Matrix Algebra.

Matrix: A matrix is a rectangular array of numbers (real or complex) enclosed by a pair of brackets and the numbers in the array are called the elements of the matrix, that is, a rectangular array of numbers of the form

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

is called a matrix. The above matrix has m rows and n columns and is called an $(m \times n)$ matrix.

The above matrix is also denoted by

$$[a_{ij}], \quad i = 1, 2, \ldots m$$
$$j = 1, 2, \ldots n$$

Example: $A = \begin{bmatrix} 1 & 0 & -5 \\ 2 & -3 & 7 \end{bmatrix}$ is a matrix of order 2×3

Matrix multiplication: Two matrices A and B are conformable for multiplication if the number of columns in A is equal to the number of rows in B.

let $A = [a_1 \ a_2 \ \cdots \ a_n]$ and $B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$

Then $AB = [a_1b_1 + a_2b_2 + \cdots + a_nb_n]$

$$= \sum_{i=1}^{n} a_ib_i$$

Example: let $A = \begin{bmatrix} 1 & -3 & 5 \\ 2 & 0 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -1 \\ -2 & 4 \\ 3 & 0 \end{bmatrix}$

$$AB = \begin{bmatrix} 1.1 + (-3).(-2) + 5.3 & 1.(-1) + (-3).4 + 5.0 \\ 2.1 + 0.(-2) + (-1).3 & 2.(-1) + 0.4 + (-1).0 \end{bmatrix}$$

$$= \begin{bmatrix} 22 & -13 \\ -1 & -2 \end{bmatrix}$$

Transpose of a matrix:

If A is an $m \times n$ matrix, then the $n \times m$ matrix obtained from the matrix A by writing its rows as columns and its columns as rows is called the transpose of A and is denoted by the symbol A^T.

That is if $A = [a_{ij}]$ is an $m \times n$ matrix then $A^T = [a_{ji}]$ is an $n \times m$ matrix.

example: Let $A = \begin{bmatrix} 1 & 0 & 5 & -7 \\ 2 & 3 & -1 & 6 \end{bmatrix}$

Then $A^T = \begin{bmatrix} 1 & 2 \\ 0 & 3 \\ 5 & -1 \\ -7 & 6 \end{bmatrix}$

Diagonal matrix:

A square matrix whose elements $a_{ij} = 0$ when $i \neq j$ is called a diagonal matrix.

For example: $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

are diagonal matrices.

A diagonal matrix whose diagonal elements are all equal is called a scalar matrix.

Identity matrix or Unit matrix: A square m...

whose elements $a_{ij} = 0$, if $i \neq j$ and $a_{ij} = 1$ if $i = j$ is called the identity matrix or unit matrix.

For examples; $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ are identity matrices of order 3 and 4 respect...

Upper and lower triangular matrices: A squa...

matrix whose elements $a_{ij} = 0$ for $i > j$ is calle... an upper triangular matrix and a square matrix whose elements $a_{ij} = 0$ for $i < j$ is called a lowe... triangular matrix.

for example: $\begin{bmatrix} 2 & 3 & 5 \\ 0 & 1 & -3 \\ 0 & 0 & 7 \end{bmatrix}$ and $\begin{bmatrix} 1 & -1 & 2 & 3 \\ 0 & -1 & 2 & 5 \\ 0 & 0 & 3 & -2 \\ 0 & 0 & 0 & 7 \end{bmatrix}$

are upper triangular matrices.

and $\begin{bmatrix} 5 & 0 & 0 \\ -1 & 2 & 0 \\ 3 & 7 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 & 0 & 0 \\ i & 2 & 0 & 0 \\ -2 & 5 & -1 & 0 \\ -3 & 7 & -i & 6 \end{bmatrix}$ are lower triangular matrices.

Inverse matrix: A square matrix A is said to be invertible if there ~~exists~~ a unique matrix B such that $AB = BA = I$ where I is the unit matrix. We call such a matrix B the inverse of A & is generally denoted by A^{-1}. Here we have to note that if B is the inverse of A, then A is the inverse of B.

Example: Let $A = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$

Then $AB = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} 4-3 & -2+2 \\ 6-6 & -3+4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

$BA = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 4-3 & 2-2 \\ -6+6 & -3+4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I.$

Therefore A and B are invertible and are inverses of each other. That is $A^{-1} = B$ and $B^{-1} = A$.

If $A = \begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$ then show that A and B are invertible and inverses of each other. That is $A^{-1} = B$ and $B^{-1} = A$.

proof:

$$AB = \begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 6+0-5 & -2+0+2 & 2+0-2 \\ 15-15+0 & -5+6+0 & 5-5+0 \\ 0-15+15 & 0+6-6 & 0-5+6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$BA = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 6-5+0 & 0-1+1 & -3+0+0 \\ -30+30+0 & 0+6-5 & 15+0-15 \\ 10-10+0 & 0-2+2 & -5+0+6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} =$$

Therefore A and B are invertible and inver of each other. That is $A^{-1} = B$ and $B^{-1} = A$.

Singular and non-singular matrices:

let D be the determinant of the square matrix A, then $D = 0$, the matrix A is called the singular matrix and if $D \neq 0$ the matrix A is called the non-sing matrix.

Example: $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ -1 & -1 & 7 \end{bmatrix}$ are singular matrices

Since $D_1 = |A| = 0$, $D_2 = |B| = 0$.

Again $A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & -1 & 0 \\ 1 & -2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 4 & -1 \\ 1 & 0 & 3 \\ 2 & -1 & 4 \end{bmatrix}$ are non-singular matrices.

Since $D_1 = |A| \neq 0 = -6$ and $D_2 = |B| = 18 \neq 0$

If $A = \begin{bmatrix} 1 & 2 & 3 \\ -2 & 5 & -1 \\ 2 & 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 5 & 3 \\ 7 & -2 & 1 \\ 2 & 0 & -3 \end{bmatrix}$

then prove that $(AB)^T = B^T A^T$.

Proof: Give $A = \begin{bmatrix} 1 & 2 & 3 \\ -2 & 5 & -1 \\ 2 & 3 & 4 \end{bmatrix}$ $\therefore A^T = \begin{bmatrix} 1 & -2 & 2 \\ 2 & 5 & 3 \\ 3 & -1 & 4 \end{bmatrix}$

Given $B = \begin{bmatrix} -1 & 5 & 3 \\ 7 & -2 & 1 \\ 2 & 0 & -3 \end{bmatrix}$ $\therefore B^T = \begin{bmatrix} -1 & 7 & 2 \\ 5 & -2 & 0 \\ 3 & 1 & -3 \end{bmatrix}$

Now $B^T A^T = \begin{bmatrix} -1 & 7 & 2 \\ 5 & -2 & 0 \\ 3 & 1 & -3 \end{bmatrix} \begin{bmatrix} 1 & -2 & 2 \\ 2 & 5 & 3 \\ 3 & -1 & 4 \end{bmatrix}$

$$= \begin{bmatrix} -1+14+6 & 2+35-2 & -2+21+8 \\ 5-4+0 & -10-10+0 & 10-6+0 \\ 3+2-9 & -6+5+3 & 6+3-12 \end{bmatrix} = \begin{bmatrix} 19 & 35 & 27 \\ 1 & -20 & 4 \\ -4 & 2 & -3 \end{bmatrix} \text{ --- (1)}$$

Again $AB = \begin{bmatrix} 1 & 2 & 3 \\ -2 & 5 & -1 \\ 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} -1 & 5 & 3 \\ 7 & -2 & 1 \\ 2 & 0 & -3 \end{bmatrix}$

$$= \begin{bmatrix} -1+14+6 & 5-4+0 & 3+2-9 \\ 2+35-2 & -10-10+0 & -6+5+3 \\ -2+21+8 & 10-6+0 & 6+3-12 \end{bmatrix}$$

$$= \begin{bmatrix} 19 & 1 & -4 \\ 35 & -20 & 2 \\ 27 & 4 & -3 \end{bmatrix}$$

$$\therefore (AB)^T = \begin{bmatrix} 19 & 35 & 27 \\ 1 & -20 & 4 \\ -4 & 2 & -3 \end{bmatrix} \text{ --- (2)}$$

Hence from (1) and (2), we get

$$(AB)^T = B^T A^T$$

(proved)

27. If $A = \begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix}$ prove that $A^2 - 3A + 2I = 0$

Proof: $A^2 = A \cdot A = \begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix}$

$$= \begin{bmatrix} 3\cdot3 + 2\cdot(-1) & 3\cdot2 + 2\cdot0 \\ -1\cdot3 + 0\cdot(-1) & (-1)\cdot2 + 0\cdot0 \end{bmatrix} = \begin{bmatrix} 7 & 6 \\ -3 & -2 \end{bmatrix}$$

$$3A = 3\begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 9 & 6 \\ -3 & 0 \end{bmatrix}$$

and $2I = 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

Now L.H.S

$$A^2 - 3A + 2I = \begin{bmatrix} 7 & 6 \\ -3 & -2 \end{bmatrix} - \begin{bmatrix} 9 & 6 \\ -3 & 0 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 7-9 & 6-6 \\ -3+3 & -2-0 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

$\therefore$ L.H.S = R.H.S (proved)

If $A = \begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -4 & -3 \end{bmatrix}$ prove that

$A^3 - 4A^2 - A + 4I = 0$

Let $A = \begin{bmatrix} 1 & -2 & 3 \\ 5 & 1 & -4 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 3 & 5 \\ 1 & 4 & -2 \end{bmatrix}$

Find the matrices $2A$, $A+B$, $A-B$.

Solution: $A = \begin{bmatrix} 1 & -2 & 3 \\ 5 & 1 & -4 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 3 & 5 \\ 1 & 4 & -2 \end{bmatrix}$

$$2A = \begin{bmatrix} 2.1 & 2.(-2) & 2.3 \\ 2.5 & 2.1 & 2.(-4) \end{bmatrix} = \begin{bmatrix} 2 & -4 & 6 \\ 10 & 2 & -8 \end{bmatrix}$$

$$A+B = \begin{bmatrix} 1 & -2 & 3 \\ 5 & 1 & -4 \end{bmatrix} + \begin{bmatrix} 2 & 3 & 5 \\ 1 & 4 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 1+2 & -2+3 & 3+5 \\ 5+1 & 1+4 & -4-2 \end{bmatrix} = \begin{bmatrix} 3 & 1 & 8 \\ 6 & 5 & -6 \end{bmatrix}$$

$$A-B = \begin{bmatrix} 1 & -2 & 3 \\ 5 & 1 & -4 \end{bmatrix} - \begin{bmatrix} 2 & 3 & 5 \\ 1 & 4 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 1-2 & -2-3 & 3-5 \\ 5-1 & 1-4 & -4-(-2) \end{bmatrix} = \begin{bmatrix} -1 & -5 & -2 \\ 4 & -3 & -2 \end{bmatrix}$$

Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & 0 \\ 2 & 1 \end{bmatrix}$

Compute the matrix products AB and BA

Solution: $AB = \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 2 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 1\cdot5 + 0\cdot2 & 1\cdot0 + 0\cdot1 \\ 0\cdot5 + 5\cdot2 & 0\cdot0 + 5\cdot1 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 0 \\ 10 & 5 \end{bmatrix}$$

$$BA = \begin{bmatrix} 5 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 5\cdot1 + 0\cdot0 & 5\cdot0 + 0\cdot5 \\ 2\cdot1 + 1\cdot0 & 2\cdot0 + 1\cdot5 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 0 \\ 2 & 5 \end{bmatrix}$$

So we see that $AB \neq BA$.

Find the inverse of the matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

Solution: Let D be the determinant of the matrix. Then

$$D = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 4-6 = -2 \neq 0.$$ So the matrix A is non-singular and hence A^{-1} exists.

Now the cofactors of D are

$A_{11} = 4, \quad A_{12} = -3, \quad A_{21} = -2, \quad A_{22} = 1$

Then $\operatorname{Adj} A = \begin{bmatrix} 4 & -3 \\ -2 & 1 \end{bmatrix}^T = \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$

$\therefore \; A^{-1} = \dfrac{\operatorname{Adj} A}{D} = \dfrac{1}{-2} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$

An

Find the inverse of the matrix

$$A = \begin{bmatrix} 2 & -1 & 3 \\ 4 & 0 & -1 \\ 3 & 3 & 2 \end{bmatrix}$$

<u>solution</u>: Let D be the determinant of the matrix.

Then $D = \begin{vmatrix} 2 & -1 & 3 \\ 4 & 0 & -1 \\ 3 & 3 & 2 \end{vmatrix} = 2(0+3)+1(8+3)+3(12-0)$

$= 6+11+36 = 53 \neq 0$

So the matrix A is non-singular and A^{-1} exists. Now the cofactors of ~~A~~ D are

$A_{11} = \begin{vmatrix} 0 & -1 \\ 3 & 2 \end{vmatrix} = 3$, $A_{12} = -11$, $A_{13} = 12$,

$A_{21} = 11$, $A_{22} = -5$, $A_{23} = -9$

$A_{31} = 1$, $A_{32} = 14$, $A_{33} = 4$.

~~Therefore,~~

There

Therefore $\text{Adj } A = \begin{bmatrix} 3 & -11 & 12 \\ 11 & -5 & -9 \\ 1 & 14 & 4 \end{bmatrix}^T$

$$= \begin{bmatrix} 3 & 11 & 1 \\ -11 & -5 & 14 \\ 12 & -9 & 4 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{D} \text{Adj } A = \frac{1}{53} \begin{bmatrix} 3 & 11 & 1 \\ -11 & -5 & 14 \\ 12 & -9 & 4 \end{bmatrix}$$

If $A = \begin{bmatrix} -1 & 2 & -3 \\ 2 & 1 & 0 \\ 4 & -2 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 2 & 1 \\ 5 & 2 & -3 \end{bmatrix}$

Find $A^{-1}B$.

Solution: Given $A = \begin{bmatrix} -1 & 2 & -3 \\ 2 & 1 & 0 \\ 4 & -2 & 5 \end{bmatrix}$

Let $D = |A| = \begin{bmatrix} -1 & 2 & -3 \\ 2 & 1 & 0 \\ 4 & -2 & 5 \end{bmatrix}$

$$= -1(5+0) - 2(10-0) - 3(-4-4)$$

$$= -5 - 20 + 24 = -1 \neq 0$$

So A is non-singular and A^{-1} exists.

cofactors of D are

$A_{11} = 5$, $A_{12} = -10$, $A_{13} = -8$

$A_{21} = -4$, $A_{22} = 7$, $A_{23} = 6$

$A_{31} = 3$, $A_{32} = -6$, $A_{33} = -5$

$$\text{Adj } A = \begin{bmatrix} 5 & -10 & -8 \\ -4 & 7 & 6 \\ 3 & -6 & -5 \end{bmatrix}^T = \begin{bmatrix} 5 & -4 & 3 \\ -10 & 7 & -6 \\ -8 & 6 & -5 \end{bmatrix}$$

$$A^{-1} = \frac{1}{D} \text{Adj } A = \frac{1}{-1}\begin{bmatrix} 5 & -4 & 3 \\ -10 & 7 & -6 \\ -8 & 6 & -5 \end{bmatrix}$$

$$= \begin{bmatrix} -5 & 4 & -3 \\ 10 & -7 & 6 \\ 8 & -6 & 5 \end{bmatrix}$$

Thus $A^{-1}B = \begin{bmatrix} -5 & 4 & -3 \\ 10 & -7 & 6 \\ 8 & -6 & 5 \end{bmatrix}\begin{bmatrix} 2 & 1 & -1 \\ 0 & 2 & 1 \\ 5 & 2 & -3 \end{bmatrix}$

$= \cancel{(-5).2 + 4.0 + (-3).5 \quad (-5).1 + 4.2 + (-3).2 \quad -5.-1 + 4.1 + -3.-3}$

$= \cancel{10.2 + (-7).0 + 6.5 \quad 10.1 + (-7).2 + 6.2 \quad 10(-1) + (-7)1 + 6.-3}$

$$= \begin{bmatrix} (-5).2 + 4.0 + (-3).5 & (-5).1 + 4.2 + (-3).2 & (-5)(-1) + 4.1 + (-3).(-3) \\ 10.2 + (-7).0 + 6.5 & 10.1 + (-7).2 + 6.2 & 10.(-1) + (-7).1 + 6.(-3) \\ 8.2 + (-6).0 + 5.5 & 8.1 + (-6).2 + 5.2 & 8.(-1) + (-6).1 + 5.(-3) \end{bmatrix}$$

$$= \begin{bmatrix} -25 & -3 & 18 \\ 50 & 8 & -35 \\ 41 & 6 & -29 \end{bmatrix}$$

Ans.