طراحی الگوریتم ها

مبحث هشتم:
کران پایین برای مرتب سازی

سجاد شیرعلی شهرضا

بهار 1402

یکشنبه، 14 اسفند 1401

Based on Stanford CS161 Slides from Summer 2020 by Karey Shi

اطلاع رسانی

- بخش مرتبط کتاب برای این جلسه: 8.1
- مهلت نظرسنجی دوم: صبح یکشنبه هفته آینده، 21 اسفند 1401
- احتمال برگزاری کلاس سه شنبه هفته آینده به صورت مجازی

کران پایین برای مرتب سازی

آیا می توان الگوریتم مرتب سازی بهتر از $O(n\lg n)$ هم طراحی کرد؟

O(n log n) ALGORITHMS WE'VE SEEN

- MergeSort
  - Worst-case $\Theta(n \log n)$ time.
- QuickSort
  - Expected: $\Theta(n \log n)$

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THE QUESTION IS…

CAN WE DO BETTER?

INTRODUCING… SPAGHETTI SORT?

Input: A sequence of real numbers

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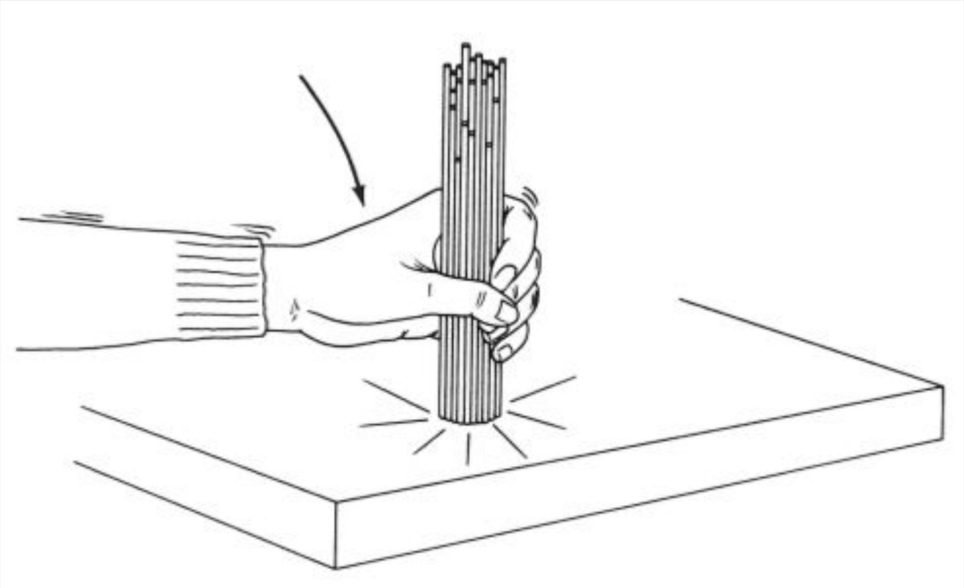

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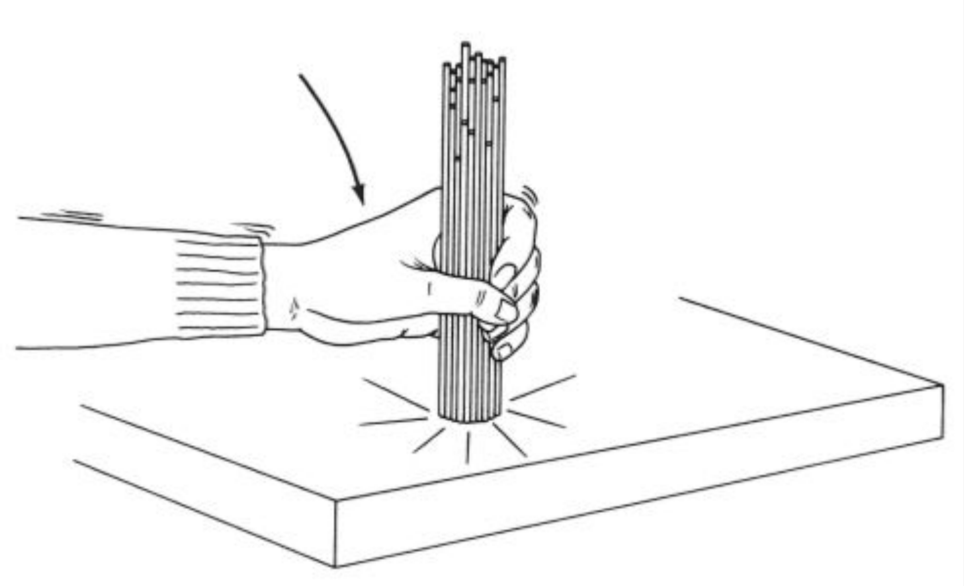
- Take all the spaghetti in your fist, and push their lower sides against the table **O(1)**
- Lower your other hand on the bundle of spaghetti - the first spaghetto you touch is the longest one. Remove it, transcribe its length, and repeat until all spaghetti have been removed. **O(n)**



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- For each number, break off a piece of spaghetti whose length is that number — **O(n)**
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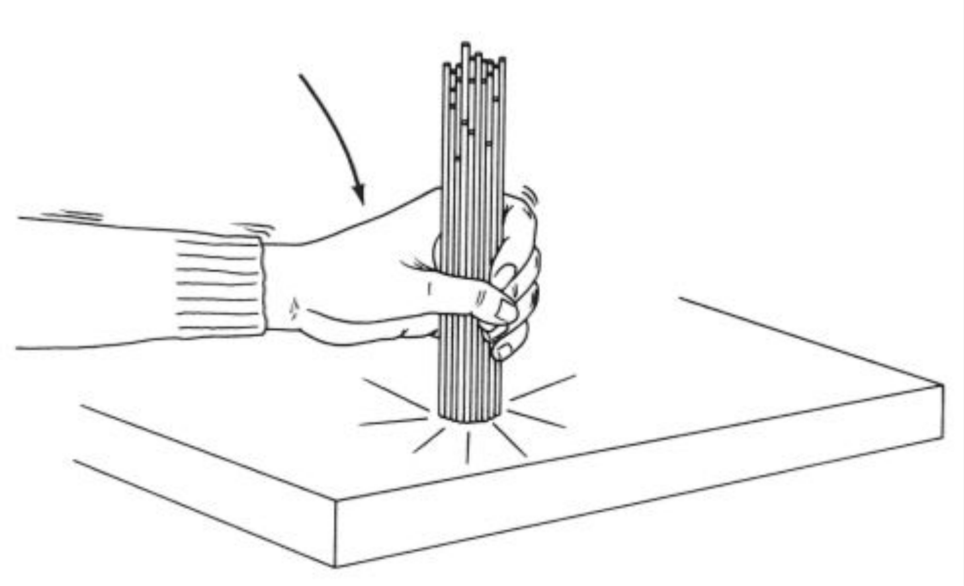
Total Runtime:O(n)



سوال؟

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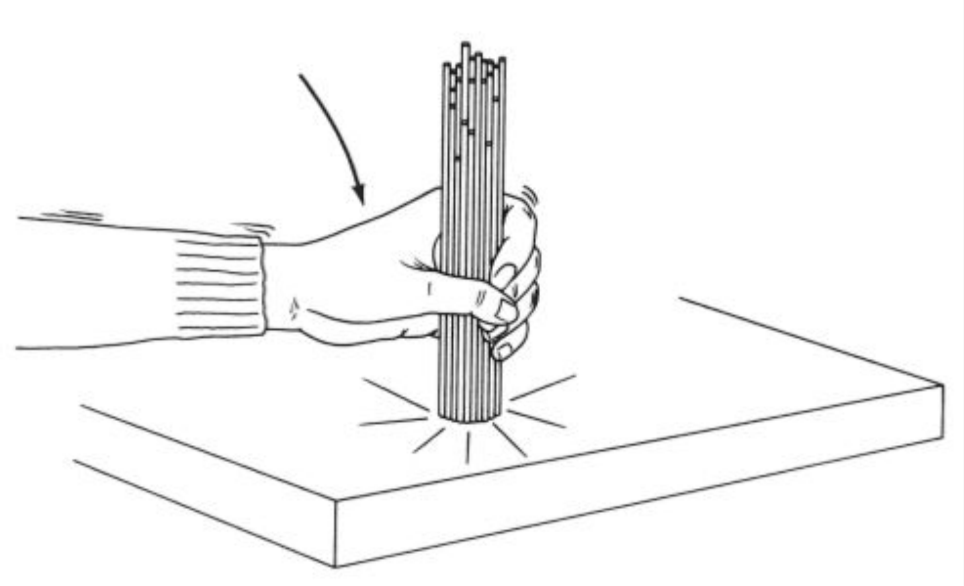

While you shouldn't take this algorithm too seriously… it does raise some important questions!

WHAT IS OUR MODEL OF COMPUTATION?

Input: array of elements

Output: sorted array

Operations allowed: comparisons

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Input: some real numbers

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Operations allowed: breaking spaghetti, dropping on tables, lowering hand

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In a CS class where we're more concerned with what computers can do, the first model seems more reasonable.

سوال؟

COMPARISON-BASED SORTING

- **You want to sort an array of items**
- **You can't access the items' values directly: you can only *compare* two items and find out which is bigger or smaller.**
- Examples: Insertion Sort, MergeSort, QuickSort

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"Comparison-based sorting algorithms" are general-purpose.

The algorithm makes no assumption about the input elements other than that they belong to some totally ordered set.

COMPARISON-BASED SORTING

In other words, the only way you can interact with the array:

For two indices i and j, is A[i] bigger than A[j]?

A[0] A[1] A[2] A[3]

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(I find it helpful to imagine that there is a *genie* who knows what the right order is, and you can only ask this genie this YES/NO question to figure out how to sort the items)

A[0] A[1] A[2] A[3]

Is A[1] bigger than A[3]?

Yes!

A Comparison-based Sorting Algorithm

All-knowing Genie

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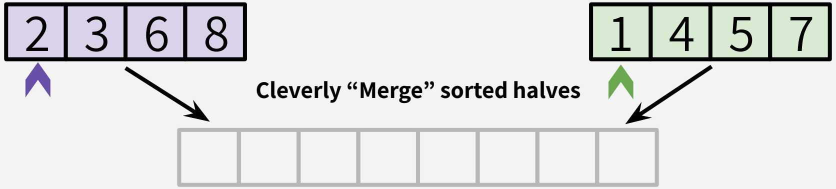

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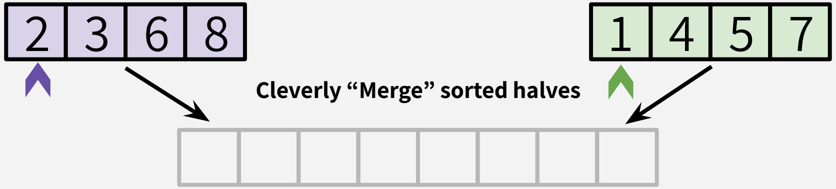

Is 2 bigger than 1 ?

MergeSort algorithm

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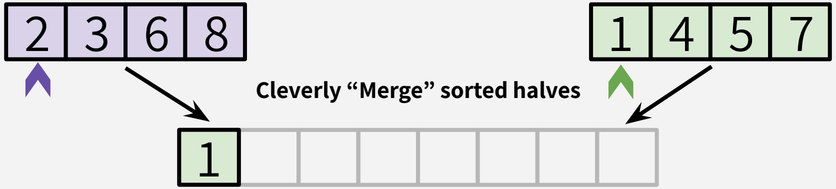

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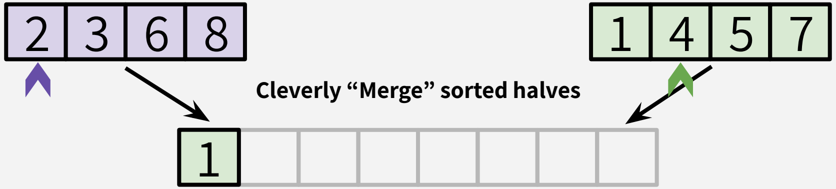

Is 2 bigger than 4 ?

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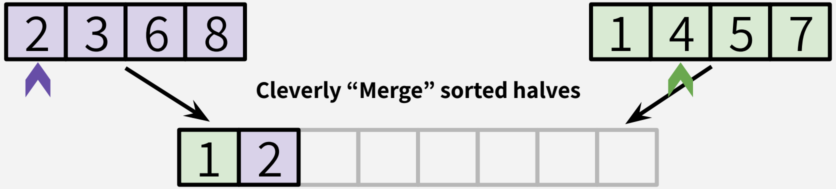

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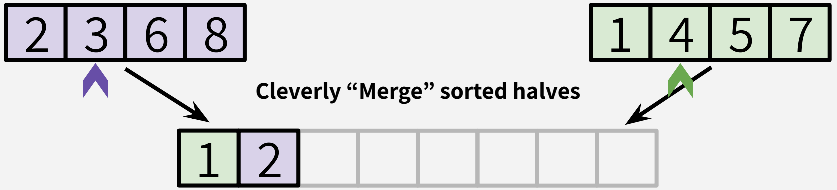

Is 3 bigger than 4 ?

MergeSort algorithm

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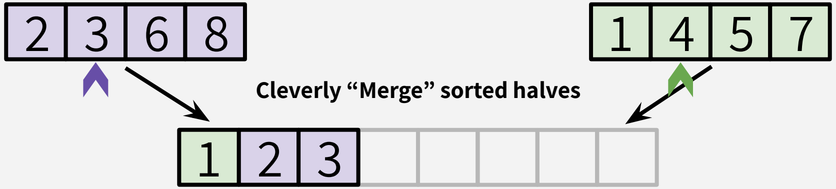

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MergeSort algorithm

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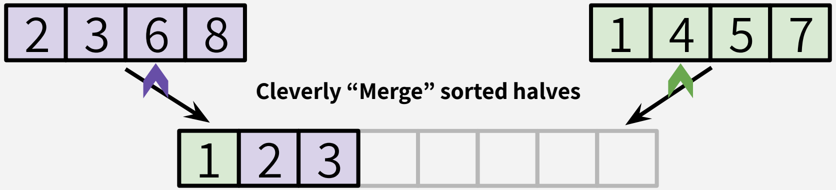

Is 6 bigger than 4 ?

MergeSort algorithm

All-knowing Genie

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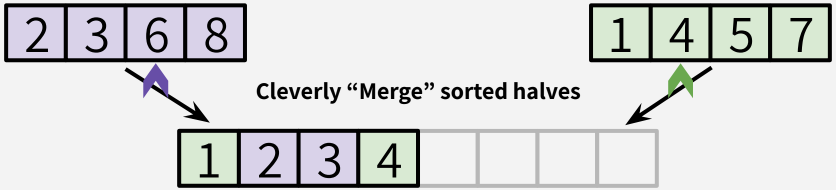

Is 6 **bigger than** 4 **?**

Yes!

MergeSort algorithm

All-knowing Genie

سوال؟

COMPARISON-BASED SORTING

Theorem:

Any deterministic comparison-based sorting algorithm must take $\Omega(n \log n)$ time.

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Think about it like this: this is the input format that your algorithm is ready to accept.

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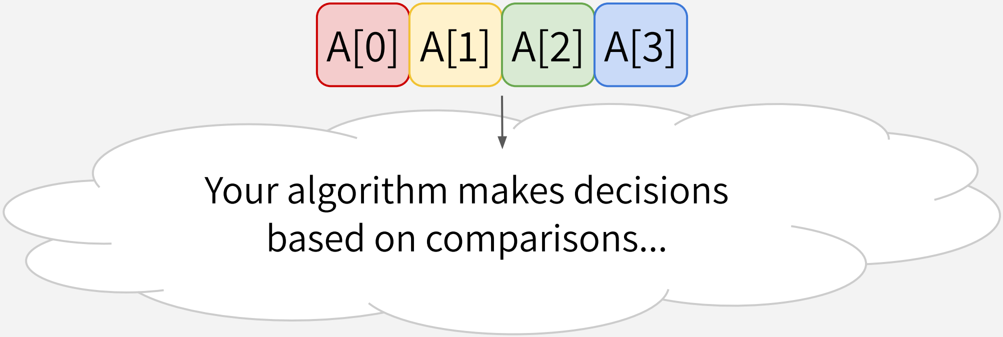

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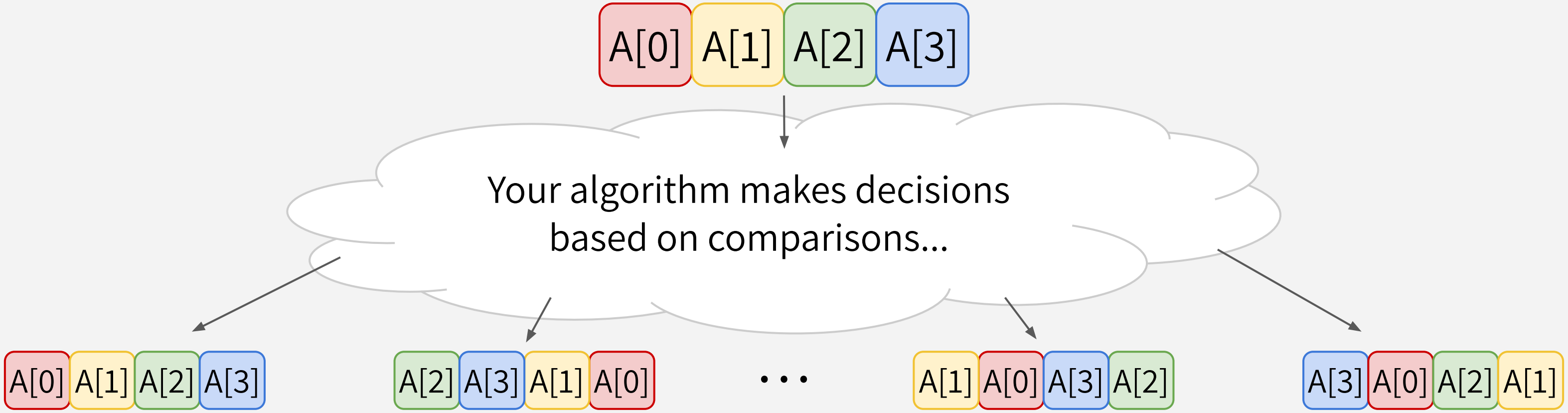

Your algorithm needs to be able to output any one of ______ possible orderings

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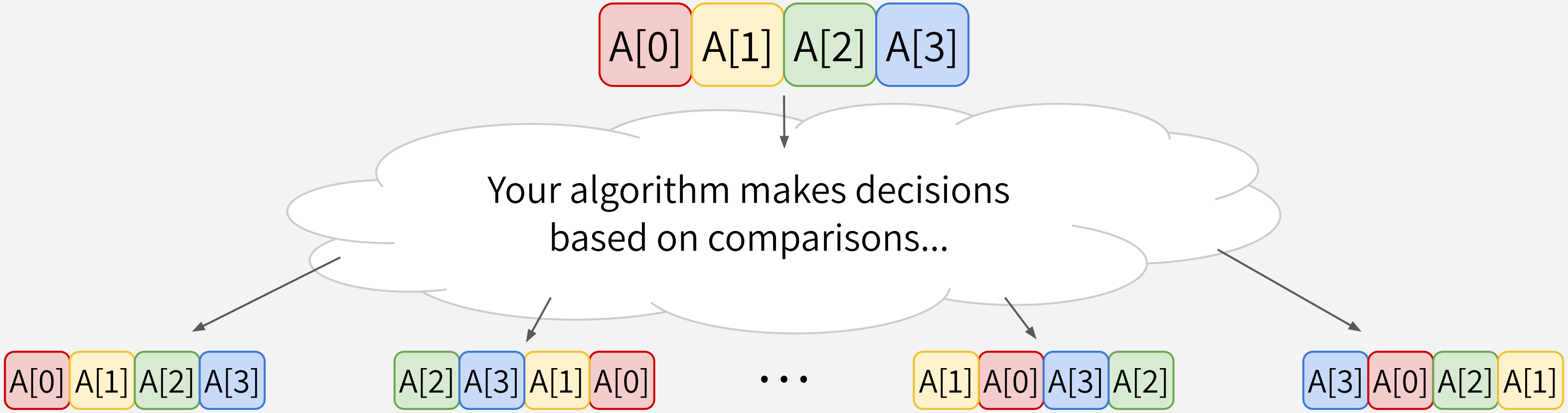


Your algorithm needs to be able to output any one of __**n!**__ possible orderings

COMPARISON-BASED SORTING

A[0] A[1] A[2] A[3]

The algorithm's execution "branches" only as a result of comparisons, since this is the only input-specific information that the algorithm receives.

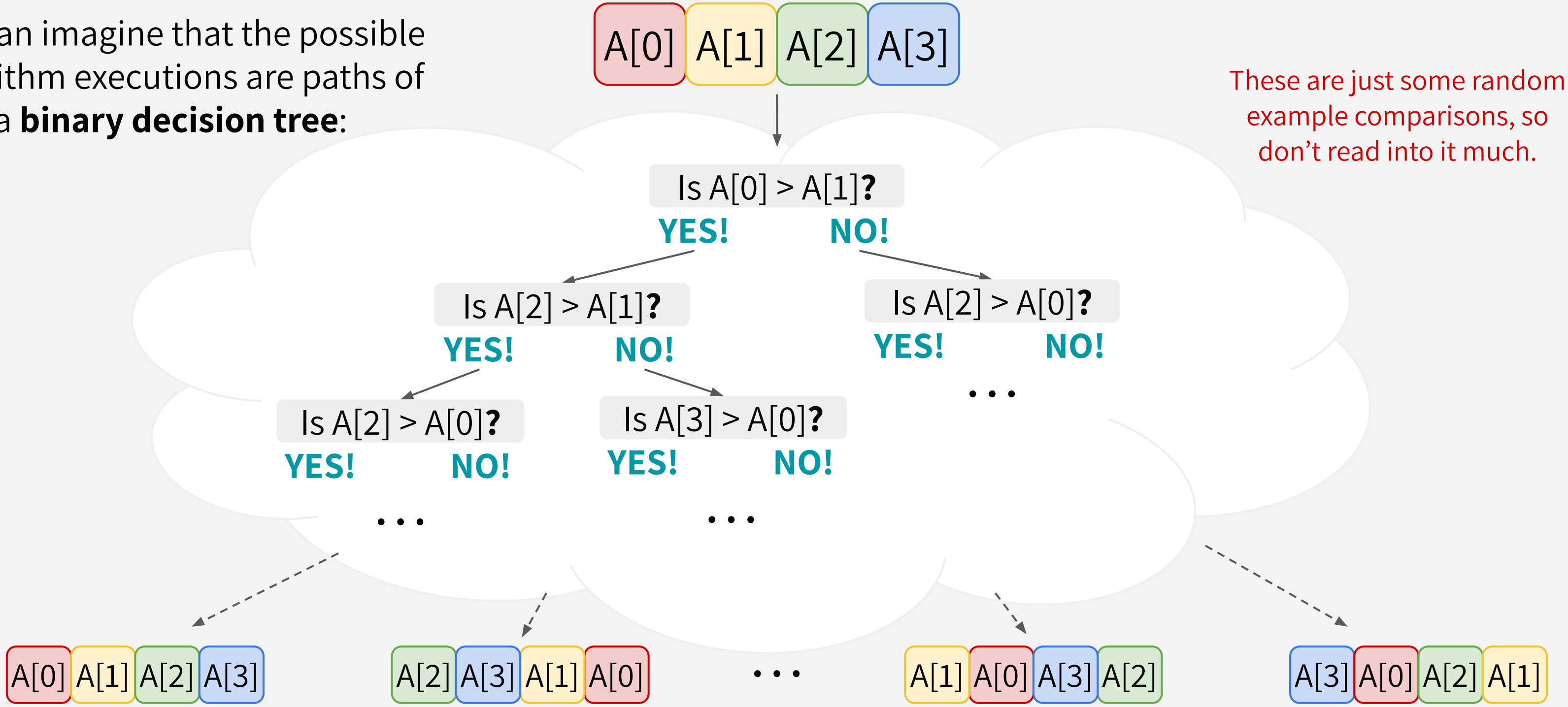
A[0] A[1] A[2] A[3]   A[2] A[3] A[1] A[0]   . . .   A[1] A[0] A[3] A[2]   A[3] A[0] A[2] A[1]

Your algorithm needs to be able to output any one of __**n!**__ possible orderings

COMPARISON-BASED SORTING

You can imagine that the possible algorithm executions are paths of a **binary decision tree**:

These are just some random example comparisons, so don't read into it much.



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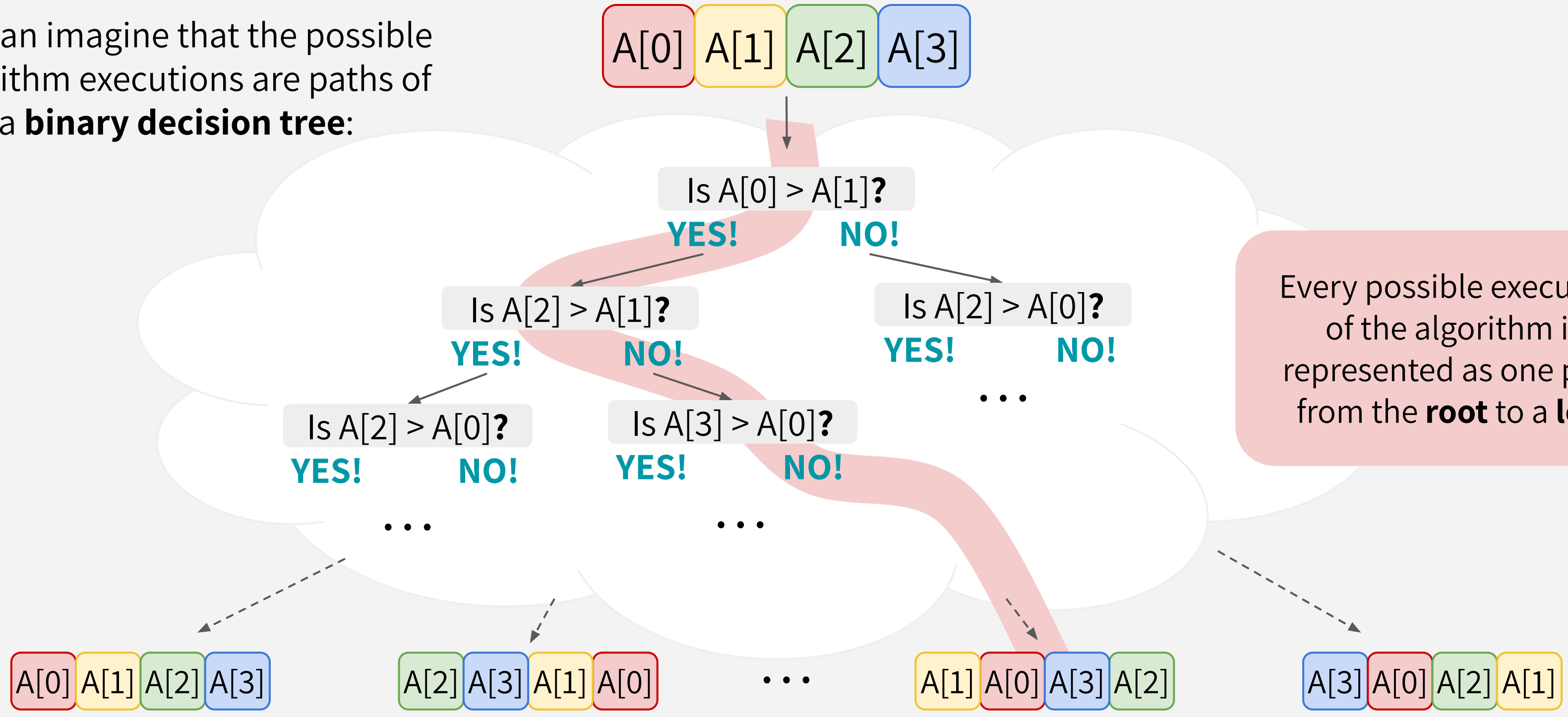

Every possible execution of the algorithm is represented as one path from the **root** to a **leaf**

Your algorithm needs to be able to output any one of __**n!**__ possible orderings

COMPARISON-BASED SORTING

You can imagine that the possible algorithm executions are paths of a **binary decision tree**:

A[0] A[1] A[2] A[3]

This is a binary tree with at least **n!** leaves.

What is the length of the longest possible path?

…very possible execution of the algorithm is …epresented as one path …rom the **root** to a **leaf**

A[0] A[1] A[2] A[3]

A[2] A[3] A[1] A[0]

…

A[1] A[0] A[3] A[2]

A[3] A[0] A[2] A[1]

Your algorithm needs to be able to output any one of __**n!**__ possible orderings

COMPARISON-BASED SORTING

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A[0] A[1] A[2] A[3]

This is a binary tree with at least **n!** leaves.

The shallowest tree with n! leaves is the completely "balanced" one, which has depth log(n!)

Thus, in all binary trees with at least n! leaves, **the longest path has length at least log(n!)**

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Your algorithm needs to be able to output any one of __**n!**__ possible orderings

COMPARISON-BASED SORTING

The longest path has length at least log(n!)

Consequently, any execution of a comparison-based sorting algorithm has to perform at least log(n!) steps.

The worst-case runtime is at least $\log(n!) = \Omega(n \log n)$.

COMPARISON-BASED SORTING

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$$
\begin{aligned}
\log(n!) &= \log 1 + \log 2 + \cdots + \log(n-1) + \log n \\
&\geq \log\left(\frac{n}{2}+1\right) + \log\left(\frac{n}{2}+2\right) + \cdots + \log(n-1) + \log n \\
&\geq \left(\frac{n}{2}\right)\log\left(\frac{n}{2}\right) \\
&= \frac{1}{2}n\big(\log n - \log 2\big) \\
&= \Omega(n \log n)
\end{aligned}
$$

PROOF RECAP

Theorem:

Any deterministic comparison-based sorting algorithm must take $\Omega(n \log n)$ time.

- Any deterministic comparison-based algorithm can be represented as a decision tree with n! leaves
- The worst-case runtime is at least the length of the longest path in the decision tree
- All decision trees with n! leaves have a longest path with length at least $\log(n!) = \Omega(n \log n)$
- So, any comparison-based sorting algorithm must have worst-case runtime at least $\Omega(n \log n)$

THE GOOD NEWS

Theorem:

Any deterministic comparison-based sorting algorithm must take $\Omega(n \log n)$ time.

This bound also applies to the expected runtime of *randomized* comparison-based sorting algorithms!
The proof is out of scope of this class, but it relies on this theorem.

This means that MergeSort is optimal!

(This is one of the cool things about proving lower bounds - we know when we can declare victory!)

THE GOOD NEWS

﴿ إِنَّ ٱلۡإِنسَٰنَ خُلِقَ هَلُوعًا ١٩ ﴾

[سُورَةُ المَعَارِجِ:١٩]

Any deterministic comparis… …n must take $\Omega(n \log n)$ time.

This bound also applies to … …-based sorting algorithms!
The pro… …heorem.

THE QUESTION IS...

CAN WE DO BETTER?

**using a model of computation that's less silly than spaghetti?*

This mea… …optimal!

(This is one of th… …out proving lower bounds - we know when we can declare victory!)

سوال؟