CSE 231: Numerical Analysis

P-2

Topics 9 – 10

Interpolation with Unequal Interval: Introduction about interpolation of Unequal interval, Derivation and problem solution of Newton General divided difference

Lecture 9

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Interpolation of Unequal interval

- The Newton's forward and backward interpolation formulae are applicable only when the values of n are given at equal intervals. In this section we study the problem of interpolation when the values of the independent variable x are given at unequal intervals.

Divided differences

The concept of divided differences: Let the function $y = f(x)$ be given at the point $x_0, x_1, x_2, \ldots, x_n$ (which need not be equally spaced) $f(x_0), f(x_1), f(x_2), \ldots, f(x_n)$ denote the $(n + 1)$ values the function at the points $x_0, x_2, \ldots, x_n$. Then the first divided differences of $f(x)$ for the arguments x_0, x_1, is defined as

$$\frac{f(x_0) - f(x_1)}{x_0 - x_1}.$$

It is denoted by $f(x_0, x_1)$ or by $\underset{x_1}{\Delta} f(x)$ or by $[x_0, x_1]$

$$\therefore\ f(x_0, x_1) = \frac{f(x_0) - f(x_1)}{x_0 - x_1}.$$

Similarly we can define

$$f(x_1, x_2) = \frac{f(x_1) - f(x_2)}{x_1 - x_2},$$

$$f(x_2, x_3) = \frac{f(x_2) - f(x_3)}{x_2 - x_3},$$

The second divided differences for the arguments $x_0, x_1, x_2, \ldots$ is defined as

$$f(x_0, x_1, x_2) = \frac{f(x_0, x_1) - f(x_1, x_2)}{x_0 - x_1},$$

similarly the third differences for the arguments $x_0, x_1, x_2, x_3 \ldots$ is defined as

Divided differences (Cont.)

$$f(x_0, x_1, x_2, x_3) = \frac{f(x_0, x_1, x_2) - f(x_1, x_2, x_3)}{x_0 - x_3}.$$

The first divided differences are called the *divided differences of order one*, the second divided differences are called the *divided differences of order two*, etc.

The divided difference table:

Argument	Entry	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
x	$f(x)$			
x_0	$f(x_0)$			
		$f(x_0, x_1)$		
x_1	$f(x_1)$		$f(x_0, x_1, x_2)$	
		$f(x_1, x_2)$		$f(x_0, x_1, x_2, x_3)$
x_2	$f(x_2)$		$f(x_1, x_2, x_3)$	
		$f(x_2, x_3)$		
x_3	$f(x_3)$			

Divided differences (Cont.)

Example 5.1 *If* $f(x) = \frac{1}{x}$, *then find f(a, b) and f(a, b, c)*

Solution

$$f(x) = \frac{1}{x},$$

$$\Rightarrow f(a,b) = \frac{f(a) - f(b)}{a - b} = \frac{\frac{1}{a} - \frac{1}{b}}{(a-b)} = \frac{b-a}{ab(a-b)} = -\frac{1}{ab}$$

and

$$f(a,b,c) = \frac{f(a,b) - f(b,c)}{a-c}$$

$$= \frac{\frac{-1}{ab} - \left(-\frac{1}{bc}\right)}{a-c} = \frac{1}{b}\left(\frac{-c+a}{ac}\right)\frac{1}{a-c} = \frac{1}{abc}$$

$$\therefore \; f(a,b) = -\frac{1}{ab},\; f(a,b,c) = \frac{1}{abc}.$$

Newton's General Divided Differences Formula

Let a function $f(x)$ be given for the $(n + 1)$ values $x_0, x_1, x_2 \ldots, x_n$ as $f(x_0), f(x_1), f(x_2), \ldots, f(x_n)$ where $x_0, x_1, x_2, \ldots, x_n$ are not necessarily equispaced. From the definition of divided difference

$$f(x, x_0) = \frac{f(x) - f(x_0)}{x - x_0}$$

$$\Rightarrow f(x) = f(x_0) + (x - x_0) f(x, x_0)$$

$$f(x, x_0, x_1) = \frac{f(x_1, x_0) - f(x_0, x_1)}{x - x_1}$$

$$\therefore f(x, x_0) = f(x_0, x_1) + (x - x_1) f(x, x_0, x_1).$$

Newton's General divided difference formula-

$$f(x) = f(x_0) + (x - x_0) f(x_0 - x_1) + \ldots +$$

$$(x - x_0)(x - x_1) \ldots (x - x_{n-1}) f(x_0, x_1, \ldots, x_n)$$

Newton's General Divided Differences Formula (Cont.)

Example 5.3 *Use Newton divided difference formula and evaluate f(6), given*

x	5	7	11	13	21
$f(x)$	150	392	1452	2366	9702

Solution

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
5	150				
		121			
7	392		24		
		265		1	
11	1452		32		0
		457		1	
13	2366		46		
		917			
21	9702				

We have $f(x_0) = 150$ $f(x_0, x_1) = 121$, $f(x_0, x_1, x_2) = 24$, $f(x_0, x_1, x_2, x_3) = 1$

Newton's General Divided Differences Formula (Cont.)

$$f(x) = f(x_0) + (x - x_0)\, f(x_0, x_1) + (x - x_0), (x - x_1)\, f(x_0, x_1, x_2) +$$
$$(x - x_0)(x - x_1)(x - x_2)\, f(x_0, x_1, x_2, x_3) +$$
$$(x - x_0)(x - x_1)(x - x_2)(x - x_3)(x - x_3)\, f(x_0, x_1, x_2, x_3, x_4) + \dots$$

$\therefore$ $$f(6) = 150 + (6 - 5)(121) + (6 - 5)\, j(6 - 7)(24) +$$
$$(6 - 5)(6 - 7)(6 - 11)\, 1 + 0 + \dots$$

$\Rightarrow$ $$f(6) = 150 + 121 - 24 + 5$$

$\therefore$ $$f(x) = 252.$$

Newton's General Divided Differences Formula (Cont.)

Example 5.4 *Find the form of the function $f(x)$ under suitable assumption from the following data.*

x	0	1	2	5
$f(x)$	2	3	12	147

Solution The divided difference table is given as under:

x	$f(x)$	Δ	Δ^2	Δ^3
0	2			
		1		
1	3		4	
		9		1
2	12		9	
		45		
5	147			

We have $x_0 = 0, f(x_0) = 2\ f(x_0, x_1) = 1, f(x_0, x_1, x_2) = 4, f(x_0, x_1, x_2, x_3) = 1.$

The Newton's divided difference interpolation formula is

$$f(x) = f(x_0) + (x - x_0)\, f(x_0, x_1) + (x - x_0)(x - x_1)\, f(x_0, x_1, x_2) + (x - x_0)(x - x_1)(x - x_2)\, f(x_0, x_1, x_2, x_3).$$

Substituting we get

$$\therefore\ f(x) = 2 + (x - 0)1 + (x - 0)(x - 1)4 + (x - 0)(x - 1)(x - 2)1$$

$$f(x) = x^3 + x^2 - x + 2.$$

CSE 231: Numerical Analysis

Topics

Interpolation with Unequal Interval: Derivation and problem solution of Lagrange's interpolation formula

Lecture 10

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Lagrange's Interpolation Formula

Let $y = f(x)$ be a function which assumes the values $f(x_0)$, $f(x_1)$, $f(x_2)$, …, $f(x_n)$ corresponding to the values $x = x_0, x_1, x_2, \ldots, x_n$, where the values of x are not equispaced. Since $(n + 1)$ values of the function are given corresponding to the $(n + 1)$ values of the independent variable x, we can represent the function $y = f(x)$ be a polynomial in x of degree n.

Let the polynomial be

$$f(x) = a_0(x - x_1)(x - x_2)\ldots(x - x_n) + a_1(x - x_0)(x - x_2)\ldots(x - x_n) +$$

$$a_2(x - x_0)(x - x_1)(x - x_3)\ldots(x - x_n) + \ldots + a_n(x - x_0)(x - x_1)\ldots(x - x_{n-1}) \quad (8)$$

Lagrange's Interpolation Formula (Cont.)

Each term in equation (8) being a product of n factors in x of degree n, putting $x = x_0$ is (8) we get

$$f(x) = a_0(x_0 - x_1)(x_0 - x_2)\dots(x_0 - x_n)$$

$$\Rightarrow a_0 = \frac{f(x_0)}{(x_0 - x_1)(x_0 - x_2)\dots(x_0 - x_n)}$$

Putting $x = x_1$ in (8) we get

$$f(x_1) = a_1(x_1 - x_0)(x_1 - x_2)\dots(x_1 - x_n)$$

$$\Rightarrow a_1 = \frac{f(x_1)}{(x_1 - x_0)(x_1 - x_2)\dots(x_1 - x_n)},$$

similarly putting $x = x_2$, $x = x_3$, $x = x_n$ in (8) we get

$$\Rightarrow a_2 = \frac{f(x_2)}{(x_2 - x_0)(x_2 - x_1)\dots(x_2 - x_n)},$$

$$\vdots$$

$$\Rightarrow a_n = \frac{f(x_n)}{(x_n - x_0)(x_n - x_1)\dots(x_n - x_{n-1})}.$$

Lagrange's Interpolation Formula (Cont.)

Substituting the values of $a_0, a_1, \ldots, a_n$ in (8) we get

$$y = f(x) = \frac{(x - x_1)(x - x_2) \ldots (x - x_n)}{(x_0 - x_1)(x_0 - x_2) \ldots (x_0 - x_n)} f(x_0) +$$

$$\frac{(x - x_0)(x - x_2) \ldots (x - x_n)}{(x_1 - x_0)(x_1 - x_2) \ldots (x_1 - x_n)} f(x_1) + \ldots + \frac{(x - x_0)(x - x_1) \ldots (x - x_{n-1})}{(x_n - x_0)(x_n - x_1) \ldots (x_n - x_{n-1})} f(x_n) \cdots \tag{9}$$

The formula given by (9) is called *Lagrange's interpolation formula*. It is simple and easy to remember but the calculations in the formula are more complicated than in Newton's divided difference formula. The application of the formula is not speedy and there is always a chance of committing some error due to the number of positive and negative signs in the numerator and denominator of each term.

Lagrange's Interpolation Formula (Cont.)

Example 5.6 *Using Lagrange's interpolation formula, find the value of y corresponding to x = 10 from the following table*

x	5	6	9	11
$y = f(x)$	12	13	14	16

Solution We have $x_0 = 5$, $x_1 = 6$, $x_2 = 9$, $x_3 = 11$, $y_0 = 5$, $y_1 = 13$, $y_2 = 14$, $y_3 = 16$.

Using Lagrange's Interpolation formula we can write

$$y = f(x) = \frac{(x - x_1)(x - x_2)(x - x_3)}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3)}(y_0) + \frac{(x - x_0)(x - x_2)(x - x_3)}{(x_1 - x_0)(x_1 - x_2)(x_1 - x_3)}(y_1) +$$

$$\frac{(x - x_0)(x - x_1)(x - x_3)}{(x_2 - x_0)(x_2 - x_1)(x_2 - x_3)} y_2 + \frac{(x - x_0)(x - x_1)(x - x_2)}{(x_3 - x_0)(x_3 - x_1)(x_3 - x_2)} y_3 .$$

Substituting we get

$$f(10) = \frac{(10 - 6)(10 - 9)(10 - 11)}{(5 - 6)(5 - 9)(5 - 11)} \times (12) + \frac{(10 - 5)(10 - 9)(10 - 11)}{(6 - 5)(6 - 9)(6 - 11)} \times (13) +$$

$$\frac{(10 - 5)(10 - 6)(10 - 11)}{(9 - 5)(9 - 6)(9 - 11)} \times (14) + \frac{(10 - 5)(10 - 6)(10 - 9)}{(11 - 5)(11 - 6)(11 - 9)} \times (16)$$

$$= 2 - \frac{13}{3} + \frac{35}{3} + \frac{16}{3} = \frac{42}{3}.$$

CSE 231: Numerical Analysis

Topics 11—12

Central difference Interpolation formulae: Gauss Forward Interpolation Formula, Gauss Backward Interpolation Formula

Lecture 11

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Gauss Forward Interpolation Formula

The Newton forward interpolation formula is

$$y = f(x) = y_0 + u\Delta y_0 + \frac{u(u-1)}{1\times 2}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{1\times 2\times 3}\Delta^3 y_0 + \ldots, \quad (1)$$

where $u = \dfrac{x - x_0}{h}$ and $x = x_0$ is the origin.

From the central difference table we have

$$\Delta^2 y_0 = \Delta^2 y_{-1} + \Delta^3 y_{-1}$$

$$\Delta^3 y_0 = \Delta^3 y_{-1} + \Delta^4 y_{-1}$$

$$\Delta^4 y_0 = \Delta^4 y_{-1} + \Delta^5 y_{-1} \ldots$$

$$\Delta^3 y_{-1} = \Delta^3 y_{-2} + \Delta^4 y_{-2}$$

$$\Delta^4 y_{-1} = \Delta^4 y_{-2} + \Delta^5 y_{-2}$$

$$\vdots$$

Gauss Forward Interpolation Formula (Cont.)

Substituting the values in (1) we get

$$y = f(x) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\left(\Delta^2 y_{-1} + \Delta^3 y_{-1}\right) + \frac{u(u-1)(u-2)}{3!}\left(\Delta^3 y_{-1} + \Delta^4 y_{-1}\right) + \frac{u(u-1)(u-2)(u-3)}{4!}\left(\Delta^4 y_{-1} + \Delta^5 y_{-1}\right) + \ldots$$

The above formula may be written as

$$y_4 = f(x) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_{-1} + \frac{(u+1)u(u-1)}{3!}\Delta^3 y_{-1} + \frac{(u+1)u(u-1)(u-2)}{4!}\Delta^4 y_{-2} + \ldots \qquad (2)$$

Equation (2) is called *Gauss's forward interpolation formula.*

Gauss Forward Interpolation Formula (Cont.)

Example 6.1 *Use Gauss forward formula to find y for x = 30 given that*

x	21	25	29	33	37
y	18.4708	17.8144	17.1070	16.3432	15.5154

Solution We construct the difference table by taking as

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
$x_0 - 2h = 21$	18.4708				
		–0.6564 (Δy_{-2})			
$x_0 - h = 25$	17.8144		–0.0510 $(\Delta^2 y_{-2})$		
		–0.7074 (Δy_{-1})		–0.0054 $(\Delta^3 y_{-2})$	
$\boxed{x_0 = 29}$	y_0 17.1070		–0.0564 $(\Delta^2 y_{-1})$		–0.002 $(\Delta^4 y_{-2})$
		–0.7638 (Δy_0)		–0.0076 $(\Delta^3 y_{-1})$	
$x + h + 33$	16.3432		–0.0640 $(\Delta^2 y_0)$		
		–0.8278 (Δy_1)			
$x_0 + 2h = 37$	15.5154				

Gauss Forward Interpolation Formula (Cont.)

Here $h = 4$, $u = \dfrac{30 - 29}{4} = \dfrac{1}{4} = 0.25.$

$u =$ 0.25 lies between 0 and 1.

$\therefore$ Gauss's forward formula is suitable. Substituting in the Gauss's interpolation formula

$$y = y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_{-1} + \frac{(u+1)u(u-1)}{3!}\Delta^3 y_{-1} + \frac{(u+1)u(u-1)(u-2)}{4!}\Delta^4 y_{-2} + \ldots$$

we get

$$y_{0.25} = f(0.25) = 17.1070 + (0.25)(-0.7638) + \frac{(0.25)\,(-0.75)}{2} \times (-0.0564) +$$

$$\frac{(1.25)\,(0.25)\,(-0.75)}{6} \times (-0.0076) + \frac{(1.25)(0.25)(-0.75)(-1.75)}{24}(-0.0022)$$

$$= 16.9216.$$

Gauss Backward Interpolation Formula

Substituting

$$\Delta y_0 = \Delta y_0 + \Delta^2 y_{-1}$$

$$\Delta^2 y_0 = \Delta^2 y_{-1} + \Delta^3 y_{-1}$$

$$\Delta^3 y_0 = \Delta^3 y_{-1} + \Delta^4 y_{-1}$$

$$\vdots$$

and

$$\Delta^3 y_{-1} = \Delta^3 y_{-2} + \Delta^2 y_{-2}$$

$$\Delta^4 y_{-1} = \Delta^4 y_{-2} + \Delta^5 y_{-2}$$

In Newton's forward interpolation formula we see

$$y = f(x) = y_0 + \frac{u}{1!}\left(\Delta y_{-1} + \Delta^2 y_{-1}\right) + \frac{u(u-1)}{2!}\left(\Delta^2 y_{-1} + \Delta^3 y_{-1}\right) + \ldots$$

$$\Rightarrow y_5 = \overset{Y_0}{\cancel{y_0 +}} \frac{u}{1!}\Delta y_{-1} + \frac{u(u+1)}{2!}\Delta^2 y_{-1} + \frac{(u+1)u(u-1)}{3!}\Delta^3 y_{-2} +$$

$$\frac{(u+2)(u+1)\,u\,(u-1)}{4!}\Delta^4 y_{-2} + \ldots$$

This is called *Gauss's Backward Interpolation formula*.

$$+ \frac{(u+2)(u+1)\,u.(u-1)(u-2)}{5!}\Delta^5 y_{-3} \ldots$$

Gauss Backward Interpolation Formula (Cont.)

Example 6.2 *Use Gauss's backward formula and find the sales for the year 1966, given that*

Year	*1931*	*1941*	*1951*	*1961*	*1971*	*1981*
Sales (in lakhs)	*12*	*15*	*20*	*27*	*39*	*52*

Solution We have $h = 10$, we take 1971 as the origin. The central difference table with origin at 1971 is

u	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
–4	12					
		3 Δy_{-4}				
–3	15		2 $(\Delta^2 y_{-4})$			
		5 (Δy_{-3})		0 $(\Delta^3 y_{-4})$		
–2	20		2 $(\Delta^2 y_{-3})$		3 $(\Delta^4 y_{-4})$	
		7 (Δy_{-2})		3 $(\Delta^3 y_{-3})$		–10 $(\Delta^5 y_{-4})$
–1	27		5 $(\Delta^2 y_{-2})$		–7 $(\Delta^4 y_{-3})$	
		12 (Δy_{-1})		–4 $(\Delta^3 y_{-2})$		
0	$\boxed{39}$ y_0		1 $(\Delta^2 y_{-1})$			
		13 (Δy_0)				
1	–52					

Gauss Backward Interpolation Formula (Cont.)

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u at 19~~9~~6 is $$u = \frac{1966 - 1971}{10} = \frac{-5}{10} = -0.5.$$

Gauss's backward formula is $$y = y_0 + \frac{u}{1!}\Delta y_{-1} + \frac{(u+1)u}{2!}\Delta^2 y_{-1} + \frac{(u+1)u(u-1)}{3!}\Delta^3 y_{-2} + \ldots$$

substituting we get $$y_{-0.5} = 39 + (0.5)(12) + \frac{(0.5)(-0.5)}{2} \times 1 + \frac{0.5 \times (-0.5) \times (-1.5)}{6} \times (-4) + \ldots$$

$$= 39 - 6 - 0.125 - 0.25$$

$$y_{1966} = 32.625.$$

$\therefore$ The sales in the year 1966 is 32.625 lakh of rupees.

CSE 231: Numerical Analysis

Topics

Central difference Interpolation formulae: Bessel's Formula, Stirling's Formula

Lecture 12

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Bessel's Formula

Changing the origin in the Gauss's backward interpolation formula from 0 to 1, we have

$$y = y_1 + (u-1)\Delta y_0 + \frac{u(u-1)}{2!}\Delta^3 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_{-1} + \dots$$

Taking the mean of the above formula and the Gauss's forward interpolation formula, we obtain

$$y_4 = \frac{1}{2}[y_0 + y_1] + \left(u - \frac{1}{2}\right)\Delta y_0 + \frac{u(u-1)}{2!}\frac{1}{2}\left[\Delta^2 y_{-1} + \Delta^2 y_0\right] + \frac{\left(u - \frac{1}{2}\right)u(u-1)}{3!}\Delta^3 y_{-1} + \dots$$

This is called *Bessel's formula.*

Bessel's Formula (Cont.)

Example 6.6 *Apply Bessel's formula to obtain Y_{25} given that $Y_{20} = 2854$, $Y_{24} = 3162$, $Y_{28} = 3544$ and $Y_{32} = 3992$.*

Solution Taking 24 as the origin we get

$$u = \frac{25 - 24}{4} = \frac{1}{4}.$$

The difference table is

X	$u = \dfrac{X - 24}{4}$	Y_u	ΔY_u	$\Delta^2 Y_u$	$\Delta^3 Y_u$
20	-1	2854			
			308 (Δy_{-1})		
24	0	3162		74 ($\Delta^2 y_{-1}$)	
			382 (Δy_0)		-8 ($\Delta^3 y_{-1}$)
28	1	3544		66 ($\Delta^2 y_0$)	
			448 (Δy_1)		
32	2	3992			

Bessel's Formula (Cont.)

The Bessel's formula is given by

$$Y_n = \frac{1}{2}(Y_0 + Y_1) + \left(u - \frac{1}{2}\right)\Delta Y_0 + \frac{u(u-1)}{2!}\frac{\left(\Delta^2 Y_{-1} + \Delta^2 Y_0\right)}{2} + \frac{\left(u - \frac{1}{2}\right)u(u-1)}{3!}\Delta^3 Y_{-1} + \ldots$$

$$\Rightarrow\ Y_{0.25} = \frac{1}{2}(3162 + 3544) + \left(\frac{1}{4} - \frac{1}{2}\right)\cdot(382) + \frac{\left(\frac{1}{4}\right)\left(\frac{1}{4} - 1\right)}{2!}\frac{(74 + 66)}{2} + \frac{\left(\frac{1}{4} - \frac{1}{2}\right)\frac{1}{4}\left(\frac{1}{4} - 1\right)}{3!}\cdot(-8)$$

$$= 3353 - 95.5 - 6.5625 - 0.0625$$

$$\Rightarrow\ Y_{0.25} = 3250.875,$$

$$\therefore\ y = 3250.875.$$

$$x = 25$$

$$\therefore\ y_{25} = 3250.875.$$

Stirling's Formula

Gauss's forward interpolation formula is

$$y_u = y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{(u+1)u(u-1)}{3!}\Delta^3 y_{-1} +$$

$$\frac{(u+1)u(u+1)(u-2)}{4!}\Delta^4 y_{-2} + \frac{(u+2)(u+1)u(u-1)(u-2)}{5!}\Delta^5 y_{-2} + \dots \quad (3)$$

Gauss's backward interpolation formula is

$$y_u = y_0 + \frac{u}{1!}\Delta y_{-1} + \frac{(u-1)u}{2!}\Delta^2 y_{-1} + \frac{(u+1)u(u-1)}{3!}\Delta^3 y_{-2} +$$

$$\frac{(u+2)u(u+1)(u-1)}{4!}\Delta^4 y_{-2} + \dots \quad (4)$$

Taking the mean of the two Gauss's formulae, we get

Stirling's Formula (Cont.)

$$y_u = y_0 + u\left[\frac{\Delta y_0 + \Delta y_{-1}}{2}\right] + \frac{u^2}{2}\Delta^2 y_{-1} +$$

$$\frac{u(u^2-1)}{3!}\frac{(\Delta^3 y_{-1} + \Delta^3 y_{-2})}{2} + \frac{u^2(u^2-1)}{4!}\Delta^4 y_{-2} + \frac{u(u^2-1)(u^2-4)}{5!}(\Delta^5 y_{-2} + \Delta^5 y_{-3}) + \dots$$

The above is called Stirling's formula. Stirling's formula gives the most accurate result for $-0.25 \le u \le 0.25$. Therefore, we have to choose x_0 such that u satisfies this inequality.

Stirling's Formula (Cont.)

Example 6.9 *Use Stirling's formula to compute $u_{12.2}$ from the following table*

x_0	10	11	12	13	14
$10^5 \log x$	23967	28060	31788	35209	38368

Solution The difference table is

	x^0	$10^5 u_x$	$10^5 \Delta u_x$	$10^5 \Delta^2 u_x$	$10^5 \Delta^3 u_x$	$10^5 \Delta^4 u_x$
-2	10	23967				
			4093 (Δy_{-2})			
-1	11	28060		−365 ($\Delta^2 y_{-2}$)		
			3728 (Δy_{-1})		58 ($\Delta^3 y_{-2}$)	
	12 (x_0)	31788 (y_0)		−307 ($\Delta^2 y_{-1}$)		−13 ($\Delta^4 y_{-2}$)
			3421 (Δy_0)		45 ($\Delta^3 y_{-1}$)	
	13	35209		−262 ($\Delta^2 y_0$)		
			3159 (Δy_1)			
	14	38368				

Stirling's Formula (Cont.)

We have $$u = \frac{x - x_0}{h} = \frac{12.2 - 12}{1} = 0.2,$$

where $x_0 = 12$ is the origin.

The Stirling's formula is

$$y_u = y_0 + u\frac{(\Delta y_0 + \Delta y_{-1})}{2} + \frac{u^2\Delta^2 y_{-1}}{2} + \frac{u(u^2-1)}{6}\frac{(\Delta^3 y_{-1} + \Delta^3 y_{-2})}{2} + \frac{u^2(u^2-1)}{24}\cdot\Delta^4 y_{-2} + \ldots$$

$$\Rightarrow\ 10^5 u_{12.2} = 31788 + (0.2)\left(\frac{3421 + 3728}{2}\right) + (0.02)(-307) - (0.016)(45 + 58) - (0.0016)(-13)$$

$$= 31788 + 714.9 - 6.1 - 1.6 + 0.000$$

$$\Rightarrow\ 10^5 u_{12.2} = 32495$$

$$\Rightarrow\quad u_{12.2} = 0.32495.$$