

ساختمان داده و الگوریتم ها (CE203)

جلسه دوم:
ضرب

سجاد شیرعلی شهرضا

بهار، 1401

شنبه، 2 مهر 1401

اطلاع رسانی

- نظرسنجی اول
 - مهلت ارسال: 8 صبح شنبه امروز (2 مهر 1401) بود!

ضرب مدرسه ای

MULTIPLICATION: THE PROBLEM

Input: 2 non-negative numbers, x and y (n digits each)

Output: the product $x \cdot y$

$$\begin{array}{r} 5678 \\ \times 1234 \\ \hline 7006652 \end{array}$$

GRADE-SCHOOL MULTIPLICATION

$$\begin{array}{r} 45 \\ \times 63 \\ \hline 135 \\ 2700 \\ \hline 2835 \end{array}$$

GRADE-SCHOOL MULTIPLICATION

Algorithm description (informal*):

compute partial products (using multiplication & “carries” for digit overflows), and add all (properly shifted) partial products together

$$\begin{array}{r} 45 \\ \times 63 \\ \hline 135 \\ 2700 \\ \hline 2835 \end{array}$$

** This is not a good example of what your algorithm descriptions should look like on HW/exams*

GRADE-SCHOOL MULTIPLICATION

$$\begin{array}{r} 45123456678093420581217332421 \\ \times 63782384198347750652091236423 \\ \hline \end{array}$$

) :

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$$\begin{array}{r} \overbrace{45123456678093420581217332421}^{n \text{ digits}} \\ \times 63782384198347750652091236423 \\ \hline \end{array}$$

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How efficient is this algorithm?

(How many single-digit operations are required?)

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(How many single-digit operations
in the worst case?)

calculating n partial products: $\sim 2n^2$ ops
(at most n multiplications & n additions per
partial product)

adding n partial products: $\sim 2n^2$ ops
(a bunch of additions & “carries”)

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$\sim 4n^2$ operations in the worst case

GRADE-SCHOOL MULTIPLICATION

n digits

$$\begin{array}{r} 4512345 \\ \times 6378 \\ \hline \end{array}$$

partial products: $\sim 2n^2$ ops
(multiplications & n additions per partial product)

$$\begin{array}{r} 17332421 \\ \times 236423 \\ \hline \end{array}$$

partial products: $\sim 2n^2$ ops
(a bunch of additions & “carries”)

THE QUESTION IS...
**CAN WE DO
BETTER?**

How efficient is this algorithm?
(How many single-digit operations in the worst case?)

$\sim 4n^2$ operations in the worst case

WHAT EXACTLY DOES “BETTER” MEAN?

Is **$1000000n$** operations better than $4n^2$?

Is **$0.000001n^3$** operations better than $4n^2$?

Is **$3n^2$** operations better than $4n^2$?

WHAT EXACTLY DOES “BETTER” MEAN?

Is $1000000n$ operations better than $4n^2$?

Is $0.000001n^3$ operations better than $4n^2$?

Is $3n^2$ operations better than $4n^2$?

- **The answers for the first two depend on what value n is...**
 - $1000000n < 4n^2$ only when n exceeds a certain value (in this case, 250000)
- **These constant multipliers are too environment-dependent...**
 - An operation could be faster/slower depending on the machine, so $3n^2$ ops on a slow machine might not be “better” than $4n^2$ ops on a faster machine

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INTRODUCING...

ASYMPTOTIC ANALYSIS

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- **Some guiding principles:** we care about how the running time/number of operations *scales* with the size of the input (i.e. the runtime's *rate of growth*), and we want some measure of runtime that's independent of hardware, programming language, memory layout, etc.

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 - Note: details like hardware/language/memory/compiler/etc. could totally be important to real world engineers, but in TheoryLand™, we want to reason about high-level algorithmic approaches rather than lower-level details

ASYMPTOTIC ANALYSIS (High Level Idea)

We'll express the asymptotic runtime of an algorithm using

BIG-O NOTATION

- We would say Grade-school Multiplication “**runs in time $O(n^2)$** ”
 - Informally, this means that the runtime “scales like” n^2
 - We'll discuss the formal definition of Big-O (math-y stuff) in next session

“big-oh of n
squared”
or

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THE POINT OF ASYMPTOTIC NOTATION

suppress constant factors and lower-order terms

too system dependent

irrelevant for large inputs

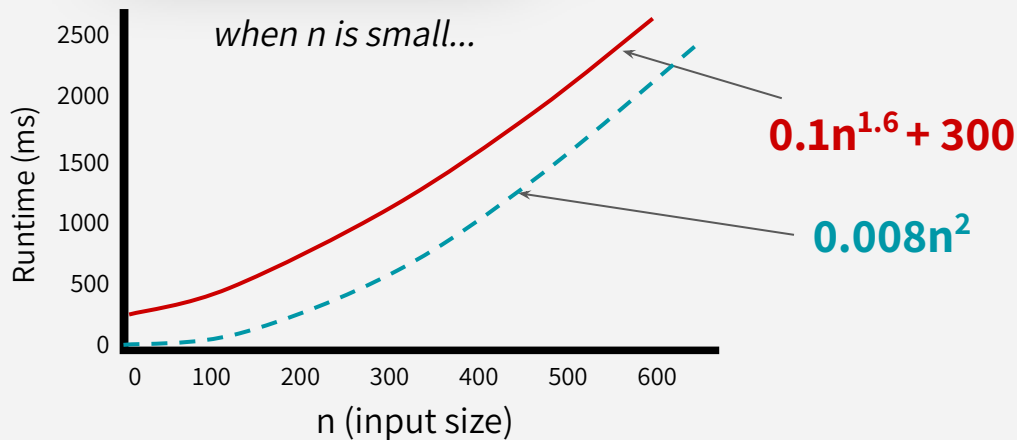
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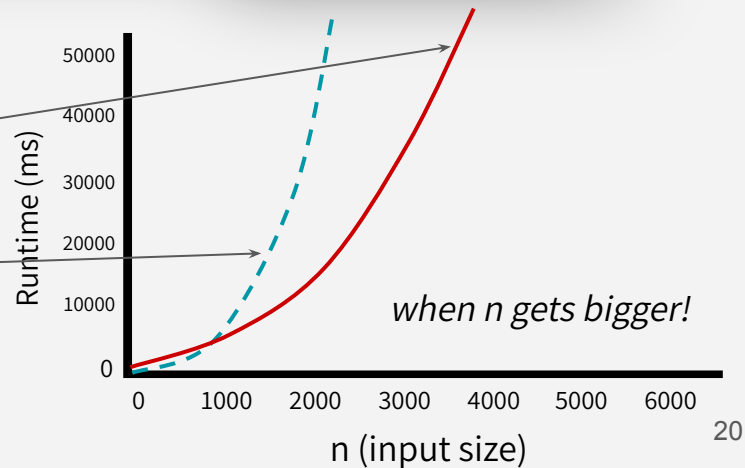
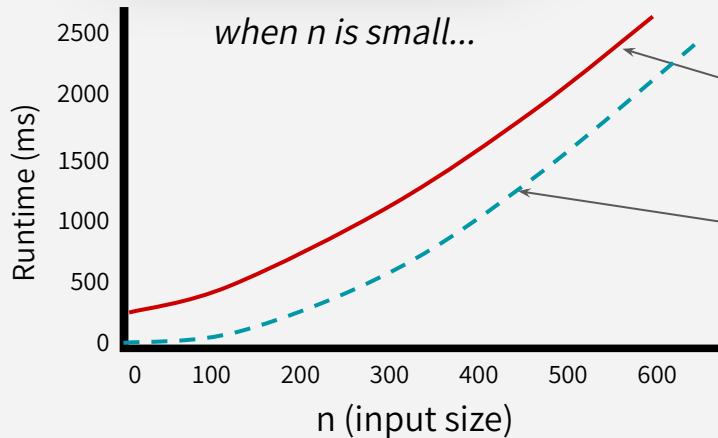
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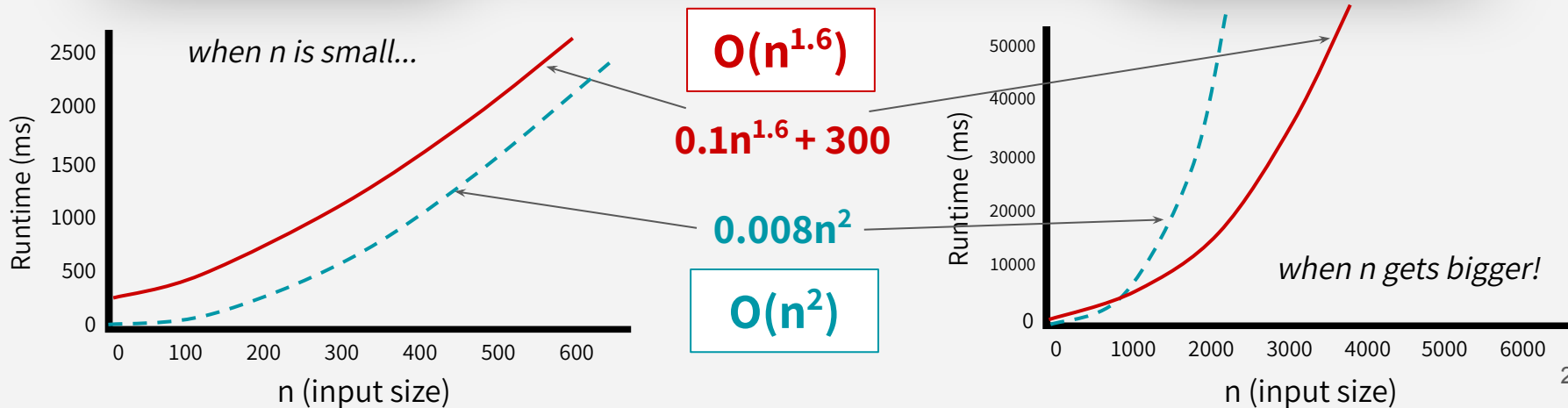
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ASYMPTOTIC ANALYSIS (High Level Idea)

- To compare algorithm runtimes in this class, we compare their Big-O runtimes
 - Ex: a runtime of $O(n^2)$ is considered “better” than a runtime of $O(n^3)$
 - Ex: a runtime of $O(n^{1.6})$ is considered “better” than a runtime of $O(n^2)$
 - Ex: a runtime of $O(1/n)$ is considered “better” than $O(1)$

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So the question is:

**Can we multiply
n-digit integers
faster than $O(n^2)$?**

*Don't worry,
we'll revisit
Asymptotic Analysis
& Big-O stuff more
formally next session!*



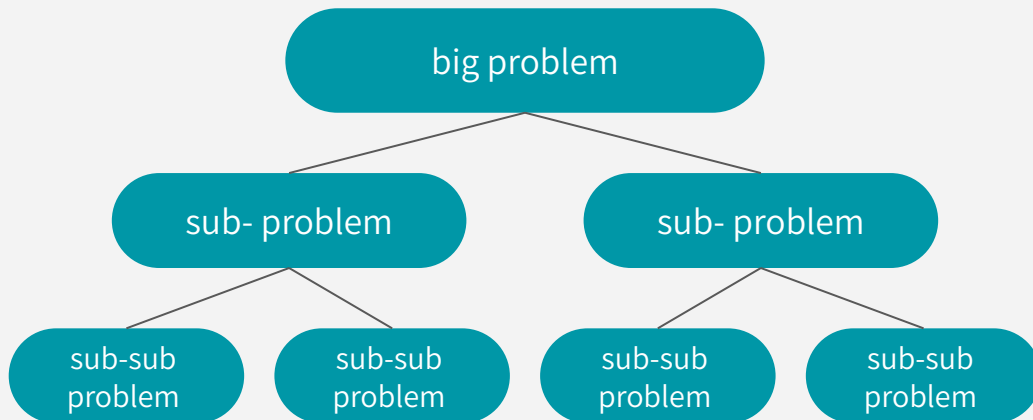
سوال؟

تقسیم و غلبه

اولین روش طراحی الگوریتم ما در این درس!

DIVIDE AND CONQUER

- **An algorithm design paradigm:**
 1. break up a problem into smaller subproblems
 2. solve those subproblems *recursively*
 3. combine the results of those subproblems to get the overall answer



MULTIPLICATION SUBPROBLEMS

- **Original large problem:** multiply 2 n -digit numbers
- **What are the subproblems?** Let's unravel some stuff...

MULTIPLICATION SUBPROBLEMS

- **Original large problem:** multiply two 4-digit numbers
- **What are the subproblems?** Let's unravel some stuff...

$$1234 \times 5678$$

$$= (12 \times 100 + 34) \times (56 \times 100 + 78)$$

$$= (12 \times 56) 100^2 + (12 \times 78 + 34 \times 56) 100 + (34 \times 78)$$

MULTIPLICATION SUBPROBLEMS

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$$1234 \times 5678$$

$$= (12 \times 100 + 34) \times (56 \times 100 + 78)$$

$$= (\underbrace{12 \times 56}_{\text{1}}) 100^2 + (\underbrace{12 \times 78}_{\text{2}} + \underbrace{34 \times 56}_{\text{3}}) 100 + (\underbrace{34 \times 78}_{\text{4}})$$

One 4-digit problem



Four 2-digit subproblems

MULTIPLICATION SUBPROBLEMS

- **Original large problem:** multiply 2 n -digit numbers
- **What are the subproblems?** More generally:

$$\begin{aligned}
 & \begin{bmatrix} x_1 \dots x_{n/2} x_{n/2+1} \dots x_n \end{bmatrix} \times \\
 & \begin{bmatrix} y_1 \dots y_{n/2} y_{n/2+1} \dots y_n \end{bmatrix} \\
 & = (\overset{\textcircled{1}}{a} \times 10^{n/2} + \overset{\textcircled{2}}{b}) \times (\overset{\textcircled{3}}{c} \times 10^{n/2} + \overset{\textcircled{4}}{d}) \\
 & = (\overset{\textcircled{1}}{a} \times \overset{\textcircled{3}}{c}) 10^n + (\overset{\textcircled{1}}{a} \times \overset{\textcircled{4}}{d} + \overset{\textcircled{2}}{b} \times \overset{\textcircled{3}}{c}) 10^{n/2} + (\overset{\textcircled{2}}{b} \times \overset{\textcircled{4}}{d})
 \end{aligned}$$

One n -digit problem
➔
Four $(n/2)$ -digit subproblems

LET'S SEE SOME PSEUDOCODE

MULTIPLY(x, y):

x & y are n-digit numbers

Note: *we're making a big assumption that n is a power of 2 just to make the pseudocode simpler*

LET'S SEE SOME PSEUDOCODE

MULTIPLY(x, y):

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if (n = 1):

return $x \cdot y$

Base case: we can just reference some memorized 1-digit multiplication tables

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LET'S SEE SOME PSEUDOCODE

MULTIPLY(x, y):

x & y are n-digit numbers

if (n = 1):

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write x as $a \cdot 10^{n/2} + b$

write y as $c \cdot 10^{n/2} + d$

a, b, c, & d are
(n/2)-digit numbers

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ac = MULTIPLY(a, c)

ad = MULTIPLY(a, d)

bc = MULTIPLY(b, c)

bd = MULTIPLY(b, d)

These are recursive calls that provide subproblem answers

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return $ac \cdot 10^n + (ad + bc) \cdot 10^{n/2} + bd$

Add them up to get our overall answer!

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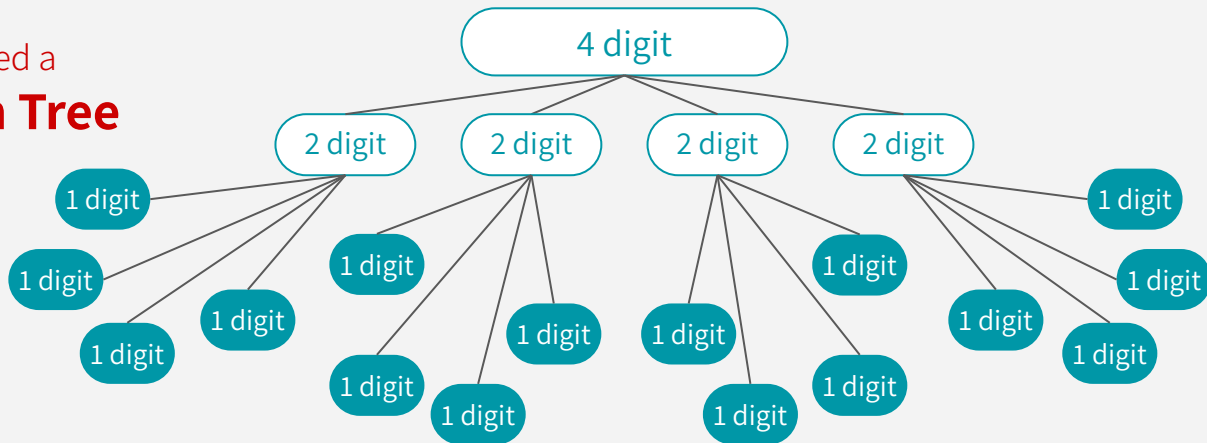
HOW EFFICIENT IS THIS ALGORITHM?

- **Let's start small:** if we're multiplying two 4-digit numbers, how many 1-digit multiplications does the algorithm perform?
 - In other words, how many times do we reach the base case where we actually perform a “multiplication” (a.k.a. a table lookup)?
 - This at least lower bounds the number of operations needed overall

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This is called a
Recursion Tree

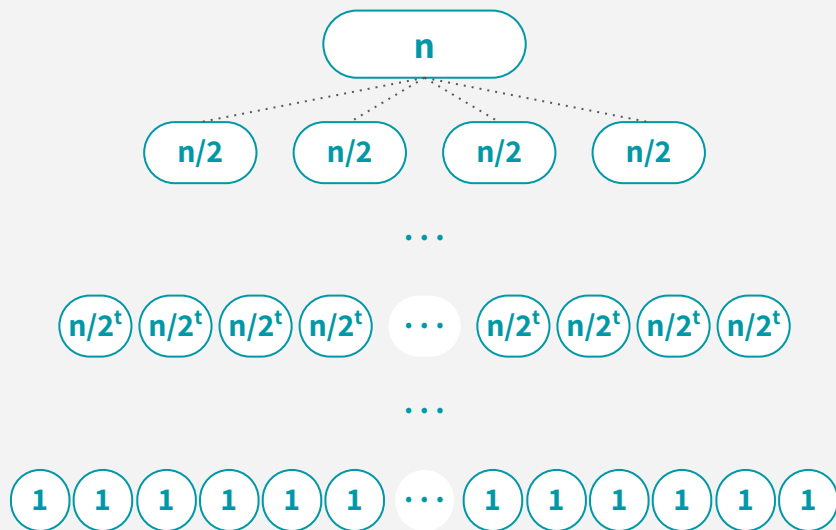


***Sixteen 1-digit
multiplications!***

HOW EFFICIENT IS THIS ALGORITHM?

- **Now let's generalize:** if we're multiplying two n -digit numbers, how many 1-digit multiplications does the algorithm perform?

Recursion Tree



Level 0: 1 problem of size n

Level 1: 4^1 problems of size $n/2$

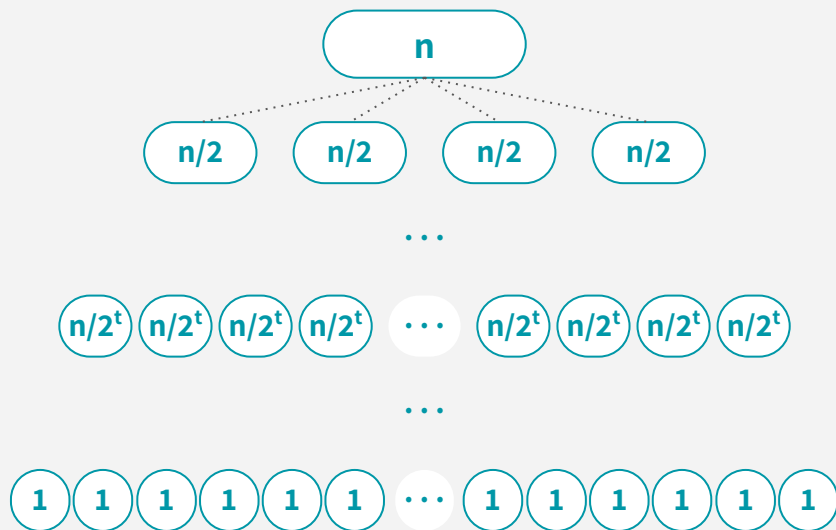
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Level $\log_2 n$: _____ problems of size 1

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Recursion Tree



Level 0: 1 problem of size n

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Level $\log_2 n$: $\frac{n^2}{4}$ problems of size 1

$\log_2 n$ levels

(you need to cut n in half $\log_2 n$ times to get to size 1)

of problems on last level (size 1)

$$= 4^{\log_2 n} = n^{\log_2 4}$$

$$= n^2$$

HOW EFFICIENT IS THIS ALGORITHM?

The running time of this
Divide-and-Conquer
multiplication algorithm
is **at least $O(n^2)$** !

We know there are already n^2 multiplications happening at the bottom level of the recursion tree, so that's why we say "at least" $O(n^2)$

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Is Divide-and-Conquer really that useless?

Karatsuba says no!!!

no!!!



روش ضرب اعداد صحیح کاراتسوبا

سه زیر مساله به جای چهارتا!

CHOOSING SUBPROBLEMS WISELY

$$\begin{aligned} & [\mathbf{x}_1 \cdots \mathbf{x}_{n/2} \mathbf{x}_{n/2+1} \cdots \mathbf{x}_n] \times [\mathbf{y}_1 \cdots \mathbf{y}_{n/2} \mathbf{y}_{n/2+1} \cdots \mathbf{y}_n] \\ &= (\mathbf{a} \times 10^{n/2} + \mathbf{b}) \times (\mathbf{c} \times 10^{n/2} + \mathbf{d}) \\ &= (\mathbf{a} \times \mathbf{c}) 10^n + (\mathbf{a} \times \mathbf{d} + \mathbf{b} \times \mathbf{c}) 10^{n/2} + (\mathbf{b} \times \mathbf{d}) \end{aligned}$$

The subproblems we choose to solve just need to provide these quantities:

\mathbf{ac}

$\mathbf{ad} + \mathbf{bc}$

\mathbf{bd}

*Originally, we assembled these quantities by computing FOUR things: **ac**, **ad**, **bc**, and **bd**.*

KARATSUBA'S TRICK

$$\text{end result} = (\text{ac})10^n + (\text{ad} + \text{bc})10^{n/2} + (\text{bd})$$

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ac & **bd** can be recursively computed as usual

ad + **bc** is equivalent to $(\mathbf{a+b})(\mathbf{c+d}) - \mathbf{ac} - \mathbf{bd}$

$$\begin{aligned} &= (\mathbf{ac} + \mathbf{ad} + \mathbf{bc} + \mathbf{bd}) - \mathbf{ac} - \mathbf{bd} \\ &= \mathbf{ad} + \mathbf{bc} \end{aligned}$$

KARATSUBA'S TRICK

$$\text{end result} = (\text{ac})10^n + (\text{ad} + \text{bc})10^{n/2} + (\text{bd})$$

ac & **bd** can be recursively computed as usual

$$\begin{aligned} \text{ad} + \text{bc} \text{ is equivalent to } & \mathbf{(a+b)(c+d) - ac - bd} \\ & = (ac + ad + bc + bd) - ac - bd \\ & = ad + bc \end{aligned}$$

So, instead of computing **ad** & **bc** as two separate subproblems, let's just compute **(a+b)(c+d)** instead!

OUR THREE SUBPROBLEMS

These *three* subproblems give us everything we need to compute our desired quantities:

- | | |
|---|-------------------|
| ① | ac |
| ② | bd |
| ③ | (a+b)(c+d) |

Assemble our overall product by combining these three subproblems:

$$\begin{array}{ccccccc} (& \text{ac} &) 10^n & + & (& \text{ad} & + & \text{bc} &) 10^{n/2} & + & (& \text{bd} &) \\ & \text{①} & & & \text{③} & - & \text{①} & - & \text{②} & & & \text{②} \end{array}$$

OUR THREE SUBPROBLEMS

These *three* subproblems give us everything we need to compute our desired quantities:

①

ac

②

bd

③

(a+b)(c+d)

(a+b) and (c+d) are both
going to be $n/2$ -digit
numbers!



This means we still
have half-sized
subproblems!

Assemble our overall product by combining these three subproblems:

$$\left(\textcolor{red}{a}\textcolor{green}{c} \right) 10^n + \left(\textcolor{red}{a}\textcolor{blue}{d} + \textcolor{yellow}{b}\textcolor{green}{c} \right) 10^{n/2} + \left(\textcolor{yellow}{b}\textcolor{blue}{d} \right)$$

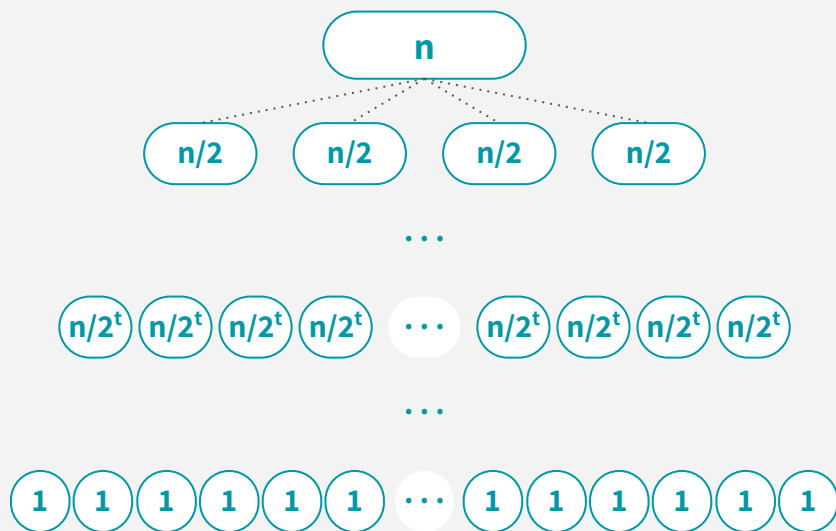
①

③ - ① - ②

②

WHAT'S THE RUNTIME?

This was the Recursion Tree + Analysis from Divide-and-Conquer Attempt 1:



Level 0: 1 problem of size n

Level 1: 4^1 problems of size $n/2$

Level t : 4^t problems of size $n/2^t$

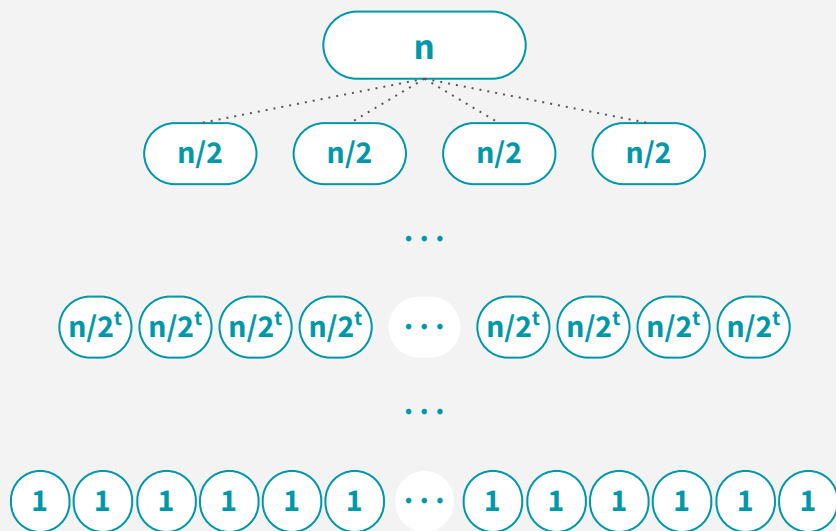
Level $\log_2 n$: n^2 problems of size 1

$\log_2 n$ levels
(you need to cut n
in half $\log_2 n$ times
to get to size 1)

**# of problems on
last level (size 1)**
 $= 4^{\log_2 n} = n^{\log_2 4}$
 $= n^2$

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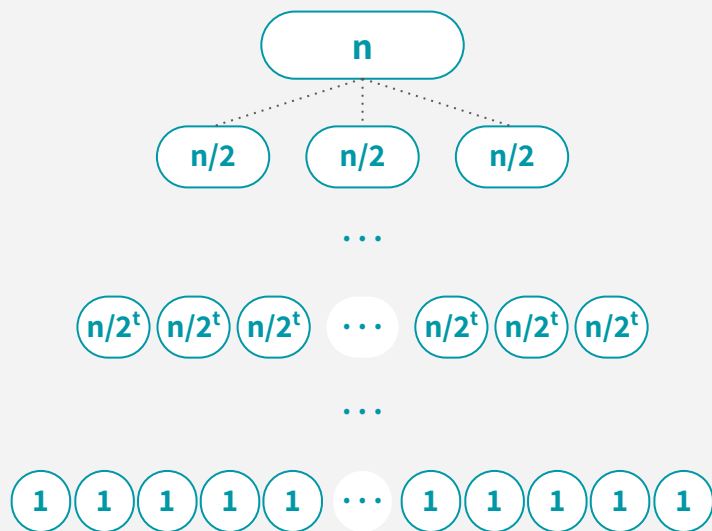
**# of problems on
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$$\begin{aligned} &= 4^{\log_2 n} = n^{\log_2 4} \\ &= n^2 \end{aligned}$$

For Karatsuba's, we'll replace the branching factor of 4 with a 3! \Rightarrow

WHAT'S THE RUNTIME?

Karatsuba Multiplication Recursion Tree



Level 0: 1 problem of size n

Level 1: 3^1 problems of size $n/2$

Level t : 3^t problems of size $n/2^t$

Level $\log_2 n$: $\underline{n^{1.6}}$ problems of size 1

$\log_2 n$ levels

(you need to cut n in half $\log_2 n$ times to get to size 1)

of problems on last level (size 1)

$$= 3^{\log_2 n} = n^{\log_2 3}$$

$$\approx n^{1.6}$$

Thus, the runtime is $O(n^{1.6})$!

WHAT'S THE RUNTIME?

Karatsuba Multiplication Recursion Tree

NOTE: I know it looks like we didn't account for the work done on higher levels in the recursion tree, but as we'll learn later, the work on the last level actually dominates *in this particular recursion tree!*

...



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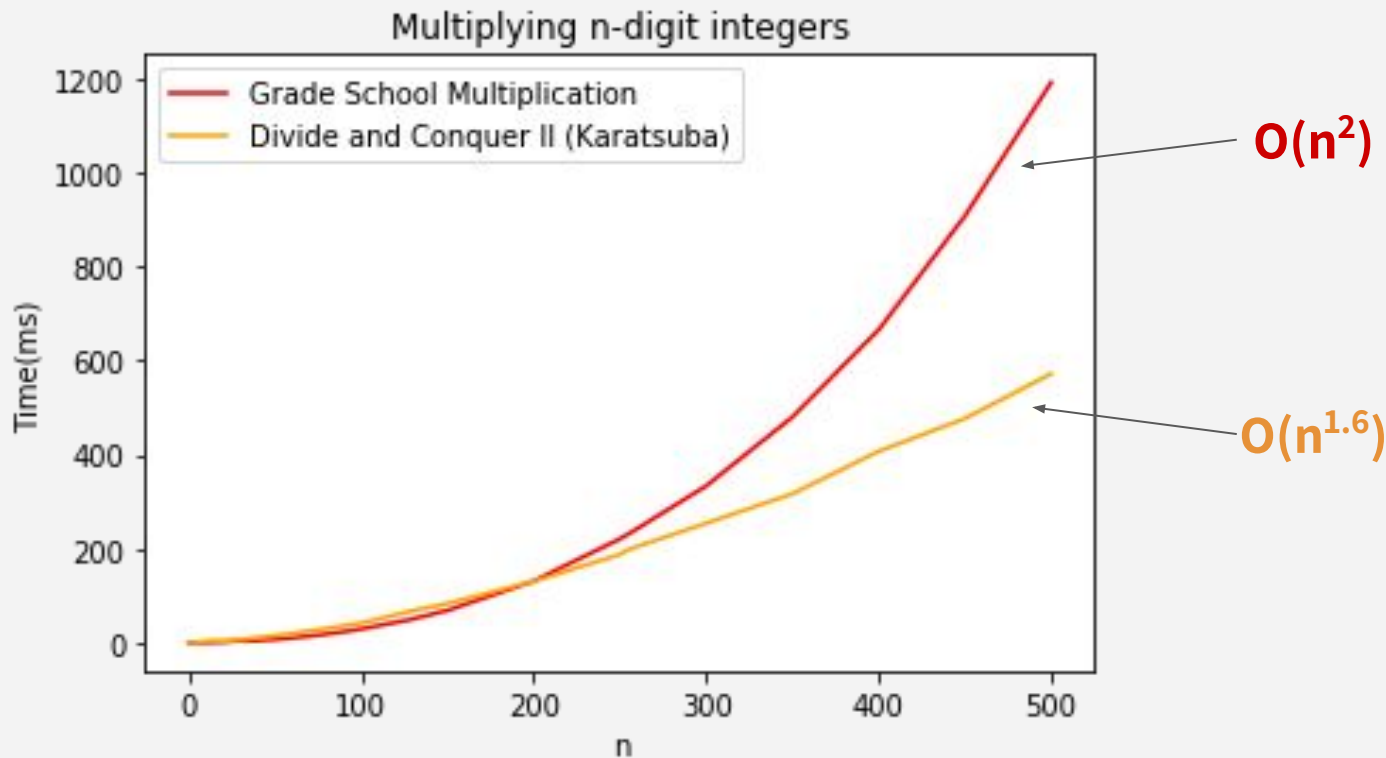
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Thus, the runtime is $O(n^{1.6})$!

IT WORKS IN PRACTICE TOO!



CAN WE DO BETTER?

- **Toom-Cook (1963):** another Divide & Conquer! Instead of breaking into three $(n/2)$ -sized problems, break into five $(n/3)$ -sized problems.
 - Runtime: $O(n^{1.465})$
- **Schönhage-Strassen (1971):** uses fast polynomial multiplications
 - Runtime: $O(n \log n \log \log n)$
- **Fürer (2007):** uses Fourier Transforms over complex numbers
 - Runtime: $O(n \log(n) 2^{O(\log^*(n))})$
- **Harvey and van der Hoeven (2019!):** crazy stuff
 - Runtime: $O(n \log(n))$

We won't expect
you to know any of
these algorithms
by the way!



سوال؟