Math Prep Course Day One

Ashhad

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1 Definite Integrals

Question 1.1

Find $\int_{-3}^{3} f(x)\,dx$ Where

$$f(x) = \begin{cases} -x & x \in [-3,-1] \\ 1 & x \in [-1,1] \\ x & x \in [1,3] \end{cases}$$

To evaluate this, split the integral into three parts

$$\begin{aligned} \int_{-3}^{3} f(x)\,dx &= \int_{-3}^{-1} -x\,dx + \int_{-1}^{1} dx + \int_{1}^{3} x\,dx \\ &= \int_{1}^{3} x\,dx + 2 + \int_{1}^{3} x\,dx \\ &= 2\left[\frac{x^2}{2}\right]_1^3 + 2 \\ &= 2\left[\frac{9}{2} - \frac{1}{2}\right] + 2 \\ &= 10 \end{aligned}$$

Question 1.2

Calculate $\int_0^{2\pi} \sin^2 x\,dx$

$$\begin{aligned} \int_0^{2\pi} \sin^2 x\,dx &= \frac{1}{2}\int_0^{2\pi} 1 - \cos(2x)\,dx \\ &= \frac{1}{2}[x - \sin(2x)]_0^{2\pi} \\ &= \frac{1}{2}[2\pi] \\ &= \pi \end{aligned}$$

Question 1.3

Calculate $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^3 x\,dx$

Since we have $\sin^3(-x) = -\sin^3 x$, we may write

$$\begin{aligned}
\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^3 x\, dx &= \int_{-\frac{\pi}{2}}^{0} \sin^3 x\, dx + \int_{0}^{\frac{\pi}{2}} \sin^3 x\, dx \\
&= \int_{0}^{\frac{\pi}{2}} \sin^3(-x)\, dx + \int_{0}^{\frac{\pi}{2}} \sin^3 x\, dx \\
&= \int_{0}^{\frac{\pi}{2}} \sin^3 x + \sin^3(-x) \\
&= 0
\end{aligned}$$

Question 1.4

Calculate $\int_0^\infty xe^{-x}\, dx$

Using Integration by parts

$$\begin{aligned}
\int_0^\infty xe^{-x}\, dx &= \left[x \int e^{-x}\, dx\right]_0^\infty - \int_0^\infty \int e^{-x}\, dx \\
&= \left[-xe^{-x}\right]_0^\infty + \int_0^\infty e^{-x} \\
&= [0-0] + [1-0] \\
&= 1
\end{aligned}$$

Question 1.5

If $f(t+a) = f(t)$ then find $\int_0^{na} f(t)\, dt$ where $\int_0^a f(t)\, dt = \frac{a}{2}$

For this, we split the integral into n integrals.

$$\int_0^{na} f(t)\, dt = \sum_{i=0}^{n-1} \int_{ia}^{(i+1)a} f(t)\, dt$$

On each integral I_i in the sum, substitute in the expression $t = t - ia$

$$\begin{aligned}
\sum_{i=0}^{n-1} \int_{ia}^{(i+1)a} f(t)\, dt &= \sum_{i=0}^{n-1} \int_0^a f(t-ia)\, dt \\
&= \sum_{i=0}^{n-1} \int_0^a f(t)\, dt \\
&= \sum_{i=0}^{n-1} \frac{a}{2} \\
&= \frac{na}{2}
\end{aligned}$$

Question 1.6

Show that

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

Is orthogonal

An orthogonal matrix is one whose rows/columns form an orthonormal set.

Let e_1 represent the first column and e_2 represent the second one. To show orthogonality, it is sufficient to check that the dot product of e_1 and e_2 is zero.

$$\begin{aligned} e_1 \cdot e_2 &= (\cos\theta - \sin\theta)\cdot(\sin\theta + \cos\theta) \\ &= (\cos\theta\sin\theta - \sin\theta\cos\theta) \\ &= 0 \end{aligned}$$

Next we must show that they are both also unit vectors

$$\begin{aligned} ||e_1|| &= (\cos\theta)^2 + (-\sin\theta)^2 = 1 \\ ||e_2|| &= (\sin\theta)^2 + (\cos\theta)^2 = 1 \end{aligned}$$

Therefore the matrix is orthogonal.

Question 1.7

Find the inverse of

$$B = \begin{bmatrix} -3 & 1 & 1 \\ 2 & 3 & -1 \\ 4 & 2 & 1 \end{bmatrix}$$

Row Operations:

1. $R_1 \leftarrow R_1/(-3)$
2. $R_2 \leftarrow R_2 - 2R_1$
3. $R_3 \leftarrow R_3 - 4R_1 \leftarrow$ Matrix 2
4. $R_2 \leftarrow R_2/(\frac{11}{3})$
5. $R_1 \leftarrow R_1 + \frac{1}{3}R_2$
6. $R_3 \leftarrow R_3 - \frac{10}{3}R_2 \leftarrow$ Matrix 3
7. $R_3 \leftarrow R_3/(\frac{29}{11})$
8. $R_1 \leftarrow R_1 + \frac{4}{11}R_3$
9. $R_2 \leftarrow R_2 - \frac{1}{11}R_3 \leftarrow$ Matrix 4

$$\left[\begin{array}{ccc|ccc} -3 & 1 & 1 & 1 & 0 & 0 \\ 2 & 3 & -1 & 0 & 1 & 0 \\ 4 & 2 & 1 & 0 & 0 & 1 \end{array}\right] \tag{1}$$

$$\left[\begin{array}{ccc|ccc} 1 & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & 0 & 0 \\ 0 & \frac{11}{3} & -\frac{1}{3} & \frac{2}{3} & 1 & 0 \\ 0 & \frac{10}{3} & \frac{7}{3} & \frac{4}{3} & 0 & 1 \end{array}\right] \tag{2}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -\frac{12}{33} & -\frac{9}{33} & \frac{1}{11} & 0 \\ 0 & 1 & -\frac{1}{11} & \frac{2}{11} & \frac{3}{11} & 0 \\ 0 & 0 & \frac{29}{11} & \frac{8}{11} & -\frac{10}{11} & 1 \end{array}\right] \tag{3}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{5}{29} & -\frac{1}{29} & \frac{4}{29} \\ 0 & 1 & 0 & \frac{6}{29} & \frac{7}{29} & \frac{1}{29} \\ 0 & 0 & 1 & \frac{8}{29} & -\frac{10}{29} & \frac{11}{29} \end{array}\right] \tag{4}$$

So the inverse is

$$\frac{1}{29}\begin{bmatrix} -5 & -1 & 4 \\ 6 & 7 & 1 \\ 8 & -10 & 11 \end{bmatrix}$$