طراحی الگوریتم ها

مبحث نوزدهم: پیچیدگی محاسبات

سجاد شیرعلی شهرضا

بهار 1402

سه شنبه، 9 خرداد 1402

Based on Rose-Hulman MA/CSSE 473 Slides from Summer 2017 by Claude Anderson

اطلاع رسانی

- آخرین مبحث سال!

زمان حل چند جمله ای

Idea

- Well-known algorithm rule: Polynomial good, exponential bad!
  - The latter is obvious, the former may need some explanation
  - We say that polynomial-time problems are **tractable**
    - I.e., exponential problems are **intractable**

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  - If f(t) and g(t) are polynomials, so is f(g(t))
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- Almost all of the algorithms we have studied in this course had polynomial time

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- Almost every optimization problem can be expressed in decision problem form

Decision Problem Example 1

- **Definition**:
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- **k-Clique Decision problem**
  - **Instance**: an undirected graph G=(V,E).
    - Note: k is some constant, independent of the problem
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- **Graph Coloring Decision Problem**
  - **Instance**: an undirected graph G=(V,E) and an integer k>0.
  - **Question**: Is there a coloring of G that uses no more than k colors?

Decision Problem Example 3

- **Definition**:
  - Suppose we have an unlimited number of bins, each with capacity 1.0, and n objects with sizes $s_1, \ldots, s_n$, where $0 < s_i \leq 1$ (all s_i rational)

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- **Bin Packing Decision Problem**
  - **Instance**: $s_1, \ldots, s_n$ as described above, and an integer k.
  - **Question**: Can the n objects be packed into k bins?

سوال؟

تقليل

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- If there is an algorithm for q, create an algorithm for p:
    - Compose T with that algorithm
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- If T is a function with polynomially bounded running time:
    - We say: p is **polynomially reducible** to q
    - We write: $\mathbf{p \leq_p q}$
- For now, reducible means polynomially reducible

Class P

- **Definition**: An algorithm is polynomially bounded if its worst-case complexity is big-O of a polynomial function of the input size n.
  - I.e. there is a single polynomial $\boldsymbol{p}$ such that for each input of size $\boldsymbol{n}$, the algorithm terminates after at most $\boldsymbol{p(n)}$ steps
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  - Informally (with slight abuse of notation), we can say that polynomially bounded optimization problems are in P

Example of a problem in P

- Minimum Spanning Tree (MST)
- **Input**: A weighted graph G=(V,E) with n vertices [each edge e is labeled with a non-negative weight w(e)], and a number k.
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- How do we know it's in P?
    - Find the MST and check whether its cost is less than k

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  - No. The size to represent k is log k

Class NP

- NP: Nondeterministic Polynomial time
- First stage: assumes a "guess" of a possible solution
- Can we verify in polynomial time whether the proposed solution is a correct solution?
    - I.e., do we have a verifier that verifies an answer in a polynomial time?

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- The problem is in NP **if and only if** there is a polynomial-time (in N) algorithm that can check a proposed solution to see if it really is a solution

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    - Output yes if it is a solution
- NP Class: class of decision problems for which there is a polynomially bounded nondeterministic algorithm
  - Examples: Graph coloring, Bin packing, Clique

سوال؟

رابطه بین P و NP

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- $P \subseteq NP$
  - Directly solve the problem to find the answer
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- $NP \subseteq Exp$
  - Generate all possible solutions
    - Will be exponential in terms of the input
  - Check them to see if one of them is the answer
  - Will be deterministic

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- Try to solve it as an extra assignment
  - Will give you extra mark for course if you do it!

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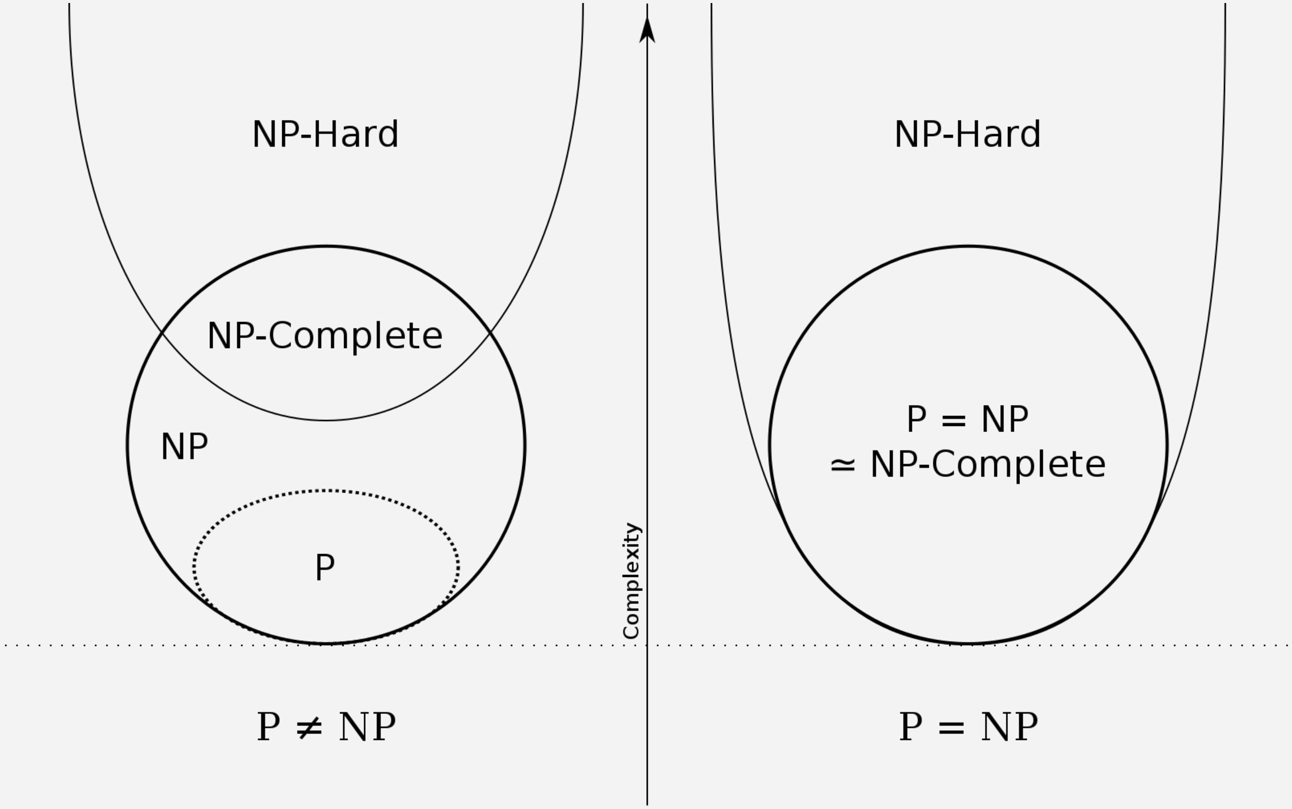
- NP – H includes all NP - C problems

Relation between P, NP, NP-H, NP-C



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  - Example: 3-satisfiability
- If $\boldsymbol{p}$ is NP-hard, and $\boldsymbol{p} \leq_{\boldsymbol{p}} \boldsymbol{q}$, then **q** is NP-hard.
  - Most NP-complete problems are shown to be NP-C by showing that 3-satisfiability (or some other known NP-complete problem) reduces to them

3-SAT

- 3-Satisfiability problem:
- A CNF (conjunctive normal form) formula is in 3-CNF if every clause has exactly three literals
- **Instance**: A 3CNF propositional formula ***f*** (containing **n** different variables)
- **Question**: Is there a truth assignment that satisfies ***f***?

$$(\neg a \vee b \vee c) \wedge (\neg b \vee a \vee c) \wedge (\neg c \vee a \vee b) \wedge (\neg d \vee a \vee b) \wedge (\neg e \vee a \vee b) \wedge (\neg a \vee b \vee d) \wedge (\neg b \vee a \vee d)$$

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