Assignment 5

MTL712

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1 Introduction

In the study of differential equations, a boundary-value problem is a differential equation subjected to constraints called boundary conditions. A solution to a boundary value problem is a solution to the differential equation which also satisfies the boundary conditions.

Linear Shooting Method

The *linear shooting method* transforms a linear boundary value problem (BVP) into a series of initial value problems (IVPs), allowing us to leverage the simpler techniques for IVPs to solve BVPs.

Consider a linear second-order differential equation of the form:

$$y'' = p(x)y' + q(x)y + r(x), \quad a \le x \le b, \quad y(a) = \alpha, \quad y(b) = \beta.$$

To solve this BVP, we decompose the solution into two separate initial value problems (IVPs):
1. **Homogeneous solution:** Solve the homogeneous equation by setting $r(x) = 0$:

$$y_1'' = p(x)y_1' + q(x)y_1, \quad y_1(a) = 0, \quad y_1'(a) = 1.$$

This provides the complementary solution $y_1(x)$, which represents the response due to the initial slope.

2. **Particular solution:** Solve the non-homogeneous equation with the actual forcing term $r(x)$:

$$y_2'' = p(x)y_2' + q(x)y_2 + r(x), \quad y_2(a) = \alpha, \quad y_2'(a) = 0.$$

This provides a particular solution $y_2(x)$ that satisfies the boundary condition at $x = a$.

The general solution to the BVP is then constructed as a linear combination of these two solutions:

$$y(x) = y_2(x) + Cy_1(x),$$

where C is a constant determined by satisfying the boundary condition at $x = b$:

$$C = \frac{\beta - y_2(b)}{y_1(b)}.$$

This method is effective for linear BVPs because it reduces the problem to solving a system of equations for a single constant C, thereby simplifying the process.

Nonlinear Shooting Method

For nonlinear boundary value problems, the shooting method is adapted to handle the nonlinearity by iteratively adjusting the initial slope. Consider a nonlinear second-order differential equation:

$$y'' = f(x, y, y'), \quad y(a) = \alpha, \quad y(b) = \beta.$$

In this approach, we guess an initial value for $y'(a) = s$ and integrate from $x = a$ to $x = b$. The value of s is iteratively adjusted until $y(b) \approx \beta$. This correction is often done using root-finding methods such as the Newton-Raphson technique, which refines the estimate for s based on how close $y(b)$ is to the target boundary condition β.

This method is particularly useful for BVPs with strong nonlinearities, where other methods like finite differences may be less efficient. However, it requires good initial guesses and a reliable root-finding algorithm to ensure convergence.

Linear Finite Difference Method

The *finite difference method* discretizes the differential equation on a mesh of points across the interval $[a, b]$. The second derivative $y''(x)$ is approximated by:

$$y''(x_i) \approx \frac{y_{i-1} - 2y_i + y_{i+1}}{h^2},$$

where h is the step size between consecutive points in the discretization.

Substituting this finite difference approximation into the differential equation at each mesh point leads to a system of linear equations. Solving this system yields approximate values of $y(x)$ at each point in the mesh. For linear BVPs, this method is straightforward and efficient, especially with a tridiagonal system solver.

Nonlinear Finite Difference Method

In the case of nonlinear BVPs, the finite difference discretization leads to a system of nonlinear equations due to the presence of nonlinear terms in the differential equation. The system of nonlinear equations can be solved using iterative methods, such as the Newton-Raphson method, which linearizes the system at each iteration until convergence.

This method is widely used because it applies directly to the discretized equations, but it may require careful handling of convergence criteria and stability, particularly in stiff problems.

Additional Points

The choice of method for solving a BVP depends on factors such as the linearity or nonlinearity of the problem, the desired accuracy, and computational efficiency:
- **Shooting methods** are effective for smooth problems where IVP solvers perform well. They are often preferred for their simplicity but may face challenges in problems with multiple solutions or in systems where the solution oscillates.
- **Finite difference methods** are well-suited for problems where the domain is complex or has boundary conditions at irregular intervals. They convert the BVP into a form that can leverage linear algebra techniques, making them particularly suitable for problems requiring high precision.
- **Iterative solvers**, such as the Newton-Raphson technique, are essential for nonlinear BVPs, as they help adjust the initial guess to achieve a solution that meets the boundary conditions accurately.

Numerical methods for BVPs are a key area in applied mathematics and engineering, as they allow for approximate solutions in cases where analytic methods are infeasible. Each method has strengths and weaknesses, and in practice, the choice depends on the specific nature of the differential equation and boundary conditions.

2 Testcases

2.1 Testcase Description

Testcase 1

Apply the Linear Shooting technique with $N = 10$ to the BVP

$$y'' = -\frac{2}{x} + \frac{2}{x^2}y + \frac{\sin(\ln x)}{x}, \quad 1 \le x \le 2, \quad y(1) = 1, \quad y(2) = 2,$$

and compare the results to those of the exact solution

$$y = c_1 x + \frac{c_2}{x^2} - \frac{3}{10}\sin(\ln x) - \frac{1}{10}\cos(\ln x),$$

where $c_2 = \frac{1}{70}[8 - 12\sin(\ln 2) - 4\cos(\ln 2)] = -0.03920701320$ and $\frac{11}{10} - c_2 = 1.1392070132$.

Testcase 2

Apply the Linear Shooting technique with (a) $h = \frac{1}{2}$ and (b) $h = \frac{1}{4}$ to the BVP

$$y'' = 4(y - x), \quad 0 \le x \le 1, \quad y(0) = 0, \quad y(1) = 2,$$

and compare the results to those of the exact solution

$$y(x) = x + \frac{e^2(e^{2x} - e^{-2x})}{e^4 - 1}.$$

Testcase 3

Apply the Linear Shooting technique with (a) $h = \frac{1}{10}$ and (b) $h = \frac{1}{20}$ to the BVP

$$y'' = 100y, \quad 0 \le x \le 1, \quad y(0) = 1, \quad y(1) = e^{-10},$$

and compare the results to those of the exact solution

$$y(x) = e^{-10x}.$$

Testcase 4

Apply the Nonlinear Shooting Method with Newton's Method with $N = 20, M = 10, \mathrm{TOL} = 10^{-5}$, to the BVP

$$y'' = \frac{1}{8}[32 + 2x^3 - (y')^2], \quad 1 \le x \le 3, \quad y(1) = 17, \quad y(3) = \frac{43}{3},$$

and compare the results to those of the exact solution

$$y = x^2 + \frac{16}{x}.$$

Testcase 5

Apply the Nonlinear Shooting Method with $h = 0.5$, to the BVP

$$y'' = -(y')^2 - y + \ln x, \quad 1 \le x \le 2, \quad y(1) = 0, \quad y(2) = \ln 2,$$

and compare the results to those of the exact solution

$$y = \ln x.$$

Testcase 6

Apply the Nonlinear Shooting Method with $N = 10, \mathrm{TOL} = 10^{-4}$ to the BVP

$$y'' = -e^{-2y}, \quad 1 \le x \le 2, \quad y(1) = 0, \quad y(2) = \ln 2,$$

and compare the results to those of the exact solution

$$y = \ln x.$$

Testcase 7

Apply the Nonlinear Shooting Method with $N = 10$, to the BVP

$$y'' = y' \cos x - y \ln y, \quad 0 \le x \le \frac{\pi}{2}, \quad y(0) = 1, \quad y\left(\frac{\pi}{2}\right) = e,$$

and compare the results to those of the exact solution

$$y = e^{\sin x}.$$

Testcase 8

Apply the Linear Finite Difference Method with (a) $h = \frac{1}{2}$ and (b) $h = \frac{1}{4}$ to the BVP

$$y'' = 4(y - x), \quad 0 \le x \le 1, \quad y(0) = 0, \quad y(1) = 2,$$

and compare the results to those of the exact solution

$$y(x) = x + \frac{e^2(e^{2x} - e^{-2x})}{e^4 - 1}.$$

Testcase 9

Apply the Linear Finite Difference Method with (a) $h = \frac{1}{10}$ and (b) $h = \frac{1}{20}$ to the BVP

$$y'' = 100y, \quad 0 \le x \le 1, \quad y(0) = 1, \quad y(1) = e^{-10},$$

and compare the results to those of the exact solution

$$y(x) = e^{-10x}.$$

Testcase 10

Apply the Nonlinear Finite Difference Method with $h = 0.5$, to the BVP

$$y'' = -(y')^2 - y + \ln x, \quad 1 \le x \le 2, \quad y(1) = 0, \quad y(2) = \ln 2,$$

and compare the results to those of the exact solution

$$y = \ln x.$$

Testcase 11

Apply the Nonlinear Finite Difference Method with $N = 10, \text{TOL} = 10^{-4}$ to the BVP

$$y'' = -e^{-2y}, \quad 1 \le x \le 2, \quad y(1) = 0, \quad y(2) = \ln 2,$$

and compare the results to those of the exact solution

$$y = \ln x.$$

Testcase 12

Apply the Nonlinear Finite Difference Method with $N = 10$, to the BVP

$$y'' = y' \cos x - y \ln y, \quad 0 \le x \le \frac{\pi}{2}, \quad y(0) = 1, \quad y\left(\frac{\pi}{2}\right) = e,$$

and compare the results to those of the exact solution

$$y = e^{\sin x}.$$

2.2 Results

Testcase 1

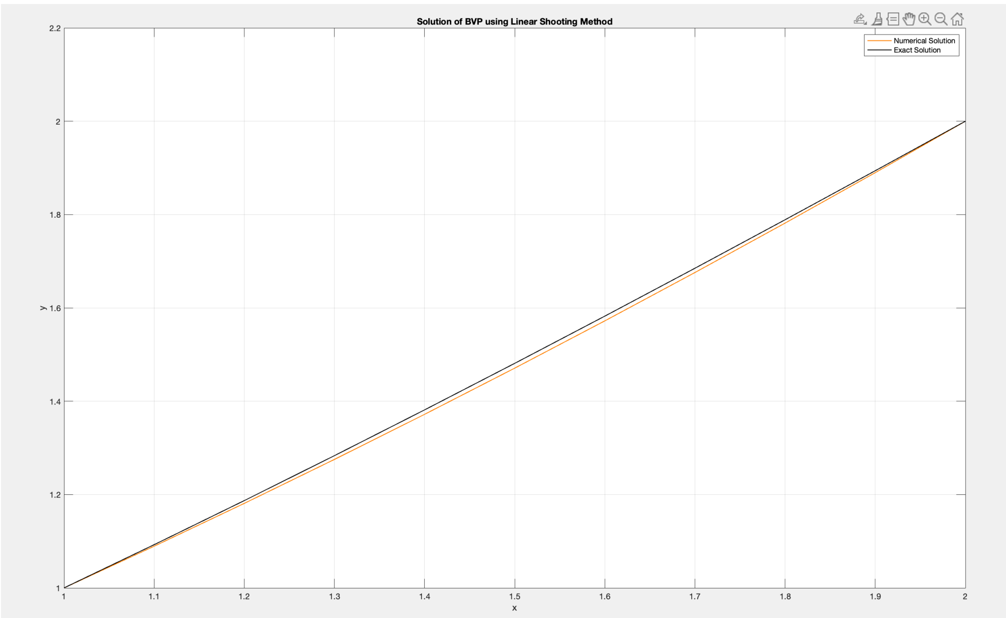

Figure 1: Testcase 1 output

Testcase 2

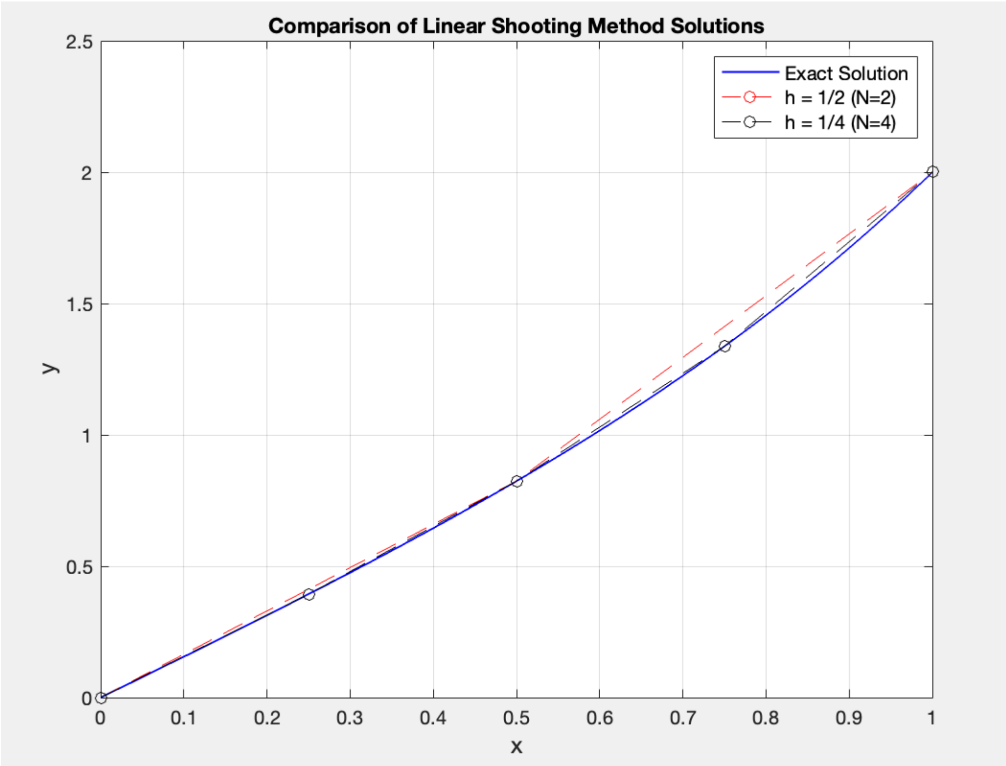

Figure 2: Testcase 2 output

Testcase 3

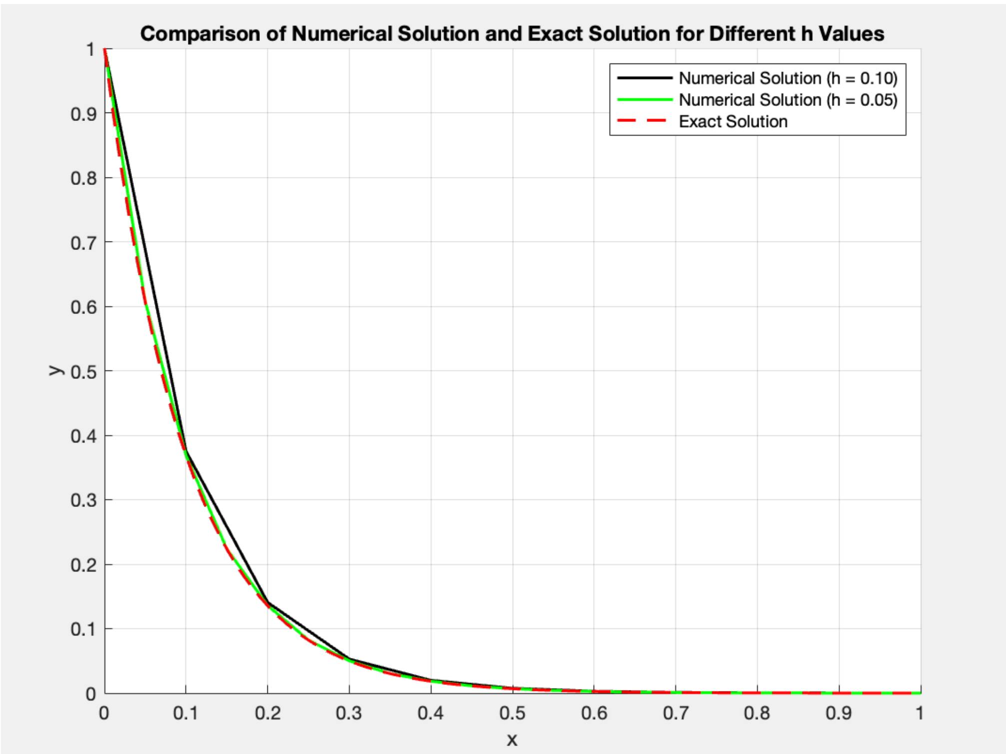

Figure 3: Testcase 3 output

Testcase 4

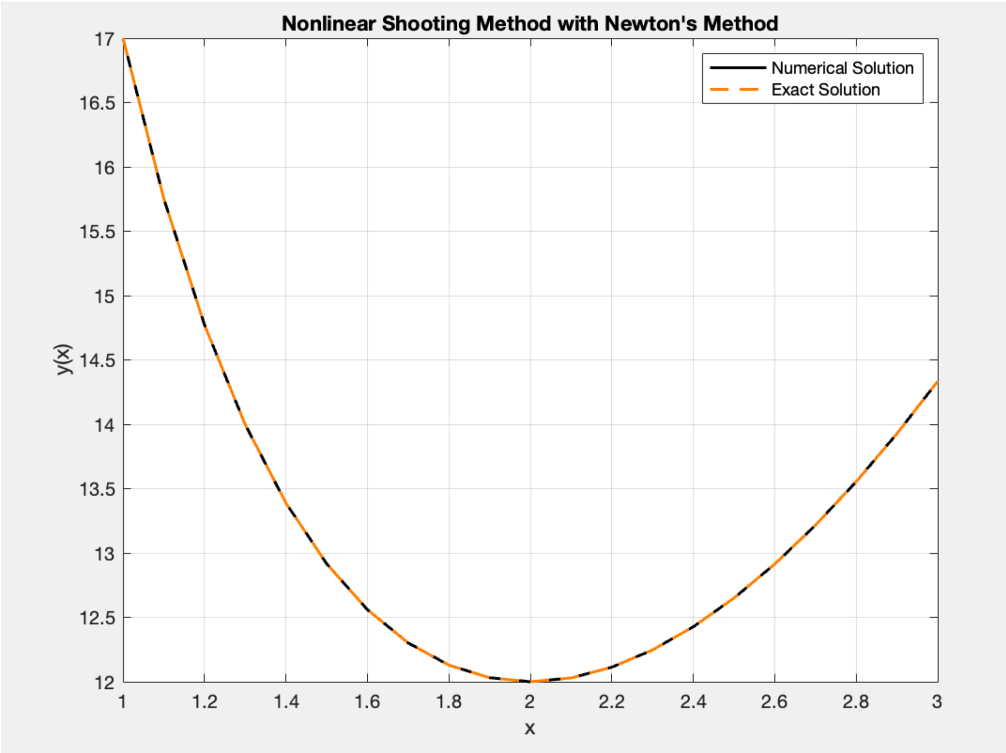

Figure 4: Testcase 4 output

Testcase 5

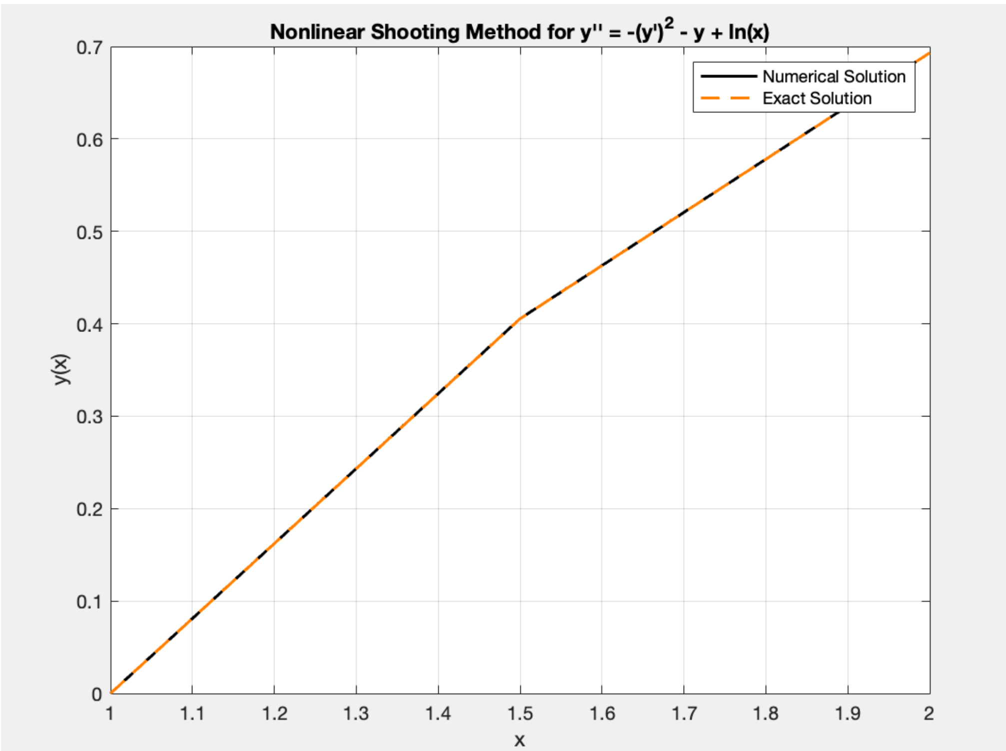

Figure 5: Testcase 5 output

Testcase 6

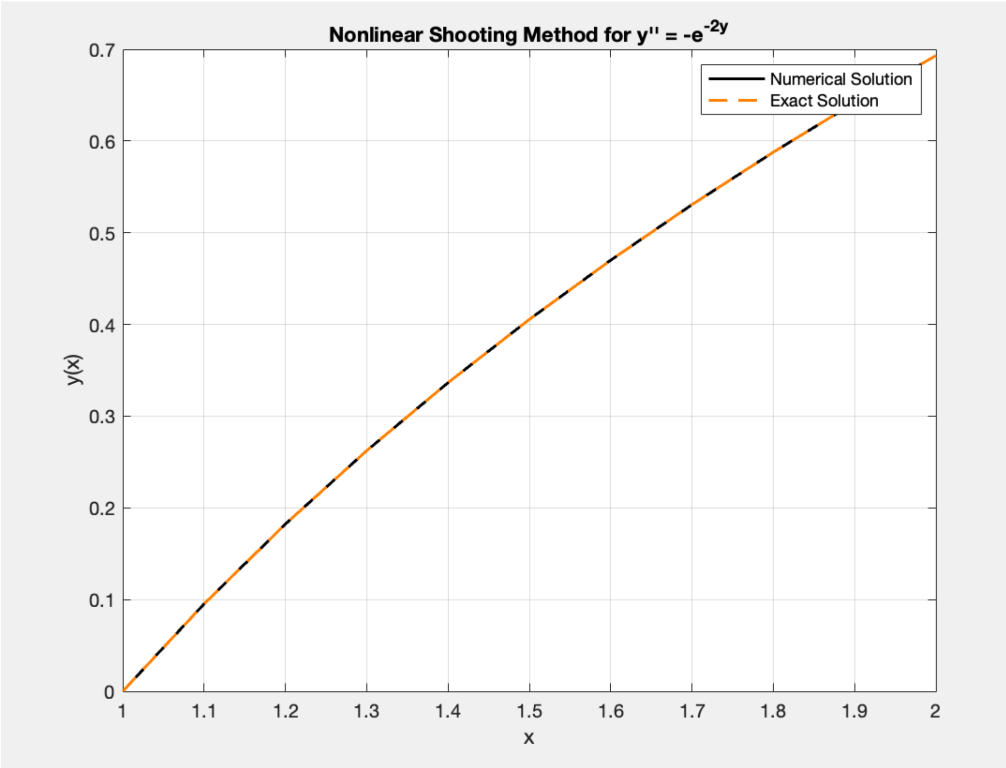

Figure 6: Testcase 6 output

Testcase 7

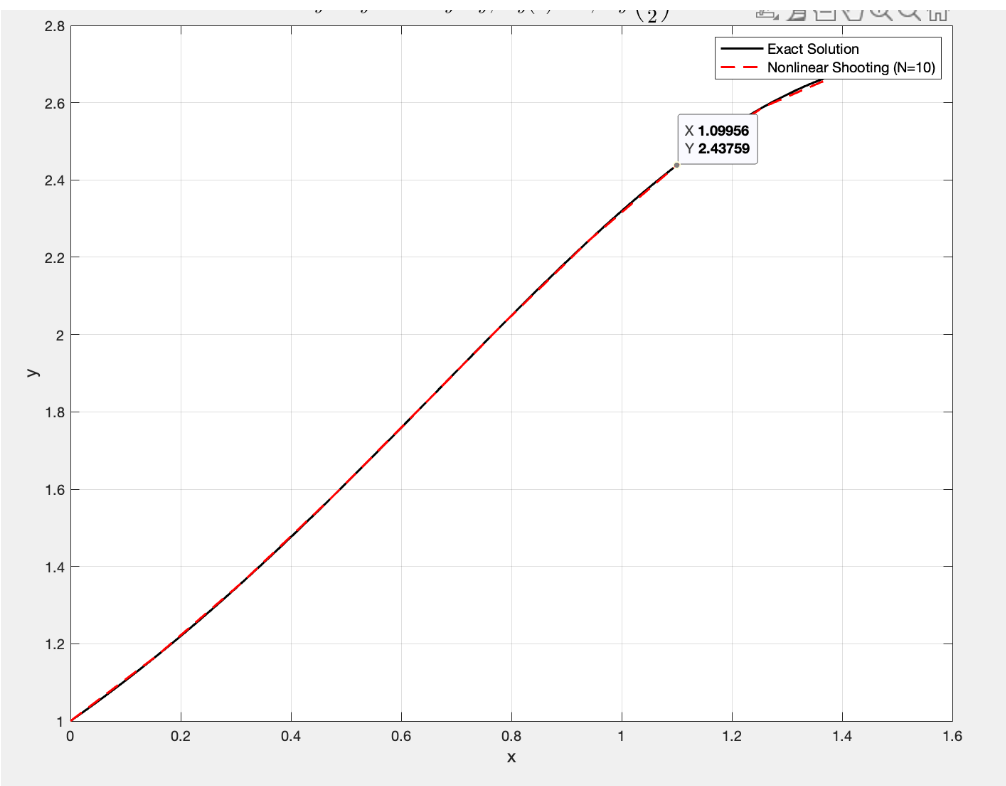

Figure 7: Testcase 7 output

Testcase 8

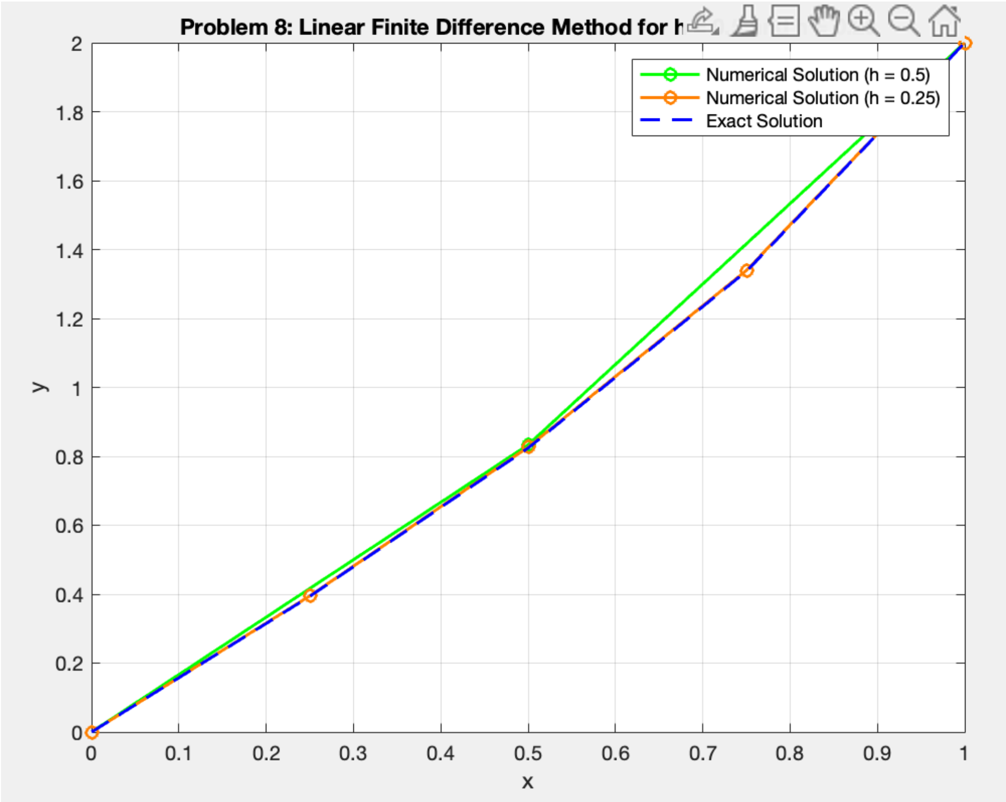

Figure 8: Testcase 8 output

Testcase 9

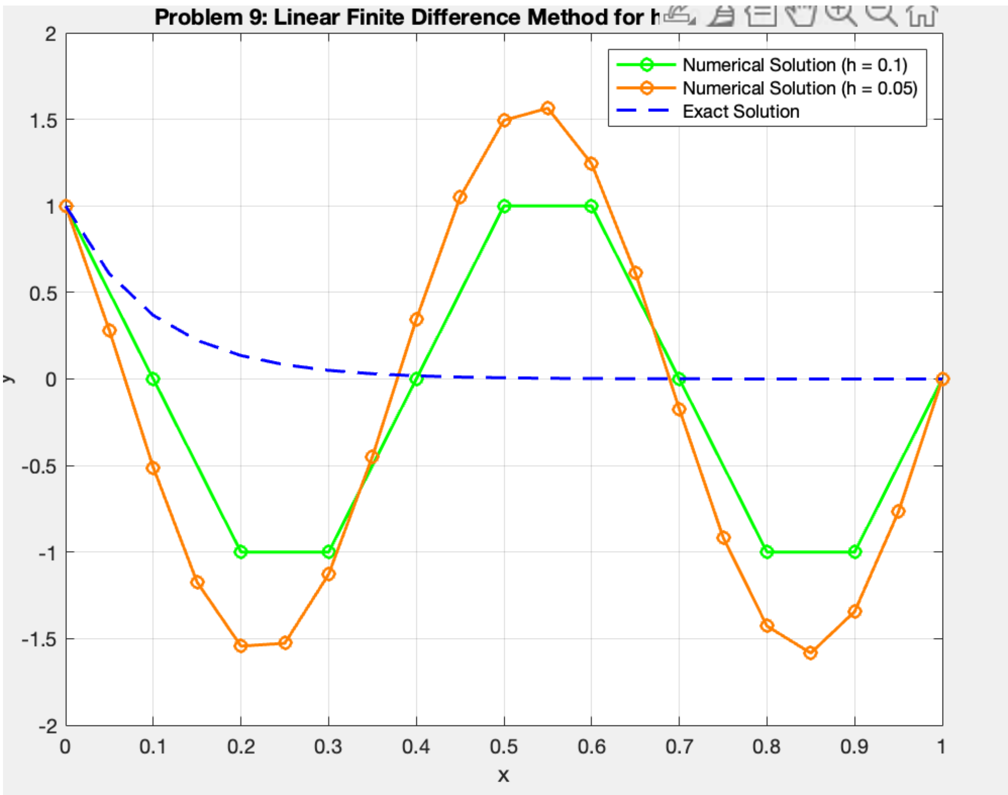

Figure 9: Testcase 9 output

Testcase 10

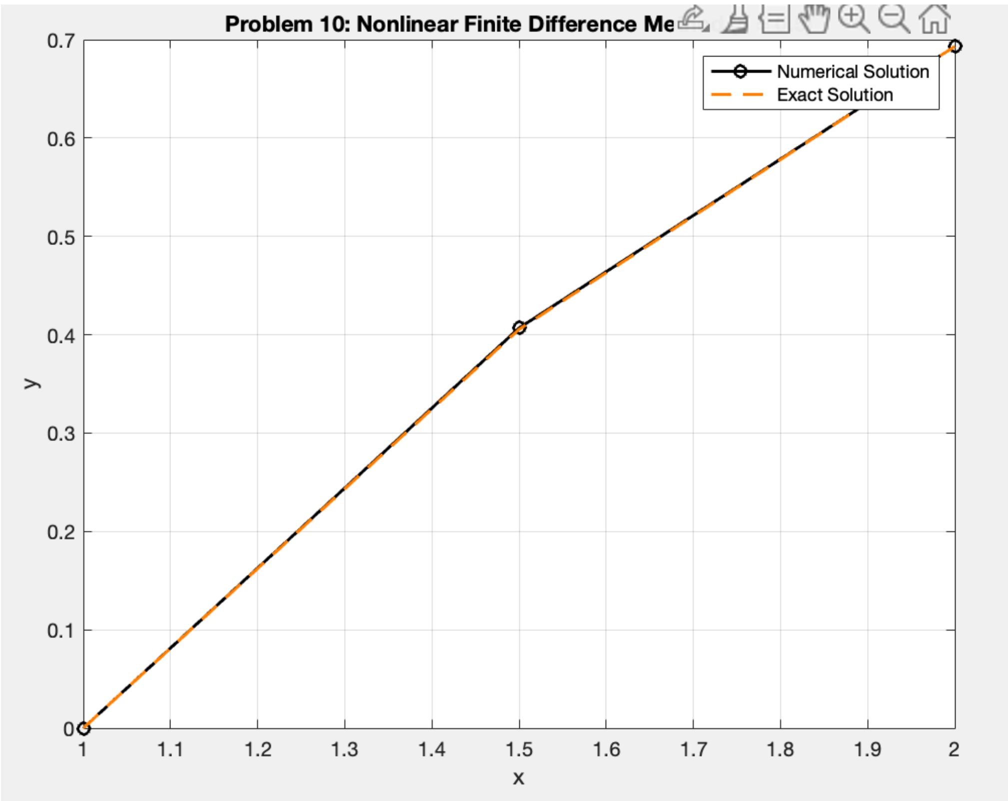

Figure 10: Testcase 10 output

Testcase 11

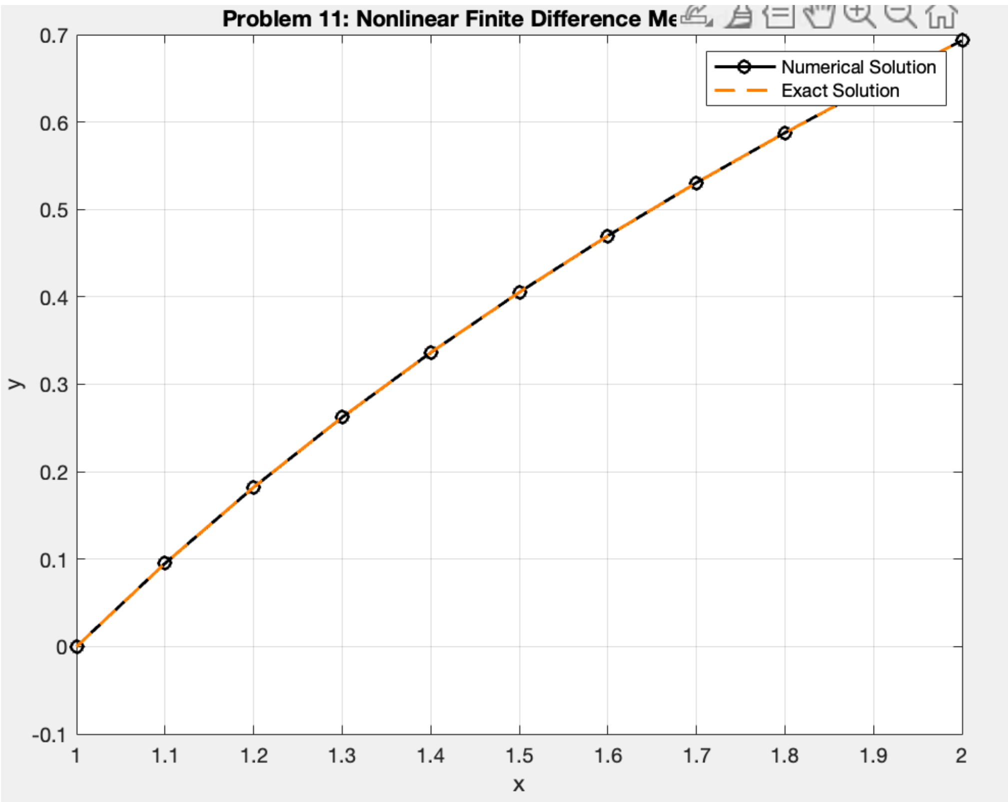

Figure 11: Testcase 11 output

Testcase 12

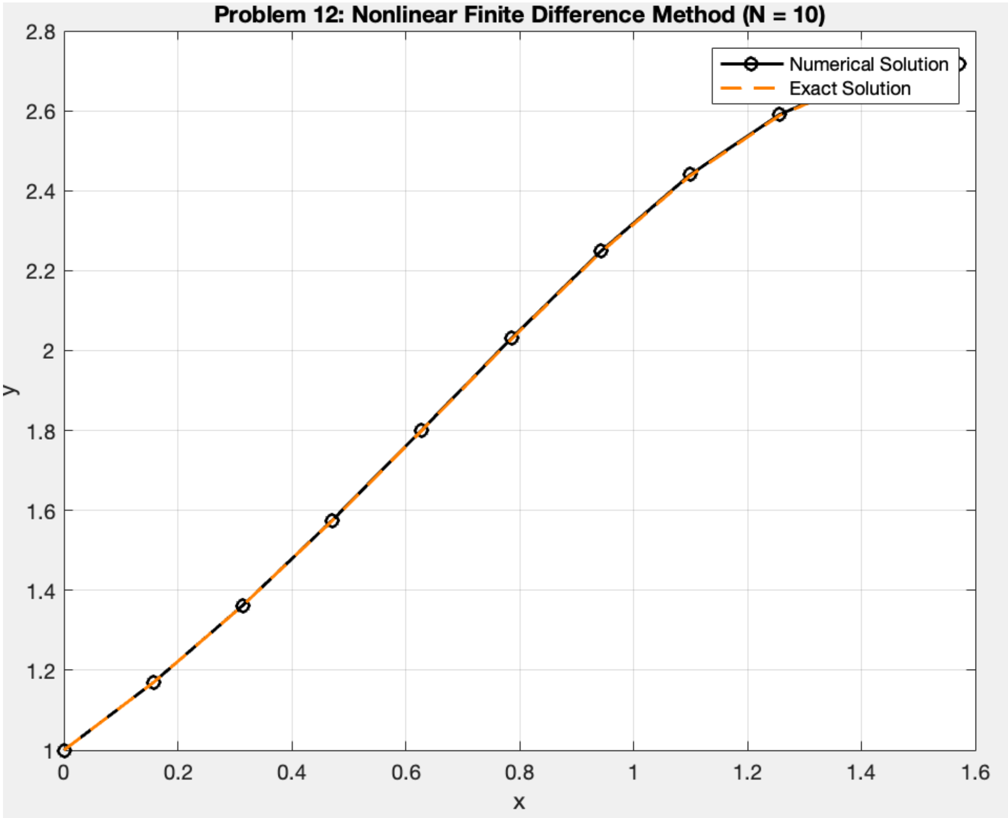

Figure 12: Testcase 12 output