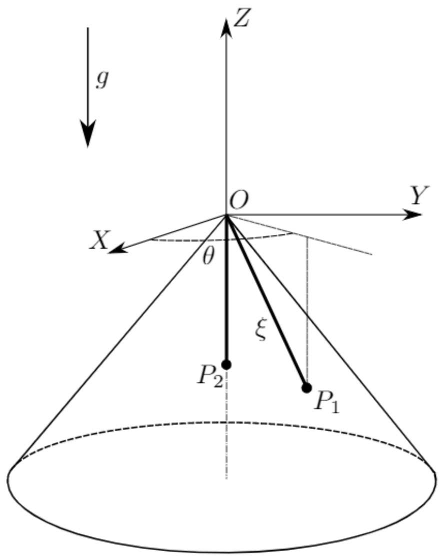
Two particles on cone with a string

Two heavy point particles P_1 and P_2 of mass m each are connected by an inextensible, massless string of total length ℓ. Particle P_1 is constrained to move without friction over the surface of a cone of semiangle α (the particle cannot leave the surface of the cone). The string passes through an aperture at the cone vertex O, and it can slide through it without friction. Particle P_2 hangs along the cone axis, and is constrained to move along it. We define an inertial reference frame $OXYZ$ as shown in the figure. We call ξ the length of the fraction of string between O and P_1, and θ the angle formed between the X axis and the projection onto the XY plane of the OP_1 vector.In the following, assume that the string is always tense and that the particles do not reach point O during motion.



Kinematics

```
import anakin.*

syms t xi(t) theta(t) ell alpha m g T(t) N(t); % declare symbolic variables

B0 = basis;
B1 = B0.rotatez(theta);
B2 = B1.rotatey(pi/2-alpha);

O0 = point;
S0 = frame(O0,B0) % inertial reference frame
```

```
S0 = 
Frame with origin with canonical position:
     0
     0
     0

And basis with canonical rotation matrix:
```

```
     1     0     0
     0     1     0
     0     0     1
```

```
S1 = frame(O0,B1)
```

```
S1 =
Frame with origin with canonical position:
     0
     0
     0

And basis with canonical rotation matrix:
[ cos(theta(t)), -sin(theta(t)), 0]
[ sin(theta(t)),  cos(theta(t)), 0]
[             0,              0, 1]
```

```
S2 = frame(O0,B2)
```

```
S2 =
Frame with origin with canonical position:
     0
     0
     0

And basis with canonical rotation matrix:
[ cos(theta(t))*sin(alpha), -sin(theta(t)), cos(theta(t))*cos(alpha)]
[ sin(theta(t))*sin(alpha),  cos(theta(t)), sin(theta(t))*cos(alpha)]
[              -cos(alpha),              0,               sin(alpha)]
```

```
P1 = particle(tensor(xi)*B2.e(1));
P1.pos
```

```
ans =
Vector with canonical components:
 cos(theta(t))*sin(alpha)*xi(t)
 sin(theta(t))*sin(alpha)*xi(t)
              -cos(alpha)*xi(t)
```

```
P1.vel
```

```
ans =
Vector with canonical components:
 cos(theta(t))*sin(alpha)*diff(xi(t), t) - sin(theta(t))*sin(alpha)*xi(t)*diff(theta(t), t)
 sin(theta(t))*sin(alpha)*diff(xi(t), t) + cos(theta(t))*sin(alpha)*xi(t)*diff(theta(t), t)
                                                                -cos(alpha)*diff(xi(t), t)
```

```
P1.accel
```

```
ans =
Vector with canonical components:
```

```
cos(theta(t))*sin(alpha)*diff(xi(t), t, t) - 2*sin(theta(t))*sin(alpha)*diff(theta(t), t)*diff(xi(t), t) - cos(the
sin(theta(t))*sin(alpha)*diff(xi(t), t, t) + 2*cos(theta(t))*sin(alpha)*diff(theta(t), t)*diff(xi(t), t) - sin(the
```

```
P2 = particle(tensor(xi-ell)*B0.e(3));
P2.pos
```

```
ans =
Vector with canonical components:
          0
          0
 xi(t) - ell
```

```
P2.vel
```

```
ans =
Vector with canonical components:
             0
             0
 diff(xi(t), t)
```

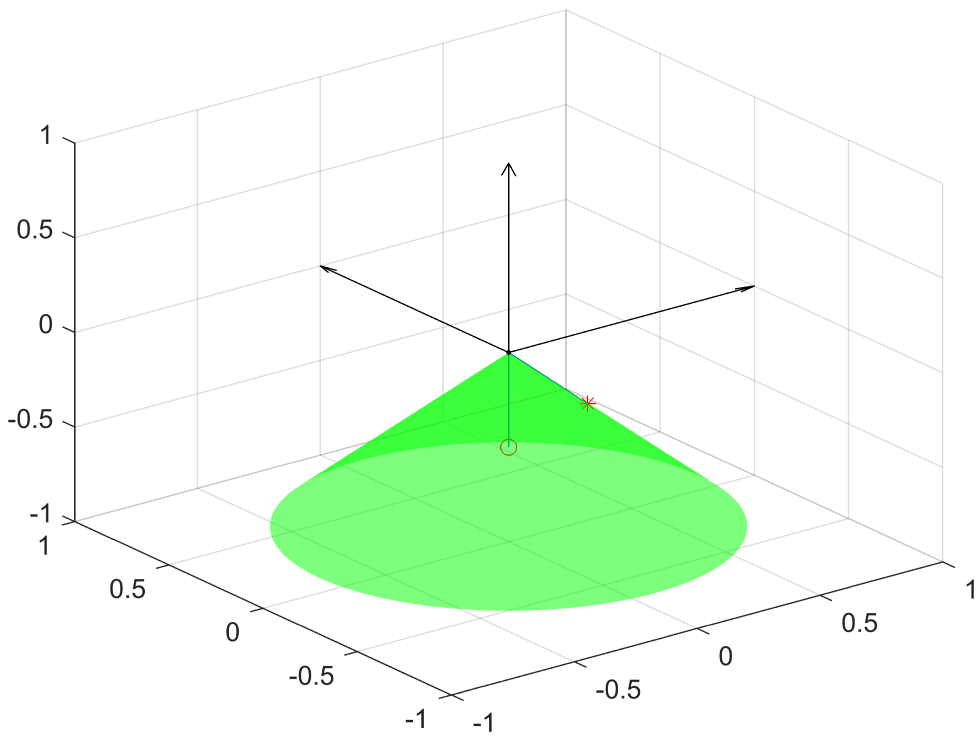
```
P2.accel
```

```
ans =
Vector with canonical components:
                0
                0
 diff(xi(t), t, t)
```

Interactive plot

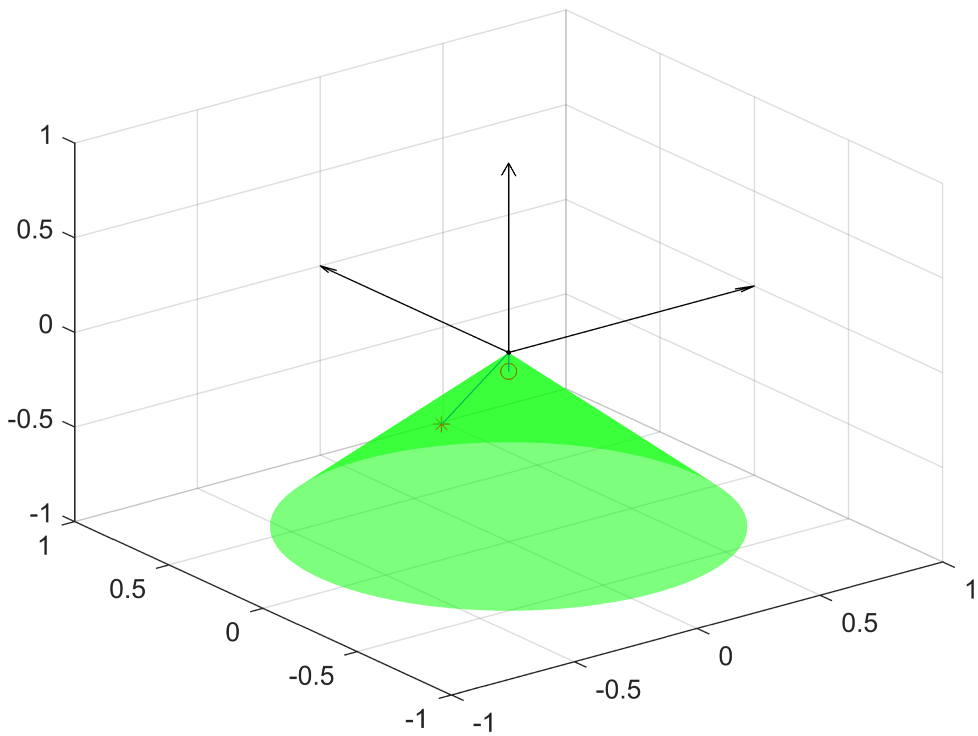
```
% Initial plot
m_ = 1;
g_ = 1;
alpha_ = 40*pi/180;
ell_ = 1;
xi_ = 0.5;
theta_ = 0;

f1 = figure(1);
a = axes;

tt = linspace(0,2*pi,100);
xx = linspace(0,1.2,100);
[xx,tt] = meshgrid(xx,tt);
XX = xx.*sin(alpha_).*cos(tt);
YY = xx.*sin(alpha_).*sin(tt);
ZZ = -xx.*cos(alpha_);
surf(XX,YY,ZZ,'LineStyle','none','FaceAlpha',0.5,'FaceColor','g')
```

```
S0.plot;

P1_ = P1.subs({xi,theta,alpha,ell},{xi_,theta_,alpha_,ell_});
P2_ = P2.subs({xi,theta,alpha,ell},{xi_,theta_,alpha_,ell_});

h1 = P1_.plot('Color','r','Marker','*');
h2 = P2_.plot('Color','r','Marker','o');
pos1 = P1_.pos.components;
pos2 = P2_.pos.components;
h3 = line([pos1(1),0,pos2(1)],[pos1(2),0,pos2(2)],[pos1(3),0,pos2(3)]);
```




```
% Interactive plot
xi_ =0.9;
theta_ =1.3;

delete(h1);
delete(h2);
delete(h3);

P1_ = P1.subs({xi,theta,alpha,ell},{xi_,theta_,alpha_,ell_});
P2_ = P2.subs({xi,theta,alpha,ell},{xi_,theta_,alpha_,ell_});

h1 = P1_.plot('Color','r','Marker','*');
h2 = P2_.plot('Color','r','Marker','o');
pos1 = P1_.pos.components;
```

```
pos2 = P2_.pos.components;
h3 = line([pos1(1),0,pos2(1)],[pos1(2),0,pos2(2)],[pos1(3),0,pos2(3)]);
```




```

```

Forces

```
% On P1
W1 = tensor(-m*g*[0,0,1])
```

```
W1 =
Vector with canonical components:
    0
    0
 -g*m
```

```
T1 = tensor(tensor(T)*(-B2.e(1)))
```

```
T1 =
Vector with canonical components:
 -cos(theta(t))*T(t)*sin(alpha)
 -sin(theta(t))*T(t)*sin(alpha)
               cos(alpha)*T(t)
```

```
N1 = tensor(tensor(N)*(B2.e(3)))
```

```
N1 =

Vector with canonical components:
 cos(theta(t))*N(t)*cos(alpha)
 sin(theta(t))*N(t)*cos(alpha)
              N(t)*sin(alpha)
```

```
P1.forces = {W1, T1, N1};

% On P2
W2 = tensor(-m*g*[0,0,1])
```

```
W2 =

Vector with canonical components:
    0
    0
 -g*m
```

```
T2 = tensor(T*[0,0,1])
```

```
T2 =

Vector with canonical components:
    0
    0
 T(t)
```

```
P2.forces = {W2, T2};
```

Equations of motion

```
eq1 = (P1.mass * P1.accel - W1 - T1 - N1) * B2.e(1)
```

```
eq1 =

Scalar with value:
T(t) - xi(t)*diff(theta(t), t)^2 + diff(xi(t), t, t) + cos(alpha)^2*xi(t)*diff(theta(t), t)^2 - g*m*cos(alpha)
```

```
eq2 = (P1.mass * P1.accel - W1 - T1 - N1) * B2.e(2)
```

```
eq2 =

Scalar with value:
sin(alpha)*(xi(t)*diff(theta(t), t, t) + 2*diff(theta(t), t)*diff(xi(t), t))
```

```
eq3 = (P1.mass * P1.accel - W1 - T1 - N1) * B2.e(3)
```

```
eq3 =

Scalar with value:
g*m*sin(alpha) - N(t) - (sin(2*alpha)*xi(t)*diff(theta(t), t)^2)/2
```

```
eq4 = (P2.mass * P2.accel - W2 - T2) * B0.e(3)
```

```
eq4 =
Scalar with value:
g*m - T(t) + diff(xi(t), t, t)
```

```
% To export to latex (not great):
% latex(eq1.components)
```

Integration of equations of motion

```
eqs1 = P1.equations(B2);
eqs2 = P2.equations(B0);
eqs = [eqs1;eqs2(3)]
```

eqs =

$$\begin{pmatrix} \sigma_2 - \sin(\alpha)^2\,\xi(t)\,\sigma_1 = g\,m\cos(\alpha) - T(t) \\ \sin(\alpha)\,\left(\xi(t)\,\frac{\partial^2}{\partial t^2}\,\theta(t) + \frac{\partial}{\partial t}\,\xi(t)\,\frac{\partial}{\partial t}\,\theta(t)\,2\right) = 0 \\ -\frac{\sin(2\,\alpha)\,\xi(t)\,\sigma_1}{2} = N(t) - g\,m\sin(\alpha) \\ \sigma_2 = T(t) - g\,m \end{pmatrix}$$

where

$$\sigma_1 = \left(\frac{\partial}{\partial t}\,\theta(t)\right)^2$$

$$\sigma_2 = \frac{\partial^2}{\partial t^2}\,\xi(t)$$

```
vars = [xi,theta,T,N]
```

vars(t) = $\begin{pmatrix} \xi(t) & \theta(t) & T(t) & N(t) \end{pmatrix}$

```
[eqs, vars] = reduceDifferentialOrder(eqs, vars)
```

eqs =

$$\begin{pmatrix} \frac{\partial}{\partial t}\,\mathrm{Dxit}(t) - \xi(t)\,\mathrm{Dtheta}_t(t)^2 \sin(\alpha)^2 + T(t) - g\,m\cos(\alpha) \\ \sin(\alpha)\,\left(\xi(t)\,\frac{\partial}{\partial t}\,\mathrm{Dtheta}_t(t) + 2\,\mathrm{Dtheta}_t(t)\,\mathrm{Dxit}(t)\right) \\ -\frac{\sin(2\,\alpha)\,\xi(t)\,\mathrm{Dtheta}_t(t)^2}{2} - N(t) + g\,m\sin(\alpha) \\ \frac{\partial}{\partial t}\,\mathrm{Dxit}(t) - T(t) + g\,m \\ \mathrm{Dxit}(t) - \frac{\partial}{\partial t}\,\xi(t) \\ \mathrm{Dtheta}_t(t) - \frac{\partial}{\partial t}\,\theta(t) \end{pmatrix}$$

vars =

$$\begin{pmatrix} \xi(t) \\ \theta(t) \\ T(t) \\ N(t) \\ \mathrm{Dxit}(t) \\ \mathrm{Dtheta}_t(t) \end{pmatrix}$$

```
f = daeFunction(eqs,vars,m,g,alpha);
F = @(t, Y, YP) f(t, Y, YP, m_, g_, alpha_);

Y0est = [0.5,0,0,0,-0.1,1];
YP0est = [0,0,0,0,0,0];
opt = odeset('RelTol', 10.0^(-7), 'AbsTol' , 10.0^(-7));
[Y0, YP0] = decic(F, 0, Y0est, [], YP0est, [], opt)
```

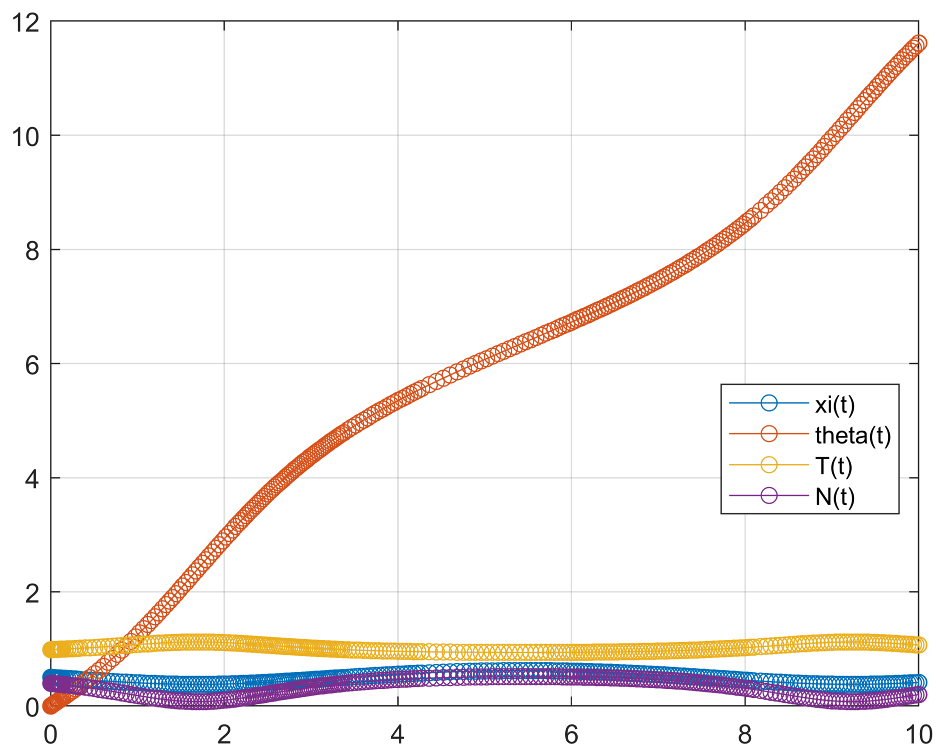
```
Y0 = 6×1
    0.5000
         0
    0.9863
    0.3966
   -0.1000
    1.0000
YP0 = 6×1
   -0.1000
    1.0000
         0
         0
   -0.0137
    0.4000
```

```
[t_,y_] = ode15i(F, [0, 10], Y0, YP0, opt);
f2 = figure(2)
```

```
f2 =
  Figure (2) with properties:

      Number: 2
```

```
        Name: ''
       Color: [0.9400 0.9400 0.9400]
    Position: [680 558 560 420]
       Units: 'pixels'

  Show all properties
```

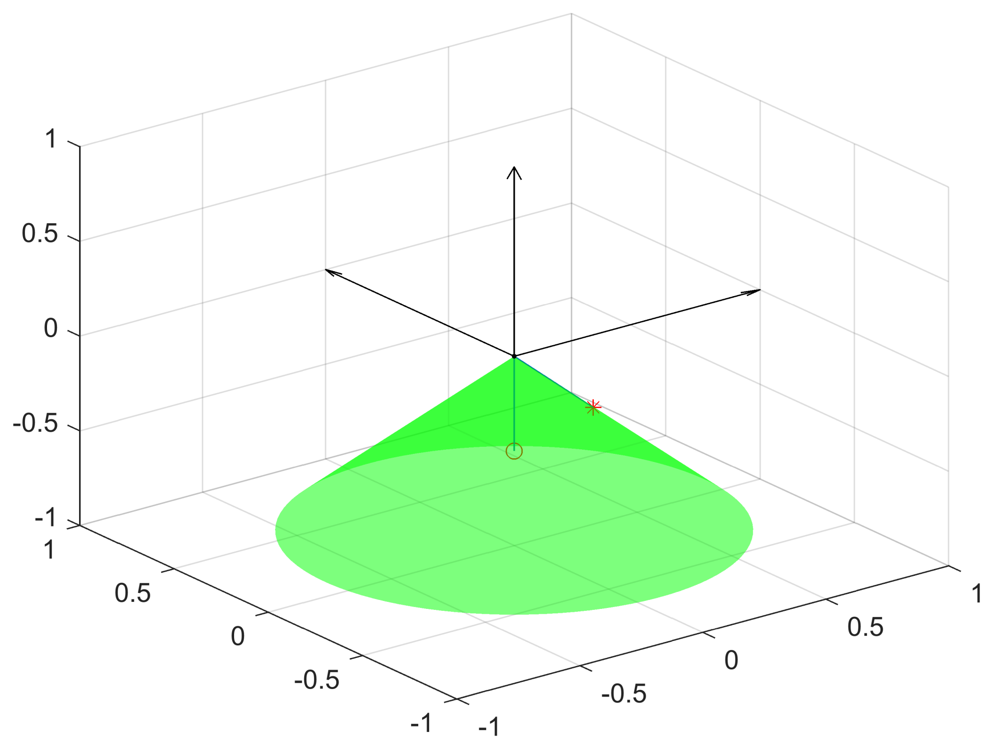
```
plot(t_,y_(:,1:4),'-o')

for k = 1:4
  S{k} = char(vars(k));
end

legend(S, 'Location', 'Best')
grid on
```



Time movie

```
xi_ = y_(:,1);
theta_ = y_(:,2);

figure(f1);

for i = 1:length(t_)
    delete(h1);
    delete(h2);
    delete(h3);
```

```
    P1_ = P1.subs({xi,theta,alpha,ell},{xi_(i),theta_(i),alpha_,ell_});
    P2_ = P2.subs({xi,theta,alpha,ell},{xi_(i),theta_(i),alpha_,ell_});

    h1 = P1_.plot('Color','r','Marker','*');
    h2 = P2_.plot('Color','r','Marker','o');
    pos1 = P1_.pos.components;
    pos2 = P2_.pos.components;
    h3 = line([pos1(1),0,pos2(1)],[pos1(2),0,pos2(2)],[pos1(3),0,pos2(3)]);

    drawnow;
    MOVIE(i) = getframe(f1);
end
```




```
f3 = figure(3);
movie(f3,MOVIE,2)
```

```
v = VideoWriter('mymovie');
```

```
Warning: No video frames were written to this file. The file may be invalid.
```

```
v.open;
writeVideo(v,MOVIE);
v.close;
```