Divided Differences

Let the function,

$$y = f(x)$$

The first diveded differences of $f(x)$ for the arguments x_0, x_1 is defined as,

$$\frac{f(x_0) - f(x_1)}{x_0 - x_1}$$

$$\therefore f(x_0, x_1) = \frac{f(x_0) - f(x_1)}{x_0 - x_1}$$

Simlilarly we can define,

$$f(x_1, x_2) = \frac{f(x_1) - f(x_2)}{x_1 - x_2}$$

$$f(x_2, x_3) = \frac{f(x_2) - f(x_3)}{x_2 - x_3}$$

The second diveded differences for the arguments $x_0, x_1, x_2 \dots$ is define as,

$$f(x_0, x_1, x_2) = \frac{f(x_0, x_1) - f(x_1, x_2)}{x_0 - x_2}$$

similarly the third differences for the arguments x_0, x_1, x_2, x_3 is defined as,

$$f(x_0, x_1, x_2, x_3) = \frac{f(x_0, x_1, x_2) - f(x_1, x_2, x_3)}{x_0 - x_3}$$

The diveded difference table:

Argument	Entry	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
x	$f(x)$			
x_0	$f(x_0)$	$f(x_0, x_1)$		
x_1	$f(x_1)$		$f(x_0, x_1, x_2)$	$f(x_0, x_1, x_2, x_3)$
		$f(x_1, x_2)$		
x_2	$f(x_2)$		$f(x_1, x_2, x_3)$	
		$f(x_2, x_3)$		
x_3	$f(x_3)$			

Example: 5.1: If $f(x) = \frac{1}{x}$, then find $f(a, b)$ and $f(a, b, c)$

Soln:

Given, $f(x) = \frac{1}{x}$

$$\therefore f(a, b) = \frac{f(a) - f(b)}{a - b} = \frac{\frac{1}{a} - \frac{1}{b}}{a - b}$$

$$= \frac{b - a}{ab(a - b)} = -\frac{1}{ab}$$

and.

$$f(a, b, c) = \frac{f(a, b) - f(b, c)}{a - c}$$

$$= \frac{-\frac{1}{ab} - \left(-\frac{1}{bc}\right)}{a - c}$$

$$= \frac{1}{abc}$$

$$\therefore f(a, b) = -\frac{1}{ab}, \quad f(a, b, c) = \frac{1}{abc}$$

Newton's General divided difference formula:-

$$f(x) = f(x_0) + (x - x_0) f(x_0 - x_1) + \dots + (x-x_0)(x-x_1)\dots(x-x_{n-1}) f(x_0, x_1, \dots x_n)$$

Example: 5.3

Use Newton divided difference formula and evaluate $f(6)$, given table

x	5	7	11	13	21
$f(x)$	150	392	1452	2366	9702

Solution:

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
5	150				
		121			
7	392		24		
		265		1	
11	1452		32		0
		457		1	
13	2366		46		
		917			
21	9702				

We have,

$f(x_0) = 150$ $\quad f(x_0, x_1) = 121$

$f(x_0, x_1, x_2) = 24$ $\quad f(x_0, x_1, x_2, x_3) = 1$

$\therefore f(x) = f(x_0) + (x - x_0) f(x_0, x_1) + (x - x_0)(x - x_1)$
$f(x_0, x_1, x_2) + (x - x_0)(x - x_1)(x - x_2)$
$f(x_0, x_1, x_2, x_3) + (x - x_0)(x - x_1)$
$(x - x_2)(x - x_3) f(x_0, x_1, x_2, x_3, x_4)$

$\therefore f(6) = 150 + (6-5)(121) + (6-5)(6-7)24 +$
$(6-5)(6-7)(6-11)1 + 0 + \cdots$

$= 150 + 121 - 24 + 5$

$\therefore f(6) = 252.$

Eample: 5.4 Find the form of the function $f(x)$ under suitable assumption from the following data.

Solⁿ: The divided difference table is given as under:

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
0	2			
		1		
1	3		4	
		9		1
2	12		9	
		45		
3	147			

We have, $x_0 = 0$, $f(x_0) = 2$, $f(x_0, x_1) = 1$

$$f(x_0, x_1, x_2) = 4 \quad f(x_0, x_1, x_2, x_3) = 1$$

The newton divided difference formula is,

$$f(x) = f(x_0) + (x - x_0) f(x_0, x_1) + (x - x_0)(x - x_1) f(x_0, x_1, x_2) + (x - x_0)(x - x_1)(x - x_2) f(x_0, x_1, x_2, x_3)$$

substituting we get,

$$\therefore f(x) = 2 + (x-0)1 + (x-0)(x-1)4 + (x-0)(x-1)(x-2)1$$

$$\therefore f(x) = x^3 + x^2 - x + 2.$$

Lagrange's Interpolation Formula

Lagrange's interpolation formula is;

$$y = f(x) = \frac{(x-x_1)(x-x_2)\ldots(x-x_n)}{(x_0-x_1)(x_0-x_2)\ldots(x_0-x_n)} f(x_0) +$$

$$\frac{(x-x_0)(x-x_2)\ldots(x-x_n)}{(x_1-x_0)(x_1-x_2)\ldots(x_1-x_n)} f(x_1) +$$

$$\ldots\ldots\ldots\ldots\ldots +$$

$$\frac{(x-x_0)(x-x_1)\ldots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\ldots(x_n-x_{n-1})} f(x_n) + \ldots$$

Example 5.6: Using Lagrange's interpolation formula find the value of y corresponding to $x = 10$ from the following table

x	5	6	9	11
$y=f(x)$	12	13	14	16

Soln.

We have, $x_0 = 5$ $\quad x_1 = 6$ $\quad x_2 = 9$ $\quad x_3 = 11$

$y_0 = 12$ $\quad y_1 = 13$ $\quad y_2 = 14$ $\quad y_3 = 16$

Using Lagrange's Interpolation formula we can write,

$$y = f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} y_0 +$$

$$\frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} y_1 +$$

$$\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} y_2 +$$

$$\frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} y_3$$

$$\therefore f(10) = \frac{(10-6)(10-9)(10-11)}{(5-6)(5-9)(5-11)} \times 12 +$$

$$\frac{(10-5)(10-9)(10-11)}{(6-5)(6-9)(6-11)} \times 13 +$$

$$\frac{(10-5)(10-6)(10-11)}{(9-5)(9-6)(9-11)} \times 14 +$$

$$\frac{(10-5)(10-6)(10-9)}{(11-5)(11-6)(11-9)} \times 16$$

$$= 2 - \frac{13}{3} + \frac{35}{3} + \frac{16}{3} = \frac{42}{3}$$

$$= 14.$$

Gauss Forward Interpolation Formula

Example 6.1: Use Gauss forward formula to find y for $x = 30$ given that

x	21	25	29	33	37
y	18.4708	17.8144	17.1070	16.3432	15.5154

Sol^n:
Constructing the difference table:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
21	18.4708				
		-0.6564 (Δy_{-2})			
			-0.0510 ($\Delta^2 y_{-2}$)		
25	17.8144			-0.0054 ($\Delta^3 y_{-2}$)	
		-0.7074 (Δy_{-1})			-0.002 ($\Delta^4 y_{-2}$)
			-0.0564 ($\Delta^2 y_{-1}$)		
29	17.1070			-0.0076 ($\Delta^3 y_{-1}$)	
		-0.7638 (Δy_0)			
			-0.0640 ($\Delta^2 y_0$)		
33	16.3432				
		-0.8278 (Δy_1)			
37	15.5154				

Here, $h = 4$ $\quad$ $p = \dfrac{x - x_0}{h} = \dfrac{30 - 29}{4} = \dfrac{1}{4} = 0.25$

gauss forward formula,

$$y = y_0 + \frac{p}{1!}\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_{-1} + \frac{(p+1)p.(p-1)}{3!}\Delta^3 y_{-1}$$

$$+ \frac{(p+1)p.(p-1)(p-2)}{4!}\Delta^4 y_{-2} + \cdots$$

We get,

$$y_{0.25} = f(0.25) = 17.1070 + (0.25)(-0.7638) + \frac{(0.25)(-0.75)}{2}$$

$$\times(0.0564) + \frac{(1.25)(0.25)(-0.75)}{6}$$

$$\times(-0.0076) + \frac{(1.25)(0.25)(-0.75)(-1.75)}{24}$$

$$\times(-0.0022)$$

$$= 16.9216$$

Gauss Backward Interpolation Formula

Example: 6.2:– Use Gauss's backward formula and find the sales for the year 1966, given that.

Year	1931	1941	1951	1961	1971	1981
Sales (in lakhs)	12	15	20	27	39	52

Solⁿ: We have, $h = 10$

We take 1971 as the origin. The central difference table with origin at 1971 is

Year(x)	Sales(Y)	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
1931	12					
		$3(\Delta y_{-4})$				
1941	15		$2(\Delta^2 y_{-4})$			
		$5(\Delta y_{-3})$		$0(\Delta^3 y_{-4})$		
1951	20		$2(\Delta^2 y_{-3})$		$3(\Delta^4 y_{-4})$	
		$7(\Delta y_{-2})$		$3(\Delta^3 y_{-3})$		$-10(\Delta^5 y_{-4})$
1961	27		$5(\Delta^2 y_{-2})$		$-7(\Delta^4 y_{-3})$	
		$12(\Delta y_{-1})$		$-4(\Delta^3 y_{-2})$		
1971	39		$1(\Delta^2 y_{-1})$			
		$13(\Delta y_0)$				
1981	52					

Here, $h = 10$, $p = \frac{x - x_0}{h} = \frac{1966 - 1971}{10} = -\frac{5}{10}$

$= -0.5$

Gauss's backward formula is,

$$y = y_0 + \frac{p \Delta y_{-1}}{1!} + \frac{(p+1) \cdot p}{2!} \Delta^2 y_{-1} + \frac{(p+1) p (p-1)}{3!} \Delta^3 y_{-2} +$$

$$\frac{(p+2)(p+1) p (p-1)}{4!} \Delta^4 y_{-2} + \frac{(p+2)(p+1) p (p-1)(p-2)}{5!} \Delta^5 y_{-3}$$

$$= 39 + (0.5)(12) + \frac{(0.5)(-0.5)}{2} \times 1 + \frac{0.5 \times (-0.5) \times (-1.5)}{6}$$

$$\times (-4) + \ldots$$

$$= 39 - 6 - 0.125 - 0.25$$

$$= 32.625$$

A.

Bessel's Formula

Bessel's Formula is,

$$y_n = \frac{y_0 + y_1}{2} + \left(p - \frac{1}{2}\right)\Delta y_0 + \frac{p(p-1)}{2!}\left(\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2}\right)$$

$$+ \frac{\left(p-\frac{1}{2}\right).p.(p-1)}{3!}\Delta^3 y_{-1} + \dots$$

Example 6.6: Apply Bessel's formula to obtain y_{25} given that $y_{20} = 2854$, $y_{24} = 3162$, $y_{28} = 3544$ and $y_{32} = 3992$.

Solⁿ: Taking 24 as the origin we get

$$P = \frac{25-24}{4}$$

$$= \frac{1}{4}$$

P.T.O

The difference table is,

x	$v=\frac{x-24}{4}$	y	Δy	$\Delta^2 y$	$\Delta^3 y$
20	-1	2854			
			$308 (\Delta y_{-1})$		
24	0	3162		$74 (\Delta^2 y_{-1})$	
			$382 (\Delta y_0)$		$-8 (\Delta^3 y_{-1})$
28	1	3544		$66 (\Delta^2 y_0)$	
			$448 (\Delta y_1)$		
32	2	3992			

The Bessel's formula is given by

$$Y_n = \frac{Y_0+Y_1}{2} + \left(P-\frac{1}{2}\right)\Delta y_0 + \frac{P(P-1)}{2!}\left(\frac{\Delta^2 y_{-1}+\Delta^2 y_0}{2}\right) +$$

$$\frac{\left(P-\frac{1}{2}\right)P(P-1)}{3!}\Delta^3 y_{-1} + \cdots$$

$$\Rightarrow y_{0.25} = \frac{1}{2}(3162+3544) + \left(\frac{1}{4}-\frac{1}{2}\right)(382) + \frac{\left(\frac{1}{4}\right)\left(\frac{1}{4}-1\right)}{2!}\left(\frac{74+66}{2}\right)$$

$$+ \frac{\left(\frac{1}{4}-\frac{1}{2}\right)\frac{1}{4}\left(\frac{1}{4}-1\right)}{3!}(-8)$$

$$= 3353 - 955 - 65625 - 0.0625$$

$$\Rightarrow Y_{0.25} = 3250.875$$

$$\therefore y = 3250.875$$

$$\therefore x = 25$$

$$\therefore y_{25} = 3250.875.$$

Stirling's Formula

Gauss's forward interpolation is

$$y_p = y_0 + \frac{p}{1!}\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-1}$$

$$+ \frac{(p+1)p(p-1)(p-2)}{4!}\Delta^4 y_{-2} + \cdots \quad \text{--- (1)}$$

Gauss's Backward interpolation is

$$y_p = y_0 + \frac{p\Delta y_{-1}}{1!} + \frac{(p+1)p}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-2}$$

$$+ \frac{(p+2)(p+1)p(p-1)}{4!}\Delta^4 y_{-2} + \cdots \quad \text{--- (2)}$$

Taking the mean of the two Gauss's formulae, we get,

$$y_p = y_0 + p\left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) + \frac{p^2}{2}\Delta^2 y_{-1} + \frac{p(p^2-1)}{3!}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) + \frac{p^2(p^2-1)}{4!}\Delta^4 y_{-2} + \frac{p(p^2-1)(p^2-4)}{5!}\left(\Delta^5 y_{-2} + \Delta^5 y_{-3}\right) + \cdots$$

The above is called Stirling's formula. Stirling's formula gives the most accurate result for

$-0.25 \leq p \leq 0.25$.

Therefore, we have to choose x_0 such that p satisfies this inequality.

Example 6.9: Use Stirling's formula to complute $P_{12.2}$ from the following table

x_0	10	11	12	13	14
$10^5 \log x$	23967	28060	31788	35209	38368

Soln: The difference table is

x	$y = 10^5 \log x$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
10	23967				
		$4093 (\Delta y_{-2})$			
11	28060		$-365 (\Delta^2 y_{-2})$		
		$3728 (\Delta y_{-1})$		$58 (\Delta^3 y_{-2})$	
$12 (x_0)$	$31788 (y_0)$		$-307 (\Delta^2 y_{-1})$		$-13 (\Delta^4 y_{-2})$
		$3421 (\Delta y_0)$		$45 (\Delta^3 y_{-1})$	
13	35209		$-262 (\Delta^2 y_0)$		
		$3159 (\Delta y_1)$			
14	38368				

we have, $p = \frac{x - x_0}{h} = \frac{12.2 - 12}{1} = 0.2$

where, $x_0 = 12$ is the origin.

The Stirling formula is,

$$y = y_0 + p\left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) + \frac{p^2 \Delta^2 y_{-1}}{2} + \frac{p(p^2-1)}{6}\,\frac{(\Delta^3 y_{-1} + \Delta^3 y_{-2})}{2}$$

$$+ \frac{p^2(p^2-1)}{24}\,\Delta^4 y_{-2} + \frac{p(p^2-1)(p^2-4)}{120}\cdot\frac{\Delta^5 y_{-2} + \Delta^5 y_{-3}}{2}$$

$$= 31788 + (0.2)\left(\frac{3421 + 3728}{2}\right) + (0.02)(-307) - (0.016)(45+58)$$

$$- (0.0016)(-13)$$

$$= 31788 + 714.9 - 6.1 + 1.6 + 0$$

$$= 32495$$

$$\Rightarrow 10^5 P_{12.2} = 32495$$

$$\therefore P_{12.2} = 0.32495$$