Landau Fermi Liquid Theory

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1.1 The Outline: Goals for today

Guiding Questions:

1. **What** is a "Fermi Liquid" and how does it differ from a non-interacting Fermi Gas?
2. **Why** does this simple picture work? Why do strong e^{-}–e^{-} interactions not destroy the free-electron model?
3. **What** are its concrete, measurable predictions (e.g., for specific heat, susceptibility, resistivity) and do they match experiment?

1.2.1 Motivation — Why Fermi Liquids Matter

Core Idea: Interacting electrons *should* behave chaotically, yet metals act as if they're almost free. So why do metals look "free"? Landau Fermi-liquid theory explains this remarkable stability.

The Puzzle: In a typical metal, electrons interact via strong **Coulomb repulsion**, comparable to their kinetic energy:

$$E_C = \frac{e^2}{4\pi\epsilon_0 r} = \frac{e^2}{a_0}\frac{1}{r_s}$$

- comparable to the kinetic (fermi) energy:

$$E_F = \frac{\hbar^2 k_F^2}{2m} = \frac{\hbar^2}{2ma_0^2}\frac{1}{r_s^2}$$

$$\langle E_k \rangle = \frac{3}{5} E_F$$

The ratio is then: $\frac{E_C}{E_F} \propto r_s$

1.2.2 Motivation — Why Fermi Liquids Matter

The ratio is then: $\frac{E_C}{E_F} \propto r_s$

This should make the system a messy, strongly correlated fluid with no sharp excitations, no simple Fermi surface, since the Coulomb repulsion E_C is not a small perturbation.

BUT: experiments show simple, "free-electron-like" properties:

- Well-defined Fermi surfaces (via quantum oscillations, seen in dHvA, ARPES)
- Linear specific heat at low T: $C_V = \gamma T$
- Temperature-independent Pauli susceptibility $\chi \approx$ const.
- Characteristic Electrical resistivity $\rho = \rho_0 + AT^2$

So how can an interacting many-body system of fermions behave almost exactly like a non-interacting gas?

1.3.1 Landau's Insight: The Quasiparticle

How can an interacting many-body system of fermions behave almost exactly like a non-interacting gas?

Landau's Postulate: The low-energy excitations of the interacting system are in one-to-one correspondence with the excitations of the non-interacting gas. These excitations are called the **quasiparticles**.

- Quasiparticles are not a bare electron. Think of as an electron "dressed" by a cloud of virtual particle-hole excitations and screening charges.
- same quantum numbers as a bare electron (charge e, spin $\frac{1}{2}$ momentum p).
- Interactions renormalize its properties:
  - effective mass $m^* \neq m_e$
  - effective g-factor $g^* \neq g_e$
  - quasiparticle residue $Z < 1$ (this is strength of the "bare electron" component)

1.3.2 Landau's Insight: The Quasiparticle

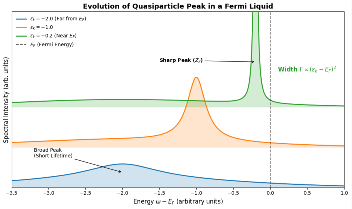
To really understand this recall Fermi Surface:

- Free Gas: A sharp boundary in $\mathbf{k}$-space at $T = 0$ separating occupied $(n_k = 1)$from empty $(n_k = 0)$ states.
- Fermi Liquid: This sharp boundary survives interactions forming the surface of quasiparticles.
- The Luttinger's Theorem (1960) later proved that the volume enclosed by the Fermi surface is unchanged by interactions, as long as the system remains a Fermi liquid.
- These quasiparticles behave as nearly independent entities, but their properties and their residual interactions are determined by the many-body system.

1.4.1 Why it works: Adiabaticity

Adiabaticity:

- turn on the interactions U slowly, from $U = 0$ (free gas) to their full strength
- the ground state and low-lying excited states of the free gas smoothly/adiabatically evolve into the ground state and low-lying excited states of the interacting Fermi Liquid
    - we effectively turn a bare electron into a quasiparticle



1.4.2 Why it works: Phase Space and Pauli Blocking

Why are Quasiparticles Stable? Why do they not decay instantly?

- Consider a quasiparticle with energy $\epsilon > E_F$ (and $T = 0$). For it to decay, it must scatter, creating, for example, two new quasiparticles and a quasi-hole.
- Pauli exclusion severely restricts the available final states since they must all be outside the Fermi sea.
- The available phase space for scattering scales as $(\epsilon - E_F)^2$.This means the lifetime (τ) gets very long as the quasiparticle approaches the Fermi surface:

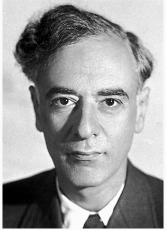
$$\frac{1}{\tau} \propto (\epsilon - E_F)^2 + (\pi k_B T)^2$$

At the Fermi surface ($\epsilon = E_F$, $T = 0$), the lifetime is infinite. This is why the Fermi surface remains sharp. We will see a quick example of this later on.

Arnold Sommerfeld (c. 1930s)

From Sommerfeld to Landau

- Sommerfeld's free-electron model explains heat capacity and conductivity in metals.
- Puzzle: why does a non-interacting model work for systems with charged particles?



Lev Landau (1940s)

2.1.2: A Timeline of Understanding: 1940s–1950s

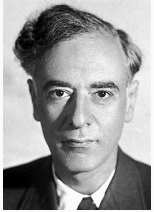
The Liquid ^{3}He Anomaly

- Liquid ^{3}He is a system of strongly interacting neutral fermions.
- Experiments showed it behaves just like a metal (linear C_V, finite compressibility).
- Implication: Interactions clearly exist, but they do not destroy the "fermionic" nature of the state.

2.1.3: A Timeline of Understanding: 1956–1958

Landau's Solution

- Lev Landau formulates Fermi Liquid Theory.
- A universal phenomenological description for interacting fermions at low T
- Key Shift: We stop tracking electrons and start tracking quasiparticles.



Lev Landau (1940s)

1. Liquid ^{3}He: The Perfect Fermi Liquid

- Strongly interacting, dense system of neutral fermions.
- Observation: At low T, it exhibits metallic behavior:
  - Linear specific heat: $C_V \propto T$
  - Finite compressibility and spin susceptibility.
- Landau Theory was essentially built to explain this system.

2.2.2: Experiments

2. **Simple Metals (Na, Cu, K)**
   - Electrons interact via long-range Coulomb repulsion.
   - Observation: Thermodynamics match predictions of nearly free electrons.
   - Robustness against Coulomb interactions was the major theoretical hurdle.
3. **APRES:Angle-Resolved Photoemission Spectroscopy**
   - Direct observation of the spectral function $A(\mathbf{k}, \omega)$.
   - We see a sharp quasiparticle peak at E_F that broadens as energy increases.

3.1.1: Building the Framework: Adiabatic Evolution

The Adiabatic Connection Consider the non-interacting ground state $|\Psi_0\rangle$ where all states with $p < p_F$ are filled:

$$|\Psi_0\rangle = \prod_{p<p_F} c^\dagger_{\mathbf{p}}|0\rangle \tag{1}$$

We turn on interactions $V(t)$ infinitely slowly ("adiabatically") from $t = -\infty$ to $t = 0$.

If no phase transition occurs, $|\Psi_0\rangle$ evolves smoothly into the interacting ground state $|\Phi_0\rangle$:

$$|\Phi_0\rangle = U(0, -\infty)|\Psi_0\rangle$$

where U is the time-evolution operator.

3.1.2: Building the Framework: Defining the Quasiparticle

- What happens to a single excited electron created by $c^\dagger_{\mathbf{p}\sigma}$ (where $|\mathbf{p}| > p_F$)?
- It evolves into a quasiparticle created by a new operator $a^\dagger_{\mathbf{p}\sigma}$:

$$a^\dagger_{\mathbf{p}\sigma} = U c^\dagger_{\mathbf{p}\sigma} U^\dagger$$

- Key Insight: This new operator $a^\dagger_{\mathbf{p}\sigma}$ is a superposition of the original electron plus a "cloud" of particle-hole pairs.
- It carries the same quantum numbers (spin, charge, momentum) but has renormalized dynamical properties (mass m^*, g-factor g^*).

3.1.3: Building the Framework: The Landau Energy Functional

The Distribution Function

- Since quasiparticles are stable near the Fermi surface, their occupation numbers $n_{\mathbf{p}\sigma}$ are good quantum numbers (approximate constants of motion).
- We describe the state of the system solely by the distribution of quasiparticles:

$$n_{\mathbf{p}\sigma} = n^{(0)}_{\mathbf{p}\sigma} + \delta n_{\mathbf{p}\sigma}$$

where $n^{(0)}_{\mathbf{p}\sigma} = \theta(p_F - p)$ is the ground state distribution.

Landau's Energy Expansion The total energy E is a functional of the occupations $n_{\mathbf{p}\sigma}$. We expand it for small deviations δn:

$$E = E_0 + \sum_{\mathbf{p}\sigma} \epsilon_{\mathbf{p}} \delta n_{\mathbf{p}\sigma} + \frac{1}{2} \sum_{\mathbf{p}\sigma,\mathbf{p}'\sigma'} f_{\mathbf{p}\sigma,\mathbf{p}'\sigma'} \delta n_{\mathbf{p}\sigma} \delta n_{\mathbf{p}'\sigma'} + \dots$$

- First Term ($\epsilon_{\mathbf{p}}$): The quasiparticle energy.

3.2.1: The Landau Parameters ($F_l^{s,a}$)

Exploiting Symmetry

For an isotropic liquid (like ^{3}He or ideal metal), $f_{\mathbf{p}\sigma,\mathbf{p}'\sigma'}$ depends only on:

1. The angle θ between momenta $\mathbf{p}$ and $\mathbf{p}'$.
2. The relative spin orientation (σ, σ').

Spin Decomposition

We split the interaction into spin-symmetric (f^s) and spin-antisymmetric (f^a) parts:

$$f_{\mathbf{p}\sigma,\mathbf{p}'\sigma'} = f^s(\theta) + f^a(\theta)\boldsymbol{\sigma} \cdot \boldsymbol{\sigma}'$$

- f^s: Interaction independent of spin (density-density like)
- f^a: Interaction dependent on spin alignment (exchange like).

3.2.2: The Landau Parameters ($F_l^{s,a}$)

Legendre Expansion

Since dependence is only on the angle $\cos\theta$, we expand in Legendre polynomials $P_l(\cos\theta)$:

$$\mathcal{N}(0) f^{s,a}(\theta) = \sum_{l=0}^{\infty} F_l^{s,a} P_l(\cos\theta)$$

- $\mathcal{N}(0) = \frac{m^* p_F}{\pi^2 \hbar^3}$: Density of states at the Fermi level.
- The Landau Parameters $F_l^{s,a}$: Dimensionless numbers that characterize the liquid.
  - F_0^s: Related to compressibility.F_1^s: Related to effective mass (m^*).
  - F_0^a: Related to magnetic susceptibility.

3.3.1: Example 6.1 — Microscopic View (Unitary Operator)

Goal: Show formal relation between bare and dressed operators.

Setup:

- Ground state $|\Psi_0\rangle$.
- Bare creation operator $c^\dagger_{\mathbf{p}\sigma}$.
- Time evolution operator $U(t) = T\exp[-i\int_{-\infty}^{t} V(t')dt']$.

Process:

1. Start with a bare electron state $c^\dagger_{\mathbf{p}\sigma}|\Psi_0\rangle$ at $t = -\infty$.
2. Evolve to $t = 0$:

$$|\Psi_{\mathbf{p}\sigma}\rangle = U(0, -\infty)c^\dagger_{\mathbf{p}\sigma}|\Psi_0\rangle$$

3. Insert identity $U^\dagger U = 1$:

$$|\Psi_{\mathbf{p}\sigma}\rangle = \left(Uc^\dagger_{\mathbf{p}\sigma}U^\dagger U\right)|\Psi_0\rangle$$

3.3.2: Example 6.1 — Microscopic View (Unitary Operator)

Goal: Show formal relation between bare and dressed operators.

Setup:

- Ground state $|\Psi_0\rangle$.
- Bare creation operator $c^\dagger_{\mathbf{p}\sigma}$.
- Time evolution operator $U(t) = T\exp[-i\int_{-\infty}^{t} V(t')dt']$.

Process:

Insert identity $U^\dagger U = 1$:

$$|\Psi_{\mathbf{p}\sigma}\rangle = \underbrace{(U c^\dagger_{\mathbf{p}\sigma} U^\dagger)}_{\text{Quasiparticle Operator } a^\dagger_{\mathbf{p}\sigma}} \underbrace{U|\Psi_0\rangle}_{\text{Interacting Ground State } |\Phi_0\rangle}$$

Result: The quasiparticle operator $a^\dagger_{\mathbf{p}\sigma}$ creates an excitation that carries momentum $\mathbf{p}$ and spin σ on top of the interacting ground state.

3.4.1: Bringing the Framework together

The Idealized System For our theory for a Neutral Fermi Liquid (like liquid ^{3}He).

- Quasiparticles move in free space.
- Interactions are short-ranged and isotropic.
- Coulomb interactions in metals are long-ranged and require special handling (discussed later).
- For low densities of quasiparticles (low T, low excitation energy), we describe the system by the deviation from equilibrium (rearrange state of the system via distribution of quasiparticle):

$$\delta n_{\mathbf{p}\sigma} = n_{\mathbf{p}\sigma} - n_{\mathbf{p}\sigma}^{(0)}$$

The Idealized System For our theory for a Neutral Fermi Liquid (like liquid ^{3}He), now looking at the Thermodynamics:

- Entropy: Under adiabatic turn-on, entropy is invariant. It retains the combinatorial form of non-interacting fermions:

$$S = -k_B \sum_{\mathbf{p}\sigma} [n_{\mathbf{p}\sigma} \ln n_{\mathbf{p}\sigma} + (1 - n_{\mathbf{p}\sigma}) \ln(1 - n_{\mathbf{p}\sigma})]$$

- Free Energy: $F = \mathcal{E} - TS$.

3.4.3: Bringing the Framework together

The Landau Energy Functional (recap)

The total energy density $\mathcal{E}$ is a functional of the distribution $n_{\mathbf{p}\sigma}$. We expand it in powers of δn:

$$\mathcal{E} = \mathcal{E}_0 + \sum_{\mathbf{p}\sigma} (E^{(0)}_{\mathbf{p}\sigma} - \mu)\, \delta n_{\mathbf{p}\sigma} + \frac{1}{2} \sum_{\mathbf{p},\mathbf{p}',\sigma,\sigma'} f_{\mathbf{p}\sigma,\mathbf{p}'\sigma'}\, \delta n_{\mathbf{p}\sigma}\, \delta n_{\mathbf{p}'\sigma'} + \dots$$

This expansion defines the two key pillars of the theory:

1. First Order Coefficient: The Quasiparticle Energy.

$$\epsilon^{(0)}_{\mathbf{p}\sigma} \equiv E^{(0)}_{\mathbf{p}\sigma} - \mu = \frac{\delta \mathcal{E}}{\delta n_{\mathbf{p}\sigma}}$$

2. Second Order Coefficient: The Interaction Function.

$$f_{\mathbf{p}\sigma,\mathbf{p}'\sigma'} = \frac{\delta^2 \mathcal{E}}{\delta n_{\mathbf{p}\sigma}\, \delta n_{\mathbf{p}'\sigma'}}$$

3.4.4: Framework: The Quasiparticle Energy & Effective Mass

Quasiparticle Energy Near the Fermi surface ($p \approx p_F$), we linearize:

$$E_p^{(0)} \approx \mu + v_F(p - p_F)$$

Effective Mass (m^*)
We define the effective mass via the Fermi velocity v_F:

$$v_F \equiv \frac{p_F}{m^*} = \left.\frac{d\epsilon_p^{(0)}}{dp}\right|_{p=p_F}$$

These are key definitions that we will need to see how the density of states changes which will allow us to make key predictions.

3.4.5: Bringing the Framework together: Density of States

Density of States

$$\mathcal{N}^*(0) = \frac{m^* p_F}{\pi^2 \hbar^3}$$

Key Prediction: Specific heat $C_V \propto \mathcal{N}^*(0) \propto m^*$.

Recall: **The Landau Interaction Function** $f_{\mathbf{p}\mathbf{p}'}$

$$f_{\mathbf{p}\sigma,\mathbf{p}'\sigma'} = \frac{\delta \epsilon_{\mathbf{p}\sigma}}{\delta n_{\mathbf{p}'\sigma'}}$$

and for **Energy Renormalization** the total energy of a quasiparticle depends on the state of the entire sea:

$$\epsilon_{\mathbf{p}\sigma} = \epsilon^{(0)}_{\mathbf{p}\sigma} + \sum_{\mathbf{p}'\sigma'} f_{\mathbf{p}\sigma,\mathbf{p}'\sigma'}\, \delta n_{\mathbf{p}'\sigma'}$$

Now lets use this framework to do another example.

3.5.1: Example 6.2 — Calculating $f_{\mathbf{pp'}}$ (Perturbation Theory)

Use first-order perturbation theory to calculate the Landau interaction parameters for a fluid of fermions with a weak interaction described by:

$$H = \sum_{p\sigma} E_p n_{p\sigma} + \frac{\lambda}{2} \sum_{p\sigma, p'\sigma', q} V(q) c^{\dagger}_{p-q\sigma} c^{\dagger}_{p'+q\sigma'} c_{p'\sigma'} c_{p\sigma}$$

where E_p is the energy of the non-interacting Fermi gas, $V(q) = \int \frac{d^3q}{(2\pi)^3} e^{-iq\cdot r} V(r)$ is the Fourier transform of the interaction $V(r)$ and $\lambda << 1$ is a very small coupling constant. Use first-order perturbation theory in λ to compute the energy of a state $\psi = |n_{p_1,\sigma_1}, n_{p_2,\sigma_2}, \ldots, n_{p_N,\sigma_N}\rangle$ to a leading order in the interaction strength λ and then read off the terms quadratic in $n_{p\sigma}$

3.5.2: Example 6.2 — Calculating $f_{\mathbf{pp}'}$ (Perturbation Theory)

Goal: Compute Landau parameters for a weakly interacting gas.

Model: Interaction $V(q)$ with coupling $\lambda \ll 1$.

1. Energy Expectation Value (First Order)

$$E = \sum_{\mathbf{p}\sigma} E_{\mathbf{p}} n_{\mathbf{p}\sigma} + \frac{\lambda}{2} \sum_{\mathbf{p}\sigma, \mathbf{p}'\sigma'} [V(0) - V(\mathbf{p} - \mathbf{p}')\delta_{\sigma\sigma'}] n_{\mathbf{p}'\sigma'} n_{\mathbf{p}\sigma}$$

   - Direct Term: $V(0)$.
   - Exchange Term: $-V(\mathbf{p} - \mathbf{p}')$ .Requires parallel spins ($\delta_{\sigma\sigma'}$).

2. Extracting the Landau FunctionWe identify the term quadratic in n:

$$f_{\mathbf{p}\sigma, \mathbf{p}'\sigma'} = \lambda [V(0) - V(\mathbf{p} - \mathbf{p}')\delta_{\sigma\sigma'}] + O(\lambda^2)$$

3.5.3: Example 6.2 — Calculating $f_{\mathbf{pp}'}$ (Perturbation Theory)

3. Spin Decomposition

- Symmetric (f^s):Direct - Exchange/2

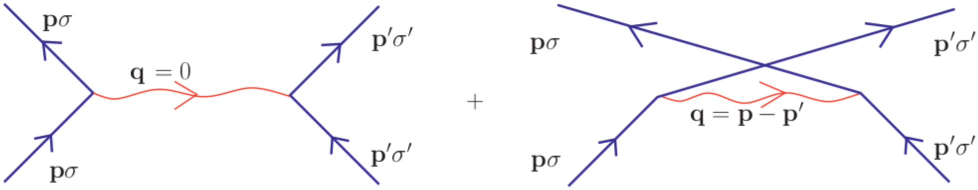
$$f^s \propto V(0) - \frac{1}{2}V(\mathbf{p} - \mathbf{p}')$$

- Antisymmetric (f^a): Exchange only

$$f^a \propto -\frac{1}{2}V(\mathbf{p} - \mathbf{p}')$$

Insight: For repulsive interactions ($V > 0$), the exchange term drives ferromagnetic tendencies ($f^a < 0$).

3.5.4: Example 6.2 — Calculating $f_{\mathbf{pp}'}$ (Perturbation Theory)



$V(\mathbf{q} = 0)$ $\qquad$ $-V(\mathbf{p} - \mathbf{p}')\delta_{\sigma\sigma'}$

$$f_{\mathbf{p}\sigma,\, \mathbf{p}'\sigma'} = V(\mathbf{q} = 0) - V(\mathbf{p} - \mathbf{p}')\delta_{\sigma\sigma'}$$

Feynman diagrams for leading-order contributions to the Landau parameter for an interaction $V(\mathbf{q})$. Wavy line represents the interaction between quasiparticles.

3.6.1: Landau Parameters — Summary

We have now made the theory usable and as a summary

Table: Free Gas vs. Fermi Liquid

Property	Non-interacting Gas	Landau Fermi Liquid
Fermi momentum	p_F	**Unchanged** (Luttinger)
Particle density	$n = k_F^3/3\pi^2$	**Unchanged**
Effective mass	m	$m^* = m\big(1 + F_1^s/3\big)$
Density of states	$\mathcal{N}(0)$	$\mathcal{N}^*(0) = (m^*/m)\,\mathcal{N}(0)$
Specific heat	$\gamma \propto \mathcal{N}(0)$	$\gamma^* \propto \mathcal{N}^*(0)$
Compressibility	κ_0	$\kappa \propto \dfrac{\mathcal{N}^*(0)}{1 + F_0^s}$
Susceptibility	χ_P	$\chi = \dfrac{\chi_P}{1 + F_0^a}$

3.6.2: Landau Parameters — Summary

Key Takeaway: The functional form of thermodynamics ($C_V \sim T$, $\chi \sim const$) is preserved; the *values* are rescaled by the Landau parameters $F_\ell^{s,a}$.
Next Step (for you): We are perfectly set up for analyzing feedback, deriving compressibility and susceptibility, and doing mass renormalizations. I won't be covering these in this presentation.

4.1.1: The Problem of Charged Liquids (Landau-Silin Theory)

The Challenge: Long-Range Forces

- Landau's original theory assumes short-range interactions (finite $f_{\mathbf{pp'}}$ at $\mathbf{q} \to 0$).
- In metals, electrons interact via the Coulomb Force:

$$V(q) = \frac{e^2}{\epsilon_0 q^2}$$

- This diverges as $\mathbf{q} \to 0$. Does the theory break down?

4.1.2: The Problem of Charged Liquids (Landau-Silin Theory)

Silin's Solution (1957)

We split the electric potential ϕ into two distinct parts:

$$\phi(\mathbf{x}) = \underbrace{\phi_P(\mathbf{x})}_{\text{Long-Range (Classical)}} + \underbrace{\delta\phi_Q(\mathbf{x})}_{\text{Short-Range (Quantum)}}$$

- ϕ_P: The macroscopic polarization field satisfying Gauss' Law. It comes from the average charge distribution.
- $\delta\phi_Q$: The local "quantum" fluctuations (exchange, correlation) localized within $\lambda \sim h/p_F$.

The New Energy Functional

The quasiparticle energy now includes the macroscopic field explicitly:

$$\epsilon_{\mathbf{p}\sigma}(\mathbf{x}) = \epsilon_{\mathbf{p}}^{(0)} + \underbrace{e\phi_P(\mathbf{x})}_{\text{Classical Field}} + \underbrace{\sum_{\mathbf{p}'} f_{\mathbf{p}\mathbf{p}'}\delta n_{\mathbf{p}'}}_{\text{Short-Range Residual}}$$

This is the key conceptual leap. We treat the dangerous $1/q^2$ part as a classical mean field, and keep the rest as our familiar Landau parameters.

4.2.1: Screening & The q-Dependent Interaction

How does this split affect the Landau parameters?

The Effective Interaction

In momentum space, the total effective interaction is the sum of the Coulomb term and the residual Landau term:

$$f^{\text{eff}}_{\mathbf{pp}'}(\mathbf{q}) = \frac{e^2}{\epsilon_0 q^2} + f^{\text{residual}}_{\mathbf{pp}'}$$

The Singularity is confined to $l = 0$

- The Coulomb term is isotropic (independent of angle between $\mathbf{p}$ and $\mathbf{p}'$)
- Therefore, it only affects the $l = 0$ symmetric parameter (F_0^s):

$$F_0^s(\mathbf{q}) = \underbrace{\frac{e^2\mathcal{N}^*(0)}{\epsilon_0 q^2}}_{\text{Diverges at } q\to 0} + F_0^s$$

All other parameters ($F_1^s, F_0^a, F_1^a, \dots$) remain finite and well-defined, identical to the neutral case.

4.2.2: Screening & The q-Dependent Interaction

Consequence: Thomas-Fermi Screening

The charge susceptibility (response to potential) becomes:

$$\chi_c(\mathbf{q}) = \frac{\chi_n}{1 + \frac{e^2}{\epsilon_0 q^2}\chi_n} \xrightarrow{q \to 0} \frac{\epsilon_0 q^2}{e^2}$$

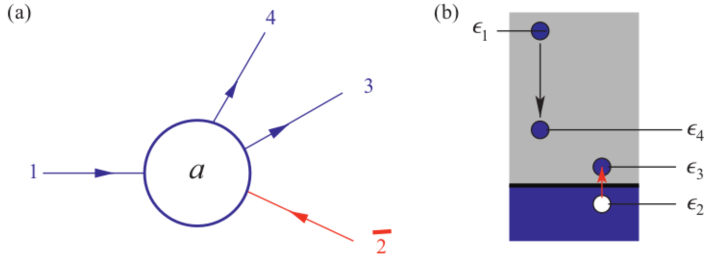
- The divergence in the denominator cancels the divergence in the force.
- The system perfectly screens long-range charge fluctuations.

The “magic” here is that the infinity in the interaction kills the response at $q = 0$. This is why metals are neutral in the bulk.

5.1.1: Why Do Quasiparticles Survive?

The Stability Problem

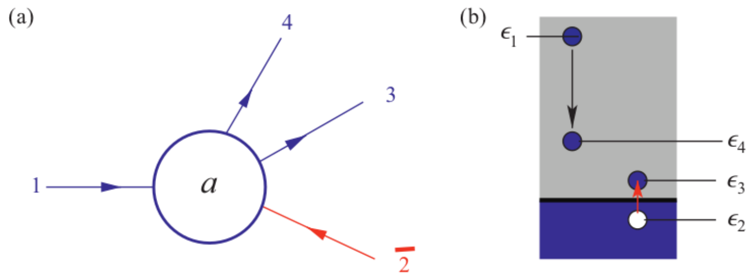
- A quasiparticle above the Fermi surface ($\epsilon > 0$) is an excited state.
- It should decay by scattering with the Fermi sea, creating lower-energy excitations.
    - The Dominant Process: A 3-body decay.



Decay of a quasiparticle into two quasiparticles and a quasihole: (a) scattering process; (b) energies of final states.

The Stability Problem

- The Dominant Process: A 3-body decay.
  - incoming quasiparticle (1) scatters off a sea electron (2)
  - creates two new quasiparticles (3,4) and leaves a hole ($\bar{2}$)



Decay of a quasiparticle into two quasiparticles and a quasihole: (a) scattering process; (b) energies of final states.

5.1.3: Why Do Quasiparticles Survive?

The Phase Space Constraint (Pauli Blocking)

- Constraint 1: Energy conservation. $\epsilon_1 + \epsilon_2 = \epsilon_3 + \epsilon_4$.
- Constraint 2: Pauli Exclusion. Final states 3 and 4 must be empty (above E_F). Initial state 2 must be filled (below E_F).
- As $\epsilon_1 \to 0$ (approaching Fermi surface), the "window" of available states shrinks drastically.
- Dimensional Analysis: Phase space scales as area $\sim \epsilon^2$.

$$\Gamma \propto \left(\frac{\epsilon}{\epsilon_F} \right)^2$$

Basically nowhere to go for the particle to scatter into.

5.2.1: The Microscopic Calculation

Setup

$$a(\mathbf{1} \to \bar{\mathbf{2}} + \mathbf{3} + \mathbf{4}) = a(\mathbf{1} + \mathbf{2} \to \mathbf{3} + \mathbf{4}) \equiv a(\mathbf{1}, \mathbf{2}; \mathbf{3}, \mathbf{4})$$

To prove this, we use Fermi's Golden Rule for the scattering rate Γ.

$$I = \frac{2\pi}{\hbar} \sum_{2,3,4} |A(1,2;3,4)|^2 \times \text{Occupancy Factors} \times \delta(\text{Energy})\delta(\text{Momentum})$$

Here I is the collision Integral denoted $I[n]$ and it is the collision term in the Boltzmann Transport Equation:

$$\left(\frac{\partial n_1}{\partial t}\right)_{\text{coll}} = I[n_1]$$

5.2.2: The Microscopic Calculation

$$\left(\frac{\partial n_1}{\partial t}\right)_{\text{coll}} = I[n_1]$$

The Collision Integral is the difference:

$$I \propto \sum (\text{Gain Term} - \text{Loss Term})$$

To calculate the quasiparticle lifetime, assume there is one extra quasiparticle in state 1 (so n_1 is slightly away from equilibrium) and you ask: "How fast does this specific excess density disappear?" So the functional derivative:

$$\Gamma \approx -\frac{\delta I}{\delta n_1}$$

gives the rate.

5.2.4: The Microscopic Calculation

$$a(\mathbf{1} \to \bar{\mathbf{2}} + \mathbf{3} + \mathbf{4}) = a(\mathbf{1} + \mathbf{2} \to \mathbf{3} + \mathbf{4}) \equiv a(\mathbf{1}, \mathbf{2}; \mathbf{3}, \mathbf{4})$$

$$I = \frac{2\pi}{\hbar} \sum_{2,3,4} |A(1,2;3,4)|^2 \times \text{Occupancy Factors} \times \delta(\text{Energy})\delta(\text{Momentum})$$

Factorization Strategy

- Angular Part (Geometry): $\langle W \rangle_\Omega$
  - Depends on the scattering angle θ and relative angle ϕ.
  - Determines the strength of the scattering (the pre-factor).
- Energy Part (Phase Space): $I(\epsilon, T)$
  - Depends only on energy conservation and Fermi functions $n(\epsilon)$.
  - Determines the scaling (the lifetime).

$$\Gamma \propto \underbrace{\langle W \rangle_\Omega}_{\text{Interaction Strength}} \times \underbrace{I(\epsilon, T)}_{\text{Phase Space}}$$

5.2.5: The Microscopic Calculation

The energy dependence comes from integrating over the energies of the three participating particles $(\epsilon_2, \epsilon_3, \epsilon_4)$.

$$I(\epsilon, T) \propto \iiint d\epsilon_2 d\epsilon_3 d\epsilon_4 \, \delta(\epsilon + \epsilon_2 - \epsilon_3 - \epsilon_4) \times [n_2(1 - n_3)(1 - n_4) + \dots]$$

The Result (Exact Calculation)

Using standard identities for Fermi functions, this integral yields the famous result:

$$I(\epsilon, T) = \frac{1}{2} \left[\epsilon^2 + (\pi k_B T)^2\right]$$

Key Implications

1. At $T = 0$: $\Gamma \propto \epsilon^2$

5.2.6: The Microscopic Calculation

The Result

Using standard identities for Fermi functions, this integral yields the famous result:

$$I(\epsilon, T) = \frac{1}{2}\left[\epsilon^2 + (\pi k_B T)^2\right]$$

Key Implications

1. At $T = 0$: $\Gamma \propto \epsilon^2$
   - As $\epsilon \to 0$ (at Fermi surface), $\Gamma \to 0$.
   - Lifetime $\tau \to \infty$. The quasiparticle is stable!
2. At $\epsilon = 0$: $\Gamma \propto T^2$
   - Thermally excited quasiparticles have a finite lifetime.

This is the most important mathematical result of the presentation.

5.3.1: Final Rate & Observables

Combining the angular average (interaction strength w) and the energy integral we get the scattering rate $\Gamma = \frac{1}{\tau}$:

$$\frac{1}{\tau} = \frac{2\pi}{\hbar}\left(\frac{w^2}{16\epsilon_F}\right)\left[\epsilon^2 + (\pi k_B T)^2\right]$$

Experimental Regimes:

1. Thermal Regime $(T \gg \epsilon)$:$\frac{1}{\tau} \propto T^2$
   - $\frac{1}{\tau} \propto T^2$
   - Resistivity: Since $\rho \sim \frac{m^*}{ne^2\tau}$, we predict:

$$\rho(T) = \rho_0 + AT^2$$

This is the standard signature of Fermi liquid behavior in metals (e.g., Cu, Al, heavy fermions)

5.3.1: Final Rate & Observables

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$$\frac{1}{\tau} = \frac{2\pi}{\hbar} \left(\frac{w^2}{16\epsilon_F} \right) \left[\epsilon^2 + (\pi k_B T)^2 \right]$$

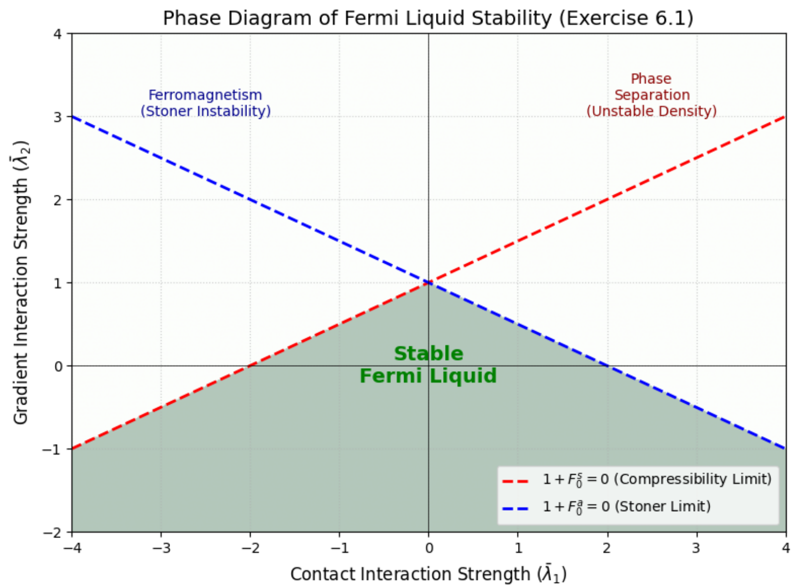
Experimental Regimes:

2. Quantum Regime ($\epsilon \gg T$):

- $\frac{1}{\tau} \propto \epsilon^2$
- Optical Conductivity: The relaxation rate depends on frequency ω (photon energy).
- $\frac{1}{\tau(\omega)} \propto (\hbar\omega)^2$.

So we conclude managing to derive the AT^2 resistivity straight from first principles.

6.1.1: Conclusion



Works Cited

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Piers Coleman, *Introduction to Many-Body Physics* 5(Cambridge University Press, 2015)