Q17

$$V = (2, 5, 1)^T$$

(i) $R_1 = \begin{bmatrix} R_y & 0 \\ 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} \cos\frac{\pi}{2} & 0 & \sin\frac{\pi}{2} & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\frac{\pi}{2} & 0 & \cos\frac{\pi}{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} R_x & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R_y & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} R_x R_y & 0 \\ 0 & 1 \end{bmatrix}$$

$R_2 = \begin{bmatrix} R_x & 0 \\ 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\left(-\frac{\pi}{2}\right) & -\sin\left(-\frac{\pi}{2}\right) & 0 \\ 0 & \sin\left(-\frac{\pi}{2}\right) & \cos\left(-\frac{\pi}{2}\right) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Net Rotation Matrix $\Rightarrow$ $R = R_2 R_1$

$$R = \begin{bmatrix} R_x R_y & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Transformation Matrix (T) $\Rightarrow$ $\begin{bmatrix} R & t \\ 0 & 1 \end{bmatrix}$

$$T = \begin{bmatrix} 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 3 \\ 0 & -1 & 0 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(ii) Let new coordinates be P'

" old " " $P = [2 \ 5 \ 1 \ 1]$

Then, $P' = TP^T$

$$= \begin{bmatrix} 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 3 \\ 0 & -1 & 0 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 1 \\ -3 \\ 1 \end{bmatrix}$$

$$P' = [0 \quad 1 \quad -3]$$

Let the new origin be O'

then, $O' = TO^T$

$$= \begin{bmatrix} 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 3 \\ 0 & -1 & 0 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ 2 \\ 1 \end{bmatrix}$$

$\therefore$ New origin $\Rightarrow$ $O' = \begin{bmatrix} -1 & 3 & 2 \end{bmatrix}$

(iii) $$h = \frac{1}{2\sin(\theta)} \begin{bmatrix} R_{32} - R_{23} \\ R_{13} - R_{31} \\ R_{21} - R_{12} \end{bmatrix}$$

Also,

$$\cos\theta = \frac{\text{trace}(R) - 1}{2}$$

$$\text{trace}(R) = 0 + 0 + 0 = 0$$

$$\therefore \cos\theta = \frac{0-1}{2} = \frac{-1}{2}$$

$$\theta = \cos^{-1}\left(\frac{-1}{2}\right)$$

$$\theta = \frac{2\pi}{3}$$

Now, $n = \dfrac{1}{2\sin\left(\frac{2\pi}{3}\right)} \begin{bmatrix} -1-0 \\ 1-0 \\ -1-0 \end{bmatrix}$

$= \dfrac{1}{2 \cdot \frac{\sqrt{3}}{2}} \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}$

$= \dfrac{1}{\sqrt{3}} \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}$

So, $\hat{n} = \dfrac{1}{\sqrt{3}} \left(-\hat{i} + \hat{j} - \hat{k} \right)$

(iv) From Rodrigues's Formulae,

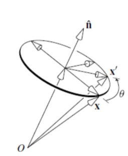
$$\vec{x}' = \vec{x} + (\sin\theta)\, \hat{n} \times \vec{x} + (1-\cos\theta)\, \hat{n} \times (\hat{n} \times \vec{x})$$

$$N = \begin{bmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{bmatrix}$$

Rodrigues's Formula

Given point $\mathbf{x}$, decompose into components parallel and perpendicular to the rotation axis

$$\mathbf{x} = \hat{\mathbf{n}}(\hat{\mathbf{n}} \cdot \mathbf{x}) - \hat{\mathbf{n}} \times (\hat{\mathbf{n}} \times \mathbf{x})$$



Only $\mathbf{x}_\perp$ is affected by the rotation, yielding *Rodrigues's formula*:

$$\mathbf{x}' = \hat{\mathbf{n}}(\hat{\mathbf{n}} \cdot \mathbf{x}) + \sin\theta\,(\hat{\mathbf{n}} \times \mathbf{x}) - \cos\theta\,\hat{\mathbf{n}} \times (\hat{\mathbf{n}} \times \mathbf{x})$$

A common variation:

$$\mathbf{x}' = \mathbf{x} + (\sin\theta)\,\hat{\mathbf{n}} \times \mathbf{x} + (1-\cos\theta)\,\hat{\mathbf{n}} \times (\hat{\mathbf{n}} \times \mathbf{x})$$

Rodrigues's Formula

Axis-angle to *R*

$$\mathbf{x}' = \mathbf{x} + (\sin\theta)\,\hat{\mathbf{n}} \times \mathbf{x} + (1-\cos\theta)\,\hat{\mathbf{n}} \times (\hat{\mathbf{n}} \times \mathbf{x})$$

$$N = \begin{pmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{pmatrix}$$

$$N\mathbf{x} = \hat{\mathbf{n}} \times \mathbf{x}$$

$$\mathbf{x}' = \mathbf{x} + (\sin\theta)N\mathbf{x} + (1-\cos\theta)N^2\mathbf{x}$$

$$R = I + (\sin\theta)N + (1-\cos\theta)N^2$$

***R* to Axis-angle**

$$\theta = \cos^{-1}\left(\frac{trace(\boldsymbol{R}) - 1}{2}\right)$$

$$n = \frac{1}{2\sin(\theta)} \begin{bmatrix} \boldsymbol{R}_{32} - \boldsymbol{R}_{23} \\ \boldsymbol{R}_{13} - \boldsymbol{R}_{31} \\ \boldsymbol{R}_{21} - \boldsymbol{R}_{12} \end{bmatrix}$$

$$= \frac{1}{\sqrt{3}} \begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix}$$

$$N\vec{x} = \hat{n} \times \vec{x}$$

$$\vec{x}' = \vec{x} + (\sin\theta) N\vec{x} + (1-\cos\theta) N^2 \vec{x}$$

$$R' = I + (\sin\theta) N + (1-\cos\theta) N^2$$

$$N^2 = \frac{1}{3} \begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix}$$

$$= \frac{1}{3} \begin{bmatrix} -2 & -1 & 1 \\ -1 & -2 & -1 \\ 1 & -1 & -2 \end{bmatrix}$$

$$\sin\theta = \frac{\sqrt{3}}{2} \qquad \cos\theta = \frac{-1}{2}$$

$$\hookrightarrow 1-\cos\theta = 1+\frac{1}{2} = \frac{3}{2}$$

$\therefore \quad I + \frac{\sqrt{3}}{2} N + \frac{1}{2} N^2$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \frac{\cancel{\sqrt{3}}}{2} \frac{1}{\cancel{\sqrt{3}}} \begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix}$$

$$+ \frac{\cancel{3}}{2} \cdot \frac{1}{\cancel{3}} \begin{bmatrix} -2 & -1 & 1 \\ -1 & -2 & -1 \\ 1 & -1 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 1+0-1 & \frac{1}{2} - \frac{1}{2} + 0 & 0 + \frac{1}{2} \; \frac{1}{2} \\ 0 - \frac{1}{2} - \frac{1}{2} & 1 + 0 - 1 & 0 + \frac{1}{2} - \frac{1}{2} \\ 0 - \frac{1}{2} + \frac{1}{2} & 0 - \frac{1}{2} - \frac{1}{2} & 1 + 0 - 1 \end{bmatrix}$$

$\therefore$

$$R' = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}$$

$\therefore \quad R' = R$

Q3)

$$\vec{x} = \underbrace{K}_{\text{Intrinsic Param.}} \underbrace{[R \mid t]}_{\text{Extrinsic Param.}} \underbrace{X}_{\text{Pt. Coordinate}}$$

Camera $C_1 \rightarrow K_1$

Camera $C_2 \rightarrow K_2$

C_1 is used as world coordinate frame. then, for C_1 the image pt. x_1 can be written as

$$x_1 = K_1 [I \mid 0] X \quad — (1)$$

Camera 2 is obtained by pure 3d-Rotation of Camera-1, ie for camera 2, $t = 0$

So, image pt x_2 for C_2 can be written as

$$x_2 = K_2 [R \mid 0] X \quad — (2)$$

From ②

$$X = (K_2 [R|0])^{-1} x_2 \quad -③$$

From ①

$$X = (K_1 [I|0])^{-1} x_1 \quad -④$$

Using ③ & ④

$$(K_1 [I|0])^{-1} x_1 = (K_2 [R|0])^{-1} x_2$$

$$(K_1 [I|0])^{-1} x_1 = [R|0]^{-1} K_2^{-1} x_2$$

then,

$$x_1 = K_1 [I|0] [R|0]^{-1} K_2^{-1} x_2$$

$$[I|0].\text{shape} \rightarrow (3,4)$$

$$[R|0]^{-1}.\text{shape} \rightarrow (4,3)$$

$$\therefore \left([I|0] [R|0]^{-1}\right).\text{shape} \rightarrow (3 \times 3)$$

$$[I|0]\,[R|0]^{-1} = R^{-1}$$

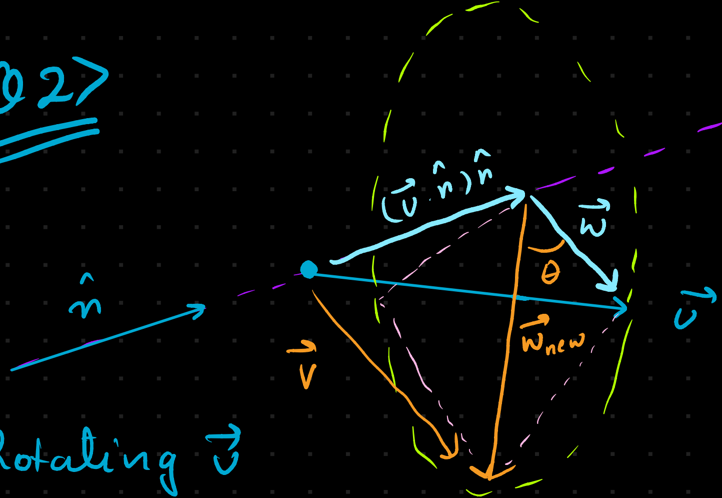
$$\therefore \quad x_1 = K_1 R^{-1} K_2^{-1} x_2$$

$$x_1 = \underbrace{\left(K_1 R^{-1} K_2^{-1}\right)}_{= H \; (3\times 3)} x_2$$

$$\therefore \quad x_1 = H x_2$$

Here, H is a (3×3) matrix.

Q2>



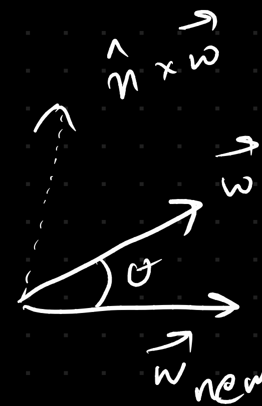
Rotating $\vec{v}$ by θ w.r.t axis $\hat{n}$

axis: $\hat{n}$

$(\vec{v}\cdot\hat{n})\hat{n}$

$$\text{Proj}(\vec{v}) = (\vec{v}\cdot\hat{n})\hat{n} \quad -①$$

$$(\vec{v}\cdot\hat{n})\hat{n} + \vec{w} = \vec{v} \quad -②$$



$$\therefore \quad \vec{w} = \vec{v} - (\vec{v}\cdot\hat{n})\hat{n}$$

$$\vec{V} = (\vec{v}\cdot\hat{n})\cdot\hat{n} + \vec{w}_{new}$$

$$\vec{w}_{new} =$$

$$\vec{V} = (\vec{v}\cdot\hat{n})\cdot\hat{n} + \frac{\vec{w}}{\|\vec{w}\|}\cdot\|\vec{w}\|\cos\theta$$

$$+ \underbrace{\frac{\vec{w}\times\hat{n}}{\|\vec{w}\|}\|\vec{w}\|\cdot\sin\theta}$$

orthogonal vector

using ②

$$= (\vec{v}\cdot\hat{n})\cdot\hat{n} + [\vec{v} - (\vec{v}\cdot\hat{n})\hat{n}]\cos\theta$$

$$+ \hat{n}\times[\vec{v} - (\vec{v}\cdot\hat{n})\hat{n}]\sin\theta$$

$$\left((\vec{v}\cdot\hat{n})\hat{n}\times\hat{n} = 0 \right)$$

$$= (\vec{v}\cdot\hat{n})\hat{n}\left[1-\cos\theta\right] + \vec{v}\cos\theta$$

$$+ \hat{n}\times\vec{v}\sin\theta$$

$$= \vec{v} + (\hat{n}\times\vec{v})\sin\theta$$

$$+ [1-\cos\theta]\,\hat{n}\times(\hat{n}\times\vec{v})$$

$$\left(\begin{array}{l} \cancel{\vec{v}} + (\hat{n}\times\vec{v})\sin\theta \\ +(1-\cos\theta)(\vec{v}\cdot\hat{n})\hat{n} - (\cancel{1}-\cos\theta)\vec{v} \end{array} \right)$$

$$\left(\vec{v}\cos\theta + (\hat{n}\times\vec{v})\sin\theta + (1-\cos\theta)(\vec{v}\cdot\hat{n})\hat{n} \right)$$

Hence, proved.