# Assignment 4

## MTL712

Ashi Veerman
2021MT10241

# 1 Introduction

## Euler's Method

The following is the Taylor series expansion of function of y(t):

$$y(t+\Delta t) = y(t) + \Delta t y'(t) + \frac{1}{2}\Delta t^2 y''(t) + \frac{1}{3!}\Delta t^3 y'''(t) + \dots$$

The Taylor series with remainder term is

$$y(t+\Delta t) = y(t) + \Delta t y'(t) + \frac{1}{2}\Delta t^2 y''(t) + \frac{1}{3!}\Delta t^3 y'''(t) + \dots + \frac{1}{n!}\Delta t^n y^{(n)}(\tau)$$

where $\tau$ is some value between $t$ and $t + \Delta t$. You can truncate this for any value of $n$.

If we truncate the Taylor series at the first term it is **Euler's Method**

$$y(t+\Delta t) = y(t) + \Delta t y'(t) + \frac{1}{2}\Delta t^2 y''(\tau),$$

we can rearrange this and solve for $y'(t)$

$$y'(t) = \frac{y(t+\Delta t) - y(t)}{\Delta t} + O(\Delta t).$$

Now we can attempt to solve IVP by replacing the derivative with a difference:

$$y((n+1)\Delta t) \approx y(n\Delta t) + \Delta t f(n\Delta t, y(n\Delta t)).$$

Start with $y(0)$ and step forward to solve for any time.

What's good about this? If the $O$ term is something nice looking, this quantity decays with $\Delta t$, so if we take $\Delta t$ smaller and smaller, this gets closer and closer to the real value.

## Taylor Series Methods

To derive these methods we start with a Taylor Expansion:

$$y(t+\Delta t) \approx y(t) + \Delta t y'(t) + \frac{1}{2}\Delta t^2 y''(t) + \dots + \frac{1}{r!} y^{(r)}(t)(\Delta t)^r.$$

Let's say we want to truncate this at the second derivative and base a method on that. The scheme is, then:

$$y_{n+1} = y_n + f_n \Delta t + \frac{f'_n}{2}\Delta t^2.$$

The Taylor series method can be written as

$$y_{n+1} = y_n + \Delta t F(t_n, y_n, \Delta t)$$

where $F = f + \frac{1}{2}\Delta t f'$. If we take the LTE for this scheme, we get (as expected)

$$LTE(t) = \frac{y(t_n+\Delta t) - y(t_n)}{\Delta t} - f(t_n, y(t_n)) - \frac{1}{2}\Delta t f'(t_n, y(t_n)) = O(\Delta t^2).$$

Of course, we designed this method to give us this order, so it shouldn't be a surprise!

So the LTE is reasonable, but what about the global error? Just as in the Euler Forward case, we can show that the global error is of the same order as the LTE. How do we do this? We have two facts,

$$y(t_{n+1}) = y(t_n) + \Delta t F(t_n, y(t_n), \Delta t),$$

and

$$y_{n+1} = y_n + \Delta t F(t_n, y_n, \Delta t)$$

where $F = f + \frac{1}{2}\Delta t f'$. Now we subtract these two

$$|y(t_{n+1}) - y_{n+1}| = |y(t_n) - y_n + \Delta t(F(t_n, y(t_n)) - F(t_n, y_n)) + \Delta t|LTE|| \leq |y(t_n) - y_n| + \Delta t|F(t_n, y(t_n)) - F(t_n, y_n)| + \Delta t|LTE|.$$

Now, if $F$ is Lipschitz continuous, we can say

$$e_{n+1} \leq (1 + \Delta t L)e_n + \Delta t|LTE|.$$

Of course, this is the same proof as for Euler's method, except that now we are looking at $F$, not $f$, and the $LTE$ is of higher order. We can do this no matter which Taylor series method we use, how many terms we go forward before we truncate.

**Advantages and Disadvantages of the Taylor Series Method**

**Advantages**

- One step, explicit
- Can be high order
- Easy to show that global error is the same order as LTE

**Disadvantages**

- Needs the explicit form of derivatives of $f$.

## Runge-Kutta Methods

Runge-Kutta methods are a family of iterative methods for approximating the solutions of ODEs. The most commonly used is the fourth-order Runge-Kutta method (RK4), which provides a good balance between accuracy and computational efficiency. The update formula for RK4 is:

$$k_1 = f(t_n, y_n), \tag{1}$$

$$k_2 = f\left(t_n + \frac{h}{2}, y_n + \frac{h}{2}k_1\right), \tag{2}$$

$$k_3 = f\left(t_n + \frac{h}{2}, y_n + \frac{h}{2}k_2\right), \tag{3}$$

$$k_4 = f(t_n + h, y_n + hk_3), \tag{4}$$

$$y_{n+1} = y_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4). \tag{5}$$

RK4 is widely used due to its accuracy and relatively simple implementation.

# 2 Testcases

## 2.1 Testcase Description

**Testcase 1**

Solve the following initial-value problems using Euler's method.

1. $y' = te^t - 2y$, $0 \leq t \leq 1$, $y(0) = 0$, with step size $h = 0.5$.
2. $y' = 1 + (t - y)^2$, $2 \leq t \leq 3$, $y(2) = 1$, with step size $h = 0.5$.
3. $y' = 1 + y/t$, $1 \leq t \leq 2$, $y(1) = 2$, with step size $h = 0.25$.
4. $y' = \cos 2t + \sin 3t$, $0 \leq t \leq 1$, $y(0) = 1$, with step size $h = 0.25$.

The exact solutions to the above IVPs are as follows.

1. $y(t) = \frac{1}{5}te^t - \frac{1}{25}e^{3t} + \frac{3}{25}e^{-2t}$,
2. $y(t) = t + \frac{1}{1-t^2}$,
3. $y(t) = t\ln t + 2t$,
4. $y(t) = \frac{1}{2}\sin 2t - \frac{1}{3}\cos 3t + \frac{4}{3}$.

**Testcase 2**

Solve the following initial-value problems using Euler's method.

1. $y' = y/t - (y/t)^2$, $1 \le t \le 2$, $y(1) = 1$, with step size $h = 0.1$.
2. $y' = 1 + y/t + (y/t)^2$, $1 \le t \le 3$, $y(1) = 0$, with step size $h = 0.2$.
3. $y' = -(y+1)(y+3)$, $0 \le t \le 2$, $y(0) = -2$, with step size $h = 0.2$.
4. $y' = -5y + 5t^2 + 2t$, $0 \le t \le 1$, $y(0) = \frac{1}{3}$, with step size $h = 0.1$.

The exact solutions to the above IVPs are as follows.

1. $y(t) = \frac{t}{1+\ln t}$,
2. $y(t) = t\tan(\ln t)$,
3. $y(t) = -3 + \frac{2}{1+e^{-2t}}$,
4. $y(t) = t^2 + \frac{1}{3}e^{-5t}$.

**Testcase 3**

The exact solution for the IVP

$$y' = \frac{2}{t}y + t^2 e^t, \quad 1 \le t \le 2, \quad y(1) = 0,$$

is $y(t) = t^2(e^t - e)$.

1. Use Euler's method with $h = 0.1$ to approximate the solution, and compare it with the actual values of $y$.
2. Use the answers generated in part (a) and linear interpolation to approximate the following values of $y$, and compare them to the actual values:
   - $y(1.04)$
   - $y(1.55)$
   - $y(1.97)$
3. Compute the value of $h$ necessary for $|y(t_i) - w_i| \le 0.1$, using

   $$|y(t_i) - w_i| \le \frac{hM}{2L}\left[e^{L(t_{i-1}-1)} - 1\right]$$

   where $L$ is the Lipschitz constant and $M$ is a constant satisfying $|y''(t)| \le M$, for all $t \in [1, 2]$.

**Testcase 4**

Solve the following initial-value problems using Taylor's method of order two:

(a) $y' = te^{3t} - 2y, \quad 0 \le t \le 1, \quad y(0) = 0$, with step size $h = 0.5$.

(b) $y' = 1 + (t-y)^2, \quad 2 \le t \le 3, \quad y(2) = 1$, with step size $h = 0.5$.

(c) $y' = 1 + y/t, \quad 1 \le t \le 2, \quad y(1) = 2$, with step size $h = 0.25$.

(d) $y' = \cos 2t + \sin 3t, \quad 0 \le t \le 1, \quad y(0) = 1$, with step size $h = 0.25$.

**Testcase 5**

Solve the following initial-value problems using Taylor's method of order two:

(a) $y' = y/t - (y/t)^2, \quad 1 \le t \le 1.2, \quad y(1) = 1$, with step size $h = 0.1$.

(b) $y' = \sin t + e^{-t}, \quad 0 \le t \le 1, \quad y(0) = 0$, with step size $h = 0.5$.

(c) $y' = (y^2 + y)/t, \quad 1 \le t \le 3, \quad y(1) = -2$, with step size $h = 0.5$.

(d) $y' = -ty + 4ty^{-1}, \quad 0 \le t \le 1, \quad y(0) = 1$, with step size $h = 0.25$.

**Testcase 6**

The exact solution for the IVP

$$y' = \frac{2}{t}y + t^2 e^t, \quad 1 \le t \le 2, \quad y(1) = 0,$$

is $y(t) = t^2(e^t - e)$.

(a) Use Taylor's method of order two with $h = 0.1$ to approximate the solution, and compare it with the actual values of $y$.

(b) Use the answers generated in part (a) and linear interpolation to approximate the following values of $y$, and compare them to the actual values of $y$:

(i) $y(1.04)$

(ii) $y(1.55)$

(iii) $y(1.97)$

(c) Use Taylor's method of order four with $h = 0.1$ to approximate the solution, and compare it with the actual values of $y$.

(d) Use the answers generated in part (c) and linear interpolation to approximate the following values of $y$, and compare them to the actual values of $y$:

(i) $y(1.04)$

(ii) $y(1.55)$

(iii) $y(1.97)$

**Testcase 7**

Runge Kutta 4th order to solve this

a. $y' = \frac{y}{t} - \left(\frac{y}{t}\right)^2; \quad 1 \le t \le 2; \quad y(1) = 1$

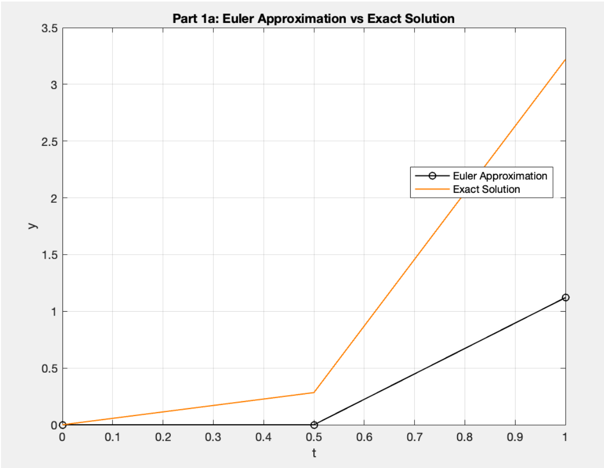
b. $y' = 1 + \frac{y}{t} + \left(\frac{y}{t}\right)^2; \quad 1 \le t \le 3; \quad y(1) = 0$

c. $y' = -(y+1)(y+3); \quad 0 \le t \le 2; \quad y(0) = -2$

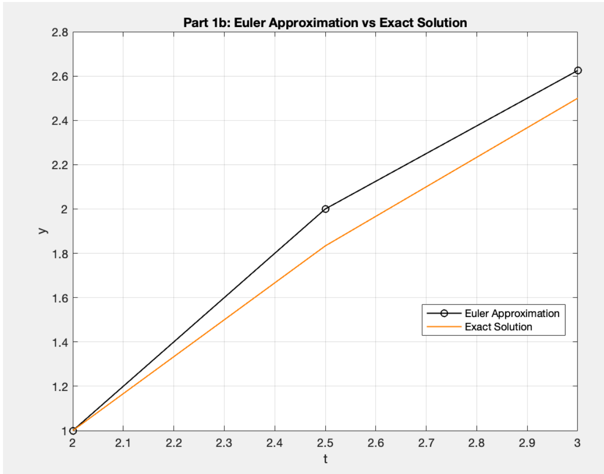
d. $y' = -5y + 5t^2 + 2t; \quad 0 \le t \le 1; \quad y(0) = \frac{1}{3}$

## 2.2 Interpretation of Results

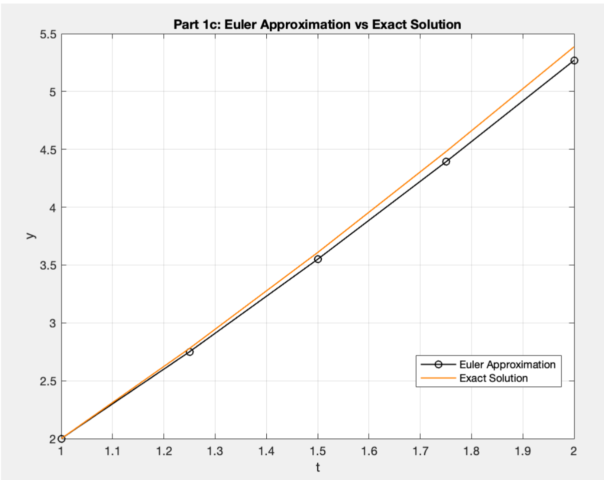
### Testcase 1



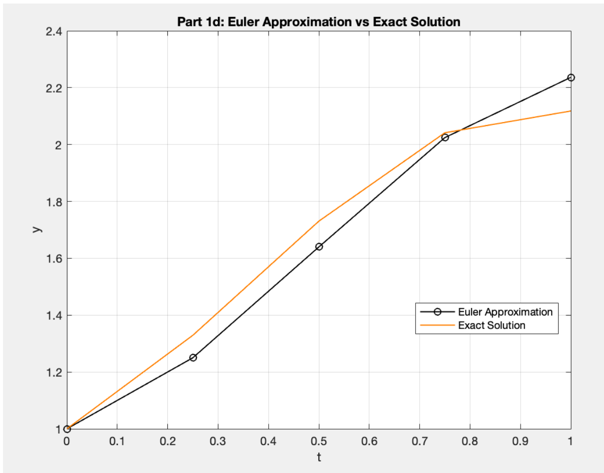

(a)




(b)




(c)

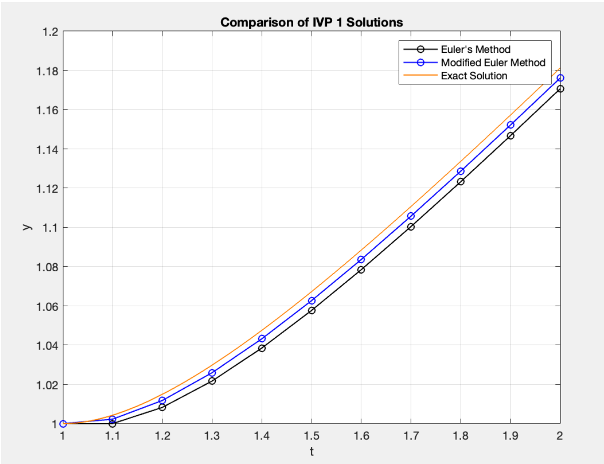



(d)

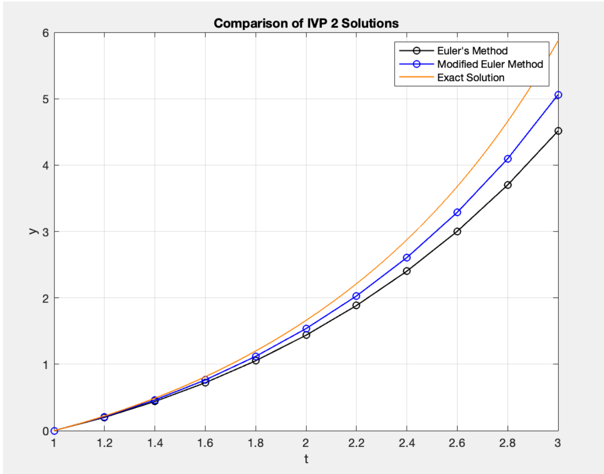
Figure 1: Plot for Testcase 1

- Euler's method provides a significant approximation of the exact solution for (b),(c) and (d) whereas for case (a) there is still a significant gap.

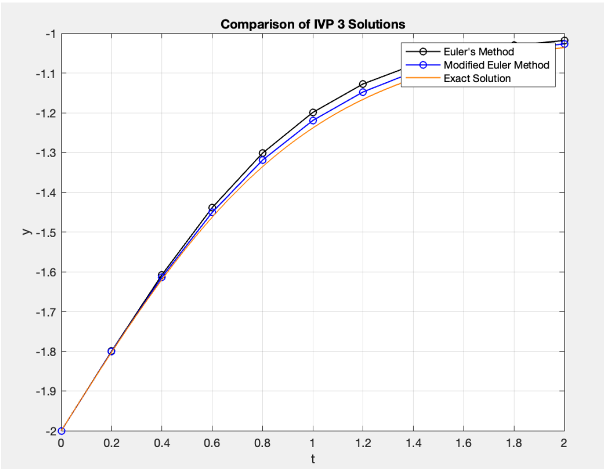
## Testcase 2



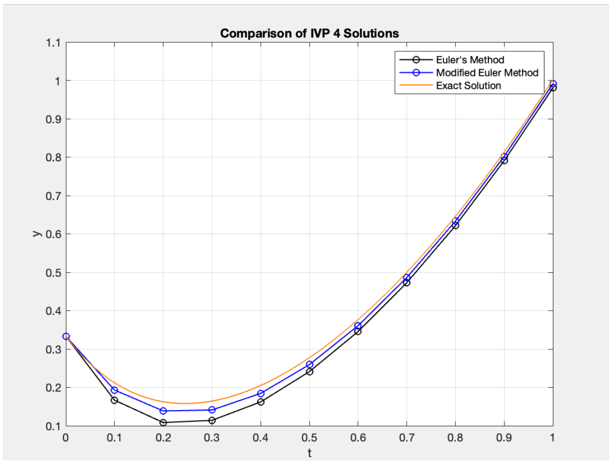

(a)




(b)




(c)

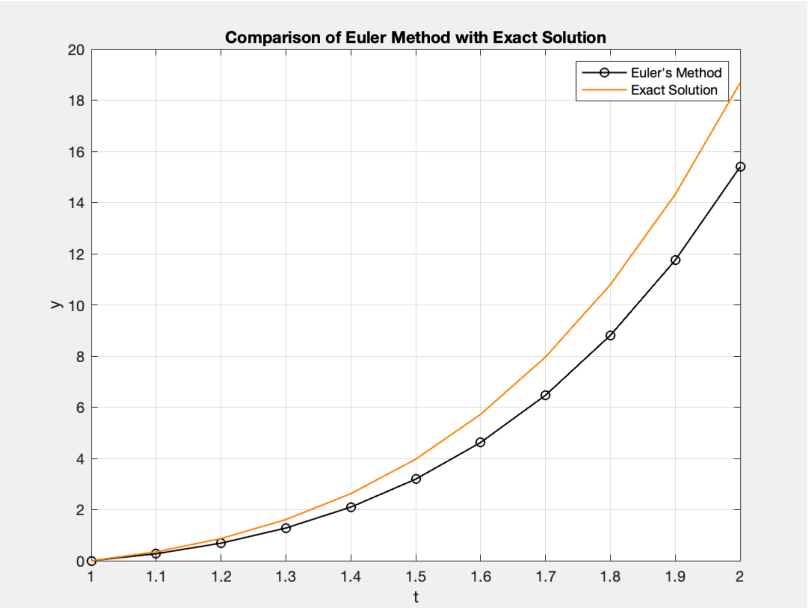



(d)

Figure 2: Plot for Testcase 2

- Euler's method provides a significant approximation of the exact solution, effectively capturing its shape with a high degree of accuracy.
- The Modified Euler method performs better than the traditional Euler method.

**Testcase 3**

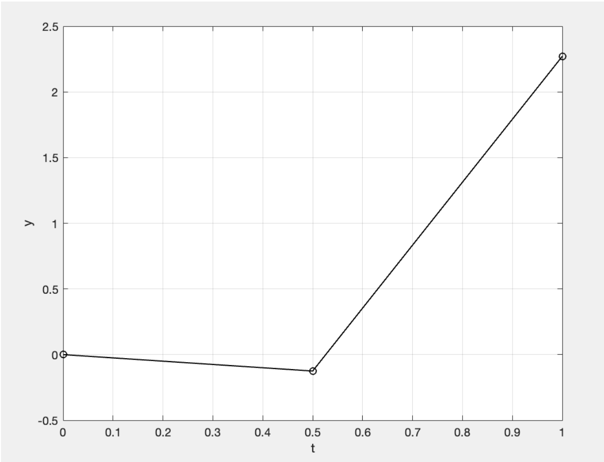



(a)

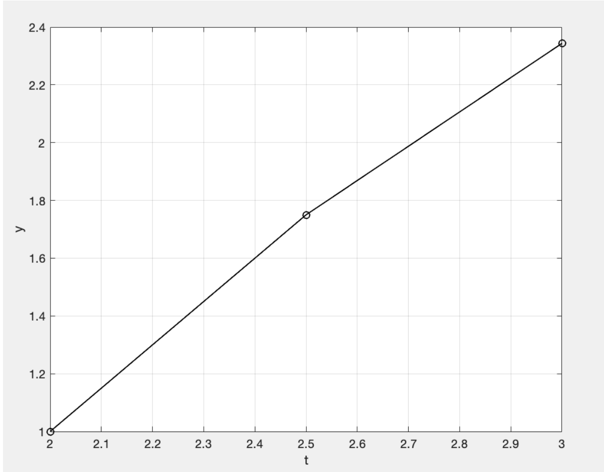
Figure 3: Plot for Testcase 3

- Euler's method provides a significant approximation of the exact solution, effectively capturing its shape with a high degree of accuracy.

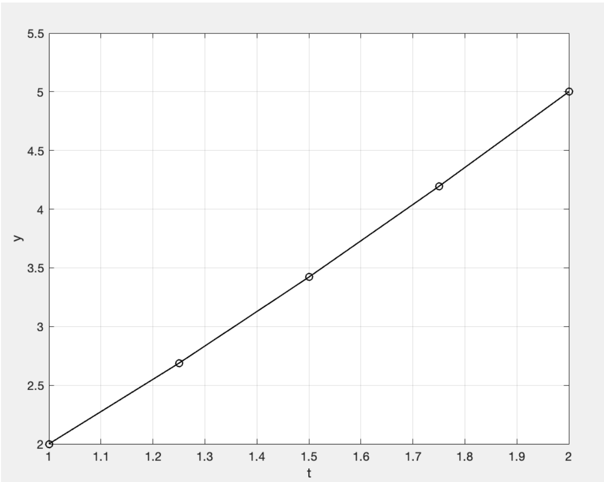
**Testcase 4**



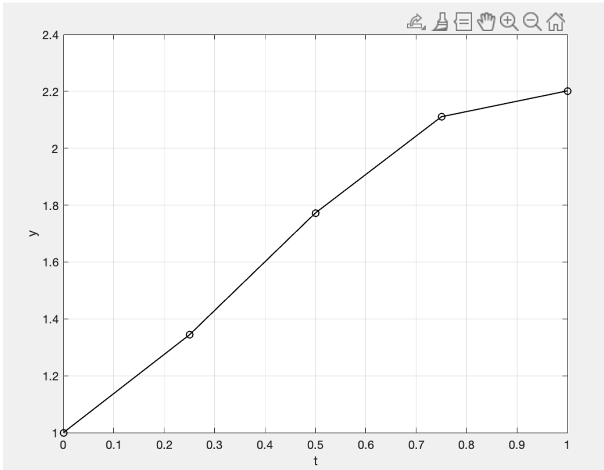

(a)




(b)




(c)

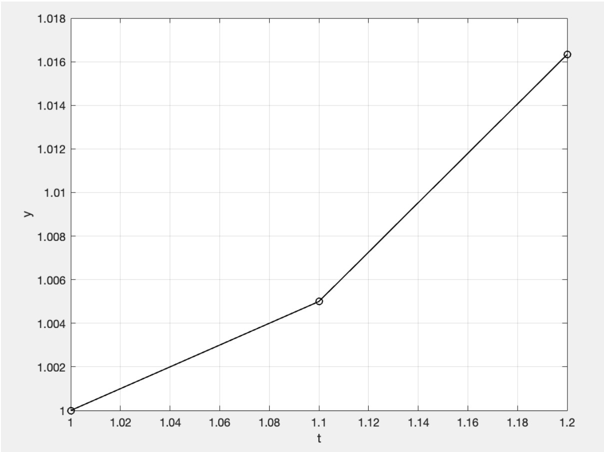



(d)

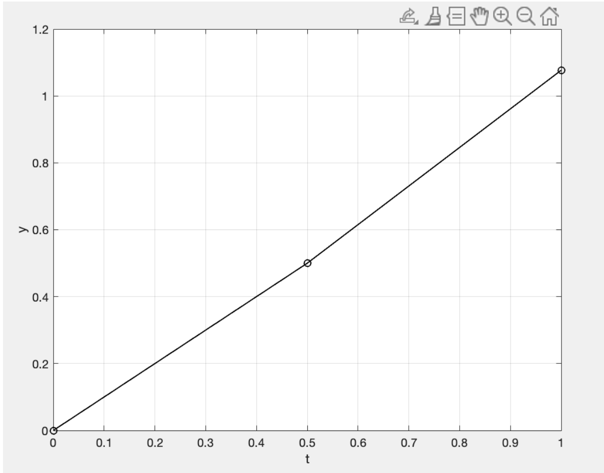
Figure 4: Plot for Testcase 4

- Above, you can see the plots for the second-order Taylor series method applied to the four initial value problems (IVPs)

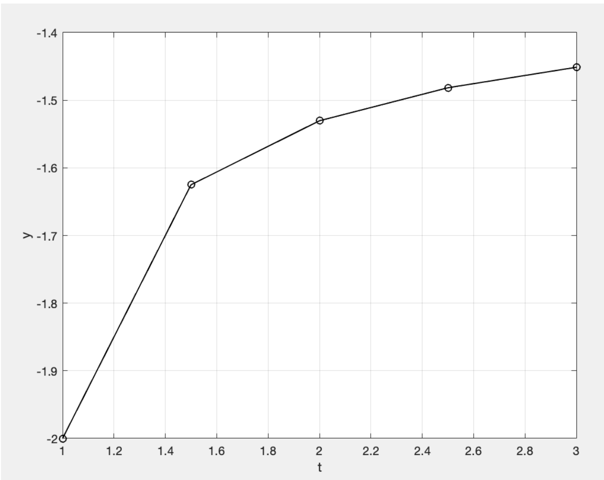
**Testcase 5**



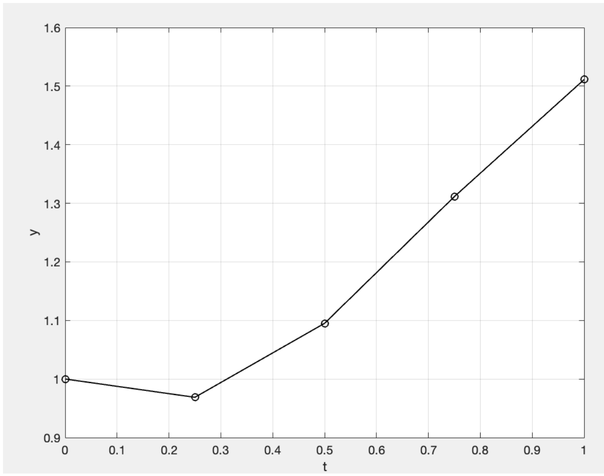

(a)




(b)




(c)

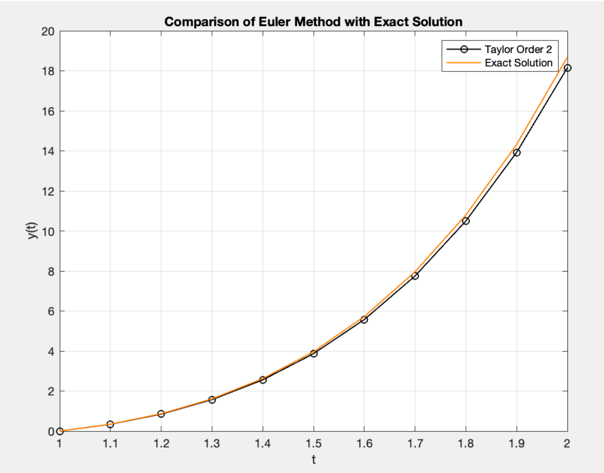



(d)

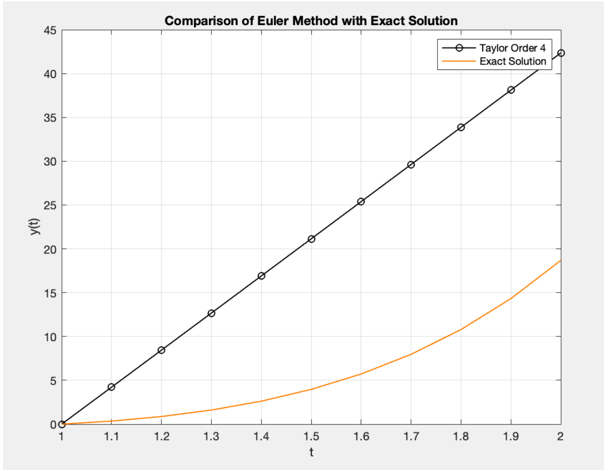
Figure 5: Plot for Testcase 5

- Above, you can see the plots for the fourth-order Taylor series method applied to the four initial value problems (IVPs)

### Testcase 6




(a)

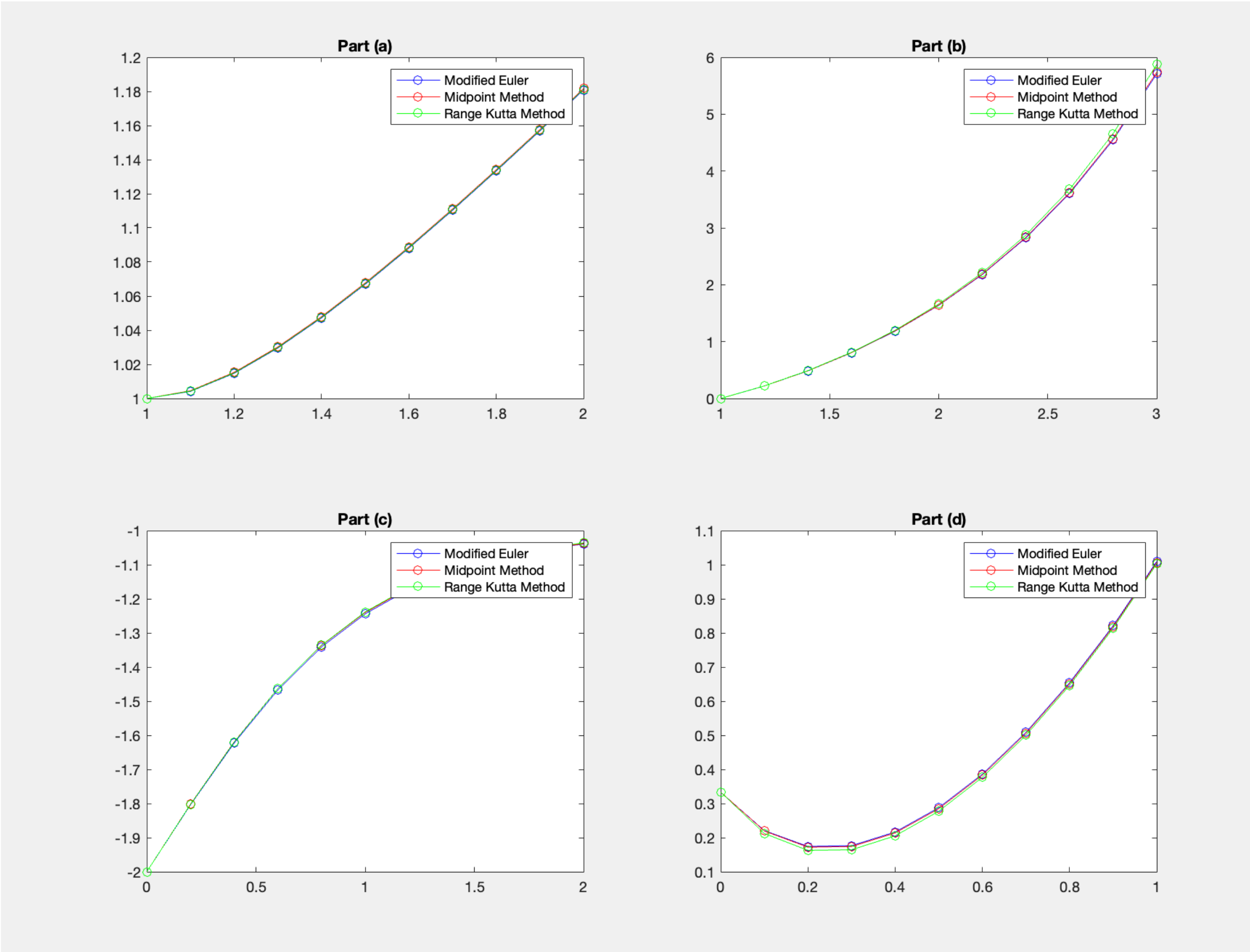



(b)

Figure 6: Plot for Testcase 6

- The second-order Taylor series method provides a more accurate approximation of the result.
- The fourth-order Taylor series method is significantly different from the exact solution.

**Testcase 7**




(a)

Figure 7: Plot for Testcase 7