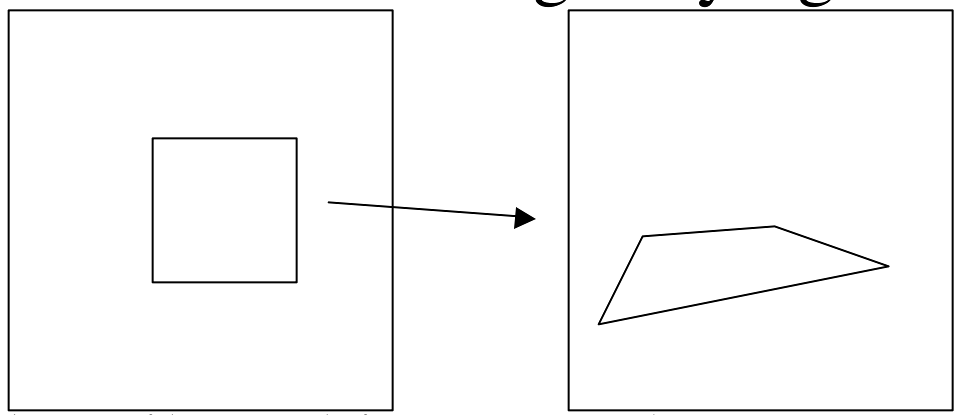
# Projective Geometry

# Euclidean versus Projective Geometry

- Euclidean geometry describes shapes "as they are"
  - Properties of objects that are unchanged by rigid motions
    - » Lengths
    - » Angles
    - » Parallelism



- Projective geometry describes objects "as they appear"
  - Lengths, angles, parallelism become "distorted" when we look at objects
  - Mathematical model for how images of the 3D world are formed.

# Overview

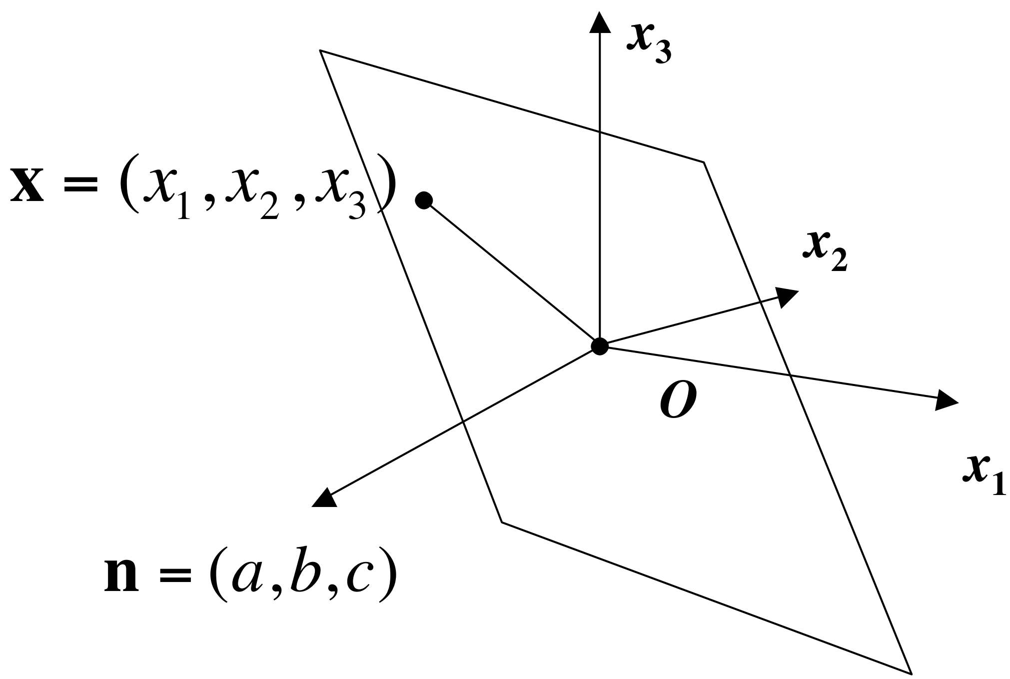
- Tools of algebraic geometry
- Informal description of projective geometry in a plane
- Descriptions of lines and points
- Points at infinity and line at infinity
- Projective transformations, projectivity matrix
- Example of application
- Special projectivities: affine transforms, similarities, Euclidean transforms
- Cross-ratio invariance for points, lines, planes

# Tools of Algebraic Geometry 1

- Plane ***passing through origin*** and perpendicular to vector $\mathbf{n} = (a, b, c)$ is locus of points $\mathbf{x} = (x_1, x_2, x_3)$ such that $\mathbf{n} \bullet \mathbf{x} = 0$

$$=> \quad a\,x_1 + b\,x_2 + c\,x_3 = 0$$

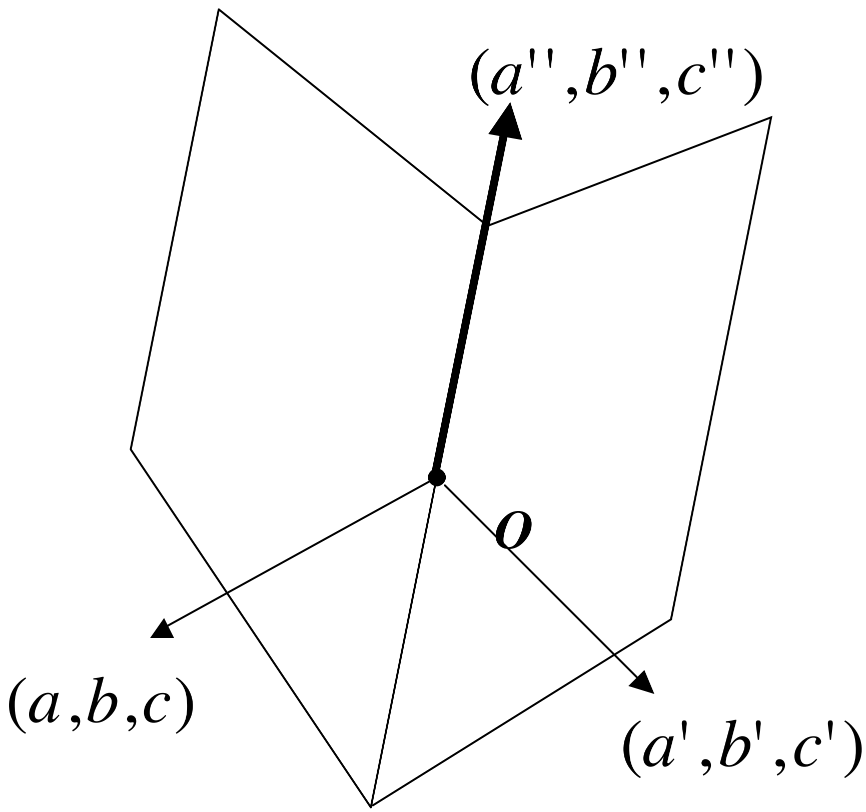
- Plane through origin is completely defined by $(a, b, c)$



# Tools of Algebraic Geometry 2

- A vector parallel to intersection of 2 planes $(a,b,c)$ and $(a',b',c')$ is obtained by cross-product

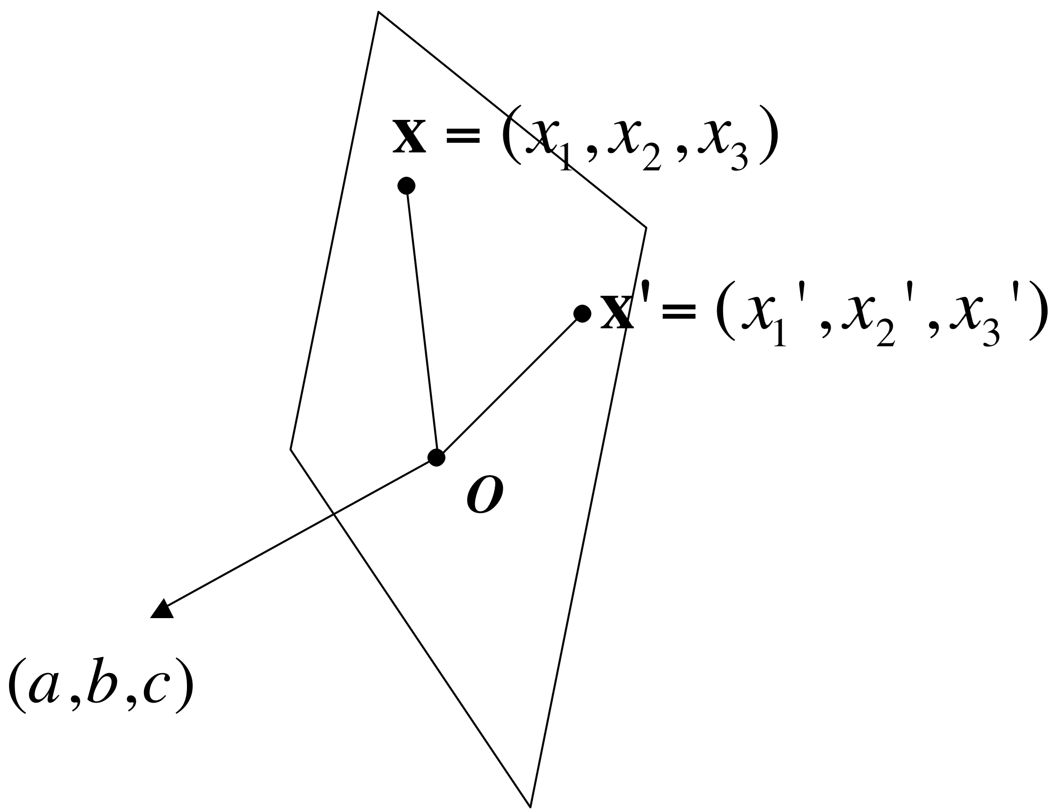
$$(a'',b'',c'') = (a,b,c) \times (a',b',c')$$



# Tools of Algebraic Geometry 3

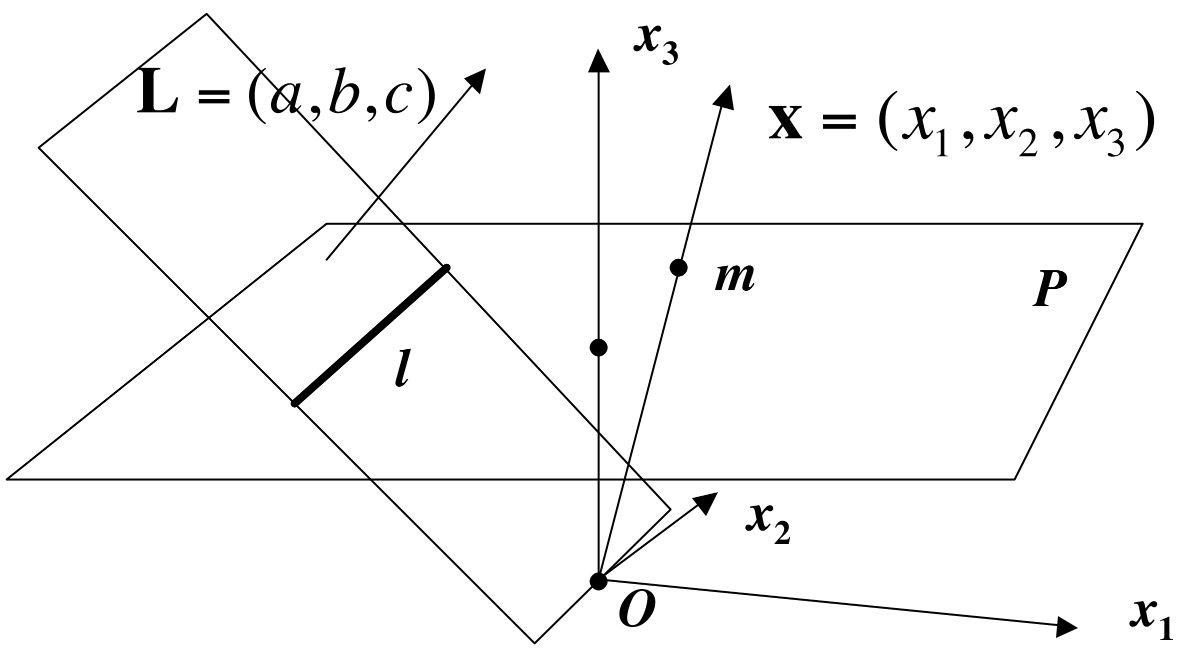
- Plane passing through two points **x** and **x'** is defined by

$$(a, b, c) = \mathbf{x} \times \mathbf{x}'$$



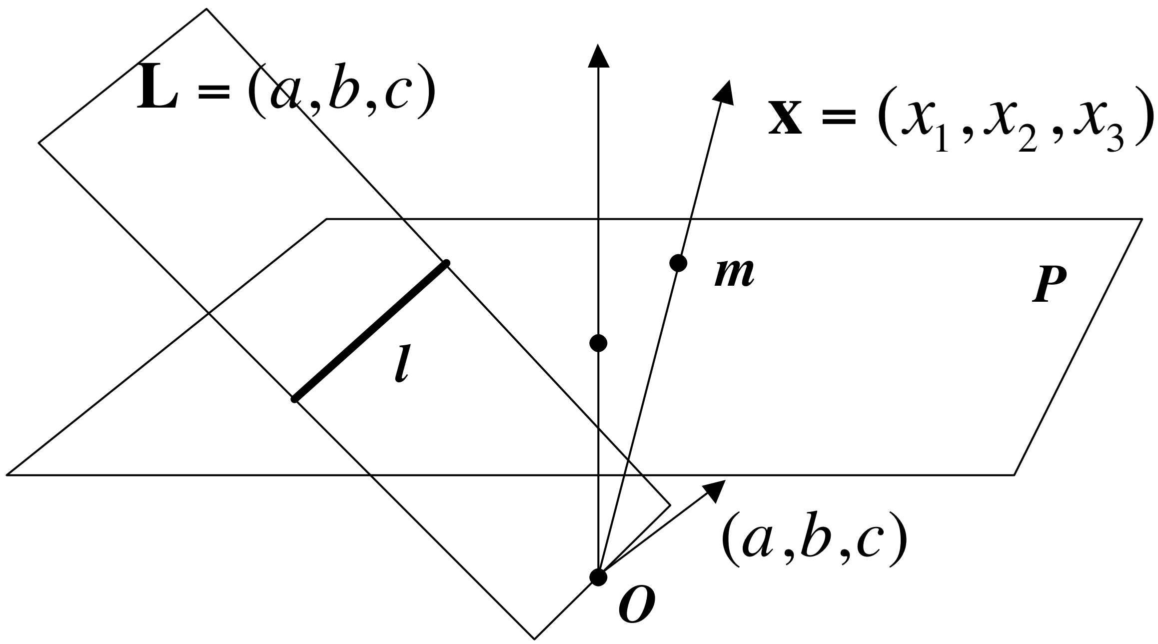
# Projective Geometry in 2D

- We are in a plane ***P*** and want to describe lines and points in ***P***
- We consider a third dimension to make things easier when dealing with infinity
  - Origin ***O*** out of the plane, at a distance equal to 1 from plane
- To each point ***m*** of the plane ***P*** we can associate a single ray $\mathbf{x} = (x_1, x_2, x_3)$
- To each line ***l*** of the plane ***P*** we can associate a single plane $(a, b, c)$



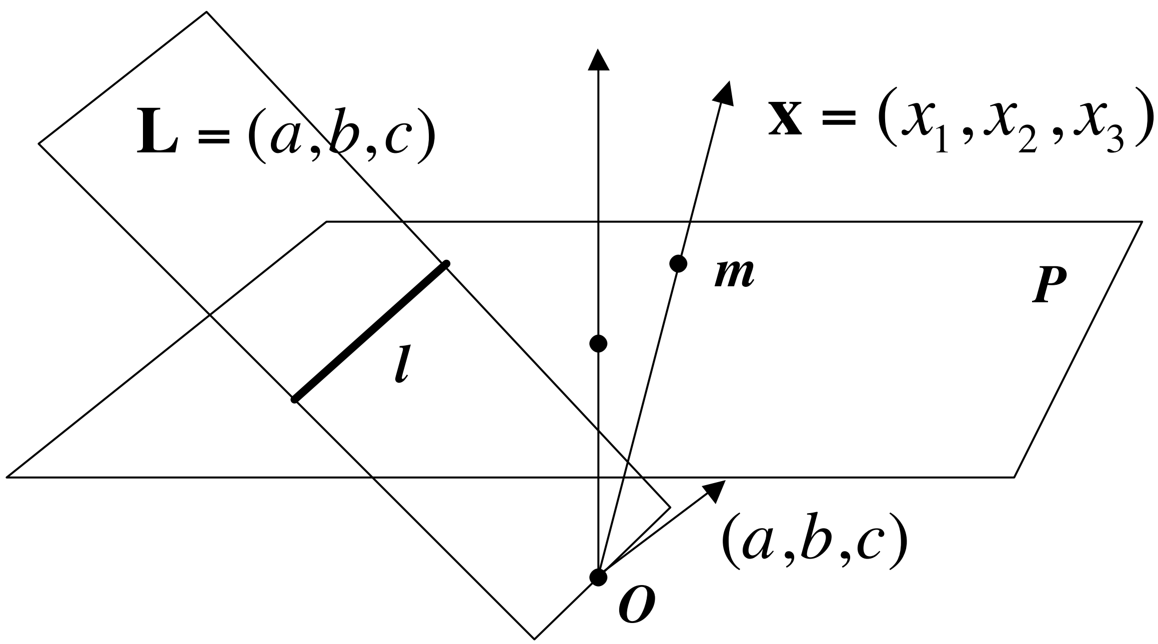
# Projective Geometry in 2D

- The rays $\mathbf{X} = (x_1, x_2, x_3)$ and $\mathbf{X} = (\lambda x_1, \lambda x_2, \lambda x_3)$ are the same and are mapped to the same point ***m*** of the plane ***P***
  - ***X*** is the coordinate vector of ***m***, $(x_1, x_2, x_3)$ are its homogeneous coordinates
- The planes $(a, b, c)$ and $(\lambda a, \lambda b, \lambda c)$ are the same and are mapped to the same line ***l*** of the plane ***P***
  - **L** is the coordinate vector of ***l***, $(a, b, c)$ are its homogeneous coordinates



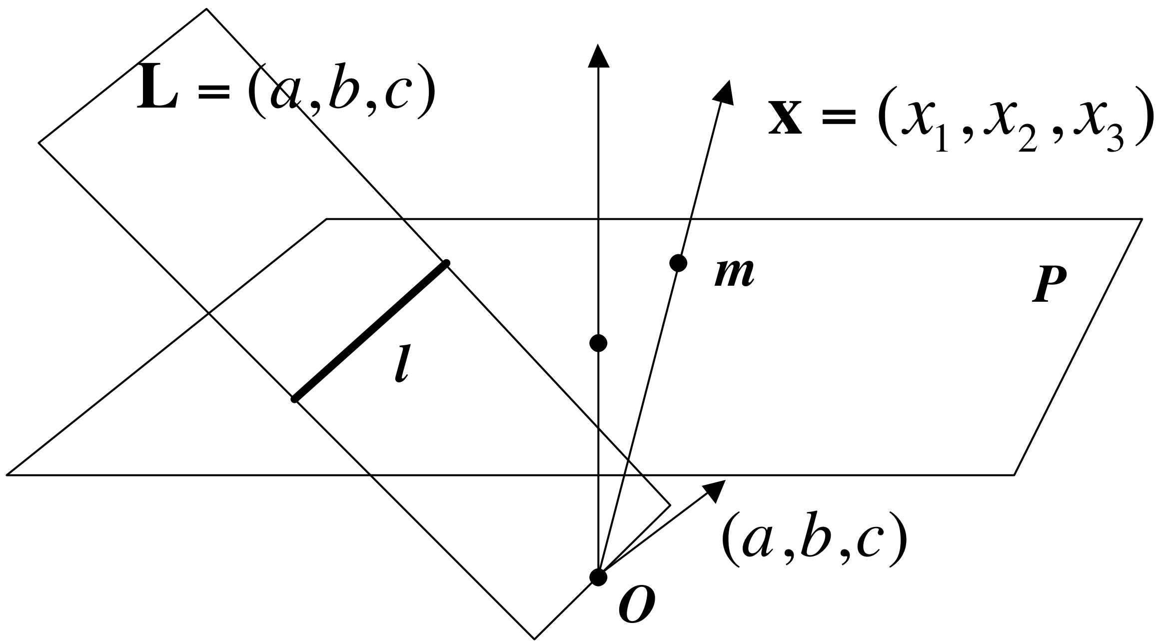
# Properties

- Point $\mathbf{X}$ belongs to line $\mathbf{L}$ if $\mathbf{L}\,.\,\mathbf{X} = 0$
- Equation of line $\mathbf{L}$ in projective geometry is $a\,x_1 + b\,x_2 + c\,x_3 = 0$
- We obtain homogeneous equations



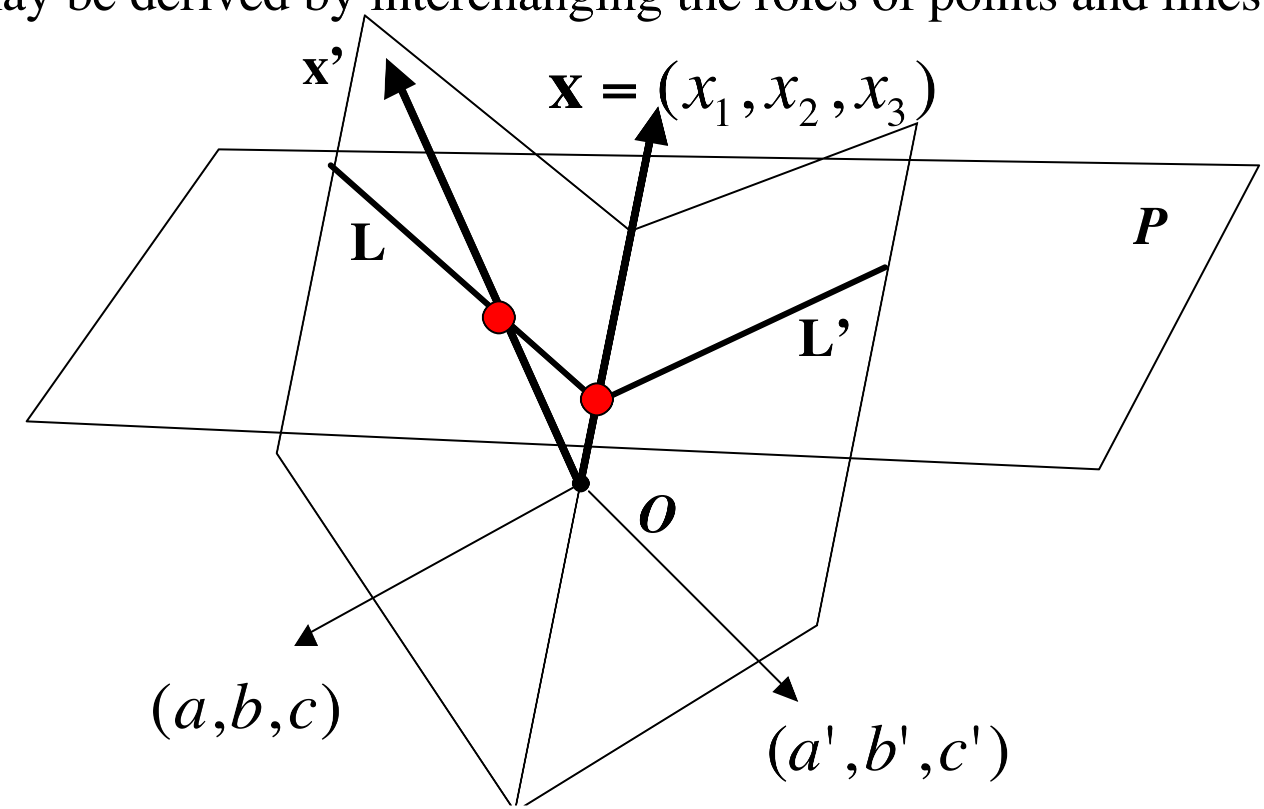
# From Projective Plane to Euclidean Plane

- How do we "land" back from the projective world to the 2D world of the plane?
  - For point, consider intersection of ray $\mathbf{x} = (\lambda x_1, \lambda x_2, \lambda x_3)$ with plane $x_3 = 1$ => $\lambda = 1/x_3$ , $\mathrm{m} = (x_1/x_3, x_2/x_3)$
- For line, intersection of plane $a x_1 + b x_2 + c x_3 = 0$ with plane $x_3 = 1$ is line ***l***: $a x_1 + b x_2 + c = 0$



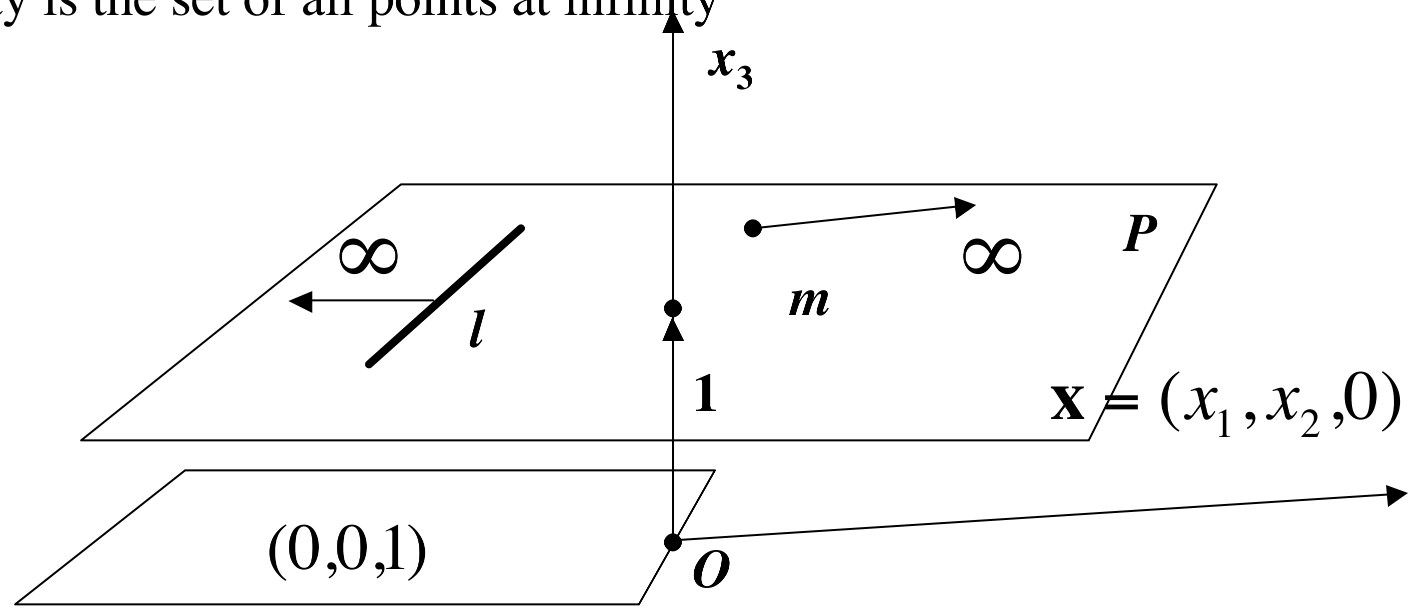
# Lines and Points

- Two lines **L = (*a, b, c*)** and **L' = (*a',b',c'*)** intersect in the point
$$\mathbf{x} = \mathbf{L} \times \mathbf{L}'$$
- The line through 2 points **x** and **x'** is $\mathbf{L} = \mathbf{x} \times \mathbf{x}'$
- Duality principle: To any theorem of 2D projective geometry, there corresponds a dual theorem, which may be derived by interchanging the roles of points and lines in the original theorem



# Ideal Points and Line at Infinity

- The points $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2, 0)$ do not correspond to finite points in the plane. They are points at infinity, also called ***ideal points***
- The line $\mathbf{L} = (0,0,1)$ passes through all points at infinity, since $\mathbf{L} \cdot \mathbf{x} = 0$
- Two parallel lines $\mathbf{L} = (\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c})$ and $\mathbf{L}' = (\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}')$ intersect at the point $\mathbf{x} = \mathbf{L} \times \mathbf{L}' = (\boldsymbol{c}' - \boldsymbol{c})(\boldsymbol{b}, -\boldsymbol{a}, 0)$, i.e. $(\boldsymbol{b}, -\boldsymbol{a}, 0)$
- Any line $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c})$ intersects the line at infinity at $(\boldsymbol{b}, -\boldsymbol{a}, 0)$. So the line at infinity is the set of all points at infinity

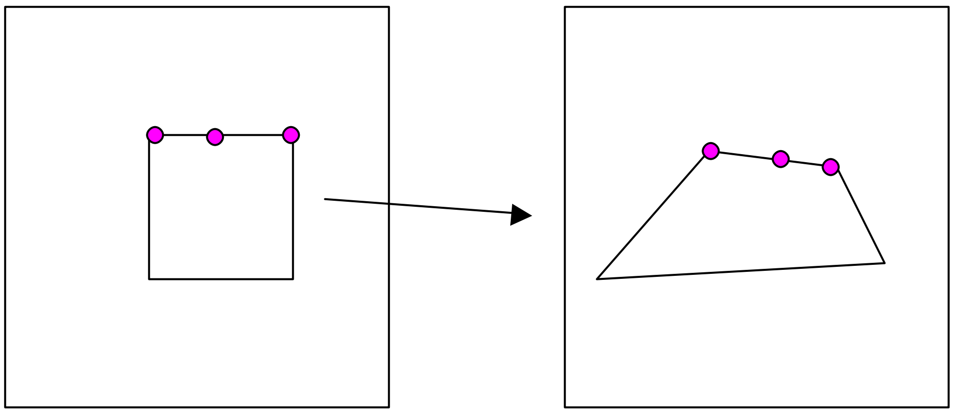


# Ideal Points and Line at Infinity

- With projective geometry, two lines always meet in a single point, and two points always lie on a single line.
- This is not true of Euclidean geometry, where parallel lines form a special case.

# Projective Transformations in a Plane

- ***Projectivity***
  - Mapping from points in plane to points in plane
  - 3 aligned points are mapped to 3 aligned points
- Also called
  - ***Collineation***
  - ***Homography***



# Projectivity Theorem

- A mapping is a *projectivity* if and only if the mapping consists of a linear transformation of homogeneous coordinates $\mathbf{x}' = \mathrm{H}\mathbf{x}$

  with $\mathbf{H}$ non singular

- *Proof*:
  - If $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_3$ are 3 points that lie on a line $\mathbf{L}$, and $\mathbf{x'}_1 = \mathbf{H}\,\mathbf{x}_1$, etc, then $\mathbf{x'}_1$, $\mathbf{x'}_2$, and $\mathbf{x'}_3$ lie on a line $\mathbf{L'}$
  - $\mathbf{L}^\mathbf{T}\,\mathbf{x}_i = 0$, $\mathbf{L}^\mathbf{T}\,\mathbf{H}^{-1}\,\mathbf{H}\,\mathbf{x}_i = 0$, so points $\mathbf{H}\,\mathbf{x}_i$ lie on line $\mathbf{H}^{-\mathbf{T}}\,\mathbf{L}$

- Converse is hard to prove, namely if all collinear sets of points are mapped to collinear sets of points, then there is a single linear mapping between corresponding points in homogeneous coordinates

# Projectivity Matrix

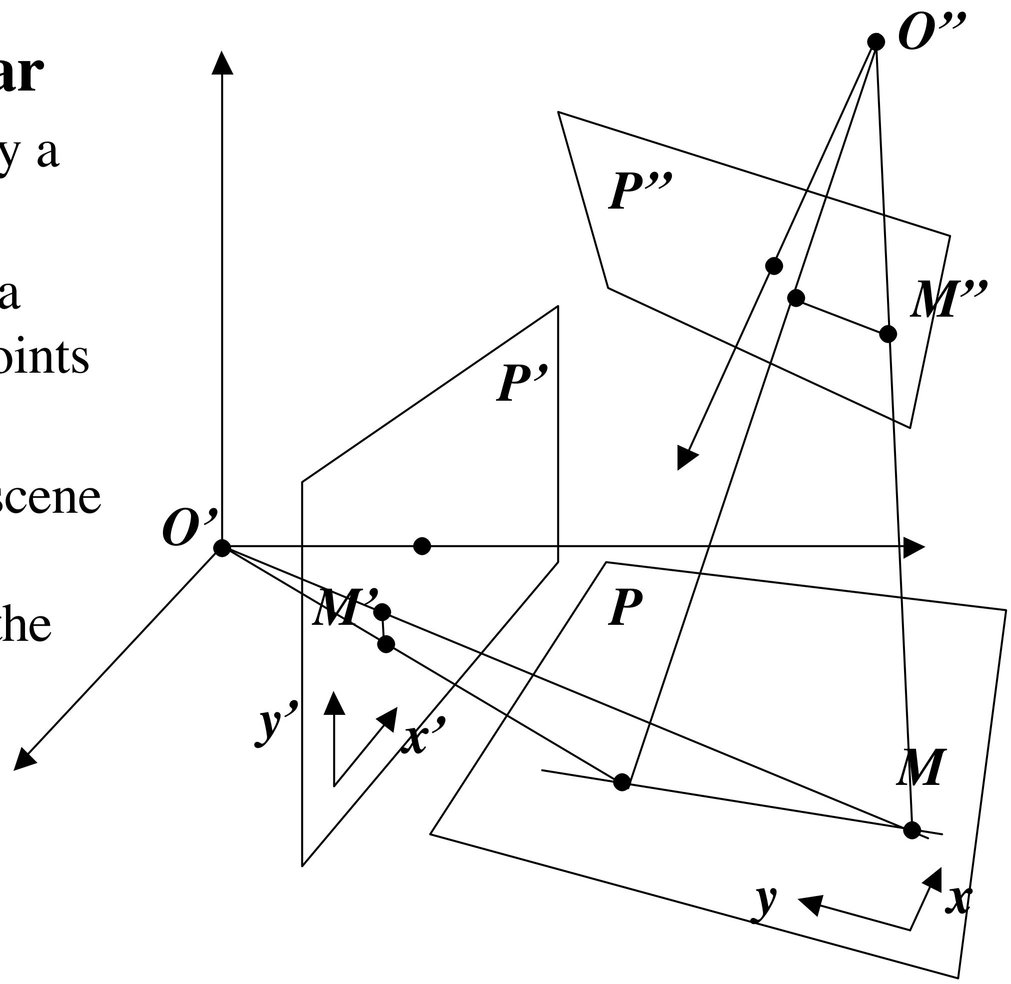
$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \qquad \mathbf{x}' = \mathrm{H}\,\mathbf{x}$$

- The matrix **H** can be multiplied by an arbitrary non-zero number without altering the projective transformation
- Matrix **H** is called a “homogeneous matrix” (only ratios of terms are important)
- There are 8 independent ratios. It follows that projectivity has 8 degrees of freedom
- A projectivity is simply a linear transformation of the rays

# Examples of Projective Transformations

- Central projection maps **planar scene** points to image plane by a projectivity
  - True because all points on a scene line are mapped to points on its image line
- The image of the same planar scene from a second camera can be obtained from the image from the first camera by a projectivity
  - True because
    $\mathbf{x'}_i = \mathbf{H'}\,\mathbf{x}_i$ , $\mathbf{x''}_i = \mathbf{H''}\,\mathbf{x}_i$

    so $\mathbf{x''}_i = \mathbf{H''}\,\mathbf{H'}^{-1}\,\mathbf{x'}_i$



# Computing Projective Transformation

- Since matrix of projectivity has 8 degrees of freedom, the mapping between 2 images can be computed if we have the coordinates of 4 points on one image, and know where they are mapped in the other image
  - Each point provides 2 independent equations

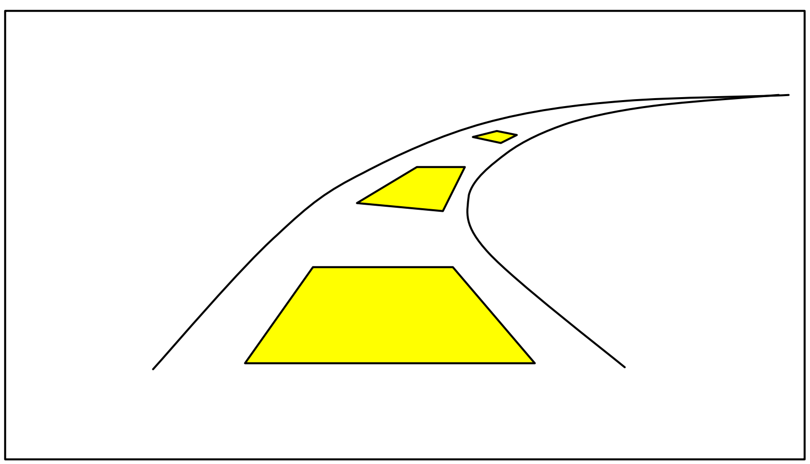
$$x' = \frac{x'_1}{x'_3} = \frac{h_{11}x + h_{12}y + h_{13}}{h_{31}x + h_{32}y + h_{33}} = \frac{h'_{11}\,x + h'_{12}\,y + h'_{13}}{h'_{31}\,x + h'_{32}\,y + 1}$$

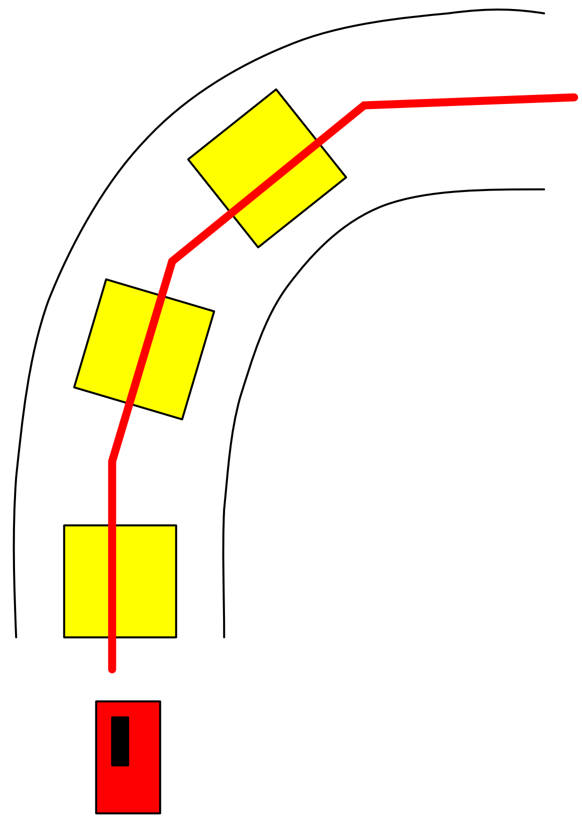
$$y' = \frac{x'_2}{x'_3} = \frac{h_{21}x + h_{22}y + h_{23}}{h_{31}x + h_{32}y + h_{33}} = \frac{h'_{21}\,x + h'_{22}\,y + h'_{23}}{h'_{31}\,x + h'_{32}\,y + 1}$$

  - Equations are linear in the 8 unknowns $h'_{ij} = h_{ij} / h_{33}$

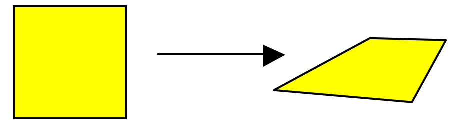
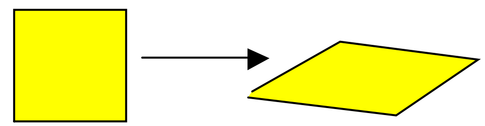
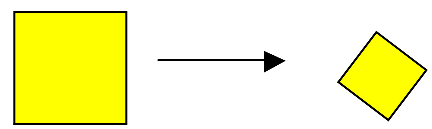
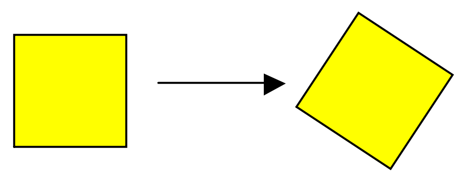
# Example of Application

- Robot going down the road
- Large squares painted on the road to make it easier
- Find road shape without perspective distortion from image
  - Use corners of squares: coordinates of 4 points allow us to compute matrix **H**
  - Then use matrix **H** to compute 3D road shape





# Special Projectivities

| | | Invariants | |
|---|---|---|---|
| Projectivity<br>8 dof | $\begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix}$ | Collinearity,<br>Cross-ratios |  |
| Affine transform<br>6 dof | $\begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_x \\ 0 & 0 & 1 \end{bmatrix}$ | Parallelism,<br>Ratios of areas,<br>Length ratios |  |
| Similarity<br>4 dof | $\begin{bmatrix} s\,r_{11} & s\,r_{12} & t_x \\ s\,r_{21} & s\,r_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$ | Angles,<br>Length ratios |  |
| Euclidean transform<br>3 dof | $\begin{bmatrix} r_{11} & r_{12} & t_x \\ r_{21} & r_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$ | Angles,<br>Lengths,<br>Areas |  |

# Projective Space $P_{\mathrm{n}}$

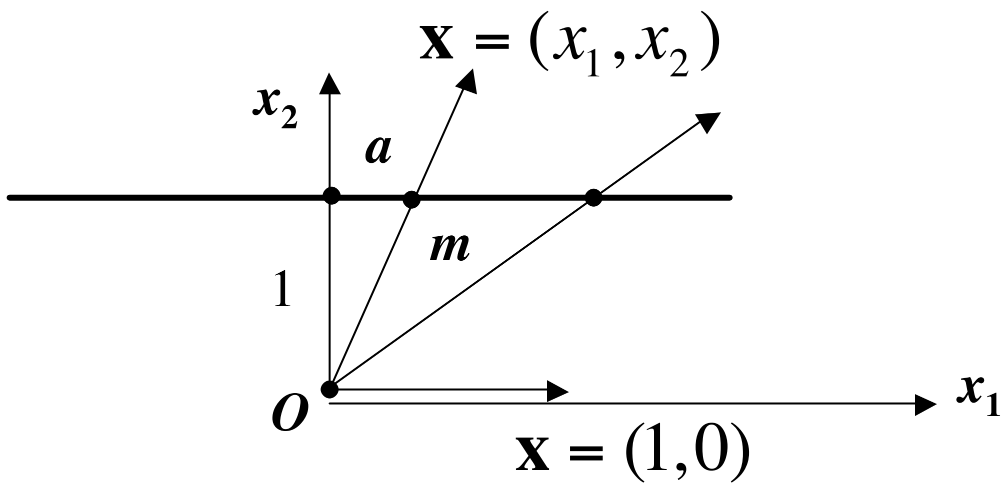
- A point in a projective space $\boldsymbol{P}_{\mathrm{n}}$ is represented by a vector of $n$+1 coordinates $\mathbf{x} = (x_1, x_2, \cdots, x_{n+1})$
- At least one coordinate is non zero.
- Coordinates are called homogeneous or projective coordinates
- Vector **x** is called a coordinate vector
- Two vectors $\mathbf{x} = (x_1, x_2, \cdots, x_{n+1})$ and $\mathbf{y} = (y_1, y_2, \cdots, y_{n+1})$ represent the same point if and only if there exists a scalar $\boldsymbol{l}$ such that

$$x_i = \lambda\, y_i$$

The correspondence between points and coordinate vectors is not one to one.

# Projective Geometry in 1D

- Points ***m*** along a line
- Add up one dimension, consider origin at distance 1 from line
- Represent ***m*** as a ray from the origin (0, 0): $\mathbf{x} = (x_1, x_2)$
- X = (1,0) is point at infinity
- Points can be written X = (***a*****, 1**), where ***a*** is abscissa along the line

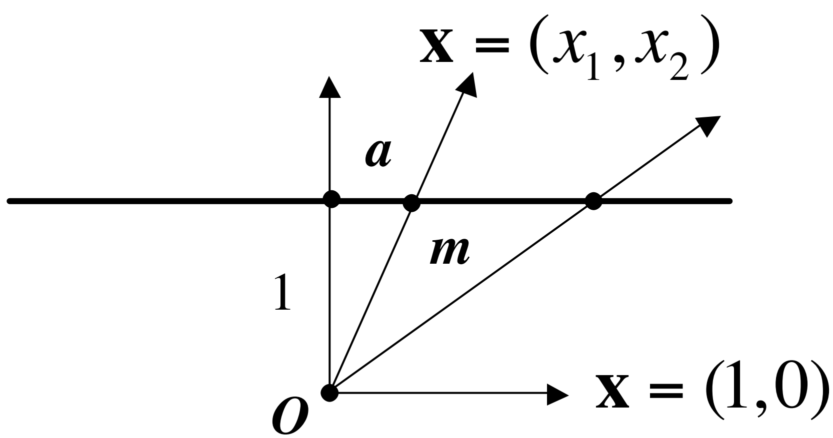


# Projectivity in 1D

- A projective transformation of a line is represented by a 2x2 matrix

$$\begin{pmatrix} x'_1 \\ x'_2 \end{pmatrix} = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \qquad \mathbf{x}' = \mathrm{H}\,\mathbf{x}$$

- Transformation has 3 degrees of freedom corresponding to the 4 elements of the matrix, minus one for overall scaling
- Projectivity matrix can be determined from 3 corresponding points



# Cross-Ratio Invariance in 1D

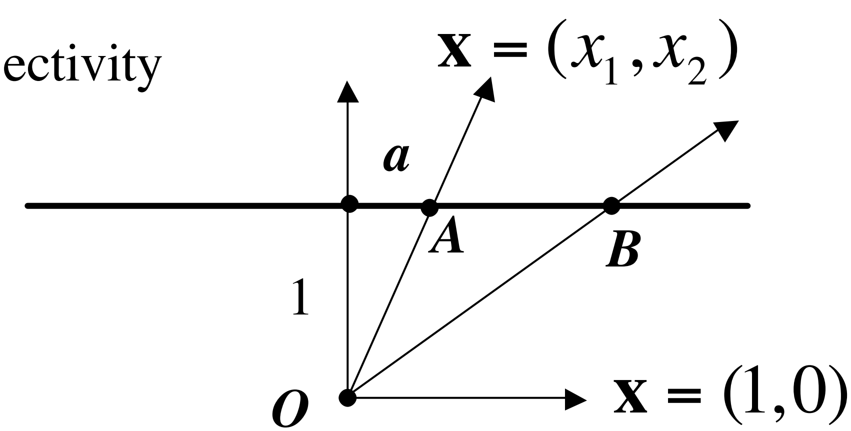
- Cross-ratio of 4 points A, B, C, D on a line is defined as

$$\text{Cross}(A,B,C,D) = \frac{|AB|}{|AD|} \div \frac{|CB|}{|CD|} \text{ with } |AB| = \det \begin{bmatrix} x_{A1} & x_{B1} \\ x_{A2} & x_{B2} \end{bmatrix}$$

- Cross-ratio is not dependent on which particular homogeneous representation of the points is selected: scales cancel between numerator and denominator. For A = (a, 1), B = (b, 1), etc, we get

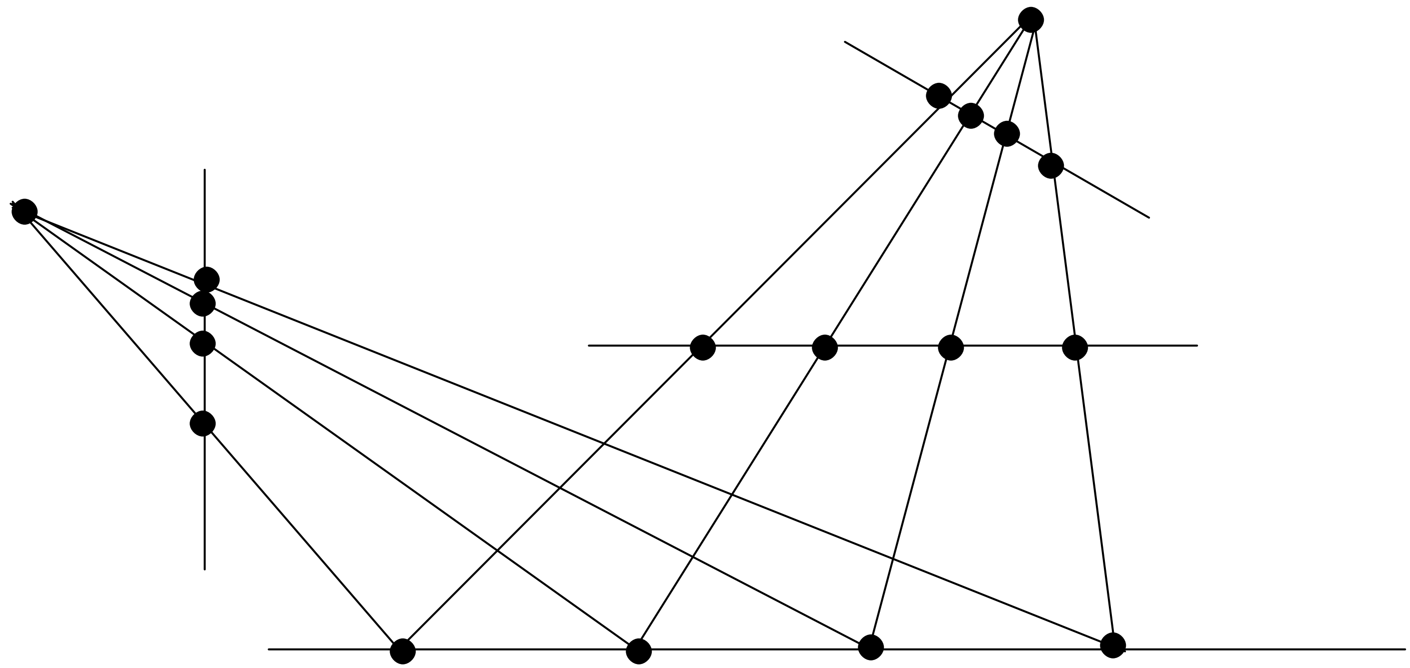
$$\text{Cross}(A,B,C,D) = \frac{a-b}{a-d} \div \frac{c-b}{c-d}$$

- Cross-ratio is invariant under any projectivity



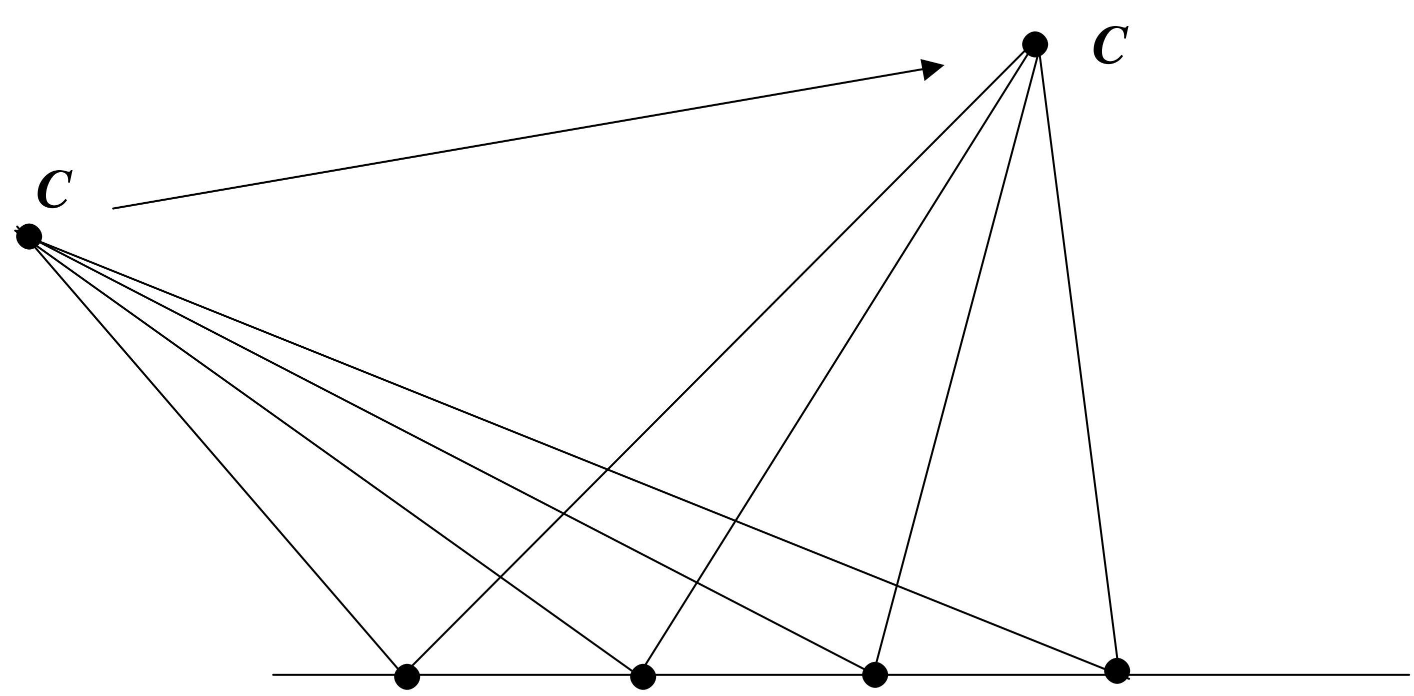
# Cross-Ratio Invariance in 1D

- For the 4 sets of collinear points in the figure, the cross-ratio for corresponding points has the same value



# Cross-Ratio Invariance between Lines

- The cross-ratio between 4 lines forming a *pencil* is invariant when the point of intersection ***C*** is moved
- It is equal to the cross-ratio of the 4 points

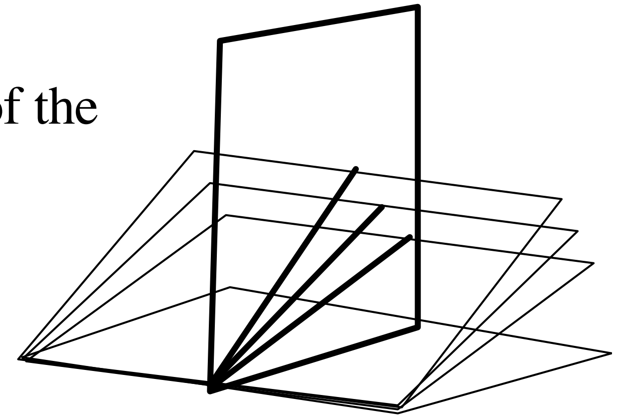


# Projective Geometry in 3D

- Space $P_3$ is called projective space
- A point in 3D space is defined by 4 numbers $(\boldsymbol{x_1}, \boldsymbol{x_2}, \boldsymbol{x_3}, \boldsymbol{x_4})$
- A plane is also defined by 4 numbers $(\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}, \boldsymbol{u_4})$
- Equation of plane is

$$\sum_{i=1}^{4} u_i\, x_i = 0$$

- The plane at infinity is the plane (0,0,0,1). Its equation is $\boldsymbol{x_4}$=0
- The points $(\boldsymbol{x_1}, \boldsymbol{x_2}, \boldsymbol{x_3}, 0)$ belong to that plane in the direction $(\boldsymbol{x_1}, \boldsymbol{x_2}, \boldsymbol{x_3})$ of Euclidean space
- A line is defined as the set of points that are a linear combination of two points $\boldsymbol{P_1}$ and $\boldsymbol{P_2}$
- The cross-ratio of 4 planes is equal to the cross-ratio of the lines of intersection with a fifth plane



# Central Projection

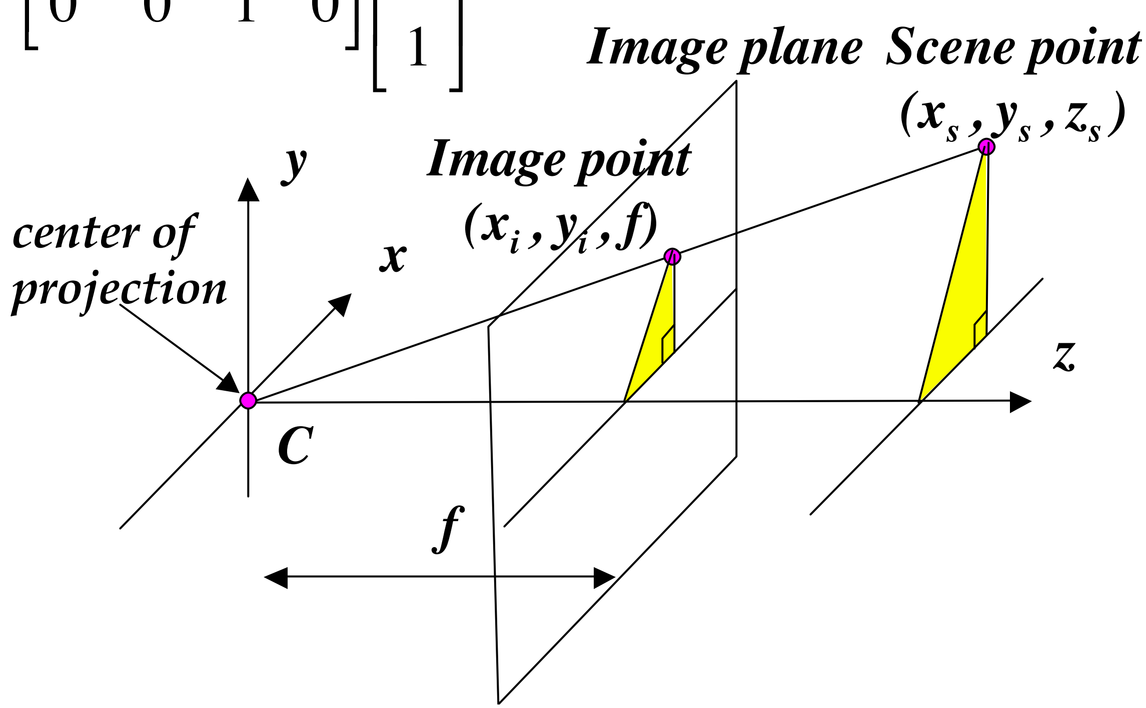
If world and image points are represented by homogeneous vectors, central projection is a linear mapping between $\boldsymbol{P}_3$ and $\boldsymbol{P}_2$:

$$x_i = f\,\frac{x_s}{z_s}$$

$$y_i = f\,\frac{y_s}{z_s}$$

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_s \\ y_s \\ z_s \\ 1 \end{bmatrix}$$

$$x_i = u / w, \quad y_i = v / w$$



# References

- Multiple View Geometry in Computer Vision, R. Hartley and A. Zisserman, Cambridge University Press, 2000
- Three-Dimensional Computer Vision: A Geometric Approach, O. Faugeras, MIT Press, 1996