IT5506 — Mathematics for Computing II

1.

(a) Given that square matrix S is denoted as $S = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}$

(i) Show that S^{-1} is exist.

(03 Marks)

$$\det(S) = 1 \cdot ((1)(2) - (1)(1)) - 1 \cdot ((2)(2) - (1)(1)) + 1 \cdot ((2)(1) - (1)(1))$$

Simplify the terms inside the parentheses:

$$\det(S) = 1 \cdot (2 - 1) - 1 \cdot (4 - 1) + 1 \cdot (2 - 1)$$

$$\det(S) = 1 \cdot 1 - 1 \cdot 3 + 1 \cdot 1$$

$$\det(S) = 1 - 3 + 1 = -1$$

Det(S) is non-zero.

(ii) Find the inverse of the matrix S by using the Gauss-Jordan method.

(07 Marks)

Start with the augmented matrix [S|I]

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 & 0 & 1 \end{array}\right]$$

…… ….. After row operations ends with

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & 3 & -1 & -1 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

$$S^{-1} = \begin{bmatrix} -1 & 1 & 0 \\ 3 & -1 & -1 \\ -1 & 0 & 1 \end{bmatrix}$$

(b) Matrix multiplication is not commutative in general, but there are instances where it satisfies the commutative property

(i) Show that for two matrices A and B, it is not always true that AB=BA

(03 Marks)

Let

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

1. Compute AB:

$$AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}$$

2. Compute BA:

$$BA = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$$

Clearly, $AB \neq BA$, demonstrating that matrix multiplication is not commutative.

Or any other counter-example

(ii) If $A = \begin{bmatrix} -4 & 4 & 4 \\ -7 & 1 & 3 \\ 5 & -3 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & 3 \end{bmatrix}$ then show that $AB = BA = 8I_3$

(04 Marks)

(iii) If $AB = BA = 8I_3$ then show that B^{-1} =(1/8)A

(02 Marks)

1. Compute AB.
2. Compute BA.

- $(AB)_{11} = (-4)(1) + (4)(1) + (4)(2) = -4 + 4 + 8 = 8$
- $(AB)_{12} = (-4)(-1) + (4)(-2) + (4)(1) = 4 - 8 + 4 = 0$
- $(AB)_{13} = (-4)(1) + (4)(-2) + (4)(3) = -4 - 8 + 12 = 0$

Second row of AB:

- $(AB)_{21} = (-7)(1) + (1)(1) + (3)(2) = -7 + 1 + 6 = 0$
- $(AB)_{22} = (-7)(-1) + (1)(-2) + (3)(1) = 7 - 2 + 3 = 8$
- $(AB)_{23} = (-7)(1) + (1)(-2) + (3)(3) = -7 - 2 + 9 = 0$

Third row of AB:

- $(AB)_{31} = (5)(1) + (-3)(1) + (-1)(2) = 5 - 3 - 2 = 0$
- $(AB)_{32} = (5)(-1) + (-3)(-2) + (-1)(1) = -5 + 6 - 1 = 0$
- $(AB)_{33} = (5)(1) + (-3)(-2) + (-1)(3) = 5 + 6 - 3 = 8$

So, the matrix AB is:

$$AB = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix}$$

Similarly calculate BA

3. Verify if both results yield $8I_3$, where I_3 is the 3×3 identity matrix and $8I_3 = 8 \times I_3$.

(iv) Represent the following system of linear equations in matrix form $x - y + z = 4, x - 2y - 2z = 9, 2x + y + 3z = 1$. Using the results from part (a)(ii) and (a)(iii), solve the system of equations.

(06 Marks)

$$\begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 9 \\ 1 \end{bmatrix}$$

Invers of $\begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & 3 \end{bmatrix}$ is (1/8) $\begin{bmatrix} -4 & 4 & 4 \\ -7 & 1 & 3 \\ 5 & -3 & -1 \end{bmatrix}$ refer (a)(iii) and (a)(ii)

Hence

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} =$$

$$\frac{1}{8}\begin{bmatrix} -4 & 4 & 4 \\ -7 & 1 & 3 \\ 5 & -3 & -1 \end{bmatrix} \begin{bmatrix} 4 \\ 9 \\ 1 \end{bmatrix} = \frac{1}{8}\begin{bmatrix} -16+36+4 \\ -28+9+3 \\ 20-27-1 \end{bmatrix} = \frac{1}{8}\begin{bmatrix} 24 \\ -16 \\ -8 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ -1 \end{bmatrix}$$

The solution is (x = 3, y = - 2, $z = -1$).

2.

(a). Let V be a vector space over the field F, and let $S \subseteq V$ be a subset of V

(i) For S to be a subspace over V, what conditions must be satisfied?

(Marks 06)

For a subset S of a vector space V over a field F to be a **subspace** of V, it must satisfy three essential conditions:

1. **Contains the Zero Vector**: The zero vector $\mathbf{0} \in V$ must be in S. In other words, $\mathbf{0} \in S$.
2. **Closed under Addition:** If $\mathbf{u}, \mathbf{v} \in S$, then the sum $\mathbf{u} + \mathbf{v}$ must also be in S. This means:

$$\mathbf{u}, \mathbf{v} \in S \quad \Rightarrow \quad \mathbf{u} + \mathbf{v} \in S.$$

3. **Closed under Scalar Multiplication**: If $\mathbf{v} \in S$ and $c \in F$ (a scalar), then the scalar multiple $c\mathbf{v}$ must also be in S. In other words:

$$\mathbf{v} \in S \quad \text{and} \quad c \in F \quad \Rightarrow \quad c\mathbf{v} \in S.$$

(ii) Any line through the origin is given by $S = \{ \begin{bmatrix} x \\ y \end{bmatrix} : ax + by = 0 \, , a, b \in \mathbb{R}^2 \}$. Is S a subspace of $\mathbb{R}^2$? Justify your answer.

(Marks 06)

1. **Contains the Zero Vector**:

The zero vector in $\mathbb{R}^2$ is $\mathbf{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Substituting $x = 0$ and $y = 0$ into the equation $ax + by = 0$ gives:

$$a(0) + b(0) = 0,$$

which is true. Therefore, the zero vector is in S.

2. **Closed under Addition:**

Let $\mathbf{u} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$ be two vectors in S, so that $ax_1 + by_1 = 0$ and $ax_2 + by_2 = 0$. We need to check if their sum $\mathbf{u} + \mathbf{v} = \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \end{pmatrix}$ is also in S, i.e., if it satisfies the equation $ax + by = 0$:

$$a(x_1 + x_2) + b(y_1 + y_2) = (ax_1 + by_1) + (ax_2 + by_2) = 0 + 0 = 0.$$

Therefore, $\mathbf{u} + \mathbf{v} \in S$, so S is closed under addition.

3. **Closed under Scalar Multiplication**:

Let $\mathbf{v} = \begin{pmatrix} x \\ y \end{pmatrix} \in S$, so that $ax + by = 0$. For any scalar $c \in \mathbb{R}$, we need to check if $c\mathbf{v} = \begin{pmatrix} cx \\ cy \end{pmatrix}$ satisfies the equation $ax + by = 0$. Substituting:

$$a(cx) + b(cy) = c(ax + by) = c(0) = 0.$$

Therefore, $c\mathbf{v} \in S$, so S is closed under scalar multiplication.

S satisfies all three conditions — it contains the zero vector, is closed under addition, and is closed under scalar multiplication S **is a subspace**

(b). Determine whether the following function T is a *linear transformation*.

$$T: \mathrm{R}^3 \rightarrow \mathrm{R}^3, \text{ with } T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x^3 \\ y^2 \\ \sin z \end{bmatrix}$$

(Marks 06)

Let $\mathbf{u} = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}$. We need to check if:

$$T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$$

First, calculate $T(\mathbf{u} + \mathbf{v})$:

$$\mathbf{u} + \mathbf{v} = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{pmatrix}$$

Now apply T:

$$T(\mathbf{u} + \mathbf{v}) = \begin{pmatrix} (x_1 + x_2)^3 \\ (y_1 + y_2)^2 \\ \sin(z_1 + z_2) \end{pmatrix}$$

Now calculate $T(\mathbf{u}) + T(\mathbf{v})$:

$$T(\mathbf{u}) = \begin{pmatrix} x_1^3 \\ y_1^2 \\ \sin(z_1) \end{pmatrix}, \quad T(\mathbf{v}) = \begin{pmatrix} x_2^3 \\ y_2^2 \\ \sin(z_2) \end{pmatrix}$$

$$T(\mathbf{u}) + T(\mathbf{v}) = \begin{pmatrix} x_1^3 \\ y_1^2 \\ \sin(z_1) \end{pmatrix} + \begin{pmatrix} x_2^3 \\ y_2^2 \\ \sin(z_2) \end{pmatrix} = \begin{pmatrix} x_1^3 + x_2^3 \\ y_1^2 + y_2^2 \\ \sin(z_1) + \sin(z_2) \end{pmatrix}$$

Clearly, $T(\mathbf{u} + \mathbf{v}) \neq T(\mathbf{u}) + T(\mathbf{v})$ in general because:

$$(x_1 + x_2)^3 \neq x_1^3 + x_2^3 \quad \text{and} \quad (y_1 + y_2)^2 \neq y_1^2 + y_2^2 \quad \text{and} \quad \sin(z_1 + z_2) \neq \sin(z_1) + \sin(z_2)$$

Thus, **additivity does not hold**.

Hence T is not a *linear transformation*.

(c). Consider given tow vector s $\vec{A} = [2 \quad 3 \quad 4]$ and $\vec{B} = [1 \quad 0 \quad -1]$ in R^3

(i) Calculate the magnitudes (lengths) of $\vec{A}$ and $\vec{B}$

(02 Marks)

The magnitude of a vector $\vec{V} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ is given by:

$$|\vec{V}| = \sqrt{v_1^2 + v_2^2 + v_3^2}$$

For $\vec{A} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$:

$$|\vec{A}| = \sqrt{2^2 + 3^2 + 4^2} = \sqrt{4 + 9 + 16} = \sqrt{29}$$

For $\vec{B} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$:

$$|\vec{B}| = \sqrt{1^2 + 0^2 + (-1)^2} = \sqrt{1 + 0 + 1} = \sqrt{2}$$

(ii) Find the unit vectors along $\vec{A}$ and $\vec{B}$

(02 Marks)

Unit vector along $\vec{A}$:

$$\hat{A} = \frac{1}{|\vec{A}|} \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$$

Unit vector along $\vec{B}$:

$$\hat{B} = \frac{1}{|\vec{B}|} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

(iii) Compute the angle between vectors $\vec{A}$ and $\vec{B}$

(03 Marks)

The angle θ between two vectors $\vec{A}$ and $\vec{B}$ can be found using the dot product formula:

$$\vec{A} \cdot \vec{B} = |\vec{A}||\vec{B}| \cos\theta$$

Rearranging for $\cos\theta$:

$$\cos\theta = \frac{\vec{A} \cdot \vec{B}}{|\vec{A}||\vec{B}|}$$

Step 1: Find the dot product $\vec{A} \cdot \vec{B}$

$$\vec{A} \cdot \vec{B} = (2)(1) + (3)(0) + (4)(-1) = 2 + 0 - 4 = -2$$

$$|\vec{A}| = \sqrt{2^2 + 3^2 + 4^2} = \sqrt{4 + 9 + 16} = \sqrt{29} \approx$$

$$|\vec{B}| = \sqrt{1^2 + 0^2 + (-1)^2} = \sqrt{1 + 0 + 1} = \sqrt{2}$$

Get cos invers to find θ