Math Prep Course Day Four

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1 Function Composition

Problem 1.1. Find The Domain of

$$f(x) = \frac{7 - \sqrt{x^2 - 9}}{\sqrt{25 - x^2}}$$

Solution. We can decompose $f(x)$ into its numerator and denominator functions and take the intersection of their domain. That can be done as follows

$$\begin{aligned} x^2 - 9 &\geq 0 \\ x^2 &\geq 9 \\ x &\geq |3| \end{aligned}$$

$$\begin{aligned} 25 - x^2 &\geq 0 \\ x^2 &\leq 25 \\ x &\leq |5| \end{aligned}$$

And so the intersection of the domains is simply $(-5, -3] \cup [3, 5)$.

Problem 1.2. Let $f(x) = \sqrt{x}$, $g(x) = \frac{4}{5-x}$, $h(x) = x^2$. Find

$$(h(h \circ g \circ f - f))(4)$$

And it's domain

Solution.

$$
\begin{aligned}
h(h \circ g \circ f - f) &= \left(\left(\frac{4}{5-\sqrt{x}}\right)^2 - \sqrt{x}\right)^2 \\
(h(h \circ g \circ f - f))(4) &= \left(\left(\frac{4}{5-\sqrt{4}}\right)^2 - \sqrt{4}\right)^2 \\
&= \left(\left(\frac{4}{5-2}\right)^2 - 2\right)^2 \\
&= \left(\left(\frac{4}{3}\right)^2 - 2\right)^2 \\
&= \frac{4}{81}
\end{aligned}
$$

The domain is clearly $[0, \infty)/\{25\}$

2 Polynomials

I've solved some of these using long division instead of the remainder theorem for my own sake. The remainder theorem is however far more efficient.

Problem 2.1. Given that $(x-3)$ is a factor of

$$f(x) = x^3 - 2x^2 + kx + 6$$

Show that $k = -5$.

Solution.

$$
\begin{array}{r|l}
 & x-3 \\
\hline
x^2 & x^3 - 2x^2 + kx + 6 \\
 & -x^3 + 3x^2 \\
\hline
x & x^2 + kx + 6 \\
 & -x^2 + 3x \\
\hline
k+3 & (k+3)x + 6 \\
 & -(k+3)x + 3k + 39 \\
\hline
 & 3k + 15
\end{array}
$$

Since $3k + 15 = 0$, we obtain $k = -5$

Problem 2.2. Evaluate

$$(x^5 - 4x^4 + 2x^3 - 3x^2 + 4x + 1) \div (x^2 + x + 1)$$

Solution.

$$
\begin{array}{r|l}
 & (x^2 + x + 1) \\
\hline
x^3 & x^5 - 4x^4 + 2x^3 - 3x^2 + 4x + 1 \\
 & -x^5 - x^4 - x^3 \\
\hline
-5x^2 & -5x^4 + x^3 - 3x^3 + 4x + 1 \\
 & 5x^4 + 5x^3 + 5x^2 \\
\hline
6x & 6x^3 + 2x^2 + 4x + 1 \\
 & -6x^3 - 6x^2 - 6x \\
\hline
-4 & -4x^2 - 2x + 1 \\
 & 4x^2 + 4x + 4 \\
\hline
 & 2x + 5
\end{array}
$$

Problem 2.3. Find the remainder

$$(4x^3 - 2x^2 + x + 1) \div (x - 1)$$

Solution.

$$
\begin{array}{r|l}
 & x - 1 \\
\hline
4x^2 & 4x^3 - 2x^2 + x + 1 \\
 & -4x^3 + 4x^2 \\
\hline
2x & 2x^2 + x + 1 \\
 & -2x^2 + 2x \\
\hline
3 & 3x + 1 \\
 & -3x + 3 \\
\hline
 & 4
\end{array}
$$

Problem 2.4.

$$f(x) = ax^3 - 7x^2 + 1$$

Has the factor $(x - 1)$. Find a.

Solution.

$$
\begin{array}{r|l}
 & x - 1 \\
\hline
ax^2 & ax^3 - 7x^2 + 1 \\
 & -ax^3 + ax^2 \\
\hline
(a-7)x & (a-7)x^2 + 1 \\
 & (7-a)x^2 + (a-7)x \\
\hline
(7-a) & (a-7)x + 1 \\
 & (7-a)x + (a-7) \\
\hline
 & a - 6
\end{array}
$$

Since the remainder must be zero, we obtain $a = 6$

Problem 2.5.

$$5x^6 - 3x^5 - x^4 + 1 = (x - 1)(x - 2) - x^2 + 3x - 1$$

Solution.

$$\begin{aligned}5x^6 - 3x^5 - x^4 + 1 &= x^2 - 3x + 2 - x^2 + 3x - 1\\ 5x^6 - 3x^5 - x^4 &= 0\\ x^4(5x^2 - 3x - 1) &= 0\end{aligned}$$

Solving for the quadratic, we obtain the solutions

$$x = \left\{0,\ \frac{3 \pm \sqrt{29}}{10}\right\}$$

3 Logarithmic and Exponential Functions

Problem 3.1.

$$\log_2 x = \log_2 x^2 - 4$$

Solution.

$$\begin{aligned}\log_2 x &= 2\log_2 x - 4\\ \log_2 x &= 4\\ x &= 2^4\\ x &= 16\end{aligned}$$

Problem 3.2.

$$2^x = 64$$

Solution.

$$\begin{aligned}2^x &= 64\\ x &= \log_2(2^6)\\ x &= 6\end{aligned}$$

Problem 3.3.

$$\ln(x-1) - \ln(x^2-1) = e$$

Solution.

$$\begin{aligned}\ln\left(\frac{x-1}{x^2-1}\right) &= e\\ \ln(x+1) &= -e\\ x+1 &= e^{-e}\\ x &= e^{-e} - 1\end{aligned}$$

However, $e^{-e} - 1 < 1$, which makes the expression $\ln(x-1)$ invalid, therefore we have an extraneous solution and the equation is unsatisfiable in $\mathbb{R}$

Problem 3.4. Find the domain and range of $x = \ln(e^x)$.

Solution. Observe that e^x is defined over all of $\mathbb{R}$ and has range $(0, \infty)$. Note that this is precisely the domain of the ln function, which has the range $(-\infty, \infty)$. Therefore the domain and range of the function are both $\mathbb{R}$.

In fact its just the constant function x since its the composition of a function with its inverse lmao.

Problem 3.5.

$$\log_2 x + \log_x 2 + 1 = 0$$

Solution.

$$\begin{aligned}
\log_2 x + \frac{\log_2 2}{\log_2 x} + 1 &= 0 && \because \log_a b = \frac{\log_c b}{\log_c a} \\
\log_2 x + \frac{1}{\log_2 x} + 1 &= 0 \\
(\log_2 x)^2 + \log_2 x + 1 &= 0 \\
y^2 + y + 1 &= 0 && \because \textit{let } y = \log_2 x
\end{aligned}$$

Since the solution is complex, I'm not going to evaluate it further.

Problem 3.6.

$$\log_5 x + \log_{10} 8 = 1$$

Solution.

$$\begin{aligned}
\log_5 x &= 1 - \log_{10} 8 \\
x &= 5^{(1 - \log_{10} 8)}
\end{aligned}$$

Problem 3.7.

$$(e^x)^2 + \ln e^{e^x} + e^{\ln_e 5} = 0$$

Solution.

$$\begin{aligned}
(e^x)^2 + e^x + 5 &= 0 \\
y^2 + y + 5 &= 0 && \because y = e^x
\end{aligned}$$

Again, the solution is complex.

Problem 3.8.

$$x^2 = e^{\ln\left(\frac{1}{5}x\right) + \ln(5x)} + \log_{10} e^{10}$$

Solution.

$$\begin{aligned} x^2 &= \left(\frac{x}{5}\right)(5x) + 10\log_{10} e \\ 0 &= 10\log_{10} e \end{aligned}$$

Thus the equation has no solutions.