JOMO KENYATTA UNIVERSITY OF AGRICULTURE AND TECHNOLOGY

DEPARTMENT OF MATHEMATICS AND ACTUARIAL SCIENCE

STA 2401: TIME SERIES ANALYSIS - CAT II
DATE: APRIL 25, 2025. TIME: 60 MINUTES

INSTRUCTIONS: ATTEMPT ALL QUESTIONS

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1. **What is the main difference between AR(p) and MA(q) modeling? (2 Marks)**

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Feature	AR(p)	MA(q)
Model Type	Autoregressive	Moving Average
Dependence	Past observations (X_{t-k})	Past erro rs (ϵ_{t-k})
ACF Pattern	Tails off gradually	Cuts off after lag q
PACF Pattern	Cuts off after lag p	Tails off gradually
Use Case	Long-term trends, memory effects	Short-term shocks, noise

2. **Suppose ϵ_t is zero-mean white noise with $\mathrm{var}(\epsilon_t) = \sigma^2$. Consider the model defined by;**

$$X_t = 0.25 + 0.75X_{t-1} + \epsilon_t \tag{1}$$

(a) **Identify the time series model in Equation (1). (1 Mark)**

(α) The model is of the form $X_t = c + \phi X_{t-1} + \epsilon_t$.

(β) This looks like an autoregressive model because X_t depends on its previous value X_{t-1}.

(γ) The order is 1 because it only depends on X_{t-1}.

Therefore, the model is an AR model of order 1. That is, an AR(1) process.

(b) **Estimate the following from the model in Equation (1)**:

(i) **Expected value of X_t (1 Mark)**

For a stationary AR(1) process, the expected value $E(X_t)$ is constant over time. The model is;

$$X_t = 0.25 + 0.75X_{t-1} + \epsilon_t$$

Taking expectations on both sides, we have;

$$E(X_t) = 0.25 + 0.75E(X_{t-1}) + E(\epsilon_t)$$

At stationary,

$$E(\epsilon_t) = 0$$

and

$$E(X_t) = E(X_{t-1}).$$

So,

$$E(X_t) = 0.25 + 0.75E(X_t)$$

Solving for $E(X_t)$, we have;

$$\begin{aligned} E(X_t) - 0.75E(X_t) &= 0.25 \\ 0.25E(X_t) &= 0.25 \\ E(X_t) &= 1 \end{aligned}$$

Therefore, the expected value is 1.

(ii) **Variance function and standard deviation of of X_t** **(3 Marks)**
For a stationary AR(1) process;

$$X_t = c + \phi X_{t-1} + \epsilon_t,$$

the variance γ_0 is given as;

$$\gamma_0 = \frac{\sigma^2}{1 - \phi^2}$$

From the question, $\phi = 0.75$ and σ^2 is the variance of ϵ_t, and we have;

$$\gamma_0 = \frac{\sigma^2}{1 - (0.75)^2} = \frac{\sigma^2}{1 - 0.5625} = \frac{\sigma^2}{0.4375} \approx 2.2857\sigma^2$$

The standard deviation is the square root of the variance, defined as;

$$\sqrt{\gamma_0} \approx \sqrt{2.2857\sigma^2} \approx 1.5119\sigma$$

Therefore, the variance is $2.2857\sigma^2$ and the standard deviation is 1.5119σ.

(iii) **Autocovariance function of X_t** **(2 Marks)**
For an AR(1) process, the autocovariance at lag k is;

$$\gamma_k = \phi^{|k|}\gamma_0$$

We already have γ_0 from part (ii). So, for any lag k, the autocovariance is;

$$\gamma_k = (0.75)^{|k|} \times \frac{\sigma^2}{0.4375}$$

Therefore, the autocovariance function is;

$$\gamma_k = (0.75)^{|k|} \times \frac{\sigma^2}{0.4375} = (0.75)^{|k|} \times 2.2857\sigma^2, \text{for} k \geq 0.$$

(c) **With reference to (b)(i, ii, and iii), comment on the model identified (1 Mark)**

(i) The mean is constant over time.
(ii) The variance is finite and constant.
(iii) The autocovariance depends only on the lag k and decays exponentially.

Therefore, the model is stationary since it has a constant mean, finite variance, and autocovariance that depends only on the lag and decays exponentially.

(d) **State the acf of the model identified in Equation (1).** **(2 Marks)**
From part (ii), and (iii), we have;

$$\gamma_0 \frac{\sigma^2}{0.4375} \approx 2.2857\sigma^2$$

and

$$\gamma_k = \phi^{|k|}\gamma_0$$

So,

$$\rho_k = \frac{\gamma_k}{\gamma_0} = \phi^{|k|} = (0.75)^{|k|}$$

Therefore, the ACF is $\rho_k = (0.75)^{|k|}$ for $k \geq 0$.

(e) **Using the result of (d), create a table of the autocorrelations, ρ_k of the process at the following lags: 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9.** **(2 Marks)**
Using the ACF $\rho_k = (0.75)^{|k|}$, we can compute ρ_k for each lag k as;

$$\begin{aligned}
\rho_0 &= (0.75)^0 = 1.0000 \\
\rho_1 &= (0.75)^1 = 0.7500 \\
\rho_2 &= (0.75)^2 = 0.5625 \\
\rho_3 &= (0.75)^3 \approx 0.4219 \\
\rho_4 &= (0.75)^4 \approx 0.3164 \\
\rho_5 &= (0.75)^5 \approx 0.2373 \\
\rho_6 &= (0.75)^6 \approx 0.1780 \\
\rho_7 &= (0.75)^7 \approx 0.1335 \\
\rho_8 &= (0.75)^8 \approx 0.1001 \\
\rho_9 &= (0.75)^9 \approx 0.0751
\end{aligned}$$

Therefore, the autocorrelations at the given lags are;

Lag k	Autocorrelation ρ_k
0	1.0000
1	0.7500
2	0.5625
3	0.4219
4	0.3164
5	0.2373
6	0.1780
7	0.1335
8	0.1001
9	0.0751

3. **Assuming the autocorrelations, ρ_k in 2(e) are from a data consisting of 400 observations, use $\alpha = 0.05$ to test for the evidence of non-randomness in the data** **(2 Marks)**

Simplification of Inequality

The inequality for investigating non-randonmess is defined as;

$$-Z_{\alpha/2} \times \frac{1}{\sqrt{n}} < \rho < Z_{\alpha/2} \times \frac{1}{\sqrt{n}} = -1.96 \times \frac{1}{\sqrt{400}} < \rho < 1.96 \times \frac{1}{\sqrt{400}}$$

Step 1: Simplify the Right-Hand Side

The term $\frac{1}{\sqrt{400}}$ can be simplified because $\sqrt{400} = 20$. Therefore:

$$-1.96 \times \frac{1}{20} < \rho < 1.96 \times \frac{1}{20}$$

Step 2: Perform the Multiplication

$$-1.96 \times 0.05 < \rho < 1.96 \times 0.05$$
$$-0.098 < \rho < 0.098$$

Step 3: Final Simplified Inequality

The final simplified inequality is defined as;

$$-0.098 < \rho < 0.098$$

Identifying Values Outside the Interval and Calculating the Percentage

Given the ρ values, use the following steps to identify values outside the interval and calculate the percentage as;

Step 1: Check Each ρ Value

(1) $\rho_0 = 1.0000 \rightarrow 1.0000 > 0.098 \rightarrow$ Outside

(2) $\rho_1 = 0.7500 \rightarrow 0.7500 > 0.098 \rightarrow$ Outside

(3) $\rho_2 = 0.5625 \rightarrow 0.5625 > 0.098 \rightarrow$ Outside

(4) $\rho_3 \approx 0.4219 \rightarrow 0.4219 > 0.098 \rightarrow$ Outside

(5) $\rho_4 \approx 0.3164 \rightarrow 0.3164 > 0.098 \rightarrow$ Outside

(6) $\rho_5 \approx 0.2373 \rightarrow 0.2373 > 0.098 \rightarrow$ Outside

(7) $\rho_6 \approx 0.1780 \rightarrow 0.1780 > 0.098 \rightarrow$ Outside

(8) $\rho_7 \approx 0.1335 \rightarrow 0.1335 > 0.098 \rightarrow$ Outside

(9) $\rho_8 \approx 0.1001 \rightarrow 0.1001 > 0.098 \rightarrow$ Outside

(10) $\rho_9 \approx 0.0751 \rightarrow -0.098 < 0.0751 < 0.098 \rightarrow$ Inside

Step 2: Count Values Outside the Interval

All ρ_k for $k = 0$ to 8 are outside the interval, except ρ_9. Therefore;

(a) Total values outside the interval is 9.

(b) Total values considered: 10

Step 3: Calculate the Percentage

$$Percentage\ outside = \left(\frac{9}{10}\right) \times 100 = 90\%$$

Conclusions

(a) Values outside the interval are: $\rho_0, \rho_1, \rho_2, \rho_3, \rho_4, \rho_5, \rho_6, \rho_7, \rho_8$

(b) Percentage outside the interval is 90%

(c) Therefore, there are 9 (90%) of the autocorrelation coefficients are lying outside the limit or internal which contribute more than 5% of the total number of autocorrelations. Hence, there is evidence of non-randomness in the data.

4. **Examine the invertibility of the process defined by**;

$$X_t = \epsilon_t - 0.8\epsilon_{t-1} + 0.15\epsilon_{t-2} \tag{2}$$

where ϵ_t is a zero-mean white noise process with variance, σ^2, using the Backward Shift Operator (BSO) (3 Marks)

Step 1: Understand the Problem

We are given a time series process defined by:

$$X_t = \epsilon_t - 0.8\epsilon_{t-1} + 0.15\epsilon_{t-2}$$

where ϵ_t is a white noise process with mean zero and variance σ^2 to examine the invertibility of this process using BSO.

Step 2: Recall Definitions

For an MA process, invertibility is determined by the roots of the MA polynomial. The BSO is denoted by B, which shifts the time index back by one period, i.e., $B\epsilon_t = \epsilon_{t-1}$.

Step 3: Rewrite the Process Using BSO

Given the process;

$$X_t = \epsilon_t - 0.8\epsilon_{t-1} + 0.15\epsilon_{t-2}$$

We can express ϵ_{t-1} and ϵ_{t-2} using the BSO as;

$$\epsilon_{t-1} = B\epsilon_t$$
$$\epsilon_{t-2} = B^2\epsilon_t$$

Substituting these into the original equation, we have;

$$X_t = \epsilon_t - 0.8B\epsilon_t + 0.15B^2\epsilon_t = (1 - 0.8B + 0.15B^2)\epsilon_t$$

Step 4: Define the MA Polynomial

The MA polynomial $\theta(B)$ is;

$$\theta(B) = 1 - 0.8B + 0.15B^2$$

Step 5: Find the Roots of the MA Polynomial

To check invertibility, we need to find the roots of $\theta(B) = 0$. That is;

$$1 - 0.8B + 0.15B^2 = 0$$

This is a quadratic equation in B. Solving for B, we have;

$$0.15B^2 - 0.8B + 1 = 0$$

Using the quadratic formula, we have;

$$B = \frac{0.8 \pm \sqrt{(-0.8)^2 - 4 \cdot 0.15 \cdot 1}}{2 \cdot 0.15}$$

$$B = \frac{0.8 \pm \sqrt{0.64 - 0.6}}{0.3}$$

$$B = \frac{0.8 \pm \sqrt{0.04}}{0.3}$$

$$B = \frac{0.8 \pm 0.2}{0.3}$$

Thus, the roots are;

$$B_1 = \frac{0.8 + 0.2}{0.3} = \frac{1.0}{0.3} \approx 3.333$$

$$B_2 = \frac{0.8 - 0.2}{0.3} = \frac{0.6}{0.3} = 2$$

Step 6: Check of Invertibility Conditions and Conclusions

(a) Root $B_1 \approx 3.333$: $|3.333| > 1$

(b) Root $B_2 = 2$: $|2| > 1$

Since both roots satisfy the condition $|B| > 1$, the given MA process is invertible.

5. **In general, the least squares estimates $(\phi_1, \phi_2, \ldots, \phi_p)$ of an AR(p) process are obtained by minimizing;**

$$SSE(\phi) = \sum_{t=1}^{n} (X_t - \phi_1 X_{t-1} - \phi_2 X_{t-2} - \cdots - \phi_p X_{t-p})^2 \tag{3}$$

and setting the partial differentials with respect to the parameter of interest to zero. For $p = 1$, show that the least squares estimate of the parameter of interest is; **(4 Marks)**

$$\hat{\phi}_1 = \frac{\sum_{t=2}^{n} X_{t-1} X_t}{\sum_{t=2}^{n} X_{t-1}^2} \tag{4}$$

Deriving the Least Squares Estimate for AR(1)

An autoregressive model of order 1 is defined as;

$$X_t = \phi_1 X_{t-1} + \epsilon_t$$

where ϵ_t is white noise with mean 0 and variance σ^2. We wish to find $\hat{\phi}_1$ that minimizes the sum of squared errors (SSE);

$$\text{SSE}(\phi_1) = \sum_{t=2}^{n} (X_t - \phi_1 X_{t-1})^2$$

Step 1: Expand SSE

$$\text{SSE}(\phi_1) = \sum_{t=2}^{n} (X_t^2 - 2\phi_1 X_t X_{t-1} + \phi_1^2 X_{t-1}^2)$$

Step 2: Differentiate SSE with respect to ϕ_1

To find the minimum, we take the derivative and set it to zero;

$$\frac{d}{d\phi_1}\text{SSE}(\phi_1) = \sum_{t=2}^{n} (-2X_t X_{t-1} + 2\phi_1 X_{t-1}^2) = 0$$

Step 3: Solve for ϕ_1

$$-2\sum_{t=2}^{n} X_t X_{t-1} + 2\phi_1 \sum_{t=2}^{n} X_{t-1}^2 = 0$$

Dividing both sides by 2, we have;

$$-\sum_{t=2}^{n} X_t X_{t-1} + \phi_1 \sum_{t=2}^{n} X_{t-1}^2 = 0$$

Rearranging, we have;

$$\phi_1 \sum_{t=2}^{n} X_{t-1}^2 = \sum_{t=2}^{n} X_t X_{t-1}$$

Hence, the estimate is defined as;

$$\hat{\phi}_1 = \frac{\sum_{t=2}^{n} X_{t-1} X_t}{\sum_{t=2}^{n} X_{t-1}^2}$$

6. **Hence, state the formula for estimating the $\hat{\sigma}^2$ of the process** **(2 Marks)**

Estimating the Variance σ^2

For an AR(1) model, the variance of the error term ϵ_t can be estimated using the residuals. The residuals are the differences between the observed values and the values predicted by the model;

$$\hat{\epsilon}_t = X_t - \hat{\phi}_1 X_{t-1}$$

The variance σ^2 is estimated by the mean squared error (MSE) of the residuals. For an AR(1) model with $n-1$ residuals (since we start from $t = 2$) and p is number of parameters to be estimated:

$$\hat{\sigma}^2 = \frac{1}{n-1-p}\sum_{t=2}^{n}\hat{\epsilon}_t^2 = \frac{1}{n-2}\sum_{t=2}^{n}(X_t - \hat{\phi}_1 X_{t-1})^2$$

7. **Using the results of *Q 5* and *Q 6*, consider the following observed time series data: $X_1 = 1.5$, $X_2 = 2.5$ and $X_3 = 3.5$, and estimate the following parameters.**

   (a) **Estimating $\hat{\phi}_1$** **(2 Marks)**
   Given the data: $X_1 = 1.5, X_2 = 2.5, X_3 = 3.5$
   Use the formula from Question 4 defined as;

$$\hat{\phi}_1 = \frac{\sum_{t=2}^{3} X_{t-1}X_t}{\sum_{t=2}^{3} X_{t-1}^2}$$

   Now, compute the Numerator and Denominator

   Numerator:

$$\begin{aligned} X_1X_2 &= 1.5 \times 2.5 = 3.75 \\ X_2X_3 &= 2.5 \times 3.5 = 8.75 \\ \text{Total} &= 3.75 + 8.75 = 12.50 \end{aligned}$$

   Denominator:

$$\begin{aligned} X_1^2 &= 1.5^2 = 2.25 \\ X_2^2 &= 2.5^2 = 6.25 \\ \text{Total} &= 2.25 + 6.25 = 8.50 \end{aligned}$$

   Thus, we have;

$$\hat{\phi}_1 = \frac{12.5.00}{8.50} \approx 1.4706$$

   (b) **Estimate $\hat{\sigma}^2$** **(2 Marks)**
   First, compute the residuals using;

$$\hat{\epsilon}_t = X_t - \hat{\phi}_1 X_{t-1}$$

(a) For $t = 2$, we have;

$$\hat{\epsilon}_2 = 2.5 - 1.4706 \times 1.5 \approx 2.5 - 2.2059 \approx 0.2941$$

(b) For $t = 3$, we have;

$$\hat{\epsilon}_3 = 3.5 - 1.4706 \times 2.5 \approx 3.5 - 3.6765 \approx -0.1765$$

Squaring the residuals, we have;
(a) $\hat{\epsilon}_2^2 \approx (0.2941)^2 \approx 0.0865$
(b) $\hat{\epsilon}_3^2 \approx (-0.1765)^2 \approx 0.0312$
For the sum of squared residuals, we have;

$$0.0865 + 0.0312 \approx 0.1177$$

Dividing by $n - 2 = 3 - 2 = 1$, we have;

$$\hat{\sigma}^2 \approx \frac{0.1177}{1} \approx 0.1177$$

8. **Define the term Spectral Density Function and hence, find the Spectral Density Function of and the AR(1) process (3 Marks)**

Definition of Spectral Density Function

The Spectral Density Function (SDF), also known as the power spectral density (PSD), is a fundamental concept in time series analysis. It describes how the variance (or power) of a stochastic process is distributed across different frequencies. Mathematically, for a stationary process $\{X_t\}$, the spectral density $f(\omega)$ is the Fourier transform of its autocovariance function $\gamma(h)$;

$$f(\omega) = \sum_{h=-\infty}^{\infty} \gamma(h) e^{-ih\omega}, \quad \omega \in [-\pi, \pi]$$

The SDF decomposes the total variance of the process into contributions from different frequency components. Peaks in the SDF indicate dominant frequencies (periodicities) in the process.

Derivation of the Spectral Density Function for AR(1) Process

Step 1: Recall the Autocovariance Function (ACVF) of AR(1)

For an AR(1) process, we have;

$$\gamma(k) = \frac{\sigma^2}{1 - \phi^2} \phi^{|k|} \quad \text{for } k = 0, \pm 1, \pm 2, \ldots$$

Step 2: Definition of SDF

The SDF $f(\lambda)$ of a stationary process is the Fourier transform of its acvf, $\gamma(k)$ is;

$$f(\lambda) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \gamma(k) e^{-ik\lambda}$$

where;

(a) λ is the frequency in radians $(-\pi \leq \lambda \leq \pi)$

(b) i is the imaginary unit

Step 3: Substitute the ACVF into the SDF Formula

Given $\gamma(k) = \frac{\sigma^2}{1-\phi^2}\phi^{|k|}$, the SDF becomes;

$$f(\lambda) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \frac{\sigma^2}{1-\phi^2} \phi^{|k|} e^{-ik\lambda}$$

Step 4: Simplify the Summation

First, factor out the constants;

$$f(\lambda) = \frac{\sigma^2}{2\pi(1-\phi^2)} \sum_{k=-\infty}^{\infty} \phi^{|k|} e^{-ik\lambda}$$

The summation can be split into three parts:

(a) m $k = 0$: $\phi^0 e^0 = 1$

(b) $k > 0$: $\sum_{k=1}^{\infty} \phi^k e^{-ik\lambda}$

(c) $k < 0$: $\sum_{k=-\infty}^{-1} \phi^{-k} e^{-ik\lambda}$ (since $|k| = -k$ for $k < 0$)

For $k < 0$, substitute $k' = -k$;

$$\sum_{k=-\infty}^{-1} \phi^{-k} e^{-ik\lambda} = \sum_{k'=1}^{\infty} \phi^{k'} e^{ik'\lambda}$$

Thus, the summation becomes;

$$1 + \sum_{k=1}^{\infty} \phi^k e^{-ik\lambda} + \sum_{k=1}^{\infty} \phi^k e^{ik\lambda}$$

Step 5: Evaluate the Infinite Sums

The sums are geometric series. For $|r| < 1$, we have;

$$\sum_{k=1}^{\infty} r^k = \frac{r}{1-r}$$

Thus;

$$\sum_{k=1}^{\infty} \phi^k e^{-ik\lambda} = \frac{\phi e^{-i\lambda}}{1-\phi e^{-i\lambda}}$$

and

$$\sum_{k=1}^{\infty} \phi^k e^{ik\lambda} = \frac{\phi e^{i\lambda}}{1-\phi e^{i\lambda}}$$

Step 6: Combine the Results

Now, the SDF becomes;

$$f(\lambda) = \frac{\sigma^2}{2\pi(1-\phi^2)}\left[1 + \frac{\phi e^{-i\lambda}}{1-\phi e^{-i\lambda}} + \frac{\phi e^{i\lambda}}{1-\phi e^{i\lambda}}\right]$$

Simplify the expression inside the brackets, we have;

$$1 + \frac{\phi e^{-i\lambda}}{1-\phi e^{-i\lambda}} + \frac{\phi e^{i\lambda}}{1-\phi e^{i\lambda}} = 1 + \frac{\phi e^{-i\lambda}(1-\phi e^{i\lambda}) + \phi e^{i\lambda}(1-\phi e^{-i\lambda})}{(1-\phi e^{-i\lambda})(1-\phi e^{i\lambda})}$$

$$= 1 + \frac{\phi e^{-i\lambda} - \phi^2 + \phi e^{i\lambda} - \phi^2}{1-\phi(e^{i\lambda}+e^{-i\lambda})+\phi^2} = 1 + \frac{2\phi\cos\lambda - 2\phi^2}{1-2\phi\cos\lambda+\phi^2} = \frac{1-\phi^2}{1-2\phi\cos\lambda+\phi^2}$$

Step 7: Final Simplification

Substitute back into $f(\lambda)$, we have;

$$f(\lambda) = \frac{\sigma^2}{2\pi(1-\phi^2)} \times \frac{1-\phi^2}{1-2\phi\cos\lambda+\phi^2} = \frac{\sigma^2}{2\pi} \times \frac{1}{1-2\phi\cos\lambda+\phi^2}$$

9. **The following results were obtained from the residual analysis of a certain AR(p) model:**

$$\textit{Jarque-Berra (JB)} = \frac{10}{6}\left(0.22^2 + \frac{(2.15-3)^2}{4}\right) \approx 0.53$$

From the Chi-Sq. distribution table, with $df = 2$ and $\alpha = 0.05$, we have; $\chi^2_{(2,0.05)} = 5.99$

(a) **What general term is used to describe the process in *Qu 9*? (1 Mark)**
Model Diagnostics or Residual Analysis

(b) **What is the purpose of carrying out this test? (1 Mark)**
The JB test is a statistical test used to determine whether a given dataset follows the normal distribution based on its skewness and kurtosis.

(c) **State the hypotheses of interest? (1 Mark)**
The hypotheses of interest are;

(a) Null Hypothesis (H_0): The data is normally distributed.
(b) Alternative Hypothesis (H_1): The data is not normally distributed.

(d) **Draw conclusion based on the results and hypotheses in (b)? (2 Marks)**
Since $JB \approx 0.53 < \chi^2_2(0.05) = 5.99$, we fail to reject the null hypothesis. This suggests that the data is normally distributed.

10. **A time series model is given by;**

$$X_t = 0.75X_{t-1} + \epsilon_t$$

Given that two previous values of the series are $X_4 = 5$ and $X_5 = 7$ find:

(a) $\hat{X}_8$ (3 Marks)

Understanding the Problem

We are given an AR(1) model defined by;

$$X_t = 0.75X_{t-1} + \epsilon_t$$

Given;

(a) $X_5 = 7$

We need to forecast X_8, denoted by $\hat{X}_8$ using the following steps:

Step 1: Understand the AR(1) Model

An AR(1) model is defined as;

$$X_t = \phi X_{t-1} + \epsilon_t$$

where;

(a) $\phi = 0.75$ (AR coefficient)

(b) ϵ_t is white noise with mean 0 and variance σ^2

Step 2: Forecasting with AR(1)

For an AR(1) model, the forecast $\hat{X}_{t+h}$ for $h \geq 1$ is given by;

$$\hat{X}_{t+h} = \phi^h X_t$$

But we need to consider the recursive nature of the AR(1) process.

Step 3: Forecast Step-by-Step

Given;

(a) $X_5 = 7$

We need $\hat{X}_8$, which is 3 steps ahead ($h = 3$).

Forecast X_6:

$$\hat{X}_6 = \phi X_5 = 0.75 \times 7 = 5.25$$

Forecast X_7:

$$\hat{X}_7 = \phi \hat{X}_6 = 0.75 \times 5.25 = 3.9375$$

Forecast X_8:

$$\hat{X}_8 = \phi \hat{X}_7 = 0.75 \times 3.9375 = 2.953125$$

ALTERNATIVELY

$X_4 = 5$ **implies 4-steps ahead forecast, where** $h = 4$

Using the general formula for an AR(1) forecast, there are h steps ahead, defined as;

$$\hat{X}_{t+h} = \phi^h X_t$$

And when we substitute, we have;

$$\hat{X}_8 = \phi^4 X_4 = (0.75)^4 \times 5 = 1.5820$$

The forecasted value is;

$$\hat{X}_8 = 1.5820$$

Particularly, for;

Forecast X_5:

$$\hat{X}_5 = \phi X_4 = 0.75 \times 5 = 3.75$$

Forecast X_6:

$$\hat{X}_6 = \phi X_5 = 0.75 \times 3.75 = 2.8125$$

Forecast X_7:

$$\hat{X}_7 = \phi \hat{X}_6 = 0.75 \times 2.8125 = 2.1094$$

Forecast X_8:

$$\hat{X}_8 = \phi \hat{X}_7 = 0.75 \times 2.1094 = 1.5821$$