Note No: 2

Topics → Determinants.

Course Title : Matrix and Linear Algebra.

Dept : CSE . EEE & Civil .

Course Instructor: Arjuman Ara (Asst. Prof.)

City University.

Arjuman Ara
Asst. prof. (BSTE)

Determinant: Each n-square matrix $A = (a_{ij})$ is assigned a special scalar called the determinant of A, denoted by det(A) or $|A|$. And is generally written in the form given below:

$$D = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

Example: Determinant $= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

Sarrus diagram:

$$\begin{array}{ccccc} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{array}$$

value: $a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{31}a_{22}a_{13} - a_{32}a_{23}a_{11} - a_{33}a_{21}a_{12}$

Minors and cofactors: Consider an n-square matrix $A = (a_{ij})$. Let M_{ij} denote the $(n-1)$ square sub matrix of A ~~ode~~ obtained by deleting its i-th row and j-th column. The determinant $|M_{ij}|$ is called the minor of the element a_{ij} of A, and we define the cofactor of a_{ij}, denoted by A_{ij}, to ~~to~~ be the signed minor

$$A_{ij} = (-1)^{i+j}\,|M_{ij}|.$$

Example: Consider the matrix

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

Minor, $|M_{23}| = \begin{vmatrix} 1 & 2 \\ 7 & 8 \end{vmatrix} = 8-14 = -6$

cofactor, $A_{23} = (-1)^{2+3} \begin{vmatrix} 1 & 2 \\ 7 & 8 \end{vmatrix} = -(-6) = 6$

Minor, $|M_{31}| = \begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix} = 12-15 = -3$

cofactor, $A_{31} = (-1)^{3+1} \begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix} = -3$

and so on.

Adjoint of Determinant: When each element in a given determinant A is replaced by its corresponding cofactor, the new determinant so formed is called adjoint or adjugate determinant of A.

This is, if $A = |a_{ij}|$, $i, j = 1, 2, \ldots n$ be a determinant of order n then the new determinant $B = |A_{ij}|$ where A_{ij} is the cofactor of a_{ij} for $i, j = 1, 2, \ldots n$ is called the adjoint or adjugate determinant of A. and is denoted by adj(A).

Example: If $a_{ij} = \begin{vmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & a_{2n} \\ - & - & - & - \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{vmatrix}$

be the cofactors of A_{ij} then the Adjoint of A will be

$$\begin{vmatrix} A_{11} & A_{21} & A_{31} & \cdots & A_{n1} \\ A_{12} & A_{22} & A_{32} & \cdots & A_{n2} \\ - & - & - & - & - \\ A_{1n} & A_{2n} & A_{3n} & \cdots & A_{nn} \end{vmatrix}$$

Let $A = \begin{pmatrix} 3 & 2 & -1 \\ 1 & 6 & 3 \\ 2 & -4 & 0 \end{pmatrix}$, find cofactors and the adjoint of A.

<u>Solution:</u> Given $A = \begin{pmatrix} 3 & 2 & -1 \\ 1 & 6 & 3 \\ 2 & -4 & 0 \end{pmatrix}$

$$A_{11} = (-1)^{1+1} \begin{vmatrix} 6 & 3 \\ -4 & 0 \end{vmatrix} = 12$$

$$A_{12} = (-1)^{1+2} \begin{vmatrix} 1 & 3 \\ 2 & 0 \end{vmatrix} = 6$$

$$A_{13} = (-1)^{1+3} \begin{vmatrix} 1 & 6 \\ 2 & -4 \end{vmatrix} = -4 - 12 = -16$$

$$A_{21} = (-1)^{2+1} \begin{vmatrix} 2 & -1 \\ -4 & 0 \end{vmatrix} = -(-4) = 4$$

$$A_{22} = (-1)^{2+2} \begin{vmatrix} 3 & -1 \\ 2 & 0 \end{vmatrix} = 2$$

$$A_{23} = (-1)^{2+3} \begin{vmatrix} 3 & 2 \\ 2 & -4 \end{vmatrix} = -(-12-4) = 16$$

$$A_{31} = (-1)^{3+1} \begin{vmatrix} 2 & -1 \\ 6 & 3 \end{vmatrix} = 6+6 = 12$$

Anjuman Ara
Asst. Prof (BSTE)

$$A_{32} = (-1)^{3+2} \begin{vmatrix} 3 & -1 \\ 1 & 3 \end{vmatrix} = -(9+1) = -10$$

$$A_{33} = (-1)^{3+3} \begin{vmatrix} 3 & 2 \\ 1 & 6 \end{vmatrix} = 18-2 = 16$$

So that the cofactors of A is

$$\begin{pmatrix} 12 & 6 & -16 \\ 9 & 2 & 16 \\ 12 & -10 & 16 \end{pmatrix}$$

and the adjoint of A is

$$Adj(A) = \begin{pmatrix} 12 & 4 & 12 \\ 6 & 2 & -10 \\ -16 & 16 & 16 \end{pmatrix}$$

Inverse or reciprocal determinant:

Let D' be the adjoint determinant of the determinant D where $D \neq 0$. Then the determinant obtained by dividing every element of D' by D is called the inverse or reciprocal determinant of D.

#

$$\therefore \bar{D} = \left| \frac{A_{ij}}{D} \right|$$

problem 1: Prove that $\begin{vmatrix} a^2 & bc & ac+c^2 \\ a^2+ab & b^2 & ac \\ ab & b^2+bc & c^2 \end{vmatrix} = 4a^2b^2c^2$

proof: L.H.S

$$\begin{vmatrix} a^2 & bc & ac+c^2 \\ a^2+ab & b^2 & ac \\ ab & b^2+bc & c^2 \end{vmatrix}$$

$$= abc \begin{vmatrix} a & c & a+c \\ a+b & b & a \\ b & b+c & c \end{vmatrix}$$

$$C_1' = C_1 - (C_2 + C_3)$$

$$= abc \begin{vmatrix} -2c & c & a+c \\ 0 & b & a \\ -2c & b+c & c \end{vmatrix}$$

$$R_3' = R_3 - R_1$$

$$= abc \begin{vmatrix} -2c & c & a+c \\ 0 & b & a \\ 0 & b & -a \end{vmatrix} = abc(-2c)(-ab-ab)$$

$$= 4a^2b^2c^2 = R.H.S$$

(proved)

CS CamScanner

Argumam Ara
Asst. prof. (BSTE)

prove that:

$$\begin{vmatrix} a+b+c & -c & -b \\ -c & a+b+c & -a \\ -b & -a & a+b+c \end{vmatrix} = 2(a+b)(b+c)(c+a)$$

<u>proof:</u> L.H.S

$$\begin{vmatrix} a+b+c & -c & -b \\ -c & a+b+c & -a \\ -b & -a & a+b+c \end{vmatrix}$$

$c_1' = c_1 + c_2, \quad c_2' = c_2 + c_3$

$$= \begin{vmatrix} a+b & -(b+c) & -b \\ a+b & b+c & -a \\ -(a+b) & b+c & a+b+c \end{vmatrix} = (a+b)(b+c)\begin{vmatrix} 1 & -1 & -b \\ 1 & 1 & -a \\ -1 & 1 & a+b+c \end{vmatrix}$$

$c_1' = c_1 + c_2$

$$= (a+b)(b+c)\begin{vmatrix} 0 & -1 & -b \\ 2 & 1 & -a \\ 0 & 1 & a+b+c \end{vmatrix}$$

$= (a+b)(b+c)(-2)\{-(a+b+c)+b\}$

$= (a+b)(b+c)(-2)\{-(c+a)\}$

$= 2(a+b)(b+c)(c+a) = $ R.H.S (proved)

prove that

$$\begin{vmatrix} 1+a^2-b^2 & 2ab & -2b \\ 2ab & 1-a^2+b^2 & 2a \\ 2b & -2a & 1-a^2-b^2 \end{vmatrix} = (1+a^2+b^2)^3$$

Proof: L.H.S

$$\cancel{(1+a^2-b^2)} \quad \begin{vmatrix} 1+a^2-b^2 & 2ab & -2b \\ 2ab & 1-a^2+b^2 & 2a \\ 2b & -2a & 1-a^2-b^2 \end{vmatrix} \cancel{= (1+a^2+b^2)^3}$$

~~proof~~ $C_1' = C_1 - bC_3$

$C_2' = C_2 + aC_3$

$$= \begin{vmatrix} 1+a^2-b^2+2b^2 & 0 & -2b \\ 2ab-2ab & 1-a^2+b^2+2a^2 & 2a \\ 2b-b+a^2b+b^3 & -2a+a-a^3-ab^2 & 1-a^2-b^2 \end{vmatrix}$$

$$= \begin{vmatrix} 1+a^2+b^2 & 0 & -2b \\ 0 & 1+a^2+b^2 & 2a \\ (1+a^2+b^2)b & -a(1+a^2+b^2) & 1-a^2-b^2 \end{vmatrix}$$

Arjuman Ara
Asst. prof. IBST

$$= (1+a^2+b^2)^2 \begin{vmatrix} 1 & 0 & -2b \\ 0 & 1 & 2a \\ b & -a & 1-a^2-b^2 \end{vmatrix}$$

$$C_3' = C_3 + 2bC_1$$

$$= (1+a^2+b^2)^2 \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 2a \\ b & -a & 1-a^2+b^2 \end{vmatrix}$$

$$= (1+a^2+b^2)^2 \left\{ 1-a^2+b^2+2a^2 \right\}$$

$$= (1+a^2+b^2)^2 (1+a^2+b^2)$$

$$= (1+a^2+b^2)^3 = R.H.S$$

(proved)

Exam

\# Consider the matrix $A = \begin{bmatrix} 1 & -1 & -3 & 4 \\ 4 & 2 & -1 & 3 \\ -2 & 3 & 1 & 2 \\ 1 & 2 & 3 & 4 \end{bmatrix}$

find minor, cofactor and adjoint of the above matrix

Sln:

$$\text{Minors} = \begin{bmatrix} 25 & -39 & 31 & 49 \\ 54 & -60 & -30 & 21 \\ 51 & 78 & 39 & 3 \\ 10 & 45 & 73 & -41 \end{bmatrix} \quad \text{Ans.}$$

$$\text{Cofactors} = \begin{bmatrix} 25 & 39 & 31 & -49 \\ -54 & -60 & 30 & 21 \\ 51 & -78 & 39 & -3 \\ -10 & 45 & -73 & -41 \end{bmatrix} \quad \text{Ans.}$$

$$\text{Adjoint} = \begin{bmatrix} 25 & -54 & 51 & -10 \\ 39 & -60 & -78 & 45 \\ 31 & 30 & 39 & -73 \\ -49 & 21 & -3 & 41 \end{bmatrix} \quad \text{Ans.}$$

Note no: 2 (2nd part)

Cramer's Rule:

If $Ax=b$ is a system of n linear equations in n unknowns such that $\det(A) \neq 0$, then the system has a unique solution. This solution is

$$x_1 = \frac{\det(A_1)}{\det(A)}, \quad x_2 = \frac{\det(A_2)}{\det(A)}, \quad \ldots, \quad x_n = \frac{\det(A_n)}{\det(A)},$$

where A_j is the matrix obtained by replacing the entries in the j-th column of A by the entries in the matrix

$$b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}.$$

Solve the following equations by Cramer's rule:

$$\begin{aligned} x_1 + \quad\quad\quad + 2x_3 &= 6 \\ -3x_1 + 4x_2 + 6x_3 &= 30 \\ -x_1 - 2x_2 + 3x_3 &= 8. \end{aligned}$$

#

Solution: Here we have.

$$A = \begin{pmatrix} 1 & 0 & 2 \\ -3 & 4 & 6 \\ -1 & -2 & 3 \end{pmatrix}, \quad \det(A) = \begin{vmatrix} 1 & 0 & 2 \\ -3 & 4 & 6 \\ -1 & -2 & 3 \end{vmatrix} = 44$$

$$A_1 = \begin{pmatrix} 6 & 0 & 2 \\ 30 & 4 & 6 \\ 8 & -2 & 3 \end{pmatrix}, \quad \det(A_1) = \begin{vmatrix} 6 & 0 & 2 \\ 30 & 4 & 6 \\ 8 & -2 & 3 \end{vmatrix} = -40$$

$$A_2 = \begin{pmatrix} 1 & 6 & 2 \\ -3 & 30 & 6 \\ -1 & 8 & 3 \end{pmatrix}, \quad \det(A_2) = \begin{vmatrix} 1 & 6 & 2 \\ -3 & 30 & 6 \\ -1 & 8 & 3 \end{vmatrix} = 72$$

$$1(90-48)-6(-9+6)+2(-24+30) = 42+18+12 = 72$$

$$A_3 = \begin{pmatrix} 1 & 0 & 6 \\ -3 & 4 & 30 \\ -1 & -2 & 8 \end{pmatrix}, \quad \det(A_3) = \begin{vmatrix} 1 & 0 & 6 \\ -3 & 4 & 30 \\ -1 & -2 & 8 \end{vmatrix} = 152$$

Therefore.

$$x_1 = \frac{\det(A_1)}{\det(A)} = \frac{-40}{44} = -\frac{10}{11}$$

$$x_2 = \frac{\det(A_2)}{\det(A)} = \frac{72}{44} = \frac{18}{11}$$

$$x_3 = \frac{\det(A_3)}{\det(A)} = \frac{152}{44} = \frac{38}{11}$$

Solve the following equations by Cramer's rule or by determinants.

1. $x-y+z=1$, $x+y-2z=0$, $2x-y-z=0$ $(1,1,1)$

2. $x+2y-z=9$, $2x-y+3z=-2$, $3x+2y+3z=9$ $(2,3,-1)$