Investigating relationship between sequences and their means

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Underlying thought

The measures of central tendencies are highly critical towards understanding the characteristics of data at hand. But it is quite an interesting question to think that why anything like mean would be this important. Is it because of the way it is calculated or is there something else which is lurking behind the camera? Now, our observation also shows us that whatever the nature of data the center of mass interpretation of the mean holds true. So, in a way if the data points are considered as point masses then mean represents their center of mass. This is a very specific property that is preserved across all data sets, if, mean is considered as a transformation of the data leading to a point value. This generates an interest in finding out similar transformations which preserve other properties of the data.

Keeping this thought in mind, I started off with a basic alternating series with power transformations applied to each term. The normalization factor which can preserve the mean value across all the positive integral powers is searched for. The general case is first conjectured by visualizing the behavior in **R** which is an open source statistical software and finally proved to arrive at the famous Bernoulli numbers which are also important in summations involving higher powers of natural numbers.

Keywords: Alternating Sequence, Faulhaber's Formulae, Bernoulli Numbers

1. INTRODUCTION

Considering the sequence $a_n = (-1)^n n$ and now taking the sequence $b_n = (\sum_{i=1}^{n} a_i)/n$, where n is any natural number. In order to avoid any confusion, first few terms of the sequence are given below:

$$a_n = -1, 2, -3, 4, -5, 6, -7 \ldots \ldots \ldots. \qquad \text{Eqn.1}$$

$$b_n = -1, \frac{1}{2}, -\frac{2}{3}, \frac{1}{2}, -\frac{3}{5}, \frac{1}{2}, -\frac{4}{7} \ldots \ldots \ldots \ldots \qquad \text{Eqn. 2}$$

As we can clearly see that in Eqn.2 every even term is ½ and odd terms are slowly tending towards -1/2 which can be seen as follows:

Taking k to be any positive natural number, following two cases exist -

Case 1: n = 2k for even n

$$b_{2k} = ((2 + 4 + 6 + 8 \ldots. +2k) - (1 + 3 + 5 + 7 \ldots \ldots.. +2k - 1))/2k$$

$b_{2k} = (k(k + 1) - k^2)/(2k) = 1/2$ (Using the sum of natural numbers formula)

Case 2: n = 2k – 1 for odd n

$$b_{2k-1} = ((2 + 4 + 6 + 8 \ldots. +2k - 2) - (1 + 3 + 5 + 7 \ldots \ldots.. +2k - 1))/(2k - 1)$$

$b_{2k-1} = \frac{k(k-1)-k^2}{2k-1} = -k/(2k - 1)$ (Using the sum of natural numbers formula)

$\lim_{k\to\infty} b_{2k-1} = -1/2$ (Using L'Hospital rule for evaluating the limit)

Clearly, as n becomes very large the two (even and odd) subsequences contained within the larger sequence converge to different values making the complete series as a whole, a divergent sequence but nevertheless there is a beautiful symmetric pattern that can be seen as sequences settle at higher values of n.

It becomes interesting to see the variation in the sequence if we make slight modifications to it which is discussed in the following section.

2. RAISING THE TERMS TO A POWER m: m > 0 and m ∈ I

The sequence b_n is modified such that

$$b_n(m) = (\sum_{i=1}^{n}(a_i)^m)\ /n \text{ , for odd m}$$

$$(\sum_{i=1}^{n}(-1)^i\ (a_i)^m/n, \text{ for even m}$$

where n is again any natural number

This ensures that irrespective of the power to which the terms are raised, alternate positive and negative signs of the terms are preserved. For better understanding of the sequence it is better to visualize the variation of the terms of the sequence for different values of m which is given below:

- x_ind denotes indices, the 'n' in our sequence we have considered
- Y-axis denotes the mean of the terms raised to respective powers mentioned in the y-label
- First Y value is plotted for x_ind=1, it looks so close to zero because of the overall scale of the graph

For m=1, the standard case:-

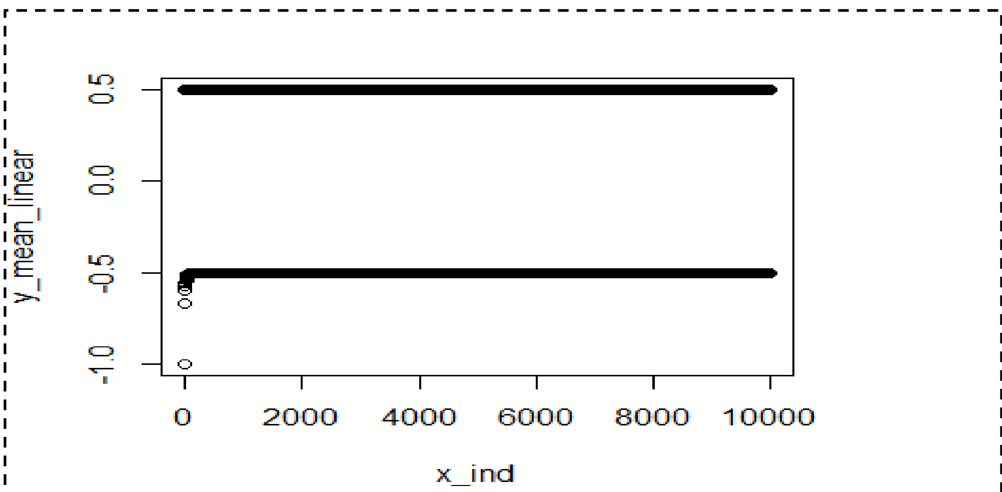

FIG. 1

For m=2, when the terms are squared:-

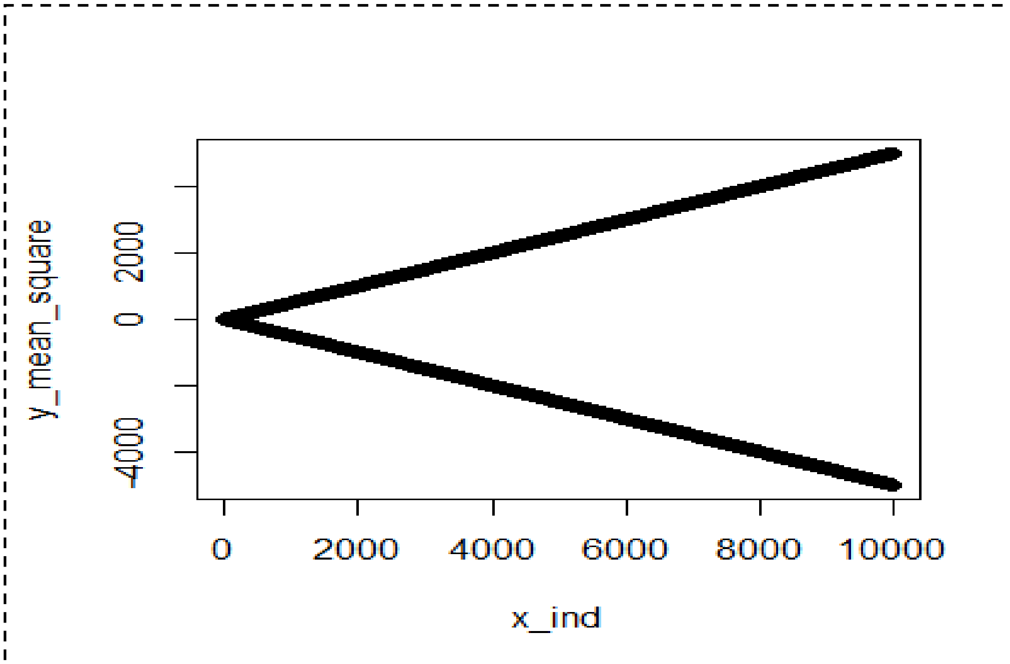

FIG. 2

For m=3, when the terms are cubed:-

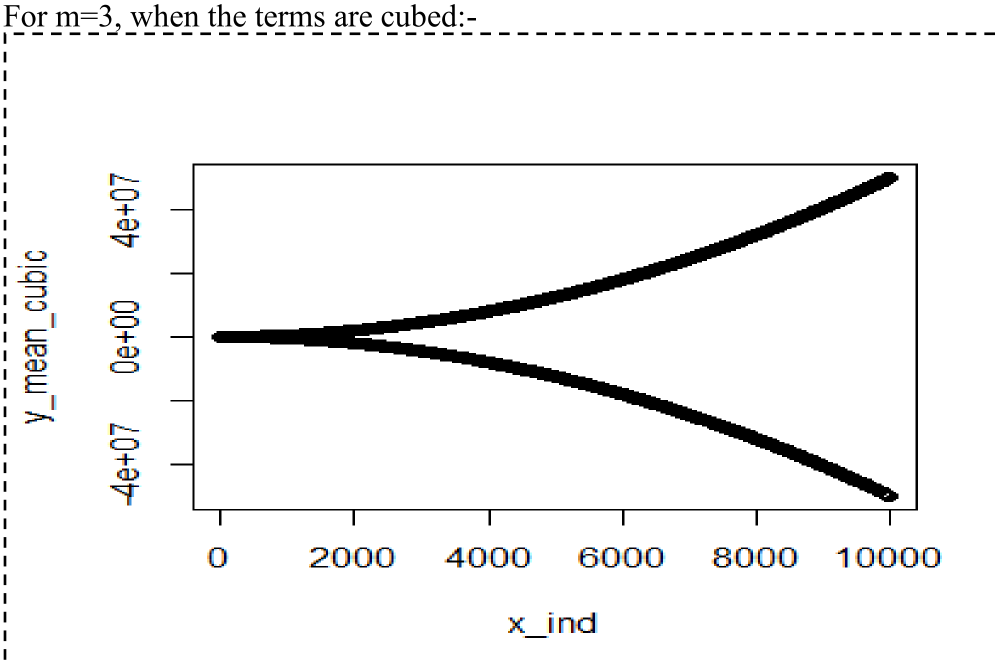

FIG. 3

For m=4, when the terms are raised to fourth power:

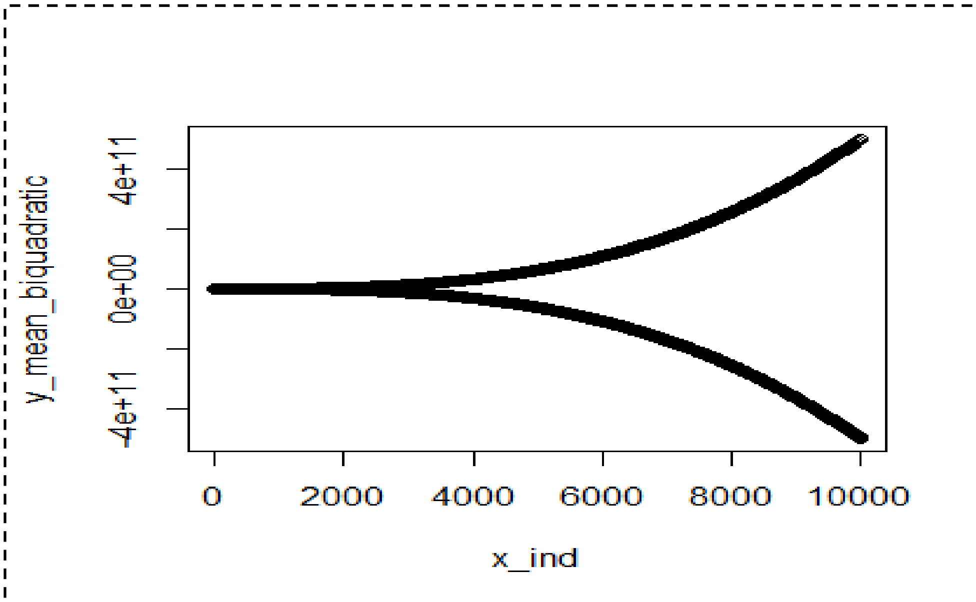

FIG. 4

As we can see from the above graphs, as the value of n increases, the series symmetrically diverges off to infinity from both sides i.e. when 'n' is even and also when 'n' is odd. As the terms of the sequence was raised to different powers still it maintained its symmetrical behavior at larger values of n but no longer remained bounded. As an exploratory exercise I tried to normalize the sequence $\boldsymbol{b_n}$ using $\boldsymbol{n^m}$ instead of just 'n' i.e. defining $\boldsymbol{b_n}$ as:

$$b_n(m) = (\textstyle\sum_{i=1}^{n}(a_i)^m) / n^m \text{ , for odd m}$$

$$(\textstyle\sum_{i=1}^{n}(-1)^i \, (a_i)^m / n^m \text{, for even m}$$

where n is any natural number

- x_ind denotes the indices
- Y-axis denotes values which are normalized by this new factor

In order to understand the variation in the terms of this new sequence, it is essential to visualize its evolution as 'n' increases in the same manner i.e. for m=1, 2, 3 and 4. It turns out to be quite strange and beautiful at the same time.

For m=1, the standard case:

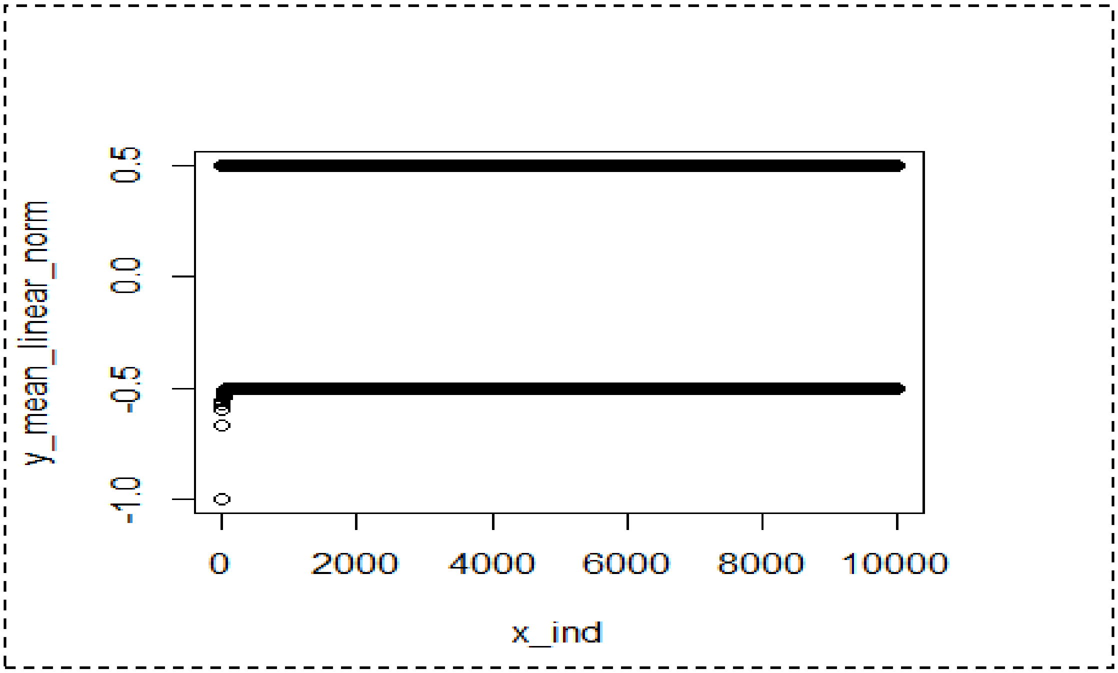

FIG. 5

For m=2, the squared case:

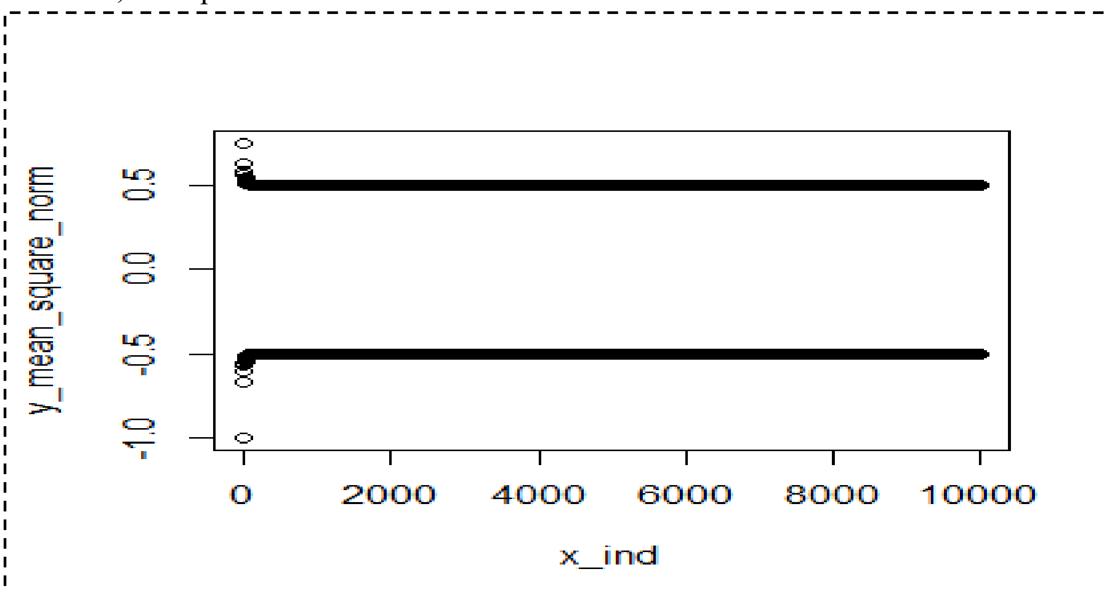

FIG. 6

For m=3, the cubic case:

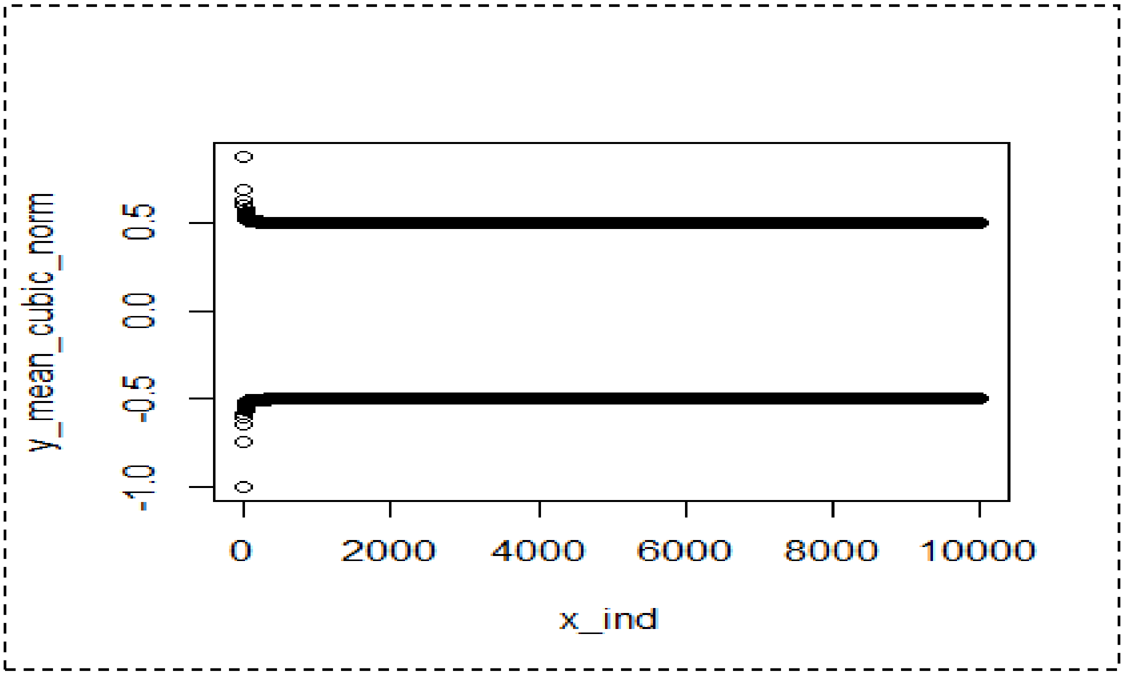

FIG. 7

For m=4, when the terms are raised to the fourth power

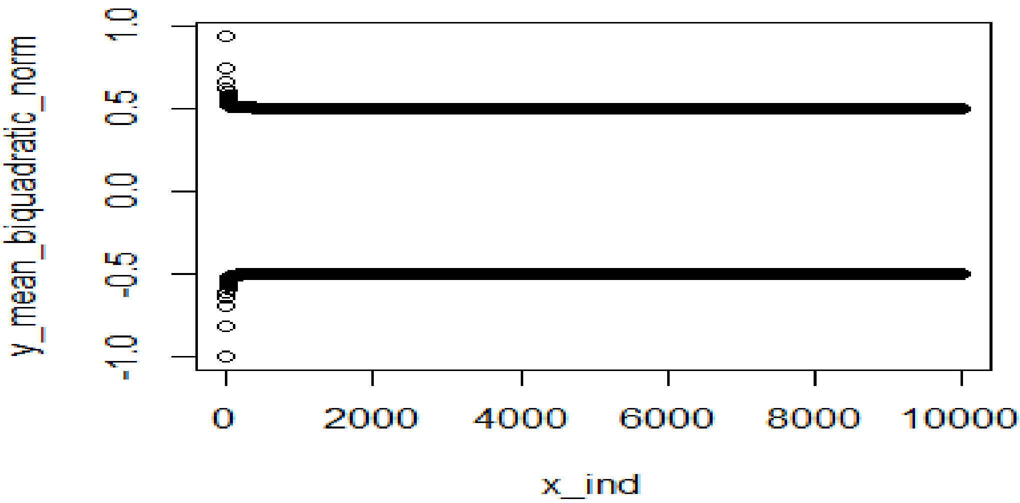

FIG. 8

Clearly, as we see for this new sequence created by normalizing it with the power factor of $\boldsymbol{n^m}$ for different **m** the sequence retains its symmetric and as well as bounded nature and moreover it converges to +1/2 and -1/2 respectively which was previously only true for the case m=1.

This motivates to search for a proof of the above result which can be stated in the form of a theorem which preserves the native steady state and as well as the symmetry, so this is called the preservation theorem and $\boldsymbol{n^m}$ is the preservation normalization factor.

Preservation Theorem: For a sequence $b_n(m)$ defined as:

$$b_n(m) = (\sum_{i=1}^{n}(a_i)^m)\,/n^m \text{ , for odd m}$$

$$(\sum_{i=1}^{n}(-1)^i\,(a_i)^m/n^m\text{, for even m}$$

where n is any natural number

$\lim_{k\to\infty} b_{f(k)} =$ ½ for n = 2k , here n = f(k) is defined as stated in the condition

-1/2 for n = 2k-1

Proof: As the terms of the sequence occur in alternate in positive and negative signs independent of the nature of 'm' whether even or odd, let us consider the following with same $\boldsymbol{b_n(m)}$

Case 1: n = 2k

b_{2k} = $((2^m + 4^m + 6^m \ldots\ldots.. (2k)^m) - (1^m + 3^m + 5^m \ldots\ldots.. (2k-1)^m))$ / $(2k)^m$

Taking 2^mcommon from the set of even terms from the above equation we get

b_{2k} = $(2^m(1^m + 2^m + 3^m \ldots\ldots.. k^m) - (1^m + 3^m + 5^m \ldots\ldots.. (2k-1)^m))$ / $(2k)^m$

These summations can be found out by using Faulhaber's formula for sum of natural numbers which can be written as:

$$\sum_{i=1}^{k} i^n = (\sum_{j=0}^{n}(-1)^j \binom{n+1}{j} B_j k^{n+1-j})/(n+1)$$

where B_j denotes Bernoulli Numbers

Now, calculating the summation for the odd terms we get

$$S = (1^m + 3^m + 5^m \ldots\ldots.. (2k-1)^m,$$

adding all the even terms both sides we have

S +$(2^m(1^m + 2^m + 3^m \ldots\ldots.. k^m)) = \sum_{i=1}^{2k} i^m$

S =$\sum_{i=1}^{2k} i^m - (2^m(1^m + 2^m + 3^m \ldots\ldots.. k^m))$

S = $(\sum_{j=0}^{m}(-1)^j \binom{m+1}{j} B_j (2k)^{m+1-j}$ - $(2^m(\sum_{j=0}^{m}(-1)^j \binom{m+1}{j} B_j k^{m+1-j})))/(m+1)$

Denoting $\sum_{j=0}^{m}(-1)^j \binom{m+1}{j} B_j (2k)^{m+1-j} = F_2$ and $\sum_{j=0}^{m}(-1)^j \binom{m+1}{j} B_j k^{m+1-j} = F_1$

Substituting all of this in b_{2k} we have:

b_{2k} = $(2^{m+1}F_1$- $F_2)/((m+1)(2k)^m)$

From the expressions of F_1 and F_2 it is clear that after j=1 the denominator will have a higher power of k as compared to that of numerator, so evaluating the first two terms we get (for j=0 and j=1)

j=0

$F_1 = B_0 k^{m+1}$
$F_2 = B_0 (2k)^{m+1}$

Substituting these values in b_{2k} we get

$b_{2k} = 2^{m+1}B_0k^{m+1}$ - $B_0(2k)^{m+1}$ = 0

j=1

$F_1 = -(m+1)B_1k^m$
$F_2 = -(m+1)B_1(2k)^m$

Substituting these values in b_{2k} we get

$b_{2k} = -(m+1)2^{m+1}B_1k^m + (m+1)B_1(2k)^m/((m+1)(2k)^m)$

$b_{2k} = -(B_1)(2k)^m/(2k)^m = -B_1$

Now, observing the behavior of b_{2k} as k becomes very large we get, as all other terms will become 0

$$\lim_{k\to\infty} b_{2k} = -B_1$$

Where B_1 is a Bernoulli number which has a value of -1/2 and which ultimately leads to

$$\lim_{k\to\infty} b_{2k} = 1/2$$

Similarly, this can be shown for cases when n = 2k - 1

Hence, proved.

3. SUMMARY AND RESULTS

- This normalization factor has a property of preserving the original characteristics of the sequence at extreme values
- Bernoulli numbers can be seen through a different perspective i.e. as the limit of convergence of an alternating sequence
- Apart from this preservation property, this normalization factor also generates the classic sequence $a_i = (-1)^i$, which can be shown as follows :

As for odd m,

$$b_n(m) = (\sum_{i=1}^{n}(a_i)^m)\,/n^m$$

Taking $\lim\limits_{m\to\infty}$ both sides we get

$$\lim_{m\to\infty} b_n(m) = -1$$

(because the last term in this case will be negative and all other terms shall reduce to zero)

For even m,

$$b_n(m) = (\sum_{i=1}^{n}(-1)^i(a_i)^m)\,/n^m$$

Taking $\lim_{m\to\infty}$ both sides we get

$$\lim_{m\to\infty} b_n(m) = 1$$

(because the last term in this case will be positive and all other terms shall reduce to zero)

APPENDIX

R Code for producing the divergent series plots

```
n <- 10000

x_ind <- seq(1:n)


# plotting the single power average

mean_seq_1 <- sapply(x_ind,function(x) if (x%%2!=0){-x} else {x})

y_mean_linear <- vector('numeric',n)

for (i in x_ind)

{

  y_mean_linear[i] <- sum(mean_seq_1[1:i])/(abs(i))

}

z_linear <- data.frame(x_ind,y_mean_linear)

plot(x_ind,y_mean_linear)
```

```
}

# plotting the square power average
mean_seq_2 <- sapply(x_ind,function(x) if (x%%2!=0){-(x^2)} else {x^2})
y_mean_square <- vector('numeric',n)
for (i in x_ind)
{
  y_mean_square[i] <- sum(mean_seq_2[1:i])/(abs(i))
}
z_square <- data.frame(x_ind,y_mean_square)
plot(x_ind,y_mean_square)

# plotting the cubic power average
mean_seq_3 <- sapply(x_ind,function(x) if (x%%2!=0){-(x^3)} else {x^3})
y_mean_cubic <- vector('numeric',n)
for (i in x_ind)
{
  y_mean_cubic[i] <- sum(mean_seq_3[1:i])/(abs(i))
}
z_cubic <- data.frame(x_ind,y_mean_cubic)
plot(x_ind,y_mean_cubic)

# plotting the biquadratic average
# plotting the cubic power average
mean_seq_4 <- sapply(x_ind,function(x) if (x%%2!=0){-(x^4)} else {x^4})
y_mean_biquadratic <- vector('numeric',n)
for (i in x_ind)
{
  y_mean_biquadratic[i] <- sum(mean_seq_4[1:i])/(abs(i))
}
```

```
z_biquadratic <- data.frame(x_ind,y_mean_biquadratic)

plot(x_ind,y_mean_biquadratic)
```

R Code for producing the normalized converging plots

```
n <- 10000

x_ind <- seq(1:n)


# plotting the single power average

mean_seq_1 <- sapply(x_ind,function(x) if (x%%2!=0){-x} else {x})

y_mean_linear_norm <- vector('numeric',n)

for (i in x_ind)

{

  y_mean_linear_norm[i] <- sum(mean_seq_1[1:i])/(abs(i))

}

z_linear <- data.frame(x_ind,y_mean_linear_norm)

plot(x_ind,y_mean_linear_norm)


# plotting the square power average

mean_seq_2 <- sapply(x_ind,function(x) if (x%%2!=0){-(x^2)} else {x^2})

y_mean_square_norm <- vector('numeric',n)

for (i in x_ind)

{

  y_mean_square_norm[i] <- sum(mean_seq_2[1:i])/(abs(i)^2)

}

z_square <- data.frame(x_ind,y_mean_square_norm)

plot(x_ind,y_mean_square_norm)


# plotting the cubic power average
```

```
mean_seq_3 <- sapply(x_ind,function(x) if (x%%2!=0){-(x^3)} else {x^3})

y_mean_cubic_norm <- vector('numeric',n)

for (i in x_ind)

{

  y_mean_cubic_norm[i] <- sum(mean_seq_3[1:i])/(abs(i)^3)

}

z_cubic <- data.frame(x_ind,y_mean_cubic_norm)

plot(x_ind,y_mean_cubic_norm)


# plotting the biquadratic average

# plotting the cubic power average

mean_seq_4 <- sapply(x_ind,function(x) if (x%%2!=0){-(x^4)} else {x^4})

y_mean_biquadratic_norm <- vector('numeric',n)

for (i in x_ind)

{

  y_mean_biquadratic_norm[i] <- sum(mean_seq_4[1:i])/(abs(i)^4)

}

z_biquadratic <- data.frame(x_ind,y_mean_biquadratic_norm)

plot(x_ind,y_mean_biquadratic_norm)
```