Numerical Method - 1

Algebraic And Transcendental Equations

Polynomial → An expression of form $f(x) = a_0x^n + a_1x^{n-1} + \dots + a_n$. is called polynomial of degree 'n'.

Example: $f(x) = x^3 + 4x^2 + 5x + 2$ is polynomial of degree 3.

Polynomial or Algebraic Eqn's

If $f(x) = 0$ then it is called polynomial eqn.

Transcendental Equation

If algebraic equation contains exponential, logarithmic or trigonometric functions, then it is called Transcendental Eqn.

Example: (i) $x \log_{10} x = 1.2$

(ii) $xe^x = 2$

(iii) $3x = \cos x + 1$.

Roots:- The value of equation that satisfies the equation is called root of that eqn.

* Eqn of nth degree has only 'n' roots.
* Every eqn of odd degree has atleast one real root.

NOTE FOR CALCULATOR

1. Always reset your Calculator
SHIFT + MODE + 3 + = (4 times)

2. Set it to correct decimal places as asked:
(i) Press MODE button until it shows. fix Sci Norm
1 2 3

Choose fix option by pressing 1
It will ask Fix 0 ~ 9?.
Choose decimal places.

3. While solving any question that involves trigonometric ratio ~~use~~ fix calculator on Radian mode. by using:

Pressing MODE until shows.
Deg Rad Gra
1 2 3

Choose Radian option using/ pressing 2 button.

4. For $\log_{10}$ use simple log and for $\log_e$ use ln

5. If 5 decimal places is asked set on 6 decimal places always fix on one digit extra.

-ION OR BOLZANO METHOD

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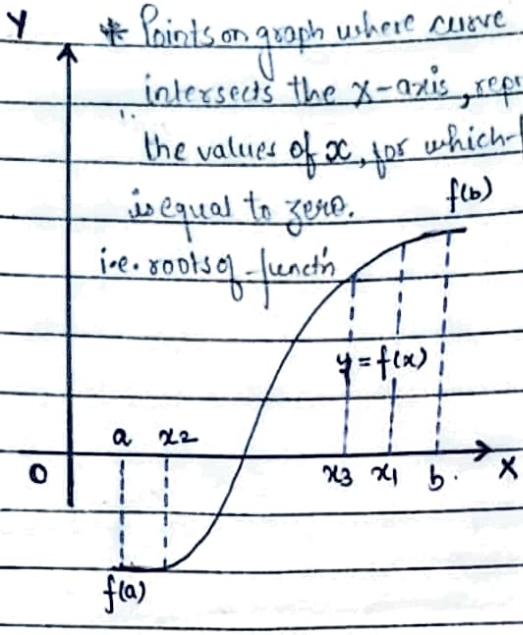
Concept of Bisection Method



* Points on graph where curve intersects the x-axis, represents the values of x, for which function is equal to zero.
i.e. roots of function

WORKING RULE

1. Let $f(x) = 0$, be the given eqn.

2. Find a and b such that,
$f(a) < 0$ (−ve) and $f(b) > 0$ (+ve) [or $f(a) \cdot f(b) < 0$]

3. Find 1st approximate root using Bisection method,

$$x_1 = \frac{a+b}{2}$$

Now, calculate $f(x_1)$ and examine it's sign.

(3.1) If $f(x_1) < 0 \Rightarrow$ root lies b/w x_1 and b and second approximate root is given by, $x_2 = \frac{x_1 + b}{2}$

(3.2) If $f(x_1) > 0 \Rightarrow$ root lies b/w a and x_1, and second approximate root is given by, $x_2 = \frac{a + x_1}{2}$

Now calculate $f(x_2)$ and check sign and repeat (3.1) and (3.2) to find desired accuracy.

Ques 1. find the root of the equation $x^3 - x - 4 = 0$ between 1 and 2, to four (three) places of decimal using Bisection Method.

Soln. Let $f(x) = x^3 - x - 4 = 0$

Here $a = 1$ and $b = 2$

$\therefore f(a) = f(1) = 1^3 - 1 - 4 = -4$, (-ve) and

$f(b) = f(2) = 2^3 - 2 - 4 = 2$. (+ve)

Clearly $f(a) \cdot f(b) < 0$. Therefore, roots of $f(x)$ lies b/w 1 and 2.

1st approximation $x_1 = \frac{a+b}{2}$

$\Rightarrow x_1 = \frac{1+2}{2} = 1.5$.

Now, Since. $f(1) = -4$ -ve

$f(2) = 2$ +ve

and $f(x_1) = f(1.5) = -2.1250$ -ve.

Therefore root lies in interval $(1.5, 2)$.

2nd approximation $x_2 = \frac{1.5+2}{2}$

$\Rightarrow x_2 = 1.7500$

and $f(1.7500) = -0.3906$

$f(x_2) = -0.3906$ (-ve)

Clearly $f(1.7500)$ is -ve and $f(2)$ is positive.

Therefore root lies in interval $(1.7500, 2)$.

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3rd approximation :

$$x_3 = \frac{1.7500 + 2}{2}$$

$x_3 = 1.8750$ and

~~and~~ $f(x_3) = f(1.8750) = 0.7168$ +ve

Since $f(1.7500)$ is negative and $f(1.8750)$ is positive.

Therefore, root lies in interval $(1.7500, 1.8750)$.

4th approximation

$$x_4 = \frac{1.7500 + 1.8750}{2}$$

$x_4 = 1.8125$ and

$f(x_4) = f(1.8125) = 0.1418$ (+ve)

Therefore, root lies b/w interval $(1.7500, 1.8125)$.

5th approximation

$$x_5 = \frac{1.7500 + 1.8125}{2}$$

$x_5 = 1.7813$, and

$f(x_5) = f(1.7813) = -0.1296$ (-ve)

Therefore root lies b/w interval $(1.7813, 1.8125)$

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6th approximation :

$$x_6 = \frac{1.7813 + 1.8125}{2}$$

$x_6 = 1.7969$ and

$f(x_6) = f(1.7969) = 0.0050$ +ve

Therefore, root lies b/w interval.

$(1.7813, 1.7969)$.

7th approximation :

$$x_7 = \frac{1.7813 + 1.7969}{2}$$

$x_7 = 1.7891$ and

$f(x_7) = f(1.7891) = -0.0624$ -ve

Therefore, root lies b/w interval

$(1.7891, 1.7969)$.

8th approximation

$$x_8 = \frac{1.7891 + 1.7969}{2}$$

$x_8 = 1.7930$ and

$f(x_8) = f(1.7930) = -0.0288$ -ve

Therefore, root lies between interval

$(1.7930, 1.7969)$

9th approximation

$$x_9 = \frac{1.7930 + 1.7969}{2}$$

$x_9 = 1.7950$ and

$f(x_9) = f(1.7950) = -0.0119$ -ve

Therefore, root lies in interval.

$(1.7950, 1.7969)$.

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10th approximation

$$x_{10} = \frac{1.7950 + 1.7969}{2}$$

$x_{10} = 1.7960$ and

$f(x_{10}) = f(1.7960) = -0.0032$ -ve

Therefore, root lies b/w interval.

$(1.7960, 1.7969)$

11th approximation

$$x_{11} = \frac{1.7960 + 1.7969}{2}$$

$x_{11} = 1.7965$ and

$f(x_{11}) = f(1.7965) = 0.0011$ +ve

Therefore, root lies b/w interval.

$(1.7960, 1.7965)$

12th approximation

$$x_{12} = \frac{1.7960 + 1.7965}{2}$$

$x_{12} = 1.7963.$

Since, $x_{10} = x_{11} = x_{12} = 1.796$ upto three decimal places, hence the root of given equation is 1.796.

Solution of Algebraic and Transcendental Equations

Polynomials → An expression of form

$$f(x) = a_0x^n + a_1x^{n-1} + \dots + a_n$$

is called Polynomial of degree 'n'

Example : ① $f(x) = x^3 + 4x^2 + 5x + 2$ (polynomial of degree 3)

② $f(x) = x^2 + 5x + 6$ (Polynomial of degree 2).

Equations If $f(x) = 0$ then this is called Polynomial equations

Eg = $x^3 + 4x^2 + 5x + 2 = 0$

(is polynomial or algebraic equation of degree 3)

Transcendental Equation → If Algebraic equation contains logarithmic terms or exponential terms or trigonometric terms then they are called Transcendental equations.

Example – ⓐ $3x = \cos x + 1$

ⓑ $xe^x = 2$

ⓒ $x \log_{10} x = 1.2$

Roots :- The value of eqn that satisfies the equation is called a root.

Ex. : $f(x) = x^2 - 5x + 6 = 0$

$= x^2 - 3x - 2x + 6 = 0$

$= x(x-3) - 2(x-3) = 0$

$\Rightarrow (x-3) = 0, (x-2) = 0$

$x = 3 \quad x = 2$

Here $x = 3, 2$ are roots of eqn

Bisection Method

Working Rule

1) Let $f(x) = 0$ — (1)
be the given equation. find a and b such that $f(a) < 0$ and $f(b) > 0$ {or $f(a) \cdot f(b) < 0$}.

(2) find 1st approx. root using bisection method.

$x_1 = \frac{a+b}{2}$, calculate $f(x_1)$ and examine its sign.

(2.1) If $f(x_1) < 0 \Rightarrow$ root lies b/w x_1 and b.

and second approximation or approx. root is given by

$$x_2 = \frac{x_1 + b}{2}$$

(2.2) If $f(x_1) > 0 \Rightarrow$ root lies b/w a and x_1

and second approx. root is given by

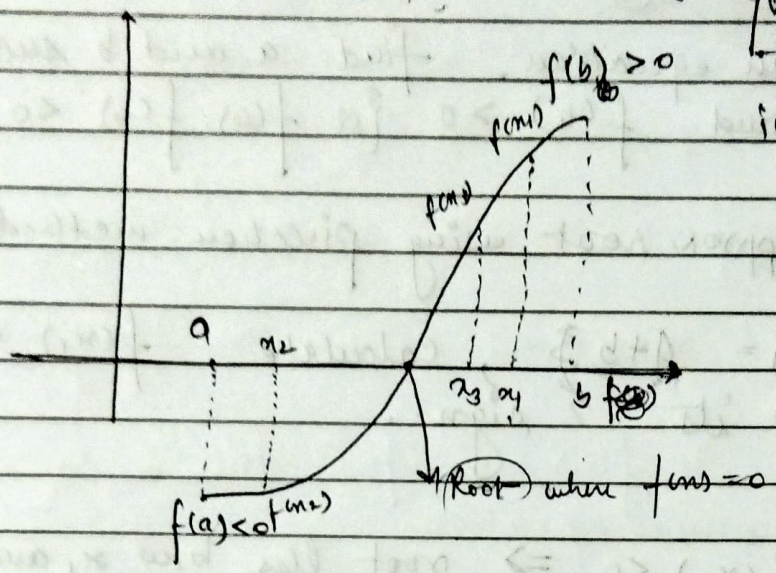
$$x_2 = \frac{a + x_1}{2}.$$

Now calculate $f(x_2)$ and repeat step (2.1) and (2.2) until required accuracy.

Concept of Bisection method.

$f(x) = 0$ given eqn.

* Root means where the function intersects x-axis.



Note

(1) In trigonometric function set Calci on Radian mode

* The points ~~at which~~ where its graph ~~intercep~~ intersects or touches the x-axis, representing the values of x for which function equals zero

i.e. x_{10} and x_{11} are same upto two decimal places the root is 0.73

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$$f(x) = x - \cos x = 0$$

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$f(0) = -1$ (A) $\qquad f(1) = 0.4597$ (B)

Since, $f(0) < 0$ and $f(1) > 0$, the root lies b/w. 0 and 1.

$$C = \frac{A+B}{2} \quad ; \quad C - \cos C.$$

use Alpha and Calc last.

No.	A (−ve)	B (+ve)	$x_0 = a+b/2$	Sign of $f(x)$.
				−0.3775.
1	0	1	0.5	−ve.
2	0.5	1	0.75	0.0183 (+ve)
3	0.5	0.75	~~0.75~~ 0.6250	−0.1860 (−ve)
4	0.625	0.75	~~0.75~~ 0.6875	−0.0853 (−ve)
5	0.6875	0.75	~~0.75~~ 0.7188	−0.0339 (−ve)
6	0.7188	0.75	~~0.75~~ 0.7344	−0.0078 (−ve)
7	0.7344	0.75	0.7422	0.0052 (+ve)
8	0.7344	0.7422	0.7383	−0.0013 (−ve)
9	0.7383	0.7422	0.7403	0.0020 (+ve)
10	0.7383	0.7403	0.7393	0.0004 (+ve)
11	0.7383	0.7393	0.7388	−0.0005 (−ve)
12	0.7388	0.7393.	0.7391	−0.0001
13.	0.7391	0.7393	0.7392	0.0002

Here. $x_{11} = x_{12} = 0.739$ upto 3 decimal places. Hence, root of given eqn is 0.739.

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Ques 2. Find the real root of the equation $x \log_{10} x = 1.2$ by Bisection Method. Correct to three decimal places.

Soln Let- $f(x) = x \log_{10} x - 1.2 = 0$

Taking $f(1) = -1.2000$

$f(2) = -0.5979$ -ve.

$f(3) = 0.2314$ +ve

Hence root lies b/w interval (2, 3).

Approximation No.	A	B	Approx. $x = \frac{a+b}{2}$	$f(x)$	Sign of $f(x)$
1	2	3	2.5000	-0.2051	-ve.
2	2.5000	3	2.7500	0.0082	+ve.
3	2.5000	2.7500	2.6250	-0.0998	-ve
4	2.6250	2.7500	2.6875	-0.0461	-ve
5	2.6875	2.7500	2.7188	-0.0191	-ve.
6	2.7188	2.7500	2.7344	-0.0054	-ve.
7	2.7344	2.7500	2.7422	0.0014	+ve
8	2.7344	2.7422	2.7383	-0.0020	-ve
9	2.7383	2.7422	2.7403	-0.0003	-ve
10	2.7403	2.7422	2.7413	0.0005	+ve
11	2.7403	2.7413	2.7408	0.0001	+ve
12	2.7403	2.7408	2.7406	-0.0001	-ve
13	2.7406	2.7408	2.7407	0.0000	—

Here, $x_{11} = x_{12} = x_{13} \approx 2.740$ upto 3 decimal places. Hence root of given function is 2.740

①

Ques. find the root of eqn x^3+x-1 which is near to 1. upto 3 decimal places.

Soln

Here, $f(x) = x^3+x-1$

Now,

$f(0) = -1$ -ve

$f(1) = 1$ +ve.

$\because$ $f(0).f(1) < 0$, therefore root lies in interval $(0,1)$

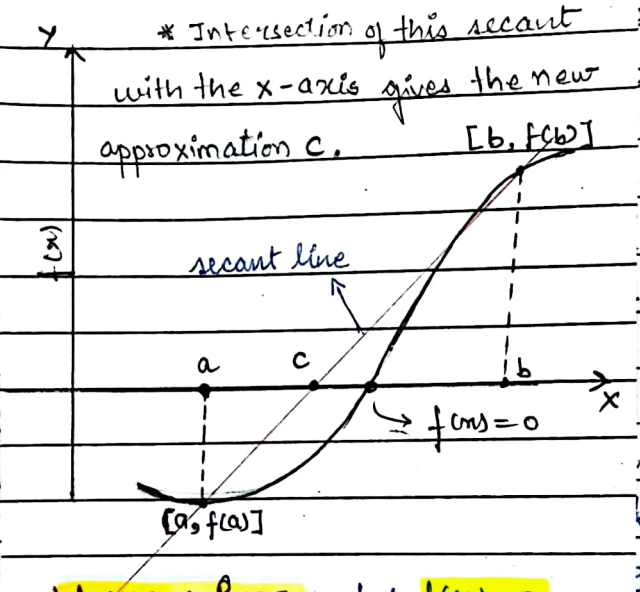
No.	a	b	$x = \frac{a+b}{2}$	$f(x)$	Sign
1	0	1	0.5000	-4.3750	-ve.
2	0.5000	1	0.7500	-4.3281	-ve.
3	0.7500	1	0.8750	-4.2051	-ve
4	0.8750	1	0.9375	-4.1135	-ve
5	0.9375	1	0.9688	-4.0596	-ve
6	0.9688	1	0.9844	-4.0305	-ve
7	0.9844	1	0.9922	-4.0154	-ve
8	0.9922	1	0.9961	-4.0078	-ve
9	0.9961	1	0.9981	-4.0039	-ve.
10	0.9981	1	0.9991	-4.0019	-ve
11	0.9991	1	0.9996	-4.0009	-ve
12	0.9996	1	0.9998	-4.0004	-ve.

Here, $x_{10} = x_{11} = x_{12} = 0.999$ upto 3 decimal places.

Therefore, root of given equation is 0.999.

Regula Falsi Method or False Position Method

Regula falsi method is similar to the Bisection method, but it improves convergence by using a weighted average based function values.



Working Rule: Let $f(x) = 0$ —— (1)

1. Select two initial points (a, b) such that $f(a)$ and $f(b)$ have opposite signs [i.e. function crosses the x-axis between them].
   i.e. $f(a) \cdot f(b) < 0$

2. Compute new approximation (c) using the formula:

$$c = \frac{a f(b) - b f(a)}{f(b) - f(a)}$$

3. Check if $f(c) \approx 0$ (or meets) the desired accuracy.

4. update a or b based on the sign of $f(c)$ and continue iterating.

5. Repeat until convergence

<u>For Calculator</u>

Que. $x \log x - 1.2$.

$A : B : C = A \log A - 1.2$:
$D = B \log B - 1.2$:

$$X = \frac{AD - BC}{D - C} :$$

$E = X \log X - 1.2$.

Here $A = a$
$B = b$
$C = f(a)$
$D = f(b)$
$x_n = X$
and $E = X \log X - 1.2$.

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Ques. find a real root of the eqn $x^3 - 9x + 1 = 0$ by Regula falsi.

Sol: Here, $f(x) = x^3 - 9x + 1 = 0$

$f(1) = -7$, $f(2) = -9$

$f(3) = 1$.

Since $f(2) \cdot f(3) < 0$ therefore root lies in interval $(2, 3)$.

No.	a	b	f(a)	f(b)	$x_2 = \frac{a f(b) - b f(a)}{f(b) - f(a)}$	f(x2)
1	2	3	-9	1	$x_1 = 2.9$	-0.7110
2	2.9	3	-0.7110	1	$x_2 = 2.9416$	-0.0215
3	2.9416	3	-0.0207	1	$x_3 = 2.9428$	-0.0006
4	2.9428	3	-0.0003	1	$x_4 = 2.9428$	0.0000

Here = $x_3 = x_4 = 2.942$, upto 3 decimal places.

So, root of given eqn is 2.942

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Ques find the real root of the equation $x \log_{10} x - 1.2 = 0$ by Regula falsi method upto 4 decimal places.

Soln Tabular form:-

No.	a	b	f(a)	f(b)	$x_n = \frac{af(b)-bf(a)}{f(b)-f(a)}$	$f(x_n)$
1	2	3	-0.59794	0.23136	2.72101	-0.01709
2	2.72101	3	-0.01709	0.23136	2.74021	-0.00038
3	2.74021	3	-0.00038	0.23136	2.74064	-0.00001
4	2.74064	3	-0.00001	0.23136	2.74065	0.0000

Complete Solution:

Here $f(x) = x \log_{10} x - 1.2 = 0$

$f(2) = 2 \log_{10} 2 - 1.2 = -0.59794$ (-ve)

$f(3) = 3 \log_{10} 3 - 1.2 = 0.23136$ (+ve)

atleast a root lies in (2,3).

taking $a = 2$ and $b = 3$.

$f(2) = -0.59794$, $f(3) = 0.23136$.

By Regula falsi method,

$$x_1 = \frac{a f(b) - b f(a)}{f(b) - f(a)}$$

$$\Rightarrow x_1 = \frac{2(0.23136) - 3(-0.59794)}{0.23136 - (-0.59794)}$$

$$\Rightarrow x_1 = 2.72102.$$

Also,

$$f(x_1) = f(2.72102) = -0.01709 < 0$$

Since $f(2.72102)$ and $f(3)$ are of opposite sign.
So the root lies between $(2.72102, 3)$

$$x_2 = \frac{2.72102\,(0.23136) - 3\,(-0.01709)}{(0.23136) - (-0.01709)}$$

$$x_2 = 2.74021.$$

Also. $f(x_2) = f(2.74021) = -0.00038 < 0$

Since $f(2.74021)$ and $f(3)$ are of opposite sign
So the root lies between $(2.74021, 3)$

$$x_3 = \frac{2.74021\,(0.23136) - 3\,(-0.00038)}{0.23136 - (-0.00038)}$$

$$x_3 = 2.74064$$

Again $f(x_3) = f(2.74064) = -0.00001 < 0$

Since $f(2.74064)$ and $f(3)$ are of opposite sign
So, the root lies between $(2.74064, 3)$.

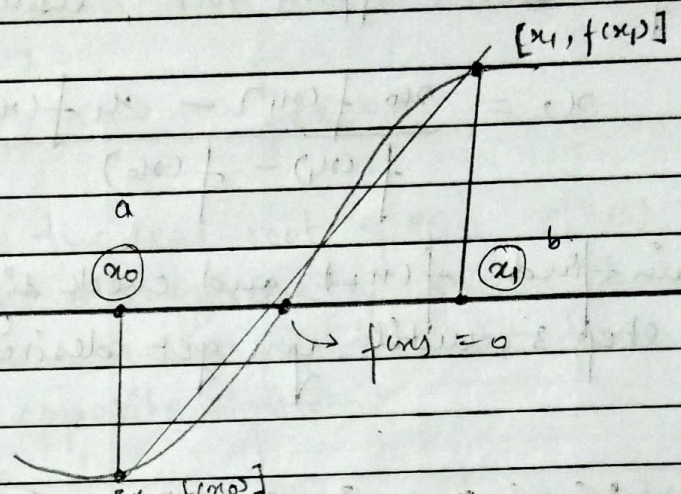
$$x_4 = \frac{2.74064\,(0.23136) - 3\,(-0.00001)}{0.23136 - (-0.00001)}$$

$$x_4 = 2.74065$$

Here. $x_3 = x_4 = 2.7406$ upto four decimal places.

Hence. 2.7406 is the required root of given eqn.

Regula falsi Method. OR Method of false Position.



formula used :

$$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)} \quad x_n = \frac{a f(b) - b f(a)}{f(b) - f(a)}$$

Working Rule :- ① Given $f(x) = 0$ —— ①

② find x_0 and x_1 such that $f(x_0) < 0$ and $f(x_1) > 0$ i.e. $f(x_0) \cdot f(x_1) < 0 \Rightarrow$ roots of ① lies in interval (x_0, x_1)

③ find first approximate root by Regula falsi method

$$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)}.$$

OR for (a, b).

$$x_n = \frac{a f(b) - b f(a)}{f(b) - f(a)}$$

find $f(x_2)$ and check its sign

(2.1) If $f(x_2) < 0$ then replace $x_0 = x_2$

(2.2) If $f(x_2) > 0$ then replace $x_1 = x_2$

and find second approx. root using

$$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)}$$

Again find $f(x_2)$ and check sign and repeat step 3, until you get desired accuracy.

① ~~$x^3 - x + 9$~~ $x^3 - 9x + 1 \rightarrow 2.9428$ ~~0.3609~~

$x^3 - 2x - 5 \rightarrow 2.09$

$x \log_{10} x - 1.2 = 0 \rightarrow 2.7406$

$x e^x = 2 \rightarrow 0.8526$

$x e^x = \cos x \rightarrow 0.5174$

$3x + \sin x - e^x \rightarrow 0.36042$

$\cos x = 3x - 1 \rightarrow 0.607$

$f(a) \cdot f(b) < 0$

$-ve$ $\quad$ $+ve$

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Ques. find ~~the~~ a real root of the equation $x^3 - 9x + 1 = 0$ by the method of false position.

Soln let $f(x) = x^3 - 9x + 1 = 0$

$f(1) = -7$ $\quad$ $f(0) = 1$

$f(2) = -9$ $\quad$ $f(3) = 1$

* To find the real root of eqn $f(x) = 0$, we consider a sufficiently small interval (a, b) where $a < b$ such that $f(a)$ and $f(b)$ will have opposite sign.

$f(x)$	a (−ve)	b (+ve)	$f(a)$	$f(b)$	$x = \frac{a f(b) - b f(a)}{f(b) - f(a)}$
−0.7110	2	3	−9	1	$x_1 = 2.9$
−0.0215	2.9	3	−0.7110	1.000	$x_2 = 2.9416$
−0.0006	2.9416	3	−0.0207	1	$x_3 = 2.9428$
0.0000	2.9428	3	−0.0003	1	$x_4 = 2.9428$

Here $x_3 = x_4 = 2.9428$

$5 \overline{)8} \; 1.6$

	a	b	f(a)	f(b)	$x_n =$	$f(x_n)$
①	2	3	-0.59734	0.23136	2.72101	-0.01709
②	2.72101	3	-0.01709	0.23136	2.74021	-0.00038
③	2.74021	3	-0.00038	0.23136	2.74064	-0.00001
④	2.74064	3	-0.00001	0.23136	2.74065	0.00000

root = 2.74065

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Ques. Find the real root of the equation $x \log_{10} x - 1.2 = 0$ by Regula-falsi method upto 4 decimal places.

Soln. $f(x) = x \log x - 1.2$

$f(1) = 1 \log 1 - 1.2 = -1.20000$

$f(2) = 2 \log 2 - 1.2 = -0.59794$ (-ve)

$f(3) = 3 \log 3 - 1.2 = \cancel{0.23}\ 0.23136$. (+ve)

Here $f(2) \cdot f(3) < 0$. Hence at least a root lies b/w 2 and 3.

Taking $a = 2$ and $b = 3$.

$f(2) = -0.59794$, $f(3) = 0.23136$.

By method of false Position

1st approx, $x_1 = \dfrac{a f(b) - b f(a)}{f(b) - f(a)}$

$x_1 = \dfrac{2 f(3) - 3 f(2)}{f(3) - f(2)} = \dfrac{2(0.23136) - 3(-0.59794)}{0.23136 - (-0.59794)}$

$x_1 = 2.72102$

Now. $f(x_1) = 2.72102 \log 2.72102 - 1.2$

$= -0.01709 < 0$

Here $A : B : C = \overset{a}{A} \log \overset{a}{A} - 1.2 \overset{f(a)}{} : D = \overset{b}{B} \log \overset{b}{B} - 1.2 \overset{f(b)}{} :$

$X = \dfrac{AD - BC}{D - C} : X \log_{10} X - 1.2$.

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Que (8) Find the root of the eqn $x^3 - 5x - 7$

$x^3 - 5x - 7 = 0$ which lies b/w 2 and 3 by method of false position.

Soln

$f(x) = x^3 - 5x - 7$

$f(2) = -9$, $f(3) = 5$

$\because f(2) \cdot f(3) < 0$, then atleast a root lies b/w. $(2, 3)$.

$= \frac{a f(b) - b f(a)}{f(b) - f(a)}$

	a	b	f(a)	f(b)	x_n	$f(x_n)$
(1)	2	3	-9	5	2.6429	-1.7547
(2)	2.6429	3	-1.7541	5	2.7356	-0.2054
(3)	2.7356	3	-0.2061	5	2.7461	-0.0225
(4)	2.7461	3	-0.0220	5	2.7472	-0.0029
(5)	2.7472	3	-0.0026	5	2.7473	-0.0003
(6)	2.7473	3	-0.0008	5	2.7473	-0.0001

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Q.4 Regula-falsi upto 3 decimal places

$f(x) = x e^{x} - \cos x$.

$f(0) = -1$. , $f(1) = 2.1780$

$\because f(0) \cdot f(1) < 0$. root lies b/w $(0, 1)$,

No.	a	b	f(a)	f(b)	x_n	$f(x_n)$
	0	1	−1	2.1780	0.3147	−0.5199
	0.3147	1	−0.5198	2.1780	0.4467	−0.2035
	0.4467	1	−0.2036	2.1780	0.4940	−0.0708
	0.4940	1	−0.0708	2.1780	0.5099	−0.0236
	0.5099	1	−0.0237	2.1780	0.5152	−0.0078
	0.5152	1	−0.0078	2.1780	0.5169	−0.0025
	0.5169	1	−0.0026	2.1780	0.5175	−0.0009
	0.5175	1	−0.0008	2.1780	0.5177	−0.0003.

NEWTON'S RAPHSON'S METHOD

Newton Raphsons method is an iterative numerical technique used to find the roots of a real-valued function. It follows the general formula.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

where, $n = 0, 1, 2, 3 \ldots$

WORKING RULE :-

1. Let $f(x)$ be the given function.
2. Calculate, $f(a) < 0$ and $f(b) > 0$, then root lies between a and b.
3. Now select an initial approximation,
$$x_0 = \frac{a+b}{2}$$
4. Calculate $f(x_n)$ and $f'(x_n)$.
5. Then apply Newton Raphson's formula.
$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$
where $n = 0, 1, 2, 3 \ldots$
6.) Repeat the steps to get desired root.

$A : B = A^4 - A - 10 : C = 4A^3 - 1 :$

$D = A - (B \div C)$.

Ques

Ques. Find a real root of equation $x^4 - x - 10 = 0$ upto 3 decimal place using Newton Raphson Method.

Soln.

$$f(x) = x^4 - x - 10 = 0, \quad f'(x) = 4x^3 - 1$$

Now,
$$f(0) = -10 < 0$$
$$f(1) = -10 < 0$$
$$f(2) = 4 > 0$$

Hence. root lies between 1 and 2.

Taking initial approximation,

$$x_0 = \frac{a+b}{2} = \frac{1+2}{2} = 1.5$$

Now, by Newton Raphson's Method., General formula.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

1st approximation

at $n=0$.

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$x_0 = 1.5$
$f(x_0) = -6.4375$
$f'(x_0) = 12.5000$
$x_1 = 2.0150$

$$x_1 = 1.5 - \left[\frac{(1.5)^4 - 1.5 - 10}{4(1.5)^3 - 1}\right]$$

$$x_1 = 1.5 - \left[\frac{-6.4375}{12.5000}\right]$$

$$x_1 = 2.0150.$$

2nd approximation at $n=1$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 2.0150 - \frac{4.4704}{31.7254}$$

$$x_2 = 1.8741$$

$x_1 = 2.0150$
$f(x_1) = 4.704$
$f'(x_1) = 31.7254$
$x_2 = 1.8741$

3rd approximation: at $n = 2$.

$x_2 = 1.8741$

$f(x_2) = 0.4618$

$f'(x_2) = 25.3292$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$x_3 = 1.8741 - \frac{0.4618}{25.3292}$$

$x_3 = 1.8559$

$$x_3 = 1.8559$$

4th approximation: at $n = 3$.

$x_3 = 1.8559$

$f(x_3) = 0.0077$

$f'(x_3) = 24.5696$

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)}$$

$$x_4 = 1.8559 - \frac{0.0077}{24.5696}$$

$x_4 = 1.8556$

$$x_4 = 1.8556.$$

Since. x_3 and $x_4 = 1.855$ upto 3 decimal place.

Hence, required root is 1.855.

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Que. 2. find a root by Newton Raphson method of equation $x^3 - 3x + 1 = 0$ correct to 3 decimal places

Soln $f(x) = x^3 - 3x + 1$ —(1)

$f'(x) = 3x^2 - 3$

Now, $f(0) = 1$ and $f(1) = -1$.

So, root lies b/w $(0, 1)$.

Initial Approximation, $x_0 = \frac{0+1}{2} = 0.5$

1st approximation: at $n = 1$.

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$x_1 = 0.5 - \frac{(-0.3750)}{(-2.2500)}$$

$$x_1 = 0.3333$$

2nd approx. at $n = 2$.

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$x_2 = 0.3333 - \frac{0.0371}{(-2.6667)}$$

$$x_2 = 0.3472.$$

3rd approx at $n = 3$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 0.3472 - \frac{0.0003}{(-2.6384)}$$

$$x_3 = 0.3473..$$

Here, $x_2 = x_3 = 0.347$ upto 3 decimal places.

Hence required root is 0.347.

Ques. Apply Newton-Raphson Method to solve
$3x - \cos x - 1 = 0$ correct to three decimal places

Soln. Here, $f(x) = 3x - \cos x - 1$, $f'(x) = 3 + \sin x$

$f(0) = -2 < 0$

$f(1) = 1 > 0$

So, root lies between $(0, 1)$.

Initial approximation

$$x_0 = \frac{0+1}{2} = 0.5$$

1st approx at $n = 0$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = \cancel{0.5 - \frac{(-0.500)}{3.009}}$$

$$\cancel{x_1 / 0.676} \quad x_1 = 0.5 - \frac{(-0.378)}{3.479}$$

$$x_1 = 0.609.$$

2nd approx at $n = 1$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 0.609 - \frac{0.007}{3.572}$$

$$x_2 = 0.607$$

3rd approx at $n = 2$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 0.607 - \frac{0.000}{3.570}$$

$$x_3 = 0.607$$

Here, $x_2 = x_3 = 0.607$ upto 3 decimal places.
Hence required root is 0.607.

~~Que.~~ Ques. Newton Raphson's

$f(x) = x \sin x + \cos x$, nearby $x = \pi$

$f(x) = x \sin x + \cos x$

$f'(x) = x \cos x + \cancel{\sin x} + \cancel{\sin x} = x \cos x$.

$x_0 = 3.1416$. value app.

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 3.1416 -$$

$x_1 = 2.8233$.

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 2.7986$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 2.7984.$$

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} = 2.7984.$$

kth root of N

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Q. Find negative root of equation $x^3 - 21x + 3500 = 0$ correct to three decimal places using Newton Raphson's method

Soln

$$f(x) = x^3 - 21x + 3500$$

$$f'(x) = 3x^2 - 21.$$

$-9 \to +ve$
$-4 \to +ve$

$$f(-15) = 440 > 0 \quad \text{and} \quad f(-16) = -260 < 0$$

$$\because f(15) \cdot f(16) < 0,$$

$\Rightarrow$ root lies b/w -15 and -16.

Initial root,

$$x_0 = \frac{-15 + (-16)}{2} = -15.5$$

1st approx. $x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)} =$

$$x_1 = -15.5 - \frac{101.62500}{699.75000}$$

$$= -15.64523.$$

2nd approx.

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$x_2 = -15.64523 - \frac{(-0.98352)}{713.31967}$$

$$x_2 = -15.64385$$

3rd approx

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$x_3 = -15.64385 - \frac{0.00077}{713.19013}$$

$$= -15.64385$$

$x_2 = x_3 = -15.6438$ upto 4 decimal place.

Hence -15.6438 is required root.

Que ~~find~~ Determine Newton Raphson iterative formula to find k^{th} root of N.

$x = \sqrt[k]{N}$

$f(x) = x^k - N = 0$

$f'(x) = kx^{k-1}$

Newton's formula.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{(x_n^k - N)}{kx_n^{k-1}}$$

$$= \frac{x_n k x_n^{k-1} - x_n^k + N}{kx_n^{k-1}}$$

$$= \frac{kx_n^k - x_n^k + N}{kx_n^{k-1}}$$

$$= \frac{(k-1)x_n^k + N}{kx_n^{k-1}}$$

$$= \frac{1}{k}\left[\frac{(k-1)x_n^k}{x_n^{k-1}} + \frac{N}{x_n^{k-1}}\right]$$

$$x_{n+1} = \frac{1}{k}\left[(k-1)x^n + \frac{N}{x_n^{k-1}}\right]$$

Difference Operators

Let $y = f(x)$ be a given function and h is any fixed value. Let $x = x_0, x_1, x_2, \ldots, x_n$ such that,

$$x_1 = x_0 + h$$
$$x_2 = x_1 + h = x_0 + 2h$$
$$x_3 = x_2 + h = x_0 + 3h$$
$$\vdots$$
$$x_n = x_{n-1} + h = x_0 + nh$$

Also,
$$y_0 = f(x_0)$$
$$y_1 = f(x_1) = f(x_0 + h)$$
$$y_2 = f(x_2) = f(x_0 + 2h)$$
$$\vdots$$
$$y_n = f(x_n) = f(x_0 + nh)$$

Note: Here $h \rightarrow$ length of interval and all x_i's are equidistant.

1. **Shifting Operator**

Symbol – E and is defined as,

$$\boxed{E f(x) = f(x+h)}$$

$$E^2 f(x) = E[E f(x)] = E f(x+h) = f(x+2h)$$

- $E^2 f(x) = f(x+2h)$
- $E^{-1} f(x) = f(x-h)$, • $E^{-2} = f(x-2h)$
- $E^{1/2} f(x) = f\left(x + \frac{h}{2}\right)$

\# $Ey_0 = E f(x_0) = f(x_0 + h) = f(x_1) = y_1$

$\Rightarrow Ey_0 = y_1$ and $Ey_1 = y_2$, $E^{-1} y_1 = y_0$

* Forward value − current value.

2. <u>Forward Difference Operator</u>

Symbol − Δ defined as.

1st forward diff

$$\boxed{\Delta f(x) = f(x+h) - f(x)}$$

$f(x-h)$

Now.

$$\Delta^2 f(x) = \Delta[\Delta f(x)]$$
$$= \Delta[f(x+h) - f(x)]$$

diff $f(x)$, $f(x+h)$

$$= \Delta f(x+h) - \Delta f(x)$$
$$= f(x+2h) - f(x+h) - \{f(x+h) - f(x)\}$$
$$= f(x+2h) - 2f(x+h) + f(x)$$
$$\Rightarrow \Delta^2 f(x) = f(x+2h) - 2f(x+h) + f(x)$$

* $\Delta y_0 = \Delta f(x_0) = f(x_0+h) - f(x_0)$
$= f(x_1) - f(x_0)$
$= y_1 - y_0$

$\Rightarrow \Delta y_0 = y_1 - y_0$

* $\Delta y_1 = y_2 - y_1$

$\Delta y_2 = y_3 - y_2$.

* $\Delta^2 y_0 = \Delta(\Delta y_0) = \Delta(y_1 - y_0) = \Delta y_1 - \Delta y_0$.

$= y_2 - y_1 - (y_1 - y_0) = y_2 - 2y_1 + y_0$

$$\Rightarrow \boxed{\Delta^2 y_0 = y_2 - 2y_1 + y_0}$$

current value − Backward value.

3) <u>Backward Difference Operator</u>

Symbol − ∇ and is defined as

$$\nabla f(x) = f(x) - f(x-h)$$

$$\nabla y_1 = \nabla f(x_1) = f(x_1) - f(x_1 - h)$$
$$= f(x_1) - f(x_0 + h - h)$$
$$= f(x_1) - f(x_0).$$
$$= y_1 - y_0.$$

$$\Rightarrow \nabla y_1 = y_1 - y_0.$$

* $\nabla y_2 = y_2 - y_1$
* $\nabla^2 y_2 = \nabla(y_2 - y_1) = \nabla y_2 - \nabla y_1$

$$= y_2 - y_1 - \{y_1 - y_0\} = y_2 - 2y_1 + y_0.$$

4.) <u>Central Difference Operator</u>

Symbol − δ, defined as.

$$\delta f(x) = f\left(x + \frac{h}{2}\right) - f\left(x - \frac{h}{2}\right)$$

5.) <u>Averaging Operator</u>

Symbol − μ and is defined as.

$$\mu f(x) = \frac{1}{2}\left[f\left(x + \frac{h}{2}\right) + f\left(x - \frac{h}{2}\right)\right]$$

Relation between Difference Operator

Prove that

1. $\Delta = E - 1$
2. $\nabla = 1 - E^{-1}$
3. $\delta = E^{1/2} - E^{-1/2}$
4. $\mu = \frac{1}{2}\left[E^{1/2} + E^{-1/2}\right]$
5. $\left[E^{1/2} + E^{-1/2}\right](1+\Delta)^{1/2} = 2 + \Delta$
6. $E = e^{hD}$, where $D = \frac{d}{dx}$
7. $hD = \log(1+\Delta) = -\log(1-\nabla)$
8. $(1+\Delta)(1-\nabla) = 1$.

Ques. Prove that, $\Delta = E - 1$

Since $\Delta f(x) = f(x+h) - f(x)$.

$= E f(x) - f(x)$ $\{\because E f(x) = f(x+h)\}$

$\Delta f(x) = (E-1) f(x)$.

{by left cancellation}

$\Rightarrow \Delta = E - 1$

Ques. Prove that, $\nabla = 1 - E^{-1}$

Since:

$\nabla f(x) = f(x) - f(x-h)$

$= f(x) - E^{-1} f(x)$

$\nabla f(x) = (1 - E^{-1}) f(x)$

By left cancell.

$\Rightarrow \nabla = 1 - E^{-1}$.

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3.) Prove that $\delta = E^{1/2} - E^{-1/2}$.

$\because \delta f(x) = f\left(x+\frac{h}{2}\right) - f\left(x-\frac{h}{2}\right)$ by defn

$\Rightarrow \delta f(x) = E^{1/2} f(x) - E^{-1/2} f(x)$.

$\delta \cancel{f(x)} = E^{1/2} - E^{-1/2} \cancel{f(x)}$

$\Rightarrow \delta = E^{1/2} - E^{-1/2}$

4. Prove that $\mu = \frac{1}{2}\left[E^{1/2} + E^{-1/2}\right]$

$\because \mu f(x) = \frac{1}{2}\left[f\left(x+\frac{h}{2}\right) + f\left(x-\frac{h}{2}\right)\right]$

$= \frac{1}{2}\left[E^{1/2} f(x) + E^{-1/2} f(x)\right]$

$\mu \cancel{f(x)} = \frac{1}{2}\left[E^{1/2} + E^{-1/2}\right] \cancel{f(x)}$.

$\mu = \frac{1}{2}\left[E^{1/2} + E^{-1/2}\right]$.

5. Prove that $\left[E^{1/2} + E^{-1/2}\right](1+\Delta)^{1/2} = 2+\Delta$

LHS.

$= \left[E^{1/2} + E^{-1/2}\right](1+\Delta)^{1/2}$

$= \left[E^{1/2} + E^{-1/2}\right] E^{1/2}$ $\{\because 1+\Delta = E\}$

$= E + 1$

$= 1 + \Delta + 1$

$= 2 + \Delta$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

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Que Prove that $(1+\Delta)(1-\nabla) = 1$.

LHS

$$= (1+\Delta)(1-\nabla)$$
$$= E \cdot E^{-1}$$
$$= 1.$$

Que Prove that $E = e^{hD}$, where $D = \frac{d}{dx}$

By defn of shifting operator.

$$E f(x) = f(x+h)$$

By Taylor's Theor.

$$\Rightarrow E f(x) = f(x) + h f'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \cdots$$

$$\Rightarrow E f(x) = f(x) + hD f(x) + \frac{h^2}{2!} D^2 f(x) + \frac{h^3}{3!} D^3 f(x) + \cdots$$

$$\Rightarrow E f(x) = \left\{1 + hD + \frac{h^2 D^2}{2!} + \frac{h^3 D^3}{3!} + \cdots\right\} f(x)$$

$$\Rightarrow E f(x) = e^{hD} f(x)$$

$$\Rightarrow E = e^{hD}.$$

Que Prove that $hD = \log(1+\Delta) = -\log(1-\nabla)$

WKT $E = e^{hD}$ — (1)

Taking log both side.

$$\log E = hD \quad \text{— (2)}$$

$$\log(1+\Delta) = hD \quad \text{— (3)}$$

from (2) $-\log E = -hD$

$$\log E^{-1} = -hD$$

$$\Rightarrow \log(1-\nabla) = -hD.$$

$$\Rightarrow -\log(1-\nabla) = hD \quad \text{— (4)}$$

from (3) & (4) $hD = \log(1+\Delta) = -\log(1-\nabla)$.

Ans. Prove that $\left(\frac{\Delta^2}{E}\right) e^x \cdot \frac{E e^x}{\Delta^2 e^x} = e^x$

LHS

$$= \left(\frac{\Delta^2}{E}\right) e^x \cdot \frac{E e^x}{\Delta^2 e^x}$$

$$= \left[\frac{(E-1)^2}{E}\right] e^x \frac{e^{x+h}}{(E-1)^2 e^x} \quad \begin{cases} \Delta = E-1, \\ E f(x) = f(x+h) \end{cases}$$

$$= \left[\frac{E^2 - 2E + 1}{E}\right] e^x \cdot \frac{e^{x+h}}{[E^2 - 2E + 1]\, e^x}$$

$$= [E - 2 + E^{-1}]\, e^x \frac{e^{x+h}}{e^{x+2h} - 2e^{x+h} + e^x}$$

$$= (e^{x+h} - 2e^x + e^{x-h}) \cdot \frac{e^x \cdot e^h}{e^{x+2h} - 2e^{x+h} + e^x}$$

$$= \frac{\cancel{(e^{x+2h} - 2e^{x+h} + e^x)} \cdot e^x}{\cancel{(e^{x+2h} - 2e^{x+h} + e^x)}}$$

$$= e^x$$

$\because \sin C + \sin D = 2\sin\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$

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Ques. Evaluate,

$$\left(\frac{\Delta^2}{E}\right)\sin(x+h) + \frac{\Delta^2 \sin(x+h)}{E\sin(x+h)}$$

Sol $= \left[\frac{(E-1)^2}{E}\right]\sin(x+h) + \frac{(E-1)^2\sin(x+h)}{\sin(x+2h)}$

$= \left[\frac{E^2-2E+1}{E}\right]\sin(x+h) + \frac{\left[E^2-2E+1\right]\sin(x+h)}{\sin(x+2h)}$

$= \left[E-2+E^{-1}\right]\sin(x+h) + \frac{\sin(x+3h) - 2\sin(x+2h) + \sin(x+h)}{\sin(x+2h)}$

$= \sin(x+2h) - 2\sin(x+h) + \sin x$

$+ \frac{2\sin(x+2h)\cos h - 2\sin(x+2h)}{\sin(x+2h)}$

$= 2\sin(x+h)\cos h - 2\sin(x+h)$

$+ \frac{2\cancel{\sin(x+2h)}\left[\cos h - 1\right]}{\cancel{\sin(x+2h)}}$

$= 2\sin(x+h)\cos h - 2\sin(x+h) + 2(\cos h - 1)$

$= 2\sin(x+h)\left[\cos h - 1\right] + 2(\cos h - 1)$

$= 2(\cos h - 1)\left[\sin(x+h) + 1\right]$: ans

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Que Evaluate

$$\Delta^2 \left[\frac{5x+12}{x^2+5x+6} \right]$$

$$= \Delta^2 \left[\frac{5x+12}{(x+2)(x+3)} \right]$$

resolving into partial fraction.

$$= \Delta^2 \left[\frac{2}{x+2} + \frac{3}{x+3} \right]$$

$$= \Delta \left[\Delta \left(\frac{2}{x+2} \right) + \Delta \left(\frac{3}{x+3} \right) \right]$$

$$\left\{ \because \Delta f(x) = f(x+h) - f(x) \right\}$$

$$= \Delta \left[2 \left(\frac{1}{x+3} - \frac{1}{x+2} \right) + 3 \left(\frac{1}{x+4} - \frac{1}{x+3} \right) \right]$$

$$= -2\Delta \left[\frac{1}{(x+2)(x+3)} \right] - 3\Delta \left[\frac{1}{(x+3)(x+4)} \right]$$

$$= -2 \left[\frac{1}{(x+3)(x+4)} - \frac{1}{(x+2)(x+3)} \right] - 3 \left[\frac{1}{(x+4)(x+5)} - \frac{1}{(x+3)(x+4)} \right]$$

$$= \frac{4}{(x+2)(x+3)(x+4)} + \frac{6}{(x+3)(x+4)(x+5)}$$

$$= \frac{2(5x+16)}{(x+2)(x+3)(x+4)(x+5)}$$

FACTORIAL NOTATION OR POLNOMIAL

The factorial polynomial of degree 'n' is denoted by the symbol $x^{(n)}$ or $x^{[n]}$
(* read it as x to the power n factorial)

Thus,

$$x^{[n]} = x(x-h)(x-2h) \ldots\ldots (x-[n-1]h)$$

Here h is constant difference and n is positive integer. in particular $h=1$, we have.

$$x^{[1]} = x$$
$$x^{[2]} = x(x-1)$$
$$x^{[3]} = x(x-1)(x-2) \quad \text{and so on.}$$

* If n is negative integer then reciprocal factorial function of order n is denoted by $x^{(-n)}$ or $x^{[-n]}$

$$x^{(-n)} = \frac{1}{(x+h)(x+2h)(x+3h)\ldots(x+nh)}$$

when $h=1$

$$x^{(-1)} = \frac{1}{(x+1)}$$

$$x^{(-2)} = \frac{1}{(x+1)(x+2)}$$

$$x^{(-3)} = \frac{1}{(x+1)(x+2)(x+3)}$$

Remarks

1. $\Delta x^{(n)} = nhx^{[n-1]}$

example:

$$\Delta x^{(3)} = \frac{d}{dx} x^{(3)} = 3x^{(2)}.$$

Slly,

$$\Delta^2 x^{(n)} = n(n-1)h^2 x^{(n-2)}$$

2. $\Rightarrow$ $\boxed{\Delta^n x^{(n)} = n!\, h^n.}$

3.) $\Delta^k x^{(n)} = 0$ for $k > n$ and.

$$\Delta x^{(n)} = \frac{d}{dx} x^{(n)} = n x^{(n-1)}$$

4. $$\frac{1}{\Delta} x^{(n)} = \int x^{(n)} dx = \frac{x^{(n+1)}}{n+1} + c.$$

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Ques. 1 Express $y = 2x^3 - 3x^2 + 3x - 10$ in factorial notation and show that $\Delta^3 y = 12$

Let $f(x) = 2x^3 - 3x^2 + 3x - 10$

Consider.

$$f(x) = A x^{(3)} + B x^{(2)} + C x^{(1)} + D. \quad \text{--- (1)}$$

where, A, B, C, D are constant, will find ~~this~~ these values using synthetic division method

1	2	-3	3	$-10 = D$
	+ 0	2	-1	
2	2	-1	$2 = C$	
	0	4		
3	2	$3 = B$		
	0			
	$2 = A$			

putting in eq (1).

required factorial notation

$$y = f(x) = 2x^{(3)} + 3x^{(2)} + 2x^{(1)} - 10. \quad \text{--- (2)}$$

$$= 2x(x-1)(x-2) + 3(x)(x-1) + 2x - 10.$$

from eq (2)

$$y = 2x^{(3)} + 3x^{(2)} + 2x^{(1)} - 10$$

$$\Delta y = 6x^{(2)} + 6x^{(1)} + 2$$

$$\Delta^2 y = 12x^{(1)} + 6.$$

$$\Delta^3(y) = 12.$$

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Ques 2. find the function whose 1st difference is $9x^2 + 11x + 5$.

Soln Given first-difference i.e $\Delta f(x)$

$$\Delta f(x) = 9x^2 + 11x + 5$$

OR

$$\Delta f(x) = Ax^2 + Bx + C \quad \text{--- (1)}$$

By synthetic Division method.

1	9	11	5 = C
	0	9	
2	9	20 = B	
	0		
3	9 = A		

putting values of A, B, C in (1).

$$\Delta f(x) = 9x^{(2)} + 20x^{(1)} + 5$$

$$\therefore f(x) = \frac{1}{\Delta}\left[9x^{(2)} + 20x^{(1)} + 5\right] \qquad \left\{\because \frac{1}{\Delta} = \int dx\right.$$

$$f(x) = 9\frac{1}{\Delta}x^{(2)} + 20\frac{1}{\Delta}x^{(1)} + 5\frac{1}{\Delta}$$

$$f(x) = 9\int x^{(2)}dx + 20\int x^{(1)}dx + 5\int dx$$

$$f(x) = 3\left(\frac{x^{(3)}}{3}\right) + 10\left(\frac{x^{(2)}}{2}\right) + 5.[x^{(1)}] + C$$

$$f(x) = 3x^{(3)} + 10x^{(2)} + 5x + C$$

$$= 3x(x-1)(x-2) + 10x(x-1) + 5x + C$$

$$f(x) = 3x^2 + x^2 + x + C$$

where c is constant

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Ques Obtain the function whose first difference is $2x^3 + 3x^2 - 5x + 4$.

Soln first Difference :- $\Delta f(x)$ i.e.

$$\Delta f(x) = 2x^3 + 3x^2 - 5x + 4$$

Let $\Delta f(x) = Ax^3 + Bx^2 + Cx + D$ — (1)

By synthetic division method.

1	2	3	-5	$4 = D$
	0	2	5	
2	2	5	$0 = C$	
	0	4		
3	2	$9 = B$		
	0			
	$2 = A$			

from eqn (1)

$$\Delta f(x) = 2x^{(3)} + 9x^{(2)} + 4$$

$$f(x) = \frac{1}{\Delta}\left[2x^{(3)} + 9x^{(2)} + 4\right]$$

$$= \frac{\cancel{2} \times x^{(4)}}{\cancel{4}\,2} + \frac{\cancel{9}^{3}\, x^{(3)}}{\cancel{3}} + 4x + k.$$

$$= \frac{x^{(4)}}{2} + 3x^{(3)} + 4x + k.$$

(in factorial notation)

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Ques 4. Express $f(x) = x^3 - 2x^2 + x - 1$ in factorial notation and show that $\Delta^4 f(x) = 0$

Soln

$$f(x) = x^3 - 2x^2 + x - 1 \quad \text{--- (1)}$$

let $f(x) = Ax^3 + Bx^2 + Cx + D$ --- (2)

By synthetic Division,

1	1	-2	1	-1 = D
	0	1	-1	
2	1	-1	0 = C	
	0	2		
3	1	1 = B		
	0			
	1 = A			

putting in eqn (2)

$$f(x) = x^{(3)} + x^{(2)} - 1. \qquad \left\{ \because \Delta = \frac{d}{dx} \right\}$$

$$\Delta f(x) = \Delta [x^{(3)} + x^{(2)} - 1]$$

$$\Delta f(x) = 3x^{(2)} + 2x.$$

$$\Delta^2 f(x) = 6x + 2$$

$$\Delta^3 f(x) = 6.$$

$$\Delta^4 f(x) = 0. \qquad \text{H.P.}$$

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Que.: Represent the function $f(x)$

$f(x) = x^4 - 12x^3 + 24x^2 - 30x + 9$ and its successive difference in factorial notation.

Soln. $f(x) = x^4 - 12x^3 + 24x^2 - 30x + 9$

Let

$$f(x) = A x^{(4)} + B x^{(3)} + C x^{(2)} + D x + E \quad \text{—— (1)}$$

1	1	-12	24	-30	$9 = E$
	0	1	-11	13	
2	1	-11	13		$-17 = D$
	0	2	-18		
3	1	-9			$-5 = C$
	0	3			
4	1				$-6 = B$
	0				
	$1 = A$				

from eq (1) $f(x) = x^{(4)} - 6x^{(3)} - 5x^{(2)} - 17x + 9$ —— (2)

↳ factorial notation.

Now successive differences from (2)

$\Delta f(x) = 4x^{(3)} - 18x^{(2)} - 10x^{(1)} - 17$

$\Delta^2 f(x) = 12x^{(2)} - 36x^{(1)} - 10$

$\Delta^3 f(x) = 24x - 36$

$\Delta^4 f(x) = 24$

$\Delta^5 f(x) = 0$

FORWARD DIFFERENCE TABLE

Arguments x	Entry $y = f(x)$	1st Difference Δy	2nd Difference $\Delta^2 y$	3rd Difference $\Delta^3 y$	4th Difference $\Delta^4 y$
x_0	y_0	Δy_0	$\Delta^2 y_0$	$\Delta^3 y_0$	$\Delta^4 y_0$
x_1	y_1	Δy_1	$\Delta^2 y_1$	$\Delta^3 y_1$	
x_2	y_2	Δy_2	$\Delta^2 y_2$		
x_3	y_3	Δy_3			
x_4	y_4				

Here $y_0 \longrightarrow$ is leading entry.

$\Delta y_0, \Delta^2 y_0, \Delta^3 y_0 \ldots \longrightarrow$ leading differences

* if given n observations then $\Delta^n y = 0$

Backward Difference table

Arg x	Entry y	1st Diff. ∇y	2nd Diff $\nabla^2 y$	3rd Diff $\nabla^3 y$	4th Diff $\nabla^4 y$
x_0	y_0				
x_1	y_1	∇y_1	$\nabla^2 y_2$	$\nabla^3 y_3$	$\nabla^4 y_4$
x_2	y_2	∇y_2	$\nabla^2 y_3$	$\nabla^3 y_4$	
x_3	y_3	∇y_3	$\nabla^2 y_4$		
x_4	y_4	∇y_4			

$\nabla y_1 = y_1 - y_0$

$\nabla^2 y_2 = \nabla y_2 - \nabla y_1$

Ques. Represent the function ~~$f(x) = x^4 - 12x^3 + 42x^2$~~

$f(x) = x^4 - 12x^3 + 42x^2 - 30x + 9$ and its successive difference in factorial notation in which interval of differencing is one.

1st Method <u>FORWARD</u> <u>Method</u>

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
0	$9 = f(0)$				
		$01 = \Delta f(0)$			
1	10		$26 = \Delta^2 f(0)$		
		27		$-36 = \Delta^3 f(0)$	
2	37		-10		$24 = \Delta^4 f(0)$
		17		-12	
3	54		-22		
		-5			
4	49				

In factorial Notation

$$f(x) = f(0) + \frac{\Delta f(0)\, x^{(1)}}{1!} + \frac{\Delta^2 f(0)\, x^{(2)}}{2!} + \frac{\Delta^3 f(0)\, x^{(3)}}{3!} + \frac{\Delta^4 f(0)\, x^{(4)}}{4!}$$

$$f(x) = 9 + x^{(1)} + \frac{26\, x^{(2)}}{2} - \frac{36 x^{(3)}}{6} + \frac{24 x^{(4)}}{24}$$

factorial notation $\boxed{f(x) = 9 + x^{(1)} + 13 x^{(2)} - 6 x^{(3)} + x^{(4)}}$

Successive Differences.

$\Delta f(x) = 1 + 26 x^{(1)} - 18 x^{(2)} + 4 x^{(3)}$

$\Delta^2 f(x) = 26 - 36 x^{(1)} + 12 x^{(2)}$

$\Delta^3 f(x) = -36 + 24 x^{(1)}$

$\Delta^4 f(x) = 24$

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2nd Method Direct Method.

Let $f(x) = x^4 - 12x^3 + 42x^2 - 30x + 9$ — (1)

and its factorial notation is denoted by.

$f(x) = A x^{(4)} + B x^{(3)} + C x^{(2)} + D x^{(1)} + E$ — (2)

from (1) and (2)

$\Rightarrow$ $x^4 - 12x^3 + 42x^2 - 30x + 9$

$= Ax(x-1)(x-2)(x-3) + Bx(x-1)(x-2) + Cx(x-1) + Dx + E$ — (3)

put $x = 0$ in (3)

$\Rightarrow \boxed{E = 9}$

put $x = 1$ in (3)

$\Rightarrow 10 = D + E$

$\Rightarrow 10 = D + 9$

$\Rightarrow \boxed{D = 1}$

put $x = 2$ in (3)

$\Rightarrow 37 = 2C + 2D + E$

$\Rightarrow 37 = 2C + 2 + 9$

$\Rightarrow 37 = 2C + 11$

$\Rightarrow 2C = 37 - 11$

$\Rightarrow 2C = 26$

$\Rightarrow \boxed{C = 13}$

put $x = 3$ $\Rightarrow 54 = 6B + 6C + 3D + E$

$\Rightarrow 54 = 6B + 78 + 3 + 9$

$\Rightarrow 54 = 6B + 90$

$\Rightarrow 6B = 90 - 54$

$\Rightarrow 6B = -36$

$\Rightarrow \boxed{B = -6}$

put $x = 4$ $\Rightarrow 49 = 24A + 24B + 12C + 4D + E$

$49 = 24A - 144 + 156 + 4 + 9$

$49 = 24A + 25$

$24A = 49 - 25$

$24A = 24$

$\boxed{A = 1}$

Hence,

$f(x) = x^{(4)} - 6x^{(3)} + 13x^{(2)} + x^{(1)} + 9$

3rd Method **Synthetic Division Method**

Let $f(x) = x^4 - 12x^3 + 42x^2 - 30x + 9$ —— (1)

and $f(x) = A x^{(4)} + B x^{(3)} + C x^{(2)} + D x^{(1)} + E$ —— (2)

be the factorial notation.

By synthetic Division.

1	1	-12	42	-30	$9 = E$
	0	1	-11	31	
2	1	-11	31	$1 = D$	
	0	2	-18		
3	1	-9	$13 = C$		
	0	3			
	1	$-6 = B$			
	0				
	$1 = A$				

put values of A, B, C, D, E in eq (2).

$f(x) = x^{(4)} - 6x^{(3)} + 13x^{(2)} + x^{(1)} + 9$

INTERPOLATION

According to Theile.

"Interpolation is the art of reading between the lines of table."

It also means insertion or filling up intermediate terms of the series. Suppose we are given the following ~~values~~ values of $y = f(x)$ for a set of values of x.

x:	x_0	x_1	x_2	x_3	—	—	x_n
y:	y_0	y_1	y_2	y_3	—	—	y_n

Thus, the process of finding the values of y corresponding to any value of x between x_0 to x_n is called "Interpolation".

Hence, interpolation is technique of estimating the value of function for any intermediate value of the independent variable, while the process of computing the value of the function outside the given range is called "Extrapolation".

Ques) Estimate the missing term in the following table.

x	0	1	2	3	4
y	1	3	9	—	81

Soln By Difference table.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0	y_0 1				
		2			
1	y_1 3		4		
		6		y_3-19	
2	y_2 9		y_3-15		$124-4y_3$
		y_3-9		$105-3y_3$	
3	y_3		$90-2y_3$		
		$81-y_3$			
4	y_4 81				

Now. Here, $\Delta^4 y = 0$. {Since 4 entries are given}

$\Rightarrow 124 - 4y_3 = 0$

$\Rightarrow 124 = 4y_3$

$\Rightarrow y_3 = \frac{124}{4} = 31$

$\Rightarrow \boxed{y_3 = 31}$ Ans.

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Que 2. find the missing value in following data.

x	2.0	2.1	2.2	2.3	2.4	2.5	2.6
y	0.135	y_1	0.111	0.100	y_4	0.082	0.074

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
2.0	0.135	$y_1 - 0.135$				
			$0.246 - 2y_1$	~~$-0.114 + 3y_1$~~		
2.1	y_1	$0.111 - y_1$		$-0.368 + 3y_1$		
			$-0.122 + y_1$		$y_4 + 0.401 - 4y_1$	
2.2	0.111	-0.011		$y_4 + 0.033 - y_1$		$-0.183 - 5y_4 + 5y_1$
			$y_4 - 0.089$		$0.238 - 4y_4 + y_1$	
2.3	0.100	$y_4 - 0.100$		$0.271 - 3y_4$		
			$0.182 - 2y_4$		$-0.543 + 6y_4$	$-0.781 + 10y_4 - y_1$
2.4	y_4	$0.082 - y_4$		$-0.272 + 3y_4$		
			$-0.09 + y_4$			
2.5	0.082	-0.008				
2.6	0.074					

No. of known values = 5.

Therefore $\Delta^5 y = 0$

$$5y_1 - 5y_4 = 0.183 \quad \text{—— (1)}$$
$$-y_1 + 10y_4 = 0.781 \quad \text{—— (2)}$$

$y_1 = 0.123$

$y_4 = 0.0904$

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Que Compute next three values.

x	0	1	2	3	4	5	6	7
y	1	-1	1	-1	1	-	-	-

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
0	1					
		-2				
1	-1		4			
		2		-8		
2	1		-4		16	
		-2		8		y_5-31
3	-1		4		y_5-15	
		2		y_5-7		y_6-5y_5+26
4	1		y_5-3		y_6-4y_5+11	
		y_5-1		y_6-3y_5+4		$y_7-5y_6+10y_5-16$
5	y_5		y_6-2y_5+1		$y_7-4y_6+6y_5-5$	
		y_6-y_5		$y_7-3y_6+3y_5-1$		
6	y_6		$y_7-2y_6+y_5$			
		y_7-y_6				
7	y_7					

Since five values are given

$$\Delta^5 y = 0$$

$\Rightarrow$ $y_5 - 31 = 0$ — (1) $\Rightarrow$ $\boxed{y_5 = 31.}$

$y_6 - 5y_5 + 26 = 0$ — (2)

$y_7 - 5y_6 + 10y_5 - 16 = 0$ — (3)

from (2)

$y_6 - 5\times 31 + 26 = 0 \Rightarrow y_6 - 129 = 0 \Rightarrow \boxed{y_6 = 129}$

from (3).

$y_7 - 5\times 129 + 10\times 31 - 16 = 0 \quad y_7 - 351 = 0 \Rightarrow \boxed{y_7 = 351}$

Ques. find the first term of the series whose second and subsequent terms are 8, 3, 0, -1, 0.

Soln

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
0	y_0					
		$8-y_0$				
1	8		$-13+y_0$			
		-5		$15-y_0$		
2	3		2		$-15+y_0$	
		-3		0		$15-y_0$
3	0		2		0	
		-1		0		
4	-1		2			
		1				
5	0					

Since numberg known data is 5

$\therefore \Delta^5 y = 0$

$\Rightarrow 15 - y_0 = 0$

$\Rightarrow \boxed{y_0 = 15}$

$\because \Delta = E - 1$

$y_0 = ?$, $y_1 = 8$, $y_2 = 3$, $y_3 = 0$, $y_4 = -1$, $y_5 = 0$

Soln. Here no. of known data = 5 values.

Therefore $\Delta^5 y = 0$.

$\Rightarrow (E-1)^5 y = 0$

$\Rightarrow [E^5 - {}^5C_1 E^4 + {}^5C_2 E^3 - {}^5C_3 E^2 + {}^5C_4 E - {}^5C_5] y = 0$

$\Rightarrow E^5 - 5E^4 + 10E^3 - 10E^2 + 5E - 1] y = 0$ — (1)

(i) putting $y = y_0$ in (1).

$[E^5 - 5E^4 + 10E^3 - 10E^2 + 5E - 1] y_0 = 0$

$\Rightarrow y_5 - 5y_4 + 10y_3 - 10y_2 + 5y_1 - y_0 = 0$

$\Rightarrow 0 - 5(-1) + 10(0) - 10(3) + 5(8) - y_0 = 0$

$\Rightarrow 5 - 30 + 40 - y_0 = 0$

$\Rightarrow 15 - y_0 = 0$

$$\boxed{y_0 = 15}$$

$\{(x+y)^n = {}^nC_0 x^n y^0 + {}^nC_1 x^{n-1} y^1 + {}^nC_2 x^{n-2} y^2 + \cdots + {}^nC_n x^0 y^n.\}$

$^nC_0 = 1$

$^nC_n = 1$

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Qu. Given that $y_5 = 4$, $y_6 = 3$, $y_7 = 4$, ~~$y_8 = 10$~~

$y_8 = 10$ and $y_9 = 24$. find value of $\Delta^4 y_5$:
(i) By using difference table (ii) without using table

Soln (i) By difference Table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
5	4				
		-1			
6	3		2		
		1		3	
7	4		5		$0 = \Delta^4 y_5$
		6		3	
8	10		8		
		14			
9	24				

Here. $\Delta^4 y_5 = 0$.

(ii) without table. $\Delta^4 y_5$ ∵ 5 entries given.

(∵ $\Delta = E - 1$) $\Delta^4 y_5 = (E-1)^4 y_5$

$\Rightarrow [E^4 - {}^4C_1 E^3 + {}^4C_2 E^2 - {}^4C_3 E + {}^4C_4] y_5$

∵ $E^n y_x = y_{x+n}$ $\Rightarrow y_9 - 4y_8 + 6y_7 - 4y_6 + y_5$

$\Rightarrow 24 - 4(10) + 6(4) - 4(3) + 4$

$\Rightarrow 24 - 40 + 24 - 12 + 4$

$= 0.$ Ans.

$\Delta^4 y_5 = 0.$

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Que Given.

x :	1	2	3	4	5
y :	2	5	10	17	26.

find value of $\nabla^2 y_5$

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$
1	$y_1 = 2$			
		3		
2	$y_2 = 5$		2	
		5		0
3	$y_3 = 10$		2	
		7		0
4	$y_4 = 17$		2	
		9	$= \nabla^2 y_5$	
5	$y_5 = 26$			

from diff table $\nabla^2 y_5 = 2$.

Without Table

$$\nabla^2 y_5 = (1 - E^{-1})^2 y_5 \qquad \left\{ \because \nabla = (1 - E^{-1}) \right.$$

$$\left. E^{-n} y_x = y_{x-n} \right\}$$

$$= (1 + E^{-2} - 2E^{-1}) y_5$$

$$= y_5 + y_3 - 2y_4.$$

$$= 26 + 10 - 2 \times 17$$

$$= 36 - 34$$

$$= 2$$

$$\boxed{\nabla^2 y_5 = 2}$$

INTERPOLATION WITH EQUAL INTERVALS

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Interpolation using Newton's forward and Backward Difference formula.

1. **Newton's forward Interpolation formula.**

$$y = f(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \cdots$$

where $p = \frac{x - x_0}{h}$

Here, $x_0 \rightarrow$ first term
$h \rightarrow$ common difference
$x \rightarrow$ target point.

2.) **Newton's Backward Interpolation formula**

$$y = f(x) = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \cdots$$

where $p = \frac{x - x_n}{h}$

Here,
$x \rightarrow$ target point
$x_n \rightarrow$ last term
$h \rightarrow$ common difference

* Though it is suggested that the forward formula should be used for interpolating y at points near x_0 and backward formula should be used for interpolating near x_n, it is not necessary.

Either of the formula ~~should be~~ can be used for interpolation.

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Ques. Using Newtons forward Interpolation formula find value of $f(1.6)$ if,

x	1	1.4	1.8	2.2
y	3.49	4.82	5.96	6.5

Soln Newton's forward Difference Table.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
$x_0 = 1$	y_0 3.49	Δy_0 1.33		
1.4	4.82		$\Delta^2 y_0$ −0.19	
		1.14		$\Delta^3 y_0$ −0.41
1.8	5.96		−0.60	
		0.54		
2.2	6.5			

Now, Newton's forward Interpolation formula.

$$y = f(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \cdots \quad \text{—— (1)}$$

Now. $p = \frac{x - x_0}{h} = \frac{1.6 - 1}{0.4} = 1.5$

$\Rightarrow \boxed{p = 1.5}$

$$f(1.6) = 3.49 + \underbrace{(1.5) \times (1.33)}_{(2.0615)} + \underbrace{\frac{(1.5)(0.5)}{2} \times (-0.19)}_{-0.07125} + \underbrace{\frac{(1.5)(0.5)(-0.5)}{6}}_{0.025625} \times (-0.41) = 5.44$$

$= 5.57$

Ques from the table, estimate the number of students who obtained marks between 40 and 45.

Marks.	30-40	40-50	50-60	60-70	70-80
No. of students	31	42	51	35	31

Soln To find the number of students who got marks between 40 and 45 i.e. difference b/w $f(40)$ and $f(45)$

Using forward difference Table.

Marks less than (x)	No. of Stud. y	Δy_0	$\Delta^2 y_0$	$\Delta^3 y_0$	$\Delta^4 y$
40	$31 = y_0$				
		$42 = \Delta y_0$			
50	$31+42=73$		$9 = \Delta^2 y_0$		
		51		$-25 = \Delta^3 y_0$	
60	$73+51=124$		-16		$37 = \Delta^4 y_0$
		35		12	
70	$124+35=159$		-4		
		31			
80	$159+31=190$				

from table. at $x = 40 \Rightarrow f(40) = 31$.

Now at $x = 45$,

Here, $p = \dfrac{x - x_0}{h} = \dfrac{45-40}{10} = 0.5$

By Newton's forward formula.

$$f(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \cdots$$

$$+ \frac{p(p-1)(p-2)(p-3)}{4!} \times \Delta^4 y_0 \quad \text{—— (1)}$$

$$f(45) = 31 + (0.5)(42) + \frac{(0.5)(-0.5)}{2} \times 9$$

$$+ \frac{(0.5)(-0.5)(-1.5)}{6} \times (-25)$$

$$+ \frac{(0.5)(-0.5)(-1.5)(-2.5)}{24} \times 37$$

$$= 31 + 21 + (-1.13) + (-1.45)$$

$$f(45) = 49.42 \approx 49$$

Hence. No. of students getting marks b/w 40 and 45.

$$= f(45) - f(40) = 49 - 31 = 18.$$

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Ques: find the number of men getting wages between Rs. 10 and Rs. 15 from foll. data

Soln. Difference Table

wage	0-10	10-20	20-30	30-40
freq.	9	30	35	42

wages less than (x)	frequency y	Δy	$\Delta^2 y$	$\Delta^3 y$
10	$9 = y_0$			
		$30 = \Delta y_0$		
20	$30+9=39$		$5 = \Delta^2 y_0$	
		35		$2 = \Delta^3 y_0$
30	$39+35=74$		7	
		42		
40	$74+42=116$			

Here, at $x=10$, $f(10) = 9$.

To find $f(15)$:- $p = \frac{x-x_0}{h} = \frac{15-10}{10} = 0.5$

Using Newton's forward formula,

$$f(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 :$$

$f(15)$

$$f(x) = 9 + (0.5)(30) + \frac{(0.5)(-0.5)(5)}{2} + \frac{(0.5)(-0.5)(-1.5)(2)}{6}$$

$$f(15) = 9 + 15 + (-0.63) + (0.13) = 23.50 \approx 24$$

Hence no. of men getting wages b/w ₹10 and ₹15

$$= f(15) - f(10) = 24 - 9 = 15$$

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Ques. Estimate $f(42)$ from following data.

x	20	25	30	35	40	45
$f(x)$	354	332	291	260	231	204

Soln. Difference table

x	$f(x)$	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$	$\nabla^5 y$
20	354					
		-22				
25	332		-19			
		-41		29		
30	291		10		-37	
		-31		-8		45
35	260		2		8	
		-29		0		
40	231		2			
		-27				
45	204					

To find $f(x)$ at $x = 42$ i.e. $f(42)$, so taking

$x_n = 45$ $\therefore$ $p = \frac{x - x_n}{h} = \frac{42-45}{5} = \frac{-3}{5}$

$\Rightarrow \boxed{p = -0.6}$

Using Newton's backward Difference formula.

$$f(x) = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n$$

$$+ \frac{p(p+1)(p+2)(p+3)}{4!}\nabla^4 y_n + \frac{p(p+1)(p+2)(p+3)(p+4)}{5!}\nabla^5 y_n.$$

$$f(42) = 204 + (-0.6)\times(-27) + \frac{(-0.6)(0.4)}{2!}\times(2) +$$

$$\frac{(-0.6)(0.4)(1.4)}{3!}\times(0) + \frac{(-0.6)(0.4)(1.4)(2.4)}{4!}\times(8)$$

$$+ \frac{(-0.6)(0.4)(1.4)(2.4)(3.4)}{5!}\times(45).$$

$$f(42) = 204 + 16.20 - 0.24 + 0 + 0.27 - 1.03$$

$$f(42) = 218.66 \approx 219.$$

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Que: from the following data find. $f(7.5)$

x	1	2	3	4	5	6	7	8
y	1	8	27	64	125	216	343	512

Soln: Difference table.

x	$f(x) = y$	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$	$\nabla^5 y$
1	1					
		7				
2	8		12			
		19		6		
3	27		18		0	
		37		6		0
4	64		24		0	
		61		6		0
5	125		30		0	
		91		6		0
6	216		36		0	
		127		6		
7	343		42			
		169				
8	512					

To find $f(x)$ at $x = 7.5$ i.e. $f(7.5) =$

Here $x = 7.5$ which is near to $x_n = 8$

$$\therefore \quad p = \frac{x - x_n}{h} = \frac{7.5 - 8}{1} = -0.5$$

$$p = (-0.5)$$

Therefore, using Newton Backwards Interpolation formula

$$f(x) = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n$$

$$f(7.5) = 512 + (-0.5)(169) + \frac{(-0.5)(0.5)}{2} \times 42$$

$$+ \frac{(-0.5)(0.5)(1.5)}{3!} \times 6 + 0.$$

$$f(7.5) = 512 - 84.50 + 5.25 - 0.38$$

$$f(7.5) = 421.875 \approx 422.$$

Ques. find the cubic polynomial which takes the following values:

x :	0	1	2	3
$f(x)$:	1	2	1	10

Hence evaluate $f(4)$.

Soln

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
0	1			
		1		
1	2		-2	
		-1		12
2	1		10	
		9		
3	10			

Here $h = 1$ and $p = \frac{x - x_0}{h} = \frac{x - 0}{1} = x$

By Newton's forward Diff. formula.

$$f(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0$$

$$f(x) = 1 + x(1) + \frac{x(x-1)}{2!} \times (-2) + \frac{x(x-1)(x-2)}{3!} \times 12$$

$$= 1 + x - x(x-1) + 2x(x-1)(x-2)$$

$$= 1 + x + x^2 - x + 2x^3 - 6x^2 + 4x$$

$$f(x) = 2x^3 - 7x^2 + 6x + 1 \quad \text{—— (1)}$$

$$f(4) = 41$$

Ques. The area 'A' of a circle of diameter 'd' is given for following values.

d :	80	85	90	95	100
A :	5026	5674	6362	7088	7854

By using appropriate interpolation formulae, find approx. values for the areas of circles of diameters 82 and 91 resp. Ans. $f(82) = 5280$, $f(91) = 6504$.

Ques. The following table gives population of town. Using Newton's interpolation formula find increase in population during 1946 to 1948 (1946 → 32.34) (1948 → 34.87)

Year	1911	1921	1931	1941	1951	1961	Ans
Pop.	12	15	20	27	39	52	2580

Ques. A third degree polynomial passes through the points (0, −1), (1, 1), (2, 1) and (3, −2), find the polynomial.

Ans: $-\frac{1}{6}x^3 - \frac{1}{2}x^2 + \frac{8}{3}x - 1$

$\Rightarrow -0.17x^3 - 0.5x^2 + 2.67x - 1$ decimal.

Ques. The pressure p of wind corresponding to velocity v is given by following data. Estimate p when v = 15.

v :	10	20	30	40
p :	1.1	2.0	4.4	7.9

Ans: $f(1.5) = 1.325$.

INTERPOLATION WITH UNEQUAL INTERVALS

① Lagranges Interpolation formula

② Newton's Divided Difference interval formula

Method - 1

LAGRANGES INTERPOLATION formula

- Let $f(x_0), f(x_1), \ldots, f(x_n)$ be the values of function $f(x)$ corresponding to the $(n+1)$ arguments $x_0, x_2, \ldots, x_n$ not necessarily equally spaced

Then,

$$y = f(x) = \frac{(x-x_1)(x-x_2)\cdots(x-x_n)}{(x_0-x_1)(x_0-x_2)\cdots(x_0-x_n)} \times y_0$$

$$+ \frac{(x-x_0)(x-x_1)\cdots(x-x_n)}{(x_1-x_0)(x_1-x_2)\cdots(x_1-x_n)} \times y_1$$

$$+ \cdots + \frac{(x-x_0)(x-x_1)\cdots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\cdots(x_n-x_{n-1})} \times y_n.$$

Ques. Given the values

x:	5	7	11	13	17
$f(x)$:	150	392	1452	2366	5202

Evaluate $f(9)$ using Lagrange's formula.

Soln. Lagrange's formula.

$$f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)(x-x_4)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)(x_0-x_4)} \times y_0$$

$$+ \frac{(x-x_0)(x-x_2)(x-x_3)(x-x_4)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)(x_1-x_4)} \times y_1$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)(x-x_4)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)(x_2-x_4)} \times y_2$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_4)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)(x_3-x_4)} \times y_3$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)}{(x_4-x_0)(x_4-x_1)(x_4-x_2)(x_4-x_3)} \times y_4$$

Here,

$x_0 = 5$, $x_1 = 7$, $x_2 = 11$, $x_3 = 13$, $x_4 = 17$

$y_0 = 150$, $y_1 = 392$, $y_2 = 1452$, $y_3 = 2366$, $y_4 = 5202$

putting $x = 9$ and given values in ① we get

$$f(9) = \frac{\overset{2}{(9-7)}\overset{-2}{(9-11)}\overset{-4}{(9-13)}\overset{+8}{(9-17)}}{\underset{-2}{(5-7)}\underset{-6}{(5-11)}\underset{-8}{(5-13)}\underset{-12}{(5-17)}} \times 150$$

$$+ \frac{\overset{4}{(9-5)}\overset{-2}{(9-11)}\overset{-4}{(9-13)}\overset{-8}{(9-17)}}{\underset{+2}{(7-5)}\underset{-4}{(7-11)}\underset{-6}{(7-13)}\underset{-10}{(7-17)}} \times 392$$

$$+ \frac{\overset{4}{(9-5)}\overset{2}{(9-7)}\overset{-4}{(9-13)}\overset{-8}{(9-17)}}{\underset{6}{(11-5)}\underset{4}{(11-7)}\underset{-2}{(11-13)}\underset{-6}{(11-17)}} \times 1452$$

$$+ \frac{\overset{4}{(9-5)}\overset{2}{(9-7)}\overset{-2}{(9-11)}\overset{-4}{(9-17)}}{\underset{8}{(13-5)}\underset{6}{(13-7)}\underset{2}{(13-11)}\underset{-4}{(13-17)}} \times 2366$$

$$+ \frac{\overset{4}{(9-5)}\overset{2}{(9-7)}\overset{-2}{(9-11)}\overset{-4}{(9-13)}}{\underset{12}{(17-5)}\underset{10}{(17-7)}\underset{6}{(17-11)}\underset{4}{(17-13)}} \times 5202$$

$$f(9) = \frac{-128}{1152} \times 150 + \frac{256}{480} \times 392 + \frac{256}{288} \times 1452$$

$$+ \frac{(-128)}{384} \times 2366 + \frac{64}{2880} \times 5202$$

$$f(9) = \frac{-19200}{1152} + \frac{100352}{480} + \frac{371712}{288}$$

$$- \frac{302848}{384} + \frac{332928}{2880}$$

$f(9) = -16.67 + 209 + 1291 - 789 + 116$

$f(9) = 810.$

Ques Apply Lagrange's formula to find $f(15)$ if :-

x	10	12	14	16	18	20
$f(x)$	2420	1942	1497	1109	790	540

Ans. $= f(15) = 1295$ approx.

Ques. Given that

x	4	5	7	10	11	13
$f(x)$	48	100	294	900	1210	2028

Evaluate $f(8)$

Ans :- $f(8) = 448.$

Method 2

Newton's Divided Difference.

Divided Difference Table :-

x	$y = f(x)$	Δy	$\Delta^2 y$
x_0	$f(x_0)$	$f(x_0, x_1) = \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}$	$f(x_0, x_1, x_2) = \dfrac{f(x_1, x_2) - f(x_0, x_1)}{x_2 - x_0}$
x_1	$f(x_1)$	$f(x_1, x_2) = \dfrac{f(x_2) - f(x_1)}{x_2 - x_1}$	$f(x_1, x_2, x_3) = \dfrac{f(x_2, x_3) - f(x_1, x_2)}{x_3 - x_1}$
x_2	$f(x_2)$	$f(x_2, x_3) = \dfrac{f(x_3) - f(x_2)}{x_3 - x_2}$	
x_3	$f(x_3)$		

$\Delta^3 y$

$$f(x_0, x_1, x_2, x_3) = \frac{f(x_1, x_2, x_3) - f(x_0, x_1, x_2)}{(x_3 - x_0)}$$

Now $f(x) = f(x_0) + (x - x_0)\Delta f(x_0)$

$$+ (x - x_0)(x - x_1)\Delta^2 f(x_0) +$$

$$+ (x - x_0)(x - x_1) \ldots (x - x_{n-1})\Delta^n f(x_0)$$

Que Evaluate $f(9)$ using Newton's Divided Difference formula.

x	5	7	11	13	17
$f(x)$	150	392	1452	2366	5202

x	y $f(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
5	150	$\frac{392-150}{7-5} = 121$ $\Delta f(x_0)$	$\frac{265-121}{11-5} = 24$ $\Delta^2 f(x_0)$	$\frac{32-24}{13-5} = 1$ $\Delta^3 f(x_0)$	0
7	392	$\frac{1452-392}{11-7} = 265$	$\frac{457-265}{13-7} = 32$	$\frac{42-32}{17-7} = 1$	
11	1452	$\frac{2366-1452}{13-11} = 457$	$\frac{709-457}{17-11} = 42$		
13	2366	$\frac{5202-2366}{17-13} = 709$			
17	5202				

By divided difference formula.

$$\therefore f(x) = f(x_0) + (x-x_0)\Delta f(x_0) + (x-x_0)(x-x_1)\Delta^2 f(x_0) + (x-x_0)(x-x_1)(x-2)\Delta^3 f(x_0) \quad \text{—— (1)}$$

Put $x=9$

$$f(9) = 150 + (9-5)(121) + (9-5)(9-7)(24) + (9-5)(9-7)(9-11)(1) + 0$$

$$f(9) = 810$$

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Qm. Using Newton Divided Difference formula, find value of $f(8)$, $f(9)$, $f(15)$ using foll. data.

x	4	5	7	10	11	13
$f(x)$	48	100	294	900	1210	2028

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
4	48	$\frac{100-48}{5-4} = 52$	$= \frac{97-52}{7-4} = 15$	$\frac{21-15}{10-4} = 1$	0	0
5	100					
		$\frac{294-100}{7-5} = 97$	$= \frac{202-97}{10-5} = 21$	$\frac{27-21}{11-5} = 1$	0	
7	294					
10	900	$\frac{900-294}{10-7} = 202$	$= \frac{310-202}{11-7} = 27$	$\frac{33-27}{13-7} = 1$		
11	1210					
		$\frac{1210-900}{11-10} = 310$	$= \frac{409-310}{13-10} = 33$			
13	2028					
		$\frac{2028-1210}{13-11} = 409$				

	x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
		y_0	Δy_0				
x_0	4	48	52	$\Delta^2 y_0$			
				15	$\Delta^3 y_0$		
x_1	5	100	97		1	0	
				21			0
x_2	7	294	202		1	0	
				27			
x_3	10	900	310		1		
				33			
x_4	11	1210	409				
x_5	13	2028					

$$f(x) = f(x_0) + (x - x_0)\,\Delta f(x_0) + (x - x_0)(x - x_1)\,\Delta^2 f(x_0) + (x - x_0)(x - x_1)(x - x_2)\,\Delta^3 f(x_0)$$

at $x = 8$

$$f(8) = 48 + (8-4)(52) + (8-4)(8-5)(15) + (8-4)(8-5)(8-7).(1)$$

$$= 48 + 208 + 180 + 12$$

$$= 448$$

$$\boxed{f(8) = 448}$$

at $x = 9$

$$f(9) = 48 + (9-4) \times (52) + (9-4)(9-5) \times (15) + (9-4)(9-5)(9-7) \times (1)$$

$$= 48 + 260 + 800 + 40$$

$$= 648$$

$$\boxed{f(9) = 648}$$

at $x = 15$

$$f(15) = 48 + (15-4)\,52 + (15-4)(15-5) \times (15) + (15-4)(15-5)(15-7) \times (1)$$

$$= 48 + 572 + 2400 + 880 = 3150$$

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Ques. Find $f'(10)$ from following data

x	3	5	11	27	34
$f(x)$	-13	23	899	17315	35606

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
3	-13				
		18			
5	23		16		
		146		1	
11	899		40		0
		1026		1	
27	17315		69		
		2613			
34	35606				

By Newtons Divided Diff formula

$$f(x) = f(x_0) + (x-x_0)\Delta f(x_0) + (x-x_0)(x-x_1)\Delta^2 f(x_0)$$

$$+ (x-x_0)(x-x_1)(x-x_2)\Delta^3 f(x_0). \quad \text{--- (1)}$$

putting values in (1)

$$f(x) = -13 + (x-3)18 + (x-3)(x-5)16 + (x-3)(x-5)(x-11)1$$

$$f(x) = x^3 - 3x^2 - 7x + 8.$$

Now differentiating.

$$f'(x) = 3x^2 - 6x - 7 \qquad \text{put } x = 10$$

$$f'(10) = 3\times(10)^2 - 6(10) - 7$$

$$= 233$$

$$\boxed{f'(10) = 233}$$