Approximating Sums of Powers of Binomial Coefficients

Ashutosh Srivastava

April 21, 2018

Situation: n players toss a fair coin k times. We shall consider the computating the probability of the event that all of them get the same number of heads form two different techniques

Method 1: The analytical technique

P(getting same number of heads) =

$$\sum_{i=0}^{k} \binom{k}{i}^{n} / 2^{nk}$$

Method 2: The normal approximation to binomial

Let X_m denote the number of heads for the mth player. As it is a binomial random variable with E[X_m] = k/2 and Var[X_m] = k/4. The normal approximation will be valid in our case if k/2 > 5 or k >10 so we take k > 10.Also as m is an integer the correction factor of 1/2 is needed.

P(X_m = m) $\approx$ P[(m-(k/2)-(1/2))/($\sqrt{k}$/2) $\leq$ Z $\leq$ (m-(k/2)+(1/2))/($\sqrt{k}$/2)]

= $(2/\sqrt{k})\ (1/\sqrt{2\pi})\ e^{-(1/2)(m-(k/2)-(1/2)/\sqrt{k}/2)^2}$

= $\sqrt{2/k\pi}\ e^{-(1/2)(m-(k/2)-(1/2)/\sqrt{k}/2)^2}$

P(getting the same number of heads) =

$$\sum_{i=0}^{k} \sqrt{2/k\pi}^{\,n}\, e^{-(n/2)(i-(k/2)-(1/2)/\sqrt{k}/2)^2}$$

$= (\sqrt{2/k\pi})^n \sum_{i=0}^{k} e^{-(i-(k-1)/2)/\sqrt{k/n})^2}$

Substituing j = i-(k-1)/2 in the aboe expression, we get

$= (\sqrt{2/k\pi})^n \sum_{j=-(k-1)/2}^{(k+1)/2} e^{-(j^2)/(k/n)}$

$= ((\sqrt{2/k\pi})^n \sum_{j=-(k-1)/2}^{(k-1)/2} e^{-(j^2)/(k/n)}) + ((\sqrt{2/k\pi})^n \sum_{j=(k-1)/2}^{(k+1)/2} e^{-(j^2)/(k/n)})$

Let θ(k) = $((\sqrt{2/k\pi})^n \sum_{j=(k-1)/2}^{(k+1)/2} e^{-(j^2)/(k/n)})$

$= ((\sqrt{2/k\pi})^n \, 2 \sum_{j=0}^{(k-1)/2} e^{-(j^2)/(k/n)}) + \theta$(k)

Now, as k gets larger θ(k) and the error due to the normal approximation of binomial probability both tends to zero first term and also the first term can be approximated by an integral of the exponential terms

$\approx ((\sqrt{2/k\pi})^n) \, 2\sqrt{\pi/(2 * 2n/k)}$

$= ((\sqrt{2/k\pi})^(n-1)) \, \sqrt{1/n}$

Finally, equating both sides

$\sum_{i=0}^{k} \binom{k}{i}^n / 2^{nk} \approx ((\sqrt{2/k\pi})^{n-1}) \, \sqrt{1/n}$

$$\boxed{\sum_{i=0}^{k} \binom{k}{i}^n \approx 2^{nk}/\sqrt{n} \, (\sqrt{2/k\pi})^{n-1} \text{ for k>10}}$$