Asher Meyers
Math Bridge Week 2

Calculus based Probability

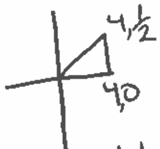
1a. $\int_0^4 cx = 1$

$\left.\frac{cx^2}{2}\right|_0^4 = 1$

$8c = 1$

$c = \frac{1}{8}$

$f(x) = \frac{x}{8}$




$4 \cdot \frac{1}{2} \cdot \frac{1}{2} = 1$ Area under line

$c = \frac{1}{8}$

b. pdf $= \frac{x}{8}$

cdf $= \frac{x^2}{16}$

$P(-1 \leq X \leq 1) =$

$= P(X \leq 1)$

$= \frac{1}{16}$

c. $\left.\frac{x^2}{16}\right|_2^4 = P(X>2)$

$= 1 - \frac{1}{4}$

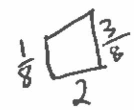
$= \frac{3}{4}$

1d. $P(X<3 \mid X>1)$

$P(X>1) = \left.\frac{x^2}{16}\right|_1^4 = 1 - \frac{1}{16} = \frac{15}{16}$

$P(1<X<3)$

$= \left.\frac{x^2}{16}\right|_1^3 = \frac{9}{16} - \frac{1}{16} = \frac{8}{16} = \frac{1}{2}$




Area $= \frac{1}{2}$ $\frac{\frac{9}{16}}{\frac{15}{16}} = \boxed{\frac{3}{5}}$

1e. $\cancel{0.5 = \frac{x^2}{16}}$

$\cancel{x^2 = 8}$

$\cancel{x = 2\sqrt{2}}$

$E(X) = \int_0^4 \frac{x^2}{8} dx = \left.\frac{x^3}{24}\right|_0^4 = \frac{4^3}{24} = \frac{16}{6} = \frac{8}{3}$

f. $Var(X) = E[X^2] - (E[X])^2$

$E(X^2) = \int_0^4 \frac{x^3}{8} dx = \left.\frac{x^4}{32}\right|_0^4 = \frac{16}{2} = 8$

$Var(X) = 8 - \left(\frac{8}{3}\right)^2 = 8 - \frac{64}{9} = \frac{8}{9}$

$\sigma_x = \sqrt{\frac{8}{9}} = \frac{2\sqrt{2}}{3}$

g. CDF $= \int_0^4 \frac{x}{8} = \frac{x^2}{16}$

2. $PDF = \frac{3}{2}x^2 \quad (-1 < x < 1)$

$CDF = \int_{-1}^{1} \frac{3}{2}x^2\,dx =$

$= \frac{3x^3}{6}\Big|_{-1}^{1}$

$= \frac{3}{6} + \frac{3}{6} = 1$

$CDF = \frac{x^3}{2}$

a. $P(-2 < X < 0)$

$= P(-1 < x < 0)$

$= \frac{x^3}{2}\Big|_{-1}^{0}$

$= \frac{1}{2}$

b. $\frac{x^3}{2}\Big|_{-0.5}^{1} = \frac{1}{2} + \frac{1}{16} = \frac{9}{16}$

c. $P(X > -0.5)$

$= \frac{9}{16}$

$P(X > 0.5)$

$= \frac{x^3}{2}\Big|_{0.5}^{1} = \frac{1}{2} - \frac{1}{16} = \frac{7}{16}$

$P(X > 0.5 \mid X > -0.5)$

$= \frac{\frac{7}{16}}{\frac{9}{16}} = \frac{7}{9}$

2 d $E(x) = \int x f(x)$

$= \int_{-1}^{1} \frac{3}{2} x^3\,dx$

$= \frac{3x^4}{2 \cdot 4}\Big|_{-1}^{1}$

$= \frac{3x^4}{8}\Big|_{-1}^{1}$

$= \frac{3}{8} - \frac{3}{8} = 0$

e. $Var(x) = E(x^2) - E(x)^2$

$E(x^2) = \int x^2 f(x)\,dx$

$= \int_{-1}^{1} \frac{3x^4}{2}$

$= \frac{3}{2 \cdot 5} x^5 \Big|_{-1}^{1}$

$= \frac{3}{10} + \frac{3}{10} = \frac{3}{5}$

$Var(x) = \frac{3}{5} - 0^2 = \frac{3}{5}$

$\sigma_x = \frac{\sqrt{3}}{\sqrt{5}}$

f. $CDF = \int_{-1}^{1} \frac{3}{2}x^2\,dx$

$= \frac{x^3}{2}$

3.

a. $CDF = \begin{pmatrix} \frac{x^2}{2} & 0<x<1 \\ 2x-\frac{x^2}{2} & 1<x<2 \end{pmatrix}$

$P(1<x<3)$

$= P(1<x<2)$

$= 2x - \frac{x^2}{2}\Big|_1^2$

$= (4-2) - (2-\frac{1}{2})$

$2 - 1.5$

$= 0.5$

b. $P(x>0.5)$

$= \cancel{P(0.5>x>}$

$P(0.5<x<2)$

$= P(0.5<x<1)$

$+ P(1<x<2)$

$P(0.5<x<1)$

$= \cancel{2x}\ \frac{x^2}{2}\Big|_{0.5}^{1} = \frac{1}{2} - \frac{1}{8}$

$\cancel{= (2 - \frac{1}{2}) - (1 - \frac{1}{8})}$

$\cancel{= 1.5 - \frac{7}{8}} = \frac{3}{8}$

$\cancel{= \frac{5}{8}}$

$P(0.5<x<2) =$

$= \cancel{\frac{5}{8}} \times \frac{4}{8} + \frac{3}{8} = \frac{7}{8}$

c. $P(x>0.5) = \frac{7}{8}$

$P(x>1) = \frac{1}{2} = \frac{4}{8}$

$P(x>1 \mid x>0.5) = \frac{4}{7} = \frac{\frac{4}{8}}{\frac{7}{8}}$

d. $E(x) = \int_0^1 x^2 + \int_1^2 2x - x^2$

$= \frac{1}{3} + \left(x^2 - \frac{x^3}{3}\right)\Big|_1^2$

$= \frac{1}{3} + (4 - \frac{8}{3}) - (1 - \frac{1}{3})$

$= \frac{1}{3} + (\frac{4}{3}) - (\frac{2}{3})$

$= 1$

e. $E(x^2) = \int_0^1 x^3 + \int_1^2 2x^2 - x^3$

$= \frac{1}{4} + \left(\frac{2x^3}{3} - \frac{x^4}{4}\right)\Big|_1^2$

$= \frac{1}{4} + (\frac{16}{3} - 4) - (\frac{2}{3} - \frac{1}{4})$

$= \frac{1}{4} + (\frac{4}{3}) - (\frac{5}{12})$

$= \frac{3}{12} + \frac{16}{12} - \frac{5}{12} = \frac{14}{12}$

$Var(x) = \frac{14}{12} - 1 = \frac{1}{6}$

$\sigma_x = \frac{\sqrt{6}}{6} = \frac{1}{\sqrt{6}}$

f. $CDF = \frac{x^2}{2} \quad 0<x<1$

$2x - \frac{x^2}{2} \quad 1<x<2$

$0 \quad$ otherwise

4. $f(x) = \lambda e^{-\lambda x}$ $\quad \lambda = g$

$E(x) = \int g x e^{-gx}\, dx = 4$

$u = x \quad du = 1$

$v = -e^{-gx} \quad dv = ge^{-gx}$

$\int xge^{-gx} = -xe^{-gx} + \int e^{-gx}$

$= -xe^{-gx} + \frac{e^{-gx}}{-g}$

$4 = -e^{-gx}\left(x + \frac{1}{g}\right)\Big|_0^{\infty}$

$\lim_{x\to\infty} \frac{x+\frac{1}{g}}{e^{gx}} = 0$

$4 = e^0\left(0 + \frac{1}{g}\right)$

$g = 0.25$

a. PDF: $f(x) = 0.25e^{-0.25x}$

b. $\int_1^3 \text{PDF} = \text{CDF}\Big|_1^3$

$\text{CDF} = -e^{-0.25x}$

$\text{CDF}\Big|_1^3 = \frac{-1}{e^{0.75}} + \frac{1}{e^{0.25}}$

$= \frac{\sqrt{e} - 1}{e^{0.75}}$

c. $\text{CDF}\Big|_3^{\infty}$

$= \frac{-1}{e^{0.25\infty}} + \frac{1}{e^{0.75}}$

$= \frac{1}{e^{0.75}}$

d. $P(x>6) = \frac{1}{e^{1.5}}$

$P(x>3) = \frac{1}{e^{0.75}}$

$P(x>6 \mid x>3) =$

$= \frac{\frac{1}{e^{1.5}}}{\frac{1}{e^{0.75}}} = \frac{e^{0.75}}{e^{1.5}} = \frac{1}{e^{0.75}}$

e. $E(x)^2 = 16$

$E(x^2) = \int g x^2 e^{-gx}$

$\text{Var}(x) = \frac{1}{g^2} = \frac{1}{\left(\frac{1}{16}\right)} = 16$

$\sigma_x = 4$

f. $\text{CDF} = F(x) = \int_a^b g e^{-gx}$

$= \frac{-1}{e^{0.25x}}\Big|_a^b$ on the interval $x > 0$