(11)

\# Find the cartesian co-ordinates of the whose polar co-ordinates are ~~(i) 5, (-π~~ :
(i) $(5, -\pi/4)$ ii) $(2, 330^\circ)$, iii) $(3, 585^\circ)$, (iv) $(6, 780^\circ)$
(v) $(2, 1395^\circ)$, (vi) $(7, 870^\circ)$, (vii) $(9, 1560^\circ)$

Solution: (i) we know,

$$x = r\cos\theta = 5\cos(-\pi/4) = 5\cos\left(\frac{\pi}{4}\right) = 5\cos 45^\circ$$

$$= 5 \cdot \frac{1}{\sqrt{2}} = \frac{5\cdot\sqrt{2}}{\sqrt{2}\cdot\sqrt{2}} = \frac{5\sqrt{2}}{2}$$

$$\therefore x = \frac{5\sqrt{2}}{2}$$

and $y = r\sin\theta = 5\sin(-\pi/4) = -5\sin 45^\circ$

$$= -5 \cdot \frac{1}{\sqrt{2}} = -\frac{5\sqrt{2}}{2}$$

$\therefore$ The cartesian co-ordinates are $\left(\frac{5\sqrt{2}}{2}, -\frac{5\sqrt{2}}{2}\right)$

(ii) $x = 2\cos 330^\circ = 2\cos(360^\circ - 30^\circ)$

$$= 2\cos 30^\circ$$

$$= \cancel{2 \cdot \frac{\sqrt{3}}{2}}$$

$$= 2 \cdot \frac{\sqrt{3}}{2} = \sqrt{3}$$

$y = 2\sin 330^\circ = 2\sin(360^\circ - 30^\circ) = -2\sin 30^\circ = $

$$= -2 \cdot \frac{1}{2} = -1$$

$\therefore$ The cartesian co-ordinates are $(\sqrt{3}, -1)$

\# Transform to cartesian co-ordinates the equation is $r(\cos 3\theta + \sin 3\theta) = 5k \sin\theta \cos\theta$.

Solution:. We have $r(\cos 3\theta + \sin 3\theta) = 5k\sin\theta\cos\theta$

$\Rightarrow r(4\cos^3\theta - 3\cos\theta + 3\sin\theta - 4\sin^3\theta) = 5k\sin\theta\cos\theta$

$\Rightarrow r\left(4 \cdot \frac{x^3}{r^3} - 3 \cdot \frac{x}{r} + 3\frac{y}{r} - 4\frac{y^3}{r^3}\right) = 5k \cdot \frac{xy}{r^2}$

$\Rightarrow \frac{4x^3}{r^2} - 3x + 3y - \frac{4y^3}{r^2} = \frac{5kxy}{r^2}$

$\Rightarrow \frac{4x^3 - 3xr^2 + 3yr^2 - 4y^3}{\cancel{r^2}} = \frac{5kxy}{\cancel{r^2}}$

$\Rightarrow 4x^3 - r^2(3x - 3y) - 4y^3 = 5kxy$

$\Rightarrow 4x^3 - (x^2+y^2)(3x-3y) - 4y^3 = 5kxy$

$\therefore 4x^3 - 3(x^2+y^2)(x-y) - 4y^3 = 5kxy$

$\therefore 4(x^3 - y^3) - 3(x^2+y^2)(x-y) = 5kxy$ Ans.

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Transform to polar co-ordinates the eqn

i) $9x^2+4y^2=36$ ii) $x^3=y^2(2a-x)$

Solution: i) putting $x=r\cos\theta$, $y=r\sin\theta$

The required equation is

$$9r^2\cos^2\theta+4r^2\sin^2\theta=36$$

~~or $r^2(4-4$~~

$$4r^2(1-\cos^2\theta)+9r^2\cos^2\theta=36$$

$$\Rightarrow 4r^2-4r^2\cos^2\theta+9r^2\cos^2\theta=36$$

$$\Rightarrow 5r^2\cos^2\theta+4r^2=36$$

$$\Rightarrow r^2(4+5\cos^2\theta)=36$$

$$\therefore (4+5\cos^2\theta)r^2=36$$

Ans

(ii) $$r^{\cancel{3}}\cos^3\theta=\cancel{r^2}\sin^2\theta(2a-r\cos\theta)$$

$$\Rightarrow r\cos^3\theta=\sin^2\theta(2a-r\cos\theta)$$

$$\Rightarrow 2a\sin^2\theta-r\sin^2\theta\cos\theta=r\cos^3\theta$$

$$\Rightarrow 2a\sin^2\theta=r\cos^3\theta+r\sin^2\theta\cos\theta=\cancel{r\cos^3\theta+r}$$

$$=r\cos^3\theta+r\cos\theta(1-\cos^2\theta)$$

$$=\cancel{r\cos^3\theta}+r\cos\theta-\cancel{r\cos^3\theta}$$

$$\therefore 2a\sin^2\theta=r\cos\theta$$ A

Change the equations to polar co-ordinates

(i) $(x^2+y^2)^2 = 2a^2xy$, (ii) $x^4 + x^2y^2 - (x+y)^2 = 0$

Ans: $r^2 = a^2\sin 2\theta$ Ans: $r = \pm(1+\tan\theta)$

Transform to cartesian co-ordinates the equations

i) $r^2 - 2r(\cos\theta - \sin\theta) - 7 = 0$ Ans: $x^2+y^2-2x+2y-7=0$

$\cos \cancel{8}(1395^\circ)$

$\sin 585^\circ$

$\sin \cancel{\tan} 780^\circ$

$\cos 870^\circ$

$\cos \cancel{\tan} 1560^\circ$

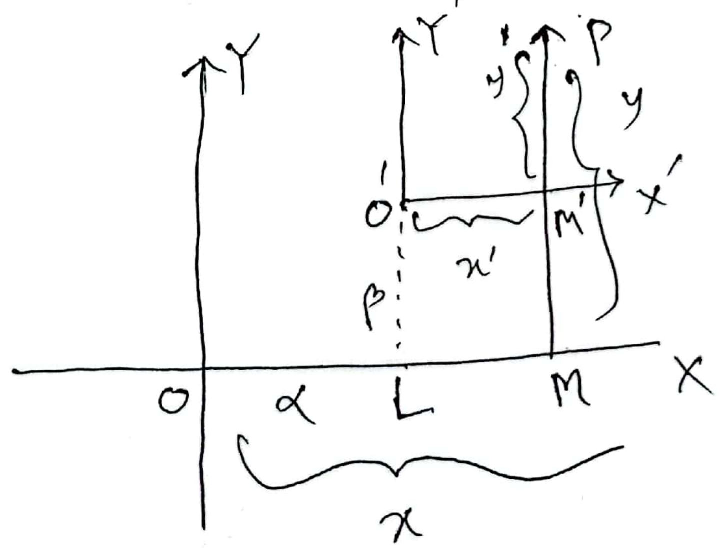
(15) # Change of Axes

Transformation of co-ordinates:

The co-ordinates of a point or the equation of a curve are always given with reference to a fixed origin and a set of axes of co-ordinates. The above co-ordinates of the equation of the curve changes when the origin is changed or the direction of axes changed or both. The process of changing the co-ordinate of a point or the equation of a curve is called transformation of co-ordinates.

(1) # Change of origin:

Find the change in the co-ordinates of a point when the origin is shifted to another point $O'(\alpha, \beta)$ where the direction of axes remain unaltered.



Let us take a new pair of axes $O'X'$ and $O'Y'$ parallel to the old pair OX and OY; Now O' being a new origin whose co-ordinate are (α, β) referred to $O'X'$ and $O'Y'$.

Let (x', y') be the co-ordinates reffered to the axes $O'X'$ and $O'Y'$ or a point P, whose co-ordinate reffered to the old axes are (x, y).

It is required to transform the co-ordinates (x, y) in terms of (x', y')

From O' and P draw $O'L$ and PM perpendiculars to OX. Let PM meet $O'X'$ in M'.

Then $OL = \alpha$, $LO' = \beta$, $OM = x$, $MP = y$

Also $O'M' = x'$ and $M'P = y'$

Therefore, $OM = OL + LM = OL + O'M'$

$\therefore x = \alpha + x'$ ——— (I)

Similarly, $MP = MM' + M'P = LO' + M'P$

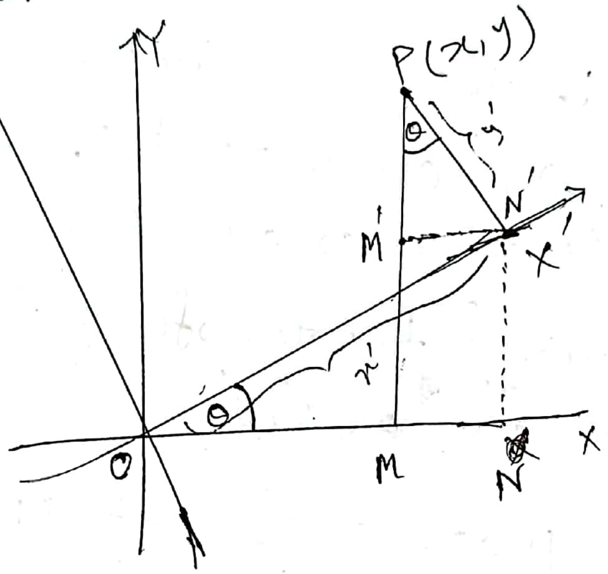
$\therefore y = \beta + y'$ ——— (II)

The transformed co-ordinates are

$$\left.\begin{aligned} x' &= x - \alpha \\ \text{and } y' &= y - \beta \end{aligned}\right\}$$

Rotation of axes (origin fixed):

(11) Find the change in the co-ordinates of a point when the direction of axes is turned through an angle θ where as the origin of co-ordinates remains the same.



Let OX and OY be the old axes and OX' and OY' set the new axes. O is the common origin for the two sets of axes. Let the angle X'OX through which the axes have rotated be represented by θ.

Let P be any point in the plane and let its co-ordinates referred to the old axes be (x, y) and referred to the new axes be (x', y').

Let us try to determine x and y in terms of x', y' and θ. Draw PM perpendicular to OX, PN' perpendicular to OX' and N'N perpendicular to OX, and N'M' parallel to OX.

Then $x = OM = ON - MN = ON - M'N'$

$= ON' \cos\theta - PN' \sin\theta$

$= x' \cos\theta - y' \sin\theta$

$y = MP = MM' + M'P = NN' + M'P$

$= ON' \sin\theta + PN' \cos\theta$

$= x' \sin\theta + y' \cos\theta$

$\cos\theta = \frac{\text{ভূমি}}{\text{অতিভুজ}} =$

$\cos\theta = \frac{ON}{ON'}$

$\therefore ON = ON' \cos\theta$

$\therefore ON = x' \cos\theta$

$\sin\theta = \frac{\text{লম্ব}}{\text{অতিভুজ}} = \frac{M'N'}{PN'}$

$\therefore M'N' = PN' \sin\theta$
$= y' \sin\theta$

Hence the formula for the rotation of the axes through an angle θ are,

$$\left.\begin{array}{l} x = x' \cos\theta - y' \sin\theta \\ \text{and} \quad y = x' \sin\theta + y' \cos\theta \end{array}\right\}$$

$\sin\theta = \frac{NN'}{ON'}$

$\therefore NN' = ON' \sin\theta = x' \sin\theta$

$\cos\theta = \frac{M'P}{PN'}$

$M'P = PN' \cos\theta$

$= y' \cos\theta$

(iii) # Change of origin with the change of the direction of axes.

Let us suppose that the system of axes be rectangular. The origin is shifted to the point (α, β) and then the axes are rotated through an angle θ. If the co-ordinates of any point be (x, y) in the old system, and (x', y') in the new system, from (I) and (II) we have

$$x = \alpha + x'\cos\theta - y'\sin\theta$$

and $$y = \beta + x'\sin\theta + y'\cos\theta.$$

Invariants:

If by the rotation of the rectangular co-ordinate axes about the origin, the expression $ax^2 + 2hxy + by^2$ changes to $a'x'^2 + 2h'x'y' + b'y'^2$, $a + b = a' + b'$ and $ab - h^2 = a'b' - h'^2$

proof:

Let (x, y) be the co-ordinates of a point P referred to a set of rectangular axes. If the axes are roated through an angle θ about the origin.

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Let the co-ordinates of the same point P be (x', y') referred to the new system of rectangular axis. Then we have

$x = x'\cos\theta - y'\sin\theta$ and

$y = x'\sin\theta + y'\cos\theta$.

By using the above transformations the expression $ax^2 + 2hxy + by^2$ becomes,

$a(x'\cos\theta - y'\sin\theta)^2 + 2h(x'\cos\theta - y'\sin\theta)(x'\sin\theta + y'\cos\theta)$
$+ b(x'\sin\theta + y'\cos\theta)^2$

$= a\{x'^2\cos^2\theta - 2x'y'\sin\theta\cos\theta + y'^2\sin^2\theta\} + 2h\,x'y'\cos^2\theta - 2hy'^2\sin\theta\cos\theta\}$
$+ \{2h\,x'^2\sin\theta\cos\theta - 2h\,x'y'\sin^2\theta$
$+ b\{x'^2\sin^2\theta + 2x'y'\sin\theta\cos\theta + y'^2\cos^2\theta\}$

$= x'^2\{a\cos^2\theta + 2h\sin\theta\cos\theta + b\sin^2\theta\} +$
$y'^2\{a\sin^2\theta - 2h\sin\theta\cos\theta + b\cos^2\theta\} +$
$2x'y'\{h(\cos^2\theta - \sin^2\theta) - (a-b)\sin\theta\cos\theta\}$

$= a'x'^2 + 2h'x'y' + b'y'^2$

where $a' = a\cos^2\theta + 2h\sin\theta\cos\theta + b\sin^2\theta$ —— (I)
$b' = a\sin^2\theta - 2h\sin\theta\cos\theta + b\cos^2\theta$ —— (II)
$h' = h(\cos^2\theta - \sin^2\theta) - (a-b)\sin\theta\cos\theta$ —— (III)

From (I) and (III) we have on simplification

$a' + b' = a + b$

Again $2a' = 2a\cos^2\theta + 4h\sin\theta\cos\theta + 2b\sin^2\theta$

$= a(1+\cos 2\theta) + 2h\sin 2\theta + b(1-\cos 2\theta)$

$= a+b + 2h\sin 2\theta + (a-b)\cos 2\theta$ —— (IV)

similarly, $2b' = a+b - 2h\sin 2\theta - (a-b)\cos 2\theta$ —— (V)

from (4), (5) and (3) we have

$4(a'b' - h'^2) = (a+b)^2 - \{2h\sin 2\theta + (a-b)\cos 2\theta\}^2$
$- \{2h\cos 2\theta - (a-b)\sin 2\theta\}^2$

$= (a+b)^2 - 4h^2 - (a-b)^2$

$= 4ab - 4h^2$

$\therefore a'b' - h'^2 = ab - h^2$

The two quantities $a+b$ and $ab-h^2$ for the expression $ax^2 + 2hxy + by^2$ are called invariants of transformation from one system of rectangular axes without change of the origin, because their values remain unchanged by the transformation.