exists a subset $A_0 = \{\alpha_1, \alpha_2, \ldots, a_N\}$ of A such that $(I_\alpha)_{\alpha\in A_0}$ is a finite subset of $(I_\alpha)_{\alpha\in A}$ that is also a cover
- The set E is said to be **compact** iff every open covering of E has a **finite subcovering**; that is
$$E \subseteq \bigcup_{j=1}^{N} I_{aj} \quad \text{or} \quad E \subseteq I_{\alpha_1} \cup I_{a_2} \cup \cdots \cup I_{a_N}$$

Definition 1.2: Convergence of Sequences and Cauchy

A sequence of real numbers (x_n) is said to **converge** to a real number $a \in \mathbb{R}$ iff for every $\epsilon > 0$ there is an $N \in \mathbb{N}$ such that

$$n \geq N \text{ implies } |x_n - a| < \epsilon$$

If (x_n) converges to a, we will write $\lim_{n\to\infty} x_n = a$, or $x_n \to a$. The number a is called the limit of the sequence (x_n). A sequence that does not converge to some real number is said to *diverge

Def 1.3: A sequence (x_n) of numbers $x_n \in \mathbb{R}$ is said to be **Cauchy** if for every $\epsilon > 0$ there is $N \in \mathbb{N}$ such that

$$|x_n - x_m| < \epsilon \quad \forall n, m \geq N$$

Thm 1.3: Let (x_n) be a sequence of real numbers. Then (x_n) is a Cauchy sequence if and only if (x_n) is a convergent sequence.
Thm 1.4: Let (x_n) be a sequence of real numbers. Then (x_n) is a Cauchy sequence iff (x_n) is a convergent sequence

Definition 1.4: Subsequences

Suppose $(x_n)_{n\in\mathbb{N}}$ is a sequence. A subsequence of this sequence is a sequence of the form $(x_{n_k})_{k\in\mathbb{N}}$ where for each k there is a positive integer n_k such that

$$n_1 < n_2 < \cdots < n_k < n_{k+1} < \cdots$$

Thus, $(x_n)_{n\in\mathbb{N}}$ is just a selection of some (possibly all) of the x_n's taken in order
Thm 1.5 (Bolzano-Weierstrass): Every bounded sequence of real numbers has a convergent subsequence

Definition 1.5: Limit Superior and Inferior

If (x_n) is a bounded sequence of real numbers we denote by

$$\limsup_{n\to\infty} x_n = \lim_{n\to\infty}\left(\sup_{k\geq n} x_k\right), \qquad \liminf_{n\to\infty} x_n = \lim_{n\to\infty}\left(\inf_{k\geq n} x_k\right)$$

Note: These are only defined for bounded sequences

- If (x_n) is not bounded from above then we write $\limsup_{n\to\infty} x_n = +\infty$
- If (x_n) is not bounded from below then we write $\liminf_{n\to\infty} x_n = +\infty$

Thm 1.6: A sequence (x_n) of real numbers is convergent if and only if $\limsup_{n\to\infty} x_n$ and $\liminf_{n\to\infty} x_n$ are real numbers and

$$\limsup_{n\to\infty} x_n = \liminf_{n\to\infty} x_n$$

Definition 1.7: Continuity

Let f be a function $f : \text{dom}(f) \to \mathbb{R}$ where $\text{dom}(f) \subset \mathbb{R}$. We say that f is **continuous** at some $a \in \text{dom}(f)$, if for any sequence (x_n) whose terms lie in $\text{dom}(f)$ and which converges to a, we have $\lim_{n\to\infty} f(x_n) = f(a)$.
If f is continuous at each $a \in S \subset \text{dom}(f)$ then we say f is continuous on S. If f is continuous of $\text{dom}(f)$ then we say f is continuous

Thm 1.10: Let $f, g : D \to \mathbb{R}$ be continuous on D, and let $\alpha \in \mathbb{R}$. Then the following functions are continuous on D :

1. α f
2. $f + g$
3. fg

Thm 1.12 ($\epsilon - \delta$ Definition of Continuity): Let f be a function $f : \text{dom}(f) \to \mathbb{R}$ where $\text{dom}(f) \subset \mathbb{R}$. Then f is continuous at $a \in \text{dom}(f)$ iff for any $\epsilon > 0$ there exists $\delta > 0$ s.t. whenever $x \in \text{dom}(f)$ and $|x - a| < \delta$ we have $|f(x) - f(a)| < \epsilon$

Thm 1.13 (Intermediate Value Theorem): Let $a < b$ real numbers and $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$. If $f(a)f(b) < 0$ then there exists at least one $c \in (a, b)$ s.t. $f(c) = 0$

Thm 1.14 (Extreme Value Theorem): Let $a < b$ real numbers and $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$. Then there exists points $c, d \in [a, b]$ s.t.

$$f(c) = \inf\{f(x) : x \in [a, b]\}, \quad f(d) = \sup\{f(x) : x \in [a, b]\}$$

That is, the function f on the interval $[a, b]$ is bounded and attains its minimal value at some point $c \in [a, b]$. Similarly, the maximal value of f is also attained at some point $d \in [a, b]$

Definition 1.6: Convergent Infinite Series

Let $S = \sum_{k=1}^{\infty} a_k$ be an infinite series a_k. For each $n \in \mathbb{N}$, the partial sum of S of order n is defined by

$$s_n = \sum_{k=1}^{n} a_k$$

S is said to **converge** iff its sequence of partial sums (s_n) converges to some $s \in \mathbb{R}$ as $n \to \infty$; that is, iff for every $\epsilon > 0$ there is an $N \in \mathbb{N}$ s.t. for all $n \geq N$ we have $|s_n - s| < \epsilon$. In this case we shall write

$$\sum_{k=1}^{\infty} a_k = s$$

and call s the **sum** or **value** of the series $\sum_{k=1}^{\infty} a_k$

- **Absolutely convergent**: the series $\sum_{k=1}^{\infty} |a_k|$ is convergent
- **Conditionally convergent**: Convergent but not absolutely

Thm 1.7 (Cauchy Criteron): Let $S = \sum_{k=1}^{\infty} a_k$ be a series. Then S is convergent iff for any $\epsilon > 0$ there exists N such that for all $m \geq n \geq N$ we have that

$$\left|\sum_{k=n+1}^{m} a_k\right| < \epsilon$$

Thm 1.8 (Rearranging Absolutely Convergent Series)
Let $S = \sum_{k=1}^{\infty} a_k$ be an absolutely convergent series. Then

- The series S is convergent
- Let $z : \mathbb{N} \to \mathbb{N}$ be a bijection. Then the series $\sum_{k=1}^{\infty} a_{z(k)}$ is convergent and
$$\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} a_{z(k)}$$

The series $\sum_{k=1}^{\infty} a_{z(k)}$ is called a **rearrangement** of the series $\sum_{k=1}^{\infty} a_k$. What we do here is add the terms of the sum in a different order to the original one, for example

$$a_3 + a_7 + a_1 + a_{100} + a_2 + \ldots$$

Since $z : \mathbb{N} \to \mathbb{N}$ is a bijection, we will miss no terms.

Thm 1.9 (Rearranging Conditionally Convergent Series)
Let $S = \sum_{k=1}^{\infty} a_k$ be any conditionally convergent series. Then there exists rearrangements $z : \mathbb{N} \to \mathbb{N}$ (where z is a bijection) such that

- For any $r \in \mathbb{R}$, $\sum_{k=1}^{\infty} a_{z(k)}$ is conditionally convergent with sum r
- The series $\sum_{k=1}^{\infty} a_{z(k)}$ diverges to $+\infty$
- The series $\sum_{k=1}^{\infty} a_{z(k)}$ diverges to $-\infty$
- The partial sums of the series $\sum_{k=1}^{\infty} a_{z(k)}$ oscillate between any two real numbers

Definition 1.8: Composition

Let $A, B \subseteq \mathbb{R}$ be nonempty, let $f : A \to \mathbb{R}$, $g : B \to \mathbb{R}$ and $f(A) \subseteq B$. The composition of g with f is the function $g \circ f : A \to \mathbb{R}$ defined by

$$(g \circ f)(x) = g(f(x)), \quad \text{for all } x \in A$$

Thm 1.11: If f is continuous at $a \in \mathbb{R}$ and g is continuous at $f(a)$ then the composition $g \circ f$ is continuous at a

2 Uniform convergence

Definition 2.1: Pointwise Convergence

Let E be a nonempty subset of $\mathbb{R}$. A sequence of functions $f_n : E \to \mathbb{R}$ is said to **converge pointwise** on E, written $f_n \to f$ pointwise on E as $n \to \infty$, iff $f(x) = \lim_{n\to\infty} f_n(x)$ exists for each $x \in E$

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f_n converges pointwise on E, as $n \to \infty$, iff for every $\epsilon > 0$ and $x \in E$ there is an $N \in \mathbb{N}$ (which may depend on x as well we ϵ) such that

$$n \geq N \quad \text{implies} \quad |f_n(x) - f(x)| < \epsilon$$

Remarks:

- The pointwise limit of continuous (or differentiable) functions is not necessarily continuous (or differentiable).
- The pointwise limit of integrable functions is not always integrable.
- There exist continuous functions f_n and f such that $f_n \to f$ pointwise on $[0,1]$ but

$$\lim_{n\to\infty} \int_0^1 f_n(x)\,dx \neq \int_0^1 \left(\lim_{n\to\infty} f_n(x)\right)\,dx$$

Def 2.2: Let E be a nonempty subset of $\mathbb{R}$. A sequence of functions $f_n : E \to \mathbb{R}$ is said to **converge uniformly** on E to a function f (notation: $f_n \to f$ uniformly on E as $n \to \infty$) if and only if for every $\epsilon > 0$ there is an $N \in \mathbb{N}$ such that for all $x \in E$

$$n \geq N \quad \text{implies} \quad |f_n(x) - f(x)| < \epsilon$$

Remark (The difference between Pointwise and Uniform): For a sequence of functions to be pointwise convergent, it is enough to have an N_n for every x_n, but for it to be uniformly convergent, it has to have **the same** N for every x in the sequence

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Def 2.2: A sequence of functions f_n is said to be **uniformly bounded** on a set E if there is a $M > 0$ such that $|f_n(x)| \leq M$ for all $x \in E$ and all $n \in N$

Definition 2.3: Convergence of series

Let f_k be a sequence of real functions defined on some set E and set

$$s_n(x) = \sum_{k=1}^{n} f_k(x), \quad x \in E,\ n \in \mathbb{N}$$

- The series $\sum_{k=1}^\infty f_k$ **converges pointwise** on E iff the sequence $s_n(x)$ converges pointwise on E as $n \to \infty$
- The series $\sum_{k=1}^\infty f_k$ **converges uniformly** on E iff the sequence $s_n(x)$ converges uniformly on E as $n \to \infty$
- The series $\sum_{k=1}^\infty f_k$ **converges absolutely** (pointwise) on E iff $\sum_{k=1}^\infty |f_k(x)|$ converges for each $x \in E$

Theorem 2.1 - 2.3: Uniform Continuity Theorems

- The following are equivalent concerning a sequence of functions $f_n : E \to \mathbb{R}$ and $f : E \to \mathbb{R}$:
  - $f_n \to f$ uniformly on E
  - $\sup_{x\in E} |f_n(x) - f(x)| \to 0$ as $n \to \infty$
  - there exists a seq $a_n \to 0$ s.t. $|f_n(x) - f(x)| \leq a_n$, $\forall x \in E$

2.1: Let E be a nonempty subset of $\mathbb{R}$ and suppose that $f_n \to f$ uniformly on E as $n \to \infty$. If each f_n is continuous at some $x_0 \in E$, then f is continuous at $x_0 \to E$

2.2: Suppose that $f_n \to f$ uniformly on a closed interval $[a,b]$. If each f_n is integrable on $[a,b]$, then so is f and

$$\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b \left(\lim_{n\to\infty} f_n(x)\right)\,dx$$

2.3: Let (a,b) be a bounded interval and suppose that f_n is a sequence of functions which converges at some $x_0 \in (a,b)$. If each f_n is differentiable on (a,b), and f_n' converges uniformly on (a,b) as $n \to \infty$, then f_n converges uniformly on (a,b) and

$$\lim_{n\to\infty} f_n'(x) = \left(\lim_{n\to\infty} f_n(x)\right)'$$

Theorem 2.4: Results of Convergent Series

Let E be a nonempty subset of $\mathbb{R}$ and let (f_k) be a sequence of real functions defined on E.

- Suppose that $x_0 \in E$ and that each f_k is continuous at $x_0 \in E$. If $f = \sum_{k=1}^\infty f_k$ converges uniformly on E, then f is continuous at $x_0 \in E$.
- Term-by-term integration: Suppose that $E = [a,b]$ and that each f_k is integrable on $[a,b]$. If $f = \sum_{k=1}^\infty f_k$ converges uniformly on $[a,b]$, then f is integrable on $[a,b]$ and

$$\int_a^b \sum_{k=1}^\infty f_k(x)\,dx = \sum_{k=1}^\infty \int_a^b f_k(x)\,dx$$

- Term-by-term differentiation: Suppose that E is a bounded, open interval and that each f_k is differentiable on E. If $\sum_{k=1}^\infty f_k(x_0)$ converges at some $x_0 \in E$, and $g = \sum_{k=1}^\infty f'(k)$ converges uniformly on E, then $f = \sum_{k=1}^\infty f_k$ converges uniformly on E, is differentiable on E, and

$$f'(x) = \left(\sum_{k=1}^\infty f_k(x)\right)' = \sum_{k=1}^\infty f_k'(x) = g(x)$$

for $x \in E$

Theorem 2.5: Weierstrass M-test

Let E be a nonempty subset of $\mathbb{R}$, let $f_k : E \to \mathbb{R}$, $k \in \mathbb{N}$, and suppose that $M_k > 0$ satisfies $\sum_{k=1}^\infty M_k < \infty$. If $|f_k(x)| \leq M_k$ for $k \in \mathbb{N}$ and $x \in E$, then $f = \sum_{k=1}^\infty f_k$ converges absolutely and uniformly on E.

Definition 3.1: Power Series and RoC

The **radius of convergence** R of the power series

$$\sum_{n=0}^\infty a_n(x-c)^n \tag{$*$}$$

is defined by

$$R = \sup\{r \geq 0 : (a_n r^n) \text{ is bounded}\}$$

unless $(a_n r^n)$ is bounded for all $r \geq 0$, where we say that $R = \infty$

Thm 3.1: Suppose the radius of convergence R of $*$ satisfies $0 < R < \infty$. If $|x-c| < R$, the power series $*$ converges absolutely. If $|x-c| > R$, the power series $*$ diverges

Theorem 3.2: Continuity of Power Series

Assume that $R > 0$. Suppose that $0 < r < R$. Then a power series converges uniformly and absolutely on $|x-c| \leq r$ to a continuous function f. Hence

$$f(x) = \sum_{n=0}^\infty a_n(x-c)^n$$

defines a continuous function $f : (c-R, c+R) \to \mathbb{R}$

Lemma 3.1: The two power series

$$\sum_{n=1}^\infty a_n(x-c)^n \text{ and } \sum_{n=1}^\infty n a_n (x-c)^{n-1}$$

have the same radius of convergence

Theorem 3.3: Differentiation of Power Series

Suppose the radius of convergence of a power series is R. Then the function

$$f(x) = \sum_{n=0}^\infty a_n(x-c)^n$$

is infinitely differentiable on $|x-c| < R$, and for such x,

$$f'(x) = \sum_{n=0}^\infty n a_n (x-c)^{n-1}$$

and the series converges absolutely, and also uniformly on $[c-r, c+r]$ for any $r < R$. Moreover,

$$a_n = \frac{f^{(n)}(c)}{n!}$$

3 Lebesgue Integration

Definition 4.0: Characteristic Function

Let E be a subset of $\mathbb{R}$. We define its **characteristic function** $\chi_E : \mathbb{R} \to \mathbb{R}$ by $\chi_E(x) = 1$ if $x \in E$ and $\chi_E(x) = 0$ if $x \notin E$. In other words,

$$\chi_E = \begin{cases} 1 & x \in E \\ 0 & x \notin E \end{cases}$$

In other words, this is a function that is 1 at all points of a bounded interval, and 0 elsewhere

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Let I be a bounded interval with endpoints a, b and $a \le b$. We call the number $b - a$ the **length of the interval** I and we denote it by $\lambda(I)$. This might also be referred to as $|I|$. That is,

$$\lambda((a, b)) = \lambda([a, b]) = \lambda((a, b]) = \lambda([a, b)) = b - a$$

From our definition of a characteristic function and the length of an interval, we have that the area of the characteristic function is a rectangle with width $\lambda(I)$ and height 1, therefore

$$\int \chi_I = 1 \cdot \lambda(I) = \lambda(I)$$

Definition 4.1: Step function

We say that $\phi : \mathbb{R} \to \mathbb{R}$ is a **step function** if there exist real numbers $x_0 < x_1 < x_2 < \cdots < x_n$ (for some $n \in \mathbb{N}$) such that

1. $\phi(x) = 0$ for $x < x_0$ and $x > x_n$
2. ϕ is constant on (x_{j-1}, x_j) for $1 \le j \le n$

We shall use the phrase "ϕ is a step function with respect to $\{x_0, x_1, \ldots, x_n\}$" to describe this situation

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Properties of Step Functions

1. The class of step functions is a vector space - i.e. if ϕ and ψ are step functions and α and β are real numbers, then $\alpha\phi + \beta\psi$ is a step function, and that if ϕ and ψ are step functions, then $\max\{\phi, \psi\}$, $\min\{\phi\psi\}$, $|\phi|$ and $\phi\psi$ are also step functions
2. If ϕ and ψ are step functions, then $\phi + \psi$ is a step function
3. ϕ is a step function if and only if it is of the form

$$\phi = \sum_{j=1}^{n} c_j \chi_{J_j}$$

for some n, c_j, and bounded intervals J_j

Def 4.2: Integral of a Step Function
If ϕ is a step function with respect to $\{x_0, x_1, \ldots, x_n\}$ which takes the value c_j on (x_{j-1}, x_j), then

$$\int \phi := \sum_{j=1}^{n} c_j (x_j - x_{j-1})$$

Therefore, using the characteristic definition of a step function, the integral is

$$\int \phi = \int \sum_{j=1}^{n} c_j \chi_{J_j} = \sum_{j=1}^{n} c_j \int \chi_{I J_j} = \sum_{j=1}^{n} c_j \lambda(J_j)$$

Definition 4.3: Lebesgue Integrals

A function $f : I \to \mathbb{R}$ is said to be **integrable** or more precisely **Lebesgue integrable** on an interval I if there exist numbers c_j and bounded intervals $J_j \subset I$, $j = 1, 2, 3, \ldots$ such that

$$\sum_{j=1}^{\infty} |c_j| \lambda(J_j) < \infty$$

and the equality

$$f(x) = \sum_{j=1}^{\infty} c_j \chi_{J_j}(x)$$

holds for all $x \in I$ at which

$$\sum_{j=1}^{\infty} |c_j| \chi_{J_j}(x) < \infty$$

We denote by $\int_I f$ the number

$$\int_I f = \sum_{j=1}^{\infty} c_j \lambda(J_j)$$

and call it the integral of f over the interval I. If the function f is not integrable on the interval I then we say that the integral of f on I does not exist. Hence if we say that the integral of f on I exists it just means that f is (Lebesgue) integrable on I.

Theorem 4.1: Lebesgue Equality

Suppose that c_j, d_j are real numbers and J_j, K_j are bounded intervals for all $j = 1, 2, 3, \ldots$, and

$$\sum_{j=1}^{\infty} |c_j| \lambda(J_j) < \infty, \qquad \sum_{j=1}^{\infty} |d_j| \lambda(K_j) < \infty$$

If

$$\sum_{j=1}^{\infty} c_j \chi_{J_j}(x) = \sum_{j=1}^{\infty} d_j \chi_{K_j}(x)$$

holds for all x such that

$$\sum_{j=1}^{\infty} |c_j| \chi_{J_j}(x) < \infty, \qquad \sum_{j=1}^{\infty} |d_j| \chi_{K_j}(x) < \infty$$

Then

$$\sum_{j=1}^{\infty} c_j \lambda(J_j) = \sum_{j=1}^{\infty} d_j \lambda(K_j)$$

Theorem 4.2: Lebesgue Integral Properties

Suppose f and g are integrable on I and α and β are real numbers. Then

1. $\alpha f + \beta g$ is integrable on I and
$$\int_I (\alpha f + \beta g) = \alpha \int_I f + \beta \int_I g$$
2. If $f \ge 0$ on I then $\int_I f \ge 0$; if $f \ge g$ on I then $\int_I f \ge \int_I g$
3. $|f|$ is integrable on I and $\left| \int_f f \right| \le \int_I |f|$
4. $\max\{f, g\}$ and $\min\{f, g\}$ are integrable on I
5. If one of the functions is bounded then the product fg is integrable on I
6. If $f \ge 0$ with $\int_I f = 0$ then any function h such that $0 \le h \le f$ on I is integrable on I

Theorem 4.3: Integrability of Sequences and Series

Suppose that $(f_n)_{n \in \mathbb{N}}$ is a sequence of functions each of which is integrable on I

1. Assume that
$$\sum_{n=1}^{\infty} \int_I |f_n| < \infty$$
Let f be a function on the interval I such that
$$f(x) = \sum_{n=1}^{\infty} f_n(x) \quad \text{for all } x \in I \text{ such that} \quad \sum_{n=1}^{\infty} |f_n(x)| < \infty$$
Then f is integrable on I and its integral on I is equal to
$$\int_I f = \sum_{n=1}^{\infty} \int_I f_n$$
2. Assume that each $f_n \ge 0$ on I and let $f(x) = \sum_{n=1}^{\infty} f_n(x)$ for all $x \in I$ (we allow for the possibility that at some points this sum is infinite). Then f is integrable on I if and only if
$$\sum_{n=1}^{\infty} \int_I f_n < \infty$$

Theorem 4.4: Monotone Convergence Theorem

Suppose that (f_n) is a monotone increasing sequence of integrable functions on an interval I. That is, $f_1(x) \le f_2(x) \le f_3(x) \le \ldots$ for all $x \in I$. For all $x \in I$, let

$$f(x) = \lim_{n \to \infty} f_n(x)$$

where we allow for the possibility that at some points this limit is infinite. Then f is integrable on I iff

$$sup_{n \in \mathbb{N}} \int_I f_n = \lim_{n \to \infty} \int_I f_n < \infty. \quad \text{Also,} \int_I f = \lim_{n \to \infty} \int_I f_n$$

Definition 4.4: Riemann Integrable Functions

Let $f:\mathbb{R}\to\mathbb{R}$. We say that f is **Riemann-integrable** if for every $\epsilon>0$ there exists step functions ϕ and ψ such that
$$\phi \le f \le \psi$$
and
$$\int \psi - \int \phi < \epsilon$$

Thm 4.5: A function $f:\mathbb{R}\to\mathbb{R}$ is Riemann-integrable iff
$$\sup\left\{\int \phi : \phi \text{ is a step function and } \phi \le f\right\}$$
$$= \inf\left\{\int \psi : \psi \text{ is a step function and } \phi \ge f\right\}$$

Def 4.5: If f is Riemann-integrable we define its Riemann integral $(R)\int f$ as the common value
$$(R)\int f := \sup\left\{\int \phi : \phi \text{ is a step function and } \phi \le f\right\}$$
$$= \inf\left\{\int \psi : \psi \text{ is a step function and } \phi \ge f\right\}$$

Theorem 4.1: Riemann lemmas

Let $f:\mathbb{R}\to\mathbb{R}$ be a bounded function with bounded support $[a,b]$. The following are equivalent:

1. f is Riemann-integrable
2. for every $\epsilon>0$ there exists $a=x_0<\cdots<x_n=b$ s.t. if M_j and m_j denote the sup and inf of f on (x_{j-1},x_j) respectively, then
$$\sum_{j=1}^{n}(M_j-m_j)(x_j-x_{j-1})<\epsilon$$
3. for every $\epsilon>0$ there exists $\alpha=x_0<\cdots<x_n=b$ s.t. with $I_j=(x_{j-1},x_j)$ for $j\ge 1$
$$\sum_{j=1}^{n}\sup_{x,y\in I_j}|f(x)-f(y)|\lambda(I_j)<\epsilon$$

Notation to aid these lemmas: For $f:\mathbb{R}\to\mathbb{R}$ a bounded function with bounded support $[a,b]$ and for $a=x_0<\cdots<x_n=b$, we let $I_j=(x_{j-1},x_j)$, $m_j:=\inf_{x\in I_j}f(x)$ and $M_j:=\sup_{x\in I_j}f(x)$. We define the **lower step function of f with respect to** $\{x_0,\dots,x_n\}$ as
$$\phi_*(x)=\sum_{j=1}^{n}m_j\chi_{I_j}(x)+\sum_{j=0}^{n}f(x_j)\chi_{\{x_j\}}(x)$$
and the **upper step function of f with respect to** $\{x_0,\dots,x_n\}$ as
$$\phi^*(x)=\sum_{j=1}^{n}M_j\chi_{I_j}(x)+\sum_{j=0}^{n}f(x_j)\chi_{\{x_j\}}(x)$$
$\phi_*(x)$ and $\phi^*(x)$ are step functions, and $\phi_*(x)\le f\le\phi^*(x)$

Suppose that $g:[a,b]\to\mathbb{R}$ and let f be defined by $f(x)=g(x)$ for $x\in[a,b]$ and $f(x)=0$ otherwise.

1. If g is continuous on $[a,b]$, then f is Riemann-integrable
2. If g is a monotone function then f is Riemann-integrable

Theorem 4.6: Connection between Riemann and Lebesgue

Suppose that $f:\mathbb{R}\to\mathbb{R}$ is Riemann-integrable. Then f is also Lebesgue integrable on $\mathbb{R}$ and moreoever
$$(R)\int f=\int f$$
where the number on the lefthand side is the value of the Riemann integral of f, while the righthand side denotes the value of the Lebesgue integral of f on $\mathbb{R}$

Theorem 4.8: Dependence on Intervals for Lebesgue

Let I and J be two intervals such that $J\subset I$.

1. If f is integrable on I then f is also integrable on the subinterval J
2. If f is integrable on J and simultaneously $f(x)=0$ for all $x\in I\backslash J$ then f is integrable on I and
$$\int_J f=\int_I f$$
3. If f is integrable on I and $f(x)\ge 0$ for all $x\in I$ then
$$\int_J f\le\int_I f$$
4. Suppose that I can be written as the union of disjoint intervals I_n, $n=1,2,3,\dots$ and let f be integrable on each of the intervals I_n. Then f is integrable on I iff
$$\sum_{n=1}^{\infty}\int_{I_n}|f|<\infty$$
If this holds, then
$$\int_I f=\sum_{n=1}^{\infty}\int_{I_n}f$$

Theorem 4.9: Addition of Intervals

If any two of these integrals
$$\int_a^b f,\quad \int_b^c f,\quad \int_a^c f$$
exist then so does the third and
$$\int_a^b f,+\int_b^c f=\int_a^c f$$

Theorem 4.10: Fundamental Theorem of Calculus

Let I be an interval and let $g:I\to\mathbb{R}$ be integrable on I. For all $x\in I$ and some fixed $x_0\in I$ let $G(x)=\int_{x_0}^x g$. Suppose g is continuous at x for some $x\in I$ [if x is an endpoint we mean one-sided continuity.] Then G is differentiable at x and $G'(x)=g(x)$. [if x if an endpoint we mean one-sided differentiable]

Suppose $f:I\to\mathbb{R}$ has continuous derivative f' on the interval I. Then for any $a,b\in I$:
$$\int_a^b f'=f(b)-f(a)$$

Lemma 4.2: Fatoux Lemma

Let (f_n) be a sequence of non-negative integrable functions on an interval I. Let
$$f(x)=\liminf_{n\to\infty}f_n(x),\quad \text{for all } x\in I$$
If $\liminf_{n\to\infty}\int_I f_n<\infty$ then f is integrable on I and
$$\int_I f\le\liminf_{n\to\infty}\int_i f_n$$

Theorem 4.12: Dominated Convergence Theorem

Let (f_n) be a sequence of integrable functions on an interval I and assume that
$$f(x)=\lim_{n\to\infty}f_n(x),\quad \text{for all } x\in I$$
. Assume also that the sequence (f_n) is **dominated** by some integrable function g, that is
$$|f_n(x)|\le g(x),\quad \text{for all } x\in I \text{ and } n=1,2,\dots,\qquad \int_I g<\infty$$
Then the function f is integrable on I and
$$\int_I f=\lim_{n\to\infty}\int_I f_n$$

Theorem 4.13

Let (a,b) be a bounded interval and suppose that $f_n:(a,b)\to\mathbb{R}$ are integrable functions which converges uniformly to a function f. Then f is integrable on (a,b) and
$$\int_a^b f=\lim_{n\to\infty}\int_a^b f_n$$

4 Fourier Series and Orthogonality

Definition 5.1: The Space L^2

Define the space $L^2 = L^2([a,b])$ as the set of measurable functions $f:[a,b]\to\mathbb{C}$ so that the function $x\mapsto |f(x)|^2$ is Lebesgue integrable, i.e.

$$\|f\|_2^2 := \int_a^b |f(x)|^2 dx < \infty$$

The quantity $\|f\|_2$ is called the **L^2-norm** of f. If $\|f\|_2 = 1$, then we say that f is **L^2-normalised**

Definition 5.2: Inner Product

For two functions $f,g\in L^2([a,b])$, we define their **inner product** by

$$\langle f,g\rangle = \int_a^b f(x)\overline{g(x)}dx$$

Theorem 5.1: Cauchy-Shwarz Inequality

Let $f,g\in L^2([a,b])$. then the function $x\mapsto f(x)\overline{g(x)}$ is Lebesgue integrable and we have

$$|\langle f,g\rangle| \le \|f\|_2\|g\|_2$$

Minkowski's Inequality: For two functions $f,g\in L^2([a,b])$,

$$\|f+g\|_2 \le \|f\|_2 + \|g\|_2$$

Definition 5.3: Convergent Sequences in L^2

Let $f, f_1, f_2, \dots$ be functions in $L^2([a,b])$. We say that the function $(f_n)_n$ converges to f in L^2 if the sequence

$$\|f_n - f\|_2 = \left(\int_a^b |f_n(x)-f(x)|^2 dx\right)^{1/2}$$

converges to zero as $n\to\infty$. We will also write $f_n\to f$ in L^2

Definition 5.4: Orthonormal Systems

A sequence $(\phi_n)_n$ of L^2 functions on $[a,b]$ is called an **orthonormal system on** $[a,b]$ if

$$\langle \phi_n,\phi_m\rangle = \int_a^b \phi_n(x)\overline{\phi_m(x)}dx = \begin{cases}0, & \text{if } n\neq m\\ 1, & \text{if } n=m\end{cases}$$

(The index n may run over any countable set. We will write $\sum_n$ to denote a sum over all the indices. In proofs we will always adopt the interpretation that n runs over $1,2,3,\dots$ without loss of genereralit y)

Theorem 5.2

Let $(\phi_n)_n$ be an orthonormal system on $[a,b]$ and $f\in L^2$. Consider

$$s_N(x) = \sum_{n=1}^N \langle f,\phi_n\rangle\phi_n(x)$$

Denote the linear span of the functions $(\phi_n)_{n=1,\dots,N}$ by X_N. Then

$$\|f-s_N\|_2 \le \|f-g\|_2$$

holds for all $g\in X_N$ with equality iff $g = s_N$

Definition 5.3: Bessel's Inequality

If $(\phi_n)_n$ is an orthonormal system on $[a,b]$ and $f\in L^2$, then

$$\sum_n |\langle f,\phi_n\rangle|^2 \le \|f\|_2^2$$

Corollary - Riemann-Lebesgue lemma in L^2. Let $(\phi_n)_{n=1,2,\dots}$ be an orthonormal system and $f\in L^2$, then

$$\lim_{n\to\infty}\langle f,\phi_n\rangle = 0$$

Definition 5.5: Complete Orthonormal Systems

An orthonormal system $(\phi_n)_n$ is called **complete** if

$$\sum_n |\langle f,\phi_n\rangle|^2 = \|f\|_2^2$$

for all $f\in L^2$

Thm 5.4: Let $(\phi_n)_n$ be an orthonormal system on $[a,b]$. Let $(s_N)_N$ be as in Theorem 5.2. Then $(\phi_n)_n$ is complete iff $(s_N)_N$ converges to f in the L^2-norm for every $f\in L^2$

Definition 5.6: Trigonometric Polynomials

A **trigonometric polynomial** is a function of the form

$$f(x) = \sum_{n=-N}^N c_n e^{2\pi i n x} \quad (x\in\mathbb{R})$$

where $N\in\mathbb{N}$ and $c_n\in\mathbb{C}$. If c_N or c_{-N} is non-zero, then N is called the **degree** of f

Observe that trigonometric polynomials are continuous functions. From Euler's identity $e^{ix} = \cos(x) + i\sin(x)$, $(x\in\mathbb{R})$ we see that every trigonometric polynomial can also be written in the form

$$f(x) = a_0 + \sum_{n=0}^N (a_n\cos(2\pi n x) + b_n\sin(2\pi n x))$$

Lemma 5.1: $(e^{2\pi i n x})_{n\in\mathbb{Z}}$ forms an orthonormal system on $[0,1]$. In particular,

1. for all $n\in\mathbb{Z}$,

$$\int_0^1 e^{2\pi i n x}dx = \begin{cases}0, & \text{if } n\neq 0\\ 1, & \text{if } n=0\end{cases}$$

2. if $f(x) = \sum_{n=-N}^N c_n e^{2\pi i n x}$ is a trigonometric polynomial, then

$$c_n = \langle f,\phi_n\rangle = \int_0^1 f(t)e^{-2\pi i n t}dt$$

Definition 5.7: Fourier Coefficient

For a 1-periodic integrable function f and $n\in\mathbb{Z}$ we define the n**th Fourier coefficieent** by

$$\widehat{f}(n) = \int_0^1 f(t)e^{-2\pi i n t}dt = \langle f,\phi_n\rangle$$

(the integral on the right exists since f is integrable and $|\phi_n|\le 1$.) The doubly infinite series

$$\sum_{n=-\infty}^{\infty}\widehat{f}(n)e^{2\pi i n x}$$

is called the **Fourier series** of f

Def 5.8 (Partial Sums): For a 1-periodic integrable function f, we define the **partial sums**

$$S_N f(x) = \sum_{n=-N}^N \widehat{f}(n)e^{2\pi i n x}$$

Note: for all $f\in L^2$ and trigonometric polynomials g of degree $\le N$, we have

$$\|f - S_N f\|_2 \le \|f-g\|_2$$

Definition 5.9: Convolution

For two 1-periodic functions $f,g\in L^2$ we define their **convolution** by

$$f*g(x) = \int_0^1 f(t)g(x-t)dt$$

(The integral on the right hand side exists by Cauchy-Shwarz)

Lemma bank

5.2 For 1-periodic functions $f,g\in L^2$,

$$f*g = g*f$$

5.3 We have

$$D_N(x) = \frac{\sin(2\pi(N+\frac{1}{2})x)}{\sin(\pi x)}$$

5.4 We have

$$K_N(x) = \frac{1}{2(N+1)}\frac{1-\cos(2\pi(N+1)x)}{\sin(\pi x)^2} = \frac{1}{N+1}\left(\frac{\sin(\pi(N+1)x)}{\sin(\pi x)}\right)^2$$

Thm 5.5 (Fejér): For every 1-periodic continuous function f,

$$K_N * f \to f$$

uniformly on $\mathbb{R}$ as $N\to\infty$

Corollary: Every 1-periodic continuous function can be uniformly approximated by trigonometric polynomials. That is, for every 1-periodic continuous f there exists a sequence $(f_n)_n$ of trigonometric polynomials so that $f_n\to f$ uniformly

Definition 5.10: Approximation of Unity

A sequence of 1-periodic integrable functions $(k_n)_n$ is called **approximation of unity** if for all 1-periodic continuous functions f we have that $f * k_n$ converges uniformly to f on $\mathbb{R}$. That is,

$$\sup_{x\in\mathbb{R}}|f * k_n(x) - f(x)| \to 0 \quad \text{as } n \to \infty$$

Thm 5.6: Let $(k_n)_n$ be a sequence of 1-periodic integrable functions such that

1. $k_n(x) \geq 0$ for all $x \in \mathbb{R}$
2. $\int_{-1/2}^{1/2} k_n(t)dt = 1$
3. For all $1/2 \geq \delta > 0$ we have
$$\int_{-\delta}^{\delta} k_n(t)dt \to 1 \quad \text{as } n \to \infty$$

Then $(k_n)_n$ is an approximation of unity

Corollary: The Fejér kernel $(K_N)_N$ is an approximation of unity

Lemma 5.5

Let f be a 1-periodic and continuous function. Then

$$\lim_{N\to\infty} \|S_N f - f\|_2 = 0$$

Theorem 5.7: Completeness of Trigonometric System

The trigonometric system is complete. In view of Theorem 5.4 this means that for every 1-periodic L^2 function f we have

$$\lim_{N\to\infty} \|S_N f - f\|_2 = 0$$

In other words, the Fourier series of f converges to f in the L^2 sense

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Corollary (Parseval's Theorem): If f, g are 1-periodic L^2 functions then

$$\langle f, g\rangle = \sum_{n=-\infty}^{\infty} \widehat{f}(n)\overline{\widehat{g}(n)}$$

In particular,

$$\|f\|_2^2 = \sum_{n=-\infty}^{\infty} |\widehat{f}(n)|^2$$

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