ASSIGNMENTS FOR THE THEORETICAL PART OF THE COURSE 'COSMOLOGY' (NWI-NM026C)

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Contents

Assignments week 41 (A. Achterberg)

1 Arrival rate of photons in an expanding universe

In special relativity a photon world line (track through space-time) satisfies

$$\mathrm{d}s^2 = c^2\,\mathrm{d}t^2 - d\ell^2 = 0\;, \tag{1.1}$$

expressing the fact that all photons move with speed $c \simeq 300,000$ km/s. Here $\mathrm{d}\ell$ is the (infinitesimal) spatial distance along the track traversed by the photon in an infinitesimal time interval time $\mathrm{d}t$. The photon is said to move along a *null geodesic*.

This special-relativistic result is replaced by the following relation in the general-relativistic theory of an expanding Universe:

$$\mathrm{d}s^2 = c^2\,\mathrm{d}t^2 - a^2(t)\,d\ell_{\rm cm}^2 = 0\;. \tag{1.2}$$

Here t is cosmic time and $a(t)$ is the scale factor that describes the universal expansion, with Hubble parameter $H(t) = (1/a)(\mathrm{d}a/\mathrm{d}t)$. Also, $\mathrm{d}\ell_{\rm cm}$ is an infinitesimal *comoving* distance element, the corresponding physical distance element at some time t being equal to

$$\mathrm{d}\ell = a(t)\,d\ell_{\rm cm}\;. \tag{1.3}$$

Objects (e.g. distant galaxies) that passively follow the Hubble expansion of the universe always have **constant** comoving coordinates, the one property that makes comoving co-ordinates so useful in practical calculations.

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a. Consider a spatially flat universe where one use cartesian comoving coordinates[1] x, y and z and where

$$d\ell^2_{\rm cm} = {\rm d}x^2 + {\rm d}y^2 + {\rm d}z^2 \ . \tag{1.4}$$

Consider a galaxy at the origin $x = y = z = 0$ that emits a photons along the x-axis at time $t_{\rm em}$. An observer somewhere on the x-axis, at comoving distance $x = x_{\rm obs}$, intercepts the photon at cosmic time $t_{\rm obs}$.

Now show that the following relation holds:

$$x_{\rm obs} = \int_{t_{\rm em}}^{t_{\rm obs}} \frac{c\,{\rm d}t}{a(t)} \ . \tag{1.5}$$

b. What is the *physical* distance $D_{\rm obs}$ from source to observer at the moment of photon reception, assuming that the observer also resides in a 'typical' galaxy with $x_{\rm obs}$ constant? Express the result in terms of the integral of question **a**.

c. Now we look at **two** photons. The first is emitted at $t_{\rm em}$ and observed at $t_{\rm obs}$. The second is emitted later, at $t_{\rm em}+\Delta t_{\rm em}$, and detected by the same observer (still residing at $x_{\rm obs}$) at time $t_{\rm obs} + \Delta t_{\rm obs}$. Now use result **a** for each of the two photons to show that the following relation holds:

$$\int_{t_{\rm em}}^{t_{\rm em}+\Delta t_{\rm em}} \frac{c\,{\rm d}t}{a(t)} = \int_{t_{\rm obs}}^{t_{\rm obs}+\Delta t_{\rm obs}} \frac{c\,{\rm d}t}{a(t)} \ . \tag{1.6}$$

[1] I will drop the subscript 'cm' on these coordinates!

d. Now assume that the interval between emission and reception of the two photons is short so that $\Delta t_{\rm em} \ll t_{\rm em}$ and $\Delta t_{\rm obs} \ll t_{\rm obs}$. Also assume that both intervals are much shorter than the age of the universe: the practical situation in any astronomical observation.

Show that a simple approximation for the two integrals then implies a relation of the form

$$\Delta t_{\rm obs} = (1 + z)\ \Delta t_{\rm em} \ , \tag{1.7}$$

with z the redshift of the photons upon reception. (You will need the relation between redshift and scale factor!)

Give a *physical* reason why, in an expanding universe, the time interval between the reception of the two photons is larger than the time interval between emission, as implied by this result.

Although these results have been derived for a flat universe, they hold generally, see Longair's book, Ch. 5.5.

2 Luminosity, flux and flux distance

Astronomers define the observed radiative flux F of an object as the photon energy received per unit time and per unit area. For instance, if a source emits $\dot{N}$ photons per second, each with a photon energy $\epsilon = hc/\lambda$, and if this emission is isotropic (equally strong in all directions), a stationary observer at a distance D from the source measures a flux[2]

$$F = \frac{\dot{N}\epsilon}{4\pi D^2} = \frac{L}{4\pi D^2} \tag{2.1}$$

Here $L = \dot{N}\epsilon$ is the luminosity (radiative power) of the source. This result uses the fact that crossing photons are uniformly distributed over a sphere of radius D, and assumes that no photons are absorbed between source and observer.

In this assignment we will consider the effect of the expansion of the universe on this result, using what we found in Assignment 1. In an expanding universe the analogue of (2.1) still holds:

$$F_{\text{obs}} = \frac{\dot{N}_{\text{obs}}\epsilon_{\text{obs}}}{4\pi D_0^2} \ . \tag{2.2}$$

Here $\dot{N}_{\text{obs}}$ is the total number of photons crossing a sphere with radius D_0 per unit time, with D_0 the physical distance between source and observer *at the moment of photon detection*. Also, ϵ_{obs} is the photon energy measured by the observer.

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a. What is the relation between the observed photon energy and the photon energy at the source if the observer measures a redshift z?

b. If no photons are lost between source and observers, what is (according to the results of Assigment 1) the relation between $\dot{N}_{\text{obs}}$ and the number of photons emitted per unit time at the source, $\dot{N}$, assuming once again that the source is at redshift z?

[2]This assignment assumes for simplicity that all photons have the same wavelength/energy. This is not essential since redshift affects all wavelengths in the same way, so the same result holds for a source emitting photons in a wide range of wavelengths, as is usually the case.

c. Observers define the *luminosity distance* D_L by

$$D_L = \sqrt{L/4\pi F_{\rm obs}}\ , \tag{2.3}$$

with $L = \dot{N}\epsilon$ as before. Show that the results of **a** and **b** imply that

$$D_L = (1+z)\ D_0\ . \tag{2.4}$$

A more sophisticated derivation of this result can be found in Longairs book, Ch. 5.5.

d. Gamma Ray Bursts are brief flashes of gamma rays, lasting typically 1-1000 seconds, that are associated with the collapse of a super-massive star in the distant universe. The most distant GRB observed to date, GRB 090423, has a redshift $z = 8.2$. For these sources one measures the *fluence* $\mathcal{F}$, the time-integrated flux:

$$\mathcal{F} = \int {\rm d}t_{\rm obs}\ F_{\rm obs}(t_{\rm obs})\ , \tag{2.5}$$

which is essentially the total photon energy of the flash that crosses a unit area on a sphere with radius D_0.

Now use the results of Assignment 1 and of **a-c** of this assignment to show that, if a GRB emits isotropically[3], a measurement of the fluence determines the total photon energy produced in the burst at the source as

$$E_{\rm iso} = (1+z)\, 4\pi D_0^2\, \mathcal{F} \tag{2.6}$$

if the GRB occurs in a galaxy with redshift z.

[3]**Not** true in reality: the emission in Gamma Ray Bursts is beamed into a narrow cone with an opening angle of a few degrees! This means that we are only able to observe a small fraction of all GRBs, namely those where the beam happens to hit the Earth.

3 Hand-in assignment: geodesics in the Robertson-Walker metric

In General Relativity, the motion of freely falling observers follow *geodesics*. The equation describing geodesic motion can be derived from a variational principle. This demands that the action $\mathcal{A}$ is stationary:

$$\delta\mathcal{A} = \delta \int_{\lambda_1}^{\lambda_2} \mathrm{d}\lambda \, \mathcal{L}(x^\mu \ , \ \dot{x}^\mu) = 0 \ , \tag{3.1}$$

where one can use as the so-called Lagrangian $\mathcal{L}(x^\mu \ , \ \dot{x}^\mu)$

$$\mathcal{L}(x^\mu \ , \ \dot{x}^\mu) = \tfrac{1}{2} g_{\mu\nu}(x) \, \dot{x}^\mu \dot{x}^\nu \quad \text{with} \quad \dot{x}^\mu \equiv \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda} \ . \tag{3.2}$$

The x^μ are a chosen set of coordinates, and $g_{\mu\nu}(x)$ is the metric tensor that sits in the distance recipe for these coordinates in curved space-time. The parameter λ that appears everywhere is an *affine parameter* that measures the position along a geodesic.

The action must be stationary for small changes δx^μ that vanish at the end points of integration λ_1 and λ_2. Standard techniques[4] yield that the following set of *Euler-Lagrange relations* that must hold for each of the x^μ (that is for $\mu = 0 \ , \ 1 \ , \ 2 \ , \ 3$):

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{\partial\mathcal{L}}{\partial\dot{x}^\mu}\right) - \frac{\partial\mathcal{L}}{\partial x^\mu} = 0 \ . \tag{3.3}$$

We will apply this to the Robertson-Walker metric of an expanding universe, where (using metric 5.35, p. 161 in Longairs book using a simplified notation)

$$\mathcal{L} = \tfrac{1}{2}\left[\left(\dot{x}^0\right)^2 - a^2(t)\left(\frac{\dot{r}^2}{1-kr^2} + r^2\,\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2\right)\right] \ , \tag{3.4}$$

employing as a coordinate set $x^0 = ct$, $x^1 = r$, $x^2 = \theta$ and $x^3 = \phi$. The constant k can take the values -1, 0 and $+1$.

[4] See for instance: G.B. Arfken & H.J. Weber: *Mathematical Methods for Physicists*, 5th Ed., Academic Press, 2001, ISBN 0-12-059826-4, Chapter 17.

a. Show that for $\mu = 0$ the Euler-Lagrange relation (3.3) leads to the following relation for $x^0 = ct(\lambda)$:

$$\frac{\mathrm{d}^2 x^0}{\mathrm{d}\lambda^2} + \frac{a}{c}\frac{\mathrm{d}a}{\mathrm{d}t}\left(\frac{\dot{r}^2}{1-kr^2} + r^2\,\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2\right) = 0\ . \tag{3.5}$$

(**Hint:** partial derivatives of x^μ only act on the *explicit* dependence of $\mathcal{L}$ on that coordinate.)

b. Show that the Euler-Lagrange relation (3.3) for $\mu = 1$ leads to the following equation for $x^1 = r(\lambda)$:

$$\frac{\mathrm{d}^2 r}{\mathrm{d}\lambda^2}+\left(\frac{2}{c}\frac{1}{a}\frac{\mathrm{d}a}{\mathrm{d}t}\right)\dot{x}^0\dot{r}+\frac{kr}{1-kr^2}\dot{r}^2-r\left(1-kr^2\right)\dot{\theta}^2-r\sin^2\theta\left(1-kr^2\right)\dot{\phi}^2 = 0\ . \tag{3.6}$$

(**Hint:** the derivative $\mathrm{d}/\mathrm{d}\lambda$ is a *total* derivative that acts both on the explicit dependence of $\mathcal{L}$ on λ (not relevant in this case) and on the *implicit* dependence on λ. For instance: $\mathrm{d}a/\mathrm{d}\lambda = (\mathrm{d}t/\mathrm{d}\lambda)(\mathrm{d}a/\mathrm{d}t) = (\dot{x}^0/c)(\mathrm{d}a/\mathrm{d}t)$ since the scale factor a is a function of time, $t = t(\lambda) = x^0(\lambda)/c$.)

c. In a similar fashion, show that $x^2 = \theta(\lambda)$ satisfies

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}\lambda^2} + \left(\frac{2}{c}\frac{1}{a}\frac{\mathrm{d}a}{\mathrm{d}t}\right)\dot{x}^0\dot{\theta} + \frac{2}{r}\,\dot{r}\dot{\theta} - \sin\theta\cos\theta\,\dot{\phi}^2 = 0\ . \tag{3.7}$$

d. Since none of the coefficients in $\mathcal{L}$ explicitly depends on $x^3 = \phi$ it is an *ignorable coordinate*. Show that the Euler-Lagrange relation (3.3) for $\mu = 3$ leads to a conservation law:

$$\frac{\partial\mathcal{L}}{\partial\dot{\phi}} = a^2(t)\,r^2\sin^2\theta\,\dot{\phi} = \text{constant}\ . \tag{3.8}$$

Upon differentiation w.r.t. λ this yields: (**no need to do this yourselves!**):

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}\lambda^2} + \left(\frac{2}{c}\frac{1}{a}\frac{\mathrm{d}a}{\mathrm{d}t}\right)\dot{x}^0\dot{\phi} + \frac{2}{r}\,\dot{r}\dot{\phi} + 2\cot\theta\,\dot{\phi}\dot{\theta} = 0\ . \tag{3.9}$$

e. The canonical form of the equation for a geodesic in General Relativity is (implied summation over the double indices α and β):

$$\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\lambda^2} + \Gamma^\mu_{\alpha\beta} \frac{\mathrm{d}x^\alpha}{\mathrm{d}\lambda} \frac{\mathrm{d}x^\beta}{\mathrm{d}\lambda} = \frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\lambda^2} + \Gamma^\mu_{\alpha\beta}\, \dot{x}^\alpha \dot{x}^\beta = 0 \; . \tag{3.10}$$

Now read off (and give) all the non-vanishing Christoffel symbols $\Gamma^\mu_{\alpha\beta}$ for the Robertson-Walker metric used in this assignment. You may put $c = 1$ here and in the rest of this exercise.

(**Hint:** be careful with cross terms of the form $\dot{x}^\alpha \dot{x}^\beta$ with $\alpha \neq \beta$ since they occur twice in the implied summation!)

f. Show that observers with fixed comoving coordinates, the fiducial observers in Cosmology who see the universe as isotropic and which have by definition $\dot{r} = \dot{\theta} = \dot{\phi} = 0$, all move on geodesics *provided* the affine parameter λ is related to cosmic time (Robertson-Walker coordinate time t) by

$$\lambda = at + b \; , \tag{3.11}$$

with a and b arbitrary constants.

The morale: comoving observers move on geodesics, and are therefore proper 'free-falling' observers. The fact that λ and coordinate time t are related linearly with arbitrary constants means that we are free to choose $a = 1$ and $b = 0$ and simply use time, putting $\lambda = t$, to track the motion through space-time of these fiducial observers. This is the way all Cosmology textbooks look at it.

Assignments week 43 (A. Achterberg)

4 Expansion laws for a flat Friedmann universe

The expansion of a flat universe with $k = 0$ is described by the equation

$$H^2 = \left(\frac{1}{a}\frac{\mathrm{d}a}{\mathrm{d}t}\right)^2 = \frac{8\pi G\rho}{3} \,. \tag{4.12}$$

Here ρ is the mass density of the universe, related to the energy density ϵ by $\rho = \epsilon/c^2$. This assignment looks at the solutions and applications of this equation.

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a. Show that a flat universe, where the density drops as a result of the expansion as a power of the scale factor,

$$\rho(t) = \rho_0 \left(\frac{a(t)}{a_0}\right)^{-n} \tag{4.13}$$

with $n \neq 0$, expands as

$$\frac{a(t)}{a_0} = \left(\frac{t}{t_0}\right)^{2/n} \,. \tag{4.14}$$

with a Hubble 'constant'

$$H(t) = \frac{2}{nt} \,. \tag{4.15}$$

Here $a_0 \equiv a(t_0)$ and $\rho_0 \equiv \rho(t_0)$ with t_0 an arbitrary reference time that can be taken to be the current age of the universe.

b. Show that [1] this solution holds *provided* that the density at t_0 is the critical density $\rho_{c0} = 3H_0^2/8\pi G$, and [2] that at all other times the density is again the critical density for that time, decreasing with time as

$$\rho_c(t) \propto \frac{1}{t^2} \; . \qquad (4.16)$$

c. What is, given a measurement $H_0 \equiv H(t_0)$ of the Hubble constant, the corresponding age of the universe in the case $n = 3$ that applies to a flat 'cold matter-dominated' universe?

Calculate that age using the measured value (as determined from the WMAP experiment in 2012)

$$H_0 = 74.3 \text{ km/s per Mpc} \qquad (4.17)$$

Note: this is an exercise in using the correct units (SI units) in order to get an age in seconds, that can be converted into an age in years by using 1 yr = 3.156×10^7 seconds. You will also need 1 Mpc = 3.086×10^{22} m.

d. The simplest models for Dark Energy interpret it as a constant energy density of the vacuum: ρ_{vac}. The *De Sitter model* describes a flat, empty universe with a non-vanishing vacuum energy ρ_{vac} in terms of the corresponding cosmological constant:

$$\Lambda \equiv 8\pi G \, \rho_{vac} \; . \qquad (4.18)$$

Show that Friedmann's equation in this case admits the expanding solution

$$\frac{a(t)}{a_0} = \exp\left(\sqrt{\frac{\Lambda}{3}} \; (t - t_0) \right) \; , \qquad (4.19)$$

and that the Hubble constant is indeed a constant in this case.

e. As a last example consider a positively curved ($k > 0$) universe filled with cold matter so that $\rho \propto a^{-3}$. Through clever scaling it is possible to put $a_0 = a(t_0) = 1$, essentially equating the comoving distance of each object to the distance at t_0. Positive curvature requires a density larger than critical, that is:

$$\Omega_0 \equiv \frac{8\pi G\rho_0}{3H_0^2} > 1 \ . \tag{4.20}$$

Now answer the following questions:

1. Use Friedmann's equation at $t = t_0$ (NOW!) with $a_0 = 1$ to show that, with this scaling, the curvature constant must equal

$$k = H_0^2 \ (\Omega_0 - 1) \ . \tag{4.21}$$

2. Show that, at an arbitrary time t, Friedmann's equation can be written as

$$\left(\frac{\dot{a}}{H_0}\right)^2 = \frac{\Omega_0}{a(t)} - (\Omega_0 - 1) \ , \tag{4.22}$$

with $\dot{a} \equiv \mathrm{d}a/\mathrm{d}t$.

3. Show that the age of this universe can be obtained by quadrature[5] as

$$t = \frac{1}{H_0} \int_0^a \frac{\mathrm{d}a'}{\sqrt{\dfrac{\Omega_0}{a'} - (\Omega_0 - 1)}} \ . \tag{4.23}$$

4. Show that one can solve version (4.22) of the Friedmann equation parametrically by introducing a parameter $0 \leq \vartheta \leq 2\pi$ and writing:

$$a(\vartheta) = \frac{\Omega_0}{2\,(\Omega_0 - 1)} \ (1 - \cos\vartheta) \quad , \quad t(\vartheta) = \frac{\Omega_0}{2H_0\,(\Omega_0 - 1)^{3/2}} \ (\vartheta - \sin\vartheta) \ . \tag{4.24}$$

5. Show that this universe recollapses (shrinks) for $\vartheta > \pi$, ending ultimately in a 'Big Crunch' at $\vartheta = 2\pi$. Calculate the maximum age of this universe.

[5] A fancy word for integration!

5 Relative speed and the Horizon distance

Consider (for simplicity) a flat Friedmann universe with metric

$$\mathrm{d}s^2 = c^2\mathrm{d}t^2 - \mathrm{d}\ell^2 = c^2\mathrm{d}t^2 - a^2(t)\left(\mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2 + r^2\sin^2\theta\,\mathrm{d}\phi^2\right) . \tag{5.1}$$

We as observers are located at the origin $r = 0$. A photon is launched from a distant source at comoving radius $r = r_\mathrm{s}$ at time $t = t_\mathrm{em}$ in the past, and (due to our coordinate choice) follows a purely radial orbit from source towards observer with $\mathrm{d}\theta = \mathrm{d}\phi = 0$. As the physical speed of the photon seen by any comoving observer is always the speed of light, we have in this case

$$\frac{\mathrm{d}\ell}{\mathrm{d}t} = -c , \tag{5.2}$$

the *peculiar speed* of the photon relative to the expanding Robertson-Walker grid of coordinates and fiducial cosmological observers. The minus sign in (5.2) expresses the fact that the photon is launched towards the origin $r = 0$ from finite (positive) r.

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a. Show that the distance from observer to photon, $D(t) = a(t)r(t)$, satisfies

$$\frac{\mathrm{d}D}{\mathrm{d}t} = H(t)D(t) - c \equiv v_\mathrm{eff} . \tag{5.1}$$

Give a *physical* interpretation of this result.

b. Beyond which distance is the effective speed such that the photon (initially) moves away from the observer?

c. Show that Eqn. (5.2) can be solved directly to give the comoving distance $r(t)$ of the photon at some time $t > t_\mathrm{em}$:

$$r(t) = r_\mathrm{s} - \int_{t_\mathrm{em}}^{t} \frac{c\,\mathrm{d}t'}{a(t')} . \tag{5.2}$$

What is the general solution for $D(t)$, given $a(t)$?
(**You should be able to write this down directly!**)

d. What is the comoving source distance r_s for photons that reach the observer (at $r = 0$) now (at $t = t_0$) if they started their journey directly following the Big Bang at $t = 0$. This distance is called the **horizon distance** r_{hor} in comoving coordinates[6].

e. Consider an universe expanding as

$$a(t) = \left(\frac{t}{t_0}\right)^{\alpha} , \tag{5.3}$$

with $\alpha < 1$. Show that with this choice of scaling factor, which satisfies $a(t_0) = 1$, the comoving and physical horizon distance at $t = t_0$ are equal to:

$$D_{hor} = r_{hor} = \frac{ct_0}{1 - \alpha} . \tag{5.4}$$

Also show, using the results of Assignment 1, that this equals

$$D_{hor} = \frac{c}{H_0} \frac{\alpha}{1 - \alpha} \tag{5.5}$$

f. Again in the universe that follows expansion law (5.3): show that photons emitted from r_s at time t_{em} reach us at t_0 if (at photon reception) the source is at a distance

$$r_s = D_s(t_0) = \frac{ct_0}{1 - \alpha} \left[1 - \left(\frac{t_{em}}{t_0}\right)^{1-\alpha}\right] . \tag{5.6}$$

[6]This calculation neglects the fact that the Early Universe, younger than ~ 10,000 years, was in fact not transparent to photons.

g. In an inflationary universe one has $H = \sqrt{\Lambda/3} \equiv H_\Lambda$ = constant (see Longair, Chapter 7.3.3) and the expansion *accelerates* as the scale factor varies with time as $a(t) \propto \exp(H_\Lambda t)$. Now show that photons produced at a distance $D > c/H_\Lambda$ will *never* reach us at $r = 0$, regardless how old the universe becomes.

What does this result mean for the number of galaxies (or clusters of galaxies) that you can see as the universe gets older and older?

6 Hand-in assignment: Redshift and particle momentum

Using a heuristic argument employing the Doppler effect I have shown that the wavelength of a photon in the expanding universe becomes longer: a photon emitted at time $t_{\rm em}$ with wavelength $\lambda_{\rm em}$ has a wavelength at time $t > t_{\rm em}$ equal to

$$\lambda(t) = \left(\frac{a(t)}{a(t_{\rm em})}\right)\lambda_{\rm em} \equiv (1+z)\lambda_{\rm em} \; , \tag{6.1}$$

with $a(t)$ the scale factor of the universe and z the corresponding redshift: $z = \frac{a(t)}{a(t_{\rm em})} - 1$. As a result, the photon energy $\epsilon = hc/\lambda$ and photon momentum $p = h/\lambda$ both decrease as the photon propagates:

$$p(t) = \frac{h}{\lambda(t)} \propto \frac{1}{1+z} \; . \tag{6.2}$$

In this assignment we look at what happens with the momentum of a *massive* particle as it freely propagates through the universe. We will do this by solving the equation of motion for a freely falling particle in space-time, and using the result to obtain the physical velocity as seen by a typical cosmological observer fixed to the Robertson-Walker coordinate grid.

Because of the isotropy and homogeneity of space in Friedmann models we may assume that the particle moves along the radial direction. We can use proper time τ as the affine parameter along the particle's world line by defining (in units where $c = 1$):

$$d\tau^2 = dt^2 - a^2(t)\,\frac{dr^2}{1-kr^2} \tag{6.3}$$

for the world line distance recipe. For use in what follows I also introduce the notation

$$\dot{t} \equiv \frac{dt}{d\tau} \;\; , \;\; \dot{r} \equiv \frac{dr}{d\tau} \; . \tag{6.4}$$

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a. Show that line element definition (6.3) can be written concisely as

$$\dot{t}^2 - a^2(t)\,\frac{\dot{r}^2}{1 - kr^2} = 1 \; . \tag{6.5}$$

(**Hint:** use world-line relations like $\mathrm{d}t = (\mathrm{d}t/\mathrm{d}\tau)\,\mathrm{d}\tau = \dot{t}\,\mathrm{d}\tau$.)

b. Answer **a** from last week's assignment 3 gives the 'equation of motion' for $t(\tau)$ when one chooses $\lambda = \tau$. With $c = 1$ it reads:

$$\frac{\mathrm{d}^2 t}{\mathrm{d}\tau^2} + a\,\frac{\mathrm{d}a}{\mathrm{d}t}\left(\frac{\dot{r}^2}{1 - kr^2}\right) = 0 \; . \tag{6.6}$$

Show that result **a** of this assignment implies that this can be rewritten as:

$$\frac{\mathrm{d}^2 t}{\mathrm{d}\tau^2} + \frac{1}{a}\frac{\mathrm{d}a}{\mathrm{d}t}\left(\dot{t}^2 - 1\right) = 0 \; . \tag{6.7}$$

c. Show that (6.7) admits the solution

$$\dot{t} = \frac{\sqrt{a^2 + Q^2}}{a} \; , \tag{6.8}$$

with Q a constant.

(**Hint:** first show that the equation is equivalent with $\mathrm{d}/\mathrm{d}\tau\left(a^2(\dot{t}^2 - 1)\right) = 0$.)

d. Show that the above results imply that

$$\frac{\dot{r}}{\sqrt{1 - kr^2}} = \frac{Q}{a^2} \; . \tag{6.9}$$

e. According to a fiducial observer, who is fixed to the comoving Robertson-Walker coordinate grid and uses time cosmic time t for his clock ticking rate, the *physical* distance associated with the coordinate distance interval $\mathrm{d}r$ equals

$$\mathrm{d}\ell = \frac{a(t)\,\mathrm{d}r}{\sqrt{1-kr^2}} \,. \tag{6.10}$$

He therefore associates with the particle a physical velocity

$$v = \frac{\mathrm{d}\ell}{\mathrm{d}t} = \frac{a(t)\,(\mathrm{d}r/\mathrm{d}t)}{\sqrt{1-kr^2}} \,. \tag{6.11}$$

Evaluate this physical velocity by expressing $\mathrm{d}r/\mathrm{d}t$ in terms of $\dot{r}$ and $\dot{t}$, and show that our results for $\dot{t}$ and $\dot{r}$ imply

$$v = \frac{Q}{\sqrt{Q^2 + a^2}} \,. \tag{6.12}$$

f. If the rest mass of the particle equals m, the momentum of the particle according to our fiducial observer is (with $c = 1$)

$$p = \frac{mv}{\sqrt{1-v^2}} \,. \tag{6.13}$$

Show that the momentum of a massive particle is subject to the same redshift law as the momentum of a massless photon, i.e.

$$p = \frac{\text{constant}}{a(t)} \propto \frac{1}{1+z} \,. \tag{6.14}$$

Assignments week 44 (A. Achterberg)

7 The (in)stability of the Einstein Universe

In reaction to Friedmann's solution of the equations for a isotropic and homogeneous universe, which allowed only expanding or contracting solutions, Einstein introduced the cosmological constant Λ into his theory in order to make static solutions possible. With a cosmological constant Friedmann's equation is

$$H^2 = \left(\frac{1}{a}\,\frac{\mathrm{d}a}{\mathrm{d}t}\right)^2 = \frac{8\pi G\rho}{3} + \frac{\Lambda}{3} - \frac{k}{a^2}\ . \tag{7.1}$$

Differentiation of this equation with respect to time and thermodynamics yields the alternative form for the 'acceleration' of the universal expansion (Longair, Ch. 7.1):

$$\frac{\mathrm{d}^2 a}{\mathrm{d}t^2} = -\frac{4\pi G}{3}\left(\rho + \frac{3P}{c^2}\right) a + \frac{\Lambda}{3}\, a\ . \tag{7.2}$$

This is the form we will mostly use in what follows. We will also assume a universe filled with cold matter so that $P = 0$ and

$$\rho = \rho_0\, a^{-3}\ , \tag{7.3}$$

assuming a scaling where $a_0 \equiv a(t_0) = 1$.

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a. Einstein favored a static universe with a = constant = 1. Show that this implies

$$\rho = \rho_0 = \frac{\Lambda}{4\pi G}\quad ,\quad k = \Lambda\ . \tag{7.4}$$

b. Now consider a small perturbation to the static Einstein solution,

$$a(t) = 1 + \delta a(t) \quad \text{with} \quad |\delta a| \ll 1 \, . \tag{7.5}$$

Show that this changes the density to

$$\rho(t) = \rho_0 \; (1 - 3 \, \delta a) + \mathcal{O}(\delta a^2) \, . \tag{7.6}$$

c. Show that, once one neglects all terms of order $\delta a^2 \ll 1$ and higher (the hallmark of what is known as *perturbation analysis*), Friedmann's equation in version (7.2) leads to a linear differential equation for δa:

$$\frac{\mathrm{d}^2 \, \delta a}{\mathrm{d}t^2} = \Lambda \, \delta a \, . \tag{7.7}$$

(**Hint:** you must use the properties of the unperturbed (static) solution to obtain this!)

d. Solve this equation, assuming that at $t = t_0$ one has $\delta a = \delta a_0$, $(\mathrm{d}\delta a/\mathrm{d}t)_{t_0} = 0$.

Why will the solution always lead to a situation where the Einstein Universe starts to evolve away from the static solution? Predict (without any calculations) what happens for $\delta a_0 > 0$ and for $\delta a_0 < 0$ in the *exact* solution.

Morale: Einstein's static Universe is dynamically unstable. Fluctuations will either cause it to start collapsing (when $\delta a < 0$) or start expanding (when $\delta a > 0$). In either case, it does not return to the static state where $\mathrm{d}a/\mathrm{d}t = 0$.

8 The Friedmann-Lemaitre Universe

The observations by WMAP of the cosmic microwave background, together with the observations of distant SN1A supernovae, have led to a shift in our understanding of the universe. These observations lead to a picture of a flat ($\Omega = 1$) universe that is (at present) dominated by a cosmological constant Λ (vacuum energy $\rho_{\mathrm{v}} = \Lambda/8\pi G$). The best fit to the observations gives the current values

$$\Omega_{\mathrm{m}0} = \frac{8\pi G \rho_{\mathrm{m}0}}{H_0^2} = 0.27 \quad , \quad \Omega_{\Lambda 0} = \frac{\Lambda}{3H_0^2} = 0.73 \; . \tag{8.1}$$

In this assignment we will solve the Friedmann equation for such a universe:

$$H^2 = \left(\frac{1}{a}\frac{\mathrm{d}a}{\mathrm{d}t}\right)^2 = \frac{8\pi G\, \rho_{\mathrm{m}}(a)}{3} + \frac{\Lambda}{3} \tag{8.2}$$

We will assume cold matter so that

$$\rho_{\mathrm{m}}(a) \propto a^{-3} \; . \tag{8.3}$$

The Hubble constant in such universe tends asymptotically (that is: for $t \to \infty$, when $a \to \infty$ and $\rho_{\mathrm{m}} \to 0$) to a constant value, equal to the de-Sitter value:

$$H(t) \;\to\; H_\Lambda = \sqrt{\frac{\Lambda}{3}} \; . \tag{8.4}$$

---

a. Let a_Λ be the scale factor at the moment where the matter density $\rho_{\rm m}$ and the vacuum density $\rho_{\rm v}$ are exactly equal:

$$\rho(a_\Lambda) = \frac{\Lambda}{8\pi G} = \rho_{\rm v} \ . \tag{8.5}$$

Define a dimensionless time variable and scale factor by

$$\tau = H_\Lambda t \ \ , \quad \mathcal{R} = \frac{a}{a_\Lambda} \ . \tag{8.6}$$

Now show, by expressing everything in terms of τ and $\mathcal{R}$, that Friedmann's equation can be written as:

$$\left(\frac{1}{\mathcal{R}}\frac{{\rm d}\mathcal{R}}{{\rm d}\tau}\right)^2 = \mathcal{R}^{-3} + 1 \ . \tag{8.7}$$

b. Define the auxiliary variable

$$z(\tau) \equiv \mathcal{R}^{3/2}(\tau) \ . \tag{8.8}$$

Show that for an expanding universe the scaled Friedmann equation (8.7) is equivalent with:

$$\frac{{\rm d}z}{{\rm d}\tau} = +\tfrac{3}{2}\ \sqrt{1+z^2} \ . \tag{8.9}$$

Why do you choose for a plus sign (rather than a minus-sign) in front of the square root?

c. Show that differential equation (8.9) is solved by[7]

$$z(\tau) = \sinh(3\tau/2)\ , \tag{8.10}$$

and that the scale factor, Ω_Λ and the Hubble parameter in this universe vary as

$$\begin{aligned} a(t) &= a_\Lambda \, \sinh^{2/3}(3H_\Lambda t/2)\ , \\ \Omega_\Lambda(t) &= 1 - \Omega_\mathrm{m}(t) = \tanh^2(3H_\Lambda t/2)\ , \\ H(t) &= H_\Lambda \, \coth(3H_\Lambda t/2)\ . \end{aligned} \tag{8.11}$$

Give the solution for the scale factor $a(t)$ in the two asymptotic limits: early times ($H_\Lambda t \ll 1$) and late times ($H_\Lambda t \gg 1$). Are these two solutions familiar, and what kind of universe do they correspond to?

d. Show that the matter density and vacuum density are equal at a time

$$t_\mathrm{eq} = \frac{2}{3H_\Lambda} \ln\left(1 + \sqrt{2}\right) = \frac{0.58}{H_\Lambda}\ . \tag{8.12}$$

e. Given the current value of Hubble's constant H_0 and the vacuum omega parameter $\Omega_{\Lambda 0}$, show that the corresponding age of the universe equals

$$t_0 = \frac{2}{3H_0} \left\{ \frac{1}{\sqrt{\Omega_{\Lambda 0}}} \ln\left(\frac{1 + \sqrt{\Omega_{\Lambda 0}}}{1 - \sqrt{\Omega_{\Lambda 0}}} \right)^{1/2} \right\}\ . \tag{8.13}$$

(**Hint:** It may help if you first prove that $H_\Lambda = \sqrt{\Omega_{\Lambda 0}}\, H_0$.)

Calculate this age, using $\Omega_{\Lambda 0} = 0.73$ and $H_0 = 70$ km/s per Mpc.

[7]The following standard integral may be useful: $\int \mathrm{d}x/\sqrt{1+x^2} = \mathrm{arcsinh}(x)$.

9 Hand-in assignment: Thinking in terms of redshift

The number of cosmological parameters that is *directly* observable are few: the (current) Hubble constant H_0 and the observed redshift z come to mind. They are related to the scale factor of the universe by

$$H_0 = \left(\frac{1}{a} \frac{\mathrm{d}a}{\mathrm{d}t} \right)_{t=t_0} \quad , \quad 1 + z = \frac{a(t_0)}{a(t)} \ , \tag{9.1}$$

with t the time of photon emission for a hypothetical photon that arrives at the observer at the current epoch t_0 with redshift z. It is therefore not surprising that most astronomers think in terms of redshift, and use redshift as a measure of distance, and (implicitly) of the time when the photons they observe were emitted. So: "at redshift z" means "at a distance/time corresponding to a present photon redshift z". Given a certain redshift z, the scale factor is $a(t) = a(t_0)/(1 + z)$ and one can calculate t from $a(t)$ provided one knows the type of universe one is living in! Remember in all this that "large z" is equivalent to "distant" and "far into the past".

In this assignment we follow that approach, and calculate a number of important quantities in terms of redshift (rather than time). We will also use a scaling where $a(t_0) = 1$, which is always possible. This means that the scale factor *at time of emission* is related to the photon redshift z *at time of observation* by

$$a(t) = \frac{1}{1 + z} \ . \tag{9.2}$$

We will now convert many of the fundamental relationships in cosmology from a formulation in terms of scale factor to a formulation in terms of redshift. In order to make direct contact with other 'observables' this formulation will use the Omega-parameters (ratio of density over critical density, $\rho_{\mathrm{co}} = 3H_0^2/8\pi G$) for matter, radiation and vacuum energy:

$$\Omega_{\mathrm{m0}} = \frac{8\pi G\, \rho_{\mathrm{m0}}}{3H_0^2} \quad , \quad \Omega_{\mathrm{r0}} = \frac{8\pi G\, \rho_{\mathrm{r0}}}{3H_0^2} \quad , \quad \Omega_\Lambda = \frac{\Lambda}{3H_0^2} = \frac{8\pi G\, \rho_{\mathrm{v}}}{3H_0^2} \tag{9.3}$$

9.1 Densities

Show that the matter and radiation density at time $t \leq t_0$ can be respectively written in terms of photon redshift of the photons originating from that time as

$$\rho_{\mathrm{m}} = \rho_{\mathrm{m0}}\,(1+z)^3 \quad , \quad \rho_{\mathrm{r}} = \rho_{\mathrm{r0}}\,(1+z)^4 \ . \tag{9.4}$$

9.2 Hubble parameter and Friedmann's equation

Show that the Hubble parameter can be written as

$$H(t) = -\frac{1}{1+z}\,\frac{\mathrm{d}z}{\mathrm{d}t} \ . \tag{9.5}$$

What sign does $\mathrm{d}z/\mathrm{d}t$ have when you (correctly) interpret t as the time of emission?

Now show that Friedmann's equation can be written as

$$\left(\frac{\mathrm{d}z}{\mathrm{d}t}\right)^2 = H_0^2(1+z)^2\,\left[\Omega_{\mathrm{m0}}(1+z)^3 + \Omega_{\mathrm{r0}}\,(1+z)^4 + \Omega_\Lambda - \Omega_{k0}\,(1+z)^2\right] \ , \tag{9.6}$$

where

$$\Omega_{k0} \equiv \Omega_{\mathrm{m0}} + \Omega_{\mathrm{r0}} + \Omega_\Lambda - 1 \ . \tag{9.7}$$

is measure of the curvature of the universe at time t_0.

Hint: Get rid of the curvature constant k by looking at the Friedmann equation at time t_0 when $a(t_0) = 1$ and $H = H_0$.

9.3 Friedmann equation in a Lemaitre Universe

Now assume that the radiation term can be neglected, in effect putting $\Omega_{r0} = 0$, and simplify the notation by writing Ω_0 rather than Ω_{m0}. Show that Friedmann's equation in this case can be written as

$$\left(\frac{dz}{dt}\right)^2 = H_0^2(1+z)^2 \left[(\Omega_0 z + 1)(1+z)^2 - \Omega_\Lambda z\,(2+z)\right] . \tag{9.8}$$

This corresponds to Eqn. 7.64 on p. 217 of Longair's book.

(**Hint:** you may need the special product $(1+z)^3 = 1 + 3z + 3z^2 + z^3$.)

9.4 Source distance as a function of observed redshift z_0

In the lectures we calculated the distance to a source *at the moment of photon reception*, which is the distance always quoted in research papers, as

$$D_0 = \int_t^{t_0} \frac{c\,dt}{a(t)} , \tag{9.9}$$

assuming $a(t_0) = 1$ as before. Now show that this distance can be written through a variable change from t to z as:

$$D_0 = \frac{c}{H_0} \int_0^{z_0} \frac{dz}{\sqrt{\Omega_{m0}(1+z)^3 + \Omega_{r0}\,(1+z)^4 + \Omega_\Lambda - \Omega_{k0}\,(1+z)^2}} . \tag{9.10}$$

Here z_0 is the observed redshift of a photon from that source, which started the journey at time t.

Hint: think carefully about the way to go from an integral over time to an integral over redshift, and what this means for the integration boundaries!

9.4.1 The small-z limit

Show that for $z_0 \ll 1$ the above integral for D_0 *always* yields the result

$$D_0 = \frac{cz_0}{H_0} + O(z_0^2) . \tag{9.11}$$

9.5 Photon travel time as a function of observed redshift z_0

Using a similar approach, converting an integral over time into one over redshift, show that photon travel time $\Delta t \equiv t_0 - t$ of a photon with observed redshift z_0 equals

$$\Delta t = \frac{1}{H_0} \int_0^{z_0} \frac{\mathrm{d}z}{(1+z)\ \sqrt{\Omega_{\mathrm{m}0}(1+z)^3 + \Omega_{\mathrm{r}0}\,(1+z)^4 + \Omega_\Lambda - \Omega_{k0}\,(1+z)^2}} . \tag{9.12}$$

9.6 Horizon distance

The horizon distance corresponds to the distance a photon can travel during the entire age of the universe. Answer the following questions:

- Assuming $a(0) = 0$, what is the redshift z_0 at the moment t_0 of photon reception;
- What then, from 3.4, is the horizon distance D_{hor}?

9.7 Distances in a flat universe

I now consider a flat universe with $\Omega_{k0} = 0$ and a negligible radiation density so that we can put $\Omega_{\mathrm{r}0} = 0$. In what follows I will simply write Ω_0 for $\Omega_{\mathrm{m}0}$. In this case we have

$$\Omega_0 + \Omega_\Lambda = 1 , \tag{9.13}$$

and result (15.160) reduces to

$$D_0 = \frac{c}{H_0} \int_0^{z_0} \frac{\mathrm{d}z}{\sqrt{\Omega_0\,(1+z)^3 + \Omega_\Lambda}} . \tag{9.14}$$

Calculate D_0 for two cases:

1. A flat, matter-dominated universe with $\Omega_0 = 1$ and $\Omega_\Lambda = 0$;
2. A flat De-Sitter universe with $\Omega_0 = 0$ and $\Omega_\Lambda = 1$.

Can you now *immediately* write down the horizon distance $D_{\rm hor}$ for the first case?

9.8 Age of a flat Lemaitre universe

The age of the universe is formally the age of a photon that starts at $t = 0$. If you answered the first question in 3.6 correctly you know that the photon reaches the observer with infinite redshift, $z_0 = \infty$, and (as always) starts with $z = 0$. It's travel time is equal to the age of the universe, so result (9.12) then yields for $\Omega_{\rm m0} = \Omega_0$, $\Omega_{\rm r0} = \Omega_{k0} = 0$:

$$\Delta t = t_0 = \frac{1}{H_0} \int_0^\infty \frac{{\rm d}z}{(1+z)\ \sqrt{\Omega_0\,(1+z)^3 + \Omega_\Lambda}} . \tag{9.15}$$

Evaluate this integral using a variable change from z to θ (see Longair, p. 218) defined by:

$$1 + z = \left(\frac{\Omega_\Lambda}{\Omega_0}\right)^{1/3} \tan^{2/3}\theta . \tag{9.16}$$

Show that this reproduces answer (8.13) that was obtained in an entirely different manner.

Hint: if you do the variable change correctly, you should end up with an expression that involves an integral of the form

$$\int \frac{{\rm d}x}{\sin x} = \ln\left[\tan\left(\frac{x}{2}\right)\right] = \ln\left[\frac{\sin x}{1+\cos x}\right] \tag{9.17}$$

Now [1] carefully figure out the integration boundaries in θ that correspond to $z = 0$ and $z = \infty$ and [2] use the fact that you can always get $\sin x$ and $\cos x$ from $\tan x$ if you use $\cos^2 x + \sin^2 x = 1$. Any sign ambiguities should work itself out.

Assignments week 45 (A. Achterberg)

10 Hand-in assignment: angular distance, early horizon size and related issues

In this assignment we continue to look at various observable quantities, and how we can calculate them using the properties of the *current* universe such as redshift, the Omega parameters and the Hubble constant H_0.

Whenever possible we will convert time-integrals into redshift integrals in the same manner as was done in last week's hand-in assignment. That means that we will keep using integral conversions of the sort

$$\int_{t_1}^{t_2} dt \cdots = \int_{z_2}^{z_1} \frac{dz}{(1+z)\ \sqrt{\Omega_{m0}(1+z)^3 + \Omega_{r0}\,(1+z)^4 + \Omega_{\Lambda 0} - \Omega_{k0}\,(1+z)^2}} \cdots . \quad (10.1)$$

Here z_1 (z_2) is the redshift of a photon observed at some fixed time t_0 if it was emitted at some earlier time t_1 (t_2). Note that we do not necessarily assume that $t_2 = t_o$ (photon observation time = current age of the universe) as we did earlier. That does **not** influence in any way the validity of the expression, which is solely based on a variable change, the definition of redshift z in terms of the scale factor $a(t)$ for the choice $a(t_0) = 1$ and Friedmann's equation.

Also do not forget that $z_2 < z_1$ if $t_2 > t_1$: photons emitted later are observed with a smaller redshift.

The ultimate aim of this assignment is to have you calculate the angular size in the sky of the horizon distance as it was at the moment that the Cosmic Microwave Background (CMWB) was born. This moment in the evolution of our universe corresponds with a redshift $z = z_{\rm CMWB} \simeq 1100$, as explained in the Lectures, or in Chapter 9 of Longair's book.

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a. If we put ourselves at the origin $r = 0$ of a Robertson-Walker metric, all photon orbits that reach us are purely radial orbits that have $(\theta\,,\phi)$ constant. Also, in a Robertson-Walker metric the way angles are measured does not explicitly depend on curvature.

Use this information to prove that a distant object of fixed linear size d that is much smaller than its distance is seen with an angular size $\Delta\theta$ on the sky that equals

$$\Delta\theta = 2\arctan\left(\frac{d}{2D}\right) \simeq \frac{d}{D} \quad \text{(assuming } d \ll D,\ \Delta\theta \ll 1\text{)}\ , \tag{10.2}$$

with D the distance of the source *at the moment of emission* of the photons. (It may help to draw a figure showing the situation at photon emission, at photon reception as well as the photon orbits.) We will use the small angle approximation ($\Delta\theta \ll 1$) from this point onwards.

b. If the source is observed at t_0 with a redshift z_0, then show that the so-called *angular distance* $D_{\rm a} \simeq d/\Delta\theta$ is related to the source distance D_0 at the moment of the observation by

$$D_{\rm a} = \frac{d}{\Delta\theta} = \frac{D_0}{1 + z_0}\ . \tag{10.3}$$

c. Use one of the results of last week's hand-in assignment to show that

$$D_{\rm a} = \frac{c}{H_0\,(1 + z_0)} \int_0^{z_0} \frac{{\rm d}z}{\sqrt{\Omega_{\rm m0}(1+z)^3 + \Omega_{\rm r0}\,(1+z)^4 + \Omega_\Lambda - \Omega_{k0}\,(1+z)^2}}\ . \tag{10.4}$$

NO COMPLICATED CALCULATION SHOULD BE REQUIRED!

d. At *any* time $t > 0$, not just t_0, photons emitted at the Big Bang arrive at an observer with redshift $z = \infty$. That means that the horizon size at some $0 < t \leq t_0$ is

$$D_{\rm hor}(t) = a(t) r = a(t) \int_0^t \frac{c\,{\rm d}t'}{a(t')} \tag{10.5}$$

Show that this allows you to obtain the horizon size "at a redshift z" from

$$D_{\rm hor}(z) = \frac{c}{H_0(1+z)} \int_z^\infty \frac{{\rm d}z'}{\sqrt{\Omega_{\rm m0}(1+z')^3 + \Omega_{\rm r0}\,(1+z')^4 + \Omega_\Lambda - \Omega_{k0}\,(1+z')^2}}\ . \tag{10.6}$$

e. The angle $\Delta\theta$ under which an object of size d is observed equals for small angles according to result **b**

$$\Delta\theta(z_0) = \frac{d}{D_{\rm a}} = (1+z_0)\,\frac{d}{D_0}\ . \tag{10.7}$$

Calculate $\Delta\theta(z_0)$ for the simple case of a flat universe dominated by cold matter, that is for $\Omega_{\rm m0} = 1$, $\Omega_{\rm r0} = \Omega_{\Lambda 0} = \Omega_{k0} = 0$.

Make a rough graph of $\Delta\theta(z_0)$ and show that the angular size, given d, first decreases with increasing redshift (distance), but then *increases* for large z_0 (i.e. large distances!). Show that the minimum angular size occurs in this universe at a redshift

$$z_{\rm min} = 1.25\ , \tag{10.8}$$

that this minimum angular size equals

$$\Delta\theta_{\rm min} = \frac{27}{8}\,\frac{H_0 d}{c} = 3.375\,\frac{H_0 d}{c}\ , \tag{10.9}$$

and that it occurs for a source that currently resides at a distance

$$D_0 = D(z=0) = \frac{2}{3}\frac{c}{H_0} = \frac{D_{\rm hor}}{3}\ . \tag{10.10}$$

Here $D_{\rm hor} = 2c/H_0$ is the horizon size for this flat, matter-dominated universe at time t_0.

Could you give a physical explanation for the fact that the every-day intuitive principle "the more distant it is, the smaller it looks" does not necessarily always apply in an expanding universe?

f. We now consider the birth of the Cosmic Microwave Background (CMWB) at $z = z_{\rm CMWB} \simeq 1100$. At that (and larger) redshifts it is a fair assumption to approximate the universe as a radiation-dominated, flat universe. That means that we only need to take the radiation term $\propto \Omega_{\rm r0}$ into account in integrals for $z \geq 1100$ as that term dominates over the other terms, and can put $\Omega_{k0} = 0$ as we are dealing with a flat universe. Show that the horizon size at the birth of the CMWB can then be calculated from (10.6) as

$$D_{\rm hor}(z_{\rm CMWB}) \simeq \frac{c}{\sqrt{\Omega_{\rm r0}}\, H_0} \frac{1}{(1 + z_{\rm CMWB})^2} \simeq 10^{-6} \frac{c}{\sqrt{\Omega_{\rm r0}}\, H_0} \ . \tag{10.11}$$

g. The distance associated with redshift $z = z_{\rm CMWB} \simeq 1100$ defines a fictitious spherical surface around each observer that corresponds to the *last scattering surface*, the surface where the CMWB photons experienced (on average) their last scattering and beyond which (i.e. at larger redshifts) the universe is opaque to visible light. The corresponding distance is D_{LS}. Using the Friedmann-Lemaitre model (a fair approximation for $z < 1100$) to calculate D_0, given $z_{\rm CMWB} \simeq 1100$, one finds roughly:

$$D_{LS} \simeq \frac{c}{H_0} \int_0^{z_{\rm CMWB}} \frac{{\rm d}z}{\sqrt{\Omega_{\rm m0}\,(1+z)^3 + \Omega_{\Lambda 0}}} \simeq \frac{3c}{H_0} \ . \tag{10.12}$$

This uses the observed values for $\Omega_{\rm m0} = 0.27$ and $\Omega_{\Lambda 0} \simeq 0.73$. The current radiation density corresponds to

$$\Omega_{\rm r0} \simeq 10^{-4} \ . \tag{10.13}$$

Use this information to calculate the angular size in the sky of the horizon distance at the moment the CMWB was born, first in radians and then in degrees!

h. As a last application we consider the **past light cone**. For simplicity I will again assume a flat universe. Photon orbits follow from the equation for *null geodesics*

$$ds^2 = c^2 dt^2 - a^2(t)\, dr^2 = 0\ , \tag{10.14}$$

which gives the comoving distance Δr traveled in the cosmic time interval $\Delta t = t_2 - t_1$ as

$$\Delta r_{\rm phot}(t_1|t_2) = \int_{t_1}^{t_2} \frac{c\ dt}{a(t)}\ . \tag{10.15}$$

The *past light cone* is usually defined as the *physical distance at the moment of emission* $t_{\rm e}$ corresponding with the *comoving distance* traveled by a photon in the time interval $t_0 - t_{\rm e}$, assuming if it received at $t = t_0$. This gives

$$D_{\rm PLC} = a(t_{\rm e})\ \Delta r_{\rm phot}(t_{\rm e}|t_0) = a(t_{\rm e}) \int_{t_{\rm e}}^{t_0} \frac{c\ dt}{a(t)}\ . \tag{10.16}$$

The reason why this particular definition is useful is that one can in principle see how many objects are within the past light cone when the light was emitted. At the time the photon is received, this distance has been stretched by the Hubble expansion by a factor $a(t_0)/a(t_{\rm e}) = 1 + z_0$. Assuming a scaling where $a(t_0) = 1$ this implies a distance $D_0({\rm PLC}) = \Delta r_{\rm phot}$, which is the basis of the remark that the choice $a(t_0) = 1$ means that the current physical distance and the comoving distance coincide. Now show that for photons received with redshift z_0 the past light cone size $D_{\rm PLC}$ at that redshift coincides with the angular distance, i.e.

$$D_{\rm PLC} = \frac{c}{H_0\,(1 + z_0)} \int_0^{z_0} \frac{dz}{\sqrt{\Omega_{\rm m0}(1+z)^3 + \Omega_{\rm r0}\,(1+z)^4 + \Omega_\Lambda - \Omega_{k0}\,(1+z)^2}}\ . \tag{10.17}$$

i. Can you now exploit the fact that $D_{\rm PLC} = D_{\rm a}$ in the following way for a matter-dominated universe with for $\Omega_{\rm m0} = 1$, $\Omega_{\rm r0} = \Omega_{\Lambda 0} = \Omega_{k0} = 0$, see assignment **e** where we did something similar for the angular size:

1. Demonstrate that $D_{\rm PLC}$ increases for small z, but decreases at large z;
2. Give the redshift where $D_{\rm PLC}$ has a maximum, and calculate $D_{\rm PLC}$ at that point.

Assignments week 46 (A. Achterberg)

11 Simple inflation model: massive scalar field

In this assignment we will consider the simplest incarnation for an inflation model, which employs a massive (mass m), spatially uniform, one-component scalar field with Lagrangian

$$\mathcal{L}(\varphi\,,\,\dot{\varphi}) = \tfrac{1}{2}\dot{\varphi}^2 - V(\varphi) = \tfrac{1}{2}\dot{\varphi}^2 - \tfrac{1}{2}m^2\,\varphi^2\ . \tag{11.1}$$

Here

$$\dot{\varphi} \equiv \frac{\partial\varphi}{\partial t}\ , \tag{11.2}$$

employing *natural units* with $c = \hbar = k_{\rm b} = 1$ and $G = 1/m_{\rm pl}^2$, with $m_{\rm pl} = \sqrt{\hbar c/G} = 1.36 \times 10^{19}$ GeV the *Planck mass*. Density and pressure associated with this field are:

$$\rho_\varphi = \tfrac{1}{2}\dot{\varphi}^2 + \tfrac{1}{2}m^2\,\varphi^2 \quad , \quad P_\varphi = \tfrac{1}{2}\dot{\varphi}^2 - \tfrac{1}{2}m^2\,\varphi^2\ . \tag{11.3}$$

Field dynamics in an expanding universe with Hubble parameter $H(t)$ is described by the 'equation of motion'. Introducing the notation $\ddot{\varphi} \equiv \partial^2\varphi/\partial t^2$:

$$\ddot{\varphi} + 3H\,\dot{\varphi} + m^2\,\varphi = 0\ . \tag{11.4}$$

The universe as a whole is described by Friedmann's equation, again in natural units:

$$\left(Ht_{\rm pl}\right)^2 = \frac{4\pi^3 g_*}{45}\left(\frac{T}{m_{\rm pl}}\right)^4 + \frac{8\pi}{3}\,\frac{\rho_\varphi}{m_{\rm pl}^4} - \kappa\left(\frac{T}{m_{\rm pl}}\right)^2\ . \tag{11.5}$$

The three terms on the right-hand side respectively correspond to the contribution of the energy density of radiation, the energy density of the scalar field and the curvature term. In these units we measure the temperature T, mass m and field amplitude φ all in GeV, and time in units of GeV^{-1}. The natural curvature parameter κ may be taken to be a constant.

---

a. In order to gain some understanding about the behavior of a scalar field in an expanding universe, consider the (admittedly) not very realistic case of H = constant. Try a quasi-oscillatory complex solution[8] of the form

$$\varphi(t) = \varphi_0 \, e^{-i\omega t} \; . \tag{11.6}$$

For which ω does this solution indeed solve (11.4)?

(**Hint:** the well-known formula for the solutions of a quadratic equation $\bar{a}x^2 + \bar{b}x + \bar{c} = 0$ also works if the three coefficients ($\bar{a}$, $\bar{b}$, $\bar{c}$) are complex numbers.)

b. Show that the solutions you found in **a** *always* decay exponentially, i.e. $|\varphi(t)| \equiv \sqrt{\varphi(t)\varphi^*(t)} \propto \exp(-\lambda t)$. Give **all** the possible values for $\lambda = \mathrm{Im}(\omega)$ for $m > 3H/2$ and for $m < 3H/2$, and find the slowest decaying solution for $m \ll 3H/2$.

The morale of this calculation: The expansion of the universe leads to the decay of the amplitude $|\varphi|$ of the scalar field. This remains true even if H depends on time.

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c. In the *slow roll approximation* one assumes

$$|\ddot{\varphi}| \ll \Big(H\,|\dot{\varphi}|\,,\; m^2\,|\varphi|\Big) \quad , \quad \tfrac{1}{2}\dot{\varphi}^2 \ll V(\phi) = \tfrac{1}{2}m^2\,\varphi^2 \; . \tag{11.7}$$

Use this approximation to show that a *variable change*, from $\varphi(t)$ to $\varphi(a(t))$ with $a(t)$ the scale factor, leads to an equation for the scalar field of the form

$$3H^2 \left(\frac{\mathrm{d}\ln\varphi}{\mathrm{d}\ln a}\right) + m^2 = 0 \; . \tag{11.8}$$

(**Hint:** you will need the definition of the Hubble parameter $H(t)$).
Also show that the general solution of this equation is a quasi-exponential decay:

$$\varphi(a) = \varphi(a_0)\,\exp\left(-\frac{m^2}{3}\int_{a_0}^{a} \frac{\mathrm{d}a'}{a'\,H^2(a')}\right) \;\; (\text{with } H(t) \rightarrow H(a(t))) \; . \tag{11.9}$$

[8] You can always take the real part of a complex solution to get a real field since we are dealing with a linear equation for $\varphi(t)$, in effect putting $\varphi(t) = \mathrm{Re}\left(\phi_0\, e^{-i\omega t}\right)$ after finding the complex solution!

d. We now do a *consistency check* on the solution to see if it satisfies the slow-roll assumption under which it was derived. Show that the second (energy) condition for the slow-roll approximation can only be satisfied if we *additionally* assume that

$$m \ll \tfrac{3}{2} H \,. \tag{11.10}$$

e. Since the temperature in an expanding universe drops as $T \propto a^{-1}$ and the radiation density as $\rho_{\rm r} \propto T^4 \propto a^{-4}$ it is likely that at some point the energy density of the scalar field starts to dominate over the effects of radiation and curvature. Friedmann's equation then simplifies in the slow-roll approximation for $\varphi(t)$ to

$$H^2 t_{\rm pl}^2 = \frac{H^2}{m_{\rm pl}^2} = \frac{8\pi}{3} \frac{V(\varphi)}{m_{\rm pl}^4} = \frac{4\pi}{3} \frac{m^2 \varphi^2}{m_{\rm pl}^4} \,. \tag{11.11}$$

Show that a simple re-interpretation of the first result of question **c** then leads to:

$$\frac{\mathrm{d} \ln a}{\mathrm{d}\varphi} = -\frac{4\pi\, \varphi}{m_{\rm pl}^2} \,. \tag{11.12}$$

Also show that the condition derived in **d** is justified as long as the scalar field amplitude remains sufficiently large:

$$\frac{|\varphi|}{m_{\rm pl}} \gg \frac{1}{\sqrt{3\pi}} \simeq 0.326 \,. \tag{11.13}$$

f. What is the solution of Eqn. (11.12) for a as a function of ϕ if $a = a_0$ for $\phi = \phi_0$ at some time t_0?

g. In the scenario of *Chaotic Inflation* (as proposed by Andei Linde) Inflation starts immediately after the Big Bang at the Planck time $t_{\rm pl} \simeq 10^{-43}$ s. At that time quantum fluctuations are large, allowing the scalar field to carry an energy density roughly equal to the Planck energy density $\rho_{\rm pl} \simeq m_{\rm pl}^4$:

$$V(\phi) = \tfrac{1}{2} m^2 \varphi_0^2 \simeq m_{\rm pl}^4 \quad \Longleftrightarrow \quad \frac{|\varphi_0|}{m_{\rm pl}} \simeq \sqrt{2}\,\frac{m_{\rm pl}}{m} \tag{11.14}$$

Assuming that inflation ends when $\varphi \simeq 0$, give the typical inflation factor, the factor by which the scale factor increases over the inflation period:

$$\mathcal{Z}_{\rm infl} = \frac{a(\varphi \simeq 0)}{a(\varphi_0)} = \frac{a(\varphi \simeq 0)}{a_0}\ . \tag{11.15}$$

Calculate $\mathcal{Z}_{\rm infl}$ for the (not unreasonable) case where the mass of the scalar field equals $m = 0.01\ m_{\rm pl}$.

Assignments week 48 (A. Achterberg)

12 Cold Dark Matter perturbations and driven baryonic fluctuations

Cold Dark Matter (CDM) is the prime candidate for the material that dominates the matter content of the Universe. In that picture ordinary (baryonic) matter is only a minority species: $\Omega_{\rm CDM} \simeq 0.20$, $\Omega_{\rm ba} \simeq 0.04$. The term 'cold' in CDM means that these particles decoupled from the rest of the universe when they where non-relativistic, that is at a temperature $T_{\rm dec}$ such that $k_{\rm b}T_{\rm dec} \ll m_{\rm cdm}c^2$. One thinks that CDM consists of WIMPS: *Weakly Interacting Massive Particles*. Several experiments are being conducted to detect such particles.

In this assignment we look at small fluctuations in the CDM density in the early universe to find out how they grow. We will describe these in terms of the fractional density perturbation,

$$\Delta_{\rm cdm}(\boldsymbol{x}\,,\,t) = \left(\frac{\delta\rho(\boldsymbol{x}\,,\,t)}{\rho}\right)_{\rm cdm} . \tag{12.1}$$

Here $\boldsymbol{x}$ is the comoving coordinate, related to the physical position vector $\boldsymbol{r}$ by $\boldsymbol{r} = a(t)\,\boldsymbol{x}$.

In the limit where the contribution of baryons to the gravitational potential can be neglected the cold dark matter density fluctuations satisfy

$$\left[\frac{{\rm d}^2}{{\rm d}t^2} + 2\left(\frac{\dot{a}}{a}\right)\frac{\rm d}{{\rm d}t} - 4\pi G\,\rho_{\rm cdm}(a)\right]\Delta_{\rm cdm}(\boldsymbol{x}\,,\,t) = 0\ . \tag{12.2}$$

Here $\dot{a} = {\rm d}a/{\rm d}t = Ha$ is the time derivative of the scale factor. The sound speed of CDM may be taken to be so small that it does not matter. The unperturbed CDM density scales in the usual manner:

$$\rho_{\rm cdm}(a) \propto a^{-3}\ . \tag{12.3}$$

We will also look at how the gravitational forces associated with CDM density fluctuations drive density fluctuations in the gas of baryons.

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a. Consider a variable change from t to $a(t)$ so that one uses the scale factor $a(t)$ rather than time to track the expansion of the universe. This leads to:

$$\frac{\mathrm{d}}{\mathrm{d}t} = \dot{a}\,\frac{\mathrm{d}}{\mathrm{d}a}\;. \tag{12.4}$$

Show that

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} = \dot{a}^2\,\frac{\mathrm{d}^2}{\mathrm{d}a^2} + \ddot{a}\,\frac{\mathrm{d}}{\mathrm{d}a}\;, \tag{12.5}$$

where $\ddot{a} \equiv \mathrm{d}^2 a/\mathrm{d}t^2$.

b. Friedmann's equation for a flat ($k = 0$), matter-dominated universe leads to:

$$\dot{a}^2 = \frac{8\pi G\rho}{3}\,a^2\;\;,\;\;\ddot{a} = -\frac{4\pi G\rho}{3}\,a\;. \tag{12.6}$$

Show that the CDM wave equation (12.2) can be rewritten as:

$$\left(\tfrac{2}{3}\,a^2\,\frac{\mathrm{d}^2}{\mathrm{d}a^2} + a\,\frac{\mathrm{d}}{\mathrm{d}a} - \frac{\rho_{\mathrm{cdm}}}{\rho}\right)\Delta_{\mathrm{cdm}}(a) = 0\;. \tag{12.7}$$

c. It is a reasonable approximation in the matter-dominated phase of the Universe to assume $\rho \simeq \rho_{\mathrm{cdm}}$ so that the last term inside the brackets of Eqn. (12.7) can be replaced by unity. Show that in this case solutions of the form

$$\Delta_{\mathrm{cdm}}(a) \propto \Delta_0\left(\frac{a}{a_0}\right)^s \tag{12.8}$$

solve Eqn. (12.7), and find the two possible values for the exponent s.

Baryons (as a minority species) feel the perturbation in gravity caused by the CDM density perturbations. This leads to baryonic density perturbations that are *driven* by the CDM fluctuations. In the limit $\rho_{\rm ba} \ll \rho_{\rm cdm}$ and for scales much larger than the *baryonic* jeans length, the baryonic density fluctuations satisfy:

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}t^2} + 2\left(\frac{\dot{a}}{a}\right) \frac{\mathrm{d}}{\mathrm{d}t} \right] \Delta_{\rm ba}(t) = 4\pi G \rho_{\rm cdm}\, \Delta_{\rm cdm}(t) \ . \tag{12.9}$$

Here $\Delta_{\rm ba}(t) = \delta\rho_{\rm ba}/\rho_{\rm ba}$ is the fractional baryonic density perturbation. The term on the right-hand side of this equation is the driving term: the effect of the gravitational acceleration $\propto \Delta_{\rm cdm}(t)$ felt by the baryons due to the gravity induced by CDM fluctuations. The self-gravity of the baryons has been neglected as we assumed $\rho_{\rm ba} \ll \rho_{\rm cdm}$.

d. Now show that for the growing solution obtained in **c** for $\Delta_{\rm cdm}$, the solution with positive exponent s, the following expression for $\Delta_{\rm ba}$ solves Eqn. (12.9) for the initial condition $\Delta_{\rm ba}(a_0) = 0$:

$$\Delta_{\rm ba}(a) = \Delta_0 \left(\frac{a}{a_0} - 1 \right) \ . \tag{12.10}$$

Hint: first convert the equation for the baryons to the form analogous to (12.7) using $\rho \simeq \rho_{\rm cdm}$! That should not be any problem now.

e. What does this mean for the relative magnitude of CDM and baryonic mass fluctuations $\delta\rho_{\rm cdm}$ and $\delta\rho_{\rm ba}$ at late times after substantial growth, that is for $a \gg a_0$?

You will need the current densities as derived from WMAP, conveniently expressed in terms of the Omega-parameters: $\Omega_{\rm cdm0} = 0.20$, $\Omega_{\rm ba0} = 0.04$.

13 WIMP physics

WIMPS (Weakly Interacting Massive Particles) are the prime candidates for Cold Dark Matter (CDM). In order for them to exist they must decouple from the rest of the universe when they are 'cold', meaning: non-relativistic. Decoupling occurs when the interaction rate Γ_X of these particles becomes smaller than the expansion rate $\dot{a}/a = H$ of the universe:

$$\Gamma_{\rm X} = n_{\rm X}\sigma_{\rm X} v \leq H \; . \tag{13.1}$$

Here the subscript "X" is used to denote WIMP quantities, with $n_{\rm X}$ their density, $\sigma_{\rm X}$ the WIMP-WIMP interaction cross section and v the typical WIMP velocity, roughly the thermal velocity for non-relativistic particles:

$$v \simeq \sqrt{\frac{k_{\rm b} T_{\rm X}}{m_{\rm X}}} \tag{13.2}$$

As long as WIMPS are strongly coupled to radiation and to (baryonic) matter their temperature equals the radiation temperature, $T_{\rm X} = T$, and their density follows a equilibrium distribution. Relative to photons and other relativistic particles, such as neutrinos ("radiation"), they then have a density

$$\frac{n_{\rm X}}{n_{\rm rad}(T)} \sim 0.04\,\mu^{3/2}\, e^{-\mu} \quad , \quad \text{with}\; \mu \equiv m_{\rm X}c^2/k_{\rm b}T \gg 1. \tag{13.3}$$

The condition $\mu \gg 1$ implies that $n_{\rm X} \ll n_{\rm rad}(T)$. The number density of radiation scales with radiation temperature T as

$$n_{\rm rad}(T) \simeq 0.2 \left(\frac{k_{\rm b}T}{\hbar c}\right)^3 \;{\rm m}^{-3} \; . \tag{13.4}$$

In this type of calculation it is useful to employ *natural units* that have $\hbar = c = k_{\rm b} = 1$. Length and time are then measured in units of $[{\rm GeV}]^{-1}$ and reaction rates, the Hubble parameter, energy, mass and temperature are all measured in [GeV]. A velocity is measured in units of c and therefore dimensionless. This implies that cross sections (an area!) have units of $[{\rm GeV}]^{-2}$, and particle densities are measured in $[{\rm GeV}]^3$. You can use this to check your calculations below.

a. In natural units we have for $T_X = T$:

$$v = \sqrt{T/m_{\rm X}} = 1/\sqrt{\mu} \quad , \quad \mu = m_{\rm X}/T \quad , \quad n_{\rm rad} \simeq 3.6\,T^3 \ . \tag{13.5}$$

Show that the interaction rate $\Gamma_{\rm X}$ (while the WIMPS remain strongly coupled) scales in natural units as

$$\Gamma_{\rm X} \simeq 0.15\,T^3\,\mu e^{-\mu}\,\sigma_{\rm X}\ {\rm GeV}\ . \tag{13.6}$$

b. This early in the history of the universe it can be considered as a flat, radiation-dominated universe. Friedmann's equation in natural units then gives

$$H^2 = \frac{4\pi^3\,g_*}{45}\left(\frac{T^4}{m_{\rm pl}^2}\right) \simeq 82.8\left(\frac{T^4}{m_{\rm pl}^2}\right)\ {\rm GeV}^2\ , \tag{13.7}$$

where $g_* \simeq 30$ roughly counts the number of relativistic species (radiation content) and $m_{\rm pl} = 1.36 \times 10^{19}$ GeV is the Planck energy.

Now show that the condition for decoupling, $\Gamma_{\rm X} < H$, can be expressed as

$$\sigma_{\rm X}\,e^{-\mu} < \frac{60.7}{m_{\rm X} m_{\rm pl}}\ {\rm GeV}^{-2}\ . \tag{13.8}$$

Note that this condition involves, apart from μ, the two fundamental masses of the problem: the WIMP mass $m_{\rm X}$ and the Planck mass $m_{\rm pl}$ that enters via Friedmann's equation through the relation $G = 1/m_{\rm pl}^2$ in natural units!

c. Using $m_{\rm X} \sim 100$ GeV, a WIMP mass in the range still allowed by current experiments, one gets $\sigma_{\rm X}\,e^{-\mu} < 4.46 \times 10^{-20}$ GeV^{-2}. What is the maximum cross section allowed (in natural units!) if we require that WIMPS decouple at $\mu = m_{\rm X}/T \simeq 10$?

Express this cross section in cm^2, using the fact that 1 GeV^{-1} corresponds to a scale $\ell_{\rm nat} = \hbar c/(1\ {\rm GeV}) \sim 2 \times 10^{-14}$ cm. Compare the result with a typical weak interaction cross section, the cross section for scattering of MeV neutrino's by electrons:

$$\sigma_{\rm W} \simeq 10^{-42}\ {\rm cm}^2\ . \tag{13.9}$$

14 Hand-in assignment: the Mészáros effect for baryonic fluctuations

Before radiation and baryonic matter (protons, and He nuclei) decouple at a redshift $z_{\rm dec} \sim 3000$, they are strongly coupled through Thomson scattering by the free electrons. This frustrates the growth of baryonic density fluctuations in a uniform radiation background ($\delta\rho_{\rm rad} = \delta\rho_{\rm cdm} = 0$), the *Mészáros effect*, as this assignment demonstrates. The growth of baryonic fluctuations can be described for large wavelengths in terms of $\Delta_{\rm ba} = \delta\rho_{\rm ba}/\rho_{\rm ba}$ as:

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}t^2} + 2\left(\frac{\dot{\tilde{a}}}{\tilde{a}}\right)\frac{\mathrm{d}}{\mathrm{d}t} - 4\pi G\,\rho_{\rm ba}\right]\Delta_{\rm ba}(t) = 0\ . \tag{14.1}$$

Friedmann's equation for a flat universe without a cosmological constant (dark energy) can be written as:

$$\left(\frac{\dot{\tilde{a}}}{\tilde{a}}\right)^2 = \frac{H_0^2}{a_{\rm eq}^3}\left(\tilde{a}^{-4} + \tilde{a}^{-3}\right)\ . \tag{14.2}$$

Since matter dominates over radiation in the current universe the present value of the Hubble parameter (Hubble constant H_0) satisfies

$$H_0^2 \simeq \frac{8\pi G\rho_{\rm mo}}{3}\ . \tag{14.3}$$

The normalized scale factor $\tilde{a} \equiv a/a_{\rm eq}$ used in (14.2) is chosen in such a way that $\rho_{\rm rad} = \rho_{\rm m}$ when $\tilde{a} = 1$. The Omega-parameters $\Omega_{\rm rad}$ and $\Omega_{\rm m}$ for radiation and matter (baryons + CDM) at the current epoch $t = t_0$, together with $\rho_{\rm m} \propto a^{-3}$, $\rho_{\rm rad} \propto a^{-4}$, determine $a_{\rm eq}$:

$$a_{\rm eq} = \frac{\rho_{\rm rad0}}{\rho_{\rm m0}} = \frac{\Omega_{\rm rad}}{\Omega_{\rm m}} \simeq \frac{8.4\times 10^{-5}}{0.28} \simeq 3\times 10^{-4}\ . \tag{14.4}$$

The subscript "0" has been dropped for convenience in the Omega-parameters

a. Show that Friedmann's equation yields the following relation:

$$\dot{\tilde{a}} = \frac{H_0}{a_{\rm eq}^{3/2}} \, \frac{\sqrt{1+\tilde{a}}}{\tilde{a}} \, . \tag{14.5}$$

b. Show by differentiation of (14.2)$\times\tilde{a}^2$ that[9]

$$\ddot{\tilde{a}} = -\frac{H_0^2}{2a_{\rm eq}^3} \left(\frac{2+\tilde{a}}{\tilde{a}^3}\right) . \tag{14.6}$$

c. Show that the variable-change trick employed in assignment 1 can be used again, now for for $\tilde{a} = a(t)/a_{\rm eq}$, so that Eqn. (14.1) becomes:

$$\left[\dot{\tilde{a}}^2 \, \frac{{\rm d}^2}{{\rm d}\tilde{a}^2} + \left(\ddot{\tilde{a}} + 2\, \frac{\dot{\tilde{a}}^2}{\tilde{a}} \right) \frac{\rm d}{{\rm d}\tilde{a}} - 4\pi G \, \rho_{\rm ba} \right] \Delta_{\rm ba} = 0 \, . \tag{14.7}$$

Hint: most of the calculation is exactly the same as in Assignment 1, so you can almost immediately write down the answer!

d. Using $\rho_{\rm ba} = \rho_{\rm ba0}/a^3$ the last term inside the rectangular brackets in (14.7) can be written as

$$4\pi G \, \rho_{\rm ba}(a) = \frac{3H_0^2}{2\, a_{\rm eq}^3} \left(\frac{\Omega_{\rm ba}}{\Omega_{\rm m}}\right) \tilde{a}^{-3} \, , \tag{14.8}$$

with $\Omega_{\rm ba} \simeq 0.04$ the current baryon Omega-parameter. Now show that relation (14.7) can be rewritten (after some algebra) as:

$$\left[\frac{{\rm d}^2}{{\rm d}\tilde{a}^2} + \frac{2+3\tilde{a}}{2\tilde{a}(1+\tilde{a})} \, \frac{\rm d}{{\rm d}\tilde{a}} - \tfrac{3}{2} \left(\frac{\Omega_{\rm ba}}{\Omega_{\rm m}}\right) \frac{1}{\tilde{a}(1+\tilde{a})} \right] \Delta_{\rm ba}(\tilde{a}) = 0 \, . \tag{14.9}$$

[9] This approach yields a much more direct route to the answer than using result **a**!

e. An illustrative (but not necessarily realistic) example considers a universe with no cold matter so that $\Omega_{\rm m} = \Omega_{\rm ba}$. Show that in this case Eqn. (14.9) has the growing solution

$$\Delta_{\rm ba}(t) = \Delta_0 \left(1 + \tfrac{3}{2}\tilde{a}(t)\right) . \tag{14.10}$$

What is the **total** amplification of the initial perturbation over the whole duration of the radiation dominated area, that is from $a = 0$ (Big Bang) to $a = a_{\rm eq}$?
(This is the case treated in Chapter 12.6 of Longair's book, who does not make a proper distinction between the baryonic and total matter densities. Longair also *implicitly* assumes at the end of his calculation that $\rho_{\rm cdm} = 0$ so that $\Omega_{\rm ba}/\Omega_{\rm m} = 1$.)

f. Consider the radiation-dominated phase $a \le a_{\rm eq}$ ($\tilde{a} \le 1$). An analytical solution of Eqn. (14.9) is not possible, but we can try a solution in terms of a power series in $\tilde{a}$, hoping that it converges rapidly for $\tilde{a} < 1$, a method known as *Frobenius' Method*:

$$\Delta_{\rm ba}(\tilde{a}) = \sum_{n=0}^{\infty} c_n\, \tilde{a}^n . \tag{14.11}$$

Substitute this solution into the equation obtained from taking Eqn. (14.9)$\times 2\tilde{a}(\tilde{a} + 1)$. Show that this leads to a requirement for the coefficients c_n and c_{n+1} of the form:

$$c_{n+1} + \frac{2n^2 + n - 3\ (\Omega_{\rm ba}/\Omega_{\rm m})}{2(n+1)^2}\, c_n = 0 . \tag{14.12}$$

Give the first four terms of the solution (i.e. up to $\tilde{a}^3$), assuming as an initial condition that $\Delta_{\rm ba}(\tilde{a} = 0) = \Delta_0$.

g. Now look at the later occurring regime of matter domination, i.e. $\tilde{a} \gg 1$. In that case Eqn. (14.9) can be approximated by:

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}\tilde{a}^2} + \frac{3}{2\tilde{a}}\frac{\mathrm{d}}{\mathrm{d}\tilde{a}} - \frac{3}{2\tilde{a}^2}\right]\Delta_{\rm ba}(\tilde{a}) = 0 . \tag{14.13}$$

Show that the growing solution in this regime satisfies:

$$\Delta_{\rm ba} \propto \tilde{a} \propto a . \tag{14.14}$$

15 Solutions to the assignments in short-hand!

1a. In this case we have $c^2 \mathrm{d}t^2 - a^2(t)\, \mathrm{d}x^2 = 0$, or equivalently:

$$dx = \pm \frac{c\mathrm{d}t}{a(t)} \,. \tag{15.1}$$

Forgetting about the sign (choosing +) we can integrate:

$$\int_0^{x_{\rm source}} \mathrm{d}x = x_{\rm obs} = \int_{t_{\rm em}}^{t_{\rm obs}} \frac{c\mathrm{d}t}{a(t)} \,. \tag{15.2}$$

This is what was asked. The subtle point is that the left-hand side is an integration over x, and the right-hand side is an integration over time, both following the same photon as it travels along the light cone.

1b. Since $x_{\rm obs}$ is the comoving distance between source and observer, the physical distance at the moment of photon reception is

$$D_{\rm obs} \equiv D(t_{\rm obs}) = a(t_{\rm obs}) x_{\rm obs} = a(t_{\rm obs}) \int_{t_{\rm em}}^{t_{\rm obs}} \frac{c\mathrm{d}t}{a(t)} \,. \tag{15.3}$$

1c. The trick is to realize that $x_{\rm obs}$ does not change, so Eqn.(1.5) can be applied to **both** photons:

$$x_{\rm obs} = \int_{t_{\rm em}}^{t_{\rm obs}} \frac{c\;\mathrm{d}t}{a(t)} = \int_{t_{\rm em}+\Delta t_{\rm em}}^{t_{\rm obs}+\Delta t_{\rm obs}} \frac{c\;\mathrm{d}t}{a(t)} \,. \tag{15.4}$$

This is equivalent with

$$\int_{t_{\rm em}+\Delta t_{\rm em}}^{t_{\rm obs}+\Delta t_{\rm obs}} \frac{c\;\mathrm{d}t}{a(t)} - \int_{t_{\rm em}}^{t_{\rm obs}} \frac{c\;\mathrm{d}t}{a(t)} = 0 \,. \tag{15.5}$$

This is only possible if

$$\underbrace{\int_{t_{\rm em}}^{t_{\rm em}+\Delta t_{\rm em}} \frac{c\;\mathrm{d}t}{a(t)}}_{\text{part missing from first integral w.r.t. second}} = \underbrace{\int_{t_{\rm obs}}^{t_{\rm obs}+\Delta t_{\rm obs}} \frac{c\;\mathrm{d}t}{a(t)}}_{\text{part extra in first integral w.r.t. second}} \,. \tag{15.6}$$

1d. If $\Delta t_{\rm em}$ and $\Delta t_{\rm obs}$ are both small, we approximate both integrals in (15.6) by (integrand) $\times$ (time interval) to get:

$$\frac{c\Delta t_{\rm em}}{a(t_{\rm em})} = \frac{c\Delta t_{\rm obs}}{a(t_{\rm obs})} \iff \Delta t_{\rm obs} = \frac{a(t_{\rm obs})}{a(t_{\rm em})}\, \Delta t_{\rm em} = (1+z)\ \Delta t_{\rm em}\ . \tag{15.7}$$

The physical reason that $\Delta t_{\rm obs} > \Delta t_{\rm em}$ is the fact that the universe has expanded a bit in the time between the emission of the first and second photon, so the photon already has to travel further when it starts. To make matters worse, that extra distance also expands (i.e. increases) proportional to $a(t)$ in the time it takes the photon to travel from source to observer.

2a. Due to redshift the wavelength of the photon grows as $\lambda = (1+z)\,\lambda_{\rm em}$, and the photon energy decreases when it is observed with redshift z as

$$\epsilon_{\rm obs} = \frac{hc}{\lambda} = \frac{hc}{(1+z)\,\lambda_{\rm em}} = \frac{\epsilon}{1+z}\ . \tag{15.8}$$

Here $\epsilon = hc/\lambda_{\rm em}$ is the photon energy at the source.

2b. Since the arrival time between two subsequent photons is stretched by a factor $1+z$ according to Assignment 1, the *arrival rate*, the number of photons crossing the sphere per unit time, is reduced by a factor $1+z$ with respect to the number of photons produced per unit time at the source. That means that the rate at which photons cross a sphere of radius D_0 per unit time equals

$$\dot{N}_{\rm obs} = \frac{\dot{N}}{1+z}\ . \tag{15.9}$$

2c. First you must calculate the observed flux:

$$F_{\rm obs} = \frac{\dot{N}_{\rm obs}\epsilon_{\rm obs}}{4\pi D_0^2} = \frac{(\dot{N}/1+z)(\epsilon/1+z)}{4\pi D_0^2} = \frac{\dot{N}\epsilon}{4\pi D_0^2\,(1+z)^2} = \frac{L}{4\pi D_0^2\,(1+z)^2}\ . \tag{15.10}$$

Then using the definition of the luminosity distance one gets:

$$D_L = \sqrt{L/4\pi F_{\rm obs}} = D_0\,(1+z)\ . \tag{15.11}$$

2d. The answer to this question is most easily seen by realizing that the observed flux is reduced by a factor $(1+z)^2$, but that the duration of the burst is increased by a factor $1+z$. Therefore, the fluence is reduced by a factor $1+z$, which is simply the reduction in energy of each individual photon due to redshift. Since the observed fluence (due to the time integration involved) essentially captures all photons that could hit the detector (assuming a 100% quantum efficiency) that energy reduction is the only factor that counts, and must be corrected for to get the energy at the source.

The measured fluence can therefore be used to estimate the energy produced in photons at the source (at given distance D_0) if it emits isotropically:

$$E_{\rm iso} = \underbrace{4\pi D_0^2}_{\text{surface area sphere}} \times \underbrace{\mathcal{F}}_{\text{energy crossing unit area}} \times \underbrace{(1+z)}_{\text{redshift correction}} \; . \tag{15.12}$$

3a-d. To answer these questions you need the following relations;

For $\mu = 0$ ($x^0 = ct$):

$$\frac{\partial \mathcal{L}}{\partial \dot{x}^0} = \dot{x}^0 \;\; , \;\; \frac{\partial \mathcal{L}}{\partial x^0} = -\frac{a}{c}\frac{\mathrm{d}a}{\mathrm{d}t}\left(\frac{\dot{r}^2}{1-kr^2} + r^2\,\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2\right) \; . \tag{15.13}$$

For $\mu = 1$ ($x^1 = r$):

$$\frac{\partial \mathcal{L}}{\partial \dot{x}^1} = \frac{\partial \mathcal{L}}{\partial \dot{r}} = -\frac{a^2\dot{r}}{1-kr^2} \;\; , \;\; \frac{\partial \mathcal{L}}{\partial x^1} = \frac{\partial \mathcal{L}}{\partial r} = -a^2\left(\frac{kr\dot{r}^2}{(1-kr^2)^2} + r\,\dot{\theta}^2 + r\sin^2\theta\,\dot{\phi}^2\right) \; . \tag{15.14}$$

For $\mu = 2$ ($x^2 = \theta$):

$$\frac{\partial \mathcal{L}}{\partial \dot{x}^2} = \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = -a^2r^2\dot{\theta} \;\; , \;\; \frac{\partial \mathcal{L}}{\partial x^2} = \frac{\partial \mathcal{L}}{\partial \theta} = -a^2r^2\;\sin\theta\cos\theta\,\dot{\phi}^2 \; . \tag{15.15}$$

For $\mu = 3$ ($x^3 = \phi$):

$$\frac{\partial \mathcal{L}}{\partial \dot{x}^3} = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = -a^2\,r^2\sin^2\theta\,\dot{\phi} \;\; , \;\; \frac{\partial \mathcal{L}}{\partial x^3} = \frac{\partial \mathcal{L}}{\partial \phi} = 0\ (!) \; . \tag{15.16}$$

The biggest conceptual problem for most of you will be the proper handling of the term

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{\partial\mathcal{L}}{\partial\dot{x}^\mu}\right) \tag{15.17}$$

in the Euler-Lagrange equations. It is a **total** derivative with respect to λ, and differentiates both the explicit dependence on λ (which happens to be absent here since λ never appears explicitly in the above expressions) as well as the implicit dependence on λ since you must assume that $x^\mu = x^\mu(\lambda)$, $\dot{x}^\mu = \dot{x}^\mu(\lambda)$ and since $a = a(t) = a(x^0(\lambda)/c)$. One way to do this is to write all possible contributions to the derivative explicitly, using the chain rule for differentiation:

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} = \frac{\partial}{\partial\lambda} + \sum_{\mu=0}^{3}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\partial}{\partial x^\mu} + \sum_{\mu=0}^{3}\frac{\mathrm{d}\dot{x}^\mu}{\mathrm{d}\lambda}\frac{\partial}{\partial\dot{x}^\mu}\ , \tag{15.18}$$

and note that

$$\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda} = \dot{x}^\mu \quad , \quad \frac{\mathrm{d}\dot{x}^\mu}{\mathrm{d}\lambda} = \frac{\mathrm{d}^2x^\mu}{\mathrm{d}\lambda^2} \equiv \ddot{x}^\mu\ . \tag{15.19}$$

In the problem at hand the $\partial/\partial\lambda$ term (differentiating the explicit dependence) never actually contributes. With recipe (15.18) the term (15.17) can be calculated for $\mu = 0\,,\ 1\,,\ 2\,,\ 3$. I will do the case $\mu = 0$ and $\mu = 1$ explicitly.

For $\mu = 0$ the calculation is rather simple, and does not even need the recipe: we have:

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{\partial\mathcal{L}}{\partial\dot{x}^0}\right) = \frac{\mathrm{d}\dot{x}^0}{\mathrm{d}\lambda} = \frac{\mathrm{d}^2x^0}{\mathrm{d}\lambda^2}\ . \tag{15.20}$$

For $\mu = 1$ one finds:

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{\partial\mathcal{L}}{\partial\dot{r}}\right) = \frac{\mathrm{d}}{\mathrm{d}\lambda}\left(-\frac{a^2\dot{r}}{1-kr^2}\right) = -\frac{a^2}{1-kr^2}\left[\ddot{r} + \left(\frac{2}{ca}\frac{\mathrm{d}a}{\mathrm{d}t}\right)\dot{x}^0\dot{r} + \frac{2kr}{1-kr^2}\dot{r}^2\right] \tag{15.21}$$

Combining this with Eqn. (15.14) yields the result asked for in **3b**. The answer of **3c** is obtained in an analogous fashion.

3e. The equations for $x^\mu(\lambda)$ ($\mu = 0\ ,\ 1\ ,\ 2\ ,\ 3$) are in the required form so that the Christoffel symbols can be read off. The only tricky point is that the cross-product $\dot{x}^\alpha\,\dot{x}^\beta$ with $\alpha \neq \beta$ appears twice: the implied summation gives for these terms, writing out the relevant part of the implied summation for once:

$$\sum_{\alpha\neq\beta}\left(\Gamma^\mu_{\alpha\beta}\,\dot{x}^\alpha\,\dot{x}^\beta + \Gamma^\mu_{\beta\alpha}\,\dot{x}^\alpha\,\dot{x}^\beta\right) = 2\sum_{\alpha\neq\beta}\Gamma^\mu_{\alpha\beta}\,\dot{x}^\alpha\,\dot{x}^\beta\ . \tag{15.22}$$

Here I have used the symmetry $\Gamma^\mu_{\alpha\beta} = \Gamma^\mu_{\beta\alpha}$.

If one takes this onto account one finds, with $c = 1$ for convenience:

$$\begin{aligned}
\Gamma^0_{11} = \frac{a(\mathrm{d}a/\mathrm{d}t)}{1-kr^2} \quad &, \quad \Gamma^1_{11} = \frac{kr}{1-kr^2} \\
\Gamma^2_{00} = a\left(\frac{\mathrm{d}a}{\mathrm{d}t}\right) r^2 \quad &, \quad \Gamma^0_{33} = a\left(\frac{\mathrm{d}a}{\mathrm{d}t}\right) r^2\sin^2\theta \\
\Gamma^1_{01} = \Gamma^1_{10} = \Gamma^2_{02} = \Gamma^2_{20} \quad &= \quad \Gamma^3_{03} = \Gamma^3_{30} = \frac{1}{a}\left(\frac{\mathrm{d}a}{\mathrm{d}t}\right) = H(t) \\
\Gamma^1_{22} = -r\left(1-kr^2\right) \quad &, \quad \Gamma^1_{33} = -r\left(1-kr^2\right)\sin^2\theta \\
\Gamma^2_{12} = \Gamma^2_{21} \quad &= \quad \Gamma^3_{13} = \Gamma^3_{31} = \frac{1}{r} \\
\Gamma^2_{33} = -\sin\theta\cos\theta \quad &, \quad \Gamma^3_{23} = \cot\theta
\end{aligned} \tag{15.23}$$

3f. It is easily checked by direct substitution that observers with fixed comoving spatial coordinates, which have $\dot{r} = \dot{\theta} = \dot{\phi} = 0$, satisfy the geodesic equations. The spatial components the satisfy

$$\frac{\mathrm{d}^2 r}{\mathrm{d}\lambda^2} = \frac{\mathrm{d}^2\theta}{\mathrm{d}\lambda^2} = \frac{\mathrm{d}^2\phi}{\mathrm{d}\lambda^2} = 0\ , \tag{15.24}$$

so that such observers have zero (comoving) velocity and zero acceleration, in other words: they stay at the same position in the RW coordinate grid!

The equation for $x^0 = t$ is obtained in analogous fashion (again with $c = 1$) and reads

$$\frac{\mathrm{d}^2 t}{\mathrm{d}\lambda^2} = 0 \iff t = a' + b'\lambda \ . \tag{15.25}$$

This relation can be inverted to read

$$t = a + b\lambda \ . \tag{15.26}$$

Since a and b (and a' and b') are arbitrary constants, we can choose $a = 0$ and $b = 1$ so that $\lambda = t$.

With that choice, the freely-falling fundamental observers in a Friedmann Cosmology satisfy

$$\frac{\mathrm{d}^2 r}{\mathrm{d}t^2} = \frac{\mathrm{d}^2 \theta}{\mathrm{d}t^2} = \frac{\mathrm{d}^2 \phi}{\mathrm{d}t^2} = 0 \ , \tag{15.27}$$

the same equation that one finds for *stationary observers*, fixed to an inertial coordinate frame $(r\ ,\ \theta\ ,\ \phi)$ in Special Relativity.

4a. Just substituting (15.95) into Friedmann's equation one finds:

$$\left(\frac{1}{a}\,\frac{\mathrm{d}a}{\mathrm{d}t}\right)^2 = \frac{8\pi G\rho_0}{3}\,\left(\frac{a(t)}{a_0}\right)^{-n} . \tag{15.28}$$

After taking the square root of both sides of Friedmann's equation this be rewritten in terms of $x \equiv a(t)/a_0$ as

$$x^{\frac{n}{2}-1}\,\mathrm{d}x = \frac{2}{n}\,d\left(x^{\frac{n}{2}}\right) = \sqrt{\frac{8\pi G\rho_0}{3}}\,\mathrm{d}t . \tag{15.29}$$

Using $a(0) = 0$ the answer is

$$x^{\frac{n}{2}} = \left(\frac{a(t)}{a_0}\right)^{n/2} = \frac{n}{2}\,\sqrt{\frac{8\pi G\rho_0}{3}}\,t . \tag{15.30}$$

Inverting this relation:

$$\frac{a(t)}{a_0} = \left[\frac{n}{2}\,\sqrt{\frac{8\pi G\rho_0}{3}}\right]^{2/n} t^{2/n} \equiv \left(\frac{t}{t_0}\right)^{2/n} . \tag{15.31}$$

This incidentally gives you t_0:

$$t_0 = \frac{2}{n}\,\sqrt{\frac{3}{8\pi G\rho_0}} . \tag{15.32}$$

4b. At $t = t_0$ we have $a/a_0 = 1$ and $H = H_0$, so Friedmann's equation just gives

$$H_0^2 = \frac{8\pi G\rho_0}{3} \iff \rho_0 = \frac{3H_0^2}{8\pi G} \equiv \rho_{c0} . \tag{15.33}$$

At any other time we have from the solution for $a(t)$:

$$H(t) = \frac{2}{nt} = H_0\,\left(\frac{t_0}{t}\right) , \quad \rho(t) = \rho_0\,\left(\frac{a(t)}{a_0}\right)^{-n} = \rho_0\,\left(\frac{t_0}{t}\right)^2 . \tag{15.34}$$

The critical density is

$$\rho_c(t) \equiv \frac{3H^2}{8\pi G} = \frac{3H_0^2}{8\pi G}\left(\frac{t_0}{t}\right)^2 = \rho_0 \left(\frac{t_0}{t}\right)^2 . \tag{15.35}$$

This shows that $\rho_c(t) = \rho(t)$ so the density is the critical density at all times, not just t_0, if it is critical at $t = t_0$.

4c. For $n = 3$, the value appropriate for a cold, flat universe, one has (see 15.34):

$$H_0 = \frac{2}{3t_0} \iff t_0 = \frac{2}{3H_0} . \tag{15.36}$$

Calculating this age is a bit tricky because the Hubble constant is given in 'practical units' for astronomers. Formally it has the dimension of $[\text{time}]^{-1}$, being velocity over distance. One way to do this is by noting that

$$1 \text{ km/s per Mpc} = 10^3 \text{ m/s per } (3.086 \times 10^{22} \text{ m}) = \frac{10^3}{3.086 \times 10^{22}} \text{ s}^{-1} \tag{15.37}$$

This is the same as

$$1 \text{ km/s per Mpc} = \frac{1}{3.086 \times 10^{19} \text{ s}} \simeq \frac{1}{10^{12} \text{ yr}} . \tag{15.38}$$

This then gives the age of a flat, matter-dominated universe with 'our' H_0 as

$$t_0 = \frac{2}{3H_0} = \frac{2}{3 \times 74.3} \times 10^{12} \text{ yr} \simeq 8.97 \times 10^9 \text{ yr} . \tag{15.39}$$

4d. In a *De Sitter Universe* Friedmann's equation is

$$\left(\frac{1}{a}\frac{\mathrm{d}a}{\mathrm{d}t}\right)^2 = \frac{8\pi G \rho_{\mathrm{vac}}}{3} = \frac{\Lambda}{3} . \tag{15.40}$$

Taking a square root of both sides we have

$$\left(\frac{1}{a}\frac{\mathrm{d}a}{\mathrm{d}t}\right) = \pm\sqrt{\frac{\Lambda}{3}} . \tag{15.41}$$

For the expanding solution we need to take the +-sign and an elementary calculation gives

$$a(t) = K \exp\left(\sqrt{\frac{\Lambda}{3}} t \right) . \tag{15.42}$$

The constant K can be found by noting that $a(t_0) \equiv a_0$ so that $K = a_0 \exp\left(-\sqrt{\frac{\Lambda}{3}} t_0\right)$. The answer follows immediately.

4e1. Friedmann's equation with positive curvature is

$$H^2 = \left(\frac{1}{a} \frac{\mathrm{d}a}{\mathrm{d}t} \right)^2 = \frac{8\pi G \rho}{3} - \frac{k}{a^2} . \tag{15.43}$$

If we put $a_0 = 1$ the density law is $\rho = \rho_0 \, a^{-3}$, while Friedmann's equation at $t = t_0$ simplifies to

$$H_0^2 = \frac{8\pi G \rho_0}{3} - k . \tag{15.44}$$

Solving for k, and using the definition of the omega-parameter Ω_0 (the ratio of density to critical density at $t = t_0$) you immediately get:

$$k = \frac{8\pi G \rho_0}{3} - H_0^2 = H_0^2 \left(\frac{8\pi G \rho_0}{3 H_0^2} - 1 \right) = H_0^2 \ (\Omega_0 - 1) \ . \tag{15.45}$$

4e2. By multiplying (15.43) with a^2 and using $\rho = \rho_0 / a^3$:

$$\dot{a}^2 = \frac{8\pi G \rho_0}{3a} - k . \tag{15.46}$$

Using the definition of Ω_0 and the result for k obtained above this can be written as

$$\dot{a}^2 = \frac{H_0^2 \Omega_0}{a} - H_0^2 \ (\Omega_0 - 1) \ . \tag{15.47}$$

Dividing both sides by H_0^2 gives the required answer.

4e3. Taking the square root of both sides of the equation obtained in the previous question one has:

$$\frac{1}{H_0}\,\frac{\mathrm{d}a}{\mathrm{d}t} = \sqrt{\frac{\Omega_0}{a} - (\Omega_0 - 1)}\;. \tag{15.48}$$

I have chosen the positive root since $\dot{a} > 0$ in an expanding universe. This can be written as a 'difference equation':

$$\frac{1}{H_0}\,\frac{\mathrm{d}a}{\sqrt{\dfrac{\Omega_0}{a} - (\Omega_0 - 1)}} = \mathrm{d}t\;. \tag{15.49}$$

Putting integral signs on both sides (an integration over a on the left-hand side, and a trivial time-integration on the right-hand side) one gets the required result:

$$\frac{1}{H_0}\int_0^a \frac{\mathrm{d}a'}{\sqrt{\dfrac{\Omega_0}{a'} - (\Omega_0 - 1)}} = \int_0^t \mathrm{d}t = t\;. \tag{15.50}$$

Note that I have made a careful distinction between integration boundaries and integrands in the scale-factor integral. Longair in his book is very sloppy in this regard!

4e4. The trick is to show that this parametrized solution satisfies Friedmann's equation (4.22). For this you need to use:

$$\mathrm{d}a = \frac{\mathrm{d}a}{\mathrm{d}\vartheta}\,\mathrm{d}\vartheta = \frac{\Omega_0}{2(\Omega_0 - 1)}\;\sin\vartheta\,\mathrm{d}\vartheta \tag{15.51}$$

and

$$\mathrm{d}t = \frac{\mathrm{d}t}{\mathrm{d}\vartheta}\,\mathrm{d}\vartheta = \frac{\Omega_0}{2H_0\,(\Omega_0 - 1)^{3/2}}\;(1 - \cos\vartheta)\;\mathrm{d}\vartheta\;. \tag{15.52}$$

Together these imply

$$\frac{1}{H_0}\,\frac{\mathrm{d}a}{\mathrm{d}t} = \sqrt{\Omega_0 - 1}\,\frac{\sin\vartheta}{1 - \cos\vartheta}\;. \tag{15.53}$$

Substituting this and $a(\vartheta)$ into (4.22) one finds that the following identity should be satisfied:

$$(\Omega_0 - 1)\ \frac{\sin^2\vartheta}{(1-\cos\vartheta)^2} = \frac{2\,(\Omega_0 - 1)}{1-\cos\theta} - (\Omega_0 - 1)\ . \tag{15.54}$$

Canceling the common factor $\Omega_0 - 1$ it is easily seen that this is indeed the case. For instance: an overall multiplication by $(1 - \cos\vartheta)^2$ yields

$$\sin^2\vartheta = 2\,(1-\cos\vartheta) - (1-\cos\vartheta)^2 = 1 - \cos^2\vartheta\ . \tag{15.55}$$

We conclude that the parametrized solution indeed solves Friedmann's equation.

4e5. The scale factor reaches a maximum when $\cos\vartheta = -1$ at $\vartheta = \pi$ and time $t_{1/2}$:

$$a_{\rm max} = \frac{\Omega_0}{\Omega_0 - 1}\quad ,\quad t_{1/2} = \frac{\pi\Omega_0}{2H_0(\Omega_0 - 1)}\ . \tag{15.56}$$

For $\vartheta > \pi$ we have $\sin\vartheta < 0$ and as a consequence $\mathrm{d}a/\mathrm{d}t < 0$: the universe starts to shrink again!

The scale factor becomes zero again at $t_{\rm max} = 2t_{1/2}$ when $\vartheta = 2\pi$ and $\cos\vartheta = 1$. So the maximum age is:

$$t_{\rm max} = \frac{\pi\,\Omega_0}{H_0\,(\Omega_0 - 1)^{3/2}}\ . \tag{15.57}$$

5a. This assignment is essentially a repeated application of the general relation between physical distance D and comoving distance r:

$$D = r\,a(t)\ . \tag{15.58}$$

The difference with the earlier use of this relation is the fact that now the comoving radius r is allowed to change: it is a function of time, i.e. $r = r(t)$. The peculiar speed of the photon is

$$\frac{\mathrm{d}\ell}{\mathrm{d}t} = a(t)\,\frac{\mathrm{d}r}{\mathrm{d}t} = -c\ . \tag{15.59}$$

This is the speed a comoving fiducial observer along the photon path measures as the photon passes him.

The *effective* speed is the time-derivative of the distance:

$$\frac{\mathrm{d}D}{\mathrm{d}t} \equiv \frac{\mathrm{d}}{\mathrm{d}t}\left(r(t)\, a(t)\right) = \frac{\mathrm{d}r}{\mathrm{d}t}\, a(t) + r(t)\, \frac{\mathrm{d}a}{\mathrm{d}t}\ . \tag{15.60}$$

Using (15.59), $r = D/a$, $H(t) = (1/a)(\mathrm{d}a/\mathrm{d}t)$ (Hubble constant) one gets:

$$\frac{\mathrm{d}D}{\mathrm{d}t} = H(t)D(t) - c\ . \tag{15.61}$$

The first term is the effect of the Hubble-expansion of the universe, which tends to increase the distance, and the second term the effect of the propagation of the photon towards the observer that tends to decrease the distance.

5b. The above relation shows that the effective velocity $\mathrm{d}D/\mathrm{d}t$ changes sign at

$$D = c/H \equiv D_{\mathrm{H}}\ . \tag{15.62}$$

For $D > D_{\mathrm{H}}$ the Hubble-expansion wins, and the photon (initially) moves away from the observer. For $D < D_{\mathrm{H}}$ the photon gets closer to the observer. It should be remembered in all this that $H(t)$ changes with time: in most model universes H decreases (and D_{H} increases) as the universe gets older!

5c. Relation (5.2) can be written as

$$a(t)\, \frac{\mathrm{d}r}{\mathrm{d}t} = -c \quad \Longleftrightarrow \quad \mathrm{d}r = -\frac{c\, \mathrm{d}t}{a(t)}\ . \tag{15.63}$$

Solving this by formal integration, taking account of the condition $r = r_{\mathrm{s}}$ at $t = t_{\mathrm{em}}$:

$$\int_{r_{\mathrm{s}}}^{r(t)} \mathrm{d}r = r(t) - r_{\mathrm{s}} = -\int_{t_{\mathrm{em}}}^{t} \frac{c\, \mathrm{d}t'}{a(t')}\ . \tag{15.64}$$

This is essentially the required answer. The physical photon distance at time t then follows as

$$D(t) = r(t)\, a(t) = r_{\mathrm{s}}\, a(t) - a(t) \int_{t_{\mathrm{em}}}^{t} \frac{c\, \mathrm{d}t'}{a(t')}\ . \tag{15.65}$$

5d. We want $r = 0$ at $t = t_0$ and for $t_{\rm em} = 0$. The general solution for $r(t)$ along the photon's orbit then yields:

$$0 = r_{\rm s} - \int_0^{t_0} \frac{c\,{\rm d}t'}{a(t')} \iff r_{\rm s} = \int_0^{t_0} \frac{c\,{\rm d}t'}{a(t')} \equiv r_{\rm hor} \, . \tag{15.66}$$

This is the maximum *comoving* distance a photon can travel during the entire age of the universe.

5e/f. Here we specialize in a universe that expands as $a(t) = (t/t_0)^{\alpha}$ with $\alpha < 1$, and has $a(t_0) = 1$. The following integral is useful:

$$I(t_{\rm em} \mid t_0) \equiv \int_{t_{\rm em}}^{t_0} \frac{c\,{\rm d}t'}{a(t')} = \int_{t_{\rm em}}^{t_0} c\,{\rm d}t' \left(\frac{t'}{t_0}\right)^{-\alpha} = \frac{ct_0}{1-\alpha}\left[1 - \left(\frac{t_{\rm em}}{t_0}\right)^{1-\alpha}\right] . \tag{15.67}$$

If we put $t_{\rm em} = 0$ we get the horizon distance. Since $a(t_0) = 1$ we have:

$$r_{\rm hor} = D_{\rm hor} = I(0 \mid t_0) = \frac{ct_0}{1-\alpha} \, . \tag{15.68}$$

This is the maximum distance over which photons can reach us, given the finite age t_0 of the universe.

The Hubble parameter for an expansion law $a(t) = (t/t_0)^{\alpha}$ is (see assignment 1a, with $\alpha = 2/n$)

$$H = \left(\frac{1}{a}\,\frac{{\rm d}a}{{\rm d}t}\right) = \frac{\alpha}{t} \, . \tag{15.69}$$

Therefore $H_0 = \alpha/t_0$ and $t_0 = \alpha/H_0$ so that

$$D_{\rm hor} = \frac{c}{H_0}\,\frac{\alpha}{\alpha - 1} \, . \tag{15.70}$$

For photons produced at an arbitrary time $t_{\rm em} > 0$ and received at time t_0 we have from answer c:

$$r_{\rm s} = D_{\rm s}(t_0) = I(t_{\rm em} \mid t_0) = \frac{ct_0}{1-\alpha}\left[1 - \left(\frac{t_{\rm em}}{t_0}\right)^{1-\alpha}\right] . \tag{15.71}$$

This is essentially the equation for the *past light cone* for this universe, but with the physical distance evaluated at t_0. As we will see later, one usually calculates the physical distance at the moment of emission $t_{\rm em}$, which is a factor $a(t_{\rm em})/a(t_0) = a(t_{\rm em}) = (t_{\rm em}/t_0)^{\alpha}$ smaller!

5g. An Inflationary Universe (De Sitter Universe) has a constant Hubble parameter: $H = \sqrt{\Lambda/3}$. Λ is the cosmological constant. If you now look at the the equation derived in **2a** for $\mathrm{d}D/\mathrm{d}t$ it follows that in this case the physical distance between observer and photon satisfies

$$\frac{\mathrm{d}D}{\mathrm{d}t} = H_\Lambda D - c = H_\Lambda \left(D - \frac{c}{H_\Lambda} \right) . \tag{15.72}$$

This shows that, *at any time*, photons produced at $D > c/H_\Lambda$, effectively move away from the observer. Since H_Λ is a constant, once beyond this distance they have $\mathrm{d}D/\mathrm{d}t > 0$, so they will *never* reach the observer at $r = D = 0$. Therefore, in such a universe c/H_Λ acts as the effective horizon distance in the following sense: if a galaxy *at the time of emission* resides at a physical distance $D(t_{\rm em}) > c/H_\Lambda$ the photons from that Galaxy will *never* reach us.

6a. In this case one can write $\mathrm{d}t = \dot{t}\,\mathrm{d}\tau$, $\mathrm{d}r = \dot{r}\,\mathrm{d}\tau$ so that

$$\mathrm{d}\tau^2 = \dot{t}^2\,\mathrm{d}\tau^2 - a^2(t)\,\frac{\dot{r}^2\,\mathrm{d}\tau^2}{1 - kr^2} = \mathrm{d}\tau^2 \left(\dot{t}^2 - a^2(t)\,\frac{\dot{r}^2}{1 - kr^2} \right) . \tag{15.73}$$

Canceling the common factor $\mathrm{d}\tau^2$ one gets the answer for question **6a.**

6b. Result **6a** can be written as

$$\frac{\dot{r}^2}{1 - kr^2} = \frac{\dot{t}^2 - 1}{a^2(t)} . \tag{15.74}$$

This can be used to get rid of the factor $\dot{r}^2/(1 - kr^2)$ in the second term on the left-hand side of equation (6.6) for $t(\tau)$, leading to the result.

6c. The quickest way to get a result is to multiply (6.7) by $a^2(t)\,\dot{t}$,

$$a^2(t)\,\dot{t}\,\ddot{t} + a\,\dot{t}\,\frac{\mathrm{d}a}{\mathrm{d}t}\left(\dot{t}^2 - 1 \right) = 0 . \tag{15.75}$$

Here I use the notation

$$\ddot{t} = \frac{\mathrm{d}\dot{t}}{\mathrm{d}\tau} = \frac{\mathrm{d}^2 t}{\mathrm{d}\tau^2} . \tag{15.76}$$

Next you should realize that

$$\dot{t}\,\frac{\mathrm{d}a}{\mathrm{d}t} = \frac{\mathrm{d}t}{\mathrm{d}\tau}\,\frac{\mathrm{d}a}{\mathrm{d}t} = \frac{\mathrm{d}a}{\mathrm{d}\tau}\ \ ,\ \ \dot{t}\,\ddot{t} = \tfrac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\dot{t}^2\right)\ . \tag{15.77}$$

This means that Eqn. (15.75) is equivalent with

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left[a^2\,\left(\dot{t}^2 - 1\right)\right] = 0\ . \tag{15.78}$$

The solution is

$$a^2\,\left(\dot{t}^2 - 1\right) = \text{constant} \equiv Q^2\ . \tag{15.79}$$

Solving for $\dot{t}$:

$$a^2\dot{t}^2 = Q^2 + a^2 \quad\Longleftrightarrow\quad \dot{t} = \frac{\sqrt{Q^2 + a^2}}{a}\ . \tag{15.80}$$

6d. From (15.74) we have, substituting for $\dot{t}$ from **6c**:

$$\frac{\dot{r}}{\sqrt{1 - kr^2}} = \frac{\sqrt{\dot{t}^2 - 1}}{a} = \frac{\sqrt{Q^2/a^2}}{a} = \frac{Q}{a^2}\ . \tag{15.81}$$

6e. Here we must use that

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{\mathrm{d}r}{\mathrm{d}\tau}\,\frac{\mathrm{d}\tau}{\mathrm{d}t} = \frac{\mathrm{d}r}{\mathrm{d}\tau}\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^{-1} = \frac{\dot{r}}{\dot{t}}\ . \tag{15.82}$$

Therefore definition (6.11) can be rewritten as

$$v = \frac{a\,(\dot{r}/\dot{t})}{\sqrt{1 - kr^2}} = \frac{Q}{\sqrt{Q^2 + a^2}}\ , \tag{15.83}$$

where I have used the results for $\dot{r}$ and $\dot{t}$.

6f. The Lorentz factor (for $c = 1$) associated with velocity v is

$$\gamma = \frac{1}{\sqrt{1 - v^2}} = \frac{1}{\sqrt{1 - \dfrac{Q^2}{Q^2 + a^2}}} = \frac{\sqrt{Q^2 + a^2}}{a} \; . \tag{15.84}$$

Therefore a fiducial observer associates with a particle of mass m a momentum

$$p = \gamma m v = m \, \frac{Q}{a} \; . \tag{15.85}$$

Since Q is a constant, this implies that the momentum decays as $p \propto a^{-1}(t)$, the same as the redshift law for massless photons!

7a. In a static universe we have $H = 0$, $\mathrm{d}^2/\mathrm{d}a^2 = 0$ so Friedmann's equation and its modified form with $a = 1$ reduce to

$$\frac{8\pi G \rho_0}{3} + \frac{\Lambda}{3} - k = 0 \;\; , \;\; -\frac{4\pi G \rho_0}{3} + \frac{\Lambda}{3} = 0 \; . \tag{15.86}$$

The second equation gives you $\rho_0 = \Lambda / 4\pi G$. Substituting this into the first equation yields $k = 2\Lambda/3 + \Lambda/3 = \Lambda$.

7b. The density scales as

$$\rho(a) = \frac{\rho_0}{a^3} = \frac{\rho_0}{(1 + \delta a)^3} \simeq \rho_0 \, (1 - 3 \, \delta a) \; . \tag{15.87}$$

The last equality follows from the general relation

$$(1 + x)^\alpha \simeq 1 + \alpha x + O(x^2) \;\; \text{for } |x| \ll 1 \; . \tag{15.88}$$

7c. Using the result from **7b** together with $\mathrm{d}^2a/\mathrm{d}t^2 = \mathrm{d}^2\,\delta a/\mathrm{d}t^2$ one has:

$$\begin{aligned}
\frac{\mathrm{d}^2\,\delta a}{\mathrm{d}t^2} &= -\frac{4\pi G\rho_0}{3}\,(1-3\,\delta a)\,(1+\delta a) + \frac{\Lambda}{3}\;(1+\delta a) \\
&= -\frac{4\pi G\rho_0}{3}\;(1-2\,\delta a) + \frac{\Lambda}{3}\;(1+\delta a) + O(\delta a^2) \\
&= -\frac{4\pi G\rho_0}{3} + \frac{\Lambda}{3} + \left(\frac{8\pi G\rho_0}{3} + \frac{\Lambda}{3}\right)\delta a + O(\delta a^2) \\
&= \Lambda\,\delta a + O(\delta a^2)
\end{aligned} \tag{15.89}$$

In the last line I have used $\rho_0 = \Lambda/4\pi G$, the density for a static Einstein universe. That gets rid of the term not involving δa (why should that happen to begin with?) and simplifies the term linear in δa.

7d. The general solution of Eqn. (7.7/15.89) is

$$\delta a(t) = A\exp(\chi t) + B\exp(-\chi t) \;\text{ with }\; \chi \equiv \sqrt{\Lambda}\;. \tag{15.90}$$

This implies

$$\delta a_0 = A\exp(\chi t_0) + B\exp(-\chi t_0) \tag{15.91}$$

while the condition $\mathrm{d}\delta a/\mathrm{d}t = 0$ at $t = t_0$ implies

$$\chi\,[\,A\exp(\chi t_0) - B\exp(-\chi t_0)\,] = 0 \quad\Longleftrightarrow\quad B = A\exp(2\chi t_0)\;. \tag{15.92}$$

Using this into the relation for δa_0 yields $A = \frac{1}{2}\delta a_0\exp(-\chi t_0)$ and it is easily seen that the full solution reads:

$$\delta a(t) = \frac{\delta a_0}{2}\;[\,\exp\left(\chi(t-t_0)\right) + \exp\left(-\chi(t-t_0)\right)\,] = \delta a_0\;\cosh\left(\chi(t-t_0)\right)\;. \tag{15.93}$$

This result shows that $|\delta a|$ always grows: if $\delta a_0 > 0$ the perturbed universe starts to expand, and keeps expanding! Ultimately, you will enter into the regime where $\rho = \rho_0/a^3 \ll \Lambda/4\pi G$.

The cosmological constant will then dominate so the universe will ultimately behave as a De-Sitter universe with

$$a(t) \sim \exp\left(\sqrt{\Lambda/3}t\right) \quad \text{for } t \gg t_0\ ,\ \chi^{-1}\ . \tag{15.94}$$

This is one case where a universe with positive k keeps expanding, unlike what happens to positive k solutions without a cosmological constant that *always* re-collapse.

When $\delta a_0 < 0$ the universe will start to collapse, ultimately shrinking to $a = 0$ ("Big Crunch").

8a. The trick is to rewrite Friedmann's equation. The important steps are:

$$\rho(a) = \rho_{\mathrm{v}}\, \mathcal{R}^{-3} = \frac{\Lambda}{8\pi G}\mathcal{R}^{-3}\ \ ,\ \ \frac{\Lambda}{3} = H_\Lambda^2 \tag{15.95}$$

and

$$H^2 = \left(\frac{1}{a}\frac{\mathrm{d}a}{\mathrm{d}t}\right)^2 = \left(\frac{1}{\mathcal{R}}\frac{\mathrm{d}\mathcal{R}}{\mathrm{d}t}\right)^2 = H_\Lambda^2\left(\frac{1}{\mathcal{R}}\frac{\mathrm{d}\mathcal{R}}{\mathrm{d}\tau}\right)^2\ . \tag{15.96}$$

Substituting this into the original Friedmann equation yields:

$$H_\Lambda^2\left(\frac{1}{\mathcal{R}}\frac{\mathrm{d}\mathcal{R}}{\mathrm{d}\tau}\right)^2 = \frac{\Lambda}{3}\left(\mathcal{R}^{-3} + 1\right) = H_\Lambda^2\left(\mathcal{R}^{-3} + 1\right)\ . \tag{15.97}$$

which is (apart from an overall factor H_Λ^2) the required equation.

8b. Using $z = \mathcal{R}^{3/2}$ (or its inverse: $\mathcal{R} = z^{2/3}$) yields

$$\frac{1}{\mathcal{R}}\frac{\mathrm{d}\mathcal{R}}{\mathrm{d}\tau} = \frac{2}{3}\left(\frac{1}{z}\frac{\mathrm{d}z}{\mathrm{d}\tau}\right)\ \ ,\ \ \mathcal{R}^{-3} + 1 = z^{-2} + 1 = \frac{1+z^2}{z^2}\ , \tag{15.98}$$

and Friedmann's equation for the Friedmann-Lemaitre universe becomes

$$\left(\frac{2}{3z}\frac{\mathrm{d}z}{\mathrm{d}\tau}\right)^2 = \frac{1+z^2}{z^2}\ . \tag{15.99}$$

Cancelling a common factor $1/z^2$, taking the square root of both sides of the equation **choosing the positive root for an expanding universe:**

$$\frac{\mathrm{d}z}{\mathrm{d}\tau} = +\tfrac{3}{2}\ \sqrt{1+z^2}\ . \tag{15.100}$$

8c. You can do this by direct substitution, but the mathematically elegant way is to reduce the equation to quadrature: from $\mathrm{d}z = (\mathrm{d}z/\mathrm{d}\tau)\,\mathrm{d}\tau$ one gets:

$$\mathrm{d}z = \tfrac{3}{2}\ \sqrt{1+z^2}\,\mathrm{d}\tau \quad \Longleftrightarrow \quad \frac{\mathrm{d}z}{\sqrt{1+z^2}} = \tfrac{3}{2}\,\mathrm{d}\tau\ . \tag{15.101}$$

Putting integral signs on both sides of the equation, and using the standard integral given in the **Hint** one gets:

$$\mathrm{arcsinh}(z) = \tfrac{3}{2}\,\tau + \mathrm{constant}\ . \tag{15.102}$$

The constant follows from the consideration that $\mathcal{R} = 0$ (and so $z = 0$) at $t = 0$ ($\tau = 0$). Since $\mathrm{arcsinh}(0) = 0$ this implies that the constant vanishes. Writing the solution as

$$z(\tau) = \mathcal{R}^{3/2}(\tau) = \left(\frac{a(t)}{a_\Lambda}\right)^{3/2} = \sinh(\tfrac{3}{2}\,\tau) = \sinh(\tfrac{3}{2}\,H_\Lambda t)\ , \tag{15.103}$$

we can immediately see that

$$a(t) = a_\Lambda\,\mathcal{R} = a_\Lambda\ \sinh^{2/3}(\tfrac{3}{2}\,H_\Lambda t)\ . \tag{15.104}$$

The rest of the relations are most easily obtained by first calculating the Hubble parameter:

$$\begin{aligned} H &= H_\Lambda\left(\frac{1}{\mathcal{R}}\frac{\mathrm{d}\mathcal{R}}{\mathrm{d}\tau}\right) = H_\Lambda\left(\frac{1}{\sinh^{2/3}\,(\frac{3}{2}\,\tau)}\right)\left(\frac{2}{3}\ \sinh^{-1/3}(\tfrac{3}{2}\,\tau)\times\frac{3}{2}\ \cosh(\tfrac{3}{2}\,\tau)\right) \\ &= H_\Lambda\ \coth(\tfrac{3}{2}\,\tau) = H_\Lambda\ \coth(\tfrac{3}{2}\,H_\Lambda t)\ . \end{aligned} \tag{15.105}$$

Then Ω_Λ follows as:

$$\Omega_\Lambda = \frac{\Lambda}{3H^2} = \left(\frac{H_\Lambda}{H}\right)^2 = \frac{1}{\coth^2(\frac{3}{2}\, H_\Lambda t)} = \tanh^2(\tfrac{3}{2}\, H_\Lambda t) \ , \tag{15.106}$$

and Ω_m can be calculated from the flatness condition $\Omega_\Lambda + \Omega_m = 1$.

At **early times** (i.e. $\tau \ll 1$) we can use

$$\sinh(3\tau/2) \simeq \tanh(3\tau/2) = \frac{1}{\cosh(3\tau/2)} \simeq \frac{3\tau}{2} \ \ , \ \ \cosh(3\tau/2) \simeq 1. \tag{15.107}$$

In that limit universe expands as

$$a(\tau) \propto \tau^{2/3} \propto t^{2/3} \ \ , \ \ H(\tau) \propto \frac{1}{\tau} \propto \frac{1}{t} \ , \tag{15.108}$$

which is the Friedmann solution for a flat, matter-dominated universe.

In the **late-time limit** $\tau \gg 1$ one has

$$\sinh(3\tau/2) \simeq \cosh(3\tau/2) \approx \tfrac{1}{2}\, e^{3\tau/2} = \tfrac{1}{2}\, e^{3H_\Lambda t/2} \ \ , \ \ \cosh(3\tau/2) \simeq \tanh(3\tau/2) \simeq 1 \ . \tag{15.109}$$

Then

$$a(t) \propto e^{H_\Lambda t} \ \ , \ \ H(t) \simeq H_\Lambda \ . \tag{15.110}$$

This corresponds to a pure De-Sitter universe.

8d. Matter density/vacuum density equality is achieved for $a = a_\Lambda$, ($\mathcal{R} = 1$), leading to the condition

$$\sinh(\tfrac{3}{2}\, H_\Lambda t_{eq}) = 1 \ . \tag{15.111}$$

Defining for convenience

$$x = \tfrac{3}{2}\, H_\Lambda t_{eq} \ , \tag{15.112}$$

the condition yields:

$$\frac{e^x - e^{-x}}{2} = 1 \; . \tag{15.113}$$

This can be rewritten after multiplication by $2e^x$ as

$$(e^x)^2 - 2\, e^x - 1 = 0 \; , \tag{15.114}$$

which has the positive root:

$$e^x = \sqrt{2} + 1 \; . \tag{15.115}$$

This implies

$$x = \tfrac{3}{2}\, H_\Lambda t_{\rm eq} = \ln\left(\sqrt{2} + 1\right) \iff t_{\rm eq} = \frac{2}{3H_\Lambda}\, \ln\left(\sqrt{2} + 1\right) = \frac{0.58}{H_\Lambda} \; . \tag{15.116}$$

8e. The current age of the universe, t_0 corresponds to the point in time where $\Omega_\Lambda(t_0) = \tanh^2(3H_\Lambda t_0/2) = \Omega_{\Lambda 0} = 0.73$ and $H(t) = H_\Lambda \coth(3H_\Lambda t_0/2) = H_0 \simeq 70$ km/s per Mpc. Since $\tanh = 1/\coth$ these two relations together immediately yield:

$$H_0 = H_\Lambda / \sqrt{\Omega_{\Lambda 0}} \; . \tag{15.117}$$

I will use this later. To get t_0 we have to solve the relation

$$\tanh\,(3H_\Lambda t_0/2) = \sqrt{\Omega_{\Lambda 0}} \; , \tag{15.118}$$

or equivalently

$$\frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{e^{2x} - 1}{e^{2x} + 1} = \sqrt{\Omega_{\Lambda 0}} \tag{15.119}$$

with $x = 3H_\Lambda t_0/2$. The solution is:

$$e^{2x} = (e^x)^2 = \frac{1 + \sqrt{\Omega_{\Lambda 0}}}{1 - \sqrt{\Omega_{\Lambda 0}}} \iff e^x = \sqrt{\frac{1 + \sqrt{\Omega_{\Lambda 0}}}{1 - \sqrt{\Omega_{\Lambda 0}}}} \; . \tag{15.120}$$

Taking the logarithm of the second relation yields

$$x = 3H_\Lambda t_0/2 = \ln\left[\sqrt{\frac{1+\sqrt{\Omega_{\Lambda 0}}}{1-\sqrt{\Omega_{\Lambda 0}}}}\right] \iff t_0 = \frac{2}{3H_\Lambda}\ln\left[\sqrt{\frac{1+\sqrt{\Omega_{\Lambda 0}}}{1-\sqrt{\Omega_{\Lambda 0}}}}\right]$$
$$= \frac{2}{3H_0\sqrt{\Omega_{\Lambda_0}}}\ln\left[\sqrt{\frac{1+\sqrt{\Omega_{\Lambda 0}}}{1-\sqrt{\Omega_{\Lambda 0}}}}\right]. \tag{15.121}$$

Using (go back to one of the earlier assignments to see how!)

$$H_0 = 70 \text{ km/s per Mpc} = \left(1.4\times 10^{10}\ \text{yr}\right)^{-1} \tag{15.122}$$

you get with $\sqrt{\Omega_{\Lambda_0}} = \sqrt{0.73} \simeq 0.85$:

$$t_0 \simeq 0.964\ H_0^{-1} = 1.35\times 10^{10}\ \text{yr}\ . \tag{15.123}$$

9.1. With the scaling $a(t_0) = 1$, the known density laws for (cold) matter and radiation and the redshift definition $1+z = a^{-1}$ you have:

$$\rho_\text{m} = \rho_\text{m0}\, a^{-3} = \rho_\text{m0}\,(1+z)^3 \ \ ,\ \ \rho_\text{r} = \rho_\text{r0}\, a^{-4} = \rho_\text{r0}\,(1+z)^4\ . \tag{15.124}$$

9.2. Just use the definition for the Hubble parameter and $a(t) = 1/(1+z)$:

$$H \equiv \left(\frac{1}{a}\,\frac{\text{d}a}{\text{d}t}\right) = (1+z)\,\frac{\text{d}(1+z)^{-1}}{\text{d}t} = -\frac{1}{1+z}\,\frac{\text{d}z}{\text{d}t}\ . \tag{15.125}$$

Here t stands for the emission time. If it increases (photon is emitted later) it arrives with a smaller redshift; If it is emitted earlier (smaller t) the redshift is larger. This implies $\text{d}z/\text{d}t < 0$.

The second part, deriving the Friedmann equation in terms of z and $\text{d}z/\text{d}t$, uses the following trick: for matter the term on the right-hand side of the Friedman equation is

$$\frac{8\pi G\rho_\text{m}}{3} = \frac{8\pi G\rho_\text{m0}\,(1+z)^3}{3} = H_0^2\,\frac{8\pi G\rho_\text{m0}}{3H_0^2}\,(1+z)^3 = H_0^2\,\Omega_\text{m0}\,(1+z)^3\ . \tag{15.126}$$

The radiation term can be treated similarly, and for the cosmological constant term you write $\Lambda/3 = H_0^2(\Lambda/3H_0^2) = H_0^2\,\Omega_{\Lambda 0}$. The curvature term requires some thought. Friedmann's equation at $t = t_0$ can be written as:

$$H_0^2 = H_0^2\;(\Omega_{\rm m0} + \Omega_{\rm r0} + \Omega_{\Lambda 0}) - k \tag{15.127}$$

since $a(t_0) = 1$ by definition. Solving for k we get:

$$k = H_0^2\,(\Omega_{\rm m0} + \Omega_{\rm r0} + \Omega_{\Lambda 0} - 1) \equiv H_0^2\,\Omega_{k0}\;. \tag{15.128}$$

Friedmann's equation,

$$H^2 = \frac{8\pi G}{3}\;(\rho_{\rm m} + \rho_{\rm r}) + \frac{\Lambda}{3} - \frac{k}{a^2}\;, \tag{15.129}$$

can now be rewritten as:

$$\left(\frac{1}{1+z}\,\frac{{\rm d}z}{{\rm d}t}\right)^2 = H_0^2\,\left[\,\Omega_{\rm m0}(1+z)^3 + \Omega_{\rm r0}\,(1+z)^4 + \Omega_\Lambda - \Omega_{k0}\,(1+z)^2\,\right] \tag{15.130}$$

Multiplying both sides with $(1+z)^2$ gives the relation asked for in this part of the assignment.

9.3. This is essentially solved by working out the powers of $1+z$ for $\Omega_{\rm r0} = 0$. In particular:

$$\begin{aligned}
&\Omega_{\rm m0}(1+z)^3 + \Omega_\Lambda - \Omega_{k0}\,(1+z)^2 = \\
&\Omega_{\rm m0}\,\left(1+3z+3z^2+z^3\right) + \Omega_{\Lambda 0} - (\Omega_{\rm m0} + \Omega_{\Lambda 0} - 1)\,\left(1+2z+z^2\right) = \\
&\Omega_{\rm m0}\,\left(z+2z^2+z^3\right) - \Omega_{\Lambda 0}\,(2z+z^2) + (1+z)^2 = \\
&(\Omega_{\rm m0}z + 1)\,(1+z)^2 - \Omega_{\Lambda 0}z\,(2+z)\;.
\end{aligned} \tag{15.131}$$

In the last step I have used $z + 2z^2 + z^3 = z\,(1+z)^2$.

9.4. To convert integrals over cosmic time t into integrals over z you use:

$$\mathrm{d}z = \left(\frac{\mathrm{d}z}{\mathrm{d}t}\right)\mathrm{d}t \iff \mathrm{d}t = \frac{\mathrm{d}t}{\dot{z}} \text{ with } \dot{z} \equiv \frac{\mathrm{d}z}{\mathrm{d}t}\,. \tag{15.132}$$

The form of the Friedmann equation derived in 3.3 gives us $\dot{z}$ (note the sign!):

$$\dot{z} = -H_0(1+z)\,\sqrt{\Omega_{\mathrm{m}0}(1+z)^3 + \Omega_{\mathrm{r}0}\,(1+z)^4 + \Omega_\Lambda - \Omega_{k0}\,(1+z)^2} \tag{15.133}$$

so that

$$\mathrm{d}t = -\frac{1}{H_0}\,\frac{\mathrm{d}z}{(1+z)\,\sqrt{\Omega_{\mathrm{m}0}(1+z)^3 + \Omega_{\mathrm{r}0}\,(1+z)^4 + \Omega_\Lambda - \Omega_{k0}\,(1+z)^2}}\,. \tag{15.134}$$

The perhaps unexpected minus sign will sort itself out when we use this in actual integrals, and is a simple consequence of $\dot{z} < 0$ if t is the time of emission (rather than reception) as is the case here. This relation will be used again and again in what follows.

We can immediately apply this to the source distance D_0. What you have to realize is that a photon emitted by the source at time t has an observed redshift z_o at t_0, while a (hypothetical) photon emitted at t_0 and observed at t_0 has no redshift at all: $z = 0$. This means in general that

$$\int_t^{t_0} \mathrm{d}t \cdots = \int_{z_\mathrm{o}}^{0} \frac{\mathrm{d}z}{\dot{z}} \cdots = -\int_0^{z_\mathrm{o}} \frac{\mathrm{d}z}{\dot{z}} \cdots\,, \tag{15.135}$$

where reversing the order of integration in the last equality introduces an extra minus sign that actually cancels pesky minus sign in $\dot{z}$ in Eqn. (15.133) (check this out!). For the distance integral that gives us D_0 this procedure, together with $a(t) = (1+z)^{-1}$ and (15.134) immediately yields:

$$\begin{aligned} D_0 &= \int_t^{t_0} \frac{c\,\mathrm{d}t}{a(t)} = c\int_0^{z_\mathrm{o}} \frac{\mathrm{d}z}{\dot{z}}\,(1+z) \\ &= \frac{c}{H_0}\int_0^{z_\mathrm{o}} \frac{\mathrm{d}z}{\sqrt{\Omega_{\mathrm{m}0}(1+z)^3 + \Omega_{\mathrm{r}0}\,(1+z)^4 + \Omega_\Lambda - \Omega_{k0}\,(1+z)^2}}\,. \end{aligned} \tag{15.136}$$

For $z_o \ll 1$ one can use

$$\Omega_{m0}(1+z)^3 + \Omega_{r0}\,(1+z)^4 + \Omega_\Lambda - \Omega_{k0}\,(1+z)^2 \simeq 1 + \tilde{\Omega}\, z \tag{15.137}$$

with

$$\tilde{\Omega} \equiv 2 + \Omega_{m0} + 2\Omega_{r0} - 2\Omega_{\Lambda 0} \tag{15.138}$$

This means that the integrand in (15.136) can be approximated by

$$\left(1 + \tilde{\Omega}\, z\right)^{-1/2} \approx 1 - \tfrac{1}{2}\tilde{\Omega}\, z + O(z^2)\;, \tag{15.139}$$

and that the distance integral for small z_0 is

$$D_0 \approx \frac{c}{H_0} \int_0^{z_o} \mathrm{d}z\, \left(1 - \tfrac{1}{2}\tilde{\Omega}\, z\right) = \frac{c z_o}{H_0} + O(z_o^2)\;. \tag{15.140}$$

9.5. One can calculate photon travel times by employing the almost trivial relation

$$\Delta t = t_0 - t = \int_t^{t_0} \mathrm{d}t\;, \tag{15.141}$$

and replacing the time integral by an integral over redshift using (15.134) and (15.135). This approach immediately gives the answer.

9.6. Photons that are produced at the Big Bang, when $t = 0$ and (formally) $a = 0$, arrive with a redshift $z_o = \infty$. This sets the upper limit in the integral for the distance to the source, so the horizon distance is defined by

$$D_{hor} \equiv \frac{c}{H_0} \int_0^{\infty} \frac{\mathrm{d}z}{\sqrt{\Omega_{m0}(1+z)^3 + \Omega_{r0}\,(1+z)^4 + \Omega_\Lambda - \Omega_{k0}\,(1+z)^2}}\;. \tag{15.142}$$

Note that this integral always converges! At different observation times one gets a different answer since the Omega parameters change when one chooses t_0 differently.

9.7. Equation (9.14) applied to the two limiting cases $\Omega_{\mathrm{m}0} \equiv \Omega_0 = 1$, $\Omega_{\Lambda 0} \equiv \Omega_\Lambda = 0$ and $\Omega_{\mathrm{m}0} \equiv \Omega_0 = 0$, $\Omega_{\Lambda 0} \equiv \Omega_\Lambda = 1$ yield the integrals

$$D_0 = \begin{cases} \dfrac{c}{H_0} \displaystyle\int_0^{z_\mathrm{o}} \dfrac{\mathrm{d}z}{(1+z)^{3/2}} = \dfrac{2c}{H_0}\left(1 - \dfrac{1}{\sqrt{1+z_\mathrm{o}}}\right) & \text{case 1: } (\Omega_0\,,\ \Omega_\Lambda) = (1\,,\ 0) \\ \dfrac{c}{H_0} \displaystyle\int_0^{z_\mathrm{o}} \mathrm{d}z = \dfrac{cz_\mathrm{o}}{H_0} & \text{case 2: } (\Omega_0\,,\ \Omega_\Lambda) = (0\,,\ 1) \end{cases} . \tag{15.143}$$

In **case 1** the horizon distance follows from putting $z_\mathrm{o} = \infty$ and one finds:

$$D_\mathrm{hor} = \frac{2c}{H_0} . \tag{15.144}$$

In **case 2** this procedure formally yields $D_\mathrm{hor} = \infty$. This may seem strange, but is put into perspective by looking at what this distance is *at the moment of emission t*. In a flat universe we have for $a(t_0) = 1$ that $D(t) = D_0/(1+z_\mathrm{o})$, so the distance to the source when the photon was emitted is

$$D(t) = \frac{D_0}{1+z_\mathrm{o}} = \frac{c}{H_0\,(1+z_\mathrm{o})} \int_0^{z_0} \frac{\mathrm{d}z}{\sqrt{\Omega_0\,(1+z)^3 + \Omega_\Lambda}} . \tag{15.145}$$

In case 2, where $H_0 = H_\Lambda = \sqrt{\Lambda/3}$ (from Friedmann!), this gives:

$$D = \frac{cz_\mathrm{o}}{H_0\,(1+z_\mathrm{o})} = \frac{cz_\mathrm{o}}{H_\Lambda\,(1+z_\mathrm{o})} \tag{15.146}$$

In the limit $z_\mathrm{o} \to \infty$ this yields a *finite* answer:

$$D(z_\mathrm{o} \to \infty) = \frac{c}{H_\Lambda} . \tag{15.147}$$

As we have seen in one of the earlier assignments this is exactly the distance that separates photons that effectively move towards the observer and ultimately reach him (for $D < c/H_\Lambda$) and those that will never reach the observer (for $D > c/H_\Lambda$) in a De-Sitter universe. Those photons starting exactly at $D = c/H_\Lambda$ take an infinite amount of time to reach the observer.

9.8 The aim is to evaluate the integral

$$t_0 = \frac{1}{H_0} \int_0^{\infty} \frac{dz}{(1+z)\ \sqrt{\Omega_0\,(1+z)^3 + \Omega_\Lambda}} \tag{15.148}$$

that gives us the age of the universe, given the (in principle observable) current values of the two Omega parameters $\Omega_0 \equiv \Omega_{m0}$ and $\Omega_\Lambda \equiv \Omega_{\Lambda 0}$ (Longairs notation!) and H_0. The suggested substitution is aimed at simplifying the square root[10]:

$$1 + z = \left(\frac{\Omega_\Lambda}{\Omega_0}\right)^{1/3} \tan^{2/3}\theta \iff \sqrt{\Omega_0\,(1+z)^3 + \Omega_\Lambda} = \frac{\sqrt{\Omega_\Lambda}}{\cos\theta}\ . \tag{15.149}$$

We also have

$$dz = \frac{2}{3}\left(\frac{\Omega_\Lambda}{\Omega_0}\right)^{1/3} \frac{d\theta}{\cos^2\theta\ \tan^{1/3}\theta}\ . \tag{15.150}$$

The integral in (15.148) then becomes

$$t_0 = \frac{2}{3\sqrt{\Omega_\Lambda} H_0} \int_{\theta_0}^{\pi/2} \frac{d\theta}{\sin\theta}\ . \tag{15.151}$$

Here the angle θ_0 corresponds to $z = 0$, and follows from:

$$\left(\frac{\Omega_\Lambda}{\Omega_0}\right)^{1/3} \tan^{2/3}\theta_0 = 1 \iff \tan\theta_0 = \sqrt{\frac{\Omega_0}{\Omega_\Lambda}} \simeq 0.608 \ \text{(from observed values)}\ . \tag{15.152}$$

At $\theta = \pi/2$ $\tan\theta$ goes to infinity so this corresponds to $z = \infty$. Using the primitive for the integral over θ given in the Assignment one finds:

$$t_0 = \frac{2}{3\sqrt{\Omega_\Lambda} H_0} \ln\left(\frac{1+\cos\theta_0}{\sin\theta_0}\right)\ . \tag{15.153}$$

[10]This is a standard procedure for integrals such as this.

What remains is expressing this result in terms of the Omega parameters using the definition of θ_0. To that end you can use the relations that follow straightforwardly from $\tan = \sin / \cos$ and $\sin^2 + \cos^2 = 1$. In symbolic notation:

$$\sin = \frac{\tan}{\sqrt{1 + \tan^2}} \quad , \quad \cos = \frac{1}{\sqrt{1 + \tan^2}} \ . \tag{15.154}$$

We then get

$$\frac{1 + \cos}{\sin} = \frac{\sqrt{1 + \tan^2} + 1}{\tan} = \left(\frac{\sqrt{1 + \tan^2} + 1}{\sqrt{1 + \tan^2} - 1} \right)^{1/2} \ . \tag{15.155}$$

In the last equality I have used $\tan^2 = (\sqrt{1 + \tan^2} + 1)(\sqrt{1 + \tan^2} - 1)$ to get a more symmetric-looking equation. Putting in $\tan\theta_0 = \sqrt{\Omega_0/\Omega_\Lambda}$ and using $\Omega_0 + \Omega_\Lambda = 1$ then yields after some straightforward algebra:

$$\frac{1 + \cos\theta_0}{\sin\theta_0} = \left(\frac{1 + \sqrt{\Omega_\Lambda}}{1 - \sqrt{\Omega_\Lambda}} \right)^{1/2} \ . \tag{15.156}$$

The expression for t_0 that follows from substituting this into (15.153) agrees with our earlier result (8.13) for the age of a Friedmann-Lemaitre universe:

$$t_0 = \frac{2}{3H_0 \sqrt{\Omega_{\Lambda_0}}} \ln \left[\left(\frac{1 + \sqrt{\Omega_{\Lambda 0}}}{1 - \sqrt{\Omega_{\Lambda 0}}} \right)^{1/2} \right] \ . \tag{15.157}$$

10a. By drawing a picture, it is easily seen that photons that hit the observer from the two ends of the object should leave with the correct direction so that they ultimately both hit the observer. That direction is set by the distance *at photon emission*, see the figure below. That the source moves further away from the observer as the photons propagate is immaterial.

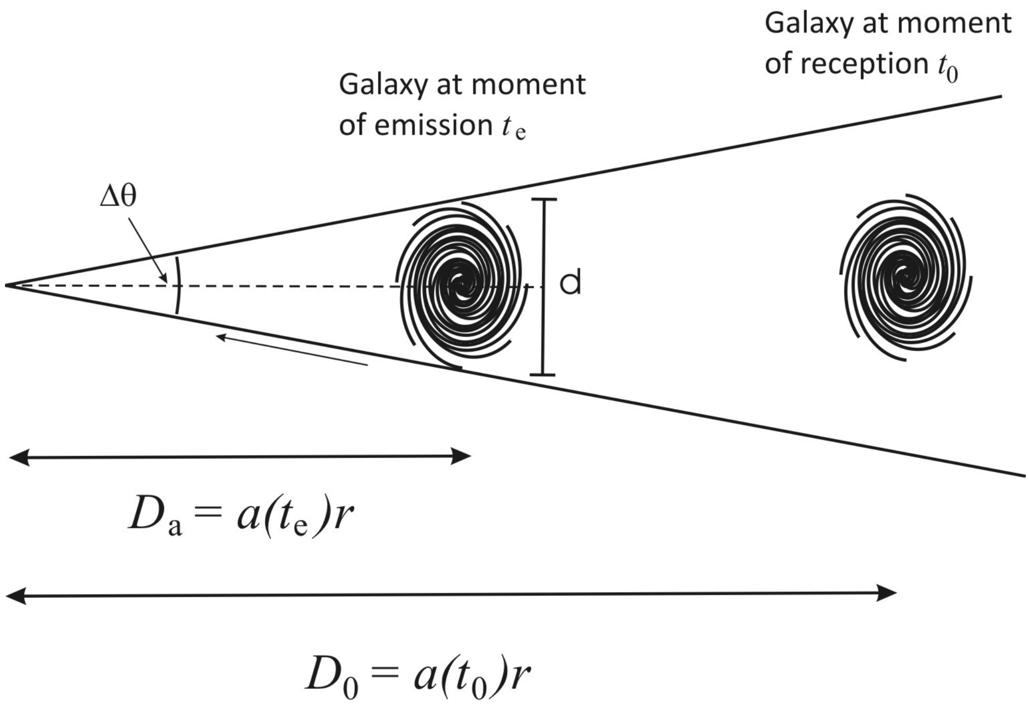

Figure 1: *The situation that applies to the determination of the angular distance D_a. Two light rays from either end of the object come in with an angle $\frac{1}{2}\Delta\theta$ with respect to the normal to the object. The figure is for a flat universe where distances scale as $D(t) = r\, a(t)$, with r the fixed comoving coordinate. However, the result derived in this assignment is generally true.*

Elementary trigonometry tells you that

$$\tan\left(\tfrac{1}{2}\Delta\theta\right) = \frac{d}{2D_a} \quad\Longleftrightarrow\quad \Delta\theta = 2\ \arctan\left(\frac{d}{2D_a}\right) = 2\ \arctan\left(\frac{d}{2a(t_e)\,r}\right) \tag{15.158}$$

10b. For small angles we have $\tan(\frac{1}{2}\Delta\theta) \simeq \frac{1}{2}\Delta\theta$ and we find:

$$D_a = a(t_{em})\,r = \left(\frac{a(t_e)}{a(t_0)}\right) a(t_0)r = \frac{D_0}{1+z} \simeq \frac{d}{\Delta\theta}\ . \tag{15.159}$$

10c. One simply uses relation (15.160) for D_0 and $D_a = D_0/(1 + z_0)$:

$$D_a = \frac{c}{H_0\,(1 + z_0)} \int_0^{z_0} \frac{dz}{\sqrt{\Omega_{m0}(1 + z)^3 + \Omega_{r0}\,(1 + z)^4 + \Omega_\Lambda - \Omega_{k0}\,(1 + z)^2}} \; . \tag{15.160}$$

10d. The only differences with earlier calculations of this type are

1. The integration boundaries that now run from redshift z to $z = \infty$ (moment of the Big Bang);
2. The fact that the (physical) distance is evaluated not at $t = t_0$ (current epoch) when $a(t_0) = 1$ by definition, but at the time corresponding to redshift z. The scale factor at that time is $a(t) = 1/(1 + z)$, which accounts for the extra factor $1/(1 + z)$ in the final answer (10.6).

10e. It is easiest if one first calculates D_0, the source distance at photon reception:

$$D_0 = \frac{c}{H_0} \int_0^{z_0} \frac{dz}{(1 + z)^{3/2}} = \frac{2c}{H_0} \left(1 - \frac{1}{\sqrt{1 + z_0}}\right) . \tag{15.161}$$

The angular distance then follows immediately:

$$D_a = \frac{D_0}{1 + z_0} = \frac{2c}{H_0\,(1 + z_0)^{3/2}} \left(\sqrt{1 + z_0} - 1\right) . \tag{15.162}$$

The angular size (in the small angle approximation) then is

$$\Delta\theta = \frac{d}{D_a} = \frac{H_0 d}{2c} \frac{(1 + z_0)^{3/2}}{\sqrt{1 + z_0} - 1} \; . \tag{15.163}$$

For small z_0 this scales as

$$\Delta\theta \simeq H_0 d/c z_0 \propto z_0^{-1} \; . \tag{15.164}$$

Since $D_0 \simeq c z_0/H_0$ for $z_0 \ll 1$ we have the intuitively obvious case where the angular size declines as 1/distance. For large z_0 on the other hand we have

$$\Delta\theta \propto z_0 \tag{15.165}$$

and the angular size increases in a counter-intuitive manner with increasing redshift.

This automatically implies that $\Delta\theta$ has a minimum. The redshift for which that that minimum angular diameter occurs follows from

$$\frac{\mathrm{d}}{\mathrm{d}z_0}\left(\frac{(1+z_0)^{3/2}}{\sqrt{1+z_0}-1}\right) = \frac{1+z_0-\frac{3}{2}\,\sqrt{1+z_0}}{\left(\sqrt{1+z_0}-1\right)^2} = 0\ . \tag{15.166}$$

That minimum occurs at a redshift $z_0 = z_*$ such that

$$\sqrt{1+z_*} = \tfrac{3}{2} \quad\Longleftrightarrow\quad z_* = \tfrac{5}{4}\ . \tag{15.167}$$

The corresponding minimum angular size, angular distance and current source distance are:

$$\Delta\theta_{\rm min} = \frac{27}{8}\,\frac{H_0 d}{c}\quad,\quad D_{\rm a} = \frac{8}{27}\,\frac{c}{H_0}\quad,\quad D_0 = (1+z_*)\,D_{\rm a} = \frac{2}{3}\,\frac{c}{H_0}\ . \tag{15.168}$$

The (somewhat far-fetched) intuitive explanation for the increasing angular size at high redshift for a universe such as this is *gravitational lensing* by the gravitational attraction of the matter in the universe as a whole. All models with $\Omega_{\rm m0} \neq 0$ produce such an increase, see the discussion and figures in Section 7.4.4. of Longairs book. The larger $\Omega_{\rm m0}$, the stronger the lensing and the smaller the redshift where the angular diameter has a minimum.

10f. In a radiation-dominated flat universe with $z > z_{\rm CMWB} \simeq 1100$ we can approximate result (10.6) for the horizon at redshift $z_{\rm CMWB}$ by

$$\begin{aligned} D_{\rm hor}(z_{\rm CMWB}) &= \frac{c}{H_0(1+z_{\rm CMWB})}\int_{z_{\rm CMWB}}^{\infty}\frac{\mathrm{d}z'}{\sqrt{\Omega_{\rm r0}\,(1+z')^4}} \\ &= \frac{c}{H_0\,\sqrt{\Omega_{\rm r0}}\,(1+z_{\rm CMWB})}\int_{z_{\rm CMWB}}^{\infty}\frac{\mathrm{d}z'}{(1+z')^2} = \frac{c}{H_0\,\sqrt{\Omega_{\rm r0}}}\,\frac{1}{(1+z_{\rm CMWB})^2}\ . \end{aligned} \tag{15.169}$$

If we put in numbers, specifically $\Omega_{\rm r0} \simeq 10^{-4}$ and $z_{\rm CMWB} \simeq 1100$, one finds $D_{\rm hor}(z_{\rm CMWB}) \simeq 10^{-4}\,(c/H_0)$.

10g. With the distance to the surface of last scattering being $D_{\rm LS} \simeq 3(c/H_0)$, and the corresponding angular distance being $D_{\rm LS}/(1 + z_{\rm CMWB})$, the angular size of the then-horizon at the moment of CMWB creation is

$$\Delta\theta_{\rm CMWB} \simeq (1 + z_{\rm CMWB})\, \frac{D_{\rm hor}(z_{\rm CMWB})}{3\,(c/H_0)} = \frac{1}{3\,\sqrt{\Omega_{\rm r0}}\,\,(1 + z_{\rm CMWB})} \simeq 3 \times 10^{-2}\ \text{radian}\ . \qquad (15.170)$$

Since one radian corresponds to $\sim$ 57.3 degrees, the angular size of the horizon is about 1.8 degrees on the sky. This means that regions separated by more than this angular distance were outside each other's horizon when the CMWB was created, and could not communicate in any way about the precise conditions, such as density or radiation temperature. There is therefore no obvious reason why these regions should look very similar (as observed, with deviations smaller than one part in 10^5 in radiation temperature, for instance). This is an example of the *horizon problem* in standard Friedmann cosmology.

10h/i. This is just a straightforward application of the recipe (10.1) with $t_1 = t_{\rm e}$ and $t_2 = t_0$ (current epoch) as well as

$$z(t_{\rm e}) = z_0\ \ ,\ \ z(t_0) = 0\ \ ,\ \ a(t_{\rm e}) = \frac{1}{1 + z_0}\ \ ,\ \ a(t) = \frac{1}{1 + z}\ . \qquad (15.171)$$

That immediately leads to result (10.17), and a comparison with expression (10.4) for the angular distance $D_{\rm a}$ show that they are indeed identical.

In **e** we proved that $\Delta\theta = d/D_{\rm a}$ has a minimum at $z = z_*$ for a flat, matter-dominated universe. That means that $D_{\rm a} = D_{\rm PLC}$ must have a *maximum* at that redshift, which equals (see above)

$$D_{\rm PLC}(z = z_* = \tfrac{5}{4}) = \frac{8}{27}\,\frac{c}{H_0}\ . \qquad (15.172)$$

11a. Substituting the trial solution into Eqn.(11.4) yields the following relation, assuming H =constant:

$$\left(-\omega^2 - 3i\,H\omega + m^2\right)\varphi_0 e^{-i\omega t} = 0\ . \tag{15.173}$$

This relation can only be satisfied for all t if the term inside the brackets vanishes:

$$\omega^2 + 3i\,H\omega - m^2 = 0\ . \tag{15.174}$$

Solving for ω:

$$\omega = \omega_\pm \equiv -\tfrac{3}{2}\,iH \pm \sqrt{m^2 - \tfrac{9}{4}\,H^2} \tag{15.175}$$

11b. All solutions have $\lambda \equiv -\mathrm{Im}(\omega) > 0$ (Note the minus sign!). Defining $\mathrm{Re}(\omega) \equiv \omega_{\mathrm{r}}$ and writing

$$\omega = \omega_{\mathrm{r}} - i\lambda\ , \tag{15.176}$$

the exponential factor becomes

$$e^{-i\omega t} = e^{-\lambda t}\, e^{-i\omega_{\mathrm{r}} t}\ . \tag{15.177}$$

This is a combination of sinusoidal behavior in time, and exponential decay in time. As a result, the field amplitude decays as $\sqrt{\varphi\varphi^*} \propto e^{-\lambda t}$. The following possibilities present themselves for λ:

$$\begin{aligned} \lambda_+ = \lambda_- &= \tfrac{3}{2}H \qquad &&\text{if } m > \tfrac{3}{2}H; \\ \lambda_\pm &= \tfrac{3}{2}H \mp \sqrt{\tfrac{9}{4}\,H^2 - m^2} \qquad &&\text{if } m < \tfrac{3}{2}H; \end{aligned} \tag{15.178}$$

In the limit $m \ll 3H/2$ the slowest decaying mode (longest surviving solution) occurs for λ_+:

$$
\begin{aligned}
\lambda_+ &= \tfrac{3}{2}H - \sqrt{\tfrac{9}{4}\,H^2 - m^2} \\
&= \tfrac{3}{2}H\left(1 - \sqrt{1 - \frac{4m^2}{9H^2}}\right) \\
&\simeq \tfrac{3}{2}H\left(1 - \left[\;1 - \frac{2m^2}{9H^2}\;\right]\right) = \frac{m^2}{3H}\;.
\end{aligned}
\tag{15.179}
$$

For comparison purposes: in the same limit the other solution has $\lambda_- \simeq 3H$ so the ratio of the two decay rates is $\lambda_+/\lambda_- = m^2/9H^2 \ll 1$.

11c. In the slow-roll approximation the field dynamics can be approximated by:

$$
3H\,\dot{\varphi} + m^2\varphi = 0\;. \tag{15.180}
$$

Using the scale factor $a(t)$ rather than time to track the expansion of the universe, in effect making the variable change $\varphi(t) \;\rightarrow \varphi(a(t))$, one can use:

$$
\dot{\varphi} = \frac{\partial\varphi}{\partial t} = \frac{\mathrm{d}a}{\mathrm{d}t}\,\frac{\partial\varphi}{\partial a}\;. \tag{15.181}
$$

The definition of the Hubble parameter yields

$$
\frac{\mathrm{d}a}{\mathrm{d}t} = H(t)\,a(t) \tag{15.182}
$$

so that

$$
\dot{\varphi} = Ha\,\frac{\partial\varphi}{\partial a} = H\,\frac{\mathrm{d}\varphi}{\mathrm{d}\ln a}\;. \tag{15.183}
$$

In the last step I have replaced the partial derivative by an ordinary derivative since φ is assumed to be uniform: it only depends on time or (in this formulation) on $a(t)$.

Substituting this into (15.180) we get:

$$3H^2 \, \frac{d\varphi}{d\ln a} + m^2 \, \varphi = \varphi \left(3H^2 \, \frac{d\ln\varphi}{d\ln a} + m^2 \right) = 0 \, . \tag{15.184}$$

The required relation follows immediately since $\varphi \neq 0$.

The formal solution follows from rewriting the equation for φ as

$$d\ln\varphi = -\frac{m^2}{3H^2(a)} \, \frac{da}{a} \, , \tag{15.185}$$

using $d\ln a = da/a$. Note that I consider the Hubble parameter now as a function of a rather than time, the same procedure as used for φ. Formally integrating,

$$\int_{\varphi_0}^{\varphi} d\ln\varphi' = -\int_{a_0}^{a} \frac{m^2}{3H^2(a')} \, \frac{da'}{a'} \tag{15.186}$$

with $\varphi_0 \equiv \varphi(a_0)$, yields the required solution (11.9) as the integral on the left-hand side gives $\ln\varphi - \ln\varphi_0 = \ln(\varphi/\varphi_0)$. It then becomes a simple matter of exponentiation to get the answer.

11d. The field will decay slowly if the factor in the exponent changes only slightly when expands by a factor ~ 2, that is when $\Delta a \sim a$. This condition gives

$$\frac{m^2}{3H^2} \ll 1 \quad \Longleftrightarrow m \ll \sqrt{3} \, H \simeq 1.7 \, H \, . \tag{15.187}$$

This is very close to the condition $m \ll \frac{3}{2} \, H$, which has a more physical interpretation in view of what we found in **b**.

Note also that if we write

$$da' = \frac{da'}{dt'} \, dt' = H(a')a' \, dt' \tag{15.188}$$

the quasi-exponential solution (11.9) for φ involves the exponent

$$\frac{m^2}{3} \int_{a_0}^{a} \frac{da'}{a' \, H^2(a')} = \frac{m^2}{3} \int_{t_0}^{t} \frac{dt'}{H(a')} \equiv \int_{t_0}^{t} dt' \, \lambda(t') \tag{15.189}$$

with

$$\lambda(t) \equiv \frac{m^2}{3H(t)} \ , \tag{15.190}$$

formally equal to the decay rate λ_+ that we found as the slowest decaying mode in the case of constant H. The only difference is that now λ depends on time, so we must use an integral: $\lambda_+ t \longrightarrow \int \lambda(t)\, \mathrm{d}t$.

This incidentally shows that the slowest decaying solution is the one that in fact satisfies the slow-roll approximation, while the other solution does not.

11e. If we look at the scale factor a as a function of φ rather than the other way around, the inverse relation, we can use

$$\frac{\mathrm{d}a}{\mathrm{d}\varphi}\,\frac{\mathrm{d}\varphi}{\mathrm{d}a} = 1 \quad \Longleftrightarrow \quad \frac{\mathrm{d}a}{\mathrm{d}\varphi} = \left(\frac{\mathrm{d}\varphi}{\mathrm{d}a}\right)^{-1} \ . \tag{15.191}$$

Relation (11.8) then gives:

$$\frac{\mathrm{d}\ln a}{\mathrm{d}\varphi} = \frac{1}{a}\,\frac{\mathrm{d}a}{\mathrm{d}\varphi} = \frac{1}{a}\left(\frac{\mathrm{d}\varphi}{\mathrm{d}a}\right)^{-1} = \frac{1}{\varphi}\,\left(\frac{\mathrm{d}\ln\varphi}{\mathrm{d}\ln a}\right)^{-1} = -\frac{3H^2 a}{m^2\varphi} \ . \tag{15.192}$$

If the scalar field dominates over the other two terms on the right-hand side of Friedmann's equation (11.5) we have (using $t_{\mathrm{pl}} = 1/m_{\mathrm{pl}}$ in natural units)

$$H^2 \simeq \frac{4\pi}{3}\,\frac{m^2\varphi^2}{m_{\mathrm{pl}}^2} \ . \tag{15.193}$$

Substituting this into (15.192) we get the required result:

$$\frac{\mathrm{d}\ln a}{\mathrm{d}\varphi} = -\frac{4\pi\,\varphi}{m_{\mathrm{pl}}^2} \ . \tag{15.194}$$

The slow-roll condition (11.10) with (15.193) for the Hubble parameter gives

$$m^2 \ll \frac{9}{4}H^2 = 3\pi\,\frac{m^2\varphi^2}{m_{\mathrm{pl}}^2} \quad \Longleftrightarrow \quad \frac{\varphi}{m_{\mathrm{pl}}} \gg \frac{1}{\sqrt{3\pi}} \ . \tag{15.195}$$

11f. The solution follows from writing

$$\mathrm{d}\ln a = -\frac{4\pi\varphi}{m_{\mathrm{pl}}^2}\,\mathrm{d}\varphi = -\frac{2\pi}{m_{\mathrm{pl}}^2}\,\mathrm{d}\left(\varphi^2\right) \tag{15.196}$$

and formally integrating:

$$\ln a - \ln a_0 = \ln\left(\frac{a}{a_0}\right) = -\frac{2\pi}{m_{\mathrm{pl}}^2}\,\left(\varphi^2 - \varphi_0^2\right)\ . \tag{15.197}$$

Here $a = a(\varphi)$ and $a_0 \equiv a(\varphi_0)$. Equivalently:

$$a(\varphi) = a_0 \exp\left[\,\frac{2\pi}{m_{\mathrm{pl}}^2}\,\left(\varphi_0^2 - \varphi^2\right)\right]\ . \tag{15.198}$$

If φ decays, $a(\varphi)$ grows quasi-exponentially. This is in fact equivalent to what we found in **e**: if a grows φ decays quasi-exponentially. Both ways of looking at this problem has it's uses.

11g. If we use the maximum value allowed for φ_0 on energetic grounds at the Planck time, that is $\varphi_0 = \sqrt{2}\,(m_{\mathrm{pl}}^2/m)$, and let the field decay to $\varphi \ll \varphi_0$ (i.e. $\varphi \simeq 0$) we get:

$$\frac{a(\varphi \simeq 0)}{a(\varphi_0)} \simeq \exp\left(\frac{2\pi\,\varphi_0^2}{m_{\mathrm{pl}}^2}\right) = \exp\left(4\pi\,\frac{m_{\mathrm{pl}}^2}{m^2}\right) \equiv \mathcal{Z}_{\mathrm{infl}}\ . \tag{15.199}$$

If we assume $m = 0.01\,m_{\mathrm{pl}}$, the inflation factor equals:

$$\mathcal{Z}_{\mathrm{infl}} = e^{1.26\times10^5} = 10^{5.5\times10^4}\ . \tag{15.200}$$

Such an enormous expansion factor would solve both the flatness problem and the horizon problem of standard cosmology.

12a. The required answer follows directly from:

$$\begin{aligned} \frac{\mathrm{d}^2}{\mathrm{d}t^2} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\dot{a}\,\frac{\mathrm{d}}{\mathrm{d}a}\right) = \frac{\mathrm{d}\dot{a}}{\mathrm{d}t}\,\frac{\mathrm{d}}{\mathrm{d}a} + \dot{a}\,\frac{\mathrm{d}^2}{\mathrm{d}t\,\mathrm{d}a} \\ &= \ddot{a}\,\frac{\mathrm{d}}{\mathrm{d}a} + \dot{a}^2\,\frac{\mathrm{d}^2}{\mathrm{d}a^2}\,. \end{aligned} \tag{15.201}$$

12b. The first step is use the results from **a** to rewrite the terms involving time-derivatives, specifically:

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} + 2\left(\frac{\dot{a}}{a}\right)\frac{\mathrm{d}}{\mathrm{d}t} = \dot{a}^2\,\frac{\mathrm{d}^2}{\mathrm{d}a^2} + \left(\ddot{a} + 2\,\frac{\dot{a}^2}{a}\right)\frac{\mathrm{d}}{\mathrm{d}a}\,. \tag{15.202}$$

Then you must use the two versions of Friedmann's equation to eliminate $\dot{a}$ and $\ddot{a}$. In particular, the term in front of $\mathrm{d}/\mathrm{d}a$ is

$$\ddot{a} + 2\,\frac{\dot{a}^2}{a} = -\frac{4\pi G\rho}{3}\,a + \frac{16\pi G\rho}{3}\,a = 4\pi G\rho\,a\,. \tag{15.203}$$

This yields:

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} + 2\left(\frac{\dot{a}}{a}\right)\frac{\mathrm{d}}{\mathrm{d}t} = 4\pi G\rho\left(\tfrac{2}{3}\,a^2\,\frac{\mathrm{d}^2}{\mathrm{d}a^2} + a\,\frac{\mathrm{d}}{\mathrm{d}a}\right)\,. \tag{15.204}$$

Substituting this into (12.2) and dividing the resulting expression by $4\pi G\rho$ gives relation (12.7). **Remember:** generally $\rho_{\mathrm{cdm}} < \rho$ as there is also radiation and baryons in the universe that contribute to the effective mass density.

12c. In the matter-dominate phase we can put $\rho \simeq \rho_{\rm cdm}$. This was the case up to the point where the universe reaches 75% of its current age, and the cosmological constant became important. In that case relation (12.7) becomes:

$$\left(\tfrac{2}{3}\, a^2 \,\frac{\mathrm{d}^2}{\mathrm{d}a^2} + a\, \frac{\mathrm{d}}{\mathrm{d}a} - 1\right) \Delta_{\rm cdm}(a) = 0 \,. \tag{15.205}$$

A solution $\Delta_{\rm cdm}(a) \propto a^s$ solves this differential equation provided s satisfies the *indicative equation*:

$$\tfrac{2}{3}\, s(s-1) + s - 1 = (\tfrac{2}{3}\, s + 1)(s-1) = 0 \quad , \quad s = 1 \text{ or } s = -3/2 \,. \tag{15.206}$$

This is a growing solution (positive $s = 1$, as a increases with time) and a decaying solution (negative $s = -3/2$). The growing solution, needed in **d**, is

$$\Delta_{\rm cdm}(a) = \Delta_0 \left(\frac{a}{a_0}\right) \,. \tag{15.207}$$

12d. If one substitutes the growing solution for $\Delta_{\rm cdm}(a)$ into (12.9) one gets:

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}t^2} + 2\left(\frac{\dot a}{a}\right) \frac{\mathrm{d}}{\mathrm{d}t}\right] \Delta_{\rm ba}(t) = 4\pi G \rho_{\rm cdm}\, \Delta_0 \left(\frac{a}{a_0}\right) \,. \tag{15.208}$$

Converting from time-derivatives to a-derivatives with $\rho \simeq \rho_{\rm cdm}$ yields, analogously to what we found in **b**:

$$\left(\tfrac{2}{3}\, a^2 \,\frac{\mathrm{d}^2}{\mathrm{d}a^2} + a\, \frac{\mathrm{d}}{\mathrm{d}a}\right) \Delta_{\rm ba}(a) = \Delta_0 \left(\frac{a}{a_0}\right) \,. \tag{15.209}$$

This equation has a particular ('driven') *linear*[11] solution with $\Delta_{\rm ba}(a_0) = 0$:

$$\Delta_{\rm ba}(a) = \Delta_0 \left(\frac{a}{a_0} - 1\right) \,, \tag{15.210}$$

as direct substitution proves.

[11] So that $\mathrm{d}^2\Delta_{\rm ba}/\mathrm{d}a^2 = 0$.

12e. For $a \gg a_0$ we have:

$$\Delta_{\rm ba}(a) \simeq \Delta_{\rm cdm}(a) = \Delta_0 \left(\frac{a}{a_0}\right) . \tag{15.211}$$

Since $\Delta_{\rm ba} = \delta\rho_{\rm ba}/\rho_{\rm ba}$ and $\Delta_{\rm cdm} = \delta\rho_{\rm cdm}/\rho_{\rm cdm}$ this (approximate) equality implies:

$$\frac{\delta\rho_{\rm ba}}{\delta\rho_{\rm cdm}} = \frac{\rho_{\rm ba}}{\rho_{\rm cdm}} \frac{\Delta_{\rm ba}}{\Delta_{\rm cdm}} \simeq \frac{\rho_{\rm ba}}{\rho_{\rm cdm}} = \frac{\Omega_{\rm ba0}}{\Omega_{\rm cdm0}} = \frac{0.04}{0.20} = 0.2 . \tag{15.212}$$

We can use the current values of the Omega-parameters for baryons and CDM since both densities scale as $\rho \propto a^{-3}$, so the density ratio remains constant as the universe expands.

The *exact* solution is also easily obtained from result **c/d**:

$$\frac{\delta\rho_{\rm ba}}{\delta\rho_{\rm cdm}} = \frac{\Omega_{\rm ba0}}{\Omega_{\rm cdm0}} \left(1 - \frac{a_0}{a}\right) . \tag{15.213}$$

13a. This is a simple matter of substitution of the set of relations in natural units into (13.1), using

$$n_{\rm X} = 0.04\, \mu^{3/2}\, e^{-\mu} \times \left(3.6\, T^3\right) \simeq 0.144\, \mu^{3/2}\, e^{-\mu}\, T^3 . \tag{15.214}$$

The numerical constant in front has been rounded to 0.15. In that case we have:

$$\Gamma_{\rm X} = n_{\rm X}\sigma_{\rm X}v = \left(0.144\, \mu^{3/2}\, e^{-\mu}\, T^3\right) \times \sigma_{\rm X} \times \mu^{-1/2} = 0.15\, T^3\, \mu e^{-\mu}\, \sigma_{\rm X}\ {\rm GeV} . \tag{15.215}$$

13b. From taking the square root on both sides of Friedmann's equation we get the Hubble parameter in natural units:

$$H \simeq 9.1\, \frac{T^2}{m_{\rm pl}}\ {\rm GeV} . \tag{15.216}$$

The decoupling condition becomes:

$$\Gamma_X = 0.15\, T^3\, \mu e^{-\mu}\, \sigma_X < H = 9.1\, \frac{T^2}{m_{pl}} \,. \tag{15.217}$$

Since $\mu = m_X/T$ we can write on the left-hand side $T^3\mu = T^2\, m_X$. Then canceling a common factor T^2 we then get after some re-arrangement:

$$\sigma_X\, e^{-\mu} < \frac{60.7}{m_X m_{pl}}\ \mathrm{GeV}^{-2} \,, \tag{15.218}$$

where $60.7 = 9.1/0.15$.

13c. Putting in $m_X \simeq 100$ GeV and $m_{pl} = 1.36 \times 10^{19}$ GeV we find for $\mu = 10$:

$$4.54 \times 10^{-5}\, \sigma_X < 4.46 \times 10^{-20} \quad \Longleftrightarrow \quad \sigma_X < 9.8 \times 10^{-16}\ \mathrm{GeV}^{-2} \,. \tag{15.219}$$

We now have to convert this result back to more conventional units, using $\hbar c/(1\ \mathrm{GeV}) \equiv \ell_{nat} = 2 \times 10^{-14}$ cm. Since the cross section is an area with dimension [length]^2 this proceeds as follows:

$$\sigma_X(\text{in cm}^2) = \ell_{nat}^2\, \sigma_X(\text{in GeV}^{-2}) \,. \tag{15.220}$$

This yields:

$$\sigma_X < 3.9 \times 10^{-43}\ \mathrm{cm}^2 \,, \tag{15.221}$$

a cross section somewhat smaller than the typical neutrino scattering cross section that one measures for MeV neutrino's in the laboratory. This shows that WIMPs indeed interact very weakly with matter if they are to be a viable candidate for Cold Dark Matter.

14a. Just rewrite (14.2) after multiplication by a^2 as

$$\dot{a}^2 = \left(\frac{\dot{\tilde{a}}}{\tilde{a}}\right)^2 = \frac{H_0^2}{a_{\rm eq}^3}\left(\tilde{a}^{-2} + \tilde{a}^{-1}\right) = \frac{H_0^2}{a_{\rm eq}^3 \tilde{a}^2}\,(1+\tilde{a})\ , \tag{15.222}$$

and take the square root of both sides:

$$\dot{\tilde{a}} = \frac{H_0}{a_{\rm eq}^{3/2}}\,\frac{\sqrt{1+\tilde{a}}}{\tilde{a}}\ . \tag{15.223}$$

14b. Differentiating both sides of the first equality in (15.222) with respect to time yields:

$$2\dot{\tilde{a}}\,\ddot{\tilde{a}} = -\frac{H_0^2}{a_{\rm eq}^3}\left(\frac{2}{\tilde{a}^3} + \frac{1}{\tilde{a}^2}\right)\dot{a}\ . \tag{15.224}$$

Canceling a factor $2\dot{\tilde{a}}$ yields

$$\ddot{\tilde{a}} = -\frac{H_0^2}{2a_{\rm eq}^3}\left(\frac{2+\tilde{a}}{\tilde{a}^3}\right)\ . \tag{15.225}$$

14c. This parallels the derivation given for **1b** where we found for $a(t)$ rather than $\tilde{a} = a/a_{\rm eq}$:

$$\frac{{\rm d}^2}{{\rm d}t^2} + 2\left(\frac{\dot{a}}{a}\right)\frac{\rm d}{{\rm d}t} = \dot{a}^2\,\frac{{\rm d}^2}{{\rm d}a^2} + \left(\ddot{a} + 2\,\frac{\dot{a}^2}{a}\right)\frac{\rm d}{{\rm d}a}\ . \tag{15.226}$$

However, any scale transformation of the sort $a \implies \lambda a \equiv a'$ with constant λ leaves the differential operator on the right-hand side unaffected in the sense that we can make the replacement $a \implies a'$: it is *form-invariant*. For instance:

$$\dot{a}^2\,\frac{{\rm d}^2}{{\rm d}a^2} = (\lambda\dot{a})^2\,\frac{1}{\lambda^2}\frac{{\rm d}^2}{{\rm d}a^2} = \dot{a}'^2\,\frac{{\rm d}^2}{{\rm d}(a')^2}\ . \tag{15.227}$$

So with $\lambda = 1/a_{\rm eq}$ we have $a' = \tilde{a}$ and we find that we can replace a by $\tilde{a}$ everywhere:

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} + 2\left(\frac{\dot{a}}{a}\right)\frac{\mathrm{d}}{\mathrm{d}t} = \dot{\tilde{a}}^2\,\frac{\mathrm{d}^2}{\mathrm{d}\tilde{a}^2} + \left(\ddot{\tilde{a}} + 2\,\frac{\dot{\tilde{a}}^2}{\tilde{a}}\right)\frac{\mathrm{d}}{\mathrm{d}\tilde{a}}\ . \tag{15.228}$$

Relation (14.7) then follows immediately from (14.1).

Another way of seeing this is to note that in the term with the time-derivatives on the left-hand side a only appears in the combination $\dot{a}/a = H$ and that

$$\frac{\dot{a}}{a} = \frac{\lambda\dot{a}}{\lambda a} = \frac{\dot{a}'}{a'} = \frac{\dot{\tilde{a}}}{\tilde{a}}\ \text{ for } \lambda = 1/a_{\rm eq}. \tag{15.229}$$

You can then retrace your steps of **1b**, using $\tilde{a}$ rather than a.

14d. This all boils down to eliminating $\ddot{a}$ and $\dot{a}$ from Eqn. (14.7), using results **a** and **b**. This is a simple exercise in algebra that anyone can do. After canceling a common factor $H_0^2/a_{\rm eq}^3$ from all terms, and dividing out a term so that the coefficient of the highest $\tilde{a}$-derivative become unity one finds Eqn. (14.9).

14e. In the case $\Omega_{\rm m} = \Omega_{\rm ba}$ (no CDM!) Eqn. (14.9) reduces to:

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}\tilde{a}^2} + \frac{2+3\tilde{a}}{2\tilde{a}(1+\tilde{a})}\,\frac{\mathrm{d}}{\mathrm{d}\tilde{a}} - \frac{3}{2\tilde{a}(1+\tilde{a})}\right]\Delta_{\rm ba}(\tilde{a}) = 0\ . \tag{15.230}$$

This can be solved by a density perturbation that is *linear* in $\tilde{a}$, a *particular solution*, so that $\mathrm{d}^2\Delta_{\rm ba}/\mathrm{d}\tilde{a}^2 = 0$. Then

$$(2+3\tilde{a})\frac{\mathrm{d}\Delta_{\rm ba}}{\mathrm{d}\tilde{a}} = 3\Delta_{\rm ba}\ , \tag{15.231}$$

which has the solution

$$\Delta_{\rm ba}(\tilde{a}) = \Delta_0\left(1 + \tfrac{3}{2}\,\tilde{a}\right)\ . \tag{15.232}$$

14f. Multiplying (14.9) by $2\tilde{a}(1+\tilde{a})$ yields:

$$\left[\, 2\tilde{a}\,(1+\tilde{a})\,\frac{\mathrm{d}^2}{\mathrm{d}\tilde{a}^2} + (2+3\tilde{a})\,\frac{\mathrm{d}}{\mathrm{d}\tilde{a}} - 3\left(\frac{\Omega_{\mathrm{ba}}}{\Omega_{\mathrm{m}}}\right)\right] \Delta_{\mathrm{ba}}(\tilde{a}) = 0\ . \tag{15.233}$$

Substituting the power series (**??**) and performing the differentiations yields a relation of the form:

$$\sum_{n=0}^{\infty} \left\{ 2n(n-1)c_n\,\tilde{a}^{n-1}(1+\tilde{a}) + nc_n\,\tilde{a}^{n-1}(2+3\tilde{a}) - 3\varpi c_n\,\tilde{a}^n \right\} = 0\ . \tag{15.234}$$

Here I have defined

$$\varpi \equiv \frac{\Omega_{\mathrm{ba}}}{\Omega_{\mathrm{m}}}\ . \tag{15.235}$$

Changing the summation indexes, so that all terms formally have the same power of $\tilde{a}$, one finds a relation of the form

$$\sum_{n=0}^{\infty} \gamma_n\,\tilde{a}^n = 0\ , \tag{15.236}$$

with

$$\gamma_n = 2(n+1)^2 c_{n+1} + \left[\, 2n^2 + n - 3\varpi \,\right] c_n\ . \tag{15.237}$$

This equation can only be satisfied for all $\tilde{a} \leq 1$ if all γ_n vanish identically:

$$\gamma_n = 0 \iff c_{n+1} + \frac{2n^2 + n - 3\ (\Omega_{\mathrm{ba}}/\Omega_{\mathrm{m}})}{2(n+1)^2}\, c_n = 0\ . \tag{15.238}$$

We now determine the first four terms in the infinite sum. The initial condition $\Delta_{\mathrm{ba}}(0) = \Delta_0$ implies immediately that $c_0 = \Delta_0$.

The next three coefficients (with $c_0 = \Delta_0$) follow from:

$$
\begin{aligned}
n = 0: &\qquad 2c_1 - 3\varpi\,\Delta_0 = 0 \\
n = 1: &\qquad 8c_2 + 3(1-\varpi)\,c_1 = 0 \\
n = 2: &\qquad 18c_3 + [\,4 + 3(2-\varpi)\,]\;c_2 = 0
\end{aligned}
\tag{15.239}
$$

This yields:

$$
c_1 = \tfrac{3}{2}\,\varpi\,\Delta_0 \;\;,\;\; c_2 = -\tfrac{9}{16}\,\varpi(1-\varpi)\,\Delta_0 \;\;,\;\; c_3 = \tfrac{1}{32}\,(7-3\varpi)(1-\varpi)\varpi\,\Delta_0\;. \tag{15.240}
$$

Note that for the special case $\varpi = \Omega_{\rm ba}/\Omega_{\rm m} = 1$ we have $c_0 = \Delta_0$, $c_1 = \frac{3}{2}\,\Delta_0$ and $c_n = 0$ for $n \geq 2$, in accordance with the result **e**. The solution for $\tilde{a} \ll 1$ can now be approximated as

$$
\Delta_{\rm ba}(\tilde{a}) \simeq \Delta_0\left(1 + \tfrac{3}{2}\,\varpi\,\tilde{a} - \tfrac{9}{16}\,\varpi(1-\varpi)\,\tilde{a}^2 + \tfrac{1}{32}\,(7-3\varpi)(1-\varpi)\varpi\,\tilde{a}^3\right)\;. \tag{15.241}
$$

14g. This is a simple matter of substituting a trial solution $\Delta_{\rm ba}(\tilde{a}) \propto \tilde{a}^s$ into (15.230), which leads to:

$$
\left(s(s-1) + \tfrac{3}{2}s - \tfrac{3}{2}\right)\,\tilde{a}^s = 0\;, \tag{15.242}
$$

which can be satisfied for all $\tilde{a}$ when

$$
s = 1 \;\;\text{and}\;\; s = -\tfrac{3}{2}\;. \tag{15.243}
$$

The growing solution (i.e. positive s!) therefore goes as $\Delta_{\rm ba}(\tilde{a}) \propto \tilde{a}$. Incidentally: this is what we also found in **12c** for the *driven* baryonic density fluctuations.