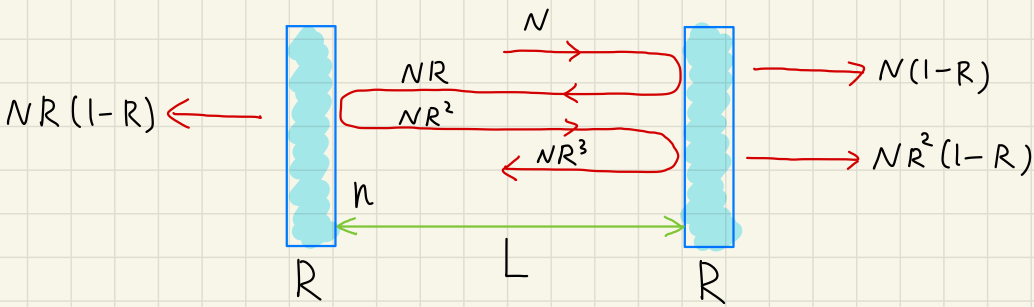
3-1 Cavity composed of two plane mirrors

(1) Putting N photons in a cavity



$$\frac{dN}{dt} = -\frac{\Delta N}{\frac{L}{\left(\frac{c}{n}\right)}} = -N\frac{c(1-R)}{nL}$$

$$\rightarrow \frac{dN}{N} = -\frac{c(1-R)}{nL}dt$$

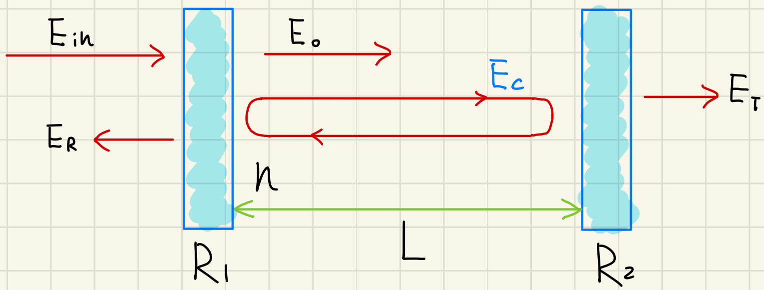
$$\rightarrow N(t) = N_0 e^{-\frac{c(1-R)}{nL}t} = N_0 e^{-\frac{t}{\tau}}.$$

Here cavity photon lifetime $\tau = \dfrac{nL}{c(1-R)}$.

We then get the cavity photon decay rate

$$\boxed{\kappa = \frac{c(1-R)}{nL}} \quad (3-1)$$

(2) A continuous input and the transmission



(a) $R_1 = r_1^2$, $R_2 = r_2^2$

(b) $E_0 = \sqrt{1-R_1}\, E_{in}$

(c) inside the cavity we have circulating field such that

$$E_c = E_0 + E_0 r_1 r_2 e^{-2i\phi} + E_0 (r_1 r_2 e^{-2i\phi})^2 + \ldots\ldots$$

$$= E_0 \left(\frac{1}{1 - r_1 r_2 e^{-2i\phi}} \right), \text{ where } \phi = nkL$$

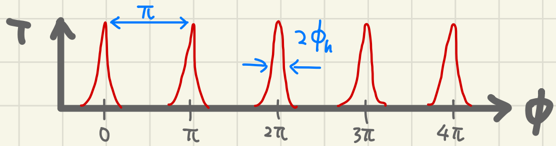
$$\rightarrow \frac{I_c}{I_0} = \frac{|E_c|^2}{|E_0|^2} = \frac{1}{1 + r_1^2 r_2^2 - 2 r_1 r_2 \cos(2\phi)}$$

$$= \frac{1}{(1 - r_1 r_2)^2 + 4 r_1 r_2 \sin^2\phi}$$

$$= \frac{1}{(1 - \sqrt{R_1 R_2})^2 + 4\sqrt{R_1 R_2}\, \sin^2\phi}$$

(d) $E_T = \sqrt{1-R_2}\, E_c$

$$\rightarrow \frac{I_T}{I_{in}} = \frac{|E_T|^2}{|E_{in}|^2}$$



$$= \boxed{T = \frac{(1-R_1)(1-R_2)}{(1-\sqrt{R_1R_2})^2 + 4\sqrt{R_1R_2}\sin^2\phi}} \quad (3-2)$$

The max transmission occurs at $\phi = m\pi$, i.e.

$$T_{max} = \frac{(1-R_1)(1-R_2)}{(1-\sqrt{R_1R_2})^2}.$$

$\frac{1}{2}T_{max}$ occurs at $\phi = m\pi \pm \phi_h$

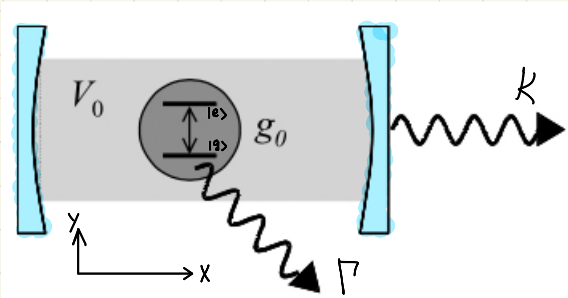
$$\sin^2\phi_h = \frac{(1-\sqrt{R_1R_2})^2}{4\sqrt{R_1R_2}}$$

$$\rightarrow 2\phi_h = 2\sin^{-1}\left[\frac{1-\sqrt{R_1R_2}}{2(R_1R_2)^{1/4}}\right] \cong \frac{1-\sqrt{R_1R_2}}{(R_1R_2)^{1/4}}$$

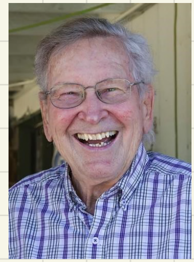
Here we define the finsse (figure of merit)

$$\boxed{F \equiv \frac{\pi}{2\phi_h} \cong \frac{\pi(R_1R_2)^{1/4}}{1-\sqrt{R_1R_2}}} \quad (3-3)$$

3-2 Atom-Cavity Coupling & Jaynes-Cummings model (1963)



Edwin Jaynes (1922-1998)



Fredrick Cummings (1931-2019)

In this section we study the atom-cavity (two level) coupling, and three parameters are important

① atom-cavity-field coupling constant g_0.

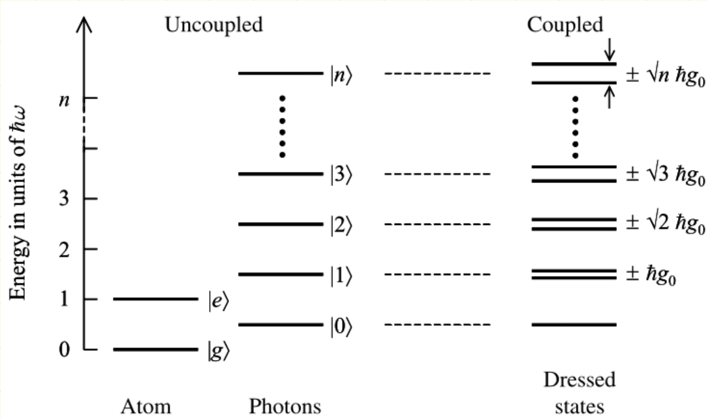
② atomic spontaneous decay rate Γ.

③ cavity decay rate κ.

Let's first look at the full Hamiltonian

$$\hat{\mathcal{H}} = \hat{\mathcal{H}}_A + \hat{\mathcal{H}}_F + \hat{\mathcal{H}}_{AF}$$

$$= \underbrace{\hbar\begin{pmatrix} 0 & 0 \\ 0 & \omega_0 \end{pmatrix} + \hbar\omega\left(\hat{a}^\dagger \hat{a} + \frac{1}{2}\right)}_{\text{Uncoupled}} + \underbrace{i\hbar\begin{pmatrix} 0 & -g_0 e^{-ikx}\hat{a}^\dagger \\ g_0^* e^{ikx}\hat{a} & 0 \end{pmatrix}}_{\text{Coupled}}$$



The Jaynes–Cummings ladder. The ladder describes the states of a coupled atom–photon system with a coupling constant of g_0. The states of the uncoupled system are labelled by whether the atom is in the ground state $|g\rangle$ or the excited state $|e\rangle$, and by the number of photons n in the mode.

and use states $|e, n-1\rangle$ and $|g, n\rangle$

construct matrix

$$\hat{H} = \hbar\begin{pmatrix} 0 & 0 \\ 0 & \omega_0 \end{pmatrix} + \hbar\omega\begin{pmatrix} n+\frac{1}{2} & 0 \\ 0 & n-\frac{1}{2} \end{pmatrix} + i\hbar\sqrt{n}\begin{pmatrix} 0 & -g_0 e^{-ikx} \\ g_0^* e^{ikx} & 0 \end{pmatrix}.$$

Let $i g_0 e^{-ikx} = \Omega$, the Hamiltonian becomes

$$\hat{H} = \hbar\begin{pmatrix} 0 & 0 \\ 0 & \omega_0 \end{pmatrix} + \hbar\omega\begin{pmatrix} n+\frac{1}{2} & 0 \\ 0 & n-\frac{1}{2} \end{pmatrix} + \hbar\begin{pmatrix} 0 & \sqrt{n}\,\Omega \\ \sqrt{n}\,\Omega^* & 0 \end{pmatrix}.$$

$$\rightarrow \boxed{\hat{H}_{JM} = \hbar\begin{pmatrix} (n+\frac{1}{2})\omega & \sqrt{n}\,\Omega \\ \sqrt{n}\,\Omega^* & (n+\frac{1}{2})\omega - \delta \end{pmatrix}} \quad (3-4)$$

which is the so-called Jaynes – Cumming's model.

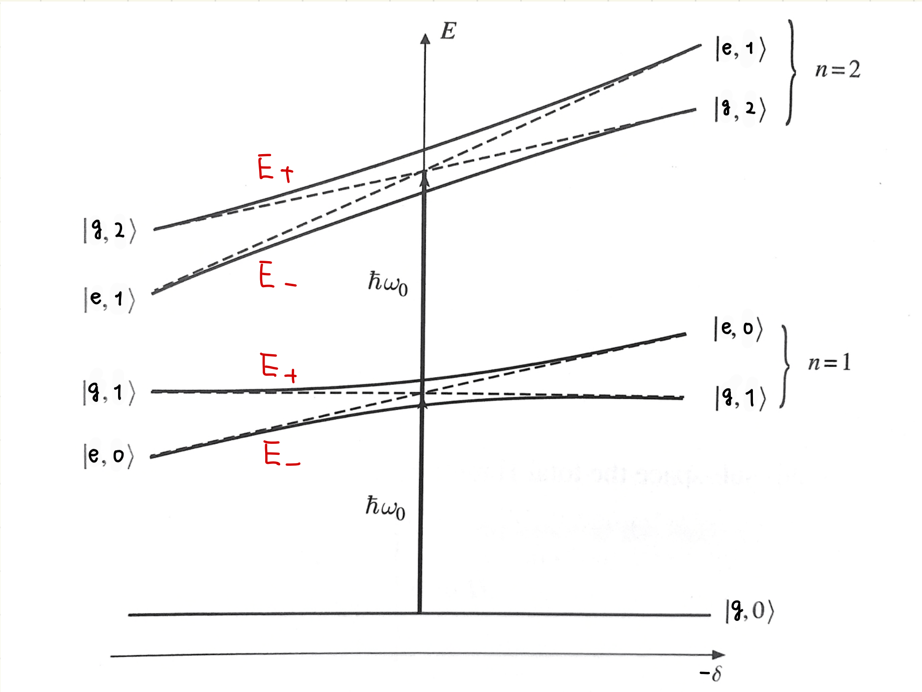
Here detuning $\delta = \omega - \omega_0$.

3-3 Eigenstate of Jaynes-Cummings model

We here diagonalize $\hat{\mathcal{H}}_{JM}$

$$\begin{vmatrix} \hbar(n+\frac{1}{2})\omega - E & \hbar\sqrt{n}\,\Omega \\ \hbar\sqrt{n}\,\Omega^* & \hbar(n+\frac{1}{2})\omega - \hbar\delta - E \end{vmatrix} = 0$$

$$\rightarrow \boxed{E_\pm = \hbar\left[(n+\frac{1}{2})\omega - \frac{1}{2}\delta \pm \frac{1}{2}\sqrt{\delta^2 + 4n|\Omega|^2}\right]} \quad (3\text{-}5)$$



$$\rightarrow \boxed{|\psi_\pm, n\rangle \propto \frac{\delta \pm \sqrt{\delta^2 + 4n|\Omega|^2}}{2\sqrt{n}\,\Omega^*}|g, n\rangle + |e, n-1\rangle}, \quad (3\text{-}6)$$

which is called dressed state and has entanglement.

3-4 Dynamics & Damping

We neglect $(n+\frac{1}{2})\hbar\omega$ in Eq. (3-4) and get the interaction Hamiltonian

$$\hat{H}_{JM} = \hbar \begin{pmatrix} 0 & \sqrt{n}\,\Omega \\ \sqrt{n}\,\Omega^* & -\delta \end{pmatrix}. \qquad (3-7)$$

$$|\psi(t)\rangle = C_e(t)\,|e, n-1\rangle + C_g(t)\,|g, n\rangle$$
$$= \begin{pmatrix} C_g(t) \\ C_e(t) \end{pmatrix}.$$

The Schrödinger equation reads

$$i\hbar\partial_t \begin{pmatrix} C_g \\ C_e \end{pmatrix} = \hbar \begin{pmatrix} 0 & \sqrt{n}\,\Omega \\ \sqrt{n}\,\Omega^* & -\delta \end{pmatrix} \begin{pmatrix} C_g \\ C_e \end{pmatrix} + \mathcal{L}\psi$$

$$\rightarrow \begin{pmatrix} \dot{C}_g \\ \dot{C}_e \end{pmatrix} = \begin{pmatrix} -i\sqrt{n}\,\Omega\, C_e \\ -i\sqrt{n}\,\Omega^* C_g + i\delta C_e - \frac{\Gamma}{2} C_e \end{pmatrix}. \qquad (3-8)$$

We phenomenologically add two damping terms to describe cavity loss and atomic decay.

When intra-cavity photon number $n=1$ and $\kappa \gg \Gamma$,

① $|\Omega| \gg \frac{\kappa}{2}$ is called strong coupling region.

② $|\Omega| \ll \frac{\kappa}{2}$ is called weak coupling region.

3-4-4 Strong coupling

For ①
$$\begin{cases} \dot{C}_g = -i\sqrt{n}\,\Omega\, C_e , & C_g(0) = 0 \\ \ddot{C}_e = -i\sqrt{n}\,\Omega^{*}\,\dot{C}_g + i\delta\,\dot{C}_e , & C_e(0) = 1 \end{cases}$$

$$\rightarrow \quad \ddot{C}_e - i\delta\,\dot{C}_e + n|\Omega|^2 C_e = 0$$

Let $C_e = e^{at}$

$$\rightarrow \quad a^2 - i\delta a + n|\Omega|^2 = 0$$

$$\rightarrow \quad a = \frac{i}{2}\left(\delta \pm \sqrt{\delta^2 + 4n|\Omega|^2}\right)$$

$$\rightarrow \begin{cases} C_e(t) = e^{\frac{i}{2}\delta t}\cos\left(\frac{t}{2}\sqrt{\delta^2+4n|\Omega|^2}\right) + \dfrac{i\delta\, e^{\frac{i}{2}\delta t}}{\sqrt{\delta^2+4n|\Omega|^2}}\sin\left(\frac{t}{2}\sqrt{\delta^2+4n|\Omega|^2}\right) & (3-9) \ \# \\ C_g(t) = \dfrac{2i\sqrt{n}\,\Omega\, e^{\frac{i}{2}\delta t}}{\sqrt{\delta^2+4n|\Omega|^2}}\sin\left(\frac{t}{2}\sqrt{\delta^2+4n|\Omega|^2}\right) & (3-10) \ \# \end{cases}$$

One can the derive the probability

$$P_e(t) = |C_e(t)|^2 = 1 - |C_g(t)|^2$$

$$= 1 - \frac{4n|\Omega|^2}{\delta^2 + 4n|\Omega|^2} \sin^2\left(\frac{t}{2}\sqrt{\delta^2 + 4n|\Omega|^2}\right)$$

For $n = 1$

(a) $|\Omega| \gg \delta$,

$$\rightarrow \begin{cases} P_e(t) = \cos^2(|\Omega| t) \\ P_g(t) = \sin^2(|\Omega| t) \end{cases} \rightarrow \text{Rabi oscillation!}$$

We get one-photon Rabi oscillation, AKA. vacuum Rabi oscillation because of the initial state $|e, 0\rangle$.

(b) $|\Omega| \ll \delta$,

$$\begin{cases} P_e(t) = 1 \\ P_g(t) = 0 \end{cases} \rightarrow \text{no decay!!}$$

The spontaneous decay is NOT an intrinsic property of an atom!!

3-4-5 Weak coupling

For ②
$$\begin{cases} \dot{C}_g = -i\sqrt{n}\,\Omega\, C_e \\ \ddot{C}_e = -i\sqrt{n}\,\Omega^* \dot{C}_g - \frac{i}{2} n^{-\frac{1}{2}} \frac{dn}{dt} \Omega^* C_g + i\delta \dot{C}_e \\ \dot{n} = -n\kappa \end{cases}$$

and initial conditions $C_g(0) = 0$, $C_e(0) = 1$.

$$\ddot{C}_e = -n|\Omega|^2 C_e + \frac{i}{2}\sqrt{n}\,\kappa\,\Omega^* C_g$$

for $\delta = 0$, $n = 1$ and

$$\dot{C}_e = -i\sqrt{n}\,\Omega^* C_g \rightarrow C_g = i\frac{1}{\sqrt{n}\,\Omega^*}\dot{C}_e$$

$$\rightarrow \ddot{C}_e + \frac{\kappa}{2}\dot{C}_e + |\Omega|^2 C_e = 0 \qquad (3-11)$$

$$\rightarrow a^2 + \frac{\kappa}{2} a + |\Omega|^2 = 0$$

$$\rightarrow a = \frac{1}{2}\left(-\frac{\kappa}{2} \pm \sqrt{\frac{\kappa^2}{4} - 4|\Omega|^2}\right)$$

$$\rightarrow C_e(t) = A e^{-\frac{t}{2}\left(\frac{\kappa}{2} + \sqrt{\frac{\kappa^2}{4} - 4|\Omega|^2}\right)} + B e^{-\frac{t}{2}\left(\frac{\kappa}{2} - \sqrt{\frac{\kappa^2}{4} - 4|\Omega|^2}\right)}$$

We now analyze the decay constant for $\kappa^2 \geq 16|\Omega|^2$ and compare it with the spontaneous decay rate in free space

$$\Gamma_{free} = \frac{p^2 w^3}{3\pi\hbar\epsilon_0 c^3} . \qquad (2.2.18)$$

To see any enhancement in cavity, we need to check the slow decay mode

$e^{-\frac{t}{2}\left(\frac{\kappa}{2}-\sqrt{\frac{\kappa^2}{4}-4|\Omega|^2}\right)}$ and its decay constant

$$\left(\frac{\kappa}{2}-\sqrt{\frac{\kappa^2}{4}-4|\Omega|^2}\right) = \frac{\kappa}{2}-\frac{\kappa}{2}\left(1-\frac{16|\Omega|^2}{\kappa^2}\right)^{\frac{1}{2}}$$
$$\simeq \frac{\kappa}{2}-\frac{\kappa}{2}+\frac{4}{\kappa}|\Omega|^2 - \cdots$$
$$\simeq \frac{4}{\kappa}|\Omega|^2 .$$

Next we have to express κ as the function of cavity finess F and quality factor Q.

From Eq. (3-3), $F = \frac{\pi\sqrt{R}}{1-R}$

$\rightarrow (1-R)^2 F^2 - \pi^2 R = 0$

$\rightarrow R = \frac{1}{2F^2}\left(2F^2+\pi^2-\pi\sqrt{4F+\pi^2}\right) \simeq 1-\frac{\pi}{F}$.

From Eq. (3-1) and define $Q \equiv \frac{2LF}{\lambda}$

$$\kappa = \frac{c(1-R)}{nL} \simeq \frac{c\pi}{nLF} = \frac{2c\pi}{n\lambda Q} . \qquad (3\text{-}12)$$

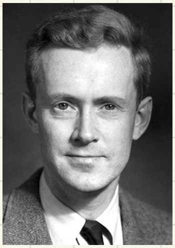
The slow decay constant becomes

$$\frac{4}{\kappa}|\Omega|^2 \cong \frac{\lambda Q}{c\pi}\left(\frac{p^2\omega}{\hbar\epsilon_0 V}\right) \quad \text{for } n=1$$

The enhancement factor reads

$$\frac{\frac{4}{\kappa}|\Omega|^2}{\Gamma_{free}} = \frac{\frac{\lambda Q}{c\pi}\left(\frac{p^2\omega}{\hbar\epsilon_0 V}\right)}{\frac{p^2\omega^3}{3\pi\hbar\epsilon_0 c^3}} = \frac{3Q\lambda^3}{4\pi^2 V}$$

$$\rightarrow \boxed{F_p \equiv \frac{3Q\lambda^3}{4\pi^2 V}} \qquad (3\text{-}13)$$



Edward Purcell (1912-1997)

is called Purcell factor.

E. Purcell was the first person who showed that spontaneous decay rate can be enhanced even in a bad cavity !!