

Math Prep Course Day Three

Ashhad Shahzad

September 2025

1 Graph sketching

Question 1.1

Find the quadratic function that passes through the points $(0,0)$, $(1,1)$ $(3,-1)$

A quadratic function is of the form $ax^2 + bx + c$. Utilizing the three points given we can establish the following system of equations

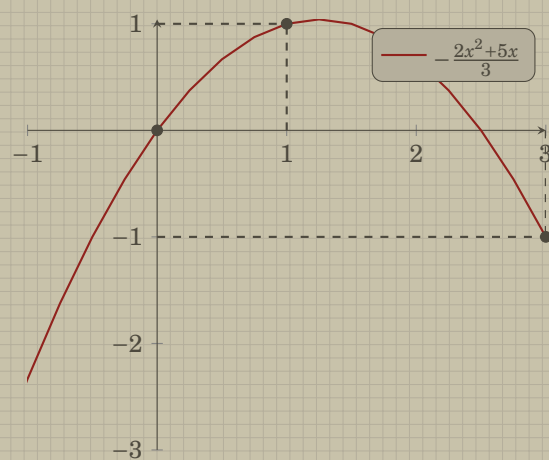
$$a(0)^2 + b(0) + c = 0 \quad (1)$$

$$a(1)^2 + b(1) + c = 1 \quad (2)$$

$$a(3)^2 + b(3) + c = -1 \quad (3)$$

From equation (1) we obtain $c = 0$. Subtracting $3 \times Eq(2)$ from (3) we obtain $6a = -4$, $a = -\frac{2}{3}$ and via re-substitution get $b = \frac{5}{3}$. Our resulting equation is therefore

$$f(x) = -\frac{2}{3}x^2 + \frac{5}{3}x$$



Question 1.2

Find the quadratic function passing through the points $(-1,1)$, $(1,-2)$, $(3,4)$

Again, we set up the following system of equations

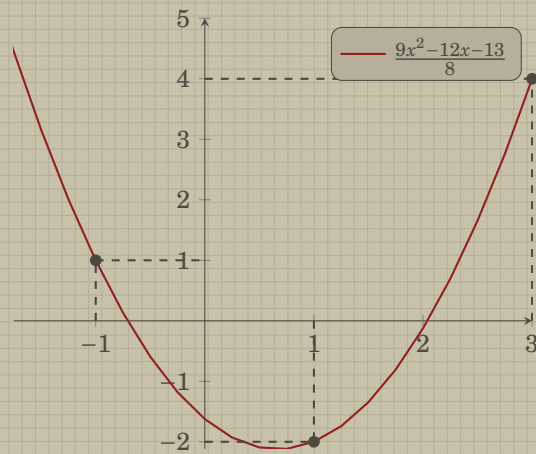
$$a(-1)^2 + b(-1) + c = 1 \quad (1)$$

$$a(1)^2 + b(1) + c = -2 \quad (2)$$

$$a(3)^2 + b(3) + c = 4 \quad (3)$$

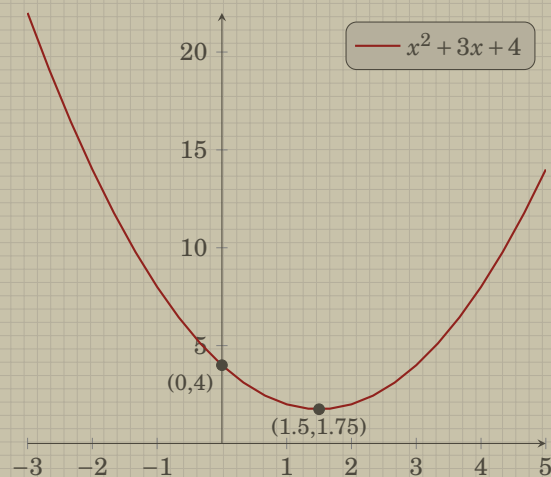
Subtracting equation (1) from equation (2) we obtain $2b = -3$, so $b = -\frac{3}{2}$. Subtracting equation (1) from equation (3) we obtain $8a + 4b = 3$, which after simplifying becomes $a = \frac{9}{8}$. Substituting the value of a and b into any of the equations we can recover $c = -\frac{13}{8}$. Our equation is this

$$f(x) = \frac{9}{8}x^2 - \frac{3}{2}x - \frac{13}{8}$$



Question 1.3

Sketch the graph of the function $f(x) = x^2 - 3x + 4$ on the interval $(-3, 5)$



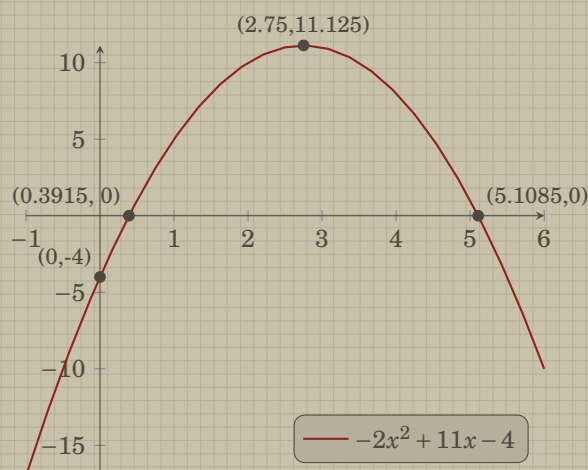
The maximum value of the graph is at $x = -3$, therefore

$$(-3)^2 - 3(-3) + 4 = 9 + 9 + 4 = 22$$

The minimum value is indicated on the graph.

Question 1.4

Draw the graph of the function $f(x) = -2x^2 + 11x - 4$



Question 1.5

Let $f(x) = ax + b$ be an arbitrary linear function. Prove that $f(f(x))$ is also linear.

By definition, a linear function is one whose only power in the variable is one. No higher or lower or fractional powers are allowed. Let us compute $f(f(x))$

$$\begin{aligned}
 f(f(x)) &= f(ax + b) \\
 &= a(ax + b) + b \\
 &= a^2x + ab + b \\
 &= a'x + b' \qquad \because \text{let } a^2 = a', ab + b = b'
 \end{aligned}$$

Note that we can perform the last step since a and b are constants. The power of x is still one in the final expression. Therefore the composition of two linear functions is linear.

Question 1.6

For the following functions

$$\begin{aligned}
 f(z) &= \sqrt{4 - z^2} \\
 g(x) &= 2x + 3
 \end{aligned}$$

Find the domain of $y = f(g(x))$ for which y is well defined.

We first immediately note that both the domain and range of $g(x)$ are the entirety of \mathbb{R} . The only restriction then comes from $f(z)$. To find the domain of $f(z)$ we must find the interval for which the square root is non-negative

$$\begin{aligned}
 4 - z^2 &\geq 0 \\
 z^2 &\leq 4 \\
 z &\leq |2|
 \end{aligned}$$

Now that we know the domain of $f(z)$, we now have to find the interval such that $-2 \leq g(x) \leq 2$. Since g is a monotone function and nicely invertible, we can find the inverse of g and plug in the values 2 and -2 to obtain

the interval for our required domain.

$$\begin{aligned}
 g(x) &= 2x + 3 \\
 g^{-1}(x) &= \frac{x-3}{2} \\
 g^{-1}(-2) &= \frac{-2-3}{2} = \frac{-5}{2} \\
 g^{-1}(2) &= \frac{2-3}{2} = \frac{-1}{2}
 \end{aligned}$$

Thus the domain of our function $y = f(g(x))$ is $[-\frac{5}{2}, -\frac{1}{2}]$

Question 1.7

Find the inverse of $f(x) = \sqrt{x} + \sqrt{x-1}$ and its applicable domain.

By observation we see that $f(x)$ has a domain $[1, \infty)$. Since it is a monotone increasing function we can find its range via plugging in the endpoints of the domain

$$\begin{aligned}
 f(1) &= \sqrt{1} + \sqrt{1-1} = \sqrt{1} \\
 \lim_{x \rightarrow \infty} f(x) &= \lim_{x \rightarrow \infty} \sqrt{x} + \lim_{x \rightarrow \infty} \sqrt{x-1} = \infty
 \end{aligned}$$

Therefore our range is $[1, \infty)$ as well and this will be the domain of our inverse function. To find the inverse function itself we must do some rather annoying algebraic manipulation

$$\begin{aligned}
 y &= \sqrt{x} + \sqrt{x-1} \\
 y^2 &= x + x - 1 + 2(\sqrt{x})(\sqrt{x-1}) \\
 y^2 - 2x + 1 &= 2\sqrt{x}\sqrt{x-1} \\
 y^4 - 4y^2x + 2y^2 + 4x^2 - 4x + 1 &= 4x^2 - 4x \\
 y^4 - 4y^2x + 2y^2 + 1 &= 0 \\
 x &= \frac{y^4 + 2y^2 + 1}{4y^2} \\
 x &= \left(\frac{y^2 + 1}{2y}\right)^2
 \end{aligned}$$

Question 1.8

Find the solution for $\frac{x+1}{x-2} \leq \frac{x+2}{x+3}$.

$$\begin{aligned}
 \frac{x+1}{x-2} - \frac{x+2}{x+3} &\leq 0 \\
 \frac{(x+1)(x+3) - (x+2)(x-2)}{(x-2)(x+3)} &\leq 0 \\
 \frac{x^2 + 4x + 3 - x^2 + 4}{(x-2)(x+3)} &\leq 0 \\
 \frac{4x + 7}{(x-2)(x+3)} &\leq 0
 \end{aligned}$$

The domain splits of the above function are $(-\infty, -3)$, $(-3, -1.75]$, $[-1.75, 2)$, $(2, \infty)$. We can again use a table to find out the final signature results of our function

<i>Interval</i>	$4x+7$	$x-2$	$x+3$	<i>Sign</i>
$(-\infty, -3)$	-	-	-	-
$(-3, -1.75]$	-	-	+	+
$[-1.75, 2)$	+	-	+	-
$(2, \infty)$	+	+	+	+

Therefore our solution set is $(-\infty, -3) \cup [-1.75, 2)$

Question 1.9

Find the solution of $\frac{3x}{x^2+2} \geq \frac{1}{x-1}$

$$\begin{aligned}\frac{3x}{x^2+2} - \frac{1}{x-1} &\geq 0 \\ \frac{3x(x-1) - (x^2+2)}{(x^2+2)(x-1)} &\geq 0 \\ \frac{2x^2 - 3x - 2}{(x^2+2)(x-1)} &\geq 0 \\ \frac{(2x+1)(x-2)}{(x^2+2)(x-1)} &\geq 0\end{aligned}$$

Since $x^2 + 2$ is always positive, we don't need to split the domain with respect to that term, for the rest we obtain $(-\infty, -\frac{1}{2}]$, $[-\frac{1}{2}, 1)$, $(1, 2]$, $[2, \infty)$. The table looks like

<i>Interval</i>	$2x+1$	$x-2$	$x-1$	<i>Sign</i>
$(-\infty, -\frac{1}{2}]$	-	-	-	-
$[-\frac{1}{2}, 1)$	+	-	-	+
$(1, 2]$	+	-	+	-
$[2, \infty)$	+	+	+	+

For positivity, our required intervals are $[-\frac{1}{2}, 1) \cup [2, \infty)$