GANITA PRAKASH

Textbook of Mathematics

GRADE

7

Part – II

0789

विद्ययाऽमृतमश्नुते

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NCERT

राष्ट्रीय शैक्षिक अनुसंधान और प्रशिक्षण परिषद्
NATIONAL COUNCIL OF EDUCATIONAL RESEARCH AND TRAINING

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FOREWORD

The National Education Policy 2020 envisages a system of education in the country that is rooted in an Indian ethos and its civilisational accomplishments in all fields of knowledge and human endeavour. At the same time, it aims to prepare students to engage constructively with the opportunities and challenges of the 21st century. The basis for this aspirational vision has been well laid out by the National Curriculum Framework for School Education (NCF-SE) 2023 across curricular areas at all stages. By nurturing students' inherent abilities across all five planes of human existence (*pañchakośhas*), the Foundational and Preparatory Stages set the stage for further learning at Middle Stage. Spanning from Grades 6 to 8, the Middle Stage serves as a critical three-year bridge between the Preparatory and Secondary Stages.

For the Middle Stage, NCF-SE 2023 aims to equip students with the skills needed to grow, as they advance in their lives. It endeavours to enhance their analytical, descriptive, and narrative capabilities, and to prepare them for the challenges and opportunities that await them. A diverse curriculum, covering nine subjects ranging from three languages — including at least two languages native to India — to Science, Mathematics, Social Science, Art Education, Physical Education and Well-being, and Vocational Education promotes their holistic development.

Such a transformative learning culture requires certain essential conditions. One of them is to have appropriate textbooks in different curricular areas, as these textbooks will play a central role in mediating between content and pedagogy — a role that will strike a judicious balance between direct instruction and opportunities for exploration and inquiry. Among the other conditions, classroom arrangement and teacher preparation are crucial to establish conceptual connections both within and across curricular areas.

The National Council of Educational Research and Training, on its part, is committed to providing students with such high-quality textbooks. Various Curricular Area Groups, constituted for this purpose, comprising notable subject-experts, pedagogues, and practising teachers as their members, have made all possible efforts to develop such textbooks. *Ganita Prakash* the textbook of Mathematics for Grade 7, Part II; aligns with the expectations of NEP 2020 and NCF-SE 2023, in respect of creating a spark for initiating mathematical thinking. This textbook designed for Grade 7 students, takes forward its journey through the world of mathematics that started in Grade 6. While Part-I set the journey recapitulating Grade 6 Mathematics, Part-II takes an advanced path on mathematical concepts and prepares students for

Grade 8. During this journey the concepts and problems emerge from daily life situations and so it is expected that students will be able to relate to them with ease. This textbook makes efforts to encourage the students to observe and explore the patterns around them and discover mathematical concepts on their own. The content attempts to integrate mathematics with other subject areas such as science, social science with cross-cutting themes like environmental education, value education, inclusive education, and Indian Knowledge Systems (IKS). Colourful illustrations and interactive exercises form the basis of this textbook that would develop a strong foundation among children in understanding more complex mathematical concepts. Throughout the book, stories, conversations and anecdotes have been incorporated to make abstract mathematical concepts more relatable and accessible to young learners. Puzzles and innovative problems will not only engage the students in thoughtfully relating the mathematical concepts to the world around them and help them in deepening their understanding of mathematics, but also prepare them to understand the concepts of the emerging field of computational thinking. The focus is on collaboration and active engagement through student-centered approach to education.

However, in addition to this textbook, students at this stage should be encouraged to explore various other learning resources. School libraries play a crucial role in making such resources available. Besides, the role of parents and teachers will also be invaluable in guiding and encouraging students to do so.

With this, I express my gratitude to all those who have been involved in the development of this textbook and hope that it will meet the expectations of all stakeholders. At the same time, I also invite suggestions and feedback from all its users for further improvement in the coming years.

Dinesh Prasad Saklani
Director
National Council of Educational
Research and Training

New Delhi
September, 2025

ABOUT THE BOOK

Mathematics helps students develop not only basic arithmetic skills, but also the crucial capacities of logical reasoning, creative problem solving, and clear and precise communication (both oral and written). Mathematical knowledge also plays a crucial role in understanding concepts in other school subjects, such as Science and Social Science, and even Arts, Physical Education, and Vocational Education. Learning Mathematics can also contribute to the development of capacities for making informed choices and decisions. Understanding numbers and quantitative arguments is necessary for effective and meaningful democratic and economic participation. Mathematics thus has an important role to play in achieving the overall aims of school education.

Mathematics at the Middle Stage is a major challenge and has to perform the dual role of being both close to the experience and environment of the child and being abstract. It must perform the dual role of developing intuition while also maintaining and emphasising rigour. It must perform the dual role of enhancing critical and logical thinking while also developing artistry and creativity and a sense of elegance and aesthetics. Finally, Mathematics must perform the dual role of providing students plenty of opportunities for exploration and discovery of concepts on their own while also teaching best-known methods in the global repertoire of mathematics.

The present textbook has made an attempt to address the above mentioned goals and challenges of learning mathematics. The writers of this book have aimed to strike a judicious balance between informal and formal definitions and methods to develop in students both intuition and rigour. The book also provides numerous opportunities for student-student and student–teacher interaction in the classroom to promote active and experiential learning. A number of questions, puzzles, and interactive exercises are posed throughout the book to encourage constant exploration. Many of the questions are open-ended to stimulate in-class discussion.

Chapter 1 of this textbook, 'Geometric Twins', introduces the concept of congruence of plane figures. Chapter 2, 'Integers — Multiplication and Division' is about how to multiply and divide positive and negative integers. Chapter 3, 'Finding Common Ground', deals with the ideas of common factors, common multiples, prime factorisation, HCF and LCM. Chapter 4, 'Decimals — Multiplication and Division' is about learning how to handle decimal numbers in practical and systematic way. It builds on earlier knowledge of decimals and extends it to operations. Chapter 5, 'Connecting the Dots', introduces the concept of central tendency (mean, median and mode). Chapter 6, 'Constructions and Tilings', introduces children to practical geometry and patterns in shapes. Chapter 7, 'Finding the Unknown', is about algebra and simple equations. It introduces children to the idea that we can use letters (like x, y) and then find their value using rules of arithmetic. In all chapters, an attempt has been made to emphasis connections with other subjects including Art, History, and Science.

By weaving storytelling and hands-on activities together, we hope that an immersive learning experience will be created that ignites curiosity and fosters a love for mathematics. It is hoped that teachers would give children the opportunity to discuss, play, engage with each other, provide logical arguments for different ideas, and find loopholes in arguments presented. This is necessary for the learners to eventually develop the ability to understand what it means to prove something and also become confident about underlying concepts. The mathematics classroom should not expect a blind application of algorithms but should rather encourage children to find many different ways to solve problems.

As per the NEP 2020, computational thinking has also been gently introduced through puzzles, games, and interactive exercises that encourage such thinking. Indian rootedness has also been kept in mind while giving contexts for different concepts. The contributions of Indian mathematicians have been given as part of a problem-solving approach to make students aware of India's rich mathematical heritage and its global contributions to mathematics.

The concepts and problems are related to daily life situations. An attempt has been made to use contexts and materials with which the students are familiar. Learning material sheets have been given at the back of the book that may be photocopied and used. At many places, exercises or activities are given to encourage peer group efforts and discussions. This textbook intends to address the learning needs of a diverse group of students in the classroom.

We have tried to link concepts learnt in initial chapters with ideas in subsequent chapters to show the connectedness and unity of mathematics. We hope that teachers will use this as an opportunity to revise these concepts in a spiralling way so that children are able to appreciate the entire conceptual structure of mathematics. We hope that teachers may give more time to the ideas of congruence of triangles, concept of central tendencies and other notions that are new to students. Many of these are the basis for further learning in mathematics.

Finally, this textbook aims to be more than just a textbook—it's a passport to a world of mathematical discovery and exploration. Whether used in the classroom or at home, we hope that it may inspire students to embark on their own mathematical adventures, empowering them to see the beauty and relevance of mathematics in everything around them. With its engaging approach and comprehensive coverage of Grade 7 mathematics concepts, this textbook hopes and aims to captivate young minds and set them on a lifelong journey of mathematical discovery.

I thank again all the writers and contributors of this textbook for their important and valuable contribution and service to the nation's mathematics teachers, learners and enthusiasts.

We look forward to your comments and suggestions regarding the book and hope that you will send interesting exercises, activities and tasks that you develop during the course of teaching and learning, to be included in future editions.

Ashutosh Wazalwar
Professor and *Academic Convener*
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CONSTITUTION OF INDIA

Part III (Articles 12 – 35)
(Subject to certain conditions, some exceptions and reasonable restrictions)
guarantees these

Fundamental Rights

Right to Equality
- before law and equal protection of laws;
- irrespective of religion, race, caste, sex or place of birth;
- of opportunity in public employment;
- by abolition of untouchability and titles.

Right to Freedom
- of expression, assembly, association, movement, residence and profession;
- of certain protections in respect of conviction for offences;
- of protection of life and personal liberty;
- of free and compulsory education for children between the age of six and fourteen years;
- of protection against arrest and detention in certain cases.

Right against Exploitation
- for prohibition of traffic in human beings and forced labour;
- for prohibition of employment of children in hazardous jobs.

Right to Freedom of Religion
- freedom of conscience and free profession, practice and propagation of religion;
- freedom to manage religious affairs;
- freedom as to payment of taxes for promotion of any particular religion;
- freedom as to attendance at religious instruction or religious worship in certain educational institutions.

Cultural and Educational Rights
- for protection of interests of minorities;
- for minorities to establish and administer educational institutions;
- saving of certain Laws 31A–31D.

Right to Constitutional Remedies
- by issuance of directions or orders or writs by the Supreme Court and High Courts for enforcement of these Fundamental Rights.

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Constitution of India

Part IV A (Article 51 A)

Fundamental Duties

It shall be the duty of every citizen of India —

(a) to abide by the Constitution and respect its ideals and institutions, the National Flag and the National Anthem;

(b) to cherish and follow the noble ideals which inspired our national struggle for freedom;

(c) to uphold and protect the sovereignty, unity and integrity of India;

(d) to defend the country and render national service when called upon to do so;

(e) to promote harmony and the spirit of common brotherhood amongst all the people of India transcending religious, linguistic and regional or sectional diversities; to renounce practices derogatory to the dignity of women;

(f) to value and preserve the rich heritage of our composite culture;

(g) to protect and improve the natural environment including forests, lakes, rivers, and wildlife, and to have compassion for living creatures;

(h) to develop the scientific temper, humanism and the spirit of inquiry and reform;

(i) to safeguard public property and to abjure violence;

(j) to strive towards excellence in all spheres of individual and collective activity so that the nation constantly rises to higher levels of endeavour and achievement;

*(k) who is a parent or guardian, to provide opportunities for education to his child or, as the case may be, ward between the age of six and fourteen years.

Note: The Article 51A containing Fundamental Duties was inserted by the Constitution (42nd Amendment) Act, 1976 S.11 (with effect from 3 January 1977).

*(k) was inserted by the Constitution (86th Amendment) Act, 2002 S.4 (with effect from 1 April 2010).

NOTE TO THE TEACHER

We hope that this textbook, *Ganita Prakash*, will serve as a strong support and guide to you in achieving the exciting task that you have before you — that of passing on the joy of learning the beautiful subject of Mathematics to the next generation.

This task calls for providing a fertile environment that allows for the flowering of mathematical thinking in the minds of students. Classrooms, where students just listen and write down whatever is being told to them or written on the board, are deficient in the conditions required for learning mathematics. Instead, classrooms need to be places where students are engaged in playing with mathematical concepts, finding and discussing patterns, and developing creative strategies together to solve problems. Students should also be posing problems to each other and discussing possible solutions. In fact, these are the very conditions that have led to the development of the entire field of mathematics so far, and so one cannot expect students to pick up mathematical thinking and understanding without these conditions.

Fortunately, it is not difficult to create such conditions in the classroom. It just requires an interesting question, problem, pattern, or challenge to be thrown open to the students on a regular basis, and sufficient time to be given to them to play with, discuss, and work on it as a class or in pairs or groups.

Along with it, an environment that accepts mistakes and acknowledges their importance in learning needs to be nurtured.

While creating the spark for initiating mathematical thinking in classrooms is not difficult, sustaining it may be challenging and may involve efforts from your side. Nevertheless, even if just the first part of throwing open a question, problem, pattern, or challenge is done at least once or twice a week, accompanied by sufficient waiting time from your side for students to play, discuss, and work on it, it can have a great positive impact on how the students view and approach mathematics.

It should be noted that this positive impact will not happen overnight. That takes time and depends on various factors such as the number of opportunities you give for problem solving, your patience, and the encouragement you give to the students.

To support you in posing problems, all the problems or questions in this book are marked using the icon ?. This icon is an indicator of a potential opportunity to start off a process of problem solving and exploration in the classroom. You will find some of the problems labelled 'Math Talk'. Such questions can especially be made as topics for classroom discussion.

An owl mascot appears at various points in the textbook to highlight important mathematical processes, ways of thinking, and problem-solving approaches. These can be brought out during classroom discussions, both where the owl is present and also in other similar situations.

To develop students' mathematical thinking and understanding of concepts, a sufficient number of problems are given. Trying to 'cover' all of them must not happen at the cost of students not getting to spend quality time on playing with and discussing them.

It is important to understand that the exploratory problems are not only for promoting problem solving skills, they also serve in strengthening procedural fluency when children start engaging in exploration.

Efforts must be made in making students independent learners. One essential aspect required for this is an ability to read and understand mathematical text. To promote this skill, students should be encouraged to read the book by themselves and in groups. Give opportunities to them to interpret what they read and express it to others. This will also address the big problem that students face in speaking mathematics and interpreting word problems.

This textbook contains a number of open-ended problems. It also contains new treatments of certain concepts. If you are not able to solve them or follow some of them immediately, it is perfectly okay! Not everyone knows everything. Along with trying to understand and reflect upon such content, it will be very useful to take it to the classroom and open it up for discussion. After the discussion, things that are clear and those that are not yet clear can be clearly summarised. This process itself can throw a lot of light on the content.

In these discussions, you can participate as a fellow seeker, and when students see a teacher seek and think to understand something, it sets a wonderful example for them.

It is hoped that you and your students will have a great and fruitful time using this textbook!

Summary of Key Points

Time for Exploration

1. It is important to routinely pose new problems, questions, patterns, or challenges to the students and give them sufficient time to play with, discuss, and work on them, individually and in groups.

2. During this time, an environment that accepts mistakes and acknowledges their importance in learning needs to be nurtured.
3. There should be a culture where students pose problems to each other and discuss with each other various ways to approach the problems.

About the Problems in the Book

1. The exploratory problems in the textbook not only promote problem solving; they also aim to strengthen procedural fluency when children start engaging in exploration.
2. Trying to 'cover' all the problems in the book must not happen at the cost of students not getting to spend quality time on playing with, discussing, and solving them.

Reading

1. Encourage students to read the book by themselves and in groups.
2. Give opportunities to them to interpret what they read and to express it to others.

Right of Not Knowing!

1. It is perfectly okay if some of the content is not understood immediately. Along with trying to understand and reflect upon such content, it can also be taken to the classroom and opened up for discussion. After the discussion, things that are clear and those that are not yet clear can be clearly summarised. In these discussions, you can participate as a fellow seeker, and when students see a teacher seek and think to understand something, it sets a wonderful example for them!
2. Learning is a continual process. Indeed, there is so much in mathematics that is still not known and requires further exploration!

A NOTE TO STUDENTS!

To be able to appreciate the art of mathematics, it is not enough to just be a passive spectator. You need to immerse yourself in its process like a detective getting into action to solve a mystery.

This is especially required when you see a new question or when a question arises from your own sense of wonder, or when you come across a new beautiful pattern. When you encounter these, pause your reading, and use your creativity to work out the question or understand and appreciate the pattern.

You will find that some questions are accompanied by their answers. Even if this is the case, it is worthwhile to work on the problems by yourself or in a group before you see the answer.

This will enrich your experience of going through the book!

Whenever there are questions coming up, you will see this icon: ?. This indicates that it is time for figuring things out!

Sometimes you will find many questions collected together in a single place under the title '**Figure it Out**'.

Some questions are marked Math Talk. These questions are meant to be discussed and worked out with your friends.

Finally, there are questions marked Try This. These questions demand more creativity to be answered, and therefore will also often be more fun to answer as a result!

CONTENTS

THE CONSTITUTION OF INDIA

PREAMBLE

WE, THE PEOPLE OF INDIA, having solemnly resolved to constitute India into a [1]**[SOVEREIGN SOCIALIST SECULAR DEMOCRATIC REPUBLIC]** and to secure to all its citizens :

JUSTICE, social, economic and political;

LIBERTY of thought, expression, belief, faith and worship;

EQUALITY of status and of opportunity; and to promote among them all

FRATERNITY assuring the dignity of the individual and the [2][unity and integrity of the Nation];

IN OUR CONSTITUENT ASSEMBLY this twenty-sixth day of November, 1949 do **HEREBY ADOPT, ENACT AND GIVE TO OURSELVES THIS CONSTITUTION.**

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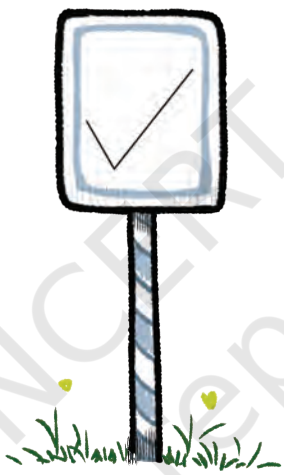
1. Subs. by the Constitution (Forty-second Amendment) Act, 1976, Sec.2, for "Sovereign Democratic Republic" (w.e.f. 3.1.1977)
2. Subs. by the Constitution (Forty-second Amendment) Act, 1976, Sec.2, for "Unity of the Nation" (w.e.f. 3.1.1977)

1 GEOMETRIC TWINS

0789CH01

1.1 Geometric Twins

The symbol on this signboard needs to be recreated on another board.

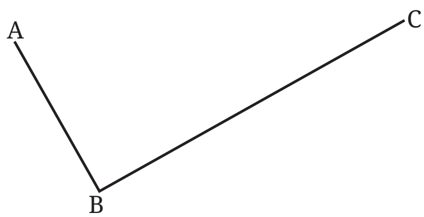


How do we do it?

One way is to trace the outline of this symbol on tracing paper to reconstruct the figure. But this is difficult for big symbols. What else can we do?

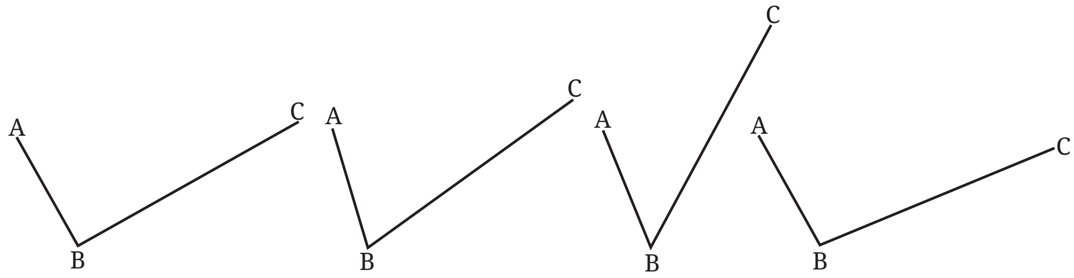
Can we take some measurements that would allow us to exactly recreate this figure? If yes, what measurements should we take?

Let us name the corner points of this symbol as shown.



Are the arm lengths AB and BC sufficient to exactly recreate this figure?

Suppose these lengths are AB = 4 cm, BC = 8 cm. We observe that several such symbols can be constructed with the same lengths.

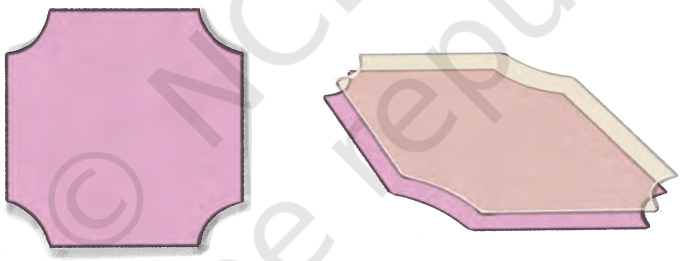


? To get the exact replica, would it help to take any other measurement?

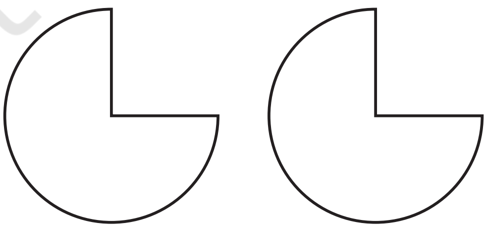
The measure of ∠ABC, along with the two arm lengths AB and BC, fix the shape and size of this figure.

? Can you draw the symbol if it is known that AB = 4 cm, BC = 8 cm, and ∠ABC = 80°?

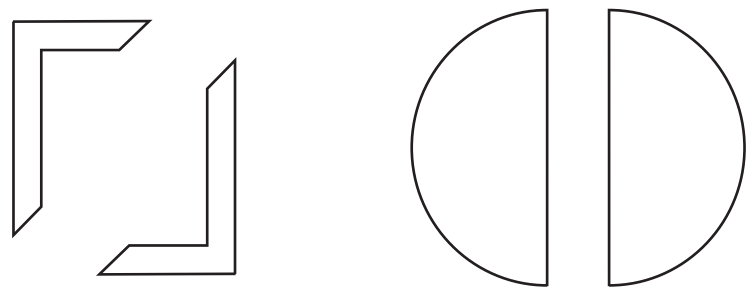
These three measurements can help us create an exact replica of the symbol on the signboard. Figures that are exact copies of each other or have the same shape and size are said to be **congruent**. Congruent figures can be superimposed exactly, one over the other.



Two congruent figures are shown below. You could use a tracing paper to trace the first figure and superimpose it on the second one. You will find that they fit exactly, one over the other.



Note that while checking for congruence, a figure can be rotated or flipped before superimposing it on the other figure. So, the following pairs of figures are also congruent to each other.



Let us get back to the symbol we saw on the signboard. Suppose there are two such symbols that look identical and we need to confirm that they are indeed congruent. Can we use their measurements to verify this?

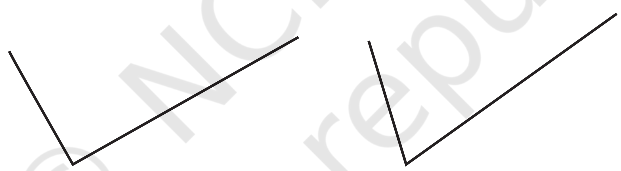
? If it is known that both symbols have the same arm lengths, can it be concluded that the two symbols are congruent?

We have seen that there can exist several such non-congruent figures with different angles between the given arm lengths. Fixing the angle determines the shape and size of the figure.

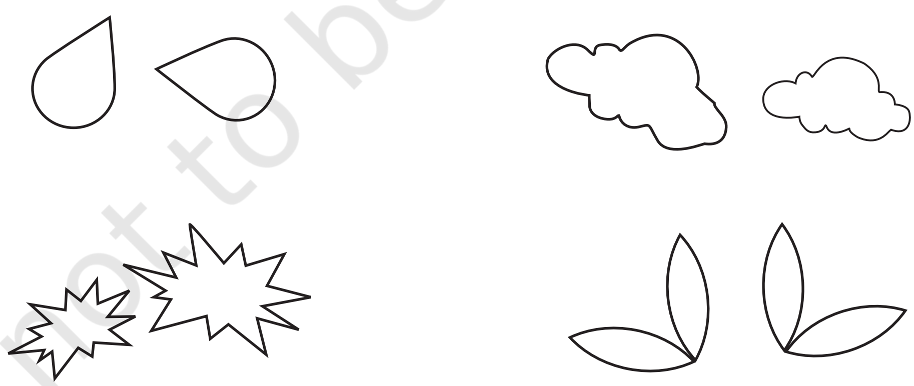
Thus, if both symbols have the same arm lengths and angle, we can be sure that the figures are congruent.

? Figure it Out

1. Check if the two figures are congruent.



2. Circle the pairs that appear congruent.



3. What measurements would you take to create a figure congruent to a given:

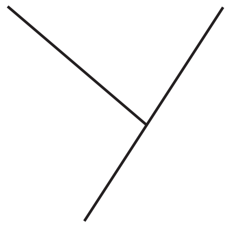
(a) Circle (b) Rectangle

Using this, state how would you check if two —

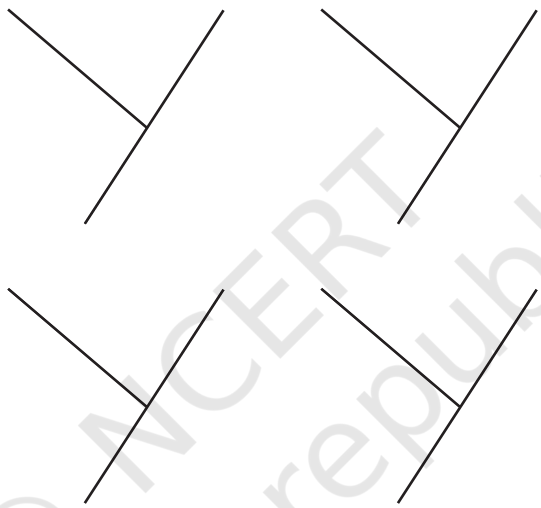
(a) Circles are congruent?

(b) Rectangles are congruent?

4. How would we check if two figures like the one below are congruent?



Use this to identify whether each of the following pairs are congruent.



1.2 Congruence of Triangles

Meera and Rabia have been asked to make a cardboard cutout identical to a triangular frame they have in school. They see that the frame is too big to be traced on a paper and replicated.

What do you think they can do?

Measuring the Sidelengths

Can certain measurements of the triangle be used for this? Using a measuring tape, the girls measure the sides of the triangle to be 40 cm, 60 cm, and 80 cm.

Then, Rabia takes out her protractor to measure the angles. She is stopped by Meera.

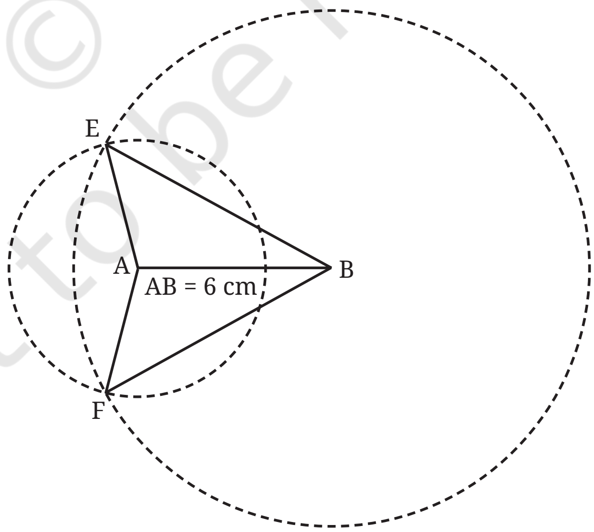
Meera: The angles of the triangle are not required! With the side lengths we have measured, we can create a triangle congruent to this one.

Do you agree with Meera?

Instead of the lengths being 40 cm, 60 cm, and 80 cm, suppose the sidelengths had been 4 cm, 6 cm, 8 cm (this triangle can fit on our page).

Is this information sufficient to replicate the triangle with the same size and shape? If yes, can you do so?

Rabia: If I were to construct this triangle, I would first draw a line segment having one of the given lengths, say 6 cm, and then draw circles from each of its end points with radii 4 cm and 8 cm. But the circles would intersect at two points, forming two triangles:
ΔABE and ΔABF



Rabia: Do these two triangles have the same shape and size? If not, then we will not be sure which of these would actually be congruent to the original triangle we are trying to replicate.

? Examine whether ΔABE and ΔABF are congruent.

For this, you could use one or more of the following methods — tracing and comparing, taking a cutout and superimposing, or observing that AB acts as a line of symmetry due to the 'sameness' of the act of construction above and below this line.

We see that ΔABE and ΔABF are congruent. From this general construction, we can see that all triangles with the same sidelengths are congruent. Hence, Meera was right when she said that the sidelengths are sufficient to construct a congruent triangle.

Thus, we have the following result:

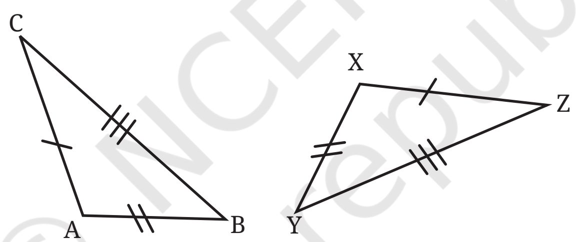
If two triangles have the same sidelengths, then they are congruent.

We call this the **SSS (Side Side Side)** condition for congruence.

Conventions to Express Congruence

The two triangles given below are congruent. How can these two triangles be superimposed? Which vertices of ΔXYZ and ΔABC should we overlap?

? This has to be done so that the equal sides overlap. Figure out how.



Overlapping Vertex A over Vertex X, Vertex B over Vertex Y and Vertex C over Vertex Z will ensure that equal sides overlap, making the triangles fit exactly over each other.

? Are there other ways of overlapping the vertices so that the triangles fit exactly over each other?

The fact that these triangles are congruent shows that their respective angles are equal:

$$\angle A = \angle X, \angle B = \angle Y \text{ and } \angle C = \angle Z$$

Thus, when two triangles are congruent, there are corresponding vertices, sides and angles which fit exactly over each other when the triangles are made to overlap. In this case, they are

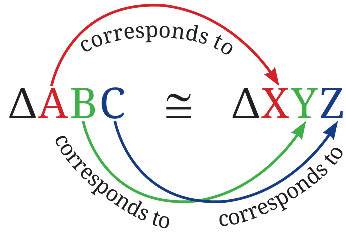
(a) Corresponding Vertices: A and X, B and Y, C and Z

(b) Corresponding Sides: AB and XY, BC and YZ, AC and XZ

(c) Corresponding Angles: $\angle A$ and $\angle X$, $\angle B$ and $\angle Y$, $\angle C$ and $\angle Z$

To capture this relation that exists when two triangles are congruent, their congruence is written as follows:

$$\Delta ABC \cong \Delta XYZ$$



By writing this, we mean that:

- the first vertex in the name of ΔABC corresponds to the first vertex in the name of ΔXYZ,
- the second vertex in the name of ΔABC corresponds to the second vertex in the name of ΔXYZ, and
- similarly with the third vertices in the names of ΔABC and ΔXYZ.

By this convention, it is **incorrect** to write for these two triangles that

$$\Delta ACB \cong \Delta XYZ.$$

However, another correct way of saying it is

$$\Delta ACB \cong \Delta XZY.$$

? Can you identify a pair of congruent triangles below? Why are they congruent?

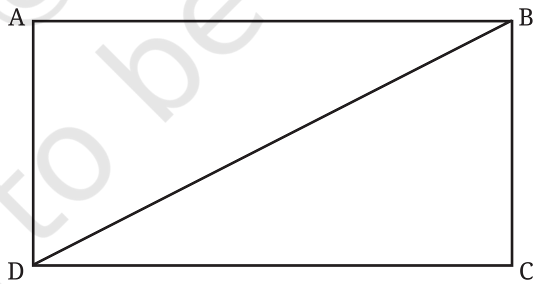


Fig.1.1

Consider ΔABD and ΔCDB. Since ABCD is a rectangle, we have

$$AB = CD$$
$$AD = CB$$

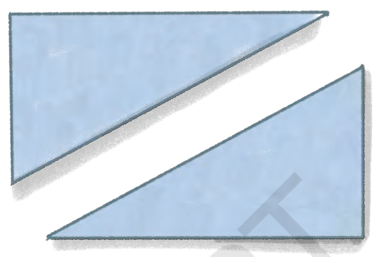
If the remaining sides of ΔABD and ΔCDB have the same length then the SSS condition is satisfied, confirming the congruence of the two triangles. Is this the case?

The remaining side is a common side BD, so the SSS condition holds. Hence, the triangles are congruent.

We know the corresponding sides of the two triangles. We have to identify the corresponding vertices. Can they be the following?

ΔABD	ΔCDB
A	C
B	B
D	D

Verify this by superimposing paper cutouts of the triangles obtained from the rectangle ABCD (Fig. 1.1).

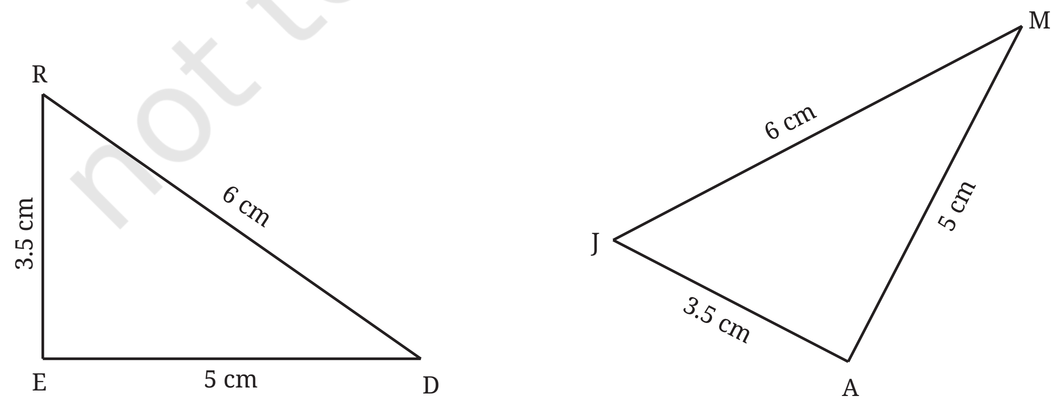


We see that this correspondence lays the side AB of ΔABD over the side CB of ΔCDB. But these sides need not be equal, and hence, this superimposition will not establish congruence.

Identify the correct correspondence of vertices and express the congruence between the two triangles.

Figure it Out

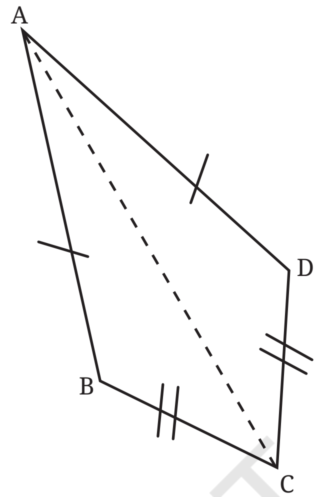
1. Suppose ΔHEN is congruent to ΔBIG. List all the other correct ways of expressing this congruence.
2. Determine whether the triangles are congruent. If yes, express the congruence.



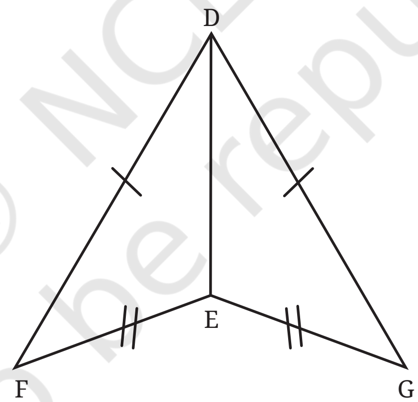
3. In the figure below, AB = AD, CB = CD.

   Can you identify any pair of congruent triangles? If yes, explain why they are congruent.

   Does AC divide ∠BAD and ∠BCD into two equal parts? Give reasons.



4. In the figure below, are ΔDFE and ΔGED congruent to each other? It is given that DF = DG and FE = GE.

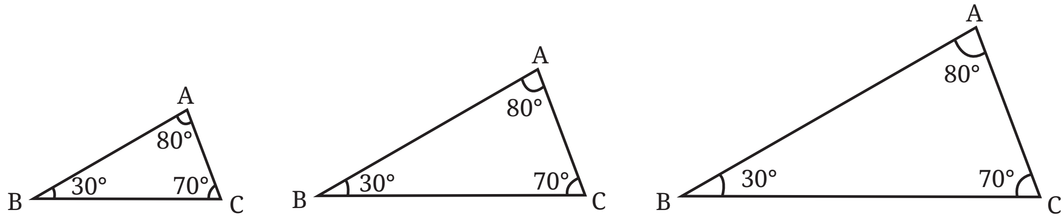


Measuring the Angles

Instead of measuring the three sidelengths of the triangular frame, if Meera and Rabia measure the three angles, can they recreate the triangle exactly?

? Suppose the angles are 30°, 70°, and 80°. Can we create an exact copy of the frame with this?

As we see, we can draw many triangles with these measurements that are not congruent.



These triangles are seen to have the same shape, but not the same size. Hence, two triangles that have the same set of angles need not be congruent.

Measuring Two Sides and the Included Angle

? ΔABC and ΔXYZ are two triangles such that

$$AB = XY = 6 \text{ cm},\ AC = XZ = 5 \text{ cm, and } \angle A = \angle X = 30°$$

Are they congruent?

To check this, we need to see if there can exist non-congruent triangles with the given measurements.

These measurements correspond to the case of two sides and the included angle. We have seen how to construct a triangle given these measurements.

? Construct a triangle having the above measurements.

Compare it with the triangles constructed by your classmates. Are the triangles all congruent? Explain why all such triangles with these measurements are congruent.

Thus, when two sides and the included angle of two triangles are equal, the two triangles are congruent.

This is referred to as the **SAS (Side Angle Side)** condition for congruence.

Measuring Two Sides and a Non-included Angle

What if two sides and a non-included angle are equal?

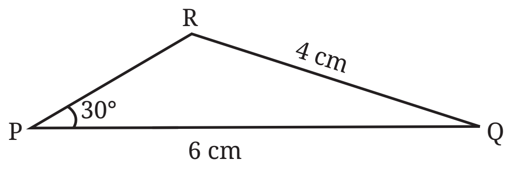
? ΔABC and ΔXYZ are two triangles such that

$$AB = XY = 6 \text{ cm},\ AC = XZ = 4 \text{ cm, and } \angle B = \angle Y = 30°$$

Are they congruent?

? Can there exist non-congruent triangles having these measurements? Construct and find out.

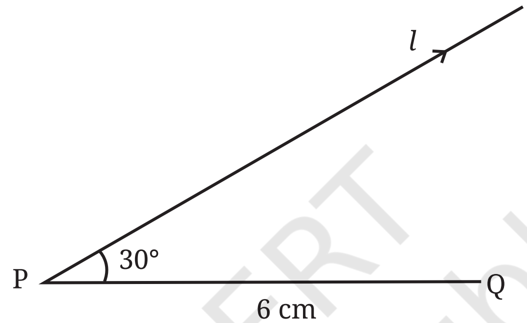
Looking at a rough diagram helps in planning the construction.



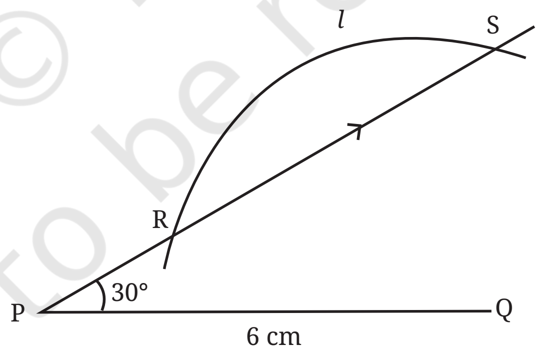
How does one construct a triangle having these measurements?

Step 1: Draw the base PQ of length 6 cm.

Step 2: Draw a line l from P that makes an angle of 30° with PQ.



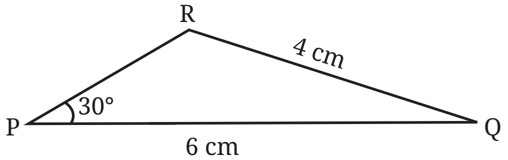
Step 3: Draw a sufficiently long arc from Q of radius 4 cm cutting the line l.

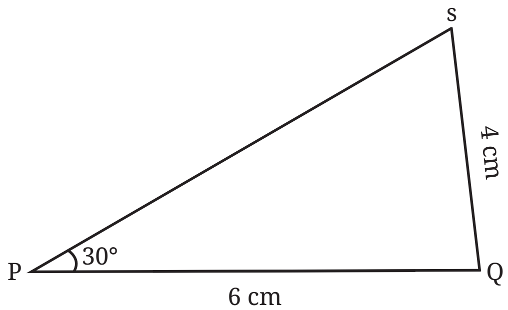


How do we find the required triangle from this figure?

A point of intersection of the arc and the line l gives the third point of the required triangle. But we see that the arc intersects the line l at two different points R and S.

Both ΔPQR and ΔPQS satisfy the given measurements.





Hence, we can draw two non-congruent triangles with the given measurements.

This is called the **SSA (Side Side Angle)** condition. We have seen that SSA condition **does not guarantee** congruence.

We have examined cases using two sides and an angle for determining congruence. Can we use two angles and a side?

Let us first take the case of two angles and the included side.

Two Angles and the Included Side

? ΔABC and ΔXYZ are two triangles with,

$$BC = YZ = 5 \text{ cm}, \angle B = \angle Y = 50° \text{ and } \angle C = \angle Z = 30°.$$

Are they congruent?

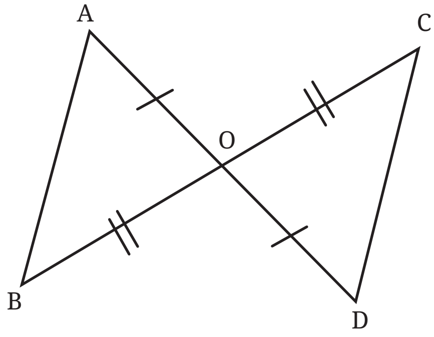
? Can there exist non-congruent triangles having these measurements? Construct and find out.

We have seen how to construct a triangle when we are given two angles and the included side.

This construction should make it clear that all the triangles having these measurements must be congruent to each other. Hence, $\Delta ABC \cong \Delta XYZ$.

This condition is referred to as the **ASA (Angle Side Angle)** condition for congruence.

? In the figure, Point O is the midpoint of AD and BC. What can one say about the lengths AB and CD?



We have,

AO = OD (as O is the midpoint of AD)

BO = OC (as O is the midpoint of BC).

? Are there any other equal sides or angles?

We also have,

$\angle AOB = \angle DOC$, as they are vertically opposite angles.

We see that the SAS condition (two sides and the included angle) is satisfied, and so we can conclude that the triangles are congruent.

What are the Corresponding Vertices?

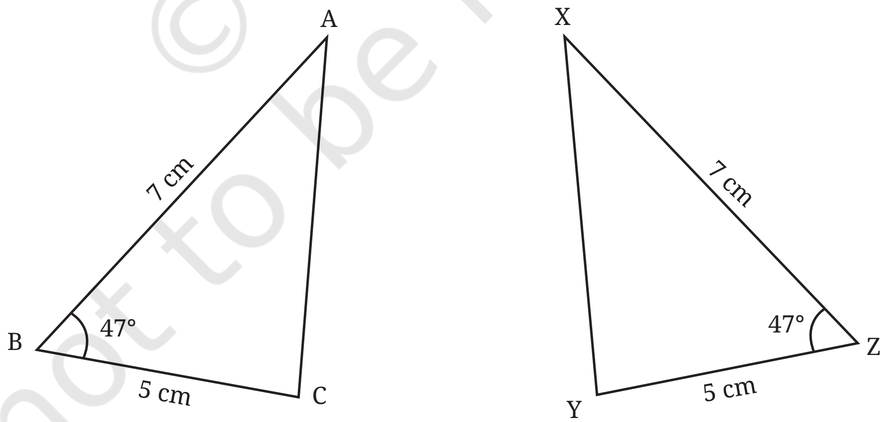
As we need AO and OD to overlap, and BO and CO to overlap for the triangles to exactly fit over each other, the corresponding vertices in ΔAOB and ΔDOC are A and D, O and O (vertex common to both the triangles), and B and C. Thus,

$$\Delta AOB \cong \Delta DOC.$$

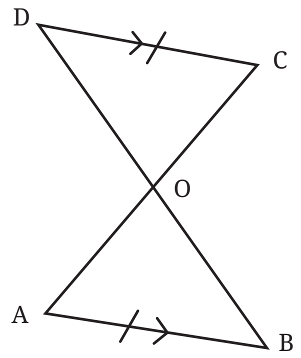
AB and DC are corresponding sides as they overlap when the triangles are superimposed. Thus, their lengths are equal.

? Figure it Out

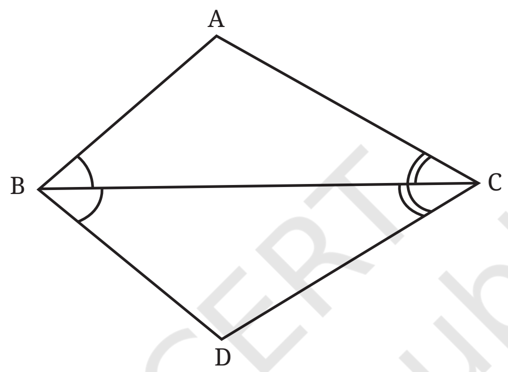
1. Identify whether the triangles below are congruent. What conditions did you use to establish their congruence? Express the congruence.



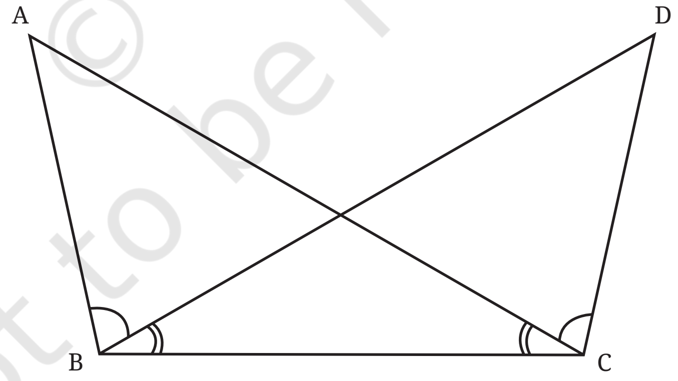
2. Given that CD and AB are parallel, and AB = CD, what are the other equal parts in this figure? (**Hint:** When the lines are parallel, the alternate angles are equal. Are the two resulting triangles congruent? If so, express the congruence.)



3. Given that ∠ABC = ∠DBC and ∠ACB = ∠DCB, show that ∠BAC = ∠BDC. Are the two triangles congruent?



4. Identify the equal parts in the following figure, given that ∠ABD = ∠DCA and ∠ACB = ∠DBC.



Measuring Two Angles and a Non-Included Side

? The following triangles ΔABC and ΔXYZ are such that ∠A = ∠X = 35°, ∠C = ∠Z = 75°, and BC = YZ = 4 cm. Are the triangles congruent? Give a reason.

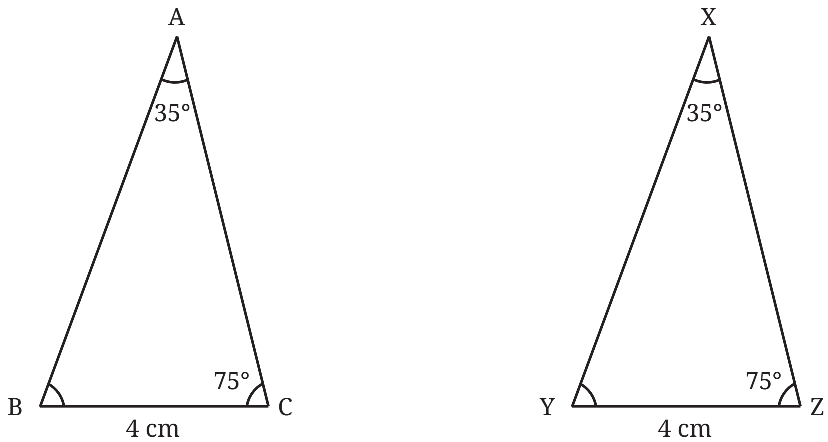


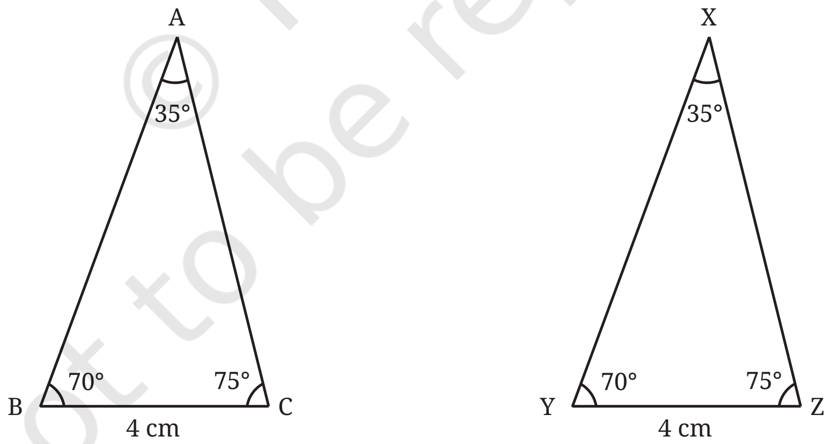
Fig.1.2

How do we proceed with this problem? Here is a method.

? What are the measures of $\angle B$ and $\angle Y$?

We know that the sum of the angles of a triangle is 180°.
So $\angle B + 35° + 75° = 180°$,
or $\angle B + 110° = 180°$
Thus, $\angle B = 70°$.
Similarly, $\angle Y$ is also 70°.
Thus, we have $\angle B = \angle Y$.

? Does this help in showing that ΔABC and ΔXYZ are congruent?



These two triangles now satisfy the ASA condition with

$$\angle B = \angle Y$$
$$BC = YZ$$
$$\angle C = \angle Z$$

So, $\Delta ABC \cong \Delta XYZ$.

In Fig. 1.2, the equalities are between two angles and the non-included side of the two triangles. This condition is referred to as the **AAS (Angle Angle Side)** condition.

As we have seen, the AAS condition guarantees congruence.

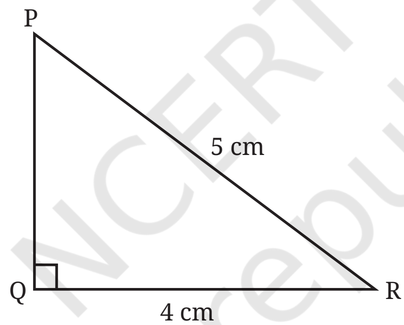
We have seen that the SSA condition doesn't always guarantee congruence. However, there are some special cases when SSA does guarantee congruence. Here is one such important case.

Measuring Two Sides in a Right Triangle

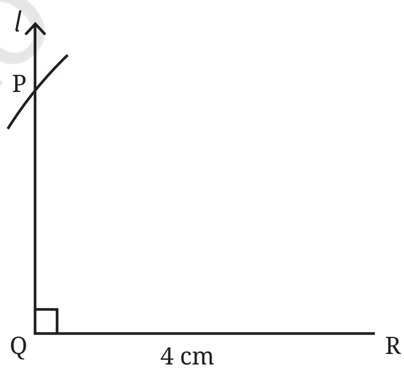
? ΔABC and ΔXYZ are right-angled triangles such that BC = YZ = 4 cm, ∠B = ∠Y = 90° and AC = XZ=5cm. Are they congruent?

? Can there exist non-congruent triangles having these measurements? Construct and find out.

Looking at the rough diagram helps in planning the construction.



Step 1: Draw the base QR of length 4 cm.
Step 2: Draw a line *l* perpendicular to QR from Q.
Step 3: From R, cut an arc on line *l* of radius 5 cm.
Step 4: Let P be the point at which the arc intersects the line *l*. Join PR.



ΔPQR is the required triangle.

Consider the downward extension of line l below QR. Would the arc from R meet this line downwards as well (as in the case of triangle construction when the sidelengths are given)? If so, would this lead to a triangle whose size and shape are different from ΔPQR, and yet has the given measurements?

It can be seen that the other triangle we get below is also congruent to ΔPQR. Why? Therefore, all triangles having these measurements will be congruent to each other.

Thus, we conclude that $\Delta ABC \cong \Delta XYZ$.

In the case that we have considered, the parts that are equal to their corresponding parts in another triangle are

(a) the right angle

(b) two other sides, one of which is opposite to the right angle. This side is called the hypotenuse.

This is called the **RHS (Right Hypotenuse Side)** condition, and is one more condition for congruence.

Conditions that are sufficient to guarantee congruence

From the discussions so far, we can see that two triangles are congruent if any of the following conditions are satisfied:

(a) SSS condition (b) SAS condition

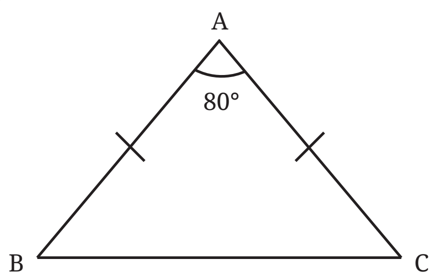
(c) ASA condition (d) AAS condition

(e) RHS condition

1.3 Angles of Isosceles and Equilateral Triangles

Congruence is a very powerful tool for studying properties of geometric figures. Let us use it to discover an important property of isosceles triangles.

ΔABC is isosceles with AB = AC, and $\angle A$ = 80. What can we say about $\angle B$ and $\angle C$?



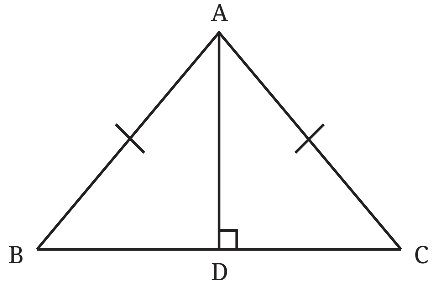
Construct the altitude from A to BC.

We have,

AB = AC (given)

∠ADB = ∠ADC = 90° (from construction)

AD is a common side of the two triangles ΔADB and ΔADC.



Thus, the triangles satisfy the RHS condition. Hence, ΔADB ≅ ΔADC.

This shows that ∠B = ∠C, as they are corresponding parts of congruent triangles.

Thus, the angles opposite to equal sides are equal.

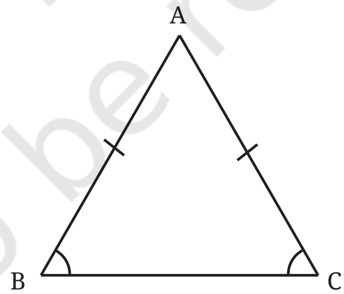
? Can you use this fact to find ∠B and ∠C?

Angles in an Equilateral Triangle

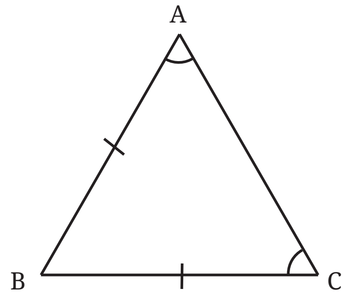
Equilateral triangles are those in which all the sides have equal lengths.

? What can we say about their angles?

We can use the recently discovered fact that angles opposite to equal sides are equal.



The sides AB and AC are equal. So ∠B = ∠C.



Similarly, the sides AB and BC are equal. So $\angle A = \angle C$.

So, all the three angles of an equilateral triangle are equal, just like their sides.

? What could be their measures?

As the three angles should add up to 180°, we have
3 × angle in an equilateral triangle = 180°.
So each angle is 60°.

? Verify this by construction.

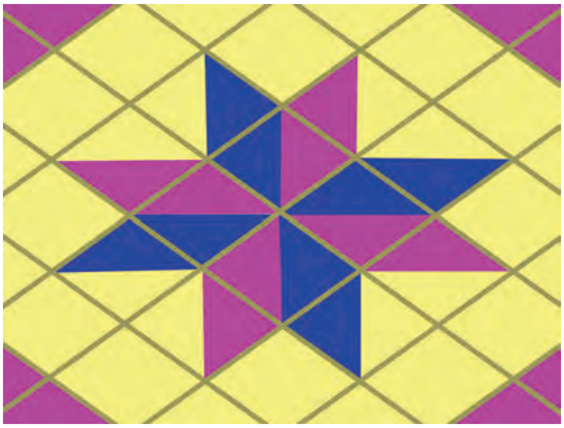
Thus, just using the notion of congruence, we have deduced that the angles of an equilateral triangle are all 60°.

Congruent Triangles in Real Life: Congruent triangles can be seen in various constructions and designs from ancient to modern times. Here are a few examples.

World-famous Louvre Museum in Paris

World-famous Egyptian Pyramid of Giza

Dome design



Rangoli design

Rabindra Setu or Howrah Bridge

Describe the congruent triangles you see in each picture.

Figure it Out

1. $\Delta AIR \cong \Delta FLY$. Identify the corresponding vertices, sides and angles.
2. Each of the following cases contains certain measurements taken from two triangles. Identify the pairs in which the triangles are congruent to each other, with reason. Express the congruence whenever they are congruent.

   (a) $AB = DE$, $BC = EF$, $CA = DF$

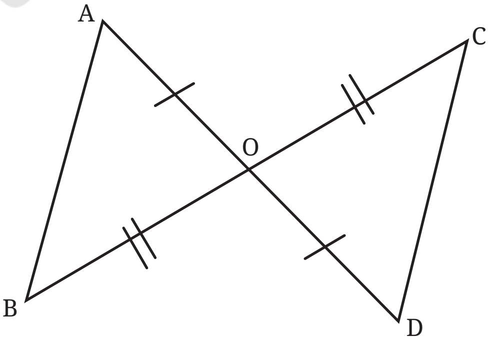
   (b) $AB = EF$, $\angle A = \angle E$, $AC = ED$

   (c) $AB = DF$, $\angle B = \angle D = 90°$, $AC = FE$

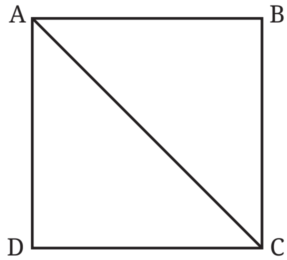
   (d) $\angle A = \angle D$, $\angle B = \angle E$, $AC = DF$

   (e) $AB = DF$, $\angle B = \angle F$, $AC = DE$

3. It is given that $OB = OC$, and $OA = OD$. Show that AB is parallel to CD. [**Hint:** AD is a transversal for these two lines. Are there any equal alternate angles?]

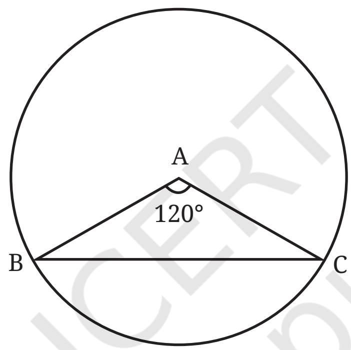


4. ABCD is a square. Show that ΔABC ≅ ΔADC. Is ΔABC also congruent to ΔCDA?

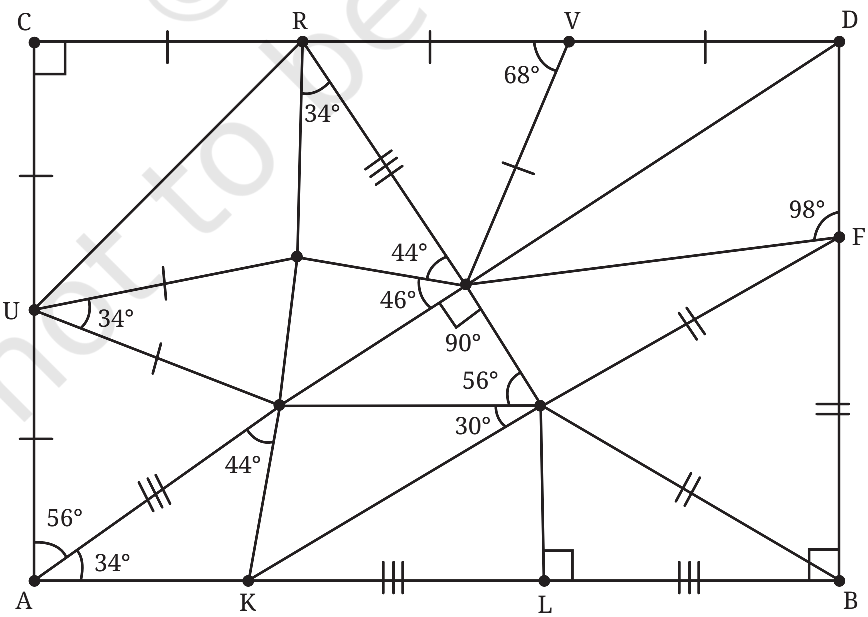


Give more examples of two triangles where one triangle is congruent to the other in two different ways, as in the case above. Can you give an example of two triangles where one is congruent to the other in six different ways?

5. Find ∠B and ∠C, if A is the centre of the circle.



6. Find the missing angles. As per the convention that we have been following, all line segments marked with a single '|' are equal to each other and those marked with a double '|' are equal to each other, etc.

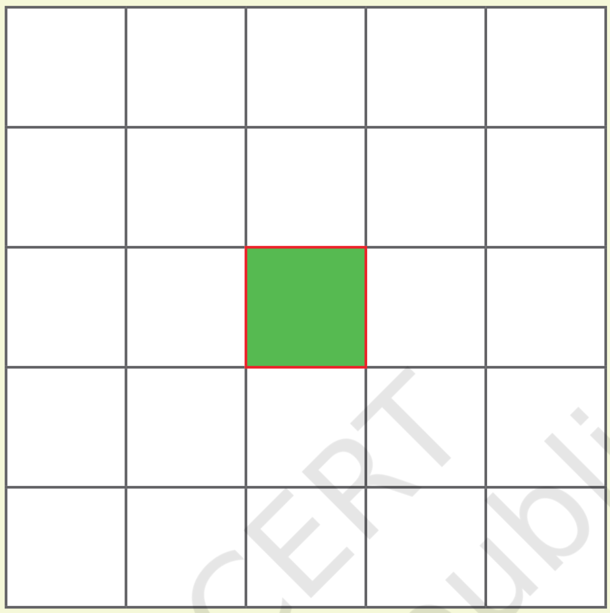


SUMMARY

- Figures that have the same shape and size are said to be **congruent**. These figures can be superimposed so that one fits exactly over the other.
- While verifying congruence, a figure can be rotated or flipped to make it fit exactly over the other figure via superimposition.
- When two triangles have the same sidelengths, we say that the **SSS (Side Side Side)** condition is satisfied. The SSS condition guarantees congruence.
- When two sides and the included angle of one triangle are equal to the two sides and the included angle of another triangle, we say that the **SAS (Side Angle Side)** condition is satisfied. The SAS condition also guarantees congruence.
- When two angles and the included side of one triangle are equal to the two angles and the included angle of another triangle, we say that the **ASA (Angle Side Angle)** condition is satisfied. The ASA condition guarantees congruence. Congruence holds even if the side is not included between the angles **AAS (Angle Angle Side)** condition.
- In a right-angled triangle, the side opposite to the right angle is called the **hypotenuse**.
- When a side and a hypotenuse of a right-angled triangle are equal to a side and the hypotenuse of another right-angled triangle, we say that the **RHS (Right Hypotenuse Side)** condition is satisfied. The RHS condition also guarantees congruence.
- Two triangles need not be congruent if two sides and a non-included angle are equal.
- In a triangle, angles opposite to equal sides are equal.
- The angles in an equilateral triangle are all 60°.

Expression Engineer!

Draw lines and split the region consisting of white squares into 6 smaller congruent regions.



2 OPERATIONS WITH INTEGERS

0789CH02

2.1 A Quick Recap of Integers

Rakesh's Puzzle: A Number Game

? Rakesh gives you a challenge.

"I have thought of two numbers", he says.

"Their **sum is 25**, and their **difference is 11**."

? Can you tell me the two numbers?

You don't need to use any formulas. Just try different pairs of numbers and then check:

1. Do the two numbers add up to 25?
2. Is the difference between them 11?

(Remember: the difference means first number – second number.)
Write your guesses like this:

First Number	Second Number	Sum	Difference
10	15	25	–5
20	5	25	15
19	6	25	13
18	7	**25**	**11**

? Did you find the right pair?

? Now that you've found the correct pair, Rakesh gives you a second challenge:

"Think of two numbers whose **sum is 25**, but their **difference is –11**."

Use the same method. Try different pairs of numbers and fill in the table again. You will notice that if you swap the numbers from the first puzzle, you get the answer to Rakesh's second puzzle. That is, the first number is 7 and the second is 18!

Figure it Out

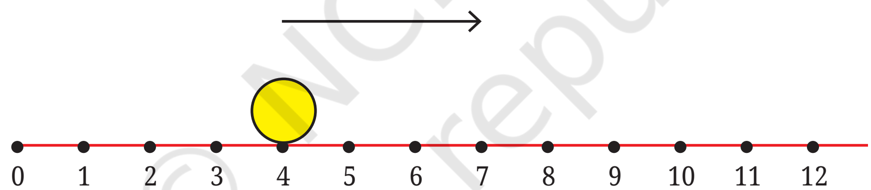
Let us try to find a few more pairs of numbers from their sums and differences:

(a) Sum = 27, Difference = 9
(b) Sum = 4, Difference = 12
(c) Sum = 0, Difference = 10
(d) Sum = 0, Difference = –10
(e) Sum = –7, Difference = –1
(f) Sum = –7, Difference = –13

Choose a partner and take turns to play this game. In each turn, one of you can think of two integers, and give their sum and difference; the other person must then figure out the integers. After some practice, you can perform this magic trick for your family members and surprise them!

Carrom Coin Integers

A carrom coin is struck to move it to the right. Each strike moves the coin a certain number of units of distance rightwards based on the force of the strike.

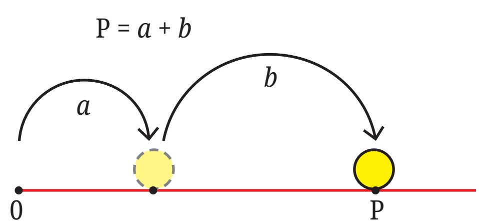


To begin with, the coin is at point 0. If the coin is struck twice, with the first strike moving it by 4 units and the second strike moving it by 3 units, what will be the final position of the coin?

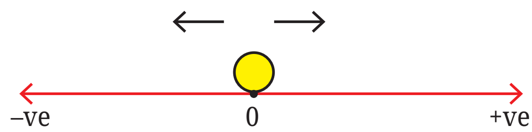
It is clear that the coin will be 4 + 3 = 7 units from 0.

If the coin is struck twice, and if the two movements are known, can you give a formula for the final position of the coin?

If the first strike moves the coin 'a' units to the right and if the second strike moves the coin 'b' units to the right, then the final position is P = a + b, where P is the distance of the coin from the starting point 0.



Now, suppose the coin can be struck to move it in either direction — left or right.



The coin is at 0. If it is struck twice (the direction of the two strikes may be the same or different) can you give a formula for the final position of the coin?

One way to do this is to consider different cases of the directions of the strikes

- both are rightward,
- both are leftward,
- the first one is rightward and the second one is leftward, and
- the first one is leftward and the second one is rightward.

An efficient way to model this situation is to use positive and negative integers. First, let us model the line on which the coin moves as a number line.

Let us consider the rightward movement positive and the leftward movement negative.

Suppose the first strike moves the coin rightward by 5 units from 0, and the second strike leftward by 7 units, then we take the

First Movement = 5 units.

Second Movement = –7 units.

What is the final position of the coin?

This can be found by simply adding the two movements: 5 + (–7) = –2.

So the coin is at –2, or it is 2 units to the left of 0.

In general, if the first strike moves the coin '*a*' units (which is positive if the strike is to the right and negative if the strike is to the left), and the second strike '*b*' units (which is positive if the strike is to the right and negative if the strike is to the left), then the final position 'P', after the two strikes, is again P = $a + b$.

Based on this new model, answer the following questions:

1. If the first movement is –4 and the final position is 5, what is the second movement?

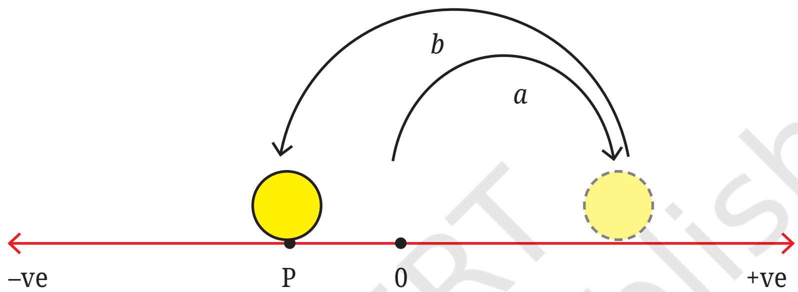
2. If there are multiple strikes causing movements in the order 1, –2, 3, –4, ..., –10, what is the final position of the coin?

By modeling the movements as numbers, both positive and negative, we are able to capture two pieces of information — the distance (magnitude) and the direction (rightwards or leftwards). For example, when we say the movement is –4, the **magnitude** is 4 and the direction is leftward.

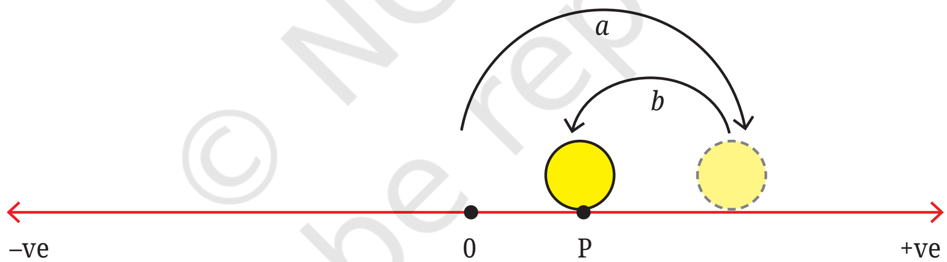
? From the figures below, what can you conclude about the magnitudes of a and b compared to each other, and what are their directions? Remember to start from 0.

Try This

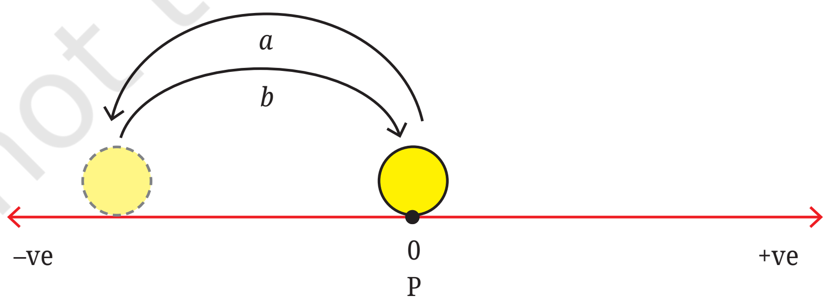
1.



2.



3.



In addition to the number line, we used the token model to understand integers. We used this model to perform addition and subtraction in Grade 6. Let us do a quick recall. We used green (+) token to represent positive 1 and red (−) token to represent negative 1, that is –1. Together they make zero, as they cancel each other out.

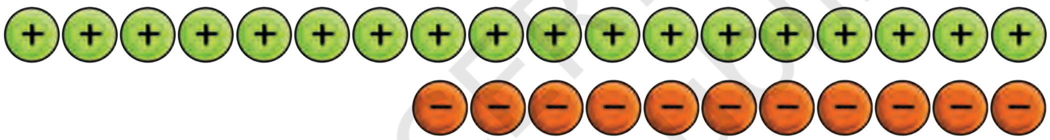
? Find (+7) – (+18).

To subtract 18 from 7, i.e., (+7) – (+18), we need to remove 18 positives from 7 positives.

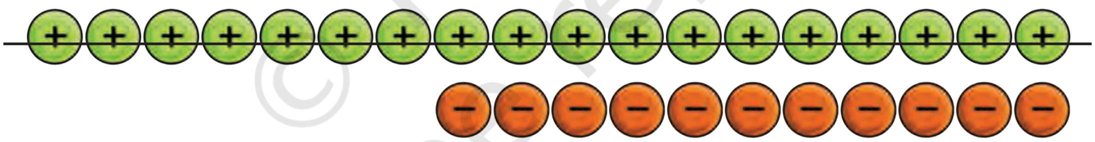
But there are not enough tokens to remove 18 positives!
We put in enough zero pairs so that we can remove 18 positives.

? How many?

We have 7 positives and we need 11 more. So we need to put in 11 zero pairs:



We can now remove 18 positives.



What is left? There are 11 negatives, meaning –11.

So, 7 – 18 = –11.

We had also seen that subtracting a number is the same as adding its additive inverse.

? Using tokens, argue out the following statements.

Math Talk

(a) 7 – 18 = 7 + (– 18) (additive inverse of 18 is – 18)

(b) 4 – (– 12) = 4 + 12 (additive inverse of – 12 is 12)

Note to the Teacher: Tokens of different shapes may be used for positive and negative numbers for visually challanged students.

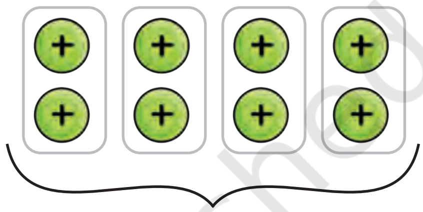
Additive inverse of an integer *a* is represented as $-a$. So the additive inverse of 18 is represented as $-(18) = -18$, and the additive inverse of -18 is represented as $-(-18) = 18$.

2.2 Multiplication of Integers

We used the token model to represent addition and subtraction of integers. We now explore how to model multiplication of integers using tokens.

Suppose we put some positive tokens into an empty bag as shown in the figure.

? How many positives are in the bag now?



Two green tokens 4 times = 8

$4 \times 2 = 8$

There are 8 positives in the bag. We can see this as adding 2 positives to the bag 4 times. Thus, the operation is, $4 \times 2 = 8$.

We have seen this kind of multiplication of positive integers before. Can we use tokens to give meaning to multiplications like $4 \times (-2)$?

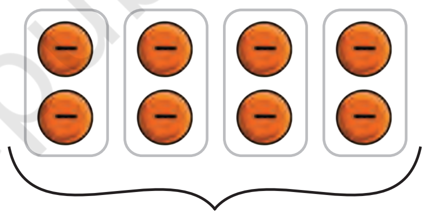
Let us see how.

For every new operation, we start with an empty bag.

$4 \times (-2)$ can be interpreted as placing 2 negatives into an empty bag 4 times. We use red tokens for negatives, so we place 2 negatives into an empty bag 4 times.

There are now 8 red tokens or 8 negatives in the bag, meaning -8.

$4 \times (-2) = (-8)$.



Two red tokens 4 times = -8

$4 \times (-2) = -8$

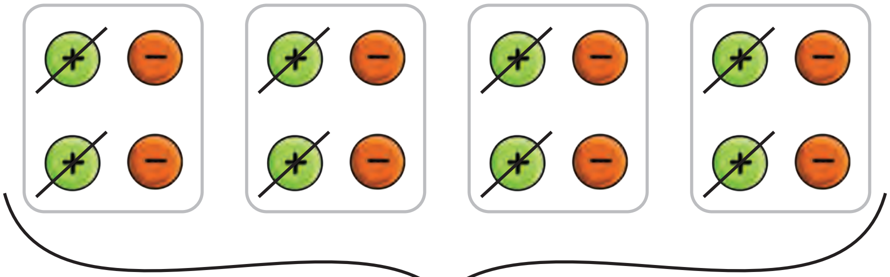
? Similarly find the values of $4 \times (-6)$ and $9 \times (-7)$? How can we interpret $(-4) \times 2$?

When the multiplier is positive, we place tokens into the bag. When the multiplier is negative, we **remove** tokens from the bag.

So, for $(-4) \times 2$, we need to remove two positives or two green tokens from the bag 4 times.

? Why are we trying to remove green tokens and not red tokens? Math Talk

But there are no tokens in the bag, because we start with an empty bag. Just as in the case of subtraction, to remove 2 positives from an empty bag, we need to first place 2 zero pairs inside and then remove the 2 positives. We need to do this 4 times.



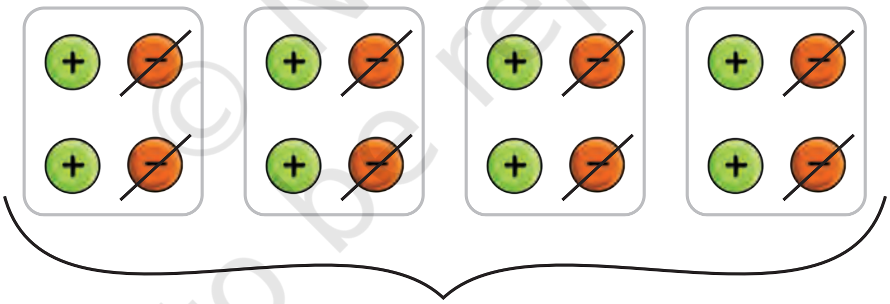
Remove 2 green tokens from the zero pairs, 4 times

$(-4) \times 2 = -8$

After removing the positives, 8 negatives are left in the bag. This is –8. This shows that $(-4) \times 2 = -8$.

? What happens when both the integers in the multiplication are negative? How do we model $(-4) \times (-2)$ with tokens?

For $(-4) \times (-2)$, we need to remove 2 negatives from the bag 4 times. Since there are no red tokens in the bag, we need to place 2 zero pairs and remove 2 negatives, and we need to do this 4 times.



Remove 2 red tokens from the zero pairs, 4 times

$(-4) \times (-2) = 8$

8 positives are left in the bag. This is + 8.

So,

$$-4 \times -2 = +8.$$

So far, we have established the following results by using tokens:

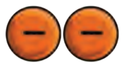
$$4 \times 2 = 8,$$
$$4 \times (-2) = -8,$$
$$(-4) \times 2 = -8, \text{ and}$$
$$(-4) \times (-2) = 8.$$

Figure it Out

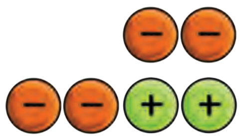
1. Using the token interpretation, find the values of:
   (a) $3 \times (-2)$ (b) $(-5) \times (-2)$
   (c) $(-4) \times (-1)$ (d) $(-7) \times 3$
2. If $123 \times 456 = 56088$, without calculating, find the value of:
   (a) $(-123) \times 456$ (b) $(-123) \times (-456)$
   (c) $(123) \times (-456)$
3. Try to frame a simple rule to multiply two integers.

Consider the numbers represented by the following tokens:

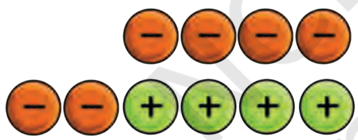
(a)



(b)



(c)



We can see that all of them represent the number (-2). Now, take 4 times each of these token sets. That is, place each set into the empty bag 4 times.

What integer do we get as the final answer in each case? Do we get different answers because the sets look different, or the same answer because they all represent -2?

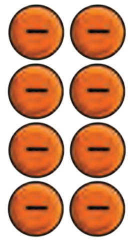
Check this for 5×4, by taking different token sets corresponding to 4.

We have seen that -4×2 is the number obtained by removing 2 positive tokens from the empty bag 4 times.

We know that removing or subtracting a number is the same as adding its inverse.

Using this, can -4×2 be defined through a process of addition of tokens instead of removal of tokens?

Since removing 2 positive tokens is the same as adding 2 negative tokens, -4×2 can also be obtained by adding 2 negative tokens to the empty bag 4 times.



$(-4) \times 2$

Patterns in Integer Multiplication

We have explored the multiplication of integers in cases where the multiplier is positive, when it is negative, when the multiplicand is positive and when it is negative. Using this understanding, let us construct a sequence of multiplications and observe the patterns.

$$\begin{array}{lr}
4 \times 3 = 12 & \\
 & -3 \\
3 \times 3 = 9 & \\
 & -3 \\
2 \times 3 = 6 & \\
 & -3 \\
1 \times 3 = 3 & \\
 & -3 \\
0 \times 3 = 0 &
\end{array}$$

? What do you notice in this pattern? Can you describe it?

We can see that, when the multiplicand is positive, for every unit **decrease** in the multiplier the product **decreases** by the multiplicand.

? Will this pattern continue when the multiplier goes below zero and becomes a negative number?

Yes indeed! The same pattern continues when the multiplier becomes a negative number.

$$\begin{array}{lr}
4 \times 3 = 12 & \\
 & -3 \\
3 \times 3 = 9 & \\
 & -3 \\
2 \times 3 = 6 & \\
 & -3 \\
1 \times 3 = 3 & \\
 & -3 \\
0 \times 3 = 0 & \\
 & -3 \\
-1 \times 3 = -3 & \\
 & -3 \\
-2 \times 3 = -6 & \\
 & -3 \\
-3 \times 3 = -9 &
\end{array}$$

? What is the pattern when the multiplicand is a negative integer?

$$\begin{array}{lr}
4 \times (-3) = -12 & \\
 & +3 \\
3 \times (-3) = -9 & \\
 & +3 \\
2 \times (-3) = -6 & \\
 & +3 \\
1 \times (-3) = -3 & \\
 & +3 \\
0 \times (-3) = 0 &
\end{array}$$

This is the inverse of the previous pattern. When the multiplicand is negative, for every unit **decrease** of the multiplier, the product **increases** by the multiplicand.

? Will this pattern continue when the multiplier goes below zero and becomes a negative integer?

Yes!

Even when the multiplier is negative, the same pattern is observed. When the multiplicand is negative, for every unit **decrease** in the multiplier, the product **increases** by the multiplicand.

We can see from these patterns that, what is true for multiplication when the integers are positive, is also true when the integers are negative.

$$\begin{array}{ll}
4 \times (-3) = -12 & \\
& \downarrow +3 \\
3 \times (-3) = -9 & \\
& \downarrow +3 \\
2 \times (-3) = -6 & \\
& \downarrow +3 \\
1 \times (-3) = -3 & \\
& \downarrow +3 \\
0 \times (-3) = 0 & \\
& \downarrow +3 \\
(-1) \times (-3) = 3 & \\
& \downarrow +3 \\
(-2) \times (-3) = 6 & \\
& \downarrow +3 \\
(-3) \times (-3) = 9 &
\end{array}$$

With this understanding of multiplication of integers, let us look at the times 3 tables when the multipliers and multiplicands are positive, and when they are negative.

1 × 3 = 3	–1 × 3 = –3	1 × –3 = –3	–1 × –3 = 3
2 × 3 = 6	– 2 × 3 = –6	2 × –3 = –6	–2 × –3 = 6
3 × 3 = 9	–3 × 3 = –9	3 × –3 = –9	–3 × –3 = 9
4 × 3 = 12	–4 × 3 = –12	4 × –3 = –12	–4 × –3 = 12
5 × 3 = 15	–5 × 3 = –15	5 × –3 = –15	–5 × –3 = 15
6 × 3 = 18	–6 × 3 = –18	6 × –3 = –18	–6 × –3 = 18
7 × 3 = 21	–7 × 3 = –21	7 × –3 = –21	–7 × –3 = 21
8 × 3 = 24	–8 × 3 = –24	8 × –3 = –24	–8 × –3 = 24
9 × 3 = 27	–9 × 3 = –27	9 × –3 = –27	–9 × –3 = 27
10 × 3 = 30	–10 × 3 = –30	10 × –3 = –30	–10 × –3 = 30

We observe the following:

- The magnitude of the product does not change with the change in the signs of the multiplier and the multiplicand.
- When both the multiplier and the multiplicand are **positive**, the product is **positive**.
- When both the multiplier and the multiplicand are **negative**, the product is **positive**.
- When one of the multiplier or the multiplicand is **positive** and the other is negative, their product is **negative**.

? Figure it Out

Find the following products.

(a) 4 × (– 3) (b) (– 6) × (– 3)

(c) (– 5) × (– 1)

(d) (– 8) × 4

(e) (– 9) × 10

(f) 10 × (– 17)

Consider the expression 1 × *a*. We know that the value of this expression is '*a*' for all positive integers.

? Is this true for all negative integers too?

Using the token model, we put '*a*' negatives into the bag just once. After this, the bag contains '*a*' negatives. For example, if '*a*' is – 5 (5 negatives), then the bag contains 5 negatives, i.e., –5. So,

$$1 \times a = a \text{ (for all integers } a\text{, both positive and negative).}$$

What is the value of the expression – 1 × *a*?

When '*a*' is positive, then from our observations on integer multiplication, the product has the same magnitude as '*a*' but is negative.

When '*a*' is negative, then the product has the same magnitude as '*a*' but is positive.

In each case, we notice that the product is the additive inverse of the multiplicand '*a*'. Thus,

$$-1 \times a = -a \text{ (for all integers a).}$$

? In the case of integers, is the product the same when we swap the multiplier and the multiplicand? Try this for some numbers.

Observe the following pairs of multiplications (fill in the blanks where needed):

3 × – 4 = –12	– 4 × 3 = –12
– 30 × 12 = _______	12 × – 30 = _______
–15 × – 8 = 120	– 8 × –15 = 120
14 × – 5 = –70	– 5 × _____ = – 70

? What do you notice in these pairs of multiplication statements?

The product is the same when we 'swap' the multiplier and multiplicand. Earlier, we have seen a similar property with addition.

? Will this always happen?

The magnitude of the product does not change when the multiplier and the multiplicand are swapped. This is because the magnitude of the product depends only on the magnitudes of the multiplier and the multiplicand, and we know that the product of two positive integers does not change when the numbers being multiplied are swapped.

Does the sign of the product change if we swap the multiplier and multiplicand?

If both are positive or both are negative, the product is positive before and after swapping. So the sign does not change in this case.

If one is positive and the other negative, the product is negative before and after swapping. So the sign does not change in this case either.

Hence, the product does not change when the multiplier and multiplicand are swapped, whatever their signs may be.

Thus, multiplication is **commutative** for integers.

In general, for any two integers, a and b, we can say that

$$a \times b = b \times a.$$

Brahmagupta's Rules for Multiplication and Division of Positive and Negative Numbers

Just like for addition and subtraction of integers, Brahmagupta in his *Brāhmasphuṭasiddhānta* (628 CE) also articulated explicit rules for integer multiplication and division. He used the notions of fortune (*dhana*) for positive values and debt (*ṛṇa*) for negative values. In his *Brāhmasphuṭasiddhānta (18.30-32),* Brahmagupta wrote:

> "The product or quotient of two fortunes is a fortune.
> The product or quotient of two debts is a fortune.
> The product or quotient of a debt and a fortune is a debt.
> The product or quotient of a fortune and a debt is a debt."

This represented the first time that rules for multiplication and division of positive and negative numbers were articulated, and was an important step in the development of arithmetic and algebra!

Example 1: An exam has 50 multiple choice questions. 5 marks are given for every correct answer and 2 negative marks for every wrong answer. What are Mala's total marks if she had 30 correct answers and 20 wrong answers?

Solution: We use positive and negative integers. The mark for each correct answer is a positive integer 5 and for each wrong answer is a negative integer -2.

Marks for 30 correct answers = 30×5.

Marks for 20 wrong answers = $20 \times (-2)$.

Thus the arithmetic expression for 30 correct answers and 20 wrong answers is:

$$30 \times 5 + 20 \times (-2)$$
$$= 150 + (-40)$$
$$= 110.$$

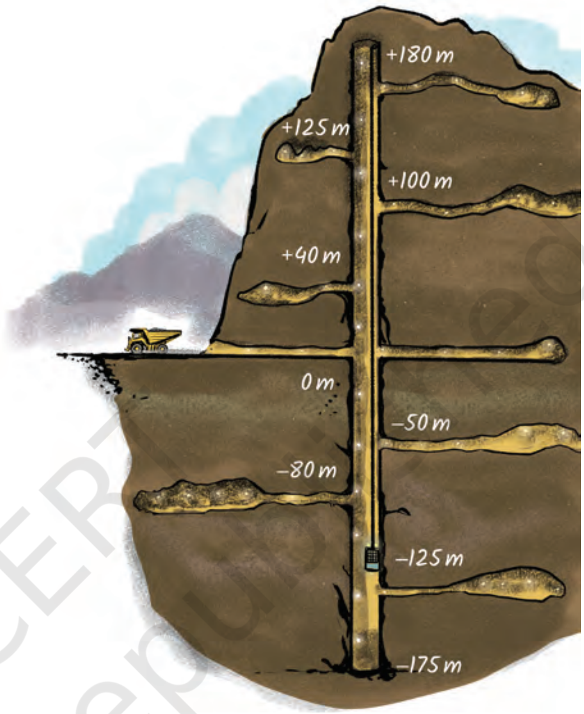
Mala got 110 marks in the exam.

What are the maximum possible marks in the exam? What are the minimum possible marks?

Example 2: There is an elevator in a mining shaft that moves above and below the ground. The elevator's positions above the ground are represented as positive integers and positions below the ground are represented as negative integers.

(a) The elevator moves 3 metres per minute. If it descends into the shaft from the ground level (0), what will be its position after one hour?

(b) If it begins to descend from 15 m above the ground, what will be its position after 45 minutes?



Solution:

Solution to part (a) of the question:

Method 1:
We can model this using subtraction.

The elevator moves at 3 metres per minute. So in one hour it moves 180 metres (60 × 3). If it started at ground level (0 metres) and descended, we should subtract 180 from 0.

$$0 - 180 = (-180).$$

So, it will reach the (–180) metre position, which is 180 metres below the ground.

Method 2:
Let us say that the speed and direction of the elevator are represented by an integer (metres per minute). It is +3 when it is moving up and it is (–3) when moving down.

Since the elevator is moving down, the speed is (–3) metres per minute. It moves for 60 minutes. So it goes

$$60 \times (-3) = (-180).$$

The position of the elevator after 60 minutes is 180 metres below the ground level.

? Find the solution to part (b) using Method 1 described above.

Solution to part (b) using Method 2:

Starting Position = 15.

Distance Travelled = The elevator moves down at the speed of 3 metres per minute for 45 minutes, that is, $(45 \times (-3))$. So,

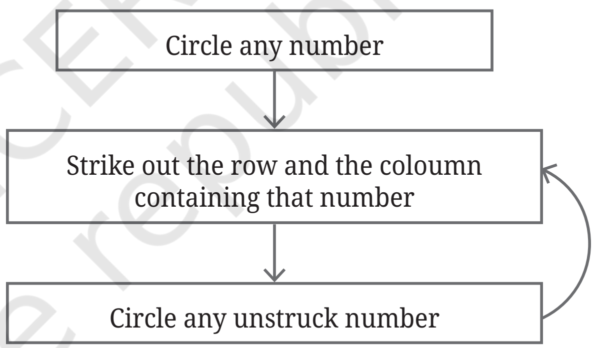
$$\begin{aligned}\text{Ending Position} &= 15 + (45 \times (-3))\\ &= 15 + (-(45\times3))\\ &= 15 + (-135)\\ &= (-120).\end{aligned}$$

The elevator will be 120 metres below the ground.

A Magic Grid of Integers

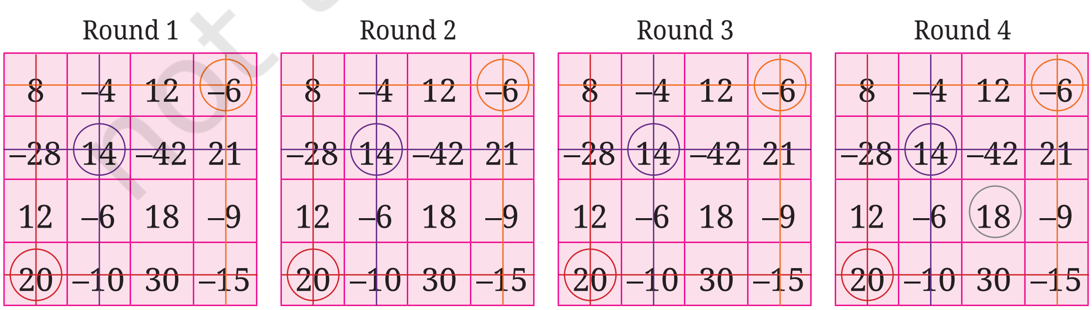
? A grid containing some numbers is given below. Follow the steps as shown until no number is left.

8	–4	12	–6
–28	14	–42	21
12	–6	18	–9
20	–10	30	–15



When there are no more unstruck numbers, stop. Multiply the circled numbers.

An example is shown below.



Try again , and choose different numbers this time. What product did you get? Was it different from the first time? Try a few more times with different numbers!

Play the same game with the grid below. What answer do you get?

8	–4	12	–6
–28	14	–42	21
12	–6	18	–9
20	–10	30	–15

Try This

What is so special about these grids? Is the magic in the numbers or the way they are arranged or both? Can you make more such grids?

Division of Integers

We have earlier seen how division can be converted into multiplication. For example, (–100) ÷ 25 can be reframed as, 'what should be multiplied to 25 to get (–100)?'. That is,

$$25 \times ? = (-100).$$

We know that

$$25 \times (-4) = (-100).$$

Therefore,

$$(-100) \div 25 = (-4).$$

Similarly, (–100) ÷ (–4) can be reframed as, 'What should be multiplied to (–4) to get (–100)?'

$$(-4) \times ? = (-100).$$

We know that

$$(-4) \times 25 = (-100).$$

Therefore,

$$(-100) \div (-4) = 25.$$

Similarly, we know that

$$(-25) \times (-2) = 50.$$

Therefore,

$$50 \div (-25) = (-2).$$

? Can you summarise the rules for integer division looking at the above pattern?

In general, for any two positive integers *a* and *b*, where $b \neq 0$, we can say that

$$a \div -b = -(a \div b),$$
$$-a \div b = -(a \div b), \text{ and}$$
$$-a \div -b = a \div b.$$

? Figure it Out

1. Find the values of:

   (a) $14 \times (-15)$ (b) $-16 \times (-5)$

   (c) $36 \div (-18)$ (d) $(-46) \div (-23)$

2. A freezing process requires that the room temperature be lowered from 32°C at the rate of 5°C every hour. What will be the room temperature 10 hours after the process begins?
3. A cement company earns a profit of ₹8 per bag of white cement sold and a loss of ₹5 per bag of grey cement sold. [Represent the profit/loss as integers.]

   (a) The company sells 3,000 bags of white cement and 5,000 bags of grey cement in a month. What is its profit or loss?

   (b) If the number of bags of grey cement sold is 6,400 bags, what is the number of bags of white cement the company must sell to have neither profit nor loss.

4. Replace the blank with an integer to make a true statement.

   (a) $(-3) \times$ _____ $= 27$ (b) $5 \times$ _____ $= (-35)$

   (c) _____ $\times (-8) = (-56)$ (d) _____ $\times (-12) = 132$

   (e) _____ $\div (-8) = 7$ (f) _____ $\div 12 = -11$

Expressions Using Integers

? What is the value of the expression $5 \times -3 \times 4$? Does it matter whether we multiply 5×-3 and then multiply the product with 4, or if we multiply -3×4 first and then multiply the product with 5?

$(5 \times -3) \times 4$	$5 \times (-3 \times 4)$
$= -15 \times 4$	$= 5 \times -12$
$= -60.$	$= -60.$

? Take a few more examples of multiplication of 3 integers and check this property. What do you observe?

We can see that the product is the same when we 'group' the multiplications in these two ways. So, integer multiplication is associative, just like integer addition.

In general, for any three integers a, b, and c,

$$a \times (b \times c) = (a \times b) \times c.$$

In the expression 5 × –3 × 4, try to multiply 5 and 4 first and then multiply the product with –3: (5 × 4) × –3.

$$5 \times 4 = 20, \text{ and } 20 \times -3 = -60.$$

This also gives the same product.

? Are there orders in which 5 × –3 × 4 can be evaluated? Will the product be the same in all these cases?

? Multiply the expression 25 × –6 × 12 in all the different orders and check if the product is the same in all cases.

The product remains the same when 3 or more numbers are multiplied in any order.

> Look at the following series of multiplications:
>
> $$-1 \times -1 = 1$$
> $$-1 \times -1 \times -1 = -1$$
> $$-1 \times -1 \times -1 \times -1 = 1$$
> $$-1 \times -1 \times -1 \times -1 \times -1 = 1$$
>
> When –1 is multiplied 2 or 4 times the product is positive. When it is multiplied 3 or 5 times the product is negative. Can you generalise these statements further?

? Using this understanding of multiplication of many integers, can you give a simple rule to find the sign of the product of many integers?

Math Talk

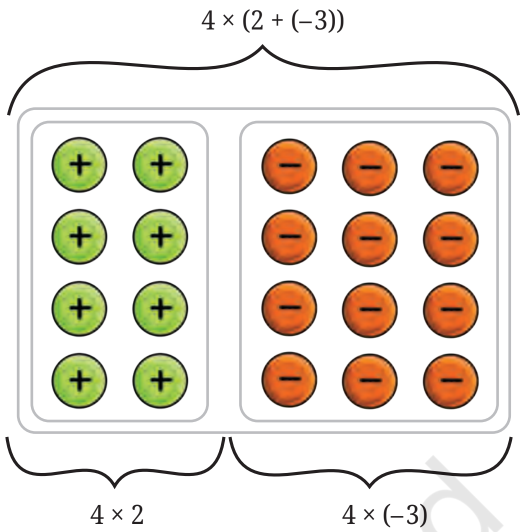
? Now, consider the expression 5 × (4 + (–2)). As in the case of positive integers, is this expression equal to 5 × 4 + 5 × (–2)?

We see that it does. Recall that we call this property the distributive property.

? Check if the distributive property holds for (–2) × (4 + (–3)) (that is, if this expression equals (–2) × 4 + (–2) × (–3)), and for a few other such expressions of your choice.

What do you observe? We see that the distributive property seems to hold for integers, as well. Will this always happen?

In the case of positive integers, we used a rectangular arrangement of objects to visually understand why the distributive property holds. We can use the same setup even in the case of integers by using green tokens for positive numbers and red tokens for negative numbers. For example, consider the following rectangular arrangement of tokens —



We see that the overall arrangement represents 4 × (2 + (– 3)), and it is clear that this also equals the sum of 4 × 2 and 4 × (–3).

Can you visually show the distributive property for an expression like –4 × (2 + (–3))? [**Hint:** Use the fact that multiplying a number by –4 is adding the inverse of the number 4 times.]

Try This

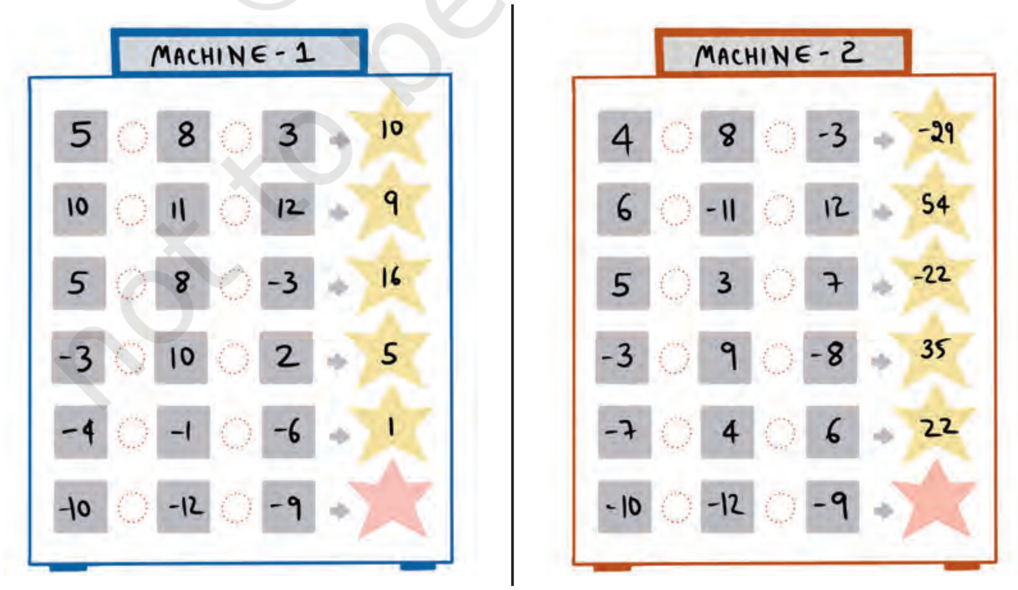
Thus, for any integers a, b, and c, we have

$$a \times (b + c) = (a \times b) + (a \times c).$$

Pick the Pattern

Two pattern machines are given below. Each machine takes 3 numbers, does some operations and gives out the result.

Find the operations being done by Machine 1.



The operation done by Machine 1 is

(first number) + (second number) – (third number).

Written as an expression, this will be $a + b - c$, where a is the first number, b is the second number, and c is the third number.

For example, 5 + 8 – 3 = 10, and (– 4) + (– 1) – (– 6) = 1.

So, the result of the last group will be, (– 10) + (– 12) – (– 9) = _______.

? Find the operations being done by Machine 2 and fill in the blank.

Math Talk

Make your own machine and challenge your peers in finding its operations.

? Figure it Out

1. Find the values of the following expressions:
   (a) (– 5) × (18 + (– 3))
   (b) (– 7) × 4 × (– 1)
   (c) (– 2) × (– 1) × (– 5) × (– 3)

2. Find the values of the following expressions:
   (a) (– 27) ÷ 9
   (b) 84 ÷ (– 4)
   (c) (– 56) ÷ (– 2)

3. Find the integer whose product with (– 1) is:
   (a) 27
   (b) – 31
   (c) – 1
   (d) 1
   (e) 0

4. If 47 – 56 + 14 – 8 + 2 – 8 + 5 = – 4, then find the value of – 47 + 56 – 14 + 8 – 2 + 8 – 5 without calculating the full expression.

5. Do you remember the Collatz Conjecture from last year? Try a modified version with integers. The rule is — start with any number; if the number is even, take half of it; if the number is odd, multiply it by – 3 and add 1; repeat. An example sequence is shown below.

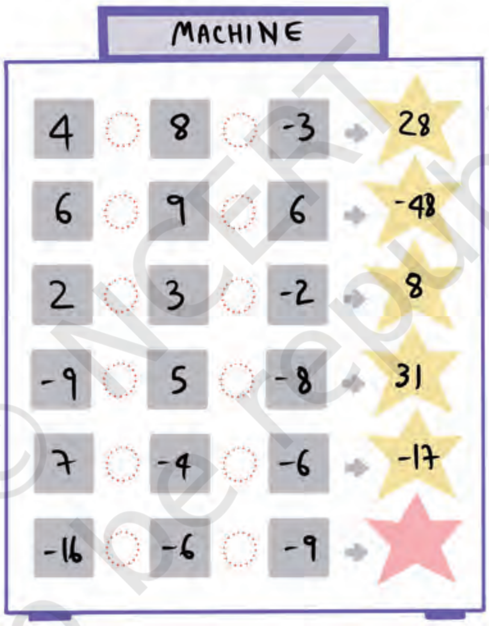
$$-7 \rightarrow 22 \rightarrow 11 \rightarrow 32 \rightarrow -16 \rightarrow -8 \rightarrow -4$$

$$\rightarrow -2 \rightarrow -1 \rightarrow 4 \rightarrow 2 \rightarrow 1$$

Math Talk

Try this with different starting numbers: (– 21), (– 6), and so on. Describe the patterns you observe.

6. In a test, (+ 4) marks are given for every correct answer and (– 2) marks are given for every incorrect answer.
   (a) Anita answered all the questions in the test. She scored 40 marks even though 15 of her answers were correct. How many of her answers were incorrect? How many questions are in the test?
   (b) Anil scored (– 10) marks even though he had 5 correct answers. How many of his answers were incorrect? Did he leave any questions unanswered?
7. Pick the pattern — find the operations done by the machine shown below.



8. Imagine you're in a place where the temperature drops by 5°C each hour. If the temperature is currently at 8°C, write an expression which denotes the temperature after 4 hours.
9. Find 3 consecutive numbers with a product of (a) – 6, (b) 120.
10. An alien society uses a peculiar currency called 'pibs' with just two denominations of coins — a+13 pibs coin and a – 9 pibs coin. You have several of these coins. Is it possible to purchase an item that costs + 85 pibs?

Yes, we can use 10 coins of +13 pibs and 5 coins of – 9 pibs to make a total of + 85. Using the two denominations, try to get the following totals:

(a) + 20 (b) + 40

(c) – 50 (d) + 8

(e) + 10 (f) – 2

(g) + 1

[**Hint:** Writing down a few multiples of 13 and 9 can help.]

(h) Is it possible to purchase an item that costs 1568 pibs?

Try This

11. Find the values of:

(a) $(32 \times (-18)) \div ((-36))$ (b) $(32\ ) \div ((-36) \times (-18))$

(c) $(25 \times (-12)) \div ((45) \times (-27))$ (d) $(280 \times (-7)) \div ((-8) \times (-35))$

12. Arrange the expressions given below in increasing order.

(a) $(-348) + (-1064)$ (b) $(-348) - (-1064)$

(c) $348 - (-1064)$ (d) $(-348) \times (-1064)$

(e) $348 \times (-1064)$ (f) 348×964

13. Given that $(-548) \times 972 = -532656$, write the values of:

(a) $(-547) \times 972$ (b) $(-548) \times 971$

(c) $(-547) \times 971$

14. Given that $207 \times (-33 + 7) = -5382$, write the value of $-207 \times (33 - 7)$ = __________.

15. Use the numbers 3, – 2, 5, – 6 exactly once and the operations '+', '–', and '×' exactly once and brackets as necessary to write an expression such that —

(a) the result is the maximum possible

(b) the result is the minimum possible

16. Fill in the blanks in at least 5 different ways with integers:

(a) $\square + \square \times \square = -36$ (b) $(\square - \square) \times \square = 12$

(c) $(\square - (\square - \square)) = -1$

SUMMARY

- When two integers are multiplied, the product is positive when both the multiplier and multiplicand are positive, or when both are negative. The product is negative if one of them is positive and the other is negative.

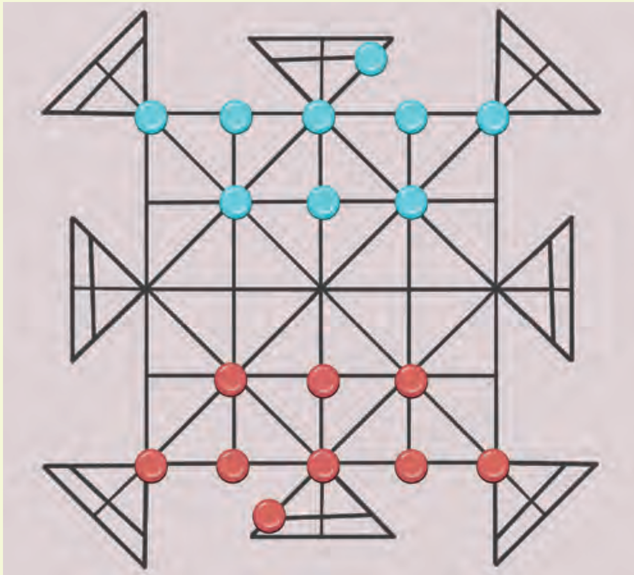
–9	–6	–3	3	3	6	9
–6	–4	–2	2	2	4	6
–3	–2	–1	1	1	2	3
–3	–2	–1	×	1	2	3
3	2	1	1	–1	–2	–3
6	4	2	2	–2	–4	–6
9	6	3	3	–3	–6	–9

- When two integers are divided, the quotient is positive when both the dividend and divisor are positive, or both are negative. The quotient is negative when one of them is positive and the other is negative.
- Integer multiplication is commutative, i.e., for any two integers a and b,
$$a \times b = b \times a.$$
- Integer multiplication is associative, i.e., for any three integers, a, b, and c,
$$a \times (b \times c) = (a \times b) \times c.$$
- Integer multiplication is distributive over addition, i.e., for any three integers, a, b, and c,
$$a \times (b + c) = (a \times b) + (a \times c).$$

IT'S PUZZLE TIME!

Terhüchü

Terhüchü is a game played in Assam and Nagaland. The board has 16 squares and diagonals are marked as shown in the following figure. This is usually roughly scratched on a large piece of stone or just drawn on mud. There are 2 players, and each player has a set of 9 coins placed as shown. The coins in one set look different from those in the other set.



Objective

The goal is to capture all the opponent's coins. The first player to do so is the winner. A player may also win by blocking any legal move by their opponent. If a draw seems unavoidable, the player with more coins wins.

Gameplay

- The starting position of the game is as shown above.
- Players take turns. In each turn, they can move a single coin along a line, in any direction, to a neighbouring vacant intersection. Or, if an opponent's coin is at a neighbouring intersection, and there is a vacant intersection just beyond it, they can jump over the opponent's coin and land in the vacant intersection.
- If a player jumps over an opponent's coin, it is considered captured and is removed from the board. Multiple captures in one move are allowed, and the direction can change after each jump.
- Inside the triangular corners, which are outside the main square, a coin may skip an intersection and move straight to the next one. That is, it can jump over an empty intersection and go to the one beyond it.

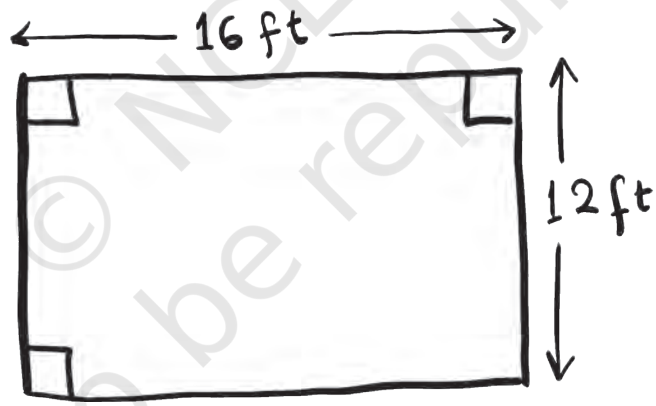
3 FINDING COMMON GROUND

0789CH03

3.1 The Greatest of All

Sameeksha is building her new house. The main room of the house is 12 ft by 16 ft. She feels that the room would look nice if the floor is covered with square tiles of the same size. She also wants to use as few tiles as possible, and for the length of the tile to be a whole number of feet. What size tiles should she buy?

Let us explore how to find the largest square tile that can be used. The breadth of the room is 12 ft and the length is 16 ft.



For the tiles to fit the breadth of the room exactly, the side of the tile should be a factor of 12. Similarly, for the tiles to fit the length of the room exactly, the side should be a factor of 16. So the side of the tile should be a factor of both 12 and 16. What are the common factors of 12 and 16?

The factors of 12 are 1, 2, 3, 4, 6, 12. The factors of 16 are 1, 2, 4, 8, 16.

The common factors are 1, 2, and 4.

So, the square tiles can have sides 1 ft, 2 ft, and 4 ft. Among these, she should use the largest sized square tile. Can you explain why?

Therefore, she needs tiles of size 4 ft.

How many tiles of this size should she purchase?

What if Sameeksha did not insist on the length of the tile to be a whole number of feet and the length could be a fractional number of feet? Would the answer change?

Try This

The **Highest Common Factor (HCF)** of two or more numbers, is the highest (or greatest) of their common factors. It is also known as the Greatest Common Divisor (GCD).

In the previous problem, 4 is the HCF of 12 and 16.

We can draw rough diagrams like the one shown on the previous page to visualise the given scenario. It may help in understanding and solving.

? Lekhana purchases rice from two farms and sells it in the market. She bought 84 kg of rice from one farm and 108 kg from the other farm. She wants the rice to be packed in bags, so each bag has rice from only one farm and all bags have the same weight that is a whole number of kg. If she wants to use as few bags as possible, what should the weight of each bag be?

To divide 84 kg of rice into bags of equal weight, we need the factors of 84. Similarly, for 108, we need the factors of 108.

Factors of 84 — 1, 2, 3, 4, 6, 7, 12, 14, 21, 28, 42, and 84.

Factors of 108 — 1, 2, 3, 4, 6, 9, 12, 18, 27, 36, 54, and 108.

Since, Lekhana wants to use bags of the same weight for both farms, the weight of the bag should be a common factor. The common factors of 84 and 108 are

1, 2, 3, 4, 6, and 12.

She can use any of these weights to pack rice from both farms in bags of equal weight. But, she wants to minimise the number of bags. Which weight should she choose to minimise the number of bags?

? Do you remember the 'Jump Jackpot' game from Grade 6 (see the chapter 'Prime Time')? Grumpy places a treasure on a number and Jumpy chooses a jump size and tries to collect the treasure. In each case below, the two numbers upon which treasures are kept are given. Find the longest jump size (starting from 0) using which Jumpy can land on both the numbers having the treasure.

(a) 14 and 30 (b) 7 and 11

(c) 30 and 50 (d) 28 and 42

? Is the longest jump size for the numbers the same as their HCF? Explain why it is so.

So far, we have been listing all the factors to find the HCF. This can become cumbersome for numbers with many factors, as you would have observed for the numbers 30 and 50, and 28 and 42.

Sometimes, we may also miss some factors which can lead to errors.

? Can this process be simplified? Can it be made more reliable?

It turns out that using prime factorisation can simplify the process. We will start by revisiting primes and prime factorisation.

Primes

Recall that a prime is a number greater than 1 that has only 1 and the number itself as its factors. Last year, we tried to find patterns amongst primes between 1 to 100. We also came across the Sieve of Eratosthenes — a method to list all primes.

Prime Factorisation

Any number can be written as a product of primes — keep rewriting composite factors until only primes are left.

Recall that we call this the prime factorisation of a given number. For example, we can find the prime factorisation of 90 as follows:

$$90 = 3 \times 3 \times 2 \times 5.$$

The number 90 could also have been factorised as 3×30 or 2×45 or in a few other different ways. Will these all lead to the same prime factors?

Remarkably, the resulting prime factors are always the same with perhaps only a change in their order. For example, if we consider factorising 3×30 further, we get

$$90 = 3 \times 30 = 3 \times 2 \times 15 = 3 \times 2 \times 3 \times 5,$$

and we have arrived at the same prime factors, just in a different order.

Note that the prime factorisation of a prime number is the prime number itself.

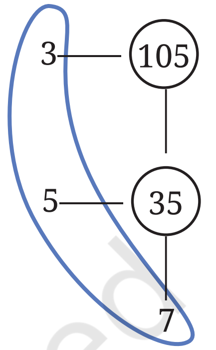
Procedure for Prime Factorisation

Can you see what is happening below?

Each circled number is the product of the numbers that are to its left and below. For example, 30 = 2 × 15, 15 = 3 × 5. Each time, a composite number is factorised so that, at least one factor is prime. We stop when we reach a prime number.

Can you write the prime factorisation of 105 and 30 using these two figures?

We collect the prime factors along the left and the one at the bottom. We then construct the prime factorisation as shown in the figure.



105 = 3 × 5 × 7

While carrying out factorisation, the circles are usually left out and the steps are carried out in this format:

$$\begin{array}{r|l} 3 & 105 \\ \hline 5 & 35 \\ \hline & 7 \end{array} \qquad \begin{array}{r|l} 2 & 30 \\ \hline 3 & 15 \\ \hline & 5 \end{array}$$

Let us call this method the **division method**.

Try finding the prime factorisation of 1200 using the method above. If we had used the earlier method, our calculation would have been as follows:

$$1200 = 40 \times 30 = 5 \times 8 \times 5 \times 6 = \ldots$$

Which calculation is easier to carry out?

Factors of a Number Using Prime Factorisation

From the prime factorisation of a number, we can construct all its factors. This can be used to simplify the procedure for finding the HCF of two numbers.

Consider the number 840 and its prime factorisation 2 × 2 × 2 × 3 × 5 × 7.

? Is 2 × 2 × 7 = 28 a factor of 840?

? If yes, what should it be multiplied by to get 840?

To answer these questions, we can reorder the prime factors of 840 as follows (recall that reordering factors does not change the product):

$$840 = (2 \times 2 \times 7) \times (2 \times 3 \times 5).$$

So,

$$840 = 28 \times 30.$$

Thus, (2 × 2 × 7) = 28 is a factor of 840, and it should be multiplied by 30 (2 × 3 × 5) to get 840.

? Similarly, is 2 × 7 = 14 a factor of 840? Why or why not?

Is 2 × 2 × 2 a factor of 840? Why or why not?

Is 3 × 3 × 3 a factor of 840? Why or why not?

Can we use this idea to list down all the possible factors of a number using just its prime factors?

? Find the factors of 225 using prime factorisation.

$$\begin{array}{r|l} 5 & 225 \\ \hline 5 & 45 \\ \hline 3 & 9 \\ \hline 3 & 3 \\ \hline & 1 \end{array}$$

Factorising 225 to its primes, we get
225 = 5 × 5 × 3 × 3.

We have seen that any 'subpart' of this factorisation gives us a factor. Let us systematically form these subparts.

Prime factors: 3, 5.

Combination of two prime factors: 3 × 3 = 9, 5 × 5 = 25, 3 × 5 = 15.

Combination of three prime factors: 3 × 3 × 5 = 45, 3 × 5 × 5 = 75.

Combination of four prime factors: 3 × 3 × 5 × 5 = 225. Adding 1 to this list of factors, we see that the factors of 225 are 1, 3, 5, 9, 15, 25, 45, 75, 225.

? Check that all the factors of 225 occur in this list.

? Figure it Out

List all the factors of the following numbers:

(a) 90 (b) 105

(c) 132 (d) 360 (this number has 24 factors)

(e) 840 (this number has 32 factors)

After observing a few prime factorisations, Anshu claims "The larger a number is, the longer its prime factorisation will be".

What do you think of Anshu's claim?

We can see that it is not true. For example, look at the prime factorisations of 96 and 121.

$$96 = 2 \times 2 \times 2 \times 2 \times 2 \times 3$$

$$121 = 11 \times 11.$$

In mathematics, statements or claims made without proof or verification are called 'conjectures'. Anshu's claim is a conjecture. We disproved this conjecture by finding a counterexample, i.e., an example where the conjecture is false.

Finding the HCF of Numbers Using Prime Factorisation

We now see how to make use of the observations made so far to find common factors and the Highest Common Factor (HCF).

Example 1: Find the common factors, and the HCF of 45 and 75.

Calculate the prime factorisation for both numbers:

$$45 = 3 \times 3 \times 5$$
$$75 = 3 \times 5 \times 5.$$

We have seen:

Factors of 45 are subparts of factors occurring in 3 × 3 × 5 and factors of 75 are subparts of factors occurring in 3 × 5 × 5.

So the common factors should be subparts of both the factorisations. Can you write them down?

While exploring or solving problems you might also get some conjectures. You can try to reason and verify these and also share with the class.

3 × 3 × 5 3 × 5 × 5	3
3 × 3 × 5 3 × 5 × 5	5
3 × 3 × 5 3 × 5 × 5	3 × 5

3, 5, 3 × 5 = 15 are subparts of both, and hence, they are the common factors along with 1. The highest among them is 15. So, their HCF = 15.

Example 2: Find the common factors, and the HCF of 112 and 84.

Calculating prime factorisations, we get,

$$112 = 2 \times 2 \times 2 \times 2 \times 7 \text{ and}$$
$$84 = 2 \times 2 \times 3 \times 7.$$

Finding the common subparts, we get

$2 \times 2 \times 2 \times 2 \times 7$ $2 \times 2 \times 3 \times 7$	2
$2 \times 2 \times 2 \times 2 \times 7$ $2 \times 2 \times 3 \times 7$	7
$2 \times 2 \times 2 \times 2 \times 7$ $2 \times 2 \times 3 \times 7$	2×2
$2 \times 2 \times 2 \times 2 \times 7$ $2 \times 2 \times 3 \times 7$	2×7
$2 \times 2 \times 2 \times 2 \times 7$ $2 \times 2 \times 3 \times 7$	$2 \times 2 \times 7$

The highest among the common factors, HCF, is $2 \times 2 \times 7 = 28$.

Example 3: Find the common factors and the HCF of 96 and 275.

We have

$$96 = 2 \times 2 \times 2 \times 2 \times 2 \times 3$$
$$275 = 5 \times 5 \times 11.$$

There is no subpart that is common amongst these two factorisations. So, 1 is the only common factor. It is also their HCF.

Figure it Out

Find the common factors and the HCF of the following numbers:

(a) 50, 60 (b) 140, 275

(c) 77, 725 (d) 370, 592

(e) 81, 243

How do we directly find the HCF without listing all the factors?

Example 4: Find the HCF of 30 and 72.

$$30 = 2 \times 3 \times 5$$
$$72 = 2 \times 2 \times 2 \times 3 \times 3$$

We need the largest common subpart to find the HCF. Clearly, it will contain only those primes that occur in both the factorisations: 2 and 3 in this case.

How many 2s will it contain?

The prime factorisation of 30 has only one 2. So, the largest common subpart should contain only one 2.

How many 3s will it contain?

The prime factorisation of 30 has only one 3. So, the largest common subpart should contain only one 3.

This has been carried out below.

$$30 = \boxed{2} \times \boxed{3} \times 5$$

$$72 = \boxed{2} \times 2 \times 2 \times \boxed{3} \times 3$$

$$\text{HCF} = 2 \times 3 = 6$$

Thus, to find the HCF, we identify the common primes and find the minimum number of times each of them appear in the factorisations of the given numbers.

Example 5: Find the HCF of 225 and 750.

$$225 = 3 \times 3 \times 5 \times 5$$

$$750 = 2 \times 3 \times 5 \times 5 \times 5$$

Common primes: 3 and 5. Let us find the largest common subpart. How many 3s will it contain?

750 contains only one 3, which is the minimum number of 3 across both numbers. So the largest common subpart should contain only one 3.

How many 5s will it contain?

225 contains the minimum number of 5s which occurs two times. So, the largest common subpart should contain two 5s, i.e., 5 × 5.

$$225 = 3 \times \boxed{3} \times \boxed{5} \times \boxed{5}$$

$$750 = 2 \times \boxed{3} \times \boxed{5} \times \boxed{5} \times 5$$

$$\text{HCF} = 3 \times 5 \times 5 = 75$$

To find the HCF of more than 2 numbers, a similar method of finding the largest common subpart of the common primes can be followed.

Figure it Out

1. Find the HCF of the following numbers:

   (a) 24, 180 (b) 42, 75, 24

   (c) 240, 378 (d) 400, 2500

   (e) 300, 800

2. Consider the numbers 72 and 144. Suppose they are factorised into composite numbers as: 72 = 6 × 12 and 144 = 8 × 18. Seeing this, can one say that these two numbers have no common factor other than 1? Why not?

3.2 Least, but not Last!

? Anshu and Guna make torans out of strips of cloth. Multiple strips are placed one next to another to make a toran. Anshu uses strips of length 6 cm and Guna uses strips of 8 cm length. If both have to make torans of the same length, what is the smallest possible length, the torans could be?

What is the length of the shortest toran that they can both make?

Anshu uses cloth strips of 6 cm each. Any toran he makes will be a multiple of 6. So, the length of the toran could be 6, 12, 18, 24, 30, 36, 42, 48, 54 cm, and so on.

Similarly, any toran Guna makes should be a multiple of 8. So, the length of the toran he makes could be 8, 16, 24, 32, 40, 48, 56, 64, 72 cm, and so on.

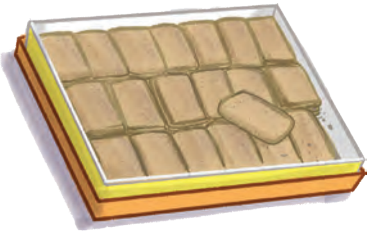
From this, we can see that if both have to make torans of the same length, the length of the toran should be a common multiple of 6 and 8.

From the two lists, we can see that 24 and 48 are two of the common multiples of 6 and 8. So, 24 cm and 48 cm are lengths of toran that Anshu and Guna can both stitch.

24 is the smallest among them. So, 24 cm is the length of the shortest toran that both can stitch. 24 is the lowest number among all the common multiples of 6 and 8.

What about the largest common multiple? Does such a number exist?

? A sweet shop gives out free *gajak* to school children on Mondays. Today is a Monday and Kabamai enjoyed eating the *gajak*. But she visits the sweet shop once every 10 days. When is the next time she would be able to get free *gajak* from Sweet shop? (Answer in number of days.)



Gajak is a sweet made from sesame seeds, jaggery and ghee

As we saw before, imagining or visualising the scenario helped us to see that it can be solved using multiples of the strips' lengths.

Since the shop gives free sweets every Monday, it will give free sweets again after

7, 14, 21, 28, 35, 42, 49, 56, 63, 70, 77, ... days.

These are multiples of 7.

Kabamai will arrive at the sweet shop again after

10, 20, 30, 40, 50, 60, 70, ... days.

These are multiples of 10.

When will Kabamai eat free sweets again? It will happen on days common to the sequences of multiples above. It can be seen that this will first happen after 70 days.

Notice that, here too, 70 is the lowest among all the common multiples of 7 and 10.

For both these problems the solution was the lowest common multiple.

The **Lowest Common Multiple (LCM)** of two or more given numbers is the lowest (or smallest or least) of their common multiples.

? Do you remember the 'Idli-Vada' game from Grade 6 (see chapter 'Prime Time')? Two numbers are chosen and whenever players come to their multiples, 'idli' or 'vada' should be called out depending on whose multiple the number is. If the number happens to be a common multiple, then 'idli-vada' should be called out. In each problem below, the two numbers corresponding to 'idli' and 'vada' are given. Find the first number for which 'idli-vada' will be called out:

(a) 4 and 6 (b) 7 and 11

(c) 14 and 30 (d) 15 and 55

Is the answer always the LCM of the two numbers? Explain.

As in the case of the HCF, the process of finding the LCM by listing down the multiples may get tedious for larger numbers, as you would have seen for questions (c) and (d) above.

Prime factorisation can simplify the process of finding the LCM as well.

? How do we find the LCM of two numbers using their prime factors?

Finding LCM through Prime Factorisation

We have seen that every factor of a number is formed by taking a subpart of its prime factorisation. We used this fact to come up with a method to find the HCF of two numbers. In a similar manner, we can come up with a method to find the LCM.

We begin by comparing the prime factorisations of a number and a multiple of that number. For example, let us take 36 and its multiple 648 (=36 × 18).

We get,

$$36 = 2 \times 2 \times 3 \times 3,$$
$$648 = 36 \times 18 = (2 \times 2 \times 3 \times 3) \times (2 \times 3 \times 3).$$

What do you observe? We can see that the prime factors of the multiple contain the prime factors of the number along with some more prime factors. Will this happen with every multiple?

Can this be used to find the LCM?

Example 6: Find the LCM of 14 and 35.

We get,

$$14 = 2 \times 7$$
$$35 = 5 \times 7.$$

Common multiples should contain each prime factor as a subpart: 2×7 as a subpart and 5×7 as a subpart.

For example,

$2 \times 7 \times 5 \times 7 \times 3$ is a common multiple of 14 and 35.

$2 \times 2 \times 5 \times 7 \times 7 \times 11$ is another common multiple.

Is $2 \times 3 \times 5 \times 7$ also a common multiple?

What is the lowest among all the common multiples of 14 and 35?

It is $2 \times 5 \times 7 = 70$ because $2 \times 5 \times 7$ contains $14 = 2 \times 7$ as well as $35 = 5 \times 7$, and removing any number from $2 \times 5 \times 7$ will give a number that is not a common multiple of 14 and 35.

Example 7: Find the LCM of 96 and 360.

We have,

$$96 = 2 \times 2 \times 2 \times 2 \times 2 \times 3$$
$$360 = 2 \times 2 \times 2 \times 3 \times 3 \times 5.$$

Every common multiple should contain the prime factors of both 96 and 360. For the LCM, we will need the smallest such number. Let us build the LCM looking at each prime factor.

The prime factors appearing in both the numbers are 2, 3 and 5.

Now, we shall find out how many occurrences there are for each prime factor.

How many 2s should the LCM contain?

The factorisation of 96 contains $2 \times 2 \times 2 \times 2 \times 2$ (five occurrences of 2s) and the factorisation of 360 contains $2 \times 2 \times 2$ (three occurrences of 2s). Choosing five occurrences of 2s as part of LCM will contain both these subparts.

Choosing more than five occurrences of 2s will give a common multiple; but it will not be the lowest. Are you able to see why?

? How many 3s should the LCM contain?

The factorisation of 96 contains 3 (one occurrence of 3) and the factorisation of 360 contains 3 × 3 (two occurrences of 3s). Choosing two occurrences of 3s as part of LCM will contain both these subparts.

? How many 5s should the LCM contain?

The factorisation of 96 doesn't have any 5s and the factorisation of 360 has one occurrence of 5. So, we choose one occurrence of 5 to be a part of the LCM.

Thus, the LCM of 96 and 360 will be 2 × 2 × 2 × 2 × 2 × 3 × 3 × 5 = 1440.

To build the LCM of two numbers, we can identify all the prime factors and find the maximum number of times each of them occur in either of the factorisations. This process can be extended to find the LCM of two or more numbers.

? Figure it Out

Find the LCM of the following numbers:

(a) 30, 72 (b) 36, 54

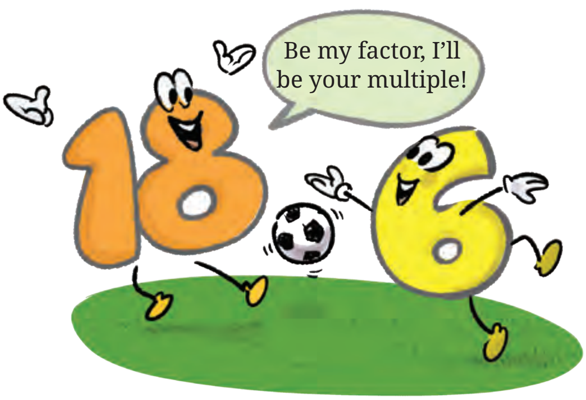
(c) 105, 195, 65 (d) 222, 370

3.3 Patterns, Properties, and a Pretty Procedure!

The HCF of 6 and 18 is 6, which is one of the two given numbers.

? Find more such number pairs where the HCF is one of the two numbers. How can we describe such pairs of numbers?

We can see that it happens when one number is a factor of the other. This also means the other number will be a multiple of the first number! Such a statement describing a pattern or a property that holds in all possible cases is called a **general statement.** This process is called **generalisation.**



Such a generalisation can also be described using algebra. Let us see how.

If n is a number, then any multiple of n can be written as a positive integer multiplied by n. For example, if we take n and $5n$ (short for $5 \times n$), then $5n$ is a multiple of n, and n is a factor of $5n$.

The HCF of n and $5n = n$.

? For number pairs satisfying this property (i.e., one of the numbers is the HCF),

(a) if m is a number, what could be the other number?

(b) if $7k$ is a number, what could be the other number?

? Figure it Out

1. Make a general statement about the HCF for the following pairs of numbers. You could consider examples before coming up with general statements. Look for possible explanations of why they hold.

Math Talk

   (a) Two consecutive even numbers

   (b) Two consecutive odd numbers

   (c) Two even numbers

   (d) Two consecutive numbers

   (e) Two co-prime numbers

Share your observations with the class.

2. The LCM of 3 and 24 is 24 (it is one of the two given numbers).

   (a) Find more such number pairs where the LCM is one of the two numbers.

   (b) Make a general statement about such numbers. Describe such number pairs using algebra.

3. Make a general statement about the LCM for the following pairs of numbers. You could consider examples before coming up with these general statements. Look for possible explanations of why they hold.

   (a) Two multiples of 3

   (b) Two consecutive even numbers

   (c) Two consecutive numbers

   (d) Two co-prime numbers

? What happens to the HCF of two numbers if both numbers are doubled? Take some pairs of numbers and explore. Are you able to see why the HCF will also double?

If both numbers are doubled, then both numbers get an extra factor of 2 in their prime factors. This 2 will be included as a factor in the largest common subpart, and so the HCF will double. For example, consider the numbers 270 and 50.

$$270 = \boxed{2} \times 3 \times 3 \times 3 \times \boxed{5}$$
$$50 = \boxed{2} \times 5 \times \boxed{5}$$

HCF = 2 × 5 = 10

Let us double these numbers to get 540 and 100.

$$540 = \boxed{2} \times \boxed{2} \times 3 \times 3 \times 3 \times \boxed{5}$$
$$100 = \boxed{2} \times \boxed{2} \times 5 \times \boxed{5}$$

HCF = 2 × 2 × 5 =20.

? Consider the following two multiples of 14— 14 × 6, 14 × 9. What is their HCF?

Clearly, 14 is a common factor. Is it also the highest common factor? To see it, let us calculate the prime factorisations.

$$14 \times 6 = \boxed{\underbrace{2 \times 7}_{14}} \times 2 \times \boxed{3}$$
$$14 \times 9 = \boxed{\underbrace{2 \times 7}_{14}} \times 3 \times \boxed{3}$$

HCF = 14 × 3 = 42.

? Here are some more numbers where both numbers are multiples of the same number. Find their HCF:

(a) 18 × 10, 18 × 15
(b) 10 × 38, 10 × 21
(c) 5 × 13, 5 × 20
(d) 12 × 16, 12 × 20

? In which of these cases is the HCF the same as the common multiplier, like problem (b) where the HCF is 10? Explore a few more examples of this type to understand when this happens.

Efficient Procedures for HCF and LCM

See the procedure on the right. Can you explain how it has been carried out?

$$\begin{array}{r|l} 2 & 84,\ 180 \\ \hline 2 & 42,\ 90 \\ \hline 3 & 21,\ 45 \\ \hline & 7,\ 15 \end{array}$$

This is similar to the procedure for prime factorisation. At each step, the two numbers are divided by a common prime factor, and the two quotients are written down in the next row. This continues till we get two numbers that do not have any common prime factors.

? How do we use this to find the HCF of 84 and 180? Explore.

[**Hint:** Observe that 84 = 2 × 2 × 3 × 7, and 180 = 2 × 2 × 3 × 15 similar to prime factorisation]

? Find the HCF in the following cases.

$$\begin{array}{r|l} 2 & 300, 150 \\ \hline 5 & 150, 75 \\ \hline 5 & 30, 15 \\ \hline 3 & 6, 3 \\ \hline & 2, 1 \end{array} \qquad \begin{array}{r|l} 2 & 630, 770 \\ \hline 5 & 315, 385 \\ \hline 7 & 63, 77 \\ \hline & 9, 11 \end{array}$$

HCF = 2 × 5 × 5 × 3 HCF = 2 × 5 × 7

This procedure not only gives the HCF but can also be used to find the LCM! Can you see how?

$$\begin{array}{r|l} 2 & 300, 150 \\ \hline 5 & 150, 75 \\ \hline 5 & 30, 15 \\ \hline 3 & 6, 3 \\ \hline & 2, 1 \end{array} \qquad \begin{array}{r|l} 2 & 630, 770 \\ \hline 5 & 315, 385 \\ \hline 7 & 63, 77 \\ \hline 3 & 9, 11 \\ \hline & 3, 11 \end{array}$$

LCM = 2 × 5 × 5 × 3 × 2 × 1 LCM = 2 × 5 × 7 × 3 × 3 × 11

? Why are these the LCMs?

[**Hint:** Will the product of the factors marked as the LCM of 300 and 150 contain the prime factorisations of both 300 and 150? Is this the smallest such number?]

Guna says "I found a better way to factorise to find HCF/LCM. This is faster than what was taught in class!"

"For the numbers 300 and 150, I can first directly divide both numbers by 50.

The HCF will be 50 × 3.

$$\begin{array}{r|l} 50 & 300, 150 \\ \hline 3 & 6, \quad 3 \\ \hline & 2, \quad 1 \end{array}$$

The LCM will be 50 × 3 × 2 × 1”.

Anshu also tried to remove the bigger common factors.

“For 630 and 770, I will divide both numbers by 10 first. Now, I can divide them by 7.

The HCF will be 10 × 7 = 70

The LCM will be 10 × 7 × 9 × 11 = 6930”.

$$\begin{array}{r|l} 10 & 630,\ 770 \\ \hline 7 & 63,\ 77 \\ \hline & 9,\ 11 \end{array}$$

Can you see why this works?

We need not restrict ourselves to dividing only one prime factor at a time. Both numbers can be divided by whatever common factor we are able to identify.

? You can try this method for these pairs of numbers.

(a) 90 and 150 (b) 84 and 132

Property Involving both the HCF and the LCM

? Which is greater — the LCM of two numbers or their product?

? You could analyse the above statement using examples. Then try to reason or prove, why the LCM is never greater than the product of the numbers. [**Hint**: Is the product also a common multiple of the two numbers?]

There is an interesting relation between the product of two numbers and their HCF and LCM.

? Consider the numbers 105 and 95. Find their LCM.

Factorising them into their primes:

$$105 = 3 \times 5 \times 7$$

$$95 = 5 \times 19$$

$$\text{LCM} = 3 \times 5 \times 7 \times 19.$$

Let us consider the product in the factorised form:

$$105 \times 95 = 3 \times 5 \times 5 \times 7 \times 19$$

Is the LCM a factor of the product? If yes, what should it be multiplied with to get the product? It can be seen that

$$105 \times 95 = \text{LCM} \times 5.$$

? Explore whether the LCM is a factor of the product in the following cases. If yes, identify the number that the LCM should be multiplied by to get the product. Do you see any pattern? Use these numbers:

(a) 45, 105 (b) 275, 352 (c) 222, 370

? Do you see that, in each case, the number by which the LCM is multiplied to get the product is actually the HCF?

Thus, our observations seem to suggest the following:

HCF × LCM = Product of the two numbers.

? Why does this happen? Can you give an explanation or proof?

Try This

[Hint: Consider the prime factorisation of the given numbers. Among their prime factors, some are common to both factorisations, and the rest occur in only one of them. Between the HCF and the LCM, see how the common and non-common prime factors get distributed. In the product, observe how these two kinds of prime factors occur. Compare them.]

? Explore whether this property holds when 3 numbers are considered.

Try This

? Figure it Out

1. In the two rows below, colours repeat as shown. When will the blue stars meet next?

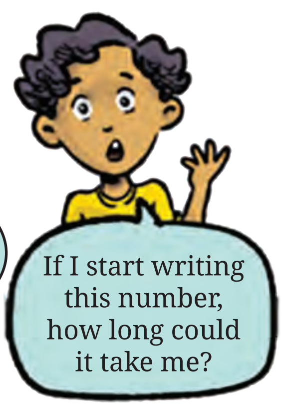
2. (a) Is 5 × 7 × 11 × 11 a multiple of 5 × 7 × 7 × 11 × 2?
   (b) Is 5 × 7 × 11 × 11 a factor of 5 × 7 × 7 × 11 × 2?
3. Find the HCF and LCM of the following (state your answers in the form of prime factorisations):
   (a) 3 × 3 × 5 × 7 × 7 and 12 × 7 × 11
   (b) 45 and 36
4. Find two numbers whose HCF is 1 and LCM is 66.
5. A cowherd took all his cows to graze in the fields. The cows came to a crossing with 3 gates. An equal number of cows passed through each gate. Later at another crossing with 5 gates again an equal number of cows passed through each gate. The same happened at the third crossing with 7 gates. If the cowherd had less than 200 cows, how many cows did he have? (Based on the folklore mathematics from Karnataka.)

6. The length, width, and height of a box are 12 cm, 18 cm, and 36 cm respectively. Which of the following sized cubes can be packed in this box without leaving gaps?
   (a) 9 cm (b) 6 cm
   (c) 4 cm (d) 3 cm
   (e) 2 cm
7. Among the numbers below, which is the largest number that perfectly divides both 306 and 36?
   (a) 36 (b) 612
   (c) 18 (d) 3
   (e) 2 (f) 360
8. Find the smallest number that is divisible by 3, 4, 5 and 7, but leaves a remainder of 10 when divided by 11. Try This
9. Children are playing 'Fire in the Mountain'. When the number 6 was called out, no one got out. When the number 9 was called out, no one got out. But when the number 10 was called out, some people got out. How many children could have been playing initially?
   (a) 72 (b) 90
   (c) 45 (d) 3
   (e) 36 (f) None of these
10. Tick the correct statement(s). The LCM of two different prime numbers (m, n) can be:
    (a) Less than both numbers
    (b) In between the two numbers
    (c) Greater than both numbers
    (d) Less than $m \times n$
    (e) Greater than $m \times n$
11. A dog is chasing a rabbit that has a head start of 150 feet. It jumps 9 feet every time the rabbit jumps 7 feet. In how many leaps does the dog catch up with the rabbit? Try This
12. What is the smallest number that is a multiple of 1, 2, 3, 4, 5, 6, 8, 9, 10? Do you remember the answer from Grade 6, Chapter 5?
13. Here is a problem posed by the ancient Indian Mathematician Mahaviracharya (850 C.E.). Add together $\frac{8}{15}$, $\frac{1}{20}$, $\frac{7}{36}$, $\frac{11}{63}$ and $\frac{1}{21}$. What do you get? How can we find this sum efficiently? Math Talk

SUMMARY

- Last year, we looked at common multiples, and common factors, and were also introduced to the amazing world of primes!
- In this chapter, we learnt a method to find the prime factorisation of a number.
- Finding all the factors of a number from its prime factorisation is easy but quite tedious — we have to list every possible subpart!
- The **Highest Common Factor (HCF)** is the highest among all the common factors of a group of numbers.
  - Every common factor is contained in the prime factorisation of the number.
  - To find the HCF, we include the minimum number of occurances of each prime across the prime factorisation of all the numbers.
- The **Lowest Common Multiple (LCM)** is the lowest among all the common multiples of a group of numbers.
  - Every common multiple contains the prime factorisation of the numbers.
  - To find the LCM, we include the highest number of occurrences of each prime across the prime factorisations of all the numbers.
- We explored more about HCF and LCM; we discovered related properties and patterns when numbers are consecutive, even, co-prime, etc.
- We learnt a procedure to get both the HCF and the LCM at the same time! We also saw how to make this even quicker!
- We learned some terms that are used when discussing mathematics, such as ‘**conjecture**’ and ‘**generalisation**’.

The largest prime found so far has 4,10,24,320 digits! It was discovered on October 12, 2024.

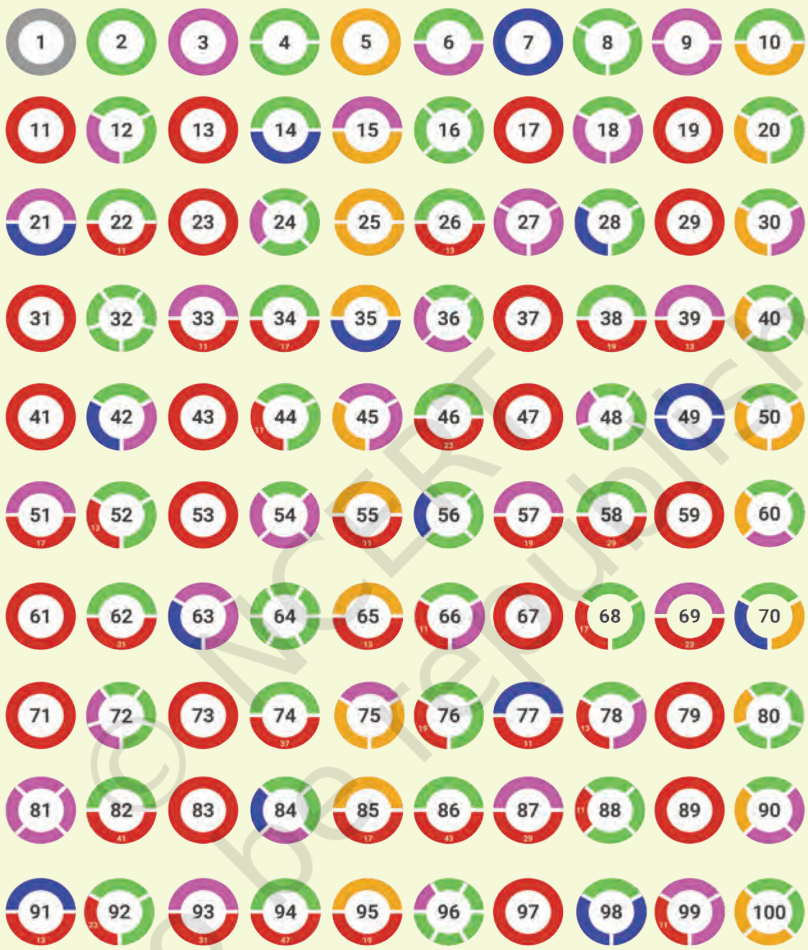


IT'S PUZZLE TIME!

Mystery Colours!

You might have noticed and wondered about these different circle designs around the page numbers on each page!

The picture below shows all the designs for the numbers from 1 to 100.



Try to decode the colour scheme for each number.
There are several interesting patterns here.
Share your observations with your classmates.
Extending this scheme, colour the page numbers from 101 – 110.

4 ANOTHER PEEK BEYOND THE POINT

0789CH04

4.1 A Quick Recap of Decimals

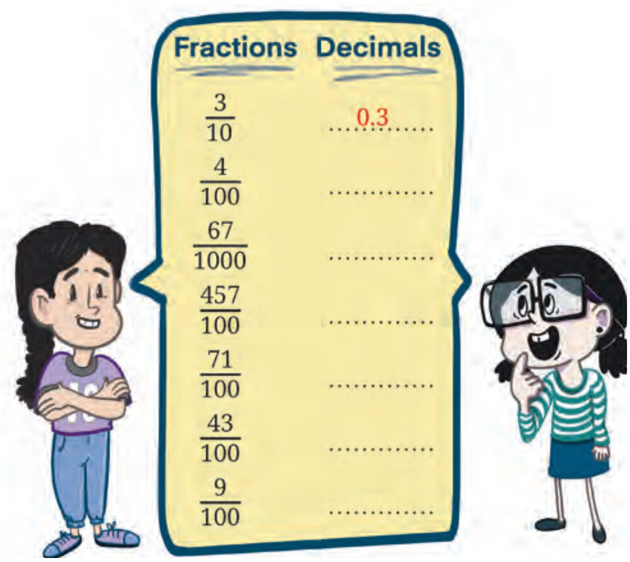
Recall that decimals are the natural extension of the Indian place value system to represent decimal fractions ($\frac{1}{10}$, $\frac{1}{100}$, $\frac{1}{1000}$, and so on) and their sums.

For example, **27.53** refers to a quantity that has:

- 2 Tens
- 7 Units (Ones)
- 5 Tenths
- 3 Hundredths

We have already learned how to multiply and divide fractions. In this chapter, we will learn how to perform these operations with decimals. You will see that the procedures for multiplying and dividing decimals are natural extensions of the procedures for multiplying and dividing counting numbers.

? Jonali and Pallabi play a game. Jonali says a fraction and Pallabi gives the equivalent decimal. Write Pallabi's answer in the blank spaces.



Jonali goes to the market to buy spices. She purchases 50 g of Cinnamon, 100 g of Cumin seeds, 25 g of Cardamom and 250 g of Pepper. Express each of the quantities in kilograms by writing them in terms of fractions as well as decimals.

The fractions Jonali gave Pallabi have denominators 10, 100, 1000, and so on.

Write the following fractions as a sum of fractions and also as decimals:

Fraction	Expanding the Numerator	Sum of one-tenths, one-hundredths, one-thousands,...	Decimals
$\frac{254}{1000}$	$\frac{200}{1000}+\frac{50}{1000}+\frac{4}{1000}$ $=\frac{2}{10}+\frac{5}{100}+\frac{4}{1000}$	0.2 + 0.05 + 0.004	0.254
$\frac{847}{10000}$			
$\frac{173}{100}$			
$\frac{23}{1000}$			

? Can you give a simple rule to divide any number by a number of the form 1 followed by zeroes — 10, 100, 1000, etc.? For example, $\frac{123}{10}$, $\frac{24}{100}$ or $\frac{678}{1000}$? Look for a pattern in the previous problems.

Math Talk

Here is one such rule. Let us consider the example 123 ÷ 10.

Step 1: Write the dividend as it is and place a decimal point at the end.

123.

Step 2: Count the number of zeroes in the divisor.

10 ⟶ 1 zero

Step 3: Move the decimal point from Step 1 left by the same number of places as the count from Step 2. Add zeroes in front if needed.

12.3

Examples:

24 ÷ 100 = 0.24	678 ÷ 1000 = 0.678
12÷1000 = 0.012	12345÷1000 = 12.345

4.2 Decimal Multiplication

Example 1: Arshad goes to a stationery shop and purchases 5 pens. If one pen costs ₹9.5 (9 rupees and 50 paisa), how much should he pay the shopkeeper?

What operation must we use here?

We have to multiply 9.5 by 5, which is the same as adding 9.5, 5 times.

That is 9.5 × 5 = 9.5 + 9.5 + 9.5 + 9.5 + 9.5 = 47.5.

We can also directly multiply the numbers by converting them into fractions.

9.5 is $\frac{95}{10}$ and 5 is $\frac{5}{1}$ as a fraction.

The cost of 5 pens = $\frac{5}{1} \times \frac{95}{10}$.

Recall that, to find the product of two fractions, we multiply the numerators and multiply the denominators.

$$\frac{5}{1} \times \frac{95}{10} = \frac{5 \times 95}{1 \times 10} = \frac{475}{10} = 47.5.$$

The cost of 5 pens is ₹47.5.

Example 2: A car travels 12.5 km per litre of petrol. What is the distance covered with 7.5 litres of petrol?

We have to multiply 12.5 by 7.5.

The distance covered = 12.5 × 7.5

$$= \frac{125}{10} \times \frac{75}{10} = \frac{125 \times 75}{10 \times 10} = \frac{9375}{100} = 93.75$$

The distance covered is 93.75 km.

Can the product of two decimals be a natural number?

Can the product of a decimal and a natural number be a natural number?

Example 3: The distance between Ajay's school and his home is 827 m. He walks to school in the morning and then walks back home in the evening, 6 days a week. How much does he walk in a week? Answer in kilometres.

Each way between school and home, Ajay walks 827 metres, i.e., 0.827 km.

So, in a day he walks,

$$0.827 \times 2 = \frac{827}{1000} \times 2 = \frac{827 \times 2}{1000} = \frac{1654}{1000} = 1.654$$

He goes to school 6 days a week. So, in a week, he walks

$$1.654 \times 6 = \frac{1654}{1000} \times 6 = \frac{1654 \times 6}{1000} = \frac{9924}{1000} = 9.924$$

Ajay walks 9.924 km a week.

Example 4: Find the area of the given rectangle.

Area of the rectangle =

$$5.7 \times 13.3 = \frac{57}{10} \times \frac{133}{10} = \frac{7581}{100} = 75.81$$

The area is 75.81 sq cm.

Observe the number of digits after the decimal point in the multiplier, the multiplicand and the product. Also note the number of zeroes in the denominator.

Examples	**No. of digits after the decimal point in:**		
	Multiplier	**Multiplicand**	**Product**
$9.5 \times 5 = \frac{95 \times 5}{10} = \frac{475}{10} = 47.5$	**1**	**0**	**1**
$12.5 \times 7.5 = \frac{125}{10} \times \frac{75}{10} = \frac{9375}{100} = 93.75$	**1**	**1**	**2**

1.64×6 $= \frac{164}{100} \times 6 = \frac{984}{100} = 9.84$	**2**	**0**	**2**
5.7×13.35 $= \frac{57}{10} \times \frac{1335}{100}$ $= \frac{57 \times 1335}{10 \times 100} = \frac{76095}{1000}$ $=76.095$	**1**	**2**	**3**

? Suppose we know that 596 × 248 = 147808, can you immediately write down the product of 5.96 × 24.8?

Math Talk

? By looking at the above examples, can you frame a rule to multiply two decimals?

Multiplication of decimals is the same as the multiplication of their corresponding fractions. When multiplying fractions, we multiply the numerators and denominators, respectively.

The product of the numerators = Product of the numbers with the decimal points removed.

Since both the denominators are of the form 1, 10, 100, 1000, ... , the product of the denominators is also of the form 1 followed by zeroes. The number of zeroes in the product is the sum of the number of zeroes in each denominator.

In the product, the decimal point is placed based on the total number of zeroes in the denominator.

So, to multiply two decimals, we can multiply the two numbers obtained by removing the decimal point, and then place the decimal point appropriately as shown below.

To evaluate 5.96 × 24.8:

$$596 \times 248 = 147808$$

$$\underbrace{5.\underbrace{96}_{\text{2 decimal places}} \times 24.\underbrace{8}_{\text{1 decimal place}}}_{2 + 1 = 3 \text{ decimal places}} = 147.\underbrace{808}_{\text{3 decimal places}}$$

Example 5: Let us use the above rule to find the product of 5.8 and 1.24.

Let us first multiply 58 and 124.

The product is 7192.

The sum of the number of digits after the decimal point in the multiplier and multiplicand is 3. So, the product is 7.192.

Verify this by converting the multiplier and multiplicand into fractions.

1 digit after decimal point

2 digits after decimal point

5.8 × 1.24

Is the Product Always Greater than the Numbers Multiplied?

Recall multiplication of two fractions. Unlike counting numbers, when two fractions are multiplied the product is not always greater than or equal to both numbers. Let us examine the case of the product of two decimals.

2.25 × 8 = 18.

In the above multiplication, the product (18) is greater than 2.25 and 8.

But in, 0.25 × 8 = 2, the product (2) is greater than 0.25 but less than 8.

In the case of 0.25 × 0.8 = 0.2, the product (0.2) is less than both 0.25 and 0.8.

When is the product of two decimals greater than both the numbers? When is it less than both the numbers?

Since decimals are just representations of fractions, the relationship between the numbers multiplied and the product are similar to fractions.

Situation	Multiplication	Relationship
Situation 1	Both numbers are greater than 1 (e.g., 3.4 × 6.5)	The product (22.1) is greater than both the numbers.
Situation 2	Both numbers are between 0 and 1 (e.g., 0.75 × 0.4)	The product (0.3) is less than both the numbers.
Situation 3	One number is between 0 and 1 and one number is greater than 1 (e.g., 0.75 × 5)	The product (3.75) is less than the number greater than 1 and greater than the number between 0 and 1.

Figure it Out

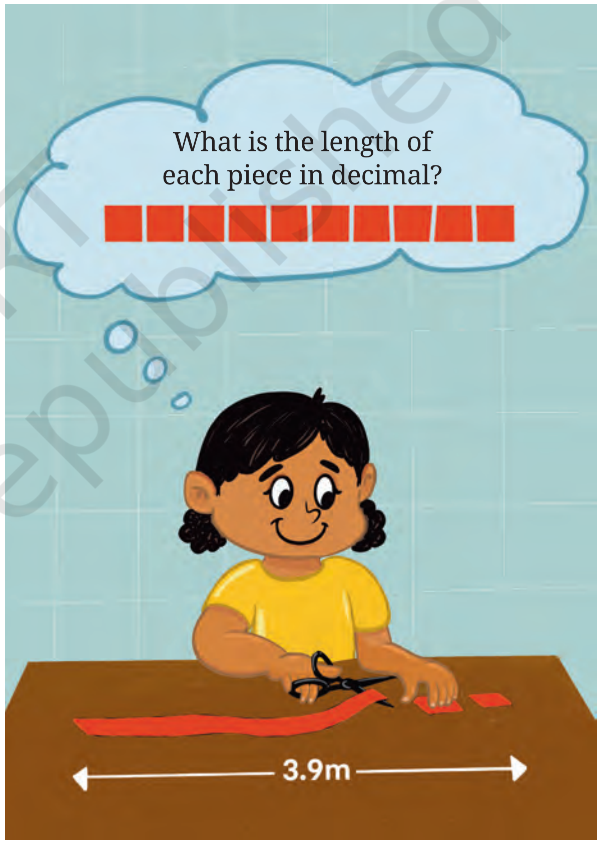
1. Recall that a tenth is 0.1, a hundredth is 0.01, and so on. Find the following products in tenths, hundredths and so on:
   (a) 6×4 tenths = 24 tenths
   (b) 7×0.3
   (c) 9×5 hundredths
2. Find the products:
   (a) 27.34×6
   (b) 4.23×3.7
   (c) 0.432×0.23
3. Thejus needs 1.65 m of cloth for a shirt. How many metres of cloth are needed for 3 shirts?
4. Meenu bought 4 notebooks and 3 erasers. The cost of each book was ₹15.50 and each eraser was ₹2.75. How much did she spend in all?
5. The thickness of a rupee coin is 1.45 mm. What is the total height of the cylinder formed by placing 36 rupee coins one over the other? Write the answer in centimeters.
6. The price of 1 kg of oranges is ₹56.50. What is the price of 2.250 kg of oranges? Can we write 56.50 as 56.5 and 2.250 as 2.25 and multiply? Will we get the same product? Why?

   Math Talk
7. Dwarakanath purchases notebooks at a wholesale price of ₹23.6 per piece and sells each notebook at ₹30/-. How much profit does he make if he sells 50 books in a week?
8. Given that $18 \times 12 = 216$, find the products:
   (a) 18×1.2 (b) 18×0.12
   (c) 1.8×1.2 (d) 0.18×0.12
   (e) 0.018×0.012 (f) 1.8×12

   In which of the cases above is the product less than 1?
9. In which of the following multiplications is the product less than 1? Can you find the answer without actually doing the multiplications?
   (a) 7×0.6 (b) 0.7×0.6
   (c) 0.7×6 (d) 0.07×0.06
10. Multiplying the following numbers by 10, 100 and 1000 to complete the table.

	× 10	× 100	× 1000
5.7			
23.02			
0.92			
0.306			
24.67			

4.3 Decimal Division



Example 6: Anuja has a 3.9 m length of ribbon and she wants to cut it into 10 equal pieces. What is the length of each piece in decimal?

Since there are 10 pieces, we can find the length of each piece by dividing 3.9 by 10.

So, what is 3.9 ÷ 10?

Let's convert 3.9 into fraction $3.9 = \frac{39}{10}$.

Hence, 3.9 ÷ 10 is the same as dividing $\frac{39}{10}$ by 10.

Recall that, dividing a fraction by a divisor is the same as multiplying the fraction by the reciprocal of the divisor. The reciprocal of 10 is $\frac{1}{10}$.

So, $\frac{39}{10} \div 10 = \frac{39}{10} \times \frac{1}{10} = \frac{39}{100} = 0.39$.

Thus, the length of each piece of ribbon is 0.39 m.

What is the length of each piece if the ribbon is cut into 100 equal pieces?

$3.9 \div 100 = \frac{39}{10} \times \frac{1}{100} = \frac{39}{1000} = 0.039$.

When the ribbon is cut into 100 equal pieces, the length of each piece is 0.039 m.

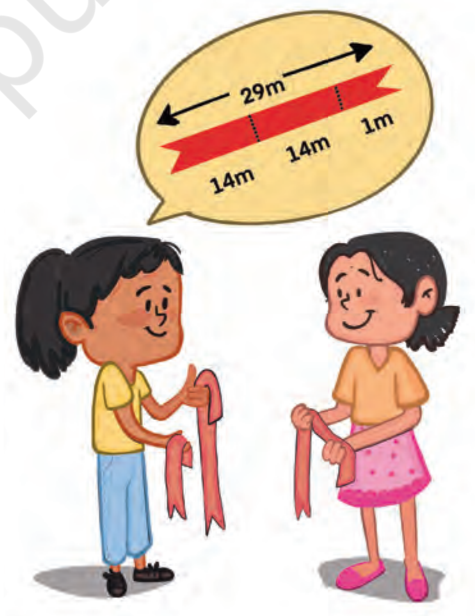
? What is 0.039 m in centimetres and millimetres?

By looking at these divisions, we can frame a simple rule for dividing decimals by 1, 10, 100, 1000, and so on.

When we divide a decimal by 1, 10, 100, 1000, and so on, we can just move the decimal point to the left by as many places as there are zeroes in the divisor!

Decimal	÷ 10	÷ 100	÷ 1000	÷ 10000
18.7	1.87	0.187	0.0187	0.00187
21.1				
0.13				
		2.146		
				0.0058

? **Example 7:** Neenu has 29 metres of red ribbon and wants to share it equally with Anu. What is the length of ribbon that each of them will get?



Since the ribbon needs to be divided into two equal parts, each girl will get a piece of 29 ÷ 2 metres.

If each of them gets 14 m, then 1 m remains. If we divide 1 m among the two, each will get another $\frac{1}{2}$ m.

? How do we convert $\frac{1}{2}$ into a decimal?

It is easy to express a fraction as a decimal if the denominator is 1, 10, 100, 1000, etc. So, can we find a fraction equivalent to $\frac{1}{2}$ with such a denominator?

$\frac{1}{2} = \frac{1 \times 5}{2 \times 5} = \frac{5}{10}$ (multiplying the numerator and denominator by 5).

We know that the fraction $\frac{5}{10}$ can be represented as a decimal 0.5.

So, each girl will get 14 m and an additional 0.5 m of ribbon.

Hence, the length of ribbon each will get is 14 + 0.5 = 14.5 m.

? Now, what if the ribbon was shared between four friends instead of 2?

So, each will get 29 ÷ 4 m, that is $\frac{29}{4}$ m.

Now, the denominator of the fraction is 4. To convert a fraction to a decimal, it helps if the denominator is of the form 1, 10, 100, 1000, and so on. Can we find a fraction equivalent to $\frac{29}{4}$ with such a denominator?

Is 4 a factor of 10? No. Is it a factor of 100? Yes. 4 × 25 = 100. So we can get an equivalent fraction of $\frac{29}{4}$ by multiplying the numerator and denominator by 25.

$$\frac{29 \times 25}{4 \times 25} = \frac{725}{100} = 7.25$$

So each of the 4 friends will get 7.25 m of ribbon.

Division Using Place Value

We have seen how to divide two counting numbers to get a decimal quotient. We first represented the division as a fraction. Then we found an equivalent fraction, with the denominator being of the form 1, 10, 100, 1000, and so on. It was then easy to represent this equivalent fraction as a decimal.

Now, let us look at the division using place value procedure to calculate the decimal quotient.

? Suppose we want to write the quotient $\frac{10}{3}$ as a decimal. Can we convert this fraction to an equivalent fraction with a denominator such as 1, 10, 100, 1000, etc.?

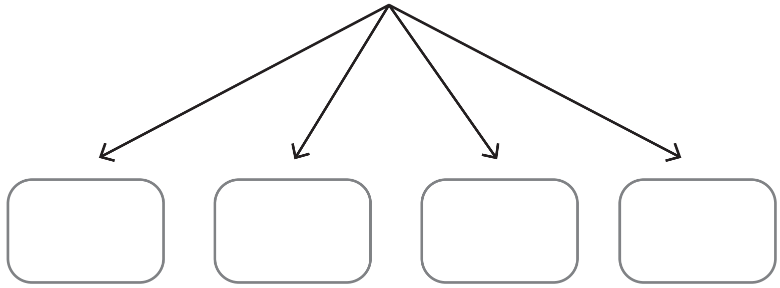
It is not possible. So, we need a more general method to divide any two counting numbers. Let us see how we can use division using place value for this.

Let us start with a quick recap of division using place value.

Example 8: Find the value of 1324 ÷ 4.

1324 ÷ 4 ⟶ Divide 1324 into 4 equal parts.

1 Thousand + 3 Hundreds + 2 Tens + 4 Ones

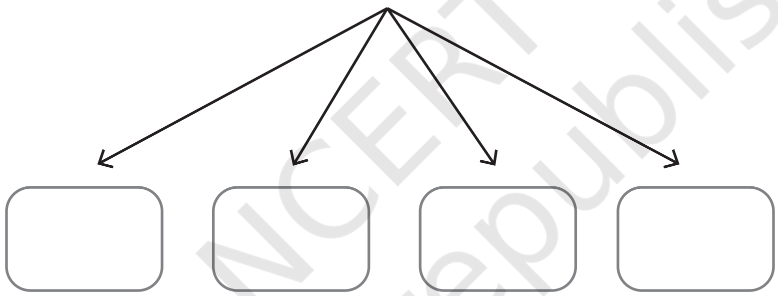


1 Thousand ÷ 4 ⟶ Not possible without regrouping.

Regroup 1 Thousand into 10 Hundreds.

10 Hundreds + 3 Hundreds = 13 Hundreds.

13 Hundreds + 2 Tens + 4 Ones



13 Hundreds ÷ 4 ⟶ Each part gets 3 Hundreds, and 1 Hundred remains.

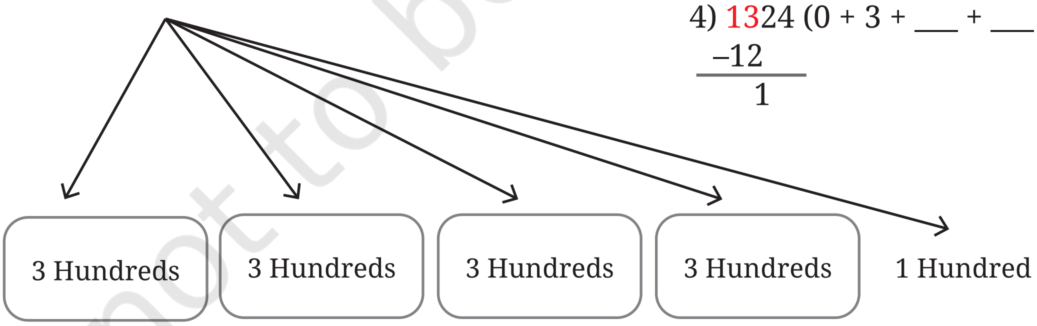
13 Hundreds + 2 Tens + 4 Ones

Th H T O

4) 1324 (0 + 3 + ___ + ___

–12

1

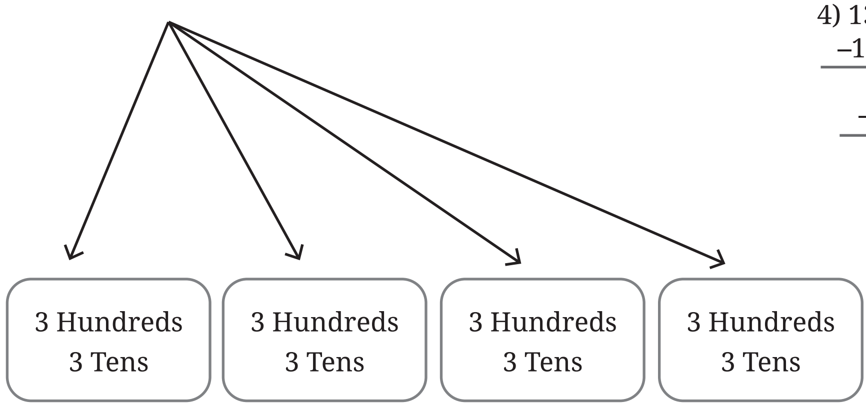


Regroup 1 Hundred into 10 Tens.

10 Tens + 2 Tens = 12 Tens.

12 Tens ÷ 4 ⟶ Each part gets 3 Tens.

12 Tens + 4 Ones

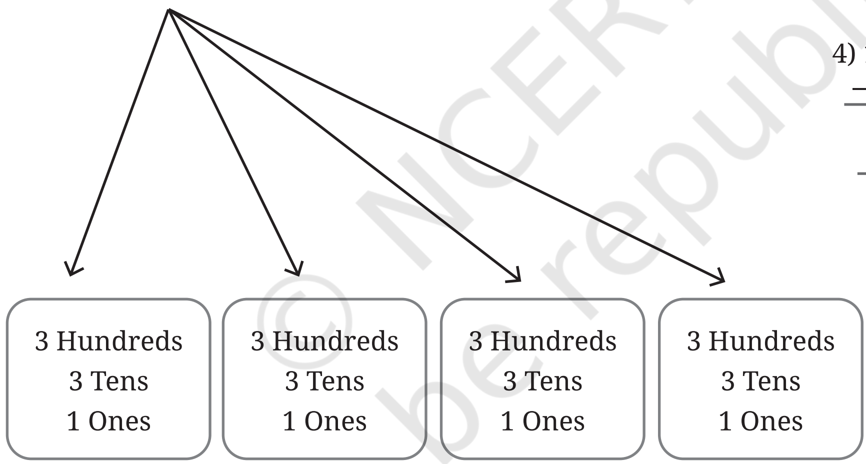


Th H T O

```
4) 1324 (0 + 3 + 3 + ___
  –12
    12
   –12
     0
```

4 Ones ÷ 4 ⟶ Each part gets 1 Ones.

4 Ones



Th H T O

```
4) 1324 (0 + 3 + 3 + 1
  –12
    12
   –12
     04
     –4
      0
```

So, 1324 ÷ 4 = 0 Thousands + 3 Hundreds + 3 Tens + 1 Ones = 331.

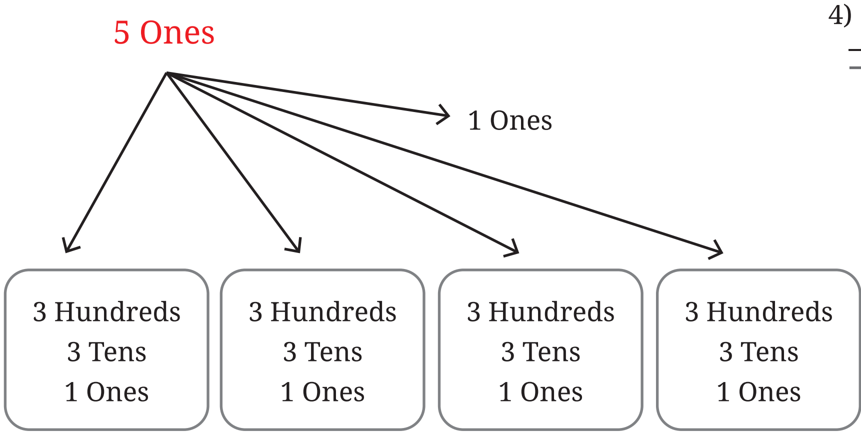
We also call this division using place values as '**long division**'.

Division with a Decimal Quotient

Now, let us use this understanding of long division to find the value of 1325 ÷ 4.

1325 ÷ 4 ⟶ Divide 1325 into 4 equal parts.

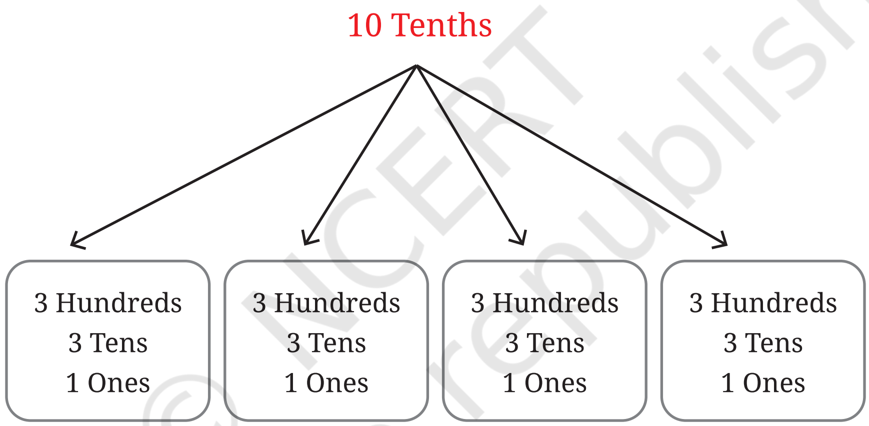
We can follow the same steps as in the previous problem.



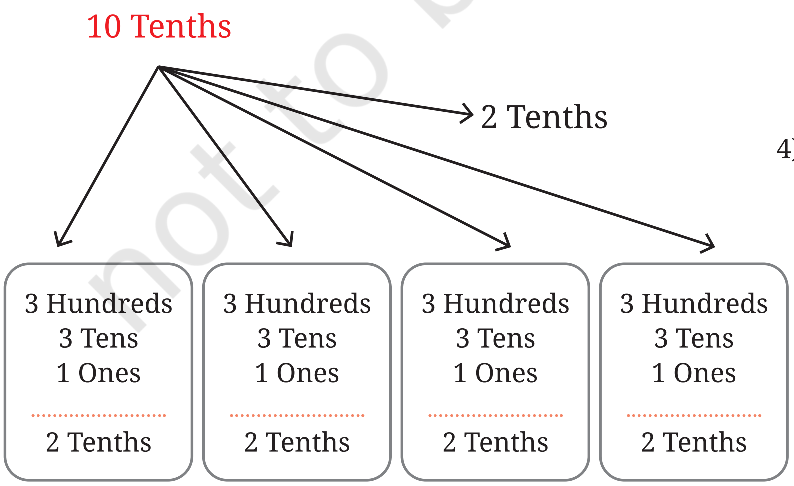
Th H T O
4) 1325 (0 + 3 + 3 + 1
–12
12
–12
05
–4
1

We are left with 1 Ones.

It is not clear how to divide 1 Ones into 4 equal parts. But we can regroup this as 10 tenths.



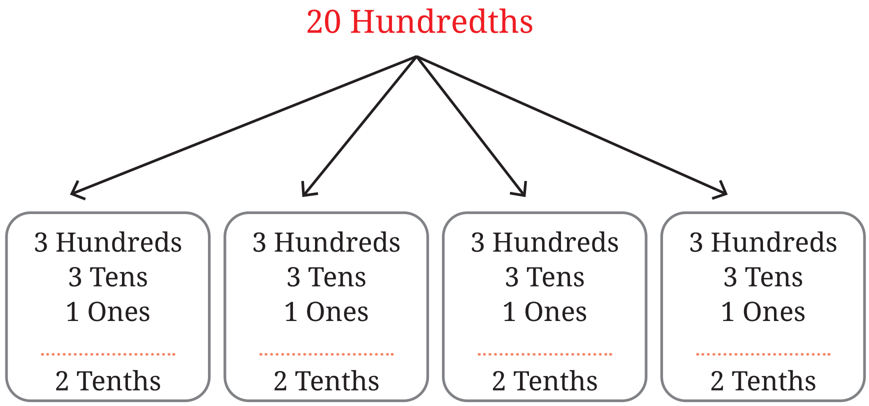
10 Tenths ÷ 4 ⟶ Each part gets 2 Tenths and 2 Tenths remain.



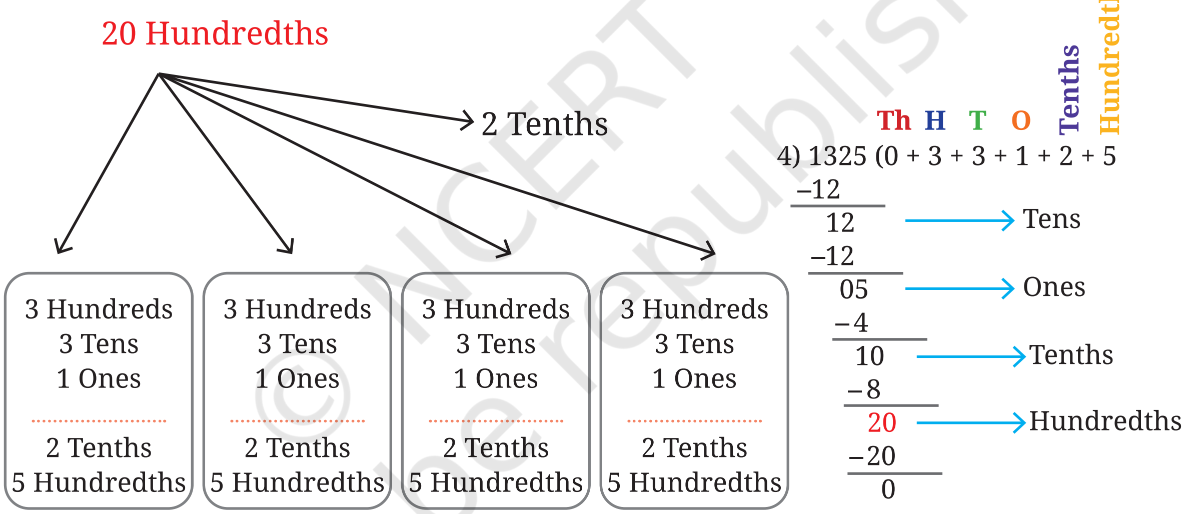
Th H T O Tenths
4) 1325 (0 + 3 + 3 + 1 + 2
–12
12 ⟶ Tens
–12
05 ⟶ Ones
–4
10 ⟶ Tenths
–8
2

We are left with 2 tenths.

To divide 2 Tenths into 4 equal parts, we have to regroup them as 20 Hundredths.



20 Hundredths ÷ 4 ⟶ Each part gets 5 Hundredths.



So, 1325 ÷ 4 = 0 Thousands + 3 Hundreds + 3 Tens + 1 Ones + 2 Tenths + 5 Hundredths. This we know is 331.25, thus,

$$1325 \div 4 = 331.25.$$

Can we verify this by finding an equivalent fraction for $\frac{1325}{4}$?

To get an equivalent fraction such that the denominator is of the form 1, 10, 100, 1000, and so on, we can multiply the numerator and denominator by 25.

$$\frac{1325 \times 25}{4 \times 25} = \frac{33125}{100} = 331.25.$$

Thus, the procedure of division using place value can be extended to find quotients with decimal values. Ones can be regrouped as tenths, tenths can be regrouped as hundredths and so on.

Example 9 : Find the value of 237 ÷ 8.

237 ÷ 8 ⟶ Divide 2 Hundreds + 3 Tens + 7 Ones into 8 equal parts.

To divide 2 Hundred into 8 equal parts we need to regrouped them as 20 Tens.

20 Tens + 3 Tens = 23 Tens.

23 Tens ÷ 8 ⟶ 2 Tens, and 7 Tens remain.

```
        H T O
8) 237 (0 2
  –16          ⟶ Tens
    7
```

7 Tens can be regrouped as 70 Ones.

70 Ones + 7 Ones = 77 Ones.

77 Ones ÷ 8 ⟶ 9 Ones, and 5 Ones remain.

```
        H T O
8) 237 (0 2 9
  –16          ⟶ Tens
    77
   –72
     5         ⟶ Ones
```

To divide 5 Ones into 8 equal parts we need to regroup them as 50 Tenths.

When we regroup Ones into Tenths, we place a decimal point in the quotient.

50 Tenths ÷ 8 ⟶ 6 Tenths, and 2 Tenths remain.

```
        H T O   Tenths
8) 237 (0 2 9 . 6
   16          ⟶ Tens
    77
    72         ⟶ Ones
     50        ⟶ Tenths
     48
      2
```

Remember, when we regroup Ones into Tenths we need to place a decimal point in the quotient.

2 Tenths cannot be divided into 8 equal parts. So we need to regroup them as 20 Hundredths.

20 Hundredths ÷ 8 ⟶ 2 Hundredths, and 4 Hundredths remain.

```
        H T O   Tenths Hundredths
8) 237 (0 2 9 . 6      2
   16          ⟶ Tens
    77
    72         ⟶ Ones
     50        ⟶ Tenths
     48
      20       ⟶ Hundredths
      16
       4
```

To divide 4 Hundredths into 8 equal parts we need to regroup them as 40 Thousandths.

40 Thousandths ÷ 8 ⟶ 5 Thousandths.

```
               H  T  O  Tenths  Hundredths  Thousandths
8) 237 (0      2  9  .  6       2           5
   16      ⟶ Tens
   77
   72      ⟶ Ones
    50     ⟶ Tenths
    48
     20    ⟶ Hundredths
     16
      40   ⟶ Thousandths
      40
       0
```

Thus, 237 ÷ 8 = 29.625.

Division with a Decimal Dividend

Example 10: A shopkeeper has 9.5 kg of sugar and he wants to pack it equally in 4 bags. What is the weight of each bag of sugar?

To find the weight of each bag we need to divide 9.5 by 4.

```
           O  Tenths  Hundredths  Thousandths
4) 9.5 (2  .  3       7           5
   8       ⟶ Ones
   15
   12      ⟶ Tenths
    30
    28     ⟶ Hundredths
     20
     20    ⟶ Thousandths
      0
```

Again, we place the decimal point in the quotient before we divide the tenths.Each bag of sugar weighs 2.375 kg.

Example 11: What is the value of 0.06 ÷ 5?

0.06 ⟶ 0 Ones + 0 Tenths + 6 Hundredths
0 Ones ÷ 5 ⟶ 0 Ones.

When we move from Ones to Tenths we need to place the decimal point in the quotient.
0 Tenths ÷ 5 ⟶ 0 Tenths.

6 Hundredths ÷ 5 ⟶ 1 Hundredths, and 1 Hundredth remains. We need to regroup 1 Hundredth into 10 Thousandths.

10 Thousandths ÷ 5 ⟶ 2 Thousandths.

So, the quotient is 0.012.

	O	Tenths	Hundredths	Thousandths	
5) 0.06 (	0 .	0	1	2	
0					⟶ Ones
00					⟶ Tenths
00					
06					⟶ Hundredths
5					
10					⟶ Thousandths
10					
0					

Figure it Out

1. Find the quotient by converting the denominator into 1, 10, 100 or 1000 and verify the solution by the long division method (division by place value).

   (a) $\frac{18}{5}$ (b) $\frac{415}{4}$

   (c) $\frac{1217}{2}$ (d) $\frac{4827}{8}$

2. Choose the correct answer:

   (a) $\frac{1526}{4} =$

   (i) 38.15 (ii) 380.15

   (iii) 381.5 (iv) 381.05

   (b) $\frac{3567}{8} =$

   (i) 4458.75 (ii) 44.5875

   (iii) 445.875 (iv) 4458.75

3. What is the quotient?

   (a) 132 ÷ 4 = (b) 13.2 ÷ 4 =

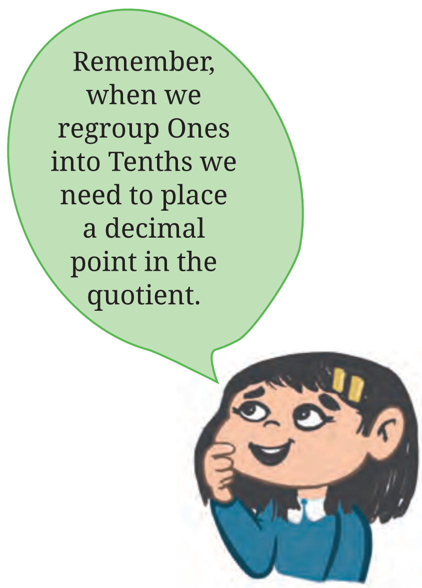
   (c) 1.32 ÷ 4 = (d) 0.132 ÷ 4 =

4. What is the quotient?

   (a) 126 ÷ 8 = (b) 12.6 ÷ 8 =

   (c) 1.26 ÷ 8 = (d) 0.126 ÷ 8 =

   (e) 0.0126 ÷ 8 =



Division with a Decimal Divisor

Example 12: Ravi went from Pune to Matheran by scooter in 2.5 hours. The distance was 126 km. What was his average speed?

We can get the average speed by dividing the distance by the time taken.

126 ÷ 2.5.

When the divisor is a decimal, we convert the divisor into a fraction.

$126 \div \frac{25}{10} = 126 \times \frac{10}{25} = \frac{1260}{25}$.

With the long division procedure we find the quotient is 50.4.

So, the average speed at which Ravi travelled was 50.4 km per hour.

Example 13: Find 4.68 ÷ 1.3.

Again converting the divisor into a fraction we get

$4.68 \div \frac{13}{10} = 4.68 \times \frac{10}{13} = \frac{46.8}{13}$.

Now, what about 4.68 ÷ 0.13?

$4.68 \div 0.13 = 4.68 \div \frac{13}{100} = 4.68 \times \frac{100}{13} = \frac{468}{13}$.

What do you notice in these cases?

When the divisor is a decimal, we can convert it into a counting number by suitably multiplying it by 10, 100, 1000, and so on. We must also multiply the dividend by the same number. Thus,

$$\frac{4.68}{0.13} = \frac{4.68 \times 100}{0.13 \times 100} = \frac{468}{13}.$$

Once we convert the divisor into a counting number, we can then follow the division using place value procedure to find the quotient.

Does This Ever End?

Can you calculate 10 ÷ 3? Try dividing using long division.

10 ⟶ 1 Tens + 0 Ones.

Step 1: Regroup 1 Tens into 10 Ones. 10 Ones ÷ 3 ⟶ 3 Ones, and 1 Ones remain.

Step 2: Regroup 1 Ones as 10 Tenths. 10 Tenths ÷ 3 ⟶ 3 Tenths, and 1 Tenths remain.

Step 3: Regroup 1 Tenths as 10 Hundredths. 10 Hundredths ÷ 3 → 3 Hundredths , and 1 Hundredths remain.

Step 4: Regroup 1 Hundredths as 10 Thousandths. 10 Thousandths ÷ 3 → 3 Thousandths, and 1 Thousandths remain. Regroup 1 Thousandths as 10 TenThousandths.

This never seems to end! Each time we divide by 3, there is a remainder of 1.

Will this process end?

In long division, we see that at each step we get a remainder of 1. So, the process will never end!

So, 10 ÷ 3 cannot be expressed using a finite number of digits in the decimal form.

```
3) 10 ( 3.333 ...
    9
   ---
   10
    9
   ---
   10
    9
   ---
   10
    9
   ---
   ...
```

10 ÷ 3 = 3.333 ...

There are decimal divisions where the quotient never ends! We will explore such numbers in greater detail in a later class.

? Can you find the quotients of 10 ÷ 9, and 100 ÷ 11?

Now divide 1 by 7 (1 ÷ 7).

Will this end?

Note all the remainders we get. It starts with 1, then 3, then 2, then 6, and so on. Let us represent this as a chain.

```
1 → 3 → 2 → 6 → 4 → 5
                    ↓
5 ← 4 ← 6 ← 2 ← 3 ← 1
↓
1 → 3 ...
```

What do you observe? Can you explain why this division never ends?

Not only do the remainders repeat in a cycle, the digits of the quotient also repeat in a cycle!

0.142857 142857 14…

```
7) 1 ( 0.142857142857...
   0
   ---
   10
    7
   ---
    30
    28
   ---
     20
     14
    ---
      60
      56
     ---
       40
       35
      ---
        50
        49
       ---
         10
          7
        ---
          30
          28
         ---
           20
           14
          ---
            60
            56
           ---
             40
             35
            ---
              50
              49
             ---
                1
```

A Magic Number: 142857

Let us consider the number 142857 that arose when dividing 1 by 7. Multiply 142857 by numbers from 1 to 6.

? What are the products? What do you notice?

You get the same number back, but with the digits cycled around! Multiply 142857 by 7. What do you observe?

Are there other such numbers? Yes!

? To find one such number, you can find 1 ÷ 17 in decimal, and use the repeating block of digits.

Try This

Are there infinitely many such "cyclic" numbers? That is, can we keep finding more cyclic numbers, or do they eventually stop? In 1927, the Austrian mathematician Emil Artin conjectured (guessed) that there must be infinitely many such numbers. However, even today, nearly a century later, this conjecture remains unsolved—despite a lot of research on the question by many mathematicians!

Dividend, Divisor, and Quotient

When we divide two counting numbers, the quotient is always less than the dividend. For example, 128 ÷ 4 = 32, and 32 (quotient) < 128 (dividend).

But what happens when we divide 128 by 0.4?

$$128 \div 0.4 = 320.$$

The quotient is greater than the dividend.

? Will the quotient be always greater than the dividend when the divisor is a decimal? Try it out with different values of the divisor.

Math Talk

Describe the relationship between the dividend, divisor, and the quotient. Create a table for capturing this relationship in different situations, like we did for multiplication.

? Figure it Out

1. Express the following fractions in decimal form:

   (a) $\frac{2}{5}$ (b) $\frac{13}{4}$

   (c) $\frac{4}{50}$ (d) $\frac{5}{8}$

2. Find the quotients:

   (a) 24.86 ÷ 1.2

   (b) 5.728 ÷ 1.52

3. Evaluate the following using the information 156 × 12 = 1872.
   (a) 15.6 × 1.2 = __________ (b) 187.2 ÷ 1.2 = __________
   (c) 18.72 ÷ 15.6 = __________ (d) 0.156 × 0.12 = __________
4. Evaluate the following:
   (a) 25 ÷ ______ = 0.025 (b) 25 ÷ ______ = 250
   (c) 25 ÷ ______ = 2.5 (d) 25 ÷ 10 = 25 × _____
   (e) 25 ÷ 0.10 = 25 × ______ (f) 25 ÷ 0.01 = 25 × ______
5. Find the quotient:
   (a) 2.46 ÷ 1.5 = (b) 2.46 ÷ 0.15 =
   (c) 2.46 ÷ 0.015 =

   Is the quotient obtained in 24.6 ÷ 1.5 the same as the quotient obtained in 2.46 ÷ 0.15?
6. A 4 m long wooden block has to be cut into 5 pieces of equal length. What is the length of each piece?
7. If the perimeter of a regular polygon with 12 sides is 208.8 cm, what is the length of its side?
8. 3 litres of watermelon juice is shared among 8 friends equally. How much watermelon juice will each get? Express the quantity of juice in millilitres.
9. A car covers 234.45 km using 12.6 litres of petrol. What is the distance travelled per litre?
10. 13.5 kg of flour (*aata*) was distributed equally among 15 students. How much flour did each student receive?

$\frac{1}{2} = 0.5$	$\frac{1}{5} = 0.2$
$\frac{1}{2 \times 2} = 0.25$	$\frac{1}{5 \times 5} = 0.04$
$\frac{1}{2 \times 2 \times 2} = 0.125$	$\frac{1}{5 \times 5 \times 5} = 0.008$
$\frac{1}{2 \times 2 \times 2 \times 2} = 0.0625$	$\frac{1}{5 \times 5 \times 5 \times 5} = 0.0016$
$\frac{1}{2 \times 2 \times 2 \times 2 \times 2} = ?$	$\frac{1}{5 \times 5 \times 5 \times 5 \times 5} = ?$

? What pattern do you observe? Why are 2 and 5 related in this way?

Math Talk

4.4 Look Before You Leap!

Did you know that it takes the Earth 365.2422 days to go around the Sun and not 365 days? For our convenience, we consider 365 days as a year in a calendar. We are talking about Gregorian calendar.

This means that, after one calendar year or 365 days, the Earth still needs 0.2422 more days to complete one full revolution around the Sun. This doesn't seem like much. But what happens after 100 such calendar years?

Using our understanding of decimal multiplication,

$$0.2422 \times 100 = 24.22 \text{ days.}$$

After 100 calendar years, the Earth will need 24.22 more days to complete its 100th revolution around the Sun.

In your Science classes, you have learnt that we experience seasons because of the Earth's tilt in axis and its revolution around the Sun. If our calendar does not accurately indicate the position of the Earth around the Sun, our seasons and our annual calendar will not match!

To correct this problem, the idea of a leap year was introduced. Every fourth year, one additional day is added to the calendar year.

Making an Adjustment

Is the year divisible by 4?

Yes → The year has 366 days

No → The year has 365 days

? Do you know which month has this extra day?

Let us see how this solution works.

Year 1	Year 2	Year 3	Year 4	Year 5	Year 6	Year 7	Year 8	...
365	365	365	366	365	365	365	366	...

Looking at the above sequence, the number of days after 4 calendar years is

$$4 \times 365 + 1 = 1461 \text{ days.}$$

What is the number of days that the Earth needs to make 4 full revolutions around the Sun?

$$4 \times 365.2422 = 1460.9688 \text{ days.}$$

? With this new scheme of adding one extra day every 4th year, what is the number of days in 100 calendar years? Can you write an expression to calculate that number?

Here is one way to form the expression. Each calendar year has 365 days. In 100 calendar years, the number of days is 100 × 365. But years that are divisible by 4 have one extra day.

? How many years are divisible by 4 in 100 years?

So, the number of days in 100 calendar years is,

$$(100 \times 365 + \frac{100}{4} \times 1) = 36{,}525 \text{ days.}$$

? Can you form different expressions for the same question?

The actual number of days that the Earth takes to go around the Sun 100 times is,

$$100 \times 365.2422 = 36{,}524.22 \text{ days.}$$

Thus, by adding a day every fourth year, after 100 years, the calendar days are more than the actual number of days taken by the Earth to go around the Sun. We have overcompensated.

So, the good people who designed calendars decided that they will not add 1 extra day in every hundredth year!

Making Another Adjustment

Is the year divisible by 100?

- Yes → The year has 365 days
- No → Is the year divisible by 4?
  - Yes → The year has 366 days
  - No → The year has 365 days

? Can you write an expression for the number of days in 100 calendar years with this new adjustment? Math Talk

We saw that there are 25 years divisible by 4 in 100 years. 100 is also divisible by 4, but we have to exclude it. So only 24 years have 366 days, and the rest (76 years) have 365 days. So the expression can be written as,

$$\left(\frac{100}{4} - \frac{100}{100}\right) \times 366 + (100 - \left(\frac{100}{4} - \frac{100}{100}\right)) \times 365$$

$$= (24 \times 366) + (76 \times 365) = 36{,}524 \text{ days.}$$

This is close to 36524.22 days but is it close enough? What happens after 1000 years with this adjustment?

If we follow this new scheme of adding 1 day every four years, but not in the 100th year, the number of calendar days in 1000 years is

$$36524 \times 10 = 3{,}65{,}240 \text{ days.}$$

The number of days the Earth takes to go around the Sun 1000 times is

$$1000 \times 365.2422 = 3{,}65{,}242.2 \text{ days.}$$

So, there is a difference of 2.2 days.

To bridge this gap, it was decided that every 400th year would be a leap year!

Is the year divisible by 400?

Yes → The year has 366 days

No → Is the year divisible by 100?

Yes → The year has 365 days

No → Is the year divisible by 4?

Yes → The year has 366 days

No → The year has 365 days

With this scheme, let us calculate the number of days in 1000 calendar years.

In 1000 calendar years, how many years are divisible by 400? 2.

In 1000 calendar years, how many years are divisible by 100 but not divisible by 400? 10 – 2 = 8.

In 1000 calendar years, how many years are divisible by 4 but not divisible by 100 and 400? 250 – 10 = 240.

The rest of the years are 1000 – (2 + 8 + 240) = 750.

So, the total number of days in 1000 calendar years is

(750 × 365) + (240 × 366) + (8 × 365) + (2 × 366) = 3,65,242 days.

The Earth needs 3,65,242.2 days to go around the Sun 1000 times. Hence, in 1000 years, the calendar year is slightly shorter (by 0.2 days) than the actual number of days the Earth takes to go around the Sun.

The calendar makers did not want to bother about a small difference that would happen in 1000 years! So, they left the scheme of leap years as is ...

Making Yet Another Adjustment

Try This

With this final scheme of leap years can you calculate the number of calendar days in 10,000 years and the number of actual days the Earth will take to make 10,000 revolutions around the Sun? What is the difference? If there is a big difference, can you suggest a way to fix this problem?

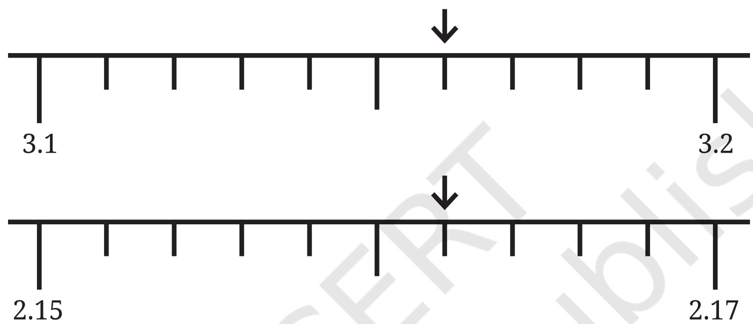
Do you wonder how people figured out that the Earth completes one revolution around the Sun in exactly 364.2422 days?

Investigate how traditional calendars in India managed to consistently align the days in the calendar with astronomical events like the Earth going around the Sun or even the positions of the stars in the sky accurately.

Try This

Figure it Out

1. A 210 gram packet of peanut *chikki* costs ₹70.5, while a 110 gram packet of potato chips costs ₹33.25. Which is cheaper?
2. Write the decimal number at the arrow mark:



3. Shyamala bought 3 kg bananas at ₹30/- per kg. She counted 35 bananas in all. She sells each banana for ₹5/-. How much profit does she make selling all the bananas?

4. A teacher placed textbooks that are 2.5 cm thick on a bookshelf. The teacher wanted to place 80 textbooks on the shelf. The bookshelf is 160 cm long. How many books could be placed on the shelf? Was there any space left? If yes, how much?

5. Fill in the following blanks appropriately:

1 cm = 10 mm 1 m = 100 cm 1 km = 1000 m	1 kg = 1000 g 1 g = 1000 mg	1 l = 1000 ml

5.5 km = ________ m	35 cm = _______ m	14.5 cm = ______ mm
68 g = _______ kg	9.02 m = _______ mm	125.5 ml = ______ l

6. The following problem was set by Sridharacharya in his book, *Patiganita*. "$6\frac{1}{4}$ is divided by $2\frac{1}{2}$, and $60\frac{1}{4}$ is divided by $3\frac{1}{2}$. Tell the quotients separately." Can you try to solve it by converting the fractions into decimals?

Math Talk

7. Fill the boxes in at least 2 different ways:

   (a) $\square \times \square = 2.4$ (b) $\square \times \square = 14.5$

8. Find the following quotients given that 756 ÷ 36 = 21:

   (a) 75.6 ÷ 3.6 (b) 7.56 ÷ 0.36
   (c) 756 ÷ 0.36 (d) 75.6 ÷ 360
   (e) 7560 ÷ 3.6 (f) 7.56 ÷ 0.36

9. Find the missing cells if each cell represents a÷b:

b ↓ a →	1517	151.7	15.17	1.517	15170
37	41				
3.7			4.1		
0.37					
0.037		4100			
370					

10. Using the digits 2, 4, 5, 8, and 0 fill the boxes $\square\square.\square \times \square.\square$ to get the:

   (a) maximum product (b) minimum product

(c) product greater than 150 (d) product nearest to 100

(e) product nearest to 5

11. Sort the following expressions in increasing order:

(a) 245.05 × 0.942368 (b) 245.05 × 7.9682

(c) 245.05 ÷ 7.9682 (d) 245.05 ÷ 0.942368

(e) 245.05 (f) 7.9682

SUMMARY

- In this chapter, we learnt procedures to perform decimal multiplication and division.
- For decimal multiplication, we first multiply the multiplier and multiplicand as counting numbers. The number of decimal digits in the product is the total number of decimal digits in the multiplier and multiplicand.
- Division of decimals uses the same procedure, i.e., division using place value (long division), as with counting numbers. The regrouping continues after the Ones place to Tenths, Hundredths, Thousandths, and so on. When the Ones are regrouped to Tenths, a decimal point is placed in the quotient.
- There are decimal divisions where the quotient never ends. After each regrouping and dividing there is always a remainder!

Hidato

	33	35					
		24	22				
			21				
	26		13	40	11		
27				9		1	
				18			
					7		
						5	

Puzzle

32	33	35	36	37			
31	34	24	22	38			
30	25	23	21	12	39		
39	26	20	13	40	11		
27	28	14	19	9	10	1	
		15	16	18	8	2	
				17	7	16	3
						5	4

Solution

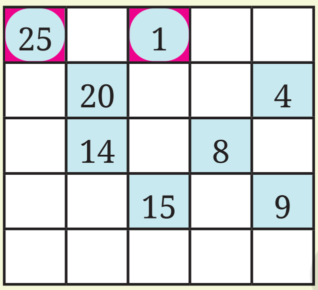
In Hidato, a grid of cells is given. It is usually square-shaped, like Sudoku or Kakuro, but it can also include hexagons or any shape that forms a tessellation. It can have inner holes (like a disc), but it is made of only one piece.

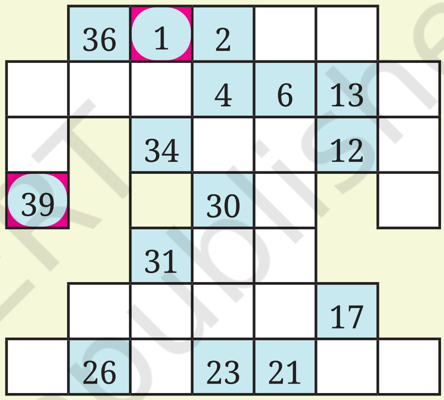
Usually, in every Hidato puzzle the lowest and the highest numbers are given on the grid.

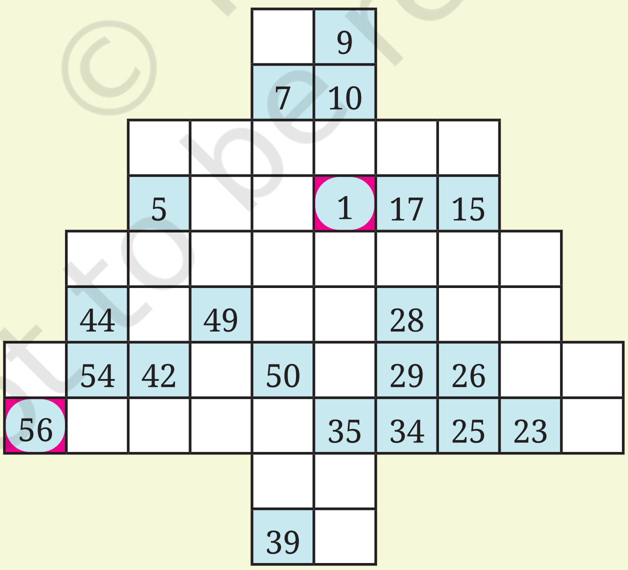
Your task is to fill the grid such that there is a continuous path of consecutive numbers from the lowest to the highest number. The next number must be in any one of the adjacent cells, including diagonally adjacent cells.

The grid comes pre-filled with some numbers (with values between the smallest and the highest) to ensure that these puzzles have a single solution.

Try solving the following Hidato puzzles.







5 CONNECTING THE DOTS...

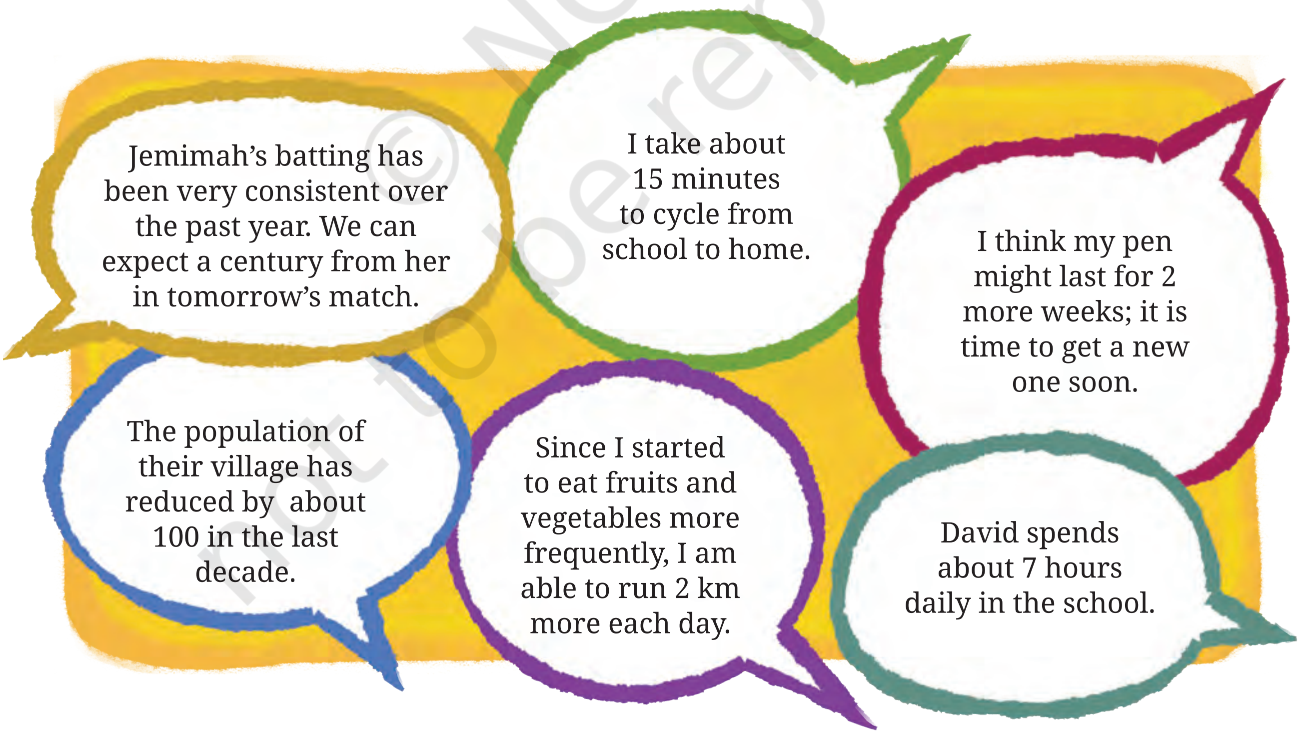
0789CH05

5.1 Of Questions and Statements

Your teacher tells you that they are meeting two of their childhood friends this evening. One is 5 feet tall and the other is 6 feet tall. What is your guess as to each friend's gender based on this information?

You might have guessed that the 5-foot-tall person is a woman and the 6-foot-tall person is a man. There is a chance that you are wrong. But experience tells us that 5-foot-tall men and 6-foot-tall women are rare. We have seen that, more often, men are taller than women.

The above is a simple example of statistical thinking. We regularly come across statements like —



We call these statistical statements. Simply put, a statistical statement is a claim or summary about some phenomenon, expressed in terms of numerical values, proportions, probabilities, or predictions.

A statistical question is a question that can be answered by collecting data. For example, "How tall are Grade 7 students in our school?" is a statistical question. We expect that not all Grade 7 students have the same height, but we can collect data, analyse it, and make conclusions about the heights that do occur. The question "Typically, are onions costlier in Yahapur or Wahapur?" is also a statistical question. Prices can vary over time. Therefore, answering this question requires us to look at data, analyse it, and come to conclusions making suitable statistical statements.

Math Talk

? Which of the following are statistical questions?

(a) What is the price of a tennis ball in India?

(b) How old are the dogs that live on this street?

(c) What fraction of the students in your class like walking up a hill?

(d) Do you like reading?

(e) Approximately how many bricks are in this wall?

(f) Who was the best bowler in the match yesterday?

(g) What was the rainfall pattern in Barmer last year?

The term statistics refers to the study of collecting, organising, analysing, interpreting, and presenting data. In this chapter, we shall encounter some statistical questions and learn how analysing data and graphs can help answer them.

5.2 Representative Values

? The runs scored by Shubman and Yashasvi in a cricket series are given in the table below. Who do you think performed better?

	Match 1	Match 2	Match 3	Match 4
Shubman	0	17	21	90
Yashasvi	67	55	18	35

Shreyas says, "Both their performances are similar since Yashasvi scored more in the first and second matches, whereas Shubman scored more in the third and fourth".

Vaishnavi says, "I think Shubman performed better because he scored the highest number of run in a match — 90!".

Shreyas says, "No! Yashasvi batted better since the total number of runs he made is 175, while Shubman made only 128".

Vaishnavi says, "Oh! Also, Yashasvi's batting is more consistent — the difference between his maximum score and minimum score is lower".

The table below shows the runs scored by these two players in another series. Who do you think performed better in this series?

	Match 1	Match 2	Match 3	Match 4	Match 5
Shubman	23	07	10	52	18
Yashasvi	26	53	02	-	15

Vaishnavi says, "Here, Shubman performed better since his total is 110 runs, while Yashasvi's total is 96 runs".

? What do you think of Vaishnavi's statement?

Shreyas says, "But Yashasvi made 96 runs in 4 matches and Shubman made 110 runs in 5 matches".

So, how do we say who performed better? It is often not simple to compare two groups of numbers and clearly say that one is better than the other.

? Can a single number act as a representative of a group of numbers? For example, can we represent Shubman's or Yashasvi's batting in this series with one number? Discuss.

Math Talk

We saw one way already — the total of the values in the group! But, if the group sizes are different, then the total may not be an appropriate measure to compare.

In some matches, a player could have scored more and in other matches less. A representative number for the group can be found by balancing out these *highs* and *lows*. For example, we can add up the runs scored in all the matches and divide the total by the number of matches played. We call this value the '**average**' or '**arithmetic mean**' of the given data.

Here, the average number of runs scored by a player in a match = [Total runs scored by the player in all the matches] ÷ [Number of matches played].

Average number of runs scored by Shubman in a match = 110 ÷ 5 = 21 runs.

Average number of runs scored by Yashasvi in a match = 96 ÷ 4 = 24 runs.

In this series, Yashasvi's average number of runs is higher than Shubman's.

The **Average** or **Arithmetic Mean** (A.M.), or simply **Mean**, is calculated as follows:

$$\text{Mean} = \frac{\text{Sum of all the values in the data}}{\text{Number of values in the data}}.$$

Average as Fair-Share

The average can also be understood as fair-share or equal-share.

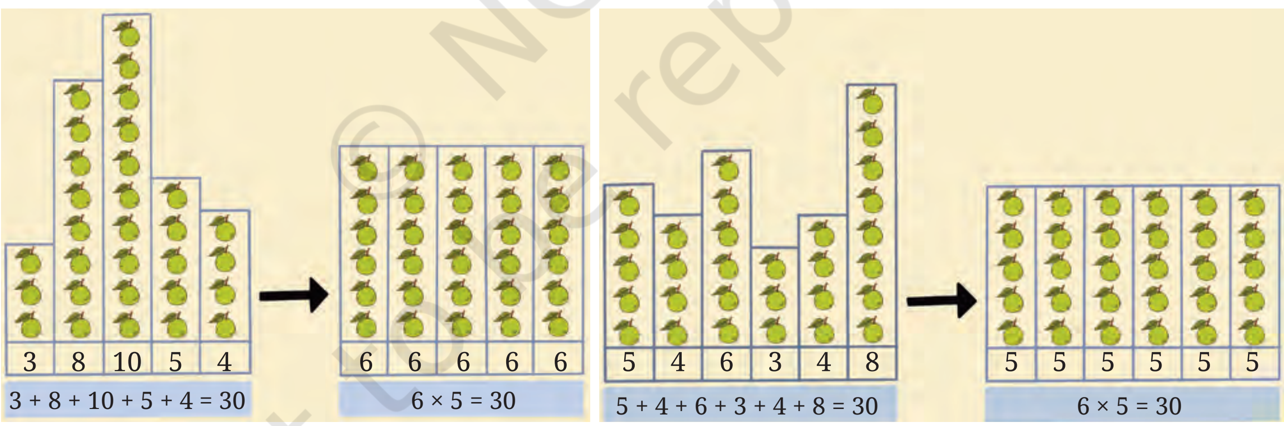
? Shreyas and 4 of his friends have collected the following numbers of guavas: 3, 8, 10, 5, and 4. Parag and 5 of his friends have collected the following numbers of guavas: 5, 4, 6, 3, 4, and 8. Each group will share their guavas equally amongst themselves. In which group will each member get a bigger share of guavas?

To find this out, we first find out how many guavas each group has collected. Then we divide this total by the number of people in the group to get each member's share.

Shreyas's group has collected 3 + 8 + 10 + 5 + 4 = 30 guavas. Each member of Shreyas's group gets 30 ÷ 5 = 6 guavas.

Parag's group has collected 5 + 4 + 6 + 3 + 4 + 8 = 30 guavas. Each member of Parag's group gets 30 ÷ 6 = 5 guavas.

So, the members of Shreyas's group get 1 more guava each than the members of Parag's group.



? Vaishnavi tracks the number of *Hibiscus* flowers blooming in her garden each day. The data for the last few days' is 2, 7, 9, 4, 3. What is the average number of *Hibiscus* flowers blooming per day in Vaishnavi's garden?

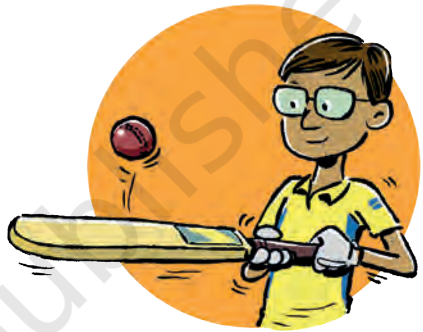
The average = (the total number of *Hibiscus* flowers bloomed) ÷ (number of days)
= (2 + 7 + 9 + 4 + 3) ÷ 5
= 5.

On an average, 5 *Hibiscus* flowers bloom daily.

In this case the average tells us the number of flowers blooming each day, if an equal number of flowers bloomed daily.

One of the terms used for the Arithmetic Mean in ancient Indian mathematics is *samamiti* (mean measure): '*sama*' means equal. Some terms used for the Arithmetic Mean in Indian texts include — *samarajju* (mean measure of a line segment) by Brahmagupta (628 CE), *samīkaraṇa* (levelling, equalising) by Mahāvīrācārya (850 CE) *sāmya* (equality, impartiality, equability towards) by Śrīpati (1039 CE) and *samamiti* (mean measure) by Bhāskarācārya (1150 CE) and Gaṇeṣa (1545 CE). The terminology shows that ancient Indian scholars perceived the Arithmetic Mean as the 'common' or 'equalising' value that is a representative measure of a collection of values.

Figure it Out



1. Shreyas is playing with a bat and a ball — but not cricket. He counts the number of times he can bounce the ball on the bat before it falls to the ground. The data for 8 attempts is 6, 2, 9, 5, 4, 6, 3, 5. Calculate the average number of bounces of the ball that Shreyas is able to make with his bat.
2. Try the activity above on your own. Collect data for 7 or more attempts and find the average.

   Try This
3. Identify a flowering plant in your neighbourhood. Track the number of flowers that bloom every day over a week during its flowering season. What is the average number of flowers that bloomed per day?
4. Two friends are training to run a 100 m race. Their running times over the past week are given in seconds — Nikhil: 17, 18, 17, 16, 19, 17, 18; Sunil: 20, 18, 18, 17, 16, 16, 17. Who on average ran quicker?
5. The enrolment in a school during six consecutive years was as follows: 1555, 1670, 1750, 2013, 2040, 2126. Find the mean enrolment in the school during this period.

Know Your Onions!

The table shows the monthly price of onions, in rupees per kilogram (kg), at two towns. Where are onions costlier, according to you?

Math Talk

Month	Yahapur
January	25
February	24
March	26
April	28
May	30
June	35
July	39
August	43
September	49
October	56
November	59
December	44

Month	Wahapur
January	19
February	17
March	23
April	30
May	38
June	35
July	42
August	39
September	53
October	60
November	52
December	42

Khushboo: 'I think Wahapur is costlier because it has the highest price of ₹60.'

Nafisa: 'I added the prices of all months in each location - Yahapur's total is 458, whereas Wahapur's total is 450.'

Vishal: 'Wahapur is costlier since it has 3 numbers in the 50s'.

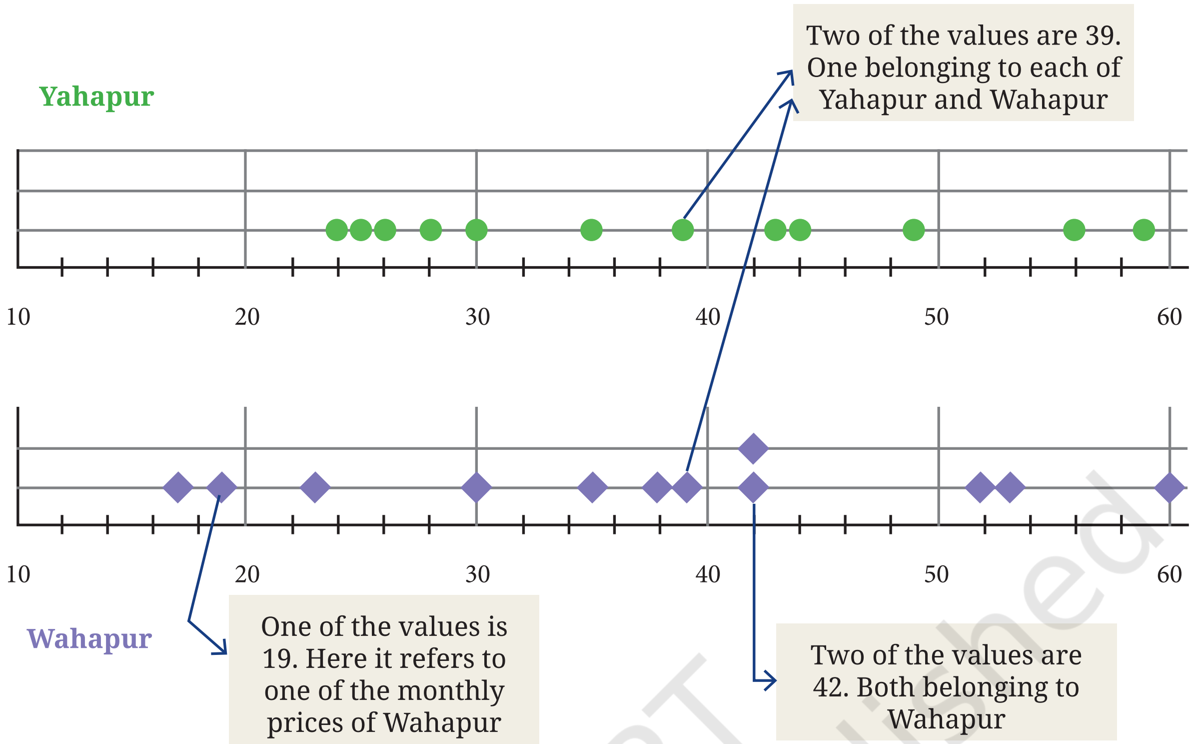
Sampat: 'I compared the prices in each month in both locations. Prices in Yahapur are higher for 6 months, prices in Wahapur are higher for 5 months, and the prices are the same for 1 month. So, I feel Yahapur is costlier.'

Jithin: 'I noticed that the difference between the highest and lowest prices in Yahapur is 59 – 24 = 35, and in Wahapur it is 60 – 17 = 43.'

Data can be described and compared by referring to its **minimum** value, **maximum** value, the average value, the sum **total** of all its values, and the difference between the maximum and minimum values.

? Can you think of any other ways to compare the data?

To study data, we can visualise it in multiple ways. One way is shown below— it is called a **dot plot**. Dot plots show data points as dots on a line, helping us visualise variability and patterns in data. In the following figure, each dot represents the monthly price of onions.



The prices in Yahapur are in green and those in Wahapur are in purple. The horizontal line shows the prices from 10 to 60 (instead of starting from 0 as there are no values from 0 – 10 or above 60). The dots on the vertical line give the number of occurrences of a data value. Notice the equal spacing between the units along the horizontal as well as the vertical lines.

? Does this visualisation capture all the data presented in the tables earlier?

? Looking at it, can we tell the price of onions in Yahapur in the month of January?

This method of presentation orders or sorts the data, but it loses the original (month-wise) sequence of the values. However, it allows us to group the data however we wish, just as Vishal did. For instance, there are 2 data values between 11 – 20 for Wahapur, while Yahapur has none. This representation makes it easier to observe the variation in the data — where and how the data is clustered or spread out. We can easily see that the prices in Wahapur are more spread out than those in Yahapur. It is also easy to spot the highest and lowest values.

We can also use the average as one of the ways to compare the prices at these two places.

? Find the average price of onions at Yahapur and Wahapur.

A statement such as, "The price of onions is ₹35 per kilo", may not trigger any further questions. But looking at variations in data, like the prices of onions over a year in Yahapur and Wahapur, can spark one's curiosity. For example, one might be curious to know more about the two locations.

You might wonder —

? What else do you wonder about?

Math Talk

You can discuss questions that you are curious about with your peers, teachers, or family members to find answers.

> Observing and trying to make sense of data can reveal interesting things. It can also trigger our curiosity in different directions.

Averages Around Us

The Arithmetic Mean is frequently used in statistics, mathematics, experimental sciences, economics, sociology, sports, biology and diverse disciplines as a representative of data. It is popular partly because the definition of the arithmetic mean is simple and easy to understand. Some statements involving averages in different scenarios are shown below:


The average rainfall per day in Jharkhand in the month of July is 37.2mm.

My scooty's average mileage this year is about 45 kilometers per liter.


Wheat yield averages 4.7 tonnes per hectare in Punjab vs. 2.9 tonnes per hectare in Bihar.


Smartphone users check their phone 58 times a day on average.

An average Indian citizen generates 0.45 kg of waste per day.

3126 is the average number of Indian long films released annually between 2017 – 2024.

Outliers and Medians

Does the average always give a reasonable summary of the values in a collection? If not, what is an alternative? Let us find out.

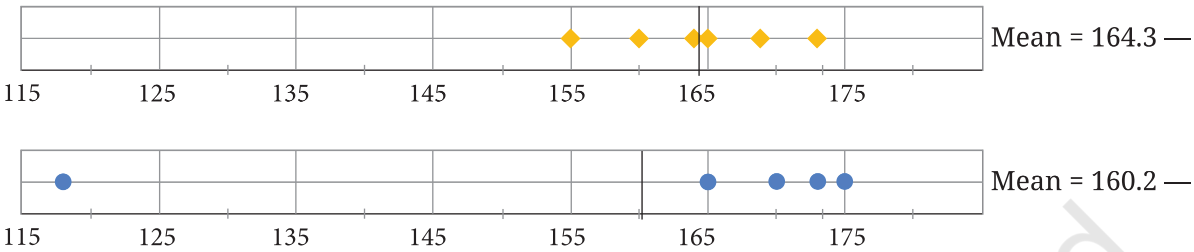
Height of a Family

The heights of the family members of Yaangba and Poovizhi are as follows:
Yaangba's family: 169 cm, 173 cm, 155 cm, 165 cm, 160 cm, 164 cm.
Poovizhi's family: 170 cm, 173 cm, 165 cm, 118 cm, 175 cm.

? Find the average height of each family. Can we say that Yaangba's family is taller than Poovizhi's family?

Math Talk

The average height of Poovizhi's family (160.2 cm) is less than that of Yaangba's family (164.3 cm). Although most members in Poovizhi's family are taller, their family's average height is less because one child is much younger and not as tall as the rest of the family. Their average height, 160.2 cm, is less than the heights of 4 out of 5 members. Here, the average doesn't seem to represent the data very well.



? Can you think of any other number that can represent the data better?

One way is to sort the data and pick the number in the middle. This number is called the **Median**.

To find the median height of Poovizhi's family, we first sort the heights —

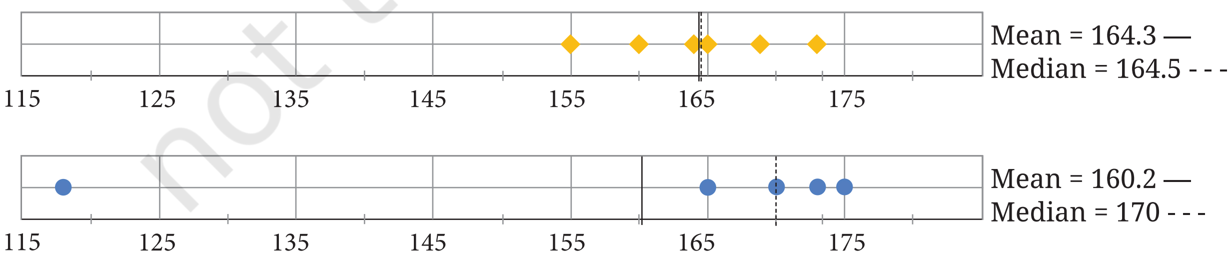
118, 165, 170, 173, 175.

The middle number in this sorted data is 170. Therefore, the median height is 170 cm.

Let us find the median height of Yaangba's family. Sorting the heights, we get

155, 160, 164, 165, 169, 173.

Since the median is the number in the middle, it will have an equal number of values less than it and greater than it. This data does not have a single middle number because it has an even number of values (6). In such cases, we take the average of the two middle numbers in the sorted data. Therefore, the median height of Yaangba's family is (164 + 165) ÷ 2 = 164.5 cm.



? In this case, does the median represent the heights of the families better than the average?

In Poovizhi's family, the height of the youngest child is quite different from the heights of the rest of the family. We call such a value an **outlier**. Outliers are values which significantly deviate from the rest of the values in the data.

Notice how the mean and the median are close to each other in Yaangba's data, in the absence of any outlier.

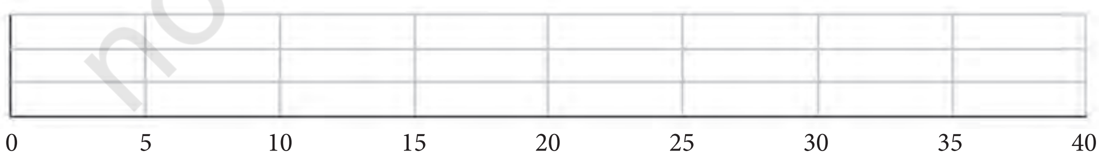
In Poovizhi's data, because of the outlier, the mean is much lower than the median.

? Find the mean and median in Poovizhi's data without the outlier value 118. What change do you notice?

Are you a bookworm?

? After the summer vacation, a class teacher asked his class how many short stories they had read. Each student answered the number of stories read on a piece of paper, as shown below. Find the mean and median number of short stories read. Before calculating them, can you guess whether the mean will be less than or greater than the median?

Mark the data, the mean, and the median on the dot plot below.



The median value 6 means that half of the class members have read 6 or more stories.

? Which of the values would you consider an outlier?

? Find the mean and median in the absence of the outlier. What change do you notice?

The average may not always be an appropriate representative of data that has outliers. A very high or a very low outlier can significantly impact the sum, thus affecting the average. For example, the 118 cm height in Poovizhi's family is an outlier at the lower end of the data. And the count of 40 short stories read is an outlier at the higher end of the data. In these cases, we saw that the median was not affected much by the outliers.

Are We on the Same Page?

? Do you read newspapers? Have you noticed how many pages a newspaper has on different days of the week — is it the same or different?

The list below shows the number of pages for a particular newspaper from Monday to Sunday: 16, 18, 20, 22, 26, 16, 10.

Mark the data, the mean, and the median on the dot plot below.

? In the three examples we considered — the heights, short-stories, and newspaper pages — observe the variability in data when: *Math Talk*

(a) the mean and median are close to each other

(b) the mean and median are comparatively far apart, with mean < median

(c) the mean and median are comparatively far apart, with mean > median

When the data is more balanced or uniformly spread out the mean, and the median appear to be close to each other. When the outlier is on the lower end, the mean appears to shift in that direction, i.e., mean < median. When the outlier is on the higher end, the mean appears to shift in that direction, i.e., mean > median.

Discuss the effect on the mean and median when outliers are present on both sides. You may take some example data to examine and explain this.

Math Talk

Mean and Median are called measures of central tendency, i.e., the tendency of the values to pile up around a particular value. In other words, they represent the 'centre' of the data.

Of Ends and the Essence

As we have just seen, the mean and the median can give different perspectives on the data. As part of analysing data, it can also be valuable to look at the variability in the given data, i.e., its extremes (minimum and maximum values).

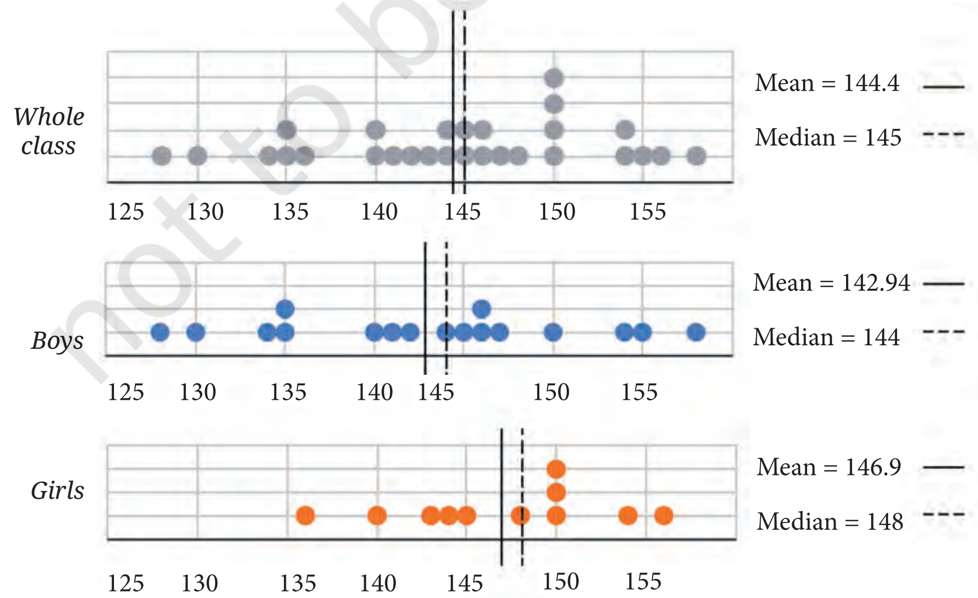
How Tall is Your Class?

Suppose you are asked the question, "How tall is your class?" What would you say?

The table below shows the heights of students in a Grade 5 class in centimeters.

Boys	147, 135, 130, 154, 128, 135, 134, 158, 155, 146, 146, 142, 140, 141, 144, 145, 150
Girls	143, 136, 150, 144, 154, 140, 145, 148, 156, 150, 150

We can visualise the data using a dot plot, identify the ends and patterns, and look at the variability. We can also find the measures of central tendency. The dot plot for the whole class, followed by the dot plots for boys and girls, respectively, are shown. The mean and the median are also shown for each collection.



? What can we infer from the dot plots and the central tendency measures?

The following points can help answer the question of how tall the class is.

- The boys' heights are more spread out and are between 128 and 158. The girls' heights lie between 136 and 156. Both the tallest and shortest in the class are boys.
- Yet, the boys' average height is less than the whole class average, and also less than the girls' average height. We can say girls are taller than boys in this class. Of course, this doesn't mean every girl is taller than every boy!
- For boys' heights, mean < median (142.94 < 144) indicating a small influence of values on the lower side. For girls' heights too, mean < median (146.9 < 148) indicating a small influence of values on the lower side.

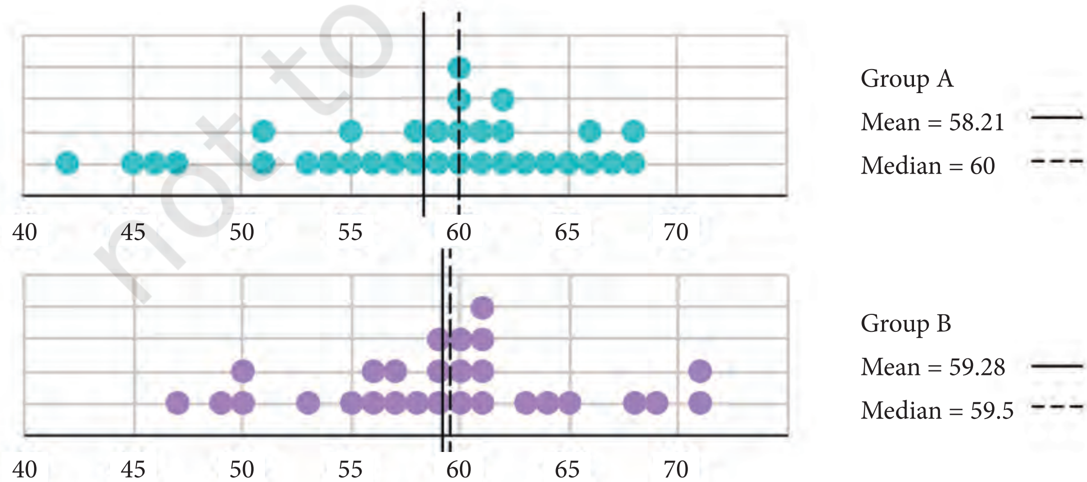
? How many students are taller than the class' average height?

? How many boys are taller than the class' average height?

How long is a minute?

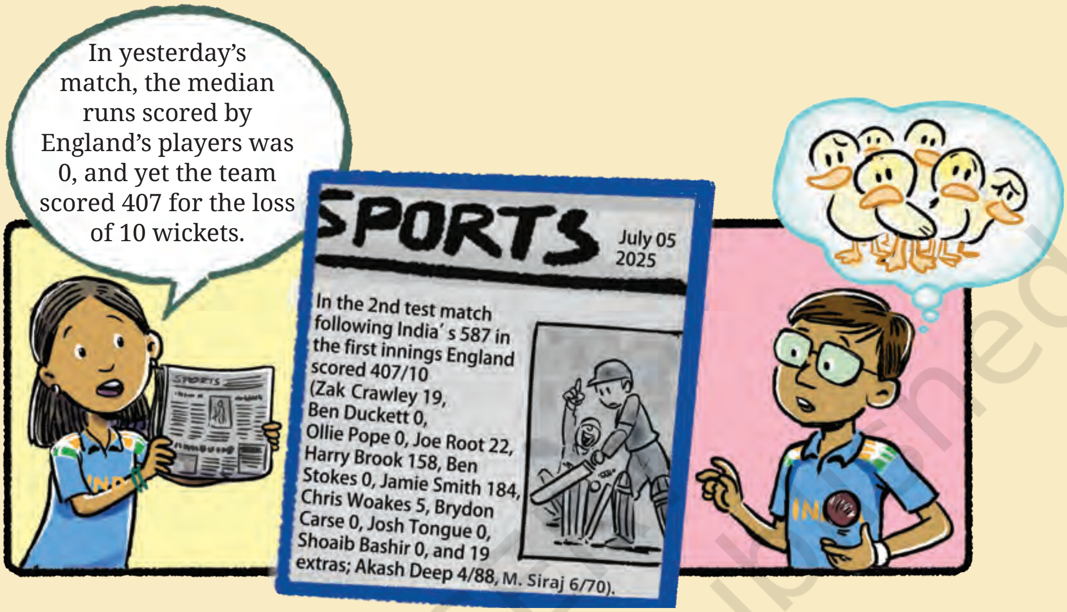
Two groups of children were asked to estimate the length of 1 minute. They start by closing their eyes and then open when they think 1 minute has passed. Of course, they are not supposed to count while their eyes are closed. The dot plots below show after how many seconds the children opened their eyes.

? Discuss how well both the groups fared at this activity. Describe and compare the variability in data and their central tendency. Math Talk



Zero Median Runs Scored!

In a cricket match, can a team's median runs scored by a player be 0 but the team's total score be 407/10?



Accounting for the extras, the average runs scored by a player in this innings is (407 – 19) ÷ 11 = 35.27.

Zero vs. No value

Suppose a player scores 57, 13, 0, 84, —, 51, 27 in a series. Notice that the player played Match 3 and scored 0 runs whereas the player did not play Match 5. So, we consider the total number of matches to be 6 and not 7. We calculate their average runs scored per match as
(57 + 13 + 0 + 84 + 51 + 27) ÷ 6.

Sita has a mango tree in her backyard. The number of mangoes the tree gave every month over the last year, from January to December, is 0, 0, 8, 24, 41, 16, 5, 0, 0, 0, 0, 0 respectively. If we want to find the mean or median number of mangoes per month, it would be appropriate to consider only the (summer) months when mangoes are expected to grow.

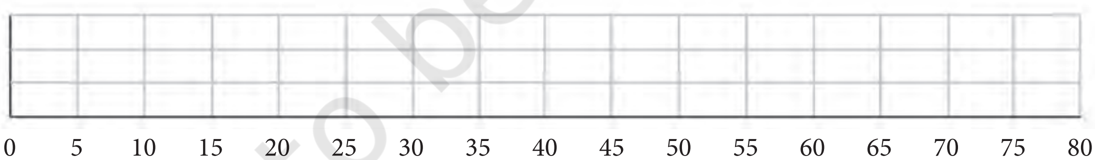
A Mean Foot

In the early 1500s in Europe, the basic unit of land measurement was the rod, defined as 16 feet long. At that time, a foot meant the length of a human foot! But foot sizes vary, so whose foot could they measure? To solve this, 16 adult males were asked to stand in a line, toe to heel, and the length of that line was considered the 16-foot rod. After the rod was determined, it was split into 16 equal sections, each representing the measure of a single foot. In essence, this was the arithmetic mean of the 16 individual feet, even though the term 'mean' was not mentioned anywhere.

Jacob Kobel's depiction of the determination of 1 foot

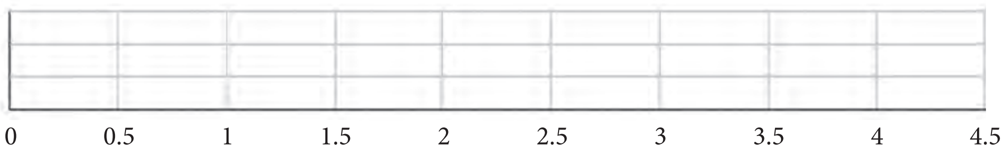
Figure it Out

1. Find the median of onion prices in Yahapur and Wahapur.
2. Sanskruti asked her class how many domestic animals and pets each had at home. Some of the students were absent. The data values are 0, 1, 0, 4, 8, 0, 0, 2, 1, 1, 5, 3, 4, 0, 0, —, 10, 25, 2, —, 2, 4. Find the mean and median. How would you describe this data?
3. Rintu takes care of a date-palm tree farm in Habra. The heights of the trees (in feet) in his farm are given as: 50, 45, 43, 52, 61, 63, 46, 55, 60, 55, 59, 56, 56, 49, 54, 65, 66, 51, 44, 58, 60, 54, 52, 57, 61, 62, 60, 60, 67. Fill the dot plot, and mark the mean and median. How would you describe the heights of these palm trees? Can you think of quicker ways to find the mean? How many trees are shorter than the average height?

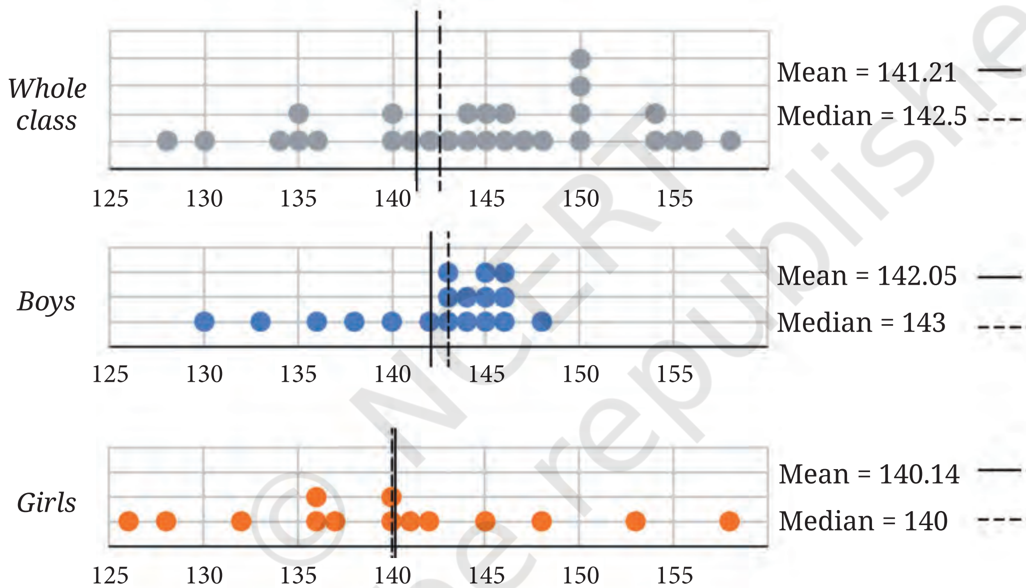


4. The daily water usage from a tap was measured. The usage in liters for the first few days are: 5.6, 8, 3.09, 12.9, 6.5, 12.1, 11.3, 20.5, 7.4.
   (a) Can the mean or median daily usage lie between 25 and 30? Justify your claim using the meaning of mean and median.
   (b) Can the mean or median be lesser than the minimum value or greater than the maximum value in a data?
5. The weights of a few newborn babies are given in kgs. Fill the dot plot provided below. Analyse and compare this data.

Boys	3.5	4.1	2.6	3.2	3.4	3.8
Girls	4.0	3.1	3.4	3.7	2.5	3.4



6. The dot plots of heights of another section of Grade 5 students of the same school are shown below. Can you share your observations? What can we infer from the dot plots and the central tendency measures?



? Compare the heights of the two sections. Share your observations.

7. The weights of some sumo wrestlers and ballet dancers are: Sumo wrestlers: 295.2 kg, 250.7 kg, 234.1 kg, 221.0 kg, 200.9 kg. Ballet dancers: 40.3 kg, 37.6 kg, 38.8 kg, 45.5 kg, 44.1 kg, 48.2 kg. Approximately how many times heavier is a sumo wrestler compared to a ballet dancer?

Math Talk

5.3 Visualising Data

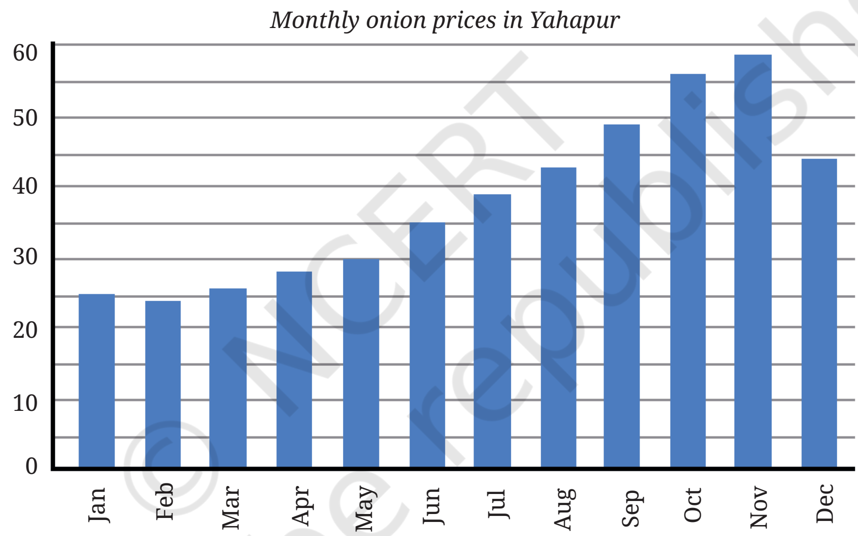
We can often understand data more clearly if it is presented as a picture. This is called **data visualisation**. Last year, we saw how to visualise data using graphs. Let us explore visualisation further.

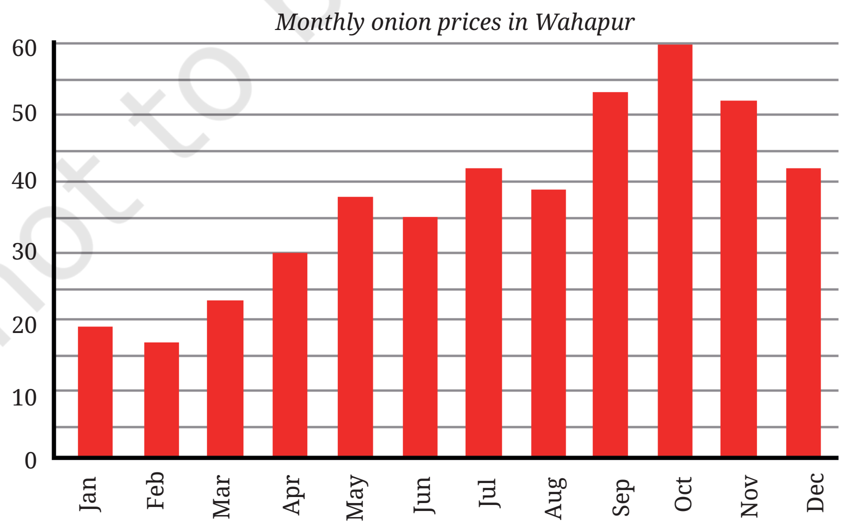
Clubbing the Columns

Earlier, we looked at the monthly onion prices in Yahapur and Wahapur.

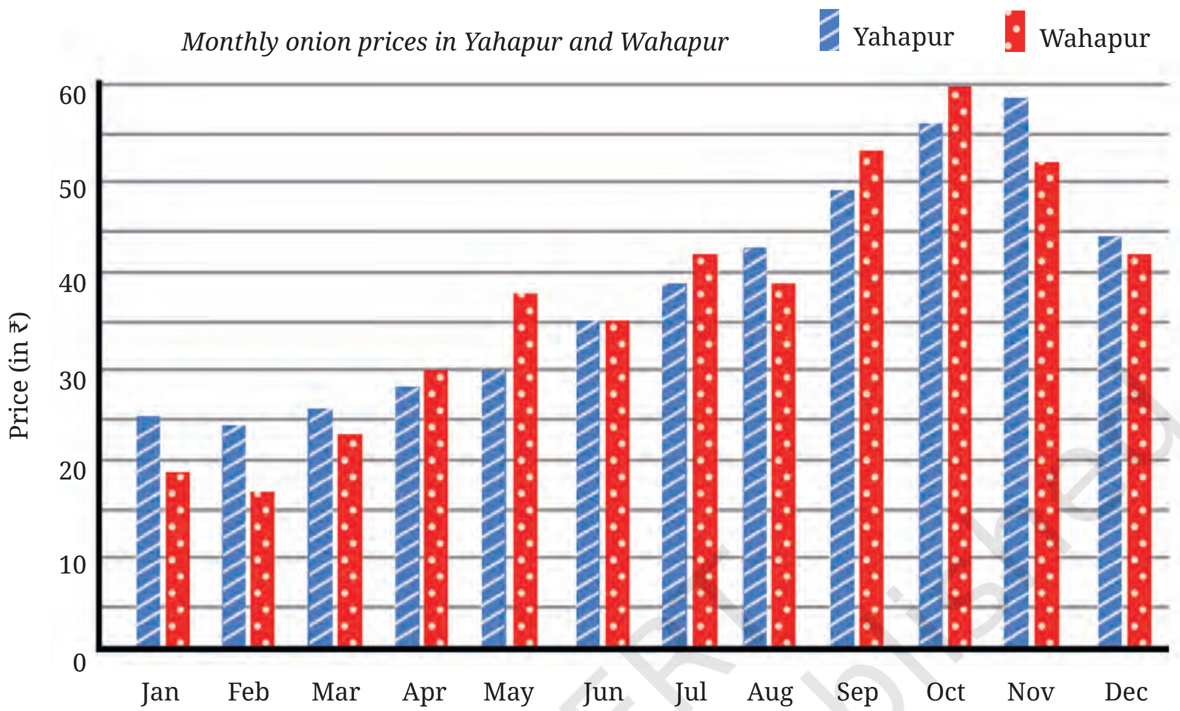
	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
Yahapur	25	24	26	28	30	35	39	43	49	56	59	44
Wahapur	19	17	23	30	38	35	42	39	53	60	52	42

Two column graphs for this data are given below.





The two graphs can also be combined into a single graph. We just draw the bars side by side! Verify if the data in the table matches the graph below.



We use different colours to clearly separate the data from the two places. This is called a **clustered column graph**. Since it has two columns in each cluster, we also call it a **double column graph**.

? What is the scale used in this graph?

The relative heights of the bars tell us where onions are costlier in each month. We can also visually estimate the difference by referring to the markings along the vertical line.

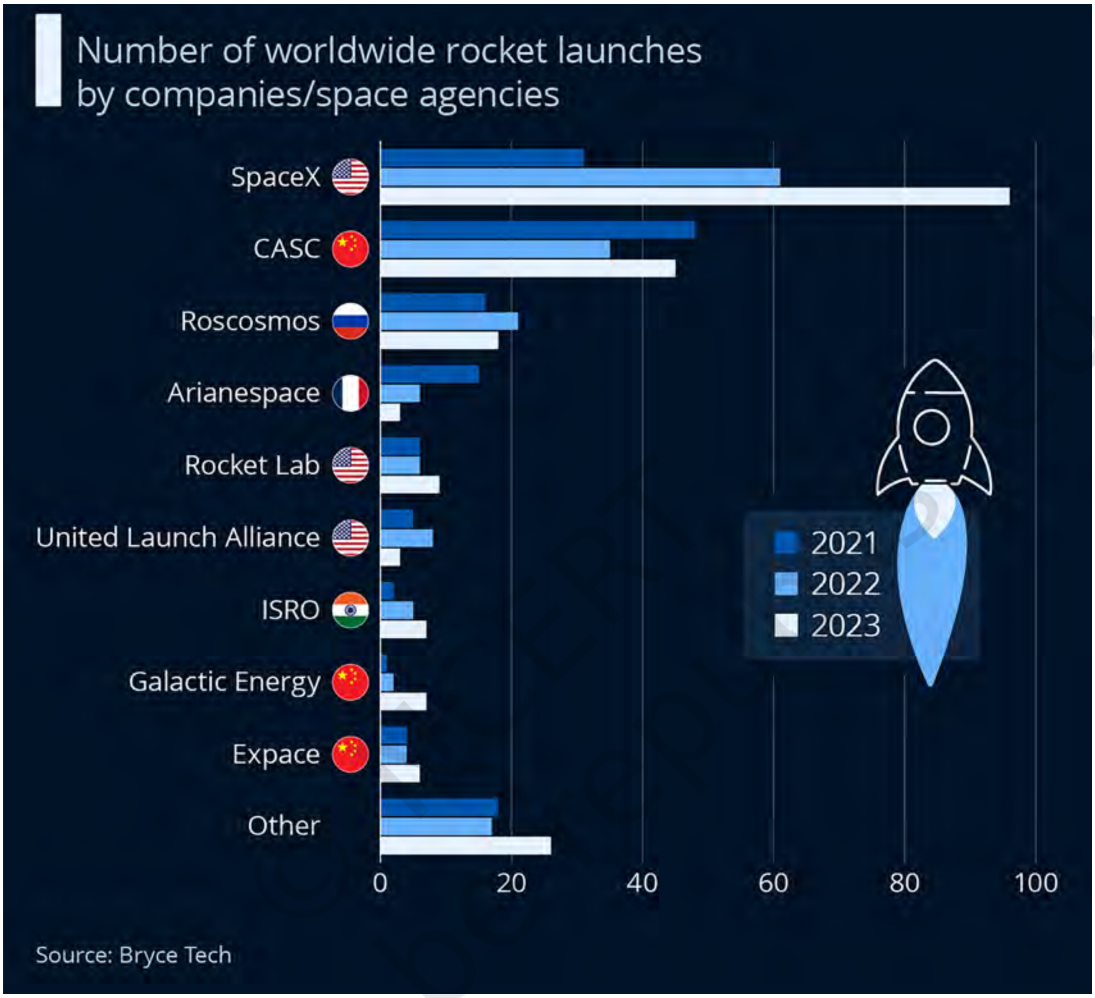
The dots and slanted lines within the bars help people who find difficulty in distinguishing colours. It is also useful when things are printed in greyscale (black-and-white).

? Is it now easier to compare month-wise prices in both places?

10...9...8...7...6...5...4...3...2...1...Take Off!

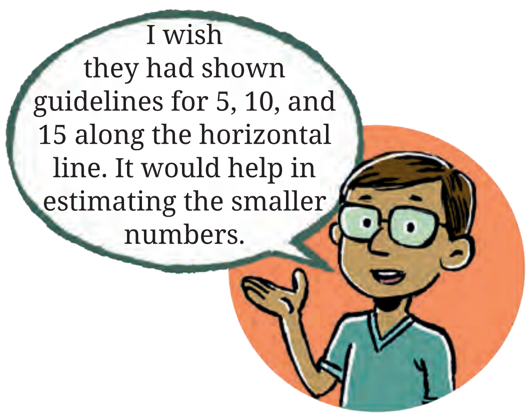
You might have heard about scientific probes (like Chandrayaan-3 launched in 2023 by ISRO or the Voyager-1 launched in 1977 by NASA), observational satellites (like Aryabhata launched in 1975 by ISRO or Sputnik-1 launched in 1957 by the Soviet Space program), or about human spaceflights to the International Space Station. All space missions are launched using rockets. Look at the graph below showing the number of worldwide rocket launches by different organisations.

? Share your observations (you may take the teacher's help to identify the countries these organisations belong to).



Source: https://www.statista.com/chart/29410/number-of-worldwide-rocket-launches-by-companies-and-space-agencies/

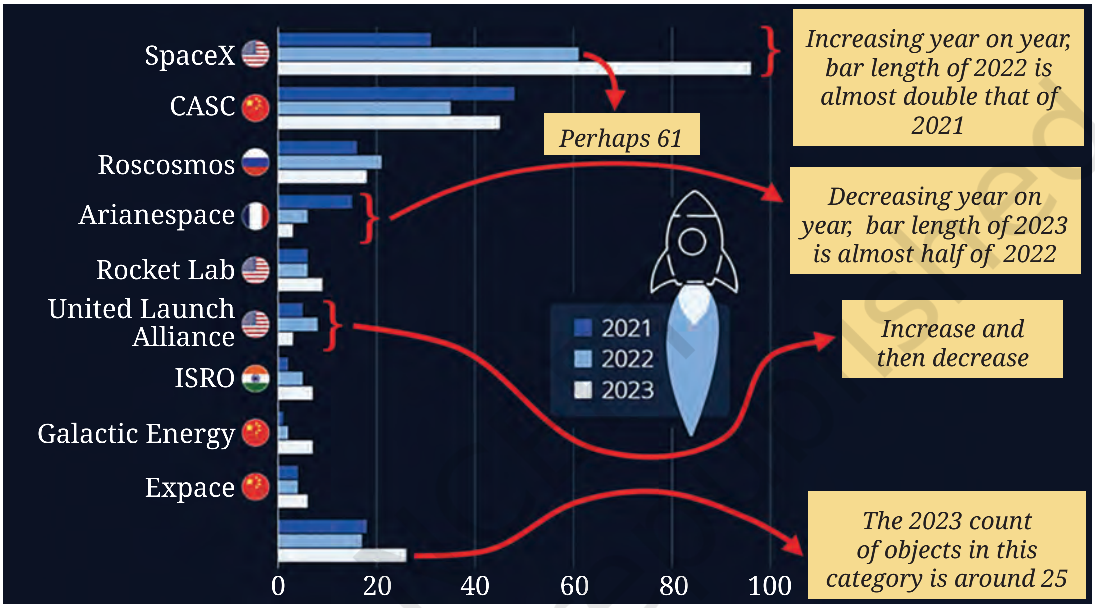
Often there is a lot of information in graphs and it may be difficult to understand. We can follow a 2-step process to simplify making sense of the data in graphs.



Step 1: Identify what is given

? Notice how the graph is organised, what scale is used, and what patterns the data shows.

- For each organisation, the numbers of rocket launches for the years 2021, 2022, and 2023 are shown as three adjacent bars. The scale used is 1 unit length = 20 rockets. Notice the numbers at the bottom.
- The 'Others' category indicates multiple organisations worldwide that are clubbed together to keep the graph short.
- Note that in the double bar graph of onion prices, the months are shown in order, i.e., January to December, to observe the change over time, whereas in this case, a change in the order of organisations does not affect the meaning.



Step 2: Infer from what is given

Analyse and interpret each of your observations.

- We can say that the USA, China, and Russia are the leading rocket launching countries in the given time period.
- SpaceX launched about twice the number of rockets in 2022 compared to 2021. And it launched about 35 more rockets in 2023 compared to 2022.
- The number of rockets launched by Arianespace decreased every year.
- United Launch Alliance launched more rockets in 2022 than in 2021. They launched fewer rockets in 2023 than in both the years 2022 and 2021.
- Other organisations launched about 25 rockets in 2023.

? Identify which of the following statements can be justified using this data.

(a) All organisations launched more rockets than the previous years.

(b) Only an organisation from the USA launched more than 50 rockets in a single year.

(c) The total number of rockets launched by France in all 3 years is less than 40.

(d) The average number of rockets launched by CASC in these 3 years is around 40.

(e) ISRO launched more rockets than Galactic Energy in these 3 years.

(f) Russia launched more than 60 rockets in these 3 years.

? List the organisations that have consistently launched more rockets every year.

? Estimate the total number of rockets launched worldwide in 2023.

(a) less than 200 (b) 200 to 400

(c) 400 to 600 (d) more than 600

We may have many questions after looking at the graph. We might wonder why USA launches so many more rockets than other countries. Or we might be eager to look at the data from previous and later years. What are you curious to know after looking at this graph?

Summer and Winter at the Same Time

The tables below show data related to weather in two cities in different countries. The numbers given are in hours. Can you guess what the data might be related to?

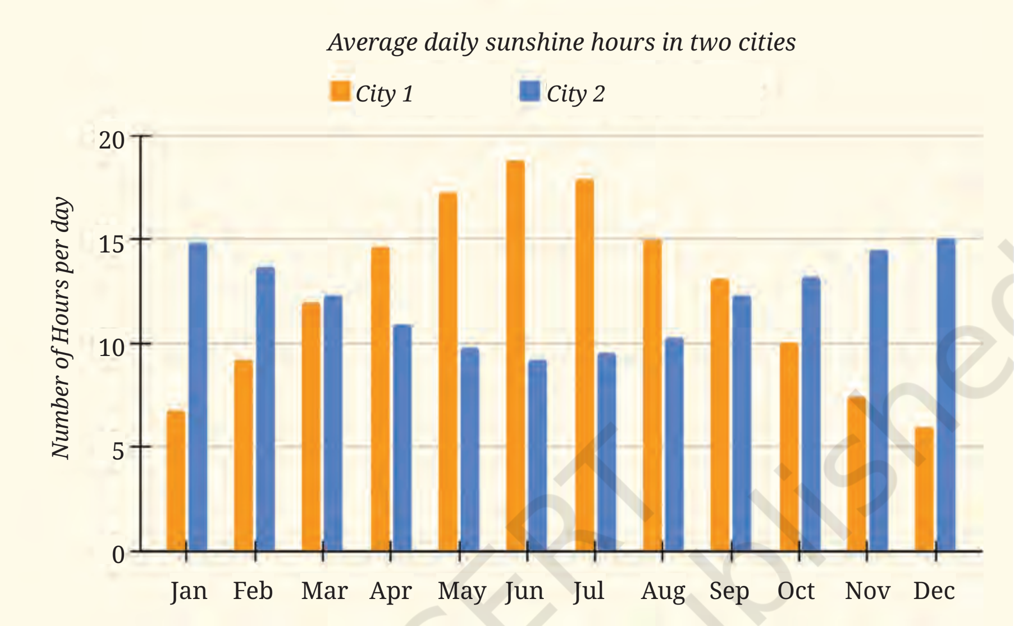
City 1

Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
210	257	372	441	536	564	555	465	394	310	222	186

City 2

Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
459	384	381	327	304	276	295	318	369	409	435	468

The data shows the monthly hours of daylight (i.e., the Sun is at least partly above the horizon) in these two cities over the year. Based on this data, a clustered bar graph showing the average daylight hours per day in each month is given below. This average is obtained by dividing the monthly daylight hours by the number of days in the month.



Let us follow the 2-step process to identify and interpret the information presented.

Step 1: Identify what is given

? Notice how the graph is organised, what scale is used, and what patterns the data shows.

- The horizontal line shows the months of the year. The vertical line shows the average daylight hours per day, using the scale 1 unit = 5 hours. The month of June has the maximum value for City 1 and the minimum value for City 2.

Step 2: Infer from what is given

? Analyse and interpret each of your observations. Share appropriate summary and conclusion statements.

- The average number of daylight hours per day in City 1 increases from January, reaching a maximum of about 17 – 18 hours in June. It then decreases, reaching a minimum of about 6 hours in December.

- The average number of daylight hours per day in City 2 decreases from January, reaching a minimum of about 9 hours in June. It then increases, reaching a maximum of about 15 hours in December.
- The maximum and minimum values in City 1 are more extreme than those of City 2. That is, the maximum number of daylight hours per day of City 1 is more than that of City 2, and the minimum number of daylight hours per day of City 1 is less than that of City 2.
- In June, City 1 experiences daylight for about $\frac{3}{4}$th of the full day (24 hours), whereas during December – January, it only experiences daylight for about $\frac{1}{4}$th of the full day.

? Does this give some idea of where these two cities are located?

City 1 and City 2 are located away from the Equator in the Northern and Southern hemispheres, respectively. City 1 is Helsinki, Finland, and City 2 is Wellington, New Zealand. These are also shown on the map. In June, the Northern Hemisphere is tilted towards the Sun, resulting in longer daylight hours; it is summertime here. Meanwhile, the Southern Hemisphere is tilted away from the Sun, leading to shorter days; it is winter time here. The inverted seasonal daylight pattern is due to the cities' location in opposite hemispheres. The large variation in the data is because they are away from the Equator.

Is there anything more that you wish to explore?

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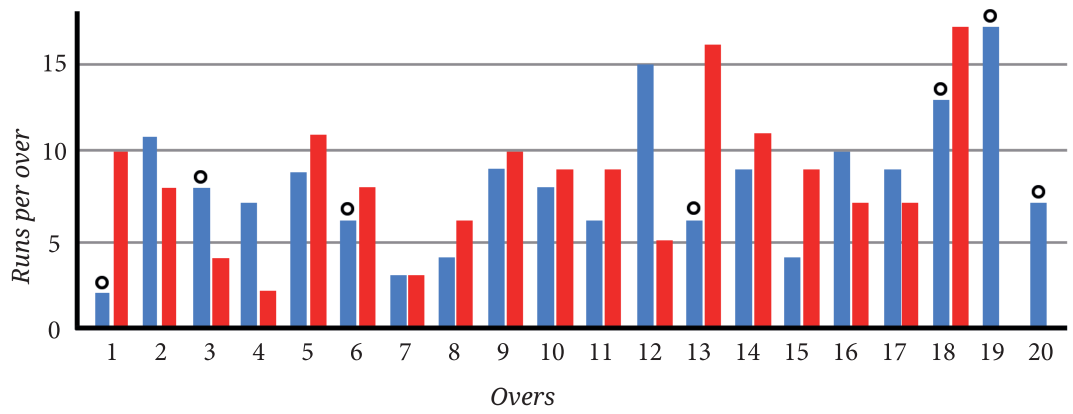
The Midnight Sun at Andøya, Norway.

All it Takes is a Minute

Have you ever missed watching a cricket match? You can catch up in a minute by looking at a graph. You might have seen graphs like the following one.



The horizontal line lists the overs starting from 1, and the vertical line indicates the runs scored in each over. The graph shows the number of runs scored per over as a double bar graph — each bar corresponding to a team. Let us call them the blue team (denoted by blue) and the red team (denoted by red). The scale used for the runs per over is 1 unit = 5 runs. The circles shown on top of the bars indicate that a wicket fell in that over.

? Answer the following questions based on the graph:

1. Can we tell who batted first? Who won the match?
2. How many runs did the blue team score in over 12?
3. In which over did the red team score the least number of runs?
4. Is it easy to tell the target set by the team batting first?

? Figure it Out

1. The following infographic shows the speeds of a few animals in air, on land, and in water. Can we call this graph a bar graph?

   Math Talk

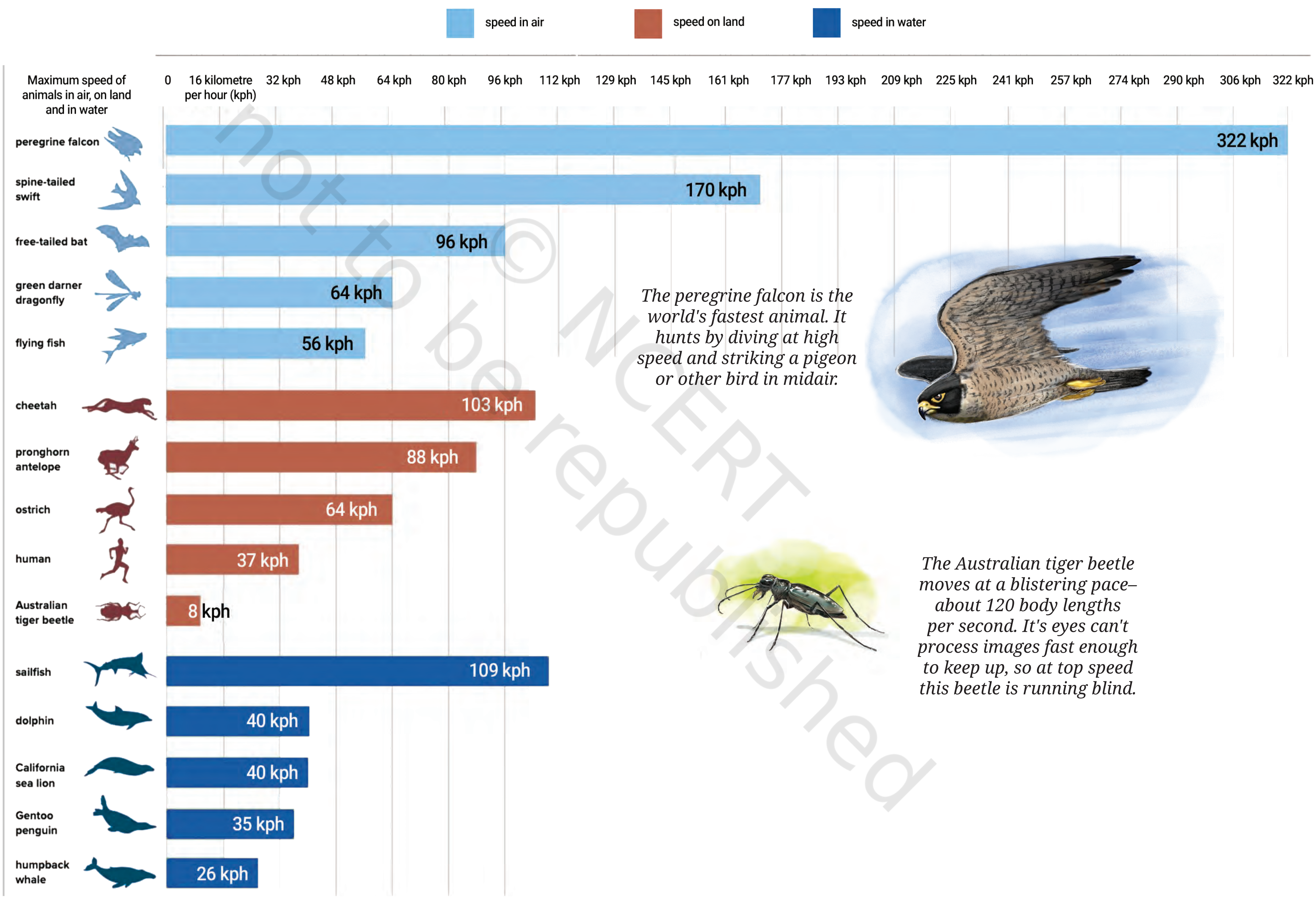
   (a) What is the scale used in this graph?

   (b) What did you find interesting in this infographic? What do you want to explore further?

   (c) Identify a pair of creatures where one's speed is about twice that of the other.

   (d) Can we say that a sailfish is about 4 times faster than a humpback whale? Can we say that a sailfish is the fastest aquatic animal in the world?

How fast?



The peregrine falcon is the world's fastest animal. It hunts by diving at high speed and striking a pigeon or other bird in midair.

The Australian tiger beetle moves at a blistering pace– about 120 body lengths per second. It's eyes can't process images fast enough to keep up, so at top speed this beetle is running blind.

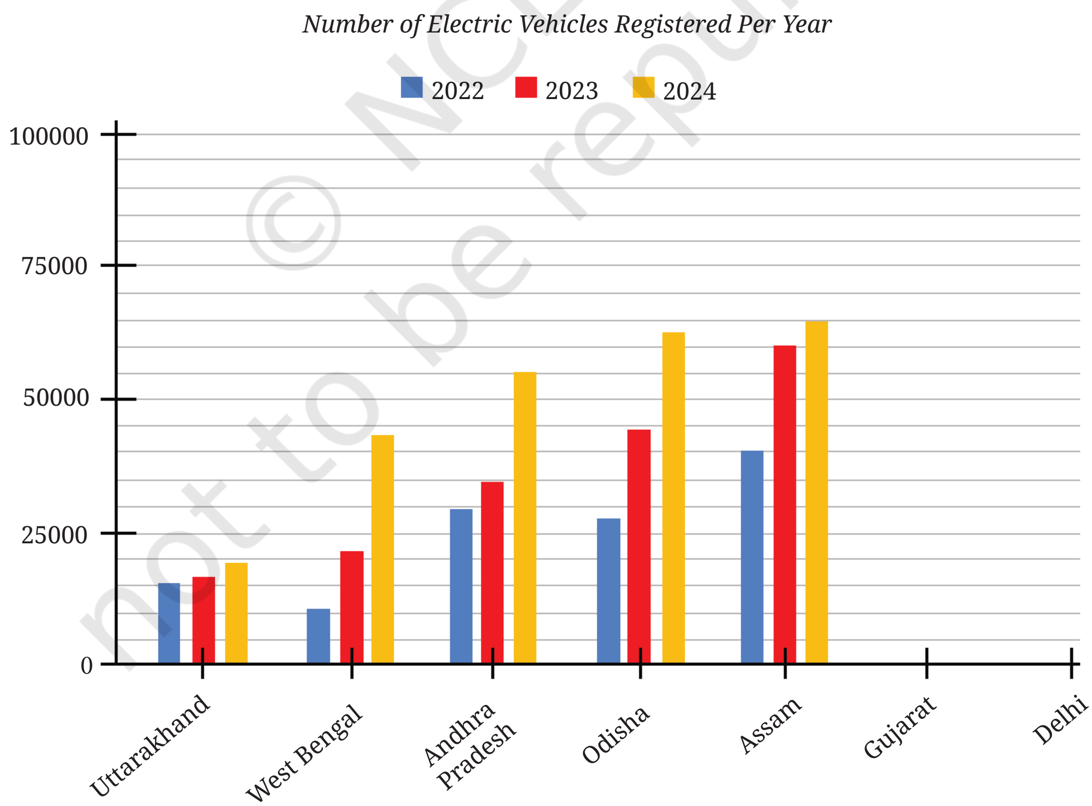
2. Preyashi asked her students 'If you were to get a super power to become aquatic (water-borne), aerial (air-borne), or spaceborne which one would you choose?'. The responses are shown below. Some chose none. Draw a double-bar graph comparing how both grades chose each option. Choose an appropriate scale.

Grade 5	w, a, a, a, w, n, s, a, n, w, a, a, a, a, a, w, w, s, a, a, n, w, a, a, n
Grade 9	n, w, s, a, s, w, s, s, a, a, w, s, s, a, s, a, n, w, s, s, a, w, a, w, a

3. The temperature variation over two days in different months in Jodhpur, Rajasthan, is given below. Draw a double-bar graph. Use the scale 1 unit = 4°C. Can you guess which two months these days might belong to?

	12 am	**3 am**	**6 am**	**9 am**	**12 pm**	**3 pm**	**6 pm**	**9 pm**
Day 1	20°C	18°C	16°C	20°C	26°C	34°C	30°C	24°C
Day 2	37°C	34°C	30°C	33°C	37°C	43°C	42°C	39°C

4. The following clustered-bar graph shows the number of electric vehicles registered in some states every year from 2022 to 2024.



(a) The data (rounded-off to thousands) for the states of Gujarat and Delhi are given in the table below. Mark the corresponding bars on the bar graph. (It is enough if you place the top of the bars between the two appropriate vertical guidelines.)

	2022	2023	2024
Gujarat	69000	89000	78000
Delhi	62000	74000	81000

(b) Notice how the graph is organised, what scale is used, and what patterns the data shows.

(c) How would you describe the change for various states between 2022 and 2024?

(d) Approximately how many more registrations did Assam get in 2023 compared to 2022?

(e) How many times more did the registrations in West Bengal increase from 2022 to 2024?

(f) Is this statement correct—'There were very few new registrations in Uttarakhand in 2023 and 2024, as the increase in the bar lengths is minimal'?

5.4 Data Detective

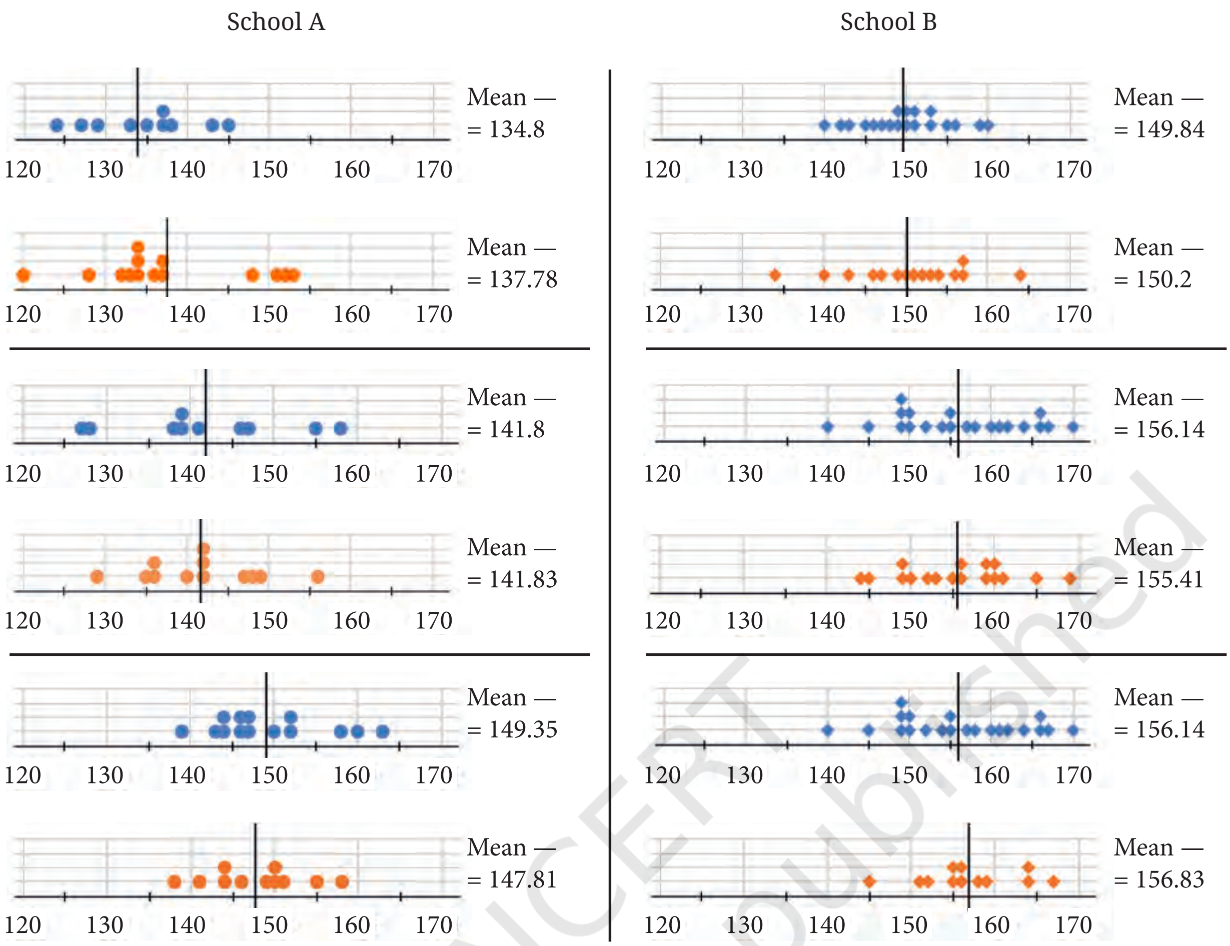
We put well-formed sentences one after the other to make a beautiful story. In the same way, well-organised and well-presented data can tell interesting stories, and can also expose new mysteries or help solve mysteries!

Telling Tall Tales

Earlier, we saw data of two Grade 5 classrooms with heights of boys and girls in each class. There, the average height of girls was more than boys in one class and vice versa in the other class.

? Following are the dot plots of heights of boys (in blue) and girls (in orange) of Grades 6, 7 and 8 (in that order) of two different schools. What do you notice? Share your observations.

Math Talk



Looking at this data, you might wonder:

"Why is there a considerable difference in heights in the same grades across these two schools?"

"Where are these schools located?"

"How tall are students in Grades 6 to 8 in my school?"

"What is the average height of all Grade 6 boys and girls?"

We see that men are taller than women in general. But what about the heights of boys and girls? Are boys taller than girls? Well, just by looking at the data of one or two schools, we cannot generalise for all children in our country, or around the world.

Let us look at some data (based on a survey) of the heights of boys and girls of different ages in India over time. The following table shows the average heights of boys and girls (in centimeters) across ages 5 to 19 in the years 1989, 1999, 2009, and 2019. In each year, the first column shows boys' heights and the second column shows girls' heights.

Age	1989		1999		2009		2019	
5	101.3	100	102.4	101.7	105.1	104	107.1	107.2
6	107.5	106	108.7	107.5	111	109.7	113.1	112.9
7	113	111.4	114.2	112.6	116.2	114.8	118.6	118
8	118.1	116.5	119.2	117.5	120.9	119.6	123.5	122.7
9	122.9	121.7	123.9	122.4	125.2	124.5	128.1	127.6
10	127.5	127.3	128.3	127.8	129.4	129.9	132.6	132.8
11	132.2	133.4	132.8	133.6	133.7	135.7	137	138.6
12	137.7	139	138	139.1	138.9	141.1	142.2	143.8
13	144.2	143.2	144.3	143.1	145.2	145.1	148.4	147.7
14	150.6	146.2	150.5	146.1	151.5	148	154.4	150.4
15	155.4	148.5	155.2	148.4	156.3	150.1	159	152.4
16	158.9	150.1	158.7	150.1	159.9	151.6	162.3	153.8
17	161.3	151.2	161.4	151.3	162.6	152.6	164.6	154.7
18	162.9	151.8	163.2	152.1	164.3	153	166	155.2
19	163.5	151.9	164.2	152.4	165.1	153	166.5	155.2

? Spend sufficient time observing the data presented in this table. Share your findings with the class.

Math Talk

These are some prompts for you to probe —

- Changes in the heights of boys or girls of a certain age from 1989 to 2019.
- The heights of boys vs. girls at different ages in a particular year.
- Changes in height between successive ages in boys and girls in 2019.

? Which of the following statements can be justified using the data?

1. The average heights of both boys and girls at every age increased from 1989 to 2019.
2. The average height of 13-year-old girls in 1989 is more than the average height of 14-year-old girls in 2009.
3. The average height of 15-year-old boys in 2019 is more than the average height of 16-year-old boys in 1989.
4. All girls aged 13 are taller than all girls aged 11.
5. Throughout the age period 5 to 19, the average boy's height is more than the average girl's height.
6. Boys keep growing even beyond age 19.

? In 2019, between which two successive ages from 5 to 19 did boys grow the most? Between which two successive ages from 5 to 19 did girls grow most?

Math Talk

? Suppose the average height of a newborn is 50 cm. Estimate the average height of young children of ages 1 to 4.

? Based on the trend observed in the table, write your estimates of the heights of boys and girls for ages 5 to 19 in the year 2029.

Whenever you see data or some graph, look closely to know the story it has to say and the mysteries it may hold.

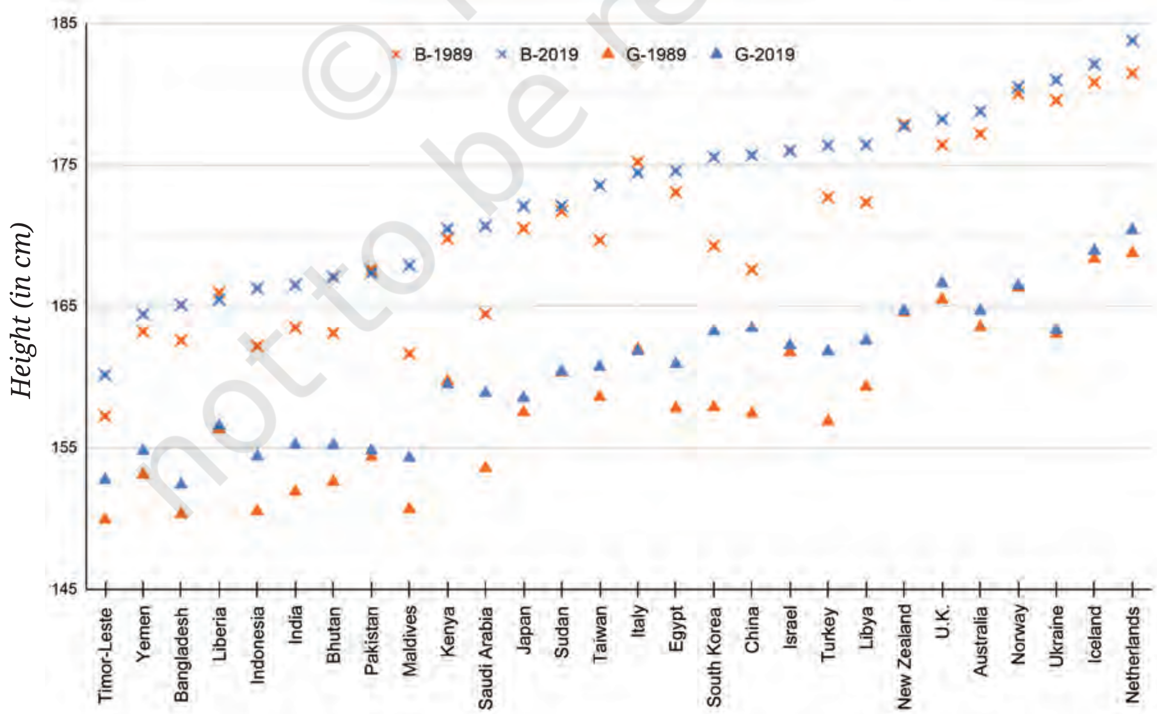
You may want to look at the data of weights. Or you might be curious to see if such patterns are present in other countries as well. You might also wonder if humans were much shorter a few centuries ago! Also, are people in some countries taller than others? The following visualisation shows the change in average heights of 19-year-old boys and girls of different countries from 1989 to 2019.

? How is the graph organised? What information is presented?

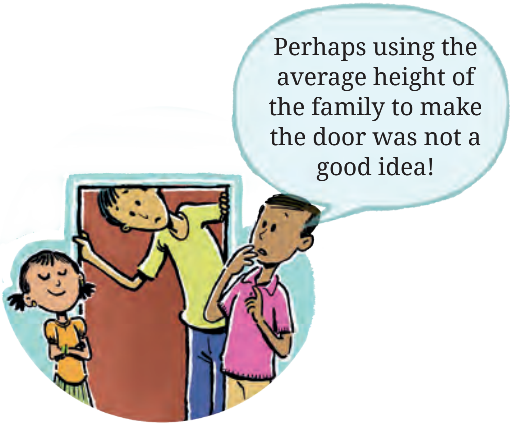
? What do you find interesting?

Math Talk

Notice that the vertical line starts from 145 cm — this helps give a closer (zoomed in) look at the heights.



A Mean Decision!



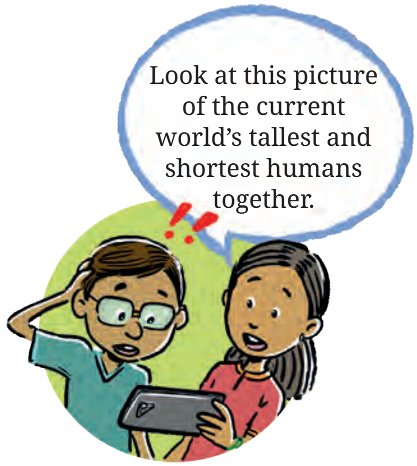
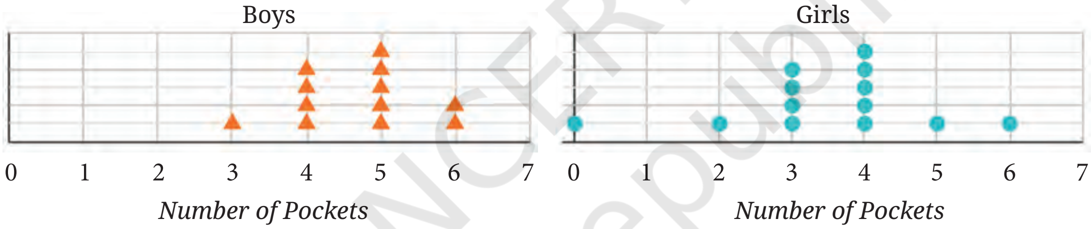


Figure it Out

1. The dot plots below show the distribution of the number of pockets on clothing for a group of boys and for a group of girls.



Based on the dot plots, which of the following statements are true?

(a) The data varies more for the boys than for the girls.

(b) The median number of pockets for the boys is more than that for the girls.

(c) The mean number of pockets for the girls is more than that for the boys.

(d) The maximum number of pockets for boys is greater than that for the girls.

2. The following table shows the points scored by each player in four games:

Player	Game 1	Game 2	Game 3	Game 4
A	14	16	10	10
B	0	8	6	4
C	8	11	Did not play	13

Now answer the following questions:

(a) Find the average number of points scored per game by A.

(b) To find the mean number of points scored per game by C, would you divide the total points by 3 or by 4? Why? What about B?

(c) Who is the best performer?

3. The marks (out of 100) obtained by a group of students in a General Knowledge quiz are 85, 76, 90, 85, 39, 48, 56, 95, 81 and 75. Another group's scores in the same quiz are 68, 59, 73, 86, 47, 79, 90, 93 and 86. Compare and describe both the groups performance using, mean and median.

4. Consider this data collected from a survey of a colony.

Favourite Sport	Cricket	Basket Ball	Swimming	Hockey	Athletics
Watching	1240	470	510	430	250
Participating	620	320	320	250	105

Choose an appropriate scale and draw a double-bar graph. Write down your observations.

5. Consider a group of 17 students with the following heights (in cm): 106, 110, 123, 125, 117, 120, 112, 115, 110, 120, 115, 102, 115, 115, 109, 115, 101. The sports teacher wants to divide the class into two groups so that each group has an equal number of students: one group has students with height less than a particular height and the other group has students with heights greater than the particular height. Suggest a way to do this. Can you guess the age of these students based on the tabular data in the 'Telling Tall Tales' section?

6. Describe the mean and median of heights of your class. You can visualise the heights on a dot plot.

7. There are two 7th grade sections at a school. Each section has 15 boys and 15 girls. In one section, the mean height of students is 154.2 cm. From this information, what must be true about the mean height of students in the other section?

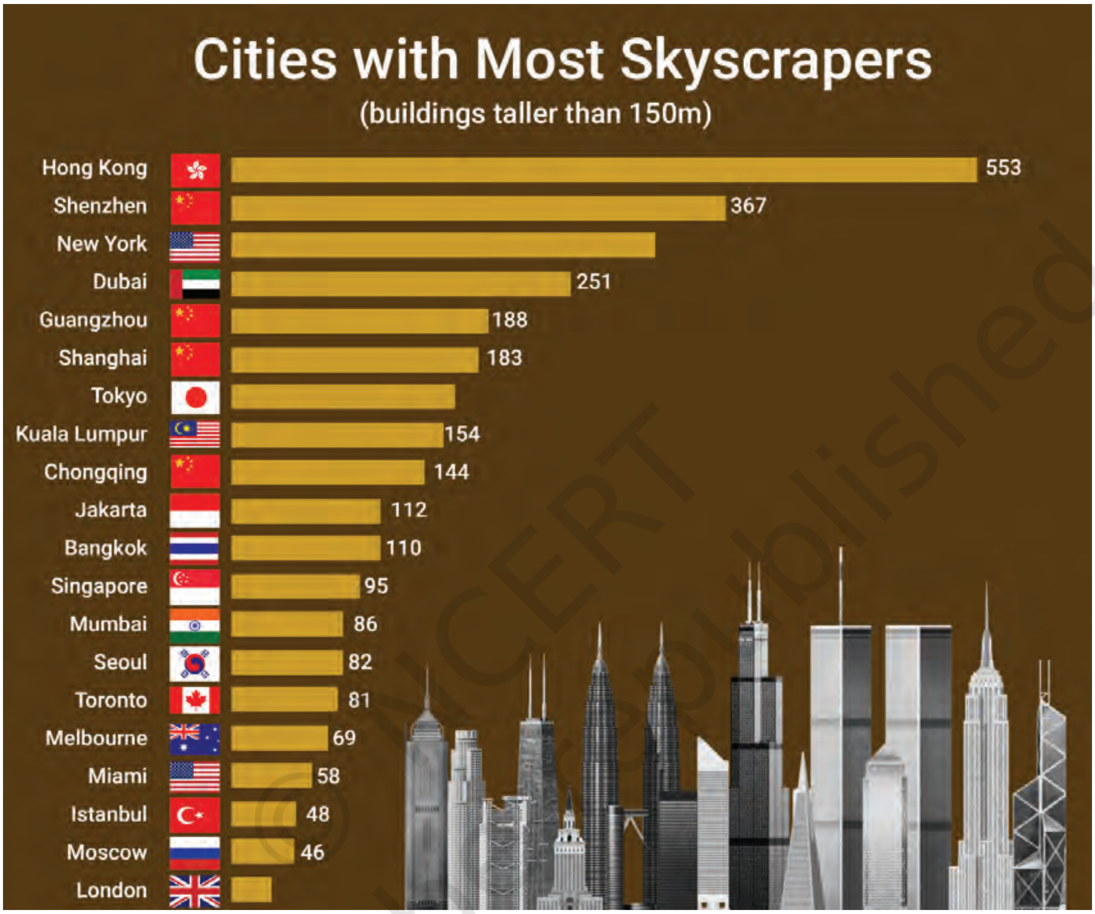
(a) The mean height of students in the other section is 154.2 cm.

(b) The mean height of students in the other section is less than 154.2 cm.

(c) The mean height of students in the other section is more than 154.2 cm.

(d) The mean height of students in the other section cannot be determined.

8. Standing tall in the storm.



(a) Write estimated values for the number of skyscrapers in New York, Tokyo, and London.

(b) Are the following statements valid?

(i) Only 12 cities have more skyscrapers than Mumbai.

(ii) Only 7 cities have fewer skyscrapers than Mumbai.

(iii) The tallest building in the world is in Hong Kong.

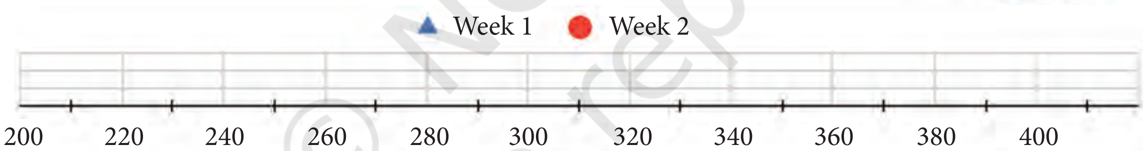
9. Estimate and then measure the objects listed in the following table. Draw a double bar graph based on the data. How accurate were your estimates? Find the average difference between the estimated and measured values.

Object	Estimate (in cm)	Measure (in cm)	Positive Difference
Length of a pen			
Length of an eraser			
Length of your plam			
Length of your geometry box			
Length of your math notebook			

10. Aditi likes solving puzzles. She recently started attempting the 'Easy' level Sudoku puzzles. The time she took (in seconds) to solve these puzzles are — 410, 400, 370, 340, 360, 400, 320, 330, 310, 320, 290, 380, 280, 270, 230, 220, 240. The first nine values correspond to Week 1 and the rest to Week 2.

(a) Construct a dot plot below showing the data for both weeks.

(b) Describe the mean, median, and any observations you may have about the data.



11. **Individual Project:** Pick at least one of the following:

(a) How Long is a Sentence? Pick any two textbooks from different subjects. Choose any page with a lot of text from each book.

(i) Use a dot plot to describe how many words the sentences have on each page.

(ii) Compare the data of both the pages using mean and median.

(b) What is in a Name? Write down the names of all of your classmates. The following are some interesting things you can do with this data!

(i) Find the mean and median name length (number of letters in a name).

(ii) Visualise the data and describe its variability and central tendency.

(iii) Which starting letters are more popular? Which are less popular?

(iv) What is the median starting letter? What does this say about the number of names starting with the letters A – M and N – Z?

(v) Plot a double-bar graph showing the number of boys' names and girls' names that:
- start and end with vowels,
- start with vowels and end with consonants,
- start with consonants and end with vowels,
- start and end with consonants.

12. **Individual project (long term):** This requires collecting data over 2 weeks or more.
In and Out: Track how many times you step out of your house in a day. Do this for a month.

(i) Describe the variability and central tendency of this data. Make a dot plot.

(ii) Do you find anything interesting about this data? Share your observations.

(iii) You can ask any of your family members or friends to do this as well.

Try This

13. **Small-group project:** Pick at least one of the following. Make groups of 8 to 10. Collect data individually as needed. Put together everyone's data and do the appropriate analysis and visualisation.

(a) Our heights vs. our family's heights: Collect the heights of your family members.

(i) Make a dot plot showing heights of just your family members. Describe its variability and central tendency.

(ii) Make a double-bar graph showing each student's height next to their family's mean height.

(iii) Look at everyone's data and share your observations.

(b) Estimating time: Check the time and close your eyes. Open them when you think 1 minute has passed (no counting). Note down after how m-any seconds you opened your eyes. Collect this data for yourself and for your family members. Repeat this activity to estimate 3 minutes.

(i) Make two dot plots (for 1 minute and 3 minutes) showing estimates of just your family members.

(ii) Mark these on the respective dot plots. Describe its variability and central tendency.

(iii) Make a double bar graph showing each family's mean 1 minute estimate and mean 3 minute estimate.

(iv) Look at everyone's data and share your observations.

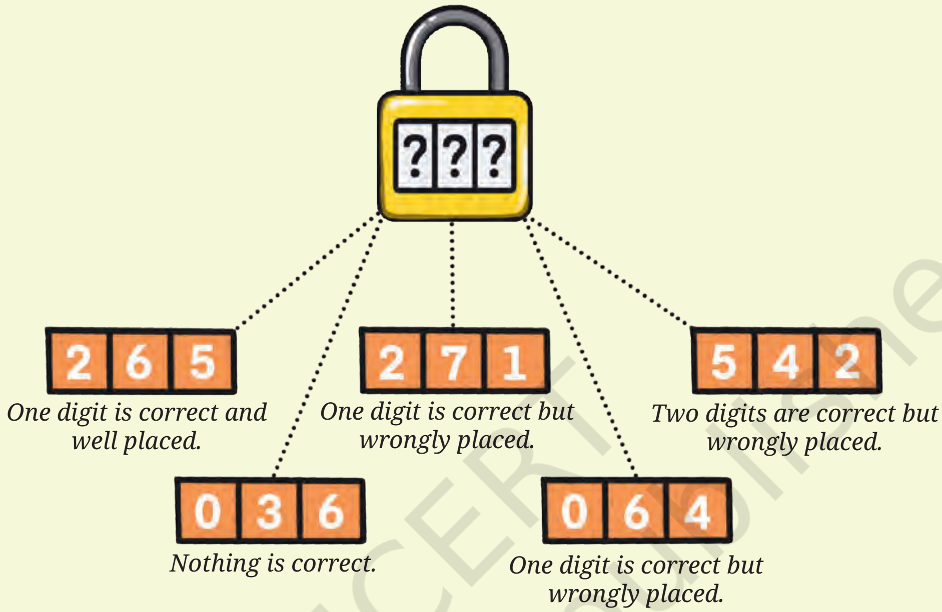
SUMMARY

- Dot plots help us get a quick glimpse of the variability of the data — minimum, maximum, range, and how data is clustered or spread out.
- The Arithmetic Mean = $\frac{\text{Sum of all the values in the data}}{\text{Number of values in the data}}$.
- The Median is the number in the middle of any sorted data. If there are an even number of values, then the median is the average of the two middle numbers.
- We can describe and compare data in several ways including by referring to the minimum, maximum, total, range, arithmetic mean, and median.
- We learnt how to read and make clustered bar graphs. These graphs can be used to compare and visualise values across categories and across time.
- Examining data can lead to new questions and directions to probe further.

IT'S PUZZLE TIME!

Connect the Dots...

A number lock has a 3-digit code. Find the code using the hints below.



6 CONSTRUCTIONS AND TILINGS

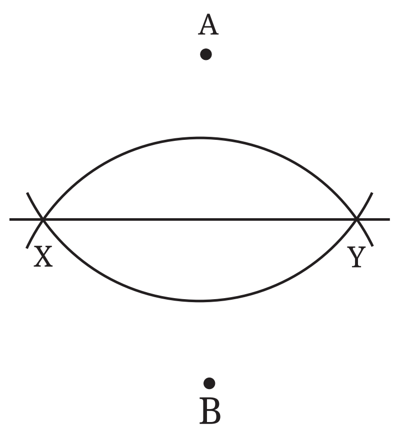
6.1 Geometric Constructions

Eyes

Do you recall the 'Eyes' construction we did in Grade 6?

Of course, eyes can be drawn freehand, but we wanted to construct them so that the lower arc and upper arc of each eye look symmetrical.

We relied on our spatial estimation to determine the two centres, A and B (see the figure), from which we drew the lower arc and upper arc respectively.



The arcs define a line XY that 'supports' the drawing though it is not part of the final figure. We can start with this supporting line and systematically find the centres A and B.

For the eye to be symmetrical, or for the supporting line to be the line of symmetry, the upper and lower arcs should have the same radius. In other words, we must have AX = BX. Since AX = AY and BX = BY, this means

$$AX = AY = BX = BY$$

How do we find such A and B?

From X and Y, draw arcs above and below XY, with the same radii.

The two points at which the arcs meet, above and below XY, give us A and B, respectively.

Use this to construct an eye.

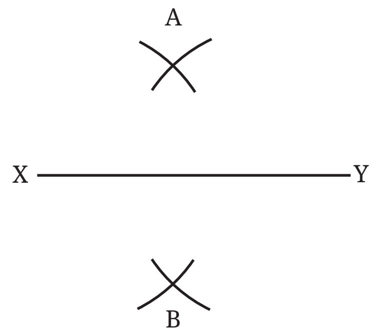
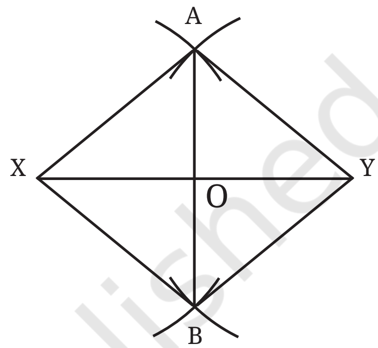


Fig. 6.1

In Fig. 6.1, join A and B with a line. Where does AB intersect XY, and what is the angle formed between them?

We observe that AB passes through the midpoint of XY, and is also perpendicular to it.

A division of a line, or any geometrical object, into two identical parts is called **bisection**. A line that bisects a given line and is perpendicular to it, is called the perpendicular bisector.

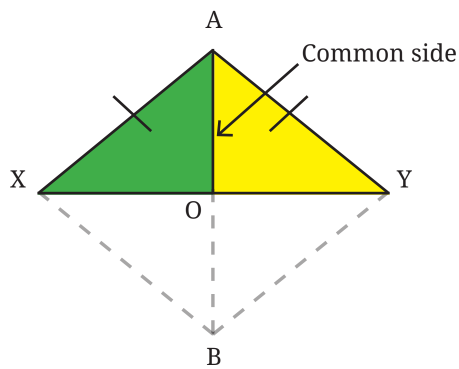


Will the line joining the two points at which the arcs meet, above and below XY, always be the perpendicular bisector of XY, i.e., when XY is of any length, and the arcs are drawn using a radius of any length?

This can be answered through congruence. Let us consider a line segment XY. Find points A and B such that AX = AY = BX = BY. Draw the lines AB, AX, AY, BX and BY. Let O be the point of intersection between AB and XY.

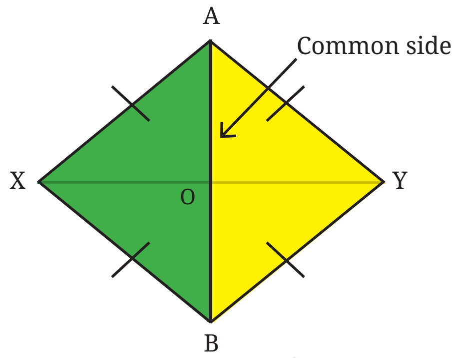
Which two triangles should be congruent for AB to be the perpendicular bisector of XY (that is, O is the midpoint of XY and AB is perpendicular to XY)?

If we show that $\Delta AOX \cong \Delta AOY$, then OX = OY, and $\angle AOX = \angle AOY$ because they are corresponding parts of congruent triangles. Since $\angle AOX$ and $\angle AOY$ together form a straight angle, we have $\angle AOX + \angle AOY = 180°$. Thus, $\angle AOX = \angle AOY = 90°$. This establishes that O is the midpoint of XY and AB is perpendicular to XY.



In ΔAOX and ΔAOY, we already know that AX = AY, and AO is common to both triangles.

If we can show that ∠XAO = ∠YAO then, by the SAS congruence condition, we can conclude that the triangles are congruent.

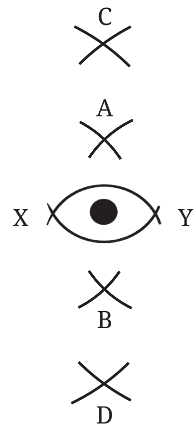


To show this, we observe that ΔABX ≅ ΔABY. This is so because AX = AY, BX = BY, and AB is common to both the triangles.

Thus, we have ∠XAB = ∠YAB, or ∠XAO = ∠YAO because they are corresponding parts of congruent triangles.

Hence, AB is the perpendicular bisector of XY.

We can have eyes of different shapes.



? How do we get these different shapes? Try!

One way is to choose two other points C and D such that CX = CY = DX = DY. An eye of a different shape can be drawn using these points.

? Will C and D lie on the perpendicular bisector AB?

The points C and D are at the same distance from both X and Y. We have just seen that joining any two such points gives the perpendicular bisector of XY. Since XY has only one perpendicular bisector, which is the line AB, the points C and D must lie on the line AB.

? Justify the following statement using the facts that we have established.

Any point that has the same distance from X and Y lies on the perpendicular bisector of XY.

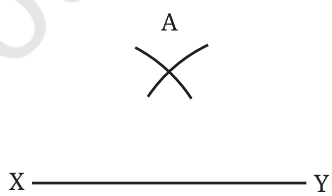
Thus, eyes of different shapes can be drawn by suitably choosing different pairs of points on the perpendicular bisector as centres to construct the upper and lower arcs of the eyes.

Construction of Perpendicular Bisector

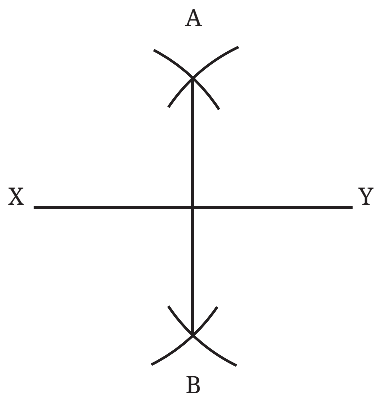
? Given a line segment XY, how do we draw its perpendicular bisector using only an unmarked ruler and a compass?

We have seen that joining any two points — one above XY and one below — that are at equal distances from X and Y, gives the perpendicular bisector of XY. This gives a method to construct the perpendicular bisector.

1. Taking some fixed radius, from X and then Y, construct two sufficiently long arcs above XY. Name the point where the arcs meet as A.



2. Using the same radius, from X and then Y, construct two sufficiently long arcs below XY. Name the point where the arcs meet as B.
3. AB is the required perpendicular bisector.



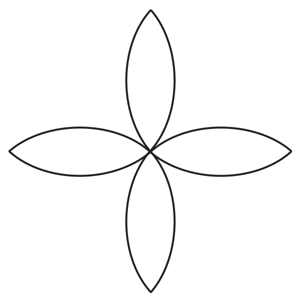
Thus, the perpendicular bisector can be constructed using the simplest geometric tools — an unmarked ruler and a compass. We will use only these two tools for all the other geometric constructions in this chapter, unless there is a need for drawing lines of specific lengths in standard units.

Figure it Out

1. When constructing the perpendicular bisector, is it necessary to have the same radius for the arcs above and below XY? Explore this through construction, and then justify your answer.

   [**Hint 1:** Any point that is of the same distance from X and Y lies on the perpendicular bisector.

   Hint 2: We can draw the whole line if any two of its points are known.]
2. Is it necessary to construct the pairs of arcs above and below XY? Instead, can we construct both the pairs of arcs on the same side of XY? Explore this through construction, and then justify your answer.
3. While constructing one pair of intersecting arcs, is it necessary that we use the same radii for both of them ? Explore this through construction, and then justify your answer.
4. Recreate this design using only a ruler and compass —



After completing the above design, you can use a colour pencil with a ruler or compass to trace its boundary. This will make the design stand out from the supporting lines and arcs.

This method of constructing the perpendicular bisector is not only geometrically exact but also a practical way to construct it accurately.

This method to find the midpoint of a line segment is more accurate than measuring the length using a marked scale.

Construction of a 90° Angle at a Given Point

? Can we extend the method of constructing the perpendicular bisector to construct a 90° angle at any point on a line? Draw a line and mark a point O on it. Construct a 90° angle at point O.

? Find a segment of this line for which O is the midpoint.

Extend the line on either side of O.

Using a compass, mark two points X and Y at equal distance from O, so that O is the midpoint of XY.

Fig. 6.2

The perpendicular bisector of XY will pass through O and is perpendicular to the given line.

In this case, do we need to draw two pairs of intersecting arcs to get the perpendicular bisector of XY?

No, we don't. We already have one point, O, lying on the perpendicular bisector.

Figures 6.2 and 6.3 describe the steps to construct a 90° angle at a given point on a line.

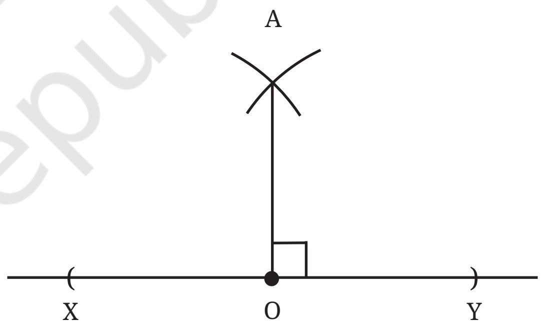


Fig. 6.3

Construction Methods in Śulba-Sūtras

Ancient mathematicians from different civilizations, including India, knew exact procedures to construct perpendiculars and perpendicular bisectors.

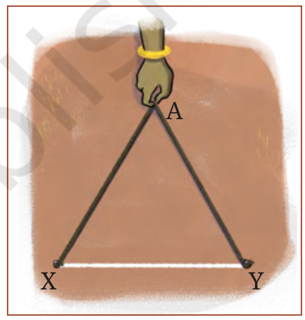
In India, the earliest known texts containing these methods are the *Śulba-Sūtras*. These are geometric texts of Vedic period dealing with the construction of fire altars for rituals. The *Śulba-Sūtras* are part of one of the six *Vedāṅgas* (a term that literally means 'limbs of the Vedas'). The *Śulbas* contain the methods that we developed earlier to construct a perpendicular and the perpendicular bisector. All the construction methods in the *Śulba-Sūtras* make use of a different kind of compass from what you would have used — a rope. A rope can be used to draw circles or arcs. It can also be stretched to form a straight line.

In addition, the *Śulba-Sūtras* also contain other methods to construct perpendicular lines. Here is an interesting construction of the perpendicular bisector using a rope (Kātyāyana-Śulbasūtra 1.2).

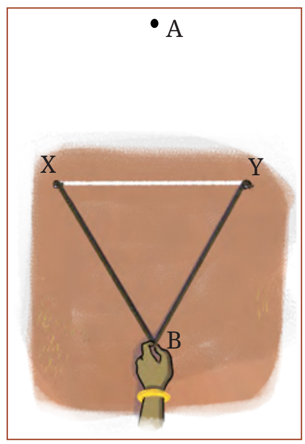
Let XY be the given line segment, drawn on the ground, for which we need to construct a perpendicular bisector. Fix a small pole or peg vertically into the ground at each point X and Y.

Take a sufficiently long rope. Make two loops at its ends. Without taking into account the parts of the rope that has gone into the loops, fold the rope into half to find and mark its midpoint.

Fasten the two loops at the ends of the rope to the poles at X and Y.



Pull the midpoint of the rope above XY, as shown in the figure, such that the two parts of the rope on either side are fully stretched. Mark this position of the midpoint as A.



Now similarly pull the midpoint of the rope below XY, as shown in the figure, such that the two parts of the rope on either side are fully stretched. Mark this position of the midpoint as B.

AB is the required perpendicular bisector.

Fig. 6.4

Figure it Out

1. Justify why AB in Fig. 6.4 is the perpendicular bisector.
2. Can you think of different methods to construct a 90° angle at a given point on a line using a rope?

Angle Bisection for a Design

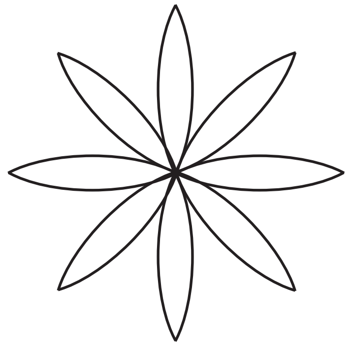
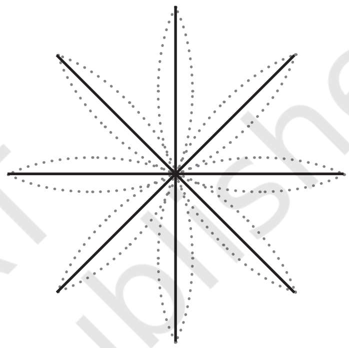


Fig. 6.5

How do we construct this figure?



The supporting lines for this figure will look like this.

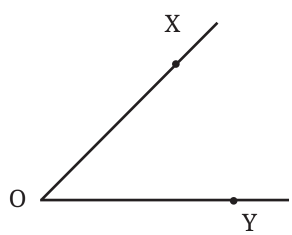
What is the angle between two adjacent lines?

We need the angle between every pair of adjacent lines to be equal. Since 360° is equally divided into 8 parts, every angle is 45°.

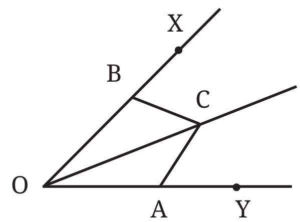
How do we construct a 45° angle using only a ruler and a compass?

We know how to construct a 90° angle. If we can divide it into two equal parts, or bisect it, then we get a 45° angle.

We will now develop a general method to bisect any angle. Consider an angle ∠XOY.



We can bisect it if we can draw two congruent triangles ΔOBC and ΔOAC as shown in the figure. Then ∠BOC = ∠AOC.

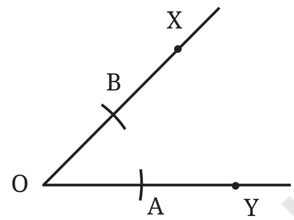


? How do we construct these congruent triangles, given the angle?

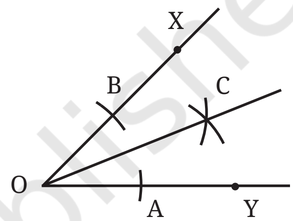
If A and B are marked such that OA = OB, and if C is chosen such that BC = AC, then by the SSS congruence condition, $\Delta OBC \cong \Delta OAC$. So we can bisect an angle as follows.

Steps for Angle Bisection

1. Mark points A and B such that OA = OB.



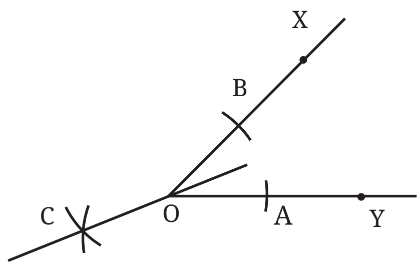
2. Choosing any sufficiently long radius, cut arcs from A and B, keeping the radius same. Mark the point of intersection as C.
3. OC bisects $\angle AOB$.



So, a 45° angle can be constructed by first constructing a 90° angle and then bisecting it.

? Figure it Out

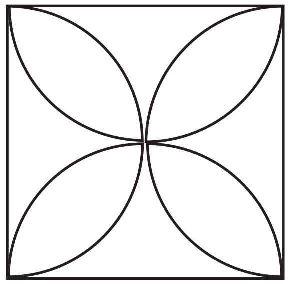
1. Construct at least 4 different angles. Draw their bisectors.
2. Construct the 8-petalled figure shown in Fig. 6.5.
3. In Step 2 of angle bisection, if arcs of equal radius are drawn on the other side, as shown in the figure, will the line OC still be an angle bisector? Explore this through construction, and then justify your answer.



4. What are the other angles that can be constructed using angle bisection? Can you construct 65.5° angle?
5. Come up with a method to construct the angle bisector using a rope.

Math Talk

6. Construct the following figure.



How do we construct the petals so that they are of the maximum possible size within a given square?

Repeating Units and Repeating Angles

? Construct the following figure.

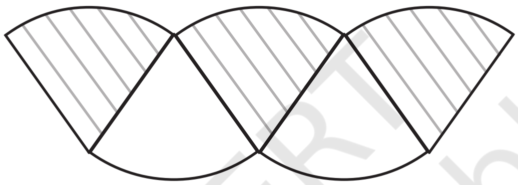
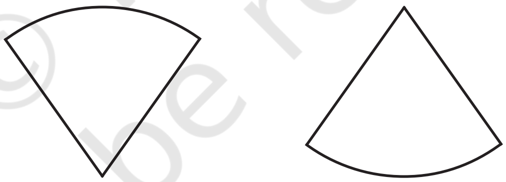


Fig. 6.6

In this figure, there is a single unit repeating itself. To construct this figure, we need to make exact copies of this unit in two different orientations.

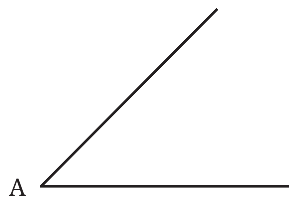


In order to make exact copies, all the units must have the same arm lengths and the same angle between the arms. We can ensure equal arm lengths using a compass, but how do we ensure equal angles?

Let us develop a method to create an exact copy of a given angle.

? Draw an angle. Create a copy of this angle using only a ruler and compass.

Math Talk

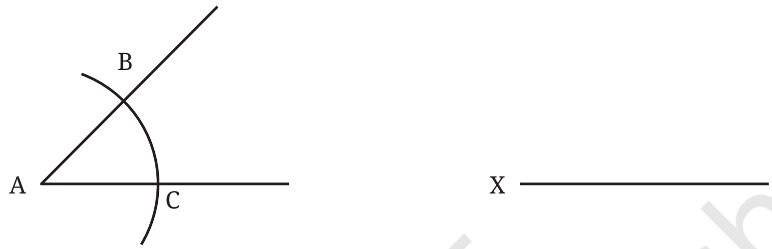


You might have developed your own method. Here is one simple approach.

Steps of Construction to Copy an Angle

1.

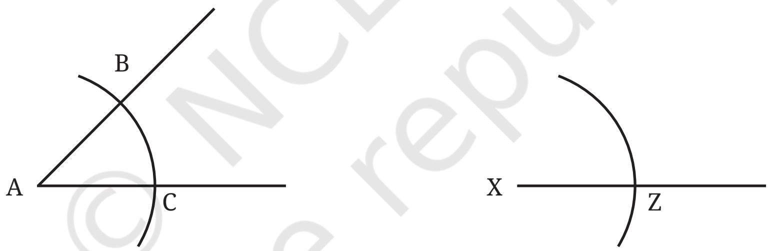
2.



Draw an arc from A.

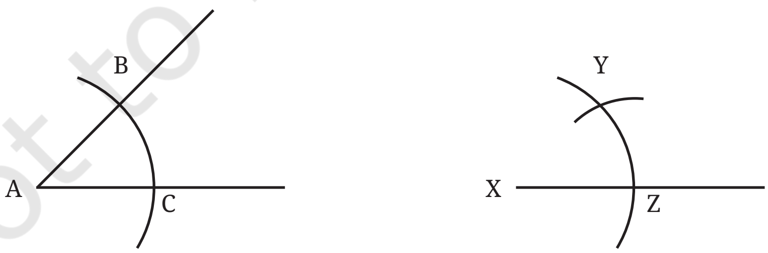
This gives us three points that form the isosceles triangle ΔABC.

3.



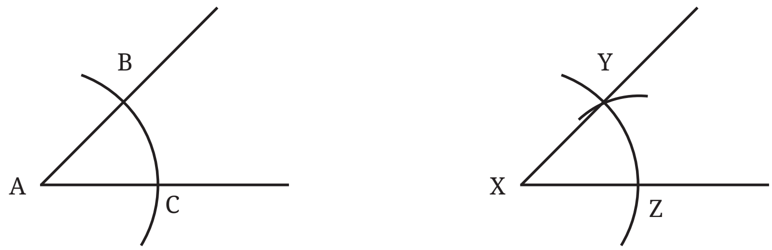
Draw an arc of the same radius from X.

4.



Measure BC using a compass. Transfer this length on the arc from Z to get YZ = BC.

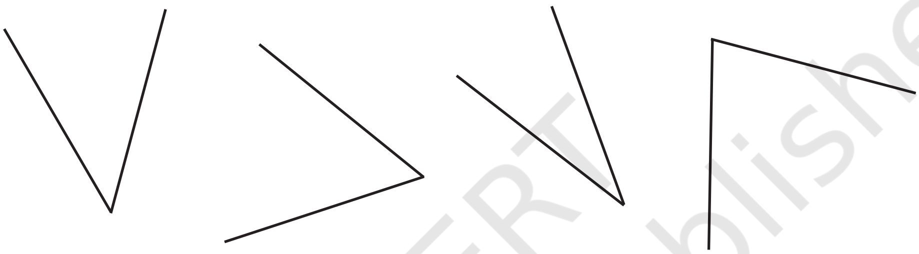
5.



By the SSS congruence condition, $\Delta ABC \cong XYZ$. So, $\angle A = \angle X$.

Figure it Out

1. Construct at least 4 different angles in different orientations without taking any measurement. Make a copy of all these angles.



2. Construct the Fig. 6.6.

This procedure to copy an angle finds an important application in constructing parallel lines using only a ruler and compass.

Construction of a Line Parallel to the Given Line

Recall that in the construction using a ruler and a set square, we constructed equal corresponding angles to get parallel lines.

How do we implement this idea using a ruler and a compass?

Suppose there is a line m to which we need to draw a parallel line. We construct a line l that intersects m. Line l will serve as a transversal to line m and to the line parallel to m that we are going to construct.

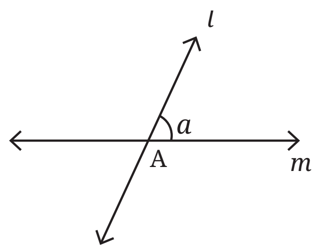
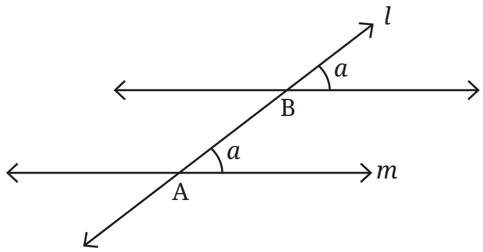


Fig. 6.7

Let us choose a point B on l through which we are going to draw the parallel line. This parallel line must make the same corresponding angle, as shown in the figure. This can be done by copying the angle between m and l.



Here is a step-by-step procedure for carrying this out.

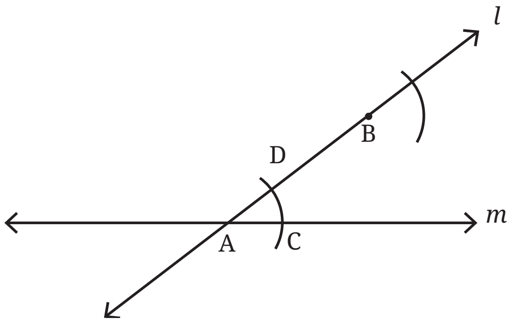


Fig. 6.8

Construct arcs of equal radius from A and B

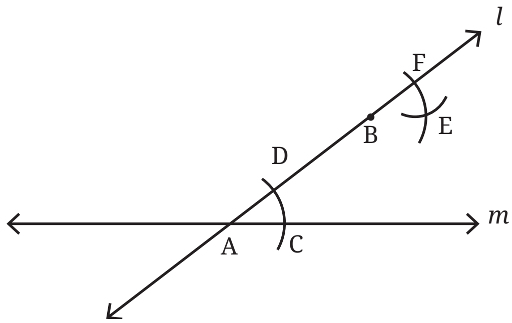


Fig. 6.9

Transfer the length CD to the arc from F

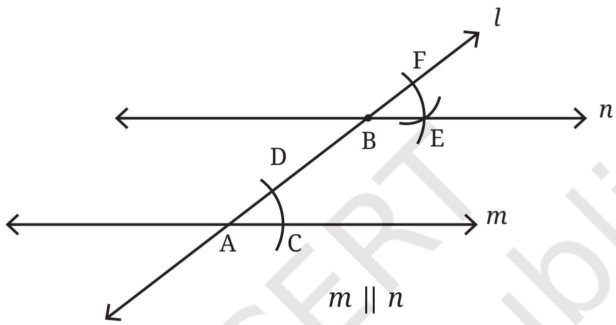
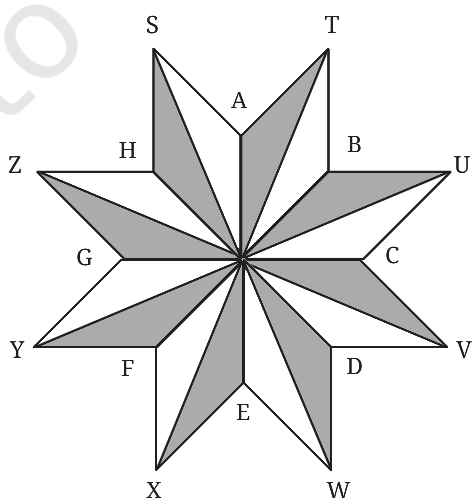


Fig. 6.10

Figures 6.7–6.10 describe a method to construct a line parallel to the given line.

Figure it Out

1. Construct 4 pairs of parallel lines in different orientations.
2. Construct the following figure.



Arch Designs

Trefoil Arch

Have you seen this kind of beautiful arch?

https://commons.wikimedia.org/w/index.php?curid=28374748
Diwan-i-Aam, Red fort

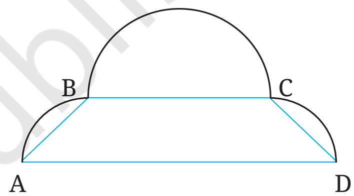
https://commons.wikimedia.org/wiki/File:Central_Park_A.jpg
Central park, New York City

? How did they make these arches?

The first step is to be able to draw them on a plane surface such as paper or stone.

? Construct this arch shape on a piece of paper.

Let us think about the support lines this figure will need.



For symmetry, we should have AB = CD, and $\angle$BAD = $\angle$CDA. How would you construct these support lines?

Construct equal angles at A and D. Mark B and C such that AB = CD.

? Use these support lines to construct an arch. If required, adjust the radii of the arcs to make the arch look more aesthetically pleasing.

A Pointed Arch

Some arches look like this.

https://en.m.wikipedia.org/wiki/File:Diwan-i-Aam,_Red_Fort,_Delhi_-_2.jpg
Diwan-i-Aam, Red Fort

? How do we construct this shape?

What supporting lines will you use to draw this arch?

Remember 'Wavy Wave' from the Grade 6 Textbook?

The supporting lines are just two line segments of equal length.

? If their midpoints are marked, will you be able to construct a pointed arch?

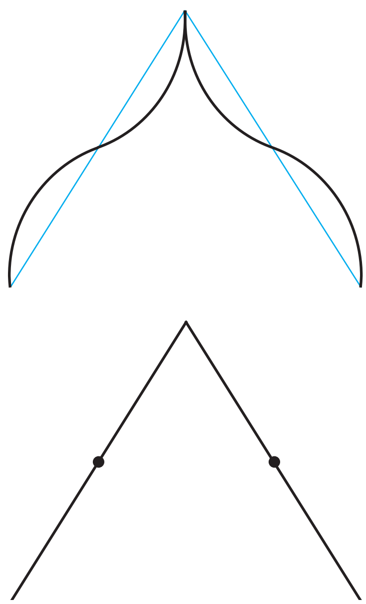


Fig. 6.11

Figure it Out

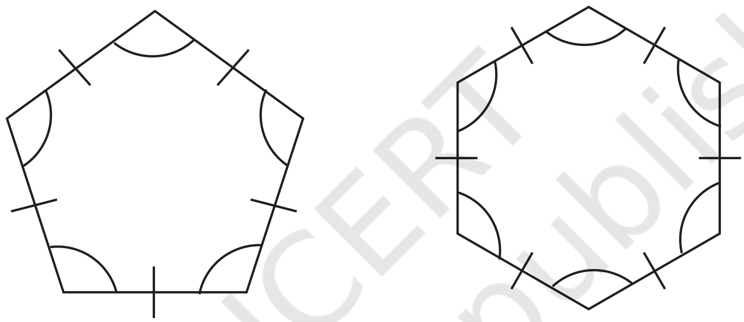
1. Use support lines in Fig. 6.11 to construct a pointed arch. Make different arches, by changing the radius of the arcs.
2. Make your own arch designs.

Regular Hexagons

Recall that a regular polygon has equal sides and equal angles. A regular polygon with 3 sides is an equilateral triangle, and a regular polygon with 4 sides is a square. We have constructed these figures earlier.

How do we construct a regular pentagon (5-sided figure) and a regular hexagon (6-sided figure)? To begin with, try to construct a pentagon and hexagon with equal sidelengths.

Try This



To construct a regular pentagon, we first need to have a better understanding of triangles and pentagons. We will discuss this in later years. However, constructing a regular hexagon is within our reach!

Can we break a regular hexagon into smaller pieces that can be constructed?

Regular Hexagon and Equilateral Triangles

What happens when we join the 'opposite' points of a regular hexagon? Since a regular hexagon has equal sides and angles, can we expect a figure like this?

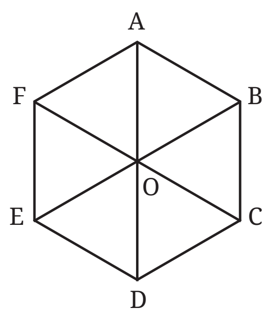


Fig. 6.12

Will all the triangles in the figure be equilateral triangles?

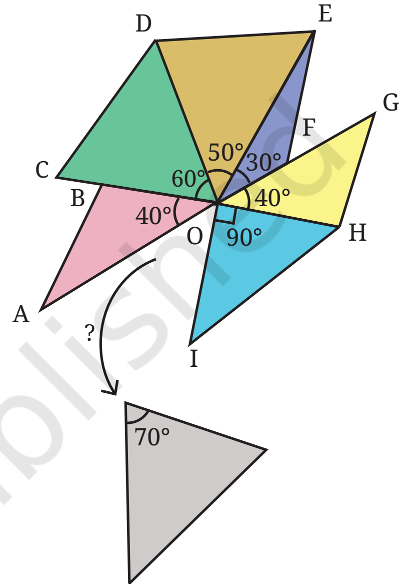
To answer these questions, we will reverse our approach.

Can six congruent equilateral triangles be placed together as in Fig. 6.12? If yes, will it result in a regular hexagon?

If six congruent equilateral triangles can indeed be placed as shown in Fig. 6.12, then the sides of the resulting hexagon are equal, and their angles are 60 + 60 = 120° (how?). So what we really need to examine is whether 6 congruent equilateral triangles can fit this way without overlapping and without leaving any gaps around the centre.

We have defined a degree by taking the complete angle around a point to be 360°. So all the angles around the centre should add up to 360°.

? Consider this figure. Will the 70° angle fit into the gap? What is the gap angle ∠AOI?



We have,

40° + 60° + 50° + 30° + 40° + 90° + gap angle = 360°.

Use this to determine whether the 70°angle fits the gap.

Thus, if there are angles that add up to 360°, their vertices can be joined together at a single point such that

(a) the angles do not overlap, and

(b) they completely cover the region around the point.

Since each angle in an equilateral triangle is 60°, six such angles add up to 360°. Therefore, six congruent triangles can be arranged as shown in Fig. 6.12.

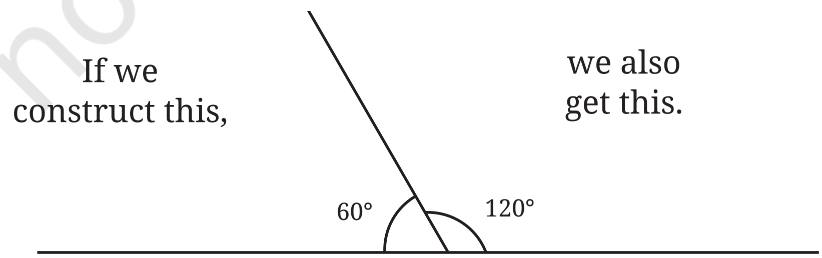
? In Fig. 6.12 can you explain why AOD, BOE and COF are straight lines?

? Construct a regular hexagon with a sidelength 4 cm using a ruler and a compass.

We can construct a regular hexagon more directly if we can construct a 120° angle using a ruler and a compass.

? How do we do it?

This can be done if we can construct a 60° angle.



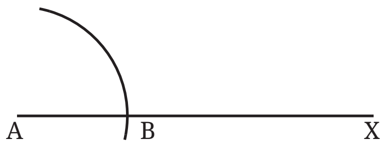
Construction of a 60° angle

? How do we construct a 60° angle?

We get a 60° angle if we construct an equilateral triangle! We can use the following steps for this.

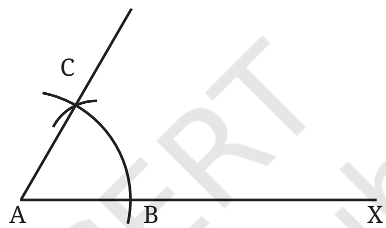
Suppose we need a 60° angle at point A on a line segment AX.

Step 1



Construct an arc with centre A and any radius.

Step 2

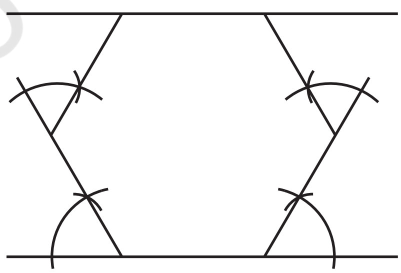


With the same radius, cut another arc from B that meets the first arc. Let C be the point at which the arcs meet.

We have $\angle CAX = 60°$.

? Why is $\angle CAX = 60°$? Is there an equilateral triangle here?

We can use these ideas to construct a regular hexagon—



? Construct a regular hexagon of sidelength 5 cm.

Related Constructions

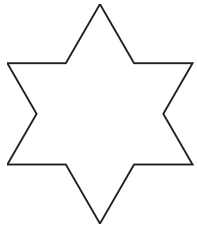
Construction of 30° and 15° angles

? How will you construct 30° and 15° angles?

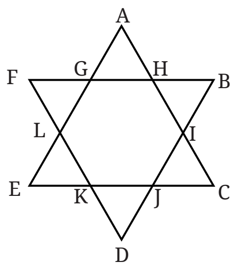
6-Pointed Star

? Construct the following 6-pointed star. Note that it has a rotational symmetry.

Math Talk



Hint:



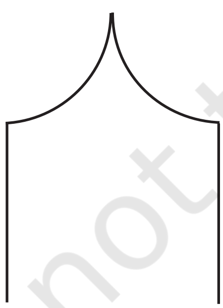
Do you see a hexagon here?

? Are the six triangles forming the 6 points of the star — ΔAGH, ΔBHI, ΔCIJ, ΔDJK, ΔELK, ΔFLG — equilateral? Why?

[**Hint:** Find the angles.]

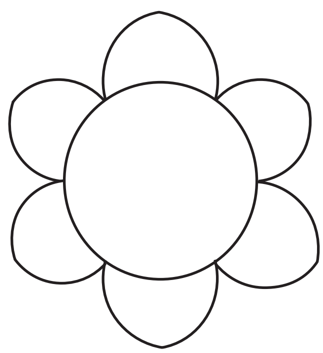
Figure it Out

1. Construct the following figures:



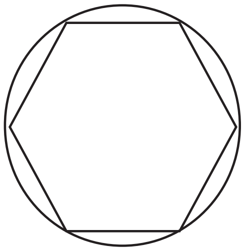
(a)

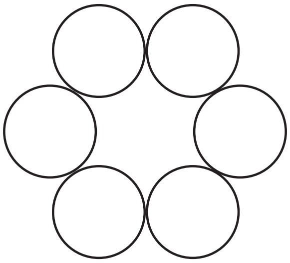
An Inflexed Arc

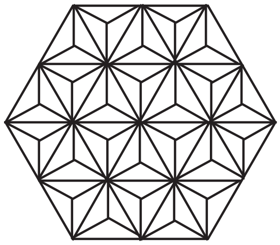


(b)

The fun part about this figure is that it can also be constructed using only a compass! Can you do it?


(c)

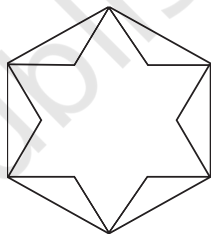

(d)


(e)

2. Optical Illusion: Do you notice anything interesting about the following figure? How does this happen? Recreate this in your notebook.

3. Construct this figure.

   [**Hint:** Find the angles in this figure.]

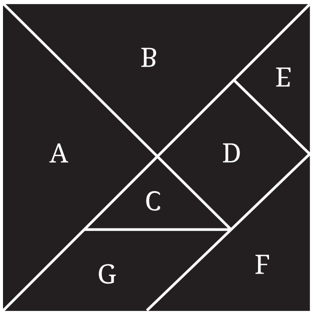


4. Draw a line l and mark a point P anywhere outside the line. Construct a perpendicular to the given line l through P.
   [**Hint:** Find a line segment on l whose perpendicular bisector passes through P.]

Try This

6.2 Tiling

Tangrams are puzzles that originated in China. They make use of 7 pieces obtained by dividing a square as shown.



For the problems ahead, we need these 7 tangram pieces. These are provided at the end of the book. Or, by looking at the figure, you could make cardboard cutouts of the pieces.

We can form interesting pieces by rearranging the tangram pieces. Here is an arrow.

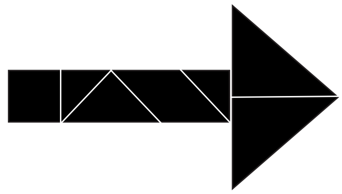
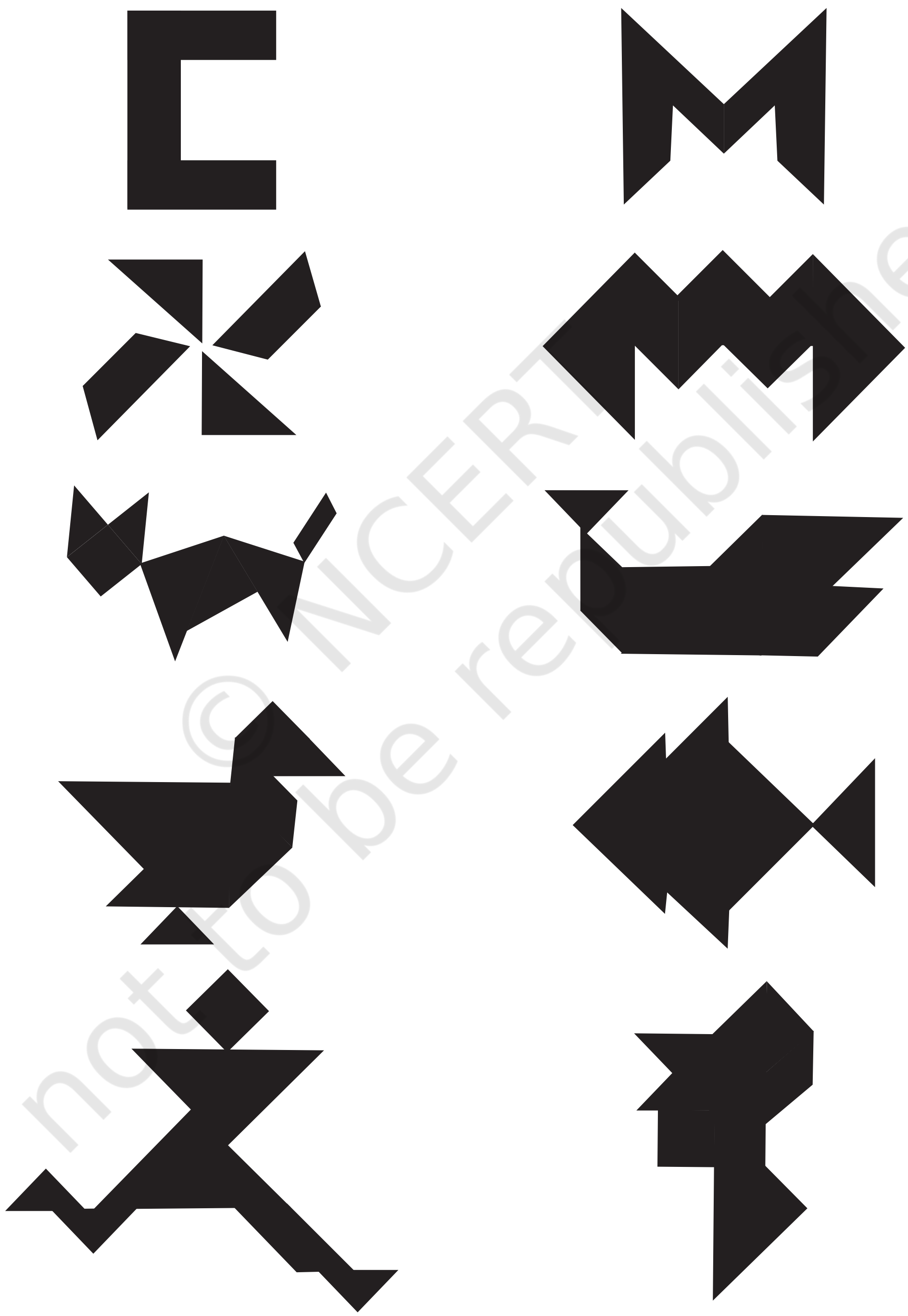


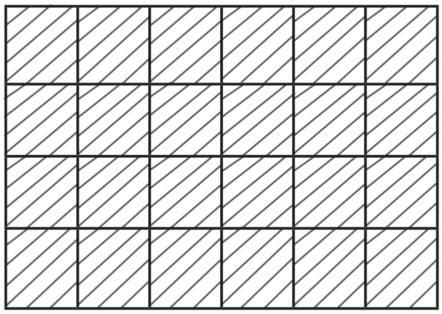
Figure it Out

How can the tangram pieces be rearranged to form each of the following figures?



Covering a region using a set of shapes, without gaps or overlaps, is called **tiling**.

Consider a rectangular grid made of unit squares —



We call this a 4 × 6 grid, since it has 4 rows and 6 columns.

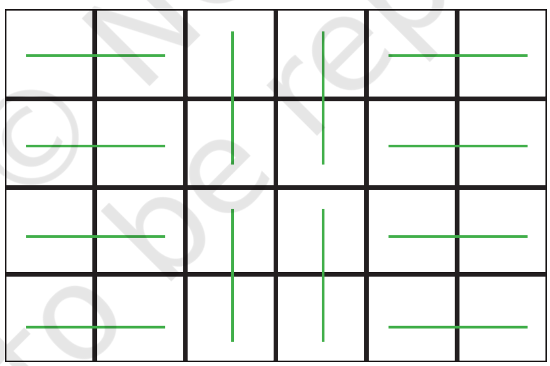
Math Talk

? Can a 4 × 6 grid be tiled using multiple copies of 2 × 1 tiles?

We are allowed to rotate a 2 × 1 tile and use it.

Vertical tile *Horizontal tile*

Here is one way.



Obviously, this is not the only tiling possible.

? Can a 4 × 7 grid be tiled using 2 × 1 tiles?

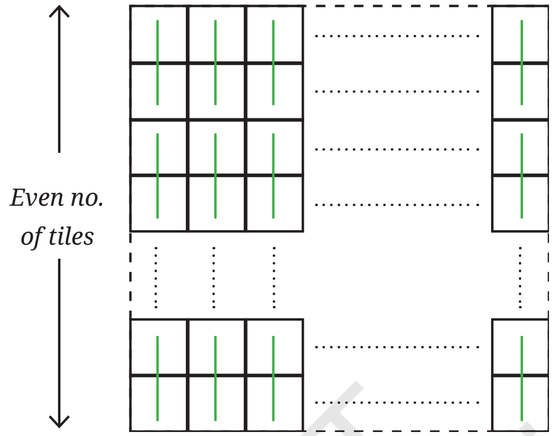
? What about a 5 × 7 grid?

To see that there is no way to tile a 5 × 7 grid using 2 × 1 tiles, observe that this grid has 35 unit squares. Each tile covers exactly 2 unit squares.

? Complete the justification.

? Is an $m \times n$ grid tileable with 2×1 tiles, if both m and n are even? If yes, come up with a general strategy to tile it.

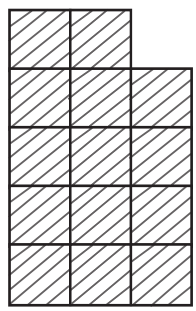
One general strategy for this case is to cover each column with vertical tiles. This is possible because the number of rows is even.



? Is an $m \times n$ grid tileable with 2×1 tiles, if one of m and n is even and the other is odd? If yes, come up with a general strategy to tile it.

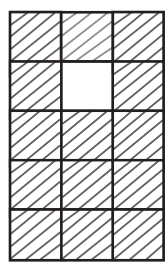
? Is an $m \times n$ grid tileable with 2×1 tiles, if both m and n are odd? Give reasons.

? Here is a 5×3 grid, with a unit square removed. Now, it has an even number of unit squares. Is it tileable with 2×1 tiles?



Math Talk

? Is the following region tileable with 2×1 tiles?



Math Talk

? What about this one?

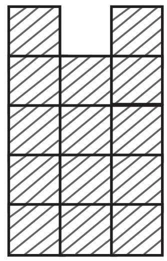


Fig. 6.13

? Were you able to tile this? How can we be sure that this is not tileable? Can you find another unit square that, when removed from a 5 × 3 grid, makes it non-tileable?

There is an interesting way to look at these questions. For any tiling problem of this kind, we can create a similar problem with the unit squares coloured black and white so that black squares have only white neighbours and white squares have only black neighbours. For the tiling problem in Fig. 6.13, we get the following.

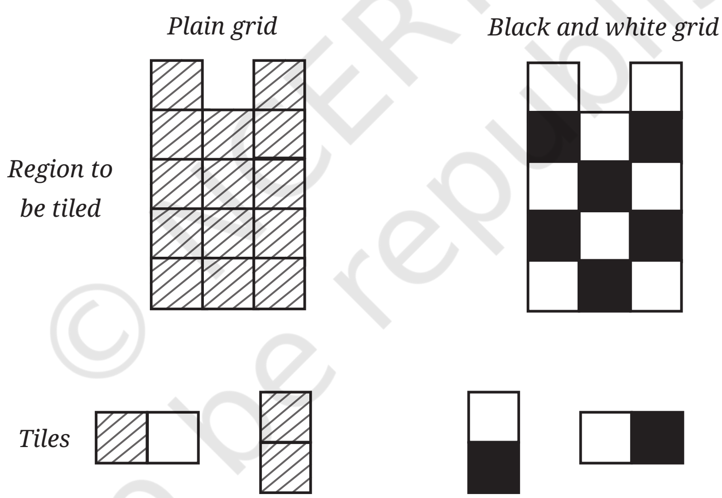


Fig. 6.14

In the black-and-white region, the problem is to tile the region with the 2 × 1 black-and-white tiles so that each black square of a tile sits on a black square of the grid, and each white square sits on a white square.

? If the plain grid is tileable, is the black-and-white-grid tileable?

? If the black-and-white grid is tileable, is the plain grid tileable?

It can be seen that the answer to both the questions is yes.

Is the black-and-white region in Fig. 6.14 tileable?

Any region tiled with black-and-white-tiles must have an equal number of black tiles and white tiles.

Since the black-and-white region in Fig. 6.14 has 8 white squares and 6 black squares, it can never be tiled with these tiles!

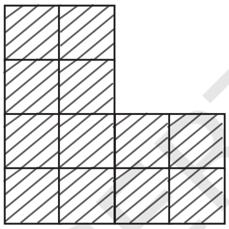
Use this idea to find another unit square that, when removed from a 5 × 3 grid, makes it non-tileable? Math Talk

Isn't it surprising how, by introducing colours and making the problem more complicated, it becomes easier to tackle? What a creative way of looking at the problem!

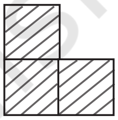
Figure it Out

Are the following tilings possible?

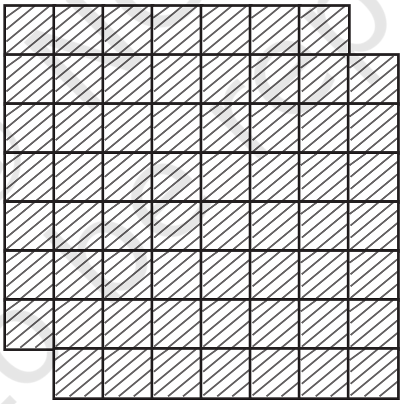
1. *Region to be tiled*



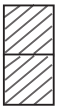
Tile



2. *Region to be tiled*



Tile

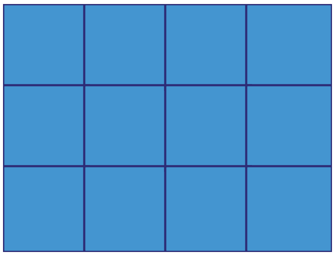


Tiling the Entire Plane

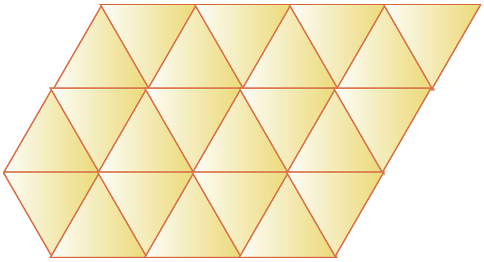
So far we have seen how to tile a given region. What about tiling the entire plane?

Can you think of a shape whose copies can tile the entire plane?

Clearly, squares can.

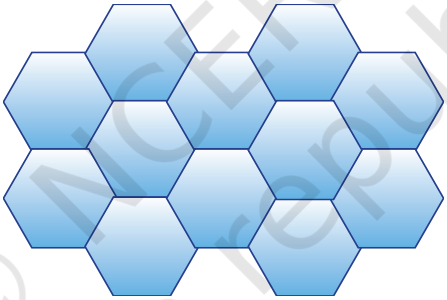


Are there other regular polygons that can tile the plane?
What about equilateral triangles?

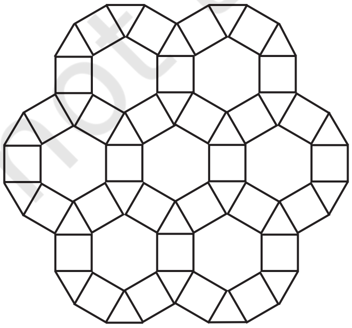


Tiling with equilateral triangles shows the possibility of tiling with another regular polygon.

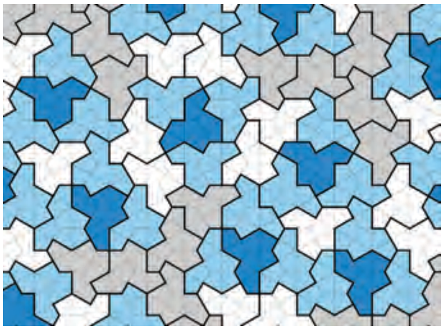
A plane can be tiled using regular hexagons as well.



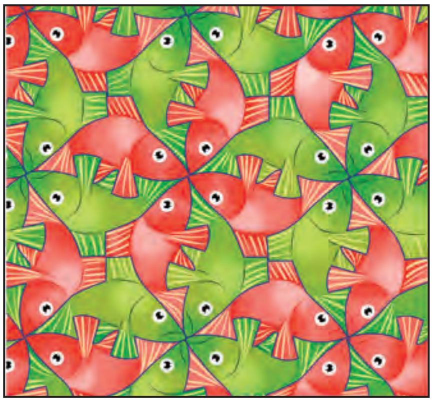
A plane can also be tiled using more than one shape, and using non-regular polygons. People such as the great Dutch artist Escher (1898 – 1972) — whose works explored mathematical themes such as tiling — have come up with creative ways of tiling a plane with animal shapes! Here are some examples.



(a)



(b)

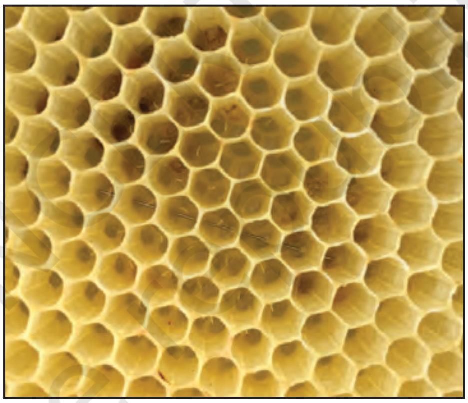


(c)

(d)

Mathematicians are still exploring various ways of tiling the plane! Tiling (b) was found as recently as 2023.

Have you seen tilings in daily life? They are often used in buildings and in designs. Tilings are found in nature too. The front face of bee hives and some wasp nests are tiled using hexagonal cells!



These cells are used by the insects to keep their eggs, larvae and pupae safe, as well as to store food. Because the region is tiled, no space is wasted.

Scientists still wonder how bees and wasps are able to make hexagonal cells. Next time you see any tiling, pay closer attention to it!

Tiling is still one of the most exciting and active areas of research in geometry.

SUMMARY

- A division of a line segment, or any geometrical quantity, into two identical parts is called **bisection**.
- Any point that is of equal distance from the two endpoints of a given line segment lies on its **perpendicular bisector.** This property can be used to construct the perpendicular bisector using a ruler and compass.
- The method of constructing the perpendicular bisector can be modified to draw a 90° angle at any point on a line using only a ruler and compass.
- An angle can be bisected and copied using the congruence properties of triangles.
- A 60° angle can be constructed using a ruler and compass by constructing an equilateral triangle.
- Covering a region using a set of shapes, without gaps or overlaps, is called **tiling**.

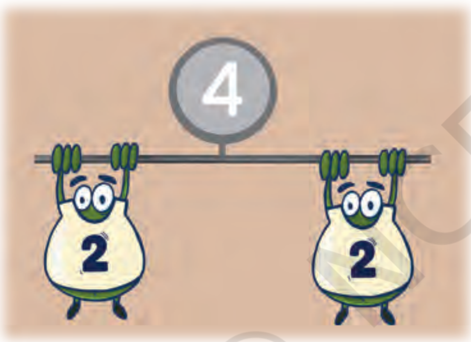
7 FINDING THE UNKNOWN

0789CH07

7.1 Find the Unknowns

Unknown Weights

We have a weighing scale that behaves as follows. The numbers represent same units of weight:



Find the unknown weights in the following cases:

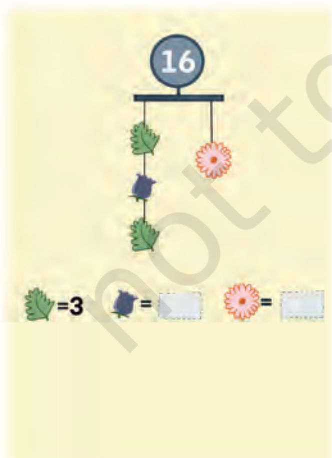


Fig. 7.1

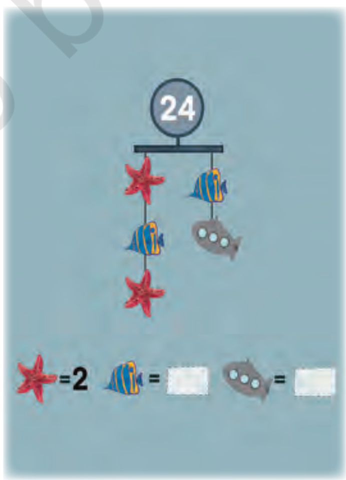


Fig. 7.2

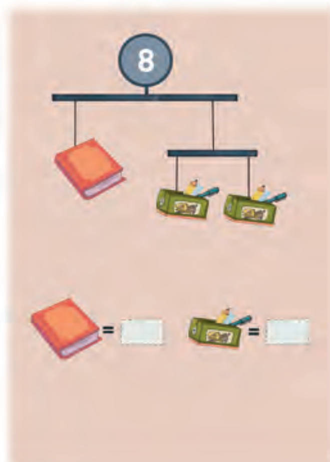


Fig. 7.3

Fig. 7.4

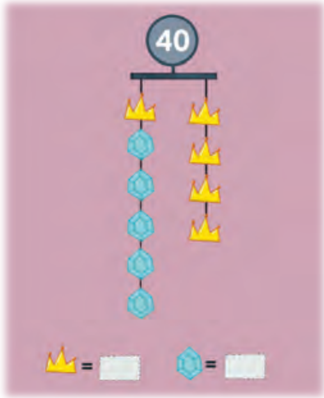


Fig. 7.5

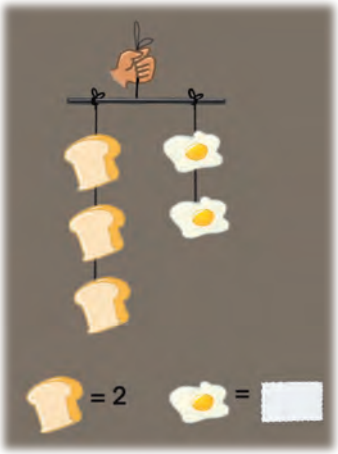


Fig. 7.6

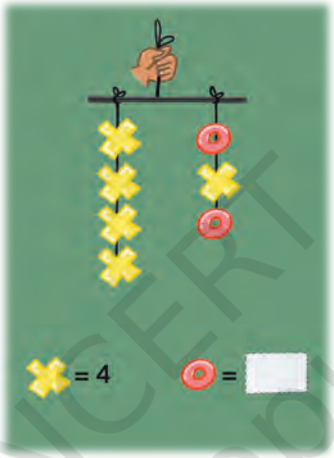


Fig. 7.7

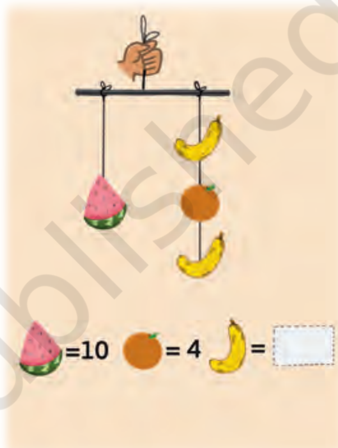


Fig. 7.8

? Discuss the answers with your classmates. Give reasons why you think your answer is right.

Math Talk

? Find the unknown weight of the sack in the following cases. In Fig. 7.10, all the sacks have the same weight.

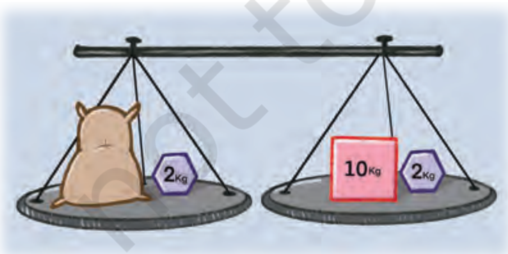


Fig. 7.9

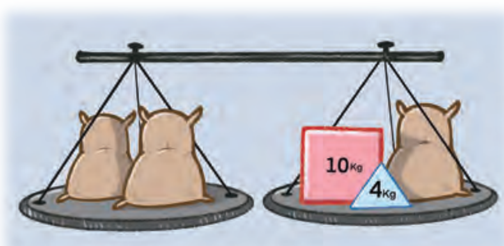


Fig. 7.10

[Hint: If we remove equal weights from both the plates, will the weighing scale still be balanced? Remove one sack from each plate for Fig. 7.10.]

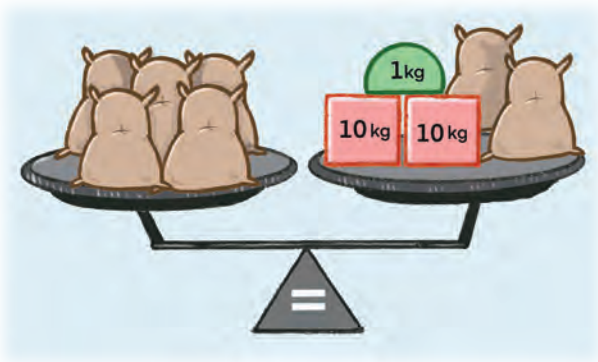


Fig. 7.11

[**Hint:** Can you remove objects so that the sacks are only on one plate?]

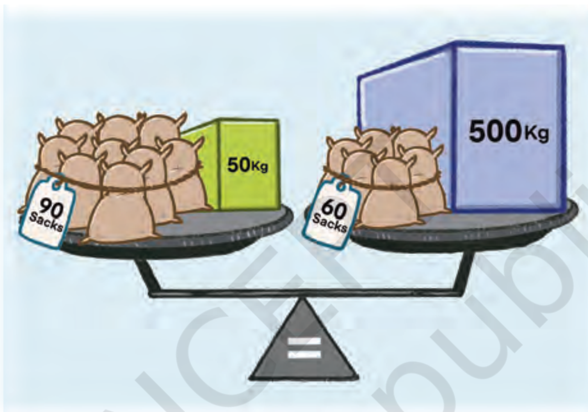


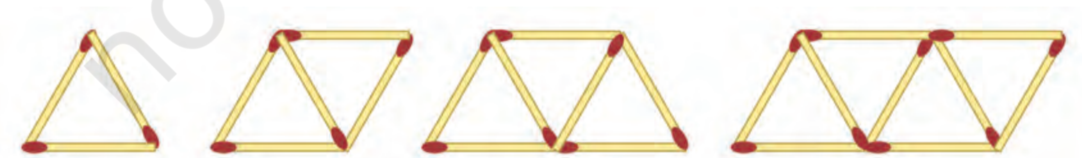
Fig. 7.12

Note to the Teacher: Encourage your students to find solutions to these problems using different strategies and methods, and ask them to compare and contrast their methods.

Let us find an unknown value in a different setting.

Matchstick Pattern

Consider this sequence of matchstick arrangements.



1 2 3 4

Recall that we have studied this sequence in an earlier chapter.

The figure shows the first 4 matchstick arrangements along with their position numbers in the sequence.

? Jasmine decides to make a matchstick arrangement that appears in this sequence, using exactly 99 sticks. What will be the position number of this arrangement in the sequence?

To answer this question, it is useful to know the number of matchsticks in each position.

We see that

the arrangement at position 1 has $2 \times (1) + 1 = 3$ matchsticks,
the arrangement at position 2 has $2 \times (2) + 1 = 5$ matchsticks,
the arrangement at position 3 has $2 \times (3) + 1 = 7$ matchsticks,
and so on.
So, the n^{th} position will have $2n+1$ matchsticks.

To answer Jasmine's question, we have to find the value of n such that $2n + 1$ has the value 99, or,

$$2n + 1 = 99.$$

? Can you find ways to get the value of n, such that $2n + 1 = 99$?

Math Talk

? Is it possible to make a matchstick arrangement that appears in this sequence using exactly 200 sticks?

A statement of equality between two algebraic expressions is called an **equation**. Nowadays, when using symbols, we write an equation as two algebraic expressions with an equal sign '=' between them.

Here are some more examples of equations:

$$3x + 4 = 7,$$

$$20 = y - 3,$$

$$\frac{a}{3} = 50,$$

$$2z + 4 = 5z - 14 \text{ etc.}$$

The process of finding the value(s) of the letter-numbers for which the equality holds, or for which the value of the Left Hand Side (LHS) of the equation becomes equal to the Right Hand Side (RHS) of the equation, is called **solving** the equation.

As we saw in the matchstick problem, framing an equation using an unknown quantity as a letter-number can help us find its value.

Left Hand Side (LHS)

Right Hand Side (RHS)

$$2n + 1 = 99$$

? For the weighing scale problems in figures 7.6, 7.7, 7.8, 7.9, 7.10, and 7.11, frame equations by using letter-numbers to denote the unknown weight.

For the problem in Fig. 7.6, let us denote the weight of one fried egg as e.

Since each slice of bread is 2, we have 2 + 2 + 2 = 6 on one side and $e + e$ on the other side. Since they are equal, we have

$$6 = e + e, \text{ or}$$

$$2e = 6.$$

For the problem in Fig. 7.7,  = 4, and we can denote the weight of one  as y. So, we have 16 on one side, and $4 + 2y$ on the other side. Thus, we have the equation

$$4 + 2y = 16.$$

? Solve the equations that you frame and check if you get the same value for the unknown weight as you got previously.

? Frame 5 equations. Find methods to solve them.

Math Talk

7.2 Solving Equations Systematically

How did you solve the various equations framed in the previous section? One way to solve an equation is to substitute different values in place of the letter-number and to check which value makes LHS = RHS. For example, consider the equation $2n + 1 = 99$.

If we substitute $n = 5$, we get LHS = $2 \times (5) + 1 = 11$. It is far away from 99, the RHS.

We can try $n = 10$. The LHS is 21. It is still not equal to 99.
Can we try $n = 30$? The LHS is now 61, still much lower than 99.
Let us try $n = 40$. The LHS is 81. We are getting closer to 99.
When $n = 50$, the LHS is 101. This is just a bit too high.
When $n = 49$, the LHS is 99!
Therefore, the solution to the equation $2n + 1 = 99$ is $n = 49$.

? Can this equation have any other solution?

This method is called the **trial and error** method.

The trial and error method can be inefficient.

? Try solving $5x - 4 = 7$ using trial and error.

Recall that in the case of the weighing scale, we didn't use the trial and error method! For some problems, finding the solution was straightforward. For others, we used the fact that when we remove equal weights from both the plates of a balanced weighing scale, it remains balanced. Do equations have a similar property?

? Consider an equation 15 + 8 = 23. If we add, subtract, multiply or divide the same number on both sides, will it still preserve the equality of LHS and RHS?

For example, you can check by adding 10 to both sides.

Since the LHS and the RHS of an equation have the same value, performing the same operation on both sides will clearly not change their equality.

In the weighing scale problems, we removed equal weights so that all the unknown weights were on only one plate of the scale. This made the problem easier to solve. We can use this same strategy to solve an equation as well! Let us first solve some arithmetic problems.

? **Example 1:** It is known that 14593 – 1459 + 145 – 14 + 88 = 13353. What is the value of 14593 – 1459 + 145 – 14?

To find the value, do we need to evaluate 14593 – 1459 + 145 – 14?
No, we can get it by subtracting 88 from 13353.

? Why can we do this?

We can do this because addition and subtraction are inverse operations. So, the value of the required expression can be found by subtracting 88 from both sides (LHS and RHS), which removes the term 88 and leaves only the expression to be evaluated on the LHS.

$$14593 - 1459 + 145 - 14 + \cancel{88} - \cancel{88} = 13353 - 88$$

? **Example 2:** It is known that 23 × 41 × 11 × 8 × 7 = 5,80,888. What is the value of the expression 23 × 41 × 11 × 8?

Using the fact that multiplication and division are inverse operations, we can simply divide 5,80,888 by 7 to get the value of the expression.

? Is this the same as dividing both sides by 7, which removes the factor 7 and leaves only the expression to be evaluated on the LHS?

Math Talk

? **Example 3:** It is known that 12345 – 5432 + 135 – 24 – (–67) = 7091. What is the value of the expression 12345 – 5432 + 135 – 24?

This problem also can be solved using the fact that addition and subtraction are inverse operations. However, we can use the following method that is perhaps easier to visualise.

To retain only the expression to be evaluated on the LHS, we need to remove the term –(–67). We can remove this term by adding (–67) to both sides of the equation.

$$12345 - 5432 + 132 - 24 - (-67) + (-67) = 7091 + (-67)$$

$$12345 - 5432 + 132 - 24 - (-\cancel{67}) + (-\cancel{67}) = 7091 + (-67)$$

Thus, $12345 - 5432 + 132 - 24 = 7024$.

Example 4: It is known that $\left(\frac{35}{113}\right) \times 24 \times 14 \times \left(\frac{8}{9}\right) = \frac{94080}{1017}$. What is the value of the expression $\left(\frac{35}{113}\right) \times 24 \times 14$?

Solution: To retain only the expression to be evaluated on the LHS we need to remove the factor $\left(\frac{8}{9}\right)$. We can do that by dividing both sides by $\left(\frac{8}{9}\right)$.

$$\left[\left(\frac{35}{113}\right) \times 24 \times 14 \times \left(\frac{8}{9}\right)\right] \div \left(\frac{8}{9}\right) = \frac{94080}{1017} \div \left(\frac{8}{9}\right)$$

$$\left(\frac{35}{113}\right) \times 24 \times 14 \times \left(\frac{\cancel{8}}{\cancel{9}}\right) \times \left(\frac{\cancel{9}}{\cancel{8}}\right) = \frac{94080}{1017} \times \left(\frac{9}{8}\right)$$

Thus,

$$\left(\frac{35}{113}\right) \times 24 \times 14 = \frac{11760}{113}.$$

Let us use these ideas to solve the equation $5x - 4 = 7$.

What can we do so that $5x$ is on one side and the equality between the LHS and the RHS still holds?

To retain only $5x$ on the LHS, we need to remove the term – 4. This can be done by adding 4 to both sides.

Thus, $5x - 4 + 4 = 7 + 4$.

Hence, $5x = 11$.

To retain only the unknown x on the LHS, we need to remove the factor 5. This can be done by dividing both sides by 5.

Thus, $\frac{5x}{5} = \frac{11}{5}$.

So, $x = \frac{11}{5}$.

Can we check that $x = \frac{11}{5}$ is the correct solution to the equation?

We can check this by substituting x with the value $\frac{11}{5}$ in the equation $5x - 4 = 7$, and checking if the LHS = RHS.

Substituting x with $\frac{11}{5}$ in the LHS we get,

$$\text{LHS} = 5\left(\frac{11}{5}\right) - 4$$

$$= \cancel{5}\left(\frac{11}{\cancel{5}}\right) - 4$$

$$= 11 - 4$$

$$= 7$$

This is the same as the RHS.

So, $\frac{11}{5}$ is indeed the correct solution to the equation.

Example 5: Solve the equation $11y + (–5) = 61$.

To retain only the unknown term $11y$ on one side, we need to remove the term –5. This can be done by subtracting (–5) from both sides.

$$11y + (–5) – (–5) = 61 – (–5).$$

That is,

$$11y = 66.$$

We can directly find the value of y, seeing that $11 \times 6 = 66$.

We can also divide both sides by 11 to find y.

$$11y \div 11 = 66 \div 11.$$

So $y = 6$ is the solution to the equation $11y + (–5) = 61$.

Can you check that this solution is correct?

Example 6: Solve $6y + 7 = 4y + 21$.

In this equation, expressions with an unknown are on both sides.

We have seen how to solve equations when the unknown term is on one side. What can be done to bring the unknown terms to the same side?

Subtracting $4y$ from both sides, we get

$$6y + 7 – 4y = 4y + 21 – 4y.$$

So,

$$2y + 7 = 21.$$

Subtracting 7 from both sides we get

$$2y + 7 - 7 = 21 - 7, \text{ which gives}$$

$$2y = 14.$$

We can directly find the value of y, seeing that $2 \times (7) = 14$.

We can also divide both sides by 2 to find y.

$$2y \div 2 = 14 \div 2, \text{ which gives}$$

$$y = 7.$$

Remember, it is always good to check your solution.

Figure it Out

1. Solve these equations and check the solutions.
   (a) $3x - 10 = 35$
   (b) $5s = 3s$
   (c) $3u - 7 = 2u + 3$
   (d) $4(m + 6) - 8 = 2m - 4$
   (e) $\frac{u}{15} = 6$

2. Frame an equation that has no solution.

Math Talk

[**Hint:** 4 more than a number, and 5 more than a number can never be equal!]

The procedure to systematically solve an equation can be made efficient if we make an observation. Consider the equation $11y + (-5) = 61$, which we solved.

As the first step, we subtracted – 5 from both sides to remove the term –5.

$$11y + (-5) - (-5) = 61 - (-5).$$

Since this action removes the term – 5 from the LHS, we could have written this step as:

$$11y = 61 - (-5).$$

Note that we could also have arrived at this step by seeing addition and subtraction as inverse operations, as in the case of Example 1.

Similarly, when we had $11y = 66$, we divided both sides by 11 to remove the factor 11 from the LHS.

$$11y \div 11 = 66 \div 11.$$

Since this action removes the factor 11 in the LHS, we can write this step as:

$$y = 66 \div 11.$$

Again, we could have arrived at this by seeing multiplication and division as inverse operations, as in the case of Example 2.

Let us write down both these ways of solving an equation.

$11y + (-5) = 61$ $11y + (-5) - (-5) = 61 - (-5).$ $11y = 66$ $11y \div 11 = 66 \div 11$ $y = 6$	$11y + (-5) = 61$ $11y = 61 - (-5)$ $11y = 66$ $y = 66 \div 11$ $y = 6$

Let us consider another equation that we solved earlier.

$6y + 7 = 4y + 21$ $6y + 7 - 4y = 4y + 21 - 4y$ $2y + 7 = 21$ $2y + 7 - 7 = 21 - 7$ $2y = 14.$ $2y \div 2 = 14 \div 2$ $y = 7$	$6y + 7 = 4y + 21$ $6y + 7 - 4y = 21$ $2y + 7 = 21$ $2y = 21 - 7$ $2y = 14.$ $y = 14 \div 2$ $y = 7$

? What happens in cases like $\frac{u}{15} = 6$?

Multiplying both sides by 15 leaves only u in the LHS —

$$u = 6 \times 15$$

$$u = 90.$$

Thus, we can make the following observations —

(a) When a term that is added or subtracted on one side of an equation is removed from that side, its additive inverse should appear as a term on the other side for the equality to hold. For example, $2y + 7 = 21$ becomes $2y = 21 - 7$.

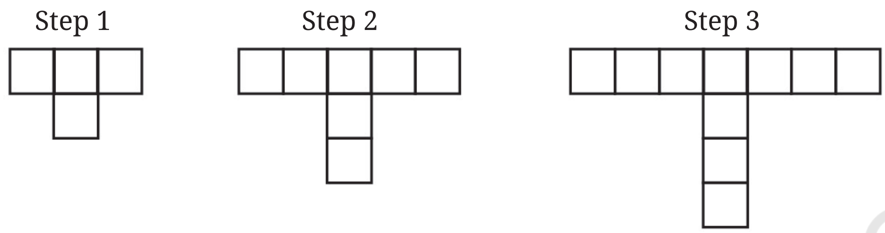
(b) If one side of an equation is the product of two or more numbers or expressions, and we remove one of the factors, then the other side should be divided by this factor for the equality to hold. For example, $2y = 14$ becomes $y = 14 \div 2$.

(c) If one side of an equation is the quotient of two numbers or expressions, and we remove the divisor, then the other side should be multiplied by this divisor for the equality to hold. For example, $\frac{u}{15} = 6$ becomes $u = 6 \times 15$.

Solving Problems

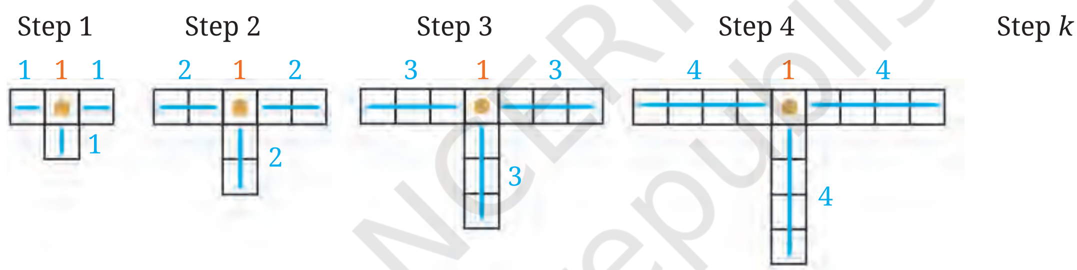
Forming and solving equations gives us the ability to find solutions to many problems in our lives. Let us see a few such examples.

Example 7: Ranjana creates a sequence of arrangements with square tiles as shown below. Can she extend the sequence and make an arrangement using 100 tiles? If yes, which step in the sequence will it be?



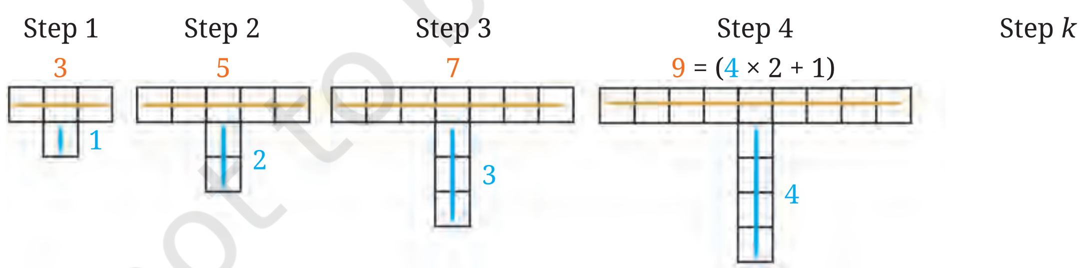
She can look at the pattern in different ways. They are shown below.

Method 1



Step 1	Step 2	Step 3	Step 4	Step k
$1 + 1 + 1 + 1$ $= 4$	$2 + 2 + 2 + 1$ $= 7$	$3 + 3 + 3 + 1$ $= 10$	$4 + 4 + 4 + 1$ $= 13$	$k + k + k + 1$ $= 3k + 1$

Method 2



Step 1	Step 2	Step 3	Step 4	Step k
$1 + 3$ $= 4$	$2 + 5$ $= 7$	$3 + 7$ $= 10$	$4 + 9$ $= 4 + (4 \times 2 + 1)$ $= 13$	$k + (2k + 1)$ $= 3k + 1$

We have the expression $3k + 1$ which gives the number of tiles needed to make an arrangement in Step k.

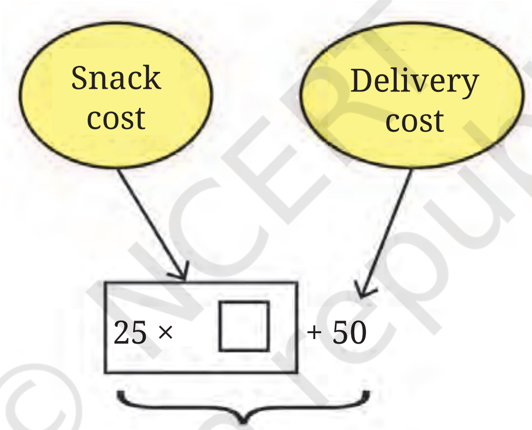
To check whether an arrangement is possible using 100 tiles at some Step k, we can solve the equation: $3k + 1 = 100$. Find the value of k.

Example 8: Madhubanti wants to organise a party. She decides to buy snacks for the party from the *chaat* shop in town. Each plate of snacks costs ₹25. The shop charges an additional fixed amount of ₹50 to deliver the snacks to Madhubanti's house.

There are 5 members in Madhubanti's family, including herself. Her parents tell her she can spend ₹500 on this party. How many friends can she invite to the party if she wants to give a plate of snacks to each person, including her family and friends?

Fatima's method of solving this problem is shown below.

She represented the situation using a rough diagram.



Out of ₹500, if we subtract the fixed delivery charge, then Madhubanti is left with ₹450.

So the question becomes "How many plates of snacks, each costing ₹25, can be bought for ₹450?".

The answer to this is $450 \div 25 = 18$.

18 plates of snacks can be bought for ₹450. Considering her 5 family members, she can invite $18 - 5 = 13$ friends to her party.

Mahesh represented the unknown quantity of the total number of people who can attend the party, including Madhubanthi and her family members, as p.

Cost incurred = $25p + 50$.

Since this cost should be 500, we have the equation

$$25p + 50 = 500.$$

Subtracting 50 from both sides, we get

$$25p = 500 - 50.$$

$$25p = 450.$$

Dividing both sides by 25, we get

$$p = 450 \div 25 = 18.$$

18 people can be at her party, including her 5 family members. That means 13 friends can be invited.

Srikanth decided to represent the unknown quantity of the total number of friends Madhubanthi can invite as f.

What will be the cost in this case?

Cost incurred = $25 (f + 5)$.

Since Madhubanthi has ₹450 for snacks, we have the equation

$$25 (f + 5) = 450.$$

Dividing both sides by 25,

$$f + 5 = 18.$$

Subtracting 5 from both sides,

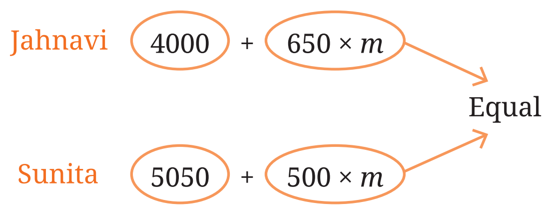
$$f = 13.$$

Example 9: Two friends want to save money. Jahnavi starts with an initial amount of ₹4000, and in addition, saves ₹650 per month. Sunita starts with ₹5050 and saves ₹500 per month. After how many months will they have the same amount of money?

Let m denote the number of months after which their savings are equal.

What are Jahnavi's savings after m months?

Jahnavi's savings = $4000 + 650\,m$.



What are Sunita's savings after m months?

Sunita's savings = $5050 + 500\,m$.

As these amounts are equal after m months, we get the following equation:

$$4000 + 650\,m = 5050 + 500\,m$$

$$4000 + 650\,m - 500\,m = 5050 \text{ (subtracting } 500\,m \text{ from both sides)}$$

$$4000 + 150\,m = 5050$$

$$150\,m = 5050 - 4000 \text{ (subtracting 4000 from both sides)}$$

$$150\,m = 1050$$

$$m = 1050 \div 150 \text{ (dividing both sides by 150)}$$

$$m = 7.$$

So, after 7 months, both will have the same amount of money.

Check the answer.

Let us solve a few equations.

Example 10: Solve $28(x + 4) + 300 = 1000$.

Here are some ways to solve this equation.

Subtracting 300 from both sides, we get $28(x + 4) = 1000 - 300$ $28(x + 4) = 700.$ Dividing both sides by 28, we get $x + 4 = 700 \div 28$ $x + 4 = 25.$ So $x = 25 - 4$, which gives $x = 21.$	$28(x + 4) + 300 = 1000$ Since 28, 300, and 1000 are divisible by 4, we get a simpler equation if we divide both sides by 4. $\frac{28(x + 4) + 300}{4} = \frac{1000}{4}$ Using the rules of fraction addition, we get $\frac{28(x + 4)}{4} + \frac{300}{4} = \frac{1000}{4}$ $7(x + 4) + 75 = 250$ Subtracting 75 from both sides, $7(x + 4) = 175$ $7x + 28 = 175$ Subtracting 28 from both sides, $7x = 147$ $x = \frac{147}{7} = 21.$	$28(x + 4) + 300 = 1000$ Simplifying the LHS, $28x + 112 + 300 = 1000$ $28x + 412 = 1000$ Subtracting 412 from both sides, we get $28x = 1000 - 412$ $28x = 588$ Dividing both sides by 28, $x = 588 \div 28$ $x = 21.$

Example 11: Riyaz created a math trick, which he tries out on his friend Akash.

Riyaz asked Akash to perform the following steps without revealing the answer to any of the intermediate steps.

1. Think of a number.
2. Subtract 3 from the number.
3. Multiply the result by 4.
4. Add 8 to the product.
5. Reveal the final answer.

The final answer revealed by Akash was 24. Using this, Riyaz correctly figured out the starting number that Akash had thought of. Find this number.

Try the steps using different numbers as the starting number. Do you see any relation between the starting number and final answer?

The answer can be found by algebraically modeling this scenario. In other words, we can form an equation using an unknown. Let the unknown starting number be x.

What are the expressions we get after each step?

Steps	Expression
Think of a number.	x
Subtract 3 from the number.	$x - 3$
Multiply the result by 4.	$4(x - 3) = 4x - 12$
Add 8 to the product.	$4x - 12 + 8 = 4x - 4$

Since Akash's final answer was 24, we have the equation:

$$4x - 4 = 24$$

$$4(x - 1) = 24$$

$$x - 1 = 6 \text{ (Dividing both sides by 4)}$$

Thus, Akash thought of the number 7.

Can you think of a simple rule that you can use to get the starting number from the final answer?

Example 12: Ramesh and Suresh have 60 marbles between them. Ramesh has 30 more marbles than Suresh. How many marbles does each boy have?

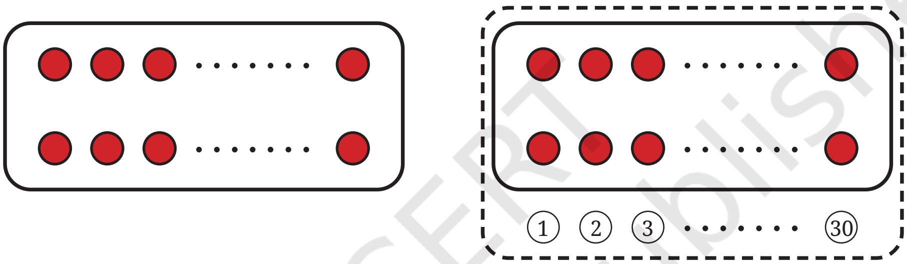
In this problem, we have two unknowns. If we denote the number of marbles with Ramesh as x and the number of marbles with Suresh as y, then what are the different equations that we have?

1. The total number of marbles is 60.

$$x + y = 60.$$

2. Ramesh has 30 marbles more than Suresh.

$$x = y + 30.$$



Suresh Ramesh

How do we find the unknowns using these equations? So far, we have only developed a strategy to solve an equation with one unknown! To get such an equation, we can do the following.

Denote the number of marbles with Suresh as y, and the number of marbles with Ramesh as $y + 30$.

Since the total number of marbles is 60, we have the equation:

$$y + (y + 30) = 60$$

$$2y + 30 = 60.$$

Use this to find both the unknowns.

Generating Equations

So far, we have solved a given equation to find the value of the letter-number. Can we do the reverse — write equations using a given value of the letter-number?

Write equations whose solution is $y = 5$. Share the equations you made with each other and discuss the methods used.

Math Talk

Two such equations are $y + 1 = 6$ and $3y = 15$.

Consider the following chains of equations, where one is obtained from the previous one by performing the same operation on both sides.

$y + 1 = 6$	$3y = 15$
$\downarrow$ Multiplying by -1	$\downarrow$ Adding 6
$-y - 1 = -6$	$3y + 6 = 21$
$\downarrow$ Adding y	$\downarrow$ Dividing by 3
$-1 = -6 + y$	$y + 2 = 7$
$\downarrow$ Adding 6	$\downarrow$ Subtracting 2
$5 = y$	$y = 5$

? Can you form a chain going from the bottom equation to the top? Compare the operations used when going from the top to the bottom and from the bottom to the top.

Math Talk

? Without calculating, can you find the value of the unknown in each equation in the chains above?

[**Hint:** We have seen that the value that satisfies an equation also satisfies the new equation obtained by performing the same operation on both sides of the original equation.]

? **Example 13:** Can you give a real-life situation that can be modelled as the equation, $100x + 75 = 250$?

Solution: If we think of these numbers as representing money, we can see that the total money is ₹250.

There are two terms that are adding to ₹250. The second term in the LHS, ₹75, is fixed and does not change. The value of the first term would change based on 'x'. So we can think of 100 as the number of units and 'x' as the cost per unit. For instance, x could represent the cost of a plate of snacks and 75 could be the delivery charge.

Figure it Out

1. Write 5 equations whose solution is $x = -2$.
2. Find the value of each unknown:
   (a) $2y = 60$
   (b) $-8 = 5x - 3$
   (c) $-53w = -15$
   (d) $13 - z = 8$
   (e) $k + 8 = 12 - k$
   (f) $7m = m - 3$
   (g) $3n = 10 + n$
3. I am a 3-digit number. My hundred's digit is 3 less than my ten's digit. My ten's digit is 3 less than my unit's digit. The sum of all the three digits is 15. Who am I?
4. The weight of a brick is 1 kg more than half its weight. What is the weight of the brick?
5. One quarter of a number increased by 9 gives the same number. What is the number?
6. Given $4k + 1 = 13$, find the values of:
   (a) $8k + 2$ (b) $4k$ (c) k (d) $4k - 1$ (e) $-k - 2$

7.3 Mind the Mistake, Mend the Mistake

The following are some equations along with the steps used to solve them to find the value of the letter-number. Go through each solution and decide whether the steps are correct. If there is a mistake, describe the mistake, correct it and solve the equation.

1
$4x + 6 = 10$
$4x = 10 + 6$
$4x = 16$
$x = 4$

2
$7 - 8z = 5$
$8z = 7 - 5$
$8z = 2$
$z = 4$

3
$2v - 4 = 6$
$v - 4 = 6 - 2$
$v - 4 = 4$
$v = 8$

4
$5z + 2 = 3z - 4$
$5z + 3z = -4 + 2$
$8z = -2$
$z = -\frac{2}{8}$

5
$15w - 4w = 26$
$15w = 26 + 4w$
$15w = 30$
$w = 2$

6
$3x + 1 = -12$
$x + 1 = -\frac{12}{3}$
$x + 1 = -4$
$x = -5$

7

$4\,(4q + 2) = 50$
$4\,(4q) = 50 - 2$
$16q = 48$
$q = 3$

8

$-\,2\,(3 - 4x) = 14$
$-6v - 8x = 14$
$-\,8x = 14 + 6$
$-\,8x = 20$
$x = -\dfrac{20}{8}$

9

$3\,(7y + 4) = 9 + 5y$
$7y + 4 = \dfrac{9}{3} + 5y$
$7y + 4 = 3 + 5y$
$7y - 5y + 4 = 3$
$2y = 4 - 3$
$y = \dfrac{1}{2}$

7.4 A Pinch of History

Forming expressions using symbols and solving equations with such expressions was an important component of mathematical explorations in ancient India. This area of mathematics was termed *bījagaṇita*, also now known as algebra. The word *bīja* means seed. Just as a tree is hidden inside a seed, the answer to a problem is hidden inside an unknown number. Solving the problem is like helping the tree grow — step by step, we discover what is hidden.

We have seen **Brahmagupta's** contributions to different areas of mathematics like integers and fractions. In Chapter 18 of his book *Brāhmasphuṭasiddhānta* (628 CE), he also explained how to add, subtract, and multiply unknown numbers using letters — similar to how we use x or y today. This chapter was one of the earliest known works in algebra in history. As renowned mathematician and Fields Medalist David Mumford remarked, 'Brahmagupta is the key person in the creation of algebra as we know it'.

In the 8th century, Indian mathematical ideas were translated into Arabic. They influenced a well-known mathematician named **Al-Khwarizmi**, who lived in present-day Iraq. Around 825 CE, he wrote a book called *Hisab al-jabr wal-muqabala*, which means 'calculation by restoring and balancing'.

These ideas spread even further. By the 12th century, Al-Khwarizmi's book was translated into Latin and brought to **Europe**, where it became very popular. The word *al-jabr* from his book gave us the word **algebra**, which we also still use today.

Similar to how we use letters from the alphabet today to represent unknowns, ancient Indian mathematicians from the time of Brahmagupta used distinct symbols like **yā, kā, nī, pī, lo**, etc., for different unknowns. The symbol *yā* was short for *yāvat-tāvat* (meaning 'as much as needed'). Others like *kā* and *nī* referred to as the first letters of the names of colours— *kālaka* (black), *nīlaka* (blue), and so on. In contrast to these unknowns, the known quantities in an expression had a specific form (*rūpa*) and were denoted by *rū*.

Here are some examples of how algebraic expressions in modern notation were written in ancient Indian notation:

Modern Notation	Ancient Indian Notation
$2x + 1$	*yā 2 rū 1* (in each term the indication of unknown/known came first)
$2x - 8$	*yā* 2 *rū* $\dot{8}$ (a dot over the number indicated that it was negative)
$3x + 4 = 2x + 8$	*yā* 3 *rū* 4 *yā* 2 *rū* 8 (the two sides of an equation were presented one below the other)

Example 16: *Bījgaṇita* by Bhāskarāchārya (1150 CE) mentions this problem.

One man has ₹300 rupees and 6 horses. Another man has 10 horses and a debt of ₹100. If they are equally rich and the price of each horse is the same, tell me the price of one horse.

Solution:

Let price of one horse = ₹ x

According to the problem

$$300 + 6x = 10x - 100$$

$$300 + 6x + (100) = 10x$$

$$400 + 6x = 10x$$

$$400 = 10x - 6x$$

$$400 = 4x$$

$$400 \div 4 = x$$

$$100 = x$$

Therefore, the price of one horse = ₹ 100.

Such problems and solutions were well understood in ancient India. In fact, a very systematic way to solve problems with single unknowns was first proposed by Aryabhata (499 CE) and Brahmagupta has outlined it in his *Brāhmasphuṭasiddhānta* (628 CE). Let us understand his method. Let us look at a few equations of the following form:

$$5x + 4 = 3x + 8, \text{ or } 3x - 6 = 2x + 4.$$

? Can we come up with a formula to solve these equations? That is, for the first equation, can we perform some operations using 5, 4, 3, and 8 that will directly give us the solution? Using a similar method, can you solve the second equation using the numbers 3, – 6, 2 and 4?

To get a formula, we can generalise equations of this form by taking the four numbers as A, B, C, and D. That is,

$$\mathrm{A}x + \mathrm{B} = \mathrm{C}x + \mathrm{D}.$$

Brahmagupta gives the solution to equations of this form with this formula:

$$x = \frac{\mathrm{D} - \mathrm{B}}{\mathrm{A} - \mathrm{C}}.$$

Using this approach, we can find the solution to the equation:

$$650m + 4000 = 500m + 5050$$

simply by calculating,

$$m = \frac{5050 - 4000}{650 - 500}.$$

? Using this formula can you solve this equation $2x + 3 = 4x + 5$?

Ancient Indian mathematicians were excellent at converting complex mathematical ideas into simple procedures so that everyone could use these procedures to solve problems!

Bījagaṇita or algebra is the branch of mathematics that uses letter symbols to solve mathematical problems. We have seen some examples in previous pages. By studying algebra, we learn to **generalise patterns**—in numbers, shapes, and situations. We express these generalisations using the language of algebra, which is both precise and powerful.

Bījagaṇita also gives us a way to justify mathematical claims (like why the sum of two odd numbers is always even) and to solve problems of many kinds.

The power of algebra was well recognised by ancient Indian mathematicians. We hope you recognise it too — and enjoy using it!

? Figure it Out

1. Fill in the blanks with integers.

   (a) $5 \times \_\_\_ - 8 = 37$

   (b) $37 - (33 - \_\_\_\_) = 35$

   (c) $-3 \times (-11 + \_\_\_\_) = 45$

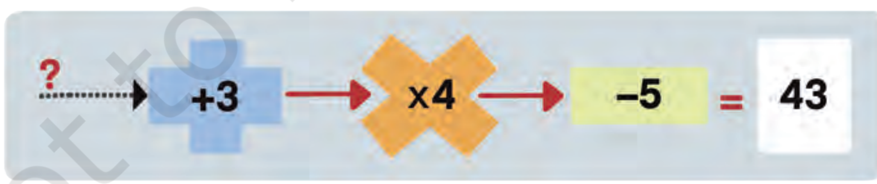
2. Ranju is a daily wage labourer. She earns ₹ 750 a day. Her employer pays her in 50 and 100 rupee notes. If Ranju gets an equal number of 50 and 100 rupee notes, how many notes of each does she have?

3. In the given picture, each black blob hides an equal number of blue dots. If there are 25 dots in total, how many dots are covered by one blob? Write an equation to describe this problem.

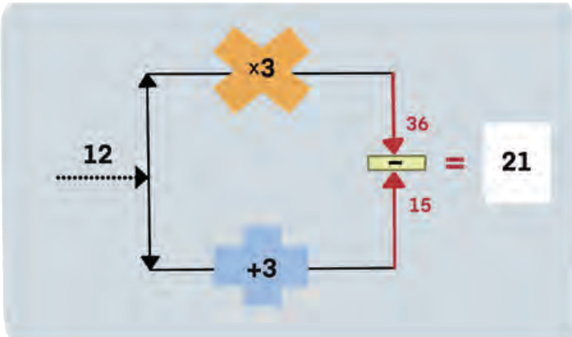
4. Here are machines that take an input, perform an operation on it and send out the result as an output.

   (a)

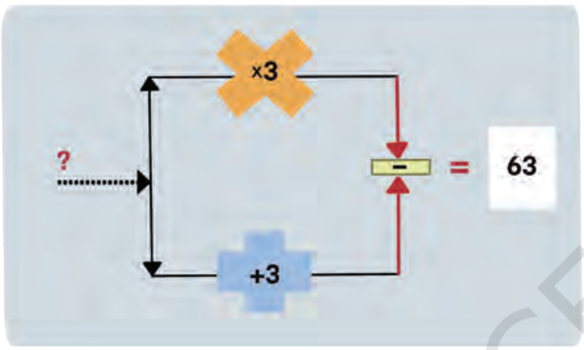
Find the inputs in the following cases:

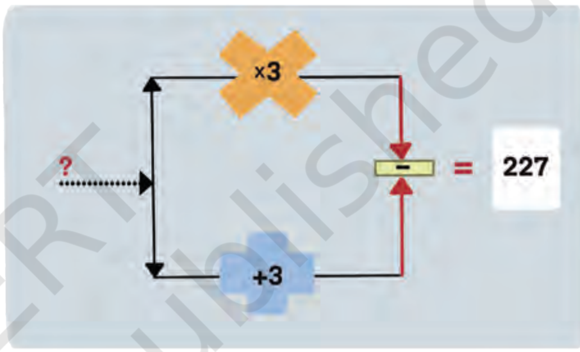


(b)

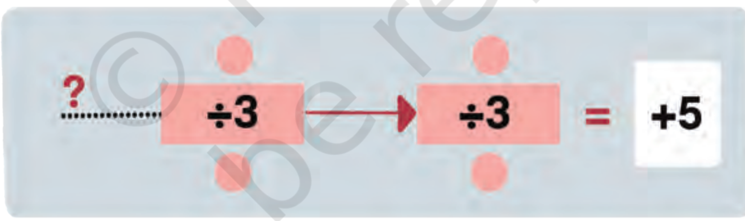


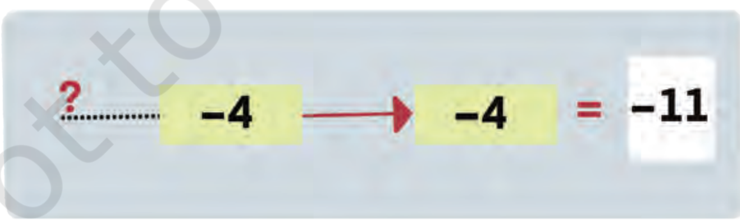
Find the inputs in the following cases:





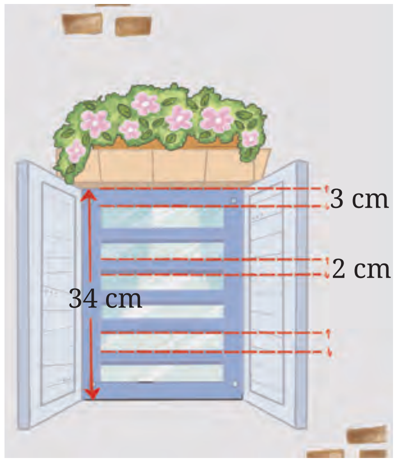
5. What are the inputs to these machines?





6. A taxi driver charges a fixed fee of ₹800 per day plus ₹20 for each kilometer traveled. If the total cost for a taxi ride is ₹2200, determine the number of kilometres traveled.

7. The sum of two numbers is 76. One number is three times the other number. What are the numbers?
8. The figure shows the diagram for a window with a grill. What is the gap between two rods in the grill?



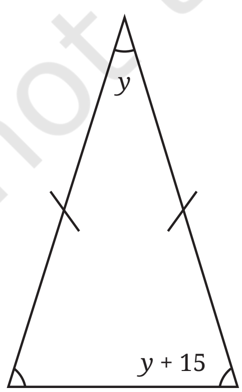
9. In a restaurant, a fruit juice costs ₹15 less than a chocolate milkshake. If 4 fruit juices and 7 chocolate milkshakes cost ₹600, find the cost of the fruit juice and milkshake.
10. Given 28p – 36 = 98, find the value of 14p – 19 and 28p – 38.
11. The steps to solve three equations are shown below. Identify and correct any mistakes.

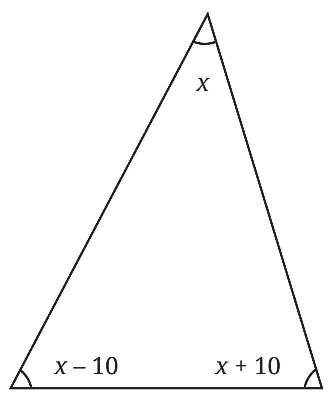
(a)
$\cancel{6}x + 9 = \overset{11}{\cancel{6}6}$
$x + 9 = 11$
$x = 11 - 9$
$x = 2$

(b)
$14y + 24 = 36$
$7y + 12 = 18$
$7y = 6$
$y = \frac{6}{7}$

(c)
$4x - 5 = 9x + 8$
$4x = 9x + 8 - 5$
$4x = 9x + 3$
$4x - 9x = 3$
$-5x = 3$
$x = \frac{-5}{3}$

12. Find the measures of the angles of these triangles.



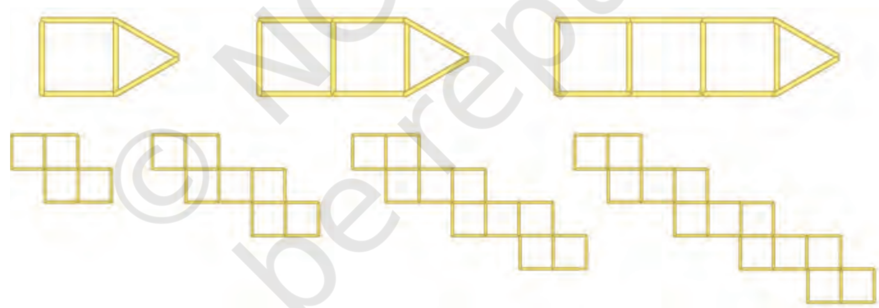


13. Write 4 equations whose solution is $u = 6$.

14. The Bakhśhāli Manuscript (300 CE) mentions the following problem. The amount given to the first person is not known. The second person is given twice as much as the first. The third person is given thrice as much as the second; and the fourth person four times as much as the third. The total amount distributed is 132. What is the amount given to the first person?

15. The height of a giraffe is two and a half metres more than half its height. How tall is the giraffe?

16. Two separate figures are given below. Each figure shows the first few positions in a sequence of arrangements made with sticks. Identify the pattern and answer the following questions for each figure:
    (a) How many squares are in position number 11 of the sequence?
    (b) How many sticks are needed to make the arrangement in position number 11 of the sequence?
    (c) Can an arrangement in this sequence be made using exactly 85 sticks? If yes, which position number will it correspond to?
    (d) Can an arrangement in this sequence be made using exactly 150 sticks? If yes, which position number will it correspond to?

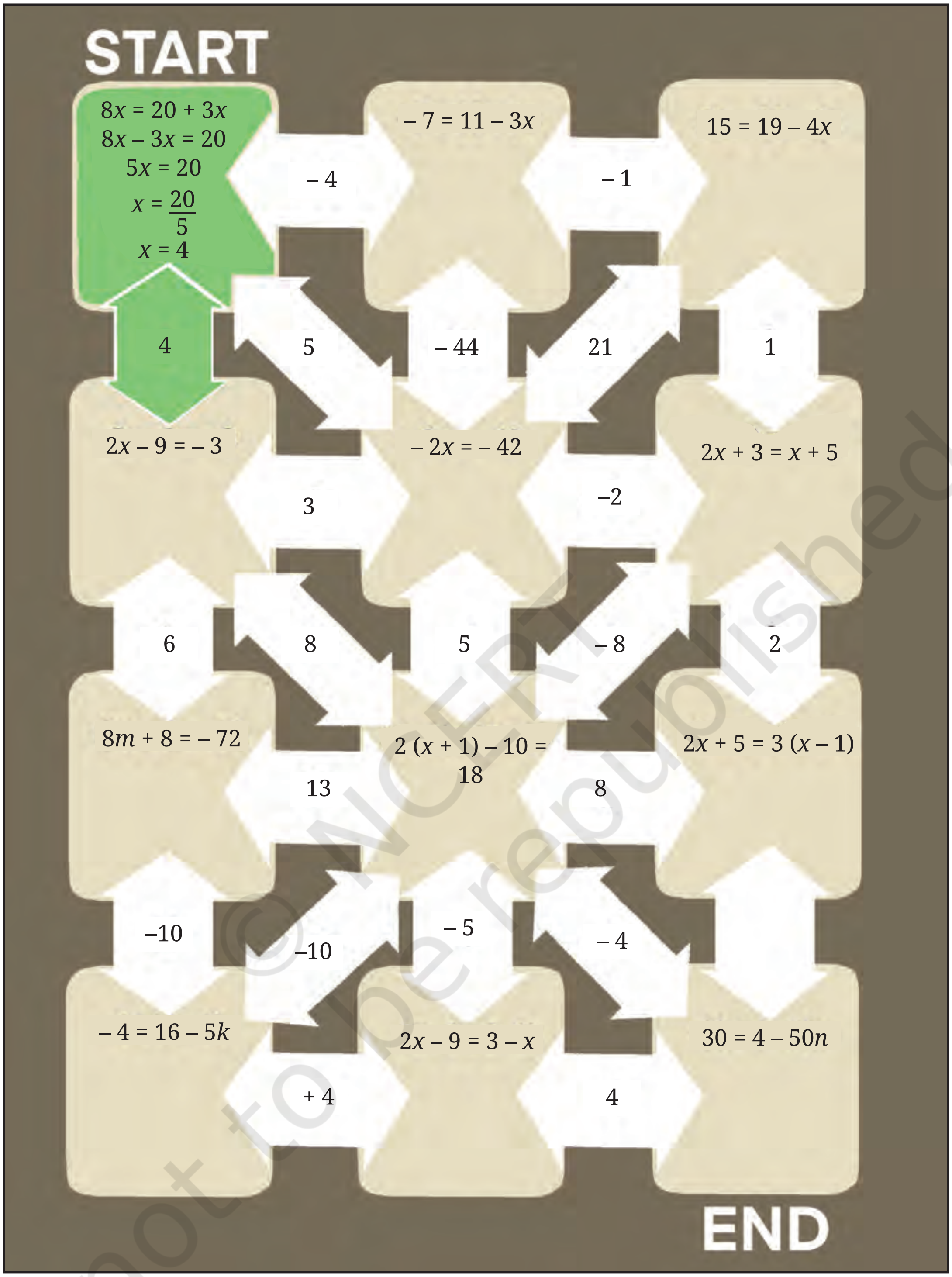


17. A number increased by 36 is equal to ten times itself. What is the number?

18. Solve these equations:

(a) $5(r + 2) = 10$	(b) $-3(u + 2) = 2(u - 1)$
(c) $2(7 - 2n) = -6$	(d) $2(x - 4) = -16$
(e) $6(x - 1) = 2(x - 1) - 4$	(f) $3 - 7s = 7 - 3s$
(g) $2x + 1 = 6 - (2x - 3)$	(h) $10 - 5x = 3(x - 4) - 2(x - 7)$

19. Solve the equations to find a path from Start to the End. Show your work in the given boxes provided and colour your path as you proceed.



20. There are some children and donkeys on a beach. Together they have 28 heads and 80 feet. How many donkeys are there? How many children are there?

Try This

SUMMARY

- An algebraic equation is a mathematical statement that indicates the equality of two algebraic expressions.
- When the same operation is performed on both sides of an equation, equality is maintained.
- Finding a solution to an equation means finding the values of the unknowns in the expressions such that the LHS is equal to the RHS.
- Equations can often be solved by performing the same operation on both sides so that the value of the unknown becomes evident.

A Magic Trick

Think of any number.
Now multiply it by 2.
Add 10.
Divide by 2.
Now subtract the original number you thought of.
Finally, add 3.

I predict that you now have 8. Am I correct?

Try the trick on your friends and family!

Can you explain why the trick works?

[**Hint:** Denote the first number thought of by x.]

Can you make your own such tricks?

TANGRAM

Note: Cut each shape along the white border.