“In the Name of God”

Engineering probabilities and Statistics **CE 40181**
Sharif University of Technology

Homework 2
Wed, Nov 08 2017

Reminders:

- **Typing the solutions has extra marks.**
- **Collaboration is permitted, but you must write solutions *by yourself without* assistance.**
- **Getting solutions from outside sources such as the Web or students not enrolled in the class is strictly forbidden.**
- **Late submissions will be treated according to the course policy.**

Problems:

1. Two random integers from $S = \{1, 2, 3\}$ are chosen. Let X be the difference of these two numbers (X can be negative).
(a) Calculate the CDF of X.
(b) Calculate the expected value of X.
(c) Calculate the variance of X.
(d) Calculate the PMF of X^2.

2. Suppose that we are generating two integer random variables x_1 and x_2, where $x_1 \in \{0, 1\}$, and $x_2 \in \{1, 2, 3, 4\}$. Consider another random variable like X, if x_1 is 0, then the value of X would be the value of x_2 and if x_1 is 1, we would choose a random number between 1 and the value of x_2 for the value of X. calculate the variance of X.

3. Let X be a discrete random variable where $X \in \{0, 1, 2, ...\}$. Prove:
$$E[X] = \sum_{i=0}^{\infty} P(X > i)$$

4. Assume twenty percent of motherboards manufactured by a company are defective. What is the probability that a sample containing 12 of these motherboards has more than 4 defective ones?

5. We have a 12×12 table with 144 cells. Each row is enumerated from 0 to 11 and so is every column.We assume that a computer is located in cell $(11, 11)$ (upper right cell). We want to send a network packet to cell $(0, 0)$. There is a router in each cell which can only send packets to its left or bottom cells. If a packet enters a cell, the router will either route

the packet down with probability p or to the left with probability $1-p$. Calculate the probability that our packet goes through cells $(7,8)$ and $(4,4)$ during the transportation.

6. Suppose that X and Y are two independent Poisson random variables with parameters μ and λ, respectively.
(a) Find the mean of X.
(b) Find the variance of X.
(c) Show that $X+Y$ is a Poisson random variable with parameter $\mu+\lambda$.

7. A company has n computers labeled from 1 to n (none of the labels are identical). The computers are randomly set up to form a line, in positions 1 to n. Every two computers are connected to one another through a cable. We know that two computers positioned in i and j , with labels A_i and A_j respectively have a slow connection if $i<j$ and $A_i > A_j$. What is the expected value of the number of slow pairs?

8. The probability distribution of X is called memoryless if and only if:

$$\forall i,j \in \{0,1,2,...\}: \quad P(X > i+j | x > i) = P(x > j)$$

Prove that the Geometric distribution is the only discrete memoryless probability distribution.

9. Let X be a Geometric random variable. Find $E[\frac{1}{2^X}]$.

10. You are going to check the security of a website. The password of the website is a permutation of $\{1,2,...,n\}$. User submits a permutation. Security system checks to see how many digits of the entered password match the system's password. If the number of matches is higher than a constant amount, the system is unlocked. Now for checking the security, we want to calculate expected value of number matches in the digits of the system's password and the entered password (the entered password has a uniform distribution over the permutations of $\{1,2,...,n\}$)

R Problem:

Where is randomness in our world? Flip a coin, roll a dice, and click on an online random generator. What makes them random? Of course! You are uncertain about the outcome of the trial. We all feel randomness around us. Therefore, we should understand it. Randomness is obvious in such events as rolling a dice or flipping a coin, while it is more tricky in computers. *Really, how do computers generate random numbers?*
In this exercise we will get to know computer random generator and employ it to simulate a few procedures.

How to generate random numbers in a computer?

In many real world problems when quite a few random numbers are needed to simulate a procedure, employing such classic random generators as coins and dice is useless. Rather, computers are used to generate random numbers. But how? During the last century, several random generators have been introduced.

Use the following instructions in order to implement the mentioned random generator in R.

(a) Linear congruential generator: The generator is defined by the recursive relation

$$X_{n+1} = (aX_n + b) \mod m,$$

where X_n is the sequence of (pseudo)random values, and
$m, 0 < m$ – the "modulus"
$a, 0 < a < m$ – the "multiplier"
$b, 0 \leq b < m$ – the "increment"
X_0, $0 \leq X_0 < m$ – the "seed" or "start value".

i. Write an R program which gets m, a, b, and X_0 and generates a sequence of random numbers of length 100.
ii. Use a cycle detection algorithm to write a code which detects a cycle in the generated sequence and prints its length.
iii. For $m = 1000$, $a = 80$, $b = 200$, and $X_0 = 80$, after printing the first 1000 elements of the sequence, report the length of its cycle.
iv. Does linear congruential generator generate a real random sequence?

(b) Read about midsquare method:

i. Write an R program which gets the seed and generates a sequence of random numbers of length 100.
ii. Use a cycle detection algorithm to write a code which detects a cycle in the generated sequence and prints its length.
iii. For seed = 100, after printing the first 1000 elements of the sequence, report the length of its cycle.
iv. Does midsquare method generate a real random sequence?

(c) The function sample() in R:

i. Write an R program which gets the seed and passes it to the set.seed() function. Then generates a sequence of random numbers between 1 and 100 of length 100 using sample function.
ii. Use a cycle detection algorithm to write a code which detects a cycle in the generated sequence and prints its length.
iii. For seed = 100, after printing the first 1000 elements of the sequence, report the length of its cycle.
iv. Does sample() function generate a real random sequence?