Assignment 2

MTL712

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1 Introduction

1.1 Godunov's Method

Godunov's first-order upwind method was discovered in 1959. In wave speed split form, Godunov's first-order upwind method is

$$u_i^{n+1} = u_i^n + C_{i+1/2}^+ \left(u_{i+1}^n - u_i^n\right) - C_{i-1/2}^- \left(u_i^n - u_{i-1}^n\right), \tag{1}$$

where

$$C_{i+1/2}^+ = -\lambda \min_{u \text{ between } u_i^n \text{ and } u_{i+1}^n} \left(\frac{f(u) - f(u_i^n)}{u_{i+1}^n - u_i^n} \right) \tag{2}$$

and

$$C_{i+1/2}^- = \lambda \max_{u \text{ between } u_i^n \text{ and } u_{i+1}^n} \left(\frac{f(u_i^n) - f(u)}{u_{i+1}^n - u_i^n} \right) \tag{3}$$

Of course, as always, there are infinitely many other wave speed split forms. However, this is the only wave speed split form with finite coefficients - in this sense, this is the unique natural wave speed split form.

1.2 Roe's Method

Our aim is to approximate the solutions of the Riemann problem. A common strategy is to linearize the nonlinear equations.

$$U_t + f(U)_x = 0$$

$$\Rightarrow \quad U_t + f'(U) \cdot U_x = 0$$

Here, f is a nonlinear flux function, and we will replace this $f'(U)$ locally by an approximation.

$$U_t + f'\left(\frac{U_j^n + U_{j+1}^n}{2}\right) U_x = 0.$$

$$\hat{A}_{j+\frac{1}{2}} = \begin{cases} \frac{f(U_{j+1}^n) - f(U_j^n)}{U_{j+1}^n - U_j^n}, & \text{if } U_{j+1}^n \neq U_j^n \\ f'(U_j^n), & \text{if } U_{j+1}^n = U_j^n. \end{cases}$$

where

$$\epsilon_{j+\frac{1}{2}}^n = \left| \hat{A}_{j+\frac{1}{2}}^n \right|$$

and

$$\epsilon_{j-\frac{1}{2}}^n = \left| \hat{A}_{j-\frac{1}{2}}^n \right|.$$

Finally, we write down the Roe solver as below:

$$U_j^{n+1} = U_j^n - \frac{\lambda}{2} \left[f\left(U_{j+1}^n\right) - f\left(U_{j-1}^n\right) \right] + \frac{\lambda}{2} \left[\epsilon_{j+\frac{1}{2}}^n \left(U_{j+1}^n - U_j^n\right) - \epsilon_{j-\frac{1}{2}}^n \left(U_j^n - U_{j-1}^n\right) \right],$$

where

$$\lambda = \frac{\Delta t}{\Delta x}.$$

2 Testcases

2.1 Testcase Description

2.1.1 Test Case 1

Here we need to find the approximate solution at $u(x, 30)$ of the differential equation:

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0$$

with initial condition:

$$u(x, 0) = -sin(\pi x)$$

2.1.2 Test Case 2

Here we need to find the approximate solution at $u(x, 4)$ of the differential equation:

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0$$

with initial condition:

$$u(x, 0) = \begin{cases} 1 & \text{if } |x| \geq \frac{1}{3} \\ 0 & \text{if } \frac{1}{3} \leq |x| \leq 1 \end{cases}$$

2.1.3 Test Case 3

Here we need to find the approximate solution at $u(x, 4)$ and $u(x, 40)$ of the differential equation :

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0$$

with initial condition:

$$u(x, 0) = \begin{cases} 1 & \text{if } |x| \geq \frac{1}{3} \\ 0 & \text{if } \frac{1}{3} \leq |x| \leq 1 \end{cases}$$

2.1.4 Test Case 4

Here we need to find the approximate solution at $u(x, 0.6)$ of the differential equation:

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}(\frac{u^2}{2}) = 0$$

with initial condition:

$$u(x, 0) = \begin{cases} 1 & \text{if } |x| \geq \frac{1}{3} \\ 0 & \text{if } \frac{1}{3} \leq |x| \leq 1 \end{cases}$$

2.1.5 Test Case 5

Here we need to find the approximate solution at $u(x, 0.3)$ of the differential equation:

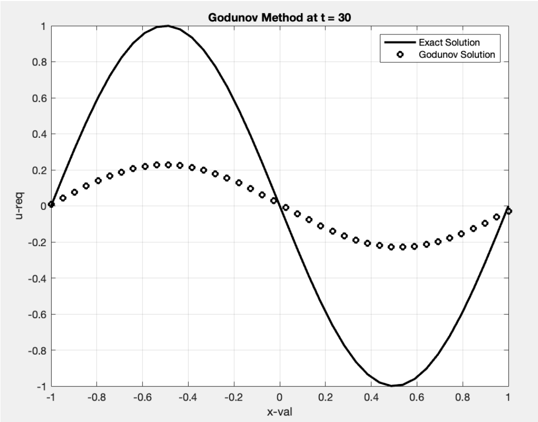
$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}(\frac{u^2}{2}) = 0$$

with initial condition:

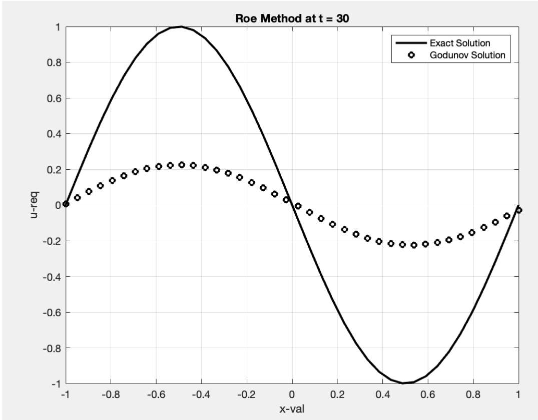
$$u(x, 0) = \begin{cases} 1 & \text{if } |x| \geq \frac{1}{3} \\ -1 & \text{if } \frac{1}{3} \leq |x| \leq 1 \end{cases}$$

2.2 Interpretation of Results

2.2.1 Testcase 1



(a) Godunov's Method

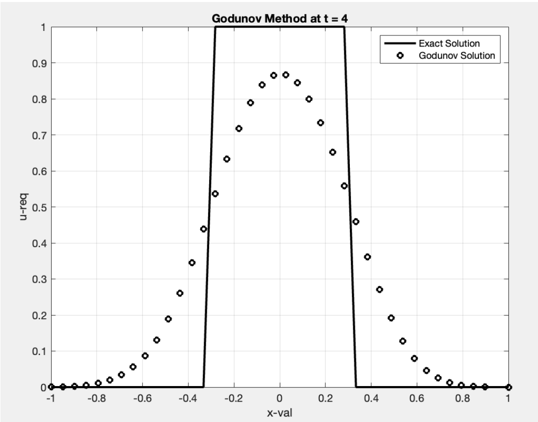


(b) Roe's Method

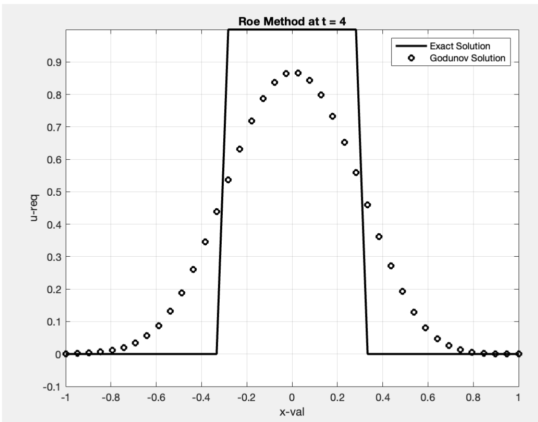
Figure 1: Plot for Testcase 1

- For both Godunov's and Roe's method, the sinusoid's shape is well preserved, and its phase is approximately correct, but its amplitude has been reduced by a significant amount.

2.2.2 Testcase 2



(a) Godunov's Method

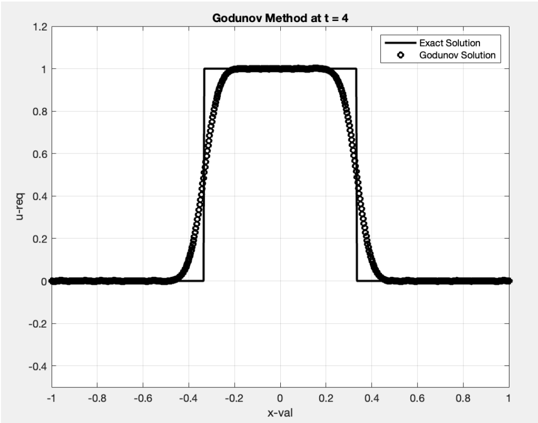


(b) Roe's Method

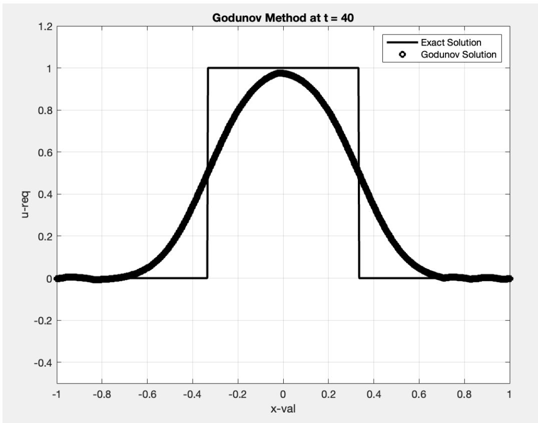
Figure 2: Plot for Testcase 2

- For both Godunov's and Roe's method, the contacts are extremely smeared and the square wave's peak has reduced. But, the solution is symmetric, properly located, and free of spurious overshoots or oscillations.

2.2.3 Testcase 3

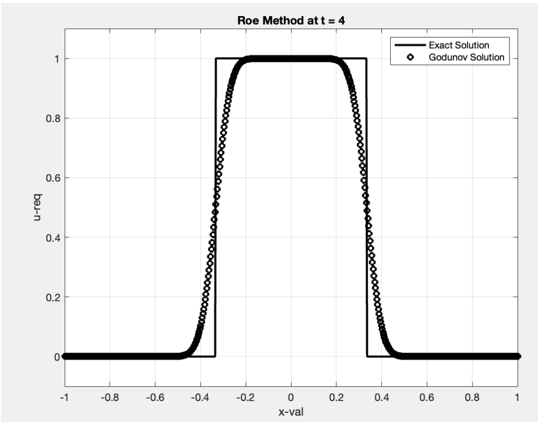


(a) t=4

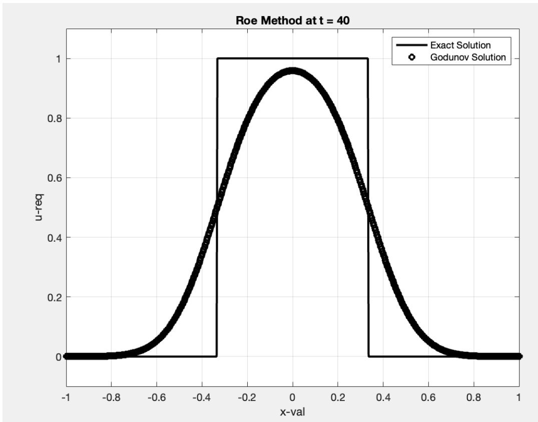


(b) t=40

Figure 3: Godunov's method



(a) t=4

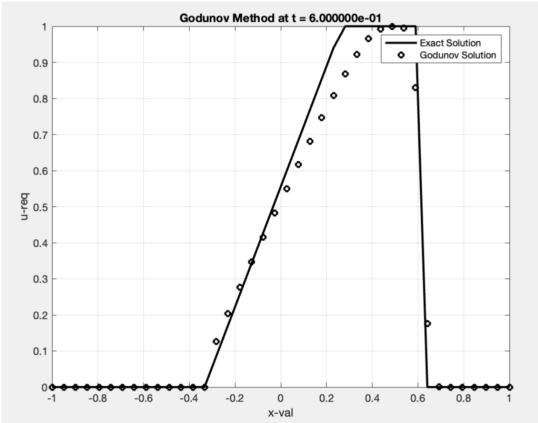


(b) t=40

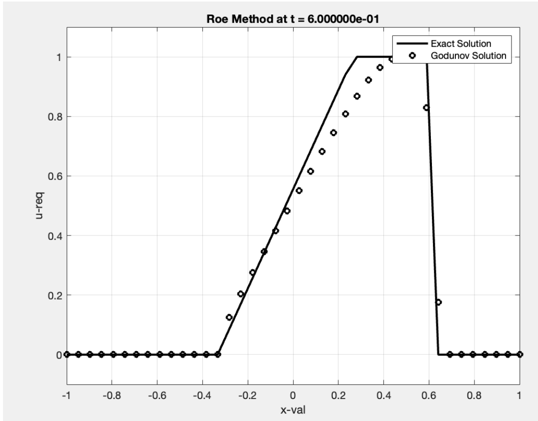
Figure 4: Roe's method

- For both Godunov's and Roe's method, increasing the number of grid points, and decreasing Δx and Δt, dramatically improve the approximation for u(x, 4),

2.2.4 Testcase 4



(a) Godunov's Method

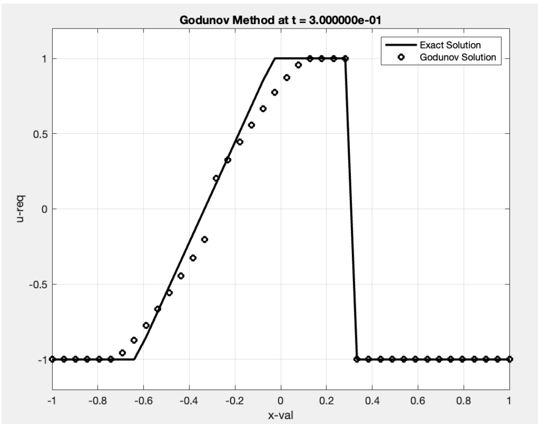


(b) Roe's Method

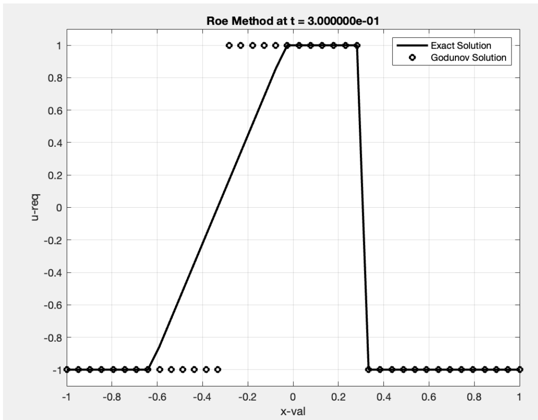
Figure 5: Plot for Testcase 4

- For both Godunov's and Roe's method, the shock is captured across only two grid points and without any spurious overshoots or oscillations. The corner at the head of the expansion fan has been slightly rounded off.

2.2.5 Testcase 5



(a) Godunov's Method



(b) Roe's Method

Figure 6: Plot for Testcase 5

- Godunov's method captures the steady shock perfectly. It partially captures the expansion fan.
- In Roe's method, the steady shock is captured perfectly. Unfortunately, like the Lax-Wendroff method, Roe's method fails to alter the initial conditions in any way, which is a total disaster for the expansion. Roe's approximate Riemann solver cannot capture the finite spread of expansion fans, and this defect carries over to Roe's first-order upwind method.
- As another way to view the situation, Godunov's first-order method has somewhat inadequate artificial viscosity at expansive sonic points, whereas Roe's first-order upwind method has drastically inadequate artificial viscosity at expansive sonic points.