System of Linear Equations

⇒ Linear Equation:

An equ^n of the form $ax+by=c$, where a, b, c are real constants is called linear equation.

In general, an equ^n is called linear if it is of the form $a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$ —— ①

where $a_1, a_2, \ldots, a_n$ are real numbers / constants and $x_1, x_2, \ldots, x_n$ are n-variables.

(i) If $b=0$, then ① is called a homogeneous linear equ^n.

(ii) If $b\neq 0$, then ① " " a non-homogeneous " ".

Examples:

1. $y - mx = 0$ → homogeneous linear equ^n.
2. $2x+3y=5$ → non-homogeneous " ".
3. $2x^2+3y=1$
4. $ax^2+2hxy+by^2=0$

$\left.\begin{matrix} 3. \\ 4. \end{matrix}\right\}$ → non-linear equ^{n}s.

Note: An equ^n which is not linear is called non linear equ^n.

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⇒ ~~Demen~~ Degenerate and Non-degenerate Linear Equⁿ

Non-degenerate equⁿ:

The general linear equation

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b.$$

is also called non-degenerate linear equation.

Degenerate equⁿ:

A linear equⁿ is said to be degenerate if it has the form $0x_1 + 0x_2 + \cdots + 0x_n = b$ ——— (*).

i.e., if every coefficient of the variable is equal to zero.

The solⁿ of such a degenerate linear equⁿ is as follows:

(i) If $b \neq 0$, then the above equⁿ (*) has no solⁿ.

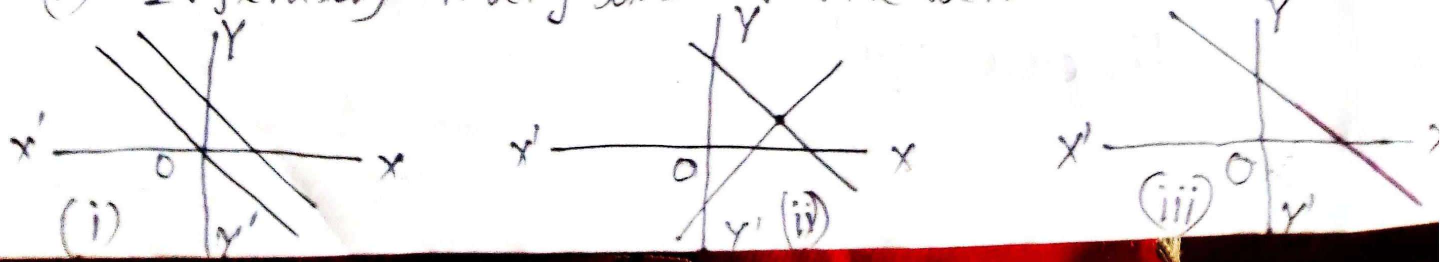
(ii) If $b = 0$, then every vector $\underline{u} = (\alpha_1, \alpha_2, \cdots, \alpha_n)$ is a solⁿ of the above equⁿ (*).

Note: There are three possible solⁿs of a linear equⁿ

(i) No solⁿ if the lines are parallel

(ii) Unique solⁿ if the lines intersect.

(iii) Infinitely many solⁿs if the lines coincide.



⇒ System of Linear Equations:

A system of linear equ^{n}s in n variables $x_1, x_2, \dots, x_n$ is a set of equations of the form

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_2 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ - - - - - - - - - - - \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{array}\right\} \text{———} (1)$$

where the coefficients of the variables a_{ij}; $i = 1, 2, \dots, m$ and $j = 1, 2, \dots, n$ and the free terms b_i, $i = 1, 2, \dots, m$ are the real numbers.

⇒ classification of system of linear equ^{n}s in terms of homogeneity.

(i) If $b_i = 0$, then (1) is called a homogeneous system.

(ii) If ~~b_i~~ at least one b_i is not zero, then the system (1) is called a non-homogeneous system.

⇒ Classification of system of linear equ^{n}s in terms o[f] solutions.

(i) A system of linear equ^{n}s (1) is called consistent if it has at least one solution.

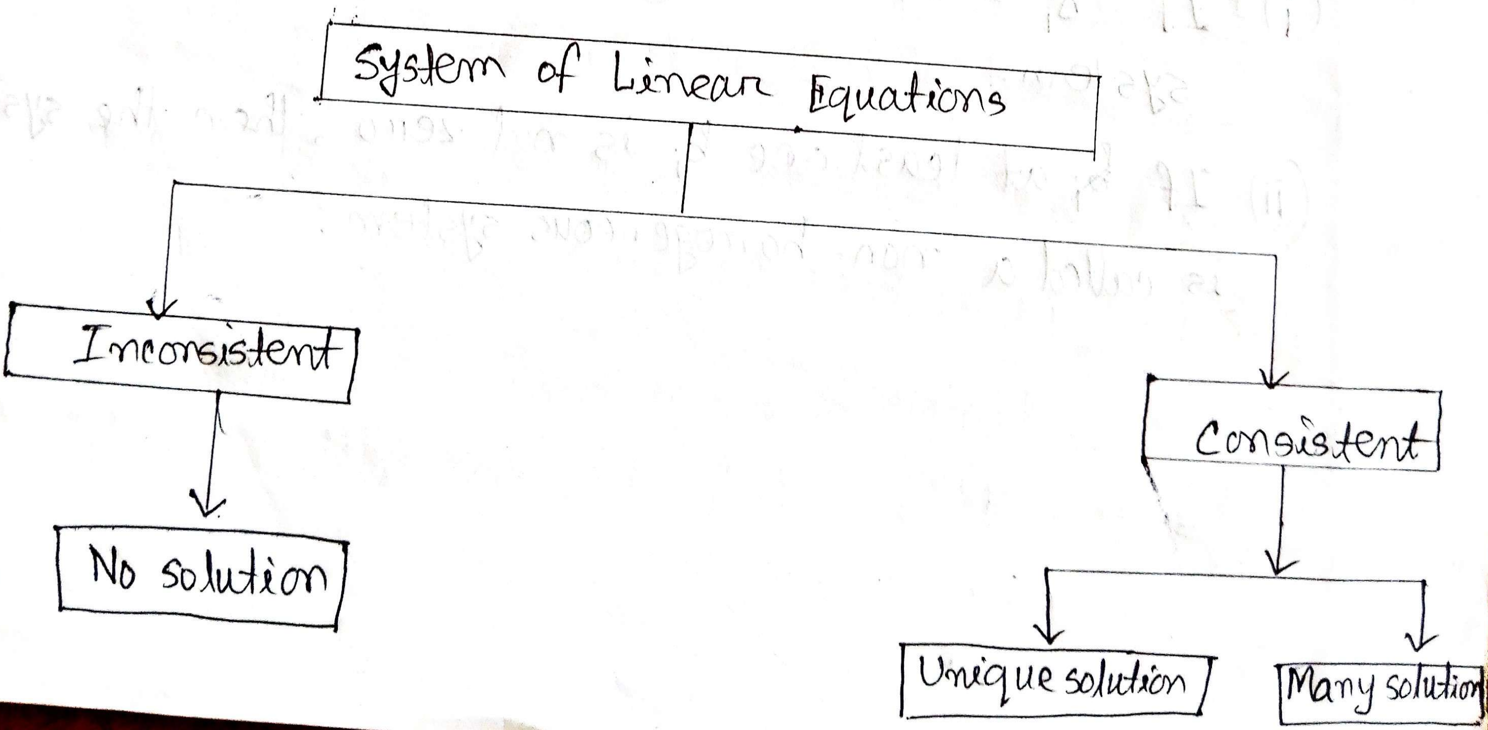
(a) A consistent system is called determinate if it has a unique sol^n.

(b) A consistent system is called ~~non~~ indeterminate if it has more than one sol^n.

(ii) A system of linear equ^{n}s (1) is called inconsistent if it has no solution.

⇒ Question:

Write down the classification of system of linear equ^{n}s in tabular form.



⇒ Echelon Matrix:

A matrix is said to be echelon matrix if it satisfies the following two properties:

(i) the 1st k rows are non-zero, the other rows are zero.

(ii) the 1st non-zero element in each non-zero row is 1 and it appears in a column to the right of the 1st non-zero element of any preceding row.

Example:

$$\begin{bmatrix} 0 & 1 & 3 & -2 \\ 0 & 0 & -13 & 11 \\ 0 & 0 & 0 & 35 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{pmatrix} 2 & 1 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$ are echelon matrix.

⇒ ~~Row~~ - Reduced echelon form:

A matrix is said to be in reduced echelon form if it satisfies the following properties

(i) the matrix is in the echelon form

(ii) the 1st non-zero element in each non-zero row is the only non-zero element in its column.

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Example:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 5 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$ are reduced echelon matri

⇒ **Augmented matrix:**

If we add constant in a coefficient matrix, then we get a new matrix called augmented matrix.

~~It~~ i.e, Let

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$
$$- - - - - - - - - - - -$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m$$

Then the augmented matrix is

$$A|B = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ - & - & - & - & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

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$\Rightarrow$ ~~So~~ Example-01:

Solve the following system of linear equations

$$3x_1 - x_2 + x_3 = -2$$
$$x_1 + 5x_2 + 2x_3 = 6$$
$$2x_1 + 3x_2 + x_3 = 0$$

using echelon matrix.

Sol^n:

The augmented matrix of the given system is

$$A|B = \left(\begin{array}{ccc|c} 3 & -1 & 1 & -2 \\ 1 & 5 & 2 & 6 \\ 2 & 3 & 1 & 0 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 5 & 2 & 6 \\ 3 & -1 & 1 & -2 \\ 2 & 3 & 1 & 0 \end{array}\right) \quad R_1 \leftrightarrow R_2$$

$$\sim \left(\begin{array}{ccc|c} 1 & 5 & 2 & 6 \\ 3-3\times1 & -1-5\times3 & 1-3\times2 & -2-3\times6 \\ 2-2\times1 & 3-5\times2 & 1-2\times2 & 0-2\times6 \end{array}\right) \quad \begin{array}{l} R_2' = R_2 - 3R_1 \\ R_3' = R_3 - 2R_1 \end{array}$$

$$\sim \left(\begin{array}{ccc|c} 1 & 5 & 2 & 6 \\ 0 & -16 & -5 & -20 \\ 0 & -7 & -3 & -12 \end{array}\right)$$

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$$\sim \left(\begin{array}{ccc|c} 1 & 5 & 2 & 6 \\ 0 & -16 & -5 & -20 \\ 0 & -7-\frac{7}{16}(-16) & -3-\frac{7}{16}(-5) & -12-\frac{7}{16}(-20) \end{array}\right) \quad R_3' = R_3 - \frac{7}{16}R_2$$

$$\sim \left(\begin{array}{ccc|c} 1 & 5 & 2 & 6 \\ 0 & -16 & -5 & -20 \\ 0 & 0 & -\frac{13}{16} & -\frac{13}{4} \end{array}\right)$$

which is in echelon form.

Now we can write

$$x_1 + 5x_2 + 2x_3 = 6 \quad \text{——— (1)}$$

$$-16x_2 - 5x_3 = -20 \quad \text{——— (2)}$$

$$-\frac{13}{16}x_3 = -\frac{13}{4} \quad \text{——— (3)}$$

From (3), we get

$$-\frac{13}{16}x_3 = -\frac{13}{4}$$

$$\Rightarrow \frac{1}{16}x_3 = \frac{1}{4}$$

$$\Rightarrow x_3 = \frac{16}{4}$$

$$\therefore x_3 = 4$$

From (2), we get

$$-16x_2 - 5\times 4 = -20$$

$$\Rightarrow -16x_2 = -20 + 20$$

$$\Rightarrow -16x_2 = 0$$

$$\therefore x_2 = 0$$

From (1), we obtain

$$x_1 + 5\times 0 + 2\times 4 = 6$$

$$\Rightarrow x_1 = 6 - 8$$

$$\therefore x_1 = -2$$

$\therefore$ $x_1 = -2$, $x_2 = 0$ and $x_3 = 4$ are the required solⁿ.

$\Rightarrow$ <u>Example-02:</u>

Prove that the following system of linear equation is inconsistent

$$x_1 + 2x_2 - 3x_3 = -1$$

$$5x_1 + 3x_2 - 4x_3 = 2$$

$$3x_1 - x_2 + 2x_3 = 7$$

Solⁿ:

The augmented matrix is

$$A|B = \left[\begin{array}{ccc|c} 1 & 2 & -3 & -1 \\ 5 & 3 & -4 & 2 \\ 3 & -1 & 2 & 7 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & -3 & -1 \\ 5-5\times1 & 3-5\times2 & -4-5(-3) & 2-5(-1) \\ 3-3\times1 & -1-3\times2 & 2-3(-3) & 7-3(-1) \end{array}\right] \begin{array}{l} \\ R_2' = R_2 - 5R_1 \\ R_3' = R_3 - 3R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & -3 & -1 \\ 0 & -7 & 11 & 7 \\ 0 & -7 & 11 & 10 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & -3 & -1 \\ 0 & -7 & 11 & 7 \\ 0 & -7-(-7) & 11-11 & 10-7 \end{array}\right] R_3' = R_3 - R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & -3 & -1 \\ 0 & -7 & 11 & 7 \\ 0 & 0 & 0 & 3 \end{array}\right]$$

which is in echelon form.

$\therefore$ We can write

$$x_1 + 2x_2 - 3x_3 = -1 \quad \text{--- (1)}$$
$$-7x_2 + 11x_3 = 7 \quad \text{--- (2)}$$
$$0 = 3 \quad \text{--- (3)}$$

From (3), we get $0 = 3$, which is not possible.

Hence the system is inconsistent (i.e, the system has no solution).

(Proved)

H.W.

Show that the following system of linear equ^{n}s

$$x_1 + 2x_2 + 3x_3 = -1$$
$$4x_1 + 5x_2 + 6x_3 = 2$$
$$7x_1 + 8x_2 + 9x_3 = 3$$

is inconsistent.

Note:

In echelon form of a matrix

(i) if no. of variables = no. of equations (or no. of non-zero rows, we get unique sol^n.

(ii) if no. of variables < no. of non-zero rows, we get infinite sol^{n}s.

(iii) if no. of variables > no. of non-zero rows, we get no sol^n.

$\lambda \rightarrow$ lamda
$\mu \rightarrow$ mu

$\Rightarrow$ Example-03:

For what values of λ and μ the following system of linear equations

$$x+y+z=6$$
$$x+2y+3z=10$$
$$x+2y+\lambda z=\mu$$

has (i) no solution (ii) more than one solution and (iii) a unique solution.

Solⁿ:

The augmented matrix is

$$A|B = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & \lambda & \mu \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & \lambda-1 & \mu-6 \end{array}\right] \quad \begin{array}{l} R_2' = R_2 - R_1 \\ R_3' = R_3 - R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & \lambda-1-2 & \mu-6-4 \end{array}\right] \quad R_3' = R_3 - R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & \lambda-3 & \mu-10 \end{array}\right] \quad \text{——— (1)}$$

which is in echelon form.

Now from the system (1), we have the following three cases:

(i) The system has no solution if

$$\lambda - 3 = 0 \quad \text{but} \quad \mu - 10 \neq 0$$

$$\Rightarrow \lambda = 3 \quad \text{but} \quad \mu \neq 10$$

(ii) The system has more than one solutions if

$$\lambda - 3 = 0 \quad \text{and} \quad \mu - 10 = 0$$

$$\Rightarrow \lambda = 3 \quad \text{and} \quad \mu = 10$$

(iii) The system has a unique solution if

$$\lambda - 3 \neq 0$$

$$\Rightarrow \lambda \neq 3$$

Ans

⇒ Example-04:

Determine the values of λ so that the following linear system in three variables x, y and z has (i) a unique solⁿ (ii) more than one solⁿ (iii) no solⁿ.

$$x + y - z = 1$$
$$2x + 3y + \lambda z = 3$$
$$x + \lambda y + 3z = 2$$

Solⁿ:

The augmented matrix is

$$A|B = \left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 2 & 3 & \lambda & 3 \\ 1 & \lambda & 3 & 2 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 2-1\times2 & 3-1\times2 & \lambda-(-1)2 & 3-1\times2 \\ 1-1 & \lambda-1 & 3-(-1) & 2-1 \end{array}\right] \quad \begin{array}{l} R_2' = R_2 - 2R_1 \\ R_3' = R_3 - R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & 1 & \lambda+2 & 1 \\ 0 & \lambda-1 & 4 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & 1 & \lambda+2 & 1 \\ 0 & (\lambda-1)-(\lambda-1) & 4-(\lambda-1)(\lambda+2) & 1-(\lambda-1)\times1 \end{array}\right] \quad R_3' = R_3 - (\lambda-1)R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & 1 & \lambda+2 & 1 \\ 0 & 0 & 4-(\lambda^2-\lambda+2\lambda-2) & 1-\lambda+1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & 1 & \lambda+2 & 1 \\ 0 & 0 & 4-\lambda^2-\lambda+2 & 2-\lambda \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & 1 & \lambda+2 & 1 \\ 0 & 0 & 6-\lambda-\lambda^2 & 2-\lambda \end{array}\right]$$

which is in echelon form.

(i) The system has a unique sol^n if

$$6-\lambda-\lambda^2 \neq 0$$

$$\Rightarrow \cancel{6-3\lambda}-(\lambda^2+\lambda-6) \neq 0$$

$$\Rightarrow \lambda^2+\lambda-6 \neq 0$$

$$\Rightarrow \lambda^2+3\lambda-2\lambda-6 \neq 0$$

$$\Rightarrow \lambda(\lambda+3)-2(\lambda+3) \neq 0$$

$$\Rightarrow (\lambda+3)(\lambda-2) \neq 0$$

$$\therefore \lambda+3 \neq 0 \text{ or } \lambda-2 \neq 0$$

$$\Rightarrow \lambda \neq -3 \text{ or } \lambda \neq 2$$

(ii) The system has more than one solⁿ if

$$6-\lambda-\lambda^2=0 \text{ and } 2-\lambda=0$$

$$\Rightarrow (\lambda+3)(\lambda-2)=0 \text{ and } 2-\lambda=0$$

$$\therefore \lambda=-3 \text{ or } \lambda=2 \text{ and } 2-\lambda=0$$

when $\lambda=2$, then $2-\lambda=\cancel{2-2}\,0$

$$\Rightarrow 2-2=0$$

$$\Rightarrow 0=0 \text{ ; which is true}$$

Hence the system has more than one solⁿ for $\lambda=2$

(iii) when $\lambda=-3$, then $2-\lambda=0$

$$\Rightarrow 2-(-3)=0$$

$$\Rightarrow 5=0 \text{ ; which is impossible}$$

$\therefore$ Hence the system has no solⁿ for $\lambda=-3$.

Ans

H.W.

Determine the values of λ such that the following system in unknowns x, y and z has (i) a unique solⁿ, (ii) no solⁿ and (iii) more than one sol

$$\lambda x+y+z=1$$

$$x+\lambda y+z=1$$

$$x+y+\lambda z=1$$

Ans: (i) $\lambda\neq1$ & $\lambda\neq-2$ (ii) $\lambda=-2$ (iii) $\lambda=1$

Remark:

Suppose $A = \left[\begin{array}{cccc|c} 2 & 0 & \cdots & 1 & 2 \\ 0 & 3 & \cdots & 4 & 3 \\ 0 & 0 & \cdots & x & y \end{array}\right]$

(i) If $x \neq 0$, then A has unique sol^n.

(ii) If $x = 0$ but $y \neq 0$, then A has no sol^n.

(iii) If $x = 0$ & $y = 0$, then A has many sol^ns / more than one sol^ns / infinite sol^ns.