طراحی الگوریتم ها

مبحث ششم:
مرتب سازی سریع

سجاد شیرعلی شهرضا

بهار 1402

یکشنبه، 30 بهمن 1401

Based on Stanford CS161 Slides from Summer 2020 by Karey Shi

اطلاع رسانی

- بخش مرتبط کتاب برای این جلسه: 5

مرتب سازی سریع

یک نمونه واقعی و کاربردی از الگوریتم های تصادفی

QUICKSORT OVERVIEW

EXPECTED RUNNING TIME

$O(n \log n)$

WORST-CASE RUNNING TIME

$O(n^2)$

QUICKSORT OVERVIEW

EXPECTED RUNNING TIME

$O(n \log n)$

WORST-CASE RUNNING TIME

$O(n^2)$

In practice, it works great! It's competitive with MergeSort (& often better in some contexts!), and it runs *in place* (no need for lots of additional memory)

QUICKSORT: THE IDEA

Let's use DIVIDE-and-CONQUER again!

Select a pivot *at random*

Partition around it

Recursively sort L and R!

QUICKSORT: THE IDEA

Select a pivot

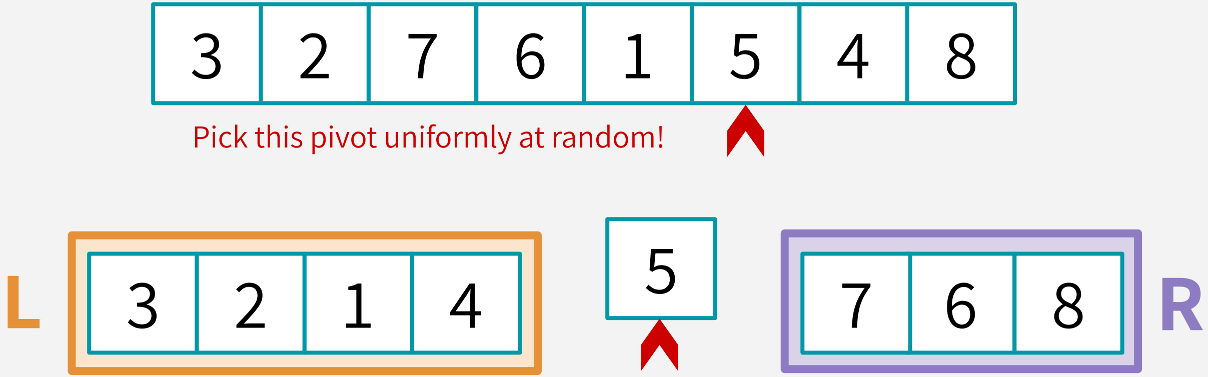
3	2	7	6	1	5	4	8

Pick this pivot uniformly at random!

QUICKSORT: THE IDEA

Select a pivot

Partition around it



Pick this pivot uniformly at random!

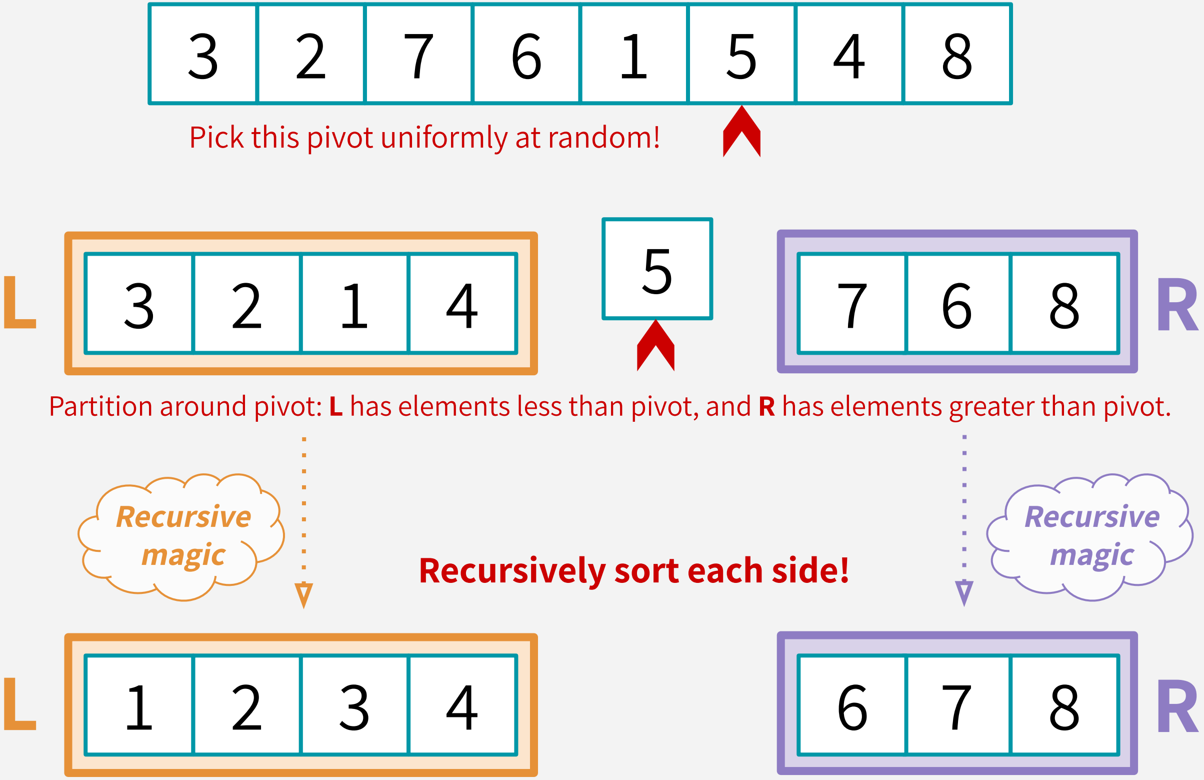
Partition around pivot: **L** has elements less than pivot, and **R** has elements greater than pivot.

QUICKSORT: THE IDEA

Select a pivot

Partition around it

Recurse!



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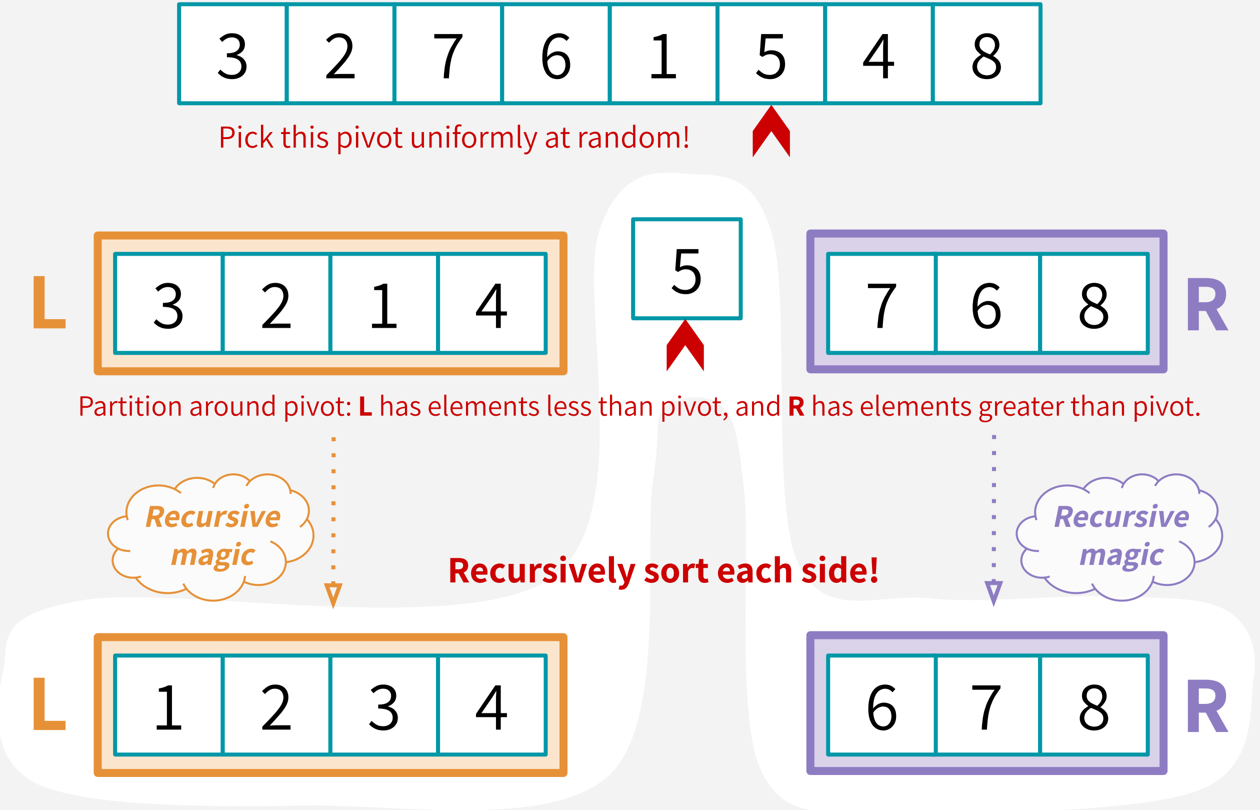

QUICKSORT: PSEUDO-PSEUDOCODE

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RECURRENCE RELATION

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Recurrence Relation for QUICKSORT

$T(n) =$

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Recurrence Relation for QUICKSORT

$$T(n) = T(|L|) + T(|R|) + O(n)$$
$$T(0) = T(1) = O(1)$$

IDEAL RUNTIME?

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In an ideal world, the pivot would split the array exactly in half, and we'd get:

$$T(n) = T(n/2) + T(n/2) + O(n)$$

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In an ideal world:

$$T(n) = 2 \cdot T(n/2) + O(n)$$

$$T(n) = \mathbf{O(n \log n)}$$

Recurrence Relation for QUICKSORT

$$) + T(|R|) + O(n)$$

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Recurrence Relation for QUICKSORT

$$T(n) = T(|L|) + T(|R|) + O(n)$$
$$T(0) = T(1) = O(1)$$

With the unluckiest randomness, the pivot would be either min(A) or max(A):

$$T(n) = T(0) + T(n-1) + O(n)$$

WORST-CASE RUNTIME

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        R (grea
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Recurrence Relation for QUICKSORT

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nin(A) or max(A):

$T(n) = T(0) + T(n-1) + O(n)$

With the worst "randomness"

$$T(n) = T(n\text{-}1) + O(n)$$

$$T(n) = \mathbf{O(n^2)}$$

(recursion tree/table or substitution method!)

سوال؟

EXPECTED RUNTIME = $O(n \log n)$

AN **INCORRECT** PROOF!

Lemma: $E[|L|] = E[|R|] = (n-1)/2$

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$$E[\,|L| + |R|\,] = n - 1$$

(because L and R make up everything except the pivot)

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$$2 \cdot E[\,|L|\,] = n - 1$$

(plugging the first line)

$$E[\,|L|\,] = (n - 1)/2$$

(Solving for $E[|L|]$)

EXPECTED RUNTIME = O(n log n)

AN **INCORRECT** PROOF:

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- $E[\,|L|\,] = E[\,|R|\,] = (n-1)/2$
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Why is this wrong?

EXPECTED RUNTIME = O(n log n)

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- Therefore, the expected running time is O(n log n)!

Why is this wrong?

Well, for starters, we can use the exact same argument to prove something false...

LET'S START WITH QUICKSORT AND …

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```

INTRODUCING **SLOW**SORT!

```
SLOWSORT(A):
    if len(A) <= 1:
        return
    pivot = either max(A) OR min(A)
    PARTITION A into:
        L (less than pivot) and
        R (greater than pivot)
    Replace A with [L, pivot, R]
    SLOWSORT(L)
    SLOWSORT(R)
```

randomly choose either!

SLOWSORT

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Recurrence Relation for SLOWSORT

$$T(n) = T(|L|) + T(|R|) + O(n)$$
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randomly choose either!

Recurrence Relation for SLOWSORT

$$T(n) = T(|L|) + T(|R|) + O(n)$$
$$T(0) = T(1) = O(1)$$

One of **|L|** or **|R|** is *always* n-1

$$T(n) = T(0) + T(n-1) + O(n)$$
$$T(n) = \mathbf{O(n^2)}$$

SLOWSORT

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Recurrence Relation for SLOWSORT

$$T(n) = T(|L|) + T(|R|) + O(n)$$
$$T(0) = T(1) = O(1)$$

Same recurrence relation as QUICKSORT!

And we also still have:

$$\mathbf{E[|L|] = E[|R|] = (n-1)/2}$$

SLOWSORT

```
SLOWSORT(A):
    if len(A)
        returr
    pivot = e
    PARTITION
        L (les
        R (gre
    Replace A
    SLOWSORT(L
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```

Recurrence Relation for
SORT

T(|R|) + O(n)

) = O(1)

ce relation as
SORT!

still have:

R|] = (n-1)/2

RED FLAG:

We could use the exact same (incorrect) proof to prove that **SLOWSort** has expected runtime **O(n log n)**, when it actually has expected runtime of **$\Theta(n^2)$**...

EXPECTED RUNTIME = O(n log n)

AN **INCORRECT** PROOF:

- $E[\,|L|\,] = E[\,|R|\,] = (n - 1)/2$
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- Therefore, the expected running time is $O(n \log n)$!

Why is this wrong?

EXPECTED RUNTIME = O(n log n)

AN

Basically:

E[f(x)**] is *not necessarily* the same as** f(**E[**x**]**)

e.g. $E[X^2]$ is not the same as $(E[X])^2$

We were reasoning about T(**E[**x**]**) instead of **E[**T(x)**]**

Why is this wrong?

EXPECTED RUNTIME = O(n log n)

Instead, to prove that the expected runtime of QuickSort is O(n log n), we're going to count the **number of comparisons** that this algorithm performs, and take the expectation of that!

How many times are any two items compared?

سوال؟

امید ریاضی زمان اجرای مرتب سازی سریع

راه حل درست: تعداد مورد انتظار مقایسه دو عنصر با همدیگر چند بار است؟

HOW MANY COMPARISONS?

3	2	7	6	1	5	4	8

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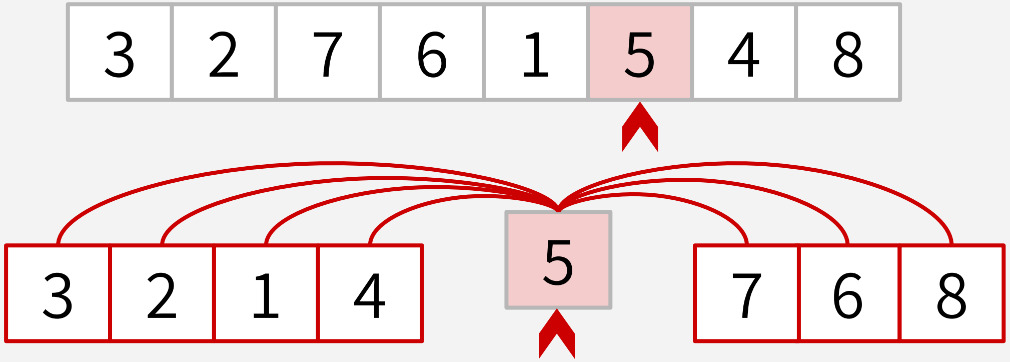

Everything is compared to 5 once in this first step… and then never again with **5**.

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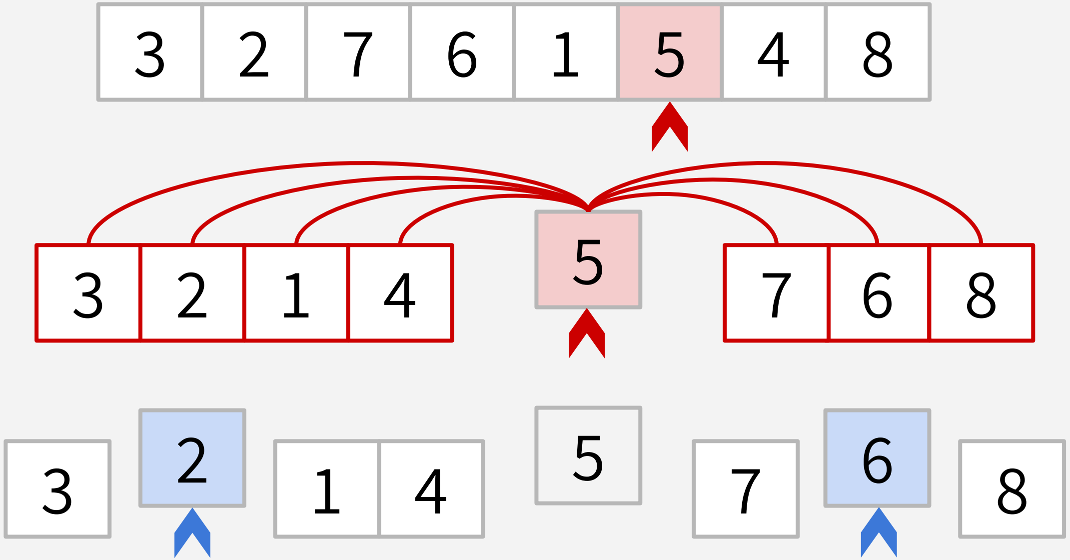

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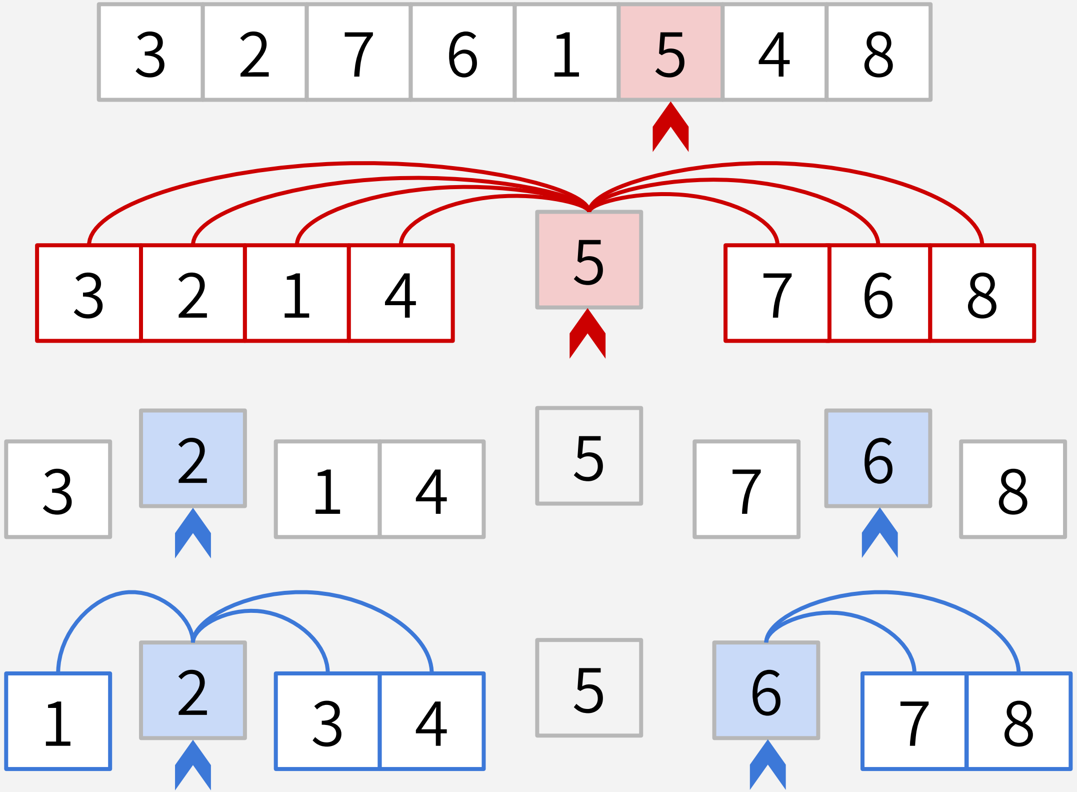

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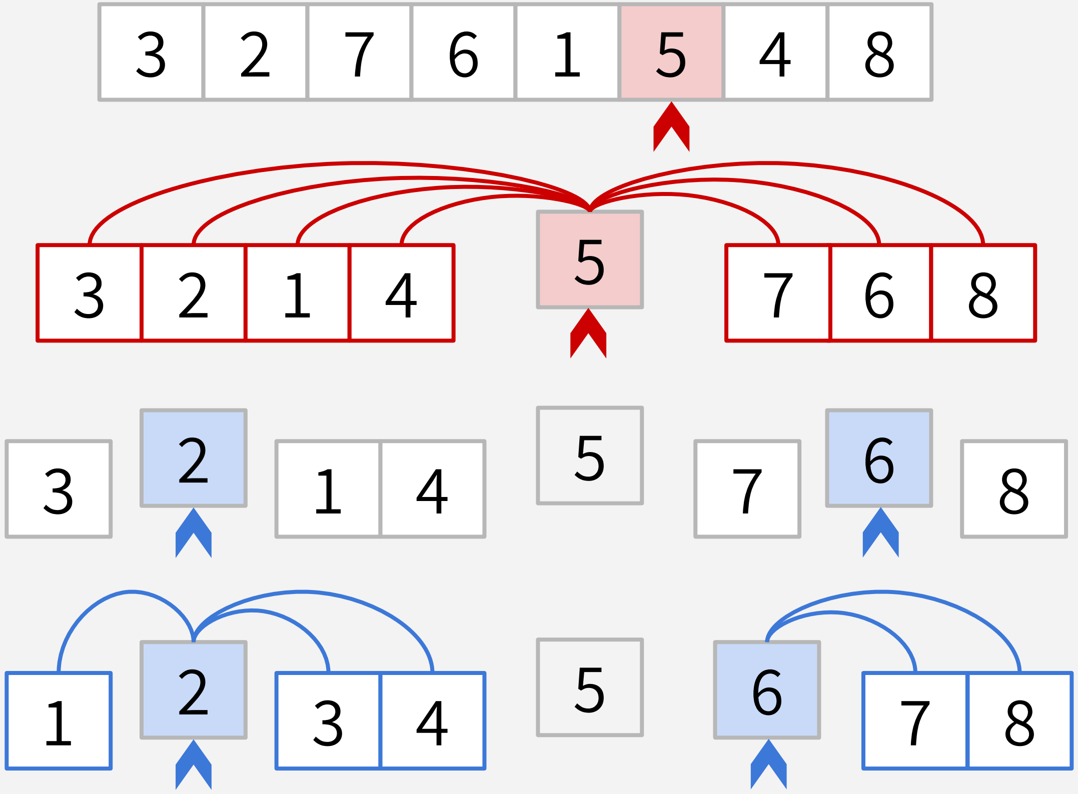


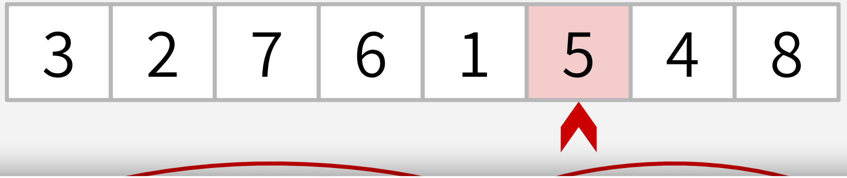
Everything is compared to 5 once in this first step… and then never again with **5**.

Only 1, 3, & 4 are compared to **2**.

And only 7 & 8 are compared with **6**.

No comparisons ever happen between two numbers on opposite sides of 5.

HOW MANY COMPARISONS?



Seems like whether or not two elements are compared has something to do with pivots...

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And only 7 & 8 are compared with **6**.

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HOW MANY COMPARISONS?

Each pair of elements is compared either **0** or **1** times.

Let $\mathbf{X_{a,b}}$ be a Bernoulli/indicator random variable such that:

$\mathbf{X_{a,b} = 1}$ if **a** and **b** are compared

$\mathbf{X_{a,b} = 0}$ otherwise

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In our example, $\mathbf{X_{2,5}}$ took on the value **1** since **2** and **5** were compared.

On the other hand, $\mathbf{X_{3,7}}$ took on the value **0** since **3** and **7** are *not* compared.

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Total number of comparisons =

$$\mathbb{E}\left[\sum_{a=0}^{n-2}\sum_{b=a+1}^{n-1} X_{a,b}\right] = \sum_{a=0}^{n-2}\sum_{b=a+1}^{n-1} \mathbb{E}[X_{a,b}]$$

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by linearity of expectation!

We need to figure out this value!

HOW MANY COMPARISONS?

So, what's $\mathbf{E[X_{a,b}]}$?

$$\mathbf{E[X_{a,b}] = 1 \cdot P(X_{a,b} = 1) + 0 \cdot P(X_{a,b} = 0) = P(X_{a,b} = 1)}$$

So, what's $\mathbf{P(X_{a,b} = 1)}$? It's the probability that **a** and **b** are compared. Consider this example:

3	2	**7**	6	1	5	4	8

$\mathbf{P(X_{3,7} = 1)}$ is the probability that **3** and **7** are compared.

3	2	**7**	6	1	5	4	8

This is exactly the probability that either 3 or 7 is first picked to be a pivot out of the highlighted entries.

1	2	**3**	4	5		**7**	8

If 4, 5, or 6 get picked as a pivot first, then 3 and 7 would be separated and never see each other again.

HOW MANY COMPARISONS?

So, what's $\mathbf{E[X_{a,b}]}$?

$\mathbf{P(X_{a,b} = 1)}$ *aka probability that **a** & **b** are compared*

=

probability that either **a** or **b** are selected as a pivot before elements between **a** and **b**.

=

$$\frac{2}{(\#\text{ elements from } \mathbf{a} \text{ to } \mathbf{b}\text{, inclusive})}$$

So … mple:

3 … ed.

3 … first … es.

1	2	**3**	4

5

7	8

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HOW MANY COMPARISONS?

So, what's **E[$X_{a,b}$]**?

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probability that either **a** or **b** are selected as a pivot before elements between **a** and **b**.

=

$$\frac{2}{\mathbf{b} - \mathbf{a} + 1}$$

So ... mple:

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3 ... first ... es.

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Total number of comparisons = $$\sum_{a=0}^{n-2} \sum_{b=a+1}^{n-1} \mathbb{E}\left[X_{a,b}\right]$$

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We just computed $E[X_{a,b}] = P(X_{a,b,} = 1)$

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$$= \sum_{a=0}^{n-2} \sum_{c=1}^{n-a-1} \frac{2}{c+1}$$

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Introduce c = b – a to make notation nicer

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$$= \sum_{a=0}^{n-2} \sum_{c=1}^{n-a-1} \frac{2}{c+1}$$

$$\leq \sum_{a=0}^{n-1} \sum_{c=1}^{n-1} \frac{2}{c+1}$$

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Introduce c = b – a to make notation nicer

Increase summation limits to make them nicer (hence the ≤)

QUICKSORT EXPECTED RUNTIME

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$$=\sum_{a=0}^{n-2}\sum_{c=1}^{n-a-1}\frac{2}{c+1}$$

$$\leq\sum_{a=0}^{n-1}\sum_{c=1}^{n-1}\frac{2}{c+1}$$

$$=2n\sum_{c=1}^{n-1}\frac{1}{c+1}$$

We just computed $E[X_{a,b}] = P(X_{a,b,} = 1)$

Introduce c = b – a to make notation nicer

Increase summation limits to make them nicer (hence the ≤)

Nothing in the summation depends on *a*, so pull 2 out

QUICKSORT EXPECTED RUNTIME

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$$= \sum_{a=0}^{n-2}\sum_{c=1}^{n-a-1}\frac{2}{c+1}$$

$$\leq \sum_{a=0}^{n-1}\sum_{c=1}^{n-1}\frac{2}{c+1}$$

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We just computed $E[X_{a,b}] = P(X_{a,b,} = 1)$

Introduce c = b – a to make notation nicer

Increase summation limits to make them nicer (hence the ≤)

Nothing in the summation depends on *a*, so pull 2 out

decrease each denominator → we get the harmonic series!

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If E[# comparisons] = O(n log n), does this mean E[running time] is also O(n log n)?

YES! Intuitively, the runtime is dominated by comparisons.

QUICKSORT

```
QUICKSORT(A):
    if len(A) <= 1:
        return
    pivot = random.choice(A)
    PARTITION A into:
        L (less than pivot) and
        R (greater than pivot)
    Replace A with [L, pivot, R]
    QUICKSORT(L)
    QUICKSORT(R)
```

Worst case runtime:
$O(n^2)$

Expected runtime:
$O(n \log n)$

BETTER WORST CASE RUNTIME

- Select a better pivot

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- How to select the median in $O(n)$?
  - Will see how to do it in the next lecture

سوال؟

مرتب سازی سریع در عمل

چگونگی پیاده سازی (و آیا واقعا کسی از آن استفاده می کند؟)

IMPLEMENTING QUICKSORT

In practice, a more clever approach (Lomuto) is used to implement PARTITION, so that the entire QuickSort algorithm can be implemented “in-place”

(i.e. via swaps, rather than constructing separate L or R subarrays)

AN EXAMPLE IN-PLACE PARTITION

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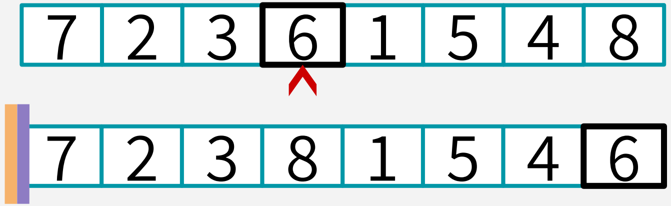
7	2	3	6	1	5	4	8

Choose pivot & swap with last element so pivot is at the end.

AN EXAMPLE IN-PLACE PARTITION



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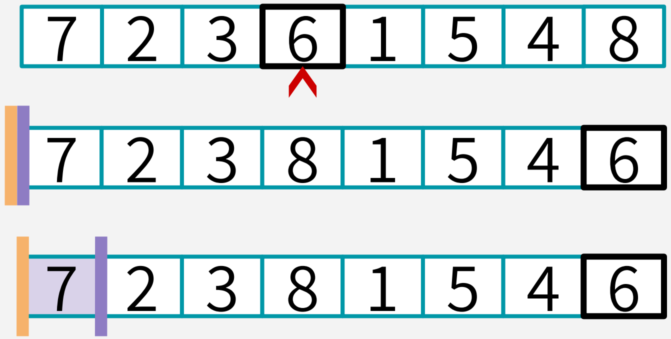

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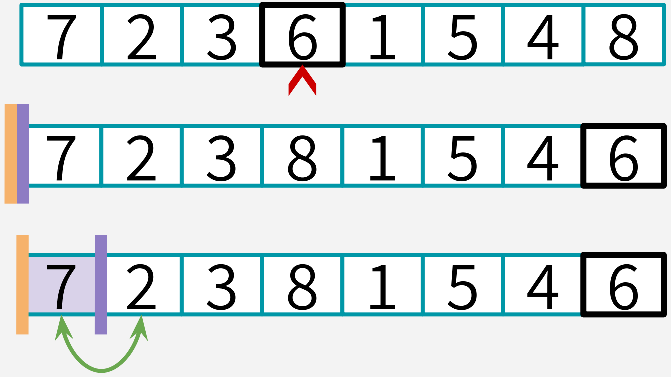

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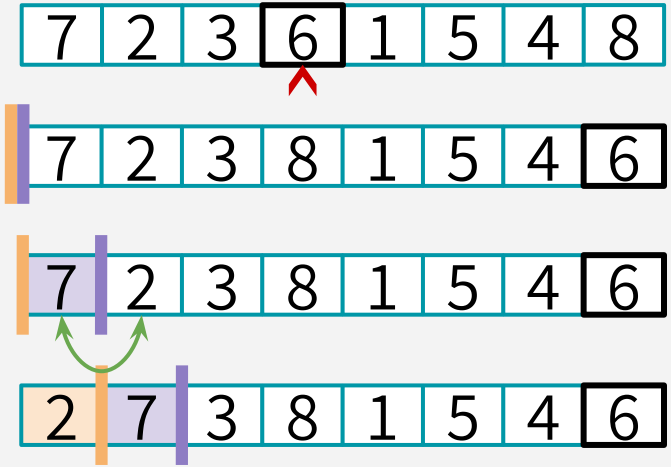

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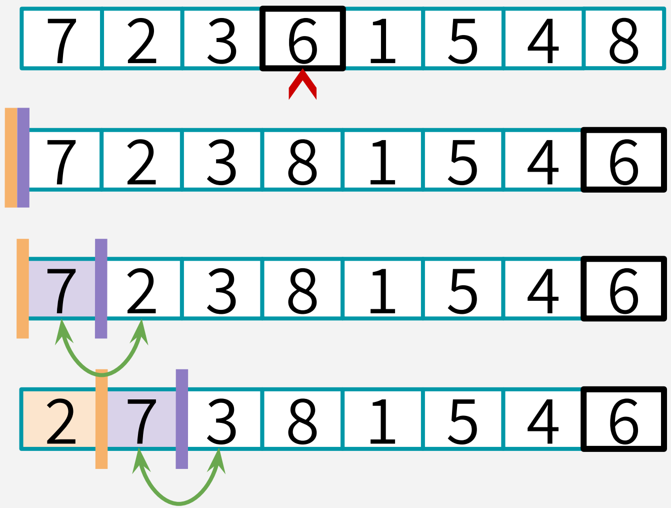

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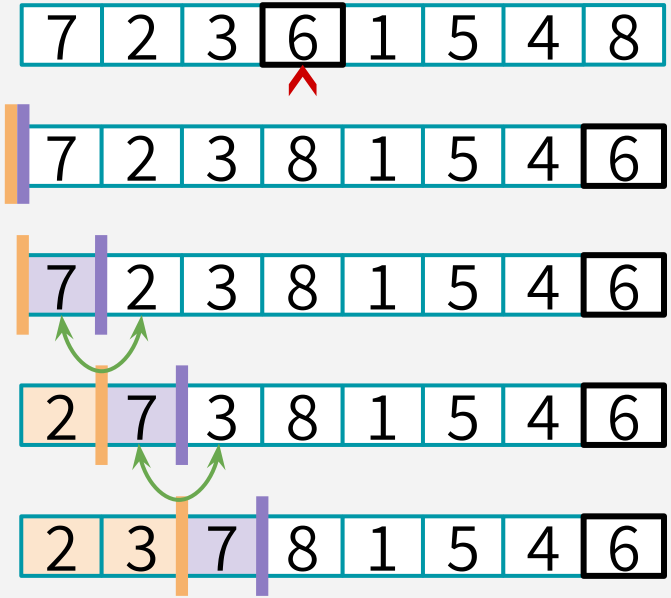

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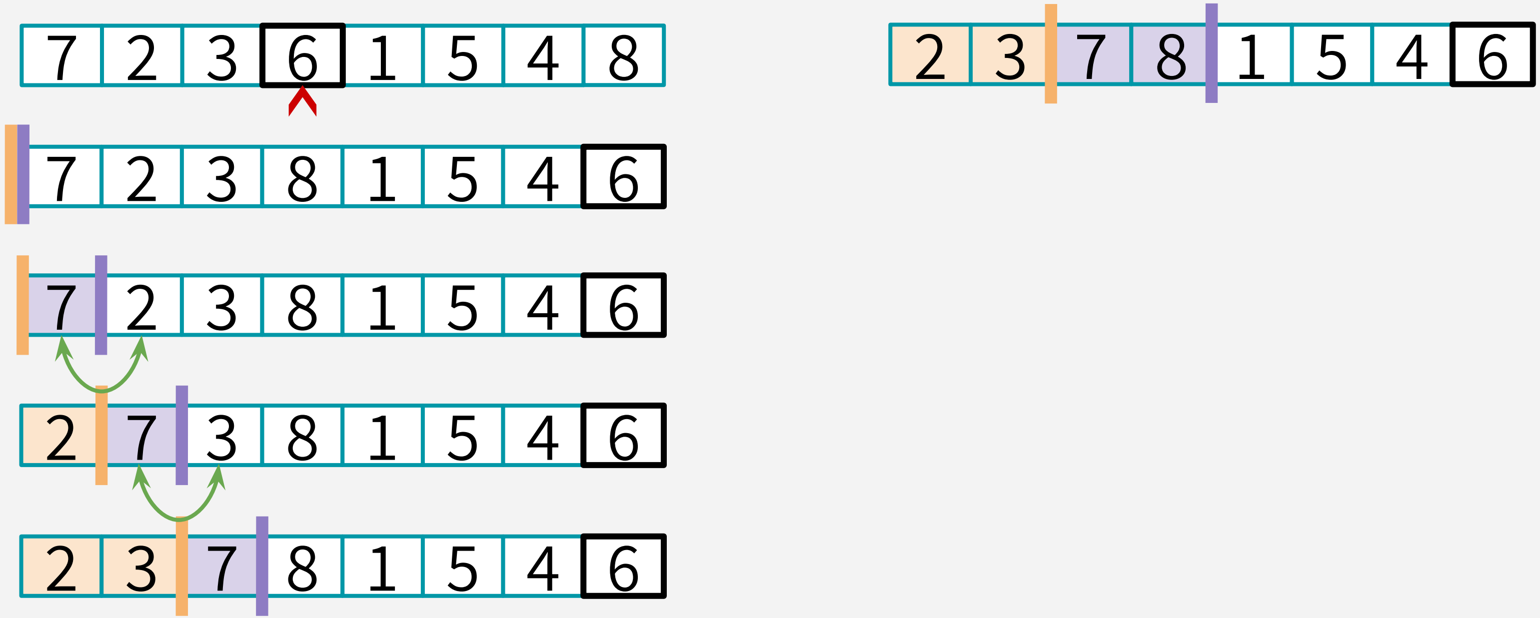

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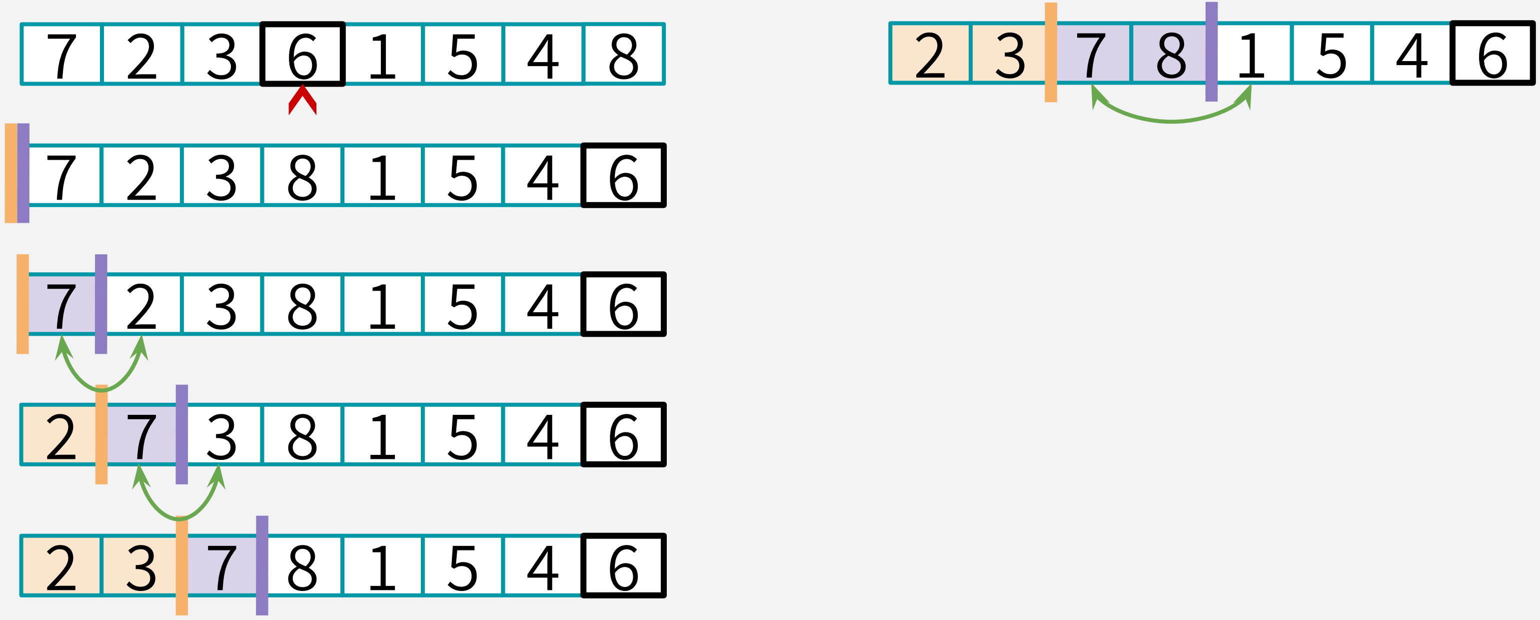

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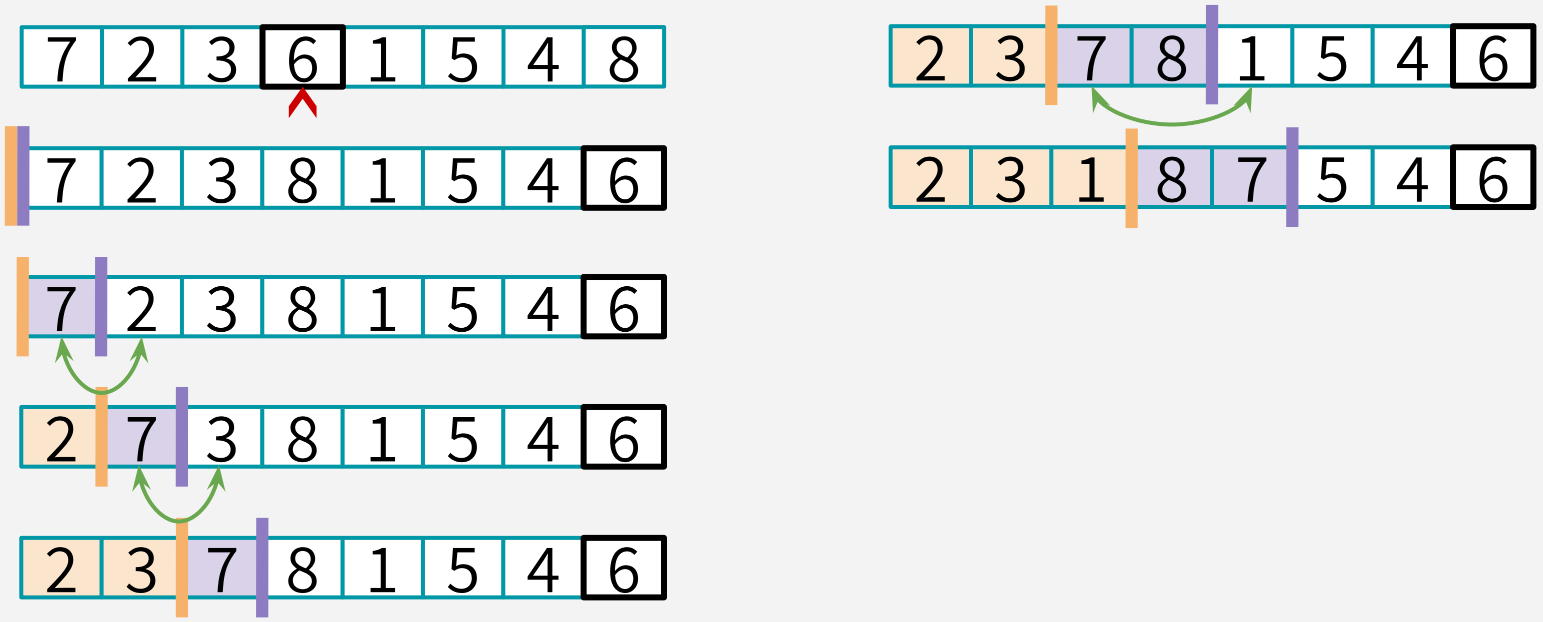

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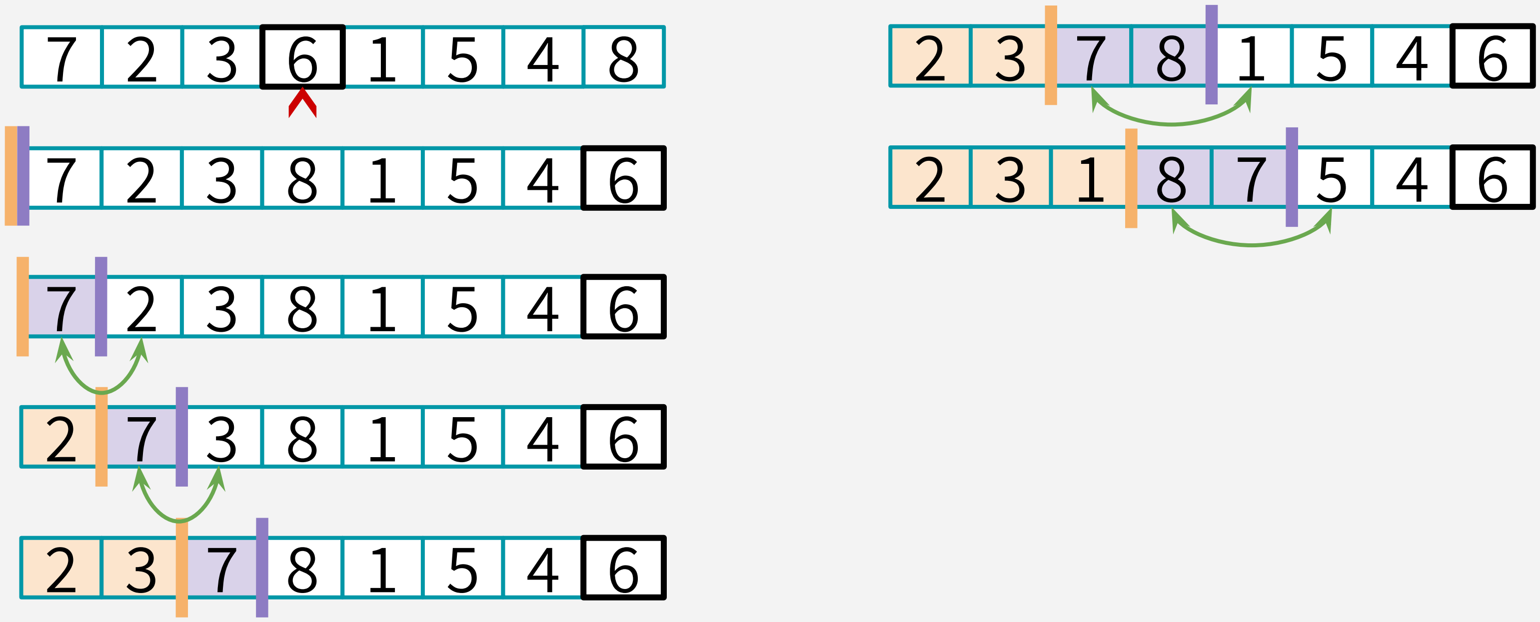

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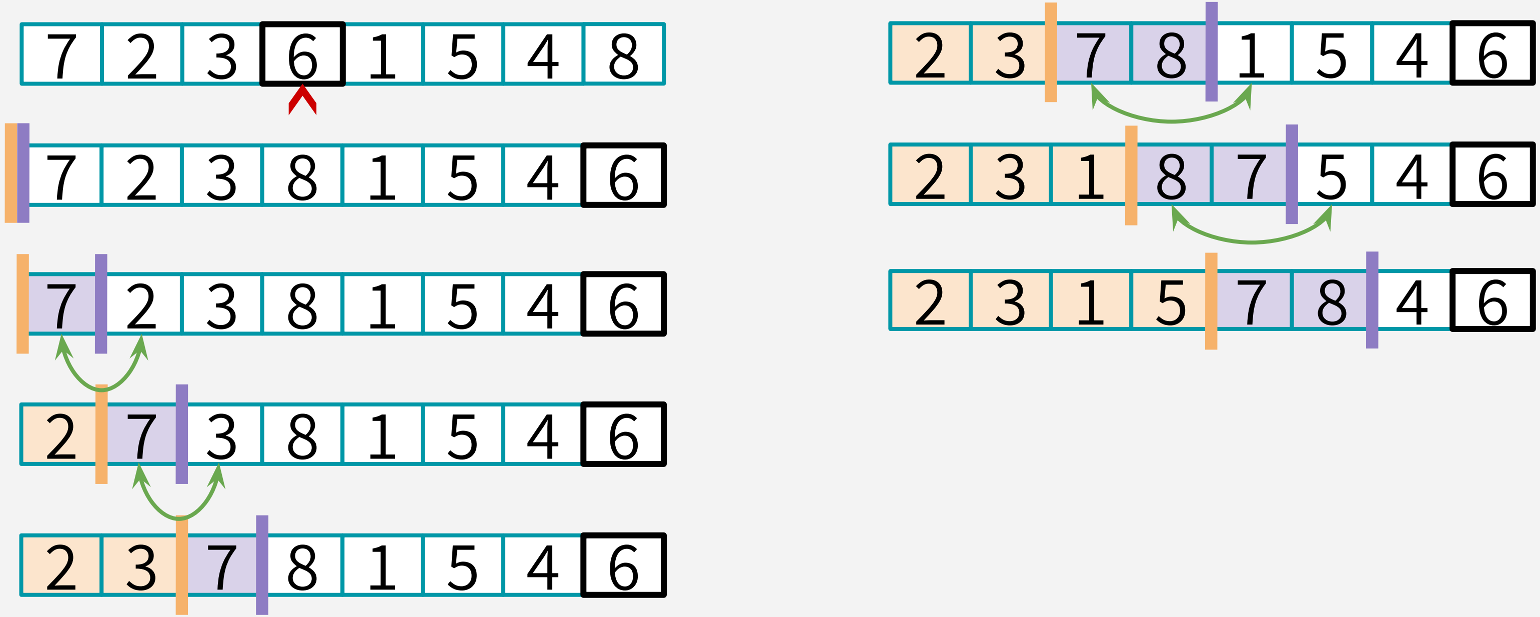

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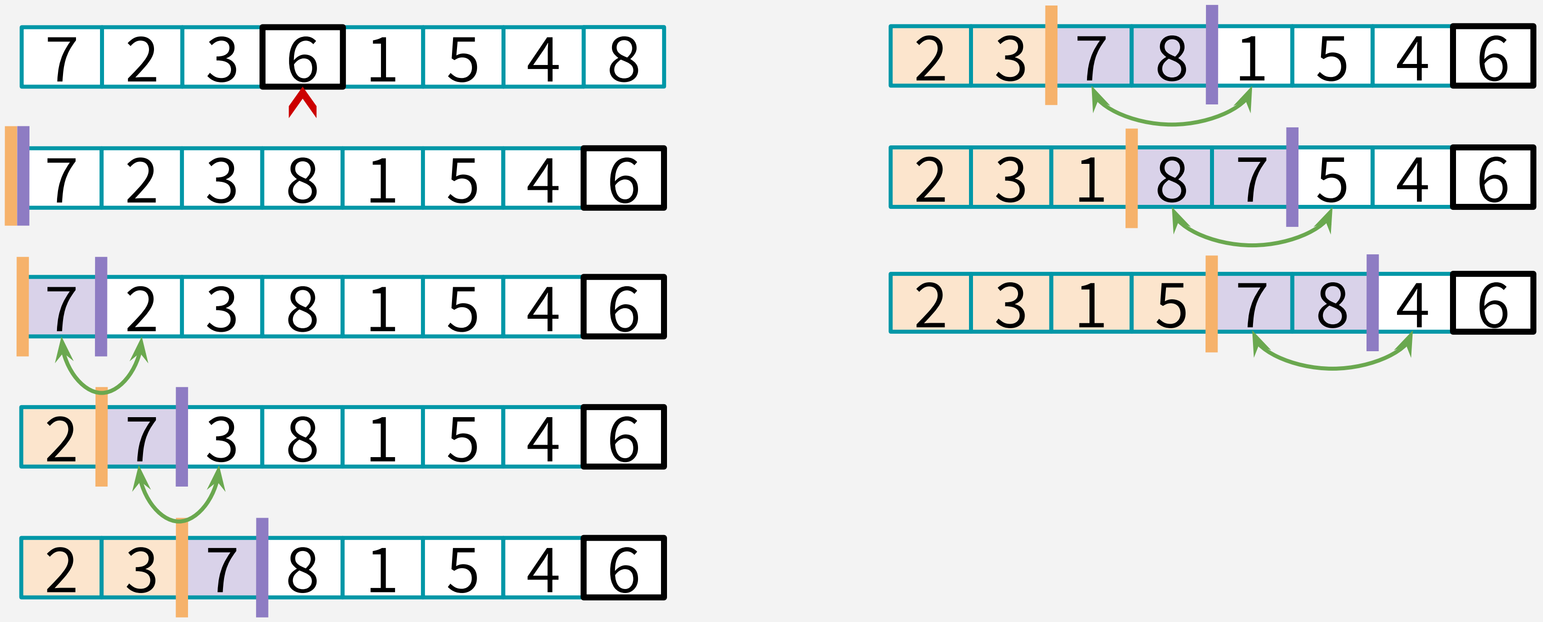

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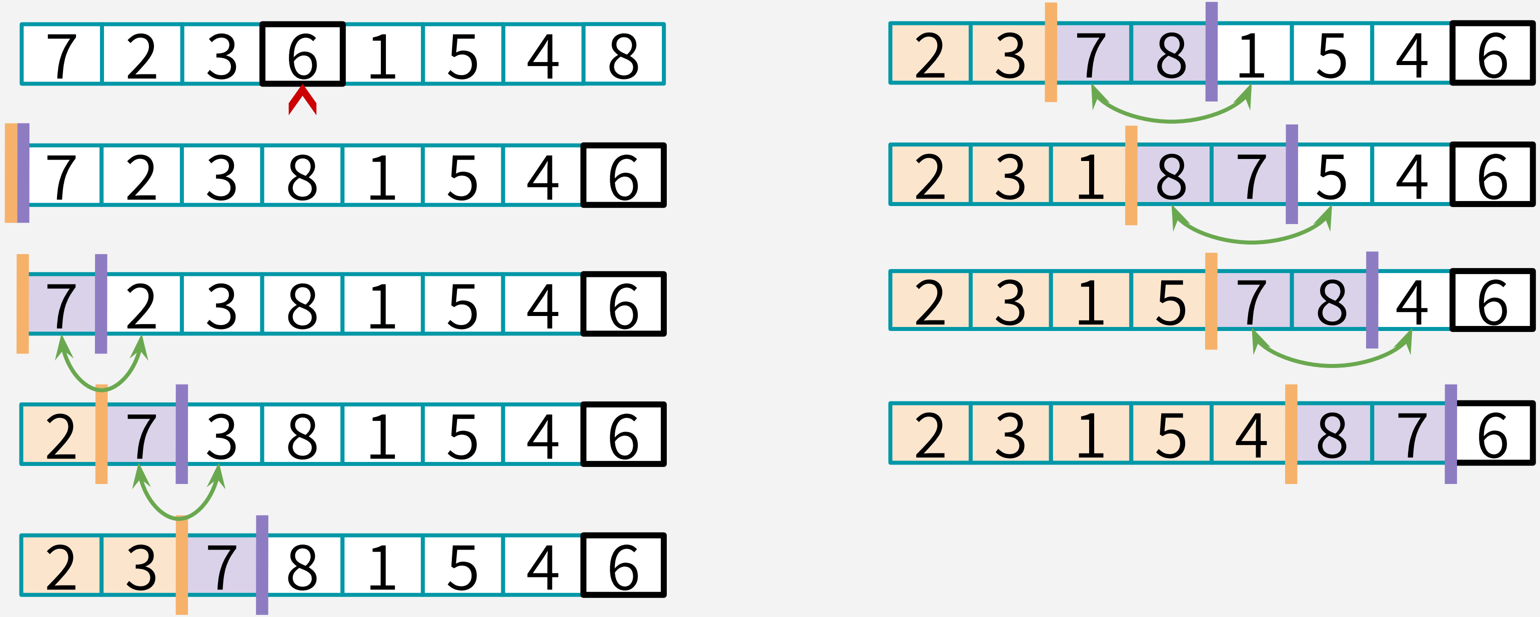

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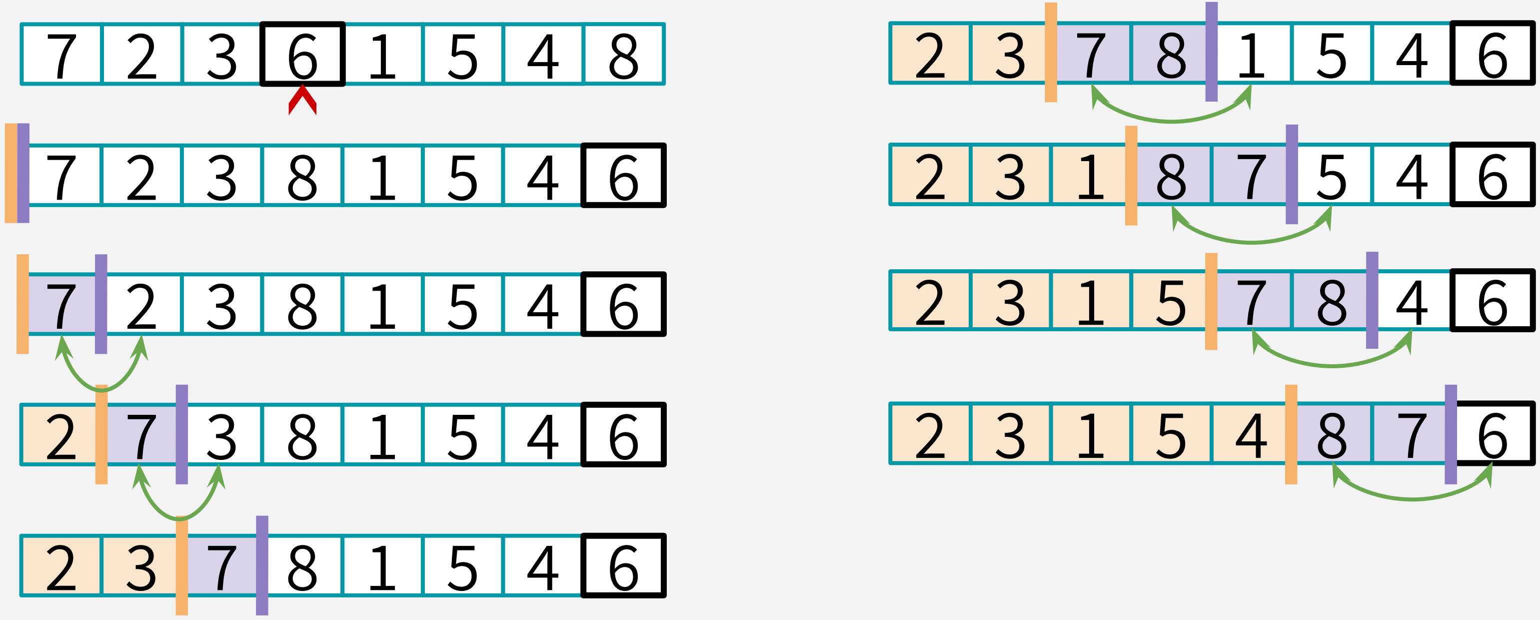

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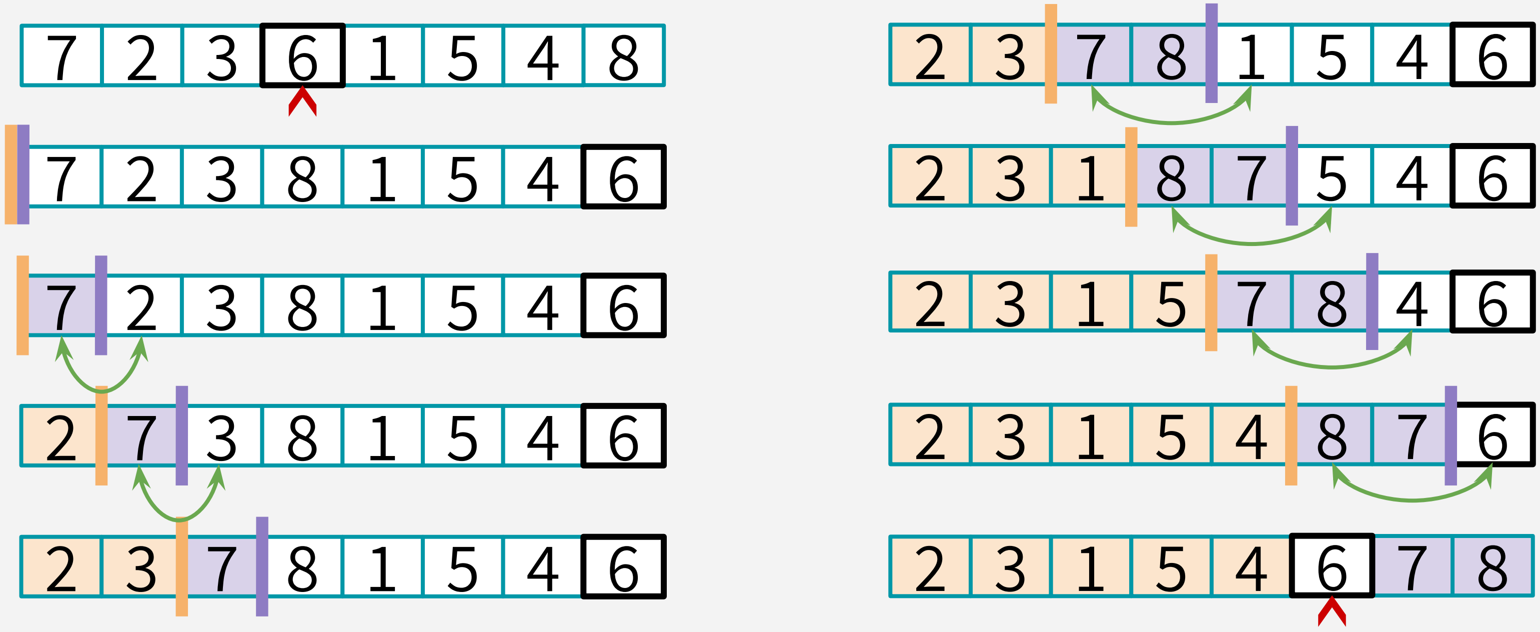

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IMPLEMENTING QUICKSORT

There's another in-place partition algorithm called Hoare Partition that's even more efficient as it performs less swaps.

What we just saw is known as Lomuto method.

(you're not responsible for knowing it in this class)

QUICKSORT vs. MERGESORT

	QuickSort (random pivot)	**MergeSort** (deterministic)
Runtime	**Worst-case: $O(n^2)$** **Expected: $O(n \log n)$**	**Worst-case: $O(n \log n)$**
Used by	Java (primitive types), C (qsort), Unix, gcc…	Java for objects, perl
In-place? **(i.e. with $O(\log n)$ extra memory)**	Yes, pretty easily!	Easy if you sacrifice runtime ($O(n\log n)$ MERGE runtime). Not so easy if you want to keep runtime & stability.
Stable?	No	Yes
Other Pros	Good cache locality if implemented for arrays	Merge step is really efficient with linked lists

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