Note No: 2
(Pair of straight lines)

CT: Analytical & vector Geometry.

CC: MAT-103

CI: AA

Program: CSE

City University, Panthopath.

Prove that the homogeneous quadratic equation $ax^2+2hxy+by^2=0$ always represents a pair of straight lines, real or imaginary, through the origin.

<u>Proof:</u> Given homogeneous quadratic equation is $ax^2+2hxy+by^2=0$. ——— (I)

If $b \neq 0$ dividing both sides of the equation (1) by $x^2 b$, we get

$$\frac{a}{b}+\frac{2h}{b}\frac{y}{x}+\left(\frac{y}{x}\right)^2=0$$

$$\therefore \left(\frac{y}{x}\right)^2+\frac{2h}{b}\frac{y}{x}+\frac{a}{b}=0 \quad \text{——— (II)}$$

Let m_1, m_2 be the roots of this quadratic equation in $\frac{y}{x}$.

Sum of the roots $= m_1+m_2 = -\frac{2h}{b}$

and product of the roots $= m_1 m_2 = \frac{a}{b}$ $\Big\}$ ——— (III)

The equation (II) must be equivalent to

$$\left(\frac{y}{x}-m_1\right)\left(\frac{y}{x}-m_2\right)=0$$

The two lines represented by (11)

i.e (1) are given

$\frac{y}{x} - m_1 = 0$ and $\frac{y}{x} - m_2 = 0$

i.e $y - m_1 x = 0$ and $y - m_2 x = 0$

which pass through the origin.

Thus the homogeneous quadratic equation $ax^2 + 2hxy + by^2 = 0$ always represents a pair of straight lines, real or imaginary, through the origin.

Prove that a homogeneous equation of the nth degree represents n straight lines, real or imaginary, which all pass through the origin.

Find the angle between the lines represented by the equation $ax^2+2hxy+by^2=0$

Soln: Given equation is

$$ax^2+2hxy+by^2=0 \quad \text{——} (1)$$

Let the lines ~~pa~~ represented by (1) be

$y-m_1x=0$ and $y-m_2x=0$.

So that (1) and $(y-m_1x)(y-m_2x)=0$ are the same.

$$\therefore m_1+m_2=-\frac{2h}{b} \text{ and } m_1m_2=\frac{a}{b} \quad \text{——} (11)$$

If θ be the angle between the staight lines.

$y-m_1x=0$ and $y-m_2x=0$

Then we get,

$$\tan\theta=\frac{m_1-m_2}{1+m_1m_2}=\frac{\sqrt{\{(m_1+m_2)^2-4m_1m_2\}}}{1+m_1m_2}$$

$$=\frac{\sqrt{\{(-\frac{2h}{b})^2-4\cdot\frac{a}{b}\}}}{1+\frac{a}{b}}=\frac{\sqrt{\{\frac{4h^2}{b^2}-\frac{4a}{b}\}}}{\frac{a+b}{b}}$$

$$=\frac{\sqrt{\{\frac{4h^2-4ab}{b^2}\}}}{\frac{a+b}{b}}=\frac{\frac{2}{b}\sqrt{\{h^2-ab\}}}{\frac{a+b}{b}}$$

$$= \frac{2}{b}\sqrt{\{h^2-ab\}} \times \frac{b}{a+b}$$

$$\therefore \tan\theta = \frac{2\sqrt{(h^2-ab)}}{a+b}$$

where θ the angle between the lines represented by the equation $ax^2+2hxy+by^2=0$.

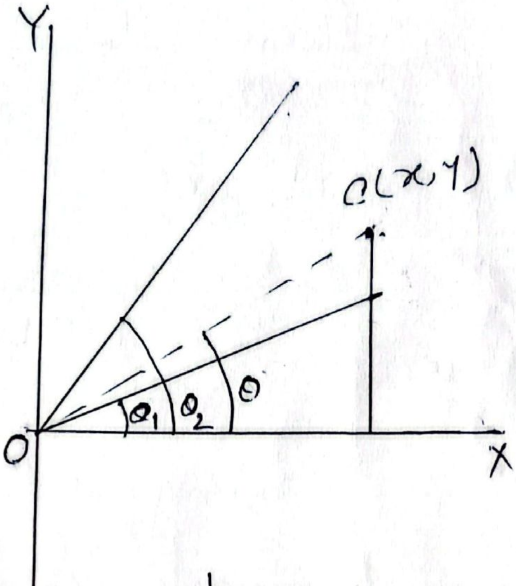
Ans.

If the straight lines are perpendicular to each other then $\theta = 90^\circ$ hence $\tan\theta = \infty$ then $a+b=0$.

Two lines represended by $ax^2+2hxy+by^2=0$ will be real if $h^2 > ab$.

Two lines of the above equation will be imaginary if $h^2 < ab$.

Find equation of the bisectors of the angles between the lines represented by $ax^2 + 2hxy + by^2 = 0$.



Sol^n: If the given lines makes angles θ_1 and θ_2 with the axis of x, then

$(y - x\tan\theta_1)(y - x\tan\theta_2) = 0$

the same as the given equation and we obtain,

$$\tan\theta_1 + \tan\theta_2 = -\frac{2h}{b} \quad \text{--- (i)}$$

$$\text{and } \tan\theta_1 \tan\theta_2 = \frac{a}{b} \quad \text{--- (ii)}$$

If θ be the angle that one of the bisectors makes with the axis of x, then

$$\theta = \frac{\theta_1 + \theta_2}{2}$$

and in either case

$$\tan 2\theta = \tan(\theta_1 + \theta_2)$$

$$\Rightarrow \frac{2\tan\theta}{1-\tan^2\theta} = \frac{\tan\theta_1+\tan\theta_2}{1-\tan\theta_1\tan\theta_2}$$

If (x, y) be any point on a bisector,

then $\frac{y}{x} = \tan\theta$

Hence

$$\frac{2 \cdot \frac{y}{x}}{1-\frac{y^2}{x^2}} = \frac{\tan\theta_1+\tan\theta_2}{1-\tan\theta_1\tan\theta_2}$$

Using (i) & (ii) we have for the required equation

$$\frac{2xy}{x^2-y^2} = \frac{-\frac{2h}{b}}{1-\frac{a}{b}}$$

$$\Rightarrow \frac{2xy}{x^2-y^2} = \frac{2h}{b} \times \frac{b}{a-b}$$

$$\Rightarrow \frac{2xy}{x^2-y^2} = \frac{2h}{a-b}$$

$$\therefore \frac{x^2-y^2}{xy} = \frac{a-b}{h}$$

Ans.

Alternative method:

Let the given equation $ax^2+2hxy+by^2=0$ represent the line $y-m_1x=0$ and $y-m_2x=0$

$$\therefore m_1+m_2=-\frac{2h}{b}, \quad m_1m_2=\frac{a}{b} \quad \text{—— (1)}$$

The equations of the required bisectors are:

$$\frac{y-m_1x}{\sqrt{1+m_1^2}} = \pm \frac{y-m_2x}{\sqrt{1+m_2^2}}$$

squaring, we have

$$(y-m_1x)^2(1+m_2^2)=(y-m_2x)^2(1+m_1^2)$$

$$\Rightarrow (y^2-2xym_1+m_1^2x^2)(1+m_2^2) = (y^2-2xym_2+m_2^2x^2)(1+m_1^2)$$

$$\Rightarrow \cancel{y^2} - 2xym_1 + m_1^2x^2 + m_2^2y^2 - 2xym_1m_2^2 + \cancel{m_1^2m_2^2x^2} =$$

$$\cancel{y^2} - 2xym_2 + m_2^2x^2 + y^2m_1^2 - 2xym_2m_1^2 + \cancel{m_1^2m_2^2x^2}$$

$$\Rightarrow m_1^2x^2+m_2^2y^2-m_2^2x^2-y^2m_1^2 = 2xym_1+2xym_1m_2^2$$

$$-2xym_2-2xym_2m_1^2$$

$$\Rightarrow x^2(m_1^2-m_2^2)-y^2(m_1^2-m_2^2) = 2xy(m_1-m_2)$$

$$-2xym_1m_2(m_1-m_2)$$

$$\Rightarrow (m_1^2-m_2^2)(x^2-y^2) = 2xy\{(m_1-m_2)-m_1m_2(m_1-m_2)\}$$

$$\Rightarrow (m_1+m_2)(m_1-m_2)(x^2-y^2) = 2xy(m_1-m_2)(1-m_1m_2)$$

$$\Rightarrow (m_1+m_2)(x^2-y^2) = 2xy(1-m_1m_2)$$

$$\Rightarrow -\frac{2h}{b}(x^2-y^2) = 2xy\left(1-\frac{a}{b}\right)$$

$$\Rightarrow -\frac{h}{b}(x^2-y^2) = xy\left(\frac{b-a}{b}\right)$$

$$\Rightarrow -h(x^2-y^2) = -xy(a-b)$$

$$\Rightarrow \frac{x^2-y^2}{xy} = \frac{a-b}{h}$$

$$\therefore \frac{x^2-y^2}{xy} = \frac{a-b}{h}$$

which in the required equation.

Ans.