Math Prep Course Day Five

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1 Trigonometric funtions

Question 1.1

Find the Maximum and minimum value of $\sin^2(\theta) + \cos^4(\theta)$

$$
\begin{aligned}
f(\theta) &= \sin^2(\theta) + (\cos^2(1 - \sin^2))\theta \\
f(\theta) &= \sin^2(\theta) + \cos^2(\theta) - \cos^2(\theta)\sin^2(\theta) \\
f(\theta) &= 1 - (\cos * \sin)^2(\theta)
\end{aligned}
$$

Since the term $(\cos * \sin)^2$ is always positive, our maximum point will be when that term is equal to zero, which gives us a maximum value of 1. Similarly, our minimum point will be when that term is maximum. For that, we may solve

$$\cos * \sin(\theta) = \frac{\sin(2\theta)}{2}$$

Since the maximum value of sin is 1, the maximum value of $(\cos * \sin)^2$ turns out to be $\frac{1}{4}$. Thus the minimum value of our function is $\frac{3}{4}$

Question 1.2

If $x + y + z = xyz$ then prove that

$$\frac{x}{1-x^2} + \frac{y}{1-y^2} + \frac{z}{1-z^2} = \frac{4xyz}{(1-x^2)(1-y^2)(1-z^2)}$$

I'm going to leverage a decent amount of shortcuts here because the calculation is so tedious if you write everything out. The notation $\sum_{cyc}$ means that you will write out three terms alternating between the variables, i.e. $\sum_{cyc} x + y$ means $(x + y) + (y + z) + (z + x)$. I highly recommend that if you aren't familiar with this notation that you explicitly write everything out and see how the steps follow, its a good exercise in notation.

$$
\begin{aligned}
S_1 &= x + y + z \\
S_2 &= xy + yz + zx \\
S_3 &= xyz
\end{aligned}
$$

Now do the cross multiplication of the LHS with respect to each denominator. Then our numerator becomes

$$
\begin{aligned}
\sum_{cyc} x(1 - y^2 - z^2 + y^2 z^2) &= \sum_{cyc} x + \sum_{cyc} xy^2 z^2 - \sum_{cyc} x(y^2 + z^2) \\
&= S_1 + S_3 \sum_{cyc} xy - \sum_{cyc} x^2(y + z) \\
&= S_3 + S_1 \sum_{cyc} xy - \sum_{cyz} x^2(y + z) \\
&= S_3 + \sum_{cyc} xyz + \sum_{cyc} (x^2 y + y^2 x) - \sum_{cyc} x^2(y + z) \\
&= 4S_3
\end{aligned}
$$

The final solution thus becomes, as desired

$$\frac{4xyz}{(1-x^2)(1-y^2)(1-z^2)}$$

Question 1.3

Find the number of solutions of

$$3\sin^2 x - 7\sin x + 2 = 0 \qquad x \in [0, 5\pi]$$

Let $\sin x = y$ and solve the quadratic

$$3y^2 - 7y + 2 = 0$$

The solution set is $\{\frac{1}{3}, 2\}$. Note that 2 is extraneous since sin can never be greater than one. Note that in its cycle, from $(0, \frac{\pi}{2})$ sin increases from 0 to 1, by IVT it must pass through $\frac{1}{3}$ and by monotonicity it does so only once. A similar conclusion can be drawn for $(\frac{\pi}{2}, \pi)$. In the latter half, the sin function becomes negative. Thus in 5π, the sin function will hit $\frac{1}{3}$ six times owing to its 2π periodicity.

Question 1.4

Find the number of solutions of

$$\tan x + \sec x = 2\cos x \qquad x \in [0, 2\pi]$$

$$\begin{aligned}
\frac{\sin x}{\cos x} + \frac{1}{\cos x} &= 2\cos x \\
\frac{\sin x}{\cos^2 x} + \frac{1}{\cos^2 x} - 2 &= 0 \\
\sin x + 1 - 2\cos^2 x &= 0 \\
2\sin^2 + \sin x - 1 &= 0 \\
2y^2 + y - 1 &= 0
\end{aligned}$$

This quadratic has the solutions $\{-1, \frac{1}{2}\}$. In terms of angles this gives us $\{\frac{3\pi}{2}, \frac{\pi}{6}, \frac{5\pi}{6}\}$. Note however that the first root in the set is extraneous since cos is zero at that value making the expression undefined. The two solutions are therefore

$$\left\{\frac{\pi}{6}, \frac{5\pi}{6}\right\}$$

Question 1.5

Solve $\tan^2 x - (1-\sqrt{3})\tan x + \sqrt{3} < 0$

$$\begin{aligned}
y^2 - (1-\sqrt{3})y + \sqrt{3} &< 0 \\
y &= \frac{1-\sqrt{3} \pm \sqrt{(4-2\sqrt{3}) - 4\sqrt{3}}}{2} \\
y &= \frac{1-\sqrt{3} \pm \sqrt{24-16\sqrt{3}}}{2}
\end{aligned}$$

Since the solutions are complex and the parabola is upward facing as the leading square coefficient is positive, the parabola lies completely above $y = 0$ and thus the inequality is never satisfied. The solution is just the set $\emptyset$

Question 1.6

Draw $y = 5\sin\left(2x + \frac{\pi}{2}\right)$