CHAPTER

2

PROBLEMS, PROBLEM SPACES, AND SEARCH

It's not that I'm so smart, it's just that I stay with problems longer.

—**Albert Einstein**
(1879 –1955), German-born theoretical physicist

In the last chapter, we gave a brief description of the kinds of problems with which AI is typically concerned, as well as a couple of examples of the techniques it offers to solve those problems. To build a system to solve a particular problem, we need to do four things:

1. Define the problem precisely. This definition must include precise specifications of what the initial situation (s) will be as well as what final situations constitute acceptable solutions to the problem.
2. Analyze the problem. A few very important features can have an immense impact on the appropriateness of various possible techniques for solving the problem.
3. Isolate and represent the task knowledge that is necessary to solve the problem.
4. Choose the best problem-solving technique(s) and apply it (them) to the particular problem.

In this chapter and the next, we discuss the first two and the last of these issues. Then, in the chapters in Part II, we focus on the issue of knowledge representation.

2.1 DEFINING THE PROBLEM AS A STATE SPACE SEARCH

Suppose we start with the problem statement "Play chess". Although there are a lot of people to whom we could say that and reasonably expect that they will do as we intended, as our request now stands it is a very incomplete statement of the problem we want solved. To build a program that could "Play chess," we would first have to specify the starting position of the chess board, the rules that define the legal moves, and the board positions that represent a win for one side or the other. In addition, we must make explicit the previously implicit goal of not only playing a legal game of chess **but** also winning the game, if possible.

For the problem "Play chess," it is fairly easy to provide a formal and complete problem description. The starting position can be described as an 8×8 array where each position contains a symbol standing for the appropriate piece in the official chess opening position. We can define as our goal any board position in which the opponent does not have a legal move and his or her king is under attack. The legal moves provide the way of getting from the initial state to a goal state. They can be described easily as a set of rules consisting of two parts: a left side that serves as a pattern to be matched against the current board position and a right side that

describes the change to be made to the board position to reflect the move. There are several ways in which these rules can be written. For example, we could write a rule such as that shown in Fig. 2.1.

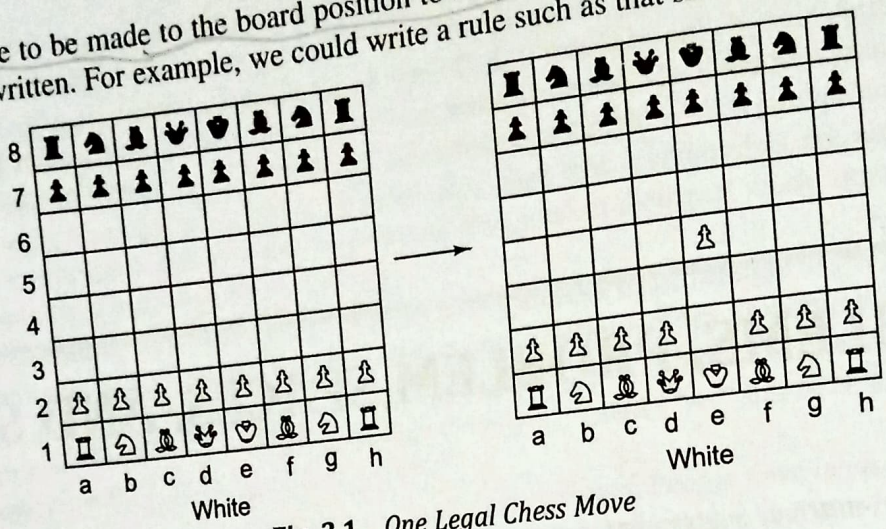

Fig. 2.1 *One Legal Chess Move*

However, if we write rules like the one above, we have to write a very large number of them since there has to be a separate rule for each of the roughly 10^{120} possible board positions. Using so many rules poses two serious practical difficulties:

- No person could ever supply a complete set of such rules. It would take too long and could certainly not be done without mistakes.
- No program could easily handle all those rules. Although a hashing scheme could be used to find the relevant rules for each move fairly quickly, just storing that many rules poses serious difficulties.

In order to minimize such problems, we should look for a way to write the rules describing the legal moves in as general a way as possible. To do this, it is useful to introduce some convenient notation for describing patterns and substitutions. For example, the rule described in Fig. 2.1, as well as many like it, could be written as shown in Fig. 2.2.[1] In general, the more succinctly we can describe the rules we need, the less work we will have to do to provide them and the more efficient the program that uses them can be.

White pawn at
Square(file e, rank 2)
AND
Square(file e, rank 3)
is empty
AND
Square(file e, rank 4)
is empty
→
move pawn from
Square(file e, rank 2)
to Square(file e, rank 4)

Fig. 2.2 *Another Way to Describe Chess Moves*

We have just defined the problem of playing chess as a problem of moving around in a *slate space*, where each state corresponds to a legal position of the board. We can then play chess by starting at an initial state, using a set of rules to move from one state to another, and attempting to end up in one of a set of final states. This state space representation seems natural for chess because the set of states, which corresponds to the set of board positions, is artificial and well-organized. This same kind of representation is also useful for naturally occurring, less well-structured problems, although it may be necessary to use more complex structures than a

[1] To be completely accurate, this rule should include a check for pinned pieces, which have been ignored here.

matrix to describe an individual state. The state space representation forms the basis of most of the AI methods we discuss here. Its structure corresponds to the structure of problem solving in two important ways:

- It allows for a formal definition of a problem as the need to convert some given situation into some desired situation using a set of permissible operations.
- It permits us to define the process of solving a particular problem as a combination of known techniques (each represented as a rule defining a single step in the space) and search, the general technique of exploring the space to try to find some path from the current state to a goal state. Search is a very important process in the solution of hard problems for which no more direct techniques are available.

In order to show the generality of the state space representation, we use it to describe a problem very different from that of chess.

A Water Jug Problem: You are given two jugs, a 4-gallon one and a 3-gallon one. Neither has any measuring markers on it. There is a pump that can be used to fill the jugs with water. How can you get exactly 2 gallons of water into the 4-gallon jug?

The state space for this problem can be described as the set of ordered pairs of integers (x, y), such that $x = 0, 1, 2, 3,$ or 4 and $y = 0, 1, 2,$ or 3; x represents the number of gallons of water in the 4-gallon jug, and y represents the quantity of water in the 3-gallon jug. The start state is $(0, 0)$. The goal state is $(2, n)$ for any value of n (since the problem does not specify how many gallons need to be in the 3-gallon jug).

The operators[2] to be used to solve the problem can be described as shown in Fig. 2.3. As in the chess problem, they are represented as rules whose left sides are matched against the current state and whose right sides describe the new state that results from applying the rule. Notice that in order to describe the operators completely, it was necessary to make explicit some assumptions not mentioned in the problem statement. We have assumed that we can fill a jug from the pump, that we can pour water out of a jug onto the ground, that we can pour water from one jug to another, and that there are no other measuring devices available. Additional assumptions such as these are almost always required when converting from a typical problem statement given in English to a formal representation of the problem suitable for use by a program.

To solve the water jug problem, all we need, in addition to the problem description given above, is a control structure that loops through a simple cycle in which some rule whose left side matches the current state is chosen, the appropriate change to the state is made as described in the corresponding right side, and the resulting state is checked to see if it corresponds to a goal state. As long as it does not, the cycle continues. Clearly the speed with which the problem gets solved depends on the mechanism that is used to select the next operation to be performed. In Chapter 3, we discuss several ways of making that selection.

For the water jug problem, as with many others, there are several sequences of operators that solve the problem. One such sequence is shown in Fig. 2.4. Often, a problem contains the explicit or implied statement that the shortest (or cheapest) such sequence be found. If present, this requirement will have a significant effect on the choice of an appropriate mechanism to guide the search for a solution. We discuss this issue in Section 2.3.4.

Several issues that often arise in converting an informal problem statement into a formal problem description are illustrated by this sample water jug problem. The first of these issues concerns the role of the conditions that occur in the left sides of the rules. All but one of the rules shown in Fig. 2.3 contain conditions that must be satisfied before the operator described by the rule can be applied. For example, the first rule says, "If the 4-gallon jug is not already full, fill it." This rule could, however, have been written as, "Fill the 4-gallon jug," since it is physically possible to fill the jug even if it is already full. It is stupid to do so since no change in the problem state results, but it is possible. By encoding in the left sides of the rules constraints that are not strictly necessary but that restrict the application of the rules to states in which the rules are most likely to lead to a solution, we can generally increase the efficiency of the problem-solving program that uses the rules.

...ter usually includes information about what

1 (x, y) if $x < 4$	$\rightarrow (4, y)$	Fill the 4-gallon jug
2 (x, y) if $y < 3$	$\rightarrow (x, 3)$	Fill the 3-gallon jug
3 (x, y) if $x > 0$	$\rightarrow (x - d, y)$	Pour some water out of the 4-gallon jug
4 (x, y) if $y > 0$	$\rightarrow (x, y - d)$	Pour some water out of the 3-gallon jug
5 $(x, y\}$ if $x > 0$	$\rightarrow (0, y)$	Empty the 4-gallon jug on the ground
6 (x, y) if $y > 0$	$\rightarrow (x, 0)$	Empty the 3-gallon jug on the ground
7 (x, y) if $x + y \geq 4$ and $y > 0$	$\rightarrow (4, y - (4 - x))$	Pour water from the 3-gallon jug into the 4-gallon jug until the 4-gallon jug is full
8 (x, y) if $x + y \geq 3$ and $x > 0$	$\rightarrow (x - (3 - y), 3)$	Pour water from the 4-gallon jug into the 3-gallon jug until the 3-gallon jug is full
9 (x, y) if $x + y \leq 4$ and $y > 0$	$\rightarrow (x + y, 0)$	Pour all the water from the 3-gallon jug into the 4-gallon jug
10 (x, y) if $x + y \leq 3$ and $x > 0$	$\rightarrow (0, x + y)$	Pour all the water from the 4-gallon jug into the 3-gallon jug
11 $(0, 2)$	$\rightarrow (2, 0)$	Pour the 2 gallons from the 3-gallon jug into the 4-gallon jug
12 $(2, y)$	$\rightarrow (0, y)$	Empty the 2 gallons in the 4-gallon jug on the ground

Fig. 2.3 *Production Rules for the Water Jug Problem*

Gallons in the 4-Gallon Jug	Gallons in the 3-Gallon Jug	Rule Applied
0	0	
0	3	2
3	0	9
3	3	2
4	2	7
0	2	5 or 12
2	0	9 or 11

Fig. 2.4 *One Solution to the Water Jug Problem*

The extreme of this approach is shown in the first tic-tac-toe program of Chapter 1. Each entry in the move vector corresponds to a rule that describes an operation. The left side of each rule describes a board configuration and is represented implicitly by the index position. The right side of each rule describes the operation to be performed and is represented by a nine-element vector that corresponds to the resulting board configuration. Each of these rules is maximally specific; it applies only to a single board configuration, and, as a result, no search is required when such rules are used. However, the drawback to this extreme approach is that the problem solver can take no action at all in a novel situation. In fact, essentially no problem *solving* ever really occurs. For a tic-tac-toe playing program, this is not a serious problem, since it is possible to enumerate all the situations (i.e., board configurations) that may occur. But for most problems, this is not the case. In order to solve new problems, more general rules must be available.

A second issue is exemplified by rules 3 and 4 in Fig. 2.3. Should they or should they not be included in the list of available operators? Emptying an unmeasured amount of water onto the ground is certainly allowed by the problem statement. But a superficial preliminary analysis of the problem makes it clear that doing so will never get us any closer to a solution. Again, we see the tradeoff between writing a set of rules that describe just the problem itself, as opposed to a set of rules that describe both the problem and some knowledge about its solution.

Rules 11 and 12 illustrate a third issue. To see the problem-solving knowledge that these rules represent, look at the last two steps of the solution shown in Fig. 2.4. Once the state (4, 2) is reached, it is obvious what to do next. The desired 2 gallons have been produced, but they are in the wrong jug. So the thing to do is to move them (rule 11). But before that can be done, the water that is already in the 4-gallon jug must be emptied out (rule 12). The idea behind these special-purpose rules is to capture the special-case knowledge that can be used at this stage in solving the problem. These rules do not actually add power to the system since the operations they describe are already provided by rule 9 (in the case of rule 11) and by rule 5 (in the case of rule 12). In fact, depending on the control strategy that is used for selecting rules to use during problem solving, the use of these rules may degrade performance. But the use of these rules may also improve performance if preference is given to special-case rules (as we discuss in Section 6.4.3).

We have now discussed two quite different problems, chess and the water jug problem. From these discussions, it should be clear that the first step toward the design of a program to solve a problem must be the creation of a formal and manipulable description of the problem itself. Ultimately, we would like to be able to write programs that can themselves produce such formal descriptions from informal ones. This process is called *operationalization*. It is not at all well-understood how to construct such programs, but see Section 17.3 for a description of one program that solves a piece of this problem. Until it becomes possible to automate this process, it must be done by hand, however. For simple problems, such as chess or the water jug, this is not very difficult. The problems are artificial and highly structured. For other problems, particularly naturally-occurring ones, this step is much more difficult. Consider, for example, the task of specifying precisely what it means to understand an English sentence. Although such a specification must somehow be provided before we can design a program to solve the problem, producing such a specification is itself a very hard problem. Although our ultimate goal is to be able to solve difficult, unstructured problems, such as natural language understanding, it is useful to explore simpler problems, such as the water jug problem, in order to gain insight into the details of methods that can form the basis for solutions to the harder problems.

Summarizing what we have just said, in order to provide a formal description of a problem, we must do the following:

1. Define a state space that contains all the possible configurations of the relevant objects (and perhaps some impossible ones). It is, of course, possible to define this space without explicitly enumerating all of the states it contains.

2. Specify one or more states within that space that describe possible situations from which the problem solving process may start. These states are called the *initial states.*
3. Specify one or more states that would be acceptable as solutions to the problem. These states are called *goal states.*
4. Specify a set of rules that describe the actions (operators) available. Doing this will require giving thought to the following issues:
   - What unstated assumptions are present in the informal problem description?
   - How general should the rules be?
   - How much of the work required to solve the problem should be precomputed and represented in the rules?

The problem can then be solved by using the rules, in combination with an appropriate control strategy, to move through the problem space until a path from an initial state to a goal state is found. Thus the process of search is fundamental to the problem-solving process. The fact that search provides the basis for the process of problem-solving does not, however, mean that other, more direct approaches cannot also be exploited. Whenever possible, they can be included as steps in the search by encoding them into the rules. For example, in the water jug problem, we use the standard arithmetic operations as single steps in the rules. We do not use search to find a number with the property that it is equal to $y - (4 - x)$. Of course, for complex problems, more sophisticated computations will be needed. Search is a general mechanism that can be used when no more direct method is known. At the same time, it provides the framework into which more direct methods for solving subparts of a problem can be embedded.

2.2 PRODUCTION SYSTEMS

Since search forms the core of many intelligent processes, it is useful to structure AI programs in a way that facilitates describing and performing the search process. Production systems provide such structures. A definition of a production system is given below. Do not be confused by other uses of the word *production*, such as to describe what is done in factories. A *production system* consists of:

- A set of rules, each consisting of a left side (a pattern) that determines the applicability of the rule and a right side that describes the operation to be performed if the rule is applied.[3]
- One or more knowledge/databases that contain whatever information is appropri- ate for the particular task. Some parts of the database may be permanent, while other parts of it may pertain only to the solution of the current problem. The information in these databases may be structured in any appropriate way.
- A control strategy that specifies the order in which the rules will be compared to the database and a way of resolving the conflicts that arise when several rules match at once.
- A rule applier.

So far, our definition of a production system has been very general. It encompasses a great many systems, including our descriptions of both a chess player and a water jug problem solver. It also encompasses a family of general production system interpreters, including:

- Basic production system languages, such as OPS5 [Brownston *et al.*, 1985] and ACT* [Anderson, 1983].
- More complex, often hybrid systems called *expert system shells*, which provide complete (relatively speaking) environments for the construction of knowledge- based expert systems.
- General problem-solving architectures like SOAR [Laird *et al.*, 1987], a system based on a specific set of cognitively motivated hypotheses about the nature of problem-solving.

[3] This convention for the use of left and right sides is natural for forward rules. As we will see later, many backward rule systems reverse the sides.

All of these systems provide the overall architecture of a production system and allow the programmer to write rules that define particular problems to be solved. We discuss production system issues further in Chapter 6.

We have now seen that in order to solve a problem, we must first reduce it to one for which a precise statement can be given. This can be done by defining the problem's state space (including the start and goal states) and a set of operators for moving in that space. The problem can then be solved by searching for a path through the space from an initial state to a goal state. The process of solving the problem can usefully be modeled as a production system. In the rest of this section, we look at the problem of choosing the appropriate control structure for the production system so that the search can be as efficient as possible.

2.2.1 Control Strategies

So far, we have completely ignored the question of how to decide which rule to apply next during the process of searching for a solution to a problem. This question arises since often more than one rule (and sometimes fewer than one rule) will have its left side match the current state. Even without a great deal of thought, it is clear that how such decisions are made will have a crucial impact on how quickly, and even whether, a problem is finally solved.

- *The first requirement of a good control strategy is that it causes motion.* Consider again the water jug problem of the last section. Suppose we implemented the simple control strategy of starting each time at the top of the list of rules and choosing the first applicable one. If we did that, we would never solve the problem. We would continue indefinitely filling the 4-gallon jug with water Control strategies that do not cause motion will never lead to a solution.
- *The second requirement of a good control strategy is that it be systematic.* Here is another simple control strategy for the water jug problem: On each cycle, choose at random from among the applicable rules. This strategy is better than the first. It causes motion. It will lead to a solution eventually. But we are likely to arrive at the same state several times during the process and to use many more steps than are necessary. Because the control strategy is not systematic, we may explore a particular useless sequence of operators several times before we finally find a solution. The requirement that a control strategy be systematic corresponds to the need for global motion (over the course of several steps) as well as for local motion (over the course of a single step). One systematic control strategy for the water jug problem is the following. Construct a tree with the initial state as its root. Generate all the offspring of the root by applying each of the applicable rules to the initial state. Fig. 2.5 shows how the tree looks at this point. Now for each leaf node, generate all its successors by applying all the rules that are appropriate. The tree at this point is shown in Fig. 2.6.[4] Continue this process until some rule produces a goal state. This process, called *breadth-first search,* can be described precisely as follows.

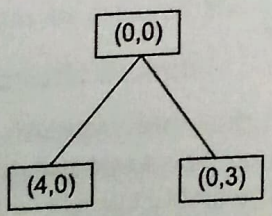

Fig. 2.5 *One Level of a Breadth-First Search Tree*

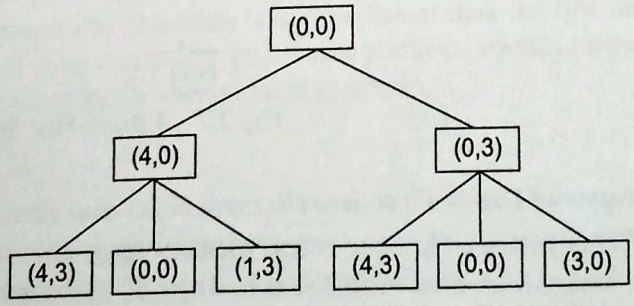

Fig. 2.6 *Two Levels of a Breadth-First Search Tree*

[4] Rule 3, 4, 11, and 12 have been ignored in constructing the search tree.

Algorithm: Breadth-First Search

1. Create a variable called *NODE-LIST* and set it to the initial state.
2. Until a goal state is found or *NODE-LIST* is empty:
   (a) Remove the first element from *NODE-LIST* and call it *E*. If *NODE-LIST* was empty, quit.
   (b) For each way that each rule can match the state described in *E* do:
      (i) Apply the rule to generate a new state,
      (ii) If the new state is a goal state, quit and return this state.
      (iii) Otherwise, add the new state to the end of *NODE-LIST*.

Other systematic control strategies are also available. For example, we could pursue a single branch of the tree until it yields a solution or until a decision to terminate the path is made. It makes sense to terminate a path if it reaches a dead-end, produces a previous state, or becomes longer than some prespecified "futility" limit. In such a case, backtracking occurs. The most recently created state from which alternative moves are available will be revisited and a new state will be created. This form of backtracking is called *chronological backtracking* because the order in which steps are undone depends only on the temporal sequence in which the steps were originally made. Specifically, the most recent step is always the first to be undone. This form of backtracking is what is usually meant by the simple term *backtracking*. But there are other ways of retracting steps of a computation. We discuss one important such way, dependency-directed backtracking, in Chapter 7. Until then, though, when we use the term backtracking, it means chronological backtracking.

The search procedure we have just described is also called *depth-first search*. The following algorithm describes this precisely.

Algorithm: Depth-First Search

1. If the initial state is a goal state, quit and return success.
2. Otherwise, do the following until success or failure is signaled:
   (a) Generate a successor, *E*, of the initial state. If there are no more successors, signal failure.
   (b) Call Depth-First Search with *E* as the initial state.
   (c) If success is returned, signal success. Otherwise continue in this loop.

Figure 2.7 shows a snapshot of a depth-first search for the water jug problem. A comparison of these two simple methods produces the following observations:

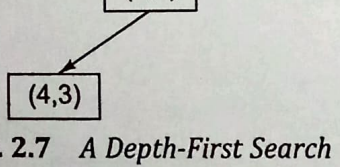

Fig. 2.7 *A Depth-First Search Tree*

Advantages of Depth-First Search

- Depth-first search requires less memory since only the nodes on the current path are stored. This contrasts with breadth-first search, where all of the tree that has so far been generated must be stored.
- By chance (or if care is taken in ordering the alternative successor states), depth-first search may find a solution without examining much of the search space at all. This contrasts with breadth-first search in which all parts of the tree must be examined to level n before any nodes on level $n + 1$ can be examined. This is particularly significant if many acceptable solutions exist. Depth-first search can stop when one of them is found.

Advantages of Breadth-First Search

- Breadth-first search will not get trapped exploring a blind alley. This contrasts with depth-first searching, which may follow a single, unfruitful path for a very long time, perhaps forever, before the path actually terminates in a state that has no successors. This is a particular problem in depth-first search if there are loops (i.e., a state has a successor that is also one of its ancestors) unless special care is expended to test for such a situation. The example in Fig. 2.7, if it continues always choosing the first (in numerical sequence) rule that applies, will have exactly this problem.
- If there is a solution, then breadth-first search is guaranteed to find it. Furthermore, if there are multiple solutions, then a minimal solution (i.e., one that requires the minimum number of steps) will be found. This is guaranteed by the fact that longer paths are never explored until all shorter ones have already been examined. This contrasts with depth-first search, which may find a long path to a solution in one part of the tree, when a shorter path exists in some other, unexplored part of the tree.

Clearly what we would like is a way to combine the advantages of both of these methods. In Section 3.3 we will talk about one way of doing this when we have some additional information. Later, in Section 12.5, we will describe an uninformed way of doing so.

For the water jug problem, most control strategies that cause motion and are systematic will lead to an answer. The problem is simple. But this is not always the case. In order to solve some problems during our lifetime, we must also demand a control structure that is efficient.

Consider the following problem.

> **The Traveling Salesman Problem:** A salesman has a list of cities, each of which he must visit exactly once. There are direct roads between each pair of cities on the list. Find the route the salesman should follow for the shortest possible round trip that both starts and finishes at any one of the cities.

A simple, motion-causing and systematic control structure could, in principle, solve this problem. It would simply explore all possible paths in the tree and return the one with the shortest length. This approach will even work in practice for very short lists of cities. But it breaks down quickly as the number of cities grows. If there are N cities, then the number of different paths among them is $1.2...(N-1)$, or $(N-1)!$. The time to examine a single path is proportional to N. So the total time required to perform this search is proportional to $N!$. Assuming there are only 10 cities, 10! is 3,628,800, which is a very large number. The salesman could easily have 25 cities to visit. To solve this problem would take more time than he would be willing to spend. This phenomenon is called *combinatorial explosion*. To combat it, we need a new control strategy.

We can beat the simple strategy outlined above using a technique called *branch-and-bound*. Begin generating complete paths, keeping track of the shortest path found so far. Give up exploring any path as soon as its partial length becomes greater than the shortest path found so far. Using this technique, we are still guaranteed to find the shortest path. Unfortunately, although this algorithm is more efficient than the first one, it still requires exponential time. The exact amount of time it saves for a particular problem depends on the order in which the paths are explored. But it is still inadequate for solving large problems.

2.2.2 Heuristic Search

In order to solve many hard problems efficiently, it is often necessary to compromise the requirements of mobility and systematicity and to construct a control structure that is no longer guaranteed to find the best answer but that will almost always find a very good answer. Thus we introduce the idea of a heuristic.[5] A

[5] The word *heuristic* comes from the Greek word *heuriskein*, meaning "to discover," which is also the origin of *eureka*, derived from Archimedes' reputed exclamation, *heurika* (for "I have found"), uttered when he had discovered a method for determining the purity of gold.

heuristic is a technique that improves the efficiency of a search process, possibly by sacrificing claims of completeness. Heuristics are like tour guides. They are good to the extent that they point in generally interesting directions; they are bad to the extent that they may miss points of interest to particular individuals. Some heuristics help to guide a search process without sacrificing any claims to completeness that the process might previously have had. Others (in fact, many of the best ones) may occasionally cause an excellent path to be overlooked. But, on an average, they improve the quality of the paths that are explored. Using good heuristics, we can hope to get good (though possibly nonoptimal) solutions to hard problems, such as the traveling salesman, in less than exponential time. There are some good general-purpose heuristics that are useful in a wide variety of problem domains. In addition, it is possible to construct special-purpose heuristics that exploit domain-specific knowledge to solve particular problems.

One example of a good general-purpose heuristic that is useful for a variety of combinatorial problems is the *nearest neighbor heuristic*, which works by selecting the locally superior alternative at each step. Applying it to the traveling salesman problem, we produce the following procedure:

1. Arbitrarily select a starting city.
2. To select the next city, look at all cities not yet visited, and select the one closest to the current city. Go to it next.
3. Repeat step 2 until all cities have been visited.

This procedure executes in time proportional to N^2, a significant improvement over $N!$, and it is possible to prove an upper bound on the error it incurs. For general-purpose heuristics, such as nearest neighbor, it is often possible to prove such error bounds, which provides reassurance that one is not paying too high a price in accuracy for speed.

In many AI problems, however, it is not possible to produce such reassuring bounds. This is true for two reasons:

- For real world problems, it is often hard to measure precisely the value of a particular solution. Although the length of a trip to several cities is a precise notion, the appropriateness of a particular response to such questions as "Why has inflation increased?" is much less so.
- For real world problems, it is often useful to introduce heuristics based on relatively unstructured knowledge. It is often impossible to define this knowledge in such a way that a mathematical analysis of its effect on the search process can be performed.

There are many heuristics that, although they are not as general as the nearest neighbor heuristic, are nevertheless useful in a wide variety of domains. For example, consider the task of discovering interesting ideas in some specified area. The following heuristic [Lenat, 1983b] is often useful:

If there is an interesting function of two arguments $f(x, y)$, look at what happens if the two arguments are identical.

In the domain of mathematics, this heuristic leads to the discovery of *squaring iff* is the multiplication function, and it leads to the discovery of an *identity* function if f is the function of set union. In less formal domains, this same heuristic leads to the discovery of *introspection* if f is the function contemplate or it leads to the notion of *suicide iff* is the function kill.

Without heuristics, we would become hopelessly ensnarled in a combinatorial ex-plosion. This alone might be a sufficient argument in favor of their use. But there are other arguments as well:

- Rarely do we actually need the optimum solution; a good approximation will usually serve very well. In fact, there is some evidence that people, when they solve problems, are not optimizers but rather are *satisficers* [Simon, 1981]. In other words, they seek any solution that satisfies some set of requirements, and as soon as they find one they quit. A good example of this is the search for a parking space. Most people stop as soon as they find a fairly good space, even if there might be a slightly better space up ahead.

- Although the approximations produced by heuristics may not be very good in the worst case, worst cases rarely arise in the real world. For example, although many graphs are not separable (or even nearly so) and thus cannot be considered as a set of small problems rather than one large one, a lot of graphs describing the real world are.[6]
- Trying to understand why a heuristic works, or why it doesn't work, often leads to a deeper understanding of the problem.

One of the best descriptions of the importance of heuristics in solving interesting problems is *How to Solve It* [Polya, 1957]. Although the focus of the book is the solution of mathematical problems, many of the techniques it describes are more generally applicable. For example, given a problem to solve, look for a similar problem you have solved before. Ask whether you can use either the solution of that problem or the method that was used to obtain the solution to help solve the new problem. Polya's work serves as an excellent guide for people who want to become better problem solvers. Unfortunately, it is not a panacea for AI for a couple of reasons. One is that it relies on human abilities that we must first understand well enough to build into a program. For example, many of the problems Polya discusses are geometric ones in which once an appropriate picture is drawn, the answer can be seen immediately. But to exploit such techniques in programs, we must develop a good way of representing and manipulating descriptions of those Fig.s. Another is that the rules are very general.

They have extremely underspecified left sides, so it is hard to use them to guide a search—too many of them are applicable at once. Many of the rules are really only useful for looking back and rationalizing a solution after it has been found. In essence, the problem is that Polya's rules have not been operationalized.

Nevertheless, Polya was several steps ahead of AI. A comment he made in the preface to the first printing (1944) of the book is interesting in this respect:

> The following pages are written somewhat concisely, but as simply as possible, and are based on a long and serious study of methods of solution. This sort of study, called *heuristic* by some writers, is not in fashion nowadays but has a long past and, perhaps, some future.

There are two major ways in which domain-specific, heuristic knowledge can be incorporated into a rule-based search procedure:

- In the rules themselves. For example, the rules for a chess-playing system might describe not simply the set of legal moves but rather a set of "sensible" moves, as determined by the rule writer.
- As a heuristic function that evaluates individual problem states and determines how desirable they are.

A *heuristic function* is a function that maps from problem state descriptions to measures of desirability, usually represented as numbers. Which aspects of the problem state are considered, how those aspects are evaluated, and the weights given to individual aspects are chosen in such a way that the value of the heuristic function at a given node in the search process gives as good an estimate as possible of whether that node is on the desired path to a solution.

Well-designed heuristic functions can play an important part in efficiently guiding a search process toward a solution. Sometimes very simple heuristic functions can provide a fairly good estimate of whether a path is any good or not. In other situations, more complex heuristic functions should be employed. Fig. 2.8 shows some simple heuristic functions for a few problems. Notice that sometimes a high value of the heuristic function indicates a relatively good position (as shown for chess and tic-tac- toe), while at other times a low value indicates an advantageous situation (as shown for the traveling salesman). It does not matter, in general, which way the function is stated. The program that uses the values of the function can attempt to minimize it or to maximize it as appropriate.

[6] For arguments in support of this, see Simon [1981].

Chess	the material advantage of our side over the opponent
Traveling Salesman	the sum of the distances so far
Tic-Tac-Toe	1 for each row in which we could win and in which we already have one piece plus 2 for each such row in which we have two pieces

Fig. 2.8 *Some Simple Heuristic Functions*

The purpose of a heuristic function is to guide the search process in the most profitable direction by suggesting which path to follow first when more than one is available. The more accurately the heuristic function estimates the true merits of each node in the search tree (or graph), the more direct the solution process. In the extreme, the heuristic function would be so good that essentially no search would be required. The system would move directly to a solution. But for many problems, the cost of computing the value of such a function would outweigh the effort saved in the search process, After all, it would be possible to compute a perfect heuristic function by doing a complete search from the node in question and determining whether it leads to a good solution. In general, there is a trade-off between the cost of evaluating a heuristic function and the savings in search time that the function provides.

In the previous section, the solutions to AI problems were described as centering on a search process. From the discussion in this section, it should be clear that it can more precisely be described as a process of heuristic search. Some heuristics will be used to define the control structure that guides the application of rules in the search process. Others, as we shall see, will be encoded in the rules themselves. In both cases, they will represent either general or specific world knowledge that makes the solution of hard problems feasible. This leads to another way that one could define artificial intelligence: the study of techniques for solving exponentially hard problems in polynomial time by exploiting knowledge about the problem domain.

2.3 PROBLEM CHARACTERISTICS

Heuristic search is a very general method applicable to a large class of problems. It encompasses a variety of specific techniques, each of which is particularly effective for a small class of problems. In order to choose the most appropriate method (or combination of methods) for a particular problem, it is necessary to analyze the problem along several key dimensions:

- Is the problem decomposable into a set of (nearly) independent smaller or easier subproblems?
- Can solution steps be ignored or at least undone if they prove unwise?
- Is the problem's universe predictable?
- Is a good solution to the problem obvious without comparison to all other possible solutions?
- Is the desired solution a state of the world or a path to a state?
- Is a large amount of knowledge absolutely required to solve the problem, or is knowledge important only to constrain the search?
- Can a computer that is simply given the problem return the solution, or will the solution of the problem require interaction between the computer and a person?

In the rest of this section, we examine each of these questions in greater detail. Notice that some of these questions involve not just the statement of the problem itself but also characteristics of the solution that is desired and the circumstances under which the solution must take place.

2.3.1 Is the Problem Decomposable?

Suppose we want to solve the problem of computing the expression

$$\int (x^2 + 3x + \sin^{2*} \cdot \cos^2 x)\, dx$$

We can solve this problem by breaking it down into three smaller problems, each of which we can then solve by using a small collection of specific rules. Figure 2.9 shows the problem tree that will be generated by the process of problem decomposition as it can be exploited by a simple recursive integration program that works as follows: At each step, it checks to see whether the problem it is working on is immediately solvable. If so, then the answer is returned directly. If the problem is not easily solvable, the integrator checks to see whether it can decompose the problem into smaller problems. If it can, it creates those problems and calls itself recursively on them. Using this technique of *problem decomposition*, we can often solve very large problems easily.

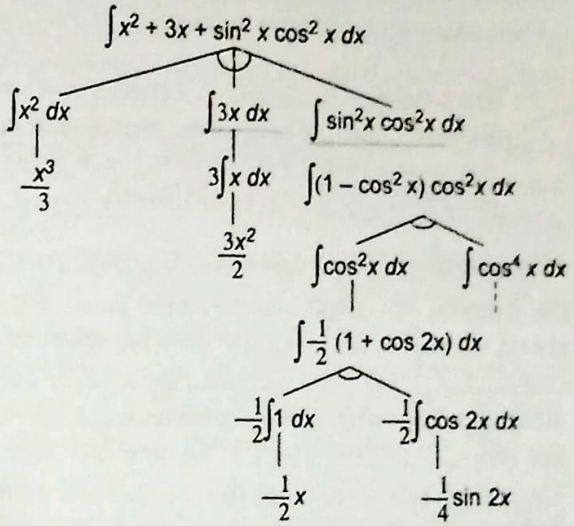

Fig. 2.9 *A Decomposable Problem*

Now consider the problem illustrated in Fig. 2.10. This problem is drawn from the domain often referred to in AI literature as the *blocks world.* Assume that the following operators are available:

1. CLEAR (*x*) [block *x* has nothing on it] → ON (*x*, Table) [pick up *x* and put it on the table]
2. CLEAR (*x*) and CLEAR (*y*) → ON (*x*, *y*) [put *x* on *y*]

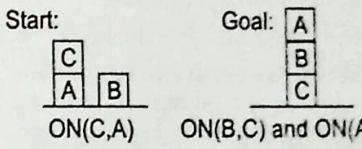

Fig. 2.10 *A Simple Blocks World Problem*

Applying the technique of problem decomposition to this simple blocks world example would lead to a solution tree such as that shown in Fig. 2.11. In the figure, goals are underlined. States that have been achieved are not underlined. The idea of this solution is to reduce the problem of getting B on C and A on B to two separate problems. The first of these new problems, getting B on C, is simple, given the start state. Simply put B on C. The second subgoal is not quite so simple. Since the only operators we have allow us to pick up single blocks at a time, we have to clear off A by removing C before we can pick up A and put it on B. This can easily be done. However, if we now try to combine the two subsolutions into one solution, we will fail. Regardless of which one we do first, we will not be able to do the second as we had planned. In this problem, the two subproblems are not independent. They interact and those interactions must be considered in order to arrive at a solution for the entire problem.

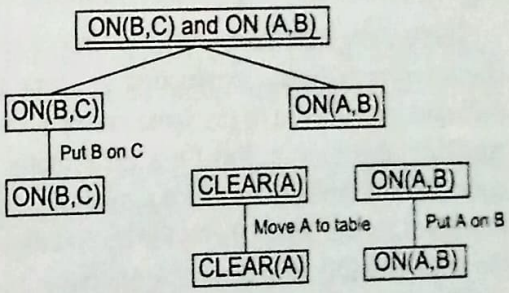

Fig. 2.11 *A Proposed Solution for the Blocks Problem*

These two examples, symbolic integration and the blocks world, illustrate the difference between decomposable and nondecomposable problems. In Chapter 3, we present a specific algorithm for problem decomposition, and in Chapter 13, we look at what happens when decomposition is impossible.

2.3.2 Can Solution Steps Be Ignored or Undone?

Suppose we are trying to prove a mathematical theorem. We proceed by first proving a lemma that we think will be useful. Eventually, we realize that the lemma is no help at all. Are we in trouble?

No. Everything we need to know to prove the theorem is still true and in memory, if it ever was. Any rules that could have been applied at the outset can still be applied. We can just proceed as we should have in the first place. All we have lost is the effort that was spent exploring the blind alley.

Now consider a different problem.

> **The 8-Puzzle:** The 8-puzzle is a square tray in which are placed, eight square tiles. The remaining ninth square is uncovered. Each tile has a number on it. A tile that is adjacent to the blank space can be slid into that space. A game consists of a starting position and a specified goal position. The goal is to transform the starting position into the goal position by sliding the tiles around.

A sample game using the 8-puzzle is shown in Fig. 2.12. In attempting to solve the 8-puzzle, we might make a stupid move. For example, in the game shown above, we might start by sliding tile 5 into the empty space. Having done that, we cannot change our mind and immediately slide tile 6 into the empty space since the empty space will essentially have moved. But we can backtrack and undo the first move, sliding tile 5 back to where it was. Then we can move tile 6. Mistakes can still be recovered from but not quite as easily as in the theorem-proving problem. An additional step must be performed to undo each incorrect step, whereas no action was required to "undo" a useless lemma. In addition, the control mechanism for an 8-puzzle solver must keep track of the order in which operations are performed so that the operations can be undone one at a time if necessary. The control structure for a theorem prover does not *need* to record all that information.

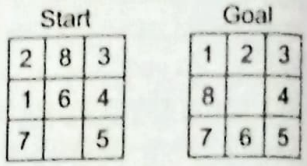

Fig. 2.12 *An Example of the 8-Puzzle*

Now consider again the problem of playing chess. Suppose a chess-playing program makes a stupid move and realizes it a couple of moves later. It cannot simply play as though it had never made the stupid move. Nor can it simply back up and start the game over from that point. All it can do is to try to make the best of the current situation and go on from there.

These three problems—theorem proving, the 8-puzzle, and chess—illustrate the differences between three important classes of problems:

- Ignorable (e.g., theorem proving), in which solution steps can be ignored
- Recoverable (e.g., 8-puzzle), in which solution steps can be undone
- Irrecoverable (e.g., chess), in which solution steps cannot be undone

These three definitions make reference to the steps of the solution to a problem and thus may appear to characterize particular production systems for solving a problem rather than the problem itself. Perhaps a different formulation of the same problem would lead to the problem being characterized differently. Strictly speaking, this is true. But for a great many problems, there is only one (or a small number of essentially equivalent) formulations that *naturally* describe the problem. This was true for each of the problems used as examples above. When this is the case, it makes sense to view the recoverability of a problem as equivalent to the recoverability of a natural formulation of it.

The recoverability of a problem plays an important role in determining the complexity of the control structure necessary for the problem's solution. Ignorable problems can be solved using a simple control structure that never backtracks. Such a control structure is easy to implement. Recoverable problems can be solved by a slightly more complicated control strategy that does sometimes make mistakes. Backtracking will be necessary to recover from such mistakes, so the control structure must be implemented using a push-down stack, in which decisions are recorded in case they need to be undone later. Irrecoverable problems, on the other hand, will need to be solved by a system that expends a great deal of effort making each decision since the decision must be final. Some irrecoverable problems can be solved by recoverable style methods used in a *planning* process, in which an entire sequence of steps is analyzed in advance to discover where it will lead before the first step is actually taken. We discuss next the kinds of problems in which this is possible.

2.3.3 Is the Universe Predictable?

Again suppose that we are playing with the 8-puzzle. Every time we make a move, we know exactly what will happen. This means that it is possible to plan an entire sequence of moves and be confident that we know what the resulting state will be. We can use planning to avoid having to undo actual moves, although it will still be necessary to backtrack past those moves one at a time during the planning process. Thus a control structure that allows backtracking will be necessary.

However, in games other than the 8-puzzle, this planning process may not be possible. Suppose we want to play bridge. One of the decisions we will have to make is which card to play on the first trick. What we would like to do is to plan the entire hand before making that first play. But now it is not possible to do such planning with certainty since We cannot know exactly where all the cards are or what the other players will do on their turns. The best we can do is to investigate several plans and use probabilities of the various outcomes to choose a plan that has the highest estimated probability of leading to a good score on the hand.

These two games illustrate the difference between certain-outcome (e.g., 8-puzzle) and uncertain-outcome (e.g., bridge) problems. One way of describing planning is that it is problem-solving without feedback from the environment. For solving certain-outcome problems, this open-loop approach will work fine since the result of an action can be predicted perfectly. Thus, planning can be used to generate a sequence of operators that is guaranteed to lead to a solution. For uncertain-outcome problems, however, planning can at best generate a sequence of operators that has a good probability of leading to a solution. To solve such problems, we need to allow for a process of *plan revision* to take place as the plan is carried out and the necessary feedback is provided. In addition to providing no guarantee of an actual solution, planning for uncertain-outcome problems has the drawback that it is often very expensive since the number of solution paths that need to be explored increases exponentially with the; number of points at which the outcome cannot be predicted.

The last two problem characteristics we have discussed, ignorable versus recoverable versus irrecoverable and certain-outcome versus uncertain-outcome, interact in an interesting way. As has already been mentioned, one way to solve irrecoverable problems is to plan an entire solution before embarking on an implementation of the plan. But this planning process can only be done effectively for certain-outcome problems. Thus one of the hardest types of problems to solve is the irrecoverable, uncertain-outcome. A few examples of such problems are:

- Playing bridge. But we can do fairly well since we have available accurate estimates of the probabilities of each of the possible outcomes.
- Controlling a robot arm. The outcome is uncertain for a variety of reasons. Someone might move something into the path of the arm. The gears of the arm might stick. A slight error could cause the arm to knock over a whole stack of things.
- Helping a lawyer decide how to defend his client against a murder charge. Here we probably cannot even list all the possible outcomes, much less assess their probabilities.

2.3.4 Is a Good Solution Absolute or Relative?

Consider the problem of answering questions based on a database of simple facts, such as the following:

1. Marcus was a man.
2. Marcus was a Pompeian.
3. Marcus was born in 40 A.D.
4. All men are mortal.
5. All Pompeians died when the volcano erupted in 79 A.D.
6. No mortal lives longer than 150 years.
7. It is now 1991 A.D.

Suppose we ask the question "Is Marcus alive?" By representing each of these facts in a formal language, such as predicate logic, and then using formal inference methods we can fairly easily derive an answer to the question.[7] In fact, either of two reasoning paths will lead to the answer, as shown in Fig. 2.13. Since all we are interested in is the answer to the question, it does not matter which path we follow. If we do follow one path successfully to the answer, there is no reason to go back and see if some other path might also lead to a solution.

	Justification
1. Marcus was a man.	axiom 1
4. All men are mortal.	axiom 4
8. Marcus is mortal.	1, 4
3. Marcus was born in 40 A.D.	axiom 3
7. It is now 1991 A.D.	axiom 7
9. Marcus' age is 1951 years.	3, 7
6. No mortal lives longer than 150 years.	axiom 6
10. Marcus is dead.	8, 6, 9
OR	
7. It is now 1991 A.D.	axiom 7
5. All Pompeians died in 79 A.D.	axiom 5
11. All Pompeians are dead now.	7, 5
2. Marcus was a Pompeian.	axiom 2
12. Marcus is dead.	11, 2

Fig. 2.13 *Two Ways of Deciding That Marcus Is Dead*

But now consider again the traveling salesman problem. Our goal is to find the shortest route that visits each city exactly once. Suppose the cities to be visited and the distances between them are as shown in Fig. 2.14.

	Boston	New York	Miami	Dallas	S.F.
Boston		250	1450	1700	3000
New York	250		1200	1500	2900
Miami	1450	1200		1600	3300
Dallas	1700	1500	1600		1700
S.F.	3000	2900	3300	1700	

Fig. 2.14 *An Instance of the Traveling Salesman Problem*

One place the salesman could start is Boston. In that case, one path that might be followed is the one shown in Fig. 2.15, which is 8850 miles long. But is this the solution to the problem? The answer is that we cannot be sure unless we also try all other paths to make sure that none of them is shorter. In this case, as can be seen from Fig. 2.16, the first path is definitely not the solution to the salesman's problem.

These two examples illustrate the difference between any-path problems and best- path problems. Best-path problems are, in general, computationally harder than any-path problems. Any-path problems can often be solved in a reasonable amount of time by using heuristics that suggest good paths to explore. (See the discussion of best-first search in Chapter 3 for one way of doing this.) If the heuristics are not perfect, the search for a solution may not be as direct as possible, but that does not matter. For true best-path problems, however, no heuristic that could possibly miss the best solution can be used. So a much more exhaustive search will be performed.

[7] Of course, representing these statements so that a mechanical procedure could exploit them to answer the question also requires the explicit mention of other facts, such as "dead implies not alive." We do this in Chapter 5.

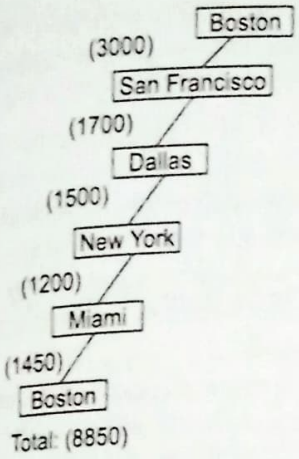

Fig. 2.15 *One Path among the Cities*

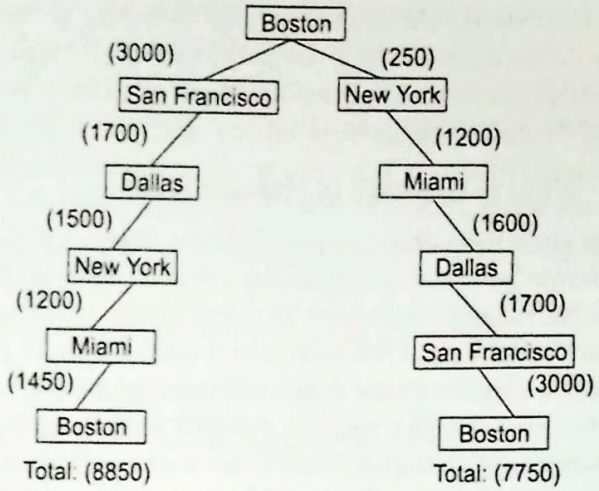

Fig. 2.16 *Two Paths Among the Cities*

2.3.5 Is the Solution a State or a Path?

Consider the problem of finding a consistent interpretation for the sentence

> The bank president ate a dish of pasta salad with the fork.

There are several components of this sentence, each of which, in isolation, may have more than one interpretation. But the components must form a coherent whole, and so they constrain each other's interpretations. Some of the sources of ambiguity in this sentence are the following:

The word "bank" may refer either to a financial institution or to a side of a river. But only one of these may have a president.

- The word "dish" is the object of the verb "eat." It is possible that a dish was eaten. But it is more likely that the pasta salad in the dish was eaten.
- Pasta salad is a salad containing pasta. But there are other ways meanings can be formed from pairs of nouns. For example, dog food does not normally contain dogs.
- The phrase "with the fork" could modify several parts of the sentence. In this case, it modifies the verb "eat." But, if the phrase had been "with vegetables," then the modification structure would be different. And if the phrase had been "with her friends," the structure would be different still.

Because of the interaction among the interpretations of the constituents of this sentence, some search may be required to find a complete interpretation for the sentence. But to solve the problem of finding the interpretation we need to produce only the interpretation itself. No record of the processing by which the interpretation was found is necessary.

Contrast this with the water jug problem. Here it is not sufficient to report that we have solved the problem and that the final state is (2, 0). For this kind of problem, what we really must report is not the final state but the path that we found to that state. Thus a statement of a solution to this problem must be a sequence of operations (sometimes called *apian*) that produces the final state.

These two examples, natural language understanding and the water jug problem, illustrate the difference between problems whose solution is a state of the world and problems whose solution is a path to a state. At one level, this difference can be ignored and all problems can be formulated as ones in which only a state is required to be reported. If we do this for problems such as the water jug, then we must redescribe our states so that each state represents a partial path to a solution rather than just a single state of the world. So this question

is not a formally significant one. But, just as for the question of ignorability versus recoverability, there is often a natural (and economical) formulation of a problem in which problem states correspond to situations in the world, not sequences of operations. In this case, the answer to this question tells us whether it is necessary to record the path of the problem-solving process as it proceeds.

2.3.6 What is the Role of Knowledge?

Consider again the problem of playing chess. Suppose you had unlimited computing power available. How much knowledge would be required by a perfect program? The answer to this question is very little—just the rules for determining legal moves and some simple control mechanism that implements an appropriate search procedure. Additional knowledge about such things as good strategy and tactics could of course help considerably to constrain the search and speed up the execution of the program.

But now consider the problem of scanning daily newspapers to decide which are supporting the Democrats and which are supporting the Republicans in some upcoming election. Again assuming unlimited computing power, how much knowledge would be required by a computer trying to solve this problem? This time the answer is a great deal. It would have to know such things as:

- The names of the candidates in each party.
- The fact that if the major thing you want to see done is have taxes lowered, you are probably supporting the Republicans.
- The fact that if the major thing you want to see done is improved education for minority students, you are probably supporting the Democrats.
- The fact that if you are opposed to big government, you are probably supporting the Republicans.
- And so on ...

These two problems, chess and newspaper story understanding, illustrate the difference between problems for which a lot of knowledge is important only to constrain the search for a solution and those for which a lot of knowledge is required even to be able to recognize a solution.

2.3.7 Does the Task Require Interaction with a Person?

Sometimes it is useful to program computers to solve problems in ways that the majority of people would not be able to understand. This is fine if the level of the interaction between the computer and its human users is problem-in solution-out. But increasingly we are building programs that require intermediate interaction with people, both to provide additional input to the program and to provide additional reassurance to the user.

Consider, for example, the problem of proving mathematical theorems. If

1. All we want is to know that there is a proof
2. The program is capable of finding a proof by itself

then it does not matter what strategy the program takes to find the proof. It can use, for example, the *resolution* procedure (see Chapter 5), which can be very efficient but which does not appear natural to people. But if either of those conditions is violated, it may matter very much how a proof is found. Suppose that we are trying to prove some new, very difficult theorem. We might demand a proof that follows traditional patterns so that a mathematician can read the proof and check to make sure it is correct. Alternatively, finding a proof of the theorem might be sufficiently difficult that the program does not know where to start. At the moment, people are still better at doing the high-level strategy required for a proof. So the computer might like to be able to ask for advice. For example, it is often much easier to do a proof in geometry if someone suggests the right line to draw into the Fig.. To exploit such advice, the computer's reasoning must be analogous to that of its human advisor, at least on a few levels. As computers move into areas of great significance to human lives, such as medical diagnosis, people will be very unwilling to accept the verdict of a program whose reasoning they cannot follow. Thus we must distinguish between two types of problems:

- Solitary, in which the computer is given a problem description and produces an answer with no intermediate communication and with no demand for an explanation of the reasoning process
- Conversational, in which there is intermediate communication between a person and the computer, either to provide additional assistance to the computer or to provide additional information to the user, or both

Of course, this distinction is not a strict one describing particular problem domains. As we just showed, mathematical theorem proving could be regarded as either. But for a particular application, one or the other of these types of systems will usually be desired and that decision will be important in the choice of a problem-solving method.

2.3.8 Problem Classification

When actual problems are examined from the point of view of all of these questions, it becomes apparent that there are several broad classes into which the problems fall. These classes can each be associated with a generic control strategy that is appropriate for solving the problem. For example, consider the generic problem of *classification*. The task here is to examine an input and then decide which of a set of known classes the input is an instance of. Most diagnostic tasks, including medical diagnosis as well as diagnosis of faults in mechanical devices, are examples of classification. Another example of a generic strategy is *propose and refine*. Many design and planning problems can be attacked with this strategy.

Depending on the granularity at which we attempt to classify problems and control strategies, we may come up with different lists of generic tasks and procedures. See Chandrasekaran [1986] and McDermott [1988] for two approaches to constructing such lists. The important thing to remember here, though, since we are about to embark on a discussion of a variety of problem-solving methods, is that there is no one single way of solving all problems. But neither must each new problem be considered totally *ab initio*. Instead, if we analyze our problems carefully and sort our problem-solving methods by the kinds of problems for which they are suitable, we will be able to bring to each new problem much of what we have learned from solving other, similar problems.

2.4 PRODUCTION SYSTEM CHARACTERISTICS

We have just examined a set of characteristics that distinguish various classes of problems. We have also argued that production systems are a good way to describe the operations that can be performed in a search for a solution to a problem. Two questions we might reasonably ask at this point are:

1. Can production systems, like problems, be described by a set of characteristics that shed some light on how they can easily be implemented?
2. If so, what relationships are there between problem types and the types of pro- duction systems best suited to solving the problems?

The answer to the first question is yes. Consider the following definitions of classes of production systems. A *monotonic production system* is a production system in which the application of a rule never prevents the later application of another rule that could also have been applied at the time the first rule was selected. A *nonmonotonic production system* is one in which this is not true. A *partially commutative production system* is a production system with the property that if the application of a particular sequence of rules transforms state x into state y, then any permutation of those rules that is allowable (i.e., each rule's preconditions are satisfied when it is applied) also transforms state x into state y. A *commutative production system* is a production system that is both monotonic and partially commutative.[8]

[8] This corresponds to the definition of a commutative production system given in Nilsson [1980].

3.2 HILL CLIMBING

Hill climbing is a variant of generate-and-test in which feedback from the test procedure is used to help the generator decide which direction to move in the search space. In a pure generate-and-test procedure, the test function responds with only a yes or no. But if the test function is augmented with a heuristic function[2] that provides an estimate of how close a given state is to a goal state, the generate procedure can exploit it as shown in the procedure below. This is particularly nice because often the computation of the heuristic function can be done at almost no cost at the same time that the test for a solution is being performed. Hill climbing is often used when a good heuristic function is available for evaluating states but when no other useful knowledge is available. For example, suppose you are in an unfamiliar city without a map and you want to get downtown. You simply aim for the tall buildings. The heuristic function is just distance between the current location and the location of the tall buildings and the desirable states are those in which this distance is minimized.

Recall from Section 2.3.4 that one way to characterize problems is according to their answer to the question, "Is a good solution absolute or relative?" Absolute solutions exist whenever it is possible to recognize a goal state just by examining it. Getting downtown is an example of such a problem. For these problems, hill climbing can terminate whenever a goal state is reached. Only relative solutions exist, however, for maximization (or minimization) problems, such as the traveling salesman problem. In these problems, there is no *a priori* goal state. For problems of this sort, it makes sense to terminate hill climibing when there is no reasonable alternative state to move to.

3.2.1 Simple Hill Climbing

The simplest way to implement hill climbing is as follows.

Algorithm: Simple Hill Climbing

1. Evaluate the initial state. If it is also a goal state, then return it and quit. Otherwise, continue with the initial state as the current state.
2. Loop until a solution is found or until there are no new operators left to be applied in the current state:
   (a) Select an operator that has not yet been applied to the current state and apply it to produce a new state.
   (b) Evaluate the new state.
      (i) If it is a goal state, then return it and quit.
      (ii) If it is not a goal state but it is better than the current state, then make it the current state.
      (iii) If it is not better than the current state, then continue in the loop.

The key difference between this algorithm and the one we gave for generate-and-test is the use of an evaluation function as a way to inject task-specific knowledge into the control process. It is the use of such knowledge that makes this and the other methods discussed in the rest of this chapter *heuristic* search methods, and it is that same knowledge that gives these methods their power to solve some otherwise intractable problems.

Notice that in this algorithm, we have asked the relatively vague question, "Is one state *better* than another?" For the algorithm to work, a precise definition of *better* must be provided. In some cases, it means a higher value of the heuristic function. In others, it means a lower value. It does not matter which, as long as a particular hill-climbing program is consistent in its interpretation.

To see how hill climbing works, let's return to the puzzle of the four colored blocks. To solve the problem, we first need to define a heuristic function that describes how close a particular configuration is to being a solution. One such function is simply the sum of the number of different colors on each of the four sides. A solution to the puzzle will have a value of 16. Next we need to define a set of rules that describe ways of transforming one configuration into another. Actually, one rule will suffice. It says simply pick a block and

[2] What we are calling the heuristic function is sometimes also called the *objective function*, particularly in the literature of mathematical optimization.

rotate it 90 degrees in any direction. Having provided these definitions, the next step is to generate a starting configuration. This can either be done at random or with the aid of the heuristic function described in the last section. Now hill climbing can begin. We generate a new state by selecting a block and rotating it. If the resulting state is better, then we keep it. If not, we return to the previous state and try a different perturbation.

3.2.2 Steepest-Ascent Hill Climbing

A useful variation on simple hill climbing considers all the moves from the current state and selects the best one as the next state. This method is called *steepest-ascent hill climbing* or *gradient search*. Notice that this contrasts with the basic method in which the first state that is better than the current state is selected. The algorithm works as follows.

Algorithm: Steepest-Ascent Hill Climbing

1. Evaluate the initial state. If it is also a goal state, then return it and quit. Otherwise, continue with the initial state as the current state.
2. Loop until a solution is found or until a complete iteration produces no change to current state:
   (a) Let *SUCC* be a state such that any possible successor of the current state will be better than *SUCC*.
   (b) For each operator that applies to the current state do:
      (i) Apply the operator and generate a new state.
      (ii) Evaluate the new state. If it is a goal state, then return it and quit. If not, compare it to *SUCC*. If it is better, then set *SUCC* to this state. If it is not better, leave *SUCC* alone.
   (c) If the *SUCC* is better than current state, then set current state to *SUCC*.

To apply steepest-ascent hill climbing to the colored blocks problem, we must consider all perturbations of the initial state and choose the best. For this problem, this is difficult since there are so many possible moves. There is a trade-off between the time required to select a move (usually longer for steepest-ascent hill climbing) and the number of moves required to get to a solution (usually longer for basic hill climbing) that must be considered when deciding which method will work better for a particular problem.

Both basic and steepest-ascent hill climbing may fail to find a solution. Either algorithm may terminate not by finding a goal state but by getting to a state from which no better states can be generated. This will happen if the program has reached either a local maximum, a plateau, or a ridge.

A *local maximum* is a state that is better than all its neighbors but is not better than some other states farther away. At a local maximum, all moves appear to make things worse. Local maxima are particularly frustrating because they often occur almost within sight of a solution. In this case, they are called *foothills*.

A *plateau* is a flat area of the search space in which a whole set of neighboring states have the same value. On a plateau, it is not possible to determine the best direction in which to move by making local comparisons.

A *ridge* is a special kind of local maximum. It is an area of the search space that is higher than surrounding areas and that itself has a slope (which one would like to climb). But the orientation of the high region, compared to the set of available moves and the directions in which they move, makes it impossible to traverse a ridge by single moves.

There are some ways of dealing with these problems, although these methods are by no means guaranteed:

- Backtrack to some earlier node and try going in a different direction. This is particularly reasonable if at that node there was another direction that looked as promising or almost as promising as the one that was chosen earlier. To implement this strategy, maintain a list of paths almost taken and go back to one of them if the path that was taken leads to a dead end. This is a fairly good way of dealing with local maxima.
- Make a big jump in some direction to try to get to a new section of the search space. This is a particularly good way of dealing with plateaus. If the only rules available describe single small steps, apply them several times in the same direction.
- Apply two or more rules before doing the test. This corresponds to moving in several directions at once. This is a particularly good strategy for dealing with ridges.

Even with these first-aid measures, hill climbing is not always very effective. It is particularly unsuited to problems where the value of the heuristic function drops off suddenly as you move away from a solution. This is often the case whenever any sort of threshold effect is present. Hill climbing is a local method, by which we mean that it decides what to do next by looking only at the "immediate" consequences of its choice rather than by exhaustively exploring all the consequences. It shares with other local methods, such as the nearest neighbor heuristic described in Section 2.2.2, the advantage of being less combinatorially explosive than comparable global methods. But it also shares with other local methods a lack of a guarantee that it will be effective. Although it is true that the hill-climbing procedure itself looks only one move ahead and not any farther, that examination may in fact exploit an arbitrary amount of global information if that information is encoded in the heuristic function. Consider the blocks world problem shown in Fig. 3.1. Assume the same operators (i.e., pick up one block and put it on the table; pick up one block and put it on another one) that were used in Section 2.3.1. Suppose we use the following heuristic function:

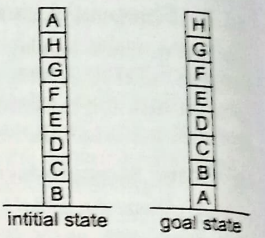

Fig. 3.1 *A Hill-Climbing Problem*

> **Local:** Add one point for every block that is resting on the thing it is supposed to be resting on. Subtract one point for every block that is sitting on the wrong thing.

Using this function, the goal state has a score of 8. The initial state has a score of 4 (since it gets one point added for blocks C, D, E, F, G, and H and one point subtracted for blocks A and B). There is only one move from the initial state, namely to move block A to the table. That produces a state with a score of 6 (since now A's position causes a point to be added rather than subtracted). The hill-climbing procedure will accept that move. From the new state, there are three possible moves, leading to the three states shown in Fig. 3.2. These states have the scores: (*a*) 4, (*b*) 4, and (*c*) 4. Hill climbing will halt because all these states have lower scores than the current state. The process has reached a local maximum that is not the global maximum. The problem is that by purely local examination of support structures, the current state appears to be better than any of its successors because more blocks rest on the correct objects. To solve this problem, it is necessary to disassemble a good local structure (the stack B through H) because it is in the wrong global context.

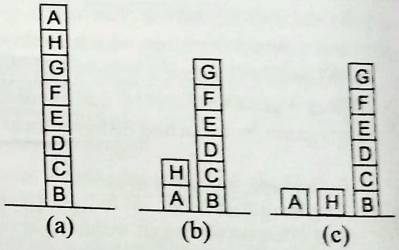

Fig. 3.2 *Three Possible Moves*

We could blame hill climbing itself for this failure to look far enough ahead to find a solution. But we could also blame the heuristic function and try to modify it. Suppose we try the following heuristic function in place of the first one:

> **Global:** For each block that has the correct support structure (i.e., the complete structure underneath it is exactly as it should be), add one point for every block in the support structure. For each block that has an incorrect support structure, subtract one point for every block in the existing support structure.

Using this function, the goal state has the score 28 (1 for B, 2 for C, etc.). The initial state has the score —28. Moving A to the table yields a state with a score of —21 since A no longer has seven wrong blocks under it. The three states that can be produced next now have the following scores: (*a*) –28, (*b*) –16, and (*c*) –15. This time, steepest-ascent hill climbing will choose move (*c*), which is the correct one. This new heuristic function captures the two key aspects of this problem: incorrect structures are bad and should be taken apart:

and correct structures are good and should be built up. As a result, the same hill climbing procedure that failed with the earlier heuristic function now works perfectly.

Unfortunately, it is not always possible to construct such a perfect heuristic function. For example, consider again the problem of driving downtown. The perfect heuristic function would need to have knowledge about one-way and dead-end streets, which, in the case of a strange city, is not always available. And even if perfect knowledge is, in principle, available, it may not be computationally tractable to use. As an extreme example, imagine a heuristic function that computes a value for a state by invoking its own problem-solving procedure to look ahead from the state it is given to find a solution. It then knows the exact cost of finding that solution and can return that cost as its value. A heuristic function that does this converts the local hill-climbing procedure into a global method by embedding a global method within it. But now the computational advantages of a local method have been lost. Thus it is still true that hill climbing can be very inefficient in a large, rough problem space. But it is often useful when combined with other methods that get it started in the right general neighborhood.

3.2.3 Simulated Annealing

Simulated annealing is a variation of hill climbing in which, at the beginning of the process, some downhill

3.3 BEST-FIRST SEARCH

Until now, we have really only discussed two systematic control strategies, breadth-first search and depth-first search (of several varieties). In this section, we discuss a new method, best-first search, which is a way of combining the advantages of both depth-first and breadth-first search into a single method.

3.3.1 OR Graphs

Depth-first search is good because it allows a solution to be found without all competing branches having to be expanded. Breadth-first search is good because it does not get trapped on dead-end paths. One way of combining the two is to follow a single path at a time, but switch paths whenever some competing path looks more promising than the current one does.

At each step of the best-first search process, we select the most promising of the nodes we have generated so far. This is done by applying an appropriate heuristic function to each of them. We then expand the chosen node by using the rules to generate its successors. If one of them is a solution, we can quit. If not, all those new nodes are added to the set of nodes generated so far. Again the most promising node is selected and the process continues. Usually what happens is that a bit of depth-first searching occurs as the most promising branch is explored. But eventually, if a solution is not found, that branch will start to look less promising than one of the top-level branches that had been ignored. At that point, the now more promising, previously ignored branch will be explored. But the old branch is not forgotten.. Its last node remains in the set of generated but unexpanded nodes. The search can return to it whenever all the others get bad enough that it is again the most promising path.

Figure 3.3 shows the beginning of a best-first search procedure. Initially, there is only one node, so it will be expanded. Doing so generates three new nodes. The heuristic function, which, in this example, is an estimate of the cost of getting to a solution from a given node, is applied to each of these new nodes. Since node D is the most promising, it is expanded next, producing two successor nodes, E and F. But then the heuristic function is applied to them. Now another path, that going through node B, looks more promising, so it is pursued, generating nodes G and H. But again when these new nodes are evaluated they look less promising than another path, so attention is returned to the path through D to E. E is then expanded, yielding nodes I and J. At the next step, J will be expanded, since it is the most promising. This process can continue until a solution is found.

Notice that this procedure is very similar to the procedure for steepest-ascent hill climbing, with two exceptions. In hill climbing, one move is selected and all the others are rejected, never to be reconsidered. This produces the straightline behavior that is characteristic of hill climbing. In best-first search, one move is selected, but the others are kept around so that they can be revisited later if the selected path becomes less

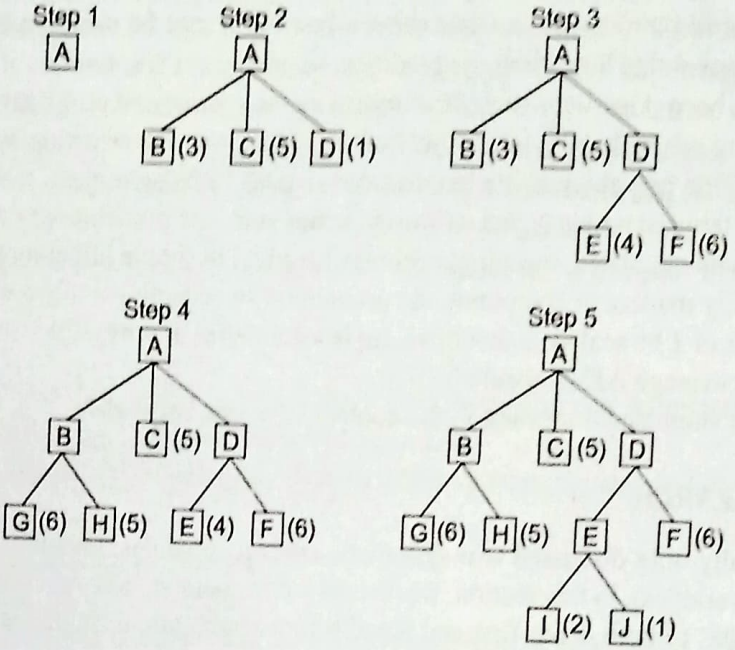

Fig. 3.3 *A Best-First Search*

promising.[3] Further, the best available state is selected in best-first search, even if that state has a value that is lower than the value of the state that was just explored. This contrasts with hill climbing, which will stop if there are no successor states with better values than the current state.

Although the example shown above illustrates a best-first search of a tree, it is sometimes important to search a graph instead so that duplicate paths will not be pursued. An algorithm to do this will operate by searching a directed graph in which each node represents a point in the problem space. Each node will contain, in addition to a description of the problem state it represents, an indication of how promising it is, a parent link that points back to the best node from which it came, and a list of the nodes that were generated from it. The parent link will make it possible to recover the path to the goal once the goal is found. The list of successors will make it possible, if a better path is found to an already existing node, to propagate the improvement down to its successors. We will call a graph of this sort an *OR graph*, since each of its branches represents an alternative problem-solving path.

To implement such a graph-search procedure, we will need to use two lists of nodes:

- *OPEN* — nodes that have been generated and have had the heuristic function applied to them but which have not yet been examined (i.e., had their successors generated). *OPEN* is actually a priority queue in which the elements with the highest priority are those with the most promising value of the heuristic function. Standard techniques for manipulating priority queues can be used to manipulate the list.
- *CLOSED* — nodes that have already been examined. We need to keep these nodes in memory if we want to search a graph rather than a tree, since whenever a new node is generated, we need to check whether it has been generated before.

We will also need a heuristic function that estimates the merits of each node we generate. This will enable the algorithm to search more promising paths first. Call this function f' (to indicate that it is an approximation to

[3] In a variation of best-first search, called *beam search*, only the n most promising states are kept for future consideration. This procedure is more efficient with respect to memory but introduces the possibility of missing a solution altogether by pruning the search tree too early.

function/that gives the true evaluation of the node). For many applications, it is convenient to define this function as the sum of two components that we call g and h'. The function g is a measure of the cost of getting from the initial state to the current node. Note that g is not an estimate of anything; it is known to be the exact sum of the costs of applying each of the rules that were applied along the best path to the node. The function h' is an estimate of the additional cost of getting from the current node to a goal state. This is the place where knowledge about the problem domain is exploited. The combined function f', then, represents an estimate of the cost of getting from the initial state to a goal state along the path that generated the current node. If more than one path generated the node, then the algorithm will record the best one. Note that because g and h' must be added, it is important that h' be a measure of the cost of getting from the node to a solution (i.e., good nodes get low values; bad nodes get high values) rather than a measure of the goodness of a node (i.e., good nodes get high values). But that is easy to arrange with judicious placement of minus signs. It is also important that g be nonnegative. If this is not true, then paths that traverse cycles in the graph will appear to get better as they get longer.

The actual operation of the algorithm is very simple. It proceeds in steps, expanding one node at each step, until it generates a node that corresponds to a goal state. At each step, it picks the most promising of the nodes that have so far been generated but not expanded. It generates the successors of the chosen node, applies the heuristic function to them, and adds them to the list of open nodes, after checking to see if any of them have been generated before. By doing this check, we can guarantee that each node only appears once in the graph, although many nodes may point to it as a successor. Then the next step begins.

This process can be summarized as follows.

Algorithm: Best-First Search

1. Start with *OPEN* containing just the initial state.
2. Until a goal is found or there are no nodes left on *OPEN* do:
   (a) Pick the best node on *OPEN*.
   (b) Generate its successors.
   (c) For each successor do:
      (i) If it has not been generated before, evaluate it, add it to *OPEN*, and record its parent.
      (ii) If it has been generated before, change the parent if this new path is better than the previous one. In that case, update the cost of getting to this node and to any successors that this node may already, have.

The basic idea of this algorithm is simple. Unfortunately, it is rarely the case that graph traversal algorithms are simple to write correctly. And it is even rarer that it is simple to guarantee the correctness of such algorithms. In the section that follows, we describe this algorithm in more detail as an example of the design and analysis of a graph-search program.

3.3.2 The A* Algorithm

The best-first search algorithm that was just presented is a simplification of an algorithm called A*, which was first presented by Hart *et al.* [1968; 1972]. This algorithm uses the same f', g, and h' functions, as well as the lists *OPEN* and *CLOSED*, that we have already described.

Algorithm: A*

1. Start with *OPEN* containing only the initial node. Set that node's g value to 0, its h' value to whatever it is, and its f' value to $h' + 0$, or h'. Set *CLOSED* to the empty list.
2. Until a goal node is found, repeat the following procedure: If there are no nodes on *OPEN*, report failure. Otherwise, pick the node on *OPEN* with the lowest f' value. Call it *BESTNODE*. Remove it from *OPEN*. Place it on *CLOSED*. See if *BESTNODE* is a goal node. If so, exit and report a solution (either *BESTNODE* if all we want is the node or the path that has been created between the initial state

and *BESTNODE* if we are interested in the path). Otherwise, generate the successors of *BESTNOD* but do not set *BESTNODE* to point to them yet. (First we need to see if any of them have already be generated.) For each such *SUCCESSOR*, do the following:

(a) Set *SUCCESSOR* to point back to *BESTNODE*. These backwards links will make it possible recover the path once a solution is found.

(b) Compute $g(SUCCESSOR) = g(BESTNODE)$ + the cost of getting from *BESTNODE* *SUCCESSOR*.

(c) See if *SUCCESSOR* is the same as any node on *OPEN* (i.e., it has already been generated but n processed). If so, call that node *OLD*. Since this node already exists in the graph, we can thr *SUCCESSOR* away and add *OLD* to the list of *BESTNODE's* successors. Now we must deci whether *OLD's* parent link should be reset to point to *BESTNODE*. It should be if the path we ha just found to *SUCCESSOR* is cheaper than the current best path to *OLD* (since *SUCCESSOR* a *OLD* are really the same node). So see whether it is cheaper to get to *OLD* via its current parent to *SUCCESSOR* via *BESTNODE* by comparing their g values. If *OLD* is cheaper (or just as chea then we need do nothing. If *SUCCESSOR* is cheaper, then reset *OLD's* parent link to point *BESTNODE*, record the new cheaper path in $g(OLD)$, and update f'(OLD).

(d) If *SUCCESSOR* was not on *OPEN*, see if it is on *CLOSED*. If so, call the node on *CLOSED* O and add *OLD* to the list of *BESTNODE's*, successors. Check to see if the new path or the old pa is better just as in step 2(*c*), and set the parent link-and g and f' values appropriately. If we ha just found a better path to *OLD*, we must propagate the improvement to *OLD's* successors. This a bit tricky. *OLD* points to its successors. Each successor in turn points to its successors, and forth, until each branch terminates with a node that either is still on *OPEN* or has no successors. to propagate the new cost downward, do a depth-first traversal of the tree starting at *OLD*, changi each node's g value (and thus also its f' value), terminating each branch when you reach eithe node with no successors or a node to which an equivalent or better path has already been found This condition is easy to check for. Each node's parent link points back to its best known pare As we propagate down to a node, see if its parent points to the node we are coming from. If s continue the propagation. If not, then its g value already reflects the better path of which it is pa So the propagation may stop here. But it is possible that with the new value of g being propagat downward, the path we are following may become better than the path through the current pare So compare the two. If the path through the current parent is still better, stop the propagation. If t path we are propagating through is now better, reset the parent and continue propagation.

(e) If *SUCCESSOR* was not already on either *OPEN* or *CLOSED*, then put it on *OPEN*, and add it the list of *BESTNODE's* successors. Compute $f'(SUCCESSOR) = g(SUCCESSOR)$ $h'(SUCCESSOR)$.

Several interesting observations can be made about this algorithm. The first concerns the role of the function. It lets us choose which node to expand next on the basis not only of how good the node itself looks (measured by h'), but also on the basis of how good the path to the node was. By incorporating g into f', we w not always choose as our next node to expand the node that appears to be closest to the goal. This is useful if w care about the path we find. If, on the other hand, we only care about getting to a solution somehow, we can defi g always to be 0, thus always choosing the node that seems closest to a goal. If we want to find a path involvi the fewest number of steps, then we set the cost of going from a node to its successor as a constant, usually 1. I on the other hand, we want to find the cheapest path and some operators cost more than others, then we set th

[4] This second check guarantees that the algorithm will terminate even if there are cycles in the graph. If there is a cycl then the second time that a given node is visited, the path will be no better than the first time and so propagation will sto

cost of going from one node to another to reflect those costs. Thus the A* algorithm can be used whether we are interested in finding a minimal-cost overall path or simply any path as quickly as possible.

The second observation involves h', the estimator of h, the distance of a node to the goal. If h' is a perfect estimator of h, then A* will converge immediately to the goal with no search. The better h' is, the closer we will get to that direct approach. If, on the other hand, the value of h' is always 0, the search will be controlled by g. If the value of g is also 0, the search strategy will be random. If the value of g is always 1, the search will be breadth first. All nodes on one level will have lower g values, and thus lower f' values than will all nodes on the next level. What if, on the other hand, h' is neither perfect nor 0? Can we say anything interesting about the behavior of the search? The answer is yes if we can guarantee that h' never overestimates h. In that case, the A* algorithm is guaranteed to find an optimal (as determined by g) path to a goal, if one exists. This can easily be seen from a few examples.[5]

Consider the situation shown in Fig. 3.4. Assume that the cost of all arcs is 1. Initially, all nodes except A are on *OPEN* (although the Fig. shows the situation two steps later, after B and E have been expanded). For each node, f' is indicated as the sum of h' and g. In this example, node B has the lowest f', 4, so it is expanded first. Suppose it has only one successor E, which also appears to be three moves away from a goal. Now $f'(E)$ is 5, the same as $f'(C)$. Suppose we resolve this in favor of the path we are currently following. Then we will expand E next. Suppose it too has a single successor F, also judged to be three moves from a goal. We are clearly using up moves and making no progress. But $f'(F) = 6$, which is greater than $f'(C)$. So we will expand C next. Thus we see that by underestimating $h'(B)$ we have wasted some effort. But eventually we discover that B was farther away than we thought and we go back and try another path.

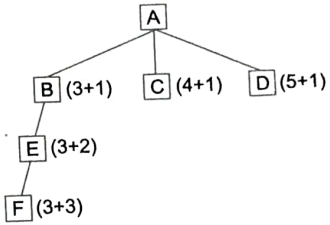

Fig. 3.4 *h' Underestimates h*

Now consider the situation shown in Fig. 3.5. Again we expand B on the first step. On the second step we again expand E. At the next step we expand F, and finally we generate G, for a solution path of length 4. But suppose there is a direct path from D to a solution, giving a path of length 2. We will never find it. By overestimating $h'(D)$ we make D look so bad that we may find some other, worse solution without ever expanding D. In general, if h' might overestimate h, we cannot be guaranteed of finding the cheapest path solution unless we expand the entire graph until all paths are longer than the best solution. An interesting question is, "Of what practical significance is the theorem that if *h!* never overestimates h then A* is admissible?" The answer is, "almost none," because, for most real problems, the only way to guarantee that *hi* never overestimates h is to set it to zero. But then we are back to breadth-first search, which is admissible but not efficient. But there is a corollary to this theorem that is very useful. We can state it loosely as follows:

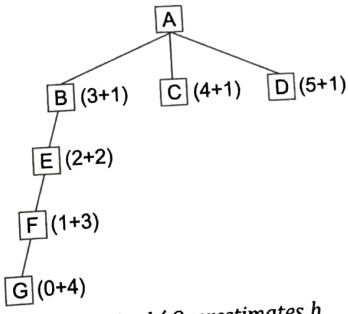

Fig. 3.5 *h' Overestimates h*

Graceful Decay of Admissibility: If h' rarely overestimates h by more than δ, then the A* algorithm will rarely find a solution whose cost is more than δ greater than the cost of the optimal solution.

The formalization and proof of this corollary will be left as an exercise.

The third observation we can make about the A* algorithm has to do with the relationship between trees and graphs. The algorithm was stated in its most general form as it applies to graphs. It can, of course, be

[5] A search algorithm that is guaranteed to find an optimal path to a goal, if one exists, is called *admissible* [Nilsson, 1980].

simplified to apply to trees by not bothering to check whether a new node is already on *OPEN* or *CLOSED*. This makes it faster to generate nodes but may result in the same search being conducted many times if nodes are often duplicated.

Under certain conditions, the A* algorithm can be shown to be optimal in that it generates the fewest nodes in the process of finding a solution to a problem. Under other conditions it is not optimal. For formal discussions of these conditions, see Gelperin [1977] and Martelli [1977].

3.3.3 Agendas

In our discussion of best-first search in OR graphs, we assumed that we could evaluate multiple paths to the same node independently of each other. For example, in the water jug problem, it makes no difference to the

3.5 CONSTRAINT SATISFACTION

Many problems in AI can be viewed as problems of *constraint satisfaction* in which the goal is to discover some problem state that satisfies a given set of constraints. Examples of this sort of problem include cryptarithmetic puzzles (as described in Section 2.6) and many real-world perceptual labeling problems. Design tasks can also be viewed as constraint-satisfaction problems in which a design must be created within fixed limits on time, cost and materials.

By viewing a problem as one of constraint satisfaction, it is often possible to reduce substantially the amount of search that is required as compared with a method that attempts to form partial solutions directly by choosing specific values for components of the eventual solution. For example, a straightforward search procedure to solve a cryptarithmetic problem might operate in a state space of partial solutions in which letters are assigned particular numbers as their valuer. A depth-first control scheme could then follow a path of assignments until either a solution or an inconsistency is discovered. In contrast, a constraint satisfaction approach to solving this problem avoids making guesses on particular assignments of numbers to letters until it has to. Instead, the initial set of constraints, which says that each number may correspond to only one letter and that the sums of the digits must be as they are given in the problem, is first augmented to include restrictions that can be inferred from the rules of arithmetic. Then, although guessing may still be required, the number of allowable guesses is reduced and so the degree of search is curtailed.

Constraint satisfaction is a search procedure that operates in a space of constraint sets. The initial state contains the constraints that are originally given in the problem description. A Goal State is any state that has been constrained "enough," where "enough" must be defined for each problem. For example, for cryptarithmetic, enough means that each letter has been assigned a unique numeric value.

Constraint satisfaction is a two-step process. First, constraints are discovered and propagated as far as possible throughout the system. Then, if there is still not a solution, search begins. A guess about something is made and added as a new constraint. Propagation can then occur with this new constraint, and so forth.

The first step, propagation, arises from the fact that there are usually dependencies among the constraints. These dependencies occur because many constraints involve more than one object and many objects participate in more than one constraint. So, for example, assume we start with one constraint, N = E + 1. Then, if we added the constraint N = 3, we could propagate that to get a stronger constraint on E, namely that E = 2. Constraint propagation also arises from the presence of inference rules that allow additional constraints to be inferred from the ones that are given. Constraint propagation terminates for one of two reasons. First, a contradiction may be detected. If this happens, then there is no solution consistent with all the known constraints. If the contradiction involves only those constraints that were given as part of the problem specification (as opposed to ones that were guessed during problem solving), then no solution exists. The second possible reason for termination is that the propagation has run out of steam and there are no further changes that can be made on the basis of current knowledge. If this happens and a solution has not yet been adequately specified, then search is necessary to get the process moving again.

At this point, the second step begins. Some hypothesis about a way to strengthen the constraints must be made. In the case of the cryptarithmetic problem, for example, this usually means guessing a particular value for some letter. Once this has been done, constraint propagation can begin again from this new state. If a solution is found, it can be reported. If still more guesses are required, they can be made. If a contradiction is detected, then backtracking can be used to try a different guess and proceed with it. We can state this procedure more precisely as follows:

Algorithm: Constraint Satisfaction

1. Propagate available constraints. To do this, first set *OPEN* to the set of all objects that must have values assigned to them in a complete solution. Then do until an inconsistency is detected or until *OPEN* is empty:
   (a) Select an object *OB* from *OPEN*. Strengthen as much as possible the set of constraints that apply to *OB*.
   (b) If this set is different from the set that was assigned the last time *OB* was examined or if this is the first time *OB* has been examined, then add to *OPEN* all objects that share any constraints with *OB*.
   (c) Remove *OB* from *OPEN*.
2. If the union of the constraints discovered above defines a solution, then quit and report the solution.
3. If the union of the constraints discovered above defines a contradiction, then return failure.
4. If neither of the above occurs, then it is necessary to make a guess at something in order to proceed. To do this, loop until a solution is found or all possible solutions have been eliminated:
   (a) Select an object whose value is not yet determined and select a way of strengthening the constraints on that object.
   (b) Recursively invoke constraint satisfaction with the current set of constraints augmented by the strengthening constraint just selected.

This algorithm has been stated as generally as possible. To apply it in a particular problem domain requires the use of two kinds of rules: rules that define the way constraints may validly be propagated and rules that suggest guesses when guesses are necessary. It is worth noting, though, that in some problem domains guessing

may not be required. For example, the Waltz algorithm for propagating line labels in a picture, which is described in Chapter 14, is a version of this constraint satisfaction algorithm with the guessing step eliminated. In general, the more powerful the rules for propagating constraints, the less need there is for guessing.

To see how this algorithm works, consider the cryptarithmetic problem shown in Fig. 3.13. The goal state is a problem state in which all letters have been assigned a digit in such a way that all the initial constraints are satisfied.

Problem:

```
  SEND
+ MORE
.....
 MONEY
```

Initial State:

No two letters have the same value.
The sums of the digits must be as shown in the problem.

Fig. 3.13 *A Cryptarithmetic Problem*

The solution process proceeds in cycles. At each cycle, two significant things are done (corresponding to steps 1 and 4 of this algorithm):

1. Constraints are propagated by using rules that correspond to the properties of arithmetic.
2. A value is guessed for some letter whose value is not yet determined.

In the first step, it does not usually matter a great deal what order the propagation is done in, since all available propagations will be performed before the step ends. In the second step, though, the order in which guesses are tried may have a substantial impact on the degree of search that is necessary. A few useful heuristics can help to select the best guess to try first. For example, if there is a letter that has only two possible values and another with six possible values, there is a better chance of guessing right on the first than on the second. Another useful heuristic is that if there is a letter that participates in many constraints then it is a good idea to prefer it to a letter that participates in a few. A guess on such a highly constrained letter will usually lead quickly either to a contradiction (if it is wrong) or to the generation of many additional constraints (if it is right). A guess on a less constrained letter, on the other hand, provides less information. The result of the first few cycles of processing this example is shown in Fig. 3.14. Since constraints never disappear at lower levels, only the ones being added are shown for each level. It will not be much harder for the problem solver to access the constraints as a set of lists than as one long list, and this approach is efficient both in terms of storage space and the ease of backtracking. Another reasonable approach for this problem would be to store all the constraints in one central database and also to record at each node the changes that must be undone during backtracking. C1, C2, C3, and C4 indicate the carry bits out of the columns, numbering from the right.

Initially, rules for propagating constraints generate the following additional constraints:

- M = 1, since two single-digit numbers plus a carry cannot total more than 19.
- S = 8 or 9, since S + M + C3 > 9 (to generate the carry) and M = 1, S + 1 + C3 > 9, so S + C3 > 8 and C3 is at most 1.
- O = 0, since S + M(1) + C3 (<= 1) must be at least 10 to generate a carry and it can be at most 11. But M is already 1, so O must be 0.
- N = E or E + 1, depending on the value of C2. But N cannot have the same value as E. So N = E + 1 and C2 is 1.

- In order for C2 to be 1, the sum of N + R + C1 must be greater than 9, so N + R must be greater than 8.
- N + R cannot be greater than 18, even with a carry in, so E cannot be 9.

At this point, let us assume that no more constraints can be generated. Then, to make progress from here, we must guess. Suppose E is assigned the value 2. (We chose to guess a value for E because it occurs three times and thus interacts highly with the other letters.) Now the next cycle begins.

The constraint propagator now observes that:

- N = 3, since N = E + 1.
- R = 8 or 9, since R + N (3) + C1 (1 or 0) = 2 or 12. But since N is already 3, the sum of these nonnegative numbers cannot be less than 3. Thus R + 3 + (0 or 1) = 12 and R = 8 or 9.
- 2 + D = Y or 2 + D = 10 + Y, from the sum in the rightmost column.

Again, assuming no further constraints can be generated, a guess is required. Suppose C1 is chosen to guess a value for. If we try the value 1, then we eventually reach dead ends, as shown in the Fig.. When this happens, the process will backtrack and try C1 = 0.

A couple of observations are worth making on this process. Notice that all that is required of the constraint propagation rules is that they do not infer spurious constraints. They do not have to infer all legal ones. For example, we could have reasoned through to the result that C1 equals 0. We could have done so by observing that for C1 to be 1, the following must hold: 2 + D = 10 + Y. For this to be the case, D would have to be 8 or 9. But both S and R must be either 8 or 9 and three letters cannot share two values. So C1 cannot be 1. If we had realized this initially, some search could have been avoided. But since the constraint propagation rules we used were not that sophisticated, it took some search. Whether the search route takes more or less actual time than does the constraint propagation route depends on how long it takes to perform the reasoning required for constraint propagation.

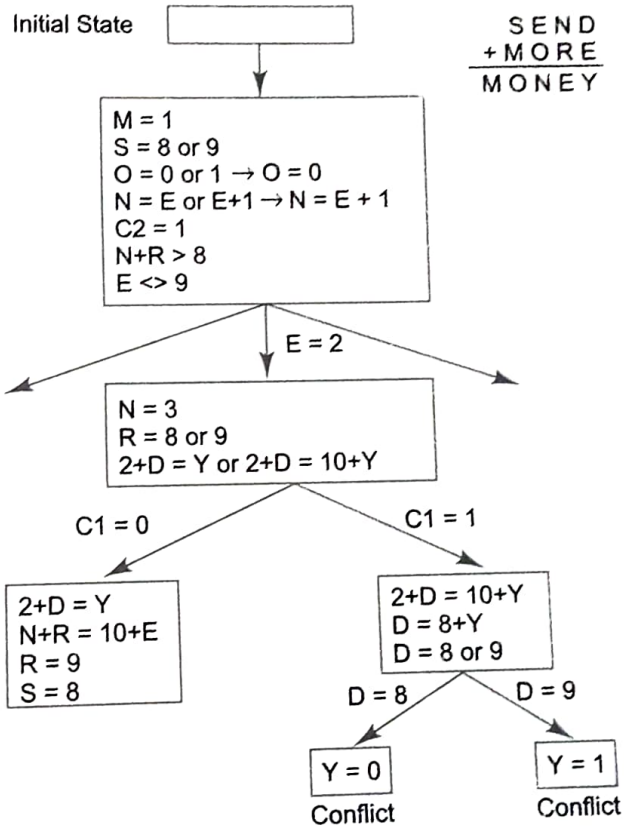

Fig. 3.14 *Solving a Cryptarithmetic Problem*

A second thing to notice is that there are often two kinds of constraints. The first kind is simple; they just list possible values for a single object. The second kind is more complex; they describe relationships between or among objects. Both kinds of constraints play the same role in the constraint satisfaction process, and in the cryptarithmetic example they were treated identically. For some problems, however, it may be useful to represent the two kinds of constraints differently. The simple, value-listing constraints are always dynamic, and so must always be represented explicitly in each problem state. The more complicated, relationship-expressing constraints are dynamic in the cryptarithmetic domain since they are different for each cryptarithmetic problem. But in many other domains they are static. For example, in the Waltz line labeling algorithm, the only binary constraints arise from the nature of the physical world, in which surfaces can meet in only a fixed number of possible ways. These ways are the same for all pictures that that algorithm may see. Whenever the binary constraints are static, it may be computationally efficient not to represent them explicitly in the state description but rather to encode them in the algorithm directly. When this is done, the only things that get propagated are possible values. But the essential algorithm is the same in both cases.

So far, we have described a fairly simple algorithm for constraint satisfaction in which chronological backtracking is used when guessing leads to an inconsistent set of constraints. An alternative is to use a more sophisticated scheme in which the specific cause of the inconsistency is identified and only constraints that depend on that culprit are undone. Others, even though they may have been generated after the culprit, are left alone if they are independent of the problem and its cause. This approach is called dependency directed backtracking (DDB). It is described in detail in Section 7.5.1.

3.6 MEANS-ENDS ANALYSIS

So far, we have presented a collection of search strategies that can reason either forward or backward, but for a given problem, one direction or the other must be chosen. Often, however, a mixture of the two directions is appropriate. Such a mixed strategy would make it possible to solve the major parts of a problem first and then go back and solve the small problems that arise in "gluing" the big pieces together. A technique known as *means-ends analysis* allows us to do that.

The means-ends analysis process centers around the detection of differences between the current state and the goal state. Once such a difference is isolated, an operator that can reduce the difference must be found. But perhaps that operator cannot be applied to the current state. So we set up a subproblem of getting to a state in which it can be applied. The kind of backward chaining in which operators are selected and then subgoals are set up to establish the preconditions of the operators is called *operator subgoaling*. But maybe the operator does not produce exactly the goal state we want. Then we have a second subproblem of getting from the state it does produce to the goal. But if the difference was chosen correctly and if the operator is really effective at reducing the difference, then the two subproblems should be easier to solve than the original problem. The means-ends analysis process can then be applied recursively. In order to focus the system's attention on the big problems first, the differences can be assigned priority levels. Differences of higher priority can then be considered before lower priority ones.

The first AI program to exploit means-ends analysis was the General Problem Solver (GPS) [Newell and Simon, 1963; Ernst and Newell, 1969]. Its design was motivated by the observation that people often use this technique when they solve problems. But GPS provides a good example of the fuzziness of the boundary between building programs that simulate what people do and building programs that simply solve a problem any way they can.

Just like the other problem-solving techniques we have discussed, means-ends analysis relies on a set of rules that can transform one problem state into another. These rules are usually not represented with complete state descriptions on each side. Instead, they are represented as a left side that describes the conditions that must be met for the rule to be applicable (these conditions are called the rule's *preconditions*) *and* a right side that describes those aspects of the problem state that will be changed by the application of the rule. A separate data structure called a *difference table* indexes the rules by the differences that they can be used to reduce.

Consider a simple household robot domain. The available operators are shown in Fig. 3.15, along with their preconditions and results. Figure 3.16 shows the difference table that describes when each of the operators is appropriate. Notice that sometimes there may be more than one operator that can reduce a given difference and that a given operator may be able to reduce more than one difference.

Suppose that the robot in this domain were given the problem of moving a desk with two things on it from one room to another. The objects on top must also be moved. The main difference between the start state and the goal state would be the location of the desk. To reduce this difference, either PUSH or CARRY could be chosen. If CARRY is chosen first, its preconditions must be met. This results in two more differences that must be reduced: the location of the robot and the size of the desk. The location of the robot can be handled by applying WALK, but there are no operators than can change the size of an object (since we did not include

Operator	*Preconditions*	*Results*
PUSH(obj, loc)	at(robot, obj)^ large(obj)^ clear(obj)^ armempty	at(obj, loc)^ at(robot, loc)
CARRY(obj, loc)	at(robot, obj)^ small(obj)	at(obj, loc)^ at(robot, loc)
WALK(loc)	none	at(robot, loc)
PICKUP(obj)	at(robot, obj)	holding(obj)
PUTDOWN(obj)	holding(obj)	¬holding(obj)
PLACE(obj1, obj2)	at(robot, obj2)^ holding(obj1)	on(obj1, obj2)

Fig. 3.15 *The Robot's Operators*

	Push	Carry	Walk	Pickup	Putdown	Place
Move object	*	*				
Move robot			*			
Clear object				*		
Get object on object						*
Get arm empty					*	*
Be holding object				*		

Fig. 3.16 *A Difference Table*

SAW-APART). So this path leads to a dead-end. Following the other branch, we attempt to apply PUSH. Figure 3.17 shows the problem solver's progress at this point. It has found a way of doing something useful. But it is not yet in a position to do that thing. And the thing does not get it quite to the goal state. So now the differences between A and B and between C and D must be reduced.

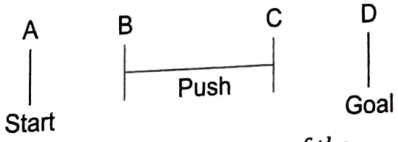

Fig. 3.17 *The Progress of the Means-Ends Analysis Method*

PUSH has four preconditions, two of which produce differences between the start and the goal states: the robot must be at the desk, and the desk must be clear. Since the desk is already large, and the robot's arm is empty, those two preconditions can be ignored. The robot can be brought to the correct location by using WALK. And the surface of the desk can be cleared by two uses of PICKUP. But after one PICKUP, an attempt to do the second results in another difference—the arm must be empty. PUTDOWN can be used to reduce that difference.

Once PUSH is performed, the problem state is close to the goal state, but not quite. The objects must be placed back on the desk. PLACE will put them there. But it cannot be applied immediately. Another difference must be eliminated, since the robot must be holding the objects. The progress of the problem solver at this point is shown in Fig. 3.18.

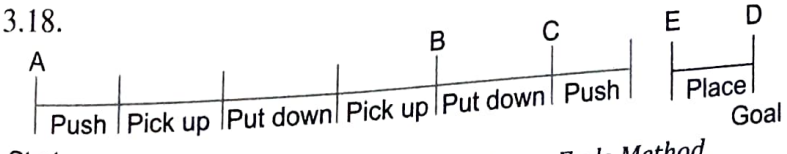

Fig. 3.18 *More Progress of the Means-Ends Method*

The final difference between C and E can be reduced by using WALK to get the robot back to the objects, followed by PICKUP and CARRY.

The process we have just illustrated (which we call MEA for short) can be summarized as follows:

Algorithm: Means-Ends Analysis (CURRENT, GOAL)

1. Compare *CURRENT* to *GOAL*. If there are no differences between them then return.
2. Otherwise, select the most important difference and reduce it by doing the following until success or failure is signaled:
   (a) Select an as yet untried operator *O* that is applicable to the current difference. If there are no such operators, then signal failure.
   (b) Attempt to apply *O* to *CURRENT*. Generate descriptions of two states: *O-START*, a state in which *O*'s preconditions are satisfied and *O-RESULT*, the state that would result if *O* were applied in *O-START*.
   (c) If
   (FIRST-PART ← *MEA(CURRENT, O-START))*
   and
   (LAST-PART ← *MEMO-RESULT, GOAL))*
   are successful, *then* signal success and return the result of concatenating *FIRST-PART, O,* and *LAST-PART*.

Many of the details of this process have been omitted in this discussion. In particular, the order in which differences are considered can be critical. It is important that significant differences be reduced before less critical ones. If this is not done, a great deal of effort may be wasted on situations that take care of themselves once the main parts of the problem are solved.

The simple process we have described is usually not adequate for solving complex problems. The number of permutations of differences may get too large. Working on one difference may interfere with the plan for reducing another. And in complex worlds, the required difference tables would be immense. In Chapter 13 we look at some ways in which the basic means-ends analysis approach can be extended to tackle some of these problems.

SUMMARY

In Chapter 2, we listed four steps that must be taken to design a program to solve an AI problem. The first two steps were:

1. Define the problem precisely. Specify the problem space, the operators for moving within the space, and the starting and goal state(s).
2. Analyze the problem to determine where it falls with respect to seven important issues.

The other two steps were to isolate and represent the task knowledge required, and to choose problem solving techniques and apply them to the problem. In this chapter, we began our discussion of the last step of this process by presenting some general-purpose, problem-solving methods. There are several important ways in which these algorithms differ, including:

- What the states in the search space(s) represent. Sometimes the states represent complete potential solutions (as in hill climbing). Sometimes they represent solutions that are partially specified (as in constraint satisfaction).

CHAPTER

12

GAME PLAYING

Every game of skill is susceptible of being played by an automaton.

—**Charles Babbage**
(1791-1871), English mathematician, philosopher, inventor and mechanical engineer

12.1 OVERVIEW

Games hold an inexplicable fascination for many people, and the notion that computers might play games has existed at least as long as computers. Charles Babbage, the nineteenth-century computer architect, thought about programming his Analytical Engine to play chess and later of building a machine to play tic-tac-toe [Bowden, 1953]. Two of the pioneers of the science of information and computing contributed to the fledgling computer game-playing literature. Claude Shannon [1950] wrote a paper in which he described mechanisms that could be used in a program to play chess. A few years later, Alan Turing described a chess-playing program, although he never built it. (For a description, see Bowden [1953].) By the early 1960s, Arthur Samuel had succeeded in building the first significant, operational game-playing program. His program played checkers and, in addition to simply playing the game, could learn from its mistakes and improve its performance [Samuel, 1963].

There were two reasons that games appeared to be a good domain in which to explore machine intelligence:

- They provide a structured task in which it is very easy to measure success or failure.
- They did not obviously require large amounts of knowledge. They were thought to be solvable by straightforward search from the starting state to a winning position.

The first of these reasons remains valid and accounts for continued interest in the area of game playing by machine. Unfortunately, the second is not true for any but the simplest games. For example, consider chess.

- The average branching factor is around 35.
- In an average game, each player might make 50 moves.
- So in order to examine the complete game tree, we would have to examine 35^{100} positions.

Thus it is clear that a program that simply does a straightforward search of the game tree will not be able to select even its first move during the lifetime of its opponent. Some kind of heuristic search procedure is necessary.

One way of looking at all the search procedures we have discussed is that they are essentially generate-and-test procedures in which the testing is done after varying amounts of work by the generator. At one extreme, the generator generates entire proposed solutions, which the tester then evaluates. At the other extreme, the generator generates individual moves in the search space, each of which is then evaluated by the tester and the most promising one is chosen. Looked at this way, it is clear that to improve the effectiveness of a search-based problem-solving program two things can be done:

- Improve the generate procedure so that only good moves (or paths) are generated.
- Improve the test procedure so that the best moves (or paths) will be recognized and explored first.

In game-playing programs, it is particularly important that both these things be done. Consider again the problem of playing chess. On the average, there are about 35 legal moves available at each turn. If we use a simple legal-move generator, then the test procedure (which probably uses some combination of search and a heuristic evaluation function) will have to look at each of them. Because the test procedure must look at so many possibilities, it must be fast. So it probably cannot do a very accurate job. Suppose, on the other hand, that instead of a legal-move generator, we use a *plausible-move generator* in which only some small number of promising moves are generated. As the number of legal moves available increases, it becomes increasingly important to apply heuristics to select only those that have some kind of promise. (So, for example, it is extremely important in programs that play the game of go [Benson *et al.*, 1979].) With a more selective move generator, the test procedure can afford to spend more time evaluating each of the moves it is given so it can produce a more reliable result. Thus by incorporating heuristic knowledge into both the generator and the tester, the performance of the overall system can be improved.

Of course, in game playing, as in other problem domains, search is not the only available technique. In some games, there are at least some times when more direct techniques are appropriate. For example, in chess, both openings and endgames are often highly stylized, so they are best played by table lookup into a database of stored patterns. To play an entire game then, we need to combine search-oriented and nonsearch-oriented techniques.

The ideal way to use a search procedure to find a solution to a problem is to generate moves through the problem space until a goal state is reached. In the context of game-playing programs, a goal state is one in which we win. Unfortunately, for interesting games such as chess, it is not usually possible, even with a good plausible-move generator, to search until a goal state is found. The depth of the resulting tree (or graph) and its branching factor are too great. In the amount of time available, it is usually possible to search a tree only ten or twenty moves (called *ply* in the game-playing literature) deep. Then, in order to choose the best move, the resulting board positions must be compared to discover which is most advantageous. This is done using a *static evaluation function*, which uses whatever information it has to evaluate individual board positions by estimating how likely they are to lead eventually to a win. Its function is similar to that of the heuristic function h' in the A* algorithm: in the absence of complete information, choose the most promising position. Of course, the static evaluation function could simply be applied directly to the positions generated by the proposed moves. But since it is hard to produce a function like this that is very accurate, it is better to apply it as many levels down in the game tree as time permits.

A lot of work in game-playing programs has gone into the development of good static evaluation functions.[1] A very simple static evaluation function for chess based on piece advantage was proposed by Turing—simply add the values of black's pieces (B), the values of white's pieces (W), and then compute the quotient W/B. A more sophisticated approach was that taken in Samuel's checkers program, in which the static evaluation function was a linear combination of several simple functions, each of which appeared as though it might be

[1] See Berliner [1979b] for a discussion of some theoretical issues in the design of static evaluation functions.

significant. Samuel's functions included, in addition to the obvious one, piece advantage, such things as capability for advancement, control of the center, threat of a fork, and mobility. These factors were then combined by attaching to each an appropriate weight and then adding the terms together. Thus the complete evaluation function had the form:

$$c_1 \times piece advantage + c_2 \times advancement + c_3 \times center control \ldots$$

There were also some nonlinear terms reflecting combinations of these factors. But Samuel did not know the correct weights to assign to each of the components. So he employed a simple learning mechanism in which components that had suggested moves that turned out to lead to wins were given an increased weight, while the weights of those that had led to losses were decreased.

Unfortunately, deciding which moves have contributed to wins and which to losses is not always easy. Suppose we make a very bad move, but then, because the opponent makes a mistake, we ultimately win the game. We would not like to give credit for winning to our mistake. The problem of deciding which of a series of actions is actually responsible for a particular outcome is called the *credit assignment problem* [Minsky, 1963]. It plagues many learning mechanisms, not just those involving games. Despite this and other problems, though, Samuel's checkers program was eventually able to beat its creator. The techniques it used to acquire this performance are discussed in more detail in Chapter 17.

We have now discussed the two important knowledge-based components of a good game-playing program: a good plausible-move generator and a good static evaluation function. They must both incorporate a great deal of knowledge about the particular game being played. But unless these functions are perfect, we also need a search procedure that makes it possible to look ahead as many moves as possible to see what may occur. Of course, as in other problem-solving domains, the role of search can be altered considerably by altering the amount of knowledge that is available to it. But, so far at least, programs that play nontrivial games rely heavily on search.

What search strategy should we use then? For a simple one-person game or puzzle, the A* algorithm described in Chapter 3 can be used. It can be applied to reason forward from the current state as far as possible in the time allowed. The heuristic function h' can be applied at terminal nodes and used to propagate values back up the search graph so that the best next move can be chosen. But because of their adversarial nature, this procedure is inadequate for two-person games such as chess. As values are passed back up, different assumptions must be made at levels where the program chooses the move and at the alternating levels where the opponent chooses. There are several ways that this can be done. The most commonly used method is the *minimax* procedure, which is described in the next section. An alternative approach is the B* algorithm [Berliner, 1979a], which works on both standard problem-solving trees and on game trees.

12.2 THE MINIMAX SEARCH PROCEDURE

The *minimax search procedure is* a depth-first, depth-limited search procedure. It was described briefly in Section 1.3.1. The idea is to start at the current position and use the plausible-move generator to generate the set of possible successor positions. Now we can apply the static evaluation function to those positions and simply choose the best one. After doing so, we can back that value up to the starting position to represent our evaluation of it. The starting position is exactly as good for us as the position generated by the best move we can make next. Here we assume that the static evaluation function returns large values to indicate good situations for us, so our goal is to *maximize* the value of the static evaluation function of the next board position.

An example of this operation is shown in Fig. 12.1. It assumes a static evaluation function that returns values ranging from – 10 to 10, with 10 indicating a win for us, –10 a win for the opponent, and 0 an even match. Since our goal is to maximize the value of the heuristic function, we choose to move to B. Backing B's value up to A, we can conclude that A's value is 8, since we know we can move to a position with a value of 8.

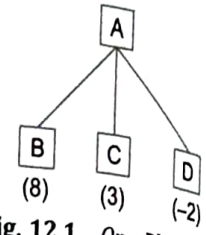

Fig. 12.1 *One-Ply Search*

But since we know that the static evaluation function is not completely accurate, we would like to carry the search farther ahead than one ply. This could be very important, for example, in a chess game in which we are in the middle of a piece exchange. After our move, the situation would appear to be very good, but, if we look one move ahead, we will see that one of our pieces also gets captured and so the situation is not as favorable as it seemed. So we would like to look ahead to see what will happen to each of the new game positions at-the next move which will be made by the opponent. Instead of applying the static evaluation function to each of the positions that we just generated, we apply the plausible-move generator, generating a set of successor positions for each position. If we wanted to stop here, at two-ply lookahead, we could apply the static evaluation function to each of these positions, as shown in Fig. 12.2.

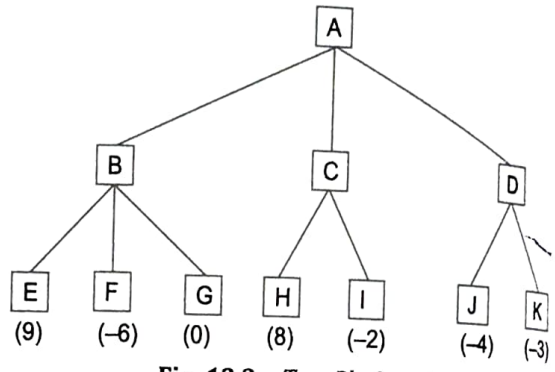

Fig. 12.2 *Two-Ply Search*

But now we must take into account that the opponent gets to choose which successor moves to make and thus which terminal value should be backed up to the next level. Suppose we made move B. Then the opponent must choose among moves E, F, and G. The opponent's goal is to *minimize* the value of the evaluation function, so he or she can be expected to choose move F. This means that if we make move B, the actual position in which we will end up one move later is very bad for us. This is true even though a possible configuration is that represented by node E, which is very good for us. But since at this level we are not the ones to move, we will not get to choose it. Figure 12.3 shows the result of propagating the new values up the tree. At the level representing the opponent's choice, the minimum value was chosen and backed up. At the level representing our choice, the maximum value was chosen.

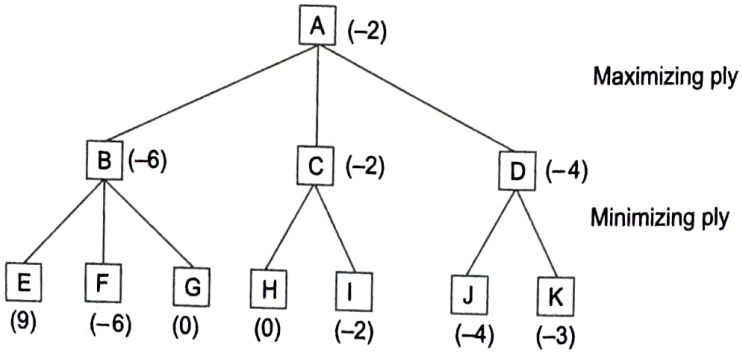

Fig. 12.3 *Backing Up the Values of a Two-Ply Search*

Once the values from the second ply are backed up, it becomes clear that the correct move for us to make at the first level, given the information we have available, is C, since there is nothing the opponent can do from there to produce a value worse than –2. This process can be repeated for as many ply as time allows, and

the more accurate evaluations that are produced can be used to choose the correct move at the top level. The alternation of maximizing and minimizing at alternate ply when evaluations are being pushed back up corresponds to the opposing strategies of the two players and gives this method the name minimax.

Having described informally the operation of the minimax procedure, we now describe it precisely. It is a straightforward recursive procedure that relies on two auxiliary procedures that are specific to the game being played:

1. MOVEGEN(*Position, Player*)—The plausible-move generator, which returns a list of nodes representing the moves that can be made by *Player* in *Position*. We call the two players PLAYER-ONE and PLAYER-TWO; in a chess program, we might use the names BLACK and WHITE instead.
2. STATIC(*Position, Player*)—The static evaluation function, which returns a num ber representing the goodness of *Position* from the standpoint of *Player*.[2]

As with any recursive program, a critical issue in the design of the MINIMAX procedure is when to stop the recursion and simply call the static evaluation function. There are a variety of factors that may influence this decision. They include:

- Has one side won?
- How many ply have we already explored?
- How promising is this path?
- How much time is left?
- How stable is the configuration?

For the general MINIMAX procedure discussed here, we appeal to a function, DEEP-ENOUGH, which is assumed to evaluate all of these factors and to return TRUE if the search should be stopped at the current level and FALSE otherwise. Our simple implementation of DEEP-ENOUGH will take two parameters, *Position* and *Depth*. It will ignore its *Position* parameter and simply return TRUE if its *Depth* parameter exceeds a constant cutoff value.

One problem that arises in defining MINIMAX as a recursive procedure is that it needs to return not one but two results:

- The backed-up value of the path it chooses.
- The path itself. We return the entire path even though probably only the first element, representing the best move from the current position, is actually needed.

We assume that MINIMAX returns a structure containing both results and that we have two functions, VALUE and PATH, that extract the separate components.

Since we define the MINIMAX procedure as a recursive function, we must also specify how it is to be called initially. It takes three parameters, a board position, the current depth of the search, and the player to move. So the initial call to compute the best move from the position CURRENT should be

```
MINIMAX(CURRENT, 0, PLAYER-ONE)
```

if PLAYER-ONE is to move, or

```
MINIMAX(CURRENT, 0, PLAYER-TWO)
```

if PLAYER-TWO is to move.

[2] This may be a bit confusing, but it need not be. In all the examples in this chapter so far (including Fig. 12.2 and 12.3), we have assumed that all values of STATIC are from the point of view of the initial (maximizing) player. It turns out to be easier when defining the algorithm, though, to let STATIC alternate perspectives so that we do not need to write separate procedures for the two levels. It is easy to modify STATIC for this purpose; we merely compute the value *of Position* from PLAYER-ONE's perspective, then invert the value if STATIC's parameter is PLAYER-TWO.

Algorithm: MINIMAX(Position, Depth, Player)

1. If DEEP-ENOUGH(*Position, Depth*), then return the structure

   VALUE = *STATIC*(*Position, Player*);
   PATH = nil

   This indicates that there is no path from this node and that its value is that determined by the static evaluation function.
2. Otherwise, generate one more ply of the tree by calling the function MOVE-GEN(Position Player) and setting SUCCESSORS to the list it returns.
3. If SUCCESSORS is empty, then there are no moves to be made, so return the same structure that would have been returned if DEEP-ENOUGH had returned true.
4. If SUCCESSORS is not empty, then examine each element in turn and keep track of the best one. This is done as follows.
   Initialize BEST-SCORE to the minimum value that STATIC can return. It will be updated to reflect the best score that can be achieved by an element of SUCCESSORS.
   For each element SUCC of SUCCESSORS, do the following:
   (a) Set RESULT-SUCC to
      MINIMAX(SUCC, *Depth* + 1, OPPOSITE(*Player*))
   This recursive call to MINIMAX will actually carry out the exploration of SUCC.
   (b) Set NEW-VALUE to - VALUE(RESULT-SUCC). This will cause it to reflect the merits of the position from the opposite perspective from that of the next lower level.
   (c) If NEW-VALUE > BEST-SCORE, then we have found a successor that is better than any that have been examined so far. Record this by doing the following:
      (i) Set BEST-SCORE to NEW-VALUE.
      (ii) The best known path is now from CURRENT to SUCC and then on to the appropriate path down from SUCC as determined by the recursive call to MINIMAX. So set BEST-PATH to the result of attaching SUCC to the front of PATH(RESULT-SUCC).
5. Now that all the successors have been examined, we know the value of Position as well as which path to take from it. So return the structure
   VALUE = BEST-SCORE
   PATH = BEST-PATH

When the initial call to MINIMAX returns, the best move from CURRENT is the first element on PATH. To see how this procedure works, you should trace its execution for the game tree shown in Fig. 12.2.

The MINIMAX procedure just described is very simple. But its performance can be improved significantly with a few refinements. Some of these are described in the next few sections.

12.3 ADDING ALPHA-BETA CUTOFFS

Recall that the minimax procedure is a depth-first process. One path is explored as far as time allows, the static evaluation function is applied to the game positions at the last step of the path, and the value can then be passed up the path one level at a time. One of the good things about depth-first procedures is that their efficiency can often be improved by using branch-and-bound techniques in which partial solutions that are clearly worse than known solutions can be abandoned early. We described a straightforward application of this technique to the traveling salesman problem in Section 2.2.1. For that problem, all that was required was storage of the length of the best path found so far. If a later partial path outgrew that bound, it was abandoned.

But just as it was necessary to modify our search procedure slightly to handle both maximizing and minimizing players, it is also necessary to modify the branch-and-bound strategy to include two bounds, one for each of the players. This modified strategy is called *alpha-beta pruning*. It requires the maintenance of two threshold values, one representing a lower bound on the value that a maximizing node may ultimately be assigned (we call this *alpha*) and another representing an upper bound on the value that a minimizing node may be assigned (this we call *beta*).

To see how the alpha-beta procedure works, consider the example shown in Fig. 12.4.[3] After examining node F, we know that the opponent is guaranteed a score of –5 or less at C (since the opponent is the minimizing player). But we also know that we are guaranteed a score of 3 or greater at node A, which we can achieve if we move to B. Any other move that produces a score of less than 3 is worse than the move to B, and we can ignore it. After examining only F, we are sure that a move to C is worse (it will be less than or equal to –5) regardless of the score of node G. Thus we need not bother to explore node G at all. Of course, cutting out one node may not appear to justify the expense of keeping track of the limits and checking them, but if we were exploring this tree to six ply, then we would have eliminated not a single node but an entire tree three ply deep.

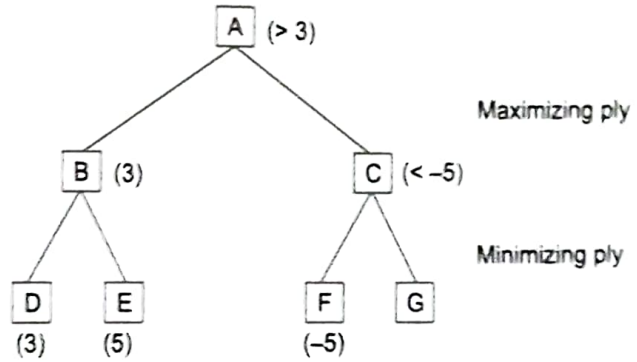

Fig. 12.4 *An Alpha Cutoff*

To see how the two thresholds, alpha and beta, can both be used, consider the example shown in Fig. 12.5. In searching this tree, the entire subtree headed by B is searched, and we discover that at A we can expect a score of at least 3. When this alpha value is passed down to F, it will enable us to skip the exploration of L. Let's see why. After K is examined, we see that I is guaranteed a maximum score of 0, which means that F is

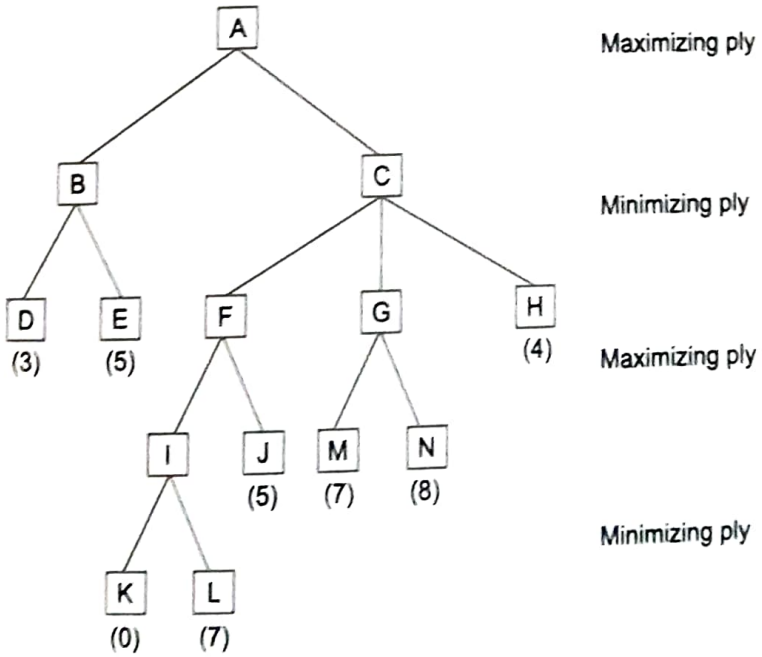

Fig. 12.5 *Alpha and Beta Cutoffs*

[3] In this figure, we return to the use of a single STATIC function from the point of view of the maximizing player.

guaranteed a minimum of 0. But this is less than alpha's value of 3, so no more branches of J need be considered. The maximizing player already knows not to choose to move to C and then to J since, if that move is made, the resulting score will be no better than 0 and a score of 3 can be achieved by moving to B instead. Now let's see how the value of beta can be used. After cutting off further exploration of I, J is examined, yielding a value of 5, which is assigned as the value of F (since it is the maximum of 5 and 0). This value becomes the value of beta at node C. It indicates that C is guaranteed to get a 5 or less. Now we must expand G. First M is examined, and it has a value of 7, which is passed back to G as its tentative value. But now 7 is compared to beta (5), which is greater, and the player whose turn it is at node C is trying to minimize. So this player will not choose G, which would lead to a score of at least 7, since there is an alternative move to F, which will lead to a score of 5. Thus it is not necessary to explore any of the other branches of G.

From this example, we see that at maximizing levels, we can rule out a move early if it becomes clear that its value will be less than the current threshold, while at minimizing levels, search will be terminated if values that are greater than the current threshold are discovered. But ruling out a possible move by a maximizing player actually means cutting off the search at a minimizing level. Look again at the example in Fig. 12.4. Once we determine that C is a bad move from A, we cannot bother to explore G, or any other paths, at the minimizing level below C. So the way alpha and beta are actually used is that search at a minimizing level can be terminated when a value less than alpha is discovered, while a search at a maximizing level can be terminated when a value greater than beta has been found. Cutting off search at a maximizing level when a high value is found may seem counterintuitive at first, but if you keep in mind that we only get to a particular node at a maximizing level if the minimizing player at the level above chooses it, then it makes sense.

Having illustrated the operation of alpha-beta pruning with examples, we can now explore how the MINIMAX procedure described in Section 12.2 can be modified to exploit this technique. Notice that at maximizing levels, only beta is used to determine whether to cut off the search, and at minimizing levels only alpha is used. But at maximizing levels alpha must also be known since when a recursive call is made to MINIMAX, a minimizing level is created, which needs access to alpha. So at maximizing levels alpha must be known not so that it can be used but so that it can be passed down the tree. The same is true of minimizing levels with respect to beta. Each level must receive both values, one to use and one to pass down for the next level to use.

The MINIMAX procedure as it stands does not need to treat maximizing and minimizing levels differently since it simply negates evaluations each time it changes levels. It would be nice if a comparable technique for handling alpha and beta could be found so that it would still not be necessary to write separate procedures for the two players. This turns out to be easy to do. Instead of referring to alpha and beta, MINIMAX uses two values, USE-THRESH and PASS-THRESH. USE-THRESH is used to compute cutoffs. PASS-THRESH is merely passed to the next level as its USE-THRESH. Of course, USE-THRESH must also be passed to the next level, but it will be passed as PASS-THRESH so that it can be passed to the third level down as USE-THRESH again, and so forth. Just as values had to be negated each time they were passed across levels, so too must these thresholds be negated. This is necessary so that, regardless of the level of the search, a test for greater than will determine whether a threshold has been crossed. Now there need still be no difference between the code required at maximizing levels and that required at minimizing ones.

We have now described how alpha and beta values are passed down the tree. In addition, we must decide how they are to be set. To see how to do this, let's return first to the simple example of Fig. 12.4. At a maximizing level, such as that of node A, alpha is set to be the value of the best successor that has yet been found. (Notice that although at maximizing levels it is beta that is used to determine cutoffs, it is alpha whose new value can be computed. Thus at any level, USE-THRESH will be checked for cutoffs and PASS-THRESH will be updated to be used later.) But if the maximizing node is not at the top of the tree, we must also consider

the alpha value that was passed down from a higher node. To see how this works, look again at Fig. 12.5 and consider what happens at node F. We assign the value 0 to node I on the basis of examining node K. This is so far the best successor of F. But from an earlier exploration of the subtree headed by B, alpha was set to 3 and passed down from A to F. Alpha should not be reset to 0 on the basis of node I. It should stay as 3 to reflect the best move found so far in the entire tree. Thus we see that at a maximizing level, alpha should be set to either the value it had at the next-highest maximizing level or the best value found at this level, whichever is greater. The corresponding statement can be made about beta at minimizing levels. In fact, what we want to say is that at any level, PASS-THRESH should always be the maximum of the value it inherits from above and the best move found at its level. If PASS-THRESH is updated, the new value should be propagated both down to lower levels and back up to higher ones so that it always reflects the best move found anywhere in the tree.

At this point, we notice that we are doing the same thing in computing PASS-THRESH that we did in MINIMAX to compute BEST-SCORE. We might as well eliminate BEST-SCORE and let PASS-THRESH serve in its place.

With these observations, we are in a position to describe the operation of the function MINIMAX-A-B, which requires four arguments, *Position, Depth, Use-Thresh,* and *Pass-Thresh.* The initial call, to choose a move for PLAYER-ONE from the position CURRENT, should be

```
MINIMAX-A-B(CURRENT,
            0,
            PLAYER-ONE,
            maximum value STATIC can compute,
            minimum value STATIC can compute)
```

These initial values for *Use-Thresh* and *Pass-Thresh* represent the worst values that each side could achieve.

Algorithm: MINIM AX-A-B(Position, Depth, Player, Use-Thresh, Pass-Thresh)

1. If DEEP-ENOUGH(*Position, Depth*), then return the structure
   VALUE = *STATIC*(*Position, Player*);
   PATH = nil
2. Otherwise, generate one more ply of the tree by calling the function MOVE- GEN(Position, Player) and setting SUCCESSORS to the list it returns.
3. If SUCCESSORS is empty, there are no moves to be made; return the same structure that would have been returned if DEEP-ENOUGH had returned TRUE.
4. If SUCCESSORS is not empty, then go through it, examining each element and keeping track of the best one. This is done as follows.
   For each element SUCC of SUCCESSORS:
   (a) Set RESULT-SUCC to
   MINIMAX-A-B(SUCC, *Depth* + 1, OPPOSITE(*Player*),
   –Pass-Thresh, –Use-Thresh).
   (b) Set NEW-VALUE to –VALUE(RESULT-SUCC).
   (c) If NEW-VALUE > *Pass-Thresh,* then we have found a successor that is better than any that have been examined so far. Record this by doing the following.
      (i) Set *Pass-Thresh* to NEW-VALUE.
      (ii) The best known path is now from CURRENT to SUCC and then on to the appropriate path from SUCC as determined by the recursive call to MINIMAX-A-B. So set BEST-PATH to the result of attaching SUCC to the front of PATH(RESULT-SUCC).

(d) If *Pass-Thresh* (reflecting the current best value) is not better than *Use-Thresh*, then we should stop examining this branch. But both thresholds and values have been inverted. So if *Pass-Thresh* >= *Use-Thresh*, then return immediately with the value

VALUE = *Pass-Thresh*

PATH = BEST-PATH

5. Return the structure

VALUE = *Pass-Thresh*

PATH = BEST-PATH

The effectiveness of the alpha-beta procedure depends greatly on the order in which paths are examined. If the worst paths are examined first, then no cutoffs at all will occur. But, of course, if the best path were known in advance so that it could be guaranteed to be examined first, we would not need to bother with the search process. If, however, we knew how effective the pruning technique is in the perfect case, we would have an upper bound on its performance in other situations. It is possible to prove that if the nodes are perfectly ordered, then the number of terminal nodes considered by a search to depth d using alpha-beta pruning is approximately equal to twice the number of terminal nodes generated by a search to depth $d/2$ without alpha-beta [Knuth and Moore, 1975]. A doubling of the depth to which the search can be pursued is a significant gain. Even though all of this improvement cannot typically be realized, the alpha-beta technique is a significant improvement to the minimax search procedure.- For a more detailed study of the average branching factor of the alpha-beta procedure, see Baudet [1978] and Pearl [1982].

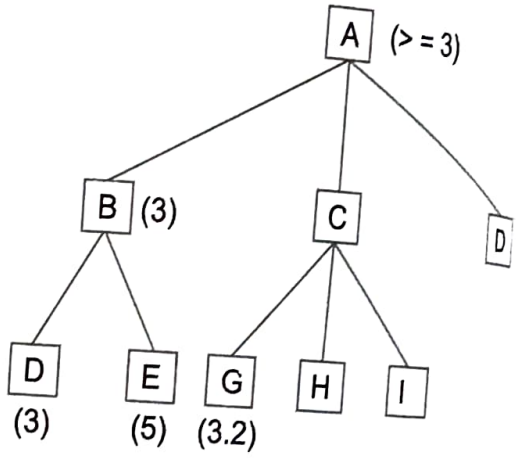

Fig. 12.6 *A Futility Cutoff*

The idea behind the alpha-beta procedure can be extended to cut off additional paths that appear to be at best only slight improvements over paths that have already been explored. In step 4(*d*), we cut off the search if the path we were exploring was not better than other paths already found. But consider the situation shown in Fig. 12.6. After examining node G, we see that the best we can hope for if we make move C is a score of 3.2. We know that if we make move B we are guaranteed a score of 3. Since 3.2 is only very slightly better than 3, we should perhaps terminate our exploration of C now. We could then devote more time to exploring other parts of the tree where there may be more to gain. Terminating the exploration of a subtree that offers little possibility for improvement over other known paths is called a *futility cutoff*.

12.4 ADDITIONAL REFINEMENTS

In addition to alpha-beta