## فرمول های بازگشتی (یا تحویل یافته) :

مثال :

$$I_n = \int x^n e^x \, dx$$

$$\begin{cases} u = x^n \\ dV = e^x \, dx \end{cases} \Longrightarrow \begin{cases} du = n x^{n-1} \, dx \\ V = e^x \end{cases}$$

$$\Longrightarrow I_n = x^n e^x - n \int x^{n-1} e^x \, dx$$

$$= x^n e^x - n I_{n-1}$$

$$\begin{cases} I_1 = x e^x - I_0 = x e^x - \int e^x \, dx = x e^x - e^x + c \\ I_2 = x^2 e^x - 2 I_1 = x^2 e^x - 2(x e^x - e^x + c) \\ \vdots \end{cases}$$

مثال:

$$I_n = \int (\sec x)^n \, dx$$

$$\begin{cases} u = (\sec x)^{n-2} \\ dv = \sec^2 x \, dx \end{cases} \Rightarrow \begin{cases} du = (n-2)\sec^{n-2} x \, tg\, x \, dx \\ v = tg\, x \end{cases}$$

$$\Rightarrow I_n = \sec^{n-2} x \; tg\, x - (n-2)\int \sec^{n-2} x \; tg^2 x \, dx$$

$$= \sec^{n-2} x \, tg\, x - (n-2)\int \sec^{n-2}(\sec^2 x - 1)\, dx$$

$$= \sec^{n-2} x \, tg\, x - (n-2) I_n + (n-2) I_{n-2}$$

$$\Rightarrow (n-1) I_n = \sec^{n-2} x \, tg\, x + (n-2) I_{n-2}$$

$$\Rightarrow I_n = \frac{1}{n-1}\left(\sec^{n-2} x \, tg\, x\right) + \left(\frac{n-2}{n-1}\right) I_{n-2}$$

$$I_2 = \int (\sec x)^2 \, dx = tg\, x + C$$

$$I_3 = \int (\sec x)^3 \, dx = \tfrac{1}{2}(\sec x \, tg\, x) + \tfrac{1}{2} I_1$$

$$\boxed{\int \sec x \, dx = Ln|\sec x + tg\, x| + C}$$

## انتگرال برخی توابع گویا :

① $\int \frac{\alpha}{ax+b}\,dx = \frac{\alpha}{a} Ln\,|ax+b| + c$

② $\int \frac{x}{x^2+a^2}\,dx = 1/2\; Ln\,(x^2+a^2) + c$

③ $\int \frac{dx}{x^2+a^2} = 1/a\; tg^{-1}(\frac{x}{a}) + c$

④ $\int \frac{x}{x^2-a^2}\,dx = 1/2\; Ln\,|x^2-a^2| + c$

⑤ $\int \frac{dx}{x^2-a^2} = 1/2a\; Ln\,\left|\frac{x-a}{x+a}\right| + c$

$\int \frac{dx}{x^2-a^2} = \int \frac{dx}{(x-a)(x+a)} = \int 1/2a\,(\frac{1}{x-a} - \frac{1}{x+a})\,dx$

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$$= \frac{1}{2a}\left(\int \frac{dx}{x-a} - \int \frac{dx}{x+a}\right)$$

$$= \frac{1}{2a}\left(\ln|x-a| - \ln|x+a|\right) + C$$

$$= \frac{1}{2a} \ln\left|\frac{x-a}{x+a}\right| + C.$$

$$\int \frac{\alpha x+\beta}{l_1 x^2+l_2 x+l_3}\,dx \underset{l_1\neq 0}{=} \frac{\alpha}{l_1}\int \frac{x+\overbrace{\frac{\beta}{\alpha}}^{\lambda}}{x^2+\underbrace{\frac{l_2}{l_1}}_{B}x+\underbrace{\frac{l_3}{l_1}}_{C}}\,dx$$

بدیهی است که آن را حذف می‌کنیم.

$$=\frac{\alpha}{l_1}\int \frac{x+\lambda}{x^2+Bx+C}\,dx = \int \frac{x+\lambda}{(x+\frac{B}{2})^2+\underbrace{C-\frac{B^2}{4}}_{\pm a^2}}\,dx$$

$$\xlongequal[du=dx]{u=x+B/2} \int \frac{u-\frac{B}{2}+\lambda}{u^2\pm a^2}\,du$$

$$\xlongequal{\lambda-B/2=b} \int \frac{u+b}{u^2\pm a^2}\,du$$

$$=\int \frac{u}{u^2\pm a^2}\,du + \int \frac{b}{u^2\pm a^2}\,du$$

## - انتگرال توابع گویا به شکل $\frac{P(x)}{Q(x)}$ :

فرض کنید $P(x)$ و $Q(x)$ دو چند جمله‌ای باشند.

اگر $\deg Q(x) > \deg P(x)$ ، آن‌گاه با تقسیم کردن $P(x)$ بر $Q(x)$ ، داریم :

$$P(x) = A(x)\,Q(x) + R(x) \;;\; \deg R(x) < \deg Q(x)$$

$$\Rightarrow \frac{P(x)}{Q(x)} = A(x) + \frac{R(x)}{Q(x)}$$

$$\Rightarrow \int \frac{P(x)}{Q(x)}\,dx = \int A(x)\,dx + \underbrace{\int \frac{R(x)}{Q(x)}\,dx}$$

محاسبه چنین انتگرال‌هایی هدف ما خواهد بود.

$$I = \int \frac{x^3 + 3x^2}{x^2+1}\, dx$$

الحل :

$$x^3 + 3x^2 = (x+3)(x^2+1) - (x+3)$$

$$\Rightarrow I = \int (x+3)\, dx - \int \frac{x+3}{x^2+1}\, dx$$

$$= \int (x+3)\, dx - \int \frac{x}{x^2+1}\, dx - 3\int \frac{dx}{x^2+1}$$

$$= \left(\frac{x^2}{2} + 3x\right) + \frac{1}{2} \ln(x^2+1) - 3\, tg^{-1} x + C$$

$$Q(x)=\left((x-a_1)^{m_1}(x-a_2)^{m_2}\cdots(x-a_j)^{m_j}\right)\left((x^2+b_1x+c_1)^{n_1}(x^2+b_2x+c_2)^{n2}\cdots(x^2+b_kx+c_k)^{n_k}\right)$$

$$(x-a_1)^{m_1}\to\frac{A_1}{(x-a_1)}+\frac{A_2}{(x-a_1)^2}+\cdots+\frac{A_{m_1}}{(x-a_1)^{m_1}}$$

$$(x^2+b_1x+c_1)^{n_1}\to\frac{B_1x+C_1}{(x^2+b_1x+c_1)}+\frac{B_2x+C_2}{(x^2+b_1x+c_1)^2}+\cdots+\frac{B_{n_1}x+C_{n_1}}{(x^2+b_1x+c_1)^{n_1}}$$

مثال

$$I=\int\frac{dx}{x^4-1}$$

$$\frac{1}{x^4-1}=\frac{1}{(x-1)(x+1)(x^2+1)}=\frac{A}{x-1}+\frac{B}{x+1}+\frac{Cx+D}{x^2+1}$$

$$=\frac{(A+B+C)x^3+(A-B+D)x^2+(A+B-C)x+(A-B-D)}{(x-1)(x+1)(x^2+1)}$$

$$\begin{cases}A+B+C=0\\A-B+D=0\\A+B-C=0\\A-B-D=1\end{cases}\Rightarrow\begin{cases}A=\frac{1}{4}\\B=\frac{-1}{4}\\C=0\\D=\frac{-1}{2}\end{cases}$$

$$\frac{1}{4}\int\frac{dx}{x-1}-\frac{1}{4}\int\frac{dx}{x+1}-\frac{1}{2}\int\frac{dx}{x^2+1}=\frac{1}{4}\ln\left|\frac{x-1}{x+1}\right|-\frac{1}{2}\tan^{-1}(x)+c$$

$$I=\int \frac{dx}{x^3+1}=?\qquad \frac{1}{x^3+1}=\frac{1}{(x+1)(x^2-x+1)}=\frac{A}{x+1}+\frac{Bx+C}{x^2-x+1}$$

$$\xrightarrow{\text{ضرب در } x+1}\quad \frac{1}{x^2-x+1}=A+\frac{(Bx+C)(x+1)}{x^2-x+1}\xrightarrow{x=-1}\boxed{A=\tfrac{1}{3}}$$

حال $x=0$ قرار می‌دهیم: $\quad 1=\frac{1}{3}+C\Rightarrow \boxed{C=\tfrac{2}{3}}$

اگر در رابطه بالا $x=1$ قرار دهیم داریم: $\quad \boxed{B=-\tfrac{1}{3}}$

$$I=\int\frac{\frac{1}{3}}{x+1}dx+\underbrace{\int\frac{-\frac{1}{3}x+\frac{2}{3}}{x^2-x+1}}_{J}=\frac{1}{3}\ln|x+1|+J$$

$$J=-\frac{1}{3}\int\frac{x-2}{x^2-x+1}\ \overset{U=x-\frac{1}{2}}{=\!=\!=}\ -\frac{1}{3}\int\frac{(U+\frac{1}{2}-2)\,dU}{U^2+\frac{3}{4}}$$

$$-\frac{1}{3}\int\frac{U\,dU}{U^2+\frac{3}{4}}-\frac{1}{3}\int\frac{-\frac{3}{2}\,dU}{U^2+\frac{3}{4}}=$$

$$-\frac{1}{6}\ln\left(U^2+\tfrac{3}{4}\right)+\left(\tfrac{1}{2}\right)\left(\tfrac{2}{\sqrt{3}}\right)\tan^{-1}\left(\frac{2U}{\sqrt{3}}\right)+C$$

$$\Rightarrow I=\frac{1}{3}\ln|x+1|-\frac{1}{6}\ln\left(\left(x-\tfrac{1}{2}\right)^2+\tfrac{3}{4}\right)+\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2(x-\frac{1}{2})}{\sqrt{3}}\right)+C$$