Assignment 1

MTL712

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1 Introduction

1.1 Lax-Friedrichs Method

The Lax-Friedrichs method is a numerical scheme used to solve hyperbolic partial differential equations (PDEs). It is an explicit method and is particularly simple to implement.

First consider FTCS discretization for the equation

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0.$$

$$\frac{u_{i,j+1} - u_{i,j}}{\Delta t} + \frac{u_{i+1,j} - u_{i-1,j}}{2\Delta x} = 0.$$

$$\Rightarrow u_{i,j+1} = u_{i,j} - \frac{\lambda}{2}\left(u_{i+1,j} - u_{i-1,j}\right),$$

where $\lambda = \frac{\Delta t}{\Delta x}$.

Now replace $u_{i,j}$ by $\frac{1}{2}(u_{i+1,j} + u_{i-1,j})$.

With this modification, FTCS becomes the Lax-Friedrichs method:

$$u_{i,j+1} = \frac{1}{2}\left(u_{i+1,j} + u_{i-1,j}\right) - \frac{\lambda}{2}\left(u_{i+1,j} - u_{i-1,j}\right).$$

1.2 Lax-Wendroff Method

The Lax-Wendroff method is a numerical scheme used to solve hyperbolic partial differential equations (PDEs). It is a second-order accurate method, providing better accuracy compared to first-order methods such as the Lax-Friedrichs method.

Consider the hyperbolic PDE in the form:

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0,$$

where $u = u(x,t)$ is the unknown function, a is a constant wave speed, x is the spatial coordinate, and t is the time.

The update equation obtained using Lax-Wendroff method is:

$$u_i^{n+1} = u_i^n - \frac{\lambda}{2}\left[f(u_{i+1}^n) - f(u_{i-1}^n)\right] + \frac{\lambda^2}{2}\left[a_{i+1/2}^n\left\{f(u_{i+1}^n) - f(u_i^n)\right\} - a_{i-1/2}^n\left\{f(u_i^n) - f(u_{i-1}^n)\right\}\right]$$

2 Testcases

2.1 Testcase Description

2.1.1 Test Case 1

Here we need to find the approximate solution at $u(x,30)$ of the differential equation:

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0$$

with initial condition:

$$u(x,0) = -sin(\pi x)$$

2.1.2 Test Case 2

Here we need to find the approximate solution at $u(x,4)$ of the differential equation:

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0$$

with initial condition:

$$u(x,0) = \begin{cases} 1 & \text{if } |x| \geq \frac{1}{3} \\ 0 & \text{if } \frac{1}{3} \leq |x| \leq 1 \end{cases}$$

2.1.3 Test Case 3

Here we need to find the approximate solution at $u(x,4)$ and $u(x,40)$ of the differential equation :

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0$$

with initial condition:

$$u(x,0) = \begin{cases} 1 & \text{if } |x| \geq \frac{1}{3} \\ 0 & \text{if } \frac{1}{3} \leq |x| \leq 1 \end{cases}$$

2.1.4 Test Case 4

Here we need to find the approximate solution at $u(x,0.6)$ of the differential equation:

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}(\frac{u^2}{2}) = 0$$

with initial condition:

$$u(x,0) = \begin{cases} 1 & \text{if } |x| \geq \frac{1}{3} \\ 0 & \text{if } \frac{1}{3} \leq |x| \leq 1 \end{cases}$$

2.1.5 Test Case 5

Here we need to find the approximate solution at $u(x,0.3)$ of the differential equation:

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}(\frac{u^2}{2}) = 0$$

with initial condition:

$$u(x,0) = \begin{cases} 1 & \text{if } |x| \geq \frac{1}{3} \\ -1 & \text{if } \frac{1}{3} \leq |x| \leq 1 \end{cases}$$

2.2 Lax-Friedrichs Method

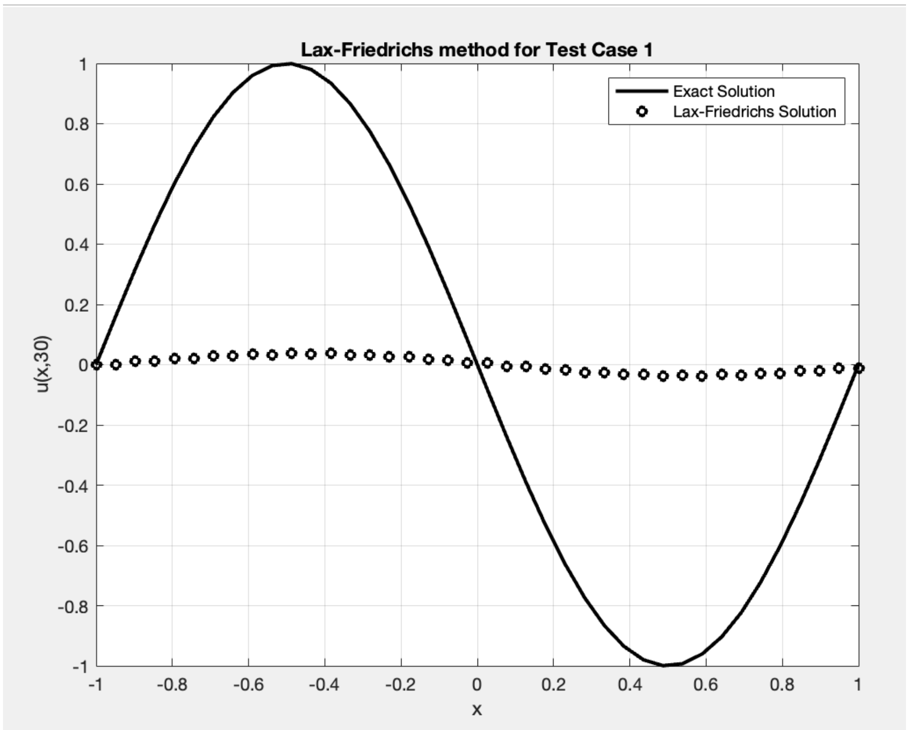

Figure 1: Testcase 1

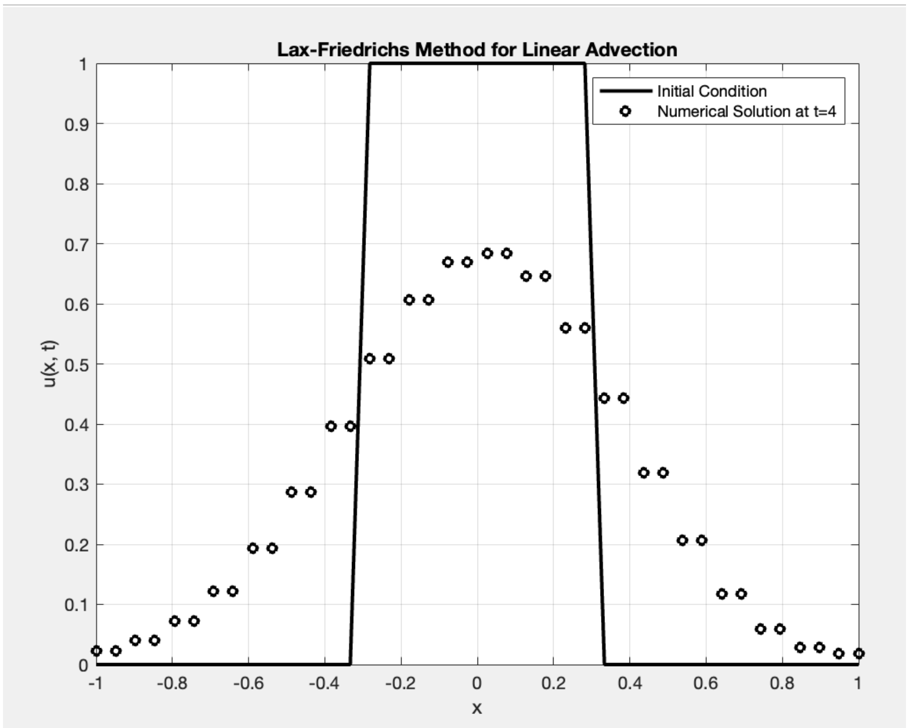

Figure 2: Testcase 2

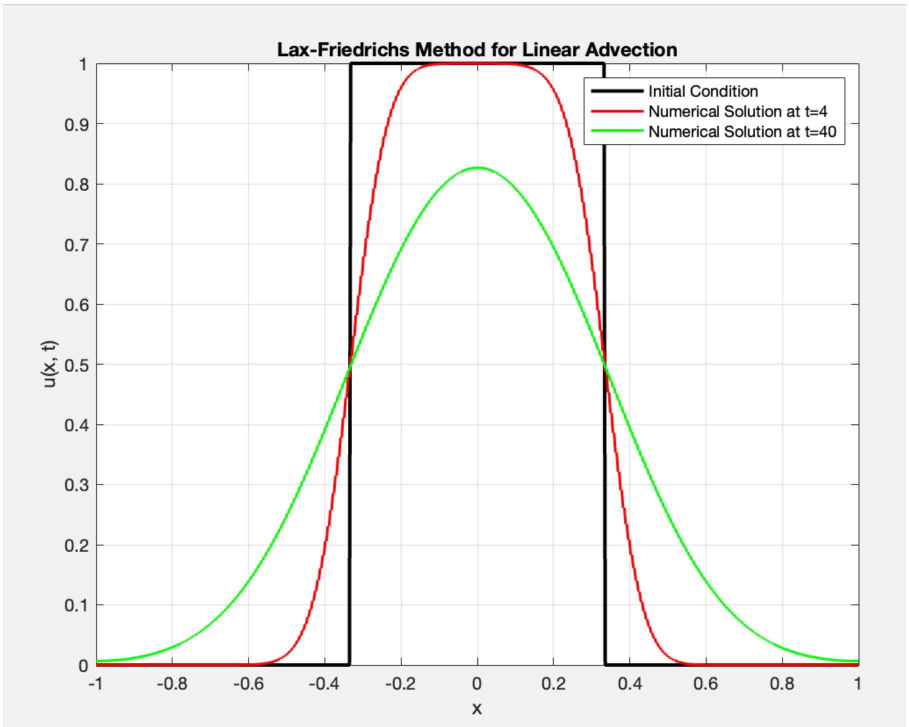

Figure 3: Testcase 3

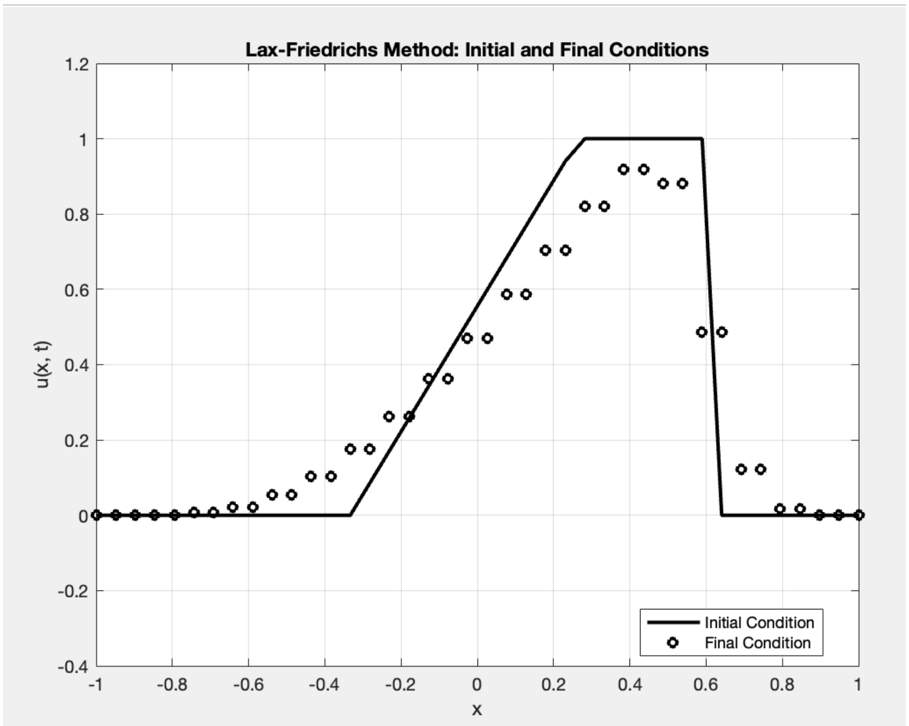

Figure 4: Testcase 4

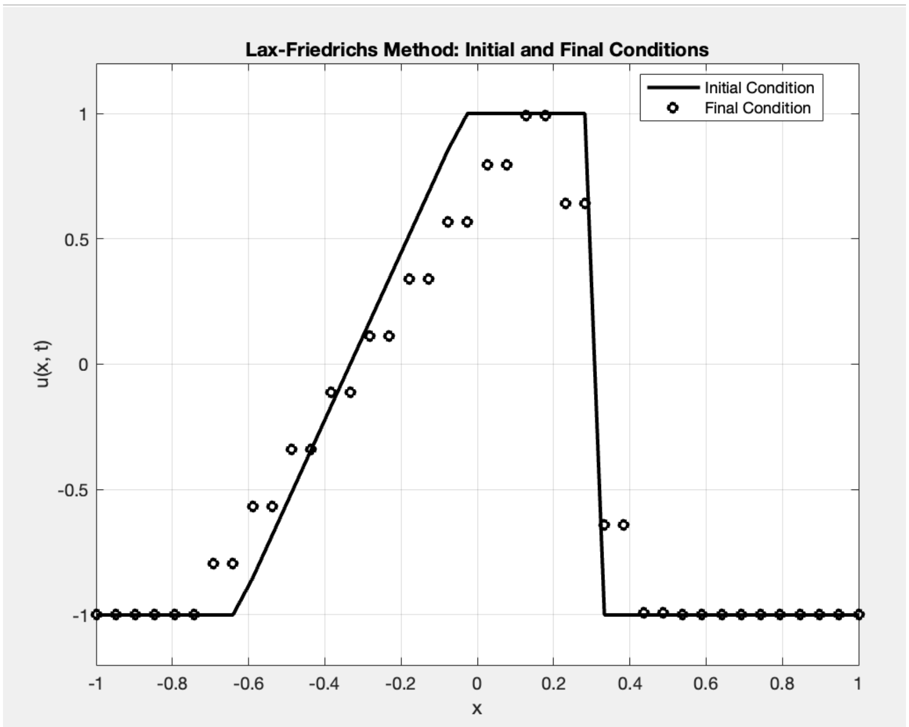

Figure 5: Testcase 5

2.3 Lax-Wendroff Method

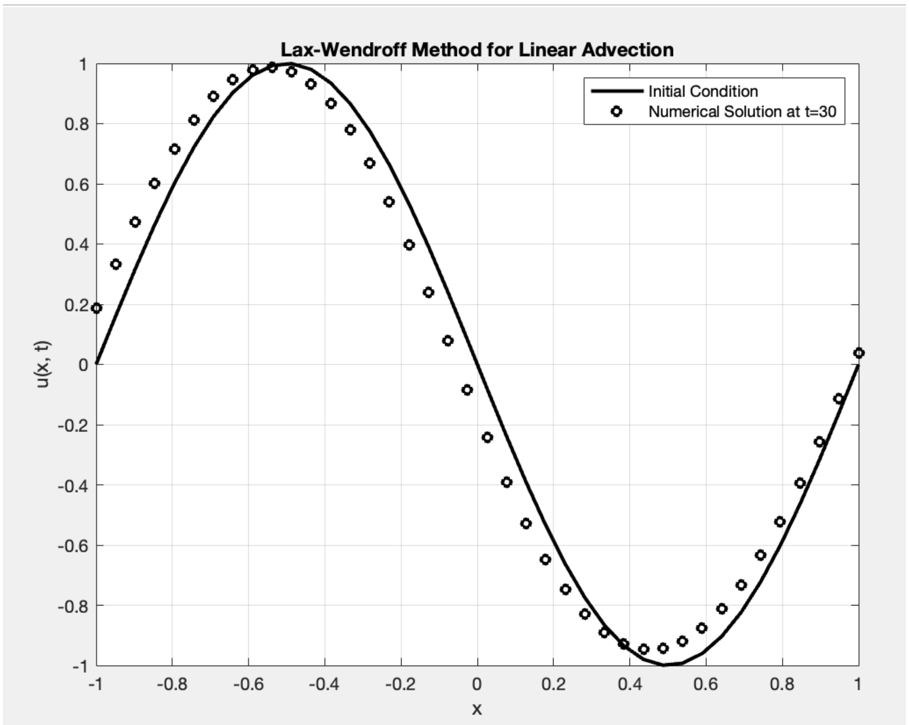

Figure 6: Testcase 1

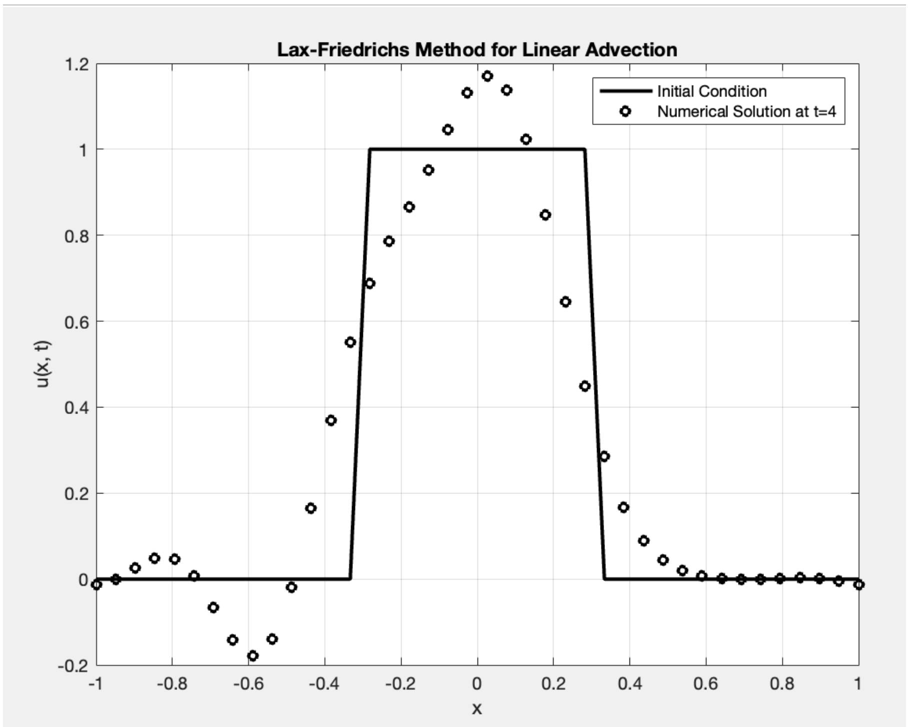

Figure 7: Testcase 2

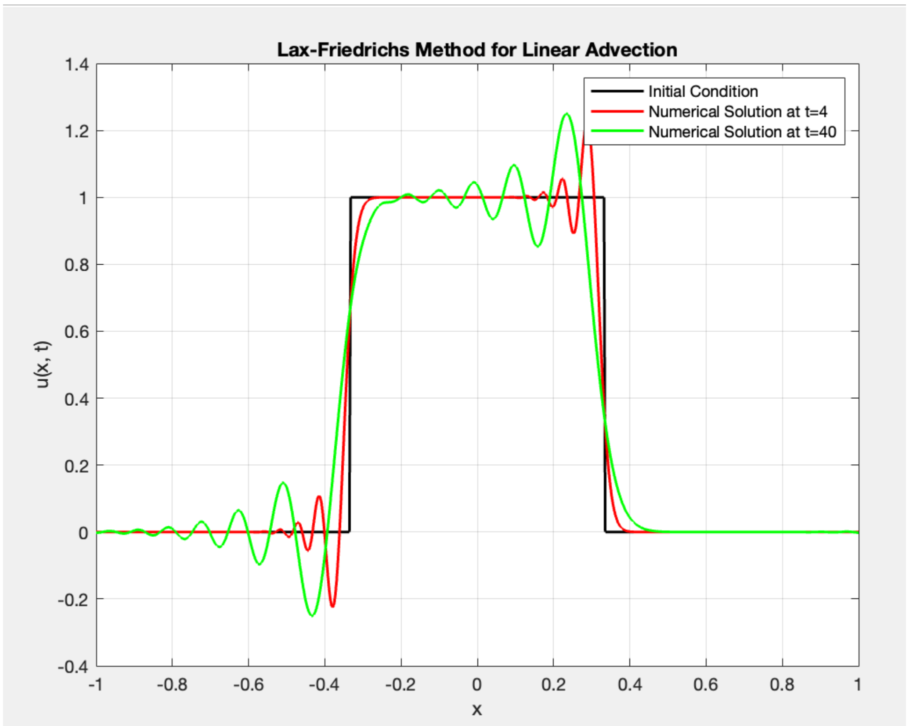

Figure 8: Testcase 3

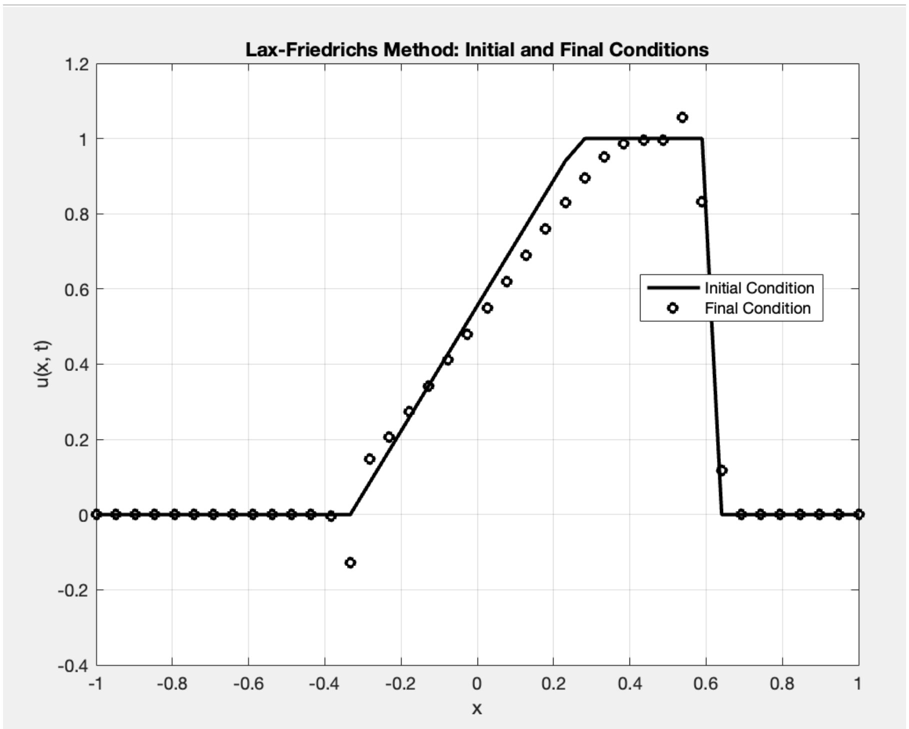

Figure 9: Testcase 4

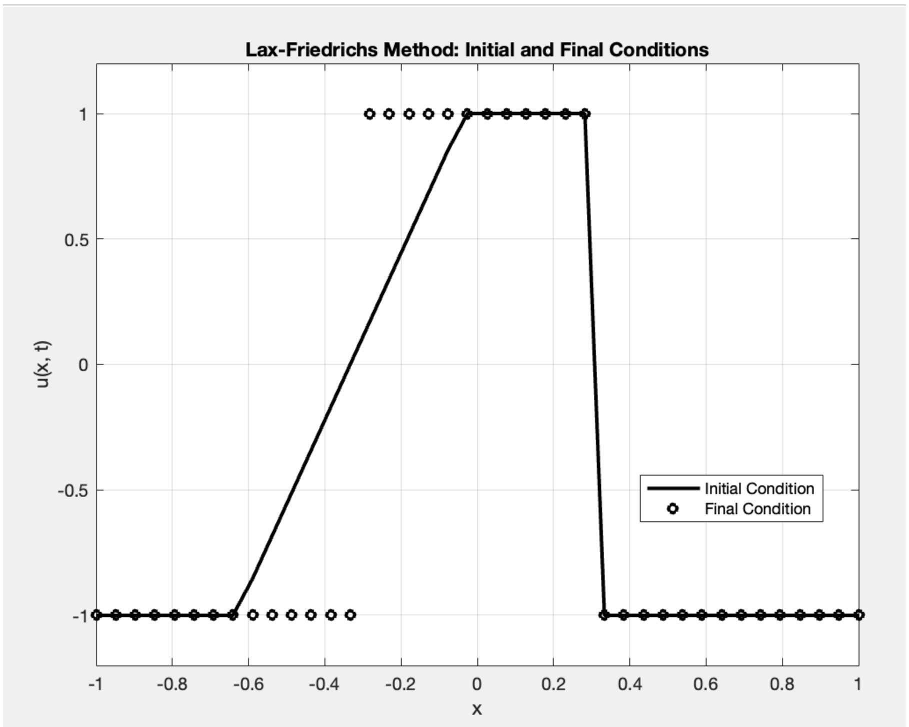

Figure 10: Testcase 5

2.4 Interpretation of Results

2.4.1 Testcase 1

- We see that Lax-Wendroff method gives a better approximation of the exact solution than Lax-Friedrich method.
- For Lax-Friedrich, the sinusoid's shape is well preserved, and the phase error is relatively small, but the amplitude is much smaller than it should be.
- For Lax-Wendroff, the sinusoid's shape and amplitude are well captured. The only visible error is a slight lagging phase error.

2.4.2 Testcase 2

- For Lax-Friedrich, the contacts are extremely smeared and the peak of the square wave has been reduced. However, the solution seems symmetric and properly located.
- For Lax-Wendroff, the solution overshoots and undershoots the exact solution. Also, the Lax-Wendroff method smears the contacts far less than the Lax-Friedrichs method.

2.4.3 Testcase 3

- The Lax Wendroff method seems to be giving a much closer approximation to the actual solution than Lax Friedrich method at many points.
- For Lax Friedrich, the approximations are symmetric and properly located. However, the peak and tails are smeared.
- For Lax Wendroff method, increasing the number of grid points creates large ringing oscillations to the left of the jump discontinuities. Increasing the final time also increases the ringing oscillations.

2.4.4 Testcase 4

- For Lax Friedrich, we see that the solution undershoots near the shock and overshoots near the tail of the expansion.
- For Lax Wendroff, we see that the solution overshoots near the shock and undershoots and overshoots near the tail of the expansion.

2.4.5 Testcase 5

- The Lax Friedrich method seems to be giving a more accurate approximation than Lax Wendroff method.
- For Lax Friedrich, the expansion fan is captured well.
- For Lax Wendroff, the expansion fan is captured as an expansion shock.