ساختمان داده و الگوریتم ها (CE203)

جلسه پنجم: حل با روش جایگذاری و قضیه اصلی

سجاد شیرعلی شهرضا

پاییز 1401

دوشنبه، 11 مهر 1401

Based on Stanford CS161 Slides from Summer 2020 by Karey Shi

اطلاع رسانی

- بخش مرتبط کتاب برای این جلسه: 4.3، 4.4، 4.5
- امتحانک اول
  - دوشنبه هفته آینده، 18 مهر
  - به صورت حضوری در کلاس
  - در ساعت کلاس
  - در صورت تغییر، از طریق سایت اطلاع رسانی خواهد شد.

زمان اجرای مرتب سازی ادغامی

چقدر سریع است؟

MERGESORT: IS IT FAST?

```
MERGESORT(A):
    n = len(A)
    if n <= 1:
        return A
    L = MERGESORT(A[0:n/2])
    R = MERGESORT(A[n/2:n])
    return MERGE(L,R)
```

CLAIM: MergeSort runs in time **O(n log n)**

AN ASIDE: $O(n \log n)$ vs. $O(n^2)$?

log(n) grows very slowly! (Much more slowly than n)

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ALL LOGARITHMS IN THIS COURSE ARE BASE 2

log(2) = 1

log(4) = 2

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log(64) = 6

log (128) = 7

…

log(4096) = 12

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log(**# particles in the universe**) < 280

AN ASIDE: $O(n \log n)$ vs. $O(n^2)$?

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log(**# particles in the universe**) < 280

n log n grows much more slowly than n^2

Punchline: A running time of $O(n \log n)$ is a LOT better than $O(n^2)$

MERGESORT: O(n log n) PROOF

Instead of counting every little operation and tracing all recursive calls, we can think about:

THE RECURSION TREE!

(and we'll add up all the work done across levels to compute the Big-O runtime)

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```
MERGE(L,R):
    result = length n array
    i = 0, j = 0
    for k in [0,...,n-1]:
        if L[i] < R[j]:
            result[k] = L[i]
            i += 1
        else:
            result[k] = R[j]
            j += 1
    return result
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n iterations, O(1) work per iteration

We can see that MERGE is **O(n)**

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MERGE(L,R):
    result = length n array
    ...
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```
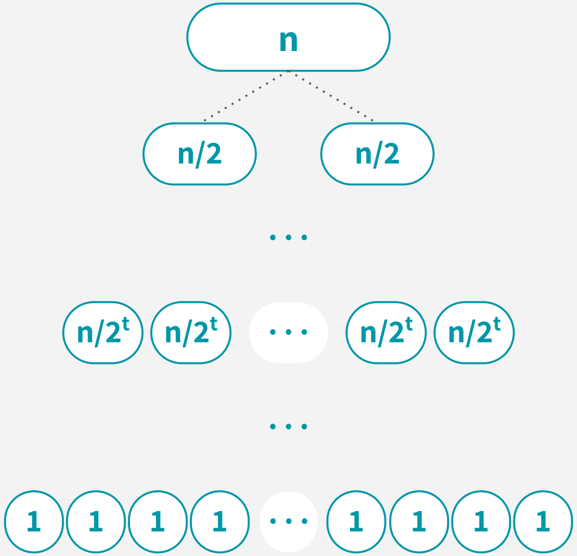

This means that within one recursive call that processes an array/subarray of length ***n***, the work done in that subproblem (creating subproblems & "merging" those results) is **O(n)**.

We can see that MERGE is **O(n)**

MERGESORT RECURSION TREE



Level	# of Problems	Size of each Problem	Work done per Problem	Total work on this level
0				
1				
…				
t				
…				
$\log_2 n$				

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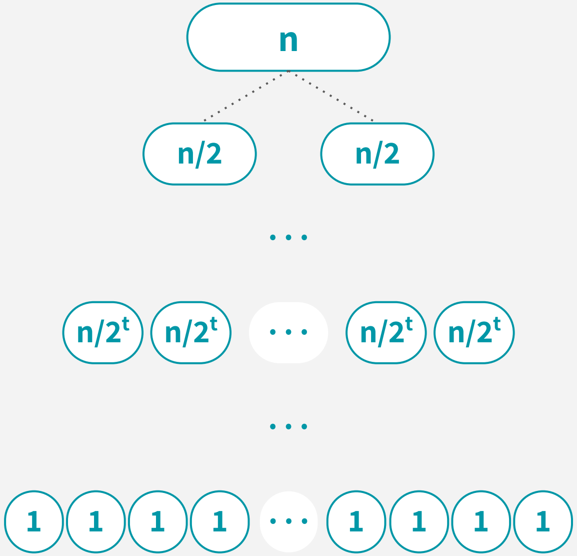

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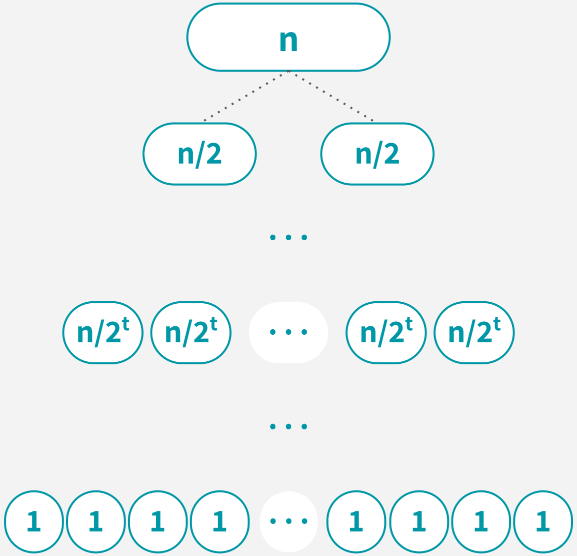

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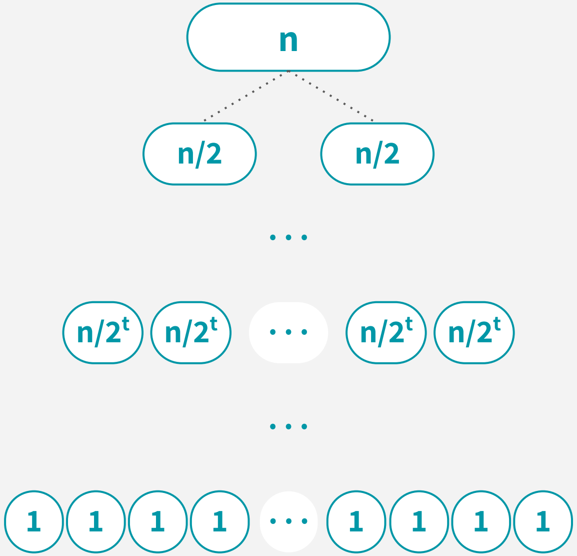
MERGESORT RECURSION TREE

If a subproblem is of size **n**, then the work done in that subproblem is **O(n)**.
⇒ **Work ≤ c · n** (c is a constant)



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...				
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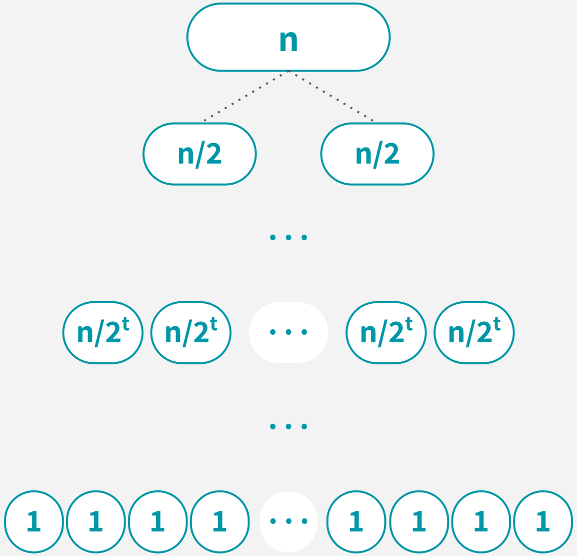

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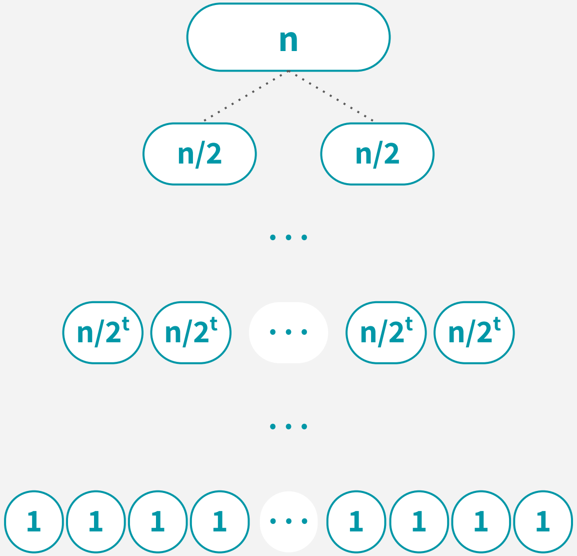

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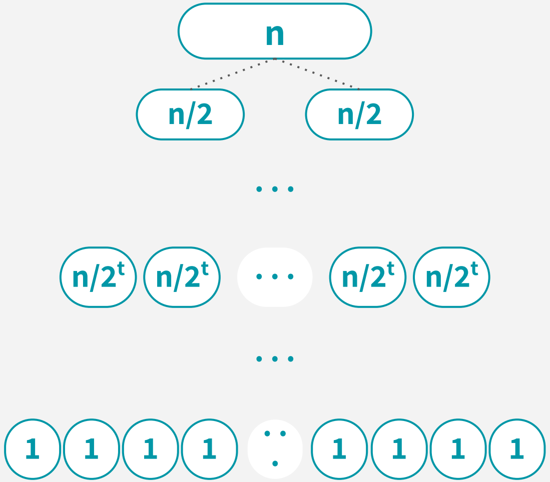
$\log_2 n$	$2^{\log_2 n} = n$	1	$c \cdot (1)$	$n \cdot c \cdot (1) =$ **O(n)**

We have ($\log_2 n + 1$) levels, each level has O(n) work total ⇒ **O(n log n)** work overall!

MERGESORT: O(n log n) RUNTIME

Using the "Recursion Tree Method" (i.e. drawing the tree & filling out the table), we showed that the runtime of MergeSort is **O(n log n)**



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سوال؟

رابطه بازگشتی

RUNTIMES FOR RECURSIVE ALGOS

Previously, we used the "Recursion Tree Method" (i.e. drawing the tree & filling out the table) to manually add up all the work in the tree and find that the runtime of MergeSort is **O(n log n)**.

Drawing the tree & doing all that adding kind of takes a lot of work...
Here's another way to reason about the runtime of a recursive algorithm like Mergesort:

INTRODUCING...

RECURRENCE RELATIONS

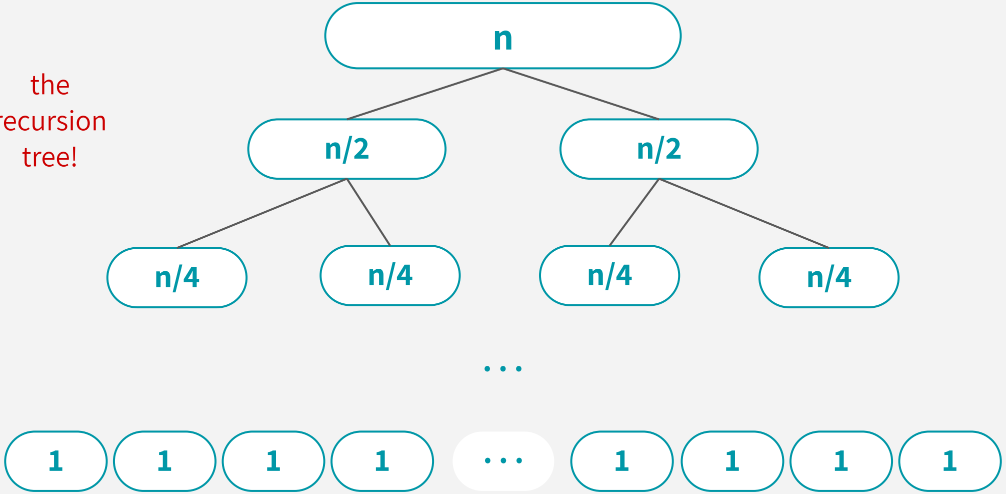
RECURRENCE RELATIONS

Basically, Recurrence Relations give us a *recursive* way to express runtimes for *recursive* algorithms!

We can then employ some math-ier approaches to analyze these recurrence relations.

RECURRENCE RELATIONS

To build the recurrence relation for MergeSort, we can think of its runtime as follows:

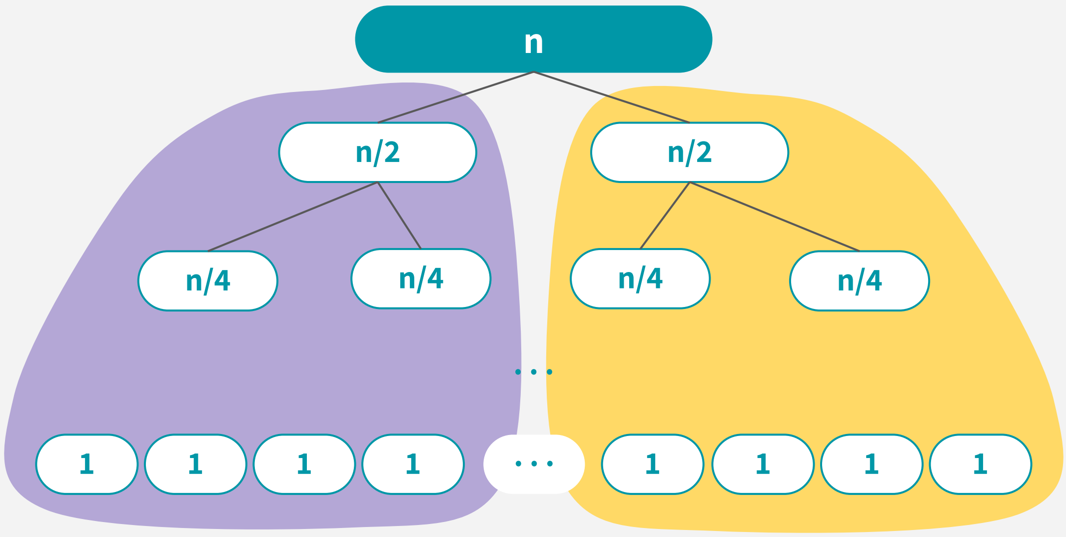


the recursion tree!

RECURRENCE RELATIONS

To build the recurrence relation for MergeSort, we can think of its runtime as follows:

Work in the whole tree =



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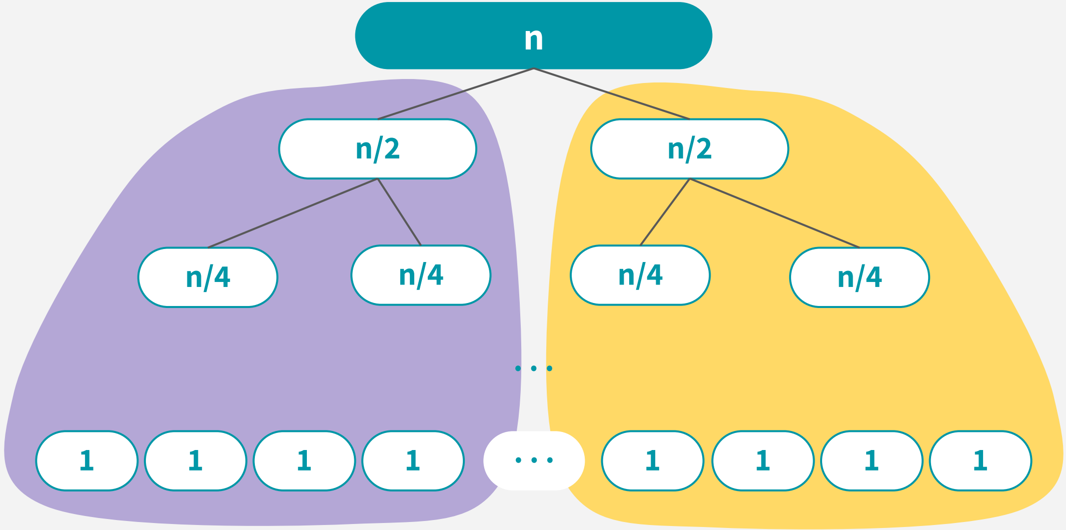

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total work in LEFT recursive call (left subtree)

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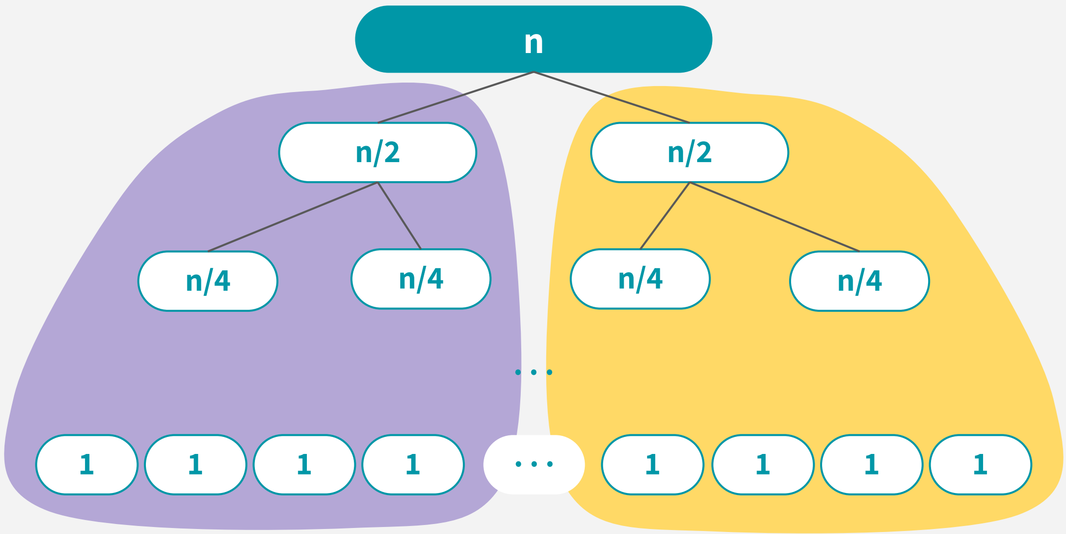

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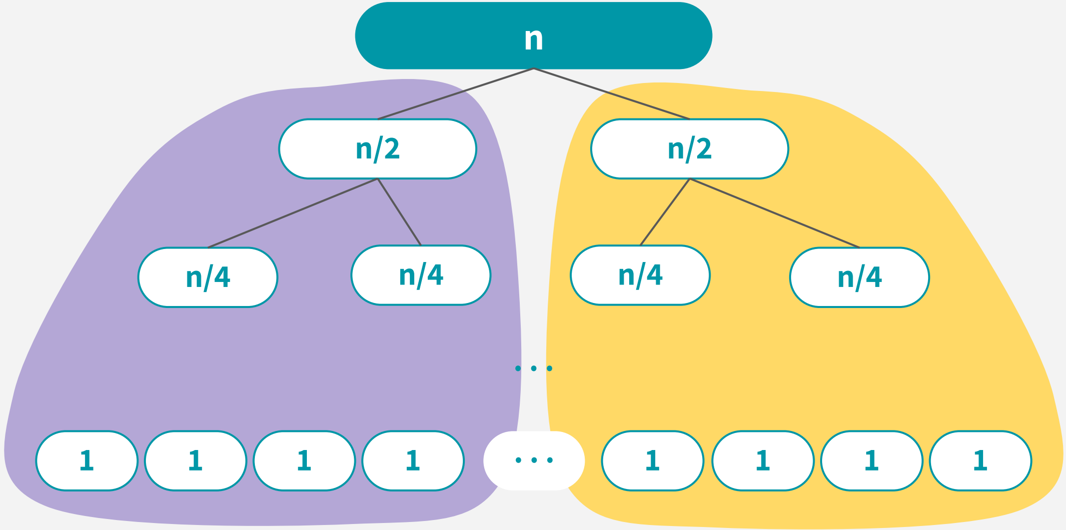

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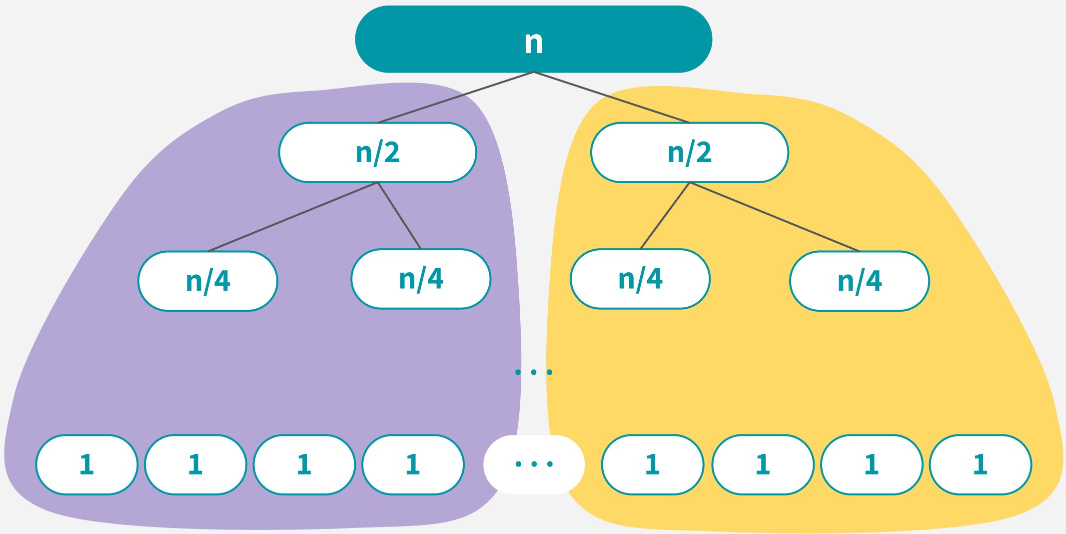

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work to create suproblems &
"merge" their solutions

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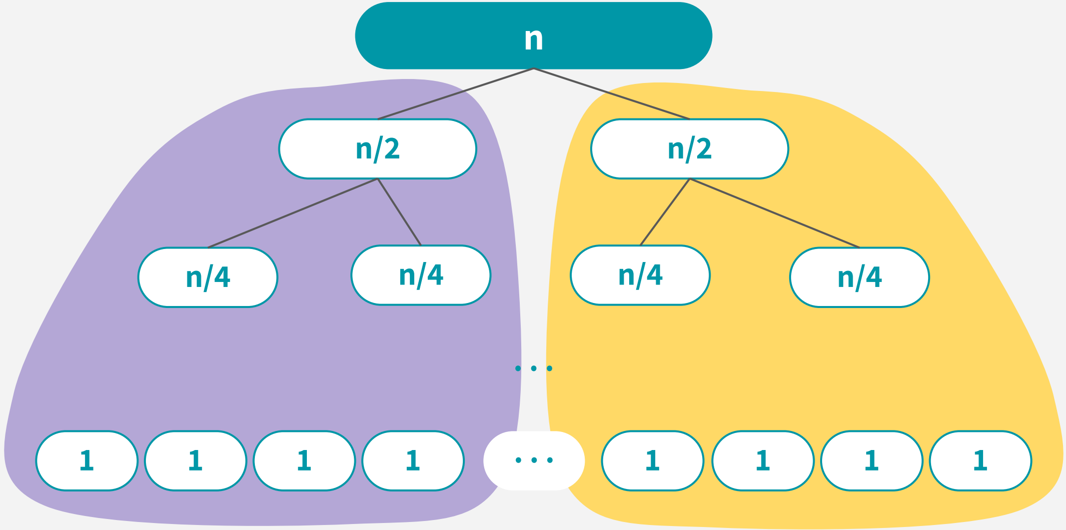

T(n) =

T(n/2)

+

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RECURRENCE RELATIONS

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A note:

We're making a simplifying assumption here that **n** is a perfect power of two (otherwise, we should use floors and ceilings).

Turns out that if we do incorporate floors and ceilings, we still get constant size subproblems at level $\lfloor \log_b n \rfloor$, and generally, the stuff we'll do in this class with Recurrence Relations will still work if we forget about floors and ceilings here.

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RECURRENCE RELATIONS

To build the recurrence relation for MergeSort, we can think of its runtime as follows:

$$\mathbf{T(n)} = \underbrace{\boxed{T(n/2)} + \boxed{T(n/2)}} + \boxed{\mathbf{O(n)}}$$

since the subproblems are equal sizes, we can also write this as ***2 · T(n/2)***

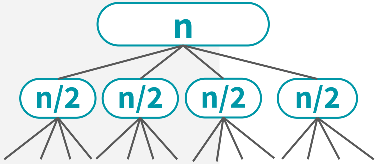
This is a *recursive* definition for T(n), so we also need a BASE CASE:

$$\mathbf{T(1) = O(1)}$$

No matter what T is, T(1) = O(1). If it's greater than O(1), then the problem size wouldn't actually be 1.

Since we already used the Recursion Tree to compute the runtime of MergeSort, we know that **T(n) = O(n log n)**.

EXAMPLE RECURRENCE RELATIONS



Useless Divide-and-Conquer Multiplication

$$T(n) = 4 \cdot T(n/2) + O(n)$$
$$T(n) = O(n^{\log_2 4}) = \mathbf{O(n^2)}$$

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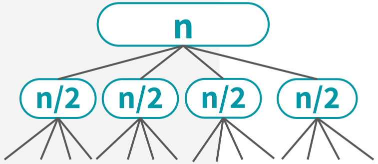

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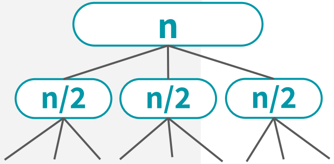

Karatsuba Integer Multiplication

$$T(n) = 3 \cdot T(n/2) + O(n)$$
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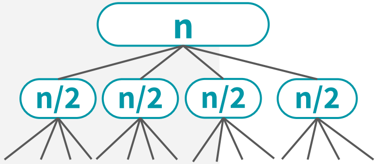

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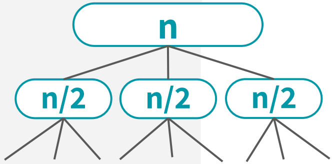

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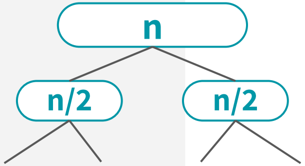

MergeSort

$$T(n) = 2 \cdot T(n/2) + O(n)$$
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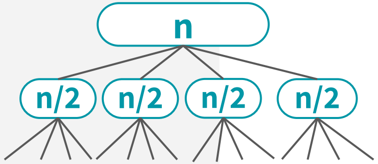

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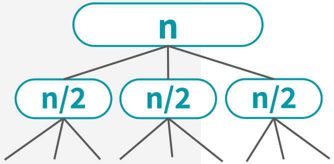

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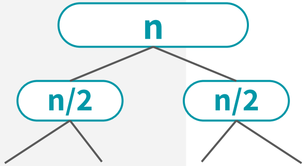

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IS THERE A
PATTERN
???

سوال؟

قضیه اصلی

فرمولی برای حل بسیاری از روابط بازگشتی
(اما نه همه آنها!)

THE MASTER THEOREM

Suppose that $\mathbf{a \geq 1}$, $\mathbf{b > 1}$, and $\mathbf{d}$ are constants (i.e. independent of $\mathbf{n}$).

Suppose $\mathbf{T(n) = a \cdot T(n/b) + O(n^d)}$. The Master Theorem states:

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$$T(n) = \begin{cases} \Theta(n^d \log n) & \text{if } a = b^d \\ \Theta(n^d) & \text{if } a < b^d \\ \Theta(n^{\log_b(a)}) & \text{if } a > b^d \end{cases}$$

a: number of subproblems (branching factor)

b: factor by which input size shrinks (shrinking factor)

d: need to do $O(n^d)$ work to create subproblems + “merge” their solutions

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USELESS DIVIDE & CONQUER MULTIPLICATION

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USELESS DIVIDE & CONQUER MULTIPLICATION	$\mathbf{T(n) = 4 \cdot T(n/2) + O(n)}$ $T(n) = O(n^{\log_2 4}) = \mathbf{O(n^2)}$	**a = 4** **b = 2** **d = 1**	$\mathbf{a > b^d}$

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b: factor by which input size shrinks (shrinking factor)
d: need to do $O(n^d)$ work to create subproblems + "merge" solutions

USELESS DIVIDE & CONQUER MULTIPLICATION	$\mathbf{T(n) = 4 \cdot T(n/2) + O(n)}$ $T(n) = O(n^{\log_2 4}) = \mathbf{O(n^2)}$	**a = 4** **b = 2** **d = 1**	$\mathbf{a > b^d}$
KARATSUBA INTEGER MULTIPLICATION	$\mathbf{T(n) = 3 \cdot T(n/2) + O(n)}$	**a = 3** **b = 2** **d = 1**	$\mathbf{a > b^d}$

MASTER THEOREM EXAMPLES

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KARATSUBA INTEGER MULTIPLICATION	$\mathbf{T(n) = 3 \cdot T(n/2) + O(n)}$ $T(n) = O(n^{\log_2 3}) \approx \mathbf{O(n^{1.6})}$	**a = 3** **b = 2** **d = 1**	$\mathbf{a > b^d}$

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MERGESORT	$\mathbf{T(n) = 2 \cdot T(n/2) + O(n)}$	$a = 2$ $b = 2$ $d = 1$	$a = b^d$

MASTER THEOREM EXAMPLES

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