國立中央大學

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Theoretical study of matter-wave emission

研究生：陳昱學

指導教授：廖文德

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Theoretical study of matter-wave emission

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Author: Yu-Hsueh, Chen

Advisor: Wen-Te, Liao

Department of Physics

National Central University

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摘要

理論上研究了玻色-愛因斯坦凝聚體（BEC）超輻射發射的控制。 BEC 的外部和內部自由度之間的耦合使得由原子物質波的干涉可以操控超輻射的空間性質。 我們在不同的動力學控制下數值求解了一個與雪茄形狀的 BEC 結合的 Maxwell-Bloch 方程（M-B 方程）的三維 Gross-Pitaevskii 方程（G-P 方程）[1]。由於缺乏伽馬射線的光學元件，通常是不可能調變伽馬射線。在這項工作中，我們先已銣原子探討發光機制和建構演算法，並在未來討論控制 BEC 的伽馬射線發射的可能性。

Abstract

The control over superradiant emission from Bose-Einstein condensate (BEC) is theoretically investigated. The coupling between external and internal degrees of freedom of a BEC allows for the manipulation of the spatial mode of superradiance emitted by an atomic coherent matter-wave. We numerically solve 3-dimensional Gross-Pitaevskii equation (G-P equation) coupled to Maxwell-Bloch equation (M-B equation) for a BEC of cigar shape under various dynamical controls [1]. In this work, we use Rubidium 87 first to make sure the process of emission light and construct the numerical algorithm. Then, we will discuss the possibility of controlling gamma-ray emission from BEC which is typically impossible due to the lack of optics in the future.

Contents

Chapter 1

Introduction

To produce controllable gamma-ray is difficult because of its extremely high frequency, causing itself to directly penetrate through most optical devices. Therefore, we attend to control the source of gamma-ray instead of using external optics. The illuminator is BEC composed of ^{135m}Cs atomic cloud, a radioactive nuclear isomer of Cesium. Why we use BEC is because we have ability to control its wave function, and we consider that if emission light carries the phase of BEC we control, it means we could control gamma-ray. The model for the light emission process is two-level system interacting with electromagnetic wave, which has been described by M-B equation. In this work, simulation of the two coupled BEC with the interaction of electromagnetic wave is numerically investigated. Through several numerical algorithms combined together for stability, the phase of emitted light is found to be controlled by two coupled BEC with different angular momentum.

Chapter 2

Background

2.1 Two-level system

There are three possible processes to describe the interaction between particles and photons, absorption, spontaneous emission, and stimulated emission. Absorption describes the incoming photon, which frequency matches to the difference of two energy level, is transformed into internal energy, causing the transition of particle from ground state to excited state. Spontaneous emission is just like the inverse process of absorption except that the transition time from excited state to ground state is much longer. However, the incoming photon could also affect the particle in excited state, leading to reduce the transition time, which is called stimulated emission. To understand all these processes, the perturbation theory is used to explain, such as treating the particle under the unperturbed Hamiltonian, and photon as time dependent perturbation, otherwise, the spontaneous emission may be strange if no one disturbs the system.

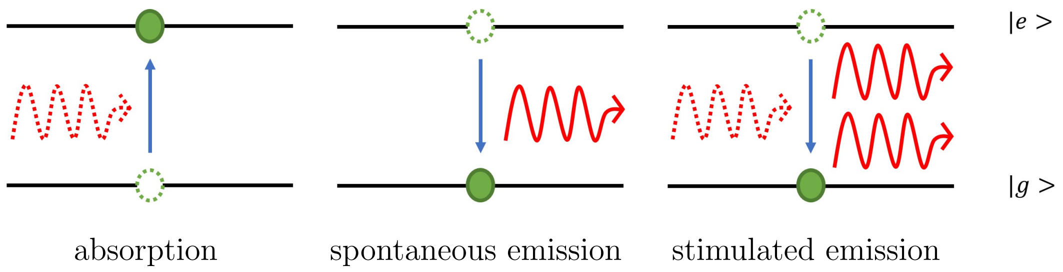


Figure 1: The dashed line describes the initial states of the system, including the particle and the photon. After the interaction, the system becomes final state as solid line.

In our simulation, we use two BEC ground state respectively with different rotary frequency for generating the phase change. One is represented for excited state as the source of emission light, and the other one is for ground state as the seed to induce the transition. Excited state or ground state means the internal energy of the particle, and we assume it has nothing to do with external energy described by G-P equation. Two-level system is associated with internal electronic states. As small perturbation, such as thermal noise or high order terms of particles interaction, induce the transition form excited state to ground state, the first light is then produced, it will lead to the following stimulated emission coherent to the first light.

2.2 Bose-Einstein condensate

BEC occurs under the condition as a group of boson is cooled down to temperature nearly close to absolute zero. Under such temperature, bosons could mostly occupy the same lowest state, meaning that most of bosons are coherent. We could treat the group of particle as a cluster, and use two clusters to produce dipole moment for light emission. This phenomenon is quite different in classical particles, whose coherence should be produce by external electric field. It's why we use BEC as the illustrator.

2.2.1 Gross-Pitaevskii equation

The equation of motion of BEC is described by Gross-Pitaevskii equation(G-P equation).

$$i\hbar\frac{\partial}{\partial t}\Psi(\mathbf{r},t) = \left(H_0 + \frac{4\pi\hbar^2 a_s N}{m}|\Psi(\mathbf{r},t)|^2\right)\Psi(\mathbf{r},t) \tag{1}$$

where $H_0 = -\frac{\hbar^2}{2m}\nabla^{\mathbf{2}} + V(\mathbf{r})$, is original Schroedinger equation. a_s is the scattering length of particle. N is the number of particles. The second term in RHS results from the interaction energy between bosons, which is proportional to density, $|\Psi|^2$, and partial number, N. It's similar to classical intuition when we consider the energy between two electric point charges, and introduce mean-field approximation to estimate the influence of one particle if N is large enough. Caused of bosons, the wave function of one particle is same as one of total particles.

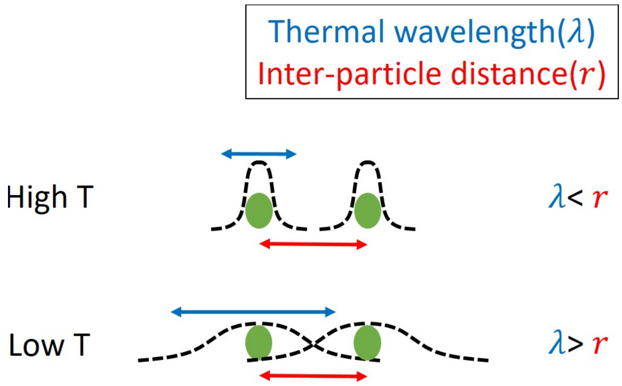
In this work, the rotation, $\mathbf{\Omega}_R \cdot \hat{\mathbf{L}}_z$, is also added to G-P equation to produce the phase of wave function. Through the rotation, there are several vortexes inside the wave function according to rotary frequency. In quantum world, the angular momentum is quantized, resulting to several identical vortexes instead of one large vortex. Besides, the external trapping potential, $V(\mathbf{r}) = \frac{1}{2}m\left(\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2\right)$, is considered to be cigar-shape harmonic trap because the emission light in long axis could be increased by more particles.

2.2.2 critical temperature

In quantum mechanics, the particle could be treat as a wave, meaning that the phenomenon of wave, such as interference, could happen to particle. At low temperature, the wavelength of particle increase to over the mean inter-particle distance. That is, particles are overlapping and indistinguishable. It's the critical temperature that describe the balance of these two length scale.

$$T_C \propto \frac{2\pi\hbar^2}{k_B m}(n)^{2/3} \tag{2}$$

where n is the density, k_B is Boltzmann constant



The critical temperature describes the competition between inter-particle distance and thermal wavelength. As the temperature is too low to distinguish the particle, the quantum effect is significant. Caused of bosons, it would condensate to form BEC.

2.2.3 Thomas-Fermi approximation

In order to check whether the ground state we generate is rational, we use Thomas-Fermi approximation (T-F approximation) to estimate that the shape is similar to the external potential. The idea of which is to consider if the system is under the critical temperature to form steady BEC, then we could neglect the kinetic energy, and treat the time derivative as some chemical potential, μ, then the equation becomes

$$\mu\Psi(\mathbf{r}) = \left(V(\mathbf{r}) + \frac{4\pi\hbar^2 a_s N}{m}|\Psi(\mathbf{r})|^2\right)\Psi(\mathbf{r}) \tag{3}$$

which we could estimate the shape of BEC ground state is just the reverse external potential. In whole simulation, we always treat $V(\mathbf{r})$ as harmonic potential.

$$|\Psi(\mathbf{r})|^2 = \left(\frac{4\pi\hbar^2 a_s N}{m}\right)^{-1}(\mu - V(\mathbf{r})) \tag{4}$$

T-F approximation could help us check whether our simulation is rational and estimate the size of system by treating where the probability density is greater than zero.

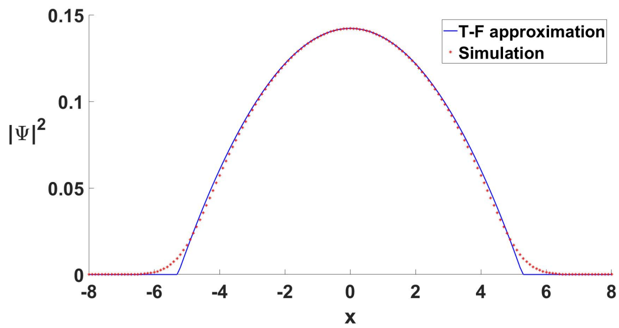


Figure 2: The difference of BEC ground state probability distribution between T-F approximation and numerical solution. We could find out they are almost the same except for the boundary. It's because there are discrete points on T-F approximation, which cause higher kinetic energy. As the results, the real ground state would be smooth on the boundary.

2.3 Maxwell-Bloch equation

The wave equation in the medium is described by M-B equation

$$\left(\frac{1}{c}\frac{\partial}{\partial t} \pm \frac{\partial}{\partial z}\right)\Omega_{\pm} = i\eta\Psi_e\Psi^*_{g\mp} \tag{5}$$

where $\eta = \frac{3N\Gamma\lambda^2}{4\pi}$ with spontaneous decay rate of exited state, Γ, and the wavelength of propagation light, λ.

$$i\hbar\frac{\partial}{\partial t}\Psi_e = \left[\mathscr{L}_{GP} - \frac{i\hbar}{2}\Gamma\right]\Psi_e - \frac{\hbar}{2}\Omega_-\Psi_{g+} \tag{6}$$

$$i\hbar\frac{\partial}{\partial t}\Psi_{g+} = \mathscr{L}_{GP}\Psi_{g+} - \frac{\hbar}{2}\Omega^*_-\Psi_e \tag{7}$$

where $\mathscr{L}_{GP} = H_0 + U_0\left(|\Psi_e|^2 + |\Psi_g|^2\right) - \mathbf{\Omega}_R \cdot \hat{\mathbf{L}}_z$ with $U_0 = \frac{4\pi\hbar^2 a_s N}{m}$
The above two equations describe the wave function of excited state and ground state with time evolution. Except G-P equation , there are three terms correspond to three processes of two-state system. In fact, no matter how these three processes run, the probability of excited state and ground state should be unchanged, so the spontaneous decay of excited state should occur in ground state. However, the emission light from spontaneous decay is random, meaning it's incoherent, so that we drop it out.

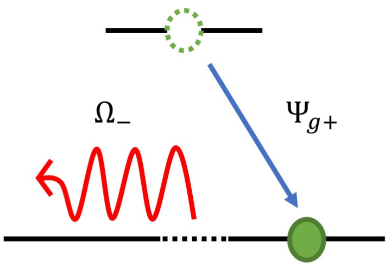


Figure 3: The demonstration of the direction of emission light. The plus sign of Ψ_{g+} and minus sign of Ω_- describes the direction due to momentum conservation.

Chapter 3

Method

3.1 BEC ground state

The final purpose is to see how the emission light is affected by BEC we produce, so the numerical solution of G-P equation is important. We use imaginary time method to obtain the ground state wave function. This method replaces original time information with imaginary time.

$$t \quad \rightarrow \quad -it \tag{8}$$

which could change the phase change of time evolution of wave function to exponential decay.

$$\Psi(t) = e^{\frac{-iEt}{\hbar}}\Psi(0) \quad \rightarrow \quad \Psi(t) = e^{\frac{-Et}{\hbar}}\Psi(0) \tag{9}$$

It means that higher energy state will decay faster than ground state, and because we normalize the wave function in each step, the wave function finally keeps higher proportion of ground state. By the way, we use normalized random distribution as out initial condition.

In iteration, we use Forward-time Central-space(FTCS) method to deal with space and time derivative, and due to imaginary time method and normalization, the algorithm is stable according to Von Neumann stability analysis. This stability analysis assumes that we could decompose the error of wave function as plane wave times some growing factor.

$$\phi_j^n = g^n e^{-ikj\delta x} \tag{10}$$

j is the number of sample point. n is the number of iteration step. k is an arbitrary wave number because the noise could be any frequency. This is a usual method to provide the restrictions of space and time numerical interval. It checks whether the numerical error on this step would become bigger or smaller according to the growing factor, g. If $|g| < 1$, the error of this step would converge to zero through iteration. So we often put it into finite differential equation to check the stability. Suppose Schroedinger equation reduce to

$$i\frac{\partial}{\partial t}\phi = \left(-\frac{\partial^2}{\partial x^2} + C\right)\phi \tag{11}$$

the growing factor is

$$g = 1 - \frac{4\delta t}{\delta x^2} sin^2(\frac{k\delta x}{2}) - C\delta t \tag{12}$$

where δt is time interval, δx is space interval, k is arbitrary wave vector for numerical error. All we have to do is to select the value of δt and δx to make sure $|g|$ is less than one. The condition to determine whether the numerical solution for ground state is done is to check whether the energy change is small enough.

3.2 Time evolution of BEC ground state

After the numerical solution of BEC ground state, it should be tested in real time evolution to check how long the probability could hold on. What's we are interested in time scale is according to the spontaneous decay rate, Γ, of excited particles. Suppose Γ is about 10^6Hz, then it's enough to see the process during 1μs. However, in real time evolution, FTCS is not a stable algorithm, besides, we can't normalize the wave function. If one checks growing factor, $|g|$, it always greater than 1. Here, we use Fourier transform (FT) for space derivative and Runge-Kutta in 4th order(RK4) for time derivative.

One advantage of FT is well known for the change from derivative operation in coordinate space to multiplication in momentum space, which could reduce instability resulting from finite difference method that always produces error. Suppose FT operator in x direction is F_x . Then we can rewrite differential equation.

$$\frac{\partial^2}{\partial x^2}\Psi = F_x^{-1}(-k_x^2 F_x(\Psi)) \tag{13}$$

We use this transformation to replace all of the space derivative in G-P equation, and combine it with RK4 for time derivative. That is, there are four steps in RK4, and each step has to do FT and inverse FT. The growing factor for G-P equation is

$$g = 1 + \frac{2}{3}(\delta t\lambda) + \frac{2}{3}(\delta t\lambda)^2 + \frac{1}{4}(\delta t\lambda)^3 + \frac{1}{24}(\delta t\lambda)^4 \tag{14}$$

where

$$\lambda = \frac{1}{i\hbar}\left(\frac{\hbar^2 k^2}{2m} + V(\mathbf{r}) + |\Psi|^2\right) \tag{15}$$

The condition to determine whether the algorithm is stable in real time evolution is to check whether the probability change is small enough.

3.3 Wave equation of emission

After checking the stability of algorithm, the interaction between particles and photons is added to the system. The numerical algorithm of light propagation of wave function have to keep the relation between spatial interval over time interval as light speed, otherwise, the numerical error would be diverge. However, it's difficult to see the process during 1μs because time interval is too small. If space interval is about 1μm, then time interval is about $3fs$. It's quite impractical to do the simulation. As the result, we claim that the steady state of light propagation is what we're interested so that the time derivative in wave equation could be neglected. Suppose the time interval is about $1ns$, comparing with $3fs$, it's like static process. The light $1ns$ before would not affect the emission light on next $1ns$.

$$\partial_z \Omega_\pm = i\eta \Psi_e \Psi_{g\mp}^* \tag{16}$$

Chapter 4

Result and Discussion

Due to the process for ^{135m}Cs is too long for numerical solution because of its long scattering length, about 2500 Bohr radius, causing large size of system, we demonstrate Rubidium, ^{87}Rb,first to make sure the numerical algorithm for whole process is reliable. The Table.1 are parameters of particles, ^{87}Rb, trapping potential, $V(\mathbf{r})$, and the relationship between dimensionless forms and SI units.

Rb^{87}	m	$1.44 \times 10^{-25}(kg)$
	a_s	$5.82(nm)$
$V(\mathbf{r})$	ω_x	$500(rad/s)$
	ω_y	$500(rad/s)$
	ω_z	$100(rad/s)$
unit length S	$S = \sqrt{\frac{\hbar}{2m\omega}}$	$0.85(\mu m)$
	dimensionless	SI unit
spatial coordinate	y	$x = Sy$
time coordinate	τ	$t = \tau/\omega_x$
wave function	ϕ	$\Psi = S^{-3/2}\phi$

Table 1: The parameter of ^{87}Rb, $V(\mathbf{r})$, and units

The unit in the following pictures would be dimensionless form according to Table.1. Our results contain two parts, the first one is the ground state of BEC, and the second one is the emission light from two-state system, which is formed by first one result.

4.1 BEC ground state

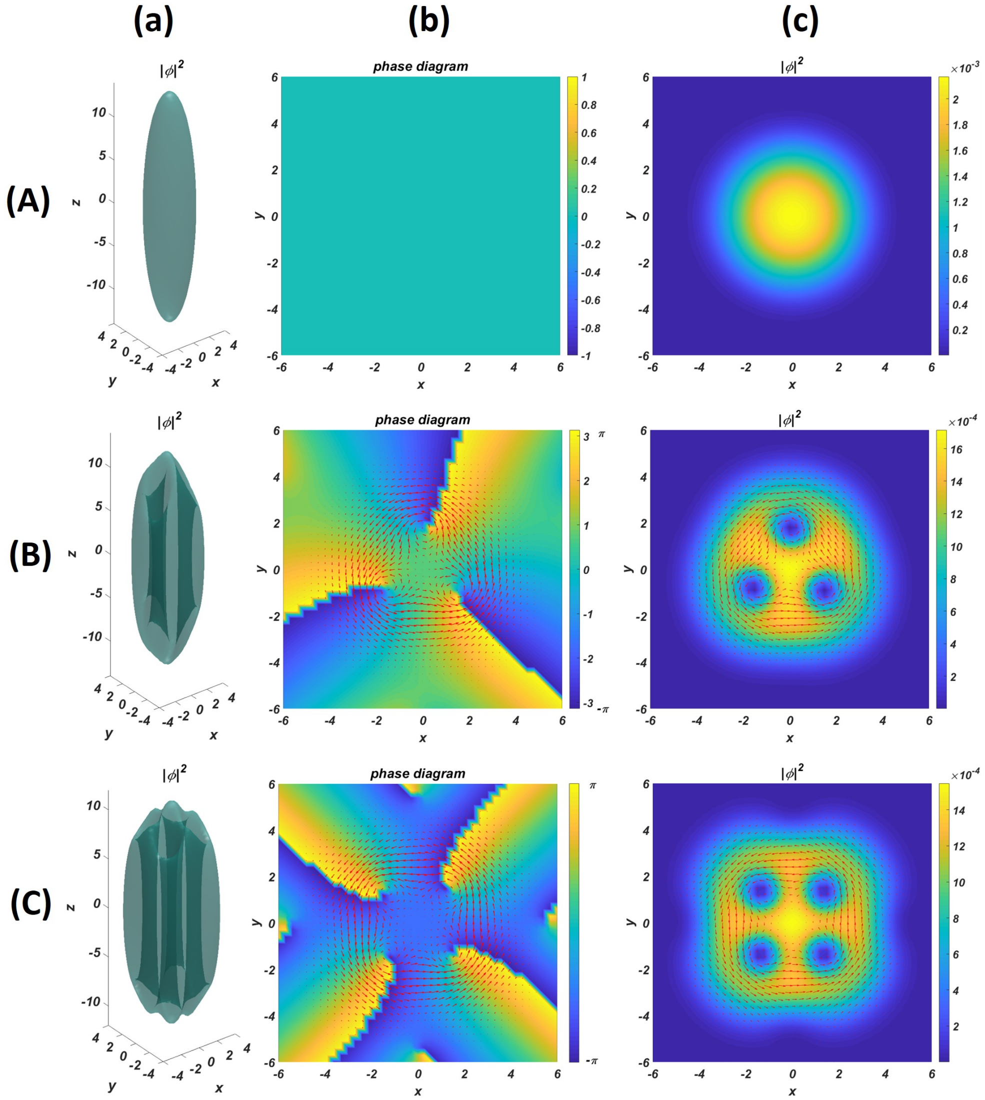


Figure 4: Particle density of 3D ground state from imaginary time method with particle number, $N = 10^4$. (a) is the isosurface of the particle density. (b) is the phase diagram at $z = 0$ plane. (c) is the particle density at $z = 0$ plane. The small red line in (b) and (c) is the vector field. Angular frequencies for (A), (B), and (C) is 0, 400, and 450 (rad/s). Trapping frequency 500(rad/s) is for x and y axis, and 100(rad/s) for z axis.

The Figure.4 shows the numerical solution of ground state with different rotary frequency. Through the phase diagram and vector field, the density of empty space inside the BEC is the vortex line. The higher the frequency, the more number of vortexes. Besides, to produce the same number of vortex, more particles would be easier to make it, that is, the rotary frequency is smaller. It'll cause rotary frequency difference if we produce state of BEC with one vortex in different particle number. The result would be the initial state for real time evolution.

4.2 Emission light

The particle number of excited state, $|e\rangle$, is $N_e = 25000$, and ground state, $|g\rangle$, is $N_g = 5000$. All the combinations of zero and one vortex separately in excited state and ground state would be demonstrated to see how the phase of BEC affect emitted light. So there are four combinations. Each combination contains the isosurface of particle density, the phase diagram, and the total probability of the emission, $|e\rangle$, and $|g\rangle$. The spontaneous decay rate, Γ, is assumed to be 1MHz, meaning that the process is considered during 1μs.

	Number of vortex	
	$\|e>$	$\|g>$
Case 1	0	0
Case 2	0	1
Case 3	1	0
Case 4	1	1

Case 1 : no vortex in $|e\rangle$, and no vortex in $|g\rangle$

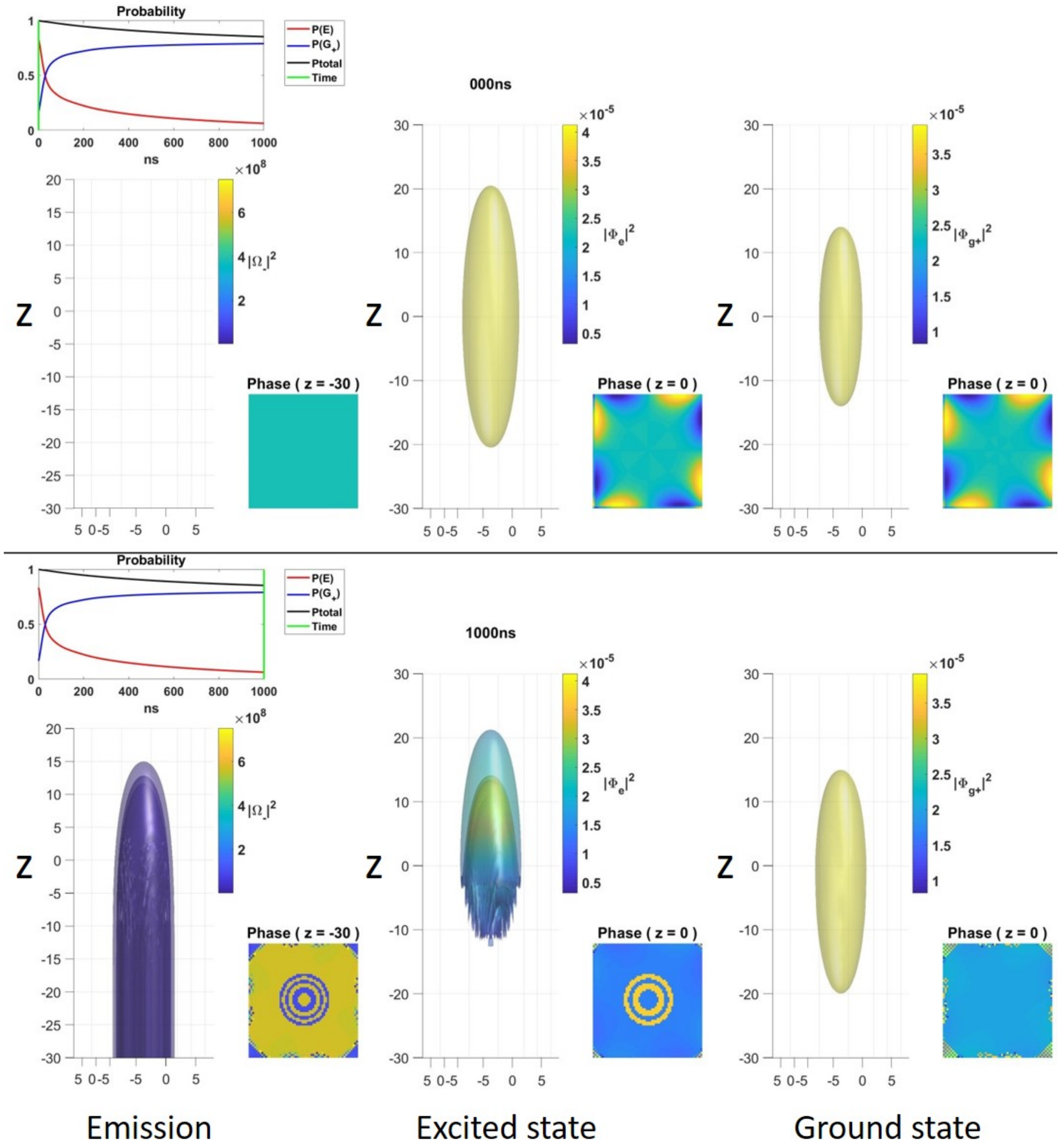


Figure 5: The left isosurface is the emission and the phase plane($z = -30$). The middle isosurface is $|e\rangle$ and the phase plane($z = 0$). The right isosurface is $|g\rangle$ and the phase plane($z = 0$). The upper half(above black line) is the initial state the system, and the lower half describes the wave function after 1μs. The small subfigure on upper left represents the total probability of $|e\rangle$ (red line), $|g\rangle$ (blue line), and their summation (black line).

Case 2 : no vortex in $|e\rangle$, and one vortex in $|g\rangle$

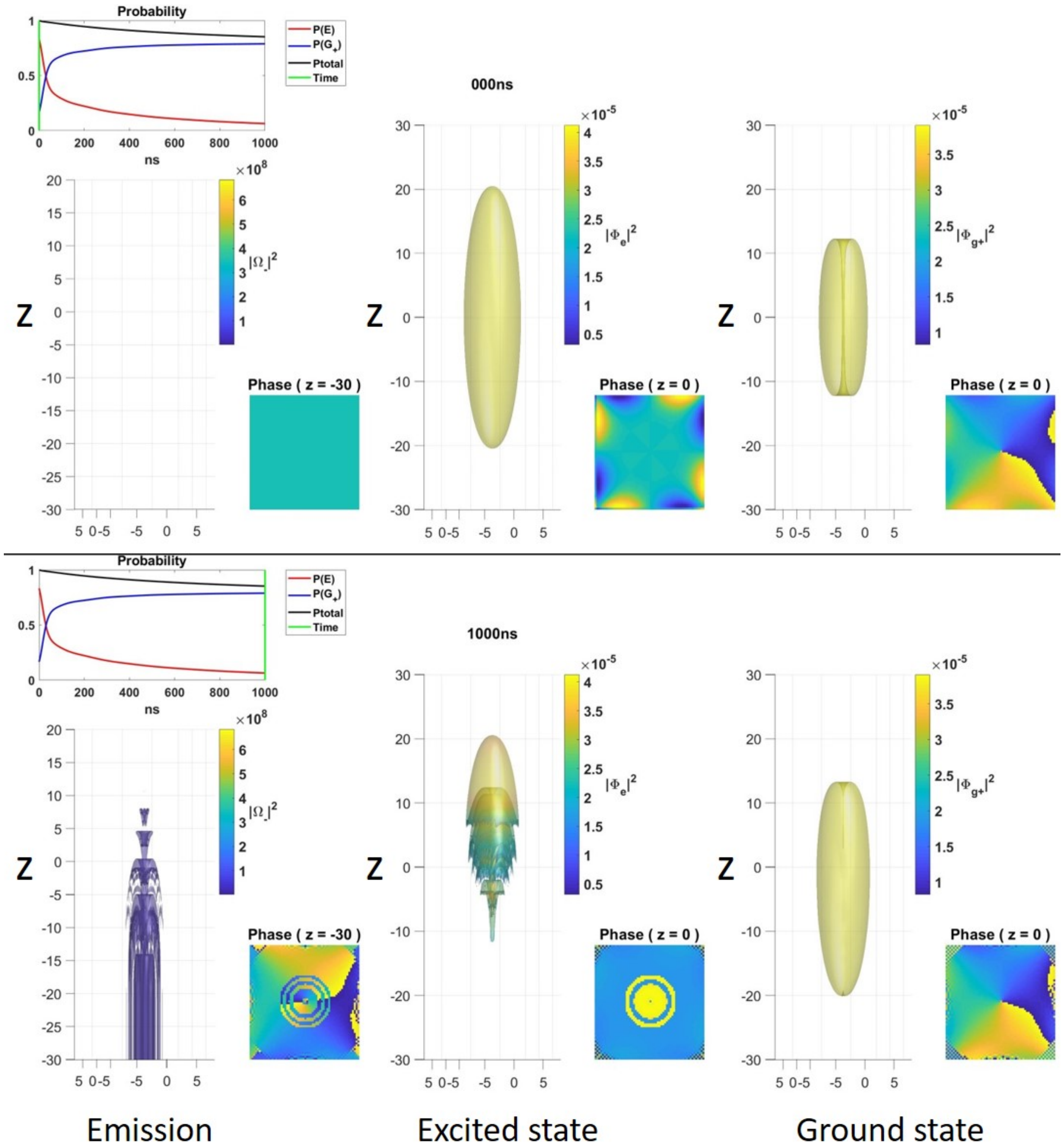


Figure 6: The left isosurface is the emission and the phase plane($z = -30$). The middle isosurface is $|e\rangle$ and the phase plane($z = 0$). The right isosurface is $|g\rangle$ and the phase plane($z = 0$). The upper half(above black line) is the initial state the system, and the lower half describes the wave function after 1μs. The small subfigure on upper left represents the total probability of $|e\rangle$ (red line), $|g\rangle$ (blue line), and their summation (black line).

Case 3 : One vortex in $|e\rangle$, and no vortex in $|g\rangle$

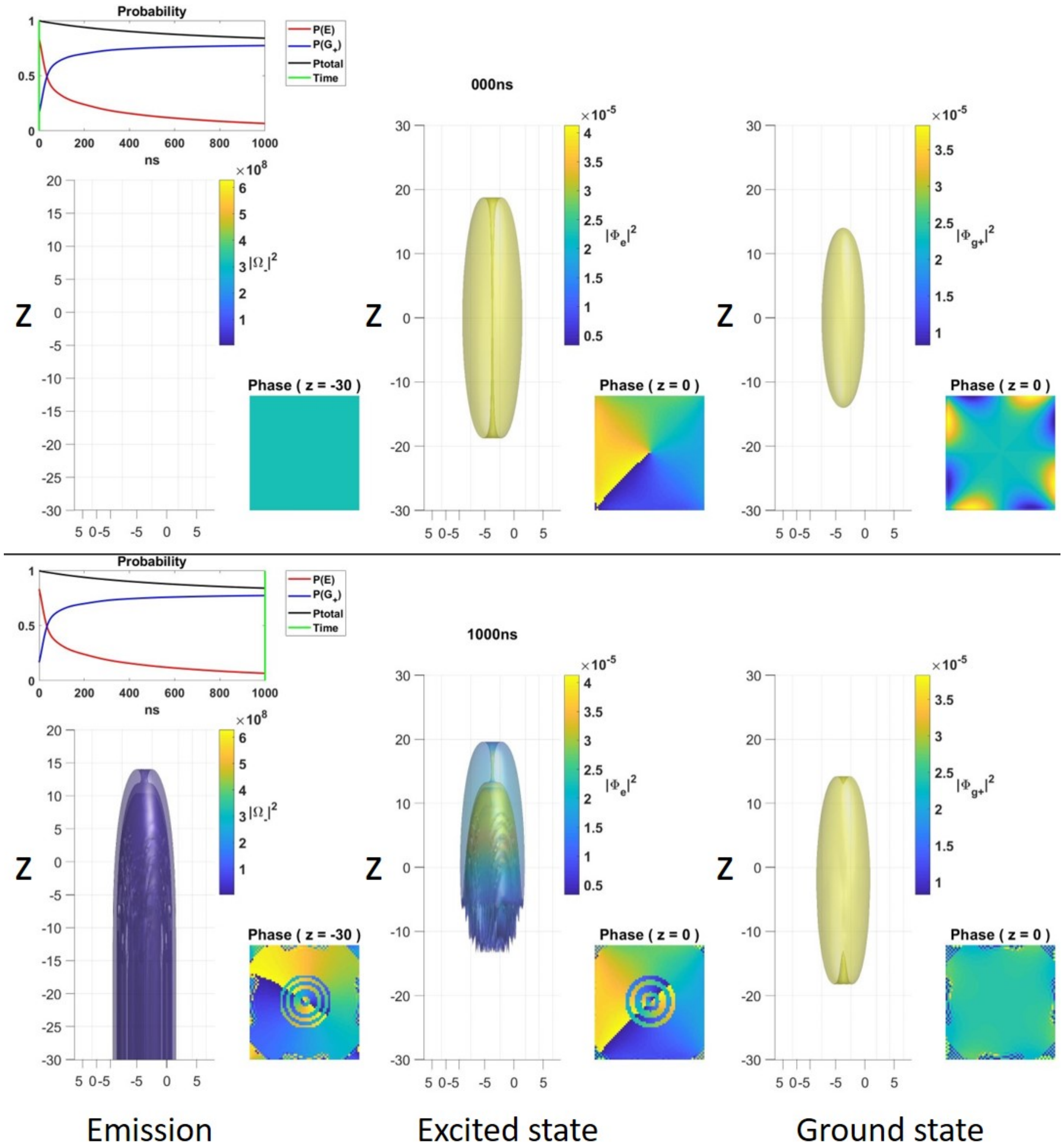


Figure 7: The left isosurface is the emission and the phase plane($z = -30$). The middle isosurface is $|e\rangle$ and the phase plane($z = 0$). The right isosurface is $|g\rangle$ and the phase plane($z = 0$). The upper half(above black line) is the initial state the system, and the lower half describes the wave function after 1μs. The small subfigure on upper left represents the total probability of $|e\rangle$ (red line), $|g\rangle$ (blue line), and their summation (black line).

Case 4 : One vortex in $|e\rangle$, and one vortex in $|g\rangle$

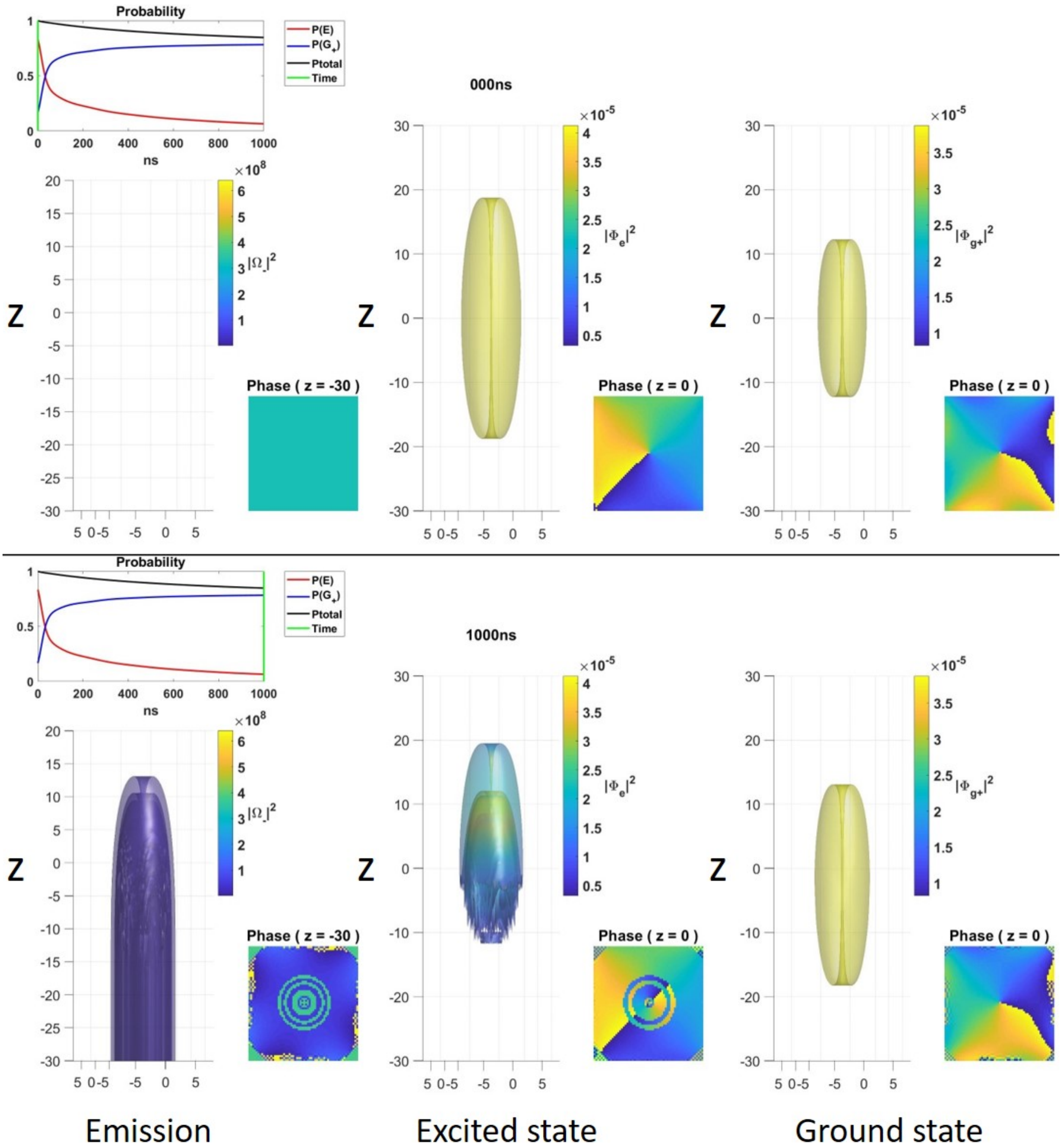


Figure 8: The left isosurface is the emission and the phase plane($z = -30$). The middle isosurface is $|e\rangle$ and the phase plane($z = 0$). The right isosurface is $|g\rangle$ and the phase plane($z = 0$). The upper half(above black line) is the initial state the system, and the lower half describes the wave function after 1μs. The small subfigure on upper left represents the total probability of $|e\rangle$ (red line), $|g\rangle$ (blue line), and their summation (black line).

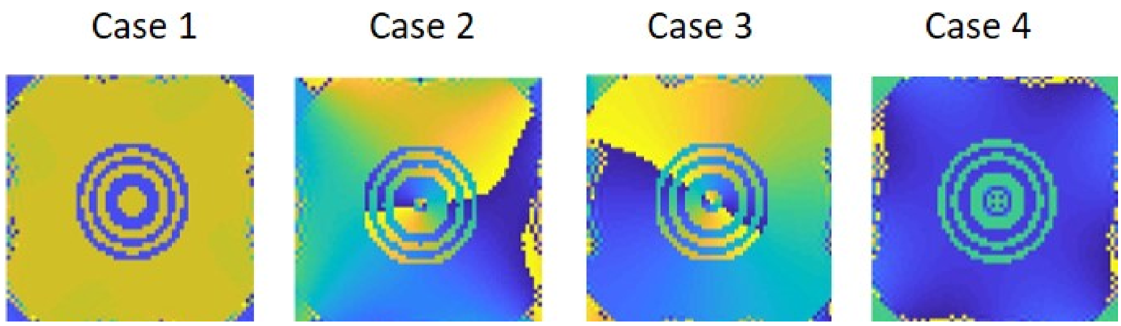


Figure 9: The phase diagram of emission from Fig. 5,6,7,8

As Fig.9 , the phase difference between $|e\rangle$ and $|g\rangle$ would produce the rotating phase into emission, as Case2 and Case3. One of $|e\rangle$ and $|g\rangle$ has one vortex line, and the other one doesn't. In view of angular momentum conservation, as one particle transits from the state with angular momentum to the state with no angular momentum, the difference of angular momentum would transfer to emission. However, if the transition is between two states with angular momentum, there is no difference so that the emission doesn't have angular momentum, as Case4. Through the phase diagram, we could also see that there are envelopes on the emission, which deserve further study on the dynamics of the system. Now on, we only focus on the phase of diagram.

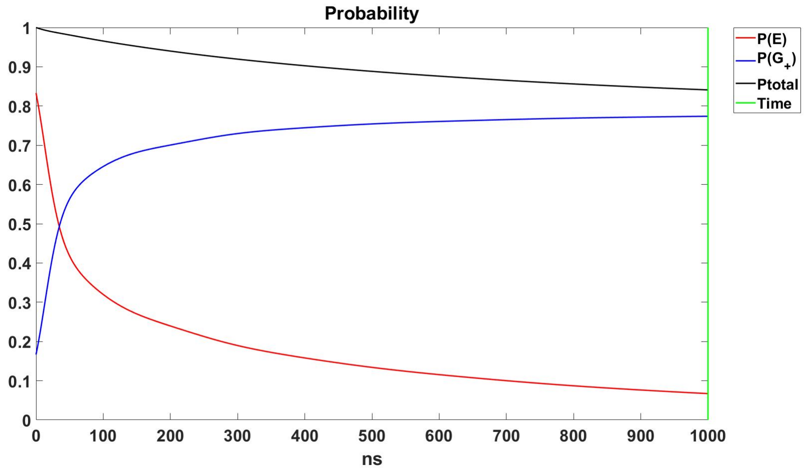


Figure 10: The probability of Case 3. Red line is the probability of the excited state. Blue line is the probability of the ground state. Black line is the total probability of two state.

We could also see that the effect of stimulated emission is greater than spontaneous decay through the figure of probability. If there is no stimulated emission, the total probability (black line) would decay to about 0.37 due to spontaneous emission. However, the probability is much higher than the value, meaning that the stimulated emission is dominant to cause particles coherent. Besides, the affluence of long axis in z direction of BEC ground state is also detected on the slope of $|e\rangle$ and $|g\rangle$ total probability. The longer the cigar-shape trapping potential, the steep the slope of probability change. The phenomenon is not related to the vortex of BEC, but the coherence of the particle composed of BEC.

Chapter 5

Conclusion

Through the modulation of ground state and excited state wave function, we could control the phase of emission light through the difference of angular momentum between two states. However, the dynamics of the emission and wave function deserves further study. There are several envelopes on the emission and wave function. Now on, we could make sure the whole numerical algorithm is reliable.

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