STA 2418

STOCHASTIC CALCULUS

Friday,
22/09/23

<u>Course Description:</u>

1. Intro to prob theory: Probability spaces
   - Random variables
   - Measurability
   - Expectations and conditional expectations
2. Stochastic process: Filtrations
   - Adapted processes and matingales
   - Examples of stochastic processes
3. Quadratic variations and covariation
4. Stopping times
5. Stochastic integration
6. Stochastic differentiation
7. Radon Nikodym, Girsanov's Theorem
8. Feynman Kac Theorem
9. Fokker Planck equations
10. Applications to financial instruments
11. Computations using statistical software

Identify δ-algebra for $\Omega = \{1, 2, 3, 4, 5, 6\}$

δ-field F is defined as any subset F of 2^{Ω} that satisfies the properties;

i) $\phi \in F$

ii) If $A_i \in F$ then $A_i^c \in F$

iii) $\bigcup_{i=1} A_i \in F$ if $A_1, A_2, \dots, A_n \in F$

$\delta \to$ sigma | $2^{\Omega} \to$ subsets | where $\Omega = \{H, T\}$ $2^{\Omega} = \{\{\emptyset\}, \{H, T\}, \{H\}, \{T\}\}$ | $A^c \to$ complement of A

① Introduction to Probability Theory

1) Probability Space:

Is a triplet given by; $(\Omega, F, \mathbb{P})$ consisting of a set of possible outcomes, Ω, a δ-field, F and a prob. measure, $\mathbb{P}$

2) Sample Space, Ω

- This is the set of all possible outcomes, eg, When flipping a coin, Ω can be head or tail $\Omega = \{H, T\}$
  Similarly, when rolling a dice, $\Omega = \{1, 2, 3, 4, 5, 6\}$
- All ē subsets of the ~~subse~~ sample space Ω have 2^{Ω} elements
- The set 2^{Ω} satisfies ē following properties;
  ① The empty set is an element, ie, $\phi \in 2^{\Omega}$
  ② If $A \in 2^{\Omega}$ then $A^c \in 2^{\Omega}$
  ③ $\bigcup_{i=1} A_i \in 2^{\Omega}$ $\bigcup \to$ Union

✓ Understood (see YT example in the back for reference)

3) δ-field, F

- This is also known as a δ-algebra and it is defined as any subset F of 2^{Ω} that satisfies the 3 properties above, that is,

① $\emptyset \in F$ ✓
② If $A \in F$ the $A^c \in F$ ✓
③ $\bigcup_{i=1} A_i \in F$ if $A_1, A_2, \dots A_n \in F$

- The δ-field provides ē knowledge about which events are considered in the probability space and it is considered the info. component of ē prob. space

4) Event

- This is any set belonging to F
- An event $\{\} \in F$ is said to have occurred ~~if ē outcome of an experiment~~ if the outcome of an ~~ele~~ experiment is an element of that subset $\in \{\}$
  $\rightarrow$ the actual outcome

$\int_{\Omega} |X(\omega)| \, d\mathbb{P}(\omega) = \int_{\mathbb{R}} |x| \, dP(x)dx) < \infty$

5) **Probability Measure, $\mathbb{P}$**

- Is a fxn that maps all the info into the interval $[0, 1]$, ie,

$\mathbb{P} : \mathcal{F} \longrightarrow [0, 1]$

- It satisfies ē following properties, ie,

① $\mathbb{P}(\Omega) = 1$

$\Rightarrow \mathbb{P}(\phi) = 0$ ⟨Assumption⟩

② If ē mutually disjoint events $A_1, A_2, --, A_n \in \mathcal{F}$, then the probability of ē union is:

$$\mathbb{P}\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \mathbb{P}(A_i) \Rightarrow \mathbb{P}\left[\bigcup_{i=1}^{3} A_i\right] = \mathbb{P}(A_1) + \mathbb{P}(A_2) + \mathbb{P}(A_3)$$

eg; $\mathbb{P}(\{H\} \cup \{T\}) = P\{H\} + P\{T\}$

6) **Random Variable**

- A r.v X is a fxn that assigns a numerical value to each state of the world; $X : \Omega \to \mathbb{R}$ $(-\infty$ to $\infty)$

7) **Measurability of a R.V**

- Given any 2 real no.s a and b ; $a, b \in \mathbb{R}$. Then if all the states of the world ; $\forall\ \omega \in \Omega$ for which X takes values btwn a and b. forms an event, then X is said to be an $\mathcal{F}$ measurable fxn.
- The event is given by $\{\omega \in \Omega : a < X(\omega) < b\} \in \mathcal{F}$

Eg; In a football match, 3 goals were scored by half time ∴ standing at half time, if someone asks you how many goals were scored in the first 30 mins you can confidently say $0 < X(30) \leq 3$

also cdf $F_x(x) = P_r(X \leq x)$ is also measurable coz it's btwn 0 and 1

8) **Independent Random Variables**

- Two r.v.s X and Y are indpdt if the occurrence of one does not change the pdf of the other, ie, for any sets $A, B \in \mathbb{R}$, then if $\{\omega : X(\omega) \in A\}$ and $\{\omega : Y(\omega) \in B\}$ then X and Y are indpdt
- When X and Y are indpdt, the marginal pdf $P_x(x)$ and $P_y(y)$ and the joint pdf $P_{x,y}(x,y)$ have ē following relations:

$\int_{-\infty}^{\infty} x f(x)\, dx = E[X] \Rightarrow \int_{-\infty}^{\infty} |x| f(x)\, dx = E[|X|]$

$a, b \in \mathbb{R}$ $\omega \in \Omega$ for X if $x > 0$ then $E[X] > 0$

$$P_{X,Y}(x,y) = P_X(x) \cdot P_Y(y)$$

9) Partition

- A partition, $(\Omega_i)\ 1 \leq i \leq n$, of Ω is a family of subsets of Ω $(\Omega_i \subset \Omega)$ that satisfy the following properties:

intersection

① $\Omega_i \cap \Omega_j = \emptyset,\ \forall i \neq j$ (meaning they never intersect ?

② $\bigcup_{i=1}^{n} \Omega_i = \Omega$

where each event Ω_i is associated with some prob. $\mathbb{P}(\Omega_i)$

10) Integrability of a R.V

- A R.V. X is said to be integrable if ~~$\int_\Omega |X(\omega)|\, d\,\mathbb{P}(\omega)$~~

$$\int_\Omega |X(\omega)|\, d\,\mathbb{P}(\omega) = \int_{\mathbb{R}} |x|\, P(x)\, dx < \infty$$

probability measure

distribution fxn

A random var. is said to be integrable if the expectation of its absolute value is less than infinity

11) Expectation

- The Expectation of an integrable r.v X is defined as:

$$E[X] = \int_\Omega X(\omega)\, d\,\mathbb{P}(\omega) \qquad \omega \in \mathbb{R}$$

$$= \int_{\mathbb{R}} x\, P(x)\, dx$$

and $E[X]$ satisfies the following properties:

① $E[cX] = c\,E[X] \quad \forall c \in \mathbb{R}$ → Homogenity

② $E[X + Y] = E[X] + E[Y]$ → Additivity

A stochastic process of the prob. space $(\Omega, F, \mathbb{P})$ is a family of random variables $\{X_t\}$ parameterized by $t \in T$ where $T \subseteq \mathbb{R}$

12) Conditional Expectations

- Let X be a r.v on a prob. space $(\Omega, F, \mathbb{P})$ and G be a δ-field contained in F, ie, $G \subset F$ then: $E[X|G]$ is known as the conditional exp. of X given G

G is a δ field contained in F

- Properties of conditional Expectations:

(x be G measurable)

① Let $X \in G$ ~~be measurable~~;

$E[X|G] = X$

② Let $X \in G$; $E[E[X|G]] = E[X]$ → law of iterated expectation ie, $E[E[X|Y]] = E[X]$

③ Let $a, b \in \mathbb{R}$, then $E[aX + bY|G] = E[aX|G] + E[bY|G]$

$= aE[X|G] + bE[Y|G]$

④ Let $X \in G$, then

$E[XY|G] = X\,E[Y|G]$

$\cancel{E[X|G] \cdot E[Y|G]}$

⑤ Let $X \in G$, then

$E[E[X|G]|H] = E[X|H]$

⑥ If $X \geq 0$, then $E[X|G] \geq 0$

⑦ Let X and G be indpdt;

$E[X|G] = E[X]$

Filtration - The ordered collections of subsets that are used to model information that is available at a given point. Is given by $\mathbb{F} = \{F_t\}_{t \in T}$

measurable → info is known with certainty

② STOCHASTIC PROCESSES

No. 68 (Me) Friday 29/09/03

- A stochastic process of ē prob space $(\Omega, F, \mathbb{P})$ is a fam. of random var $\{X_t\}$ parameterized by $t \in T$ where $T \subseteq \mathbb{R}$
  $\{X_t\}_{t \in T}$; $\subseteq$ is a subset
- If T is an interval, say, from a & b $T \in [a,b]$ then we say that X_t is a stochastic process in continuous time.
- If T is discrete, $T = \{1, 2, \ldots\}$, then we say that X_t is a stochastic process in discrete time.

Path of X_t

- This is ē evolution in time of a given state of ē world, $\omega \in \Omega$ given by ē fxn;

$$t \longmapsto X_t(\omega)$$

and is called a path or a realization of X_t

$\mathcal{F}_T$, $\mathcal{F}_t$, 0, t, T

Filtration

- Assuming that all ē info accumulated upto time t is contained in a sigma, δ-field $\mathcal{F}_t$ then it means that $\mathcal{F}_t$ contains all ē info of events that have occurred upto time t.
- $\therefore \mathcal{F}_s \subset \mathcal{F}_t \subset \mathcal{F}_T \quad \forall\ s \leq t \leq T$ $\quad \mathbb{F} = \{\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_{45}\}$
- $\therefore$ The family $\mathbb{F} = \{\mathcal{F}_t\}_{t \in T}$ is called a filtration
  ordered collections of subsets that are used to model the ~~info. that is available at a given pnt.~~

Adapted Process

- A stoch. process X_t is said to be adapted to ē filtration of $\mathcal{F}_t$ if X_t is $\mathcal{F}_t$ measurable for all $(\forall)$ $t \in T$ $\langle X_t \in \mathcal{F}_t \ \forall\, t \in T \rangle$
  ↳ you can account for every min

Martingale

- A process X_t, $t \in T$ is called a martingale with respect to ē filtration of $\mathcal{F}_t$ if: ~~X_t is $\mathcal{F}_t$ measurable~~
  i) X_t is integrable;

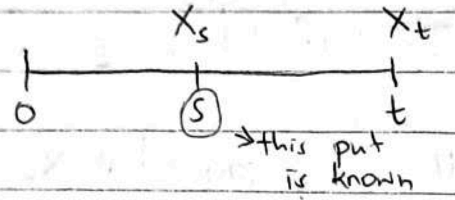
$$E[|X_t|] < \infty$$

X_t is integrable, $E[|X_t|] < \infty$

ii) X_t is adapted to ē filtration of $\mathcal{F}_t$

$$X_t \in \mathcal{F}_t \quad \forall\, t \in T$$

iii) $E[X_t | \mathcal{F}_s] = X_s \quad \forall s \leq t$



(*) If it is not $= X_s$, then it is not a martingale; at this pnt check where it is $< X_s$ or $> X_s$ are hence conclude on it being a super-martingale or a sub-martingale. respectively

Note:

If X_t satisfies ē 1st 2 properties & $E[X_t | \mathcal{F}_s] < X_s$ then X_t is called a super martingale

If $E[X_t | \mathcal{F}_s] > X_s$ then X_t is called a sub martingale

Examples of Stochastic Processes

(a) Wiener Process

- Also known as Brownian Motion
- WP, W_t is a process adapted to ē filtration of $\mathcal{F}_t$ such that

i) The process starts at ē origin, ie, $W_0 = 0$

ii) W_t is an $\mathcal{F}_t$ martingale with a finite ~~mean~~ variance;

$$E[W_t^2] < \infty \quad \forall\, t \geq 0$$

$\Rightarrow$ If the $E[W_t^2] < \infty$ then $E[W_t^2] - (E[W_t])^2 < \infty$ due to the 1st property of a martingale

iii) It has stationary & indpdt increments
can be +ve or -ve

ie,

stationarity: If $W_t - W_s$ only depends on ē interval $t-s$

this could be the explanation for independence increments

Example:

We can say that we have stationary increments if ē prob:

$$Pr(W_{t+s} - W_s \leq a) = Pr(W_t - W_0 \leq a)$$
$$= Pr(W_t \leq a) \quad \text{because } W_0 = 0$$

iv) The increments $W_t - W_s$ follow a normal dist;

$$W_t - W_s \sim N(0, t-s)$$

v) W_t is continuous in time

Simulating a Wiener Process in R - simulate variability

→ There is no stochasticity ⟨variability⟩ at time zero, ie, $W_0 = 0$

→ To move the increment from a Continuous state to a discrete state; ie; from $W_t - W_s \sim N(0, t-s)$ to $W_{t+\Delta t} - W_t \sim N(0, \Delta_t)$

$t + \Delta t - t = \Delta t$

Let $Y \sim N(0, \Delta_t)$ and $Z \sim N(0,1)$ (standard normal)

$$Y = Z\sqrt{\Delta_t}$$

from where??

$E[Z] = 0$

Are we letting $Y = Z\sqrt{\Delta_t}$?

$Z =$

$$E[Y] = \sqrt{\Delta_t}\, E[Z] = 0$$

$$Var(Y) = \Delta_t\, Var(Z) = \Delta_t$$

$Var(Z) = 1$

$$W_{t+\Delta_t} - W_t = \underbrace{Z\sqrt{\Delta_t}}_{Y}$$

$$W_{t+\Delta_t} = W_t + Z\sqrt{\Delta_t}$$

In R

```
W = 0          empty vector   W = {   }
W[1] = 0 ; n=100              W = {0, . . . }
dt = 1/365
for ( i in 2 : n)
{
  W[i] = W[i-1] + sqrt(dt) * rnorm(1)
}
W
```

```
W <- NULL
W[1] <- 0
dt < 1/365
n <- 100
for (i in 2:n) {
W[i] <- W[i-1]
       + (sqrt(dt)) * rnorm(1)
}
W
```

t = 0 : (n - 1)
plot (t, W, type = "l", xlab = "Time in Days", ylab = "Wiener Process",
main = "Sample path of a Wiener Process")

(*) x and y lengths differ

Proposition 1:

- A Wiener Process is a martingale w.r.t ē filtration $\mathcal{F}_t$

Proof

s is a fxn of t

$$E[W_t \mid \mathcal{F}_s] = W_s \quad \forall\, s \le t$$

but

$$E[W_t \mid \mathcal{F}_s] = E[W_t + W_s - W_s \mid \mathcal{F}_s]$$

$$= E[W_s + (W_t - W_s) \mid \mathcal{F}_s]$$

$$= E[W_s \mid \mathcal{F}_s] + E[W_t - W_s \mid \mathcal{F}_s]$$

why??

since W_s is $\mathcal{F}_s$ measurable its like $E[x|G]$ when x is G measurable it is $= X$

$$= W_s + \cancel{E[W_t - W_s]}^{\,0}$$

Note: $W_t - W_s \sim N(0, t-s)$

$$= W_s$$

Proposition 2:

Let $0 \le s \le t$. Then

i) $Cov(W_s, W_t) = s$

ii) $Corr(W_s, W_t) = \sqrt{\frac{s}{t}}$ correlation

Proof $Cov(X,Y) = E[XY] - E[X] \cdot E[Y]$

$$Corr(X,Y) = \frac{Cov(XY)}{\sigma(X) \cdot \sigma(Y)}$$

(i) $Cov(W_s, W_t + W_s - W_s) = Cov(W_s, W_s + (W_t - W_s))$

$$= Cov(W_s, W_s) + Cov(W_s, W_t - W_s)$$

$$= Var(W_s) + \underbrace{Cov(W_s - W_0, W_t - W_s)}_{\text{coz of indpendence of increments} = \text{zero } (0)}$$

Note: $Cov(X,X) = Var(X)$

$$= Var(W_s)$$

stationarity implies $W_t - W_s$ depend only on $t-s$
and $W_t - W_s \sim N(0, t-s)$, ie, $Var(W_t - W_s) = t-s$

✱ Because of stationarity:

$$Var(W_s) = Var(W_s - W_0) \quad ; \text{ooz } W_0 = 0, \; E[W_s]=0$$

$$= s - 0$$

$$= s$$

$$Var(W_t) = Var(W_t - W_0)$$

$$= t - 0$$

$$= t$$

(II) $Corr(W_s, W_t) = \dfrac{s}{\sqrt{s} \cdot \sqrt{t}}$

$\therefore \sigma^2(W_s) = s$ & $\sigma^2(W_t) = t$

$\sigma(W_s) = \sqrt{s}$ & $\sigma(W_t) = \sqrt{t}$

$$= \sqrt{\frac{\cancel{s}^2 s}{\cancel{s} t}} = \sqrt{\frac{s}{t}}$$

Question ①

Show that $Y_t = W_t^2 - t$ is a ~~Wiener Process~~ a Martingale given that W_t is

✱ W_t being a WP brings in stationarity & ē rest of the properties

Proof

Hint: $E[Y_t | \mathcal{F}_s] = Y_s \quad \forall s \leq t$

$$= W_s^2 - s$$

$x^2 - y^2 = (x+y)(x-y)$

$$E[Y_t | \mathcal{F}_s] = E[W_t^2 - t \mid \mathcal{F}_s]$$

$$= E[W_t^2 - t + W_s^2 - W_s^2 \mid \mathcal{F}_s]$$

$$= E[W_t^2 - W_s^2 + W_s^2 - t \mid \mathcal{F}_s]$$

$$= E[W_t^2 - W_s^2 \mid \mathcal{F}_s] + E[W_s^2 - t \mid \mathcal{F}_s]$$

$$= E[(W_t - W_s)(W_t + W_s) \mid \mathcal{F}_s] + E[W_s^2 | \mathcal{F}_s] - E[t | \mathcal{F}_s]$$

since $E[X_t | \mathcal{F}_s] = X_s$ then $E[t|\mathcal{F}_s] = s$

$$= (W_t + W_s)\underbrace{E[W_t - W_s \mid \mathcal{F}_s]}_{\text{zero}} + W_s^2 - s$$

$$= 0 + W_s^2 - s$$

$$= W_s^2 - s$$

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Solution

We need to show that $E[Y_t | \mathcal{F}_s] = Y_s \quad \forall \; s \leq t$

$$= E[W_t^2 - t \mid \mathcal{F}_s]$$

$$= E[W_t^2 \mid \mathcal{F}_s] - t$$

$$= E[(W_t + W_s - W_s)^2 \mid \mathcal{F}_s] - t$$

$$= E[(W_s + (W_t - W_s))^2 \mid \mathcal{F}_s] - t$$

$$= E[W_s^2 + 2W_s(W_t - W_s) + (W_t - W_s)^2 \mid \mathcal{F}_s] - t$$

$$= E[W_s^2 \mid \mathcal{F}_s] + 2E[W_s(W_t - W_s) \mid \mathcal{F}_s] + E[(W_t - W_s)^2 \mid \mathcal{F}_s] - t$$

$$= E[W_s^2 \mid \mathcal{F}_s] + 2E[(W_s - W_0)(W_t - W_s) \mid \mathcal{F}_s] + E[(W_t - W_s)^2 \mid \mathcal{F}_s] - t$$

$$= W_s^2 + 2E[(W_s - W_0) \mid \mathcal{F}_s] \circledast E[(W_t - W_s \mid \mathcal{F}_s] + E[(W_t - W_s)^2] - t$$

$$= W_s^2 + 2(W_s - W_0)\underbrace{E[W_t - W_s]}_{0} + t - s - t$$

$$= W_s^2 + 0 - s$$

$$= W_s^2 - s$$

$$= Y_s$$

$Y = e^{x} \sim \log N$ | $Var(Y) = e^{\sigma^2}(e^{\sigma^2}-1)$

(b) Geometric Brownian Motion (GBM)

is a lognormal dist

- The process $X_t = e^{W_t}$; $W_t \sim N(0,t)$

$E[X_t] = e^{\frac{1}{2}t}$

$Var[X_t] = e^{t}(e^{t}-1)$

(Q) $E[Y_t|\mathcal{F}_s] = Y_s$

Question:

(i) Show that X_t is not a martingale (2) $Y_s = e^{cW_s - \frac{1}{2}c^2 s}$

also, (ii) Show that $Y_t = e^{cW_t - \frac{1}{2}c^2 t}$ $\forall c \in \mathbb{R}$ is a martingale (ii)

(I)

(I) We need to show that $E[X_t|\mathcal{F}_s] \neq X_s = e^{W_s}$

$E[X_t|\mathcal{F}_s] = E[e^{W_t}|\mathcal{F}_s]$

$= E[e^{W_t - W_s + W_s}|\mathcal{F}_s]$

$= E[e^{W_t - W_s}\cdot e^{W_s}|\mathcal{F}_s]$

$= E[e^{W_t - W_s}|\mathcal{F}_s]\cdot E[e^{W_s}|\mathcal{F}_s]$

but $W_t - W_s \sim N(0, t-s)$

$\therefore e^{W_t - W_s} \sim \log N\left(e^{0+\frac{1}{2}(t-s)}, e^{2(t-s)} - e^{(t-s)}\right)$

$\Rightarrow E[e^{W_t - W_s}] = e^{\frac{1}{2}(t-s)}$

$= E[e^{W_t - W_s}]\cdot e^{W_s}$

$= e^{\frac{1}{2}(t-s)}\cdot e^{W_s}$

since $s < t$, $t-s$ is a positive number and so is its exponential

$\Rightarrow e^{\frac{1}{2}(t-s)}\cdot e^{W_s} > e^{W_s}$

or $e^{W_s + \frac{1}{2}(t-s)} > e^{W_s}$

$\therefore E[X_t|\mathcal{F}_s] > X_s$ implying X_t is a submartingale

(II)

$Y_t = e^{cW_t - \frac{1}{2}c^2 t}$

$E[Y_t|\mathcal{F}_s] = E[e^{cW_t - \frac{1}{2}c^2 t}|\mathcal{F}_s]$

$= E[e^{c(W_t - W_s + W_s) - \frac{1}{2}c^2 t}|\mathcal{F}_s]$

$= E[e^{c(W_t - W_s)}\cdot e^{cW_s}|\mathcal{F}_s]\cdot E[e^{-\frac{1}{2}c^2 t}|\mathcal{F}_s]$

$= E[e^{c(W_t - W_s)}|\mathcal{F}_s]\cdot e^{cW_s}\cdot e^{-\frac{1}{2}c^2 t}$

$c(W_t - W_s) \sim N(0, c^2[t-s])$

$e^{c(W_t - W_s)} \sim \log N$

$E[e^{c(W_t - W_s)}] = e^{\frac{1}{2}c^2(t-s)}$

$= E[e^{c(W_t - W_s)}]\cdot e^{cW_s}\cdot e^{-\frac{1}{2}c^2 t}$

$= e^{\frac{1}{2}c^2(t-s)}\cdot e^{cW_s}\cdot e^{-\frac{1}{2}c^2 t}$

$= \cancel{e^{\frac{1}{2}c^2 t}}\cdot e^{-\frac{1}{2}c^2 s}\cdot e^{cW_s}\cdot \cancel{e^{-\frac{1}{2}c^2 t}}$

$= e^{cW_s - \frac{1}{2}c^2 s} = Y_s$

$\therefore Y_t$ is a martingale

$$X_t = W_t - tW_1$$
$$= W_t - tW_t + tW_t - tW_1$$
$$= (1-t)W_t + t(W_t - W_1)$$
$$= (1-t)W_t - t(W_1 - W_t)$$

(c) <u>Brownian Bridge</u>

- This is the process $X_t = W_t - tW_1$ that is fixed both at zero and at 1 (so it lies btwn 0 and 1) $0 \le t \le 1$

-

$$X_t = W_t - tW_1$$
$$= W_t + tW_t - tW_t - tW_1$$
$$= W_t - tW_t + tW_t - tW_1$$
$$= (1-t)W_t + t(W_t - W_1)$$
since $1 \geq t$
$$= (1-t)W_t - t(W_1 - W_t)$$

$$E[X_t] = (1-t)\underbrace{E[W_t]}_{0} - t\underbrace{E[W_1 - W_t]}_{0}$$
because $1 > t \Rightarrow 0 \le t \le 1$
$$= 0$$

$$Var(X_t) = (1-t)^2 \underbrace{Var(W_t)}_{t} + t^2 \underbrace{Var(W_1 - W_t)}_{(1-t)}$$
$$= (1-t)^2 t + t^2(1-t)$$
$$= (1-t)t[\cancel{1-t+t}]^{1}$$
$$= (1-t)t$$

$$\Rightarrow X_t \sim N(0, t(1-t)$$

(d) Brownian Motion with Drift

- Is ē process $Y_t = \mu t + W_t$ → we add this to add volatility to ē eq,n

μt: the linear part; W_t: stochasticity

$$E[Y_t] = \mu t + \underbrace{E[W_t]}_{0}$$

$$= \mu t$$

$$Var(Y_t) = Var(W_t)$$

$$= t$$

$$Y_t \sim N(\mu t, t)$$

(e) Poisson Process

- A poisson process describes the no. of occurrences of a certain event before time t

Eg; the no of electrons arriving at a node until time t. or
the no. of cars arriving at a petrol station until time t etc

- A poisson process is a stoch. process N_t that satisfies the following properties:
  - (i) $N_0 = 0$; It starts at the origin
  - (ii) It has stationary & indpdt increments
  - (iii) It is right continuous in time, ie, it can never confer -ve values (is right skewed)

Δt is discrete change
dt is continuous change

λ and $t-s$

(iv) The increments $N_t - N_s$, $0 \le s \le t$ follows a poisson $\sim Poisson(\lambda(t-s))$ $\Rightarrow P_r(N_t - N_s) = k = \frac{\lambda^k (t-s)^k}{k!} e^{-\lambda(t-s)}$

(v) N_t is an adapted process

This condition is sometimes replaced by the condition
$P_r(N_t - N_s = 1) = \lambda(t-s) + o(t-s)$

and

(*) when $t-s$ is very small we tend to zero, ie, $O(t-s)$

$$P_r(N_t - N_s \geq 2) = o(t-s)$$

→ prob of observing 2 or more jumps, eg in a sec, is zero

- The prob that a jump of size 1 occurs in an $\underbrace{\text{infinitesimal interval}}_{\text{very small interval}}$ dt is λdt ie, $\lambda(t-s)$
and
- The prob that at least 2 events occur in an infinitesimal interval is zero
- That is, dN_t may take values zero and 1 and hence:

$$P_r(dN_t = 1) = \lambda dt$$

$$P_r(dN_t = 0) = 1 - \lambda dt$$

- The parameter λ is called the rate of the process and it is assumed that ō events occur at a certain constant rate λ

Proposition:
(i) N_t is not a martingale
(ii) $Cov(N_s, N_t) = \lambda_s$
(iii) $Corr(N_s, N_t) = \sqrt{\frac{s}{t}}$ $\forall s \le t$

Proof

(i) We need to show that $E[N_t | F_s] \neq N_s \quad \forall s < t$

but

$$E[N_t | F_s] = E[N_t + N_s - N_s | F_s]$$

$$= E[N_s | F_s] + E[N_t - N_s | F_s]$$

$$= N_s + E[N_t - N_s]$$

$$= N_s + \lambda(t-s)$$

$\Rightarrow$ is a submartingale (since $E[N_t | F_s] > N_s$)

(ii) $$Cov(N_s, N_t) = Cov(N_s, N_t + N_s - N_s)$$

$$= Cov(N_s, N_s) + Cov(N_s, N_t - N_s)$$

$$= Cov(N_s, N_s) + Cov(N_s - N_0, N_t - N_s)$$

$$= \underbrace{Var(N_s)}_{= \lambda(s-0)} + \underbrace{Cov(N_s - N_0, N_t - N_s)}_{\text{zero}}$$

$$= \lambda s$$

(iii) $$Corr(N_s, N_t) = \frac{Cov(N_s, N_t)}{\sqrt{Var(N_s) \cdot Var(N_t)}}$$

$$= \frac{\lambda s}{\sqrt{\lambda s \cdot \lambda t}} = \frac{\lambda s}{\sqrt{\lambda^2 s t}} = \frac{\lambda s}{\lambda \sqrt{st}} = \frac{s}{\sqrt{st}}$$

$$= \sqrt{\frac{s^2}{st}} = \sqrt{\frac{s}{t}}$$

$X_t = W_t - tW_1$

μ_t 1

(e) Compensated Poisson Process

- This is a process $M_t = N_t - \lambda t$

Question ③

Show that M_t is a martingale

③ Stopping Times

- Consider ē prob. space $(\Omega, F, \mathbb{P})$ and the filtration $\mathbb{F} = \{F_t\}$ such that $F_s \subset F_t \subset F_T$ for $\forall s < t$. Then assuming a decision to stop playing a game before or at time s is determined by ē info, F_s available at times s.
- This decision can be modelled by a rv $\tau : \Omega \to [0, \infty]$ also known as e stopping time that satisfy the event $\{\omega \in \Omega : \tau(\omega) \leq s\} \in F_s$
- This means that given the info F_s we know whether this event has occurred or not

Note: The possibility that $\tau = \infty$ is included since the decision to continue the game forever is also a possible event

- However, we require the restriction that $P_r(\tau < \infty) = 1$ ie, $\tau : \Omega \to [0, \infty)$

Example

Let F_t be ē info available upto time t regarding the evolution of a stock. Assuming ē price of the stock at time zero is \$50 per share, determine whether the following decisions are stopping times:

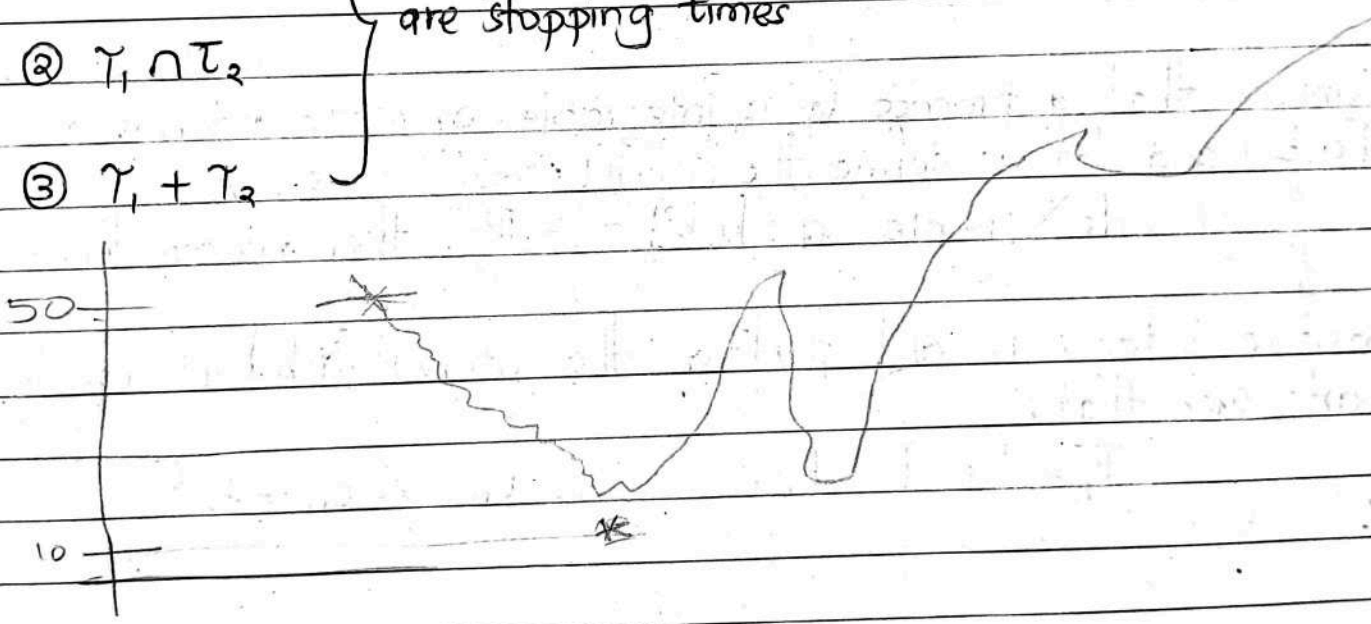
Buy stock after it reaches \$10 per share again and 2nd

α 1) Buy ē stock when it reaches for the 1st time \$10 per share
α 2) Sell ē stock when it reaches the max level it will ever be.
√ 3) Sell the stock at the end of the year
√ 4) ~~Give~~ Keep the stock either until the initial investment doubles or until the end of the year.
α 5) Keep the stock until the initial investment doubles.

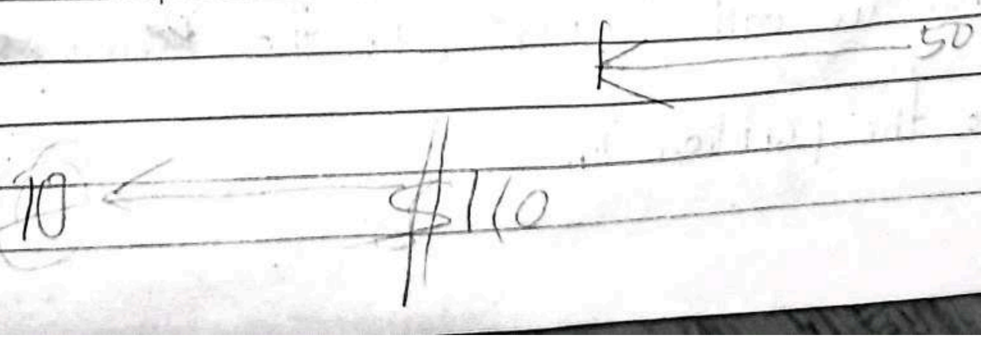
Remark

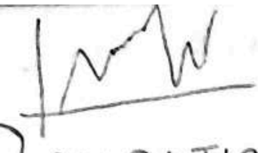
Let τ_1, τ_2 be 2 stopping times, then ;

$$\left.\begin{array}{l} ① \ \tau_1 \cup \tau_2 \\ \\ ② \ \tau_1 \cap \tau_2 \\ \\ ③ \ \tau_1 + \tau_2 \end{array}\right\} \text{are stopping times}$$



$\tau : \Omega \to [0, \infty]$ is called a
if satisfies ~~event~~ $\{\omega \in \Omega : \tau(\omega) \leq s\} \in \mathcal{F}_s$





④ QUADRATIC VARIATIONS AND COVARIATION

- Let X_t be a stoch. process then the quadratic variation, $\langle X \rangle_t$:

$$\langle X \rangle_t = \lim_{n\to\infty} \sum_{i=0}^{n-1} \Big[X(i\Delta t) - X\big((i-1)\Delta t\big) \Big]^2$$

$\hookrightarrow$ or $\sum_{i=1}^{n}$

$\Delta X_t = X_{i\Delta t} - X_{i\Delta t - \Delta t}$ (very small change)

$i\Delta t - \Delta t = (i-1)\Delta t$

and let X_t and Y_t be 2 stoch processes then the covariation is:

$$\langle X, Y \rangle_t = \lim_{n\to\infty} \sum_{i=1}^{n} \underbrace{\Big[X(i\Delta t) - X\big((i-1)\Delta t\big) \Big]}_{\Delta X_i} \underbrace{\Big[Y(i\Delta t) - Y\big((i-1)\Delta t\big) \Big]}_{\Delta Y_i}$$

Riemann Integral

- Suppose that a process g is integrable over the interval a and b $[a,b]$ and let us define the Riemann Integral as $\int_a^b g(s)\,ds$ where $g : [a,b] \to \mathbb{R}$ then we can define a positive integer n and partition the interval $[a,b]$ into equidistant parts such that:

$$\Pi_n = \{ a = t_0 < t_1 < \dots < t_{n-1} < t_n = b \}$$

$$\Delta t = \frac{b-a}{n}$$

- For $i = 1, 2, \dots, n$, $\Delta t_i = t_i - t_{i-1}$ and let $t_i^* \in [t_{i-1}, t_i]$ be distinguished points such that:

$$\tau_n^* = \{ t_1^*, t_2^*, \dots, t_n^* \}$$

$t_i^* : \tau_n^* = \{ t_1^*, t_1^*, \dots, t_n^* \}$ are a set of distinguished pnts

are a set of distinguished pnts.

- Then we call $\sum_{i=1}^{n} g(t_i^*)\Delta t_i$ the Riemann sum of g corresponding to the partition Π_n

$\langle X \rangle_t = \lim_{n\to\infty} \sum_{i=1} [X(i) - X(i-1)]^2$ | $(dW_t)^2 = dt$ $\langle W \rangle_t = t$ | $d\left(\int_0^t (dW_s)^2\right) = dt$ | $\int_0^t (dW_s)^2 = t$

$= \int_a^b (dX_s)^2$ since $\Delta X_i = X_i - X_{i-1}$ and as we move from discrete to cont $\Delta X_i \searrow dX$

- Therefore, if $I = \int_a^b g(s)\,ds < \infty$ exists then $I = \lim_{n\to\infty} \sum_{i=1}^{n} g(i\Delta t)\Delta t$

Friday 13/10/23

Example

Quadratic Variation of a Wiener Process

Show that the Quadratic Variation of a WP over the interval $[0,t]$ is t $\langle W \rangle_t = t$ Recall: $(dW_t)^2 = dt$

$\Delta t = \frac{t-0}{n} = \frac{t}{n}$

Soln

By definition, $\langle W \rangle_t = \lim_{n\to\infty} \sum_{i=1}^{n} \left[W(i\Delta t_i) - W((i-1)\Delta t_i) \right]^2$

but $\Delta t_i = \frac{t-0}{n} = \frac{t}{n}$

$\langle W \rangle_t = \lim_{n\to\infty} \sum_{i=1}^{n} \left[W\left(i\frac{t}{n}\right) - W\left((i-1)\frac{t}{n}\right) \right]^2$ Let this $= S_n$

$= S_n$

Hence we may want to see how S_n behaves when $n\to\infty$. That is, by checking its assymptotic properties (mean & Var ; I think)

$E[S_n] = \lim_{n\to\infty} \sum_{i=1}^{n} E\left[\left\{ W\left(i\frac{t}{n}\right) - W\left((i-1)\frac{t}{n}\right) \right\}^2 \right]$

$\Rightarrow i\frac{t}{n} = t \quad (i-1)\frac{t}{n} = s$

If we let $W\left(i\frac{t}{n}\right) - W\left((i-1)\frac{t}{n}\right) = Y_i$

$E[S_n] = \lim_{n\to\infty} \sum_{i=1}^{n} E[Y_i^2]$

$i\frac{t}{n} - (i-1)\frac{t}{n}$

$= \cancel{i\frac{t}{n}} - \cancel{i\frac{t}{n}} + \frac{t}{n} = \frac{t}{n}$

Recall: $W_t - W_s \sim N(0, t-s)$

$\therefore Y_i \sim N(0, t/n)$

$E(X-\mu)^4 \neq E[X^4]$

$W_t^4 = W_0 + 6\int_0 W_s^2\, ds$

$E[W_t^4] = \int_0^t 6\, E[W_s^2]\, ds$

Let $Y_i = W\left(\frac{it}{n}\right) - W\left((i-1)\frac{t}{n}\right)$

$\Rightarrow Y_i \sim N(0, t/n)$

$$E[S_n] = \lim_{n\to\infty} \sum_{i=1}^{n} E[Y_i^2]$$

since $Var(Y_i) = t/n$
$= E[Y_i^2] - (\underbrace{E[Y_i]}_{0})^2$
$= E[Y_i^2]$
then $E[Y_i^2] = t/n$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{t}{n}$$

$$= \lim_{n\to\infty} \frac{t}{n} \sum_{i=1}^{n} 1$$

$$= \lim_{n\to\infty} \frac{t}{n} \cdot n$$

$$= \lim_{n\to\infty} t$$

$= t \qquad \Rightarrow$ Hence shown.

NO. You must continue onto the variance to fully solidify your proof.

$$Var(S_n) = \lim_{n\to\infty} \sum_{i=1}^{n} Var(Y_i^2)$$

But $Var(Y) = E[Y^2] - (E[Y])^2$

$\Rightarrow Var(Y^2) = E[(Y^2)^2] - (E[Y^2])^2$

$= E[Y^4] - (E[Y^2])^2$

$Y_i \sim N(0, t/n) \longrightarrow Var(Y_i) = E[Y_i^2] - (\underbrace{E[Y_i]}_{0})^2$

$E[Y_i^2] = Var(Y_i) = \frac{t}{n}$

$E[Y_i^4] = 3\left(\frac{t}{n}\right)^2$ how?

$E[Y_i^2] = t/n$ | $3\left(\frac{t}{n}\right)^2 - \frac{t}{n}$

$\frac{t}{n}\left(3\frac{t}{n} - 1\right)$

$$\Rightarrow Var(Y_i^2) = E[Y_i^4] - (t/n)^2$$

$$= 3\left(\frac{t}{n}\right)^2 - \left(\frac{t}{n}\right)^2$$

$$= 2\left(\frac{t}{n}\right)^2$$

$$\therefore Var(S_n) = \lim_{n\to\infty} \sum_{i=1}^{n} 2\left(\frac{t}{n}\right)^2$$

$$= 2 \lim_{n\to\infty} \frac{t^2}{n^2} \sum_{i=1}^{n} 1$$

$$= 2t^2 \lim_{n\to\infty} \frac{1}{n^2} \cdot n$$

$$= 2t^2 \underbrace{\lim_{n\to\infty} \frac{1}{n}}_{0}$$

$$= 0$$

$\Rightarrow E[S_n] = t$ → ⟨deterministic fxn⟩

$Var(S_n) = 0$. A zero variance implies sth is a constant AND the expectation of a constant is itself $\Rightarrow S_n = t$

Question ④ Show that: | Find Covariations:

(i) $dt \cdot dM_t = 0$ → $\langle t, M\rangle_t$

(ii) $dW_t \cdot dM_t = 0$ → $\langle W, M\rangle_t$

(iii) $dt \cdot dW_t = 0$ → $\langle t, W\rangle_t$

$\langle t, M\rangle_t = \lim_{n\to\infty} \sum_{i=1} t(i\Delta t) - t((i-1)\Delta t)$

for the interval $[0, t]$

$W_t \in \mathcal{F}_t$

⑤ STOCHASTIC INTEGRATION

- Consider the Brownian Motion, W_t. Then a process F_t is called non participating if it is indpt of any future increment $W_u - W_t \ \forall \ s < t < u$
- Consequently, the process F_t is indpt of the behaviour of the WP in the future, ie, it can't anticipate the future
- If $\mathcal{F}_t$ denotes the info upto time t, then for any $\mathcal{F}_t$ adapted process, F_t is non-participating.

Itô Integral

- Consider the interval $0 \le a < b$ and let $F_t = f(t, W_t)$ be a non-participating process with ~~the expectation of~~ $E\left[\int_a^b F_t^2 \, dt\right] < \infty$ Then by dividing the interval $[a,b]$

into n equal parts using the partition pnts $a = t_0 < t_1 < \ldots < t_n$ and considering the partial ~~parts~~ sums $S_n = \sum_{i=1}^{n} F_{t_i} (\underbrace{W_{t_{i+1}} - W_{t_i}}_{\text{future increment}})$

- Then since F_t is non-participating, the r.v.s (F_{t_i} and $(W_{t_{i+1}} - W_{t_i})$) are indpt.

Therefore the Itô's integral is the limit of the partial sums S_n as $n \to \infty$

$$m_s = \lim_{n \to \infty} S_n$$

$$= \int_a^b F_t \, dW_t$$

$\to$ is the change in W_t, ie, $W_{t_{i+1}} - W_{t_i}$

$S_n = \sum_{i=1}^{n} F_{t_i} (W_{t_{i+1}} - W_{t_i})$

$E\left[\int f(t,W_t)\,dW_t \cdot \int (t,W_t)\,dW_t\right] = E\int f(t,W_t)\cdot g(t,W_t)\,dt$ $\quad \frac{600}{1200} = \frac{30}{60}$

Existence of an Ito^ Integral

- The Ito^ integral exists if the process F_t satisfies the following properties:

(i) The path $t \longmapsto F_t(\omega)$ is continuous on the interval $[a,b]$ $\forall\, \omega \in \Omega$

(ii) The process F_t is non-participating over the interval $[a,b]$

(iii) $E\left[\int_a^b F_t^2\,dt\right] < \infty$

Properties of Ito^ Integrals

Let ~~F_t~~ $f(t,W_t)$ $g(t,W_t)$ be 2 non-participating process and $C \in \mathbb{R}$ be a constant.

(i) Additivity

$$\int_a^b [f(t,W_t) + g(t,W_t)]\,dW_t$$

$$= \int_a^b f(t,W_t)\,dW_t + \int_a^b g(t,W_t)\,dW_t$$

(ii) ~~Homogenity~~ homogeneity

$$\int_a^b c f(t,W_t)\,dW_t = c\int_a^b f(t,W_t)\,dW_t$$

(iii) Partition

$$\int_s^T f(t,W_t)\,dW_t = \int_s^u f(t,W_t)\,dW_t + \int_u^T f(t,W_t)\,dW_t \quad \forall\ \cancel{a \le s \le T}\ \ s \le u < T$$

(iv) Zero Mean dW_t

$$E\left[\int_a^b f(t, W_t)\right] = 0$$

(v) Isometry

$$E\left[\int_a^b \{f(t, W_t)\, dW_t\}^2\right] = E\left[\int_a^b \{f(t, W_t)\}^2 (dW_t)^2\right]$$

$$= E\left[\int_a^b \{f(t, W_t)\}^2\, dt\right]$$

(*) implying $(dW_t)^2 = dt$

(vi) Covariance

$$E\left[\left\{\int_a^b f(t, W_t)\, dW_t\right\}\left\{\int_a^b g(t, W_t)\, dW_t\right\}\right]$$

$$= E\left[\int_a^b f(t, W_t)\, g(t, W_t)\, (dW_t)^2\right]$$

$$= E\left[\int_a^b f(t, W_t)\, g(t, W_t)\, dt\right]$$

We replace $f(t, W_t)$ with $f(t)$
↳ from Itô integral ∴ when t is known $f(t)$ can be a constant since t is deterministic

Wiener Integral

↳ is a form of an Itô integral

- Is the mean squared limit of partial sums

$$S_n = \sum_{i=1}^{n} f(t_i)\left[W_{t_{i+1}} - W_{t_i}\right]$$

$$\Rightarrow \lim_{n\to\infty} S_n$$

$$= \int_a^b f(t)\, dW_t$$

$$E\left[\int_a^b f(t)\, dW_t\right] = \int_a^b f(t)\, \underbrace{E[dW_t]}_{0} = 0$$

makes sense since it is an Itô integral

$$Var\left[\int_a^b f(t)\, dW_t\right] = E\left[\left\{\int_a^b f(t)\, dW_t\right\}^2\right] - \underbrace{\left(E\left[\int_a^b f(t)\, dWt\right]\right)^2}_{0}$$

$$= E\left[\int_a^b (f(t))^2\, dt\right]$$

$$= \int_a^b (f(t))^2\, dt$$

since it is a constant ↓ coz of no W_t

Wiener Integral is the mean squared limit of the partial sums $S_n = \int_a^b f(t)\left[W_{t_{i+1}} - W_{t_i}\right] = \int_a^b f(t)\, dW_t$

$$E\left[\int_a^b f(t)\, dW_t\right] = \int_a^b f(t)\, \underbrace{E[dW_t]}_{0} = 0$$

$$Var\left(\int_a^b f(t)\, dW_t\right) = E\left[\int_a^b \left(f(t)\, dW_t\right)^2\right] = E\left[\int_a^b (f(t))^2\, dt\right]$$

$$= \int_a^b (f(t))^2\, dt$$

⑥ STOCHASTIC DIFFERENTIATION

- Most stochastic process are not differentiable and hence derivatives such as $\frac{dW_t}{\partial t}$ do not make sense in stochastic calc
- However, the only quantities allowed to be used are the infinitesim changes of the process such as dW_t

Definitions:

Infinitesimal change of a process;

→ The change in the process X_t btwn time t and $t+\Delta t$ is given by $X_{t+\Delta t} - X_t$ (which is $= \Delta X_t$) and when Δt is infinitesimally small, we obtain the infinitesimal change of process X_t as $dX_t = X_{t+dt} - X_t$

Basic rules of Stochastic Differentiation

I) Constant Multiple Rule

Let X_t be a stochastic process and c be a constant, then
$d(cX_t) = c\,dX_t$

II) Sum Rule

Let X_t and Y_t be 2 stochastic process, then
$d(X_t + Y_t) = dX_t + dY_t$

III) Difference Rule

Let X_t and Y_t be 2 stochastic processes, then
$d(X_t - Y_t) = dX_t - dY_t$

$$d\left(\frac{X_t}{Y_t}\right) = \frac{Y_t\,dX_t - X_t\,dY_t - dX_t \cdot dY_t}{Y_t^2} + \frac{X_t}{Y_t^3}(dY_t)^2$$

iv) Product Rule

Let X_t & Y_t be 2 stoch processes, then

$$d(X_t Y_t) = X_t\,dY_t + Y_t\,dX_t + \underbrace{dX_t \cdot dY_t}$$

the covariation is what we are adding from the basic calculus of $d(xy) = x\,dy + y\,dx$

v) Quotient Rule

Let X_t & Y_t be 2 stoch processes, then

$$d\left(\frac{X_t}{Y_t}\right) = \frac{Y_t\,dX_t - X_t\,dY_t - dX_t \cdot dY_t}{Y_t^2} + \underbrace{\frac{X_t}{Y_t^3}}_{\text{Quad. Varation btwn } X_t \text{ \& } Y_t}(dY_t)^2$$

provided $Y_t \neq 0 \ \forall\, t \geq 0$

Example

$$d(W_t \cdot W_t) = W_t\,dW_t + W_t\,dW_t + dW_t \cdot dW_t$$
$$= 2W_t\,dW_t + dt$$

Using basic rules

(i) $d(W_t^2) = 2W_t\,dW_t + dt$

(ii) $d(tW_t) = W_t\,dt + t\,dW_t$

Sln

$$t\,dW_t + W_t\,dt + \underbrace{dt\,dW_t}_{\text{zero}}$$

(i) $d(W_t^2) = d(W_t \cdot W_t)$

$$= W_t\,dW_t + W_t\,dW_t + (dW_t)^2$$

$$= 2W_t\,dW_t + \underbrace{dt}_{\text{from quad variation of a WP}}$$

(ii) $d(tW_t) = t\,dW_t + W_t\,dt + \underbrace{dt\,dW_t}_{0}$

$$= t\,dW_t + W_t\,dt$$

NOTE: $dt\,dW_t = 0$

$C^{1,2}$ once differentiable with t & twice differentiable with x
$f(t,x)$

(*) diffussion must not be equal to zero

Itô's Formula

- Assume that the process X_t has a stoch. eqn given by

$$dX_t = \underbrace{a(t,W_t)dt}_{\text{Drift Term}} + \underbrace{b(t,W_t)\,dW_t}_{\text{Diffussion Term}}$$

(SDE) stoch. differential eqn

(*) R package sde

where a and b are adapted processes.

- Let f be a $C^{1,2}$ fxn then we can define a new process Z $Z = f(t,X_t)$ that has the SDE given by:

$$df(t,X_t) = \left[\frac{\partial f}{\partial t} + a(t,W_t)\frac{\partial f}{\partial x} + \frac{1}{2}b^2(t,W_t)\frac{\partial^2 f}{\partial x^2}\right]dt + b(t,W_t)\frac{\partial f}{\partial x}\,dW_t$$

with the multiplication table given by
$$\begin{cases}(dW_t)^2 = dt\\ dt * dW_t = 0\\ (dt)^2 = 0\end{cases}$$

OR;
$$df(t,X_t) = \frac{\partial f}{\partial t}\cdot dt + \frac{\partial f}{\partial x}\cdot dX_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(dX_t)^2$$

Example

Using Itô's Formula find:

(i) $d(\sin W_t)$

Sln

$X_t = W_t$

$\therefore dX_t = dW_t$

∂t $\quad \partial x$ $\quad 2\ \partial x^2$

$$\text{Let } f(t,x) = \sin x, \quad \frac{\partial f}{\partial t} = 0, \quad \frac{\partial f}{\partial x} = \cos x, \quad \frac{\partial^2 f}{\partial x^2} = -\sin x$$

$$d(\sin X_t) = 0\,dt + \cos x_t\, dX_t + \frac{1}{2}(-\sin x_t)(dX_t)^2$$

Let $X_t = W_t \Rightarrow dX_t = dW_t$

$$d(\sin W_t) = \cos W_t\, dW_t - \frac{1}{2}\sin W_t\,(dW_t)^2$$

$$= \cos W_t\, dW_t - \frac{1}{2}\sin W_t\, dt$$

(ii) $d(W_t^2)$

<u>Solution</u>

Let $X_t = W_t \Rightarrow dX_t = dW_t$

Let $f(t,x) = x^2 \Rightarrow f_t = 0, \ f_x = 2x, \ f_{xx} = 2$

$$d(X_t^2) = 0\,dt + 2X_t\, dX_t + \frac{1}{2}(2)(dX_t)^2$$

$$= 2X_t\, dX_t + dt$$

$$\Rightarrow d(W_t^2) = 2W_t\, dW_t + dt$$

(iii) Find an expression for

(a) $\int_0^t \sin W_s \, dW_s$

<u>Solution</u>

Since $\int \sin x \, dx = -\cos x$

Let $f(t,x) = -\cos x$, $f_t = 0$, $f_x = \sin x$, $f_{xx} = \cos x$

$$d(-\cos X_t) = 0dt + \sin X_t \, dX_t + \frac{1}{2}\cos X_t (dX_t)^2$$

$$-d(\cos X_t) = \sin X_t \, dX_t + \frac{1}{2}\cos X_t (dX_t)^2$$

then, $-d(\cos W_t) = \sin W_t \, dW_t + \frac{1}{2}\cos W_t (dW_t)^2$

Integrating over $[0,t]$

$$-\cancel{\int_0^t d}\cos W_s = \int_0^t \sin W_s \, dW_s + \frac{1}{2}\int_0^t \cos W_s \, ds$$

$-\cos W_t + \underbrace{\cos W_0}_{\cos 0 = 1}$

$$-\cos W_s \Big|_0^t = \int_0^t \sin W_s \, dW_s + \frac{1}{2}\int_0^t \cos W_s \, ds$$

$$1 - \cos W_t = \int_0^t \sin W_s \, dW_s + \frac{1}{2}\int_0^t \cos W_s \, ds$$

$$\int_0^t \sin W_s \, dW_s = 1 - \cos W_t - \frac{1}{2}\int_0^t \cos W_s \, ds$$

(iv) (b) $\int_0^t W_t \, dW_t$

Solution

$\int x\,dx = \frac{x^2}{2}$

Let $f(t,x) = \frac{x^2}{2}$, $f_t = 0$, $f_x = \frac{2x}{2} = x$, $f_{xx} = 1$

$\Rightarrow d\left(\frac{X_t^2}{2}\right) = 0\,dt + X_t\,dX_t + \frac{1}{2}(1)(dX_t)^2$

Let $X_t = W_t \Rightarrow dX_t = dW_t$

$$d\left(\frac{W_t^2}{2}\right) = W_t\,dW_t + \frac{1}{2}(dW_t)^2$$

$$= W_t\,dW_t + \frac{1}{2}dt$$

Integrating over $[0,t]$

$$\int_0^t d\left(\frac{W_s^2}{2}\right) = \int_0^t W_s\,dW_s + \frac{1}{2}\int_0^t ds$$

$$\frac{W_s^2}{2}\Big|_0^t = \int_0^t W_s\,dW_s + \frac{1}{2}s\Big|_0^t$$

$$\frac{W_t^2}{2} - \frac{W_0^2}{2} = \int_0^t W_s\,dW_s + \frac{1}{2}(t-0)$$

$$\Rightarrow \int_0^t W_s\,dW_s = \frac{W_t^2}{2} - \frac{1}{2}t$$

$$E[X_t] = X_s = \int_0^s V_u) dW_u$$

Question (5)

Show that:

(i) $X_t = \int_0^t V(u)\, dW_u$ is a martingale.

is not a new formula

(ii) If X_t is given as defined in (i) above, show that $M_t = X_t^2 - \frac{t^3}{3}$ is a martingale

is a new formula so you can invoke Itô's formula

Solution

(i) $E[X_t | \mathcal{F}_s] = X_s \quad \forall\, 0 \le s \le t$ for it to be a martingale

$$\Rightarrow E[X_t|\mathcal{F}_s] = E\left[\int_0^t V_u\, dW_u \,\Big|\, \mathcal{F}_s\right]$$

$$= \underbrace{E\left[\int_0^s V_u\, dW_u \,\Big|\, \mathcal{F}_s\right]}_{\text{it is known (certain)}} + E\left[\int_s^t V_u\, dW_u \,\Big|\, \mathcal{F}_s\right]$$

$$= \int_0^s V_u\, dW_u + \cancel{E\left[\int_s^t V_u\, dW_u\right]}^{\,0}$$

why is it equal to zero?? Because of zero mean property of an Itô integral

$$= \int_0^s V_u\, dW_u = X_s$$

(ii) $E[M_t | \mathcal{F}_s] = M_s$ where $M_s = X_s^3 - \frac{t^3}{3}$

Hint

$$dX_t = d\int_0^t V_u\, dW_u = V_u\, dW_u \Big|_0^t$$

$$\Rightarrow dX_t = V_t\, dW_t$$

$$M_t = X_t^2 - \frac{t^3}{3} \rightarrow f(t, x)$$

Integrate from $[s,t]$

$f(t,X) = X_t^2 - \frac{t^3}{3}$; $f_t = -t^2$, $f_x = 2X_t$, $f_{xx} = 2$

$d(M_t) = -t^2 dt + 2X_t \, dX_t + \frac{1}{\cancel{2}} \cdot \cancel{2}^{\,1} (dX_t)^2$

Integrate from $[s,t]$ → (Replace X_t with W_t first) → Or don't

$\int_s^t d(M_t) = -\int_s^t v^2 dv + 2\int_s^t W_v dW_v + \int_s^t dv$

$W_v^2 - \frac{v^3}{3} \Big|_s^t$

$M_t - M_s = -\frac{1}{3} \cdot v^3 \Big|_s^t + \frac{\cancel{2}}{\cancel{2}} W_v^2 \Big|_s^t + v \Big|_s^t$

$M_t - M_s = -\frac{1}{3}(t^3 - s^3) + \cancel{W_t^2} - \cancel{W_s^2}^{\,0} + t - s$

$M_t - M_s = -\frac{1}{3} t^3 + \frac{s^3}{3} + t - s$

$t - s = 0$!! Tears

$M_t = M_s - \frac{2}{3} s^3 + \frac{2}{3} t^3$

$\boxed{\int_s^t V(u)\,du \qquad -\frac{2s^3}{3} + \frac{t^3}{3}}$

$f(t,x,y,z) = c^{1,2,2,2}$

Fri
27/10/23

Multi-Dimensional Itô's Formula

- Consider a vector $\underline{X} = (X_1, X_2, ---, X_n)$ where the component X_i has an SDE given by:

$$dX_i(t) = \mu_i(t)dt + \sum_{j=1}^{n} \sigma_{ij}\, dW_j(t) \quad ; \; i = 1, 2, ---$$

and $\underline{W}$ is a vector containing Wiener processes given by:

$$\underline{W} = (W_1, W_2, ---, W_n)$$

- If we define drift vector, $\mu = (\mu_1; \mu_2; ---; \mu_n)$ and the $n \times d$ diffusion matrix σ to be: n by d

$$\sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} & --- & \sigma_{1d} \\ \sigma_{21} & \sigma_{22} & --- & \sigma_{2d} \\ | & | & & | \\ | & | & --- & | \\ \sigma_{n1} & \sigma_{n2} & --- & \sigma_{nd} \end{bmatrix}$$

where the X dynamics can be expressed as:

$$dX_i(t) = \mu_i(t)dt + \sigma_i(t)\, dW_i(t) \quad ; \; i = 1, 2, ---, n$$

then we can define a process: $Z_t = f(t, \underline{X}_t)$ where $\underline{X} = (X_1, X_2, -$
where f is a $C^{1,2}$ fxn. 2 is vector dependent on how many variables we have shown top page

(i) INDEPENDENT WIENER PROCESS

- The sd of z for indpt wiener processes will be given by:

$$df(t, \underline{X}_t) = dZ_t$$

$$= \left(\frac{\partial f}{\partial t} + \sum_{i=1}^{n} \mu_i \frac{\partial f}{\partial x_i} + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} \frac{\partial^2 f}{\partial x_i \partial x_j} \right) dt + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dW_i$$

OR;

$$df(t, \underline{X}_t) = dZ_t$$

$$= \frac{\partial f}{\partial t} . dt + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} . dX_i + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j} (dX_i \cdot dX_j)$$

with the formal multiplication table:

$\langle t \rangle$

$$(dt)^2 = 0$$
$$dt \cdot dW_t = 0$$
$$dW_i \cdot dW_j = 0 \quad \forall\ i \neq j \quad \text{independent} \qquad \sigma_{ij}\ dt$$
$$(dW_i)^2 = dt$$

(ii) CORRELATED WIENER PROCESS

$$df(t, \underline{X}_t) = dZ_t$$

$$= \left(\frac{\partial f}{\partial t} + \sum_{i=1}^{n} \mu_i \frac{\partial f}{\partial x_i} + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_i \sigma_j \rho_{ij} \frac{\partial^2 f}{\partial x_i \partial x_j} \right) dt + \sum_{i=1}^{n} \sigma_i \frac{\partial f}{\partial x_i} dW_i$$

OR;

$$df(t, \underline{X}_t) = dZ_t$$

Same as the one for indpt Wiener process

$$= \frac{\partial f}{\partial t} . dt + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} . dX_i + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j} d(X_i \cdot dX_j)$$

$$df(t, \underline{x}_t) = \frac{\partial f}{\partial t} \cdot dt + \sum_{i=1} \frac{\partial f}{\partial x_i} dx_i + \frac{1}{2} \sum_{i=1} \sum_{j=1} \frac{\partial^2 f}{\partial x_i \partial x_j} (dx_i \cdot dx_j)$$

$$\frac{1}{2} \sum_{i=1}^{2} \left(\frac{\partial^2 f}{\partial x_i \partial x_1} dx_i dx_1 + \frac{\partial^2 f}{\partial x_i \partial x_2} dx_i dx_2 \right)$$

with the formal multiplication table:

$$= \frac{1}{2} \left[\frac{\partial^2 f}{\partial x_1 \partial x_1} (dx_1)^2 + \frac{\partial^2 f}{\partial x_2 \partial x_1} (dx_1 dx_2) + \frac{\partial^2 f}{\partial x_1 \partial x_2} (dx_1 dx_2) + \frac{\partial^2 f}{\partial x_2 \partial x_2} (dx_2)^2 \right]$$

$$(dt)^2 = 0$$
$$dt \cdot dW_t = 0$$
$$dW_i \cdot dW_j = \rho_{ij}\, dt \quad \forall\ i \neq j$$
$$(dW_i)^2 = dt$$

Example

Using Itô's Formula, prove the product rule:

$$d(X_t Y_t) = X_t\, dY_t + Y_t\, dX_t + (dX_t \cdot dY_t)$$

Proof

To apply the formula $X = X_1$ and $Y = X_2$

Let $f(t, x, y) = xy$

$$f_t = 0,\ \underbrace{f_x}_{\frac{\partial f}{\partial x_1}} = y,\ \underbrace{f_{xx}}_{\frac{\partial^2 f}{\partial x_1 \partial x_1}} = 0,\ \underbrace{f_y}_{\frac{\partial f}{\partial x_2}} = x,\ \underbrace{f_{yy}}_{\frac{\partial^2 f}{\partial x_2 \partial x_2}} = 0$$

$$\underbrace{f_{xy}}_{\frac{\partial^2 f}{\partial x_1 \partial x_2}} = 1,\ \underbrace{f_{yx}}_{\frac{\partial^2 f}{\partial x_2 x_1}} = 1 \Rightarrow \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

Since this: $= 2 \frac{\partial^2 f}{\partial x \partial y}$

$$d(\underset{x_1}{X_t} \underset{x_2}{Y_t}) = 0\, dt + \frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2 + \frac{1(2)}{2} \frac{\partial^2 f}{\partial x_1 \partial x_2} \text{----}$$

$$= 0\, dt + Y dX + X dY + \frac{1}{2} \cdot 2\, dX \cdot dY$$

$$= Y_t\, dX_t + X\, dY_t + dX_t \cdot dY_t$$

Hence proven

(*) Try Quotient Rule using this Itô's Formula

> Integration Technique
> Exact Solution
> Inspection
> Method of Variation

Methods of Solving Stochastic Differential Equations (SDEs)

- Let X_t be a continuous stochastic process, then the SDE of X_t is:

$$dX_t = \underbrace{a(t, X_t, W_t)dt}_{\text{Drift term}} + \underbrace{b(t, X_t, W_t)dW_t}_{\text{Diffusion term}}$$

(i) INTEGRATION TECHNIQUE

This technique is suitable for solving SDEs of the form:

$$dX_t = a(t, W_t)dt + b(t, W_t)dW_t$$

Example

Solve for:

(i) $dX_t = a(t)dt + b(t)dW_t$

Sln

Integrating from $[0,t]$;

$$\int_0^t dX_s = \int_0^t a(s)ds + \int_0^t b(s)dW_s$$

$$X_t = X_0 + \int_0^t a(s)ds + \int_0^t b(s)dW_s$$

(ii) $dX_t = dt + W_t dW_t \quad ; \quad X_0 = 1$

Sln

Integrating from $[0,t]$,

$$\int_0^t dX_s = \int_0^t ds + \int_0^t W_s dW_s$$

$$X_t - X_0 = (t - 0) + \int_0^t W dW_s$$

But $X_0 = 1$

$\therefore$

$$X_t = 1 + t + \int_0^t W_s dW_s$$

Evaluating $\int_0^t W_s dW_s$ gives:

$$\int x ds = \frac{x^2}{2}$$

Let $f(t,x) = \frac{x^2}{2} \Rightarrow f_t = 0,\ f_x = \frac{2x}{2} = x,\ f_{xx} = 1$

$$d\left(\frac{x^2}{2}\right) = 0dt + xdx + \frac{1}{2}(1)(dx)^2$$

Let $X = W \Rightarrow dX = dW$

$$d\left(\frac{W_t^2}{2}\right) = \cancel{0dt} + W_t dW_t + \frac{1}{2}dt$$

Integrating from $[0,t]$;

$$\int_0^t d\left(\frac{W_s^2}{2}\right) = \int_0^t W_s dW_s + \frac{1}{2}\int_0^t d_s$$

$$\frac{W_t^2}{2} - \frac{W_0^2}{2} = \int_0^t W_s dW_s + \frac{1}{2}t$$

But $W_0 = 0 \Rightarrow \int_0^t W_s dW_s = \frac{W_t^2}{2} - \frac{t}{2}$

$$\Rightarrow X_t = 1 + t + \left(\frac{W_t^2}{2} - \frac{t}{2}\right)$$

$$= 1 + \frac{t}{2} + \frac{W_t^2}{2}$$

$$\therefore X_t = 1 + \frac{1}{2}(W_t^2 + t)$$

$$X_1 = \underline{1} + \frac{1}{2}(W_1^2 + 1)$$ ⊛ if you know value of W_1 then you can forecast values of X_t eg; X_{100}

QUESTION

$$dX_t = (W_t - 1)dt + W_t^2 dW_t \quad ; X_0 = 0$$

Integrating from $[0,t]$

$$\int_0^t dX_t = \int_0^t W_s ds - \int_0^t ds + \int_0^t W_s^2 dW_s$$

$$X_t = sW_s\Big|_0^t - s\Big|_0^t + \int_0^t W_s^2 \, dW_s$$

$$X_t = tW_t - t + \int_0^t W_s^2 dW_s$$

Solving for $\int_0^t W_s^2 dW_s$

$$\int x^2 dx = \frac{x^3}{3}$$

$$f(t,x) = \frac{x^3}{3} \quad ; f_t = 0 \, ; f_x = \frac{3x^2}{3} = x^2$$
$$f_{xx} = 2x$$

$$d\left(\frac{X^3}{3}\right) = 0dt + x^2 dx + \frac{1}{2} 2x(dx)^2$$

Replacing x with W_t

$$d\left(\frac{W_t^3}{3}\right) = W_t^2 dW_t + W_t(dW_t)^2$$

$$d\left(\frac{W_t^3}{3}\right) = W_t^2 dW_t + W_t dt$$

Integrating from $[0,t]$

$$\int_0^t d\left(\frac{W_s^3}{3}\right) = \int_0^t W_s^2 dW_s + \int_0^t W_s ds$$

$$\frac{W_t^3}{3} = \int_0^t W_s^2 dW_s + sW_s\Big|_0^t$$

$$\int_0^t W_s^2 dW_s = \frac{W_t^3}{3} - tW_t$$

$$\therefore X_t = tW_t - t + \frac{W_t^3}{3} - tW_t$$

$$X_t = \frac{1}{3}W_t^3 - t$$

(ii) EXACT SOLUTION TECHNIQUE

- The SDE given by: (no fxn of X)

$dX_t = a(t, W_t)dt + b(t, W_t)\,dW_t$ is called exact if there is a differentiable fxn $f(t,x)$ where:

$$a(\cancel{f}t, x) = \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}$$

$$b(t,x) = \frac{\partial f}{\partial x}$$

- The sln of an Exact SDE is given by:

$$X_t = f(t, W_t) + c$$

Example

1. Solve for the SDE given by: ~~$dX_t = e^t(1+W_t^2)dt$~~

$dX_t = e^t(1+W_t^2)dt + (1+2e^tW_t)dW_t$ using exact sln technique

Sln

Let $x = W$;

$$a(t,x) = e^t(1+x^2) \quad \text{and} \quad b(t,x) = 1+2e^tx$$

$$\text{Let } b(t,x) = \frac{\partial f}{\partial x} = 1+2e^tx$$

$$\frac{\partial^2 f}{\partial x^2} = 2e^t$$

$$f(t,x) = \int \frac{\partial f}{\partial x} = \int (1 + 2e^t x)\, dx$$

a constant

$$= x + \frac{2e^t x^2}{2} + T(t)$$

$$\Rightarrow f = x + x^2 e^t + T(t)$$

$$\frac{\partial f}{\partial t} = x^2 e^t + T'(t)$$

$$\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}$$

$$a(t,x) = e^t(1 + x^2) = e^t + x^2 e^t$$

which is $= x^2 e^t + T'(t) + \frac{1}{2}(2e^t)$

$$= x^2 e^t + e^t + T'(t)$$

If $T'(t) = 0$ ~~then~~ $\Rightarrow$ $T(t) =$ a constant, c

$$\Rightarrow f(t,x) = x + \cancel{2}x^2 e^t + c$$

Replacing x with W_t ;

$$\therefore X_t = W_t + W_t^2 e^t + c$$

$$\underbrace{X_0 = W_0 + W_0^2 e^0 + c = 0}_{\text{Implies } X_0 = 0} \Rightarrow c = 0$$

$$\Rightarrow X_t = W_t + W_t^2 e^t$$

2. Solve for $dX_t = dt + W_t dW_t$ using the exact sln technique
$X_0 = 1$

Sln; $a(t,x) = 1$ and $b(t,x) = x = \frac{\partial f}{\partial x}$

$\frac{\partial^2 f}{\partial x^2} = 1$

$f(t,x) = \int \frac{\partial f}{\partial x} = \int x \, dx = \frac{x^2}{2} + T(t)$

$\Rightarrow \frac{\partial f}{\partial t} = T'(t)$

$\therefore a(t,x) = 1$

$= \frac{\partial f}{\partial t} + \frac{1}{2} \frac{\partial^2 f}{\partial x^2}$

$= T'(t) + \frac{1}{2}(1)$

$= T'(t) + 1/2 = 1$

$\therefore T'(t) = 1 - 1/2 = 1/2$

$\Rightarrow T(t) = \int \frac{1}{2} dt = \frac{t}{2}$

$\Rightarrow f(t,x) = \frac{x^2}{2} + \frac{t}{2}$

$X_t = f(t, W_t) + c = \frac{W_t^2}{2} + \frac{t}{2} + c$

but $X_0 = 1$

$$\Rightarrow X_0 = \frac{W_0^2}{2} + \frac{0}{2} + c = 1$$

$$\Rightarrow c = 1 \quad \Rightarrow X_t = \frac{W_t^2}{2} + \frac{t}{2} + 1$$

Question

$$dX_t = (W_t - 1)dt + W_t^2 dW_t$$

$X_0 = 0$

$a(t, W_t) = \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}$

$b(t, W_t) = \frac{\partial f}{\partial x}$

using exact sln technique

$a(t, x) = x - 1$ & $b(t, X) = x^2$

$\frac{\partial f}{\partial x} = x^2$; $\frac{\partial^2 f}{\partial x^2} = 2x$

$f(t, x) = \int \frac{\partial f}{\partial x} = \int x^2 dx = \frac{x^3}{3} + T(t)$

$\frac{\partial f}{\partial t} = T'(t)$; $a(t, x) = x - 1$

$= T'(t) + \frac{1}{2} \cdot 2x$

$= T'(t) + x$

If $T'(t) = -1$ then $T(t) = \int -1 dt = -t$

$\Rightarrow f(t, x) = \frac{x^3}{3} - t$

$x_t = f(t, W_t) - t + c$

$= \frac{W_t^3}{3} - t + c$

$X_0 = 0$

$= \frac{W_0^3}{3} - 0 + c = 0$

$c = 0$

Hence $X_t = \frac{1}{3} W_t^3 - t$

Correct

<same as integration technique>

Rat Question

(i) State whether or not the SDE
$dX_t = (1+W_t^2)dt + (t^4+W_t^2)dW_t$ is exact or not

(ii) Find the sln of X_t

$a(t,x) = 1 + X^2$

It is not exact

$b(t,x) = t^4 + x^2 = \frac{\partial f}{\partial x}$

Integrating from $[0,t]$,

$$\int_0^t dX_s = \int_0^t (1+W_s^2)ds + \int_0^t (s^4 + W_s^2)dW_s$$

$$X_t - X_0 = \int_0^t ds + \int_0^t W_s^2 ds + \int_0^t s^4 dW_s + \int_0^t W_s^2 dW_s$$

Evaluating $\int_0^t W_s^2 dW_s$ gives

$f(t,x) = \frac{x^3}{3}$; $f_t = 0$, $f_x = \frac{3x^2}{3} = x^2$, $f_{xx} = 2x$

$$d\left(\frac{X^3}{3}\right) = 0dt + x^2 dx + \frac{1}{2} \cdot 2x(dx)^2$$

Replacing X with W

$$d\left(\frac{W_t^3}{3}\right) = W_t^2 dW_t + W_t dt$$

Integrating from $[0,t]$

$$\int_0^t d\left(\frac{W_s^3}{3}\right) = \int_0^t W_s^2 dW_s + \int_0^t W_s ds$$

$$\frac{W_t^3}{3} = \int_0^t W_s^2 dW_s + \int_0^t W_s ds$$

$$\Rightarrow \int_0^t W_s^2\,dW_s = \frac{W_t^3}{3} - \int_0^t W_s\,ds$$

$$\Rightarrow X_t - X_0 = t + \int_0^t W_s^2\,ds + \int_0^t s^4\,dW_s + \frac{W_t^3}{3} - \int_0^t W_s\,ds$$

$$= t + sW_s^2\Big|_0^t + \int_0^t s^4\,dW_s + \frac{W_t^3}{3} - sW_s\Big|_0^t$$

$$= t + tW_t^2 + \int_0^t s^4\,dW_s + \frac{W_t^3}{3} - tW_t$$

$$\boxed{X_t = X_0 + t + tW_t^2 - tW_t + \frac{W_t^3}{3} + \int_0^t s^4\,dW_s}$$

$$\cancel{\int_0^t s^4\,dW_s}$$

$$\cancel{f(t,x) = \frac{t^5}{5}}$$

$$\cancel{f_t = \frac{5t^4}{5} = t^4,\ f_x = 0,\ f_{xx} = 0}$$

$$\cancel{d\left(\frac{t^5}{5}\right) = t^4\,dt}$$

~~Integrating to $[0,t]$,~~

$$\cancel{\int_0^t d\left(\frac{s^5}{5}\right) = \int_0^t s^4\,ds}$$

$$\cancel{\frac{t^5}{5} = \frac{t^5}{5} - 0}$$

$$X_t = X_0 + t + tW_t^2 - tW_t + \frac{W_t^3}{3} + t^4W_t$$

yuima → R package

Monday 30/10/20..

(III) ~~Stochastic~~ Integration By Inspection Technique

- Under this technique, you use the integration technique where you take opportunities to apply product or quotient rule formulas such that,

$$d(f(t)\cdot X_t) = f(t)\,dX_t + X_t\,df(t)$$

⊛ product b/t'n deterministic & stochastic term does not have covariation ie; $dt \cdot dW_t$

$$d\left(\frac{X_t}{f(t)}\right) = \frac{f(t)\,dX_t - X_t\,df(t)}{(f(t))^2}$$

- Where ~~where~~ $f(t)$ is a deterministic process and X_t is a Stochastic process

Example

1. $\begin{cases} dX_t = (W_t + 3t^2)dt + t\,dW_t \\ X_0 = a \end{cases}$

Sol

$$dX_t = W_t\,dt + 3t^2dt + t\,dW_t$$

$$= \underbrace{d(tW_t)}_{t\,dW_t + W_t\,dt} + \underline{3t^2\,dt}$$

$$\boxed{3t^2\,dt = d(t^3)}$$

$$= d(tW_t) + d(t^3)$$

$$= d(tW_t + t^3)$$

Integrating from $[0,t]$

$$\int_0^t x\,ds = \int_0^t d(s\,dW_s + s^3)$$

$$X_t = X_0 + (sW_s + s^3)\Big|_0^t$$

the ~~technique~~ technique is ~~ont~~ used to reduce computational time

$\Rightarrow X_t = a + tW_t + t^3$

$2 \begin{cases} dX_t = t\,dt + W_t^2\,dt + 2t\,W_t\,dW_t \\ X_0 = a \end{cases}$

<u>Trial:</u>

$dX_t = t\,dt + 2d\left(t\,W_t^2\right)$

<u>Sln</u>

to demonstrate $W_t = x$

$\underline{x^2\,\underline{dt}} \qquad 2t\,x\,dx$

$= \underline{t}\ \overline{2x\,dx}$

dt is the derivative of t

and $2x\,dx$ is the " of x^2

$dX_t = t\,dt + d\left(t \cdot W_t^2\right)$

Integrating from $[0,t]$

$$\int_0^t dX_s = \int_0^t s\,ds + \int_0^t d\left(sW_s^2\right)$$

$$\Rightarrow X_t = X_0 + \frac{s^2}{2}\Big|_0^t + sW_s^2\Big|_0^t$$

$$= a + \frac{t^2}{2} + tW_t^2$$

(*) Try Integration Tech from beginning

$\int_0^t sW_s\,dW_s$ Let $X = W_t$, $dX = dW_t$ $\quad \int s\,x\,dx = t.\frac{x^2}{2}$

$t.\frac{x^2}{2} = f(t,x) \;;\; f_t = \frac{x^2}{2} \;;\; f_x = xt \;;\; f_{xx} = t$

$d\left(t \cdot \frac{x^2}{2}\right) = \frac{x^2}{2}\,dt + xt\,dx + \frac{1}{2}t(dx)^2$

Replacing x with W_t

$d\left(t.\frac{W_t^2}{2}\right) = \frac{W_t^2}{2}\,dt + tW_t\,dW_t + \frac{1}{2}t\,dt$ $\Big\}$ Then integrate over $[0,t]$

(*) if you don't ha[ve] the drift te[rm] you can still u[se] ~~the method~~.

(IV) METHOD OF VARIATION OF PARAMETERS

- Is used to solve SDEs of the form:

$$dX_t = a(t, X_t, W_t)dt + \underbrace{g(t, W_t)}_{\text{participating process}} X_t \, dW_t$$

Example

1. Consider an SDE of the form:

$$dX_t = \alpha X_t dW_t$$ no dt, so it is driftless

Sln $\quad g(t, W_t) = \alpha$ $\quad \ln(f(x))$ $\quad \frac{f'(x)}{f(x)}$

$$\frac{dX_t}{X_t} = \alpha dW_t \qquad \ln(X_t) \quad = \frac{1}{X_t} dX_t$$

$$d \ln X_t = \alpha dW_t$$

Taking $f(t,x) = \ln x$; $f_t = 0$, $f_x = \frac{1}{x}$, $f_{xx} = \frac{-1}{x^2}$

$$df(t,x) = d\ln X_t$$

$$= 0dt + \frac{1}{x_t} dX_t + \frac{1}{2}\left(-\frac{1}{X_t^2}\right)(dX_t)^2$$

$dX_t = \alpha X_t dW_t$

$(dX_t)^2 = \alpha^2 X_t^2 dt$

$$d\ln X_t = \frac{1}{X_t}(\alpha X_t dW_t) - \frac{1}{2}\left(\frac{1}{X_t^2}\right)\left(\alpha^2 X_t^2 dt\right)$$

$$= \alpha dW_t - \frac{1}{2}\alpha^2 dt$$

By Integration Technique

Integrating over $[0,t]$

$$\int_0^t d\ln X_s = \alpha \int_0^t dW_s - \frac{1}{2}\int_0^t \alpha^2 ds$$

$$\underbrace{\ln X_t - \ln X_0}_{\ln\left(\frac{X_t}{X_0}\right)} = \alpha\left(W_t - W_0\right) - \frac{1}{2}\alpha^2(t-0)$$

Introducing exponential;

$$\frac{X_t}{X_0} = e^{\alpha W_t - \frac{1}{2}\alpha^2 t}$$

$$\Rightarrow X_t = X_0\, e^{\alpha W_t - \frac{1}{2}\alpha^2 t}$$

2 $dX_t = X_t W_t dW_t$

Sln

$$\frac{dX_t}{X_t} = W_t dW_t = d\ln X_t$$

If $f(t,x) = \ln x$; $f_t = 0$, $f_x = \frac{1}{x}$, $f_{xx} = \frac{-1}{x^2}$

$$d\ln X_t = \frac{1}{X_t} dX_t - \frac{1}{2X_t^2}\left(\underbrace{X_t^2 W_t^2}_{(dX_t)^2} dt\right)$$

and $dX_t = X_t W_t dW_t$

$$= W_t dW_t - \frac{1}{2} W_t^2 dt$$

(No fxn of X) ~~Hence can~~ use integration technique

By Integration Technique

Integrating over $[0, t]$

$$\int_0^t d\ln X_s = \alpha \int_0^t dW_s - \frac{1}{2} \int_0^t \alpha^2 ds$$

$$\underbrace{\ln X_t - \ln X_0}_{\ln\left(\frac{X_t}{X_0}\right)} = \alpha (W_t - W_0) - \frac{1}{2} \alpha^2 (t - 0)$$

Introducing exponential;

$$\frac{X_t}{X_0} = e^{\alpha W_t - \frac{1}{2}\alpha^2 t}$$

$$\Rightarrow X_t = X_0 \, e^{\alpha W_t - \frac{1}{2}\alpha^2 t}$$

2 $dX_t = X_t W_t dW_t$

Sln

$$\frac{dX_t}{X_t} = W_t dW_t = d\ln X_t$$

If $f(t, x) = \ln x$; $f_t = 0$, $f_x = \frac{1}{x}$, $f_{xx} = \frac{-1}{x^2}$

$$d\ln X_t = \frac{1}{X_t} dX_t - \frac{1}{2X_t^2} (\underbrace{X_t^2 W_t^2}_{(dX_t)^2} dt)$$

and $dX_t = X_t W_t dW_t$

$$= W_t dW_t - \frac{1}{2} W_t^2 dt$$

(No fxn of X)

~~Hence can~~ use integration technique

Going back to eqn (i) ;

$$\ln\left(\frac{X_t}{X_0}\right) = \frac{W_t^2}{2} - \frac{t}{2} - \frac{1}{2}\underbrace{\int_0^t W_s^2\, ds}_{sW_s^2 \Big|_0^t = tW_t^2}$$

$$= \frac{1}{2}\left(W_t^2 - t - tW_t^2\right).$$

Taking exponential ;

$$X_t = X_0\, e^{\frac{1}{2}\left(W_t^2 - t - tW_t^2\right)}$$

Question

$$dX_t = \mu X_t\, dt + \sigma X_t\, dW_t$$

$$X_t = X_0\, e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t} \Big\} \text{Sln}$$

✓ Done

COMMON

① Vasicek Model

$$dX_t = \theta(\mu - X_t)\,dt + \sigma\,dW_t$$

→ Assumes normality
You can't use variation of Parameters ~~tee~~ method to solve this

② Orstein Uhlenbeck Process (OU)

Vasic Model is a special case of this process

$$dX_t = -\theta X_t\,dt + \sigma\,dW_t$$ (no μ here)

② Geometric Brownian Motion (GBM) Model

<used for prediction>

$$dX_t = \theta X_t\,dt + \sigma X_t\,dW_t$$

→ when you divide by X_t it assumes normality
and the Model itself assumes the log normal property

③ Cox - Ingersoll-Ross (CIR) Model

- predicting dynamics of interest rates

$$dX_t = \theta(\mu - X_t)dt + \sigma\sqrt{X_t}\,dW_t$$ (*) Added the $\sqrt{X_t}$ to Vasicek

→ Goes to ē Chi-square distribution

④ Chan, Karoyli, Longstaff, Sanders (CKLS)

$$dX_t = \theta(\mu - X_t)dt + \sigma X_t^{\alpha}\,dW_t$$

- also ideal for modelling interest rates

⑤ Hyperbolic Process

$$dX_t = -\frac{\theta X_t}{\sqrt{1+X_t^2}}\,dt + dW_t$$

A) SIMULATING SDEs in R USING YUIMA PACKAGE

(also sde package)

```
library (yuima)
model = setModel (drift = " ", diffusion= " ", state.var = " ",
        time.var= " ", solve.var= " ", xinit =    )
X = simulate (model)
plot (X)
```

↳ always x

```
X@data@original.data
```

If you let simulate (model) = Y then Y@data@original.data

```
# changing the sampling grid

grid = setSampling (Terminal=   , n=   )

Z = simulate (model, grid)
```

0 0.5 1

Simulate

1. $dX_t = 0.1\,X_t\,dt + 0.2\,X_t\,dW_t$

$X_0 = 100$

2. $dY_t = -\frac{\theta Y_t}{\sqrt{1+Y_t^2}}\,dt + dW_t$ where $\theta = 0.1$

$Y_0 = 0.5$

Range ; $Y \in [0, 100]$

$dt = 0.5$

Sln

1. model= setModel (drift = "0.1 * x", diffussion = "0.2 * x", state.var=
time.var = "t", solve.var = "Y", xinit = 100)

Z = simulate (model)

plot (Z)

Z@data@original.data

2. model = setModel (drift = "(-theta * Y)/sqrt(1+Y^2)", diffusion = "1",
stat.var = "Y", time.var = "t", solve.var = "Y", xinit=

Y = simulate (model)

plot (Y)

Y@data@original.data

Simulating Wiener Process with YUIMA :

model= setModel (drift = "0", diffusion = "1", state.var = "x",
time.var = "t", solve.var = "x", xinit = 0)

Z = simulate (model)

Z

(v) INTEGRATING FACTORS

Thursday 9th Nov 2023

- This method is applied to SDEs of the form:

$$dX_t = f(t, X_t)dt + g(t) X_t \, dW_t$$

where $f(t)$ and $g(t)$ are deterministic fxns

- Considering the integrating ~~fxn~~ factor given by:

$$\rho_t = e^{-\int_0^t g(s)\,dW_s + \frac{1}{2}\int_0^t g^2(s)\,ds} \quad \text{--- (i)}$$

then the SDE;

(*) Find out how (i) can become (ii)

$$d(\rho_t X_t) = \rho_t \, f(t, X_t)dt \quad \text{--- (ii)}$$

which can be solved by Integration or Exact Sln Technique

Solve for the SDE using Integrating Factors technique:

$$dX_t = r\,dt + \alpha X_t \, dW_t$$

<u>Sln</u>

$$f(t,x) = r \; ; \; g(t) = \alpha \; ; \; \rho_t = e^{-\int_0^t \alpha\, dW_s + \frac{1}{2}\int_0^t \alpha^2 ds}$$

$$\Rightarrow \rho_0 = e^{-\alpha \underbrace{W_0}_{=0} + \frac{1}{2}\alpha^2(0)} \qquad \rho_t = e^{-\alpha W_t + \frac{1}{2}\alpha^2 t}$$

$$= 1$$

$$d(\rho_t X_t) = r \rho_t \, dt$$

Integrating from $[0, t]$;

$$\int_0^t d(\rho_s X_s) = r \int_0^t \rho_s \, ds$$

$$\Rightarrow \rho_t X_t = \rho_0 X_0 + r \int_0^t \rho_s \, ds$$

$$X_t = \frac{X_0}{\rho_t} + \frac{r}{\rho_t} \int_0^t \rho_s \, ds$$

$$X_t = X_0 \underbrace{e^{\alpha W_t - \frac{1}{2}\alpha^2 t}}_{1/\rho_t} + r e^{\alpha W_t - \frac{1}{2}\alpha^2 t} \int_0^t e^{-\alpha W_s + \frac{1}{2}\alpha^2 s} \, ds$$

Example 2 ⟨The GBM model⟩

$$dX_t = \mu X_t \, dt + \sigma X_t \, dW_t$$

$f(t, X_t) = \mu X_t$

$g(t) = \sigma$

Sln

$$f(t,x) = \mu x \; ; \; g(t) = \sigma$$

$$\rho_t = e^{-\int_0^t g(s)\, dW_s + \frac{1}{2}\int_0^t g^2(s)\, ds}$$

$$= e^{-\int_0^t \sigma \, dW_s + \frac{1}{2}\int_0^t \sigma^2 \, ds}$$

$$= e^{-\sigma W_t + \frac{1}{2}\sigma^2 t} = e^{-(\sigma W_t - \frac{1}{2}\sigma^2 t)}$$

$$\rho_0 = 1$$

$$d(\rho_t X_t) = \rho_t f(t, X_t)\, dt = \mu X_t \rho_t \, dt$$

(*) there is a fxn of x in the drift term he[nce] integration con[stant] work here

$$\frac{d(\rho_t X_t)}{\rho_t X_t} = \mu \, dt$$

$$= d \ln(\rho_t X_t)$$

Integrating from [0,t];

$$\int_0^t d\ln(\rho_s X_s) = \int_0^t \mu \, ds$$

$$\ln(X_t \rho_t) - \ln(X_0 \overset{1}{\rho_0}) = \mu t$$

$$\ln\left(\frac{X_t \rho_t}{X_0}\right) = \cancel{e}\, \mu t$$

$$\Rightarrow \frac{X_t \rho_t}{X_0} = e^{\mu t}$$

and $X_t = \dfrac{X_0}{\rho_t} e^{\mu t}$

$$X_t = X_0 \, e^{\sigma W_t - \frac{1}{2}\sigma^2 t} \cdot e^{\mu t}$$

$$= X_0 \, e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t}$$

(VI) LINEAR SDE TECHNIQUE

- Consider the SDE with a drift term that is linear with X_t; ie,

$$dX_t = [a(t)X_t + b(t)]dt + \sigma(t, W_t)dW_t$$

$$dX_t = a(t)X_t\,dt + b(t)dt + \sigma(t, W_t)dW_t \quad \text{step 1}$$

$$dX_t - a(t)X_t\,dt = b(t)dt + \sigma(t, W_t)dW_t \quad \text{step 2}$$

; take the one that's linear in parameter to the LHS

- Let $A(t) = \int_0^t a(s)ds$

such that Integrating factor:

$$IF = e^{-A(t)}$$

multiplying with SDE gives:

$$\underbrace{\{dX_t - a(t)X_t\,dt\}e^{-A(t)}}_{d(X_t e^{-A(t)})} = \{b(t)dt + \sigma(t, W_t)dW_t\}e^{-A(t)}$$

if one the RHS there is no fxn of X, then you can use Integration Technique to solve for X_t

Sidenote (*) Showing this;

Let $I_t = e^{-A(t)}$

$$dA(t) = d\int_0^t a(s)\,ds = a(t)dt$$

$$\Rightarrow dI_t = -dA(t)\,e^{-A(t)}$$

$$= -a(t)\,e^{-A(t)}dt$$

Showing how LHS = $d(X_t e^{-A(t)})$

$$d(X_t I_t) = I_t dX_t + X_t dI_t$$
$$= e^{-A(t)}dX_t + X_t\left((-a(t))e^{-A(t)}dt\right)$$
$$= e^{-A(t)}\left[dX_t - a(t)e^{-A(t)}dt\right]$$

Hence Shown

- Integrating from $[0,t]$

$$\int_0^t d\left(X_s e^{-A(s)}\right) = \int_0^t e^{-A(s)} b(s)\, ds + \int_0^t \sigma(s,W_s) e^{-A(s)}\, dW_s$$

$$X_t e^{-A(t)} = X_0 + \int_0^t e^{-A(s)} b(s)\, ds + \int_0^t \sigma(s,W_s) e^{-A(s)}\, dW_s$$

$$X_t = X_0 e^{A(t)} + e^{A(t)} \int_0^t e^{-A(s)} b(s)\, ds + e^{A(t)} \int_0^t \sigma(s,W_s) e^{-A(s)}\, dW_s$$

since $A(0)=0$

$\Rightarrow e^{-A(0)} = 1$

Example

Solve for $dX_t = (2X_t + 1)\,dt + e^{2t}\, dW_t$ using the linear SDE Technique

Sln

$$dX_t = 2X_t\, dt + dt + e^{2t} dW_t$$

$$dX_t - 2X_t\, dt = dt + e^{2t}\, dW_t$$

$$\boxed{a(t) = 2\cancel{X_t} \;;\; b(t) = 1 \text{ and } \sigma(t, W_t) = e^{2t}}$$

$$A(t) = \int_0^t a(s)\, ds = \int_0^t 2\, ds = 2t$$

$$e^{-A(t)} = e^{-2t} \Rightarrow IF$$

Then $\underbrace{\left[dX_t - 2X_t\, dt\right]}_{d\left(X_t \cdot e^{-A(t)}\right)} e^{-2t} = \left[dt + e^{2t} dW_t\right] e^{-2t}$

$$d(X_t e^{-2t}) = e^{-2t} dt + dW_t$$

Integrating from $[0,t]$;

$$\int_0^t d(X_s e^{-2s}) = \int_0^t e^{-2s} ds + \int_0^t dW_s$$

$$\Rightarrow X_t e^{-2t} = X_0 + \frac{e^{-2s}}{-2}\Big|_0^t + (W_t - \underbrace{W_0}_{0})$$

$$= X_0 - \frac{1}{2}\left(e^{-2t} - 1\right) + W_t$$

$$X_{\cancel{2}t} = X_0 e^{2t} - \frac{1}{2} e^{2t}\left(e^{-2t} - 1\right) + W_t$$

$$= X_0 e^{2t} - \frac{1}{2}\left(1 - e^{2t}\right) + W_t$$

GIRSANOV'S THEOREM AND RISK-NEUTRAL MEASURES

- Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a prob space where $\mathcal{F}$ is a σ algebra generated by the Brownian motion W_t

(a) Radon-Nikodym Theorem
Is a result in measure theory that expresses the rship btwn 2 measures defined on the same measurable space
- It allows us to derive a new measure q from one that already exists, say p
- The measure q is calculated as follows:

$$Q(A) = \int_A f \, dP$$

where Q is a new measure defined on any measurable subset A and f is the density at a given pnt.

Note: The Radon-Nikodym Theorem states that;

Under certain conditions, any measure Q can be expressed ~~as~~ w.r.t another measure P on the same space.

- The fxn f is called the Radon-Nikodym Derivative and it is given by:

$$f = \frac{dQ}{dP}$$

$$dQ(A) = d\int_A f \, dP$$

$$dQ = f \, dp \Rightarrow f = \frac{dQ}{dP}$$

(b) Girsanov's Theorem

- Under the measure P, the exp of X is given by

$$E[X] = \int_{\Omega} X(\omega)\, dP(\omega) \quad \text{--- (i)}$$

- If we consider the process:

$$M_t = e^{-\int_0^t g(u)\, dW_u - \frac{1}{2}\int_0^t g^2(u)\, du}$$ is a martingale since $E[M_t \mid F_s] = M_s$ (from cat 1)

Then,
by the Radon-Nikodym Theorem, we can define a new measure Q s.t $dQ = M_T\, dP$; $Q : \underset{\text{any information}}{F} \to \mathbb{R}$

$$Q(A) = \int_A dQ$$

$$= \int_A M_T\, dP$$

$M_0 = 1$ (*) you integrate from zero to zero and get 0 then $e^0 = 1$

$$M_t \geq 0 \Rightarrow Q(A) = \int_A M_T\, dP \geq 0$$

$$Q(\Omega) = \int_{\Omega} M_T\, dP = E[M_T] \text{ from --- (i)}$$

$$= E[M_T \mid F_0]$$

$$= M_0$$

$$= 1$$

- Hence Q is a probability on F and $\therefore (\Omega, F, Q)$ is a probability space

<which is a consequence of> $M_t = e^{-\int_0^t g(u)\,dW_u - \frac{1}{2}\int_0^t g^2(u)\,du}$

(c) Lemma (Girsanov's Theorem)

- Let X_t be an Ito process that has the following dynamics:

is silently under P already!

$$dX_t = g(t)\,dt + dW_t^P$$
$$X_0 = 0$$

where $g(t)$ is a bounded fxn then ē exp. process is given by:

$$M_t = e^{-\int_0^t g(u)\,dW_u - \frac{1}{2}\int_0^t g^2(u)\,du}$$

then, X_t is an $\mathcal{F}_t$-martingale under Q

$$\int_s^t dX_s = \int_s^t g_s\,ds + \int_s^t dW_s$$

$$X_t = X_s + (W_t - W_s) + \int_s^t g_s\,ds$$

$$E[X_t|\mathcal{F}_s] = E[X_s|\mathcal{F}_s]\; X_s + E[(W_t - W_s)|\mathcal{F}_s]$$

$$= X_s + \int_s^t g_s\,ds \neq X_s$$

It is NOT a martingale under P

Proof for under Q;

If X_t is a martingale under Q, $\quad f = \frac{dQ}{dP}$

$$E^Q[X_t|\mathcal{F}_s] = X_s \qquad dQ = f\,dP$$

However, $E^Q[X] = \int_\Omega X\,dQ$

$$= \int_\Omega X M_T\,dP$$

$$= E^P[XM_T]$$

we want to show that:

$$E^Q[X_t|\mathcal{F}_s] = X_s \;\textcircled{=}\; E^P[X_t M_t|\mathcal{F}_s] = X_s M_s$$

$(dX_t)^2 = dt$

$dX_t = g(t)\,dt + dW_t$

Let,
$$L_t = \int_0^t g(u)\,dW_u + \tfrac{1}{2}\int_0^t g^2(u)\,du$$

$$M_t = e^{-L_t} = f(t, L_t)$$

$$dL_t = g(t)\,dW_t + \frac{1}{2} g^2(t)\,dt$$

$$\approx g_t\,dW_t + \underbrace{\frac{1}{2} g_t^2\,dt}_{=0}$$

since $(dt)^2 = 0$ when you square

$$(dL_t)^2 = g_t^2\,dt$$

$$M_t = e^{-L_t} = f(t, L)$$

$$f_t = 0 \;;\; f_L = -e^{-L_t} \;;\; f_{LL} = e^{-L_t}$$

By Ito's formula,

$$dM_t = 0\,dt - e^{-L_t}\,dL_t + \frac{1}{2} e^{-L_t}(dL_t)^2$$

$$dM_t = -M_t\,dL_t + \frac{1}{2} M_t (dL_t)^2$$

$$= -M_t\left(g_t\,dW_t + \tfrac{1}{2} g_t^2\,dt\right) + \frac{1}{2} M_t\left(g_t^2\,dt\right)$$

$$dM_t = -M_t\,g_t\,dW_t$$

$$dX_t = g_t\,dt + dW_t$$ X_t has a drift term hence it is not a martingale.

$$dX_t \cdot dM_t = -g_t M_t\,dt$$

$$d(M_t X_t) = M_t\,dX_t + X_t\,dM_t + dX_t \cdot dM_t$$

$$= M_t(\cancelto{0}{g_t\,dt} + dW_t) + X_t(-g_t M_t\,dW_t) - \cancelto{0}{g_t M_t\,dt}$$

$$d(M_t X_t) = M_t(1 - X_t g_t)\,dW_t$$

(*) Note: Martingales are driftless.
⇒ If you have a drift term just know that Stoch. process is NOT a martingale.

Integrating from $[s,t]$;

$$\int_s^t d(M_u X_u) = \int_s^t M_u(1 - X_u g_u)\,dW_u$$

$$M_t X_t = M_s X_s + \int_s^t M_u(1 - X_u g_u)\,dW_u$$

$$\Rightarrow E^P[M_t X_t \mid \mathcal{F}_s] = E^P[M_s X_s \mid \mathcal{F}_s] + E^P\left[\int_s^t M_u(1 - X_u g_u)\,dW_u \,\Big|\, \mathcal{F}_s\right]$$

$$= M_s X_s + E\left[\cancel{\int_s^t M_u(1 - X_u g_u)\,dW_u}\right]$$

By zero mean property

due to integrability (all info is known upto $\mathcal{F}_s$)

$$\Rightarrow E^P[M_t X_t \mid \mathcal{F}_s] = M_s X_s \;=\; E^Q[X_t \mid \mathcal{F}_s] = X_s$$

hence X_t is a martingale under Q

(d) Feynman-Kac Theorem

- Assuming that F is a sln to the boundary value problem:

$$\begin{cases} \dfrac{\partial F}{\partial t} + \mu \dfrac{\partial F}{\partial x} + \dfrac{1}{2} \cdot \sigma^2 \dfrac{\partial^2 F}{\partial x^2} - rF = 0 \\ F(t, x) = \Phi(x) \end{cases}$$

from Ito's formula; $a(t, W_t) = \mu$ and $b(t, W_t) = \sigma$

then F will have a presentation given by:

$$F(T, x) = e^{-r(T-t)} E\left[\Phi(X_T)\right]$$

where $\begin{cases} dX_s = \mu\, ds + \sigma\, dW_s \\ X_t = x \end{cases}$? → Always and?

Example

Solve for the PD given by:

$$\frac{\partial F}{\partial t} + \overset{0\frac{\partial f}{\partial x}}{} \frac{1}{2} \cdot \sigma^2 \frac{\partial^2 F}{\partial x^2} = 0 \quad \text{---(i)} \qquad \Rightarrow a(t) = 0$$

$$F(t, x) = x^2 = \Phi(x) \quad \text{---(ii)} \qquad \Rightarrow E[\Phi(X_T)] = E[X^2]$$

<u>Sln</u>

$dX_s = 0\, ds + \sigma\, dW_s$

$$dX_s = \sigma\, dW_s \qquad \Big| \qquad (dX_s)^2 = \sigma^2 dt$$
$$X_t = x$$

The sln for this boundary problem should be given by:

$$F(T, x) = E[\Phi(X_T)] \quad \text{since } r = 0$$
$$= E[X_T^2] \quad \text{---} \circledast$$

Let $f(t,x) = x^2$

$\Rightarrow f_t = 0 \; ; \; f_x = 2x \; ; \; f_{xx} = 2$

$$d(X_s^2) = 2X_s\,dX_s + \frac{1}{2}(2)(dX_s)^2$$

$$= 2X_s(\sigma\, dW_s) + (\sigma^2 dt)$$

$$d(X_s^2) = 2\sigma X_s\, dW_s + \sigma^2 dt$$

Integrating w.r.t $[t, T]$

$$\underbrace{\int_t^T d X_s^2}_{= X_T^2 - X_t^2} = 2\sigma \int_t^T X_s\, dW_s + \sigma^2 \int_t^T ds$$

$$X_T^2 = X_t^2 + \sigma^2(T-t) + 2\sigma \int_t^T X_s\, dW_s$$

expectation of this $= 0$ coz of zero mean property

Our goal is to get/solve $E[X_T^2]$ (eqn ⊛)

$$\Rightarrow E[X_T^2] = x^2 + \sigma^2(T-t)$$

$$= F(T, x)$$

To confirm, you can Replace T with t :

$F(t, x)$ will be $= x^2 + \sigma^2(t-t) = x^2$ ✓ ---- (ii) Hence you're good so far.

QED

Then, $\frac{\partial F}{\partial t} = -\sigma^2$; $\frac{\partial F}{\partial x} = 2x$ and $\frac{\partial^2 F}{\partial x^2} = 2$

Replacing in the (i) ; $-\sigma^2 + \frac{1}{2}\sigma^2(2) = -\sigma^2 + \sigma^2 = 0$ ✓ Hence correct

Hence solved.

QED

Question

$$\frac{\partial F}{\partial t} + \frac{1}{2}\sigma^2\frac{\partial^2 F}{\partial x^2} + \frac{1}{2}\beta^2\frac{\partial^2 F}{\partial y^2} = 0$$

$$F(t, x, y) = xy$$

① Write notes on Fokker Planck
② Download dataset sth from investing.com
③ Install packages "quantmod"

(e) Fokker Planck Equations

- This is a partial differential equation that describes the time evolution of the probability density fxn of the velocity of a particle under the influence of drag forces and random forces such as the Brownian motion.
- It has multiple applications in information theory, graph theory, data science, finance, economies and so on.
- It's also known as Kolmogorov Forward equations / Smoluschovoski Eqns

ONE-DIMENSIONAL STOCHASTIC PROCESS

- Let X be an Itô process with an SDE given by:

$dX_t = a(t, X_t)dt + b(t, X_t)dW_t$ where $a(t, X_t)$ is the drift coefficient and $D(t, X_t) = \frac{1}{2}b^2(t, X_t)$ is the diffusion coeff

- The Fokker Planck Eqn for probability density $p(t,x)$ of the random variable X_t is given by:

$$\frac{\partial p(t,x)}{\partial t} = -\frac{\partial(a(t,x)\,p(t,x))}{\partial x} + \frac{\partial^2[D(t,x)\,p(t,x)]}{\partial x^2}$$

Example

1. Consider the SDE given by $dX_t = dW_t$. Find the Fokker Planck eqn

Sln

Drift term, $a(t,x) = 0$
Diffusion term, $b(t,x) = 1$

$$D(t,x) = \frac{1}{2}$$

$$\frac{\partial p(t,x)}{\partial t} = \frac{1}{2} \frac{\partial^2 p(t,x)}{\partial x}$$

Then solve using ODE

2. Consider the SDE $dX_t = -aX_t dt + \sigma dt$. Find the Fokker Planck equation

$a(t,x) = -ax$
$b(t,x) = \sigma$
$D(t,x) = \frac{\sigma^2}{2}$

$$\frac{\partial p(t,x)}{\partial t} = -a \frac{\partial (x\, p(t,x))}{\partial x} + \frac{1}{2} \sigma^2 \frac{\partial^2 p(t,x)}{\partial x^2}$$

$$= -a \left[x \frac{\partial p(t,x)}{\partial x} + p(t,x \right] + \frac{1}{2} \sigma^2 \frac{\partial^2 p(t,x)}{\partial x}$$

HIGHER DIMENSION STOCHASTIC PROCESS

- Let $\underline{X}$ be an n-dimension stochastic process with an SDE given by:

$$d\underline{x}(t) = a(t, \underline{X}_t)\,dt + b(t, \underline{X}_t)\,dW_t$$

where $a(t, \underline{X}_t)$ is the dimensional drift vector and $dt\,\underline{X}_t$ is a drift coefficient matrix

- Then the probability density $p(t,x)$ satisfies the Fokker Plank Equation given by:

$$\partial p(t,x) = -\sum_{i=1}^{N} \frac{\partial\,[a(t,x)p(t,x)]}{\partial x_i} + \sum_{i=1}^{N}\sum_{j=1}^{N} \frac{\partial^2 [D_{ij}(t,x)p(t,x)]}{\partial x_i\,\partial x_j}$$

Where $\underline{a} = [a_1, a_2, ---, a_N]$

$$D = \frac{1}{2}\, b \cdot b^T$$

Such that $D_{ij}(t,x) = \frac{1}{2}\sum_{k=1}^{N} \sigma_{ik}(t,x)\,\sigma_{jk}(t,x)$

Question

Let $dX_t = \mu X_t\,dt + \sigma X_t\,dW_t$
$dY_t = \theta Y_t\,dt + \gamma X_t\,dW_t$

Find the Fokker Plank equation for the stochastic process $Z_t = X_t \cdot Y_t$