第一章　反时间非局部 NLS 方程-jyw

1.1　2024.09.21

1.1.1　原谱问题的解

设 Lax 对为：

$$\begin{aligned} \Phi_x &= U\Phi = (i\lambda\sigma_3 + P)\Phi \\ \Phi_t &= V\Phi = -(2i\lambda^2\sigma_3 + 2\lambda P + i(P^2 + p_x)\sigma_3)\Phi \end{aligned} \tag{1.1.1}$$

其中

$$\begin{gathered} P = \begin{pmatrix} 0 & p & p(-t) \\ -r_1 & 0 & 0 \\ -r_2 & 0 & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & & \\ & -1 & \\ & & -1 \end{pmatrix}, \\ \begin{cases} r_1 = ap^* + bp(-t), \\ r_2 = ap(-t) + b^*p^*, \quad a \in \mathbb{R}, b \in \mathbb{C} \end{cases} \end{gathered} \tag{1.1.2}$$

将相容性条件 $U_t - V_x - [U, V] = 0$ 代入上式可得

$$\begin{aligned} &P_t + 2\lambda Px + iPP_x\sigma_3 + iP_xP\sigma_3 + iP_{xx}\sigma_3 \\ &- [-2\lambda^3 I + 2i\lambda^2\sigma_3 P - \lambda\sigma_3(P^2 + P_x)\sigma_3 + 2i\lambda^2 P\sigma_3 + 2\lambda P^2 + i(P^3 + iPP_x)\sigma_3] \\ &- 2\lambda^3 + 2i\lambda^2 P\sigma_3 - \lambda(p^2 + P_x) + 2i\lambda^2\sigma_3 P + 2\lambda P^2 + i(P^2 + P_{xx}) = 0. \end{aligned} \tag{1.1.3}$$

故可化简为

$$P_t + iP_{xx}\sigma_3 - 2iP^3\sigma_3 = 0, \implies iP_t - P_{xx} + 2P^3\sigma_3 = 0. \tag{1.1.4}$$

则上式可写为

$$\begin{aligned} &\begin{pmatrix} 0 & ip_t & ip_t^*(-t) \\ -iap_t^* - ibp_t(-t) & 0 & 0 \\ -iap_t(-t) - ib^*p_t^* & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & -p_{xx} & -p_{xx}^*(-t) \\ -ap_{xx}^* - bp_{xx}(-t) & 0 & 0 \\ -ap_{xx}(-t) - b^*p_{xx}^* & 0 & 0 \end{pmatrix} \\ &\qquad + \begin{pmatrix} 0 & 2p\Delta & 2p^*(-t)\Delta \\ 2r_1\Delta & 0 & 0 \\ 2r_2\Delta & 0 & 0 \end{pmatrix} = 0 \end{aligned} \tag{1.1.5}$$

其中 $\Delta = pr_1 + p^*(-t)r_2$, 则有

$$\begin{aligned} &(12) : ip_t - p_{xx} + 2p\Delta = 0 \\ &(13) : ip_t(-t) + p_{xx}(-t) + 2p^*(-t)\Delta = 0 \\ &(21) : -iap_t^* - ibp_t(-t) + ap_{xx}^* + bp_{xx}(-t) + 2r_1\Delta = 0 \\ &(31) : -ap_t(-t) - ib^*p_t^* + ap_{xx}(-t) + b^*p_{xx}^* + 2r_2\Delta = 0 \end{aligned} \tag{1.1.6}$$

显然有 $(12)^*(-t) = (13), a(12)^* + b(13)^* = (21), b(12)^* + a(13)^* = (31)$.

1.1.2 伴随问题的解

若 $\Phi_!$ 是 $\lambda = \lambda_1$ 时原谱问题的解,$\exists A$ 使得 $\Phi_1^\dagger A$ 是 $\lambda = \lambda_1$ 时伴随问题的解[1], 即

$$\text{原问题:} \begin{cases} \Phi_x = U\Phi \\ \Phi_t = V\Phi \end{cases}, \quad \text{伴随问题:} \begin{cases} \Psi_x = -\Psi U \\ \Psi_t = -\Phi V \end{cases} \tag{1.1.7}$$

我们想要得到

$$(\Phi_1^\dagger A)_x = -\Phi_1^\dagger A U(\lambda_1^*) = -\Phi_1^\dagger A(i\lambda_1^*\sigma_3 + P) \tag{1.1.8}$$

对 $\Phi_1^\dagger A$ 求偏导可得

$$(\Phi_1^\dagger A)_x = \Phi_1^\dagger U^\dagger \lambda_1 A = \Phi_1^\dagger(-i\lambda_1^*\sigma_3 + P^T)A \tag{1.1.9}$$

由 (1.1.8) 和 (1.1.9) 式可得 $-AP = P^T A, A = (a_{ij})_{3\times3}$ 展开可得

$$\begin{aligned} &-\begin{pmatrix} -(aa_{12} - b^*a_13)p^* + (-ba_{12} - aa_{13})p(-t) & a_{11}p & a_{11}p^*(-t) \\ -(aa_{22} - b^*a_{23})p^* + (-ba_{22} - aa_{23})p(-t) & a_{21}p & a_{21}p^*(-t) \\ (-aa_{32} - b^*a_{23})p^* + (-ba_{32} - a_{33})p(-t) & a_{31}p & a_{31}p^*(-t) \end{pmatrix} \\ &\qquad = \begin{pmatrix} (-aa_{21} - ba_{31})p + (-b^*a_{21} - aa_{31})p^*(-t) & a_{11}p^* & a_{11}p(-t) \\ (-aa_{22} - ba_{32})p + (-b^*a_{22} - aa_{32})p^*(-t) & a_{12}p^* & a_{21}p(-t) \\ (-aa_{23} - ba_{33})p + (-b^*a_{23} - aa_{33})p^*(-t) & a_{13}p^* & a_{13}p(-t) \end{pmatrix}^T \end{aligned} \tag{1.1.10}$$

故有

$$\begin{aligned} a_{12} = a_{21} = a_{13} = a_{31} &= 0 \\ aa_{23} + ba_{33} &= 0 \\ b^*a_{22} + aa_{32} &= 0 \\ ba_{22} + aa_{23} &= 0 \\ aa_{32} + ba_{33} &= 0 \end{aligned} \Longrightarrow \begin{cases} a_{33} = a \\ a_{32} = -b^* \\ a_{23} = -b \\ a_{33} = a \end{cases} \tag{1.1.11}$$

故有 $(\Phi_1^\dagger A)_t = V^T(\lambda_1)A$,

$$\begin{aligned} \text{左侧} &= \Phi_1^\dagger A\big(2i\lambda_1^*\sigma_3 + 2\lambda_1^* P + i(P^2 + P_x)\sigma_3\big) = -\Phi_1^* AV(\lambda_1^*) \\ \text{右侧} &= \Phi_1^\dagger\big(2i\lambda_1^*\sigma_3 - 2\lambda_1^* P^\dagger + i\sigma_3(P^{\dagger 2} + P_x^t)\big)A \end{aligned} \tag{1.1.12}$$

故有 $V^\dagger(\lambda_1)A = -AV(\lambda_1)$.

1.2 可积条件

接下来考虑 (1.1.4) 的一般情形. 设

$$P = \begin{pmatrix} 0 & p & q \\ -r_1 & 0 & 0 \\ -r_2 & 0 & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & & \\ & -1 & \\ & & -1 \end{pmatrix}$$

[1]这里 † 表示 Hermitian 转置

则得到

$$\begin{aligned} &(12): ip_t + p_{xx} + 2p\Delta = 0 \\ &(13): iq_t + q_{xx} + 2q\Delta = 0 \\ &(21): ir_{1t} + r_{1xx} + 2r_1\Delta = 0 \\ &(31): ir_{2t} + r_{2xx} + 2r_2\Delta = 0 \end{aligned} \tag{1.2.13}$$

其中 $\Delta = pr_1 + qr_2$, 设

$$\begin{cases} r_1 = ap_1 + bq_1, \\ r_2 = cp_1 + dq_1 \end{cases} \tag{1.2.14}$$

参数 p_1 可取 $p, p^*, p(-x), p^*(-x), p(-t), p^*(-t)$, q_1 同理. 在 (12) 到 (21) 的约化中 $ip_t \to ir_{1t}$, 考虑到约化符号于是 p 只能取 $p^*, p^*(-x), p^*(-t)$, q 同理.

1.2.1 若 $p_1 = p^*, q_1 = q^*$

则

$$\begin{cases} r_1 = ap^* + bq^* \\ r_2 = cp^* + dq^* \end{cases} \tag{1.2.15}$$

(1.2.13) 式变为

$$\begin{aligned} &(12): ip_t + p_{xx} + 2p\Delta = 0 \\ &(13): iq_t + q_{xx} + 2q\Delta = 0 \\ &(21): -iap^* - ibq^* + ap^*_{xx} + bq^*_{xx} + 2ap^*\Delta + 2bq^*\Delta = 0 \\ &(31): -icp^*_t - ibq^*_t + cp^*_{xx} + bq^*_{xx} + 2ap^*\Delta + 2bq^*\Delta = 0 \end{aligned} \tag{1.2.16}$$

要约化 (12)(13), 需要 $\Delta^* = \Delta$, 其中

$$\begin{aligned} \Delta &= app^* + bpq^* + cp^*q + dqq^* \\ \Delta^* &= a^*p^*p + b^*p^*q + c^*pq^* + d^*q^*q \end{aligned} \tag{1.2.17}$$

则可得 $a, d \in \mathbb{R}, b = c^*$, 此时有

$$\begin{aligned} (21) &= a(12)^* + b(13)^* \\ (31) &= b^*(12)^* + d(13)^* \end{aligned}$$

方程约化为

$$\begin{aligned} ip_t + p_{xx} + 2p(app^* + bpq^* + b^*p^*q + dqq^*) = 0 \\ iq_t + q_{xx} + 2q(app^* + bpq^* + b^*p^*q + dqq^*) = 0 \end{aligned}$$

则 q 可取 $p, p^*, p(-x), p^*(-x), p(-t), p^*(-t).p(-x,-t), p^*(-x,-t)$, 由 (12) $\to$ (13) 的限制, 故只考虑 $p(-x), p^*(-t), p^*(-x,-t)$

$q = p^*(-t)$

此时 (1.2.15) 化为

$$\begin{cases} r_1 = ap^* + bp(-t) \\ r_2 = b^*p^* + dp(-t) \end{cases}$$

要 (12), (13) 等价, 需 $\Delta^*(-t)=\Delta$, 则

$$\Delta = app^* + bpp(-t) + b^*p^*p^*(-t) + dp(-t)p^*(-t)$$
$$\Delta^*(-t) = ap^*(-t)p(-t) + b^*p^*(-t)p^* + bp(-t)p + dp^*p$$

为满足 $\Delta^*(-t)=\Delta$, 需 $a=d\in\mathbb{R}$, 此时 $(13)=(12)^*(-t)$. 方程约化为

$$ip_t + p_{xx} + 2p\big(app^* + bpp^*(-t) + b^*pp^*(-t) + ap(-t)p^*(-t)\big) = 0$$

$q=p(-x)$

此时 (1.2.15) 化为

$$\begin{cases} r_1 = ap^* + bp(-x) \\ r_2 = b^*p^* + dp(-x) \end{cases}$$

$$\Delta = app^* + bpp(-x) + b^*p^*p^*(-x) + dp(-x)p^*(-x)$$
$$\Delta^*(-x) = ap^*(-x)p(-x) + b^*p^*(-x)p^* + bp(-x)p + dp^*p$$

欲使 (12)(13) 等价, 需 $\Delta(-x)=\Delta$, 需 $a=d, b\in\mathbb{R}$. 此时有 (13) = (12)(-x), 方程约化为

$$ip_t + p_{xx} + 2p\big(app^* + bpp^*(-x) + b^*pp^*(-x) + ap(-x)p^*(-x)\big) = 0$$

$q=p^*(-x,-t)$

$$\begin{cases} r_1 = ap^* + bq(-x,-t) \\ r_2 = b^*p^* + dq(-x,-t) \end{cases}$$

若 (12), (13) 等价, 需 $\Delta^*(-x,-t)=\Delta$

$$\Delta = app^* + bpp(-x,-t) + b^*p^*p^*(-x,-t) + dp(-x,-t)p^*(-x,-t)$$
$$\Delta^*(-x,-t) = ap^*(-x,-t)p(-x,-t) + b^*p^*(-x,-t)p^* + bp(-x,-t)p + dp^*p$$

则只需 $a=d$, 此时有 $(13)=(12)^*(-x,-t)$. 方程约化为

$$ip_t + p_{xx} + 2p\big(app^* + bpp^*(-x,-t) + b^*pp^*(-x,-t) + ap(-x,-t)p^*(-x,-t)\big) = 0$$

1.2.2 $p_1=p^*(-x), q_1=q^*(-x)$

$$\begin{cases} r_1 = ap^*(-x) + bq^*(-x) \\ r_2 = cp^*(-x) + dq^*(-x) \end{cases} \tag{1.2.18}$$

要让 (1.2.13) 中 (12)(13) 等价, 只需 $\Delta^*(-x)=\Delta$,

$$\Delta = ap^*(-x)p + bq^*(-x)p + c^*p^*(-x)q + dq^*(-x)q$$
$$\Delta^*(-x) = a^*pp^*(-x) + b^*qp^*(-x) + c^*pq^*(-x) + d^*qq^*(-x)$$

只需 $a,d\in\mathbb{R}, b=c^*\in\mathbb{C}$, 此时有

$$(21) = a(12)^*(-x) + b(13)^*(-x)$$
$$(31) = b^*(12)(-x) + d(12)^*(-x)$$

方程约化为

$$
\begin{aligned}
ip_t + p_{xx} + 2p\big(app^*(-x) + bpq^*(-x) + b^*p^*(-x)q + dq^*(-x)q\big) = 0 \\
iq_t + q_{xx} + 2q\big(app^*(-x) + bpq^*(-x) + b^*p^*(-x)q + dq^*(-x)q\big) = 0
\end{aligned}
$$

则 q 可取 $p^*(-t), p(-x), p^*(-x,-t)$.

$q = p^*(-t)$

$$
\begin{cases}
r_1 = ap^*(-x) + bp(-x,t) \\
r_2 = b^*p^*(-x) + dp(-x,-t)
\end{cases}
$$

想要 (12) (13) 等价, 需 $\Delta^*(-t) = \Delta$,

$$
\begin{aligned}
\Delta &= app^*(-x) + bpp(-x,-t) + b^*p^*(-x)p^*(-t) + dp(-x,-t)p^*(-t) \\
\Delta^*(-t) &= ap^*(-t)p + bp^*(-t)p(-x) + bp(-x,-t)p + dp^*(-x)p
\end{aligned}
$$

故只需 $a = d$. 方程约化为

$$
ip_t + p_{xx} + 2p\big(app^*(-x) + bpp(-x,-t) + b^*p^*(-x)p^*(-t) + ap(-x,-t)p^*(-t)\big) = 0
$$

$q = p(-x)$

$$
\begin{cases}
r_1 = ap^*(-x) + bp^* \\
r_2 = b^*p^*(-x) + dp^*
\end{cases}
$$

想要 (12) (13) 等价, 需 $\Delta(-x) = \Delta$,

$$
\begin{aligned}
\Delta &= app^*(-x) + bpp^* + b^*p^*(-x)p{-}x + dp^*p(-x) \\
\Delta(-x) &= ap(-x)p^* + bp(-x)p^*(-x) + b^*pp + dp^*(-x)p
\end{aligned}
$$

故只需 $a = d, b \in \mathbb{R}$, 此时 $(13) = (12)(-x)$. 方程约化为

$$
ip_t + p_{xx} + 2p\big(app^*(-x) + bpp^* + bp^*(-x)p(-x) + ap^*p(-x)\big) = 0
$$

$q = p^*(-x,-t)$

$$
\begin{cases}
r_1 = ap^*(-x) + bp(-t) \\
r_2 = b^*p^*(-t) + dp(-t)
\end{cases}
$$

想要 (12) (13) 等价, 需 $\Delta(-x,-t) = \Delta$,

$$
\begin{aligned}
\Delta &= app^*(-x) + bpp(-t) + b^*p^*(-x)p^*(-x,-t) + dp(-t)p^*(-x,-t) \\
\Delta^*(-x,-t) &= a^*p^*(-x,-t)p(-t) + b^*p^*(-x,-t)p(-x) + bp(-t)p + dp^*(-x)p
\end{aligned}
$$

故 $\Delta(-x,-t) = \Delta$ 只需 $a = d$. 方程约化为

$$
ip_t + p_{xx} + 2p\big(app^*(-x) + bpp(-t) + b^*p^*(-x)p^*(-x,-t) + ap(-t)p^*(-x,-t)\big) = 0
$$

1.2.3 $p_1 = p(-t), q_1 = q(-t)$

$$\begin{cases} r_1 = ap(-t) + bq(-t) \\ r_2 = cp(-t) + dq(-t) \end{cases} \tag{1.2.19}$$

要约化 (12) (13), 需要 $\Delta(-t) = \Delta$, 其中

$$\begin{aligned} \Delta &= app(-t) + bpq(-t) + cp(-t)q + dqq(-t) \\ \Delta(-t) &= ap(-t)p + bp(-t)q + cpq(-t) + dq(-t)q \end{aligned} \tag{1.2.20}$$

则 $\Delta(-t) = \Delta$, 只需 $b = c$, 此时有

$$\begin{aligned} (21) &= a(12)(-t) + b(13)(-t) \\ (31) &= b(12)(-t) + d(13)(-t) \end{aligned}$$

方程约化为

$$\begin{aligned} ip_t + p_{xx} + 2p\big(app(-t) + bpq(-t) + bp(-t)q + dqq(-t)\big) = 0 \\ iq_t + q_{xx} + 2q\big(app(-t) + bpq(-t) + bp(-t)q + dqq(-t)\big) = 0 \end{aligned}$$

q 取值同上

$q = p^*(-t)$

$$\begin{cases} r_1 = ap(-t) + bp^* \\ r_2 = bp(-t) + dp^* \end{cases}$$

想要 (12) (13) 等价, 需 $\Delta^*(-t) = \Delta$,

$$\begin{aligned} \Delta &= app(-t) + bpp^* + bp(-t)p^*(-t) + dp^*p^*(-t) \\ \Delta^*(-t) &= a^*p^*(-t)p^* + b^*p^*(-t)p(-t) + b^*p^*p + d^*p(-t)p \end{aligned}$$

故 $\Delta^*(-t) = \Delta$ 只需 $a^* = d, b \in mathbbR$. 方程约化为

$$ip_t + p_{xx} + 2p\big(app(-t) + bpp^* + bp(-t)p^*(-t) + a^*p^*p^*(-t)\big) = 0$$

$q = p(-x)$

$$\begin{cases} r_1 = ap(-t) + bp(-x,-t) \\ r_2 = bp(-t) + dp(-x,-t) \end{cases}$$

想要 (12) (13) 等价, 需 $\Delta(-x) = \Delta$,

$$\begin{aligned} \Delta &= app(-t) + bpp(-x,-t) + bp(-t)p(-x) + dp(-x,-t)p(-x) \\ \Delta(-x) &= ap(-x)p(-x,-t) + bp(-x)p(-x,-t) + bp(-x,-t)p + dp(-t)p \end{aligned}$$

故 $\Delta^*(-t) = \Delta$ 只需 $a = d$. 方程约化为

$$ip_t + p_{xx} + 2p\big(app(-t) + bpp(-x,-t) + bp(-t)p(-x) + ap(-x)p(-x,-t)\big) = 0$$

$q=p^*(-x,-t)$

$$\begin{cases} r_1=ap(-t)+bp^*(-x) \\ r_2=bp(-t)+dp^*(-x) \end{cases}$$

想要 (12) (13) 等价, 需 $\Delta^*(-x,-t)=\Delta$,

$$\begin{aligned} \Delta &= app(-t)+b^*pp^*(-x)+b^*p(-t)p^*(-x,-t)+dp^*(-x)p^*(-x,-t) \\ \Delta^*(-x,-t) &= a^*p^*(-x,-t)p(-x)+bp^*(-x,-t)p(-t)+bp^*(-x)p+d^*p(-t)p \end{aligned}$$

故 $\Delta^*(-x,-t)=\Delta$ 只需 $a^*=d, b\in\mathbb{R}$. 方程约化为

$$ip_t+p_{xx}+2p\big(app(-t)+bpp^*(-x)+bp(-t)p^*(-x,-t)+a^*p^*(-x)p^*(-x,-t)\big)=0$$

1.2.4 $p_1=p(-x), q_1=q(-x,-t)$

有

$$\begin{cases} r_1=ap(-x,-t)+bq(-x,-t) \\ r_2=cp(-x,-t)+dq(-x,-t) \end{cases} \tag{1.2.21}$$

要约化 (12) (21), 需要 $\Delta(-x,-t)=\Delta$, 其中

$$\begin{aligned} \Delta &= app(-x,-t)+bpq(-x,-t)+cp(-x,-t)q+dq(-x,-t)q \\ \Delta(-x,-t) &= ap(-x,-t)p+bp(-x,-t)q+cpq(-x,-t)+dqq(-x,-t) \end{aligned} \tag{1.2.22}$$

则 $\Delta(-x,-t)=\Delta$, 只需 $b=c$, 此时有

$$\begin{aligned} (21) &= a(12)(-x,-t)+b(13)(-x,-t) \\ (31) &= b(12)(-x,-t)+d(13)(-x,-t) \end{aligned}$$

方程约化为

$$\begin{aligned} ip_t+p_{xx}+2p\big(app(-x,-t)+bpq(-x,-t)+bp(-x,-t)q+dqq(-x,-t)\big)=0 \\ iq_t+q_{xx}+2q\big(app(-x,-t)+bpq(-x,-t)+bp(-x,-t)q+dqq(-x,-t)\big)=0 \end{aligned}$$

$p_1=p(-x,-t), q=q^*(-t), q_1=q(-x,-t)=p^*(-x)$

$$\begin{cases} r_1=ap(-x,-t)+bp^*(-x) \\ r_2=bp(-x,-t)+dp^*(-x) \end{cases}$$

想要 (12) (13) 等价, 需 $\Delta(-t)=\Delta$,

$$\begin{aligned} \Delta &= app(-x,-t)+bpp^*(-x)+bp^*(-t)p(-x,-t)+dp^*(-x)p^*(-t) \\ \Delta^*(-t) &= a^*p^*p^*(-x)+b^*p^*(-t)p(-x,-t)+b^*pp^*(-x,-t)+d^*pp(-x.-t) \end{aligned}$$

故 $\Delta^*(-t)=\Delta$ 只需 $a^*=d, b\in\mathbb{R}$, 此时有 $(13)=(12)^*(-t)$. 方程约化为

$$ip_t+p_{xx}+2p\big(app(-x,-t)+bpp(-t)+bp(-x,-t)p(-x)+ap(-t)p(-x)\big)=0$$

$p_1 = p(-x,-t), q = p(-x), q_1 = q(-x,-t) = p(-t)$

$$\begin{cases} r_1 = ap(-x,-t) + bp(-t) \\ r_2 = bp(-x,-t) + dp(-t) \end{cases}$$

想要 (12) (13) 等价, 需 $\Delta(-x) = \Delta$,

$$\Delta = app(-x,-t) + bpp(-t) + bp(-x)p(-x,-t) + dp(-x)p(-t)$$
$$\Delta(-x) = ap(-x)p(-t) + bp(-x)p(-x,-t) + bpp(-t) + dpp(-x.-t)$$

故 $\Delta(-x) = \Delta$ 只需 $a = d$, 此时有 $(13) = (12)(-x)$. 方程约化为

$$ip_t + p_{xx} + 2p\big(app(-x,-t) + bpp(-t) + bp(-x,-t)p(-x) + ap(-t)p(-x)\big) = 0$$

$p_1 = p(-x,-t), q = p^*(-x,-t), q_1 = q(-x,-t) = p^*$

$$\begin{cases} r_1 = ap(-x,-t) + bp^* \\ r_2 = bp(-x,-t) + dp^* \end{cases}$$

想要 (12) (13) 等价, 需 $\Delta^*(-x,-t) = \Delta$,

$$\Delta = app(-x,-t) + bpp^* + bp^*(-x,-t)p(-x,-t) + dp^*p(-x,-t)$$
$$\Delta^*(-x,-t) = a^*p^*(-x,-t)p^* + b^*p^*(-x,-t)p(-x,-t) + b^*pp^* + d^*p(-x.-t)p$$

故 $\Delta^*(-x,-t) = \Delta$ 只需 $a = d^*b \in \mathbb{R}$, 此时有 $(13) = (12)^*(-x,-t)$. 方程约化为

$$ip_t + p_{xx} + 2p\big(app(-x,-t) + bpp^* + bp(-x,-t)p^*(-x,-t) + a^*p^*p^*(-x,-t)\big) = 0$$

由上节, 我们得到

$$a_{32} = -\frac{b^*}{a}a_{22} = -\frac{b^*}{a}a_{33}, \quad a_{23} = -\frac{b}{a}a_{22} = -\frac{b}{a}a_{33}, \quad a_{11} = aa_{22} + b^*a_{23}$$

若取 $a_{22} = a_{33} = m$, 则

$$A = \begin{pmatrix} (a - \frac{|b|^2}{a})m & 0 & \\ 0 & m & -\frac{b}{a}m \\ 0 & -\frac{b^*}{a}m & m \end{pmatrix}$$

欲求 $B\Phi_1^*, B\Phi_1^*(-t)$ 谁是伴随问题, 已知

$$\begin{cases} \Phi_1 & \text{对应于 } \lambda_1\text{, 其中 } \Phi_1 \text{ 为列向量} \\ \Phi_1^\dagger A & \text{对应于 } \lambda_1^*\text{, 其中 } \Phi_1^* A \text{ 为行向量} \end{cases}$$

由上可得 $B\Phi_1^*(-t)$ 为伴随问题的解 (下面第一个括号为原谱问题的解, 第二个为伴随谱问题的解)

$$\begin{cases} \Phi_1 & \lambda_1 \\ \Phi_1^\dagger A & \lambda_1^* \end{cases} \Longrightarrow \begin{cases} B\Phi_1^*(-t) & -\lambda_1^* \\ \Phi_1^T(-t)BA & \lambda_1 \end{cases}$$

易得, $\Phi_1^T(-t)BA$ 是伴随问题在 $\lambda = -\lambda_1$ 的解.

1.3 Darboux 变换

由 AKNS, 若 Φ_1 是原谱问题在 $\lambda=\lambda_1$ 的解, Ψ 是伴随问题在 $\lambda=\mu_1$ 的解, 则 DT 可写为

$$T_1 = I + \frac{\mu_1-\lambda_1}{\lambda-\mu_1}\frac{\Phi_1\Psi_1}{\Psi_1\Phi_1}, \quad T_1^{-1} = I - \frac{\mu_1-\lambda_1}{\lambda-\lambda_1}\frac{\Phi_1\Psi_1}{\Psi_1\Phi_1}$$

思路:

1. Φ_1 是原谱问题的特解, T_1 是其 DT, 则 T_1^{-1} 是伴随问题的 DT
2. $\hat{\Psi}_1=\Psi_1 T_1^{-1}$ 的伴随问题的特解, T_2 是其 DT, 则 T_2^{-1} 是原谱问题的 DT.

即在 Twofold DT 中, 记 $T=T_2^{-1}T_1$, s.t. $\Phi_1 \xrightarrow{T_1} \widetilde{\Phi}_1 \xrightarrow{T_2^{-1}} \widehat{\Phi}_1$

证明. 先考虑原谱问题的 DT, 设 $T_1 = \lambda T_1 + T_0$, 其中 $T_{11}=(g_{ij})_{3\times3}, T_{10}=(h_{ij})_{3\times3}$. 由 $P\hat{h}i_{1x} = \hat{U}_1 P\hat{h}i_1, P\hat{h}i_1 = T\Phi$ 可得

$$T_{1x}\Phi_1 + T_1U_1\Phi_1 = \hat{U}_1T_1\Phi_1, \implies T_{1x}+T_1U = \hat{U}_1T \tag{1.3.23}$$

□

代入 T_1, U 有

$$\lambda T_{11x} + T_{10x} + \mathrm{i}\lambda^2 T_{11}\sigma_3 + \mathrm{i}\lambda T_{10}\sigma_3 + \lambda T_{11}P + T_{10}P = \mathrm{i}\lambda^2\sigma_3 T_{11} + \mathrm{i}\lambda\sigma_3 T_{10} + \lambda\hat{P}T_{11} + \hat{P}T_{10} \tag{1.3.24}$$

比较 λ 的同次幂可得

$$\begin{aligned}
\lambda^2 :& \mathrm{i}T_{11}\sigma_3 = \mathrm{i}\sigma_3 T_{11} \\
\lambda^1 :& T_{11x} + \mathrm{i}T_{10}\sigma_3 + T_{11}P = \mathrm{i}\sigma_3T_{10} + \hat{P}T_{11} \\
\lambda^0 :& T_{10x} + T_{10}P = \hat{P}T_{10}
\end{aligned} \tag{1.3.25}$$

故有

$$\begin{aligned}
\lambda^2 :& g_{12}=g_{21}=g_{13}=g_{31}=0 \\
\lambda^1 :& g_{11x}=g_{22x}=g_{23x}=g_{33x}=0
\end{aligned} \implies \begin{aligned}
-\mathrm{i}h_{12}+g_{11}p &= \mathrm{i}h_{12}+g_{22}\hat{p}+g_{33}\hat{q} \\
-\mathrm{i}h_{13}+g_{11}q &= \mathrm{i}h_{13}+g_{23}\hat{p}+g_{33}\hat{q} \\
-\mathrm{i}h_{21}+g_{11}r_1-g_{23}r_2 &= \mathrm{i}h_{21}+g_{11}\hat{r}_1 \\
-\mathrm{i}h_{31}+g_{32}r_1-g_{33}r_2 &= \mathrm{i}h_{31}+g_{11}\hat{r}_2
\end{aligned} \tag{1.3.26}$$

由 λ^0 可得:

$$\begin{aligned}
h_{11x}-r_1h_{12}-r_2h_{13} &= h_{21}\hat{p}+h_{31}\hat{q} \\
h_{12x}-h_{12}p &= h_{22}\hat{p}+h_{32}\hat{q} \\
h_{13x}-h_{11}q &= h_{23}\hat{p}+h_{33}\hat{q} \\
h_{21x}-r_1h_{22}-r_2h_{23} &= -\hat{r}_1h_{11} \\
&\left.\begin{aligned}
h_{22x}+h_{21}p &= -\hat{r}_1h_{12} \\
h_{23x}+h_{21}q &= -\hat{r}_1h_{13} \\
h_{31x}+r_1h_{32}-r_2h_{33} &= -\hat{r}_2h_{11} \\
h_{32x}+h_{31}p &= -p_2h_{12} \\
h_{33x}+h_{31}q &= -\hat{r}_2h_{13}
\end{aligned}\right\}(*)
\end{aligned} \tag{1.3.27}$$

故可取 $g_{11}=1, g_{22}=g_{23}=g_{32}=g_{33}=0$, 则有

$$h_{12}=\frac{p}{2\mathrm{i}}, h_{21}=\frac{\hat{r}_1}{-2\mathrm{i}}, h_{13}=\frac{q}{2\mathrm{i}}, h_{31}=\frac{\hat{r}_2}{-2\mathrm{i}} \tag{1.3.28}$$

带入 (*) 可得 $h_{22x}=h_{23x}=h_{32x}=h_{33x}=0$, 则

$$T_1=\begin{pmatrix}\lambda+h_{11} & \frac{p}{2\mathrm{i}} & \frac{q}{2\mathrm{i}}\\ h_{21} & 1 & 0\\ h_{31} & 0 & 1\end{pmatrix} \tag{1.3.29}$$

由 $T_1\Phi_1|_{lambda=\lambda_1}=0$, 不妨设 $\Phi_1=(\phi_1,\phi_2,\phi_3)^T$, 解得

$$h_{21}=-\frac{\phi_2}{\phi_1}, h_{31}=-\frac{\phi_3}{\phi_1}, h_{11}=-\lambda_1-\frac{p\phi_1+q\phi_3}{2\mathrm{i}\phi_1}, \tag{1.3.30}$$

故

$$T_1=\begin{pmatrix}\lambda-\lambda_1-\frac{p\phi_1+q\phi_3}{2\mathrm{i}\phi_1} & \frac{p}{2\mathrm{i}} & \frac{q}{2\mathrm{i}}\\ -\frac{\phi_2}{\phi_1} & 1 & 0\\ -\frac{\phi_3}{\phi_1} & 0 & 1\end{pmatrix},\quad T_1^{-1}=\frac{1}{\lambda-\lambda_1}\begin{pmatrix}1 & 0 & -\frac{q}{2\mathrm{i}}\\ \frac{\phi_2}{\phi_1} & \lambda-\lambda_1-\frac{p\phi_2}{2\mathrm{i}\phi_1} & -\frac{q\phi_2}{2\mathrm{i}}\\ \frac{\phi_3}{\phi_1} & -\frac{p\phi_3}{2\mathrm{i}\phi_1} & \lambda-\lambda_1-\frac{p\phi_3}{2\mathrm{i}\phi_1}\end{pmatrix}$$

同理可得

$$T_1=\begin{pmatrix}\lambda-\mu_1+\frac{\mu_1-\lambda_1}{\Delta}(x_2\phi_2+x_3\phi_3) & \frac{p}{2\mathrm{i}}-\frac{\mu_1-\lambda_1}{\Delta}\phi_1x_2 & \frac{q}{2\mathrm{i}}-\frac{mu_1-\lambda_1}{\Delta}\phi_1x_3\\ -\frac{\phi_2}{\phi_1} & 1 & 0\\ -\frac{\phi_3}{\phi_1} & 0 & 1\end{pmatrix}$$

$$T_1^{-1}=\frac{1}{\lambda-\lambda_1}\begin{pmatrix}1 & -\frac{p}{2\mathrm{i}}-\frac{\mu_1-\lambda_1}{\Delta}\phi_1x_2 & -\frac{q}{2\mathrm{i}}-\frac{\mu_1-\lambda_1}{\Delta}\phi_1x_2\\ \frac{\phi_2}{\phi_1} & 1 & 0\\ \frac{\phi_3}{\phi_1} & 0 & 1\end{pmatrix}$$

其中 $\Delta=\phi_1x_1+\phi_2x_2+\phi_3x_3$. 故

$$\begin{aligned}T=T_2^{-1}T_1&=\frac{1}{\lambda-\mu_1}\begin{pmatrix}\lambda-\mu_1+\frac{\mu_1-\lambda_1}{\Delta}\phi_1x_1 & \frac{\mu_1-\lambda_1}{\Delta}\phi_1x_2 & \frac{\mu_1-\lambda_1}{\Delta}\phi_1x_3\\ \frac{\mu_1-\lambda_1}{\Delta}\phi_2x_1 & \lambda-\mu_1-\frac{\mu_1-\lambda_1}{\Delta}\phi_2x_2 & \frac{\mu_1-\lambda_1}{\Delta}\phi_2x_3\\ \frac{\mu_1-\lambda_1}{\Delta}\phi_3x_1 & \frac{\mu_1-\lambda_1}{\Delta}\phi_3x_2 & \lambda-\mu_1+\frac{\mu_1-\lambda_1}{\Delta}\phi_3x_3\end{pmatrix}\\ &=I+\frac{\mu_1-\lambda_1}{\lambda-\mu_1}\frac{\Phi_1\Psi_1}{\Delta}=I+\frac{\mu_1-\lambda_1}{\lambda-\mu_1}\frac{\Phi_1\Psi_1}{\Psi_1\Phi_1}\end{aligned}$$

第二章　1+2 维可积系统

2.1　KP 方程及其 Darboux 变换

定义 2.1. KP 方程:

$$u_{xt} = (u_{xxx} + 6uu_x)_x + 3\alpha^2 u_{yy} \tag{2.1.1}$$

其中 $\alpha = \pm 1$, 为 KP I 方程, $\alpha = \pm \mathrm{i}$, 为 KP II 方程

这里为令 $v_x = u$, 将 (2.1.1) 对 x 积分一次, 并去 $c = 0$, 得到另一种形式

$$v_{xt} = v_{xxx} + 6v_x v_{xx} + 3\alpha v_{yy} \tag{2.1.2}$$

KP 方程的 LaX 对为

$$\begin{cases} \phi_y = \alpha^{-1}\phi_{xx} + \alpha^{-1}u\phi \\ \phi_t = 4\phi_{xx} + 6u\phi_x + 3(\alpha v_y + u_x)\phi \end{cases} \tag{2.1.3}$$

则 (2.1.1) 是 (2.1.3) 的可积条件, 即 Lax 对相容性 ($\phi_{yt} = \phi_{ty}$) 可推出方程. 不妨设

$$\begin{cases} \phi_x = U\phi \\ \phi_t = V\phi \end{cases} \implies V_t - V_x + [U, V] = 0 \implies U = \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}$$

利用 (2.1.3) 带入, 则只剩下 ϕ 和 ϕ_x.

定义 2.2. KP 方程的 Darboux 变换

$$\begin{cases} \phi' = \phi_x - \frac{h_x}{h}\phi \\ u' = u + 2(\frac{h_x}{h})_x \end{cases}$$

其中 h 为 Lax 的一个解

则

$$(U, \phi) \to (U', \phi') \to \dots \left\{ \phi'_y = \text{第一次, 未完待续} \right.$$

例 2.1. 对 $\alpha = 1, u_{xt} = (u_{xxx} + 6uu_x)_x + 3u_{yy}$, 可取

$$h = e^{\lambda x + \lambda^2 y + 4\lambda^3 t} + 1 \tag{2.1.4}$$

其中