CS7301: Advanced Topics in Optimization for Machine Learning

Lecture 2.1: Convex Functions Continued

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January 29, 2020

Outline

- Covered Last time: Convex Sets and Definitions of Convex Functions
- This Lecture:
  - Properties of Convex Functions
  - Subgradients
  - First order and Second order Convexity
  - Local and Global Minima

Properties of Convex Functions

- **Non-negative weighted sum:** $f = \sum_{i=1}^{n} \alpha_i f_i$ is convex if each f_i for $1 \leq i \leq n$ is convex and $\alpha_i \geq 0, 1 \leq i \leq n$.

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  - Any norm of an affine function, $f(x) = ||Ax + b||$, is convex.

Composition with Scalar Functions

- Composition of $g : \mathbb{R}^n \to \mathbb{R}$ and $h : \mathbb{R} \to \mathbb{R}$.

$$f(x) = h(g(x))$$

- f is convex if a) g convex, h convex and non-decreasing or b) g concave, h convex and non-increasing
- Proof idea: Take double derivative and try to show that $\nabla^2 f \geq 0$ (easier to prove this for $m = 1$).
- Examples:
  - $f(x) = \exp(f(x))$ is convex if f is convex
  - $1/g(x)$ is convex if g is concave.

Composition with Vector Functions

- Composition of $g : \mathbb{R}^n \to \mathbb{R}^k$ and $h : \mathbb{R}^k \to \mathbb{R}$.

$$f(x) = h(g(x)) = h(g_1(x), \cdots, g_k(x))$$

- f is convex if a) g_i's convex, h convex and non-decreasing in each argument or b) g_i concave, h convex and non-increasing in each argument
- Examples:
  - $f(x) = \sum_i \log(g(x))$ is concave if g is concave and positive
  - $\log \sum_{i=1}^k \exp(g_i(x)$ is convex if g_i is convex.

Pointwise Maximums and Supremums

Following functions are convex, but may not be differentiable everywhere.

- **Pointwise maximum:** If $f_1, f_2, \ldots, f_m$ are convex, then $f(\mathrm{x}) = max\{f_1(\mathrm{x}), f_2(\mathrm{x}), \ldots, f_m(\mathrm{x})\}$ is also convex. For example:

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  - Sum of r largest components of $\mathrm{x} \in \Re^n$ $f(\mathrm{x}) = x_{[1]} + x_{[2]} + \ldots + x_{[r]}$, where $x_{[1]}$ is the i^{th} largest component of x, is a convex function.

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- **Pointwise supremum:** If $f(\mathrm{x}, \mathrm{y})$ is convex in x for every $\mathrm{y} \in \mathcal{S}$, then $g(\mathrm{x}) = \sup\limits_{\mathrm{y} \in \mathcal{S}} f(\mathrm{x}, \mathrm{y})$ is convex. For example:

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- **Pointwise supremum:** If $f(\mathrm{x}, \mathrm{y})$ is convex in x for every $\mathrm{y} \in \mathcal{S}$, then $g(\mathrm{x}) = \sup_{\mathrm{y} \in \mathcal{S}} f(\mathrm{x}, \mathrm{y})$ is convex. For example:
  - The function that returns the maximum eigenvalue of a symmetric matrix X, *viz.*, $\lambda_{max}(X) = \sup_{\mathrm{y} \in \mathcal{S}} \frac{\|X\mathrm{y}\|_2}{\|\mathrm{y}\|_2}$ is a convex function of the symmetrix matrix X.

Which of the Following Loss Functions are Convex?

- L1/L2 Reg Logistic Regression: $L(\theta) = \sum_{i=1}^n \log(1 + \exp(-y_i \theta^T x_i)) + \lambda \|\theta\|$
- L1/L2 Reg SVMs: $L(\theta) = \sum_{i=1}^n \max\{0, 1 - y_i \theta^T x_i\} + \lambda \|\theta\|$
- L1/L2 Reg Multi-class Logistic Regression: $L(\theta_1, \cdots, \theta_k) = \sum_{i=1}^n -\theta_{y_i}^T x_i + \log(\sum_{c=1}^k \exp(\theta_c^T x_i))\} + \sum_{i=1}^c \lambda \sum_{j=1}^m \|\theta_j\|$
- L1/L2 Reg Least Squares (Lasso): $L(\theta) = \sum_{i=1}^n (\theta^T x_i - y_i)^2 + \lambda \|\theta\|$
- Matrix Completion: $L(X) = \sum_{i=1}^n ||y_i - A_i(X)||_2^2 + ||X||_*$
- Soft-Max Contextual Bandits: $L(\theta) = \sum_{i=1}^n \frac{r_i}{p_i} \frac{\exp(\theta^T x_i^{a_i})}{\sum_{j=1}^k \exp(\theta^T x_i^j)} + \lambda \|\theta\|$

The Direction Vector

- Consider a function $f(\mathrm{x})$, with $\mathrm{x} \in \Re^n$.
- We start with the concept of the direction at a point $\mathrm{x} \in \Re^n$.
- We will represent a vector by x and the k^{th} component of x by x_k.
- Let u^k be a unit vector pointing along the k^{th} coordinate axis in $\Re^n$;
- $u_k^k = 1$ and $u_j^k = 0, \ \forall j \neq k$
- An arbitrary direction vector v at x is a vector in $\Re^n$ with unit norm (*i.e.*, $||\mathrm{v}|| = 1$) and component v_k in the direction of u^k.

Directional derivative and the gradient vector

Let $f : \mathcal{D} \rightarrow \Re$, $\mathcal{D} \subseteq \Re^n$ be a function.

Definition

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$$D_{\mathrm{v}} f(\mathrm{x}) = \lim_{h \rightarrow 0} \frac{f(\mathrm{x} + h\mathrm{v}) - f(\mathrm{x})}{h} \tag{1}$$

provided the limit exists.

Directional Derivative

As a special case, when $\mathrm{v} = \mathrm{u}^k$ the directional derivative reduces to the partial derivative of f with respect to x_k.

$$D_{\mathrm{u}^k} f(\mathrm{x}) = \frac{\partial f(\mathrm{x})}{\partial x_k}$$

If $f(\mathrm{x})$ is a differentiable function of $\mathrm{x} \in \Re^n$, then f has a directional derivative in the direction of any unit vector v, and

$$D_{\mathrm{v}} f(\mathrm{x}) = \sum_{k=1}^{n} \frac{\partial f(\mathrm{x})}{\partial x_k} v_k = \nabla f^T v \tag{2}$$

Sublevel Sets of Convex Functions

- Lets define *sub-level sets* of a convex function as follows:

Definition

[Sublevel Sets]: Let $\mathcal{D} \subseteq \Re^n$ be a nonempty set and $f : \mathcal{D} \to \Re$. The set

$$L_\alpha(f) = \{x | x \in \mathcal{D},\ f(x) \leq \alpha\}$$

is called the $\alpha-$sub-level set of f.

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Now if a function f is convex, its $\alpha-$sub-level set is a convex set.

Convex Function $\Rightarrow$ Convex Sub-level sets

Theorem

Let $\mathcal{D} \subseteq \Re^n$ be a nonempty convex set, and $f : \mathcal{D} \to \Re$ be a convex function. Then $L_\alpha(f)$ is a convex set for any $\alpha \in \Re$.

Proof: Consider $x_1, x_2 \in L_\alpha(f)$. Then by definition of the level set, $x_1, x_2 \in \mathcal{D}$, $f(x_1) \leq \alpha$ and $f(x_2) \leq \alpha$. From convexity of $\mathcal{D}$ it follows that for all $\theta \in (0, 1)$, $x = \theta x_1 + (1 - \theta)x_2 \in \mathcal{D}$. Moreover, since f is also convex,

$$f(x) \leq \theta f(x_1) + (1 - \theta)f(x_2) \leq \theta\alpha + (1 - \theta)\alpha = \alpha$$

which implies that $x \in L_\alpha(f)$. Thus, $L_\alpha(f)$ is a convex set. $\square$

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The converse of this theorem does not hold. To illustrate this, consider the function $f(x) = \frac{x_2}{1+2x_1^2}$. The 0-sublevel set of this function is $\{(x_1, x_2) \mid x_2 \leq 0\}$, which is convex. However, the function $f(x)$ itself is not convex.

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A function is called quasi-convex if all its sub-level sets are convex

Convex Sub-level sets $\implies$ Convex Function

A function is called quasi-convex if all its sub-level sets are convex sets. Every quasi-convex function is not convex!

Consider the Negative of the normal distribution $-\frac{1}{\sigma\sqrt{2\pi}} exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$.

This function is quasi-convex but not convex.

Consider the simpler function $f(x) = -exp(-(x-\mu)^2)$.

- Then $f'(x) = 2(x-\mu)exp(-(x-\mu)^2)$
- And $f''(x) = 2exp(-(x-\mu)^2) - 4(x-\mu)^2 exp(-(x-\mu)^2) = (2 - 4(x-\mu)^2)exp(-(x-\mu)^2)$ which is < 0 if $(x-\mu)^2 > \frac{1}{2}$,
- Thus, the second derivative is negative if $x > \mu + \frac{1}{\sqrt{2}}$ or $x < -\mu - \frac{1}{\sqrt{2}}$.
- Recall from discussion of convexity of $f : \Re \to \Re$ if the derivative is not non-decreasing everywhere $\implies$ function is not convex everywhere.

To prove that this function is quasi-convex, we can ....

Proof that the function is Quasi-Convex

1. Inspect the $L_\alpha(f)$ sublevel sets of this function: $L_\alpha(f) = \{x| - exp(-(x-\mu)^2) \leq \alpha\} = \{x|exp(-(x-\mu)^2) \geq -\alpha\}$.
2. Since $exp(-(x-\mu)^2)$ is monotonically increasing for $x < \mu$ and monotonically decreasing for $x > \mu$, the set $\{x|exp(-(x-\mu)^2) \geq -\alpha\}$ will be a contiguous closed interval around μ and therefore a convex set.
3. Thus, $f(x) = -exp(-(x-\mu)^2)$ is quasi-convex (and so is its generalization - the negative of the normal density function).

- One can similarly prove that the negative of the multivariate normal density function is also quasi-convex, by inspecting its sub-level sets, which are nothing but **ellipsoids**.

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Convex Functions and Their Epigraphs

Let us further the connection between convex functions and sets by introducing the concept of the *epigraph* of a function.

Definition

[Epigraph]: Let $\mathcal{D} \subseteq \Re^n$ be a nonempty set and $f : \mathcal{D} \to \Re$. The set $\{(\mathrm{x}, f(\mathrm{x})|\mathrm{x} \in \mathcal{D}\}$ is called graph of f and lies in $\Re^{n+1}$. The epigraph of f is a subset of $\Re^{n+1}$ and is defined as

$$epi(f) = \{(\mathrm{x}, \alpha)|f(\mathrm{x}) \leq \alpha,\ \mathrm{x} \in \mathcal{D},\ \alpha \in \Re\} \tag{3}$$

In some sense, the epigraph is the set of points lying above the graph of f.

Eg: Recall affine functions of vectors: $\mathrm{a}^T\mathrm{x} + b$ where $\mathrm{a} \in \Re^n$. Its epigraph is $\{(\mathrm{x}, t)|\mathrm{a}^T\mathrm{x} + b \leq t\} \subseteq \Re^{n+1}$ which is a half-space (a convex set).

Convex Functions and Their Epigraphs (contd)

There is a one to one correspondence between the convexity of function f and that of the set $epi(f)$, as stated in the following result.

Theorem

Let $\mathcal{D} \subseteq \Re^n$ be a nonempty convex set, and $f : \mathcal{D} \to \Re$. Then

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Let $\mathcal{D} \subseteq \Re^n$ be a nonempty convex set, and $f : \mathcal{D} \to \Re$. Then f is convex if and only if $epi(f)$ is a convex set.

Proof: f **convex function** $\implies$ $epi(f)$ **convex set**

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Proof: f **convex function** $\implies$ $epi(f)$ **convex set**

Let f be convex. For any $(x_1, \alpha_1) \in epi(f)$ and $(x_2, \alpha_2) \in epi(f)$ and any $\theta \in (0, 1)$,

$$f(\theta x_1 + (1-\theta)x_2) \leq \theta f(x_1) + (1-\theta)f(x_2)) \leq \theta\alpha_1 + (1-\theta)\alpha_2$$

Since $\mathcal{D}$ is convex, $\theta x_1 + (1-\theta)x_2 \in \mathcal{D}$. Therefore, $(\theta x_1 + (1-\theta)x_2, \theta\alpha_1 + (1-\theta)\alpha_2) \in epi(f)$. Thus, $epi(f)$ is convex if f is convex. This proves the necessity part.

Convex Functions and Their Epigraphs (contd)

$epi(f)$ convex set $\implies$ f convex function

To prove sufficiency, assume that $epi(f)$ is convex. Let $x_1, x_2 \in \mathcal{D}$. So, $(x_1, f(x_1)) \in epi(f)$ and $(x_2, f(x_2)) \in epi(f)$. Since $epi(f)$ is convex, for $\theta \in (0, 1)$,

$$(\theta x_1 + (1-\theta)x_2, \theta\alpha_1 + (1-\theta)\alpha_2) \in epi(f)$$

which implies that $f(\theta x_1 + (1-\theta)x_2) \leq \theta f(x_1) + (1-\theta)f(x_2))$ for any $\theta \in (0, 1)$. This proves the sufficiency. $\square$

First-Order Convexity Conditions: The complete statement

Theorem

1. *For differentiable* $f : \mathcal{D} \to \Re$ *and convex set* $\mathcal{D}$, f *is convex* **iff**, *for any* $\mathrm{x}, \mathrm{y} \in \mathcal{D}$,

$$f(\mathrm{y}) \geq f(\mathrm{x}) + \nabla^T f(\mathrm{x})(\mathrm{y} - \mathrm{x})$$

2. f *is strictly convex* **iff**, *for any* $\mathrm{x}, \mathrm{y} \in \mathcal{D}$, *with* $\mathrm{x} \neq \mathrm{y}$,

$$f(\mathrm{y}) > f(\mathrm{x}) + \nabla^T f(\mathrm{x})(\mathrm{y} - \mathrm{x})$$

3. f *is strongly convex* **iff**, *for any* $\mathrm{x}, \mathrm{y} \in \mathcal{D}$, *and for some constant* $c > 0$,

$$f(\mathrm{y}) \geq f(\mathrm{x}) + \nabla^T f(\mathrm{x})(\mathrm{y} - \mathrm{x}) + \frac{1}{2} c ||\mathrm{y} - \mathrm{x}||^2$$

Second Order Conditions of Convexity

- Recall the Hessian of a continuous function:

$$\nabla^2 f(w) = \begin{pmatrix} \frac{\partial^2 f}{\partial w_1^2} & \frac{\partial^2 f}{\partial w_1 \partial w_2} & \cdots & \frac{\partial^2 f}{\partial w_1 \partial w_n} \\ \frac{\partial^2 f}{\partial w_2 \partial w_1} & \frac{\partial^2 f}{\partial w_2^2} & \cdots & \frac{\partial^2 f}{\partial w_2 \partial w_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial w_n \partial w_1} & \frac{\partial^2 f}{\partial w_n \partial w_2} & \cdots & \frac{\partial^2 f}{\partial w_n^2} \end{pmatrix}$$

- f is convex if and only if, a) $dom(f)$ is convex, and for all $x \in dom(f)$, $\nabla^2 f(x) \succcurlyeq 0$ (i.e. $\nabla^2 f(x)$ is positive semi-definite).
- In one dimension, this means f is convex iff $f''(x) \geq 0$

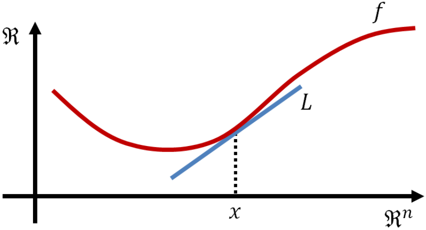
Monotonicity of Gradients

Theorem

A function f is convex if and only if $dom(f)$ is convex and for all $x, y \in dom(f), (\nabla f(x) - \nabla f(y))^T(x - y) \geq 0$

- This directly follows from the first order characterization of convexity
- Note that $f(x) \geq f(y) + \nabla f(y)^T(x - y)$ and $f(y) \geq f(x) + \nabla f(x)^T(y - x)$.
- Adding both the inequalities above we get the result!
- Note that the 1D monotonicity statement we saw earlier in the class is a special case of this!

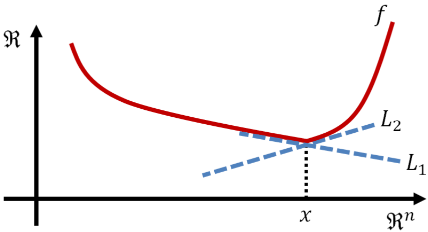
(Sub)Gradients and Convexity



To say that a function $f : \Re^n \mapsto \Re$ is differentiable at x is to say that there is a single unique linear tangent that under estimates the function:

$$f(y) \geq f(x) + \triangledown f(x)^T (y - x), \;\; \forall x, y$$

(Sub)Gradients and Convexity (contd)



In this figure we see the function f at x has many possible linear tangents that may fit appropriately. Then a **subgradient** is any $h \in \Re^n$ (same dimension as x) such that:

$$f(y) \geq f(x) + h^T(y - x), \ \ \forall y$$

Thus, intuitively, if a function is differentiable at a point x then it has a unique subgradient at that point ($\bigtriangledown f(x)$).

(Sub)Gradients and Convexity (contd)

- A **subdifferential** is the closed convex set of all subgradients of the convex function f:

$$\partial f(\mathrm{x}) = \{\mathrm{h} \in \Re^n : \mathrm{h} \text{ is a subgradient of } f \text{ at } \mathrm{x}\}$$

  Note that this set is guaranteed to be nonempty unless f is not convex.

- **Pointwise Maximum:**. if $f(\mathrm{x}) = max_{i=1\ldots m} f_i(\mathrm{x})$, then

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- **General pointwise maximum:** if $f(\mathrm{x}) = max_{s \in S} f_s(\mathrm{x})$, then under some regularity conditions (on S, f_s), $\partial f(\mathrm{x}) = conv\left(\bigcup_{s:f_s(\mathrm{x})=f(\mathrm{x})} \partial f_s(\mathrm{x})\right)$

Subgradient of $\max\{w^T x, 0\}$

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- This function is not differentiable on the hyper-plane $H = \{x | w^T x = 0\}$.

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- In other words, $\partial f(x) = \{\lambda w\}$ for $\lambda \geq 0$.

Subgradients for the ‘Lasso’ Problem in Machine Learning

We use Lasso ($\min_x f(x)$) as an example to illustrate subgradients of affine composition:

$$f(x) = \frac{1}{2}||y - x||^2 + \lambda||x||_1$$

The subgradients of $f(x)$ are

Subgradients for the 'Lasso' Problem in Machine Learning

We use Lasso ($\min_x f(x)$) as an example to illustrate subgradients of affine composition:

$$f(x) = \frac{1}{2}||y - x||^2 + \lambda||x||_1$$

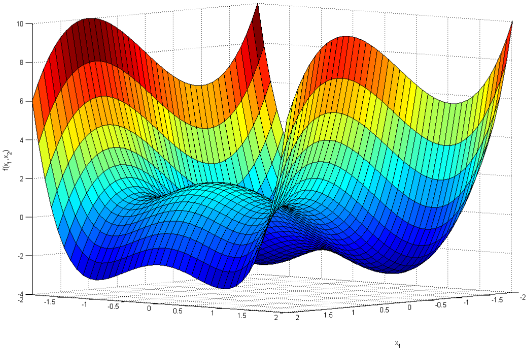
The subgradients of $f(x)$ are

$$h = x - y + \lambda s,$$

where $s_i = sign(x_i)$ if $x_i \neq 0$ and $s_i \in [-1, 1]$ if $x_i = 0$.

Local Minima

Figure below shows the plot of $f(x_1, x_2) = 3x_1^2 - x_1^3 - 2x_2^2 + x_2^4$. As can be seen in the plot, the function has several local maxima and minima.



Definition of Local Minima

Definition

A Point x is a local minimum of a function f, if there exists an $\epsilon > 0$ such that

$$\forall\, z \in \mathcal{D},\ ||z - x|| < \epsilon \Rightarrow f(z) \geq f(x)$$

More on Local Minima

- If a function f is differentiable, and x is a local minima, then $\nabla f(x) = 0$.
- If f is not differentiable, then there counld be a local minima x with non-zero (sub)-gradient. Example: $f(x_1, x_2) = |x_1 - x_2|$. However, we can say that if x is a local minima, then $0 \in \partial f(x)$.
- Is the converse true? I.e. if x is s.t. $\nabla f(x) = 0$, then x is a local minima of f?
- No. For example, $f(x_1, x_2) = x_1^2 - x_2^2$. Such points are called saddle points!

Convexity and Global Minimum

Fundamental characteristics: **Let us now prove them**

1. Any point of local minimum point is also a point of global minimum.
2. For any stricly convex function, the point corresponding to the gobal minimum is also unique.

Convexity: Local and Global Minimum

Theorem

Let $f : \mathcal{D} \to \Re$ be a convex function on a convex domain $\mathcal{D}$. Any point of locally minimum solution for f is also a point of its globally minimum solution.

Proof: Suppose $x \in \mathcal{D}$ is a point of local minimum and let $y \in \mathcal{D}$ be a point of global minimum. Thus,

Convexity: Local and Global Minimum

Theorem

Let $f : \mathcal{D} \rightarrow \Re$ be a convex function on a convex domain $\mathcal{D}$. Any point of locally minimum solution for f is also a point of its globally minimum solution.

Proof: Suppose $\mathrm{x} \in \mathcal{D}$ is a point of local minimum and let $\mathrm{y} \in \mathcal{D}$ be a point of global minimum. Thus, $f(\mathrm{y}) < f(\mathrm{x})$. Since x corresponds to a local minimum, there exists an $\epsilon > 0$ such that

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Consider a point z

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$$\forall\, z \in \mathcal{D},\ ||z - x|| < \epsilon \Rightarrow f(z) \geq f(x)$$

Consider a point $z = \theta y + (1 - \theta)x$ with $\theta = \frac{\epsilon}{2||y-x||}$. Since x is a point of local minimum (in a ball of radius ϵ), and since $f(y) < f(x)$, it must be that

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Consider a point $z = \theta y + (1 - \theta)x$ with $\theta = \frac{\epsilon}{2||y-x||}$. Since x is a point of local minimum (in a ball of radius ϵ), and since $f(y) < f(x)$, it must be that $||y - x|| > \epsilon$. Thus, $0 < \theta < \frac{1}{2}$ and $z \in \mathcal{D}$. Furthermore, $||z - x|| = \frac{\epsilon}{2}$.

Convexity: Local and Global Minimum (contd.)

Since f is a convex function

Convexity: Local and Global Minimum (contd.)

Since f is a convex function

$$f(\mathrm{z}) \leq \theta f(\mathrm{x}) + (1-\theta) f(\mathrm{y})$$

Since $f(\mathrm{y}) < f(\mathrm{x})$, we also have

$$\theta f(\mathrm{x}) + (1-\theta) f(\mathrm{y}) < f(\mathrm{x})$$

The two equations imply that $f(\mathrm{z}) < f(\mathrm{x})$, which contradicts our assumption that x corresponds to a point of local minimum. That is f cannot have a point of local minimum, which does not coincide with the point y of global minimum. $\square$

Since any locally minimum point for a convex function also corresponds to its global minimum, we will drop the qualifiers 'locally' as well as 'globally' while referring to the points corresponding to minimum values of a convex function.

Strict Convexity and Uniqueness of Global Minimum

For any stricly convex function, the point corresponding to the gobal minimum is also unique, as stated in the following theorem.

Theorem

Let $f : \mathcal{D} \to \Re$ be a strictly convex function on a convex domain $\mathcal{D}$. Then f has a unique point corresponding to its global minimum.

Proof: Suppose $x \in \mathcal{D}$ and $y \in \mathcal{D}$ with $y \neq x$ are two points of global minimum. That is $f(x) = f(y)$ for $y \neq x$. The point $\frac{x+y}{2}$ also

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Proof: Suppose $x \in \mathcal{D}$ and $y \in \mathcal{D}$ with $y \neq x$ are two points of global minimum. That is $f(x) = f(y)$ for $y \neq x$. The point $\frac{x+y}{2}$ also belongs to the convex set $\mathcal{D}$ and since f is strictly convex, we must have

$$f\left(\frac{x+y}{2}\right) < \frac{1}{2}f(x) + \frac{1}{2}f(y) = f(x)$$

which is a contradiction. Thus, the point corresponding to the minimum of f must be unique. □

Does Global Minima Always Exist?

- Does the global minimum always exist?
- Not necessarily even if f is bounded from below (e.g. $f(x) = e^x$)
- Weierstrass Theorem: Let f be a convex function and suppose there is a nonempty and bounded sublevel set $L_\alpha(f)$. Then f has a global minima.
- Since f is continuous, it attains a minimum over a closed and bounded ($=$ compact) set $L_\alpha(f)$ at some x^*. Note that x^* is also a global minimum as firstly, $f(x^*) \leq f(x), \forall x \in L_\alpha(f)$. Next since, $f(x^*) \leq \alpha$, it follows that for any $x \notin L_\alpha(f)$, $f(x) > \alpha \geq f(x^*)$

Critical Points are Global Minima for Convex Functions

- Lemma: Suppose that f is convex and differentiable over an open domain $dom(f)$. Let $x \in dom(f)$. Then if $\nabla f(x) = 0$ (i.e. a critical point), then x is a global minima.
- Proof: Suppose $\nabla f(x) = 0$. Then from the first order characterization of convex functions, $\forall y \in dom(f), f(y) \geq f(x) + \nabla f(x)^T(y - x) \geq f(x)$. Hence x is a global minima.
- Note that this cannot be extended to non-differentiable convex functions since the global minima may not be a differentiable point (for example: $f(x) = ||x||_1$).
- No Saddle points for convex functions!

Convex Optimization Problem

- Formally, a convex optimization problem is an optimization problem of the form

$$\begin{aligned} &\textit{minimize } f(\mathrm{w}) \\ &\textit{subject to } c \;\in\; C \end{aligned}$$

  where f is a convex function, X is a convex set, and w is the optimization variable.
- if $X = dom(f)$, this becomes unconstrained optimization.
- A special case (f is a convex function, g_i are convex functions, and h_i are affine functions, and x is the vector of optimization variables):

$$\begin{aligned} \textit{minimize } f(\mathrm{w}) & \\ \textit{subject to } g_i(\mathrm{w}) &\leq 0,\; i = 1, ..., m \\ h_i(\mathrm{w}) &= 0, i = 1, ..., p \end{aligned}$$

Optimality Conditions for Constrained Optimization

- Lemma: Suppose that f is convex and differentiable over an open domain $dom(f)$. Let $X \subseteq dom(f)$ be a convex set. A point x^* is a minimizer of f over X if and only if

$$\nabla f(x^*)^T(x - x^*) \geq 0, \forall x \in X$$

- Nice geometric interpretation:

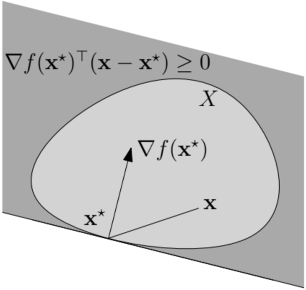

Figure 1: Caption

Linear and Quadratic Programs

- Linear Program (LP) is a special case of a convex optimization problem:

$$\begin{aligned} minimize\ c^T x \\ subject\ to\ Ax \leq b \end{aligned}$$

- Another special case is Quadratic Programs (QP):

$$\begin{aligned} minimize\ 1/2x^T Qx \\ subject\ to\ Ax \leq b \end{aligned}$$

- The QP is a convex optimization problem only if Q is positive semi-definite,