OAS - Assignment - 2 : Roll no: 2021102016

1. The fluid eq^n is given as:

$$\dot{\rho} + \frac{3\dot{a}}{a}(P+\rho) = 0$$

For cold matter, $P = 0$

$$\Rightarrow \dot{\rho} + \frac{3\dot{a}}{a}\rho = 0$$

or $$\frac{d\rho}{\rho} = -\frac{3\,da}{a}$$

Integrating both sides:

$\ln\rho = -3\ln a + C$

or $\rho = e^{C} e^{-3\ln a}$

Let $C = \ln K \Rightarrow \ln\rho = \ln a^{-3} + \ln K$ (K is any constant)

$$\Rightarrow \rho = K a^{-3}$$

$$\Rightarrow \boxed{\rho \propto a^{-3}}$$

2. For matter dominated universe, $K=0$, $\Lambda = 0$ & no radiation,

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G\rho}{3} \quad \text{(Friedmann eq}^n\text{)}$$

From Q1, we know $\rho = \rho_0\left(\frac{a_0}{a}\right)^3$

$$\Rightarrow \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho_0\left(\frac{a_0}{a}\right)^3$$

Let $H_0^2 = \frac{8\pi G\rho_0}{3}$

$$\Rightarrow \left(\frac{\dot{a}}{a}\right)^2 = H_0^2\left(\frac{a_0}{a}\right)^3$$

or $\frac{\dot{a}}{a^{-1/2}} = H_0 a_0^{3/2}$ (Taking square root)

Integrating both sides

$$\int a^{1/2}\,da = H_0 a_0^{3/2}\int dt \Rightarrow \frac{2}{3}a^{3/2} = H_0 a_0^{3/2}\,t$$

or $a(t) = a_0\left(\frac{3}{2}H_0 t\right)^{2/3}$

Thus, $a(t) \propto t^{2/3}$

$\Rightarrow$ In matter dominated universe,

$$\frac{a(t)}{a(t_0)} = \left(\frac{t}{t_0}\right)^{2/3} \quad \text{or} \quad \boxed{a(t) = a_0\left(\frac{t}{t_0}\right)^{2/3}}$$

3. Friedmann eq^n

$$H^2 = H_0^2\left[\Omega_\Lambda^\circ + \Omega_K^\circ\left(\frac{a_0}{a}\right)^2 + \Omega_M^\circ\left(\frac{a_0}{a}\right)^3 + \Omega_R^\circ\left(\frac{a_0}{a}\right)^4\right]$$

Let $x = \frac{a}{a_0} = \frac{1}{1+z}$ $\Rightarrow$ $da = a_0 dx$ or $da = \frac{a}{x}dx \Rightarrow \frac{da}{a} = \frac{dx}{x}$

Also, $H = \frac{\dot{a}}{a} = \frac{da}{a\,dt}$, we can write

$$H = \frac{dx}{x\,dt}$$

$$\Rightarrow dt = \frac{dx}{xH} = \frac{dx}{xH_0\sqrt{\Omega_\Lambda^\circ + \Omega_K^\circ x^{-2} + \Omega_M^\circ x^{-3} + \Omega_R^\circ x^{-4}}}$$

To integrate, at $t=0$, $z=\infty$, $\Rightarrow x=0$, at present $z=0$, $\Rightarrow x=1$

Thus, $$t = \int_0^1 \frac{dx}{H_0 x\sqrt{\Omega_\Lambda^\circ + \Omega_K^\circ x^{-2} + \Omega_M^\circ x^{-3} + \Omega_R^\circ x^{-4}}}$$

On calculating this integral we get $t = 13.8$ billion yrs

(code submitted)