Math Prep Course Day One

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1 Differential Calculus

Question 1.1

Find $\lim_{x\to 4} \frac{x^2-x-12}{x-4}$

$$\lim_{x\to 4} \frac{(x-4)(x+3)}{x-4}$$
$$=7$$

Question 1.2

Find $\lim_{x\to 3} \frac{x+2}{x-3}$

Observe that

$$\lim_{x\to 3^+} \frac{x+2}{x-3} = +\infty$$
$$\lim_{x\to 3^-} \frac{x+2}{x-3} = -\infty$$

Since the limits do not agree, and there exists no factorization or other simplification, the limit simply does not exist.

Question 1.3

Find $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ Where $f(x) = 4x^2 - x$

$$\lim_{h\to 0} \frac{(4x^2+8xh+4h^2-x-h)-(4x^2-x)}{h}$$
$$\lim_{h\to 0} \frac{8xh-h-4h^2}{h}$$
$$\lim_{h\to 0} 8x-1-4h$$
$$8x-1$$

Question 1.4

Find $\lim_{x\to \pm\infty} \frac{4x^3+20x^2}{x^4-1}$

Here we can simply utilize the fact that the denominator grows faster than the numerator, therefore for $x \to \pm\infty$ the numerator becomes negligible compared to the denominator, thus the limit evaluates to zero.

Question 1.5

Find $\lim_{x\to \infty} \frac{\sin x}{x}$

For this it is sufficient to note that $\sin x \leq |1|$.

$$\lim_{x\to\infty} \frac{\sin x}{x} \leq \lim_{x\to\infty} \frac{|1|}{x}$$
$$\lim_{x\to\infty} -\frac{1}{x} \leq \lim_{x\to\infty} \frac{\sin x}{x} \leq \lim_{x\to\infty} \frac{1}{x}$$
$$0 \leq \lim_{x\to\infty} \frac{\sin x}{x} \leq 0$$

By the sandwich theorem, we can now conclude that the function approaches zero as x tends to infinity.

Question 1.6

Find $\lim_{x\to 0} \frac{\sin x}{x}$

Using the Taylor expansion of the sin function around zero we obtain

$$\lim_{x\to 0} \frac{x - \frac{x^3}{3!} + \frac{x^5}{5!} \dots}{x}$$
$$\lim_{x\to 0} 1 - \frac{x^2}{3!} + \frac{x^4}{5!} \dots$$
$$1$$

It is important to note that since we have utilized the Taylor expansion around zero, this formula only holds true when $\frac{x^2}{3!} << 1$.

Question 1.7

Find $\frac{d^n(y)}{dx^n}$ for $y = \frac{1}{3+x}$

By playing around with repeated applications of the derivative operator we expect the general form of the solution to look something like

$$\frac{d^n(y)}{dx^n} = (-1)^n \frac{n!}{(3+x)^{n+1}}$$

We can verify this by induction. It is obvious that this holds for n=0, where by convention we take $(-1)^0 = 1$ and $0! = 1$. For the inductive case we observe that

$$\begin{aligned}
\frac{d}{dx}\frac{d^n(y)}{dx^n} &= \frac{d}{dx}(-1)^n \frac{n!}{3+x^{n+1}} \\
&= (-1)^n n! \frac{d}{dx} \frac{1}{(3+x)^{n+1}} \\
&= (-1)^n n! \frac{-(n+1)}{(3+x)^{n+2}} \\
&= (-1)^{n+1} \frac{(n+1)!}{(3+x)^{n+2}}
\end{aligned}$$

Question 1.8

If $x^2 + 2xy + 3y^2 = 2$ find y' and y'' when y=1.

First we must find the value of x when $y=1$

$$\begin{aligned}x^2+2x+1&=0\\(x+1)^2&=0\\x&=-1\end{aligned}$$

Next we find y'

$$\begin{aligned}\frac{d}{dx}(x^2+2xy+3y^2-2)&=2x+2y+2xy'+6yy'\\&=2x+2y+y'(2x+6y)\\y'&=-\frac{x+y}{x+3y}\\y'(1)=0\end{aligned}$$

Finally, we find y''

$$\begin{aligned}y'&=-\frac{x+y}{x+3y}\\y''&=-\frac{1+y'}{x+3y}-\frac{(x+y)(1+3y')}{(x+3y)^2}\\y''(1)&=-\frac{1+0}{-1+3}-0\\&=-\frac{1}{2}\end{aligned}$$

Question 1.9

If $f(x)=x^4-2x^3-x^2-4x+3$ then for $f'(x)=0$ find the value of x

Solving for $f'(x)$ we obtain

$$\frac{df}{dx}=4x^3-6x^2-2x-4$$

Unfortunately I was unable to find a way to factorize this besides either using a graphing calculator or running some python code, whatever the case, the only real root is $x=2$ and the others are complex.

Question 1.10

Find the absolute minima and maxima for $f(x)=x^3+2x^2+x-1$ on $x\in[-1,1]$

$$\frac{df}{dx}=3x^2+4x+1$$

The roots are $(3x+1)(x+1)$. We can now simply plug them into the original equation to find which one is the minima as both roots fall between $[-1,1]$.

$$f(-1) = -1 + 2 - 1 - 1 = -1$$
$$f\left(-\frac{1}{3}\right) = -\frac{31}{27}$$

Next we must calculate $f(1)$

$$f(1) = 1 + 2 + 1 - 1 = 3.$$

Now, using the double derivative test on both inflection points, note that $f''(-1) = -2$, hence it is a local maxima and the function is decreasing from this point forward, then $f''\left(-\frac{1}{3}\right) = 2$, therefore the function is increasing after that point. Since the function is only monotone increasing on $\left[-\frac{1}{3}, 1\right])$ in the interval, we can conclude that the maxima must either be on $f(-1)$ (where the function was at its local maxima) or at $f(1)$ (since that was the highest point of the function when function next increased after its minima). Indeed, $f(1)$ is the maxima.

Similarly, we may conclude that $f\left(-\frac{1}{3}\right)$ is the absolute minimum, since it is the only local minimum.