Hands-on Introduction to Numerical Relativity

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Abstract

This document provides a brief hands-on introduction to solving nonlinear equations in the context of a massive scalar field in spherical symmetry.

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1 Massive Scalar Field

We start with the elliptic equation for the conformal factor ψ:

$$\partial_r^2\psi + \frac{2}{r}\,\partial_r\psi + 2\pi\psi^5\rho = 0, \tag{1}$$

where the energy density ρ is

$$\rho = \tfrac{1}{2}\left(\Pi^2 + \frac{(\partial_r\Phi)^2}{\psi^4} + m^2\Phi^2\right). \tag{2}$$

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1.1 Newton–Raphson Iteration

We may solve the nonlinear PDE iteratively using the multidimensional Newton–Raphson method.

$$F(a + d\psi) \approx F(a) + J(a)\,d\psi = 0, \tag{3}$$

$$J(a)\,d\psi = -F(a), \tag{4}$$

where $J(a)$ is the Jacobian of F with respect to ψ. The iteration scheme is then

$$a_{n+1} = a_n + d\psi, \tag{5}$$

with the true solution given by

$$\psi = \lim_{n\to\infty} a_n.$$

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1.2 Linearisation

Let us expand around a background a with a perturbation $d\psi$, i.e.

$$\psi = a + d\psi.$$

Then

$$\rho = \tfrac{1}{2}\left(\Pi^2 + \frac{(\partial_r\Phi)^2}{(a+d\psi)^4} + m^2\Phi^2\right), \tag{6}$$

$$\partial_r^2(a+d\psi) + \frac{2}{r}\partial_r(a+d\psi) + 2\pi(a+d\psi)^5\rho = 0. \tag{7}$$

Taking only terms linear in $d\psi$, we obtain

$$\begin{aligned}\partial_r^2 d\psi + \frac{2}{r}\partial_r d\psi + \pi(\Pi^2 + m^2\Phi^2)(5a^4 d\psi) + \pi d\psi(\partial_r\Phi)^2 \\ = -\left(\partial_r^2 a + \frac{2}{r}\partial_r a + \pi(\partial_r\Phi)^2 + \pi(\Pi^2 + m^2\Phi^2)a^5 + \pi a(\partial_r\Phi)^2\right).\end{aligned} \tag{8}$$

We solve this using an iterative Newton Raphson method

1.3 Boundary Conditions

At the outer boundary (large r), we impose a Robin boundary condition. At the origin, regularity requires that the solution be even, which implies

$$\partial_r\psi\big|_{r=0} = 0. \tag{9}$$

1.4 Pseudospectral method

In the pseudospectral method, we approximate the solution as a smooth function that can be expanded in terms of a complete set of orthogonal basis functions (e.g. Chebyshev or Fourier polynomials). The choice of basis depends on the symmetries of the problem and the boundary conditions. Here we use Chebyshev polynomials, expanding the solution as

$$\psi(x) = \sum_{n=0}^{N} a_n T_n(x),$$

where $T_n(x)$ denotes the Chebyshev polynomial of degree n.

If L is a linear differential operator, then

$$L\psi(x) = \sum_{n=0}^{N} a_n\,(LT_n(x)).$$

Thus, the action of L on the solution reduces to its action on the basis functions.

The Chebyshev polynomials are defined by

$$T_n(\cos\theta) = \cos(n\theta), \qquad \theta = \cos^{-1}(x).$$

From this definition, their derivatives follow as

$$T_n'(x) = \frac{n\sin(n\theta)}{\sin\theta} = nU_{n-1}(x),$$

$$T_n''(x) = \frac{n\left(xT_n(x) - nU_{n-1}(x)\right)}{1-x^2} = T_n'(x)\frac{\cos(\theta)}{\sin^2(\theta)} - n^2\frac{T_n(x)}{\sin^2(\theta)}$$

where $U_{n-1}(x)$ are Chebyshev polynomials of the second kind. These relations allow us to evaluate derivatives of the expansion directly at collocation points.

To enforce the PDE, we choose the Gauss–Lobatto collocation points

$$x_i = \cos\left(\frac{\pi i}{N-1}\right), \qquad i = 0, 1, \dots, N-1.$$

Since these lie in $[-1, 1]$, we map them to the physical domain $[0, R]$ using the affine transformation

$$r = \frac{R}{2}(x+1), \qquad x = \frac{2r}{R} - 1,$$

where R is chosen as the maximum radial coordinate of interest. Under this mapping, derivatives transform via the chain rule:

$$\frac{d}{dr} = \frac{2}{R}\frac{d}{dx}, \qquad \frac{d^2}{dr^2} = \left(\frac{2}{R}\right)^2\frac{d^2}{dx^2}.$$

Thus, for $N-2$ interior collocation points, the PDE is imposed in strong form:

$$L\psi(r_i) = f(r_i), \qquad i = 1, \dots, N-2,$$

while the boundary conditions are enforced separately at $r = 0$ and $r = R$, yielding two additional equations. Altogether, this provides N equations for the N unknown coefficients $\{a_n\}$.

In matrix form, the pseudospectral discretization produces a linear system

$$H\mathbf{a} = \mathbf{G},$$

where H encodes the action of the operator L on the Chebyshev basis evaluated at the collocation points, and $\mathbf{G}$ incorporates the source terms and boundary conditions.

$$\begin{bmatrix} LT_0(x_1) & LT_1(x_1)... & LT_N(x_1) \\ LT_0(x_2) & LT_1(x_2)... & LT_N(x_2) \\ . & . & . \\ LT_0(x_{N-2}) & LT_1(x_{N-2})... & LT_N(x_{N-2}) \\ \text{Evaluate} & \text{Boundary} & \text{Conditions} \\ . & . & . \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ . \\ . \\ . \\ a_N \end{bmatrix} = \text{Source terms + Boundary conditions results}$$

If the PDE is well-posed,the matrix will be non singular(if Boundary conditions and suitable bases are taken)(Babuška–Lax–Milgram theorem(Recheck)) Note that each row in the H matrix is of the equation that is satisfied at a specific point,boundary condition equation if its the boundary and PDE equation if its at the intermediate points hence the ordering of the row doesnt matter as long as the RHS ordering is correct.

1.5 Final

Finally using equation 8 and plugging in all these,we can formulate it introduction

$$J(a)d\psi = -F(a)$$

and using boundary condition at left boundary(Either just take only even basis functions in the first place)

$$\partial_r \psi|_{r=0} = 0$$

We can set this condition into the equation by (taking a as the first guess)

$$J(a) = \sum_{n=0}^{N} \partial_r T_n \quad F(a) = \sum_{n=0}^{N} \partial_r T_n(0) a_n - 0$$

and at the right boundary, we set up robin boundary condition

$$\partial_r \psi|_{r=R} = \frac{1 - \psi|_{r=R}}{r} \quad F(a) = \sum_{n=0}^{N} \partial_r T_n(R) a_n - \frac{1}{R} + \sum_{n=0}^{N} a_n T_n(R)/R$$

$$J(a) = \sum_{n=0}^{N} \partial_r T_n(R) + \sum_{n=0}^{N} T_n(R)/R$$

Solving this equations with some stopping condition involving the norm of the tolerence we can solve the PDE