درس معادلات دیفرانسیل

استاد: دکتر مختار ایزدی

تمرینات سری ۱ «تحویلی»

1 $y'' + 4y' + 5y = 0 \qquad y = e^{-2x}(A\cos x + B\sin x)$

Ans.

$\xrightarrow{\text{مشتق}} y' = -2\underbrace{e^{-2x}(A\cos x + B\sin x)}_{y} + e^{-2x}(-A\sin x + B\cos x)$

$y' = -2y + e^{-2x}(-A\sin x + B\cos x) \rightarrow y' + 2y = e^{-2x}(-A\sin x + B\cos x)$

$\xrightarrow{\text{مشتق}} y'' + 2y' = -2\underbrace{e^{-2x}(-A\sin x + B\cos x)}_{y'+2y} + \underbrace{e^{-2x}(-A\cos x - B\sin x)}_{-y}$

$y'' + 2y' = -2(y' + 2y) - y \rightarrow y'' + 4y' + 5y = 0$

حکم اثبات و تابع داده شده جواب معادله دیفرانسیل است.

2 a) $\displaystyle\int \frac{x^2+2}{(x+1)^3(x-2)}dx = \int \frac{x^2-4+6}{(x+1)^3(x-2)}dx = \int \frac{(x-2)(x+2)}{(x-2)(x+1)^3} + 6\int \frac{dx}{(x+1)^3(x-2)}$

$\displaystyle = \int \frac{\overset{x+1+1}{x+2}}{(x+1)^3}dx + 6\int \frac{dx}{(x+1)^3(x-2)} = \int \frac{dx}{(x+1)^2} + \int \frac{dx}{(x+1)^3} + 6\int \frac{dx}{(x+1)^3(x-2)}$

$\displaystyle \frac{-1}{x+1} + \frac{-1}{2(x+1)^2} + c + 6I = \boxed{\frac{1}{27}\left(\ln^{|x-2|} - \ln^{|x+1|}\right) - \frac{1}{3}\left(\frac{1}{x+1}\right) + \frac{1}{2(x+1)^2} + C^*}$

$\displaystyle I = \int \frac{dx}{(x+1)^3(x-2)} = \int \frac{A}{x-2} + \frac{B}{x+1} + \frac{Cx+D}{(x+1)^2} + \frac{Ex^2+Fx+G}{(x+1)^3}dx$

$A = \frac{1}{27} \quad B = \frac{-1}{27} \quad C = 0 \quad D = \frac{-1}{9} \quad E = 0 \quad F = 0 \quad G = \frac{-1}{3}$

$\displaystyle I = \frac{1}{27}\int \frac{dx}{x-2} - \frac{1}{27}\int \frac{dx}{x+1} - \frac{1}{9}\int \frac{dx}{(x+1)^2} - \frac{1}{3}\int \frac{dx}{(x+1)^3} =$

روز سلامتی

$\displaystyle I = \frac{1}{27}\ln^{x-2} - \frac{1}{27}\ln^{x+1} + \frac{1}{9(x+1)} + \frac{1}{6(x+1)^2} + C'$

b) $\int \frac{x}{(x^2+1)(x-1)} \, dx =$

$$\int \frac{x-1+1}{(x^2+1)(x-1)} \, dx = \int \frac{1}{x^2+1} + \frac{1}{(x^2+1)(x-1)} \, dx = \int \frac{1}{x^2+1} - \frac{1}{2}\left(\frac{x+1}{x^2+1}\right) + \frac{1}{2}\left(\frac{1}{x-1}\right) dx$$

$$= \int \frac{dx}{x^2+1} - \frac{1}{2}\left(\int \frac{x}{x^2+1} \, dx + \int \frac{1}{x^2+1} \, dx\right) + \frac{1}{2}\int \frac{1}{x-1} \, dx$$

$$= \frac{1}{2}\int \frac{dx}{x^2+1} + \frac{1}{2}\int \frac{dx}{x-1} - \frac{1}{4}\int \frac{2x}{x^2+1} \, dx =$$

$$= \frac{\operatorname{arctg}(x)}{2} + \frac{\ln(x-1)}{2} - \frac{\ln(x^2+1)}{4}$$

معادلات دیفرانسیل

دکتر مختار ایزدی

I) مسیرهای قائم دسته منحنی‌های زیر را بیابید.

$r^{-1} = \sin 2\theta + C$

$\xrightarrow{d/d\theta \ \text{مشتق}} \frac{-r'}{r^2} = 2\cos 2\theta \longrightarrow \frac{-1}{r} = \frac{r}{r'}(2\cos 2\theta) \xrightarrow{\frac{r}{r'} \to \frac{-r'}{r}} \frac{-1}{r} = \frac{-r'}{r}(2\cos 2\theta)$

$\frac{1}{2\cos 2\theta} = r' = \frac{dr}{d\theta} \longrightarrow \frac{d\theta}{2\cos 2\theta} = dr \xrightarrow{\int} \int \frac{2d\theta}{4\cos 2\theta} = \int dr$

$\longrightarrow r = \frac{\ln(tg\, 2\theta + \sec 2\theta)}{4}$

II) نشان دهید دسته منحنی $x^2 = 4C(y+C)$ خودمتعامد است.

$\xrightarrow{d/dx} 2x = 4Cy' \to x = 2Cy' \xrightarrow{\text{توان } 2} x^2 = 4C^2 y'^2 = 4C(y+C)$

$\xrightarrow{\text{ساده کردن}} Cy'^2 = y + C \to Cy'^2 - C = y \to C(y'^2 - 1) = y$

$x = 2Cy' \to \begin{cases} C = \frac{x}{2y'} \\ y' = \frac{x}{2C} \end{cases} \xrightarrow{\text{جایگذاری}} \frac{x}{2y'}(y'^2 - 1) = y \to \frac{xy'}{2} - \frac{x}{2y'} = y$ معادله دیفرانسیل اولیه

معادله دیفرانسیل جدید

$\xrightarrow[y' \to \frac{-1}{y'}]{\text{جایگذاری}} \frac{x}{2}\left(\frac{-1}{y'}\right) - \frac{x}{2\left(\frac{-1}{y'}\right)} = \boxed{\frac{xy'}{2} - \frac{x}{2y'} = y} \xrightarrow{\text{حل}} \frac{x}{2y'}(y'^2 - 1) = y$

$\xrightarrow[y' = \frac{x}{2C}]{} \frac{x}{2\left(\frac{x}{2C}\right)}\left(\frac{x^2}{4C^2} - 1\right) = y \to C\left(\frac{x^2 - 4C^2}{4C^2}\right) = y \to x^2 - 4C^2 = 4Cy$

$\boxed{x^2 = 4C(y+C)}$