Math Prep Course Day One

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1 Logic Exercises

Problem 1.1. Truth table for $P \wedge (Q \vee R) \iff (P \wedge Q) \vee (P \wedge R)$

Table for the left hand side

P	Q	R	$Q \vee R$	$P \wedge (Q \vee R)$
T	T	T	T	T
T	T	F	T	T
T	F	T	T	T
T	F	F	F	F
F	T	T	T	F
F	T	F	T	F
F	F	T	T	F
F	F	F	F	F

Table for the Right hand side

P	Q	R	$P \wedge Q$	$P \wedge R$	$(P \wedge Q) \vee (P \wedge R)$
T	T	T	T	T	T
T	T	F	T	F	T
T	F	T	F	T	T
T	F	F	F	F	F
F	T	T	F	F	F
F	T	F	F	F	F
F	F	T	F	F	F
F	F	F	F	F	F

Final table

$P \wedge (Q \wedge R)$	$(P \wedge Q) \vee (P \wedge R)$	$P \wedge (Q \wedge R) \iff (P \wedge Q) \vee (P \wedge R)$
T	T	T
T	T	T
T	T	T
F	F	T
F	F	T
F	F	T
F	F	T
F	F	T

Thus the two logical statements are equivalent for all truth values.

Problem 1.2. Prove or disprove

$$\forall x \in \mathbb{R},\ x^2 > 0$$

Solution. For the universal quantifier $\forall$, a solution by contradiction is sufficient. We show that there exists an element $x \in \mathbb{R}$ such that $x^2 \ngtr 0$.

Take $x = 0$, then $x^2 = 0 \ngtr 0$, and since we have $\exists x \in \mathbb{R},\ x^2 \ngtr 0$ then the statement $\forall x \in \mathbb{R},\ x^2 > 0$ cannot simultaneously hold, thus contradiction.

Problem 1.3. Prove or disprove

$$\forall n \in \mathbb{N},\ n^2 > 2$$

Solution. Following the same pattern as last time, we choose the natural number $n = 1$ and obtain $n^2 = 1 \ngtr 2$, thus obtaining our contradiction.

Problem 1.4. Prove or Disprove

$$\forall x \in \mathbb{R},\ \exists y,\ y < x$$

Solution. For any real number $x \in \mathbb{R}$ we know that $x - 1$ is a well-defined real number. We also know that $x - 1 < x$. Therefore, for each real number x, there exists at least the number $y = x - 1$ such that $y < x$, thus the proposition holds true.

2 Set Operations

Problem 2.1. Take the following sets

$$\begin{aligned} A &= \{-1, 0, 3, 4, 5, 6\} \\ B &= \{0, 1, 2, 3, 4\} \\ C &= \{0, 2, 4\} \end{aligned}$$

Find $A \cap C,\ A \cap B,\ B \cap C,\ A \cap B \cap C$

Solution.

$$\begin{aligned} A \cap B &= \{0, 3, 4\} \\ A \cap C &= \{0, 4\} \\ B \cap C &= \{0, 2, 4\} \\ A \cap B \cap C &= \{0, 4\} \end{aligned}$$

3 Summation and Product Formulae

Problem 3.1. Evaluate

$$\sum_{x=2}^{4}(x^2 - x)$$

Solution. Use $x^2 - x = x(x-1)$ Then our expression becomes via substitution

$$\begin{aligned}\sum_{x=2}^{4} x(x-1) &= 2(2-1) + 3(3-1) + 4(4-1)\\ &= 2 + 6 + 12\\ &= 20\end{aligned}$$

Problem 3.2. Evaluate

$$\prod_{x=0}^{5} x^2$$

Solution. Since $0^2 = 0$, the first term $x = 0$ reduces the entire product to zero. More formally we may write

$$\begin{aligned}\prod_{x=5}^{5} x^2 &= 0^2 \times \prod_{x=1}^{5} x^2\\ &= 0\end{aligned}$$

Problem 3.3. Evaluate

$$\prod_{x=1}^{4} x^2 - (x-1)^2$$

Solution. Simplifying $x^2 - (x-1)^2 = 2x - 1$

$$\begin{aligned}\prod_{x=1}^{4}(2x-1) &= (2-1)(4-1)(6-1)(8-1)\\ &= 105\end{aligned}$$

Problem 3.4. Evaluate

$$\sum_{x=1}^{100} x$$

Solution. This question is best solved with the aid of a useful and rather famous identity $\sum_{x=1}^{n} = \frac{n(n-1)}{2}$. But if you think it is unfair to rely on prior knowledge consider this solution:

$$\begin{aligned}\sum_{x=1}^{100} x &= \sum_{x=1}^{49} x + \sum_{x=1}^{49}(100 - x) + 50 + 100\\ &= \sum_{x=1}^{49} 100 + 150\\ &= 5050\end{aligned}$$

4 Binomial Theorem

Problem 4.1. Evaluate

$$(10.1)^3$$

Solution. Expand the term inside the cube to $(10+0.1)^3$

$$\begin{aligned}
(10+0.1)^3 &= \sum_{i=0}^{3} \binom{i}{3} (10)^i (0.1)^{3-i} \\
&= \sum_{i=0}^{3} \binom{i}{3} (10)^i (10^{-1})^{3-i} \\
&= \sum_{i=0}^{3} \binom{i}{3} (10)^i (10)^{i-3} \\
&= \sum_{i=0}^{3} \binom{i}{3} (10)^{2i-3} \\
&= (10)^{-3} + 3(10)^{-1} + 3(10) + (10)^3
\end{aligned}$$

Problem 4.2. Evaluate

$$(99)^3$$

Solution. Expand the term inside the cube to $(100+(-1))^3$

$$\begin{aligned}
(100-1)^3 &= \sum_{i=0}^{3} \binom{i}{3} (100)^i (-1)^{3-i} \\
&= (-1)^3 + 3(100)(-1)^2 + 3(100)^2(-1) + (100)^3
\end{aligned}$$

5 Inequalities

Problem 5.1.

$$||x+2| - |x-2|| > 2$$

Solution. We begin by first splitting the intervals of $|x+2|$ and $|x-2|$ respectively to simplify the equation. Thus we obtain

$$|x+2| = \begin{cases} x+2, & x \geq -2 \\ -x-2, & x \leq -2 \end{cases}$$

Similarly

$$|x-2| = \begin{cases} x-2, & x \geq 2 \\ 2-x, & x \leq 2 \end{cases}$$

Thus our intervals become $\{(-\infty,-2], (-2,2),\ [2,\infty)\}$. Evaluating our functions on these intervals, we obtain

$$\begin{aligned}
(-x-2) - (2-x) &= -4 & x &\in (-\infty,-2] \\
(x+2) - (2-x) &= 2x & x &\in (-2,2) \\
(x+2) - (x-2) &= 4 & x &\in [2,\infty)
\end{aligned}$$

Notice that for the first and last case, the absolute value is always greater than 2. Therefore we direct our attention to the middle case where we impose the condition

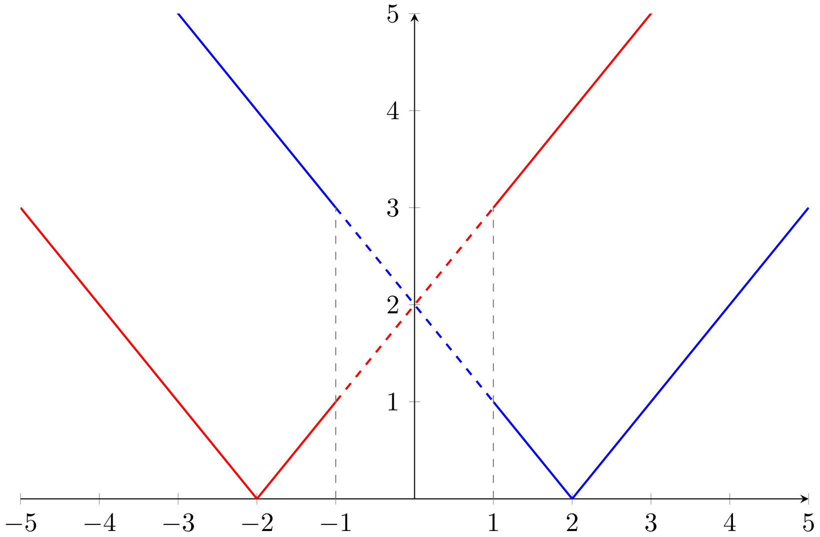
$$|2x| > 2 \qquad\qquad x \in (-2, 2)$$
$$|x| > 1$$

The solutions to which in the interval $(-2, 2)$ are $(-2, -1)$ and $(1, 2)$.

Our solution set is therefore $(-\infty, -2] \cup (-2, -1) \cup (1, 2) \cup [2, \infty)$. Simplifying, we obtain

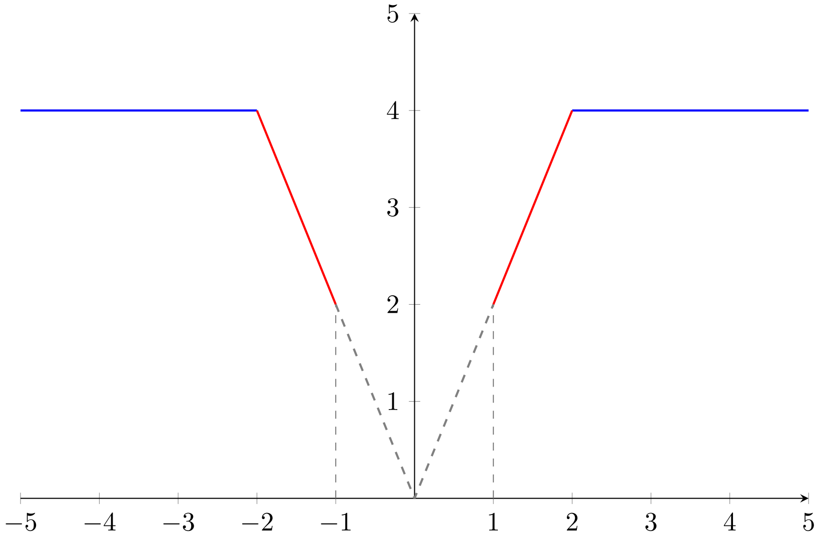
$$(-\infty, -1) \cup (1, \infty)$$

Individual graphs



The blue line indicates the function $|x - 2|$ whilst the red indicates $|x + 2|$

Graph of $||x + 2| - |x - 2||$



Problem 5.2.

$$(x-3)(x+2)x < 0$$

Solution. The roots of the polynomial given are

$$\begin{aligned} x &= 0 \\ x+2 &= 0 \implies x = -2 \\ x-3 &= 0 \implies x = 3 \end{aligned}$$

Thus our intervals become $\{(-\infty,-2),(-2,0),(0,3),(3,\infty)\}$[1] Evaluating our polynomials on the following intervals we observe that

$$(x-3)(x+2)x < 0 \qquad x \in (-\infty,-2)$$

Since all three terms are negative here, this interval satisfies our condition.

$$(x-3)(x+2)x < 0 \qquad x \in (-2,0)$$

Here only the $x-3$ and x terms are negative, thus we have an overall positive evaluation. This interval does not satisfy our condition.

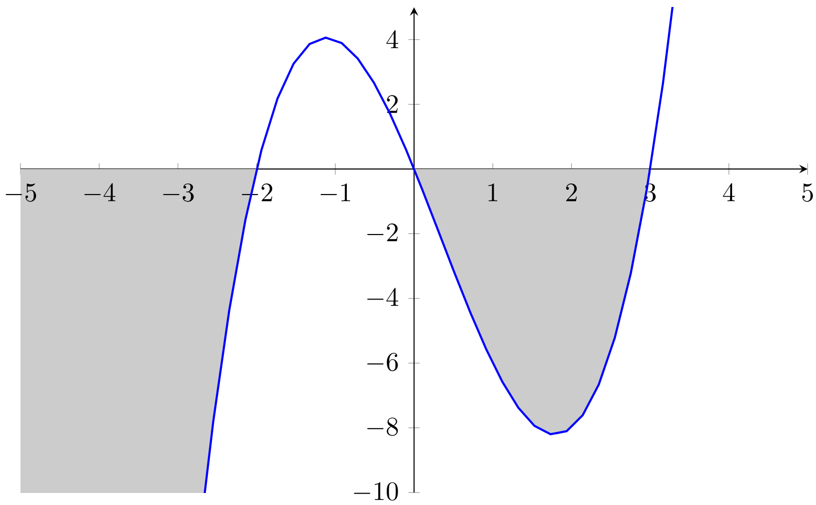
$$(x-3)(x+2)x < 0 \qquad x \in (0,3)$$

Since only the $x-3$ term is negative, this interval too satisfies our condition.

$$(x-3)(x+2)x < 0 \qquad x \in (3,\infty)$$

Beyond this point all our values are positive, thus our condition is not satisfied. The interval satisfying our requirement are therefore

$$(-\infty,-2) \cup (0,3)$$



[1] We exclude the roots themselves since the polynomial is zero at those points and thus not less than zero as the condition requires

Problem 5.3.

$$-2 < 1 - \frac{1}{x} < 0$$

Solution. Simplify the expression to $-2 < \frac{x-1}{x} < 0$

$$\begin{aligned} -2x < x - 1 < 0 \\ 2x > 1 - x > 0 \end{aligned}$$

Since the final > 0 condition ensures that $1 - x$ is positive, we can divide the equation by it without having to check for its signature. Therefore

$$\frac{2x}{1-x} > 1 > 0$$

The last inequality is trivial so we omit it. We now check when $\frac{2x}{1-x} > 1$ or rather, $\frac{3x-1}{1-x} > 0$. Observe that $1 - x$ is positive on the domain $(-\infty, 1)$ and negative otherwise[2], similarly $3x - 1$ is positive on the domain $(\frac{1}{3}, \infty)$ and negative otherwise.

All in all this gives us the three domains $\{(-\infty, \frac{1}{3}), (\frac{1}{3}, 1), (1, \infty)\}$. Evaluating the expression we observe that

$$\frac{3x-1}{1-x} \qquad x \in \left(-\infty, \frac{1}{3}\right)$$

The numerator is negative whilst the denominator is positive, thus this does not satisfy our condition.

$$\frac{3x-1}{1-x} \qquad x \in \left(\frac{1}{3}, 1\right)$$

The numerator is positive and so is the denominator, thus this interval satisfies our condition.

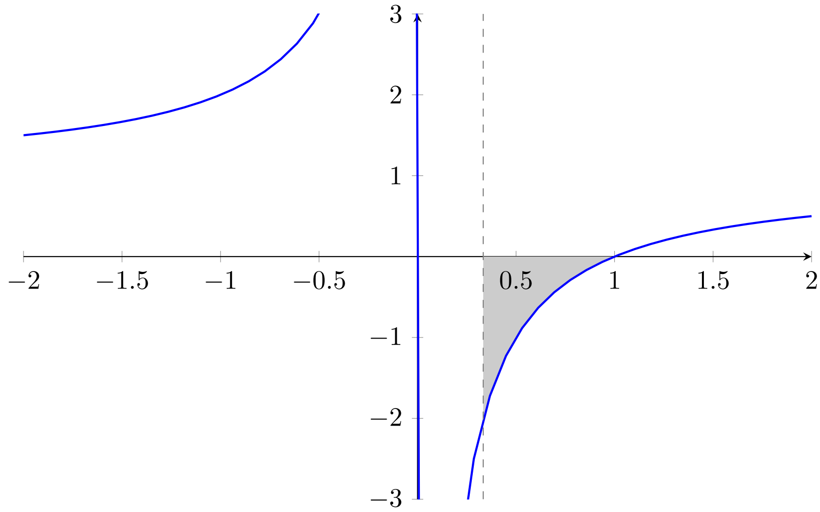
$$\frac{3x-1}{1-x} \qquad x \in (1, \infty)$$

Here, the numerator is positive but the denominator is negative, thus this interval too does not satisfy our condition.

The only interval is therefore

$$\left(\frac{1}{3}, 1\right)$$

[2] We omit the point 1 since our expression is undefined at that point


3
2
1
−1
−2
−3
−2
−1.5
−1
−0.5
0.5
1
1.5
2