The Universe Across Scales

SC1.308 - Spring 2024-25

Assignment for Module 3: Relativity & Quantum Mechanics

Submission Deadline: May 5, 2025

1. Let's get a feel for the ubiquitous constant quantity we encounter—the speed of light c. We all know what a lightyear is, it is the distance that light travels in a year. Let us make up some other quantities and understand them.
   (a) How many meters are in a light-day?
   (b) How many kilometers is a light-microsecond?
   (c) How many years in a light-parsec? (1 parsec = 3.09×10^{16} m)
   (d) How many seconds in a light-fermi? (1 fermi = 10^{-15} m, $\sim$ the size of the atomic nucleus.)

2. Observer O assigns the following spacetime coordinates to an event:

   $(x, y, z, t) = (100$ km, 10 km, 55 km, 2 $ms)$

   (a) What are the coordinates of this event in frame O', which moves in the positive x direction with speed $0.95c$?

   Assume that the origins of these frames coincide at $t = t' = 0$.

   (b) Check your answers by using the inverse Lorentz transformation equations to obtain the original data.

3. We encountered a particle called muon in the class. They are the heavier counterparts of electrons and are produced when high-energy cosmic rays strike the top of the atmosphere. Despite having a short half-life (1.56 μs in their rest frame) they are detected on the surface of the earth. Let us see how.

   The muons travel very close to the speed of light, say at $0.98c$. Assume the distance from the edge of the atmosphere to the surface of the earth to be $\sim$ 10 km.

   Non-relativistic scenario:

   (a) How many half-lives would it take for a muon to reach the ground from the edge of the atmosphere?

   (b) The number of muons surviving after t half-lives, N, can be calculated as

   $$\frac{N}{N_0} = 2^{-t},$$

   where N_0 is the number of muons we start with.

   Suppose 1 million muons are entering the atmosphere, all of them travelling at a speed of $0.98c$ towards the earth. How many of them would survive for us to detect them at the surface?

   With Relativity:

   (c) Calculate the Lorentz factor (γ) for the muons.

   (d) What is the half-life of a muon as measured by an observer standing on the surface of the earth?

   (e) Using the same formula given in (b), calculate the number of muons reaching the surface of the earth in the same scenario.

   (f) What would an observer from a muon's reference frame see? Explain.

4. (a) What potential difference would accelerate an electron to speed c according to classical physics?

   (b) With this potential difference, what speed would the electron actually attain?

5. An electron is moving at a speed of $0.95c$ in a vacuum tube of length 3 meters, as measured in the rest frame of the laboratory.

   Calculate the proper length of the vacuum tube for the electron.

6. A clock moves along the x-axis at $0.6c$ and reads zero as it passes the origin.

   (a) Calculate the clock's Lorentz factor.

   (b) What time does the clock read as it passes $x = 240$ m?

7. An aeroplane whose rest length is 40 m is moving at a uniform velocity with respect to the earth at a speed of 630 m/s.

(a) By what fraction of its rest length will it appear to be shortened to an observer on Earth?

(b) How long would it take by earth clocks for the aeroplane's clock to fall behind by 1 μs?

8. Let O be a stationary observer and O' be an observer who is moving with respect to O along the x-axis. Observer O detects two flashes of light: one at $x_1 = 1200$ m and another at $x_2 = 480$ m, 5 μs after the first one. O' detects both the flashes at the same location x'.

   (a) Calculate the speed of O'.

   (b) Is O' moving in the positive or negative x-direction?

   (c) Which flash occurs first for O'?

   (d) What is the time interval between the flashes, as measured by O'?

9. Observer O' sees a particle moving with a velocity of $0.40c$. Observer O sees that O' move with a velocity $0.60c$ with respect to them.

   What is the velocity of the particle as observed by O?

10. As you read this page on a screen, a cosmic ray proton passes along the left–right width of the page with relative speed v and a total energy of 14 nJ. Take that left–right width to be 21.0 cm.

    (a) What is the width according to the proton's reference frame?

    (b) Calculate the time elapsed for the proton's journey in your frame.

    (c) Calculate the time elapsed for the proton's journey in the proton's rest frame.

11. Let m be the mass of a particle, p be the magnitude of its momentum and T its kinetic energy.

    (a) Show that $m = \dfrac{(pc)^2 - T^2}{2Tc^2}$.

    (b) Show that as the speed of the particle tends to zero, the right-hand side reduces to m.

    (c) How much more massive is this particle compared to an electron, if its $T = 55$ MeV and $p = 121$ MeV/c? (Mass of the electron = 0.510 MeV/c^2.)

Spacetime Diagram

Spacetime diagrams are a neat way to visualise the concepts of relativity. Let us consider a two-dimensional cartesian system, with ct along the vertical axis and x along the horizontal axis (we ignore the y and z coordinates for simplicity). Let us work with a set of units where the speed of light, $c = 1$; this set of units is called natural units. Clearly, both axes have units of distance.

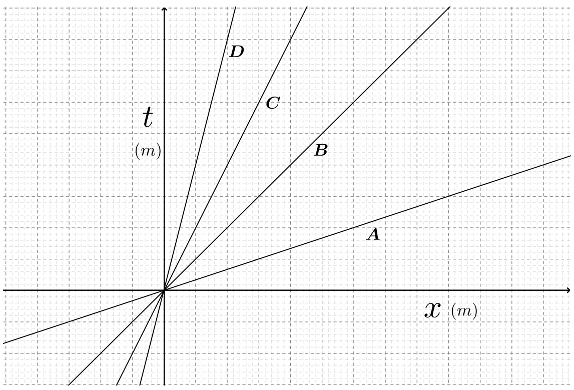

Figure 1: Spacetime diagram in natural units, i.e., $c = 1$. Lines $A, B, C,$ and D are worldlines of particles $A, B, C,$ and D, respectively. See more at, e.g., Wikipedia or here.

The key point is that a line on the spacetime diagram, $x \equiv x(t)$, shows the position of a particle at different times. This is called the particle's *world line*.

Answer the following questions:

1. How is the slope of a particle's world line related to its velocity?

2. Redraw the spacetime diagram in Figure 1 and identify the world lines of light, the particle(s) travelling faster than light and the particle(s) travelling slower than light.

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12. At time t = 0 a particle is represented by the wave function

$$\Psi(x,0) = \begin{cases} Ax/a, & \text{if } 0 \le x \le a \\ A(b-x)/(b-a), & \text{if } a \le x \le b \\ 0, & \text{otherwise} \end{cases}$$

where A, a, and b are constants.

   (a) Normalize Ψ (that is, find A in terms of a and b).
   (b) Sketch $\Psi(x,0)$ as a function of x.
   (c) Where is the particle most likely to be found, at $t = 0$?
   (d) What is the probability of finding the particle to the left of a? Check your result in the limiting cases $b = a$ and $b = 2a$.

13. Show that when you add a constant V_0 to the potential energy (constant means independent of x and t) the wave function picks up a time-dependant phase factor: $\exp(-iV_0t/\hbar)$.

14. Show that for a 1D infinite potential well

$$\frac{d}{dt}\int_{-\infty}^{\infty} \Psi_1^* \Psi_2 \, dx = 0$$

for any two solutions to the Schrodinger equation.

15. At time t = 0 a particle is represented by the wave function

$$\Psi(x,0) = \begin{cases} A(a^2 - x^2), & \text{if } -a \le x \le a \\ 0, & \text{otherwise} \end{cases}$$

where A, a, are constants.

   (a) Normalize Ψ .
   (b) What is the expectation value of x?
   (c) What is the expectation value of p?
   (d) What is the expectation value of x^2?
   (e) What is the expectation value of p^2?
   (f) Find the uncertainty in x.
   (g) Find the uncertainty in p.
   (h) Check that your result is consistent with the uncertainty principle.

16. Show that there is no acceptable solution to the time-independent Schrodinger equation for the infinite square well with E=0.

17. A particle of mass m in the infinite square well (of width a) starts out in the state

$$\Psi(x,0) = \begin{matrix} A, & \text{if } 0 \le x \le a/2 \\ 0, & \text{if } a/2 \le x \le a \end{matrix}$$

for some constant A, so it is (at t=0) equally likely to be found at any point in the left half of the well. What is the probability that a measurement of the energy (at some later point t) would yield the value $\pi^2\hbar^2/2ma^2$?

18. Show that the wave function of a particle in the infinite square well returns to its original form after time $T = 4ma^2/\pi\hbar$. That is $\Psi(x,T) = \Psi(x,0)$ for any state (not just a stationary state).

19. Explain why you can copy a single quantum state but cannot copy a linear combination of states.

20. For a spin-half system in a Stern-Gerlach experiment with an angle θ between the two magnets, what angle makes spin-up with respect to the second magnet twice as likely as spin-down?

21. A photon-entanglement experiment takes a millisecond to make a measurement of each pair. How far apart would you need to put the stations to make sure there isn't some causal influence between Alice's measurement and Bob's?

22. Estimate the uncertainty in the position of

   (a) a neutron moving at 10^6 ms^{-1}.
   (b) a 50 kg person moving at 2 ms^{-1}.

23. Explain why an electron cannot reside inside the nucleus.