Therefore, $\rho(h) = \begin{cases} 1, & h=0 \\ 0.366, & h=\pm 1 \\ -0.1307, & h=\pm 2 \\ 0, & h>2 \end{cases}$

Exercise

Determine if the processes are invertible and get the ACF

1. $X_t = 2e_t + 3e_{t-1} + 3e_{t-2} - 2e_{t-3}$
2. $X_t = e_t - 1.3e_{t-1} + 0.4e_{t-2}$

4. AUTOREGRESSIVE PROCESS — 14/04/2026

- Let the sequence e_t be a purely random process with mean, 0 and variance δ^2 Then the process X_t is said to be an autoregressive process of order p i.e AR(p) if it can be written in the form e_t - innovation

$$X_t = \alpha_1 X_{t-1} + \alpha_2 X_{t-2} + \alpha_3 X_{t-3} + \dots + \alpha_p X_{t-p} + e_t$$

- An AR(1) is simply given by $X_t = \alpha X_{t-1} + e_t \dots ①, |\alpha|<1$
- We can express AR(1) as a sum of infinite MA processes as follows by successfully replacing the X_ts by in the AR(1) process

$$X_t = \alpha X_{t-1} + e_t$$
$$X_{t-1} = \alpha X_{t-2} + e_{t-1}$$
$$X_{t-2} = \alpha X_{t-3} + e_{t-2}$$

Therefore, we can wreunte X_t as follows

$$\begin{aligned} X_t &= \alpha[\alpha X_{t-2} + e_{t-1}] + e_t \\ &= \alpha^2 X_{t-2} + \alpha e_{t-1} + e_t \\ &= \alpha^2[\alpha X_{t-3} + e_{t-2}] + \alpha e_{t-1} + e_t \\ &= \alpha^3 X_{t-3} + \alpha^2 e_{t-2} + \alpha e_{t-1} + e_t \\ &= e_t + \alpha e_{t-1} + \alpha^2 e_{t-2} + \alpha^3 X_{t-3} \\ &= e_t + \alpha e_{t-1} + \alpha^2 e_{t-2} + \dots + \alpha^s e_{t-s} + \alpha^{s+1} X_{t-(s+1)} \end{aligned}$$

In general,

$$X_t = e_t + \alpha e_{t-1} + \alpha^2 e_{t-2} + \dots + \alpha^s e_{t-s} + \alpha^{s+1} X_{t-(s+1)}$$
$$X_t = \sum_{j=0}^{s} \alpha^j e_{t-j} + \alpha^{s+1} X_{t-(s+1)}$$

$$X_t - \sum_{j=0}^{s} \alpha^j e_{t-j} = \alpha^{s+1} X_{t-(s+1)}$$

Squaring both sides and taking the expectation

$$E\left[X_t - \sum_{j=0}^{s} \alpha^j e_{t-j}\right]^2 = \alpha^{2(s+1)} E\left[X^2_{t-(s+1)}\right] \quad \dots (2)$$

From equation 1,

$$X_t = \alpha X_{t-1} + e_t \quad , \quad |\alpha| < 1$$

$$E(X_t) = \alpha E[X_{t-1}] + \cancel{E[e_t]}^{\,0} \quad , \quad e_t \sim iid(0, \delta^2)$$

$$E(X_t) = \alpha E[X_{t-1}]$$

Assuming that X_t is stationary,

$$E(X_t) = E(X_{t-1})$$

$$E(X_t) = \alpha E[X_{t-1}]$$

$$E(X_t) - \alpha E[X_{t-1}] = 0$$

But since $\alpha < 1$, $1 - \alpha \neq 0$

$$(1-\alpha) E(X_t) = 0 \text{, Therefore } E(X_t) = 0$$

$$E(X_t) = E(X_{t-1}) \text{ and } Var(X_t) = Var(X_{t-1})$$

Therefore, $E[X_{t-(s+1)}] = E(X_t)$

$$Var(X_t) = E(X_t^2) - [E(X_t)]^2$$

$$Var[X_{t-(s+1)}] = E[X^2_{t-(s+1)}] - \underbrace{[E\{X_{t-(s+1)}\}]^2}_{0}$$

$$\therefore Var[X_{t-(s+1)}] = E[X^2_{t-(s+1)}] = \delta^2 = Var(X_t)$$

$$E\left[X_t - \sum_{j=0}^{s} \alpha^j e_{t-j}\right]^2 = \alpha^{2(s+1)} Var(X_t)$$

$$= \alpha^{2(s+1)} \delta^2 \quad , \quad |\alpha| < 1 \quad \dots (3)$$

* α is a fraction

As $s \to \infty$, $E\left[X_t - \sum_{j=0}^{s} \alpha^j e_{t-j}\right]^2 = 0$

$$X_t = \sum_{j=0}^{s} \alpha^j e_{t-j} \quad \checkmark$$

This means that when $|\alpha| < 1$, then X_t converges in probability. Therefore, the AR(1) process converges to an infinite MA process of random elements with weights α^j

$$E(X_t) = E\left[\sum_{j=0}^{\infty} \alpha^j e_{t-j}\right] = \sum_{j=0}^{\infty} \alpha^j E[e_{t-j}] \quad , \quad e_t \sim iid(0, \delta^2)$$

Hence $E(X_t) = 0$

$$Var(X_t) = Var\left[\sum_{j=0}^{\infty} \alpha^j e_{t-j}\right] = \sum_{j=0}^{\infty} \alpha^{2j} Var(e_{t-j}) = \sum_{j=0}^{\infty} \alpha^{2j} \delta^2$$

Hence $Var(X_t) = \delta^2 \sum_{j=0}^{\infty} \alpha^{2j} = \delta^2 [1 + \alpha^2 + \alpha^4 + \alpha^6 + \dots]$, for $|\alpha| < 1$

$$\gamma(0) = Var(X_t) = \delta^2\left[\frac{1}{1-\alpha^2}\right] = \frac{\delta^2}{1-\alpha^2}, \quad |\alpha| < 1$$

~~Var~~ The autocovariance :-

$\gamma(h) = E[X_t X_{t+h}]$ and $X_t = \sum_{j=0}^{\infty} \alpha^j e_{t-j}$

$$X_{t+h} = \sum_{i=0}^{\infty} \alpha^i e_{t-(i-h)}$$

$$X_t X_{t+h} = \sum_{j=0}^{\infty} \alpha^j e_{t-j} \sum_{i=0}^{\infty} e_{t-(i-h)} \alpha^i$$

$$= \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} \alpha^j \alpha^i e_{t-j} e_{t-(i-h)}$$

$$E[X_t X_{t+h}] = \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} \alpha^j \alpha^i E[e_{t-j} e_{t-(i-h)}]$$

If $j = i-h$, then $i = j+h$

$$E[X_t X_{t+h}] = \sum_{j=0}^{\infty} \sum_{j+h=0}^{\infty} \alpha^j \alpha^{j+h} E[e_{t-j} e_{t-j}]$$

$$= \sum_{j=0}^{\infty} \sum_{j+h=0}^{\infty} \alpha^j \alpha^{j+h} \delta^2$$

$$= \sum_{j=0}^{\infty} \alpha^j \alpha^{j+h} \delta^2 = \delta^2 \sum_{j=0}^{\infty} \alpha^j \alpha^{j+h} = \delta^2 \sum_{j=0}^{\infty} \alpha^h \alpha^{2j}$$

$$E[X_t X_{t+h}] = \delta^2 \alpha^h \sum_{j=0}^{\infty} \alpha^{2j}$$

$$= \delta^2 \alpha^h [1 + \alpha^2 + \alpha^4 + \alpha^6 + \dots]$$

$$= \delta^2 \alpha^h \left[\frac{1}{1-\alpha^2}\right] = \frac{\delta^2 \alpha^h}{1-\alpha^2}, \quad |\alpha| < 1$$

The autocorrelation function, $\rho(h)$

$$\rho(h) = \frac{\gamma(h)}{\gamma(0)} = \frac{\delta^2 \alpha^h}{1-\alpha^2} \cdot \frac{1-\alpha^2}{\delta^2} = \alpha^h, \text{ for } |\alpha| < 1$$

- The AR process is stationary in the weak sense because mean = 0, $Var(X_t) = \frac{\delta^2}{1-\alpha^2}$, which is independent of time,

Autocovariance function, $E(X_t X_{t+h}) = \frac{\delta^2 \alpha^h}{1-\alpha^2}$, is independent of time

but dependent on the lag h

Autoregressive process of order 2

For an AR(2), we have

$$X_t = \alpha_1 X_{t-1} + \alpha_2 X_{t-2} + e_t$$

To get the ACF of this process, we multiply the process by X_{t-h} and take the expectation, then divide by the variance of X_t

$$Var(X_t) = \delta(0) = \frac{\delta^2}{1-\alpha^2}$$

$$X_t = \alpha_1 X_{t-1} + \alpha_2 X_{t-2} + e_t$$

$$* \quad X_t X_{t-h} = X_{t-h}[\alpha_1 X_{t-1} + \alpha_2 X_{t-2} + e_t] \cdot \cancel{\frac{(1-\alpha^2)}{\delta^2}}$$

Taking the expectation on both sides.

$$\frac{E[X_t X_{t-h}]}{Var(X_t)} = \frac{E[\alpha_1 X_{t-1} X_{t-h} + \alpha_2 X_{t-2} X_{t-h} + e_t X_{t-h}]}{Var(X_t)}$$

$$E[X_t X_{t-h}] = \alpha_1 E[X_{t-1} X_{t-h}] + \alpha_2 E[X_{t-2} X_{t-h}] + E[\cancel{e_t X_{t-h}}]^{0}$$

$$\frac{E[X_t X_{t-h}]}{Var(X_t)} = \alpha_1 \frac{E[X_{t-1} X_{t-h}]}{Var(X_t)} + \alpha_2 \frac{E[X_{t-2} X_{t-h}]}{Var(X_t)}$$

$$\rho(h) = \alpha_1 \rho(h-1) + \alpha_2 \rho(h-2), \quad h > 0 \quad \dots (1)$$

The set of equations given by equation 1 is known as the Yule Walker Equation. To solve this equation, we let $m^h = \rho(h)$ be the trial solution of equation 1. Therefore, equation 1 becomes :-

$$\rho(h) = \alpha_1 \rho(h-1) + \alpha_2 \rho(h-2), \quad h > 0$$

$$m^h = \alpha_1 m^{h-1} + \alpha_2 m^{h-2}$$

$$m^h - \alpha_1 m^{h-1} - \alpha_2 m^{h-2} = 0$$

$$m^{h-2}[m^2 - \alpha_1 m - \alpha_2] = 0, \quad m^{h-2} \neq 0$$

Hence $m^2 - \alpha_1 m - \alpha_2 = 0 \quad \dots (2)$

Equation 2 is known as an Auxillary / characteristic equation.
Let Π_1 and Π_2 be the roots of the auxillary equation 2. Then the general solution of the Yule Walker equation is :-

$$\rho(h) = A_1 \Pi_1^h + A_2 \Pi_2^h$$

where A_1 and A_2 are constants to be determined.

$$\Pi_1 \text{ or } \Pi_2 = \frac{\alpha_1 \pm \sqrt{\alpha_1^2 - 4(1)(-\alpha_2)}}{2} = \frac{\alpha_1 \pm \sqrt{\alpha_1^2 + 4\alpha_2}}{2} < 1$$

~~If the process is~~ At this point, we can test whether AR(2) process is stationary i.e the process will be stationary if the absolute value of the roots are less than 1 i.e $|\pi_1 \text{ or } \pi_2| < 1$

$$\frac{\alpha_1 \pm \sqrt{\alpha_1^2 + 4\alpha_2}}{2} < 1$$

CASE 1: when $\alpha_1^2 + 4\alpha_2 > 0$

In this case, both π_1 and π_2 are real and distinct. Then, the ACF which is given by, $\rho(h)$,

$$\rho(h) = A_1\pi_1^h + A_2\pi_2^h$$

when $h=0$, $\rho(0) = A_1 + A_2$

$$1 = A_1 + A_2, \text{ hence } A_2 = 1 - A_1 \quad \ldots (a)$$

when $h=1$, $\rho(1) = A_1\pi_1 + A_2\pi_2$

From Yule Walker equation 1,

$$\rho(h) = \alpha_1\rho(h-1) + \alpha_2\rho(h-2).$$

$\rho(-h) = \rho(h).$

$$\rho(1) = \alpha_1\rho(0) + \alpha_2\rho(-1) = \alpha_1\rho(0) + \alpha_2\rho(1)$$

but $\rho(0) = 1$, ~~and~~ $\cancel{\rho(1) = A_1\pi_1 + A_2\pi_2}$.

$$\rho(1) = \alpha_1 + \alpha_2\rho(1)$$

$$\rho(1) - \alpha_2\rho(1) = \alpha_1.$$

$$(1-\alpha_2)\rho(1) = \alpha_1.$$

$$\rho(1) = \frac{\alpha_1}{1-\alpha_2} \quad \ldots b)$$

$$A_1\pi_1 = \frac{\alpha_1}{1-\alpha_2} - A_2\pi_2$$

$$A_2 = 1 - A_1$$

$$A_1\pi_1 + (1-A_1)\pi_2$$

$$A_1\pi_1 + \pi_2 - A_1\pi_2$$

$$A_1(\pi_1 - \pi_2) + \pi_2 = \frac{\alpha_1}{1-\alpha_2}$$

$$A_1 = \frac{\alpha_1 - \pi_2}{(1-\alpha_2)(\pi_1 - \pi_2)}$$

But also, $\rho(1) = A_1\pi_1 + A_2\pi_2$

$$A_1\pi_1 + A_2\pi_2 = \frac{\alpha_1}{1-\alpha_2}$$

Solving a and b by substitution.

$$A_1 = \frac{\alpha_1 - \pi_2 - \alpha_2\pi_2}{(1-\alpha_2)(\pi_1 - \pi_2)} \quad \text{and} \quad A_2 = \frac{\pi_1 - \alpha_1 - \alpha_2\pi_1}{(1-\alpha_2)(\pi_1 - \pi_2)}$$

And $\rho(h) = A_1\pi_1^h + A_2\pi_2^h = \dfrac{\alpha_1 - \pi_2 - \alpha_2\pi_2}{(1-\alpha_2)(\pi_1-\pi_2)}\pi_1^h + \dfrac{\pi_1 - \alpha_1 - \alpha_2\pi_1}{(1-\alpha_2)(\pi_1-\pi_2)}\pi_2^h$

CASE 2: When $\alpha_1^2 + 4\alpha_2 < 0$

In this case, the roots are complex. Therefore

$$\pi_1 \text{ or } \pi_2 = \frac{\alpha_1 \pm \sqrt{-(\alpha_1^2 + 4\alpha_2)}}{2} = \frac{\alpha_1}{2} \pm \frac{\sqrt{-1}}{2}\sqrt{\alpha_1^2 + 4\alpha_2} = \frac{\alpha_1}{2} \pm \frac{i}{2}\sqrt{\alpha_1^2 + 4\alpha_2}$$

$$\Pi_1 = \frac{\alpha_1}{2} + \frac{i}{2}\sqrt{\alpha_1^2 + 4\alpha_2} = re^{i\theta}$$

$$\Pi_2 = \frac{\alpha_1}{2} - \frac{i}{2}\sqrt{\alpha_1^2 + 4\alpha_2} = re^{-i\theta}$$

where r is the square root of $-\alpha_2$

$$\theta = \tan^{-1}\left[\frac{\frac{1}{2}(\alpha_1^2 + 4\alpha_2)^{1/2}}{\alpha_1/2}\right]$$

The ACF is given by, $\rho(h) = A_1\Pi_1^h + A_2\Pi_2^h$, will become :-

$$\rho(h) = A_1(re^{i\theta})^h + A_2(re^{-i\theta})^h = A_1 r^h e^{i\theta h} + A_2 r^h e^{-i\theta h}$$

$$= r^h\left[A_1 e^{i\theta h} + A_2 e^{-i\theta h}\right]$$

But $e^{i\theta h} = \cos\theta h + i\sin\theta h$

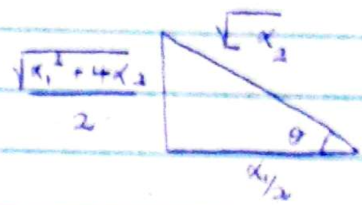
$e^{-i\theta h} = \cos\theta h - i\sin\theta h$

$$\therefore \rho(h) = r^h\left[A_1(\cos\theta h + i\sin\theta h) + A_2(\cos\theta h - i\sin\theta h)\right]$$

$$= r^h\left[(A_1 + A_2)\cos\theta h + i(A_1 - A_2)\sin\theta h\right] \text{, but } A_1 + A_2 = 1$$

$$= r^h\left[\cos\theta h + i(A_1 - A_2)\sin\theta h\right]$$

Since $i(A_1 - A_2)$ is unknown, let $h = 1$



$$\rho(1) = \frac{\alpha_1}{1-\alpha_2} = r\left[\cos\theta + i(A_1 - A_2)\sin\theta\right]$$

$$\frac{\alpha_1}{1-\alpha_2} = (-\alpha_2)^{1/2}\left[\cos\theta + i(A_1 - A_2)\sin\theta\right]$$

SOHCAH TOA

$$i(A_1 - A_2) = \frac{\cancel{\alpha_1}\ \alpha_1}{(1-\alpha_2)(-\alpha_2)^{1/2}\sin\theta} - \frac{\cos\theta}{\sin\theta}$$

Note :- $\cos\theta = \frac{\alpha_1}{2} \cdot \frac{1}{\sqrt{-\alpha_2}} = \frac{\alpha_1}{2\sqrt{-\alpha_2}}$

$$2\cos\theta = \frac{\alpha_1}{\cancel{2}\sqrt{-\alpha_2}}$$

Therefore $i(A_1 - A_2) = \frac{2\cos\theta}{(1-\alpha_2)\sin\theta} - \frac{\cos\theta}{\sin\theta}$, but $\frac{\cos\theta}{\sin\theta} = \cot\theta$

$$i(A_1 - A_2) = \frac{2\cot\theta}{1-\alpha_2} - \cot\theta = \frac{2\cot\theta - \cot\theta + \alpha_2\cot\theta}{1-\alpha_2}$$

$$i(A_1 - A_2) = \frac{\cot\theta + \alpha_2\cot\theta}{1-\alpha_2} = \frac{(1+\alpha_2)\cot\theta}{1-\alpha_2}$$

Therefore, $\rho(h) = r^h \left[\cos \theta h + \frac{(1+\alpha_2)}{1-\alpha_2} \cot \theta \sin \theta h \right]$

$$\rho(h) = (-\alpha_2)^{1/2} \left[\cos \theta h + \frac{(1+\alpha_2)}{1-\alpha_2} \cot \theta \sin \theta h \right]$$

Example 1

Consider the AR(2) process given by $X_t = \frac{1}{4} X_{t-1} + \frac{3}{64} X_{t-2} + e_t$. Is the process stationary and if so find its ACF * autocorrelation function

solution

$$X_t = \frac{1}{4} X_{t-1} + \frac{3}{64} X_{t-2} + e_t$$

$\rho(h) = \frac{1}{4} \rho(h-1) + \frac{3}{64} \rho(h-2)$, where $\alpha_1 = \frac{1}{4}$ and $\alpha_2 = \frac{3}{64}$

Let $m^h = \rho(h)$ be the trial solution of the Yule Walker equation

$$m^h = \frac{1}{4} m^{h-1} + \frac{3}{64} m^{h-2}.$$

$$m^h - \frac{1}{4} m^{h-1} - \frac{3}{64} m^{h-2} = 0$$

$m^{h-2} \left[m^2 - \frac{1}{4} m - \frac{3}{64} \right] = 0$, but since $m^{h-2} \neq 0$.

$$m^2 - \frac{1}{4} m - \frac{3}{64} = 0$$

Π_1 or $\Pi_2 = \frac{\alpha_1 \pm \sqrt{\alpha_1^2 + 4\alpha_2}}{2}$, and $\alpha_1 = 1/4$ and $\alpha_2 = 3/64$

$$\Pi_1 \text{ or } \Pi_2 = \frac{1/4 \pm \sqrt{(1/4)^2 + 4(1)(3/64)}}{2} = \frac{1/4 \pm \sqrt{1/16 + 3/16}}{2} = \frac{1/4 \pm 1/2}{2}$$

$$\Pi_1 = \frac{1/4 + 1/2}{2} = \frac{3}{8} \quad \text{and} \quad \Pi_2 = \frac{1/4 - 1/2}{2} = -\frac{1}{8}$$

The process is stationary since the absolute values of Π_1 and Π_2 are less than 1

$$\rho(h) = A_1 \Pi_1^h + A_2 \Pi_2^h = A_1 (3/8)^h + A_2 (-1/8)^h$$

when $h = 0$, $\rho(h) = A_1 + A_2 = 1$, implying $A_1 = 1 - A_2 \ldots ⓐ$

when $h=1$, $\rho(h) = \frac{3}{8}A_1 - \frac{1}{8}A_2 = \frac{\alpha_1}{1-\alpha_2} = \frac{1/4}{1-3/64} = \frac{16}{61}$

$$\rho(h) = \frac{3}{8}A_1 - \frac{1}{8}A_2 = \frac{16}{61} \quad \ldots b)$$

Solving equations a and b by substitution -

$$\frac{3}{8}A_1 - \frac{1}{8}A_2 = \frac{16}{61}, \quad \text{but } A_1 = 1 - A_2.$$

$$\frac{3}{8}(1-A_2) - \frac{1}{8}A_2 = \frac{16}{61}$$

$$\frac{3}{8} - \frac{3}{8}A_2 - \frac{1}{8}A_2 = \frac{16}{61}$$

$$-\frac{4}{8}A_2 = \frac{16}{61} - \frac{3}{8}$$

Hence $A_2 = \frac{55}{244}$ and $A_1 = \frac{189}{244}$

Therefore, $\rho(h) = \frac{189}{244}\left(\frac{3}{8}\right)^h + \frac{55}{244}\left(\frac{-1}{8}\right)^h$

Example 2

Consider the AR(2) process $X_t = X_{t-1} - \frac{1}{3}X_{t-2} + e_t$. Is the process stationary and if so find the ACF.

solution

$$X_t = X_{t-1} - \frac{1}{3}X_{t-2} + e_t$$

$$\rho(h) = \rho(h-1) - \frac{1}{3}\rho(h-2)$$

Let m^h be the trial solution of the Yule Walker equation

$$m^h = m^{h-1} - \frac{1}{3}m^{h-2}$$

$$m^h - m^{h-1} + \frac{1}{3}m^{h-2} = 0$$

$$m^{h-2}\left[m^2 - m + \frac{1}{3}\right] = 0, \text{ but since } m^{h-2} \neq 0$$

$1-4/3 = 1-1\frac{1}{3} =$

$$m^2 - m + 1/3 = 0$$

$$\Pi_1 \text{ or } \Pi_2 = \frac{\alpha_1 \pm \sqrt{\alpha_1^2 + 4\alpha_2}}{2} = \frac{1 \pm \sqrt{1 + 4(-1/3)}}{2} = \frac{1 \pm \sqrt{-1/3}}{2} = \frac{1 \pm i\sqrt{1/3}}{2}$$

$$\Pi_1 = \frac{1}{2} + \frac{i}{2}\sqrt{1/3} \quad \text{and} \quad \Pi_2 = \frac{1}{2} - \frac{i}{2}\sqrt{1/3}$$

Since the absolute values of the roots are less than 1, the process X_t is stationary. To get the ACF:-

$$\rho(h) = (-\alpha_2)^{h/2}\left[\cos\theta h + \frac{(1+\alpha_2)}{1-\alpha_2}\cot\theta \sin\theta h\right], \quad \alpha_1 = 1 \text{ and } \alpha_2 = -1/3$$

$$= (1/3)^{h/2}\left[\cos\theta h + \frac{(1-1/3)}{1+1/3}\cot\theta\sin\theta h\right]$$

$2/3 \times 3/4$ $$= (1/3)^{h/2}\left[\cos\theta h + 1/2\cot\theta\sin\theta h\right]$$

$$\cos\theta = \frac{\alpha_1/2}{\sqrt{-\alpha_2}} \quad \text{hence} \quad \theta = \cos^{-1}\left[\frac{\alpha_1/2}{\sqrt{1/3}}\right] = 30^\circ \qquad \tan^{-1}\left[\frac{1/2\sqrt{-(\alpha_1^2+4\alpha_2)}}{\alpha_1/2}\right]$$

$$\therefore \rho(h) = (1/3)^{h/2}\left[\cos 30h + 1/2\cot 30\sin 30h\right]$$

$$= (1/3)^{h/2}\left[\cos 30h + 0.866\sin 30h\right]$$

SPECTRAL ANALYSIS

17/04/2025

In spectral analysis we assume in the beginning that the time series X_t is stationary. Here, the time series can be looked upon as consisting a finite number of harmonic oscilations of the type $\cos\lambda t$ or $\sin\lambda t$ with a wavelength equal to $\frac{2\pi}{\lambda}$. Thus, the time series can be written as a func

of the sum of sin or cos otherwise known as Fourier series:

$$X_t = \sum_{h=-\infty}^{\infty}(a_j\cos\lambda_j t + b_j\sin\lambda_j t) \text{ where } -\pi \leq \lambda_j \leq \pi$$

Therefore, the time series X_t is a series of harmonic oscillations. To study harmonic oscillation, we need to apply the consept of spectral density function/power spectrum.

- The power spectrum is the natural tool of studying the pr[o]perties of a time series with the frequency domain.
- Let the time series be denoted by X_t such that $t = 0, \pm 1, \pm 2, \ldots$

Define the autocovariance function, $\delta(h) = \operatorname{cov}(X_t X_{t+h})$

$$\sigma(h) = E(X_t X_{t+h})$$

assuming stationarity i.e $E(X_t) = 0$

The Fourier series transform of the sequence $\sigma(h)$, where $h = 0, \pm 1, \pm 2, \ldots$ will be given by;

Eq 1: $$f(\lambda) = \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} \sigma(h) e^{-ih\lambda}$$, where λ - wavelength/frequency of the harmonic oscillation measured in radians, $\lambda\ (0 \Leftrightarrow 0.5)$

The above $f(\lambda)$ function is what we call the Power spectrum or spectral density function. This function is an even function

$$f(\lambda) = f(-\lambda)$$

This equation can be expressed in various ways :-

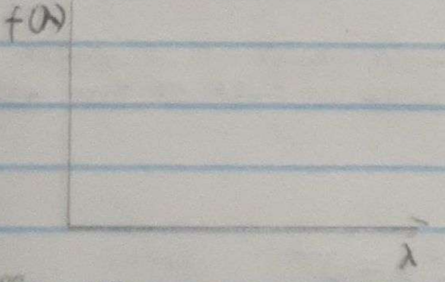
1. $$f(\lambda) = \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} \sigma(h) e^{-ih\lambda}$$

2. $$F(\lambda) = \frac{1}{2\pi} \left[\delta(0) + \sum_{h=1}^{\infty} \sigma(h) e^{-ih\lambda} + \sum_{h=1}^{\infty} \sigma(h) e^{ih\lambda} \right]$$

3. $$F(\lambda) = \frac{1}{2\pi} \left[\delta(0) + 2 \sum_{h=1}^{\infty} \cos h\lambda \right]$$

* $e^{-ih\lambda} = \cos \lambda h - i \sin \lambda h$
* $e^{ih\lambda} = \cos \lambda h + i \sin \lambda h$

4. $$F(\lambda) = \frac{1}{2\pi} \left[\sum_{h=-\infty}^{\infty} \delta(h) \cos h\lambda \right]$$ → assuming that X_t is discrete



The graph is known as a spectogram

* We are assuming SDF is continuous

$$\delta(h) = \int_{-\pi}^{\pi} F(\lambda) \cos \lambda h \, d\lambda$$

At $h = 0$, $$\delta(0) = \int_{-\pi}^{\pi} F(\lambda)\, d\lambda = \delta^2$$

$$1 = \int_{-\pi}^{\pi} \frac{F(\lambda)}{\delta^2} d\lambda$$ → This implies that $\frac{F(\lambda)}{\delta^2}$ is a pdf

Thus, when $F(\lambda)$ is divided by the variance, $\operatorname{Var}(X_t) = \delta(0)$, we obtain a normalised spectral density function.

$$F(\lambda) = \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} \gamma(h) e^{-ih\lambda} \quad \ldots ①$$

From equation 1, to obtain the corresponding normalised spectral density function $f(\lambda) = \frac{F(\lambda)}{\sigma^2}$

autocorrelation function

$$f^*(\lambda) = \frac{f(\lambda)}{\sigma^2} = \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} \boxed{\frac{\gamma(h)}{\sigma^2}} e^{-ih\lambda}$$

$$= \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} \rho(h) e^{-ih\lambda}$$

* Band spectral density regression of exports on the ... of Kenya

From equation 4, to obtain the corresponding normalised spectral density function, $f(\lambda) = \frac{1}{2\pi} \sum_{h=-\infty} \gamma(h) \cos \lambda h$.

$$f^*(h) = \frac{f(\lambda)}{\sigma^2} = \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} \frac{\gamma(h)}{\sigma^2} \cos \lambda h$$

$$= \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} \rho(h) \cos \lambda h$$

Example 1

Find the SPF of the process, e_t where $t = 0, \pm 1, \pm 2, \ldots$ where the process e_t is stationary, $E(e_t) = 0$ and $Var(e_t) = \sigma^2$.

* MA process of order 0

solution

$$f(\lambda) = \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} \gamma(h) e^{-ih\lambda}$$

* The lag h, should not be greater than the order then since our order = 0, then h = 0.

$$\gamma(h) = \sigma^2 \sum_{j=0}^{q-h} \beta_j \beta_{j+h}$$

At $h = 0$, $\gamma(0) = \sigma^2 \sum_{j=0}^{0} \beta_j^2 = \sigma^2 (\beta_0^2) = \sigma^2$ $\quad \beta_0 = 1$.

$$f(\lambda) = \frac{1}{2\pi} \sum_{h=0}^{0} \gamma(h) e^{-ih\lambda} = \frac{1}{2\pi} [\gamma(0) e^0] = \frac{1}{2\pi} \sigma^2 = \frac{\sigma^2}{2\pi}$$

Hence, $f(\lambda) = \begin{cases} \frac{\sigma^2}{2\pi}, & \text{for } h = 0 \\ 0, & h > 0 \end{cases}$

The corresponding normalised spectral density function is :-

$$f^*(\lambda) = \frac{f(\lambda)}{\sigma^2} = \frac{1}{2\pi} \sum_{h=0}^{0} \frac{\sigma(h)}{\sigma^2} e^{-ih\lambda}$$

$$= \frac{1}{2\pi} \left[\frac{\sigma^2}{\sigma^2} e^{0} \right] = \frac{1}{2\pi}.$$

Hence, $f^*(\lambda) = \begin{cases} 1/2\pi, & \text{for } h=0. \\ 0, & \text{for } h>0 \end{cases}$

Question

1. Consider an AR(1) process, given by $X_t = \alpha_1 X_{t-1} + e_t$ $|\alpha| < 1$. Show that the normalised SDF of the process X_t is given by:-

$$f^*(\lambda) = \frac{1-\alpha^2}{2\pi(1 - 2\alpha\cos\lambda + \alpha^2)}$$

2. Consider an MA process of order 2 given by $X_t = e_t + e_{t-1} + e_{t-2}$. Find the SDF and its corresponding NSDF of the process X_t

Remem!

* When asked to find the statistical properties of a time series, you are required to find the mean, $E(X_t)$ whether it is constant or dependent on time and Variance, $var(X_t)$ → you can use these outcomes to determine whether the process is stationary or not.