

Dynamics and Vector Calculus Notes

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1 Couple Oscillations and normal modes

some diagram idk

where x_1 and x_2 are displacements from equilibrium

For mass 1

- Force to the left: $-k_1 x_1$
- Force to the right: $-k_2(x_2 - x_1)$

$$m \frac{d^2 x_1}{dt^2} = -k_1 x_1 + k_2(x_2 - x_1) - k_3 x_2$$

Write this in matrix form

$$m \frac{d^2}{dt^2} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \implies m \frac{d^2 x}{dt^2} = -Kx$$

Definition 1.0.1: Normal Mode Solution

Normal Mode Solution: All co-ordinates (here x_1, x_2) oscillate with the same frequency

$$x(t) = \cos(\omega t - \phi) \underline{b}$$

\underline{b} is constant vector, ω to be determined

sub in matrixeq??

$$\begin{aligned} -m\omega^2 \cos(\omega t - \phi) \underline{b} + K \cos(\omega t - \phi) \underline{b} &= 0 \\ -m\omega^2 \underline{b} + K \underline{b} &= 0 \rightarrow K \underline{b} = \lambda \underline{b} \quad \lambda = m\omega^2 \end{aligned}$$

where λ is eigenvalue, and b is eigenvector

For simplicity, take $k_1 = k_2 = k_3 = k$

Then,

$$K = \begin{pmatrix} 2k & -k \\ -k & 2k \end{pmatrix} \quad (K - \lambda I) \underline{b} = 0 \implies |K - \lambda I| = 0$$

$$\begin{vmatrix} 2k - \lambda & -k \\ -k & 2k - \lambda \end{vmatrix} = 0 \implies (2k - \lambda)^2 - k^2 = 0$$

This is called the "Characteristic Equation"

$$(2k - \lambda) = \pm k \quad \lambda = 2k \mp k$$

Therefore, $\lambda = k, 3k$

$$\text{Mode A: } \lambda_A = k \quad (K - kI) \underline{b} = 0$$

$$\begin{pmatrix} k & -k \\ -k & k \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = 0 \quad \underline{b}_A = C t \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Usually, choose a constant s.t. $\underline{b} \cdot \underline{b} = 1$

$$\text{Mode B: } \lambda_B = 3k \quad (K - 3kI) \underline{b} = \begin{pmatrix} -k & -k \\ -k & -k \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = 0$$

and some stuff more i forgot to write

[diagram thing]

$$\text{Normal mode } \underline{x}(t) = \underline{b} \cos(\omega t - \phi) \rightarrow (K - \lambda I) \underline{b} = 0 \quad \lambda = m\omega^2$$

$$\lambda_A = k, \underline{b}_A = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \lambda_B = 3k, \underline{b}_B = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

So we have 2 independent solutions

General solution: $\underline{x}(t) = A\underline{b}_A \cos(\omega_A t - \phi_A) + B\underline{b}_B \cos(\omega_B t - \phi_B)$

So there are 4 constants A, B, ϕ_A, ϕ_B to be fixed

1.1 Motion in Normal modes

$$\text{Mode A } x_1 = x_2 \quad \text{"in phase"} \quad \omega_A = \left(\frac{k}{m}\right)^2$$

$$\text{Mode B } x_1 = -x_2 \quad \text{"antiphase"} \quad \omega_B > \omega_A$$

Normal Co-ordinates Take scalar product

$$(1, 1) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_1 + x_2 = 2A \cos(\omega_A t - \phi_A)$$

$$(1, -1) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_1 - x_2 = 2B \cos(\omega_B t - \phi_B)$$

Define

$$z_1 = \frac{1}{\sqrt{2}}(x_1 + x_2) = \alpha^1 \cos(\omega_A t - \phi) \quad z_1 + \omega_A^2 z_1 = 0 \quad (\text{SHO})$$

$$z_2 = \frac{1}{\sqrt{2}}(x_1 - x_2) = \beta^1 \cos(\omega_B t - \phi) \quad z_2 + \omega_B^2 z_2 = 0 \quad (\text{SHO})$$

z_1 and z_2 are each independent simple harmonic motions, and energy is preserved in each one

$$E_A = \frac{1}{2}m(\dot{z}_1)^2 + \frac{1}{2}kz_1^2 = \text{constant in time}$$

1.2 Summary: properties of Normal Modes

- $\underline{x}_\alpha = A_\alpha \underline{b}_\alpha \cos(\omega_\alpha t - \phi_\alpha)$
- All coordinates oscillate at the same frequency
- constants A_α, ϕ_α are fixed by ic (???)
- General motion is superposition of normal modes
- Normal coordinates $z_\alpha = \underline{b}_\alpha \cdot \underline{x}$
- Transforming to the normal coordinates \rightarrow diagonalise k (see notes i.e. ask alice or fiona for them)
- Energy in each normal mode conserved, mode with lowest ω is the most symmetric

1.3 Coupled Pendulum

[a diagram]

pendulum thing

$$ml \frac{d^2 \theta}{dt^2} = -ml\omega_0^2 \theta \quad \omega_0 = \left(\frac{g}{l}\right)^{1/2}$$

Add in the force from the spring extension:

$$x_2 - x_1 = l(\sin \theta_2 - \sin \theta_1) \approx l(\theta_2 - \theta_1)$$

$$\begin{aligned} m \frac{d^2 \theta_1}{dt^2} &= -m\omega_0^2 \theta_1 + k(\theta_2 - \theta_1) \\ m \frac{d^2 \theta_2}{dt^2} &= -m\omega_0^2 \theta_2 + k(\theta_2 - \theta_1) \end{aligned}$$

Putting it in vector form thing

$$m \frac{d^2}{dt^2} \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} = - \begin{pmatrix} m\omega_0^2 + k & -k \\ -k & m\omega_0^2 + k \end{pmatrix} \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}$$

Normal mode $\underline{\theta} = \underline{b} \cos(\omega t) \quad -m\omega_0^2 \underline{b} + K\underline{b} = 0$

eigenvalue problem $K\underline{b} = \lambda \underline{b} \quad \lambda = m\omega^2$

$$\det(K - \lambda \mathbb{K}) = 0 \quad \begin{vmatrix} m\omega_0^2 + k - \lambda & -k \\ -k & m\omega_0^2 - k - \lambda \end{vmatrix} = 0$$

$$(m\omega_0^2 + k - \lambda)^2 - k^2 = 0 \quad \lambda_\Delta = m\omega_0^2 \quad \lambda_B = m\omega_0^2 + 2k$$

Eigenvectors

$$\begin{aligned} \lambda_A = m\omega_0^2 &= \begin{pmatrix} k & -k \\ -k & k \end{pmatrix} \underline{b}_A = 0 \quad \underline{b}_A = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{inphase} \quad \omega = \omega_0 \\ \lambda_B = m\omega_0^2 + 2k &= \begin{pmatrix} k & -k \\ -k & -k \end{pmatrix} \underline{b}_B = 0 \quad \underline{b}_B = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \text{antiphase} \quad \omega_B^2 = \omega_0^2 + \frac{2k}{m} \end{aligned}$$

1.3.1 Mass Matrix

$$m_1 x_1 = (k_1 + k_2)x_1 + k_2 x_2$$

$$m_2 x_2 = k_2 x_1 - (k_3 + k_2)x_2$$

NOTE: have defo missed some double dot x at some points

Write this as $M\ddot{\underline{x}} = -K\underline{x} \quad M = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}$

Normal mode $\underline{x}(t) = \underline{b} \cos(\omega t - \pi) \quad -\omega^2 M \underline{b} = -K \underline{b} \quad (K - \lambda M) \underline{b} = 0$

$(K - \lambda M) \underline{b} = 0$ for nontrivial sol $\det(K - \lambda M) = 0$

$$\lambda = \omega^2 \quad \begin{vmatrix} k_1 + k_2 - \lambda m_1 & -k_2 \\ -k_2 & k_1 k_2 - \lambda m_2 \end{vmatrix} = 0$$

\Rightarrow quadratic for λ . For equal k ,

$$(2k - \lambda m_1)(2k - \lambda m_2) - k^2 = 0 \quad (\text{quadratic})$$

1.4 Line integral :)

idk what's happening but line integral

$$\begin{aligned}\Gamma(x) &= \int_a^{a+\frac{\pi}{2}} \sin^2 \lambda d\lambda = \int_a^{a+\frac{\pi}{2}} \cos^2 \lambda d\lambda = \frac{1}{2} \int_a^{a+\frac{\pi}{2}} \sin^2 \lambda + \cos^2 \lambda d\lambda = \frac{\pi}{4} \\ &= \frac{\pi}{4} - \frac{1}{3} \sin^3 \lambda \cos \lambda \Big|_0^{\frac{\pi}{2}} - \frac{1}{3} \underbrace{\int_0^{\frac{\pi}{2}} 4\lambda d\lambda}_I = \frac{\pi}{4} - \frac{I}{3} = I\end{aligned}$$

random facts

- $\underline{\Delta} \times \underline{\Delta} \phi = 0$
- $\underline{\Delta} \cdot (\underline{\Delta} \times \underline{a}) = 0$

1.5 Surface Integrals

(shoutout to the generalised stoke's theorem, he got me fr fr)

Definition 1.5.1: Parametric form of the surface integral

$$\begin{aligned}\underline{\Delta} &= \underline{\Delta}(u, v) \\ &= x_1(u, v)\underline{e}_1 + x_2(u, v)\underline{e}_2 + x_3(u, v)\underline{e}_3\end{aligned}$$

Example: Sphere (in spherical coordinates) idk how to draw diagrams

$$\begin{aligned}x_1(\theta, \phi) &= \sin \theta \cos \phi \\ x_2(\theta, \phi) &= \sin \theta \sin \phi \\ x_3(\theta, \phi) &= \cos \theta\end{aligned}$$

$$d\underline{r} = \underbrace{\partial_X \underline{r} du}_{d\underline{r}_u} + \underbrace{\partial_Y \underline{r} dv}_{d\underline{r}_v}$$

$$d\underline{S} = d\underline{r}_u \times d\underline{r}_v = \text{"area of infinitesimal parallelogram"}$$

Actual line integral equation

Definition 1.5.2: Line Integral Equation

$$\int_S \underline{a} \cdot d\underline{S} = \iint \underline{a} \cdot (\partial_u \underline{r} \times \partial_v \underline{r}) du dv$$

Remarks

- $\underline{\vec{n}} \propto \partial_u \underline{r} \times \partial_v \underline{r}$
- ambiguity in orientation ($u \leftrightarrow v$ interchange)
 - (circle): closed surface, choose $\underline{\vec{u}}$ outwards
 - open surface: can do either - choose one. In Stoke's theorem, there will be a double ambiguity
 - * Orientation of S

* Direction of line integral

- For applications, $\partial_u \underline{r} \delta_v \underline{r}$. e.g. spherical coords: $\partial_\theta \underline{r} \times \partial_\phi \underline{r} \propto \partial_r \underline{r}$

Definition 1.5.3: somethinglinear coordinates I , flux & surface

$$(x_1, x_2, x_3) \rightarrow (x, y, z)$$

Definition 1.5.4: Plan polar coordinates

$$\begin{aligned} x_1(\rho, \phi) &= \rho \cos \phi & \rho &= \sqrt{x_1^2 + y_2^2} \in [0, \infty) \\ x_2(\rho, \phi) &= \rho \sin \phi & \phi &= \arctan \frac{x_2}{x_1} \in [0, \infty) \end{aligned}$$

$$\underline{\Gamma}(\rho, \phi) = x_1(\rho, \phi) \underline{e}_1 + x_2(\rho, \phi) \underline{e}_2$$

$$\underline{e}_\rho = \frac{\partial_\rho \underline{r}}{|\partial_\rho \underline{r}|} = \cos \phi \underline{e}_1 + \sin \phi \underline{e}_2 = \frac{1}{\rho} \underline{r} \text{ (special case final line)}$$

Remarks

- $\underline{r} \neq \rho \underline{e}_\rho + \phi \underline{e}_\phi$
- generally, $\underline{a} = a_\rho \underline{e}_\rho + a_\phi \underline{e}_\phi$, $\alpha_{\rho, \phi} = \underline{a} \cdot \underline{e}_{\rho, \phi}$
- new aspect: $\{\underline{e}_\rho, \underline{e}_\phi\}$ is position dependent

Definition 1.5.5: Cylindrical Coordinates

$$\begin{aligned} x_1 &= \rho \cos \phi \\ x_2 &= \rho \sin \phi \\ x_3 &= z \end{aligned}$$

$$\begin{aligned} \underline{\Gamma}(\rho, \phi, z) &= \rho \cos \phi \underline{e}_1 + \rho \sin \phi \underline{e}_2 + z \underline{e}_3 \\ &= \rho \underline{e}_\rho + z \underline{e}_z \text{ (special case)} \end{aligned}$$

Note: $\{\underline{e}_\rho, \underline{e}_\phi, \underline{e}_z\}$ forms a right-hand orthogonal basis