Assignment 3

MTL712

Ashi Veerman
2021MT10241

1 Introduction

Euler's Method

The following is the Taylor series expansion of function of y(t):

$$y(t+\Delta t) = y(t) + \Delta t y'(t) + \frac{1}{2}\Delta t^2 y''(t) + \frac{1}{3!}\Delta t^3 y'''(t) + \dots$$

The Taylor series with remainder term is

$$y(t+\Delta t) = y(t) + \Delta t y'(t) + \frac{1}{2}\Delta t^2 y''(t) + \frac{1}{3!}\Delta t^3 y'''(t) + \dots + \frac{1}{n!}\Delta t^n y^{(n)}(\tau)$$

where τ is some value between t and $t+\Delta t$. You can truncate this for any value of n.

If we truncate the Taylor series at the first term it is **Euler's Method**

$$y(t+\Delta t) = y(t) + \Delta t y'(t) + \frac{1}{2}\Delta t^2 y''(\tau),$$

we can rearrange this and solve for $y'(t)$

$$y'(t) = \frac{y(t+\Delta t) - y(t)}{\Delta t} + O(\Delta t).$$

Now we can attempt to solve IVP by replacing the derivative with a difference:

$$y((n+1)\Delta t) \approx y(n\Delta t) + \Delta t f(n\Delta t, y(n\Delta t)).$$

Start with $y(0)$ and step forward to solve for any time.

What's good about this? If the O term is something nice looking, this quantity decays with Δt, so if we take Δt smaller and smaller, this gets closer and closer to the real value.

Taylor Series Methods

To derive these methods we start with a Taylor Expansion:

$$y(t+\Delta t) \approx y(t) + \Delta t y'(t) + \frac{1}{2}\Delta t^2 y''(t) + \dots + \frac{1}{r!}y^{(r)}(t)(\Delta t)^r.$$

Let's say we want to truncate this at the second derivative and base a method on that. The scheme is, then:

$$y_{n+1} = y_n + f_n \Delta t + \frac{f'_n}{2}\Delta t^2.$$

The Taylor series method can be written as

$$y_{n+1} = y_n + \Delta t F(t_n, y_n, \Delta t)$$

where $F = f + \frac{1}{2}\Delta t f'$. If we take the LTE for this scheme, we get (as expected)

$$LTE(t) = \frac{y(t_n + \Delta t) - y(t_n)}{\Delta t} - f(t_n, y(t_n)) - \frac{1}{2}\Delta t f'(t_n, y(t_n)) = O(\Delta t^2).$$

Of course, we designed this method to give us this order, so it shouldn't be a surprise!

So the LTE is reasonable, but what about the global error? Just as in the Euler Forward case, we can show that the global error is of the same order as the LTE. How do we do this? We have two facts,

$$y(t_{n+1}) = y(t_n) + \Delta t F(t_n, y(t_n), \Delta t),$$

and

$$y_{n+1} = y_n + \Delta t F(t_n, y_n, \Delta t)$$

where $F = f + \frac{1}{2}\Delta t f'$. Now we subtract these two

$$|y(t_{n+1}) - y_{n+1}| = |y(t_n) - y_n + \Delta t(F(t_n, y(t_n)) - F(t_n, y_n)) + \Delta t|LTE|| \le |y(t_n) - y_n| + \Delta t|F(t_n, y(t_n)) - F(t_n, y_n)| + \Delta t|LTE|.$$

Now, if F is Lipschitz continuous, we can say

$$e_{n+1} \le (1 + \Delta t L)e_n + \Delta t|LTE|.$$

Of course, this is the same proof as for Euler's method, except that now we are looking at F, not f, and the LTE is of higher order. We can do this no matter which Taylor series method we use, how many terms we go forward before we truncate.

Advantages and Disadvantages of the Taylor Series Method

Advantages

- One step, explicit
- Can be high order
- Easy to show that global error is the same order as LTE

Disadvantages

- Needs the explicit form of derivatives of f.

2 Testcases

2.1 Testcase Description

Testcase 1

Solve the following initial-value problems using Euler's method.

1. $y' = te^t - 2y$, $0 \le t \le 1$, $y(0) = 0$, with step size $h = 0.5$.
2. $y' = 1 + (t - y)^2$, $2 \le t \le 3$, $y(2) = 1$, with step size $h = 0.5$.
3. $y' = 1 + y/t$, $1 \le t \le 2$, $y(1) = 2$, with step size $h = 0.25$.
4. $y' = \cos 2t + \sin 3t$, $0 \le t \le 1$, $y(0) = 1$, with step size $h = 0.25$.

The exact solutions to the above IVPs are as follows.

1. $y(t) = \frac{1}{5}te^t - \frac{1}{25}e^{3t} + \frac{3}{25}e^{-2t}$,
2. $y(t) = t + \frac{1}{1-t^2}$,
3. $y(t) = t\ln t + 2t$,
4. $y(t) = \frac{1}{2}\sin 2t - \frac{1}{3}\cos 3t + \frac{4}{3}$.

Testcase 2

Solve the following initial-value problems using Euler's method.

1. $y' = y/t - (y/t)^2$, $1 \le t \le 2$, $y(1) = 1$, with step size $h = 0.1$.
2. $y' = 1 + y/t + (y/t)^2$, $1 \le t \le 3$, $y(1) = 0$, with step size $h = 0.2$.
3. $y' = -(y + 1)(y + 3)$, $0 \le t \le 2$, $y(0) = -2$, with step size $h = 0.2$.
4. $y' = -5y + 5t^2 + 2t$, $0 \le t \le 1$, $y(0) = \frac{1}{3}$, with step size $h = 0.1$.

The exact solutions to the above IVPs are as follows.

1. $y(t) = \frac{t}{1+\ln t}$,
2. $y(t) = t\tan(\ln t)$,
3. $y(t) = -3 + \frac{2}{1+e^{-2t}}$,
4. $y(t) = t^2 + \frac{1}{3}e^{-5t}$.

Testcase 3

The exact solution for the IVP

$$y' = \frac{2}{t}y + t^2 e^t, \quad 1 \le t \le 2, \quad y(1) = 0,$$

is $y(t) = t^2(e^t - e)$.

1. Use Euler's method with $h = 0.1$ to approximate the solution, and compare it with the actual values of y.
2. Use the answers generated in part (a) and linear interpolation to approximate the following values of y, and compare them to the actual values:
   - $y(1.04)$
   - $y(1.55)$
   - $y(1.97)$
3. Compute the value of h necessary for $|y(t_i) - w_i| \le 0.1$, using

$$|y(t_i) - w_i| \le \frac{hM}{2L}\left[e^{L(t_{i-1}-1)} - 1\right]$$

   where L is the Lipschitz constant and M is a constant satisfying $|y''(t)| \le M$, for all $t \in [1, 2]$.

Testcase 4

Solve the following initial-value problems using Taylor's method of order two:

(a) $y' = te^{3t} - 2y, \quad 0 \le t \le 1, \quad y(0) = 0$, with step size $h = 0.5$.

(b) $y' = 1 + (t - y)^2, \quad 2 \le t \le 3, \quad y(2) = 1$, with step size $h = 0.5$.

(c) $y' = 1 + y/t, \quad 1 \le t \le 2, \quad y(1) = 2$, with step size $h = 0.25$.

(d) $y' = \cos 2t + \sin 3t, \quad 0 \le t \le 1, \quad y(0) = 1$, with step size $h = 0.25$.

Testcase 5

Solve the following initial-value problems using Taylor's method of order two:

(a) $y' = y/t - (y/t)^2, \quad 1 \le t \le 1.2, \quad y(1) = 1$, with step size $h = 0.1$.

(b) $y' = \sin t + e^{-t}, \quad 0 \le t \le 1, \quad y(0) = 0$, with step size $h = 0.5$.

(c) $y' = (y^2 + y)/t, \quad 1 \le t \le 3, \quad y(1) = -2$, with step size $h = 0.5$.

(d) $y' = -ty + 4ty^{-1}, \quad 0 \le t \le 1, \quad y(0) = 1$, with step size $h = 0.25$.

Testcase 6

The exact solution for the IVP

$$y' = \frac{2}{t}y + t^2 e^t, \quad 1 \le t \le 2, \quad y(1) = 0,$$

is $y(t) = t^2(e^t - e)$.

(a) Use Taylor's method of order two with $h = 0.1$ to approximate the solution, and compare it with the actual values of y.

(b) Use the answers generated in part (a) and linear interpolation to approximate the following values of y, and compare them to the actual values of y:

- (i) $y(1.04)$
- (ii) $y(1.55)$
- (iii) $y(1.97)$

(c) Use Taylor's method of order four with $h = 0.1$ to approximate the solution, and compare it with the actual values of y.

(d) Use the answers generated in part (c) and linear interpolation to approximate the following values of y, and compare them to the actual values of y:

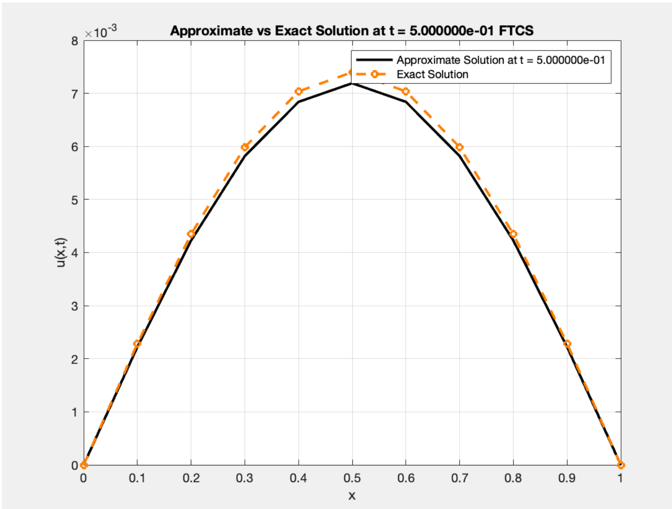
(i) $y(1.04)$

(ii) $y(1.55)$

(iii) $y(1.97)$

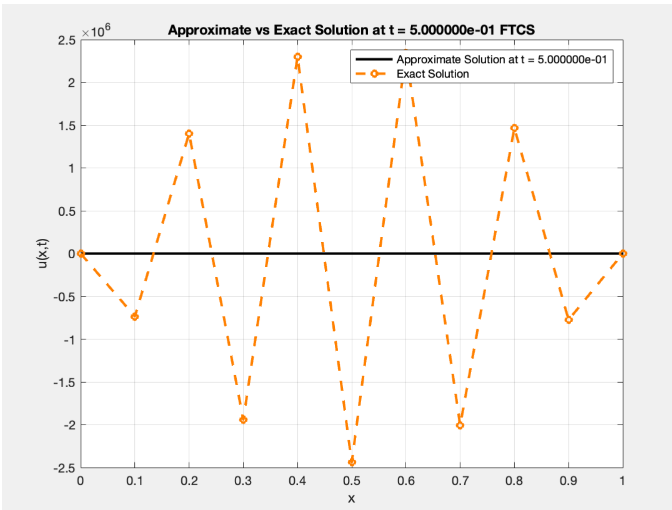
2.2 Interpretation of Results

Testcase 1



(a)

$h = 0.1, k = 0.0005, FTCS$



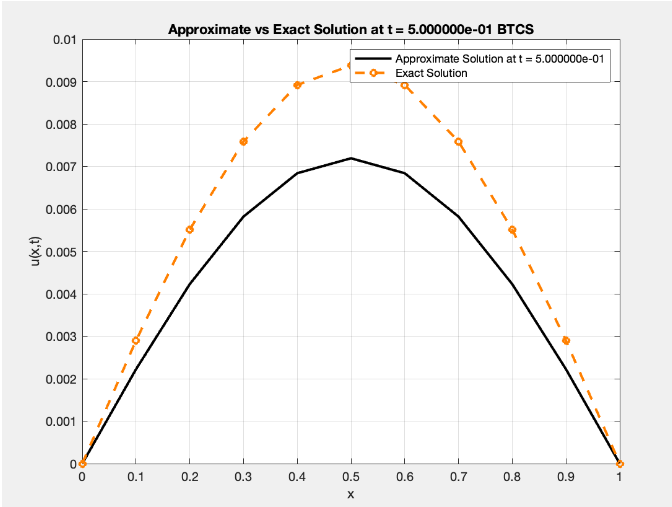
(b)

$h = 0.1, k = 0.01, FTCS$

Figure 1: Plot for Testcase 1

- When we increase the value of k in the FTCS method, the approximation becomes unstable, leading to large fluctuations in the graph (as seen by the spikes in graph 2). Therefore, the value of k can significantly impact the quality of the approximation.

Testcase 2



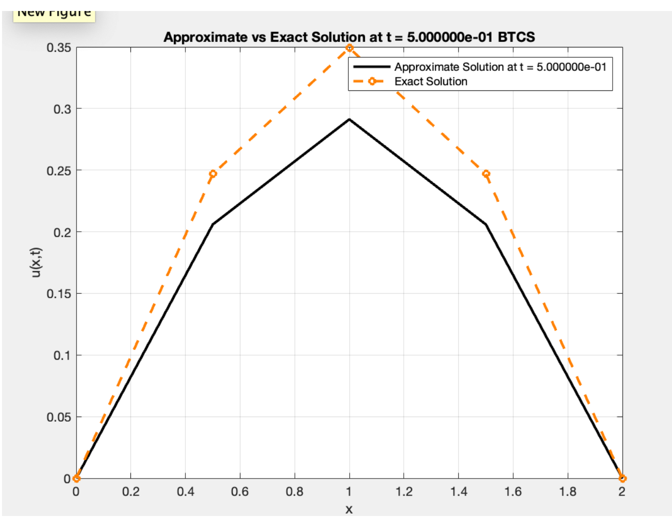
(a)

$h = 0.1, k = 0.01, BTCS$

Figure 2: Plot for Testcase 2

- In this case, it is evident that although there is still a significant difference between the exact and approximate solutions, the approximate solution closely resembles the exact solution. This indicates greater stability under the same conditions and k value as FTCS in test case 1. Therefore, it is evident that BTCS is more stable compared to FTCS.

Testcase 3



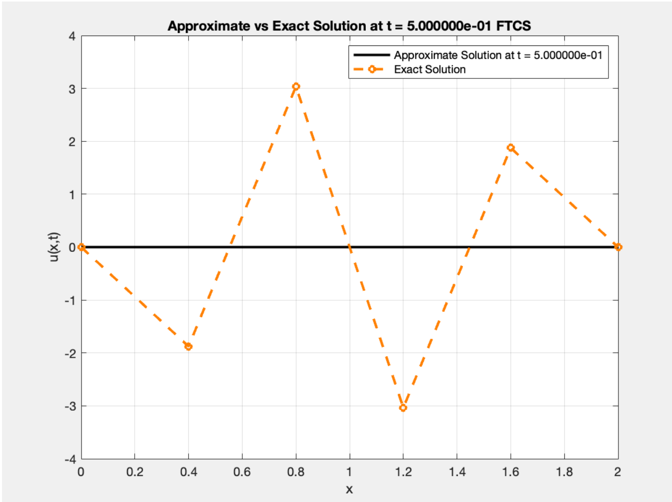
(a)

$h = 0.5, k = 0.1, BTCS$

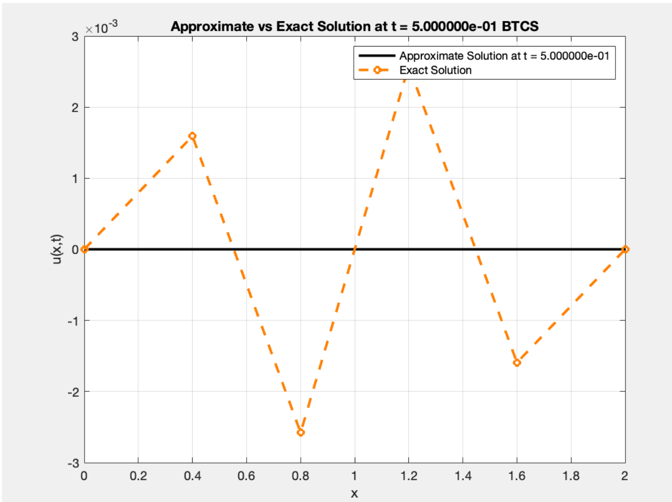
Figure 3: Plot for Testcase 3

- Test case 3 also demonstrates that while the approximate solution is still far from the exact solution, it is able to capture the shape of the exact solution quite well, even for high values of k and h. This indicates a certain level of stability.

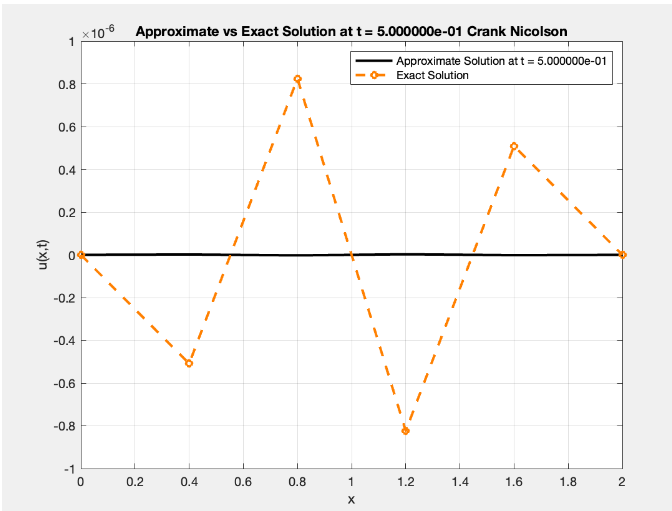
Testcase 4



(a) h = 0.4, k = 0.1, FTCS

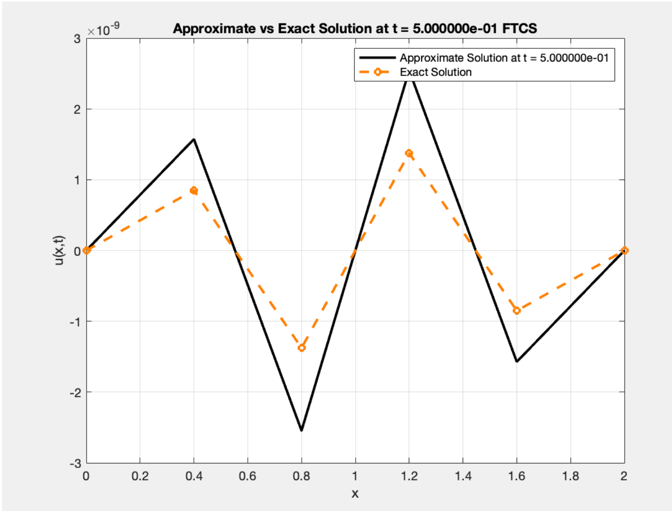


(b) h = 0.4, k = 0.1, BTCS

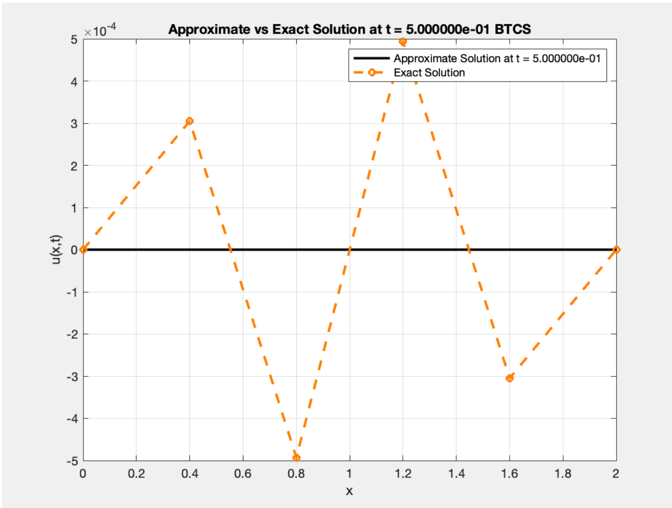


(c) h = 0.4, k = 0.1, Crank Nicolson

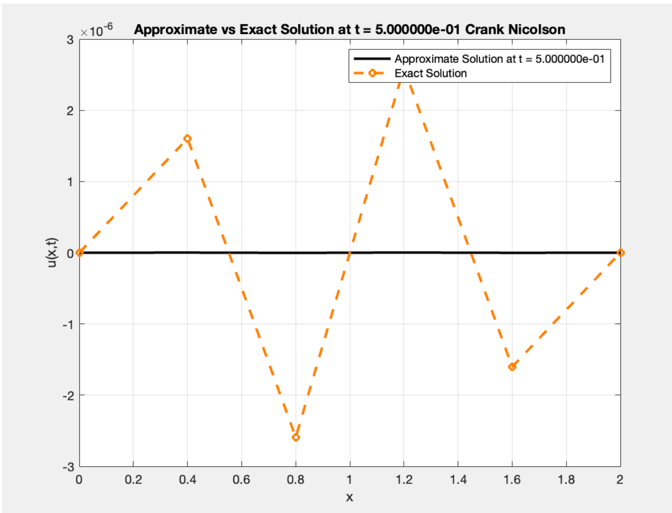
Figure 4: Plot for Testcase 4(a)



(a) h = 0.4, k = 0.05, FTCS



(b) h = 0.4, k = 0.05, BTCS

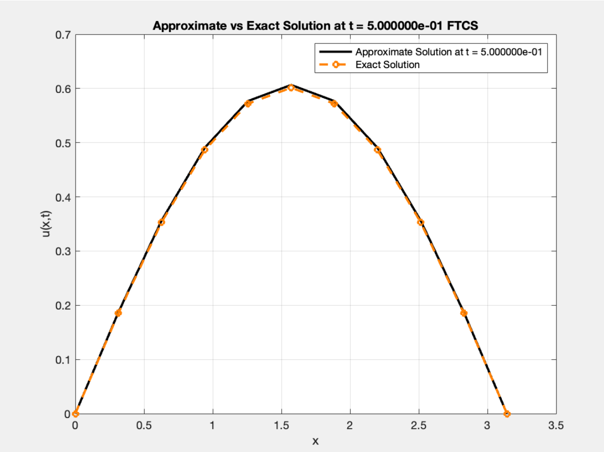


(c) h = 0.4, k = 0.05, Crank Nicolson

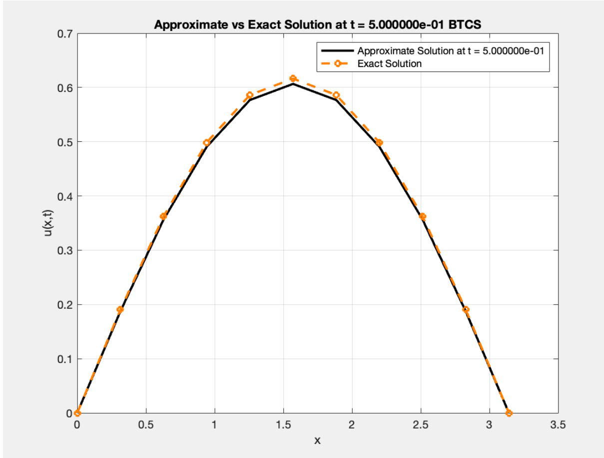
Figure 5: Plot for Testcase 4(b)

- In both parts (a) and (b), BTCS and Crank Nicolson appear to have performed similarly.
- However, with a smaller k value, as seen in part (b), it is observed that FTCS begins to approximate the exact solution quite well.

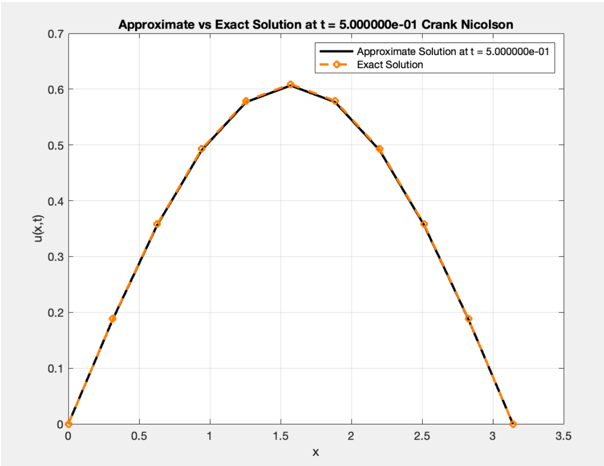
Testcase 5



(a) h = $\pi/10$, k = 0.05, FTCS



(b) h = $\pi/10$, k = 0.05, BTCS



(c) h = $\pi/10$, k = 0.05, Crank Nicolson

Figure 6: Plot for Testcase 5

- The Crank-Nicolson method provides the best approximation of the exact solution.
- While both FTCS and BTCS are close, we can observe that FTCS slightly overshoots the exact solution, while BTCS slightly undershoots it.
- Test case 5 provides a side-by-side comparison of all three methods.