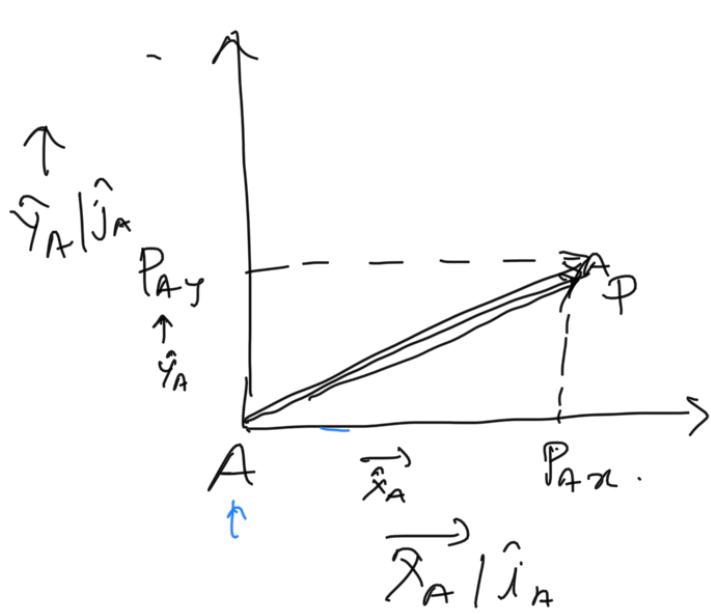


LECTURE -2.

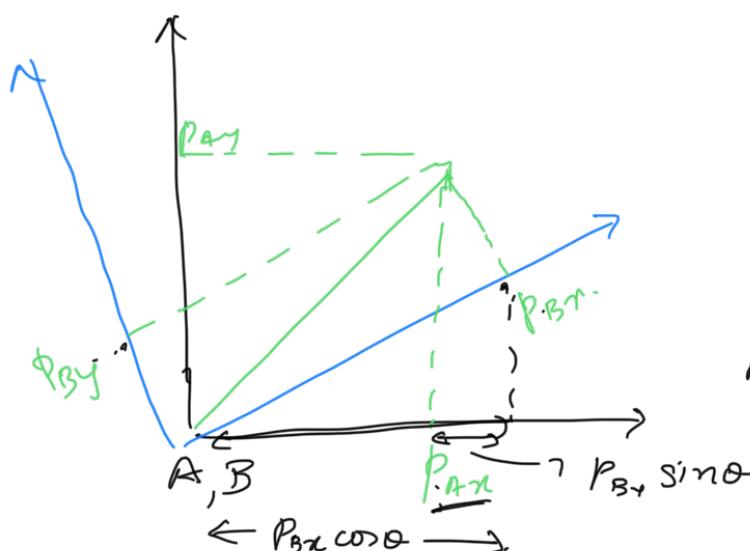
Coordinate Transform:



$${}^A \vec{P} = \begin{bmatrix} P_{Ax} \\ P_{Ay} \\ P_{Az} \end{bmatrix} \quad \rightarrow (1)$$

$$= \begin{bmatrix} \hat{x}_A & \hat{y}_A & \hat{z}_A \\ \text{unit vectors} \end{bmatrix} \begin{bmatrix} P_{Ax} \\ P_{Ay} \\ P_{Az} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} P_{Ax} \\ P_{Ay} \\ P_{Az} \end{bmatrix} \quad \hookrightarrow (2)$$



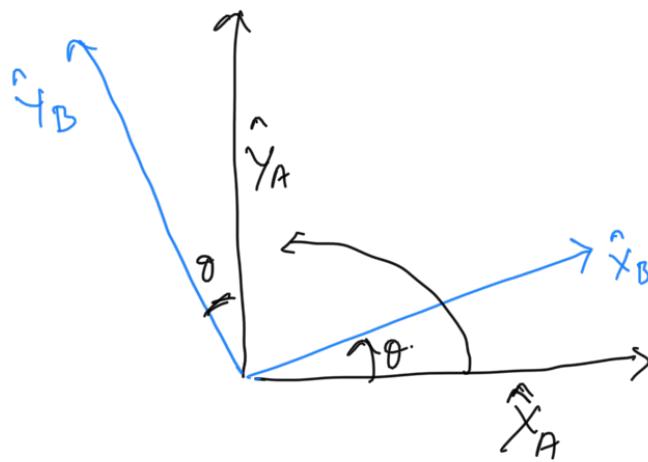
$${}^A \underline{\underline{P}} = [P_{Ax} \ P_{Ay} \ P_{Az}]^T$$

$${}^B \underline{\underline{P}} = [P_{Bx} \ P_{By} \ P_{Bz}]^T$$

What is ${}^B \underline{\underline{P}}$ in A?

$${}^B \underline{\underline{P}} = \begin{bmatrix} \hat{x}_B & \hat{y}_B & \hat{z}_B \\ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \end{bmatrix} \begin{bmatrix} P_{Bx} \\ P_{By} \\ P_{Bz} \end{bmatrix}$$

$$\text{Then } {}^A P = \begin{bmatrix} {}^A \hat{x}_B & {}^A \hat{y}_B & {}^A \hat{z}_B \end{bmatrix} \begin{bmatrix} P_{Bx} \\ P_{By} \\ P_{Bz} \end{bmatrix} \xrightarrow{\substack{\leftarrow {}^A B_2 \\ \longrightarrow (3)}} \\ = \begin{bmatrix} \begin{bmatrix} \hat{x}_B, \hat{x}_A \\ \hat{x}_B, \hat{y}_A \\ \hat{x}_B, \hat{z}_A \end{bmatrix} & \begin{bmatrix} \hat{x}_B, \hat{x}_A \\ \hat{y}_B, \hat{x}_A \\ \hat{y}_B, \hat{z}_A \end{bmatrix} & \begin{bmatrix} \hat{z}_B, \hat{x}_A \\ \hat{z}_B, \hat{y}_A \\ \hat{z}_B, \hat{z}_A \end{bmatrix} \end{bmatrix} \begin{bmatrix} P_{Bx} \\ P_{By} \\ P_{Bz} \end{bmatrix} \xrightarrow{\substack{\longrightarrow (4) \\ \underbrace{\quad}_{R_B^A}}}$$

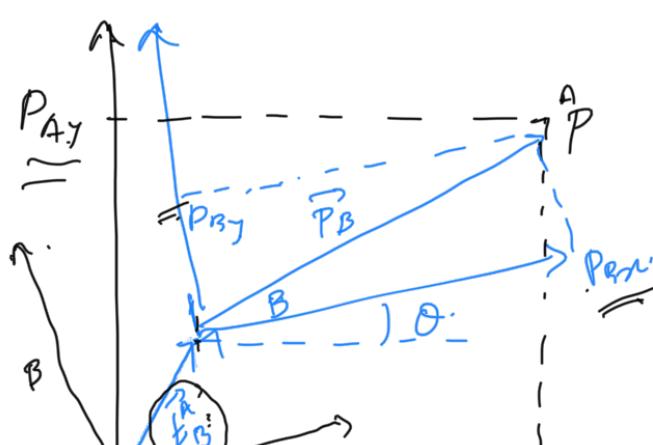


R_B^A or rotation
of frame B
with respect to A.

R_B^A is also the ROTATION
MATRIX of B wrt A.

$$R_B^A = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{\substack{\text{can also be} \\ \text{obtained from} \\ \text{trigonometry.}}}$$

In presence of both Rotation and.
Translation



$$\overrightarrow{AP} = \overrightarrow{R_B^A} \overrightarrow{P_B} + \overrightarrow{t_B^A} \xrightarrow{\substack{\longrightarrow (5) \\ \left[\begin{array}{l} \text{origin of B's frame} \\ \text{in A.} \end{array} \right]}}$$

~~A~~ $\xrightarrow{P_{A,x}}$ $\overset{A}{t}_B^A$ is the translation of B wrt A . ${}^A p = {}^A t_B^A$

$$\boxed{{}^A P = {}^T_B^A {}^B P \rightarrow (6).} \quad {}^A \vec{P} = {}^T_B^A {}^B \vec{P}$$

$4 \times 1 \quad 4 \times 4 \quad 4 \times 1$

T_B^A is the homogeneous transform matrix

$$P_A = \begin{bmatrix} P_{A,x} \\ P_{A,y} \\ P_{A,z} \\ 1 \end{bmatrix} = {}^T_B^A \begin{bmatrix} P_B \\ P_{B,y} \\ P_{B,z} \\ 1 \end{bmatrix}$$

$$T_B^A = \begin{bmatrix} R_B^A & {}^T_B^A \\ 0 & 1 \end{bmatrix} \rightarrow (7)$$

$P_B = \begin{bmatrix} P_B \\ P_{B,y} \\ P_{B,z} \\ 1 \end{bmatrix}$

$$T_B^A = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & t_{B,x} \\ \sin\theta & \cos\theta & 0 & t_{B,y} \\ 0 & 0 & 1 & t_{B,z} \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \begin{array}{l} \cos\theta = \cos(\theta) \\ \sin\theta = \sin(\theta) \end{array} \rightarrow (8)$$

$${}^A \vec{P} = \begin{bmatrix} P_{A,x} \\ P_{A,y} \\ P_{A,z} \\ 1 \end{bmatrix} \quad \mathbb{R}^3 \quad {}^B \vec{P} = \begin{bmatrix} P_{B,x} \\ P_{B,y} \\ P_{B,z} \\ 1 \end{bmatrix}$$

$\mathbb{R}^3 \hookrightarrow \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$

$$M: \mathbb{R}^3 \rightarrow \mathbb{P}^3$$

Homogenous Coordinates.

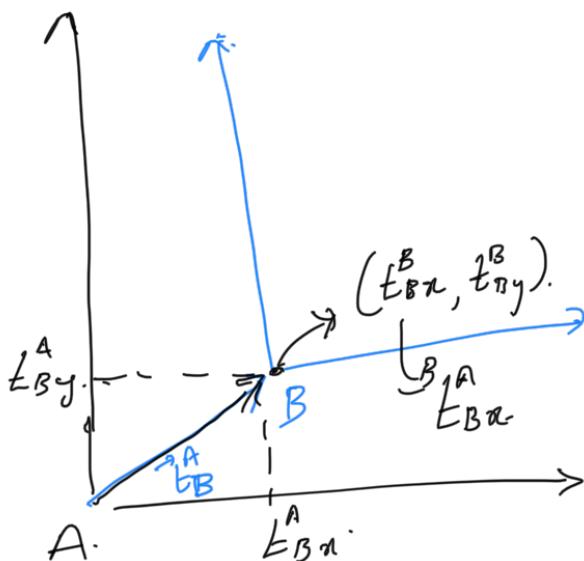
$$\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$T_C^A = T_B^A T_C^B \rightarrow (9).$$

Concatenate frames elegantly.

The non homogeneous equations
lead to many concatenation
of frames.

Q: Given \vec{t}_B^A what is \vec{t}_A^B in terms of
 \vec{E}_B^A and R_B^A



$$T_c^A = T_B^A \cdot T_C^B$$

\vec{E}_B^A is a vector in A,

\vec{t}_B^A is the same vector in B.

$$\vec{t}_B^A = T_A^B \vec{E}_B^A \quad \rightarrow (10)$$

$$= \begin{bmatrix} R_A^B & \vec{t}_A^B \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \vec{t}_B^A \\ 1 \end{bmatrix} \quad \rightarrow (11)$$

$$[T_A^B]^{-1} = [R_B^A]^T \begin{bmatrix} -R_B^A \\ \vec{t}_B^A \end{bmatrix}$$

non homogeneous $R_A^B \vec{t}_B^A + \vec{t}_A^B \rightarrow (12)$

But $\vec{t}_B^A = 0$ Why ?? $= \begin{bmatrix} [R_B^A]^T & -[R_B^A]^T \vec{t}_B^A \\ 0 & 1 \end{bmatrix}$

$$0_{3x1} = R_A^B \vec{t}_B^A + \vec{t}_A^B \rightarrow (13)$$

$$E_A^B = -R_A^B E_B^A = -R_B^{AT} E_B^A.$$

L-2(14)

$$\underline{R_A^B = [R_B^A]^{-1} = [R_B^A]^T. \text{ Why?}}$$

$$\underline{R_A^B} \cdot \underline{R_A^B}^\top = I_{3 \times 3}$$

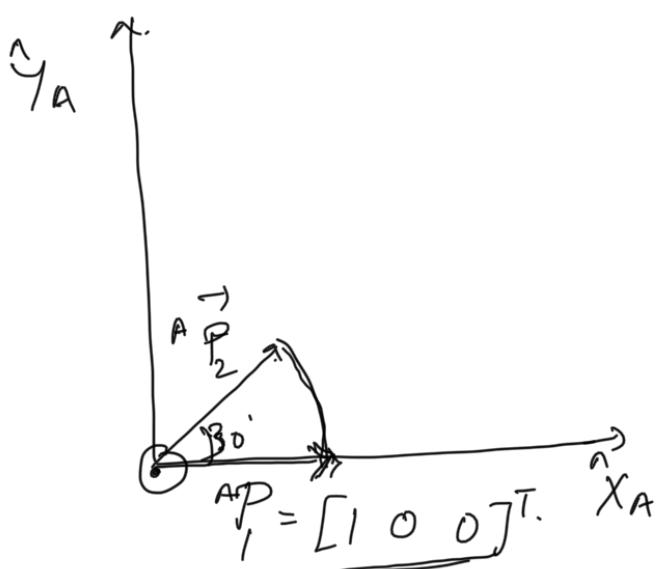
J. T. Craig's Book
2nd Chapter

The Equivalence of Operators and Transforms:

$${}^A P_2 = R_2(\theta) \cdot {}^A P_1$$

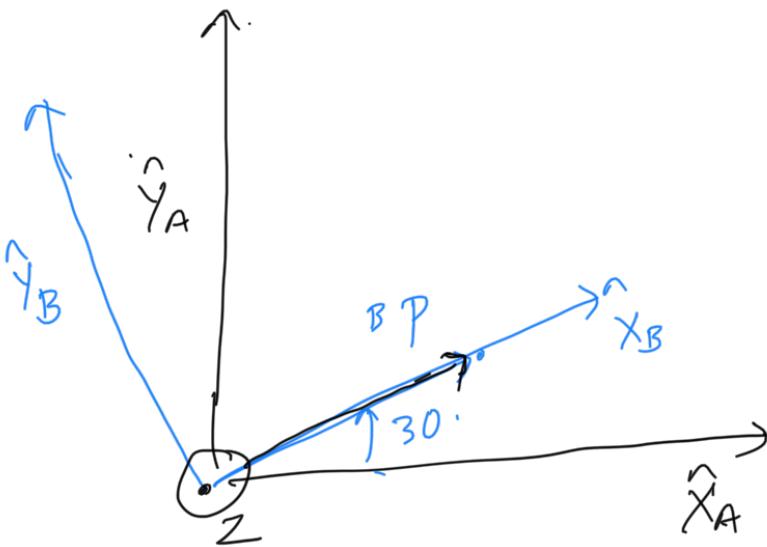
$$= \begin{bmatrix} c_{30} & -s_{30} \\ s_{30} & c_{30} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

$$= \begin{bmatrix} \sqrt{3}/2 \\ 1/2 \end{bmatrix}$$



Here $R_{2\alpha}(\theta)$ is a rotation operator
 that operates on $\underline{\overset{4}{P}_1}$ and takes it
 to $\underline{\overset{4}{P}_2}$.

This is same as:



$${}^B P = [1 \ 0 \ \phi]^T$$

$${}^A P = R_B^A {}^B P$$

$$= \begin{bmatrix} \cos 30 & -\sin 30 \\ \sin 30 & \cos 30 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= [\sqrt{3}/2 \ \ 0]^T$$

Representing Rotation as Euler Angle:

Z-Y-X Euler Angles:

$$\begin{matrix} R_1 & R_2 & R_3 \\ |R_1|^2 = 1 & R_1 \cdot R_2 = 0 \end{matrix}$$

→ Description of B w.r.t A. as a sequence of Euler Angle rotations.

→ Initially A and B coincide.

i). Rotate B about \hat{Z}_A or \hat{Z}_B by an angle α to get intermediate frame B1.

$$R_{B1}^B \text{ or } R_B^{A1} = R_z(\alpha)$$

$$= \begin{bmatrix} \cos \alpha & -\sin \alpha & 0 \\ \sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow (1).$$

ii). Now rotate B1 about \hat{Y}_{B1} by β

to obtain B_2

$$R_{B_2}^{B_1} = R_y(\beta)$$

$$= \begin{bmatrix} c\beta & 0 & s\beta \\ 0 & 1 & 0 \\ -s\beta & 0 & c\beta \end{bmatrix} \xrightarrow{(2)}$$

3). Now rotate B_2 about x_{B_2} by an angle γ to obtain B_3 .

$$R_{B_3}^{B_2} = R_x(\gamma) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\gamma & -s\gamma \\ 0 & s\gamma & c\gamma \end{bmatrix} \xrightarrow{(3)}$$

Then $R_B^A = R_{B_1}^{B_1} R_{B_2}^{B_1} R_{B_3}^{B_2} \xrightarrow{(4)}$

$$= \begin{bmatrix} cd & -sd & 0 \\ sd & cd & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\beta & 0 & s\beta \\ 0 & 1 & 0 \\ -s\beta & 0 & c\beta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\gamma & -s\gamma \\ 0 & s\gamma & c\gamma \end{bmatrix}$$

$\hookrightarrow (5)$

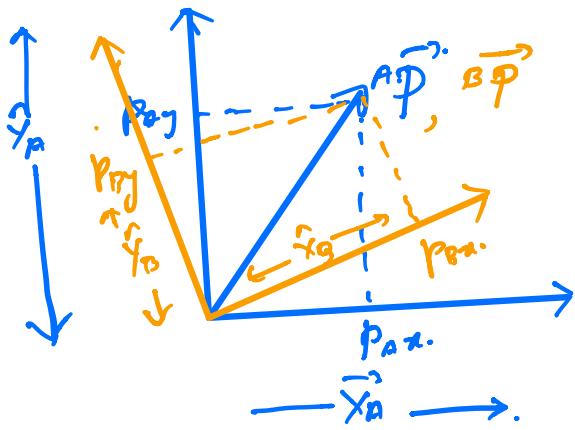
Any Rotation matrix that describes a frame wrt another such as R_B^A can be decomposed into a SEQUENCE of 3 rotation about the moving cartesian axes.

$\overset{A}{\vec{P}}$ → vector in A.

R_B^A → rotation of {B} w.r.t {A}.

T_B^A → Homogeneous transform of B w.r.t A.

$\overset{A}{\vec{t}_B}$ → translation of B w.r.t A and this is a vector in A. and if you want you can write it as $\overset{A}{\vec{t}_B}$ or just $\overset{A}{t_B}$.



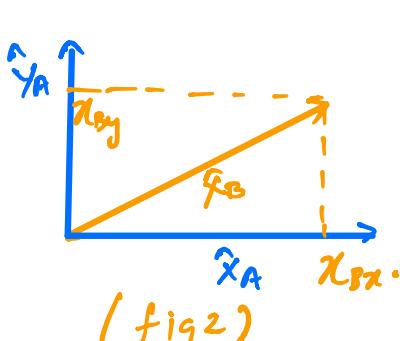
$\overset{A}{\vec{P}}$ is a l.c of the unit vectors that make {A} or

$$\overset{A}{\vec{P}} = \begin{bmatrix} \hat{x}_A & 0 \\ 0 & \hat{y}_A \end{bmatrix} \begin{bmatrix} P_{Ax} \\ P_{Ay} \end{bmatrix}$$

$$= p_{Ax} \hat{x}_A + p_{Ay} \hat{y}_A \rightarrow (1)$$

$$\overset{B}{\vec{P}} = \begin{bmatrix} \hat{x}_B & 0 \\ 0 & \hat{y}_B \end{bmatrix} \begin{bmatrix} P_{Bx} \\ P_{By} \end{bmatrix}$$

$$\overset{B}{\vec{P}} = \begin{bmatrix} \hat{x}_B \\ 0 \end{bmatrix} P_{Bx} + \begin{bmatrix} 0 \\ \hat{y}_B \end{bmatrix} P_{By} \rightarrow (2).$$



$$\hat{x}_B = \begin{bmatrix} \hat{x}_A & 0 \\ 0 & \hat{y}_A \end{bmatrix} \begin{bmatrix} x_{Bx} \\ x_{By} \end{bmatrix} = x_{Bx} \begin{bmatrix} \hat{x}_A \\ 0 \end{bmatrix} + x_{By} \begin{bmatrix} 0 \\ \hat{y}_A \end{bmatrix}$$

~~$$\text{Then } \begin{bmatrix} \hat{x}_B \\ 0 \end{bmatrix} P_{Bx} = \left(x_{Bx} \begin{bmatrix} \hat{x}_A \\ 0 \end{bmatrix} + x_{By} \begin{bmatrix} 0 \\ \hat{y}_A \end{bmatrix} \right) P_{Bx}$$~~

Repeat again: (Use figures above)

$$\overset{B}{\vec{P}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} P_{Bx} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} P_{By} = \begin{bmatrix} P_{Bx} \\ P_{By} \end{bmatrix} \rightarrow (1).$$

where $\hat{x}_B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\hat{y}_B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \rightarrow (2)$.

$$\text{or } \vec{P}_B = P_{Bx} \hat{x}_B + P_{By} \hat{y}_B \rightarrow (3)$$

Now ${}^A\hat{x}_B = \begin{bmatrix} x_{Bx} \\ x_{By} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} x_{Bx} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} x_{By} \rightarrow (4)$

see fig 2

where $\hat{x}_A = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \hat{y}_A$

$$\text{or } \vec{P}_B = P_{Bx} \left(\begin{bmatrix} x_{Bx} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ x_{By} \end{bmatrix} \right) + P_{By} \hat{y}_B \rightarrow (5)$$

$${}^A\hat{y}_B = \begin{bmatrix} y_{Bx} \\ y_{By} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} y_{Bx} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} y_{By} \rightarrow (6)$$

Then $\vec{P}_B = P_{Bx} \begin{bmatrix} x_{Bx} \\ x_{By} \end{bmatrix} + P_{By} \begin{bmatrix} y_{Bx} \\ y_{By} \end{bmatrix} \rightarrow (7)$

similar to fig 2 component of \hat{y}_B in (4)

$$\vec{P}_{B||} = \begin{bmatrix} x_{Bx} & y_{Bx} \\ x_{By} & y_{By} \end{bmatrix} \begin{bmatrix} P_{Bx} \\ P_{By} \end{bmatrix} \rightarrow (8)$$

$$\begin{bmatrix} P_{Ax} \\ P_{Ay} \end{bmatrix} = \begin{bmatrix} \hat{x}_B \cdot \hat{x}_A & \hat{y}_B \cdot \hat{x}_A \\ \hat{x}_B \cdot \hat{y}_A & \hat{y}_B \cdot \hat{y}_A \end{bmatrix} \begin{bmatrix} P_{Bx} \\ P_{By} \end{bmatrix} \rightarrow (9)$$

$$\vec{P}_A = R_B^A \vec{P}_B \longrightarrow (10)$$

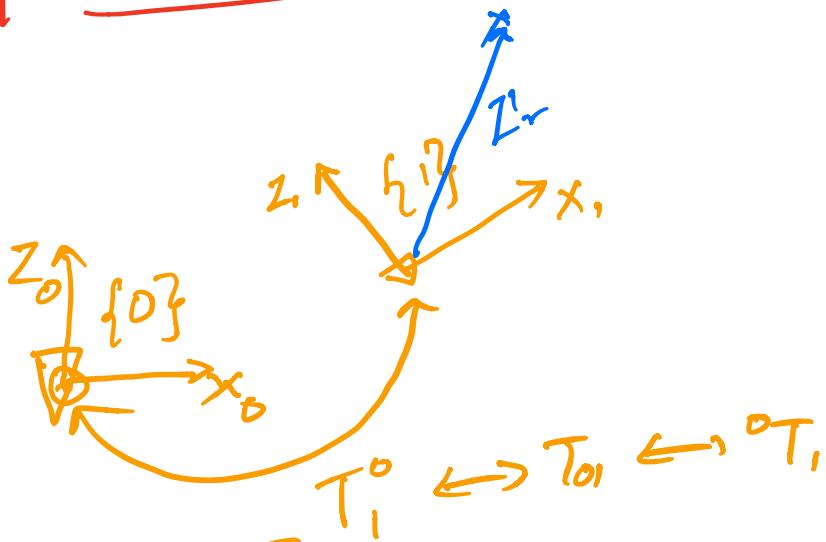
The LHS of (10) is actually P_B as represented in A or actually P_A .

Here $R_B^A \in SO(2)$. (in (10))

can be extended to 3 dimensional case

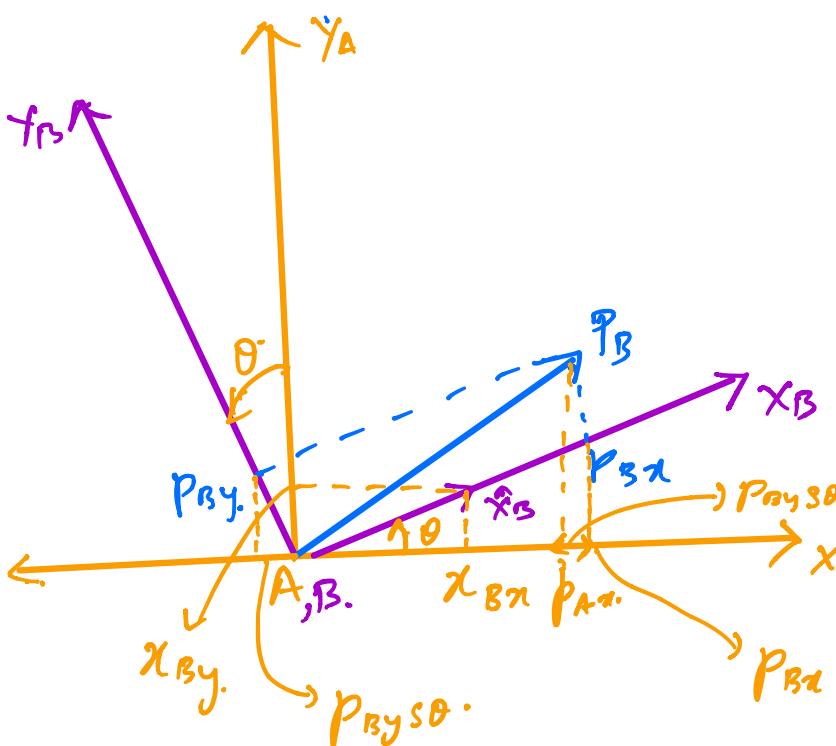
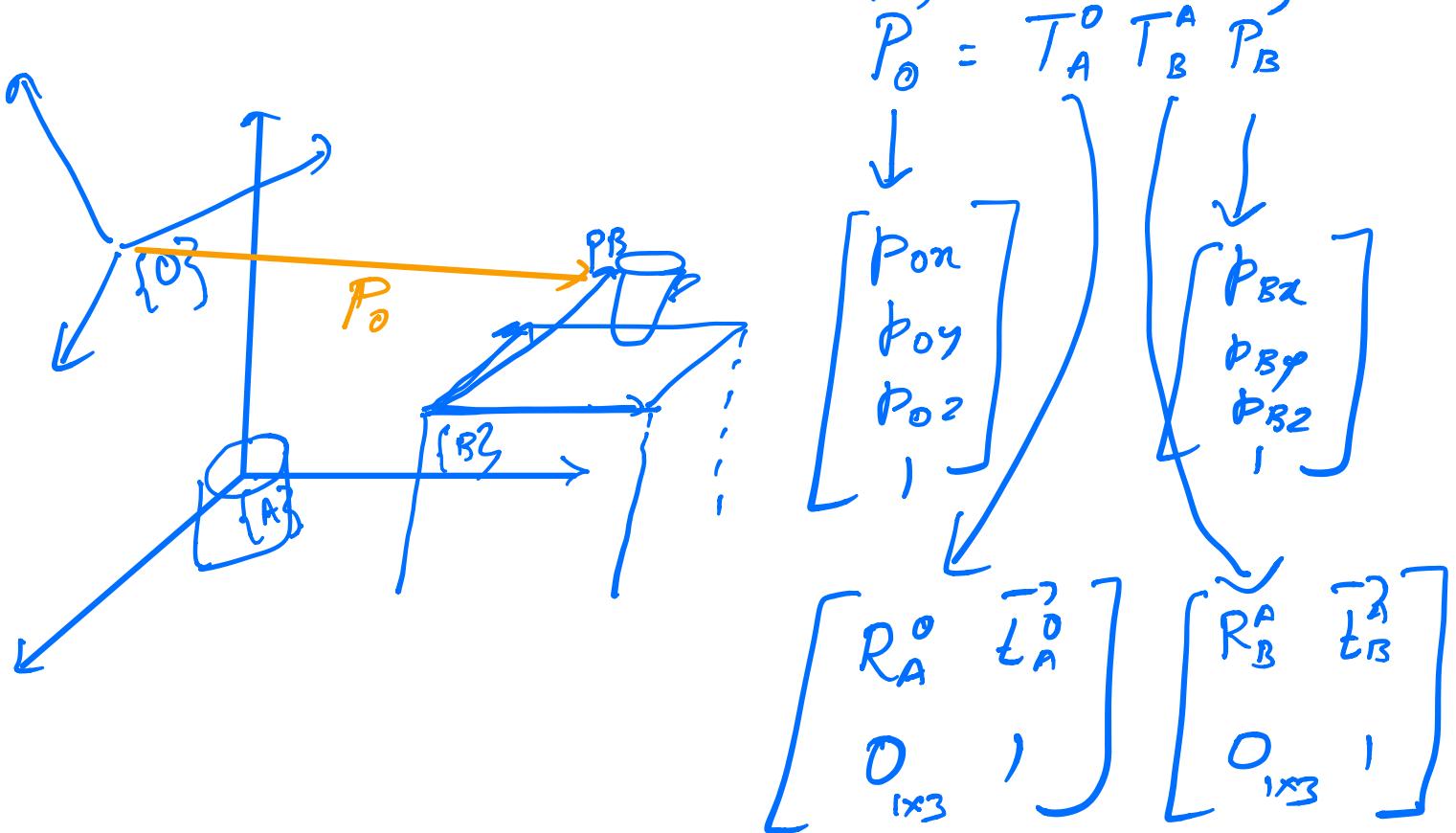
$$\text{as } {}^A P = \begin{bmatrix} P_{Ax} \\ P_{Ay} \\ P_{Az} \end{bmatrix} = \begin{bmatrix} \hat{x}_B \cdot \hat{x}_A & \hat{y}_B \cdot \hat{x}_A & \hat{z}_B \cdot \hat{x}_A \\ \hat{x}_B \cdot \hat{y}_A & \hat{y}_B \cdot \hat{y}_A & \hat{z}_B \cdot \hat{y}_A \\ \hat{x}_B \cdot \hat{z}_A & \hat{y}_B \cdot \hat{z}_A & \hat{z}_B \cdot \hat{z}_A \end{bmatrix} \begin{bmatrix} P_{Bx} \\ P_{By} \\ P_{Bz} \end{bmatrix}$$

or $\boxed{{}^A \vec{P} = R_B^A \vec{P}_B}$



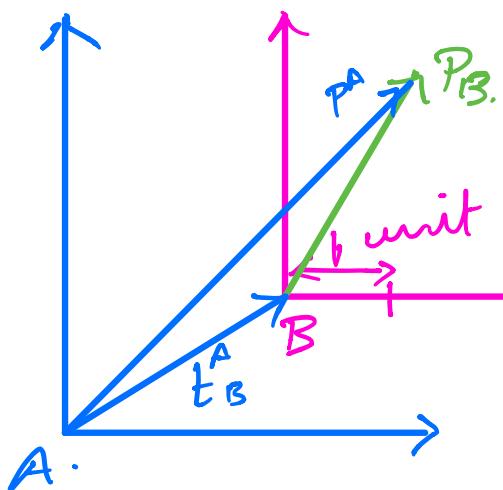
homogeneous transform matrix $(4,4) \leftrightarrow 4 \times 4 \leftrightarrow [4,4]$

$$T_1^0 = \begin{bmatrix} R_1^0 & t_{21} \\ 0_{3 \times 3} & 1_{1 \times 1} \end{bmatrix}$$



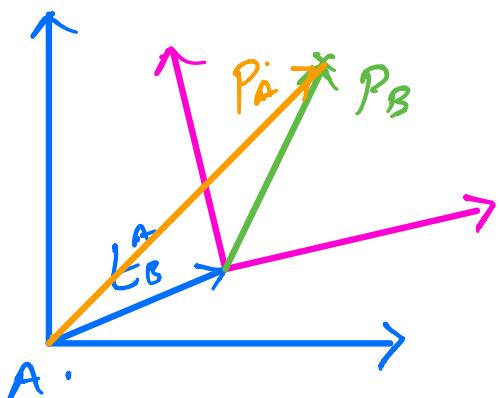
From (q)

Scaling Transformation :



Then point P_B in B
is $t_B^A + sP_B$ in A.

$$P_A = sP_B + t_B^A \rightarrow (1)$$



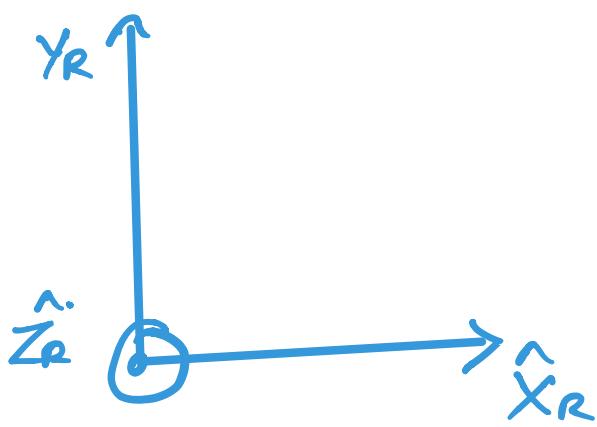
In the presence of
rotation.

$$P_A = sR_B^A P_B + t_B^A \rightarrow (2)$$

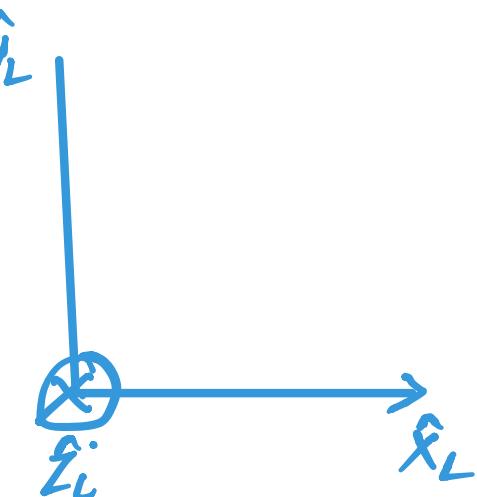
Left Handed to Right Handed

Transformations

RHC:



LHC



Clearly from above

$$\begin{aligned}\hat{x}_L &= \hat{x}_R \\ \hat{y}_L &= \hat{y}_R \\ \hat{z}_L &= -\hat{z}_R\end{aligned}$$

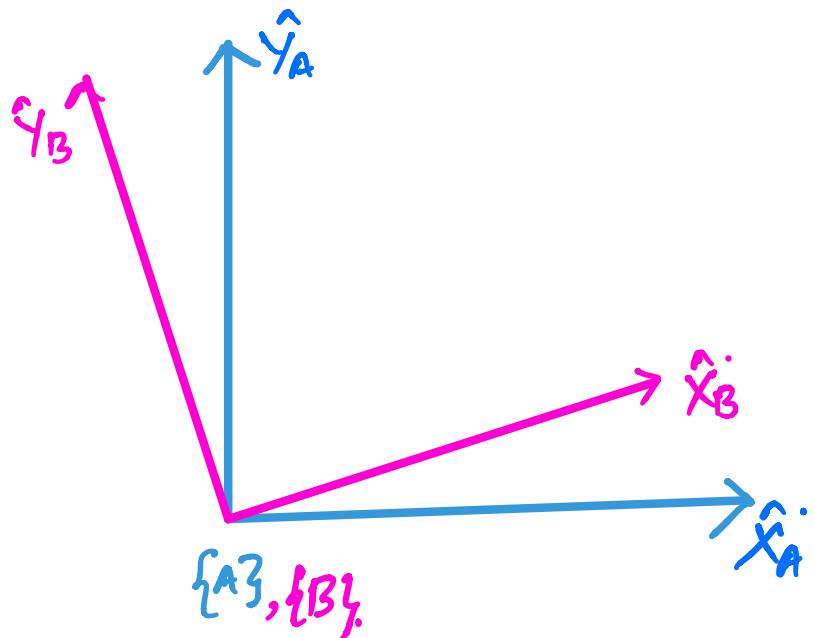
Or more formally:

$$\begin{bmatrix} \hat{x}_R \\ \hat{y}_R \\ \hat{z}_R \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \hat{x}_L \\ \hat{y}_L \\ \hat{z}_L \end{bmatrix}$$

$\underbrace{\quad}_{\gamma_{RL}}$

γ_{RL} is the transformation from LHC to RHC.

Understanding Rotation Matrix:



Write \hat{x}_B, \hat{y}_B in terms of \hat{x}_A, \hat{y}_A

$$\hat{x}_B = [\hat{x}_B \cdot \hat{x}_A] \hat{x}_A + [\hat{x}_B \cdot \hat{y}_A] \hat{y}_A \rightarrow (1)$$

$$\hat{y}_B = [\hat{y}_B \cdot \hat{x}_A] \hat{x}_A + [\hat{y}_B \cdot \hat{y}_A] \hat{y}_A \rightarrow (2)$$

Then R_{BA} or the Rotation Matrix of frame $\{{\text{A}}^3$ wrt $\{{\text{B}}^3\}$ is

$$R_{BA} \text{ or } R_A^B = \begin{bmatrix} \hat{x}_B \cdot \hat{x}_A & \hat{x}_B \cdot \hat{y}_A \\ \hat{y}_B \cdot \hat{x}_A & \hat{y}_B \cdot \hat{y}_A \end{bmatrix}$$

$$\text{or } R_A^B = \begin{bmatrix} \hat{x}_A \cdot \hat{x}_B & \hat{y}_A \cdot \hat{x}_B & \hat{z}_A \cdot \hat{x}_B \\ \hat{x}_A \cdot \hat{y}_B & \hat{y}_A \cdot \hat{y}_B & \hat{z}_A \cdot \hat{y}_B \\ \hat{x}_A \cdot \hat{z}_B & \hat{y}_A \cdot \hat{z}_B & \hat{z}_A \cdot \hat{z}_B \end{bmatrix}$$

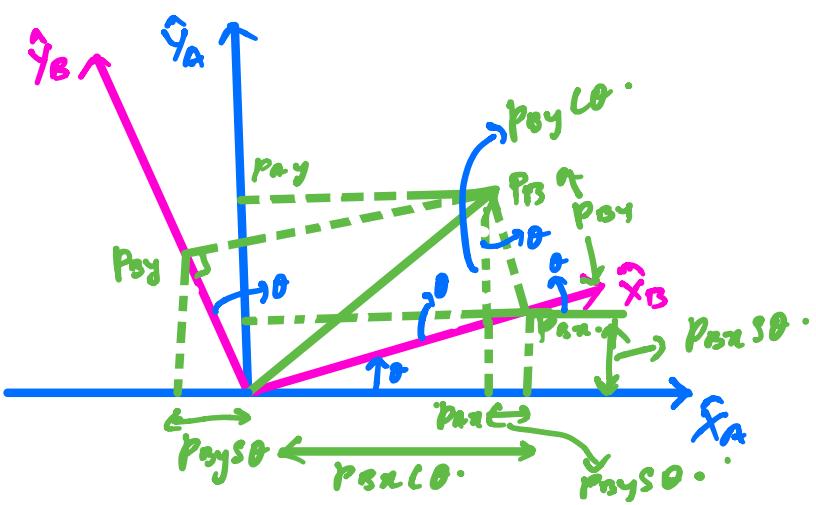
→ (3)

$$= [\hat{x}_A^B \quad \hat{y}_A^B \quad \hat{z}_A^B] \text{ (column vectors)}$$

The row vector expresses $\{B\}$ in terms of $\{A\}$. The column vector expresses $\{A\}$ in $\{B\}$ or R_{BA} .

Evidently $R_{BA} = R_{AB}^T$.

Through Trigonometry :



$$P_{Ax} = P_{Bx} \cos \theta - P_{By} \sin \theta \quad \rightarrow (1).$$

$$P_{Ay} = P_{Bx} \sin \theta + P_{By} \cos \theta \quad \rightarrow (2).$$

Then

$$\begin{bmatrix} P_{Ax} \\ P_{Ay} \end{bmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} P_{Bx} \\ P_{By} \end{bmatrix}$$

$\overbrace{R_B^A \text{ or } R_{AB}}$

R_B^A or R_{AB} .

$$\begin{bmatrix} P_{Ax} \\ P_{Ay} \end{bmatrix} = \begin{bmatrix} \hat{x}_B \cdot \hat{x}_A & \hat{y}_B \cdot \hat{x}_A \\ \hat{x}_B \cdot \hat{y}_A & \hat{y}_B \cdot \hat{y}_A \end{bmatrix} \begin{bmatrix} P_{Bx} \\ P_{By} \end{bmatrix} \rightarrow (3).$$

$\overbrace{R_B^A \text{ or } R_{AB}}$

Note that (3) & (4) are ONE & the SAME.

In \mathbb{R}^3

$$\vec{P}_A^4 \begin{bmatrix} P_{Ax} \\ P_{Ay} \\ P_{Az} \end{bmatrix} = \begin{bmatrix} \hat{x}_B \cdot \hat{x}_A & \hat{y}_B \cdot \hat{x}_A & \hat{z}_B \cdot \hat{x}_A \\ \hat{x}_B \cdot \hat{y}_A & \hat{y}_B \cdot \hat{y}_A & \hat{z}_B \cdot \hat{y}_A \\ \hat{x}_B \cdot \hat{z}_A & \hat{y}_B \cdot \hat{z}_A & \hat{z}_B \cdot \hat{z}_A \end{bmatrix} \begin{bmatrix} P_{Bx} \\ P_{By} \\ P_{Bz} \end{bmatrix}$$

$\overbrace{R_B^A \text{ or } R_{AB}}$