

Math Prep Course Day Three

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September 2025

1 Graph sketching

Problem 1.1. Find the quadratic function that passes through the points $(0, 0)$, $(1, 1)$ $(3, -1)$

Solution. A quadratic function is of the form $ax^2 + bx + c$. Utilizing the three points given we can establish the following system of equations

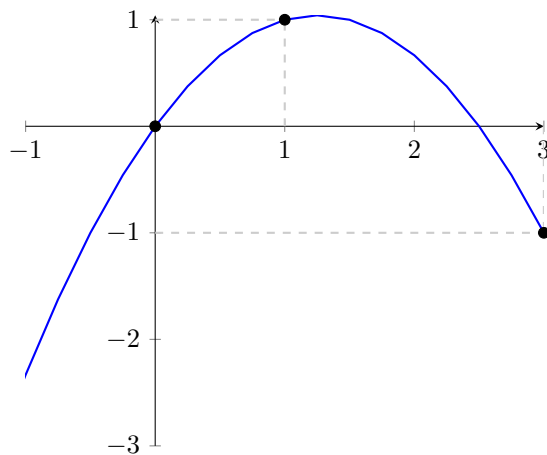
$$a(0)^2 + b(0) + c = 0 \quad (1)$$

$$a(1)^2 + b(1) + c = 1 \quad (2)$$

$$a(3)^2 + b(3) + c = -1 \quad (3)$$

From equation (1) we obtain $c = 0$. Subtracting $3 \times Eq(2)$ from (3) we obtain $6a = -4$, $a = -\frac{2}{3}$ and via re-substitution get $b = \frac{5}{3}$. Our resulting equation is therefore

$$f(x) = -\frac{2}{3}x^2 + \frac{5}{3}x$$



Problem 1.2. Find the quadratic function passing through the points $(-1, 1)$, $(1, -2)$, $(3, 4)$

Solution. Again, we set up the following system of equations

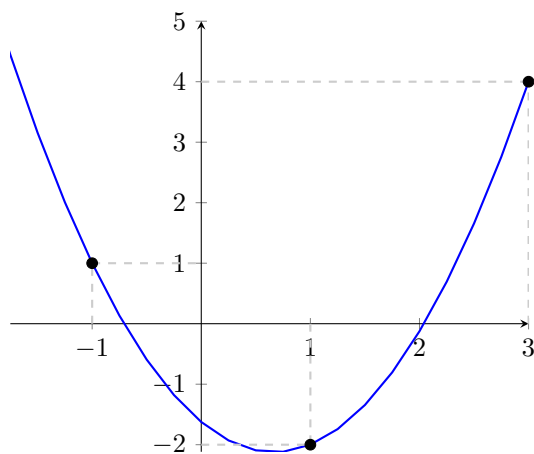
$$a(-1)^2 + b(-1) + c = 1 \quad (1)$$

$$a(1)^2 + b(1) + c = -2 \quad (2)$$

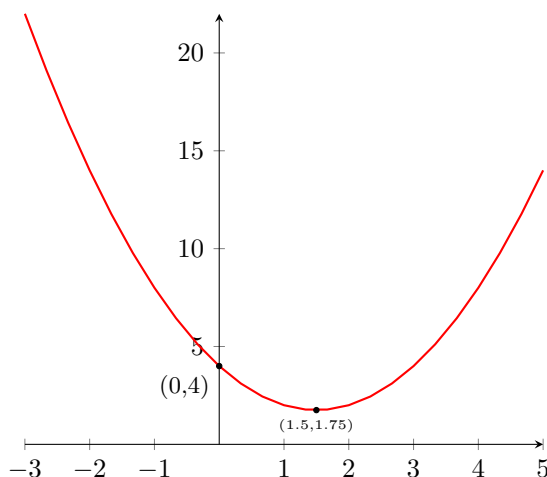
$$a(3)^2 + b(3) + c = 4 \quad (3)$$

Subtracting equation (1) from equation (2) we obtain $2b = -3$, so $b = -\frac{3}{2}$. Subtracting equation (1) from equation (3) we obtain $8a + 4b = 3$, which after simplifying becomes $a = \frac{9}{8}$. Substituting the value of a and b into any of the equations we can recover $c = -\frac{13}{8}$. Our equation is this

$$f(x) = \frac{9}{8}x^2 - \frac{3}{2}x - \frac{13}{8}$$



Problem 1.3. Sketch the graph of the function $f(x) = x^2 - 3x + 4$ on the interval $(-3, 5)$



Solution. lmao

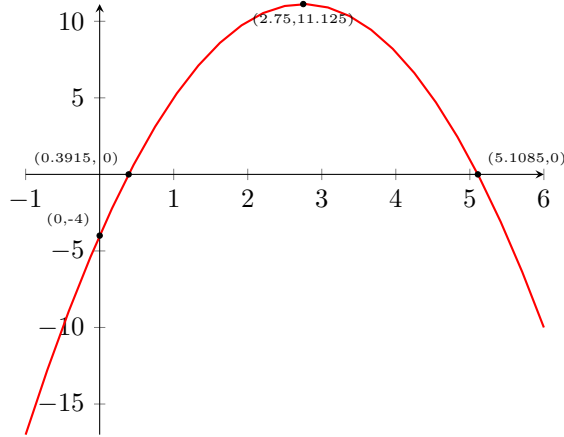
The maximum value of the graph is at $x = -3$, therefore

$$(-3)^2 - 3(-3) + 4 = 9 + 9 + 4 = 22$$

The minimum value is indicated on the graph.

Problem 1.4. Draw the graph of the function $f(x) = -2x^2 + 11x - 4$

Solution.



Problem 1.5. Let $f(x) = ax + b$ be an arbitrary linear function. Prove that $f(f(x))$ is also linear.

Solution. By definition, a linear function is one whose only power in the variable is one. No higher or lower or fractional powers are allowed. Let us compute $f(f(x))$

$$\begin{aligned} f(f(x)) &= f(ax + b) \\ &= a(ax + b) + b \\ &= a^2x + ab + b \\ &= a'x + b' \end{aligned} \quad \therefore \text{let } a^2 = a', \quad ab + b = b'$$

Note that we can perform the last step since a and b are constants. The power of x is still one in the final expression. Therefore the composition of two linear functions is linear.

Problem 1.6. For the following functions

$$\begin{aligned} f(z) &= \sqrt{4 - z^2} \\ g(x) &= 2x + 3 \end{aligned}$$

Find the domain of $y = f(g(x))$ for which y is well defined.

Solution. We first immediately note that both the domain and range of $g(x)$ are the entirety of \mathbb{R} . The only restriction then comes from $f(z)$. To find the domain of $f(z)$ we must find the interval for which the square root is non-negative

$$\begin{aligned} 4 - z^2 &\geq 0 \\ z^2 &\leq 4 \\ z &\leq |2| \end{aligned}$$

Now that we know the domain of $f(z)$, we now have to find the interval such that $-2 \leq g(x) \leq 2$. Since g is a monotone function and nicely invertible, we can find the inverse of g and plug in the values 2 and -2 to obtain the interval for our required domain.

$$\begin{aligned} g(x) &= 2x + 3 \\ g^{-1}(x) &= \frac{x-3}{2} \\ g^{-1}(-2) &= \frac{-2-3}{2} = \frac{-5}{2} \\ g^{-1}(2) &= \frac{2-3}{2} = \frac{-1}{2} \end{aligned}$$

Thus the domain of our function $y = f(g(x))$ is $[-\frac{5}{2}, \frac{-1}{2}]$

Problem 1.7. Find the inverse of $f(x) = \sqrt{x} + \sqrt{x-1}$ and its applicable domain.

Solution. By observation we see that $f(x)$ has a domain $[1, \infty)$. Since it is a monotone increasing function we can find its range via plugging in the endpoints of the domain

$$\begin{aligned} f(1) &= \sqrt{1} + \sqrt{1-1} = \sqrt{1} \\ \lim_{x \rightarrow \infty} f(x) &= \lim_{x \rightarrow \infty} \sqrt{x} + \lim_{x \rightarrow \infty} \sqrt{x-1} = \infty \end{aligned}$$

Therefore our range is $[1, \infty)$ as well and this will be the domain of our inverse function. To find the inverse function itself we must do some rather annoying algebraic manipulation

$$\begin{aligned} y &= \sqrt{x} + \sqrt{x-1} \\ y^2 &= x + x - 1 + 2(\sqrt{x})(\sqrt{x-1}) \\ y^2 - 2x + 1 &= 2\sqrt{x}\sqrt{x-1} \\ y^4 - 4y^2x + 2y^2 + 4x^2 - 4x + 1 &= 4x^2 - 4x \\ y^4 - 4y^2x + 2y^2 + 1 &= 0 \\ x &= \frac{y^4 + 2y^2 + 1}{4y^2} \\ x &= \left(\frac{y^2 + 1}{2y} \right)^2 \end{aligned}$$

Problem 1.8. Find the solution for $\frac{x+1}{x-2} \leq \frac{x+2}{x+3}$.

Solution.

$$\begin{aligned}\frac{x+1}{x-2} - \frac{x+2}{x+3} &\leq 0 \\ \frac{(x+1)(x+3) - (x+2)(x-2)}{(x-2)(x+3)} &\leq 0 \\ \frac{x^2 + 4x + 3 - x^2 + 4}{(x-2)(x+3)} &\leq 0 \\ \frac{4x+7}{(x-2)(x+3)} &\leq 0\end{aligned}$$

The domain splits of the above function are $(-\infty, -3)$, $(-3, -1.75)$, $(-1.75, 2)$, $(2, \infty)$. We can again use a table to find out the final signature results of our function

<i>Interval</i>	$4x+7$	$x-2$	$x+3$	<i>Sign</i>
$(-\infty, -3)$	-	-	-	-
$(-3, -1.75)$	-	-	+	+
$(-1.75, 2)$	+	-	+	-
$(2, \infty)$	+	+	+	+

Therefore our solution set is $(-\infty, -3) \cup (-1.75, 2)$

Problem 1.9. Find the solution of $\frac{3x}{x^2+2} \geq \frac{1}{x-1}$

Solution.

$$\begin{aligned}\frac{3x}{x^2+2} - \frac{1}{x-1} &\geq 0 \\ \frac{3x(x-1) - (x^2+2)}{(x^2+2)(x-1)} &\geq 0 \\ \frac{2x^2 - 3x - 2}{(x^2+2)(x-1)} &\geq 0 \\ \frac{(2x+1)(x-2)}{(x^2+2)(x-1)} &\geq 0\end{aligned}$$

Since $x^2 + 2$ is always positive, we don't need to split the domain with respect to that term, for the rest we obtain $(-\infty, -\frac{1}{2})$, $(-\frac{1}{2}, 1)$, $(1, 2)$, $(2, \infty)$. The table looks like

<i>Interval</i>	$2x+1$	$x-2$	$x-1$	<i>Sign</i>
$(-\infty, -\frac{1}{2})$	-	-	-	-
$(-\frac{1}{2}, 1)$	+	-	-	+
$(1, 2)$	+	-	+	-
$(2, \infty)$	+	+	+	+

For positivity, our required intervals are $(-\frac{1}{2}, 1) \cup (2, \infty)$