Math Prep Course Day One

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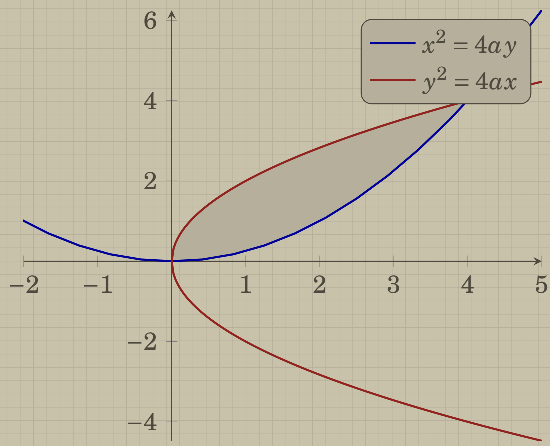
1 Graphing

Question 1.1

Find the area between the curves $y^2 = 4ax$ and $x^2 = 4ay$



First we must find the intersection points, we can obtain them as follows

$$\begin{aligned} y^2 &= 4ax \\ x &= \frac{y^2}{4a} \\ x^2 &= 4ay \\ \frac{y^4}{16a^2} &= 4ay \\ \frac{y^4 - 64a^3y}{16a^2} &= 0 \\ y(y^3 - 64a^3) &= 0 \end{aligned}$$

Resolving it further we get $y = 0,\ y = 4a$. Resubstituting to find x, we obtain $x = 0,\ x = 4a$ as well.

Given that we only have two intersection points, we can simply compute the integral of the two functions over the interval $[0, 4a]$ and take the absolute value of the difference (since area will always be positive)

So integrating $y^2 = 4ax$, note that we only need the positive right branch, thus our function becomes $y = 2\sqrt{ax}$

$$\begin{aligned} F(x) &= \int_0^{4a} 2\sqrt{ax}dx \\ &= 2\sqrt{a}\int_0^{4a} \sqrt{x} \\ &= 2\sqrt{a}\left[\frac{2x^{\frac{3}{2}}}{3}\right]_0^{4a} \\ &= \frac{4}{3}\sqrt{a}\ [8a^{\frac{3}{2}} - 0] \\ &= \frac{32a^2}{3} \end{aligned}$$

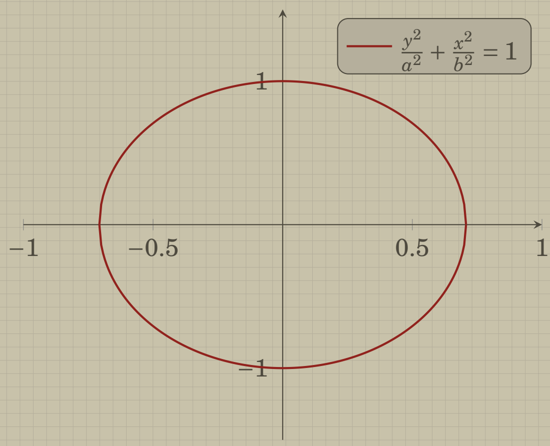
Now we integrate $x^2 = 4ay$.

$$
\begin{aligned}
f(x) &= \frac{x^2}{4a} \\
F(x) &= \frac{1}{4a}\int_0^{4a} x^2 \\
&= \frac{1}{4a}\left[\frac{x^3}{3}\right]_0^{4a} \\
&= \frac{1}{12a}[64a^3 - 0] \\
&= \frac{16a^2}{3}
\end{aligned}
$$

Thus the difference is $\frac{16a^2}{3}$ and that is precisely the area between the two curves.

Question 1.2

Sketch the curve $\frac{y^2}{a^2} + \frac{x^2}{b^2} = 1$ and find the value for which the tangent is parallel to the y-axis



In order to find the tangents parallel to the y-axis, we must find where the derivative of the function equals (or approaches) $\pm\infty$.

$$
\begin{aligned}
a^2x^2 + b^2y^2 &= a^2b^2 \\
2a^2x + 2b^2yy' &= 0 \\
y' &= -\frac{2a^2x}{2b^2y}
\end{aligned}
$$

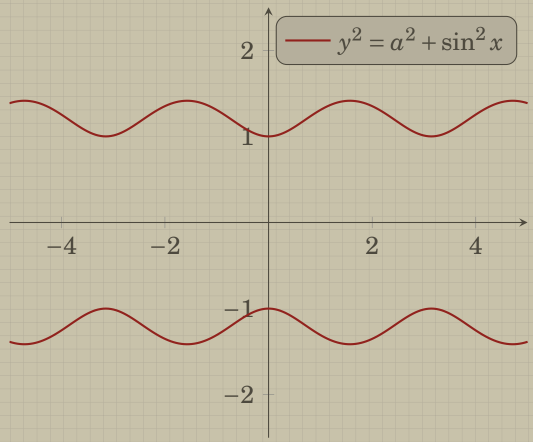
Thus that happens when $y = 0$. Plugging this into the equation of the ellipse to find the corresponding x values we obtain

$$
\begin{aligned}
0 + \frac{x^2}{b^2} &= 1 \\
x^2 &= b^2 \\
x &= \pm b
\end{aligned}
$$

The two tangent lines are therefore $x = \pm b$.

Question 1.3

Draw the curve $y^2 = a^2 + \sin^2 x$ where $a^2 > 0$.



2 Indefinite Integrals

Question 2.1

Integrate

$$\frac{x}{4 - x^2}$$

Using the substitution $x = 2\sin\theta$ we obtain

$$\begin{aligned}
\int \frac{x}{4-x^2}dx &= \int \frac{2\sin\theta}{4-4\sin^2\theta}(2\cos\theta d\theta) \\
&= \frac{4}{4}\frac{\sin\theta}{\cos^2\theta}\cos\theta d\theta \\
&= \frac{\sin\theta}{\cos\theta}d\theta \\
&= -\ln|\cos\theta| \\
&= -\ln|\sqrt{1-\sin^2\theta}| \\
&= -\ln\left|\sqrt{1-\frac{x^2}{4}}\right| \\
&= -\frac{1}{2}\ln\left|1-\frac{x^2}{4}\right| + C
\end{aligned}$$

Question 2.2

Integrate

$$\frac{2x}{4 - x^2}$$

Using the last question,

$$\begin{aligned}\int \frac{2x}{4-x^2} &= 2\int \frac{x}{4-x^2} \\ &= -\ln\left|1-\frac{x^2}{4}\right| + C\end{aligned}$$

Question 2.3

Integrate

$$x\sin x$$

We can solve this by using integration by parts

$$\begin{aligned}\int x\sin x &= x\int \sin x\,dx - \int\int \sin x\,dx \\ &= -x\cos x + \sin x + C\end{aligned}$$

Question 2.4

Integrate

$$e^{3x}x^2$$

Again, this may be solved by repeated applications of IBP

$$\begin{aligned}\int e^{3x}x^2 &= x^2\int e^{3x} - \int\int e^{3x}2x \\ &= x^2\frac{e^{3x}}{3} - \frac{2}{3}\int e^{3x}x \\ &= x^2\frac{e^{3x}}{3} - \frac{2}{3}\left[x\int e^{3x} - \int\int e^{3x}\right] \\ &= x^2\frac{e^{3x}}{3} - \frac{2}{3}\left[x\frac{e^{3x}}{3} - \frac{e^{3x}}{9}\right] \\ &= \frac{e^{3x}}{3}\left[x^2 - \frac{2x}{3} + \frac{2}{9}\right] + C\end{aligned}$$

Question 2.5

Integrate

$$\frac{1}{(x-1)(x-3)(x+3)}$$

To integrate this we first resolve by partial fractions

$$\begin{aligned}
\frac{1}{(x-1)(x-3)(x+3)} &= \frac{A}{x-1}+\frac{B}{x-3}+\frac{C}{x+3} \\
1 &= A(x^2-9)+B(x^2+2x-3)+C(x^2-4x+3) \\
2C &= B \\
A+B+C &= 0 \\
A+3C &= 0 \\
1+9A+3B-3C &= 0 \\
1+9A+3C &= 0 \\
1+8A &= 0 \\
A &= -\frac{1}{8} \\
C &= \frac{1}{24} \\
B &= \frac{1}{12}
\end{aligned}$$

Now we can integrate this

$$\begin{aligned}
\int \frac{1}{(x-1)(x-3)(x+3)} &= -\frac{1}{8}\int \frac{1}{x-1}+\frac{1}{12}\int \frac{1}{x-3}+\frac{1}{24}\int \frac{1}{x+3} \\
&= -\frac{1}{8}\ln|x-1|+\frac{1}{12}\ln|x-3|+\frac{1}{24}\ln|x+3| \\
&= \frac{1}{24}[-3\ln|x-1|+2\ln|x-3|+\ln|x+3|]+C
\end{aligned}$$