

# طراحی الگوریتم ها

## مبحث دوازدهم: برنامه نویسی پویا

**سجاد شیرعلی شمرضا**

**بهار، 1401**

**یک شنبه، 15 فروردین 1402**

## اطلاع رسانی

- بخش مرتبط کتاب برای این جلسه: 15
- یادآوری امتحانک دوم: یکشنبه دو هفته دیگر، یکشنبه 27 فروردین 1402

# مقدمه ای بر برنامه نویسی پویا

**یک روش طراحی الگوریتم**

# DYNAMIC PROGRAMMING

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We'll see an example of DP today:  
Longest Common Subsequence (LCS)  
We will see two more examples of DP next week for shortest path  
problems: Bellman-Ford and Floyd-Warshall algorithms

But first, an overview of DP!

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e.g. **Lots of different entries in the row  $d^{(k)}$  may ask for  $d^{(k-1)}[v]$**

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We will see a way later to implement **Bellman-Ford** using a top-down approach.

# DYNAMIC PROGRAMMING

## Why “dynamic programming”?

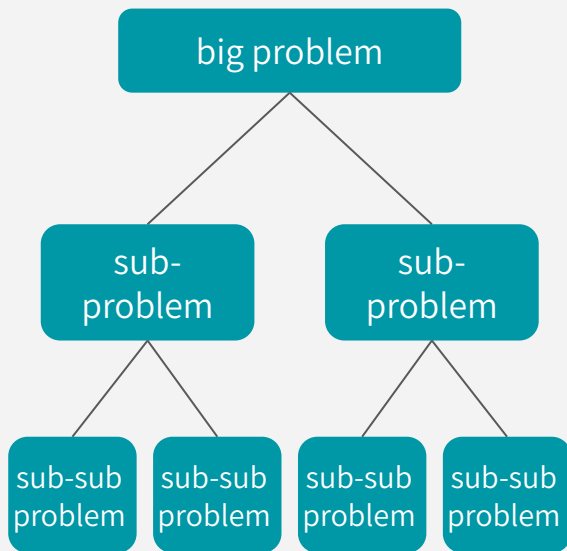
Richard Bellman invented the term in the 1950's. He was working for the RAND corporation at the time, which was employed by the Air Force, and government projects needed flashy non-mathematical non-researchy names to get funded and approved.

*“It’s impossible to use the word dynamic in a pejorative sense...  
I thought dynamic programming was a good name.  
It was something not even a Congressman could object to.”*

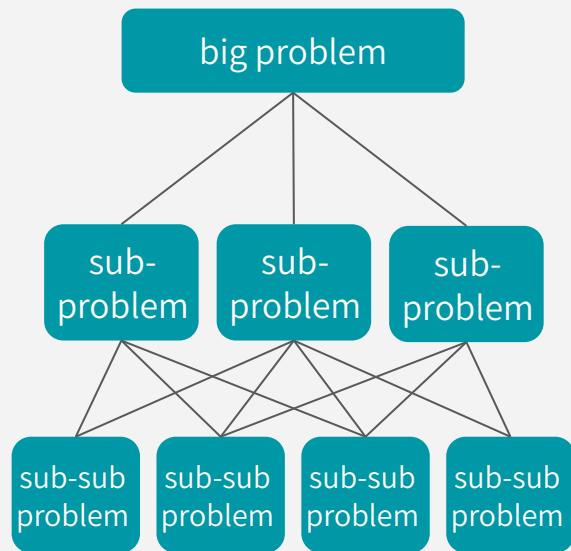


# DIVIDE & CONQUER vs DP

## DIVIDE-AND-CONQUER



## DYNAMIC PROGRAMMING





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سوال؟



پیدا کردن بزرگ ترین  
زیر رشته مشترک

# LONGEST COMMON SUBSEQUENCE

A sequence **Z** is a **SUBSEQUENCE** of **X** if **Z** can be obtained from **X** by deleting symbols

**BDFH** is a subsequence of **AB**C**DEFGH**

**C** is a subsequence of **AB**C**DEFGH**

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A sequence **Z** is a **LONGEST COMMON SUBSEQUENCE (LCS)** of **X** and **Y**  
if **Z** is a subsequence of both **X** and **Y**  
and any sequence longer than **Z** is not a subsequence of at least one of **X** or **Y**.

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**TASK:** Given sequences **X** and **Y**, find the length of their LCS, **Z**.

(Later, we'll also output **Z**, but we'll start off with the length)

**ABDFGH** is the LCS of

**ABCDEFGH** and **ABDFGHI**

# APPLICATIONS OF LCS

## **Bioinformatics!**

Detect similarities  
between DNA or  
protein sequences

## **Computational linguistics!**

Extract similarities in  
words/word-forms  
and determine how  
words are related

## **the diff unix command!**

Identify differences  
between the contents  
of two files

and so much more...



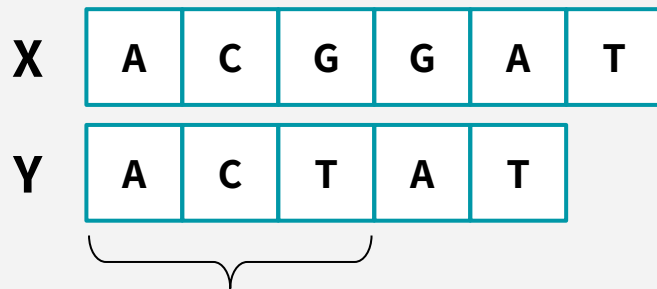
سوال؟

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# STEP 1: OPTIMAL SUBSTRUCTURE

**SUBPROBLEM:** Find the length of LCS's of *prefixes* to X and Y.

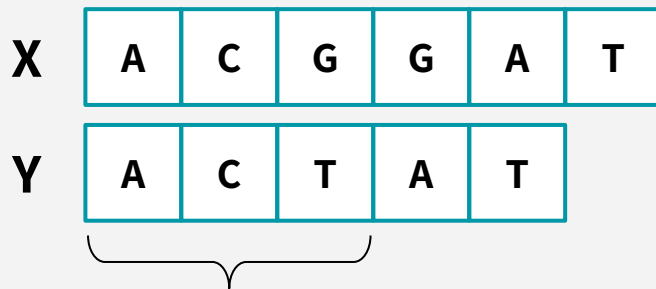


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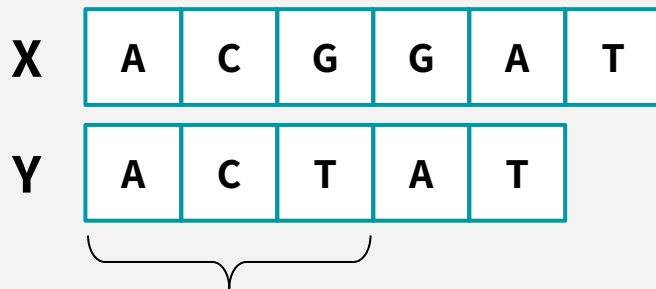
**Examples:**

$C[2,3] = 2$  (LCS of  $X_2$  and  $Y_3$  is AC)  
 $C[5,4] = 3$  (LCS of  $X_5$  and  $Y_4$  is ACA)

Let  $C[i, j] = \text{length\_of\_LCS}(X_i, Y_j)$

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**Why is this a good choice?**

# STEP 1: OPTIMAL SUBSTRUCTURE

Let  $C[i, j] = \text{length\_of\_LCS}(X_i, Y_j)$

Consider the ends of our prefixes,  $X[i]$  and  $Y[j]$ . We have two cases:

**Case 1:  $X[i] = Y[j]$**

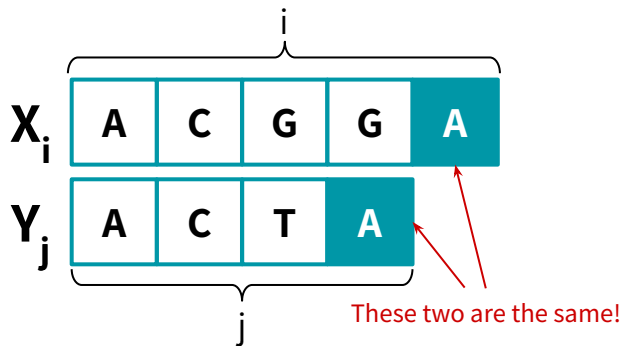
**Case 2:  $X[i] \neq Y[j]$**

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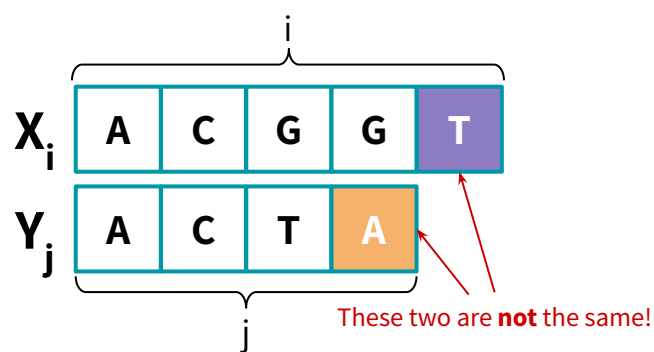
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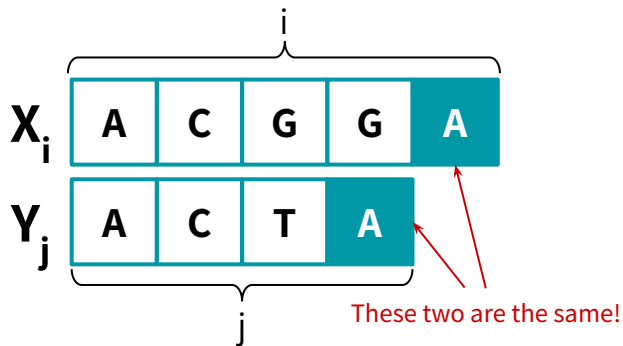


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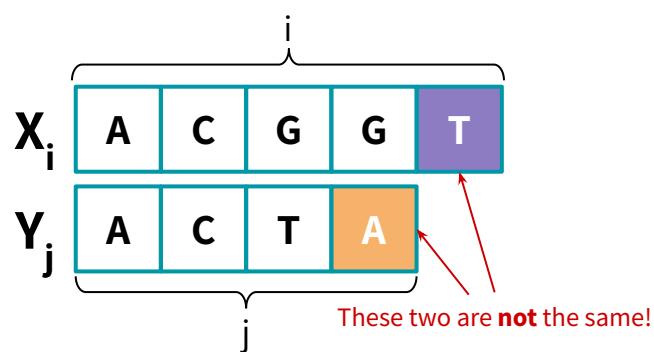
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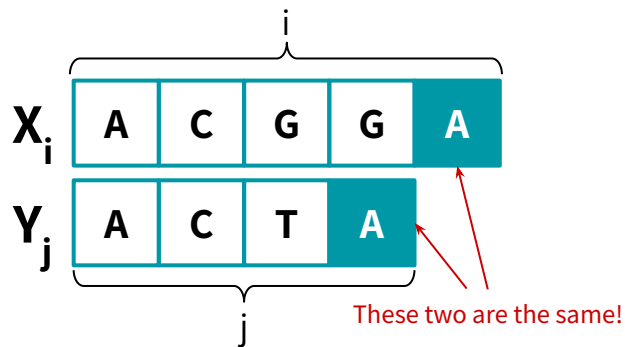
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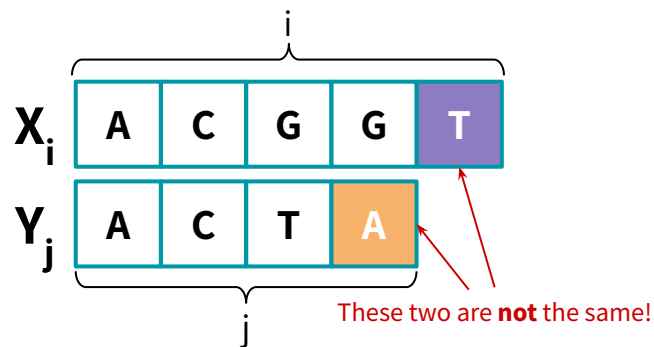
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Then,  $C[i, j] = 1 + C[i-1, j-1]$

because  $\text{LCS}(X_i, Y_j) = \text{LCS}(X_{i-1}, Y_{j-1})$  followed by **A**.

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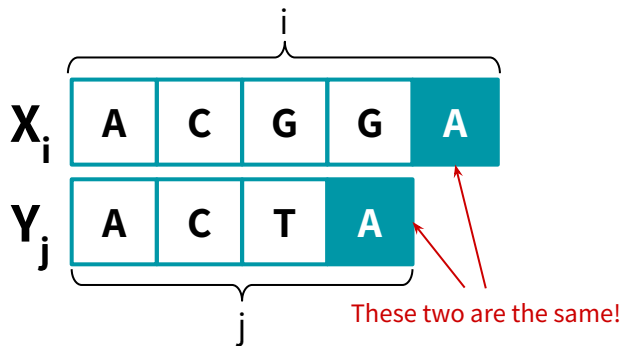


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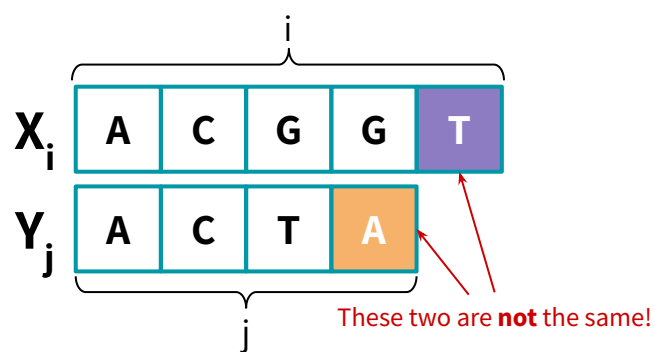
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Then,  $C[i, j] = \max\{ C[i-1, j], C[i, j-1] \}$

Give **A** a chance to “match”:  $\text{LCS}(X_i, Y_j) = \text{LCS}(X_{i-1}, Y_j)$

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$Y_j$	A	C	T	A

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if  $X[i] = Y[j]$  and  $i, j > 0$

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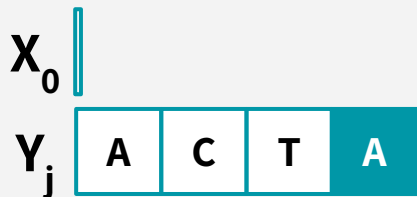
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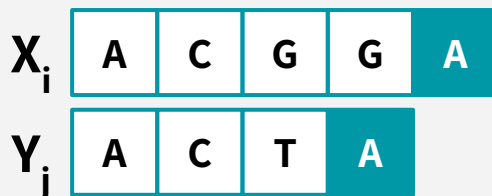
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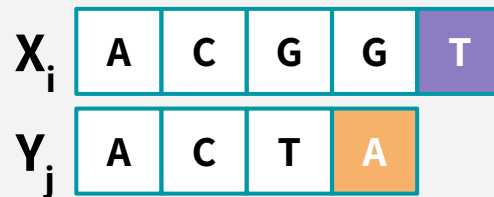
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## STEP 3: WRITE A DP ALGORITHM

We'll store answers to our subproblems  $C[i, j]$  in a table (this is our cache)!

Now that we've defined our recursive formulation, translating to appropriate pseudocode is straightforward: establish your base cases & define your cases!

**We'll do this in a bottom-up fashion. Why?**

We know we need answers to shorter prefixes first, and it's pretty easy to iterate in order and fill out answers to smaller prefixes before building up to longer prefixes (ultimately getting our final answer  $C[m, n]$  where  $|X| = m$ , and  $|Y| = n$ )



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Final answer

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**Runtime: ?**

# STEP 3: WRITE A DP ALGORITHM

$$C[i,j] = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ C[i-1,j-1] + 1 & \text{if } X[i] = Y[j] \text{ and } i, j > 0 \\ \max\{ C[i-1,j], C[i,j-1] \} & \text{if } X[i] \neq Y[j] \text{ and } i, j > 0 \end{cases}$$

**LCS(X,Y):** len(X) = m & len(Y) = n

Initialize an (m+1) x (n+1) 0-indexed array C

Make sure that  
our base cases  
are set up

$C[i,0] = C[0,j] = 0$  for all  $i=0, \dots, m$  and  $j=0, \dots, n$

for  $i = 1, \dots, m$  and  $j = 1, \dots, n$ :

if  $X[i] = Y[j]$ :

$C[i,j] = C[i-1,j-1] + 1$  ← Case 1

else:

$C[i,j] = \max\{ C[i,j-1], C[i-1,j] \}$  ← Case 2

Final answer

return  $C[m,n]$

**Runtime:  $O(mn)$**

Constant amount of work to fill  
out each of the  $mn$  entries in C



سوال؟



# EXAMPLE

		Y				
		A	C	T	G	
X	A	0	0	0	0	0
	C	0				
	G	0				
	G	0				
	A	0				

Initialize an  $(m+1) \times (n+1)$  0-indexed array  $C$   
 $C[i,0] = C[0,j] = 0$  for all  $i=0,\dots,m$  and  $j=0,\dots,n$

**Fill in our base  
cases first**

# EXAMPLE

		Y				
		A	C	T	G	
X	A	0	0	0	0	0
	C	0	1			
	G	0				
	G	0				
	A	0				

for  $i = 1, \dots, m$  and  $j = 1, \dots, n$ :

if  $X[i] = Y[j]$ :

$C[i,j] = C[i-1,j-1] + 1$

else:

$C[i,j] = \max\{ C[i,j-1], C[i-1,j] \}$

**$X[i] = Y[j]$**

$C[i,j] = C[i-1,j-1] + 1$

# EXAMPLE

		Y				
		A	C	T	G	
X	A	0	0	0	0	0
	C	0	1	1		
	G	0				
	G	0				
	A	0				

```
for i = 1,...,m and j = 1,...,n:
```

```
  if X[i] = Y[j]:
```

```
    C[i,j] = C[i-1,j-1] + 1
```

```
  else:
```

```
    C[i,j] = max{ C[i,j-1], C[i-1,j] }
```

**$X[i] \neq Y[j]$**

$C[i,j] = \max\{ C[i,j-1], C[i-1,j] \}$

# EXAMPLE

		Y				
		A	C	T	G	
X	A	0	0	0	0	0
	C	0	1	1	1	
	G	0				
	G	0				
	A	0				

```
for i = 1,...,m and j = 1,...,n:
```

```
  if X[i] = Y[j]:
```

```
    C[i,j] = C[i-1,j-1] + 1
```

```
  else:
```

```
    C[i,j] = max{ C[i,j-1], C[i-1,j] }
```

**$X[i] \neq Y[j]$**

$C[i,j] = \max\{ C[i,j-1], C[i-1,j] \}$

# EXAMPLE

		Y				
		A	C	T	G	
X	A	0	0	0	0	0
	C	0	1	1	1	1
	G	0				
	G	0				
	A	0				

```
for i = 1,...,m and j = 1,...,n:
```

```
  if X[i] = Y[j]:
```

```
    C[i,j] = C[i-1,j-1] + 1
```

```
  else:
```

```
    C[i,j] = max{ C[i,j-1], C[i-1,j] }
```

**$X[i] \neq Y[j]$**

$C[i,j] = \max\{ C[i,j-1], C[i-1,j] \}$

# EXAMPLE

		Y				
		A	C	T	G	
X	A	0	0	0	0	0
	C	0	1	1	1	1
	G	0	1			
	G	0				
	A	0				

```
for i = 1,...,m and j = 1,...,n:
```

```
  if X[i] = Y[j]:
```

```
    C[i,j] = C[i-1,j-1] + 1
```

```
  else:
```

```
    C[i,j] = max{ C[i,j-1], C[i-1,j] }
```

**$X[i] \neq Y[j]$**

$C[i,j] = \max\{ C[i,j-1], C[i-1,j] \}$

# EXAMPLE

		Y				
		A	C	T	G	
X	A	0	0	0	0	0
	C	0	1	1	1	1
	G	0	1	2		
	G	0				
	A	0				

for  $i = 1, \dots, m$  and  $j = 1, \dots, n$ :

if  $X[i] = Y[j]$ :

$C[i,j] = C[i-1,j-1] + 1$

else:

$C[i,j] = \max\{ C[i,j-1], C[i-1,j] \}$

$X[i] = Y[j]$

$C[i,j] = C[i-1,j-1] + 1$

# EXAMPLE

		Y				
		A	C	T	G	
X	A	0	0	0	0	0
	C	0	1	1	1	1
	G	0				
	G	0				
	A	0				

```
for i = 1,...,m and j = 1,...,n:
```

```
  if X[i] = Y[j]:
```

```
    C[i,j] = C[i-1,j-1] + 1
```

```
  else:
```

```
    C[i,j] = max{ C[i,j-1], C[i-1,j] }
```

**$X[i] \neq Y[j]$**

$C[i,j] = \max\{ C[i,j-1], C[i-1,j] \}$



# EXAMPLE

		Y				
		A	C	T	G	
X	A	0	0	0	0	0
	C	0	1	1	1	1
	G	0	1	2	2	2
	G	0				
	A	0				

```
for i = 1,...,m and j = 1,...,n:
```

```
  if X[i] = Y[j]:
```

```
    C[i,j] = C[i-1,j-1] + 1
```

```
  else:
```

```
    C[i,j] = max{ C[i,j-1], C[i-1,j] }
```

**$X[i] \neq Y[j]$**

$C[i,j] = \max\{ C[i,j-1], C[i-1,j] \}$

# EXAMPLE

		Y				
		A	C	T	G	
X	A	0	0	0	0	0
	C	0	1	1	1	1
	G	0	1	2	2	2
	G	0				
	A	0				

```
for i = 1,...,m and j = 1,...,n:
```

```
  if X[i] = Y[j]:
```

```
    C[i,j] = C[i-1,j-1] + 1
```

```
  else:
```

```
    C[i,j] = max{ C[i,j-1], C[i-1,j] }
```

**$X[i] \neq Y[j]$**

$C[i,j] = \max\{ C[i,j-1], C[i-1,j] \}$

# EXAMPLE

		Y				
		A	C	T	G	
X	A	0	0	0	0	0
	C	0	1	1	1	1
	G	0	1	2	2	2
	G	0	1	2		
	A	0				

for  $i = 1, \dots, m$  and  $j = 1, \dots, n$ :

if  $X[i] = Y[j]$ :

$C[i,j] = C[i-1,j-1] + 1$

else:

$C[i,j] = \max\{ C[i,j-1], C[i-1,j] \}$

**$X[i] \neq Y[j]$**

$C[i,j] = \max\{ C[i,j-1], C[i-1,j] \}$

# EXAMPLE

		Y				
		A	C	T	G	
X	A	0	0	0	0	0
	C	0	1	1	1	1
	G	0	1	2	2	2
	G	0				
	A	0				

```
for i = 1,...,m and j = 1,...,n:
```

```
  if X[i] = Y[j]:
```

```
    C[i,j] = C[i-1,j-1] + 1
```

```
  else:
```

```
    C[i,j] = max{ C[i,j-1], C[i-1,j] }
```

**$X[i] \neq Y[j]$**

$C[i,j] = \max\{ C[i,j-1], C[i-1,j] \}$

# EXAMPLE

		Y				
		A	C	T	G	
X	A	0	0	0	0	0
	C	0	1	1	1	1
	G	0	1	2	2	2
	G	0	1	2	2	3
	A	0				

for  $i = 1, \dots, m$  and  $j = 1, \dots, n$ :

if  $X[i] = Y[j]$ :

$C[i,j] = C[i-1,j-1] + 1$

else:

$C[i,j] = \max\{ C[i,j-1], C[i-1,j] \}$

**$X[i] = Y[j]$**

$C[i,j] = C[i-1,j-1] + 1$

# EXAMPLE

		Y				
		A	C	T	G	
X	A	0	0	0	0	0
	C	0	1	1	1	1
	G	0	1	2	2	2
	G	0	1	2	2	3
	A	0				

for  $i = 1, \dots, m$  and  $j = 1, \dots, n$ :

if  $X[i] = Y[j]$ :

$C[i,j] = C[i-1,j-1] + 1$

else:

$C[i,j] = \max\{ C[i,j-1], C[i-1,j] \}$

**$X[i] \neq Y[j]$**

$C[i,j] = \max\{ C[i,j-1], C[i-1,j] \}$

# EXAMPLE

		Y				
		A	C	T	G	
X	A	0	0	0	0	0
	C	0	1	1	1	1
	G	0	1	2	2	2
	G	0	1	2	2	3
	A	0				

for  $i = 1, \dots, m$  and  $j = 1, \dots, n$ :

if  $X[i] = Y[j]$ :

$C[i,j] = C[i-1,j-1] + 1$

else:

$C[i,j] = \max\{ C[i,j-1], C[i-1,j] \}$

**$X[i] \neq Y[j]$**

$C[i,j] = \max\{ C[i,j-1], C[i-1,j] \}$

# EXAMPLE

		Y				
		A	C	T	G	
X	A	0	0	0	0	0
	C	0	1	1	1	1
	G	0	1	2	2	2
	G	0	1	2	2	
	A	0				

```
for i = 1,...,m and j = 1,...,n:
```

```
  if X[i] = Y[j]:
```

```
    C[i,j] = C[i-1,j-1] + 1
```

```
  else:
```

```
    C[i,j] = max{ C[i,j-1], C[i-1,j] }
```

**$X[i] \neq Y[j]$**

$C[i,j] = \max\{ C[i,j-1], C[i-1,j] \}$



# EXAMPLE

		Y				
		A	C	T	G	
X	A	0	0	0	0	0
	C	0	1	1	1	1
	G	0	1	2	2	2
	G	0	1	2	2	3
	A	0				

for  $i = 1, \dots, m$  and  $j = 1, \dots, n$ :

if  $X[i] = Y[j]$ :

$C[i,j] = C[i-1,j-1] + 1$

else:

$C[i,j] = \max\{ C[i,j-1], C[i-1,j] \}$

**$X[i] = Y[j]$**

$C[i,j] = C[i-1,j-1] + 1$

# EXAMPLE

		Y				
		A	C	T	G	
X	A	0	0	0	0	0
	C	0	1	1	1	1
	G	0	1	2	2	3
	G	0	1	2	2	3
	A	0	1			

for  $i = 1, \dots, m$  and  $j = 1, \dots, n$ :

if  $X[i] = Y[j]$ :

$C[i,j] = C[i-1,j-1] + 1$

else:

$C[i,j] = \max\{ C[i,j-1], C[i-1,j] \}$

**$X[i] = Y[j]$**

$C[i,j] = C[i-1,j-1] + 1$

# EXAMPLE

		Y				
		A	C	T	G	
X	A	0	0	0	0	0
	C	0	1	1	1	1
	G	0	1	2	2	3
	G	0	1	2	2	3
	A	0	1	2		

for  $i = 1, \dots, m$  and  $j = 1, \dots, n$ :

if  $X[i] = Y[j]$ :

$C[i,j] = C[i-1,j-1] + 1$

else:

$C[i,j] = \max\{ C[i,j-1], C[i-1,j] \}$

**$X[i] \neq Y[j]$**

$C[i,j] = \max\{ C[i,j-1], C[i-1,j] \}$

# EXAMPLE

		Y				
		A	C	T	G	
X	A	0	0	0	0	0
	C	0	1	1	1	1
	G	0	1	2	2	3
	G	0	1	2	2	3
	A	0	1	2	2	

```
for i = 1,...,m and j = 1,...,n:
```

```
  if X[i] = Y[j]:
```

```
    C[i,j] = C[i-1,j-1] + 1
```

```
  else:
```

```
    C[i,j] = max{ C[i,j-1], C[i-1,j] }
```

**$X[i] \neq Y[j]$**

$C[i,j] = \max\{ C[i,j-1], C[i-1,j] \}$

# EXAMPLE

		Y				
		A	C	T	G	
X	A	0	0	0	0	0
	C	0	1	1	1	1
	G	0	1	2	2	3
	G	0	1	2	2	3
	A	0	1	2	2	3

```
for i = 1,...,m and j = 1,...,n:
```

```
  if X[i] = Y[j]:
```

```
    C[i,j] = C[i-1,j-1] + 1
```

```
  else:
```

```
    C[i,j] = max{ C[i,j-1], C[i-1,j] }
```

**$X[i] \neq Y[j]$**

$C[i,j] = \max\{ C[i,j-1], C[i-1,j] \}$

# EXAMPLE

		Y				
		A	C	T	G	
X	A	0	0	0	0	0
	C	0	1	1	1	1
	G	0	1	2	2	3
	G	0	1	2	2	3
	A	0	1	2	2	3

```
for i = 1,...,m and j = 1,...,n:  
  if X[i] = Y[j]:  
    C[i,j] = C[i-1,j-1] + 1  
  else:  
    C[i,j] = max{ C[i,j-1], C[i-1,j] }
```

**So the LCM of X and Y  
has length 3.**



سوال؟

# LCS: RECIPE FOR APPLYING DP

- 1. Identify optimal substructure.** What are your overlapping subproblems?
- 2. Define a recursive formulation.** Recursively define your optimal solution in terms of sub-solutions. *Always write down this formulation.*
- 3. Use dynamic programming.** Turn the recursive formulation into a DP algorithm.
- 4. If needed, track additional information.** You may need to solve a related problem, e.g. step 3 finds you an optimal *value/cost*, but you need to recover the actual optimal *solution/path/subset/substring/etc.* Go back and modify your algorithm in step 3 to make this happen.



## STEP 4: FIND ACTUAL LCS

		Y				
		A	C	T	G	
X	A	0	0	0	0	0
	C	0	1	1	1	1
	G	0	1	2	2	3
	G	0	1	2	2	3
	A	0	1	2	2	3

**Suppose we want to  
recover the actual LCS.**

How can we construct the actual LCS  
given this table C that we just filled out?

## STEP 4: FIND ACTUAL LCS

		Y				
		A	C	T	G	
X	A	0	0	0	0	0
	C	0	1	1	1	1
	G	0	1	2	2	3
	G	0	1	2	2	3
	A	0	1	2	2	3

**Suppose we want to recover the actual LCS.**

How can we construct the actual LCS given this table C that we just filled out?

**We'll start at  $C[m,n]$  and work backwards to trace out how we ended up with a 3 as our answer!**

If we see that the last character in X matches the character in Y, then we mark that character as part of our LCS and take a *diagonal step* backwards.

Otherwise, if the characters don't match, we just simply take a step towards the larger adjacent entry

## STEP 4: FIND ACTUAL LCS

		Y				
		A	C	T	G	
X	A	0	0	0	0	0
	C	0	1	1	1	1
	G	0	1	2	2	3
	G	0	1	2	2	3
	A	0	1	2	2	3

RECOVER\_LCS(X, Y, C):

// C is already filled out

L = []

i = m

j = n

while i > 0 and j > 0:

    if X[i] = Y[j]:

        append X[i] to the beginning of L

        i = i-1

        j = j-1

    else if C[i,j] = C[i,j-1]:

        j = j-1

    else:

        i = i-1

return L

## STEP 4: FIND ACTUAL LCS

		Y				
		A	C	T	G	
X	A	0	0	0	0	0
	C	0	1	1	1	1
	G	0	1	2	2	3
	G	0	1	2	2	3
	A	0	1	2	2	3

RECOVER\_LCS(X, Y, C):

// C is already filled out

L = []

i = m

j = n

while i > 0 and j > 0:

if X[i] = X[j]:

append X[i] to the beginning of L

i = i-1

j = j-1

else if C[i,j] = C[i,j-1]:

j = j-1

else:

i = i-1

return L

**Runtime?**

## STEP 4: FIND ACTUAL LCS

		Y				
		A	C	T	G	
X	A	0	0	0	0	0
	C	0	1	1	1	1
	G	0	1	2	2	3
	G	0	1	2	2	3
	A	0	1	2	2	3

RECOVER\_LCS(X, Y, C):

// C is already filled out

L = []

i = m

j = n

while i > 0 and j > 0:

if X[i] = X[j]:

append X[i] to the beginning of L

i = i-1

j = j-1

else if C[i,j] = C[i,j-1]:

j = j-1

else:

i = i-1

return L

**This extra subroutine  
takes  $O(m+n)$  time!**

## STEP 4: FIND ACTUAL LCS

		Y				
		A	C	T	G	
X	A					
	C					
	G					
	G					
	A					
		0	1	2	2	3

RECOVER\_LCS(X, Y, C):

// C is already filled out

**Note:** Sometimes, you don't need to track more info in your original DP algorithm from Step 3 (all we did here was do some reverse engineering). Other times, you may want to augment your Step 3 algorithm to keep track of extra information (e.g. info that might tell you which subproblem contributed the most, hints to help you reverse engineer, etc.)

$i = i - 1$   
return L

**Outline**  
**takes  $O(m+n)$  time!**

## STEP 4: FIND ACTUAL LCS

		Y				
		A	C	T	G	
X	A	0	0	0	0	0
	C	0	1	1	1	1
	G	0	1	2	2	3
	G	0	1	2	2	3
	A	0	1	2	2	3

LCS of X and Y:

## STEP 4: FIND ACTUAL LCS

		Y				
		A	C	T	G	
X	A	0	0	0	0	0
	C	0	1	1	1	1
	G	0	1	2	2	3
	G	0	1	2	2	3
	A	0	1	2	2	3

**$X[5] \neq Y[4]$**

We don't add anything to our LCS.  
But we can go up  $\rightarrow C[4,4]$

LCS of X and Y:



## STEP 4: FIND ACTUAL LCS

		Y				
		A	C	T	G	
X	A	0	0	0	0	0
	C	0	1	1	1	1
	G	0	1	2	2	3
	G	0	1	2	2	3
	A	0	1	2	2	3

$$X[4] = Y[4]$$

We can add “G” to our LCS  
We go diagonally back  $\rightarrow C[3,3]$

LCS of X and Y:

G

## STEP 4: FIND ACTUAL LCS

		Y				
		A	C	T	G	
X	A	0	0	0	0	0
	C	0	1	1	1	1
	G	0	1	2	2	2
	G	0	1	2	2	3
	A	0	1	2	2	3

**$X[3] \neq Y[3]$**

We don't add anything to our LCS.  
But we can go up  $\rightarrow C[2,3]$

(Going left is okay too since it's a tie. How you choose to break ties might result in different LCS's when there are multiple. In this example, there's actually only one LCS so we we'll end up with the same LCS either way)

LCS of X and Y:

G

## STEP 4: FIND ACTUAL LCS

		Y				
		A	C	T	G	
X	A	0	0	0	0	0
	C	0	1	1	1	1
	G	0	1	2	2	3
	G	0	1	2	2	3
	A	0	1	2	2	3
		0	1	2	2	3

**$X[2] \neq Y[3]$**

We don't add anything to our LCS.  
But we can go left  $\rightarrow C[2,2]$

LCS of X and Y:

G

## STEP 4: FIND ACTUAL LCS

		Y				
		A	C	T	G	
X	A	0	0	0	0	0
	C	0	1	1	1	1
	G	0	1	2	2	3
	G	0	1	2	2	3
	A	0	1	2	2	3

$$X[2] = Y[2]$$

We can add “C” to our LCS.  
We go diagonally back  $\rightarrow$  C[1,1]

LCS of X and Y:

C

G

## STEP 4: FIND ACTUAL LCS

		Y				
		A	C	T	G	
X	A	0	0	0	0	0
	C	0	1	1	1	1
	G	0	1	2	2	3
	G	0	1	2	2	3
	A	0	1	2	2	3

**$X[1] = Y[1]$**

We can add “A” to our LCS.  
We’re done!

LCS of X and Y:

A

C

G



سوال؟

# LCS: RECIPE FOR APPLYING DP

1. **Identify optimal substructure.** What are your overlapping subproblems?
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3. **Use dynamic programming.** Turn the recursive formulation into a DP algorithm.
4. **If needed, track additional information.** You may need to solve a related problem, e.g. step 3 finds you an optimal *value/cost*, but you need to recover the actual optimal *solution/path/subset/substring/etc.* Go back and modify your algorithm in step 3 to make this happen.
5. **Can we do better?** Any wasted space? Other things to optimize?  
(We won't focus on this step too much in lecture/assignments/exams, but in practice, this is definitely a very important step to always consider!)

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  - The general LCS problem is *NP-hard*, so performing much better is an open problem!



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