# Math Prep Course Day Three

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## 1 Graph sketching

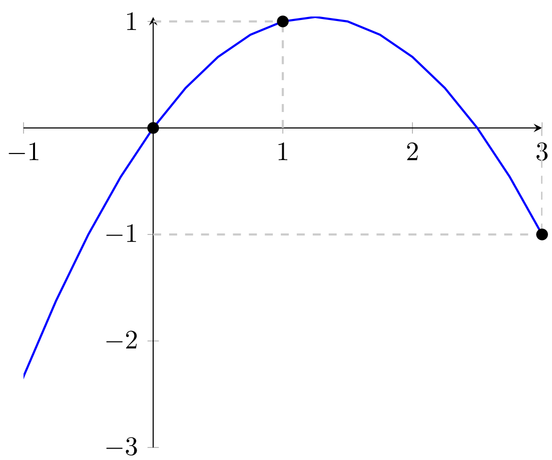
**Problem 1.1.** Find the quadratic function that passes through the points $(0, 0),\ (1, 1)\ (3, -1)$

**Solution.** A quadratic function is of the form $ax^2 + bx + c$. Utilizing the three points given we can establish the following system of equations

$$a(0)^2 + b(0) + c = 0 \tag{1}$$
$$a(1)^2 + b(1) + c = 1 \tag{2}$$
$$a(3)^2 + b(3) + c = -1 \tag{3}$$

From equation (1) we obtain $c = 0$. Subtracting $3 \times Eq(2)$ from (3) we obtain $6a = -4$, $a = -\frac{2}{3}$ and via re-substitution get $b = \frac{5}{3}$. Our resulting equation is therefore

$$f(x) = -\frac{2}{3}x^2 + \frac{5}{3}x$$



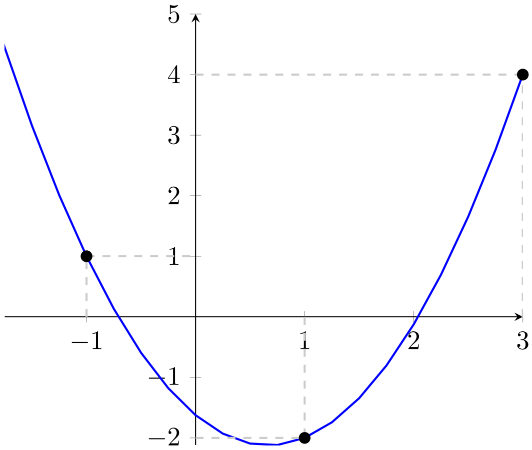

**Problem 1.2.** Find the quadratic function passing through the points $(-1, 1),\ (1, -2),\ (3, 4)$

**Solution.** Again, we set up the following system of equations

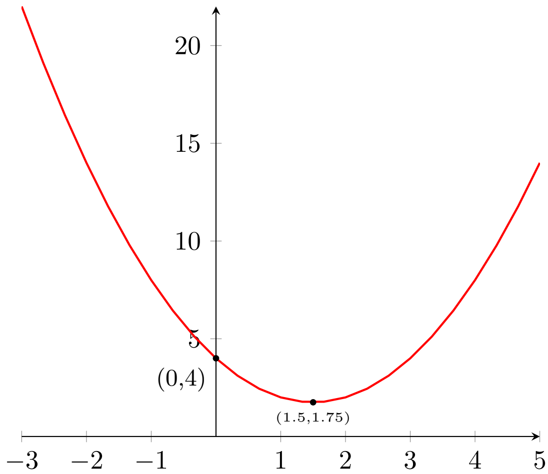
$$a(-1)^2 + b(-1) + c = 1 \tag{1}$$
$$a(1)^2 + b(1) + c = -2 \tag{2}$$
$$a(3)^2 + b(3) + c = 4 \tag{3}$$

Subtracting equation (1) from equation (2) we obtain $2b = -3$, so $b = -\frac{3}{2}$. Subtracting equation (1) from equation (3) we obtain $8a + 4b = 3$, which after simplifying becomes $a = \frac{9}{8}$. Substituting the value of $a$ and $b$ into any of the equations we can recover $c = -\frac{13}{8}$. Our equation is this

$$f(x) = \frac{9}{8}x^2 - \frac{3}{2}x - \frac{13}{8}$$




**Problem 1.3.** Sketch the graph of the function $f(x) = x^2 - 3x + 4$ on the interval $(-3, 5)$




**Solution.** lmao

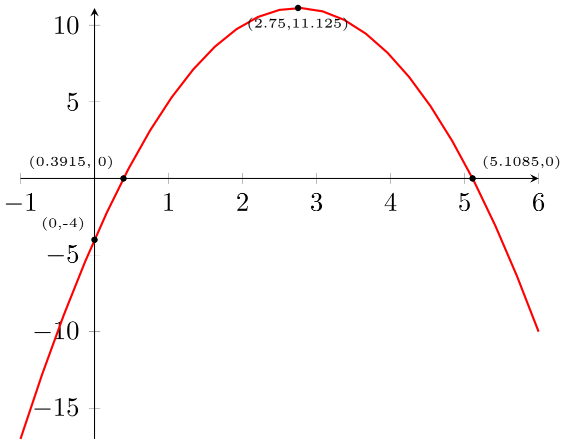
The maximum value of the graph is at $x = -3$, therefore

$$(-3)^2 - 3(-3) + 4 = 9 + 9 + 4 = 22$$

The minimum value is indicated on the graph.

**Problem 1.4.** Draw the graph of the function $f(x) = -2x^2 + 11x - 4$

**Solution.**




**Problem 1.5.** Let $f(x) = ax + b$ be an arbitrary linear function. Prove that $f(f(x))$ is also linear.

**Solution.** By definition, a linear function is one whose only power in the variable is one. No higher or lower or fractional powers are allowed. Let us compute $f(f(x))$

$$\begin{aligned} f(f(x)) &= f(ax+b) \\ &= a(ax+b)+b \\ &= a^2x+ab+b \\ &= a'x+b' & \because \textit{let } a^2 = a',\ ab+b = b' \end{aligned}$$

Note that we can perform the last step since $a$ and $b$ are constants. The power of $x$ is still one in the final expression. Therefore the composition of two linear functions is linear.

**Problem 1.6.** For the following functions

$$\begin{aligned} f(z) &= \sqrt{4-z^2} \\ g(x) &= 2x+3 \end{aligned}$$

Find the domain of $y = f(g(x))$ for which $y$ is well defined.

**Solution.** We first immediately note that both the domain and range of $g(x)$ are the entirety of $\mathbb{R}$. The only restriction then comes from $f(z)$. To find the domain of $f(z)$ we must find the interval for which the square root is non-negative

$$\begin{aligned} 4-z^2 &\geq 0 \\ z^2 &\leq 4 \\ z &\leq |2| \end{aligned}$$

Now that we know the domain of $f(z)$, we now have to find the interval such that $-2 \leq g(x) \leq 2$. Since $g$ is a monotone function and nicely invertible, we can find the inverse of $g$ and plug in the values 2 and -2 to obtain the interval for our required domain.

$$\begin{aligned} g(x) &= 2x + 3 \\ g^{-1}(x) &= \frac{x-3}{2} \\ g^{-1}(-2) &= \frac{-2-3}{2} && = \frac{-5}{2} \\ g^{-1}(2) &= \frac{2-3}{2} && = \frac{-1}{2} \end{aligned}$$

Thus the domain of our function $y = f(g(x))$ is $\left[-\frac{5}{2}, \frac{-1}{2}\right]$

**Problem 1.7.** Find the inverse of $f(x) = \sqrt{x} + \sqrt{x-1}$ and its applicable domain.

**Solution.** By observation we see that $f(x)$ has a domain $[1, \infty)$. Since it is a monotone increasing function we can find its range via plugging in the endpoints of the domain

$$\begin{aligned} f(1) &= \sqrt{1} + \sqrt{1-1} = \sqrt{1} \\ \lim_{x\to\infty} f(x) &= \lim_{x\to\infty} \sqrt{x} + \lim_{x\to\infty} \sqrt{x-1} = \infty \end{aligned}$$

Therefore our range is $[1, \infty)$ as well and this will be the domain of our inverse function. To find the inverse function itself we must do some rather annoying algebraic manipulation

$$\begin{aligned} y &= \sqrt{x} + \sqrt{x-1} \\ y^2 &= x + x - 1 + 2(\sqrt{x})(\sqrt{x-1}) \\ y^2 - 2x + 1 &= 2\sqrt{x}\sqrt{x-1} \\ y^4 - 4y^2x + 2y^2 + 4x^2 - 4x + 1 &= 4x^2 - 4x \\ y^4 - 4y^2x + 2y^2 + 1 &= 0 \\ x &= \frac{y^4 + 2y^2 + 1}{4y^2} \\ x &= \left(\frac{y^2+1}{2y}\right)^2 \end{aligned}$$

**Problem 1.8.** Find the solution for $\frac{x+1}{x-2} \leq \frac{x+2}{x+3}$.

**Solution.**

$$\begin{aligned}
\frac{x+1}{x-2}-\frac{x+2}{x+3} &\leq 0\\
\frac{(x+1)(x+3)-(x+2)(x-2)}{(x-2)(x+3)} &\leq 0\\
\frac{x^2+4x+3-x^2+4}{(x-2)(x+3)} &\leq 0\\
\frac{4x+7}{(x-2)(x+3)} &\leq 0
\end{aligned}$$

The domain splits of the above function are $(-\infty,-3),\ (-3,-1.75),\ (-1.75,2),\ (2,\infty)$. We can again use a table to find out the final signature results of our function

| *Interval* | $4x+7$ | $x-2$ | $x+3$ | *Sign* |
|---|---|---|---|---|
| $(-\infty,-3)$ | $-$ | $-$ | $-$ | $-$ |
| $(-3,-1.75)$ | $-$ | $-$ | $+$ | $+$ |
| $(-1.75,2)$ | $+$ | $-$ | $+$ | $-$ |
| $(2,\infty)$ | $+$ | $+$ | $+$ | $+$ |

Therefore our solution set is $(-\infty,-3)\cup(-1.75,2)$

**Problem 1.9.** Find the solution of $\frac{3x}{x^2+2}\geq\frac{1}{x-1}$

**Solution.**

$$\begin{aligned}
\frac{3x}{x^2+2}-\frac{1}{x-1} &\geq 0\\
\frac{3x(x-1)-(x^2+2)}{(x^2+2)(x-1)} &\geq 0\\
\frac{2x^2-3x-2}{(x^2+2)(x-1)} &\geq 0\\
\frac{(2x+1)(x-2)}{(x^2+2)(x-1)} &\geq 0
\end{aligned}$$

Since $x^2+2$ is always positive, we don't need to split the domain with respect to that term, for the rest we obtain $\left(-\infty,-\frac{1}{2}\right),\ \left(-\frac{1}{2},1\right),\ (1,2),\ (2,\infty)$. The table looks like

| *Interval* | $2x+1$ | $x-2$ | $x-1$ | *Sign* |
|---|---|---|---|---|
| $\left(-\infty,-\frac{1}{2}\right)$ | $-$ | $-$ | $-$ | $-$ |
| $\left(-\frac{1}{2},1\right)$ | $+$ | $-$ | $-$ | $+$ |
| $(1,2)$ | $+$ | $-$ | $+$ | $-$ |
| $(2,\infty)$ | $+$ | $+$ | $+$ | $+$ |

For positivity, our required intervals are $\left(\frac{1}{2},1\right)\cup(2,\infty)$