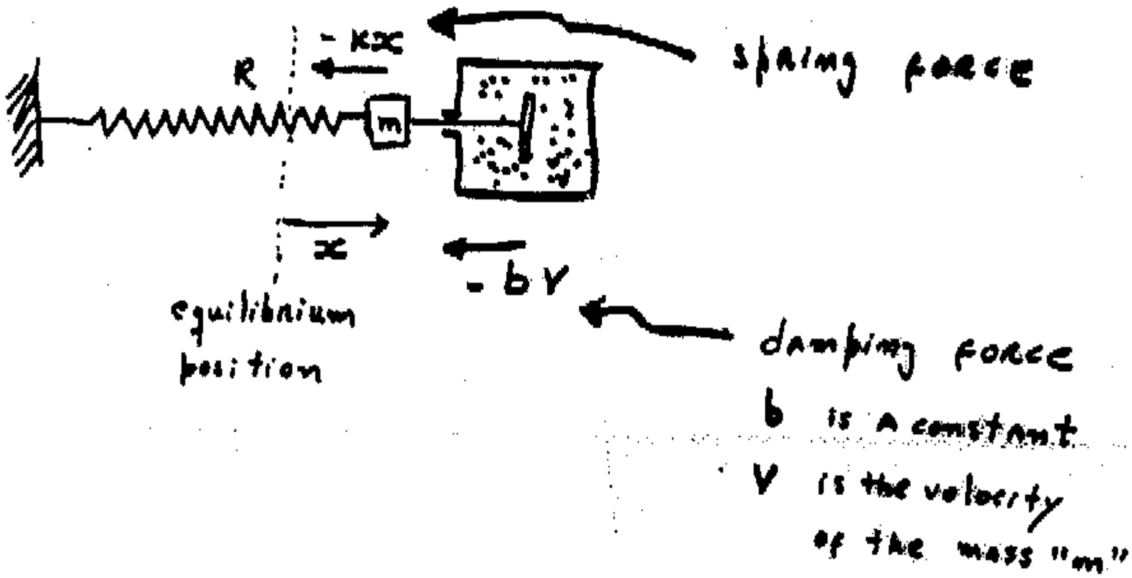
Lecture Notes A. La Rosa

PH-213

DAMPED SIMPLE HARMONIC MOTION



Forces acting on the mass "m" $\quad F = -kx - bv$

$$F = ma = m\frac{d^2x}{dt^2} \quad \text{implies}$$

$$m\frac{d^2x}{dt^2} + bv + kx = 0$$

$$\boxed{m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0}$$

- If b were zero, we know the solution is of the form $x(t) = A\cos(w_0 t + \phi)$.

- From real experience, we know that the amplitude of the oscillations decreases over time. So, maybe A depends on time in the following form $A = e^{-\alpha t}$ (α unknown)

- Also, the angular frequency of the oscillations may not be equal to $w_0 = \sqrt{K/m}$ any more, because the damping effect may tend to slow down the motion.

- Thus, tentatively, for the equation

$$\frac{d^2x}{dt^2} + \frac{b}{m}\frac{dx}{dt} + \frac{K}{m}x = 0 \qquad (1)$$

let's try a solution of the form.

$$x = A_0 e^{-\alpha t}\cos(wt + \phi)$$

α, w and ϕ, A_0 unknown.

(2)

Replacing expression 2 in expression 1, one obtains

$$\underbrace{\left(-w^2+\alpha^2-\alpha\frac{b}{m}+\frac{k}{m}\right)}\cos(wt+\phi)\ +$$

$$+\underbrace{\left(2w\alpha-w\frac{b}{m}\right)}\sin(wt+\phi) =$$

If we required that
these 2 coefficients vanish
(both equal to zero) then
this relationship would be
valid at all times t.

That is exactly what we are going to do

$$\begin{aligned} 2w\alpha - w\frac{b}{m} &= 0 \\ -w^2+\alpha^2-\alpha\frac{b}{m}+\frac{k}{m} &= 0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} \alpha &= \frac{b}{2m} \\ w^2 &= \frac{k}{m}-\left(\frac{b}{2m}\right) \end{aligned}$$

Proof: $x = e^{-at}\cos(wt + \phi)$

$$\dot{x} = e^{-\alpha t}\left[-w\sin(wt+\phi)\right] + \left(-\alpha e^{-\alpha t}\right)\cos(wt+\phi)$$

$$= -w e^{-\alpha t}\sin(wt+\phi) - \alpha e^{-\alpha t}\cos(wt+\phi)$$

$$\ddot{x} = -w e^{-\alpha t} w\cos(wt+\phi) + w\alpha e^{-\alpha t}\sin(wt+\phi)$$

$$+ w\alpha e^{-\alpha t}\sin(wt+\phi) + \alpha^2 e^{-\alpha t}\cos(wt+\phi)$$

$$\ddot{x} = (-w^2+\alpha^2)e^{-\alpha t}\cos(wt+\phi) + 2w\alpha e^{-\alpha t}\sin(wt+\phi)$$

$$\ddot{x} + \frac{b}{m}\dot{x} + \frac{k}{m}x = (-w^2+\alpha^2)e^{-\alpha t}\cos(wt+\phi) + 2w\alpha e^{-\alpha t}\sin(wt+\phi)$$

$$-\alpha\frac{b}{m}e^{-\alpha t}\cos(wt+\phi) - w\frac{b}{m}e^{-\alpha t}\sin wt$$

$$+\frac{k}{m}e^{-\alpha t}\cos wt+\phi$$

$$\underbrace{\left(-w^2+\alpha^2-\alpha\frac{b}{m}+\frac{k}{m}\right)}\cos(wt+\phi) + \left(2w\alpha - w\frac{b}{m}\right)\sin(wt+\phi) = 0$$

$$\boxed{\alpha = \frac{b}{2m}}$$

$$w^2 = \alpha^2 - \alpha\frac{b}{m} + \frac{k}{m}$$

$$= \frac{b^2}{4m^2} - \frac{b^2}{2m^2} + \frac{k}{m} = \frac{k}{m} - \frac{1}{4}\left(\frac{b}{m}\right)^2 \Rightarrow \boxed{w^2 = \frac{k}{m} - \left(\frac{b}{2m}\right)^2}$$

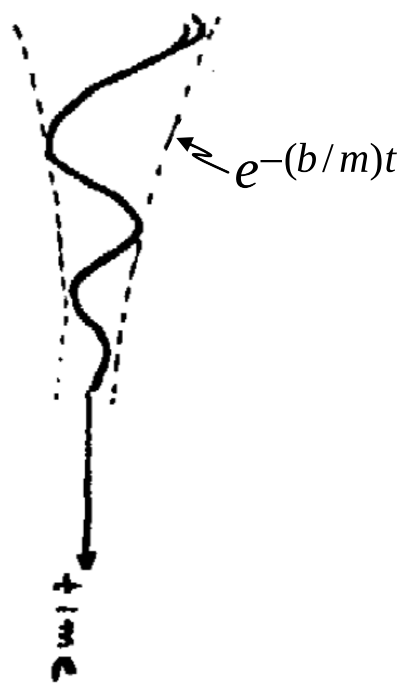
We have found a solution for the equation

$$\frac{d^2x}{dt^2}+\frac{b}{m}\frac{dx}{dt}+\omega_o^2x=0 \qquad \left(\omega_o^2=\sqrt{k/m}\right)$$

It is

$$x = A_o e^{-\frac{b}{m}t}\cos(\omega t+\phi)$$

where $\omega^2 = \omega_o^2 - \left(\frac{b}{m}\right)^2$



amplitude of the oscillations decreases with time.

Summary:

For the equation

$$\frac{d^2x}{dt^2} + \frac{b}{m}\frac{dx}{dt} + \omega_o^2 x = 0$$

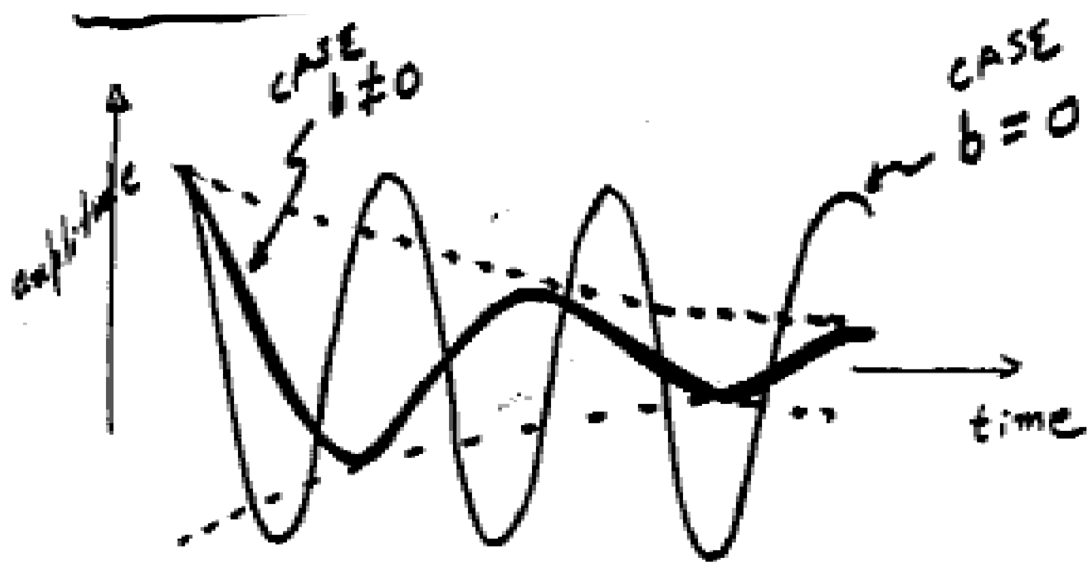
Damped oscillator

we have found a solution of the form

$$x(t) = A_o e^{-\alpha t}\,\mathrm{Cos}(\omega t + \phi)$$

where $\alpha = b/2m$ and $\omega^2 = \omega_o^2 - (b/2m)^2$

A_o and ϕ to be determined by the initial conditions.



- For small b $\left(\Leftrightarrow \text{for } \frac{b}{2m} \ll \omega_0\right)$ $\longrightarrow$ $\omega^2 = \omega_0^2 - \left(\frac{b}{2m}\right)^2$

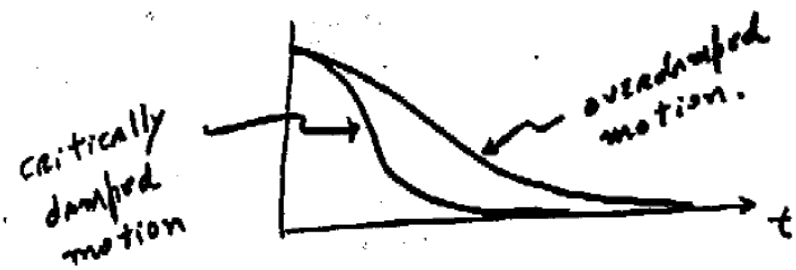
  $\omega \approx \omega_0$

- As b increases $\longrightarrow$ ω decreases

Eventually, b increases as to make $\omega = 0$ → This happens when $b = b_c$

$= 2m\omega_0$

"CRITICALLY DAMPED MOTION



- If $b > b_c$ $\longrightarrow$ system does not oscillate
  overdamped motion.

What happens to the energy of a damped oscillator?

Here we will consider the simple case of small "b": $\frac{b}{2m} < \omega_0$

- For this case $\omega \approx \omega_0$
  and $x \approx \underbrace{e^{-\frac{b}{2m}t} A_0}\cos(\omega_0 t + \phi)$

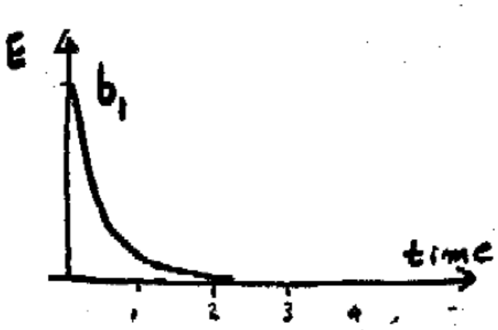
- Energy $= \frac{1}{2}K(\text{Amplitude})^2$

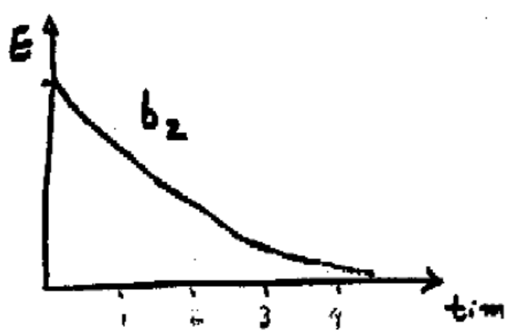
$$= \frac{1}{2}K\left[A_0 e^{-(b/2m)t}\right]^2$$

$$= \frac{1}{2}K A_0^2 e^{-\frac{b}{m}t}$$

$f(t) = e^{-2t}$

$g(t) = e^{-t}$





$b_1 < b_2$

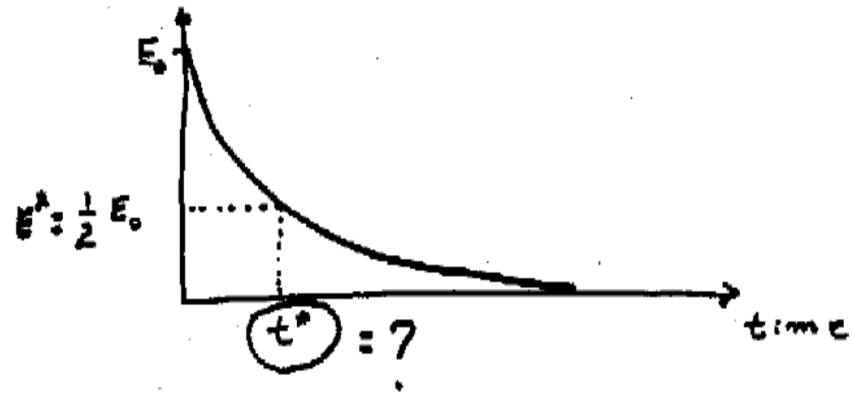
$b_1 > b_2$

?

For a damped oscillator

$$\text{Amplitude} \longrightarrow A_0 e^{-\frac{b}{2m}t}, \qquad A_0 = \text{const}$$

$$\text{Energy} \longrightarrow \frac{1}{2} k A_0^2 e^{-\frac{b}{m}t}$$



QUESTION: How long does it take for the mechanical energy to drop to one-half its initial value? We are looking for t^*

$$\text{At } t=0 \longrightarrow E = \frac{1}{2} k A_0^2 = E_0$$

$$\text{At } t=t^* \longrightarrow E^* = \frac{1}{2} k A_0^2 e^{-\frac{b}{m}t^*}$$

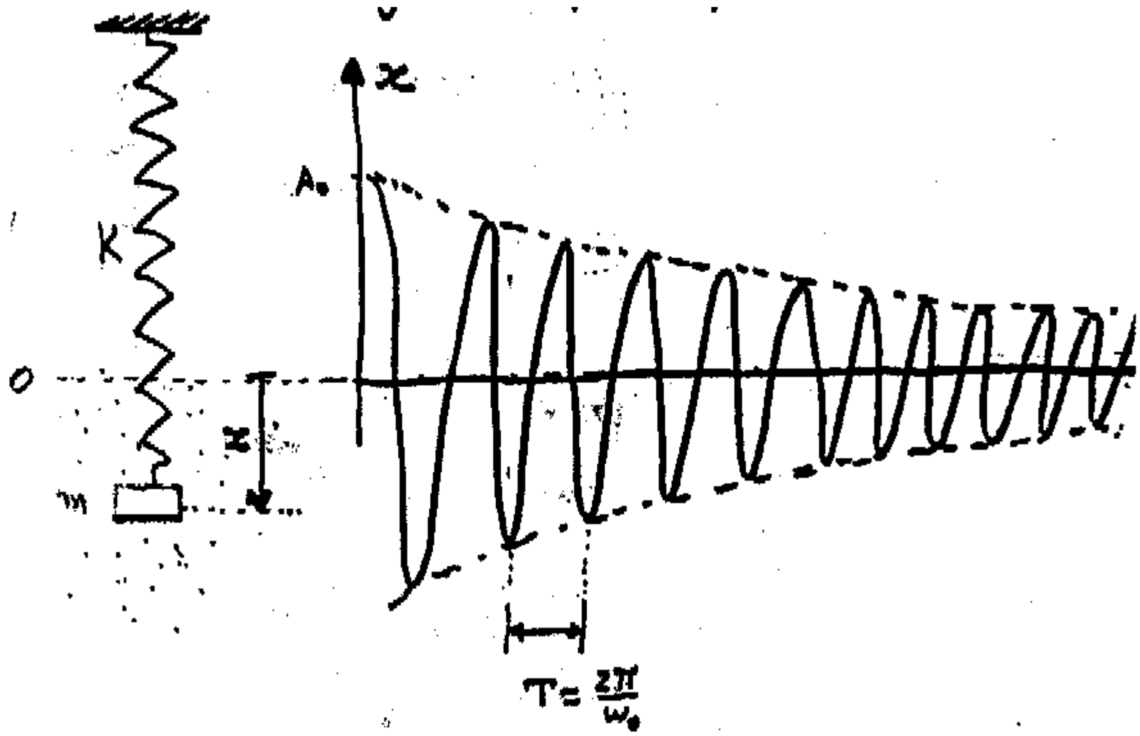
Condition $E^* = \frac{1}{2} E_0$ implies $e^{-\frac{b}{m}t^*} = \frac{1}{2} \longrightarrow$ solve for t^*

$$\ln e^{-\frac{b}{m}t} = \ln \frac{1}{2}$$

$$-\frac{b}{m}t^{*} = \ln 1 - \ln 2$$

$$t^{*} = \frac{m}{b} \ln 2$$

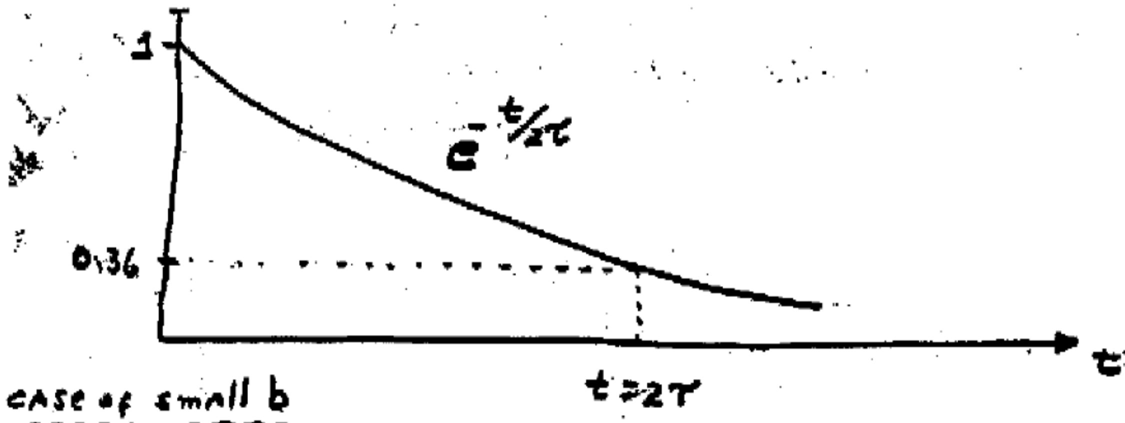
The Quality factor Q
Characterization of the damped oscillations



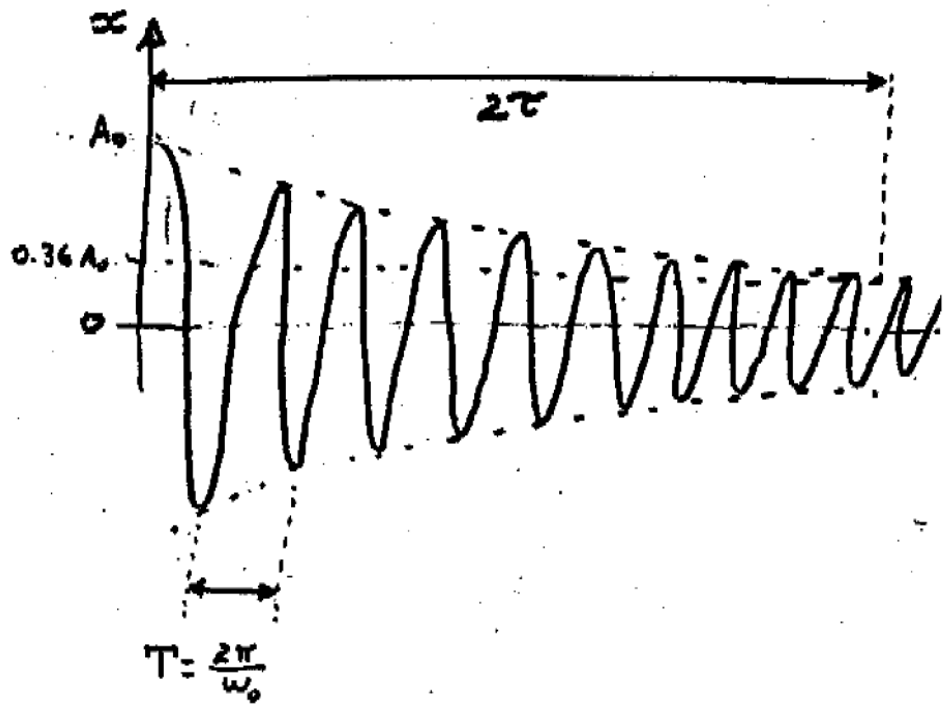
$$x = A_0 e^{-\frac{t}{2\tau}} \cos(\omega t + \alpha_0) = x(t)$$

where $\tau = \frac{m}{b}$

$$\omega^2 = \omega_0^2 - \left(\frac{b}{2m}\right)^2$$



case of small b



$\omega_0 = \sqrt{k/m}$

$\tau = \frac{m}{b}$

A damped oscillator is often described by its Q, which is defined as

$$Q = \omega_0 \tau \qquad \text{Definition of the}$$
$$= 2\pi \frac{\tau}{T} \qquad \text{Quality Factor}$$
$$(\text{where } \tau = m/b)$$

Q is related to the fractional energy loss per cycle.

Let's see how:

$$\text{Amplitude} \longrightarrow A_0 e^{-\frac{t}{2\tau}}$$

$$\text{Energy} \longrightarrow \frac{1}{2} k (\text{Amplitude})^2$$

$$E = \frac{1}{2} k A_0^2 e^{-\frac{t}{\tau}}$$

$$\frac{dE}{dt} = -\frac{1}{2} k A_0^2 \frac{1}{\tau} e^{-t/\tau}$$

$$= -\frac{1}{\tau} E$$

$$\frac{|dE|}{E} = \frac{dt}{\tau}$$

fractional change in energy during an interval of time dt

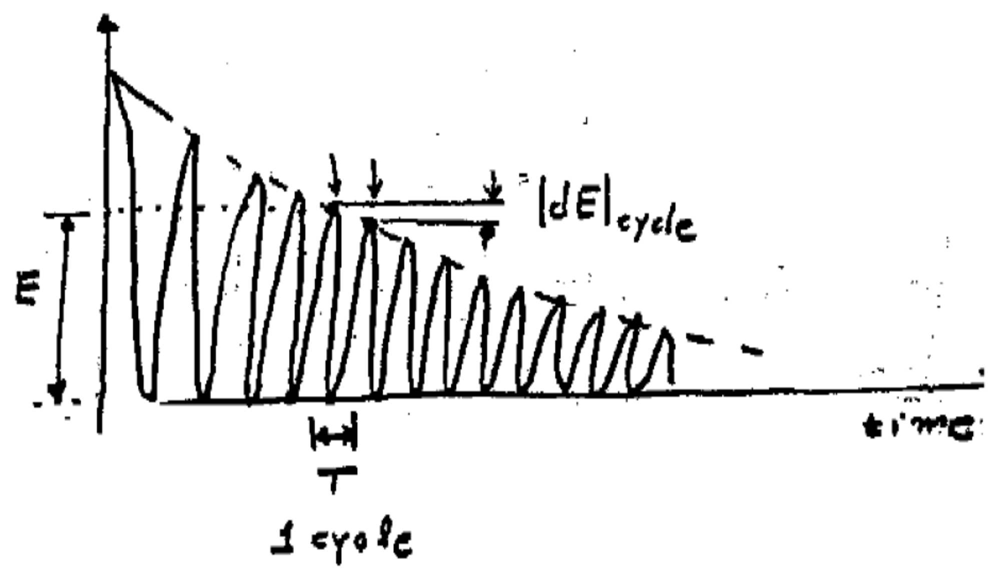
when $dt = T$

$$\boxed{\frac{|dE|_{cycle}}{E} = \frac{T}{\tau} = \frac{2\pi}{Q}}$$

$T = \frac{2\pi}{\omega_0}$

$\tau = \frac{m}{b}$

$Q = \omega_0 \tau$
$= \omega_0 \frac{m}{b}$

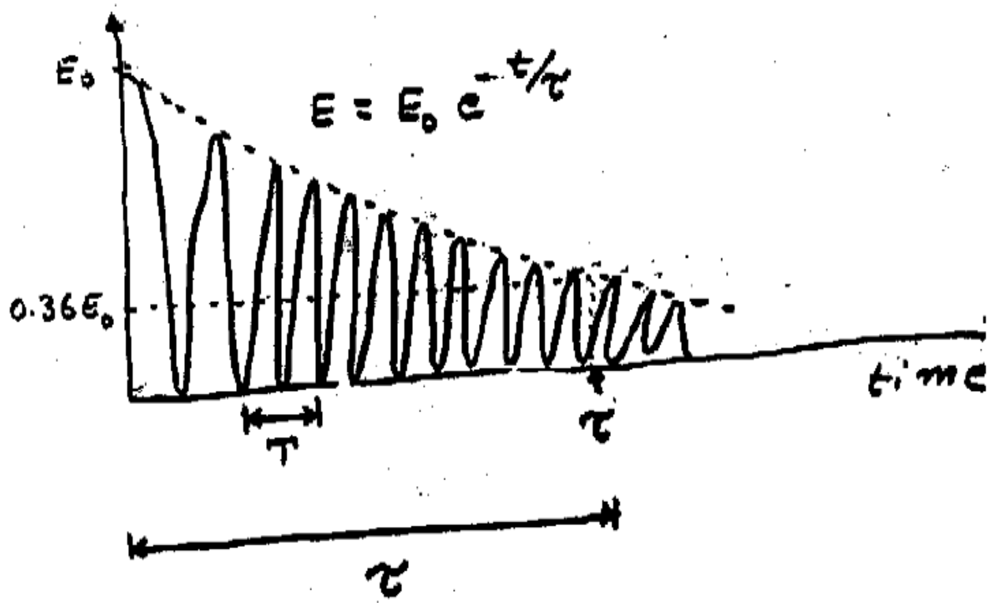


notice: The lower damping (lower b)

the higher Q

the lower loss of energy per cycle in the oscillator

Notice also



Since $Q = 2\pi \frac{\tau}{T}$, the Quality Factor is a measurement of the number of cycles it takes the oscillator to decrease its energy to 36%

$$E = E_0 \, e^{-t/\tau}$$
$$= E_0 \, e^{-t\omega_0/Q}$$

$$\frac{|\Delta E|_{cycle}}{E} = \frac{T}{\tau}$$
$$= \frac{2\pi}{Q}$$

If $Q = 600 \longrightarrow \frac{(\Delta E)_{cycle}}{E} = \frac{2\pi}{600} \simeq$ **0.01** 1% loss per cycle

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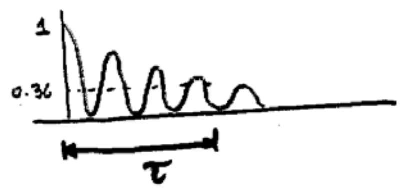
A piano or violin string sings for $\sim$ 1 sec after a pluck



$Q = \omega_0 \tau$

$\tau = 1\,sec$ in this case

Since the frequency $f_0 = \omega_0/2\pi$ of that string is of several hundred, THEN $\omega_o \sim (2\pi)\ 330\ Hz = 2 \times 10^3$

such string must have a Q of the order 2×10^3

$(\Delta E)_{cycle}/E = 2\pi/Q \sim 3 \times 10^{-3}$
0.3 % loss per cycle

atomic transition

$\tau \approx 10^{-8}$ sec duration

$Q = \omega_0 \tau$
visible light

so,
$Q \approx 10^7$