

Math Prep Course Day One

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1 Logic Exercises

Problem 1.1. Truth table for $P \wedge (Q \vee R) \iff (P \wedge Q) \vee (P \wedge R)$

Table for the left hand side

P	Q	R	$Q \vee R$	$P \wedge (Q \vee R)$
T	T	T	T	T
T	T	F	T	T
T	F	T	T	T
T	F	F	F	F
F	T	T	T	F
F	T	F	T	F
F	F	T	T	F
F	F	F	F	F

Table for the Right hand side

P	Q	R	$P \wedge Q$	$P \wedge R$	$(P \wedge Q) \vee (P \wedge R)$
T	T	T	T	T	T
T	T	F	T	F	T
T	F	T	F	T	T
T	F	F	F	F	F
F	T	T	F	F	F
F	T	F	F	F	F
F	F	T	F	F	F
F	F	F	F	F	F

Final table

$P \wedge (Q \vee R)$	$(P \wedge Q) \vee (P \wedge R)$	$P \wedge (Q \vee R) \iff (P \wedge Q) \vee (P \wedge R)$
T	T	T
T	T	T
T	T	T
F	F	T
F	F	T
F	F	T
F	F	T
F	F	T

Thus the two logical statements are equivalent for all truth values.

Problem 1.2. Prove or disprove

$$\forall x \in \mathbb{R}, x^2 > 0$$

Solution. For the universal quantifier \forall , a solution by contradiction is sufficient. We show that there exists an element $x \in \mathbb{R}$ such that $x^2 \not> 0$.

Take $x = 0$, then $x^2 = 0 \not> 0$, and since we have $\exists x \in \mathbb{R}, x^2 \not> 0$ then the statement $\forall x \in \mathbb{R}, x^2 > 0$ cannot simultaneously hold, thus contradiction.

Problem 1.3. Prove or disprove

$$\forall n \in \mathbb{N}, n^2 > 2$$

Solution. Following the same pattern as last time, we choose the natural number $n = 1$ and obtain $n^2 = 1 \not> 2$, thus obtaining our contradiction.

Problem 1.4. Prove or Disprove

$$\forall x \in \mathbb{R}, \exists y, y < x$$

Solution. For any real number $x \in \mathbb{R}$ we know that $x - 1$ is a well-defined real number. We also know that $x - 1 < x$. Therefore, for each real number x , there exists at least the number $y = x - 1$ such that $y < x$, thus the proposition holds true.

2 Set Operations

Problem 2.1. Take the following sets

$$A = \{-1, 0, 3, 4, 5, 6\}$$

$$B = \{0, 1, 2, 3, 4\}$$

$$C = \{0, 2, 4\}$$

Find $A \cap C$, $A \cap B$, $B \cap C$, $A \cap B \cap C$

Solution.

$$A \cap B = \{0, 3, 4\}$$

$$A \cap C = \{0, 4\}$$

$$B \cap C = \{0, 2, 4\}$$

$$A \cap B \cap C = \{0, 4\}$$

3 Summation and Product Formulae

Problem 3.1. Evaluate

$$\sum_{x=2}^4 (x^2 - x)$$

Solution. Use $x^2 - x = x(x - 1)$ Then our expression becomes via substitution

$$\begin{aligned}\sum_{x=2}^4 x(x - 1) &= 2(2 - 1) + 3(3 - 1) + 4(4 - 1) \\ &= 2 + 6 + 12 \\ &= 20\end{aligned}$$

Problem 3.2. Evaluate

$$\prod_{x=0}^5 x^2$$

Solution. Since $0^2 = 0$, the first term $x = 0$ reduces the entire product to zero. More formally we may write

$$\begin{aligned}\prod_{x=0}^5 x^2 &= 0^2 \times \prod_{x=1}^5 x^2 \\ &= 0\end{aligned}$$

Problem 3.3. Evaluate

$$\prod_{x=1}^4 x^2 - (x - 1)^2$$

Solution. Simplifying $x^2 - (x - 1)^2 = 2x - 1$

$$\begin{aligned}\prod_{x=1}^4 (2x - 1) &= (2 - 1)(4 - 1)(6 - 1)(8 - 1) \\ &= 105\end{aligned}$$

Problem 3.4. Evaluate

$$\sum_{x=1}^{100} x$$

Solution. This question is best solved with the aid of a useful and rather famous identity $\sum_{x=1}^n x = \frac{n(n+1)}{2}$. But if you think it is unfair to rely on prior knowledge consider this solution:

$$\begin{aligned}\sum_{x=1}^{100} x &= \sum_{x=1}^{49} x + \sum_{x=1}^{49} (100 - x) + 50 + 100 \\ &= \sum_{x=1}^{49} 100 + 150 \\ &= 5050\end{aligned}$$

4 Binomial Theorem

Problem 4.1. Evaluate

$$(10.1)^3$$

Solution. Expand the term inside the cube to $(10 + 0.1)^3$

$$\begin{aligned}(10 + 0.1)^3 &= \sum_{i=0}^3 \binom{3}{i} (10)^i (0.1)^{3-i} \\&= \sum_{i=0}^3 \binom{3}{i} (10)^i (10^{-1})^{3-i} \\&= \sum_{i=0}^3 \binom{3}{i} (10)^i (10)^{i-3} \\&= \sum_{i=0}^3 \binom{3}{i} (10)^{2i-3} \\&= (10)^{-3} + 3(10)^{-1} + 3(10) + (10)^3\end{aligned}$$

Problem 4.2. Evaluate

$$(99)^3$$

Solution. Expand the term inside the cube to $(100 + (-1))^3$

$$\begin{aligned}(100 - 1)^3 &= \sum_{i=0}^3 \binom{3}{i} (100)^i (-1)^{3-i} \\&= (-1)^3 + 3(100)(-1)^2 + 3(100)^2(-1) + (100)^3\end{aligned}$$

5 Inequalities

Problem 5.1.

$$||x + 2| - |x - 2|| > 2$$

Solution. We begin by first splitting the intervals of $|x + 2|$ and $|x - 2|$ respectively to simplify the equation. Thus we obtain

$$|x + 2| = \begin{cases} x + 2, & x \geq -2 \\ -x - 2, & x \leq -2 \end{cases}$$

Similarly

$$|x - 2| = \begin{cases} x - 2, & x \geq 2 \\ 2 - x, & x \leq 2 \end{cases}$$

Thus our intervals become $\{(-\infty, -2], (-2, 2), [2, \infty)\}$. Evaluating our functions on these intervals, we obtain

$$\begin{array}{ll}(-x - 2) - (2 - x) = -4 & x \in (-\infty, -2] \\(x + 2) - (2 - x) = 2x & x \in (-2, 2) \\(x + 2) - (x - 2) = 4 & x \in [2, \infty)\end{array}$$

Notice that for the first and last case, the absolute value is always greater than 2. Therefore we direct our attention to the middle case where we impose the condition

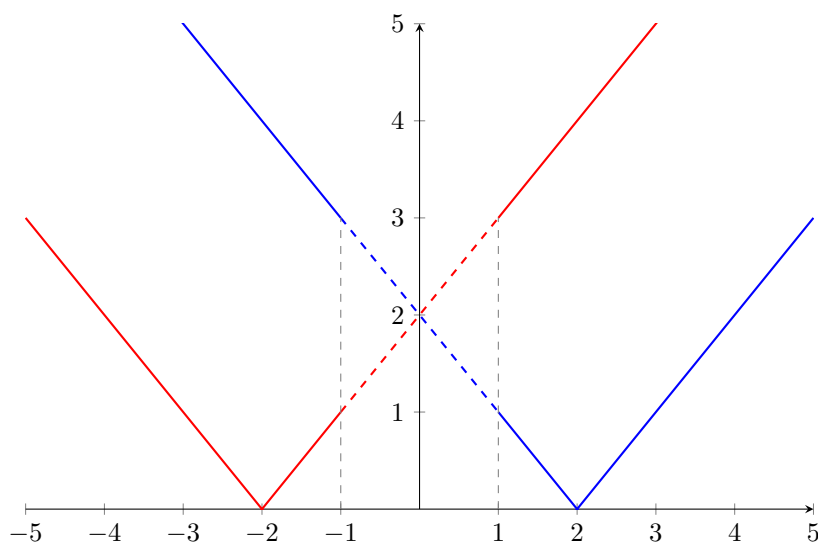
$$\begin{aligned} |2x| &> 2 & x &\in (-2, 2) \\ |x| &> 1 \end{aligned}$$

The solutions to which in the interval $(-2, 2)$ are $(-2, -1)$ and $(1, 2)$.

Our solution set is therefore $(-\infty, -2] \cup (-2, -1) \cup (1, 2) \cup [2, \infty)$. Simplifying, we obtain

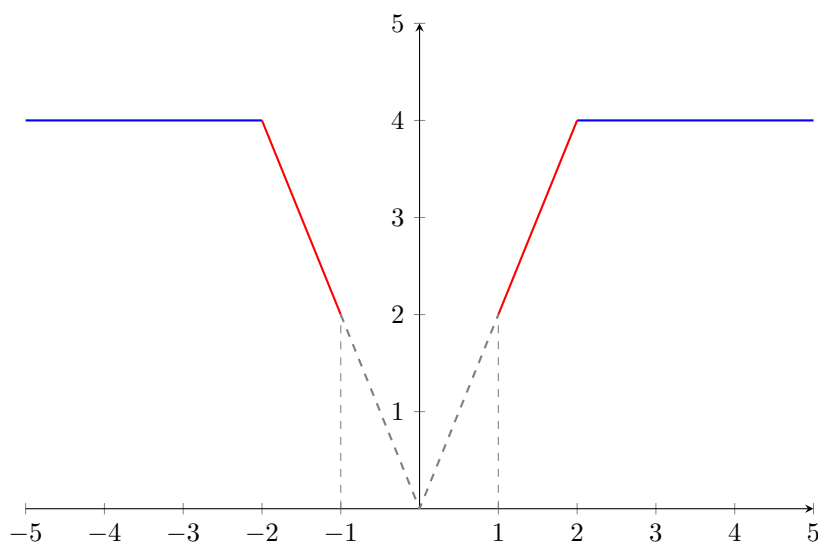
$$(-\infty, -1) \cup (1, \infty)$$

Individual graphs



The blue line indicates the function $|x - 2|$ whilst the red indicates $|x + 2|$

Graph of $||x + 2| - |x - 2||$



Problem 5.2.

$$(x - 3)(x + 2)x < 0$$

Solution. The roots of the polynomial given are

$$\begin{aligned} x &= 0 \\ x + 2 = 0 &\implies x = -2 \\ x - 3 = 0 &\implies x = 3 \end{aligned}$$

Thus our intervals become $\{(-\infty, -2), (-2, 0), (0, 3), (3, \infty)\}$ ¹ Evaluating our polynomials on the following intervals we observe that

$$(x - 3)(x + 2)x < 0 \qquad x \in (-\infty, -2)$$

Since all three terms are negative here, this interval satisfies our condition.

$$(x - 3)(x + 2)x < 0 \qquad x \in (-2, 0)$$

Here only the $x - 3$ and x terms are negative, thus we have an overall positive evaluation. This interval does not satisfy our condition.

$$(x - 3)(x + 2)x < 0 \qquad x \in (0, 3)$$

Since only the $x - 3$ term is negative, this interval too satisfies our condition.

$$(x - 3)(x + 2)x < 0 \qquad x \in (3, \infty)$$

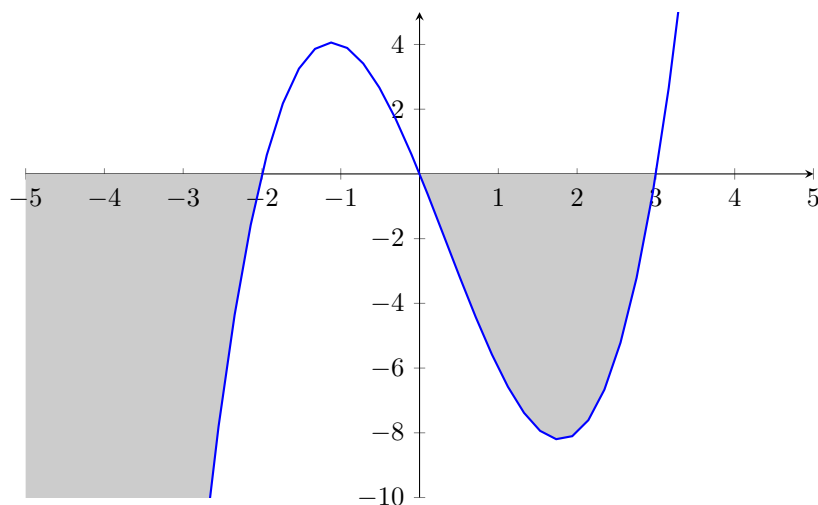
Beyond this point all our vales are positive, not satisfying our condition. We can sum up our results in the following table (Courtesy of Sherlock for the idea!)

<i>Interval</i>	$x - 3$	$x + 2$	x	<i>Product</i>
$(-\infty, -2)$	—	—	—	—
$(-2, 0)$	—	+	—	+
$(0, 3)$	—	+	+	—
$(3, \infty)$	+	+	+	+

Our solution is therefore

$$(-\infty, -2) \cup (0, 3)$$

¹We exclude the roots themselves since the polynomial is zero at those points and thus not less than zero as the condition requires



Problem 5.3.

$$-2 < 1 - \frac{1}{x} < 0$$

Solution. Simplify the expression to $-2 < \frac{x-1}{x} < 0$

$$-2x < x - 1 < 0$$

$$2x > 1 - x > 0$$

Since the final > 0 condition ensures that $1 - x$ is positive, we can divide the equation by it without having to check for its signature. Therefore

$$\frac{2x}{1-x} > 1 > 0$$

The last inequality is trivial so we omit it. We now check when $\frac{2x}{1-x} > 1$ or rather, $\frac{3x-1}{1-x} > 0$. Observe that $1 - x$ is positive on the domain $(-\infty, 1)$ and negative otherwise², similarly $3x - 1$ is positive on the domain $(\frac{1}{3}, \infty)$ and negative otherwise.

All in all this gives us the three domains $\{(-\infty, \frac{1}{3}), (\frac{1}{3}, 1), (1, \infty)\}$. We must now evaluate our numerator and denominator on each domain. Again, we can use a handy table to quickly arrange and summarize our results.

<i>Interval</i>	$3x - 1$	$1 - x$	<i>Product</i>
$(-\infty, \frac{1}{3})$	-	+	-
$(\frac{1}{3}, 1)$	+	+	+
$(1, \infty)$	+	-	-

The only valid interval is therefore

$$\left(\frac{1}{3}, 1\right)$$

²We omit the point 1 since our expression is undefined at that point

