STA 2401 - TIME SERIES ANALYSIS

Course Outline

1. The four components of economic. Time Series - Seasonal variations, Cycle events, Random variation
2. Single sinusoidal noise
3. Least square solution with zero and non-zero trend and the matrix formulation for generalised least squares.
3 4. Complex sinasoidal and Fourier transform for continuous range of frequencies [spectial analysis]
4. Use filters in ~~terms~~ time series. Method of moving averages and variate differences
5. Autoregressive processes: correlogram, periodogram and spectral analysis.

Introduction

An arrangement of statistical data in accordance with time.

Time series is a collection of observations met sequentially with time. It is an arrangement of statistical data in accordance with time.

- Mathematically we denote a time series by a random variable, X_t where X_t is the observation at time, t.
- One important characteristic of a time series is that the observations are correlated. Examples of time series:-

a. Economic time series e.g interest rates, stock prices, export sales

b. Physical time series: Many types of time series occur in physical sciences, particularly meterological, marine science and geophysics e.g rainfall, humidity, temperature, floods observed in successive months, days or years.

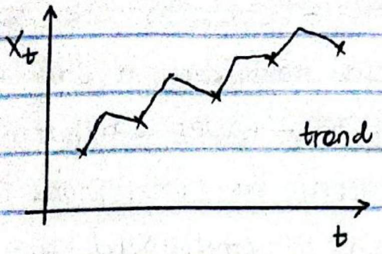
c. Demographic time series: Concerned with population e.g census.

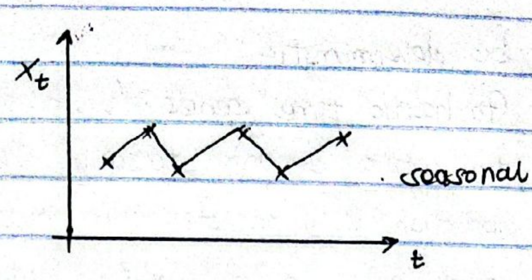
Objectives of Time Series

→ plotting the data.

1. For descriptive purpose: When presented with time series data, the first step in analysis is usually to plot the data and obtain descriptive measures of the time series, features such as trend and seasonal effects. Seasonal effects can sometimes be easily seen from such plots.

The difference btn
prediction and forecasting





The plot will enable; look for unusual observations (outliers) which are inconsistent with the rest of the data.

2. For explanation purpose : When observations are taken for two variables or more. It may lead to a deeper understanding of the mechanism which developed the series i.e we develop a structural model which governs the generation of such a series.
3. For ~~control~~ prediction purpose : Given an observed time series, future values of the time series may be predicted. This is an important task in shares forecasting and in the analysis of economic and industrial time series.
4. For control purpose: Using the structural model, one may seek to control the system by either, generating warning signs or signals for future events by examining what will happen ~~if~~ in the ~~input~~ output of the system ~~is~~ altered or changed when the input is altered or changed.
5. For statistical model building : Several jointly dependent variables are considered to come up with a statistical model.

$y = a + bx$

prediction - using past information / data to determine a future outcome
forecasting - using past and present data to determine an outcome.

✓ Terminologies used in Time Series

- Continuous time series : A time series is said to be continuous when observations are made continually in time i.e values are in intervals / range
- Discrete time series: This is when observations are taken at specific time usually equally spaced ⇒ periodically.
- Deterministic time series: When successful observations are dependent, future values may be predicted from the past observations, and if a time series can be predicted correctly or exactly then such a time series is said to

be deterministic.

- Stochastic time series: Most time series are stochastic in the sense that the future is only partially determined by past values so that the exact prediction is impossible and, must be replaced by the idea that future values have a probability distribution, which is conditioned with the knowledge of the past values.

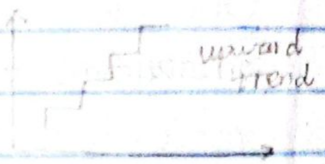
Components of a time series

There are several forces at work which affect the observed time series. These can be physically classified into the following categories:-

1. Trend/secular
2. Seasonal variations → regular movements
3. Cyclic variations
4. Random/irregular movements

1. Trend/secular

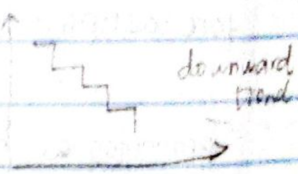
This is the general tendency of the data to increase or decrease over a long time period. Examples include:-

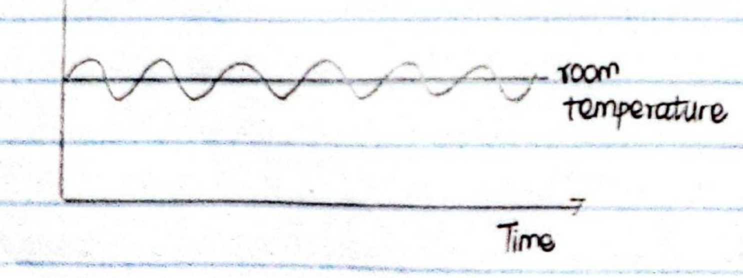


a. Data concerning population over time
b. Data concerning death over time

Apart from upward and downward trends we may also have time series whose values fluctuate along a constant reading which does not change with time

e.g. Time series of temperature readings





2. Seasonal Variations

These are regular movements in a time series with a period less than one year e.g sales and profits in a store.

Seasonal variations may be attributed to two causes:-

a) natural forces e.g the production of some commodities depend on amount of rainfal. eg increase in sales of raincoats or umbrellas during rain periods.

b) manmade forces e.g the sales and profits of some commodities/goods are related eg sales of flowers during valentines

3. Cyclic Variation

These are oscillatory movements in a time series with a period of oscillation which is more than one year. e.g business cycles which represent intervals of prosperity, recession, depression and recovery.

4. Random / Irregular Movements

These are observations which are purely random and unpredictable and therefore they are beyond human control e.g a pandemic, floods, drought, fires, earthquakes, hurricanes, terrorist attacks

Approaches to time series analysis

The analysis of Time Series in the time domain (classical approach

The main diagnostic tool to look at is the auto-correlation function (ACF) which helps us to analyse the time series through time.

Analysis of time series in the frequency domain

The main diagnostic here is the power spectrum/spectrum density function which describes how the variations in the time series may be accounted for by variation in different frequencies.

Models in Time Series

There are two main models:-

Additive model

Multiplicative model

1. ADDITIVE MODEL

According to this, a time series can be decomposed as:-

$$X_t = T_t + S_t + C_t + e_t$$

where T_t - trend / secular movement

S_t - seasonal variation

C_t - cyclic variation

e_t - regular movement.

In the annual data, the C_t component does not appear in the additive model because it takes more than a year

2. MULTIPLICATIVE MODEL

It is represented by

$$X_t = T_t \times S_t \times C_t \times e_t$$

Analysis of Time Series containing a trend / secular movements

It depends on whether one wants to measure / estimate the trend or remove the trend in order to analyse the local fluctuations.

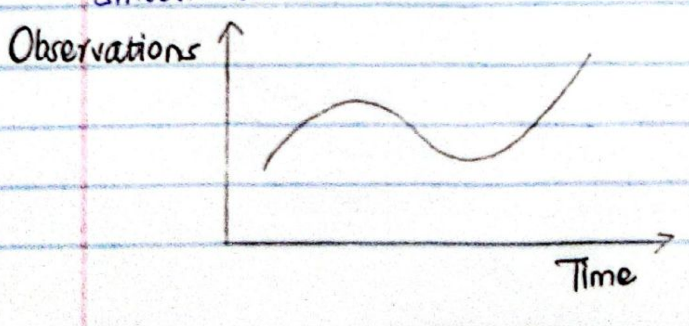
a. Measure the trend

There are several methods that can be used to measure the trend by either of the following methods:-

1. Graphical / free-hand method
2. Method of semi-averages
3. Curve fitting method
4. Moving averages method.

1. GRAPHICAL / FREE-HAND METHOD

The time series X_t is plotted against time (The main reason we plot our data is to identify the outliers) to obtain a free-hand smooth curve.



It gives us a general idea about the trend i.e are we having an upward/downward trend or a constant.

Merits

1. It does not involve complex mathematical techniques
2. It is a flexible and simple way of studying the trend

Demerits

1. It is subjective as different trend curves may be obtained by different persons.
2. It does not measure the trend but gives the direction of the trend.

2. METHOD OF CURVE FITTING

This method aims at coming up with a curve or a line of best fit i.e a line that minimizes deviations from the data points. There are various types of curves used in practice which are as follows :-

a. Straight line : $X_t = a + bt + e_t$

b. Second degree polynomial : $X_t = a + bt + \cancel{e_t}\, C_t^2 + e_t$

c. Exponential curve : $X_t = ab^t + e_t$

d. Growth curve

→ Modified exponential curve : $X_t = a + bc^t + e_t$

→ Gombertz curve : $X_t = ab^{c^t} + e_t$

→ Logistic curve : $X_t = \dfrac{K}{1 + e^{a+bt}} + e_t$

- We normally fit a straight line if the values of the time series are decreasing or increasing by equal absolute amounts i.e

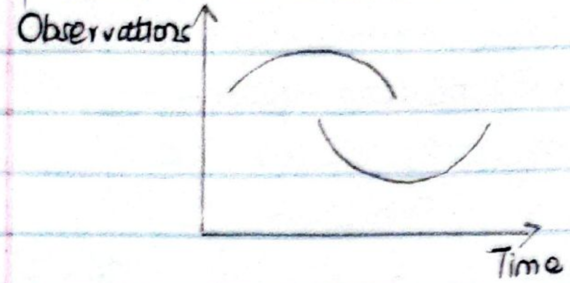
$$X_t - X_{t-1} = \text{a constant}$$

- We fit an exponential curve if the series is decreasing or increasing by a constant percentage i.e

$$\frac{X_t}{X_{t-1}} = \text{a constant percentage}$$

$$\frac{X_{t-1}}{X_{t-2}} = \text{a constant percentage}$$

- We fit a quadratic curve / a second degree polynomial if the time plot shows a curvature that is either upwards or downwards.



ii a. <u>How to fit a straight line</u>

Let the series be $X_t = a + b_t + e_t$

where a - intercept $\qquad e_t$ - error term

b - slope / gradient

To fit the straight line we use the ordinary least square method i.e make e_t the subject of the formula, then square both sides and take the summation

$$e_t = X_t - a - b_t$$
$$e_t^2 = (X_t - a - b_t)^2$$
$$\sum_{i=1}^{n} e_t^2 = \sum_{i=1}^{n} (X_t - a - b_t)^2$$

Let Q be $\sum_{i=1}^{n} e_t^2$ (square of error terms / deviations)

The aim is to estimate a and b using OLS method.

$$Q = \sum_{i=1}^{n} (X_t - a - b_t)^2$$

Estimating a, the intercept :-

$$\frac{dQ}{da} = \frac{-2\sum_{i=1}^{n}(X_t - a - b_t)}{-2} = \frac{0}{-2}$$

$$\sum_{i=1}^{n}(X_t - a - b_t) = 0$$
$$\sum_{i=1}^{n} X_t - \sum_{i=1}^{n} a - b\sum_{i=1}^{n} t = 0$$
$$\sum_{i=1}^{n} X_t - na - b\sum_{i=1}^{n} t = 0$$
$$\sum_{i=1}^{n} X_t = na + b\sum_{i=1}^{n} t \quad \ldots \text{①}$$

$$\frac{\sum X_t - b\sum t}{n} = \frac{na}{n}$$
$$\frac{\sum X_t}{n} - \frac{b\sum t}{n} = \hat{a}$$

Estimating b, the slope / gradient :-

$$\frac{dQ}{db} = \frac{-2t\sum_{i=1}^{n}(X_t - a - bt)}{-2} = \frac{0}{-2}$$

$$t\sum_{i=1}^{n}(X_t - a - bt) = 0$$

$$\sum_{i=1}^{n} tX_t - a\sum_{i=1}^{n} t - b\sum_{i=1}^{n} t^2 = 0$$

$$\sum_{i=1}^{n} tX_t = a\sum_{i=1}^{n} t + b\sum_{i=1}^{n} t^2 \quad \ldots ②$$

Equations 1 and 2 are known as **normal equations** which are solved simultaneously to obtain the estimates of a and b.

$$\sum_{i=1}^{n} tX_t = a\sum_{i=1}^{n} t + b\sum_{i=1}^{n} t^2 \quad \times n$$

$$\sum_{i=1}^{n} X_t = na + b\sum_{i=1}^{n} t \quad \times \sum_{i=1}^{n} t$$

$$n\sum_{i=1}^{n} tX_t = na\sum_{i=1}^{n} t + bn\sum_{i=1}^{n} t^2$$

$$-\quad \sum_{i=1}^{n} t\sum_{i=1}^{n} X_t = an\sum_{i=1}^{n} t + b\left(\sum_{i=1}^{n} t\right)^2$$

$$n\sum_{i=1}^{n} tX_t - \sum_{i=1}^{n} t\sum_{i=1}^{n} X_t = bn\sum_{i=1}^{n} t^2 - b\left(\sum_{i=1}^{n} t\right)^2$$

$$n\sum_{i=1}^{n} tX_t - \sum_{i=1}^{n} t\sum_{i=1}^{n} X_t = b\left[n\sum_{i=1}^{n} t^2 - \left(\sum_{i=1}^{n} t\right)^2\right]$$

$$\hat{b} = \frac{n\sum_{i=1}^{n} tX_t - \sum_{i=1}^{n} t\sum_{i=1}^{n} X_t}{n\sum t^2 - (\sum t)^2}$$

Now $\hat{a}$ can be estimated from equation 1:-

$$\sum X_t = na + b\sum t$$

$$\frac{na}{n} = \frac{\sum X_t - b\sum t}{n}$$

$$\hat{a} = \frac{\sum X_t}{n} - \frac{\hat{b}\sum t}{n}$$

Therefore:- $\hat{a} = \bar{X}_t - \hat{b}\bar{t}$, where $\bar{X}_t = \frac{\sum X_t}{n}$ and $\bar{t} = \frac{\sum t}{n}$

The fitted straight line is $\hat{X}_t = \hat{a} + \hat{b}t$

iii) <u>How to fit an exponential curve</u>

The time series is given by $X_t = ab^t + e_t$.

Taking the natural logarithms on both sides we have:-

$$\log X_t = \log(ab^t) + \log e_t \qquad y = \log X_t$$

$$\cancel{\log X_t = t\log ab + \log e_t} \qquad A = \log a$$

$$\log X_t = \log a + \log b^t + \log e_t \qquad B = \log b$$

$$\log X_t = \log a + t\log b + \log e_t \qquad E_t = \log e_t$$

$$Y_t = \cancel{Bt}\, A + Bt + E_t \quad \ldots ①$$

Making E_t the subject and using OLS:-

$$E_t = Y_t - A - Bt$$
$$E_t^2 = (Y_t - A - Bt)^2$$
$$\Sigma E_t^2 = \Sigma (Y_t - A - Bt)^2$$
$$Q = \Sigma (Y_t - A - Bt)^2$$
$$\hat{B} = \frac{n \Sigma t Y_t - \Sigma t \Sigma Y_t}{n \Sigma t^2 - (\Sigma t)^2} \quad \text{and} \quad \hat{A} = \bar{Y}_t - \hat{B}\bar{t}$$

$\hat{A} = \log \hat{a}$	$\hat{B} = \log \hat{b}$
$e^{\hat{A}} = e^{\log \hat{a}}$	$e^{\hat{B}} = e^{\log \hat{b}}$
$\hat{a} = e^{\hat{A}}$	$\hat{b} = e^{\hat{B}}$

The fitted exponential curve is :- $\hat{X}_t = \hat{a}\hat{b}^t$

Example 1

The table below shows figures of production in thousand tonnes for sugar.

Year	1963	1965	1966	1967	1968	1969	1972
production	67	88	94	85	91	98	90

Fit a straight line using OLS and tabulate the values

Eliminate the trend and find out which component is left

Obtain monthly increase in sugar production

Estimate production of sugar in 1970

solution

$\hat{X}_t = \hat{a} + \hat{b}t$

where $\hat{a} = \frac{\sum X_t}{n} - \hat{b}\frac{\sum t}{n}$ and $\hat{b} = \frac{n\sum t X_t - \sum t \sum X_t}{n\sum t^2 - (\sum t)^2}$

Time, t = Year - middle year (1967)

Year	X_t	t	t^2	tX_t	$\hat{X}_t$	$X_t - \hat{X}_t$
1963	67	-4	16	-268	78.498	
1965	88	-2	4	-176	82.878	
1966	94	-1	1	-94	85.068	
1967	85	0	0	0	87.258	
1968	91	1	1	91	89.448	
1969	98	2	4	196	91.638	
1972	90	5	25	450	98.208	
	613	1	51	199		

$\sum X_t = 613$ $\sum t^2 = 51$ and $n = 7$

$\sum t = 1$, $(\sum t)^2 = 1$ $\sum tX_t = 199$

Therefore :- $\hat{b} = \frac{7 \times 199 - 1 \times 613}{7 \times 51 - 1} = 2.19$

$\hat{a} = \frac{613}{7} - 2.19 \times \frac{1}{7} = 87.258$

Therefore the fitted straight line is :-

$\hat{X}_t = 87.258 + 2.19t$ → an increase in time causes a significant increase in sugar production.

The sugar production is expected to increase by 2.19 every year.

b. To eliminate the trends we have to evaluate $X_t - \hat{X}_t$

Here only the cyclic variations remain because observations are made for more than one year.

c. Monthly increase in sugar production :-

$\frac{2.19}{12} = 0.1825$ tonnes per month

d. $t = 1970 - 1967 = 3$, $t = 3$

$\hat{X}_t = 87.258 + 2.19(3)$

$\hat{X}_t = 87.258 + 6.57$

$\hat{X}_t = 93.828$

Example 2

Consider the table below which shows the population in millions of a given country.

Year	1929	1939	1949	1959	1969	1979	1989
Population	25	25	27.9	31.9	36.1	43.9	54.7

a. Fit an exponential curve given by $\hat{X}_t = ab^t + e^t$ and tabulate the trend values.

b. Estimate the population size in the year 1999

solution

a. $\hat{A} = \frac{\sum Y_t}{n} - \hat{B}\frac{\sum t}{n}$ and $\hat{B} = \frac{n\sum tY_t - \sum t \sum Y_t}{n\sum t^2 - (\sum t)^2}$ | $A = \log \hat{a}$

$\hat{a} = e^{\hat{A}}$

$\hat{A} = \bar{Y}_t - \hat{B}\bar{t}$ $\hat{B} = \log \hat{b} \Rightarrow \hat{b} = e^{\hat{B}}$ | $Y_t = \log X_t$

Year	X_t population	t	Y_t	tY_t	t^2
1929	25	-3	3.219	-9.657	9
1939	25	-2	3.219	-6.438	4
1949	27.9	-1	3.329	-3.329	1
1959	31.9	0	3.463	0	0
1969	36.1	1	3.586	3.586	1
1979	43.9	2	3.782	7.564	4
1989	54.7	3	4.002	12.006	9
~~199~~		0	24.6	3.732	28

$$t = \frac{\text{Year} - \text{mid year}}{10} = \frac{\text{Year} - 1959}{10}$$

$$\hat{B} = \frac{7 \times 3.732 - 0 \times 24.6}{7 \times 28 - 0} = \frac{7 \times 3.732}{7 \times 28} = 0.133$$

$$\hat{A} = \frac{24.6}{7} - 0.133 \times \frac{0}{7} = \frac{24.6}{7} = 3.514$$

1.142

$$\hat{b} = e^{\hat{B}} = e^{0.133} = 1.142$$

$$\hat{a} = e^{\hat{A}} = e^{3.514} = 33.58$$

The fitted exponential curve, $\hat{X}_t = 33.58(1.142)^t$.

b. The population size in 1999

$$t = \frac{1999 - 1959}{10} = \frac{40}{10} = 4$$

$$\hat{X}_t = 33.58(1.142)^t = 33.58(1.142)^4 = 57.114$$

iii) Growth Curve

In growth curves, the number of parameters is not equivalent to the number of parameters (variables), hence OLS method can't be used and some special techniques are required.

a. Modified exponential curve

$X_t = a + bc^t + e_t$ .... ①

where a, b and c are parameters to be estimated

- We use the method of 3 selected points i.e we choose 3 ordinates

X_1, X_2, X_3 which correspond to three equidistant time values t_1, t_2, t_3 respectively where $t_2 - t_1 = t_3 - t_2$.

- Assuming the error term, we will have ;

$$X_1 = a + bc^{t_1} \quad \ldots 2$$

$$X_2 = a + bc^{t_2} \quad \ldots 3$$

$$X_3 = a + bc^{t_3} \quad \ldots 4$$

Subtracting equation 2 from equation 3 we have ;-

$$X_2 - X_1 = bc^{t_2} - bc^{t_1} = bc^{t_1}[c^{t_2 - t_1} - 1] \ldots 5$$

Subtracting equation 3 from equation 4 we have ;-

$$X_3 - X_2 = bc^{t_3} - bc^{t_2} = bc^{t_2}[c^{t_3 - t_2} - 1] \ldots 6$$

Divide equation 6 by equation 5 we have

$t_3 - t_2 = t_2 - t_1$

$$\frac{X_3 - X_2}{X_2 - X_1} = \frac{bc^{t_2}[c^{t_3 - t_2} - 1]}{bc^{t_1}[c^{t_2 - t_1} - 1]} = c^{t_2 - t_1} \frac{[c^{t_3 - t_2} - 1]}{c^{t_2 - t_1} - 1}$$

$$\frac{X_3 - X_2}{X_2 - X_1} = c^{t_2 - t_1}$$

Therefore ; $$\hat{c} = \left[\frac{X_3 - X_2}{X_2 - X_1}\right]^{\frac{1}{t_2 - t_1}}$$ ✓

$$\left[\frac{X_3 - X_2}{X_2 - X_1}\right]^{t_2 - t_1}$$

From equation 5, can be written as :-

$$X_2 - X_1 = bc^{t_1}[c^{t_2 - t_1} - 1]$$

$$X_2 - X_1 = b\left[\frac{X_3 - X_2}{X_2 - X_1}\right]^{\frac{t_1}{t_2 - t_1}}\left[\left(\frac{X_3 - X_2}{X_2 - X_1}\right)^{\frac{t_2 - t_1}{t_2 - t_1}} - 1\right]$$

$$\frac{(X_3 - X_2) - (X_2 - X_1)}{X_2 - X_1}$$

$$X_3 - X_2 - X_2 - X_1$$

$$X_3 - 2X_2 - X_1$$

$$X_2 - X_1 = b\left[\frac{X_3 - X_2}{X_2 - X_1}\right]^{t_1/t_2 - t_1}\left[\frac{X_3 - X_2 - X_2 + X_1}{X_2 - X_1}\right]$$

$$\frac{(X_2 - X_1)^2}{X_3 - 2X_2 + X_1}$$

$$\frac{(X_2 - X_1)^2}{X_3 - 2X_2 + X_1} = b\left[\frac{X_3 - X_2}{X_2 - X_1}\right]^{t_1/t_2 + t_1}$$

$$\left(\frac{X_3 - X_2}{X_2 - X_1}\right)^{-t_1/t_2 - t_1}$$

$$\left(\frac{X_2 - X_1}{X_3 - X_2}\right)^{t_1/t_2 - t_1}$$

Therefore $$\hat{b} = \left[\frac{(X_2 - X_1)^2}{X_3 - 2X_2 + X_1}\right]\left[\frac{X_2 - X_1}{X_3 - X_2}\right]^{t_1/t_2 - t_1}$$ ✓

From equation 2

$$X_1 = a + bc^{t_1}$$

$$\hat{X}_1 = \hat{a} + \left[\frac{(X_2 - X_1)^2}{X_3 - 2X_2 + X_1}\right]\left[\frac{X_2 - X_1}{X_3 - X_2}\right]^{t_1/t_2 - t_1}\left[\frac{X_3 - X_2}{X_2 - X_1}\right]^{t_1/t_2 - t_1}$$

$$\dot{X}_1 = \hat{a} + \left[\frac{(X_2 - X_1)^2}{X_3 - 2X_2 + X_1}\right]$$

Therefore; $\hat{a} = \dot{X}_1 - \left[\frac{(X_2 - X_1)^2}{X_3 - 2X_2 + X_1}\right]$ $\quad X_2^2 - 2X_1X_2 + X_1^2$

$$\hat{a} = \frac{X_1X_3 - 2X_1X_2 + X_1^2 - (X_2 - X_1)^2}{X_3 - 2X_2 + X_1}$$

$$\hat{a} = \frac{X_1X_3 - 2X_1X_2 + X_1^2 - X_2^2 + 2X_1X_2 - X_1^2}{X_3 - 2X_2 + X_1}$$

$$\hat{a} = \frac{X_1X_3 + X_2^2}{X_3 - 2X_2 + X_1} \checkmark$$

And therefore the modified exponential curve is $\hat{X}_t = \hat{a} + \hat{b}\hat{c}^t$

b. Fitting a Gombertz Curve

The curve is given by; $X_t = ab^{c^t} + e_t$

Taking natural logarithms on both sides we have

$$\log X_t = \log ab^{c^t} + \log e_t$$

$$\log X_t = \log a + \log b^{c^t} + \log e_t$$

$$\log X_t = \log a + c^t \log b + \log e_t$$

$$Y_t = A + BC^t + E_t \quad \ldots *$$

This equation is in the form of modified exponential curve and so the previous method can be used to estimate A, B and C parameters.

c. Fitting a Logistic Curve

Here; $X_t = \frac{k}{1+e^{a+bt}} + e_t$

$$\frac{1}{X_t} = \frac{1+e^{a+bt}}{k} + \frac{1}{e_t} = \frac{1}{k} + \frac{e^{a+bt}}{k} + \frac{1}{e_t} = \frac{1}{k} + \frac{e^a}{k} \times e^{bt} + \frac{1}{e_t}$$

$$M = A + BC^t + E_t \quad \ldots **$$

where $M = \frac{1}{X_t}$, $A = \frac{1}{k}$, $B = \frac{e^a}{k}$, $C = e^b$, $E_t = \frac{1}{e_t}$

$\hat{a} = \log_e \hat{B}\hat{k}$, $\hat{b} = \log_e \hat{c}$, $\hat{k} = \frac{1}{\hat{A}}$, $\hat{X}_t = \frac{1}{\hat{M}}$

2 5x18

Equation ** is in the form of the modified exponential curve and therefore the previous method can be used to estimate the parameters.

Assignment

a. Measurement of seasonal variation
   1. Method of simple averages
   2. Ratio to trend methods
   3. Ratio to moving averages
   4. Link relative method

b. Reasons why we study seasonal variation

3. METHOD OF SEMI-AVERAGES

In this method, the semi-averages are calculated to find out the trend values. The data is divided into two equal halves and arithmetic mean of the two sets of values of X_t is plotted against the centre of the relative timespan.

- If the number of observations is even, the division of ΣX_t into halves will be straight forward. However, if the number of observations is odd, then the middle value/item i.e $\left(\frac{n+1}{2}\right)^{th}$ is dropped.

→ the arithmetic mean of the two sets of data

- The two points obtained are joined through a straight line which shows the trend. The trend values can be ~~ran~~ read from the graph corresponding to each time period.

Merits

1. The method is simple, easy and quick.
2. It smooths out the seasonal variations
3. It gives ~~the~~ a better approximation to the trend because it uses a mathematical model
4. It is objective because anyone applying the method to the given data will get identical trend values

Demerits

1. It is only applicable when the trend is linear and inappropriate when the trend is non-linear.
2. The arithmetic mean is ~~not~~ greatly affected by very large or very small values.
3. This method is crude

Example 1 [Even number of observations]

Fit a trend line by the method of semi-averages for the given data

n

$\frac{9+1}{2} = \frac{10}{2}$

(5)

Year	1990	1991	1992	1993	1994	1995	1996	1997	1998
Sales	15	11	20	10	15	25	35	30	

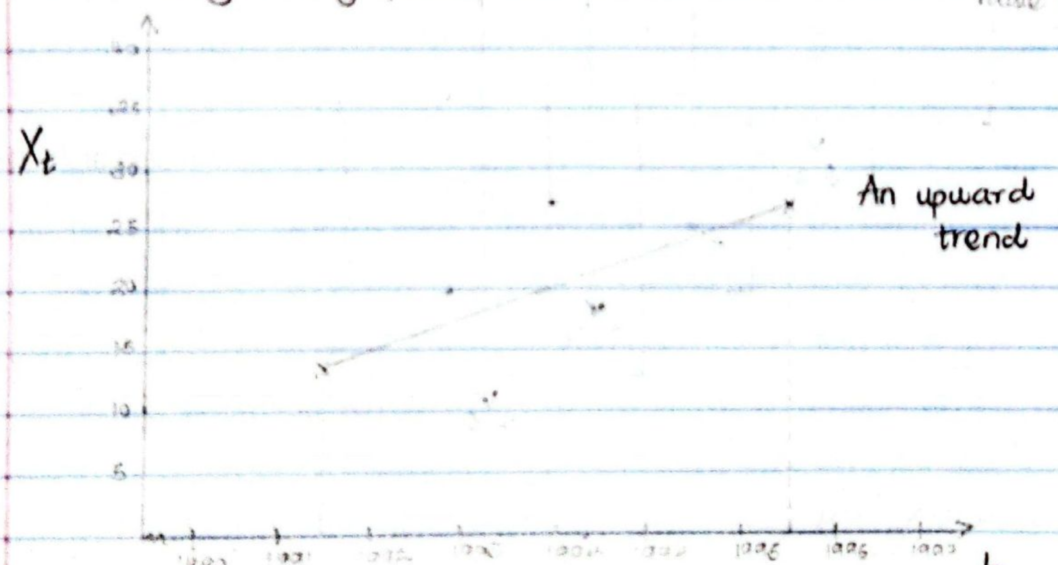
solution

The arithmetic mean for the two sets of data:-

$$Am_1 = \frac{15 + 11 + 20 + 10}{4} = 14 \qquad Am_2 = \frac{15 + 25 + 35 + 30}{4} = 26.25$$

Then plotting a graph :-

- For odd values, the middle value is ignored



4. METHOD OF MOVING AVERAGES - A method used to smoothen out random variations due to irregular components in the time series

This method measures the trend by smoothing out fluctuations of the data by means of moving average.

- The moving average of extent M is a series of successful average of m terms starting with the 1st, 2nd, 3rd etc
- The first average is the mean of the first term. The 2nd average is

the mean of m terms starting with the second to m+1 terms etc

- If the extent of M is an odd number i.e $m = 2k+1$, the 1st average is placed against the mean value of term interval it covers i.e $k+1$
- For instance consider the following data where $m = 3$

$m = 2k+1$
$3 = 2k+1$
$2 = 2k$
$k = 1$

t	observations	3-point moving average
1	3	
2	4	$\frac{3+4+10}{3} = 5.67$
3	10	$\frac{4+10+8}{3} = 7.33$
4	8	$\frac{10+8+7}{3} = 8.33$
5	7	$\frac{8+7+6}{3} = 7.$
6	6	$\frac{7+6+9}{3} = 7.33$
7	9	

Approximately an upward trend

- Again if m is even i.e $m = 2k$, the first moving average is placed between two values of the time interval it covers. For instance consider the following observations where $m = 4$.

t	1	2	3	4	5	6	7
Observations	3	4	10	8	7	6	9

6.25

7.25

7.75

7.5

An upward trend.

✓Bop
✓BFE

Demerits of moving average

1. Loss of data points - the method discards some initial and final values

* Maths for science
* Probs II
* TOE
* Real Analysis

STATIONARY TIME SERIES

27/03/2025

A time series is said to be stationary if it has no change in mean, no systematic change in variance and if strictly there is no periodic variation.

- Most probability theory of a time series is concerned with stationarity and for this reason, time series analysis often requires one to change a non-stationary time series into a stationary time series to be able to use the theory.

The following are the mathematical definition of a stationary time series:-

1. Stationarity in strict sense / First order stationarity

A time series is said to be stationary in the strict sense if the probability distribution of $X_{t_1}, X_{t_2}, X_{t_3}, \ldots, X_{t_n}$ is the same as the probability distribution of $X_{t_1+h}, X_{t_2+h}, X_{t_3+h}, \ldots, X_{t_n+h}$ for all values of $t_1, t_2, \ldots, t_n$ and h, where h is the distance between the observations.

- This means that the probability structure of the time series does not change with time.

2. Stationarity in weak sense / The Second order stationarity / covariance

A time series is said to be stationary in the weak sense if its mean is a constant and the autocovariance function is independent of time, t, but depends on the distance, h, between the observations.

- The autocovariance function denoted by $\delta(h)$ is defined by:-

$$\delta(h) = Cov(X_t, X_{t+h}) = E[X_t X_{t+h}] - E(X_t)E(X_{t+h})$$

Assumming the mean = 0, a constant ;-

$$\delta(h) = Cov(X_t X_{t+h}) = E(X_t X_{t+h})$$

- Since the autocovariance variance is influenced by a the units of ~~the~~ measurement, in order to compare the basic property of a time series it is often ~~used for~~ useful to use a function that is not influenced by the units of measurements. Such a function is called the **autocorrelation function** denoted $\rho(h)$ defined as

$$\rho(h) = \frac{Cov(X_t X_{t+h})}{\sqrt{Var(X_t)Var(X_{t+h})}} = \frac{\delta(h)}{\delta(0)}$$

Example 1

Let $X_t = a + bt + e_t$ where $e_t \sim N(0, \delta^2)$. Show whether the process is stationary.

solution

$E(X_t) = E(a + bt + e_t) = E(a) + E(bt) + E(e_t)$

$= a + bE(t) + 0$

$= a + bE(t)$

The mean is not constant and therefore the process is not stationary

Example 2

Let $X_t = e_1 \sin \lambda t + e_2 \cos \lambda t$, $t = 0, \pm 1, \pm 2, \dots$ be a time series where $e_i \sim iN(0, \delta^2)$ for $i = 1, 2, \dots$. Check for the stationarity of X_t

solution

$E(X_t) = E[e_1 \sin \lambda t + e_2 \cos \lambda t] = E(e_1) \sin \lambda t + E(e_2) \cos \lambda t$

but $e \sim N(0, \delta^2)$, the mean is zero, therefore

$E(X_t) = 0$

$Var(e_1) = E(e_1^2) - [E(e_1)]^2$

Va

$Cov(X_t X_{t+h})$

And $X_{t+h} = e_1 \sin \lambda (t+h) + e_2 \cos \lambda (t+h)$

Then, $Cov(X_t X_{t+h}) = E(X_t X_{t+h}) - E(X_t) E(X_{t+h})$, but $E(X_t) = 0$

$Cov(X_t X_{t+h}) = E(X_t X_{t+h})$

$= E[(e_1 \sin \lambda t + e_2 \cos \lambda t)(e_1 \sin \lambda [t+h] + e_2 \cos \lambda [t+h])]$

Therefores.

$Cov(X_t X_{t+h}) = E[e_1^2 \sin \lambda t \sin \lambda (t+h) + e_1 e_2 \sin \lambda t \cos \lambda (t+h) + e_1 e_2 \cos \lambda t \sin \lambda (t+h) + e_2^2 \cos \lambda t \cos \lambda (t+h)]$

$= E(e_1^2) \sin \lambda t \sin \lambda (t+h) + E(e_1 e_2) \sin \lambda t \cos \lambda (t+h) + E(e_1 e_2) \cos \lambda t \sin \lambda (t+h) + E(e_2^2) \cos \lambda t \cos \lambda (t+h)$

$Var(e_1) = E(e_1^2) - [E(e_1)]^2$, but $E(e_1) = 0$

$Var(e_1) = E(e_1^2) = \delta^2$

$Cov(X_t X_{t+h}) = \delta^2 \sin \lambda t \sin \lambda (t+h) + \delta^2 \cos \lambda t \cos \lambda (t+h)$

$= \delta^2 [\sin \lambda t \sin \lambda (t+h) + \cos \lambda t \cos \lambda (t+h)]$

Recall: $\sin(a+b) = \sin a \cos b + \sin b \cos a$

$\sin(a-b) = \sin a \cos b - \sin b \cos a$

$\cos(a+b) = \cos a \cos b - \sin a \sin b$

$\cos(a+b) = \cos a \cos b + \sin a \sin b$

$\tan(a+b) = \dfrac{\tan a + \tan b}{1 - \tan a \tan b}$

$\tan(a-b) = \dfrac{\tan a - \tan b}{1 + \tan a \tan b}$

From the identities :- Let $A = \lambda t$ and $B = \lambda(t+h)$

$Cov(X_t X_{t+h}) = E(X_t X_{t+h}) = \delta^2 [\cos(\lambda t - \lambda[t+h])]$

$= \delta^2 [\cos(\lambda t - \lambda t - \lambda h)] = \delta^2 [\cos(-\lambda h)]$

Therefore :-

$Cov(X_t X_{t+h}) = \delta^2 \cos \lambda h$ → This is a function that is only dependent to h

Since the first and second conditions are met, we conclude that the time series is stationary in the weak sense.

Differencing

This is a process of transforming a non-stationary process to stationary process. It is particularly useful in removing the trend component by simply differencing a given time series until it becomes stationary.

Example 1

Use differencing to convert the time series $X_t = a + bt + e_t$ for $e_t \sim N(0, \delta^2)$ from a non-stationary to a stationary time series.

solution

Let Δ be a differencing operator defined by

$\Delta X_t = X_t - X_{t-1} = Y_t$

$X_{t-1} = a + b(t-1) + e_{t-1}$

$\Delta X_t = a + bt + e_t - [a + b(t-1) + e_{t-1}]$

$= a + bt + e_t - a - bt + b - e_{t-1}$

$= b + e_t - e_{t-1} = Y_t$

$E(Y_t) = E(b + e_t - e_{t-1}) = E(b) + E(e_t) - E(e_{t-1})$, but $E(e_t) = E(e_{t-1})$

$E(Y_t) = b$

$Var(Y_t) = Var(b + e_t - e_{t-1}) = Var(b) + Var(e_t) - Var(e_{t-1})$

$= \delta^2 + \delta^2 = 2\delta^2$

Since the two conditions are met, we conclude that the time series is stationary in the weak sense.

Exercise

Consider a trend of second degree polynomial given by $X_t = a + bt + ct^2 + e_t$ where $e_t \sim N(0, \delta^2)$. Show whether the process is stationary and if not remove non-stationarity by applying the differencing technique

difference n times to make the series stationary

solution

To check the stationarity of the time series :-

$E(X_t) = E\left[a + bt + ct^2 + e_t\right] = E(a) + bE(t) + cE(t^2) + E(e_t)$

$E(X_t) = \cancel{a + bt + ct^2}\ a + bE(t) + cE(t)^2 + 0$

$E(X_t) = a + bE(t) + cE(t^2)$

The mean is not a constant therefore the process is not stationary

To convert the process from non-stationary to stationary :-

$\Delta X_t = X_t - X_{t-1} = Y_t$

$X_{t-1} = a + b(t-1) + c(t-1)^2 + e_{t-1}$

$= a + bt - b + c(t^2 + 2t + 1) + e_{t-1}$

$= a + bt - b + ct^2 - 2ct + c + e_{t-1}$

$\cancel{E(X_t) = E[a + bt - b + ct^2 - 2ct + c + e_{t-1}]}$

$\cancel{= E(a) + bE(bt)}$

$E\ \Delta X_t = a + bt + ct^2 + e_t - a - bt + b - ct^2 + 2ct - c - e_{t-1}$

$= b + 2ct - c + e_t - e_{t-1} = Y_t$

$E(Y_t) = E(b + 2ct - c + e_t - e_{t-1})$

$= E(b) + 2cE(t) - E(c) + E(e_t) - E(e_{t-1})$

$= b + 2cE(t) - E(c)$

Again, the mean is not a constant, it is a function of time, t, hence we apply differencing

$\Delta Y_t = Y_t - Y_{t-1}$

$Y_{t-1} = b + 2c(t-1) - c + e_{t-1} - e_{t-2}$

$\Delta Y_t = b + 2ct - c + e_t - e_{t-1} - [b + 2ct - 2c - c + e_{t-1} - e_{t-2}]$

$= \cancel{b} + \cancel{2ct} - \cancel{c} + e_t - e_{t-1} - \cancel{b} - \cancel{2ct} + 2c + \cancel{c} - e_{t-1} + e_{t-2}$

$= 2c + e_t + e_{t-2} - 2e_{t-1}$

$E(Y_t) = E(2c + e_t + e_{t-2} - 2e_{t-1})$

$= E(2c) + E(e_t) + E(e_{t-2}) - 2E(e_{t-1})$

But $E(e_t) = E(e_{t-2}) = E(e_{t-1}) = 0$

$E(Y_t) = E(2c) = 2c$

$Var(Y_t) = Var(2c + e_t - 2e_{t-1} + e_{t-2})$

$= Var(2c) + Var(e_t) + Var(-2e_{t-1}) + Var(e_{t-2})$

$= 0 + \delta^2 + 4\delta^2 + \delta^2 = 6\delta^2$

Since the two conditions are met, we conclude that the time series is stationary in the weak sense.

7/04/2025

CORRELOGRAM

A correlogram provides an objective ~~crit~~ criterion for exploring the nature of an internal structure of a time series

- It is obtained by plotting $\rho(h)$ against h
- The shape of the correlogram helps us to make inference about a time series. It identifies the probability model generating the time series

Useful time series processes

Here we describe several different types of stochastic processes which are useful in coming up with a model for a time series

1. WHITE NOISE / PURELY RANDOM PROCESS

A stochastic process is called a white noise if it consists of a ~~serie~~ sequence of a random variable e_t which are mutually independent and identically distributed with mean, μ, and variance, δ^2.

- A white noise is a process which has no memory i.e the value of the process at time t is uncorrelated with all the past values upto $t-1$ hence it can be used for forecasting future observations.

* Its autocovariance function, $\delta(h)$ is denoted by:-

$$\delta(h) = \begin{cases} \delta^2, & \text{for } h = 0 \\ 0, & h > 0 \end{cases}$$

Autocorrelation function, $\rho(h)$

$$\rho(h) = \begin{cases} 1, & \text{for } h = 0 \\ 0, & h > 0 \end{cases}$$

2. RANDOM WALK PROCESS

Suppose that e_t is a white noise process with mean, μ and variance δ^2. A process X_t is said to be a random walk if this process can be written as :-

$$X_t = X_{t-1} + e_t \text{, with } X_0 = 0.$$

Thus, for :-

$$t=1, \quad X_1 = X_0 + e_1 = 0 + e_1 = e_1.$$

$$t=2, \quad X_2 = X_1 + e_2 = e_1 + e_2.$$

$$t=3, \quad X_3 = X_2 + e_3 = e_1 + e_2 + e_3.$$

$$t=4, \quad X_4 = X_3 + e_4 = e_1 + e_2 + e_3 + e_4.$$

$$\vdots$$

$$t=t, \quad X_t = e_1 + e_2 + e_3 + e_4 + \dots + e_t.$$

In general, $X_t = \sum_{i=1}^{t} e_i$, which is a linear combination of the white noise

Now; $E(X_t) = \sum_{i=1}^{t} E(e_i)$, where $e_i \sim iid\, N(\mu, \delta^2)$.

$$= \sum m = tm, \text{ function of } t.$$

$$Var(X_t) = \sum Var(e_i) = \sum \delta^2 = t\delta^2, \text{ a function of } t.$$

Therefore, a random walk process is not stationary. We convert the process to a stationary process by differencing.

$$X_t = X_{t-1} + e_t.$$

$$\Delta X_t = X_t - X_{t-1}$$

$$= X_{t-1} + e_t - X_{t-1} = e_t$$

$$E(\Delta X_t) = E(e_t) = \mu$$

$$Var(\Delta X_t) = Var(e_t) = \delta^2$$

Both of which are constants

3. MOVING AVERAGE PROCESS

Let e_t be a pure random process with mean, 0 and variance δ^2. Then the process X_t is said to be a moving average process of order q, denoted as $MA(q)$ if

$$X_t = \beta_0 + \beta_1 e_{t-1} + \beta_2 e_{t-2} + \dots + \beta_q e_{t-q}.$$

$$= \sum_{j=0}^{q} \beta_j e_{t-j}$$

To check whether the process is stationary

$E(X_t) = \sum_{j=0}^{q} B_j E(e_{t-j})$, but $E(e_{t-j}) = 0$

$= 0$, which is a constant

$Var(X_t) = \sum B_j^2 Var(e_{t-j}) = \sum B_j^2 \delta^2$, which is a constant

Hence the process is stationary

Autocovariance Function, $\delta(h) = E(X_t X_{t+h}) - E(X_t) E[X_{t+h}]$

$$X_t = \sum B_j e_{t-j}$$

$$X_{t+h} = \sum B_k e_{t+h-k} = \sum B_k e_{t-(k-h)}$$

$$X_t X_{t+h} = \sum B_j e_{t-j} \sum B_k e_{t-(k-h)}$$

$$= \sum_{j=0}^{q} \sum_{k=0}^{q} B_j B_k e_{t-j} e_{t-(k-h)}$$

$$E(X_t X_{t+h}) = \sum \sum B_j B_k E[e_{t-j} e_{t-(k-h)}]$$

Let $k-h=j$, hence $k=j+h$

Then, $E(X_t X_{t+h}) = \sum_{j=0}^{q} \sum_{j+h=0}^{q} B_j B_{j+h} E[e_{t-j} e_{t-j}]$

$$E(X_t X_{t+h}) = \sum_{j=0}^{q} \sum_{j+h=0}^{q} B_j B_{j+h} E[e_{t-j}^2]$$

$= \sum \sum B_j B_{j+h} \delta^2$, which is a constant

When $j \neq k-h$,

$E(X_t X_{t+h}) = \sum_{j=0}^{q} \sum_{k=0}^{q} B_j B_k E[e_{t-j} e_{t-(k-h)}]$, the $e_t \sim iid N(0, \delta^2)$

$= \sum_{j=0}^{q} \sum_{k=0}^{q} B_j B_k E[e_{t-j}] E[e_{t-(k-h)}] = 0$, ~~due~~

Hence $E(X_t X_{t+h}) = 0$, due to independence.

Autocorrelation function, $\rho(h) = \dfrac{\delta(h)}{\delta(0)}$

But $\delta(h) = \sum_{j=0}^{q} \sum_{k=0}^{q} B_j B_k E[e_{t-j} e_{t-(k-h)}]$, Let $j=k-h$, $k=j+h$

$$= \sum_{j=0}^{q} \sum_{k=0}^{q} B_j B_{j+h} E[e_{t-j} e_{t-j}]$$

$$= \sum_{j=0}^{q-h} B_j B_{j+h} \delta^2$$

$$= \delta^2 \sum_{j=0}^{q-h} B_j B_{j+h}$$

Again, $\delta(0) = \delta^2 \sum_{j=0}^{q} B_j B_j = \delta^2 \sum_{j=0}^{q} B_j^2$

~~&1~~ ~~the~~ Thus, the autocorrelation function.

$$\text{Autocorrelation function}, \ \rho(h) = \frac{\delta(h)}{\delta(0)} = \begin{cases} 1, & \text{for } h = 0 \\ \dfrac{\sum_{j=0}^{q-h} \beta_j \beta_{j+h}}{\sum_{j=0}^{q} \beta_j^2}, & \text{for } 0 < h < q \\ 0, & \text{for } h > q \end{cases}$$

Therefore, $\delta(h)$ is a constant and mean is zero, hence the moving average is stationary.

Example 1

Consider the moving average process of order 2

$$X_t = \beta_0 e_t + \beta_1 e_{t-1} + \beta_2 e_{t-2}.$$

Find the autocovariance function and autocorrelation function

solution

$q = 2$, $\beta_0 = \beta_0$, $\beta_1 = \beta_1$ and $\beta_2 = \beta_2$

a. The Autocovariance function

$$\delta(h) = \delta^2 \sum_{j=0}^{q-h} \beta_j \beta_{j+h}$$

when $h = 0$, $\delta(0) = \delta^2 \sum_{j=0}^{2} \beta_j \beta_j = \delta^2 \sum_{j=0}^{2} \beta_j^2$

$j = 0$.

$$\delta(0) = \delta^2 [\beta_0\beta_0 + \beta_1\beta_1 + \beta_2\beta_2] = \delta^2 [\beta_0^2 + \beta_1^2 + \beta_2^2]$$

$h = 1$, $\delta(1) = \delta^2 \sum_{j=0} \beta_j \beta_{j+h}$

$$= \delta^2 [\beta_0\beta_1 + \beta_1\beta_2].$$

$h = 1$, $\delta(2) = \delta^2 \sum_{j=0}^{0} \beta_j \beta_{j+h} = \delta^2 [\beta_0 \beta_2]$

b. The autocorrelation function.

$$\rho(h) = \begin{cases} 1, & h = 0 \\ \dfrac{\sum_{j=0}^{q-h} \beta_j \beta_{j+h}}{\sum_{j=0}^{q} \beta_j^2}, & 0 < h < q \\ 0, & h > q \end{cases}$$

when $h = 0$, $\rho(h) = 1$.

when $h = 1$

$$\rho(h=1) = \frac{\delta(h=1)}{\delta(0)} = \frac{\delta^2 [\beta_0\beta_1 + \beta_1\beta_2]}{\delta^2 [\beta_0^2 + \beta_1^2 + \beta_2^2]}$$

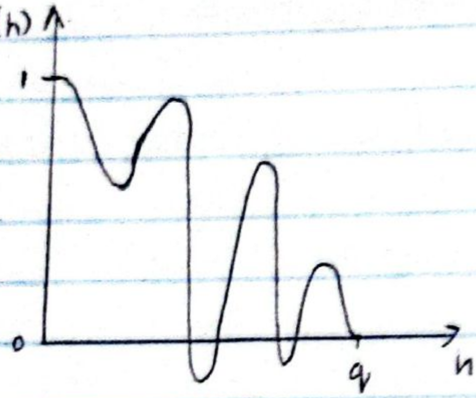
$$= \frac{\beta_0\beta_1 + \beta_1\beta_2}{\beta_0^2 + \beta_1^2 + \beta_2^2}$$

when $h=2$, $\rho(h=2) = \dfrac{\gamma(h=2)}{\gamma(h=0)} = \dfrac{\sigma^2[\beta_0\beta_2]}{\sigma^2[\beta_0^2+\beta_1^2+\beta_2^2]} = \dfrac{\beta_0\beta_2}{\beta_0^2+\beta_1^2+\beta_2^2}$

$\rho(h>2) = 0$

Therefore,
$$\rho(h) = \begin{cases} 1, & h=0 \\ \dfrac{\beta_0\beta_1+\beta_1\beta_2}{\beta_0^2+\beta_1^2+\beta_2^2}, & h=\pm 1 \\ \dfrac{\beta_0\beta_2}{\beta_0^2+\beta_1^2+\beta_2^2}, & h=\pm 2 \\ 0, & h>2 \end{cases}$$

Plotting a correlogram, correlation lies between $-1<r<1$



- There are no restrictions imposed on β_j for a moving average process to be stationary, hence there may be different outcomes from the same correlation function

- No restrictions on the β_j are required for an MA process to be stationary
- However, it is generally desirable to impose restrictions on β_j to ensure that the processes satisfy an invertibility condition (which ensures there is a unique MA process for a given autocorrelation function)

Q1: Consider 2 MA processes of order 1 i.e. $q=1$

Model A $\Rightarrow X_t = e_t + \theta e_{t-1}$, where $\beta_0 = 1$ and $\beta_1 = \theta$

Model B $\Rightarrow X_t = e_t + \frac{1}{\theta} e_{t-1}$, where $\beta_0 = 1$ and $\beta_1 = \frac{1}{\theta}$

$$\rho(h) = \begin{cases} 1, & h=0 \\ \dfrac{\sum \beta_j \beta_{j+h}}{\sum \beta_j^2}, & h=\pm 1 \\ 0, & h>1 \end{cases}$$

solution

When $h=0$, $\rho(h=0)=1$

When $h=1$, $\rho(h=1) = \dfrac{\beta_0 \beta_1}{\beta_0^2 + \beta_1^2} = \dfrac{1 \cdot \theta}{1+\theta^2} = \dfrac{\theta}{1+\theta^2}$

Therefore, $\rho(h)_A = \begin{cases} 1, & h=0 \\ \dfrac{\theta}{1+\theta^2}, & h=\pm 1 \\ 0, & h>1 \end{cases}$

Again, for model 2

$h=0$, $\rho(h=0)=1$

$h=1$, $\rho(h=1) = \dfrac{\beta_0 \beta_1}{\beta_0^2 + \beta_1^2 + \cancel{\beta_2^2}} = \dfrac{1 \times 1/\theta}{1 + 1/\theta^2} = \dfrac{1/\theta}{1+1/\theta^2} = \dfrac{1/\theta}{\frac{\theta^2+1}{\theta^2}}$

$\rho(h=1) = 1/\theta \times \theta^2/\theta^2+1 = \dfrac{\theta}{1+\theta^2}$

Therefore, $\rho(h)_B = \begin{cases} 1, & h=0 \\ \dfrac{\theta}{1+\theta^2}, & h=\pm 1 \\ 0, & h>1 \end{cases}$

Invert model A to make e_t the subject

$e_t = X_t - \theta e_{t-1}$

$e_{t-1} = X_{t-1} - \theta e_{t-2}$

Therefore, $e_t = X_t - \theta [X_{t-1} - \theta e_{t-2}]$

$= X_t - \theta X_{t-1} + \theta^2 e_{t-2}$

$e_t = X_t - \theta X_{t-1} + \theta^2 [X_{t-2} - \theta e_{t-2}]$

$= X_t - \theta X_{t-1} + \theta^2 X_{t-2} - \theta^3 e_{t-2}$

$= X_t - \theta X_{t-1} + \theta^2 X_{t-2} - \theta^3 X_{t-3} + \theta^4 X_{t-4} - \theta^5 X_{t-5} + \dots$

Therefore, $X_t = e_t + \theta X_{t-1} - \theta^2 X_{t-2} + \theta^3 X_{t-3} - \theta^4 X_{t-4} + \theta^5 X_{t-5} - \dots$

For model B, $e_t = X_t - \dfrac{1}{\theta} e_{t-1}$

$X_t = e_t + \dfrac{1}{\theta} X_{t-1} - \dfrac{1}{\theta^2} X_{t-2} + \dfrac{1}{\theta^3} X_{t-3} + \dots$

If $|\theta| < 1$, model A converges

The inverted series of model A converges, while that of model B diverges for $|\theta|<1$. Model A is said to be invertible while model B not. Therefore, the property of invertibility ensures there is a unique MA process for a given ACF.

- To test for invertibility for an MA process of order q, let $B^i e_t = e_{t-i}$, $\forall_i$ be a backward shift operator.

$MA(q)$, $X_t = \beta_0 e_t + \beta_1 e_{t-1} + \beta_2 e_{t-2} + \ldots + \beta_q e_{t-q}$

Applying a backward shift operator to this MA

$X_t = \beta_0 B^0 e_t + \beta_1 B^1 e_t + \beta_2 B^2 e_t + \ldots + \beta_q B^q e_t$

$X_t = [\beta_0 B^0 + \beta_1 B^1 + \beta_2 B^2 + \ldots + \beta_q B^q] e_t$

$X_t = \phi(B) e_t$

where $\phi(B)$ is a polynomial of order q in B

Then, we say that an $MA(q)$ is invertible if the roots of the equation $\phi(B) = 0$, all lie outside the unit circle (diameter = 1) ie the absolute values of the roots are > 1

→ For model A, $X_t = e_t + \theta e_{t-1}$

Using the backward shift operator, $B^i e_t = e_{t-i}$

$X_t = [B^0 e_t + \theta B^1 e_t] = [B^0 + \theta B^1] e_t = [1 + \theta B] e_t$

$1 + \theta B = 0$; hence $B = -\frac{1}{\theta}$

$|\theta| < 1$, $B = \frac{1}{\theta}$, which lies outside a unit circle and therefore the process is inverted.

→ For model B, $X_t = e_t + \frac{1}{\theta} e_{t-1}$

Applying the backward shift operator, $B^i e_t = e_{t-i}$

$X_t = [\beta_0 B^0 e_t + \beta_1 B^1 e_t] = [B^0 e_t + \frac{1}{\theta} B^1 e_t] = \left[1 + \frac{1}{\theta} B\right] e_t$

$1 + \frac{1}{\theta} B = 0$	$\lvert\theta\rvert < 1$
$\frac{1}{\theta} B = -1$	$B = \theta$, which lies inside the unit circle and
$B = -\theta$	therefore Model B is not invertible.

Example 1

Determine whether the following processes are invertible and also find the autocorrelation function

$X_t = e_t + 0.7\, e_{t-1} - 0.2\, e_{t-2}$, $\beta_0 = 1$, $\beta_1 = 0.7$, $\beta_2 = -0.2$.

solution

The backward shift operator is $B^1 e_t = e_{t-1}$

$X_t = (\beta_0 B^0 e_t + \beta_1 B^1 e_t + \beta_2 B^2 e_t)$

$X_t = (B^0 e_t + 0.7 B^1 e_t + 0.2 B^2 e_t)$

$X_t = (1 + 0.7B - 0.2B^2) e_t = \phi(B)t$.

$1 + 0.7B - 0.2B^2 = 0 \Rightarrow 0.2B^2 - 0.7B - 1$

$$B = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{0.7 \pm \sqrt{0.7^2 - 4 \times 0.2 \times -1}}{2 \times 0.2}$$

$$B = \frac{0.7 \pm 1.13578}{0.4}$$

$$B = \frac{0.7 + 1.13578}{0.4} \quad \text{or} \quad B = \frac{0.7 - 1.13578}{0.4}$$

$B = 4.589$ or -1.08945

$|B| > 1$, hence X_t is invertible.

$$\rho(h) = \begin{cases} 1, & h = 0 \\ \dfrac{\sum_{j=0}^{q-h} \beta_j \beta_{j+h}}{\sum_{j=0}^{q} \beta_j^2}, & h = \pm 1, \pm 2 \\ 0, & h > 2 \end{cases}$$

when $h = 1$, $\rho(h=1) = \dfrac{\beta_0 \beta_1 + \beta_1 \beta_2}{\beta_0^2 + \beta_1^2 + \beta_2^2}$, $\beta_0 = 1$, $\beta_1 = 0.7$, $\beta_2 = -0.2$.

$$\rho(h=1) = \frac{1 \times 0.7 + 0.7 \times -0.2}{1^2 + 0.7^2 + (-0.2)^2} = \frac{0.7 - 0.14}{1.53} = 0.366$$

when $h = 2$, $\rho(h=2) = \dfrac{\sum_{j=0}^{0} \beta_j \beta_{j+2}}{\sum_{j=0}^{2} \beta_j^2} = \dfrac{\beta_0 \beta_2 + B}{\beta_0^2 + \beta_1^2 + \beta_2^2}$

$$\rho(h=2) = \frac{-0.2}{1.53} = -0.1307$$

Therefore, $\rho(h) = \begin{cases} 1, & h=0 \\ 0.366, & h=\pm 1 \\ -0.1307, & h=\pm 2 \\ 0, & h>2 \end{cases}$

Exercise

Determine if the processes are invertible and get the ACF

1. $X_t = 2e_t + 3e_{t-1} + 3e_{t-2} - 2e_{t-3}$
2. $X_t = e_t - 1.3e_{t-1} + 0.4e_{t-2}$