Joint Probability Distributions ①

If x & y are two discrete random variables. we define the joint probability function x & y by

$f(X = x, Y = y) = f(x, y)$, where $f(x, y)$ satisfy the conditions

(i) $f(x, y) \geq 0$ (ii) $\sum_x \sum_y f(x, y) = 1$

The second condition means that the sum over all the values of x & y is equal to one.

suppose $X = \{x_1, x_2, x_3 \dots x_m\}$ & $Y = \{y_1, y_2, \dots y_n\}$ then $P(X = x_i, Y = y_j) = f(x_i, y_j)$ denoted by J_{ij}

It should be observed that f is a function on the cartesian product of the sets x & y

$X \times Y = \{(x_1, y_1), (x_2, y_2) \dots (x_m, y_n)$

f is also referred to as joint probability density function of x & y in the respective order. The set of values of this function

$f(x_i y_j) = J_{ij}$ for $i = 1, 2, 3 \dots m$ & $j = 1, 2 \dots n$

is called the joint probability distribution of x & y. These values are represented in the form of a two way table called the joint probability table.

X \ Y	y_1	y_2	......	y_n	sum
x_1	J_{11}	J_{12}		J_{1n}	$f(x_1)$
x_2	J_{21}	J_{22}		J_{2n}	$f(x_2)$
⋮					
x_m	J_{m1}	J_{m2}		J_{mn}	$f(x_m)$
sum	$g(y_1)$	$g(y_2)$	....	$g(y_n)$	1

Mariginal probability distribution

In the joint probability table $f(x_1), f(x_2), \dots f(x_m)$ respectively represents the sum of all the entries in the first row, second row, ... mth row. $g(y_1), g(y_2), \dots g(y_n)$ respectively represents the sum of all the entries in the first column, second column, ... nth column.

i.e $f(x_1) = J_{11} + J_{12} + \dots + J_{1n}$; $g(y_1) = J_{11} + J_{21} + \dots J_{m1}$

$f(x_2) = J_{21} + J_{22} + \dots + J_{2n}$; $g(y_2) = J_{12} + J_{22} + \dots + J_{m2}$

⋮

$f(x_m) = J_{m1} + J_{m2} + \dots + J_{mn}$; $g(y_n) = J_{1n} + J_{2n} + \dots + J_{mn}$

$\{f(x_1), f(x_2), \dots f(x_m)\}$ & $\{g(y_1), g(y_2) \dots g(y_n)\}$ are called marginal probability distribution of X & Y respectively.

It should be noted that

$f(x_1) + f(x_2) + \dots + f(x_m) = 1$ &

$g(y_1) + g(y_2) + \dots + g(y_n) = 1$

i.e $\sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i, y_j) = \sum_{1}^{m} \sum_{1}^{n} J_{ij} = 1$

It means that the total of all the entries in the joint probability table is equal to 1.

Independent random variable

The discrete random variables X & Y are said to independent random variables if

$$f(x_i)\, g(y_j) = J_{ij}$$

otherwise X & Y are said to be dependent

OR covariance$(X, Y) = 0$. then X & Y are said to independent random variables.

Expectation, Variance, and Covariance

If X is a discrete random variable taking values $x_1, x_2 \ldots x_n$ having probability function $f(x)$ then the expectation of X denoted by $E(X)$ or μ_x is defined by the relation.

$$\mu_x = E(X) = \sum_{i=1}^{n} x_i f(x_i)$$

$$\mu_y = E(Y) = \sum_{j=1}^{n} y_j f(y_j)$$

If X & Y are two discrete random variables having the joint probability function $f(x, y)$ then the expectation of X & Y are defined as follows

$$\mu_x = E(X) = \sum_{x} \sum_{y} x f(x, y) = \sum_{i} x_i f(x_i)$$

$$\mu_y = E(Y) = \sum_{x} \sum_{y} y f(x, y) = \sum_{j} y_j g(y_j)$$

Further $E(XY) = \sum_{ij} x_i y_j J_{ij}$

Variance of $X = V(X) = \sigma_x^2 = E(X^2) - [E(X)]^2$

Variance of $Y = V(Y) = \sigma_y^2 = E(Y^2) - [E(Y)]^2$

Standard deviation of $X = \sigma_x = \sqrt{V(X)}$

Covariance of $X, Y = \mathrm{cov}(X, Y) = E(XY) - E(X)E(Y)$

Correlation of X & $Y = \rho(X, Y) = \dfrac{\mathrm{cov}(X \cdot Y)}{\sigma_x \sigma_y}$

Note: If X & Y are independent random variables then

(i) $E(XY) = E(X) \cdot E(Y)$

(ii) $\mathrm{cov}(X \cdot Y) = 0$ & hence $\rho(X, Y) = 0$.

Problems:

1. The joint distribution of 2 random variables X & Y is as follows

X \ Y	−4	2	7
1	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{8}$
5	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{8}$

Compute the following
(a) Mariginal distributions of X & Y
(b) E(X) & E(Y) (c) E(XY) (d) σ_x & σ_y
(e) COV(XY) (f) $\rho(X,Y)$

Soln:
(a) Mariginal distributions of X & Y are

x_i	1	5
$f(x_i)$	$\frac{1}{2}$	$\frac{1}{2}$

y_j	−4	2	7
$g(y_j)$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{4}$

(b) $\mu_x = E(X) = \sum x_i f(x_i) = (1)(\frac{1}{2}) + 5(\frac{1}{2}) = 3$

$\mu_y = E(Y) = \sum y_j g(y_j) = (-4)(\frac{3}{8}) + 2(\frac{3}{8}) + 7(\frac{1}{4}) = 1$

(c) $E(XY) = \sum x_i y_j J_{ij}$

$= (1)(-4)(\frac{1}{8}) + (1)(2)(\frac{1}{4}) + (1)(7)(\frac{1}{8})$
$+ 5(-4)(\frac{1}{4}) + (5)(2)(\frac{1}{8}) + (5)(7)(\frac{1}{8})$

$E(XY) = \frac{3}{2}$

(d) $\sigma_x^2 = E(X^2) - [E(X)]^2$, $\sigma_y^2 = E(Y^2) - [E(Y)]^2$

Now $E(X^2) = \sum x_i^2 f(x_i) = (1)(\frac{1}{2}) + (25)(\frac{1}{2}) = 13$

$E(Y^2) = \sum y_j^2 g(y_j) = (16)(\frac{3}{8}) + (4)(\frac{1}{8}) + (49)(\frac{1}{4}) = \frac{79}{4}$

$\sigma_x^2 = (13) - (3)^2 = 4$ $\boxed{\sigma_x = 2}$

$\sigma_y^2 = \frac{79}{4} - (1)^2 = \frac{75}{4}$ $\boxed{\sigma_y = 4.33}$

(e) $COV(X\,Y) = E(XY) - E(X)E(Y)$

$= \frac{3}{2} - (3)(1) = -\frac{3}{2}$

$COV(X,Y) = -\frac{3}{2} \neq 0$

OR

$f(x_1)g(y_1) = \left(\frac{1}{2}\right)\left(\frac{3}{8}\right)$

$= \frac{3}{16}$

$J_{11} = \frac{1}{8}$

$f(x_1)g(y_1) \neq J_{11}$

∴ we conclude that X & Y are dependent random variables.

(f) correlation of X & Y $= \rho(X\,Y) = \frac{COV(X,Y)}{\sigma_x \sigma_y}$

$= \frac{-3/2}{(2)(4.33)}$

$\boxed{\rho(X,Y) = -0.1732}$

(2) The joint probability distribution table for two random variables X & Y is as follows

X \ Y	−2	−1	4	5
1	0.1	0.2	0	0.3
2	0.2	0.1	0.1	0

Determine the marginal distributions of X & Y. Also compute

(a) E(X) & E(Y) (b) S.D of X & Y (c) E(XY)

(d) COV(X,Y) (e) correlation of X & Y

Further verify that X & Y are are dependent random variables. Also find $P(X+Y \geq 0)$

Soln: Marginal distributions of X & Y

x_i	1	2
$f(x_i)$	0.6	0.4

y_j	-2	-1	4	5
$g(y_j)$	0.3	0.3	0.1	0.3

(a) $E(X) = \mu_x = \sum x_i f(x_i) = (1)(0.6) + (2)(0.4) = 1.4$

$E(Y) = \mu_y = \sum y_j g(y_j) = (-2)(0.3) + (-1)(0.3) + 4(0.1) + 5(0.3)$

$\boxed{E(Y) = 1}$

(c) $E(XY) = \sum x_i y_j J(i,j)$

$= (1)(-2)(0.1) + (1)(-1)(0.2) + (1)(4)(0)$

$+ (1)(5)(0.3) + 2(-2)(0.2) + 2(-1)(0.1)$

$+ 2(4)(0.1) + 2(5)(0)$

$E(XY) = 0.9$

$E(X^2) = \sum x_i^2 f(x_i) = (1)(0.6) + (4)(0.4) = 2.2$

$E(Y^2) = \sum y_j^2 g(y_j) = 4(0.3) + 1(0.3) + 16(0.1) + 25(0.3)$

$= 10.6$

$\sigma_x^2 = E(X^2) - [E(X)]^2 = 2.2 - (1.4)^2 = 0.24$, $\boxed{\sigma_x = 0.49}$

$\sigma_y^2 = E(Y^2) - [E(Y)]^2 = 10.6 - 1^2 = 9.6$ $\boxed{\sigma_y = 3.1}$

(d) $cov(X,Y) = E(XY) - E(X)E(Y) = (0.9) - (1.4)(1)$

$cov(XY) = -0.5 \neq 0$

Hence we conclude that X & Y are dependent random variables.

(e) $\rho(XY) = \frac{cov(X,Y)}{\sigma_x \sigma_y} = \frac{-0.5}{(0.49)(3.1)} = -0.3$

$X = \{x_1, x_2\} = \{1, 2\}$, $Y = \{y_1, y_2, y_3, y_4\} = \{-2, -1, 4, 5\}$

Also $J_{11} = 0.1$ $J_{12} = 0.2$ $J_{13} = 0$ $J_{14} = 0.3$

$J_{21} = 0.2$ $J_{22} = 0.1$ $J_{23} = 0.1$ $J_{24} = 0$

$X + Y > 0$ is possible when (X, Y) take the values

$(x_1, y_3) = (1, 4)$, $(x_1, y_4) = (1, 5)$

$(x_2, y_2) = (2, -1)$ $(x_2, y_3) = (2, 4)$ $(x_2, y_4) = (2, 5)$

$P(X + Y > 0) = J_{13} + J_{14} + J_{22} + J_{23} + J_{24}$

$= 0 + 0.3 + 0.1 + 0.1 + 0$

$\boxed{P(X + Y > 0) = 0.5}$

(3) Suppose X & Y are ~~independent random~~ variables with the following respective distributions. Find the joint distribution of X & Y. Also verify that $\mathrm{cov}(X, Y) = 0$.

x_i	1	2
$f(x_i)$	0.7	0.3

y_i	-2	5	8
$g(y_i)$	0.3	0.5	0.2

Soln: The joint distribution table is as follows

X \ Y	-2	5	8	$f(x_i)$
1	0.21	0.35	0.14	0.7
2	0.09	0.15	0.06	0.3
$g(y_i)$	0.3	0.5	0.2	1

$E(X) = \sum_i x_i f(x_i) = (1)(0.7) + 2(0.3) = 1.3$

$E(Y) = \sum_j y_j g(y_j) = (-2)(0.3) + 5(0.5) + 8(0.2) = 3.5$

$$E(XY) = \sum_{i,j} x_i y_j J_{ij}$$

$$= (1)(-2)(0.21) + (1)(5)(0.35) + (1)(8)(0.14)$$
$$+ (2)(-2)(0.09) + (2)(5)(0.15) + (2)(8)(0.06)$$

$$E(XY) = 4.55$$

$$\text{COV}(XY) = E(XY) - E(X)E(Y)$$
$$= 4.55 - (1.3)(3.5)$$
$$\text{COV}(X,Y) = 0$$

~~Thus~~ X & Y are independent random variables

4. X and Y are independent random variables. X, take values 2, 5, 7 with probability $\frac{1}{2}, \frac{1}{4}, \frac{1}{4}$ respectively. Y take values 3, 4, 5 with the probability $\frac{1}{3}, \frac{1}{3}, \frac{1}{3}$

(a) Find the Joint probability distribution of X & Y
b) S.T the covariance of X & Y is Equal to zero
c) Find the probability distribution of $Z = X + Y$

Solution: Given data

x_i	2	5	7
$f(x_i)$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$

y_j	3	4	5
$g(y_j)$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$

The joint distribution table is as follows

X \ Y	3	4	5	$f(x_i)$
2	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{2}$
5	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{4}$
7	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{4}$
$g(y_j)$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	1

(b) $E(X) = \sum_i x_i f(x_i)$

$= (2)(\frac{1}{2}) + 5(\frac{1}{4}) + 7(\frac{1}{4}) = 4$

$E(Y) = \sum_j y_j g(y_j) = 3(\frac{1}{3}) + 4(\frac{1}{3}) + 5(\frac{1}{3}) = 4$

$E(XY) = \sum_{i,j} x_i y_j J_{ij}$

$= (2)(3)(\frac{1}{6}) + (2)(4)(\frac{1}{6}) + (2)(5)(\frac{1}{6})$

$+ (5)(3)(\frac{1}{12}) + (5)(4)(\frac{1}{12}) + (5)(5)(\frac{1}{12})$

$+ (7)(3)(\frac{1}{12}) + (7)(4)(\frac{1}{12}) + (7)(5)(\frac{1}{12})$

$E(XY) = 16$

$\text{COV}(X,Y) = E(XY) - E(X)E(Y) = 16 - (4)(4) = 0$

(c) $Z = X + Y$

Let $z_i = x_i + y_i$ & hence $\{z_i\} = \{5, 6, 7, 8, 9, 10, 11, 12\}$

The corresponding probabilities are

$\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12} + \frac{1}{12} = \frac{1}{6}, \frac{1}{12}, \frac{1}{12}$

The probability distributions $Z = X + Y$ is as follows

Z	5	6	7	8	9	10	11	12
P(Z)	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{12}$	$\frac{1}{12}$

We note that $\sum P(Z) = 1$.

(5) The joint probability function for two discrete random variables X & Y is given by $f(x,y) = c(2x+y)$ where x & y can assume all integral values such that $0 \le x \le 2$, $0 \le y \le 3$ and $f(x,y) = 0$ otherwise. Find (i) the value of the constant c (ii) $P(X=2, Y=1)$ (iii) $P(X \ge 1, Y \le 2)$ (iv) $P(X+Y \le 1)$ (v) $P(X+Y>1)$, find mean

Soln: Here X takes values $x_1=0$, $x_2=1$, $x_3=2$ & Y takes the values $y_1=0$, $y_2=1$, $y_3=2$, $y_4=3$.

The values of $f(x,y)$ for allowable values of X & Y may be displayed in the following joint prob table.

X \ Y	0	1	2	3
0	0	c	2c	3c
1	2c	3c	4c	5c
2	4c	5c	6c	7c

$f(x,y) = c(2x+y)$

$f(0,0) = c(0,0) = 0$

$f(1,0) = c(2+0) = 2$

Accordingly, we find that $\sum_{j} \sum_{i} J_{ij} = 1$

$\sum_{i} \sum_{j} J_{ij} = (0+c+2c+3c) + (2c+3c+4c+5c) + (4c+5c+6c+7c) \Rightarrow 42c = 1$

Since $\sum_{i} \sum_{j} J_{ij} = 1$, we should have $42c = 1 \Longrightarrow \boxed{c = \frac{1}{42}}$

Next, we note that

(i) $P(X=2, Y=1) = P(x_3, y_2) = J_{32} = 5c = \frac{5}{42}$

(ii) $P(X \ge 1, Y \le 2) = P(X=1, Y \le 2) + P(X=2, Y \le 2)$

$= P(X=1, Y=0) + P(X=1, Y=1) + P(X=1, Y=2)$
$+ P(X=2, Y=0) + P(X=2, Y=1) + P(X=2, Y=2)$
$= P_{21} + P_{22} + P_{23} + P_{31} + P_{32} + P_{33}$
$= 2c + 3c + 4c + 4c + 5c + 6c = 24c = \frac{24}{42}$

$\boxed{P(X \geq 1, Y \leq 2) = \frac{4}{7}}$

(iv) $P(X+Y \leq 1)$

We note that $X+Y \leq 1$ when (i) $X=0$ & $Y=0$ (ii) $X=0$ & $Y=1$ & (iii) $X=1$ & $Y=0$. Therefore

$P(X+Y \leq 1) = P(0.0) + P(0,1) + P(1.0) = P_{11} + P_{12} + P_{21} = 0 + c + 2c$

$= 3c = 3\frac{1}{42}$ $\quad = 0 + c + 2c$

$P(X+Y \leq 1) = \frac{1}{14}$.

(v) $P(X+Y > 1) = 1 - P(X+Y \leq 1)$ (or) $1 - \frac{3}{42} = \frac{39}{42}$
$= 1 - \frac{1}{14} = \frac{13}{14}$

To find mean

mean $= E(X) = \sum x_i f(x_i)$.

Mariginal distribution of X & Y

x_i	0	1	2
$f(x_i)$	$\frac{1}{7}$	$\frac{7}{21}$	$\frac{11}{21}$

y_j	0	1	2	3
$g(y_j)$	$\frac{1}{7}$	$\frac{3}{14}$	$\frac{2}{7}$	$\frac{5}{14}$

$E(X) = 0 + \frac{7}{21} + \frac{22}{21} = \frac{29}{21}$

$E(Y) = 0 + \frac{3}{14} + \frac{4}{7} + \frac{15}{14} = \frac{26}{14}$

6 A coin is tossed 3 times. Let X be equal to 0 or 1 according as a head or a tail occurs on the first toss. Let Y be equal to the total no of heads which occur. Determine (i) the marginal distribution of X & Y (ii) the joint distribution of X & Y (iii) $E(X), E(Y), E(XY)$ & ~~$E(X+Y)$~~ (iv) σ_x & σ_y (v) $\operatorname{cov}(X,Y)$ & $\rho(X,Y)$

Soln: For the given random experiment, the 8 possible outcomes are

HHH, HHT, HTH, HTT, THH, THT, TTH, TTT

Among these outcomes, H appears in the first toss in the outcomes HHH, HHT, HTH, HTT, which are 4 in number. For these outcomes, $X=0$ & the corresponding prob is $\frac{4}{8}$. In the remaining 4 outcomes T appears in the first toss & for these outcomes $X=1$ and the corresponding prob is $\frac{4}{8}$. Therefore, the distribution of X is given by the following table

x	0	1
P(x)	$\frac{4}{8}$	$\frac{4}{8}$

Next, we note that

$Y=0$ for the outcome TTT

$Y=1$ for the 3 outcome HTT, THT, TTH

$Y=2$ for the 3 outcome HHT, HTH, THH

$Y=3$ for the outcome HHH

Therefore, the distribution of Y is given by the following table

y	0	1	2	3
P(y)	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

we find that

$P_{11} = P(X = x_1 \;\&\; Y = y_1) = P(X=0, Y=0) = 0$

$\because$ there is no outcome in which the first toss is H & the total no H's is zero.

$J_{12} = P(X = x_1 \;\&\; Y = y_2) = P(X=0, Y=1) = P(HTT) = \frac{1}{8}$

$J_{13} = P(X = x_1 \;\&\; Y = y_3) = P(X=0, Y=2) = P(HHT, HTH) = \frac{2}{8}$

$J_{14} = P(X=0, Y=3) = P(HHH) = \frac{1}{8}$

$J_{21} = P(X=1, Y=0) = P(TTT) = \frac{1}{8}$

$J_{22} = P(X=1, Y=1) = P(THT, TTH) = \frac{2}{8}$

$P_{23} = P(X=1, Y=2) = P(THH) = \frac{1}{8}$

$P_{24} = P(X=1, Y=3) = 0$, because there is no outcome in which the first toss is T but the total number of H's is 3.

The joint distribution of X & y is

x \ y	0	1	2	3
0	0	$\frac{1}{8}$	$\frac{2}{8}$	$\frac{1}{8}$
1	$\frac{1}{8}$	$\frac{2}{8}$	$\frac{1}{8}$	0

$\mu_x = E(X) = 0 \times \frac{4}{8} + 1 \times \frac{4}{8} = \frac{4}{8}$

$\mu_y = E(Y) = 0 + \frac{3}{8} + \frac{6}{8} + \frac{3}{8} = \frac{3}{2}$

$$E(X,Y) = 0(0)(0) + 0(1)(\tfrac{1}{8}) + 0(2)(\tfrac{2}{8}) + 0(3)(\tfrac{1}{8})$$
$$+ 1(0)(\tfrac{1}{8}) + 1(1)(\tfrac{2}{8}) + 1(2)(\tfrac{1}{8}) + 1(3)(0)$$
$$= 0 + 0 + 0 + 0 + 0 + 0 + \tfrac{2}{8} + \tfrac{2}{8} + 0$$
$$= \tfrac{1}{4} + \tfrac{1}{4} = \tfrac{1+1}{4} = \tfrac{2}{4}$$
$$\boxed{E(X,Y) = \tfrac{1}{2}}$$

$$E(X^2) = (0)^2(\tfrac{4}{8}) + 1^2(\tfrac{4}{8}) = \tfrac{1}{2} \Rightarrow \boxed{E(X^2) = \tfrac{1}{2}}$$

$$E(Y^2) = 0(\tfrac{1}{8}) + 1^2(\tfrac{3}{8}) + 2^2(\tfrac{3}{8}) + 9(\tfrac{1}{8})$$
$$= 0 + \tfrac{3}{8} + \tfrac{12}{8} + \tfrac{9}{8} = \tfrac{24}{8} = 3 \quad \boxed{E(Y^2) = 3}$$

$$\sigma_x^2 = E(X^2) - [E(X)]^2 = \tfrac{1}{2} - (\tfrac{1}{2})^2 = \tfrac{1}{2} - \tfrac{1}{4} = \tfrac{1}{4}$$
$$\boxed{\sigma_x^2 = \tfrac{1}{4}} \Rightarrow \boxed{\sigma_x = \tfrac{1}{2}}$$

$$\sigma_y^2 = E(Y)^2 - [E(Y)]^2 = 3 - (\tfrac{3}{2})^2 = 3 - \tfrac{9}{4} = \tfrac{12-9}{4} = \tfrac{3}{4}$$
$$\boxed{\sigma_y^2 = \tfrac{3}{4}} \quad \boxed{\sigma_y = \tfrac{\sqrt{3}}{2}}$$

$$\mathrm{cov}(X.Y) = E(X,Y) - E(X)E(Y)$$
$$= \tfrac{1}{2} - \tfrac{1}{2}(\tfrac{3}{2}) = -\tfrac{1}{4} \neq 0$$

∴ X & Y are dependent random variables

$$\rho(X.Y) = \frac{\mathrm{cov}(X.Y)}{\sigma_x \sigma_y} = \frac{-\tfrac{1}{4}}{(\tfrac{1}{2})(\tfrac{\sqrt{3}}{2})} = \frac{-1}{\sqrt{3}}$$

(7) A fair coin is tossed twice. Let X & Y be defined as follows (i) $X=1$ if the first toss is H & $X=0$ otherwise
(ii) $Y=1$ if both tosses are H & $Y=0$ otherwise.
Find the joint probability distribution of X & Y. Show that X & Y are not independent.

Ans: $S = \{HH, HT, TH, TT\}$

X \ Y	0	1
0	$\frac{2}{4} = \frac{1}{2}$	0
1	$\frac{1}{4}$	$\frac{1}{4}$

$X = \{0, 1\}$
$Y = \{0, 1\}$
$X = 1$ first toss H
$Y = 1$ Both tosses are H

Marginal distribution of X & Y are

x_i	0	1
$f(x_i)$	$\frac{1}{2}$	$\frac{1}{2}$

y_j	0	1
$g(y_j)$	$\frac{3}{4}$	$\frac{1}{4}$

$$f(x_1)\, g(y_1) = J_{11}$$
$$\left(\frac{1}{2}\right)\left(\frac{3}{4}\right) \neq \frac{1}{2}$$

$\therefore$ ~~X~~ Y are indedendent

$f(x_1) = \frac{1}{2}$
$g(y_1) = \frac{3}{4}$
$J_{11} = \frac{2}{4} = \frac{1}{2}.$

(8) A fair coin is tossed twice. Let X & Y be defined as follows. (i) $X=1$ if the first toss is H & $X=0$ otherwise (ii) $Y=1$ if the second toss is H & $Y=0$ otherwise. Find the joint probability distribution of X & Y.

Ans: All possible outcomes are

$$S=\{HH, HT, TH\ TT\}$$

Joint probability distribution table

X\Y	0	1
0	$\frac{1}{4}$	$\frac{1}{4}$
1	$\frac{1}{4}$	$\frac{1}{4}$

(i) $X=1$ first toss is H

(ii) $Y=1$ if the second toss is H

(9) Two marbles are selected at random from a box containing 3 blue, 2 red & 3 green marbles. If X is the number of blue marbles & Y is the number of red marbles selected, find (i) The marginal distribution of X & Y (ii) S.T the random variables X & Y are statistically independent.

Ans: $X=\{0, 1, 2\}$ $Y=\{0, 1, 2\}$.

Total marbles $= 3(B)+2(R)+3(G) = 8$ marbles

X: no of blue marbles

Y: no of red marbles

(ii) Marginal distribution of X & Y

x_i	0	1	2
$f(x_i)$	$\frac{10}{28}$	$\frac{15}{28}$	$\frac{3}{28}$

y_j	0	1	2
$g(y_j)$	$\frac{15}{28}$	$\frac{12}{28}$	$\frac{1}{28}$

(i) Joint probability distribution

X\Y	0	1	2
0	$\frac{3C_2}{8C_2}$	$\frac{2C_1 \times 3C_1}{8C_2}$	$\frac{2C_2}{8C_2}$
1	$\frac{3C_1 \times 3C_1}{8C_2}$	$\frac{3C_1 \times 2C_1}{8C_2}$	0
2	$\frac{3C_2}{8C_2}$	0	0

$\therefore$

X\Y	0	1	2
0	$\frac{3}{28}$	$\frac{6}{28}$	$\frac{1}{28}$
1	$\frac{9}{28}$	$\frac{6}{28}$	0
2	$\frac{3}{28}$	0	0

I (i) $X=0$ (Blue) then either 2R (or) 2(G) (or) 1R & 1G

$$P(X=0) = \frac{2C_2}{8C_2} + \frac{3C_2}{8C_2} + \frac{2C_1 \times 3C_1}{8C_2} = \frac{10}{28}$$

$$\boxed{P(X=0) = \frac{10}{28}}$$

(ii) $X=1$ (Blue) then either 1R and 1G

$$P(X=1) = \frac{3C_1 \times 2C_1}{8C_2} + \frac{3C_1 \times 3C_1}{8C_2} = \frac{3\times2}{28} + \frac{3\times3}{28}$$

$$\boxed{P(X=1) = \frac{15}{28}}$$

(iii) $X=2$ (Blue) $\boxed{P(X=2) = \frac{3C_2}{8C_2} = \frac{3}{28}}$

∴

II (i) $Y=0$ (Red) then either 2B (or) 2(G) (or) 1B & 1G

$$P(Y=0) = \frac{3C_2}{8C_2} + \frac{3C_2}{8C_2} + \frac{3C_1 \times 3C_1}{8C_2}$$

$$= \frac{3C_1 + 3C_2 + (3C_1 \times 3C_1)}{8C_2} = \frac{15}{28}$$

(ii) $Y=1$ (Red) then 1B (or) 1G

$$P(Y=1) = \frac{2C_1 \times 3C_1}{8C_2} + \frac{2C_1 \times 3C_1}{8C_2} = \frac{12}{28}$$

(iii) $Y=2$ (Red) $P(Y=2) = \frac{2C_2}{8C_2} = \frac{1}{28}$

(10) Roll of pair dice. Let X donote the smaller and Y larger outcome on the dice. Find the disstribution of X & Y

Ans (1,1) (1,2) (1,3) (1 4) (1,5) (1 6)

(2,1) (2 2) (2 3) (2,4) (2,5) (2 6)

(3,1) (3,2) (3,3) (3,4) (3,5) (3,6)

(4,1) (4,2) (4,3) (4,4) (4,5) (4,6)

(5,1) (5,2) (5,3) (5,4) (5,5) (5,6)

(6,1) (6,2) (6,3) (6,4) (6,5) (6,6)

Total no of outcomes $= 6^2 = 36$

X : smaller the out come of X

Y : larger the out come of Y

X\Y	1	2	3	4	5	6
1	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{2}{36}$
2	0	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{2}{36}$
3	0	0	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{2}{36}$
4	0	0	0	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{2}{36}$
5	0	0	0	0	$\frac{1}{36}$	$\frac{2}{36}$
6	0	0	0	0	0	$\frac{1}{36}$

Marginal distribution of ~~X~~

x_i	1	2	3	4	5	6
$f(x)$	$\frac{11}{36}$	$\frac{9}{36}$	$\frac{7}{36}$	$\frac{5}{36}$	$\frac{3}{36}$	$\frac{1}{36}$

Marginal distribution of Y

y_j	1	2	3	4	5	6
$g(y_j)$	$\frac{1}{36}$	$\frac{3}{36}$	$\frac{5}{36}$	$\frac{7}{36}$	$\frac{9}{36}$	$\frac{11}{36}$

11) Consider an experiment consisting of 2 throws of a fair die. Let X be the no of 4's and Y be the no of 5's obtained in the two throws. Find the joint probability distribution of X & Y. Also find P(2X+Y<3) (21)

Ans. The total possible outcomes is $6^2 = 36$

X: be the no of 4's
Y: be the no of 5's
$X = \{0, 1, 2\}$ $Y = \{0, 1, 2\}$

X \ Y	0	1	2
0	$\frac{16}{36}$	$\frac{8}{36}$	$\frac{1}{36}$
1	$\frac{8}{36}$	$\frac{2}{36}$	0
2	$\frac{1}{36}$	0	0

$$P(2X+Y<3) = \frac{16}{36} + \frac{8}{36} + \frac{1}{36} + \frac{8}{36}$$

$$= \frac{11}{12}.$$

Markov chains

(i) Stochastic process

Any stochastic process involving a random variable x is called as a Markovian process if the next value of x depends only on the current state of x and not the previous state.

A Markovian process is completely described when the transition probabilites from one state to another are completely obtained. These probabilites given in the form of a matrix is called "Transition probability matrix" which is given as shown

$$\begin{array}{c|ccc} & 1 & 1 & 2 \\ \hline 1 & 0.2 & 0.6 & 0.2 \\ 2 & 0.3 & 0.4 & 0.3 \\ 3 & 0.4 & 0 & 0.6 \end{array} \qquad \text{Total} = 1$$

(ii) Markov chain:

Defn: A stochastic process which is such that the generation of the prob distribution depend only on the present state is called a markov process

If any stochastic process involving a random variable x_i ($i = 0, 1, 2 \ldots$) is such that, x_i depends only on x_{i-1} and affects only x_{i+1} is called a 'MARKOVIAN CHAIN' OR PROCESS. Then the markov process is known as a markov chain.

Stationary markovian chain:

Any stationary matrix, the transition probabilities donot change with respect to time. Hence the stationary matrix is said to give a steady state transition probability for a stationary markovian chain

Stochastic matrix:

A square matrix $P = (P_{ij})$ having every row in the form of a probability vector is called a stochastic matrix.

Regular stochastic matrix

A stochastic matrix P is said to be a regular stochastic matrix if all the entries in some positive integral power of P are positive.

Irreducible markov chain

A Markov chain is said to be irreducible if its transition matrix is a regular stochastic matrix.

Fixed probability vector.

Given a regular stochastic matrix P of order m, if there exists a probability vector V of order m such that

$$VP = V$$

Then V is called a fixed probability vector of P. It can be proved that V exists and is unique.

Types of stochastic process

1. Discrete-state, discrete parameter process
2. Discrete state, continuous parameter process
3. continuous state discrete parameter process
4. continuous state, continuous parameter process.

Problems:

① Verify that the matrix $P=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ \frac{1}{2} & \frac{1}{2} & 0\end{bmatrix}$

is a regular stochastic matrix. Also find the associated unique fixed probability vector.

Soln: Given matrix P, Each element is non-negative and the sum of the elements in each row is equal to 1. Therefore P is a stochastic matrix.

consider $P^2=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ \frac{1}{2} & \frac{1}{2} & 0\end{bmatrix}\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ \frac{1}{2} & \frac{1}{2} & 0\end{bmatrix}=\begin{bmatrix}0 & 0 & 1\\ \frac{1}{2} & \frac{1}{2} & 0\\ 0 & \frac{1}{2} & \frac{1}{2}\end{bmatrix}$

$P^3=\begin{bmatrix}0 & 0 & 1\\ \frac{1}{2} & \frac{1}{2} & 0\\ 0 & \frac{1}{2} & \frac{1}{2}\end{bmatrix}\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ \frac{1}{2} & \frac{1}{2} & 0\end{bmatrix}=\begin{bmatrix}\frac{1}{2} & \frac{1}{2} & 0\\ 0 & \frac{1}{2} & \frac{1}{2}\\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2}\end{bmatrix}$

$P^4=\begin{bmatrix}\frac{1}{2} & \frac{1}{2} & 0\\ 0 & \frac{1}{2} & \frac{1}{2}\\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2}\end{bmatrix}\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ \frac{1}{2} & \frac{1}{2} & 0\end{bmatrix}=\begin{bmatrix}0 & \frac{1}{2} & \frac{1}{2}\\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2}\\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4}\end{bmatrix}$

$P^5=\begin{bmatrix}0 & \frac{1}{2} & \frac{1}{2}\\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2}\\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4}\end{bmatrix}\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ \frac{1}{2} & \frac{1}{2} & 0\end{bmatrix}=\begin{bmatrix}\frac{1}{4} & \frac{1}{4} & \frac{1}{2}\\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4}\\ \frac{1}{8} & \frac{3}{8} & \frac{1}{2}\end{bmatrix}$

We note that all entries in P^5 are positive. Hence 'P' is a regular stochastic matrix.

Next we have to find $V=(a, b, c)$ where $a+b+c=1$ such that $VP=V$

$$[a, b, c]\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ \frac{1}{2} & \frac{1}{2} & 0\end{bmatrix}=[a, b, c]$$

$$\left[\frac{c}{2}, a+\frac{c}{2}, b\right]=[a, b, c]$$

$$\frac{c}{2}=a, \quad a+\frac{c}{2}=b, \quad b=c$$

$\therefore \boxed{c=2a}$ & $b=c$
$\boxed{b=2a}$

$\therefore a+b+c=1$

$a+2a+2a=1$

$5a=1 \Rightarrow \boxed{a=\frac{1}{5}}, \boxed{b=\frac{2}{5}} \boxed{c=\frac{2}{5}}$

$\therefore \left(\frac{1}{5}, \frac{2}{5}, \frac{2}{5}\right)$ is the required fixed probability vector.

② Prove that the Markov chain with transition matrix

$$P=\begin{bmatrix}0 & \frac{2}{3} & \frac{1}{3}\\ \frac{1}{2} & 0 & \frac{1}{2}\\ \frac{1}{2} & \frac{1}{2} & 0\end{bmatrix}$$

is irreducible. & determine the steady state probababilities?

Solu: We note that the given matrix P is a stochastic matrix (being a transition matrix).

$$P^2=\begin{bmatrix}\frac{1}{2} & \frac{1}{6} & \frac{1}{3}\\ \frac{1}{4} & \frac{7}{12} & \frac{1}{6}\\ \frac{1}{4} & \frac{1}{3} & \frac{5}{12}\end{bmatrix}$$

The entries in this matrix are $P^{(2)}$ which are all positive. Hence the given matrix P is regular stochastic matrix. Consequently it follows that the given Markov chain is irreducible.

Next we have to find $V = V(a \cdot b\, c)$ where $a+b+c=1$, such that $VP = V$

$$[a\ b\ c]\begin{bmatrix} 0 & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} = [a \cdot b \cdot c]$$

$$\left[\frac{b}{2}+\frac{c}{2},\ \frac{2a}{3}+\frac{c}{2},\ \frac{a}{3}+\frac{b}{2}\right] = [a, b\ c]$$

$$\frac{b}{2}+\frac{c}{2}=a, \quad \frac{2a}{3}+\frac{c}{2}=b, \quad \frac{a}{3}+\frac{b}{2}=c$$

x'y 2 x'y 6 x'y 6

$$b+c=2a \qquad 4a+3c=6b \qquad 2a+3b=6c$$

$$\therefore \boxed{2a-b-c=0} \to ① \qquad \boxed{4a-6b+3c=0} \to ② \qquad \boxed{2a+3b-6c=0} \to ③$$

From ② x'y by 2 $\Rightarrow$ $8a-12b+6c=0$

$\boxed{\text{Eqn ② x'y by ②}}$

$$2a+3b-6c=0 \quad (\text{Eqn ③})$$

Adding $10a-9b=0$

$$\Rightarrow 9b=10a \Rightarrow \boxed{b=\frac{10}{9}a}$$

consider $2a+3b-6c=0$

$$2a+\frac{30a}{9}=6c \Rightarrow 6c=\frac{18a+30a}{9}$$

$$6c=\frac{48a}{9} \Rightarrow \boxed{c=\frac{8a}{9}}$$

$a+b+c=1$

$a+\frac{10a}{9}+\frac{8a}{9}=1 \quad | \times^{ly} 9$

$9a+10a+8a=9$

$27a=9$, $b=\frac{10a}{9}$ $c=\frac{8}{9}a$

$\boxed{a=\frac{9}{27}}$ $\boxed{b=\frac{10}{27}}$ $\boxed{c=\frac{8}{27}}$

$\therefore \left(\frac{9}{27},\frac{10}{27},\frac{8}{27}\right)$ is the required steady state probability vector

(3) Find the unique fixed probability vector for the regular stochastic matrix

$$\begin{bmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

consider $P=\begin{bmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$

we have to find $V=V(a,b)$ where $a+b=1$ such that $VP=V$

$$[a \ b]\begin{bmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}=[a \ b]$$

$$\left[\frac{3}{4}a+\frac{1}{2}b,\ \frac{1}{4}a+\frac{1}{2}b\right]=[a.\ b]$$

$\frac{3}{4}a + \frac{1}{2}b = a$ $\quad \times^{ly} 4$ $\qquad \frac{1}{4}a + \frac{b}{2} = b$

$3a + 2b = 4a$

$\boxed{a = 2b}$ $\qquad$ consider $a + b = 1$

$2b + b = 1$

$\boxed{3b = 1} \Rightarrow \boxed{b = \frac{1}{3}}$

$\boxed{a = \frac{2}{3}}$

Thus $\left(\frac{2}{3}, \frac{1}{3}\right)$ is the unique fixed probability vector.

(4) Find the unique fixed probability vector for regular stochastic matrix

$$P = \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{6} & \frac{1}{2} & \frac{1}{3} \\ 0 & \frac{2}{3} & \frac{1}{3} \end{bmatrix}$$

we have to find $v = v(a, b, c)$, where $a + b + c = 1$ such that $vP = v$

$$[a, b, c] \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{6} & \frac{1}{2} & \frac{1}{3} \\ 0 & \frac{2}{3} & \frac{1}{3} \end{bmatrix} = [a, b, c]$$

$$\left[\frac{b}{6}, \; a + \frac{b}{2} + \frac{2c}{3}, \; \frac{b}{3} + \frac{c}{3}\right] = [a, b, c]$$

$\frac{b}{6} = a \qquad a + \frac{b}{2} + \frac{2c}{3} = b, \qquad \frac{b}{3} + \frac{c}{3} = c$

$\therefore \boxed{b = 6a}$

$b + c = 3c$

$2c = b$

$2c = 6a$

$\boxed{c = 3a}$

i.e $a+b+c=1$

$a+6a+3a=1$

$10a=1$ $\quad$ $b=6a$ $\quad$ $c=3a$

$\therefore \boxed{a=\frac{1}{10}}$ $\quad$ $\boxed{b=\frac{6}{10}}$ $\quad$ $\boxed{c=\frac{3}{10}}$

Hence $v=\left(\frac{1}{10}, \frac{6}{10}, \frac{3}{10}\right)$ is the unique fixed probability vector.

(5) Which of the stochastic matrices are regular

(i) $\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{bmatrix}$ $\quad$ (ii) $\begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ 0 & 1 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$

<u>Soln;</u>

(i) consider $P=\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{bmatrix}$

$$P^2=P\cdot P=\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{bmatrix}$$

$$P^2=\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{3}{8} & \frac{3}{8} & \frac{1}{4} \end{bmatrix}$$

$\therefore P$ is not a regular stochastic matrix

(ii) consider $P = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ 0 & 1 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$

$$P^2 = P \cdot P = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ 0 & 1 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ 0 & 1 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$$

$$P^2 = \begin{bmatrix} \frac{3}{8} & \frac{1}{2} & \frac{1}{8} \\ 0 & 1 & 0 \\ \frac{1}{4} & \frac{5}{8} & \frac{1}{8} \end{bmatrix}$$

$\therefore$ It is not regular stochastic matrix (or '1' is in main diagonal)

(6) Find the unique fixed probability vector of the regular stochastic matrix

$$P = \begin{bmatrix} \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 \\ \frac{2}{3} & \frac{1}{3} & 0 & 0 \\ \frac{1}{6} & \frac{1}{6} & 0 & \frac{2}{3} \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{bmatrix}$$

<u>Ans</u>

we have ~~to~~ find $V = V(a, b, c, d)$ where $a+b+c+d=1$ such that

$$VP = V$$

$$\begin{bmatrix} a & b & c & d \end{bmatrix} \begin{bmatrix} \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 \\ \frac{2}{3} & \frac{1}{3} & 0 & 0 \\ \frac{1}{6} & \frac{1}{6} & 0 & \frac{2}{3} \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} a & b & c & d \end{bmatrix}$$

$$\left[\frac{a}{4}+\frac{2}{3}b+\frac{c}{6}+\frac{d}{2},\ \frac{a}{2}+\frac{b}{3}+\frac{c}{6},\ \frac{a}{4},\ \frac{2}{3}c+\frac{1}{2}d\right]=[a,b,c,d]$$

$$\therefore \boxed{\frac{a}{4}+\frac{2}{3}b+\frac{c}{6}+\frac{d}{2}=a} \quad \boxed{\frac{a}{2}+\frac{b}{3}+\frac{c}{6}=b}$$

$$\boxed{\frac{a}{4}=c} \quad \boxed{\frac{2}{3}c+\frac{d}{2}=d}$$

consider (i) $\boxed{c=\frac{a}{4}}$

(ii) $\frac{2}{3}c+\frac{d}{2}=d$ $\times^{ly}$ 6

$4c+3d=6d$

$4c=3d$

$d=\frac{4}{3}c$ But $\boxed{c=\frac{a}{4}}$

$d=\frac{4}{3}\cdot\frac{a}{4}$ $\boxed{d=\frac{a}{3}}$

(iii) $\frac{a}{2}+\frac{b}{3}+\frac{c}{6}=b$ $\times^{ly}$ 6

$3a+2b+c=6b$

$3a+c=4b$

$4b=3a+c$ But $c=\frac{a}{4}$

$4b=3a+\frac{a}{4}$

$4b=\frac{12a+a}{4} \Rightarrow \boxed{b=\frac{13a}{16}}$

$\therefore \quad a+b+c+d=1$

$$a+\frac{13a}{16}+\frac{a}{3}+\frac{a}{4}=1 \qquad \times^{ly}\ \frac{16\times3}{48}$$

$$48a+39a+16a+12a=48$$

$$115a=48$$

$$\boxed{a=\frac{48}{115}} \qquad b=\frac{13a}{16} \qquad c=\frac{a}{4} \qquad d=\frac{a}{3}$$

$$b=\frac{13}{16}\times\frac{48}{115} \qquad \boxed{c=\frac{12}{115}} \qquad d=\frac{1}{3}\cdot\frac{48}{115}$$

$$\boxed{b=\frac{39}{115}} \qquad \boxed{d=\frac{16}{115}}$$

$\therefore \left(\frac{48}{115}, \frac{39}{115}, \frac{12}{115}, \frac{16}{115}\right)$ is the unique fixed probability vector

(7) Find the unique fixed probability vector

$$P=\begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & 0 & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \end{bmatrix}$$

Ans

To find $V=V(a, b, c, d)$ where $a+b+c+d=1$ such that

$$VP=V$$

$$[a\ b\ c\ d]\begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & 0 & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \end{bmatrix} = [a\ b\ c\ d]$$

$$\left[\frac{b}{2}+\frac{c}{2}+\frac{d}{2},\ \frac{a}{2}+\frac{c}{2}+\frac{d}{2},\ \frac{a}{4}+\frac{b}{4},\ \frac{a}{4}+\frac{b}{4}\right] = [a, b, c, d]$$

$\therefore \frac{b}{2}+\frac{c}{2}+\frac{d}{2}=a,\quad \frac{a}{2}+\frac{c}{2}+\frac{d}{2}=b,\quad \frac{a}{4}+\frac{b}{4}=c$

x'ly 2   x'ly 2

$\frac{a}{4}+\frac{b}{4}=d$ | x'ly 4

$b+c+d=2a,\quad a+c+d=2b \quad a+b=4c$

$\therefore \boxed{2a-b-c-d=0} \quad \boxed{a-2b+c+d=0} \quad \boxed{a+b=4c}$

$\boxed{a+b=4d}$

(i) consider $a+b=4c$

Substract $a+b=4d$

$0+0=4c-4d$

$4d=4c$

$\boxed{d=c}$

(ii) consider $2a-b-c-d=0$

$a-2b+c+d=0$ But $d=c$

$2a-b-2c=0$

$a-2b+2c=0$

Adding

$3a-3b+0=0$

$\boxed{b=a}$

$\therefore a+b=4c$ But $b=a$

$a+a=4c$

$4c=2a$

$\boxed{c=\frac{a}{2}}$ $d=c$ $\boxed{d=\frac{a}{2}}$ & $\boxed{b=a}$

$\therefore a+b+c+d=1$

$a+a+\frac{a}{2}+\frac{a}{2}=1$

$2a+a=1$

$\boxed{3a=1}$

$\boxed{a=\frac{1}{3}}$ $b=a$ $\boxed{b=\frac{1}{3}}$ $c=\frac{a}{2}$ $c=\frac{1}{2}\times\frac{1}{3}$ $\boxed{c=\frac{1}{6}}$ $d=\frac{a}{2}$ $d=\frac{1}{2}\times\frac{1}{3}$ $\boxed{d=\frac{1}{6}}$

$\therefore \left(\frac{1}{3},\frac{1}{3},\frac{1}{6},\frac{1}{6}\right)$ is the unique fixed probability vector

(8) Find the unique fixed probability vector

$$P=\begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{4} & 0 & \frac{1}{4} \\ 0 & 0 & 0 & 1 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$$

9) S.T $P = \begin{bmatrix} 0 & \frac{3}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 0 \end{bmatrix}$ is regular stochastic matrix & find the unique fixed probability vector.

Ans. $P^2 = \begin{bmatrix} 0 & \frac{3}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & \frac{3}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} \frac{3}{8} & \frac{5}{8} & 0 \\ \frac{1}{4} & \frac{5}{8} & \frac{1}{8} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$

$P^3 = \begin{bmatrix} \frac{3}{8} & \frac{5}{8} & 0 \\ \frac{1}{4} & \frac{5}{8} & \frac{1}{8} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} 0 & \frac{3}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} \frac{5}{16} & \frac{19}{32} & \frac{3}{32} \\ \frac{5}{16} & \frac{10}{16} & \frac{1}{16} \\ \frac{1}{4} & \frac{5}{8} & \frac{1}{8} \end{bmatrix}$

$\therefore$ It is regular stochastic matrix

To find $V = V(a, b, c)$ where $a+b+c=1$

such that $VP = V$

$$[a \; b \; c] \begin{bmatrix} 0 & \frac{3}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 0 \end{bmatrix} = [a \; b \; c]$$

$$\left[\frac{b}{2}, \; \frac{3}{4}a + \frac{b}{2} + c, \; \frac{a}{4}\right] = [a \; b \; c]$$

$\frac{b}{2} = a, \quad \frac{3}{4}a + \frac{b}{2} + c = b \quad \frac{a}{4} = c$

$\therefore \boxed{b = 2a} \qquad \boxed{c = \frac{a}{4}}$

$a+b+c=1 \Rightarrow a + 2a + \frac{a}{4} = 1$

$3a + \frac{a}{4} = 1$ x'ly 4

$12a + a = 4$

$13a = 4$ $\Rightarrow$ $\boxed{a = \frac{4}{13}}$

$b = 2a$

$\boxed{b = \frac{8}{13}}$

$c = \frac{a}{4}$

$c = \frac{1}{4} \cdot \frac{4}{13}$

$\boxed{c = \frac{1}{13}}$

$\therefore$ $\left(\frac{4}{13}, \frac{8}{13}, \frac{1}{13}\right)$ is the unique fixed probability vector.

probability vector.

(4) State classification:-

The classification of the state of a Markov chain are

(1) Absorbing State:

A state 'i' is called an absorbing state if the transition probabilities p_{ij} are such that
$$p_{ij} = \begin{cases} 1 & \text{for } j = i \\ 0 & \text{otherwise} \end{cases}$$

(2) Transient State

A state i is said to be a transient state if the system is in this state at some step (as it has to be) and there is a chance (i.e there is a non-zero probability) that it will not return to that state.

(3) Recurrent State:-

A state i is said to be a recurrent state if starting from state i, the system does eventually return to the same state. (Here it is implicit that the probability of return is one)

④ Periodic state:

Let i be the recurrent state so that $P_{ii}^{(k)} > 0$ for some $k \geq 1$. Also, let d_i be the greatest common divisor of the set of positive integers k such that $P_{ii}^{(k)} > 0$. Then d_i is called the period of the state i.

A recurrent state i is said to be periodic if $d_i > 1$ and aperiodic if $d_i = 1$.

Problems:

(10) A student's study habits are as follows. If he studies one night, he is 70% sure not to study the next night. On the other hand if he does not study one night, he is 60% sure not to study the next night. In the long run how often does he study?

Soln: The state space of the system is 2 i.e (study, not study).

The transition matrix is

$$P = \begin{array}{c c} & \begin{array}{cc} S & NS \end{array} \\ \begin{array}{c} S \\ NS \end{array} & \begin{bmatrix} 0.3 & 0.7 \\ 0.4 & 0.6 \end{bmatrix} \end{array}$$

In order to find the happening in the long run we have to find the unique fixed prob vector v of P. i.e to find $V = V(a, b)$ such that $VP = V$ where $a + b = 1$

38 (a)

$VP = V$

$$[a \quad b]\begin{bmatrix} 0.3 & 0.7 \\ 0.4 & 0.6 \end{bmatrix} = [a \quad b]$$

$$[0.3a + 0.4b \quad 0.7a + 0.6b] = [a \quad b]$$

$$0.3a + 0.4b = a, \quad 0.7a + 0.6b = b$$

$$\frac{3a}{10} + \frac{4}{10}b = a \qquad x^{ly}\ 10$$

$$3a + 4b = 10a \Rightarrow 4b = 7a \Rightarrow \boxed{b = \frac{7}{4}a}$$

$$\therefore a + b = 1$$

$$a + \frac{7}{4}a = 1 \quad x^{ly}\ 4$$

$$4a + 7a = 4 \Rightarrow 11a = 4$$

$$\boxed{a = \frac{4}{11}} \qquad b = \frac{7}{4}a = \frac{7}{4}\cdot\frac{4}{11}$$

$$\boxed{b = \frac{7}{11}}$$

$\therefore \left(\frac{4}{11}, \frac{7}{11}\right)$ is the unique fixed prob vector. In the long run the student will study $\frac{4}{11} \times 100 = 36.36\%$ of the time.

10a A man's smoking habit's are as follows. If he smokes filter cigarettes one week, he switches to nonfilter cigarettes the next week proba-bility 0.2. On the other hand if he smokes non filter cigarettes one week, there is a prob of 0.7 that he will smoke non filter cigarettes the next week as well. In the long run how often does he smoke filter cigarettes.

Ans: The state space of the system is 2 i.e (Filter, nonfilter)

$$P = \begin{array}{c} \\ F \\ NF \end{array}\begin{array}{c} \begin{array}{cc} F & NF \end{array} \\ \begin{bmatrix} 0.8 & 0.2 \\ 0.3 & 0.7 \end{bmatrix} \end{array} = \frac{1}{10}\begin{bmatrix} 8 & 2 \\ 3 & 7 \end{bmatrix}$$

we have to find the unique fixed probability vector $V = (a, b)$ such that $VP = V$ where $a+b=1$

$$[a.\ b]\frac{1}{10}\begin{bmatrix} 8 & 2 \\ 3 & 7 \end{bmatrix} = [a.b]$$

$$[8a+3b, \cancel{2a}+7b] = [10a, 10b]$$

$$8a+3b = 10a, \quad 2a+7b = 10b$$

$$3b = 2a \qquad \therefore a+b=1$$

$$\boxed{b = \frac{2}{3}a} \qquad a + \frac{2a}{3} = 1$$

$$3a + 2a = 3$$

$$5a = 3$$

$$\boxed{a = \frac{3}{5}} \quad \boxed{b = \frac{2}{5}}$$

$$\therefore V = \left(\frac{3}{5}, \frac{2}{5}\right)$$

In the long run, he will smoke filter cigarettes $\frac{3}{5}$ or 60% of the time.

(21) Three boys A, B and C are throwing a ball to each other. A always throws the ball to B, & B always throws the ball to C, C is just as likely to throw the ball to B or A. Prepare the transition matrix and determine the long run fixed probability vector.

Soln: The state space of the system is [A, B, C], The transition matrix is

$$P = \begin{array}{c} \\ A \\ B \\ C \end{array}\begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} \end{array}$$

To find $V = V(a, b, c)$, such that $VP = V$ where $a+b+c = 1$.

Answer from problem No (1)

$V = \left(\frac{1}{5}, \frac{2}{5}, \frac{2}{5}\right)$ is the required fixed probability vector.

(12) Three boys A, B, C are throwing ball to each other. A always throw the ball to B and B always throws the ball to C. C is just as likely to throw the ball to B as to A. If 'C' was the first person to throw the ball find the probabilities that after three throws.

(i) A has the ball (ii) B has the ball

(iii) C has the ball.

Soln: state space $= [A, B, C]$ & the associated transition matrix is

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$$

Initially if C has the ball, the associated initial probability vector is given by $p^{(0)} = (0, 0, 1)$

Since the probabilities are desired after 3 throws we have to find $p^{(3)} = p^{(0)} P^3$. To find P^3

$$P^3 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{bmatrix}$$

$$P^{(3)} = P^{(0)} P^3 = [0, 0, 1] \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{bmatrix} = \left[\frac{1}{4}, \frac{1}{4}, \frac{1}{2}\right]$$

$$P^{(3)} = \left[\frac{1}{4}, \frac{1}{4}, \frac{1}{2}\right] = \left[P_A^{(3)}, P_B^{(3)}, P_C^{(3)}\right]$$

Thus after 3 throws the probability that the ball is with A is $\frac{1}{4}$, with B is $\frac{1}{4}$ & with C is $\frac{1}{2}$.

(13) A salesman's territory consists of 3 cities A, B & C. He never sells in the same city for 2 consecutive days. If he sells in city A, then the next day he sells in the city B. However if he sells in either B or C, then the next day he is twice as likely to sell in the city A as in the other city. In the long run, how often does he sell in each of the cities.

Soln: The state space of the system is [city A, city B, city C]. The transition matrix is

$$P = \begin{array}{c c} & \begin{array}{ccc} A & B & C \end{array} \\ \begin{array}{c} A \\ B \\ C \end{array} & \begin{bmatrix} 0 & 1 & 0 \\ \frac{2}{3} & 0 & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & 0 \end{bmatrix} \end{array}$$

$\frac{1}{3} \frac{1}{3} \frac{1}{3}$

We find the fixed probability vector $V = V(a, b, c)$ where $a+b+c=1$, such that $VP = V$

$$[a, b, c] \begin{bmatrix} 0 & 1 & 0 \\ \frac{2}{3} & 0 & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & 0 \end{bmatrix} = [a, b, c]$$

$$\left[\frac{2b}{3}+\frac{2c}{3},\ a+\frac{c}{3},\ \frac{b}{3}\right]=[a.b.c]$$

$$\frac{2b}{3}+\frac{2c}{3}=a,\quad a+\frac{c}{3}=b,\quad \frac{b}{3}=c$$

$$\boxed{b=3c}$$

$2b+2c=3a$ but $b=3c$

$6c+2c=3a$

$8c=3a$

$\boxed{c=\frac{3}{8}a}$ $\boxed{b=\frac{9}{8}a}$

$a+b+c=1$

$a+\frac{3a}{8}+\frac{9}{8}a=1$

$8a+3a+9a=8$

$20a=8$

$\boxed{a=\frac{8}{20}}$ $\boxed{b=\frac{9}{20}}$ $\boxed{c=\frac{3}{20}}$

$\therefore$ probability $=\left(\frac{8}{20},\frac{9}{20},\frac{3}{20}\right)$

In the long run he sells 40% in city A
45% in city B
15% in city C.

(14) A gambler's luck follows a pattern. If he wins a game, the probability of winning the next game is 0.6. However if he loses the game, the probability of losing the next game is 0.7. There is an even chance of gambler winning the first game. If so

(i) What is the prob of he winning the 2nd game

(ii) What is the prob of he winning the 3rd game

(iii) In the long run, how often he will win?

Soln: The State Space Of the System [win, lose]

The transition matrix,

$$P = \begin{array}{c c} & \begin{array}{cc} W & L \end{array} \\ \begin{array}{c} W \\ L \end{array} & \begin{bmatrix} 0.6 & 0.4 \\ 0.3 & 0.7 \end{bmatrix} \end{array} = \frac{1}{10}\begin{bmatrix} 6 & 4 \\ 3 & 7 \end{bmatrix}$$

probability of winning the first game is $\frac{1}{2}$

$\therefore$ initial probability vector $p^{(1)} = \left(\frac{1}{2}, \frac{1}{2}\right)$

(i) Now $p^{(2)} = p^{(1)} P = \frac{1}{2}[1, 1] \cdot \frac{1}{10}\begin{bmatrix} 6 & 4 \\ 3 & 7 \end{bmatrix} = \begin{bmatrix} \frac{9}{20} & \frac{11}{20} \end{bmatrix}$

Thus the prob of he winning the second game is $\frac{9}{20}$

(ii) $p^{(3)} = p^{(2)} P = \begin{bmatrix} \frac{9}{20}, \frac{11}{20} \end{bmatrix} \cdot \frac{1}{10}\begin{bmatrix} 6 & 4 \\ 3 & 7 \end{bmatrix} = \begin{bmatrix} \frac{87}{200}, \frac{113}{200} \end{bmatrix}$

Thus the prob of he winning the third game is $\frac{87}{200}$.

(iii) we have to find the fixed probability vector $V = V(a, b)$ such that $VP = V$ where $a + b = 1$

$$[a \cdot b] \cdot \begin{bmatrix} \frac{6}{10} & \frac{4}{10} \\ \frac{3}{10} & \frac{7}{10} \end{bmatrix} = [a \cdot b]$$

$$[6a + 3b, 4a + 7b] = [10a, 10b]$$

$6a + 3b = 10a$, $4a + 7b = 10b$

$3b = 4a$

$\boxed{b = \frac{4a}{3}}$

i.e $a + b = 1$

$a + \frac{4a}{3} = 1$

$3a + 4a = 3$

$7a = 3$

$\boxed{a = \frac{3}{7}}$, $\boxed{b = \frac{4}{7}}$

Hence $v = \left(\frac{3}{7}, \frac{4}{7}\right)$

Thus in the long run he wins $\frac{3}{7}$ of the time.

(15) Each year a man trades his car for a new car in 3 brands of the popular company Maruti Udyog Limited. If he has a 'Standard' he trades it for 'Zen'. If he has a 'Zen' he trades it for a 'Esteem'. If he has a 'Esteem' he is just as likely to trade it for a new 'Esteem' or for a 'Zen' or a 'Standard' one. In 1996 he bought his first car which was Esteem.

(i) Find the probability that he has
  (a) 1998 Esteem (b) 1998 Standard
  (c) 1999 Zen (d) 1999 Esteem.

(ii) In the long run, how often will he have a Esteem.

Soln: The state space of the system is [A, B, C] where A: Standard, B: Zen, C: Esteem.

The associated transition matrix is as follows

$$P = \begin{array}{c} \\ A(S) \\ B(Z) \\ C(E) \end{array} \begin{array}{c} \begin{array}{ccc} A(S) & B(Z) & C(E) \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix} \end{array} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

(i) With 1996 as the first year, 1998 is to be regarded as 2 years after and 1999 was 3 years after.

We need to compute $\underline{P^2 \,\&\, P^3}$.

ii) In 1996 he bought his first car which was esteen. $\therefore\ p^{(0)} = \begin{array}{ccc} S & Z & E \\ \end{array}$ $\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$

To find P^2 & P^3

$$P^2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{9} & \frac{4}{9} & \frac{4}{9} \end{bmatrix}$$

$$\therefore p^{(2)} = p^{(0)} P^2 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{9} & \frac{4}{9} & \frac{4}{9} \end{bmatrix}$$

$$= \begin{array}{c} \begin{bmatrix} \frac{1}{9} & \frac{4}{9} & \frac{4}{9} \end{bmatrix} \\ \begin{array}{ccc} S & Z & E \end{array} \end{array}$$

$\therefore$ 1998 Esteem $= \frac{4}{9}$, 1998 Standard $= \frac{1}{9}$

$$P^3 = P^2 P = \begin{bmatrix} 0 & 0 & 1 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{9} & \frac{4}{9} & \frac{4}{9} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{9} & \frac{4}{9} & \frac{4}{9} \\ \frac{4}{27} & \frac{7}{27} & \frac{16}{27} \end{bmatrix}$$

$$p^{(3)} = p^{(0)} P^3 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{9} & \frac{4}{9} & \frac{4}{9} \\ \frac{4}{27} & \frac{7}{27} & \frac{16}{27} \end{bmatrix} =$$

$$= \begin{array}{c} \begin{bmatrix} \frac{4}{27} & \frac{7}{27} & \frac{16}{27} \end{bmatrix} \\ \begin{array}{ccc} S & Z & E \end{array} \end{array}$$

1999 Zen: $\frac{7}{27}$ 1999 Esteem: $\frac{16}{27}$

(ii) To find the unique fixed probability vector $V = V(a, b, c)$ such that

$VP = V$ where $a+b+c=1$

$$[a\ b\ c]\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix} = [a\ b\ c]$$

$$\left[\frac{c}{3},\ a+\frac{c}{3},\ b+\frac{c}{3}\right] = [a\ b\ c]$$

$\frac{c}{3} = a,$ $\quad a+\frac{c}{3} = b$ $\quad b+\frac{c}{3} = c$

$\boxed{c = 3a}$ $\quad \times^{l}$ by 3

$3a + c = 3b$ $\quad$ But $c = 3a$

$3a + 3a = 3b$

$6a = 3b$

$\boxed{b = 2a}$

$a+b+c=1$

$a+2a+3a=1$

$6a = 1$ $\quad b = 2a$ $\quad c = 3a$

$\boxed{a = \frac{1}{6}}$ $\quad \boxed{b = \frac{2}{6}}$ $\quad \boxed{c = \frac{3}{6}}$

$V = \left(\frac{1}{6}\ \frac{2}{6}, \frac{3}{6}\right) = [p^{s}, p^{2}\ p^{E}]$.

In the long run probability of he having Esteem is $p^E = \frac{1}{2}$. Thus in the long run 50% of the time he will have Esteem.

16 Every year, a man trades his car for a new car. If he has a Maruti, he trades it for an Ambassador. If he has Ambassador, he trades it for a santro. However, if he has a santro, he is just as likely to trade it for a new santro as to trade it for a maruti or an Ambassador. In 2000 he bought his first car which was a santro. (i) Find the probability that he has a) 2002 santro (b) 2002 maruti (c) 2003 Ambassador (d) 2003 santro (ii) In the long run, how often will he have a santro.

Ans. Associated transition matrix

$$P = \begin{array}{c} \\ M \\ A \\ S \end{array} \begin{array}{c} \begin{array}{ccc} M & A & S \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix} \end{array}$$

Initial condition is

$$p^{(0)} = \begin{array}{c} \begin{array}{ccc} M & A & S \end{array} \\ \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \end{array}$$

(i) To find P^2 & P^3

(a) 2002 santro $= \frac{4}{9}$, (b) 2002 Maruti $= \frac{1}{9}$

(c) 2003 Ambassador : $\frac{7}{27}$, (d) 2003 santro : $\frac{16}{17}$

(ii) $[p^M \; p^A \; p^S] = \left[\frac{1}{6} \; \frac{2}{6} \; \frac{3}{6}\right]$

(17) A software Engineer goes to his office everyday by motor bike (or) by car. He never goes by bike on two consecutive days, but if he goes by car on a day then he is equally likely to go by car or by bike the next day. Find the total probability matrix of the Markov chain. If car is used on the first day of the week (1) find the probability that ~~after 4 days~~ (a) bike is used (b) car is used, on the fifth day. (2) If the prob that bike is used on the Ist week Find the probs that on the fifth day bike is used

Ans: The state space of the system is [Bike, Car]. The transition matrix

$$P = \begin{array}{c} \\ B \\ C \end{array} \begin{array}{c} \begin{array}{cc} B & C \end{array} \\ \begin{bmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \end{array}$$

(i) The initial prob distribution is $p^{(1)} = [0, 1]$

The prob distribution of the mode of transport on the fifth day is $p^{(5)} = p^{(1)} P^{(4)}$

$$P^2 = \begin{bmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{3}{4} \end{bmatrix}$$

$$P^3 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{3}{4} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & \frac{3}{4} \\ \frac{3}{8} & \frac{5}{8} \end{bmatrix}$$

$$P^4 = \begin{bmatrix} \frac{1}{4} & \frac{3}{4} \\ \frac{3}{8} & \frac{5}{8} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{3}{8} & \frac{5}{8} \\ \frac{5}{16} & \frac{11}{16} \end{bmatrix}$$

$p^{(5)} = p^{(1)} p^4 = [0\ 1] \begin{bmatrix} \frac{3}{8} & \frac{5}{8} \\ \frac{5}{16} & \frac{11}{16} \end{bmatrix}$

$p^{(5)} = \begin{bmatrix} \frac{5}{16} & \frac{11}{16} \end{bmatrix}$
$\quad\quad\ \ \text{B} \quad \text{C}$

on the fifth day, the prob of using the bike is $\frac{5}{16}$ & prob of using the car is $\frac{11}{16}$

(2) Given $p^{(1)} = \begin{bmatrix} \frac{5}{6} & \frac{1}{6} \end{bmatrix}$

$p^{(5)} = p^{(1)} p^4 = \begin{bmatrix} \frac{5}{6} & \frac{1}{6} \end{bmatrix} \begin{bmatrix} \frac{3}{8} & \frac{5}{8} \\ \frac{5}{16} & \frac{11}{16} \end{bmatrix}$

$= \begin{array}{c} \begin{bmatrix} \frac{35}{96} & \frac{61}{96} \end{bmatrix} \\ \text{B} \quad\quad \text{C} \end{array}$

prob that on the fifth day, bike is used is $= \frac{35}{96}$

(18) Two boys B_1, B_2 & two girls G_1 & G_2 are throwing ball from one to the other. Each boy throws the ball to the other boy with probability $\frac{1}{2}$ & to each girl with probability $\frac{1}{4}$. On the other hand each girl throws the ball to each boy with probability $\frac{1}{2}$ & never to the other girl. In the long run how often does each receive the ball.

Ans: State Space $[B_1, B_2, G_1, G_2]$

$$P = \begin{array}{c|c} & \begin{array}{cccc} B_1 & B_2 & G_1 & G_2 \end{array} \\ \begin{array}{c} B_1 \\ B_2 \\ G_1 \\ G_2 \end{array} & \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & 0 & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \end{bmatrix} \end{array}$$

Ans: $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{6}, \frac{1}{6}\right)$

(19) A habitual gambler visits either of two clubs A & B every day. He never visits club A on two consecutive days. But if he visits club B on a day, then the next day he is as likely to visit club B or club A. Find the transition matrix for the chain of his visits. Show that it is a regular stochastic matrix & find its fixed probability vector.

Ans: $$P = \begin{array}{c|c} & \begin{array}{cc} A & B \end{array} \\ \begin{array}{c} A \\ B \end{array} & \begin{bmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \end{array}$$

The first row corresponds to the fact that he never goes to club A on 2 consecutive days which ⇒ he is sure to visit B. The prob of going A is $\frac{1}{2}$ & prob of going B is also $\frac{1}{2}$

To find P^2

$$P^2 = \begin{bmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$P^2 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{3}{4} \end{bmatrix}$$ (regular stochastic matrix)

(ii) To find $V = V(a, b)$ where $a + b = 1$ ∋'

$$VP = V$$

$$[a \ b] \begin{bmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} = [a \ b]$$

$$\left[\frac{b}{2}, \ a + \frac{b}{2}\right] = [a \ b]$$

$$\frac{b}{2} = a, \quad a + \frac{b}{2} = b$$

$$\Rightarrow \boxed{b = 2a} \qquad \therefore a + b = 1$$

$$a + 2a = 1$$

$$3a = 1 \qquad b = 2a$$

$$\boxed{a = \frac{1}{3}} \qquad \boxed{b = \frac{2}{3}}$$

$\therefore \left(\frac{1}{3}, \frac{2}{3}\right)$ is the unique fixed prob vector.

(20) Suppose an urn contains 2 white marbles and urn B contains 4 red marbles. At each step of the process, a marble is selected at random from each urn & the two marbles selected are interchanged. Let X_n denote the number of red marbles in urn A after n interchanges. (i) Find the transition matrix P. (ii) What is the prob that there are 2 red marbles in urn A after 3 steps. (iii) In the long run, what is the prob that there are 2 red marbles in urn A (iv) What is the stationary distribution of the system.

Solⁿ. There are 3 states a_0, a_1, a_2

2W	4R
A	B

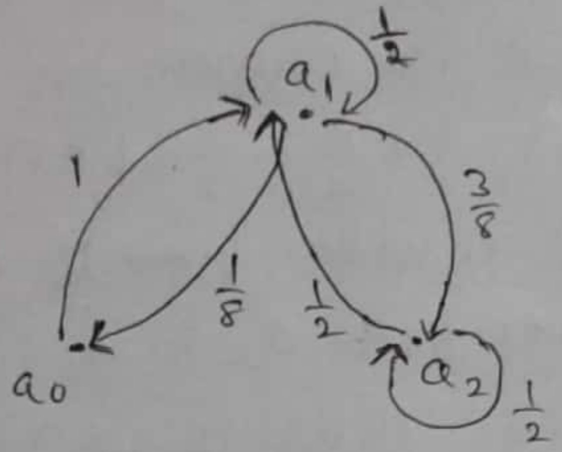
State a_0

1W 1R	3R 1W
A	B

State a_1

2R	2R 2W
A	B

State a_2



(i) If the system is in the state a_0, then the white marble from A and a red from B must be selected. So that the system will now move to state a_1. Accordingly the first row of the transition matrix is $(0, 1, 0)$

(ii) Now suppose the system is in a_1. It can move to state a_0, iff red from A and white from B with probability $\frac{1}{2} \cdot \frac{1}{4} = \frac{1}{8}$. Thus $P_{10} = \frac{1}{8}$. The system can move from a_1 to a_2 iff white from A and red from B with probability $\frac{1}{2} \cdot \frac{3}{4} = \frac{3}{8}$ i.e $P_{12} = \frac{3}{8}$

$\therefore P_{11} = 1 - \frac{1}{8} - \frac{3}{8} = \frac{1}{2}$

$\therefore$ 2nd row of TCM $\left[\frac{1}{8}, \frac{1}{2}, \frac{3}{8}\right]$

(iii) Finally, suppose the system is in state a_2. Note that the system can never move from state a_2 to a_1. However it may remain in a_2 itself, if a red from A and red from B is chosen. In this case the probability is $\frac{1}{1} \times \frac{2}{4} = \frac{1}{2}$. Lastly if a red from A and white from B is chosen then system moves from a_2 to a_1 with probability $\frac{2}{4} = \frac{1}{2}$. Thus third row of the transition matrix is $(0, \frac{1}{2}, \frac{1}{2})$

$$P = \begin{array}{c} \\ a_0 \\ a_1 \\ a_2 \end{array} \begin{array}{c} \begin{array}{ccc} a_0 & a_1 & a_2 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{8} & \frac{1}{2} & \frac{3}{8} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix} \end{array}$$

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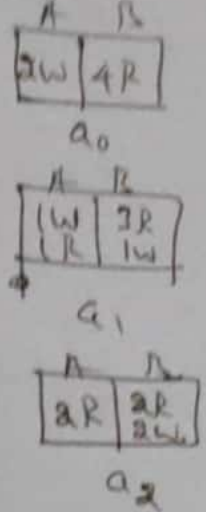
Ans. (i) Transition matrix is

$$P = \begin{array}{c} \\ a_0 \\ a_1 \\ a_2 \end{array}\begin{array}{c} \begin{array}{ccc} a_0 & a_1 & a_2 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{8} & \frac{1}{2} & \frac{3}{8} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \end{array}$$

Initial prob is $p^{(1)} = [1, 0, 0]$



$\frac{1}{2}\cdot\frac{1}{4} = \frac{1}{8}$

$\frac{1}{2}\cdot\frac{3}{4} = \frac{3}{8}$

(ii) To find $p^{(3)} = p^{(1)} P^{3}$

$$P^2 = \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{8} & \frac{1}{2} & \frac{3}{8} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{8} & \frac{1}{2} & \frac{3}{8} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{8} & \frac{1}{2} & \frac{3}{8} \\ \frac{1}{16} & \frac{9}{16} & \frac{6}{16} \\ \frac{1}{16} & \frac{1}{2} & \frac{7}{16} \end{bmatrix}$$

$$P^3 = \begin{bmatrix} \frac{1}{8} & \frac{1}{2} & \frac{3}{8} \\ \frac{1}{16} & \frac{9}{16} & \frac{6}{16} \\ \frac{1}{16} & \frac{1}{2} & \frac{7}{16} \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{8} & \frac{1}{2} & \frac{3}{8} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{16} & \frac{9}{16} & \frac{6}{16} \\ \frac{9}{128} & \frac{17}{32} & \frac{51}{128} \\ \frac{1}{16} & \frac{17}{32} & \frac{13}{32} \end{bmatrix}$$

$$\therefore p^{(3)} = p^{(1)} P^3 = [1\ 0\ 0]\begin{bmatrix} \frac{1}{16} & \frac{9}{16} & \frac{6}{16} \\ \frac{9}{128} & \frac{17}{32} & \frac{51}{128} \\ \frac{1}{16} & \frac{17}{32} & \frac{13}{32} \end{bmatrix}$$

$$= \left[\frac{1}{16}\ \ \frac{9}{16}\ \ \frac{6}{16}\right]$$

$\therefore$ Probability that there are 2 red in A i.e in state a_2 after 3 steps is $\frac{6}{16}$

OR

OR) $p^{(0)} = [1, 0, 0]$ (5)

$$p^{(1)} = p^{(0)} P = [1\ 0\ 0]\begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{8} & \frac{1}{2} & \frac{3}{8} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} = [0, 1, 0]$$

$$p^{(2)} = p^{(1)} P = [0\ 1\ 0]\begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{8} & \frac{1}{2} & \frac{3}{8} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} = \left[\frac{1}{8}\ \frac{1}{2}, \frac{3}{8}\right]$$

$$p^{(3)} = p^{(2)} P = \left[\frac{1}{8}\ \frac{1}{2}\ \frac{3}{8}\right]\begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{8} & \frac{1}{2} & \frac{3}{8} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} = \left[\frac{1}{16}, \frac{9}{16}, \frac{6}{16}\right]$$

(iii) TO find $V = V(a, b, c)$ where $a+b+c=1$

$\exists'$ $VP = V$

$$[a\ b\ c]\begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{8} & \frac{1}{2} & \frac{3}{8} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} = [a\ b\ c]$$

$$\left[\frac{b}{8}, \ a+\frac{b}{2}+\frac{c}{2} \quad \frac{3b}{8}+\frac{c}{2}\right] = [a.\ b.\ c]$$

$\frac{b}{8} = a$ $\quad$ $a+\frac{b}{2}+\frac{c}{2} = b$ $\quad$ $\frac{3b}{8}+\frac{c}{2} = c$

$\boxed{b = 8a}$

x'g 2

$2a+b+c = 2b$

$c = b-2a$

$c = 8a-2a \Rightarrow \boxed{c = 6a}$

$a+b+c=1$

$a+8a+6a=1 \Rightarrow 15a = 1 \Rightarrow \boxed{a = \frac{1}{15}}$

$\boxed{b = \frac{8}{15}}$ $\quad$ $\boxed{c = \frac{6}{15}}$

$\therefore$ $V = \left(\frac{1}{15}, \frac{8}{15}\ \frac{6}{15}\right)$ is fixed prob vector.

(iv) The unique fixed prob vector $\left(\frac{1}{15}, \frac{8}{15}, \frac{6}{15}\right)$ is the stationary distribution since P^n approaches V in the long run.

(21) A Students study habits are as follows, if he studies one night, he is 70% sure not to study the next night. on the other hand if he does not study one night, he is 60% sure not to study the next night supposing that he studies on Monday night, find the probability that he does not study on the next friday night. In the long run how often does he study.

Ans: The state space of the system is 2 (study, not study)

$$P = \begin{array}{c c} & \begin{array}{cc} S & NS \end{array} \\ \begin{array}{c} S \\ NS \end{array} & \begin{bmatrix} 0.3 & 0.7 \\ 0.4 & 0.6 \end{bmatrix} \end{array}$$

The initial prob vector is $p^{(1)} = [1, 0]$

To find $p^{(5)} = p^{(1)} P^4$.

To find P^4 :-

$$P^2 = \begin{bmatrix} 0.37 & 0.63 \\ 0.36 & 0.64 \end{bmatrix} \quad P^3 = \begin{bmatrix} 0.363 & 0.637 \\ 0.364 & 0.636 \end{bmatrix}$$

$$P^4 = \begin{bmatrix} 0.3637 & 0.6363 \\ 0.3637 & 0.6363 \end{bmatrix}$$

$$p^{(5)} = [1, 0] = \begin{bmatrix} 0.3637 & 0.6363 \\ 0.3637 & 0.6363 \end{bmatrix} = \begin{array}{c} \begin{bmatrix} 0.3637 & 0.6363 \end{bmatrix} \\ \begin{array}{cc} S & NS \end{array} \end{array}$$

$\therefore P(\text{not study}) = 0.6363$

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5C(a)

In the long run, $VP = V$ $\ni'$ $a+b=1$

$\ni'$ $VP = V$

$$[a \ b]\begin{bmatrix}0.3 & 0.7\\ 0.4 & 0.6\end{bmatrix} = [a \ b]$$

$0.3a + 0.4b = a, \quad 0.7a + 0.6b = b$

$\frac{3a}{10} + \frac{4b}{10} = a$

$3a + 4b = 10a$

$4b = 7a \Rightarrow \boxed{b = \frac{7}{4}a}$

$\therefore a + b = 1$

$a + \frac{7}{4}a = 1 \quad \times^{l_y} \Rightarrow 4a + 7a = 4$

$11a = 4 \Rightarrow \boxed{a = \frac{4}{11}}$

$b = \frac{7}{4}a$

$b = \frac{7}{4}\cdot\frac{4}{11}$

$\boxed{b = \frac{7}{11}}$

i.e in the long run, he studies

$\frac{4}{11} \times 100 \equiv 36.36\%$ of the time.

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(22) A house wife buys three brands of soaps A, B, C. She never buys the same brand on successive weeks. If she buys brands A in a week, the buys brand B in the next week. If she buys the brand other than A in a week, then in the next week she is 3 times as likely to buy brand A as the other brand. supposing that she has brought brand B in the first week. Find the prob of her buying each of the 3 brands in the fourth week. (56)

Ans: The transition matrix is

$$P = \begin{array}{c} \\ A \\ B \\ C \end{array} \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ \frac{3}{4} & 0 & \frac{1}{4} \\ \frac{3}{4} & \frac{1}{4} & 0 \end{bmatrix} \end{array}$$

Initial probability vector is $p^{(1)} = [0, 1, 0]$

To find $p^{(4)} = p^{(1)} P^3$:

To find P^3

$$P^2 = \begin{bmatrix} 0 & 1 & 0 \\ \frac{3}{4} & 0 & \frac{1}{4} \\ \frac{3}{4} & \frac{1}{4} & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ \frac{3}{4} & 0 & \frac{1}{4} \\ \frac{3}{4} & \frac{1}{4} & 0 \end{bmatrix} = \begin{bmatrix} \frac{3}{4} & 0 & \frac{1}{4} \\ \frac{3}{16} & \frac{13}{16} & 0 \\ \frac{3}{16} & \frac{3}{4} & \frac{1}{16} \end{bmatrix}$$

$$P^3 = P^2 P = \begin{bmatrix} \frac{3}{4} & 0 & \frac{1}{4} \\ \frac{3}{16} & \frac{13}{16} & 0 \\ \frac{3}{16} & \frac{3}{4} & \frac{1}{16} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ \frac{3}{4} & 0 & \frac{1}{4} \\ \frac{3}{4} & \frac{1}{4} & 0 \end{bmatrix}$$

$$P^3 = \begin{bmatrix} \frac{3}{16} & \frac{13}{16} & 0 \\ \frac{39}{64} & \frac{3}{16} & \frac{13}{64} \\ \frac{39}{64} & \frac{13}{64} & \frac{3}{16} \end{bmatrix}$$

$$\therefore P^{(4)} = P^{(1)} P^3$$

$$= \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{3}{16} & \frac{13}{16} & 0 \\ \frac{39}{64} & \frac{3}{16} & \frac{13}{64} \\ \frac{39}{64} & \frac{13}{64} & \frac{3}{16} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{39}{64}, & \frac{3}{16} & \frac{13}{16} \end{bmatrix} = \begin{bmatrix} P^A & P^B & P^C \end{bmatrix}$$

$$P^A = \frac{39}{64}, \quad P^B = \frac{3}{16} \quad P^C = \frac{13}{16}.$$

JOINT PROBABILITY DISTRIBUTION AND MARKOV CHAINS

✓ 1. Find (a) marginal distributions f(x) and g(y), (b) E(X) and E(Y), (c) Cov (X,Y), (d) σ_x, σ_y and (e) $\rho(X,Y)$ for the following joint distribution, (f) Are X and Y independent random variables?

X \ Y	-4	2	7
1	1/8	1/4	1/8
5	1/4	1/8	1/8

2. The joint probability distribution table for two random variables X and Y is as follows. Determine the marginal probability distribution of X and Y. Also compute(a) Expectations of X , Y and XY , (b) Standard deviations of X and Y, (c) Covariance of X and Y, (d) Correlations of X and Y.

X \ Y	-2	-1	4	5
1	0.1	0.2	0	0.3
2	0.2	0.1	0.1	0

3. The joint probability distribution of X and Y is given below

X \ Y	3	4	5
2	1/6	1/6	1/6
5	1/12	1/12	1/12
7	1/12	1/12	1/12

Find i) marginal distribution of X and y, ii) E(X),E(Y),E(XY), iii) Cov (X,Y) iv) $\rho(X,Y)$. Are X and Y independent?

4. Find the joint distribution of X and Y, which are independent random variables with the following respective distribution. Show that Cov (X,Y)=0.

x_i:	1	2
$f(x_i)$:	0.7	0.3

y_i	-2	5	8
$g(y_i)$	0.3	0.5	0.2

5. X and Y are independent random variables. X takes values 2,5,7 with probability ½, ¼, ¼ respectively. Y take values 3,4,5 with the probability 1/3, 1/3, 1/3 . (a) Find the joint probability distribution of X and y. (b) Show that the covariance of X and Y is equal to zero. (c) Find the probability distribution of Z = X+Y.

6. Given the following joint distribution of the random variables X and Y, find the corresponding marginal distribution. Also compute the covariance and the correlation of the random variables X and Y.

X \ Y	1	3	9
2	1/8	1/24	1/12
4	1/4	1/4	0
6	1/8	1/24	1/12

7. Determine (a) marginal distributions of X and Y (b) Cov (X,Y), (c) $\rho(X,Y)$ for the following joint distribution, (d) Determine whether X and Y are independent.

X \ Y	-3	2	4
1	0.1	0.2	0.2
3	0.3	0.1	0.1

8. If X and Y are independent random variables, find the joint distribution of X and Y with the following respective distribution of X and Y.

x_i:	1	2
$f(x_i)$:	0.6	0.4

y_i:	5	10	15
$g(y_i)$:	0.2	0.5	0.3

9. A Fair coin is tossed three times Let X denote 0 or 1 according as a head or a tail occurs on the first toss. Let Y denote the number of heads which occur. (a) Find the marginal distributions of X and Y. (b) Determine the joint distribution of X and Y and (c) Cov(X,Y).

10. A fair coin is tossed twice. Let X and Y be defined as follows i) X = 1 if the first toss is H and X = 0 otherwise. ii) Y = 1 if both tosses are H and Y = 0 otherwise. Find the joint probability distribution of X and Y. Show that X and Y are not independent.

11. A fair coin is tossed twice. Let X and Y be defined as follows i) X = 1 if the first toss is H and X = 0 otherwise. Ii) Y = 1 if the second toss is H and Y = 0 otherwise. Find the joint probability distribution of X and Y.

12. Given the joint distribution (a) Determine the marginal distributions of X and Y, (b) Are X and Y independent.

x Y	0	1	2
0	0.1	0.4	0.1
1	0.2	0.2	0

13. Find the marginal distributions of X and Y , if the joint distribution is

x Y	1	2	3
1	0.05	0.05	0.1
2	0.05	0.1	0.35
3	0	0.2	0.1

14. Two marbles are selected at random from a box containing 3 blue, 2 red and 3 green marbles. If X is the number of blue marbles and Y is the number of red marbles selected, find (a) the marginal distribution of X and Y. (b) Show that the random variables X and Y are not statistically independent.

15. Roll a pair of dice. Let X denote the smaller and Y larger outcome on the dice. Find the distribution of X and Y.

16. The joint probability distribution of two discrete random variables X and Y is give by $f(x,y)=k(2x+y)$ where x and y are integers such that $0\leq x\leq 2,\ 0\leq y\leq 3$ ()a Find the value of the constant k. (b) Find the marginal probability distributions of X and Y. (c) Show that the random variables X and Y are dependent

MARKOV CHAINS

1. Which vectors are probability vectors

a) $\left(\frac{1}{4},\frac{3}{2},\frac{-1}{4},\frac{1}{2}\right)$

b) $\left(\frac{1}{12},\frac{1}{2},\frac{1}{6},0,\frac{1}{4}\right)$

c) $\left(\frac{5}{2},0,\frac{8}{3},\frac{1}{6},\frac{1}{6}\right)$

d) $(3,0,2,5,3)$

e) $\left(\frac{1}{2},\frac{1}{3},0,\frac{-1}{5}\right)$

f) $(3,4,5,0)$

g) $\left(\frac{1}{4}, \frac{1}{2}, 0, \frac{1}{4}\right)$

2. Find scalar multiple of each vector, which is a probability vector.

a) $\left(2, \frac{1}{2}, 0, \frac{1}{4}, \frac{3}{4}, 0, 1\right)$

b) $\left(\frac{1}{3}, 2, \frac{1}{2}, 0, \frac{1}{4}, \frac{2}{3}\right)$

c) $\left(\frac{1}{2}, \frac{2}{3}, 0, 2, \frac{5}{6}\right)$

d) $(1, 2, 3, 4, 5, 6)$

3. Which matrices are stochastic

a) $\begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \end{bmatrix}$

b) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

c) $\begin{bmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{4} \end{bmatrix}$

d) $\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$

e) $\begin{bmatrix} 0 & 1 \\ -\frac{1}{2} & \frac{3}{2} \end{bmatrix}$

f) $\begin{bmatrix} 1/3 & 2/3 & 4/3 \\ 1/2 & 1 & 1/2 \end{bmatrix}$

g) $\begin{bmatrix} 15/16 & 1/16 \\ 2/3 & 4/3 \end{bmatrix}$

h) $\begin{bmatrix} 1 & 0 \\ 1/2 & 1/2 \end{bmatrix}$

i) $\begin{bmatrix} 1/2 & -1/2 \\ 1/4 & 3/4 \end{bmatrix}$

4. Which of the stochastic matrices are regular?

a) $\begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$

b) $\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{bmatrix}$

c) $\begin{bmatrix} 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \end{bmatrix}$

d) $\begin{bmatrix} 1/2 & 1/2 \\ 0 & 1 \end{bmatrix}$

e) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

f) $\begin{bmatrix} 1/2 & 1/4 & 1/4 \\ 0 & 1 & 0 \\ 1/2 & 1/2 & 0 \end{bmatrix}$

g) $\begin{bmatrix} 0 & 0 & 1 \\ 1/2 & 1/4 & 1/4 \\ 0 & 1 & 0 \end{bmatrix}$

5. Show that the Markov chain with the transition matrix $\begin{bmatrix} 1 & 0 \\ 1/4 & 3/4 \end{bmatrix}$ is not irreducible.

6. Show that the Markov chain with the transition matrix $P = \begin{pmatrix} 0 & 3/4 & 1/4 \\ 1/2 & 1/2 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ is irreducible

7. Show that $v = (b, a)$ is fixed point of the stochastic matrix $P = \begin{pmatrix} 1-a & a \\ b & 1-b \end{pmatrix}$

8. Find the unique fixed probability vector t of $P = \begin{pmatrix} 0 & 3/4 & 1/4 \\ 1/2 & 1/2 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

9. Find the unique fixed probability vector of each matrix

a) $\begin{bmatrix} 2/3 & 1/3 \\ 2/5 & 3/5 \end{bmatrix}$

b) $\begin{bmatrix} 1/4 & 3/4 \\ 5/6 & 1/6 \end{bmatrix}$

c) $\begin{bmatrix} 0.2 & 0.8 \\ 0.5 & 0.5 \end{bmatrix}$

d) $\begin{bmatrix} 3/4 & 1/4 \\ 1/2 & 1/2 \end{bmatrix}$

e) $\begin{bmatrix} 1/3 & 2/3 \\ 1/4 & 3/4 \end{bmatrix}$

f) $\begin{bmatrix} 0 & 1 & 0 \\ 1/6 & 1/2 & 1/3 \\ 0 & 2/3 & 1/3 \end{bmatrix}$

g) $\begin{bmatrix} 0 & 1/2 & 1/2 \\ 1/3 & 2/3 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

h) $\begin{bmatrix} 0 & 1/2 & 1/4 & 1/4 \\ 1/2 & 0 & 1/4 & 1/4 \\ 1/2 & 1/2 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \end{bmatrix}$

i) $\begin{bmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/4 & 1/4 \end{bmatrix}$

j) $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix}$

10. Show that $P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix}$ is a regular stochastic matrix. Also find the associated unique fixed probability vector.

11. Find the unique fixed probability vector t of $P = \begin{pmatrix} 0 & 1/2 & 1/2 & 0 \\ 1/2 & 1/4 & 0 & 1/4 \\ 0 & 0 & 0 & 1 \\ 0 & 1/2 & 0 & 1/2 \end{pmatrix}$, $P = \begin{bmatrix} 0 & 1 \\ 1/2 & 1/2 \end{bmatrix}$

12. Find the unique fixed probability vector t of $P=\begin{pmatrix} 1/4 & 1/2 & 1/4 & 0 \\ 2/3 & 1/3 & 0 & 0 \\ 1/6 & 1/6 & 0 & 2/3 \\ 1/2 & 0 & 0 & 1/2 \end{pmatrix}$.

13. Every year, a man trades his car for a new car. If he has a Maruti, he trades it for an Ambassador. If he has an Ambassador, he trades it for a Santro. However, if he has a Santro, he is just as likely to trade it for a new Santro as to trade it for a Maruti or an Ambassador. In 2000 he bought his first car, which was a Santro. (i)Find the probability that he has (a) 2002 Santor, (b) 2002 Maruti;, (c) 2003 Ambassador, (d) 2003 Santro. And (ii) In the long run, how often will he have a Santro.

14. Suppose an urn A contains 2 white marbles and urn B contains 4 red marbles. At each step of the process, a marble is selected at random from each urn and the two marbles selected are interchanged. Let X_n denote the number of red marbles in urn A after n interchanges. (i) Find the transition matrix P. (ii) What is the probability that there are 2 red marbles in urn A after 3 steps. (iii) In the long run, what is the probability that there are 2 red marbles in urn A. (iv) What is the stationary distribution of the system.

15. A man's smoking habits are as follows. If he smokes filter cigarettes one week, he switches to non filter cigarettes the next week with probability 0.2. On the other hand it he smokes non filter cigarettes one week, there is a probability of 0.7 that he will smoke non filter cigarettes the next week as well. In the long run how often does he smoke filter cigarettes.

16. A saleman's territory consists of 3 cities A,B and C. He never sells in the same city on successive days. If he sells in city A, then the next day he sells in city B. However it he sells in either B or C, then the next day he is twice as likely to sell in city A as in other city. In the long run, how often does he sell in each of the cities?

17. A student's study habits areas follows. If he studies one night, he is 70% sure not to study the next night. On the other hand if he does not study one night, he is 60% sure not to study the next night. In the long run how often does he study?

18. Three boys A,B,C are throwing ball to each other. A always throws the ball to B and B always throws the ball to C. C is just as likely to throw the ball to B as to A. If C was the first person to throw the ball find the probabilities that after three throws.

19. A software engineer goes to his office everyday by motorbike or by car. He never goes by bike on two consecutive days, but if he goes by car on a day then he is equally likely to go by car or by bike the next day. Find the total probability matrix of the Markov chain. If car is used on the first day of the week find the probability that after 4 days (a) bike is used (b) car is used.

20. A professor drives or takes the college bus to work. If he drives to work, then the next day he takes the bus with the probability 0.2. on the other hand, if he takes the bus, then the next day he drives with the probability 0.3. Find the transition matrix for the mode of transport.

21. A habitual gambler visits either of two clubs A and B every day. He never visits club A on two consecutive days. But it he visits club B on a day, then the next day he is as likely to visit club B or Club A. Find the transition matrix for the chain of his visits. Show that it is a regular stochastic matrix and find its fixed probability vector.

22. An engineer goes to his work place every day be bike or by car. He never goes by bike on two consecutive days: but if he goes by car on a day then he is equally likely to go by car or by bike on the next day. Find the transition matrix for the chain of the mode of transport he uses. If the probability that the bike is used on the first day of a week is 5/6. Find the probability that on the fifth day (i) bike is used (ii) car is used.

23. A student's study habits areas follows. If he studies one night, he is 70% sure not to study the next night. On the other hand if he does not study one night, he is 60% sure not to study the next night. Supposing that he studies on Monday night, find the probability that he does not study on the next Friday night. In the long run how often does he study?

24. A company executive changes his car every year. If he has a Maruti he changes over to Fiat. If he has a Fiat, he changes over to Santro. However, if he has a Santro, he is just as likely to change over to new Santro, or Fiat or Maruti. In 2008 he bought his first car, which was a Santro i) Find the probability that he has a) 2010 Santro, b) 2010 Maruti, c) 2011 Fiat , d) 2011 Santro. Ii) In the long run, how often will he have a Santro.