# COM2109

# Automata

Robert Hierons

# Composing Regular Languages

# Motivation

How do we build large/complex finite automata?

How do we build large/complex finite automata that are correct?

Classic solution:

Compose (combine) simpler finite automata.

For regular languages $L_1$ and $L_2$ this will allow us to prove that:

$$\left.\begin{array}{rl} \text{Union:} & L_1 \cup L_2 \\ \text{Concatenation:} & L_1 L_2 \\ \text{Star:} & L_1{}^* \\ \text{Reversal:} & L_1{}^R \\ \text{Complement:} & \overline{L_1} \\ \text{Intersection:} & L_1 \cap L_2 \end{array}\right\} \text{Are regular Languages}$$

# Show: Regular languages are closed under

Union: $L_1 \cup L_2$

Concatenation: $L_1 L_2$

Star: $L_1{}^*$

Reversal: $L_1{}^R$

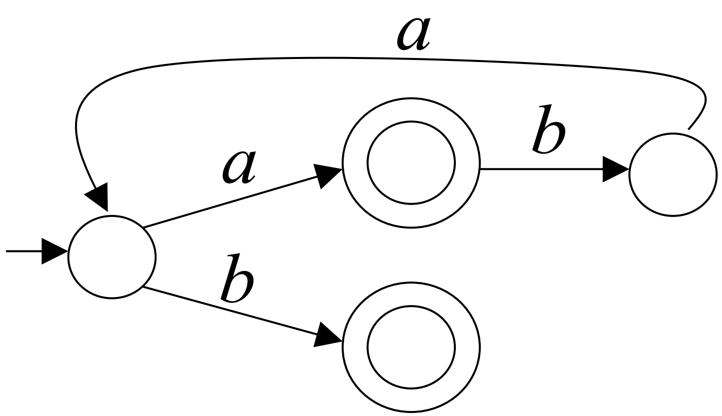
Complement: $\overline{L_1}$

Intersection: $L_1 \cap L_2$

Non-determinism makes proofs easier

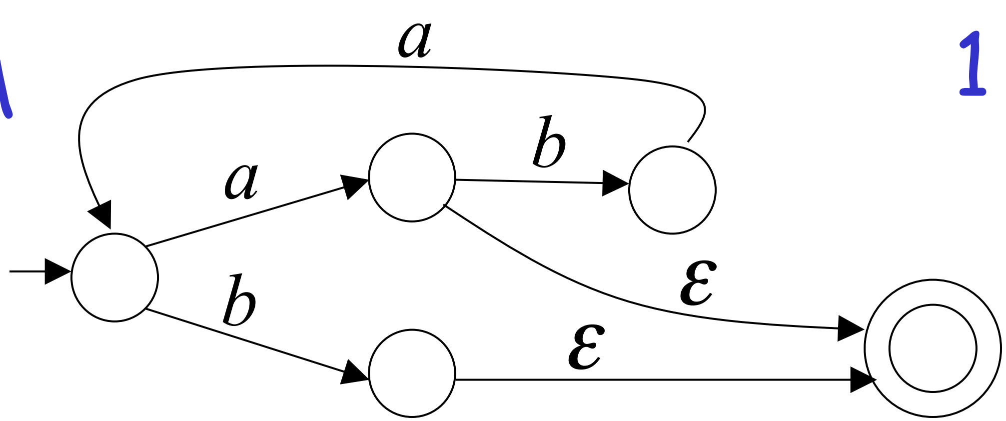
# A useful transformation: use one accept state

NFA




2 accept states

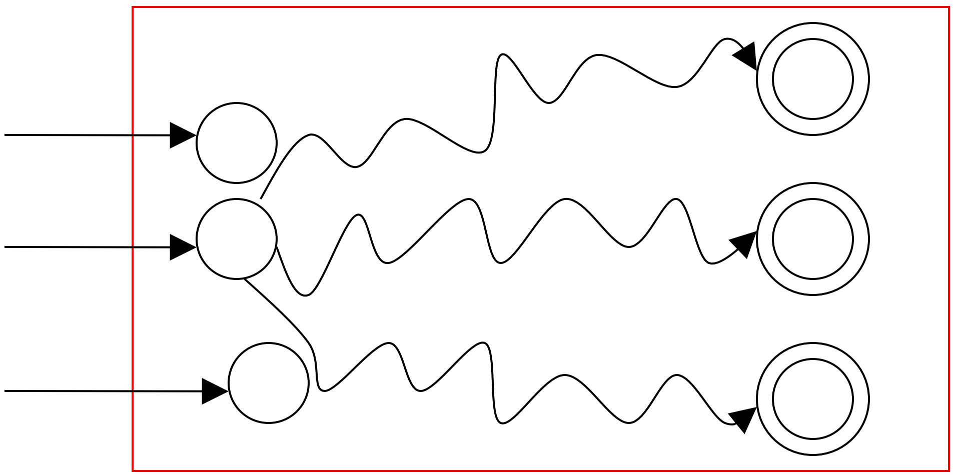
Equivalent
NFA




1 accept state

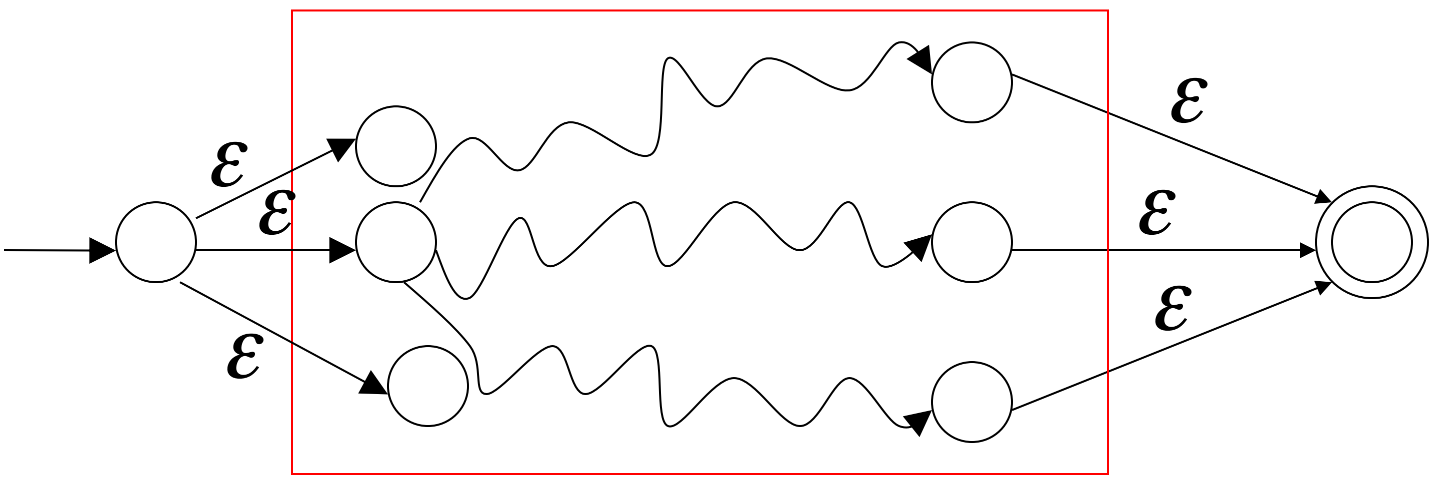
# In General

NFA



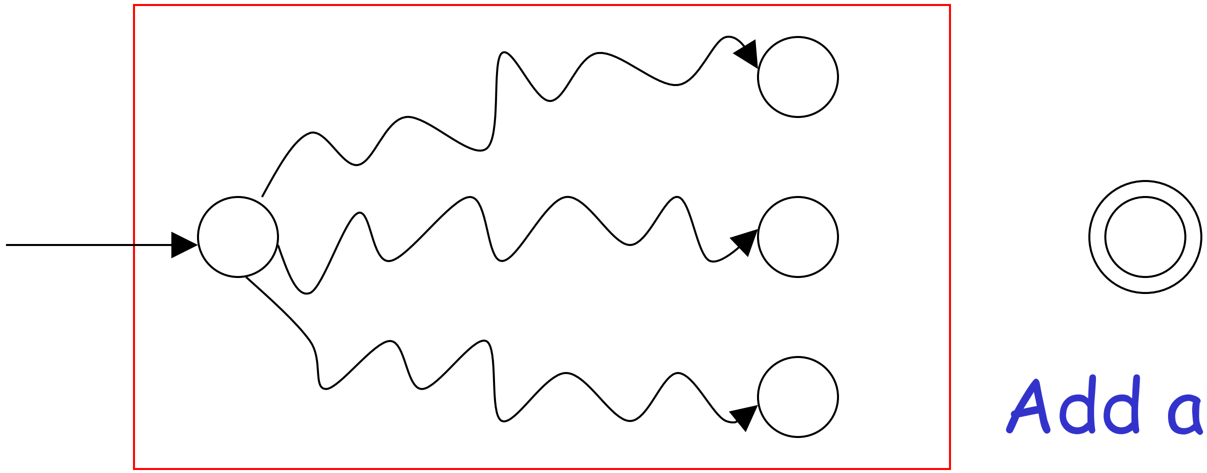
Single Initial and accepting states

Equivalent NFA



# Special/extreme case

## NFA without accepting state



Add an accepting state without transitions

# So

We can assume that all finite automata considered:

- Have a single initial state
- Have a single final state

We only need to know how to compose such finite automata.

# Take two languages

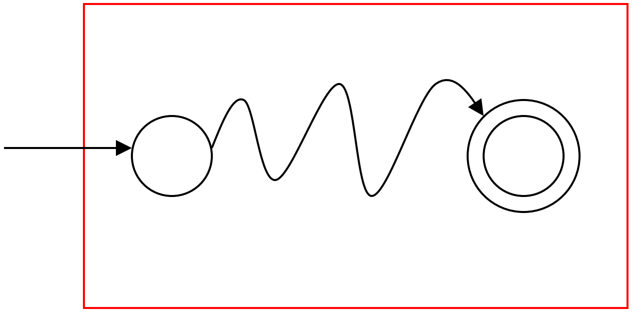
Regular language $L_1$

Regular language $L_2$

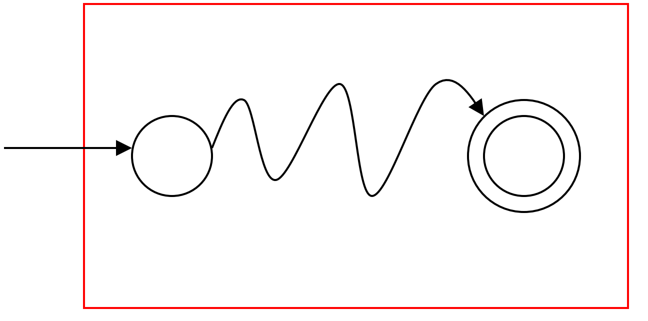
$$L(M_1) = L_1$$

$$L(M_2) = L_2$$

NFA $M_1$



NFA $M_2$



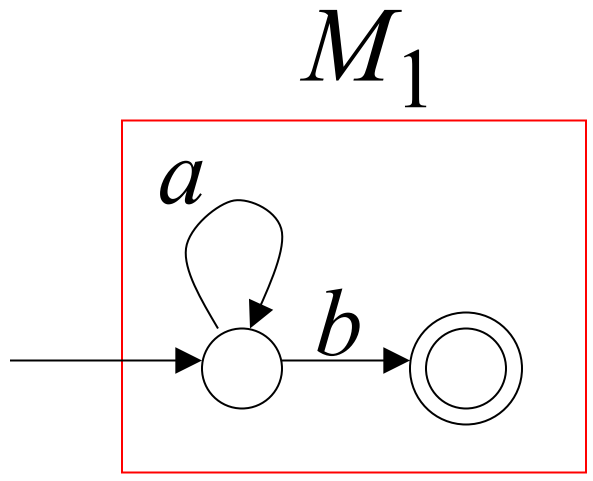
Single accepting state

Single accepting state

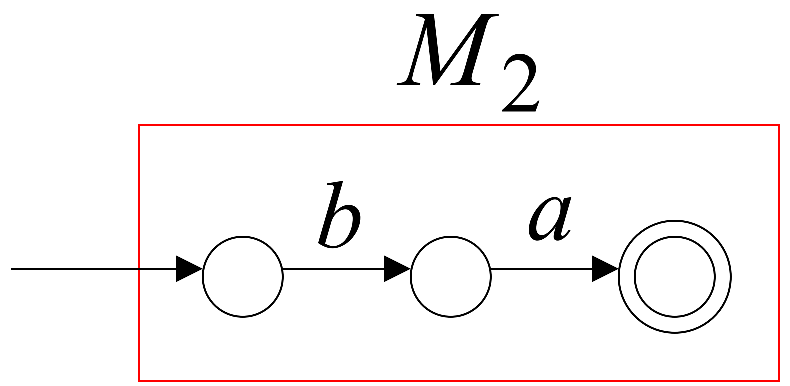
# Example

$n \geq 0$

$L_1 = \{a^n b\}$

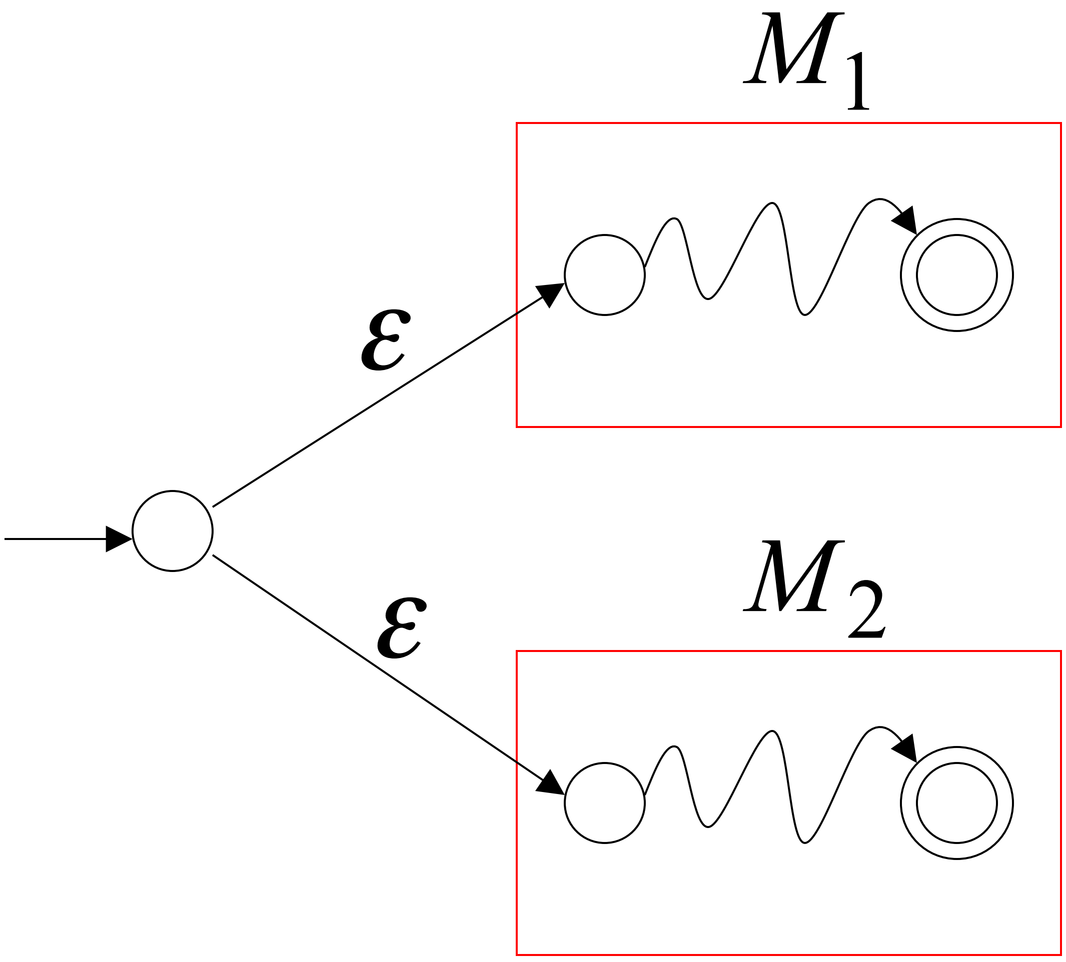



$L_2 = \{ba\}$



# Union

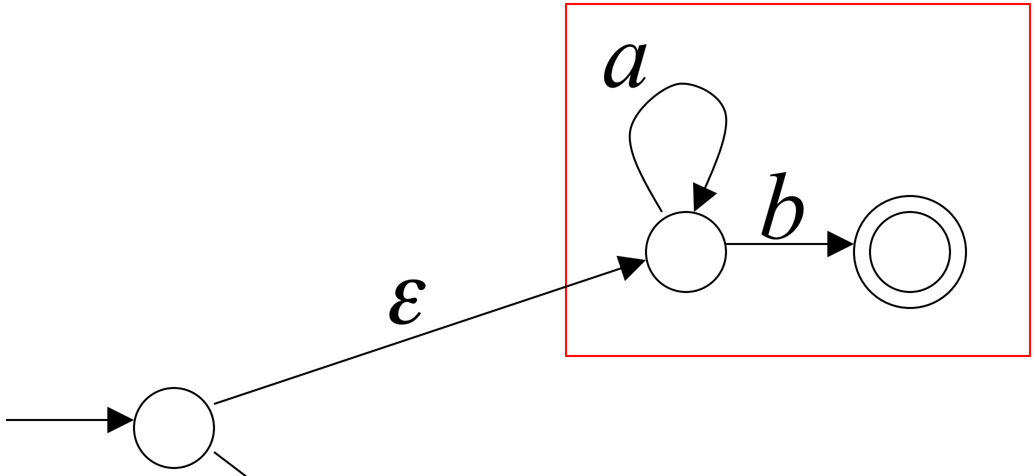
NFA for $L_1 \cup L_2$



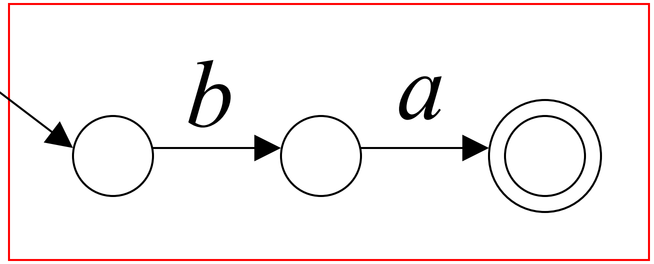
# Example

NFA for $L_1 \cup L_2 = \{a^n b\} \cup \{ba\}$

$$L_1 = \{a^n b\}$$

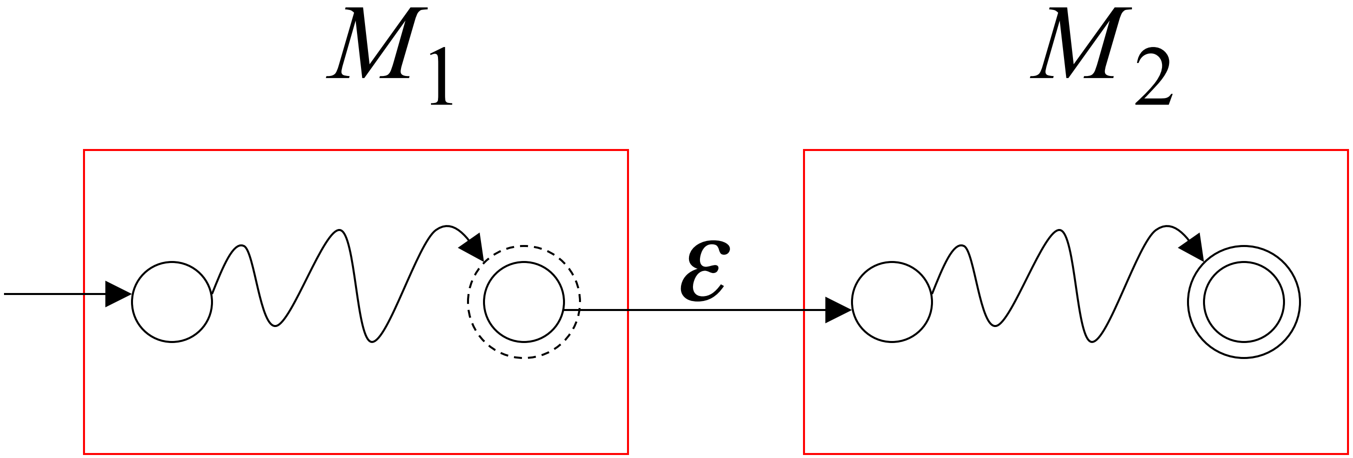



$$L_2 = \{ba\}$$



# Concatenation

NFA for $L_1 L_2$

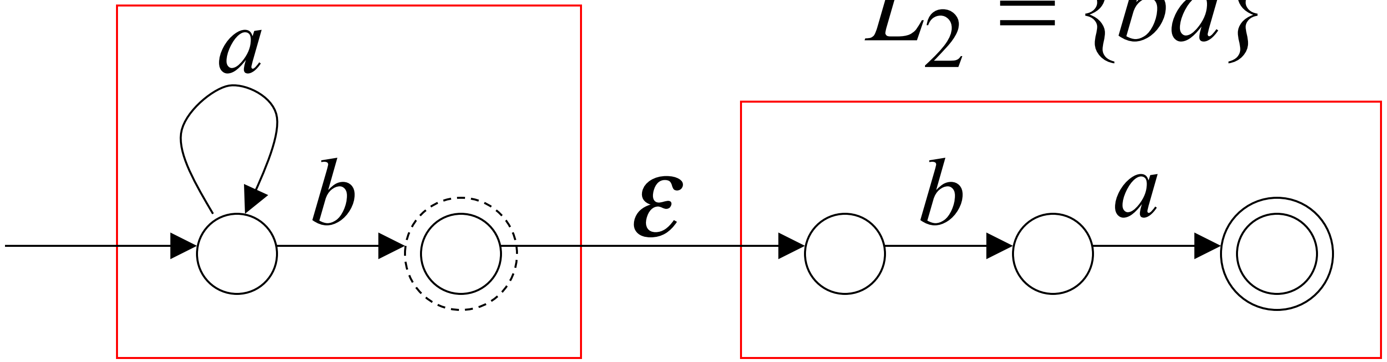


# Example

NFA for $L_1 L_2 = \{a^n b\}\{ba\} = \{a^n bba\}$

$L_1 = \{a^n b\}$

$L_2 = \{ba\}$

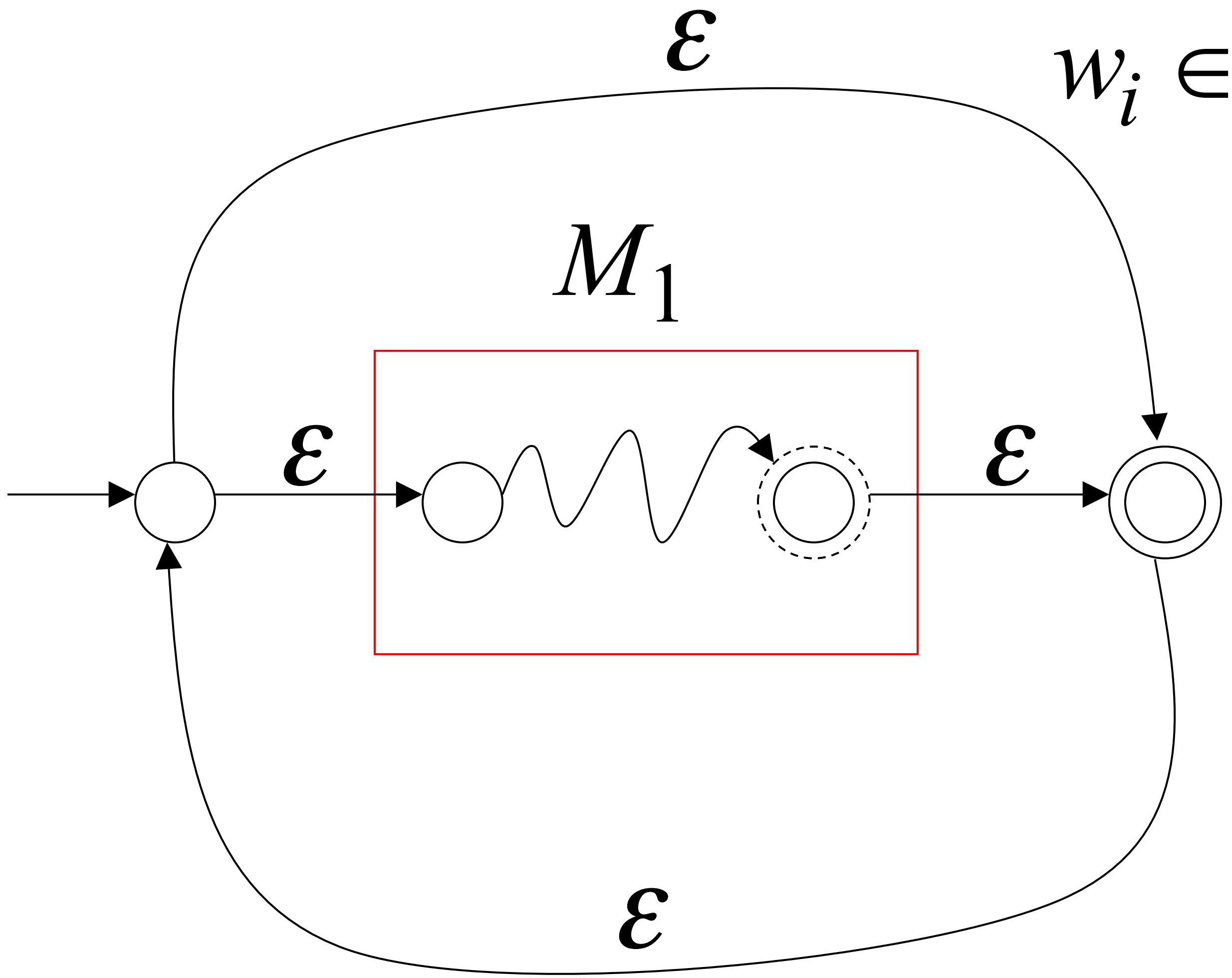


# Star Operation

NFA for $L_1{}^*$

$w = w_1 w_2 \cdots w_k$

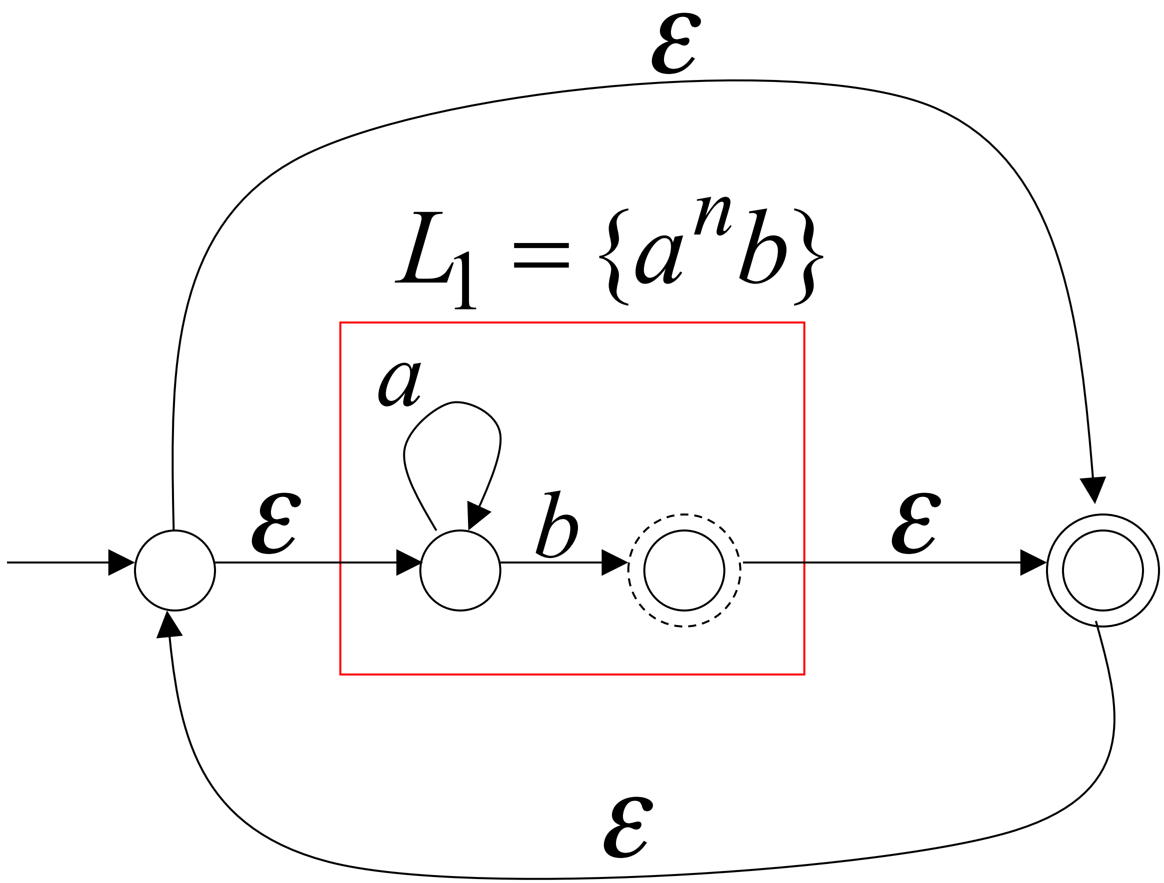
$w_i \in L_1$




$\varepsilon \in L_1{}^*$

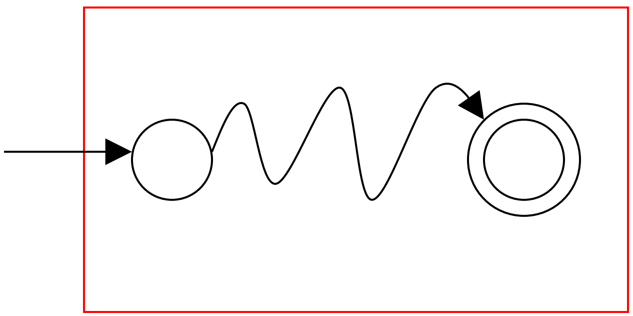
# Example

NFA for $L_1{}^* = \{a^n b\}^*$



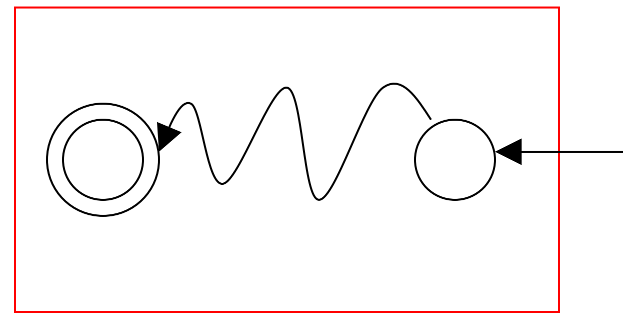
# Reverse

$L_1$ $M_1$



NFA for

$L_1^R$ $M_1'$



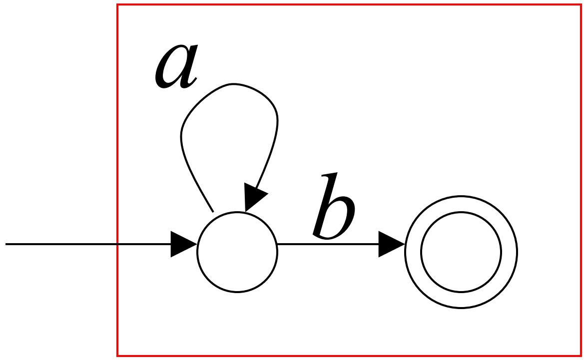
1. Reverse all transitions

2. Make initial state accepting state and vice versa

# Example

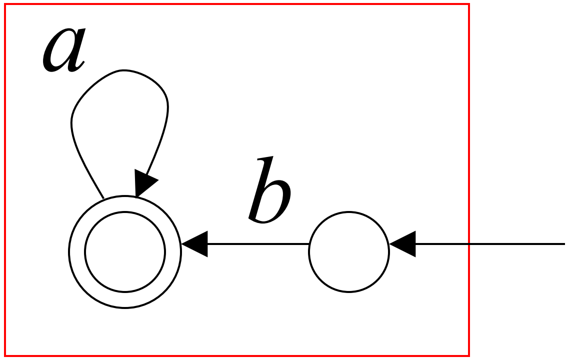
$$L_1 = \{a^n b\}$$

$M_1$



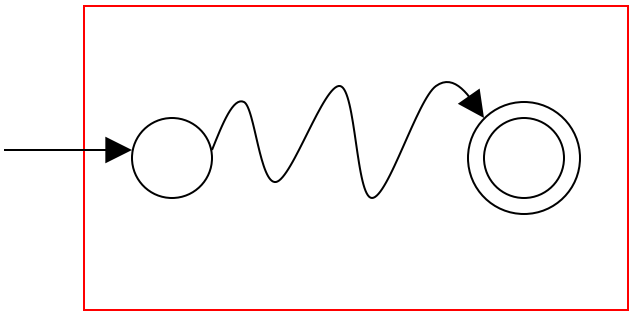

$$L_1^R = \{ba^n\}$$

$M_1'$

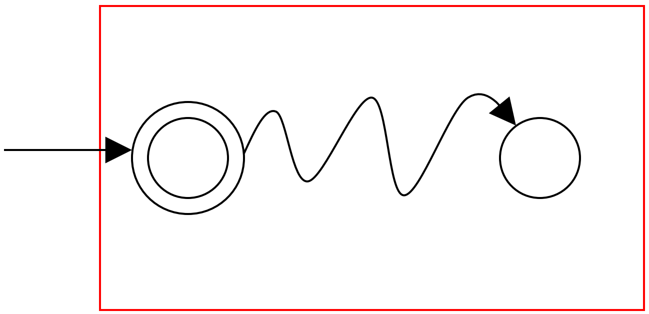


# Complement

$L_1$ $M_1$



$\overline{L_1}$ $M_1'$



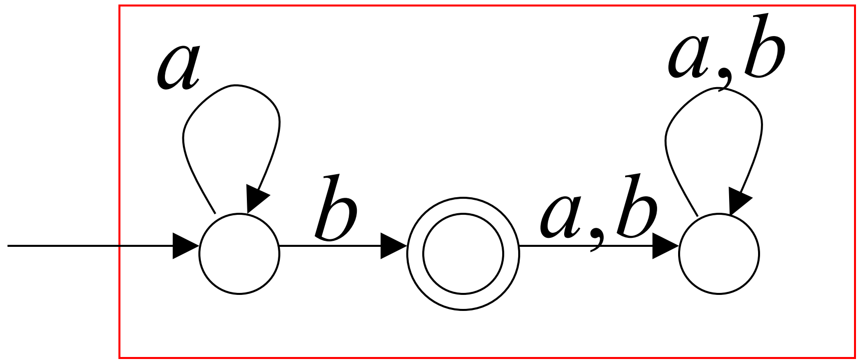
1. Take the DFA that accepts $L_1$

2. Make accepting states non-final, and vice-versa

# Example

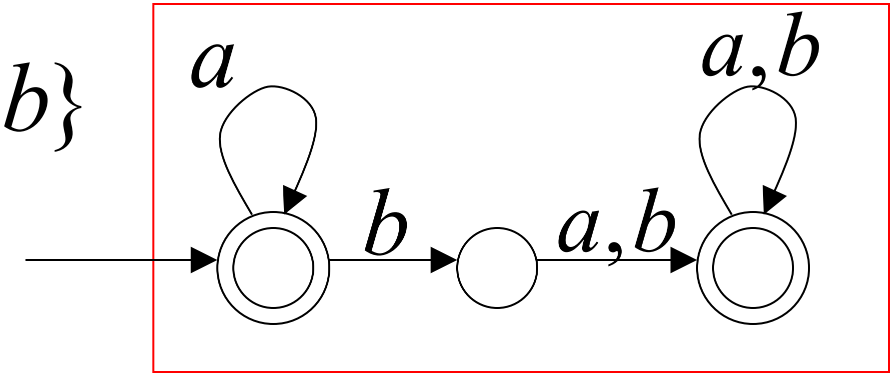
$M_1$

$$L_1 = \{a^n b\}$$




$M_1'$

$$\overline{L_1} = \{a,b\}^* - \{a^n b\}$$



# Intersection

$L_1$ regular

$L_2$ regular

$\Bigg\}$ We show $\Rightarrow$ $L_1 \cap L_2$ regular

DeMorgan's Law: $L_1 \cap L_2 = \overline{\overline{L_1} \cup \overline{L_2}}$

$L_1\ ,\ L_2$ regular

$\Rightarrow$ $\overline{L_1}\ ,\ \overline{L_2}$ regular

$\Rightarrow$ $\overline{L_1} \cup \overline{L_2}$ regular

$\Rightarrow$ $\overline{\overline{L_1} \cup \overline{L_2}}$ regular

$\Rightarrow$ $L_1 \cap L_2$ regular

# Example

$L_1 = \{a^n b\}$ regular

$L_2 = \{ab, ba\}$ regular

$\Rightarrow$

$L_1 \cap L_2 = \{ab\}$

regular

# Another Proof for Intersection Closure

Machine $M_1$

DFA for $L_1$

Machine $M_2$

DFA for $L_2$

Construct a new DFA $M$ that accepts $L_1 \cap L_2$

$M$ simulates in parallel $M_1$ and $M_2$

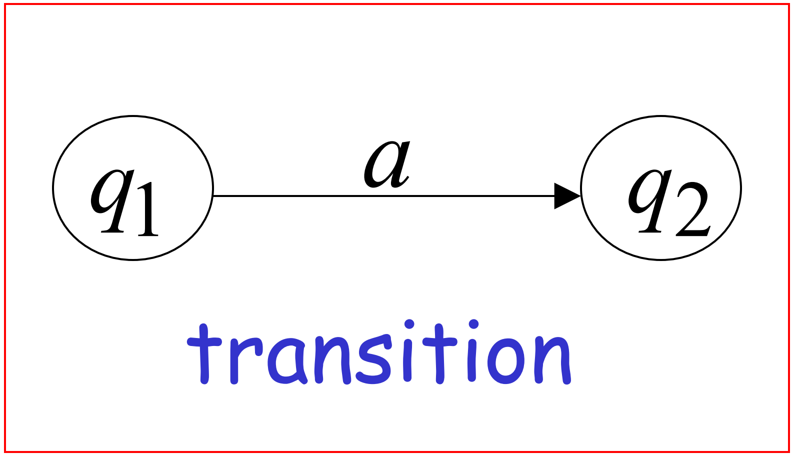
States in $M$

$q_i, p_j$

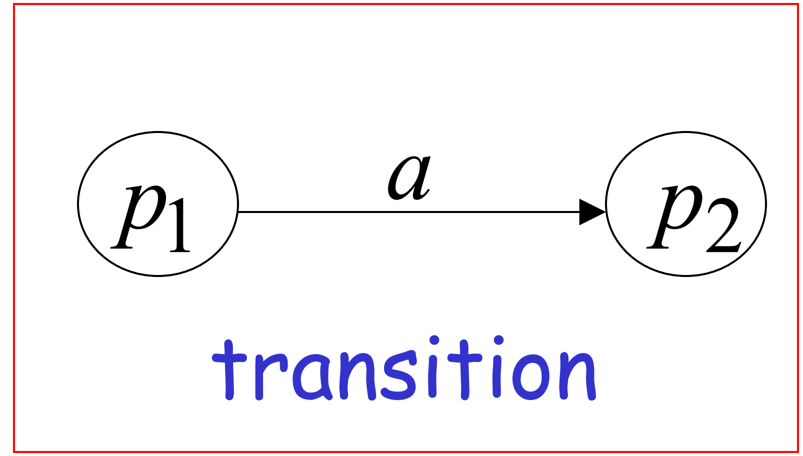
State in $M_1$

State in $M_2$

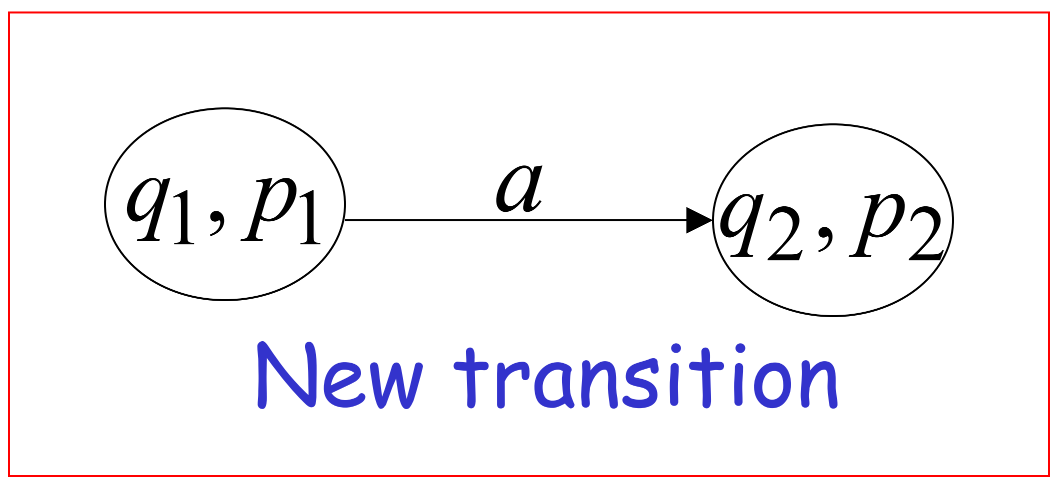
DFA $M_1$

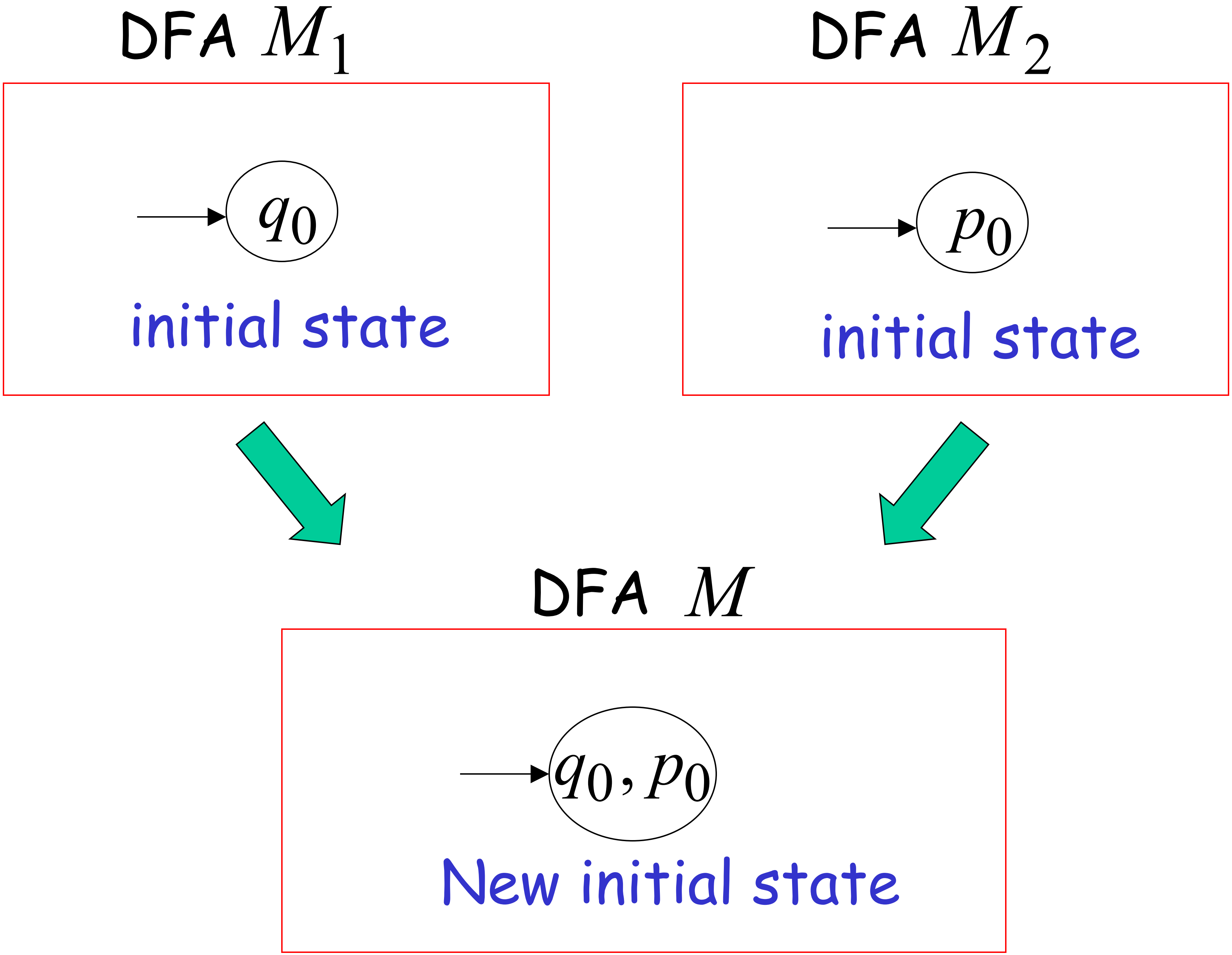


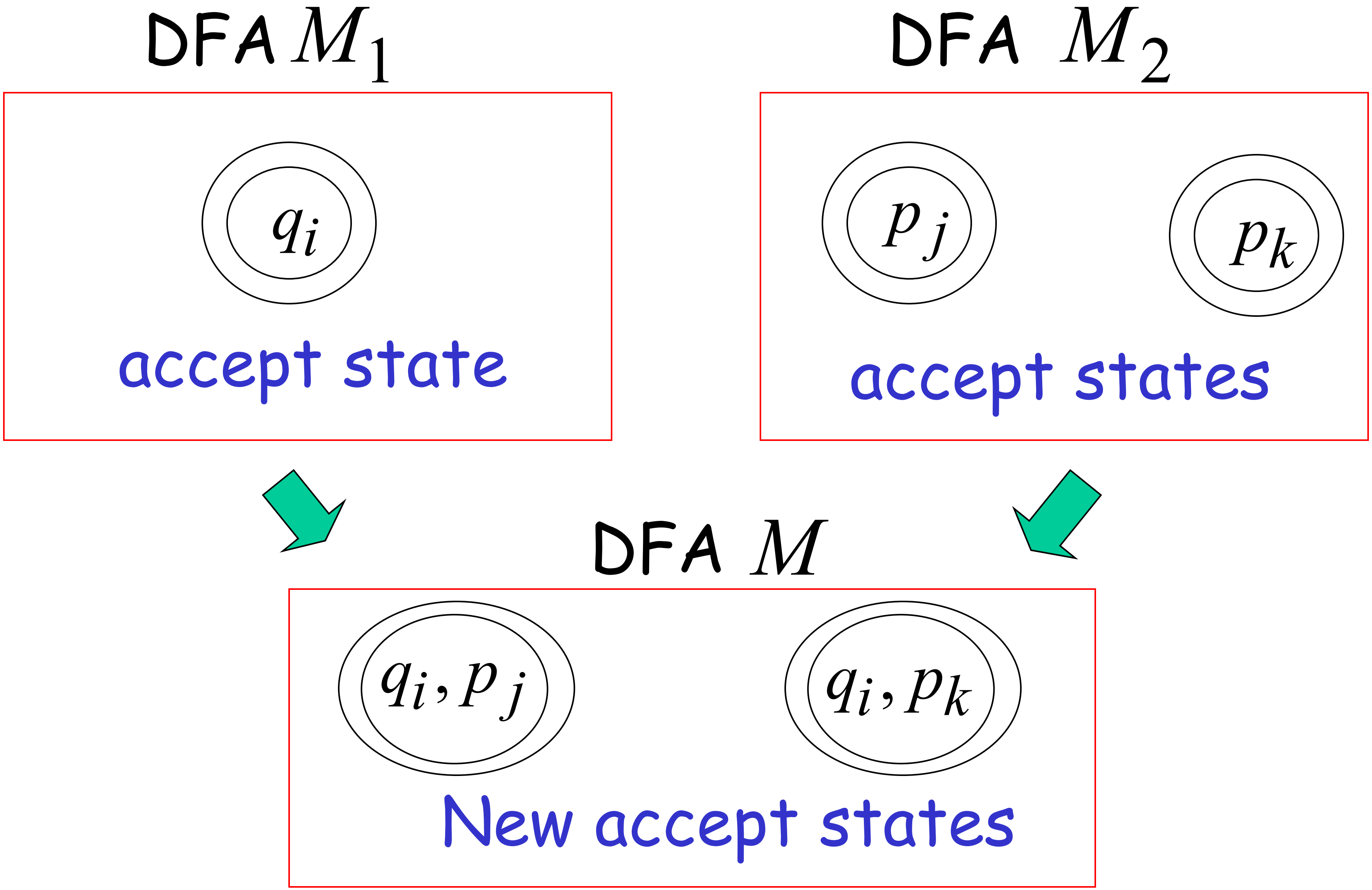

DFA $M_2$




DFA $M$




DFA $M_1$
$q_0$
initial state
DFA $M_2$
$p_0$
initial state
DFA $M$
$q_0, p_0$
New initial state



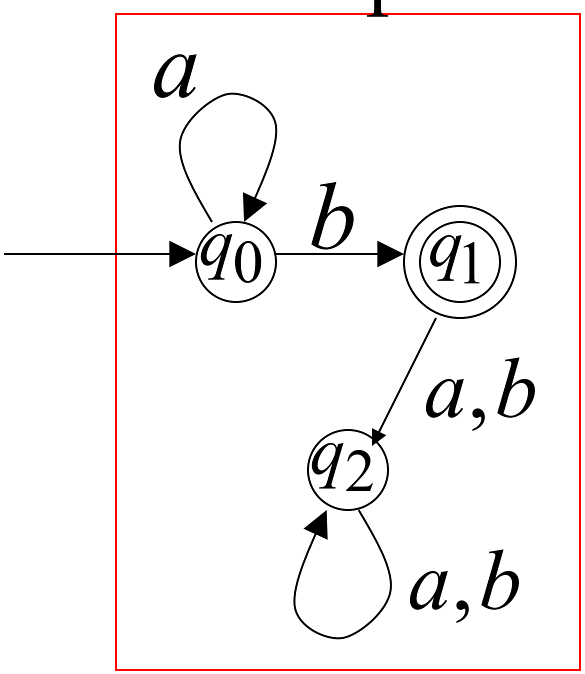

Both constituents must be accepting states

# Example:

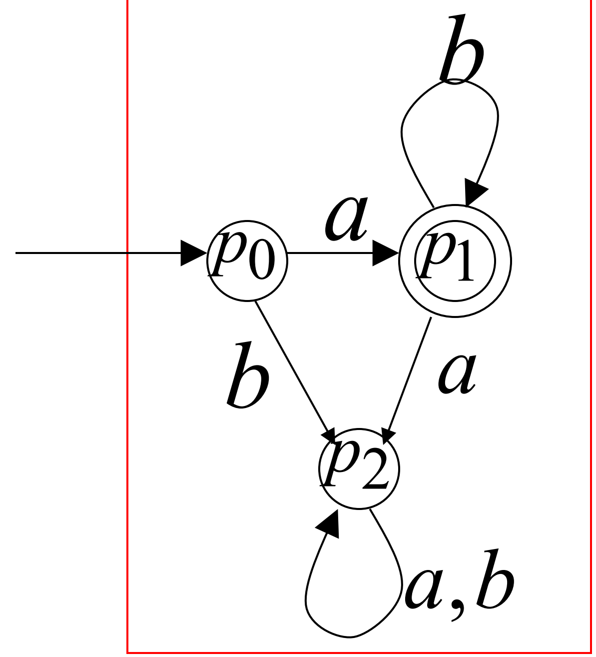
$$L_1 = \{a^n b\} \quad n \geq 0$$

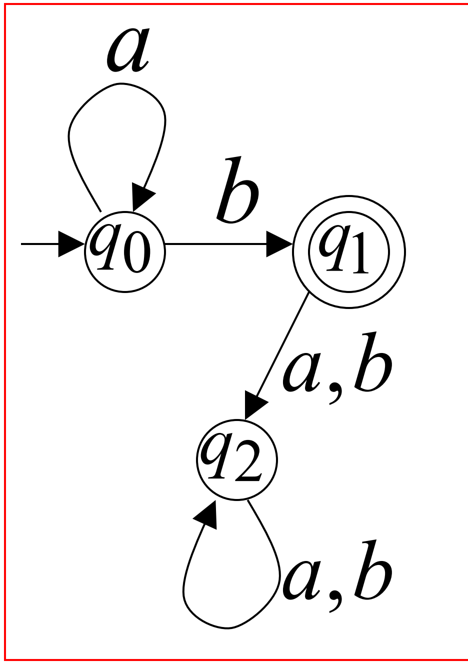
$$L_2 = \{ab^m\} \quad m \geq 0$$

$M_1$



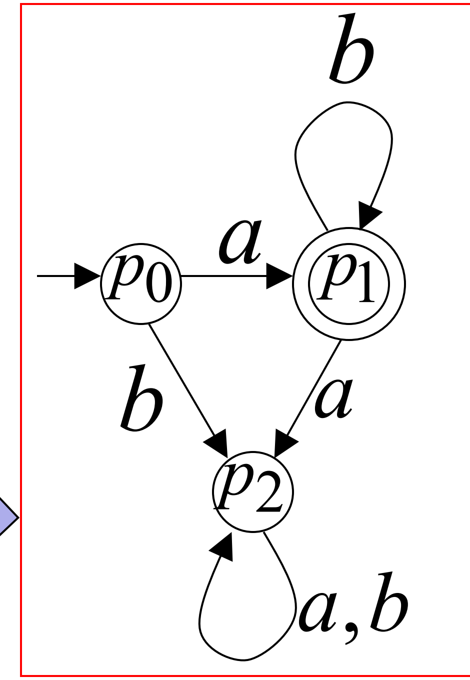

$M_2$

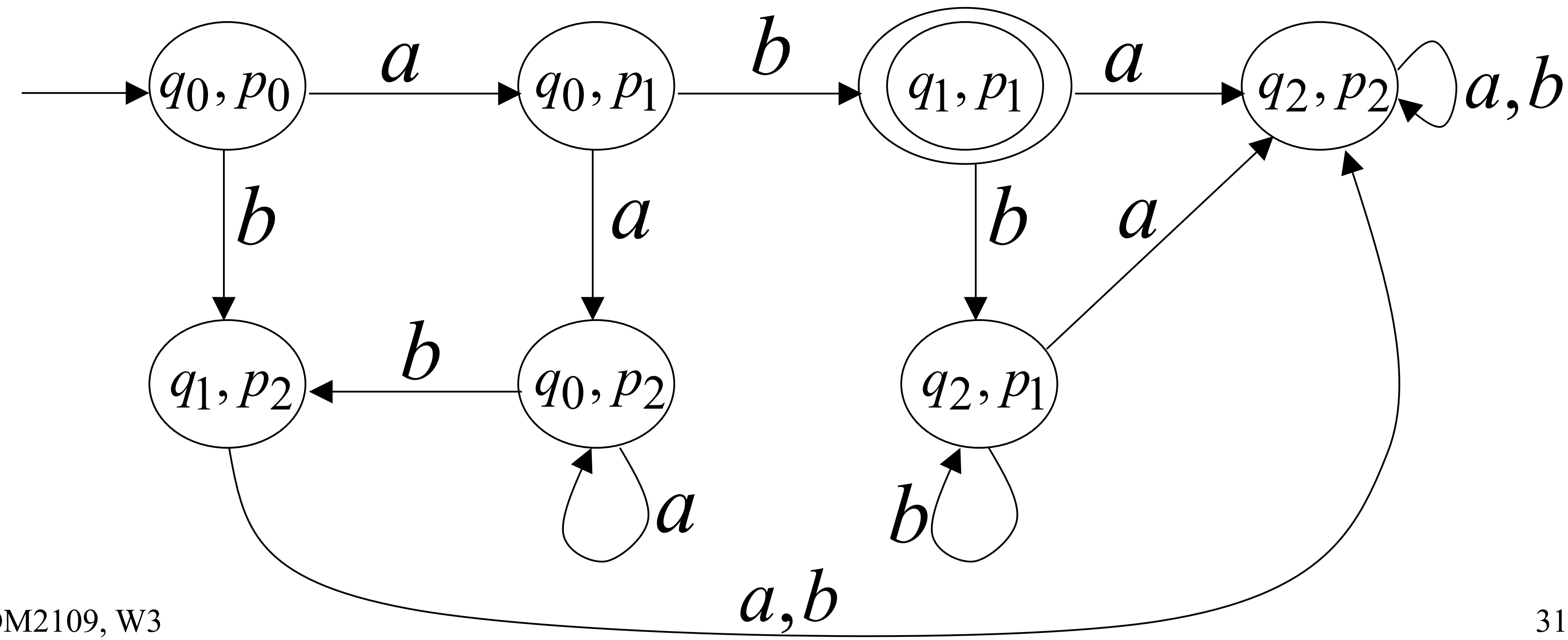





# Automaton for intersection

$$L = \{a^n b\} \cap \{ab^n\} = \{ab\}$$



$M$ simulates in parallel $M_1$ and $M_2$

$M$ accepts string $w$ if and only if:

$M_1$ accepts string $w$

and $M_2$ accepts string $w$

$$L(M) = L(M_1) \cap L(M_2)$$

# Summary

We have many ways of combining regular languages

These lead to regular languages

Observation: for some parts, it was easier to deal with DFA, for others we need NFA.