Fractals

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Introduction

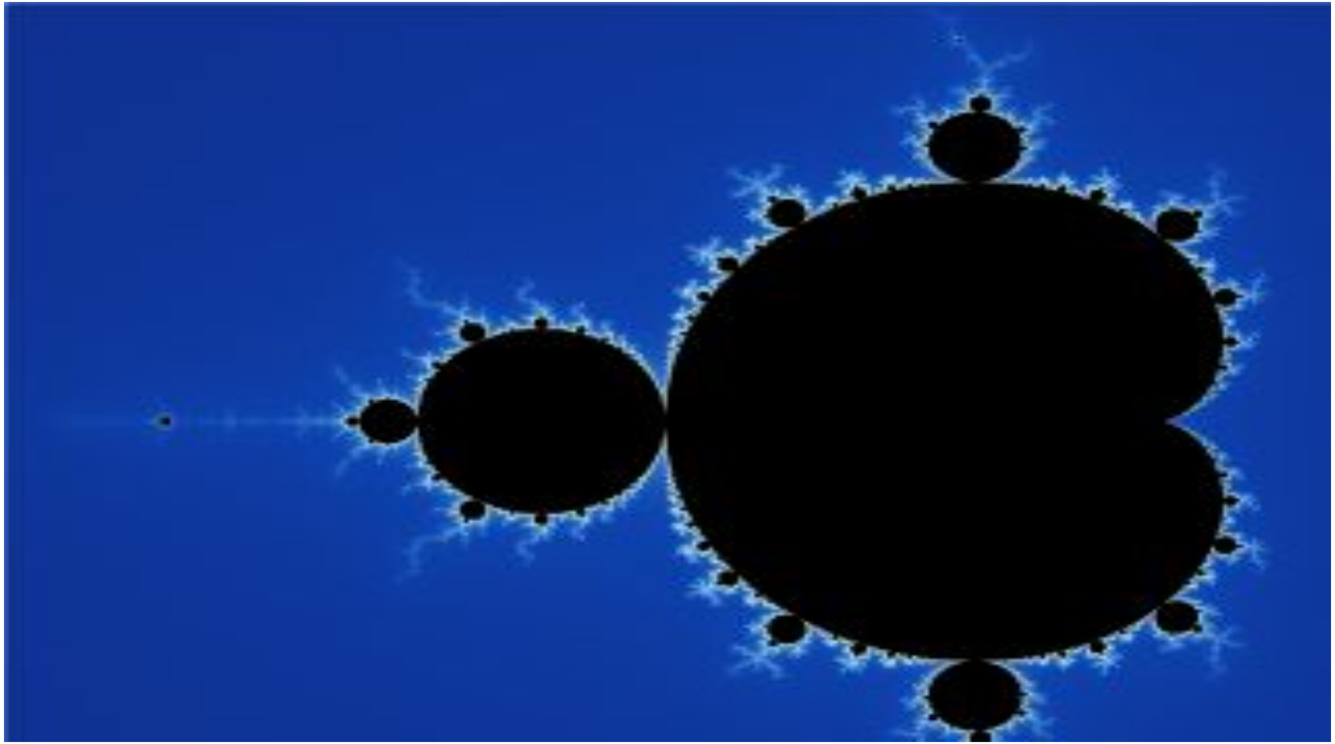
- What do we mean by fractals?
- These are mathematical objects that are self-similar at different scales .This means that they have similar patterns that repeat themselves when we zoom in or zoom out of the objects.
- Some of the very common examples of fractals includes Mandelbrot set, Sierpinski triangle and the koch snowflake.

Fractal dimension

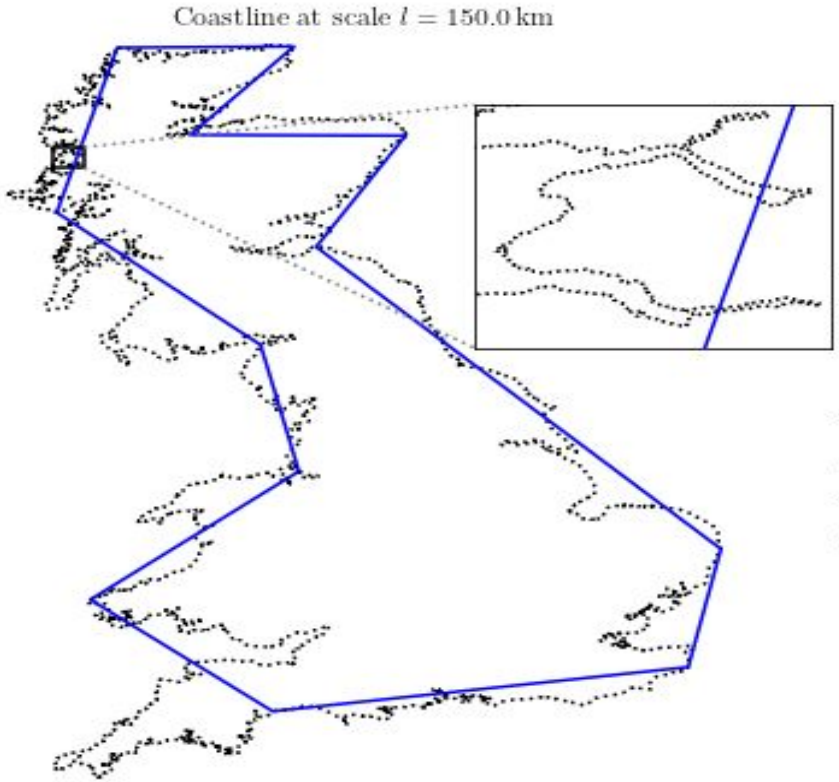
- In mathematics, a **fractal dimension** is a term invoked in the science of geometry to provide a rational statistical index to quantify the level of complexity in a given pattern.
- A fractal pattern changes with the scale at which it is measured. It has also been mythologized as a measure of the space-filling capacity of a pattern that tells how a fractal scales differently and in a fractal dimension, i.e. one that does not have to be an integer.

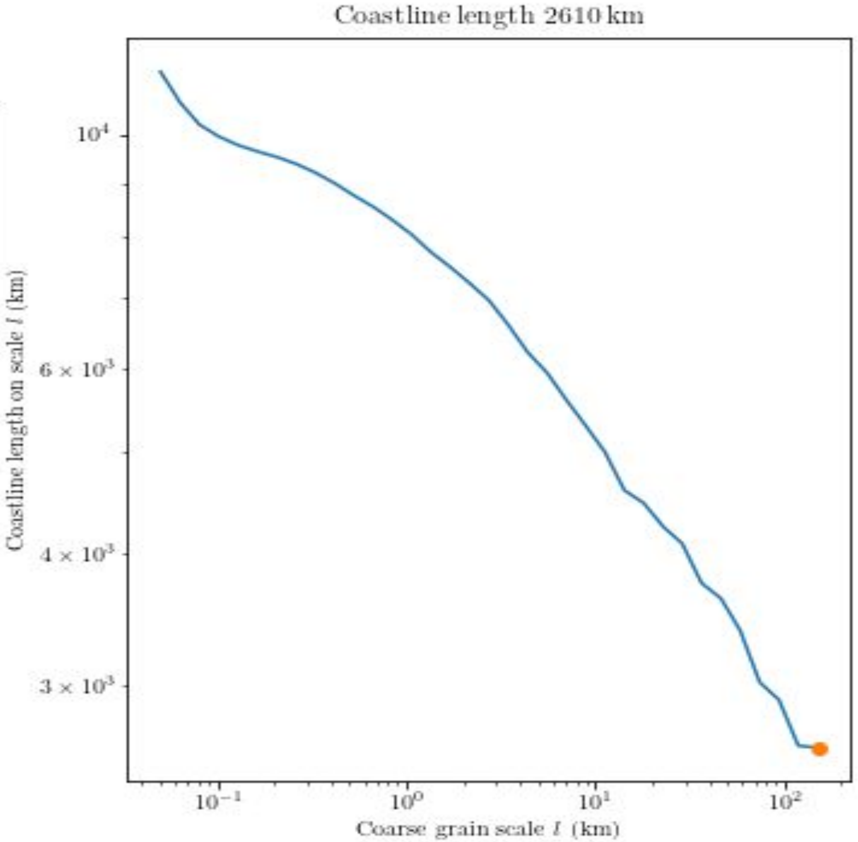
Mandelbrot Set:

It is a complex mathematical set that exhibits self-similarity and complexity at different scales. It has a fractal dimension of approximately 2, meaning that its structure is highly irregular and complex.

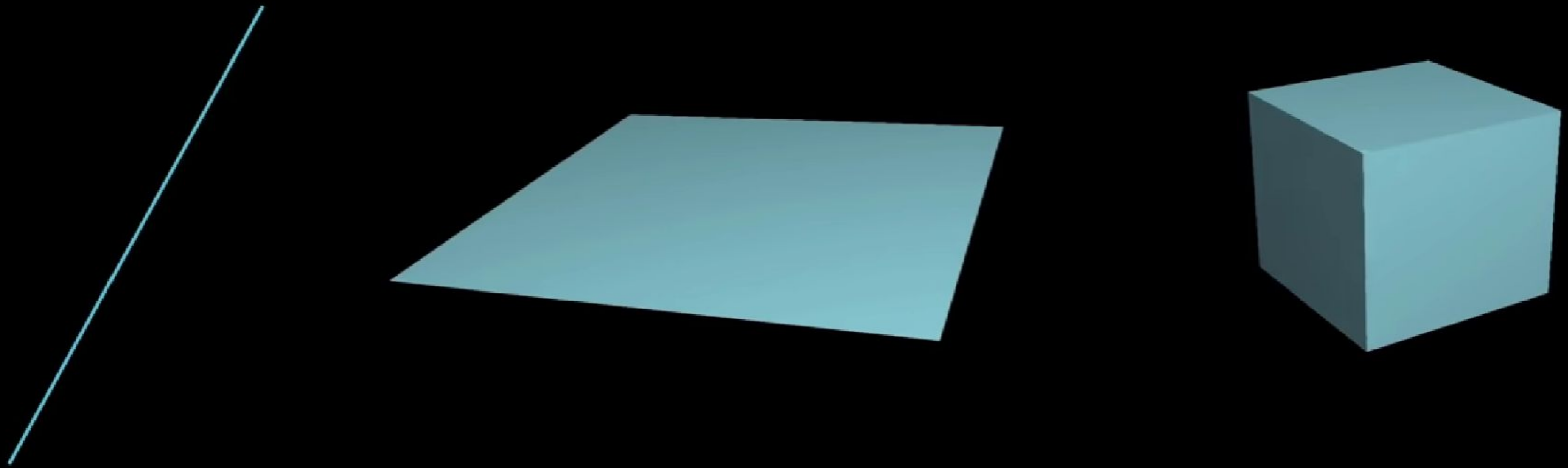


Coastline of Great Britain : A pure fractal





UNDERSTANDING DIMENSION

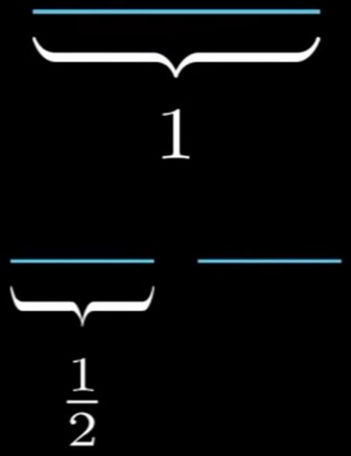


1-dimensional　　2-dimensional　　3-dimensional

Line

Scaling factor: $\frac{1}{2}$

Mass scaling factor: $\frac{1}{2}$



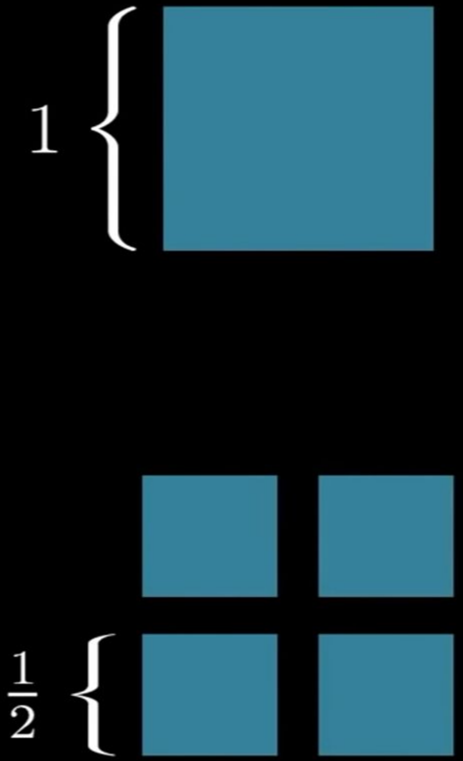
Dimension : 1

Square

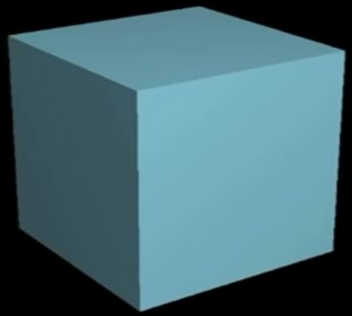
Scaling factor: $\frac{1}{2}$

Mass scaling factor: $\frac{1}{4} = \left(\frac{1}{2}\right)^2$

Dimension : 2

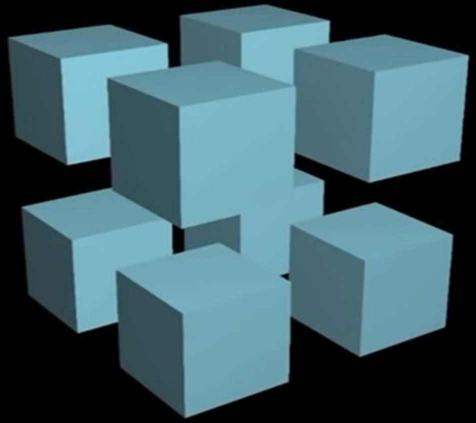


Cube



Scaling factor: $\frac{1}{2}$

Mass scaling factor: $\frac{1}{8} = \left(\frac{1}{2}\right)^3$



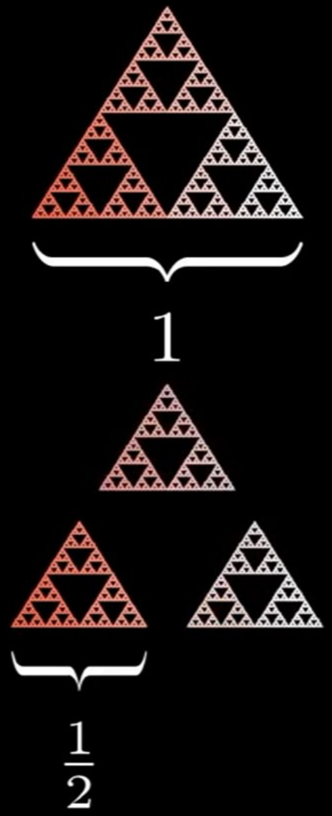
Dimension : 3

Sierpinski

Scaling factor: $\frac{1}{2}$

Mass scaling factor: $\frac{1}{3}$

Dimension : D



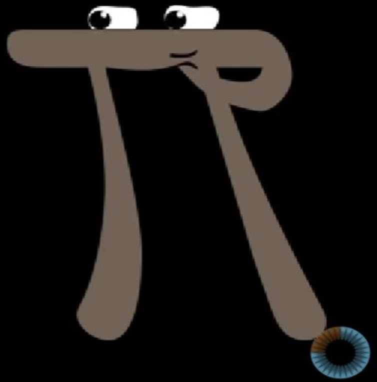
2-dimensional

Length: L

Mass: M

Length: sL

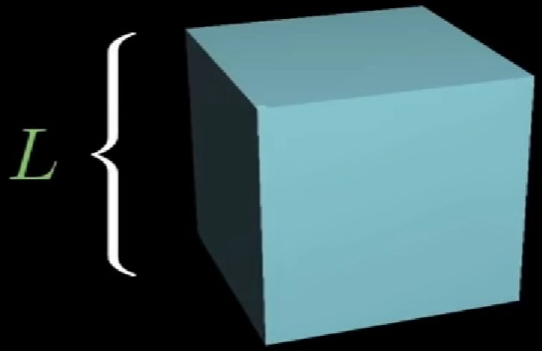
Mass: s^2M



3-dimensional

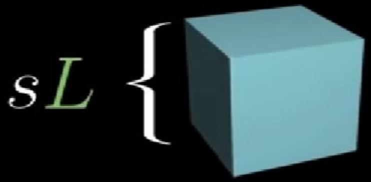
Length: L

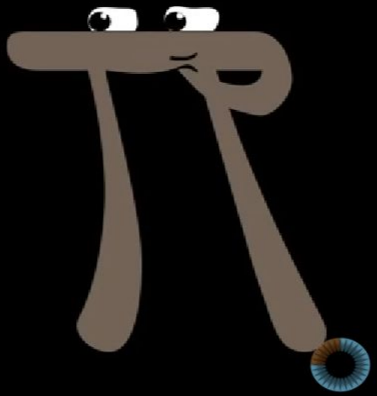
Mass: M



Length: sL

Mass: $s^3 M$



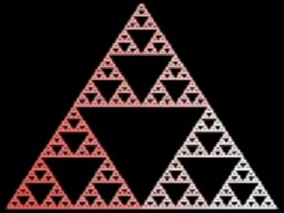


D-dimensional

Length: L

Mass: M

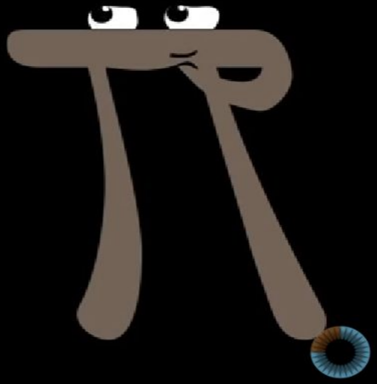
L



Length: $\left(\frac{1}{2}\right)L$

$\left(\frac{1}{2}\right)L$

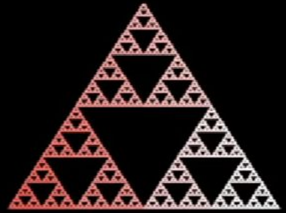
Mass: $\left(\frac{1}{2}\right)^{D}M$



D-dimensional

Length: L

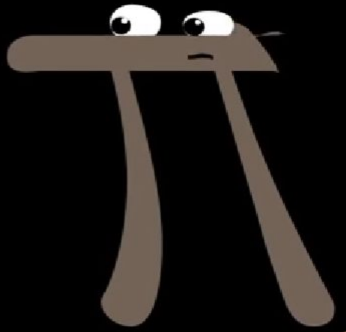
Mass: M

L 

Length: $\left(\frac{1}{2}\right)L$

$\left(\frac{1}{2}\right)L$ 

Mass: $\left(\frac{1}{2}\right)^{D} M = \left(\frac{1}{3}\right) M$

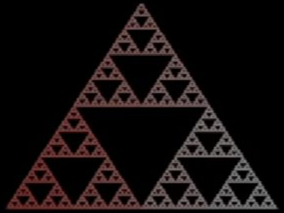


D-dimensional

$$\left(\frac{1}{2}\right)^{D} = \left(\frac{1}{3}\right)$$

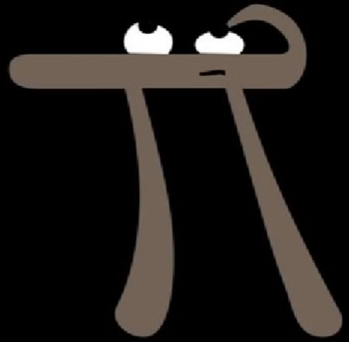
Length: L

Mass: M

L 

Length: $\left(\frac{1}{2}\right)L$

$\left(\frac{1}{2}\right)L$ 



Mass: $\left(\frac{1}{2}\right)^{D} M = \left(\frac{1}{3}\right) M$

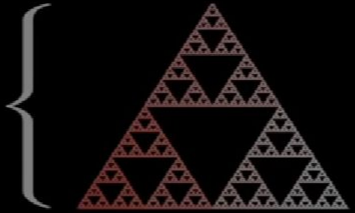
D-dimensional

$$\left(\frac{1}{2}\right)^{D} = \left(\frac{1}{3}\right)$$

$$2^{D} = 3$$

Length: L

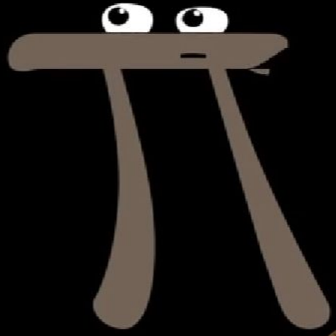
Mass: M

L 

Length: $\left(\frac{1}{2}\right)L$

$\left(\frac{1}{2}\right)L$ 

Mass: $\left(\frac{1}{2}\right)^{D} M = \left(\frac{1}{3}\right) M$

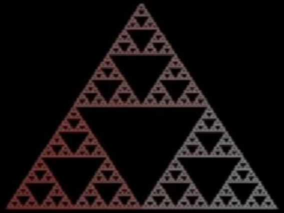


D-dimensional

Length: L

Mass: M

L



$$\left(\tfrac{1}{2}\right)^{D} = \left(\tfrac{1}{3}\right)$$

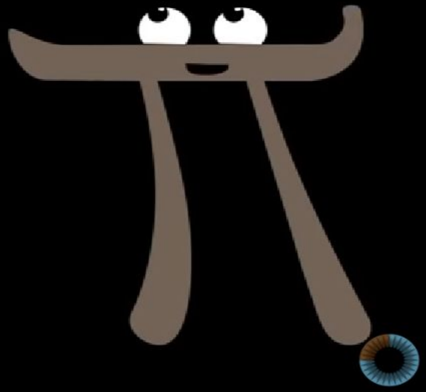
$$2^{D} = 3$$

$$\log_2(3) \approx 1.585$$

Length: $\left(\tfrac{1}{2}\right)L$

$\left(\tfrac{1}{2}\right)L$

Mass: $\left(\tfrac{1}{2}\right)^{D} M = \left(\tfrac{1}{3}\right) M$

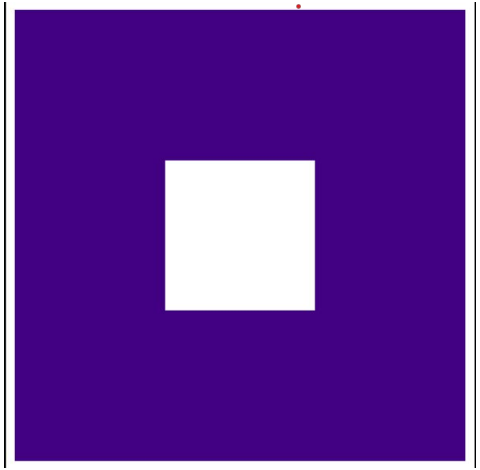
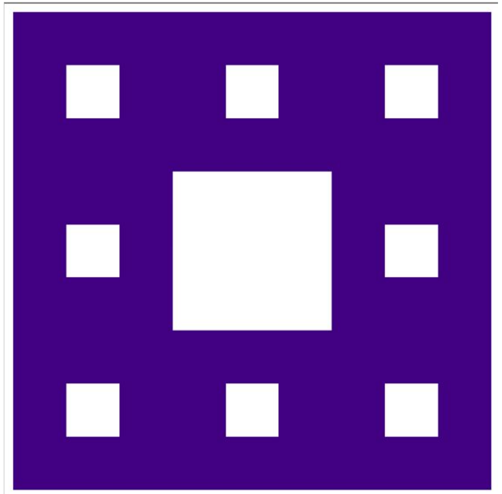
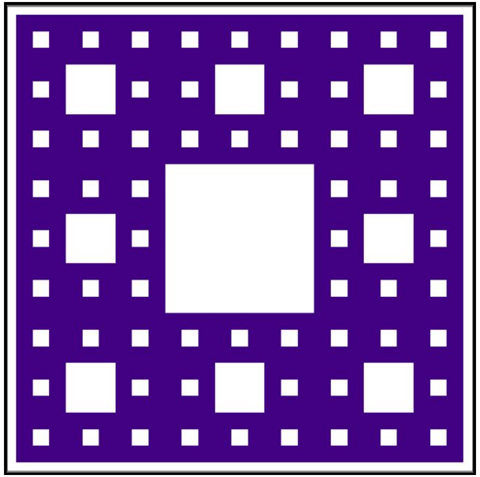
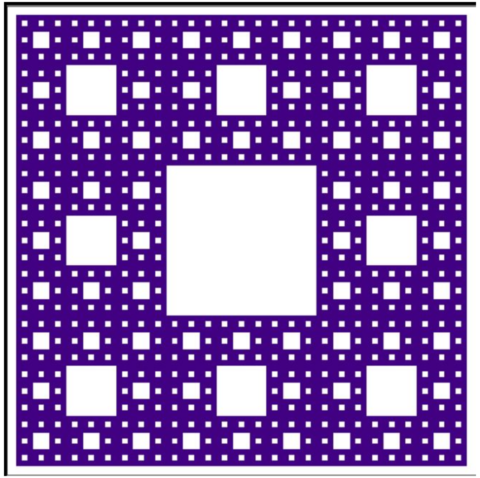
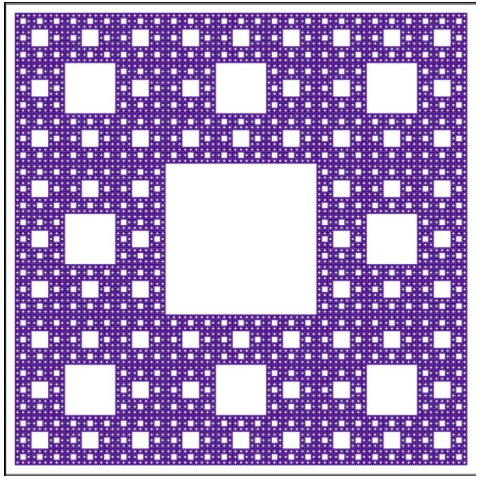


SEIRPINSKI CARPET

- The **Sierpiński carpet** is a plane fractal first described by Wacław Sierpiński in 1916.

CONSTRUCTION

- The construction of the Sierpiński carpet begins with a square.
- The square is cut into 9 congruent subsquares in a 3-by-3 grid,, and the central subsquare is removed.
- The same procedure is then applied recursively to the remaining 8 subsquares, ad infinitum.
- The Hausdorff dimension of carpet is log(8) / log(3) ≈ 1.8928.

Working of Fractal Dimension Algorithm

For a $M \times M$ pixel block image, let $I(x, y)$ be the pixel intensity and x; y 1⁄4 0; 1; 2......M−1. The following items are defined:

- Scale : distance Δr between two pixels. The unit of distance is pixel.
- NSR : normalized scale range vector. It consists of reference integer scale and is monotonic increasing. $NSR = [ndr(1), ndr(2), \ldots, ndr(k) \ \ldots ndr(n)]$, where k is an integer $ndr(k) = k$, and the n value depends on practical considerations. In general, for an $M \times M$ image, we can select it to be M, then $NSR = [1, 2, 3, 4 \ldots, M]$

- NPN: normalized pixel pair number vector. It consists of elements which are the numbers of pixel pairs with scale (distance) values whose integer parts are the same integer reference scale.

$$NPN = \left[npn(1), npn(2), \ldots., npn(k), \ldots., npn(n) \right]$$

where npn(k) is the total number of pixel pairs with distance Δr and $k < \Delta r < k + 1$.

- NMSID: normalized multi-scale intensity difference vector. It consists of different absolute intensity difference averages around each normalized reference scale (NSR).

$$NMSID = \left[ndi(1), ndi(2), \ldots ndi(k), \ldots, ndi(n) \right]$$

where is the average of absolute intensity difference of all pixel pairs with distance values whose integer parts are .

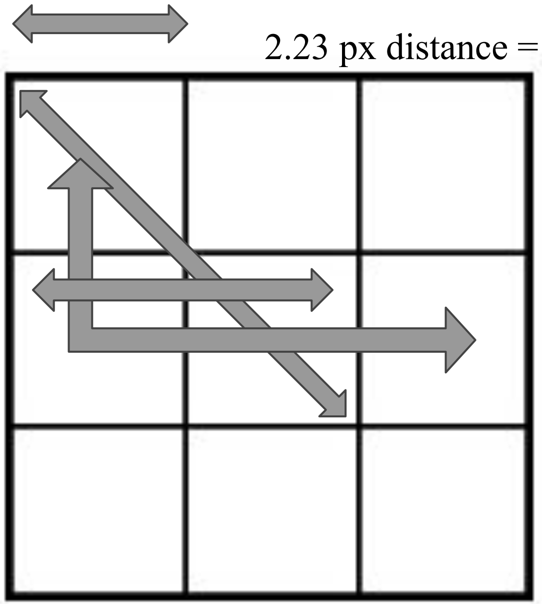
❑ NSR

1 pixel unit distance = NSR(1)

2.23 px distance = NSR(2)

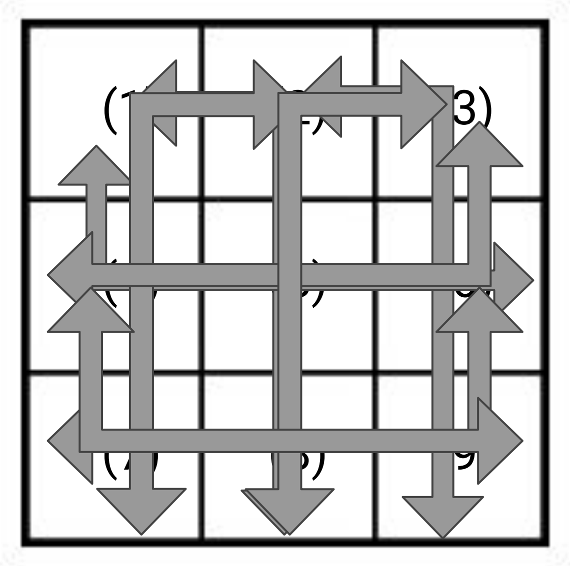
1.414 px unit distance = NSR(1)

2 px distance = NSR(2)



And so on ….

❑ NPN



NPNset(2)={(1,8)

(1,6)

(2,7)

(2,9)

(3,4)

(3,8)

(6,7)

(4,9)

}

Applications:

- Image processing and computer vision :

  Used for feature extraction, image segmentation, and pattern recognition.

- Medical Imaging :

  Used in medical imaging for the analysis of X-ray, CT, MRI, and other medical images. For example, fractal dimensions can be used to detect and diagnose diseases such as osteoporosis and cancer.

- Material sciences :

  Used in material sciences to study the microstructure and mechanical properties of materials such as ceramics, metals, and polymers.

Bibliography

1. 3b1b Youtube channel
2. Wikipedia
3. University of California State Notes

THANK YOU!!