طراحی الگوریتم ها

مبحث شانزدهم:
برش کمینه

سجاد شیرعلی شهرضا

بهار 1402

سه شنبه، 19 اردیبهشت 1402

Based on Stanford CS161 Slides from Summer 2021 by Karey Shi

اطلاع رسانی

- بخش مرتبط کتاب برای این جلسه: 26
- مهلت ارسال تمرین سوم: صبح یکشنبه 24 اردیبهشت

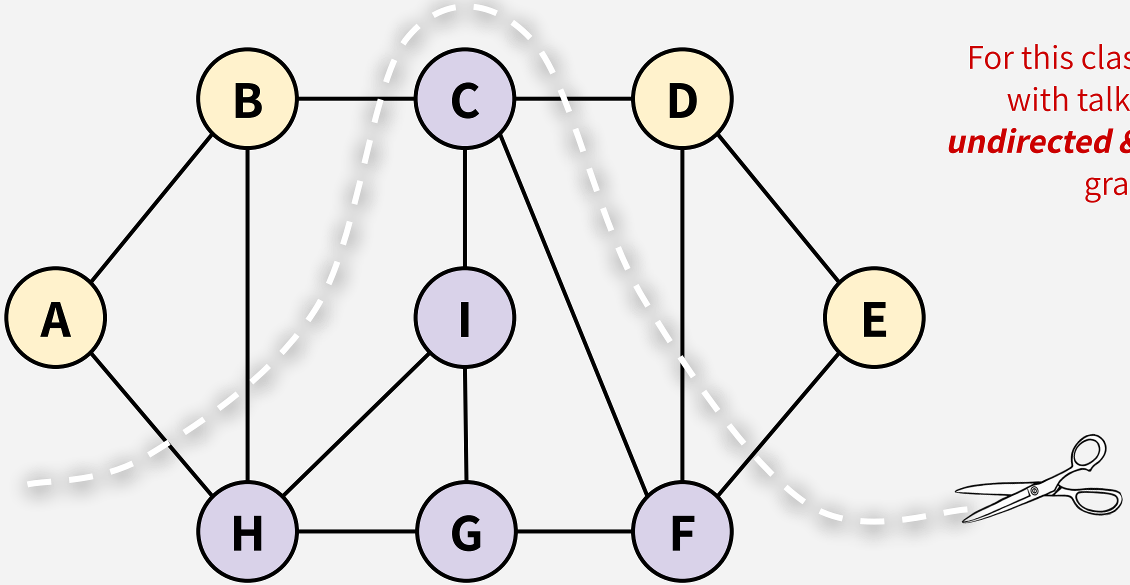
برش کمینه

یک برش کمینه در گراف چیست؟

RECALL: CUTS IN GRAPHS

A **cut** is a partition of the vertices into two nonempty parts.

For this class, we'll stick with talking about ***undirected & unweighted*** graphs.

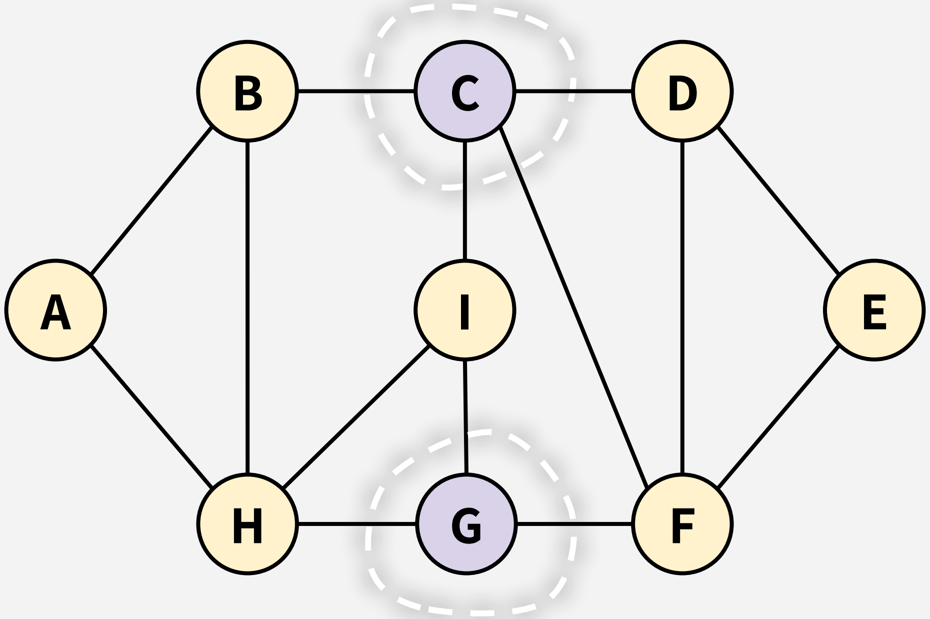


e.g. this is the cut **"{A,B,D,E} and {C,I,F,G,H}"**

RECALL: CUTS IN GRAPHS

A **cut** is a partition of the vertices into two nonempty parts.

This is also a cut!



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EDGES THAT "CROSS" A CUT

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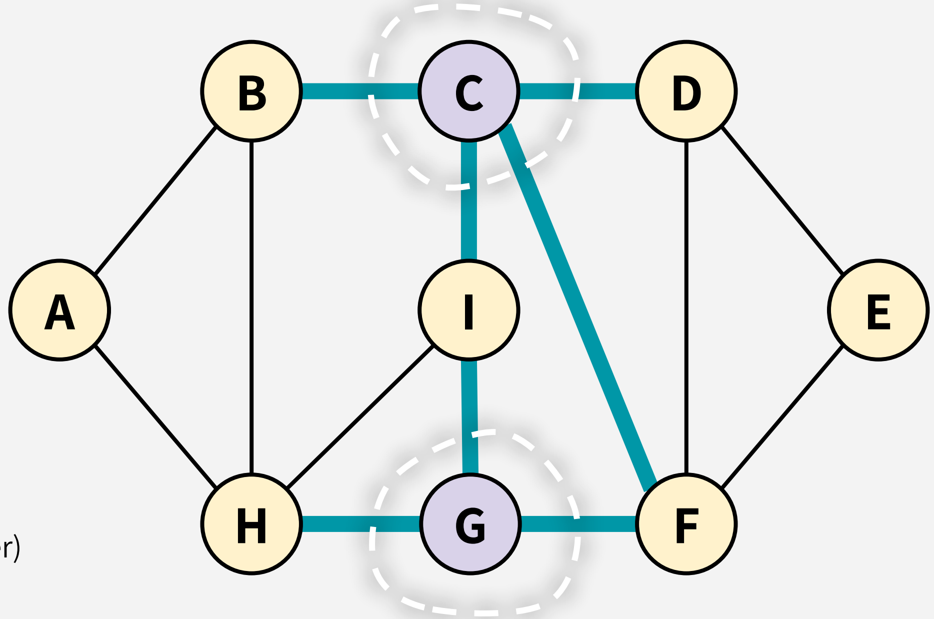
And the **teal edges** are the ones "**crossing the cut**"
(they go from one part to the other)



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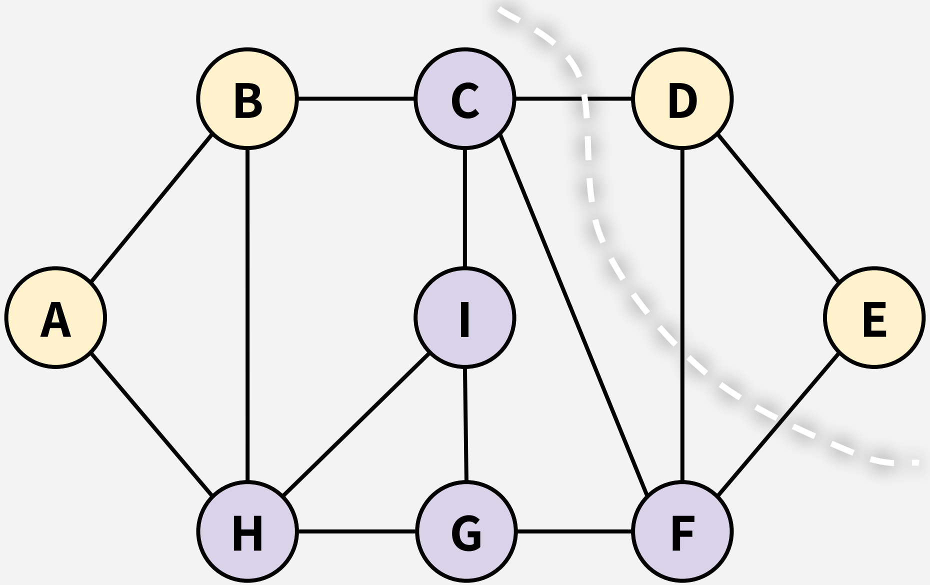

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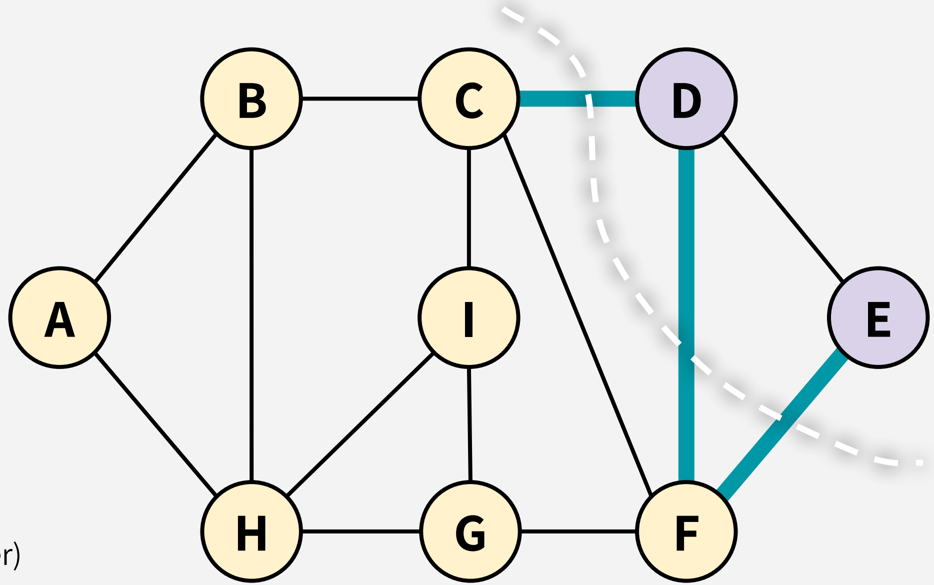
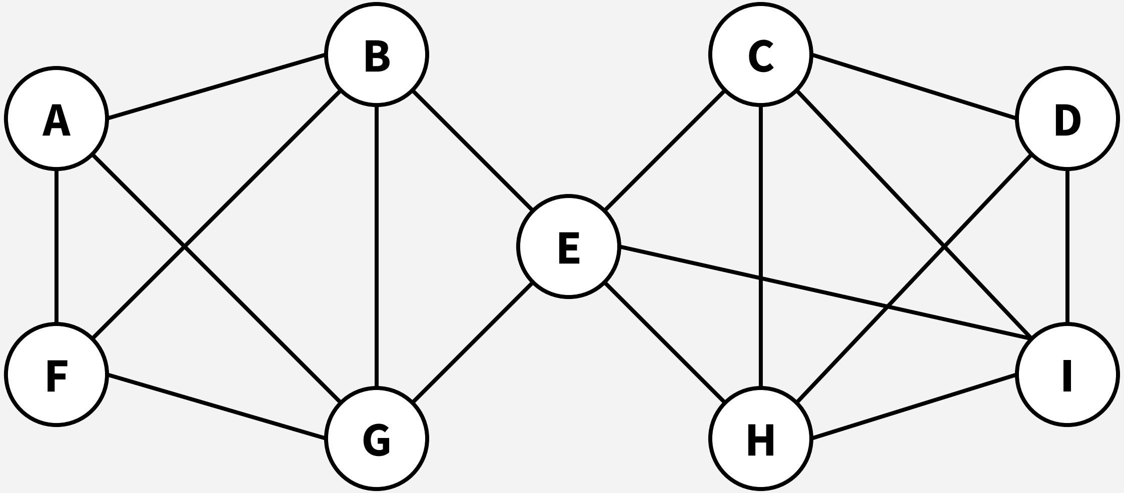


For this class, we'll stick with talking about ***undirected & unweighted*** graphs.

(GLOBAL) MINIMUM CUT

A (global) **minimum cut** is a cut that has the fewest edges possible crossing it.

What is the (global) minimum cut in this graph?



Note: we emphasize "global" because later in class, we'll talk about *s–t minimum cuts*, which is a cut that separates specific nodes s and t (a slightly different concept).

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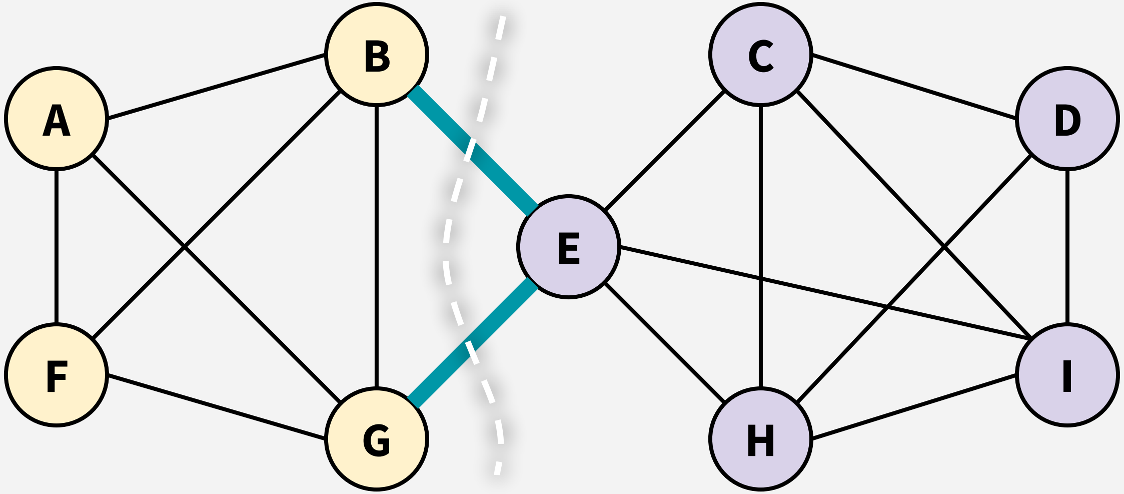

This cut is the (global) minimum cut (it has only 2 edges crossing it)!

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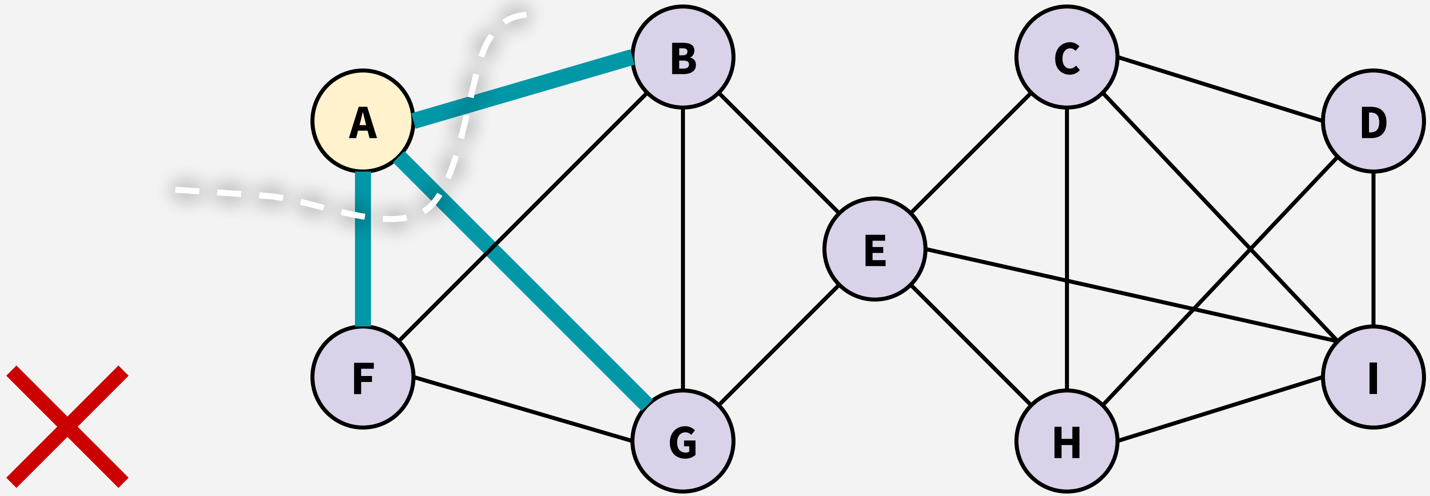
What is the (global) minimum cut in this graph?

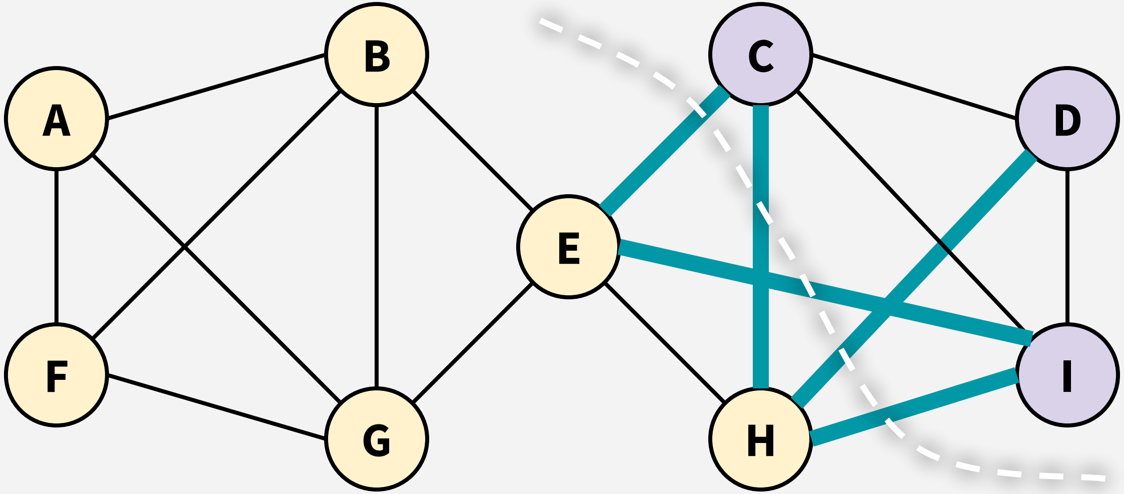


This is ***NOT*** a global min-cut (3 edges cross it, so it's not better than the one on the last slide)

(GLOBAL) MINIMUM CUT

A (global) **minimum cut** is a cut that has the fewest edges possible crossing it.

What is the (global) minimum cut in this graph?



This is also ***NOT*** a global min-cut (5 edges cross it, so it's not better than the cut we already found)

APPLICATIONS?

There are several immediate use cases for finding global minimum cuts!

Clustering

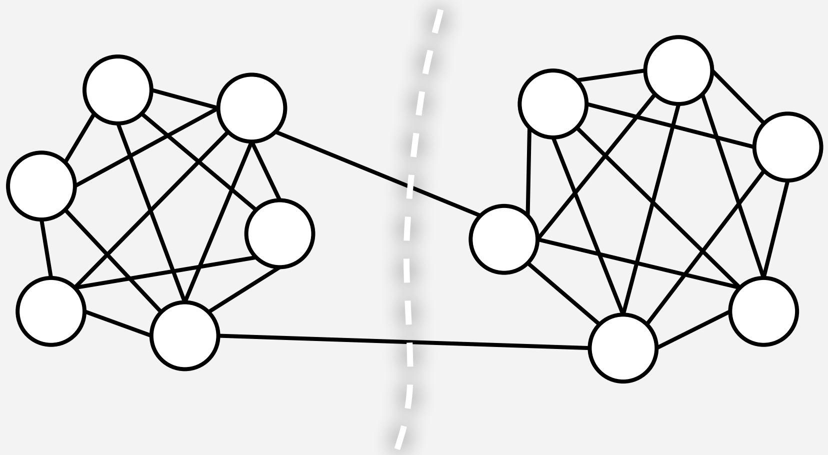
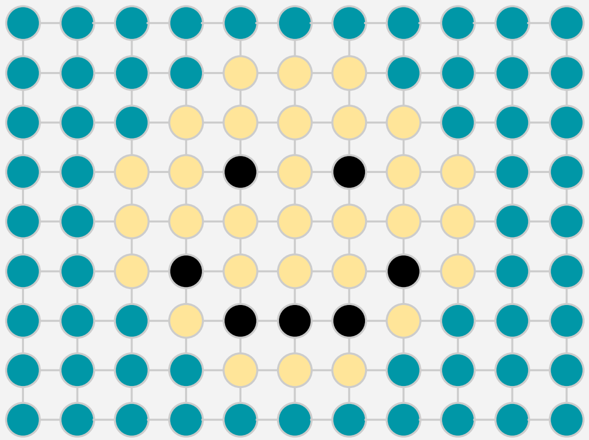


Image Segmentation*



*Although we're only working with unweighted graphs for now, we could represent an image using large edge weights between similar pixels

سوال؟

الگوریتم کارگِر

یک الگوریتم تصادفی برای یافتن برش کمینه سراسری در گراف بدون جهت

LAS VEGAS vs. MONTE CARLO

LAS VEGAS ALGORITHMS

Guarantees correctness!

But the runtime is a random variable.
(i.e. there's a chance the runtime could take awhile)

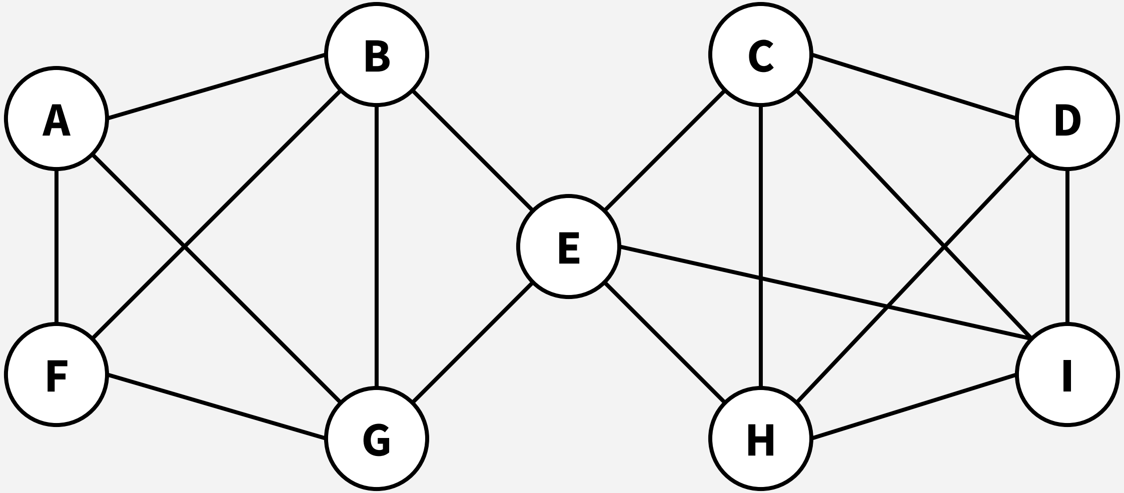
MONTE CARLO ALGORITHMS

Correctness is a random variable.
(i.e. there's a chance the output is wrong)

But the runtime is guaranteed!

KARGER'S ALGORITHM

Pick a random edge, **contract** it, and repeat until you only have 2 vertices left.



As always, let's see an example of the algorithm in action!

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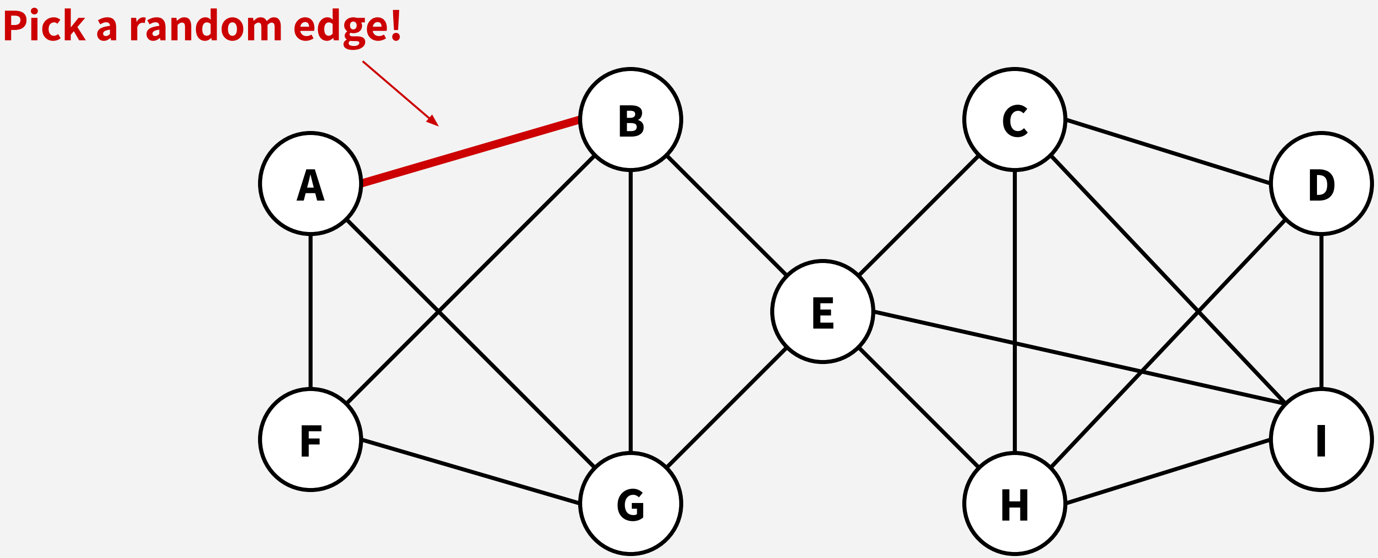

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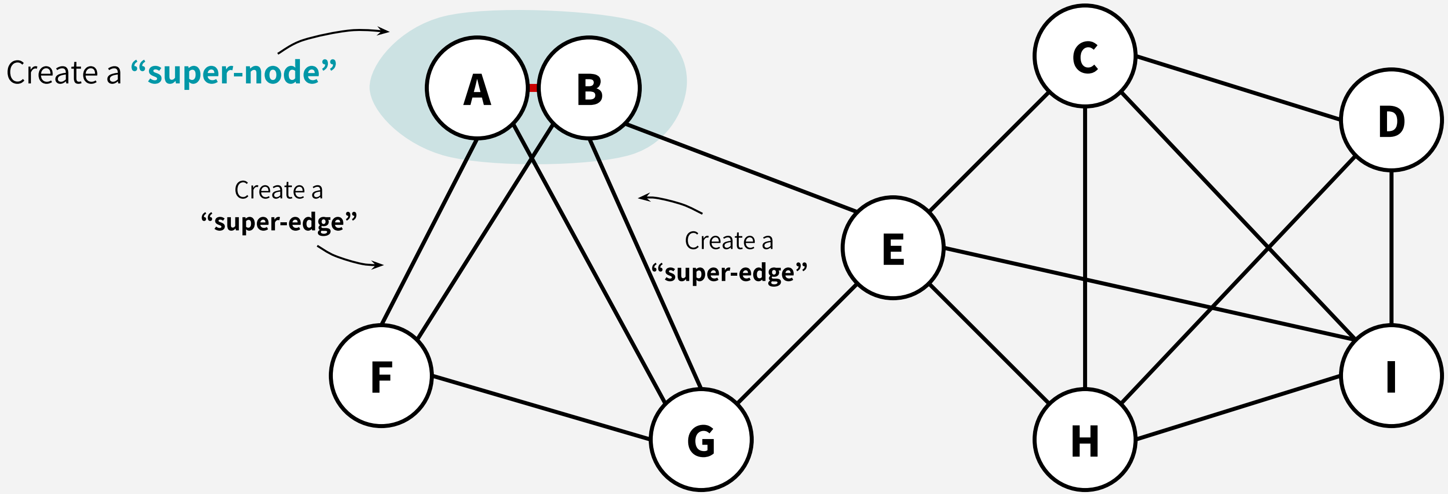
Pick a random edge, **contract** it, and repeat until you only have 2 vertices left.

Now perform an "edge contraction"!



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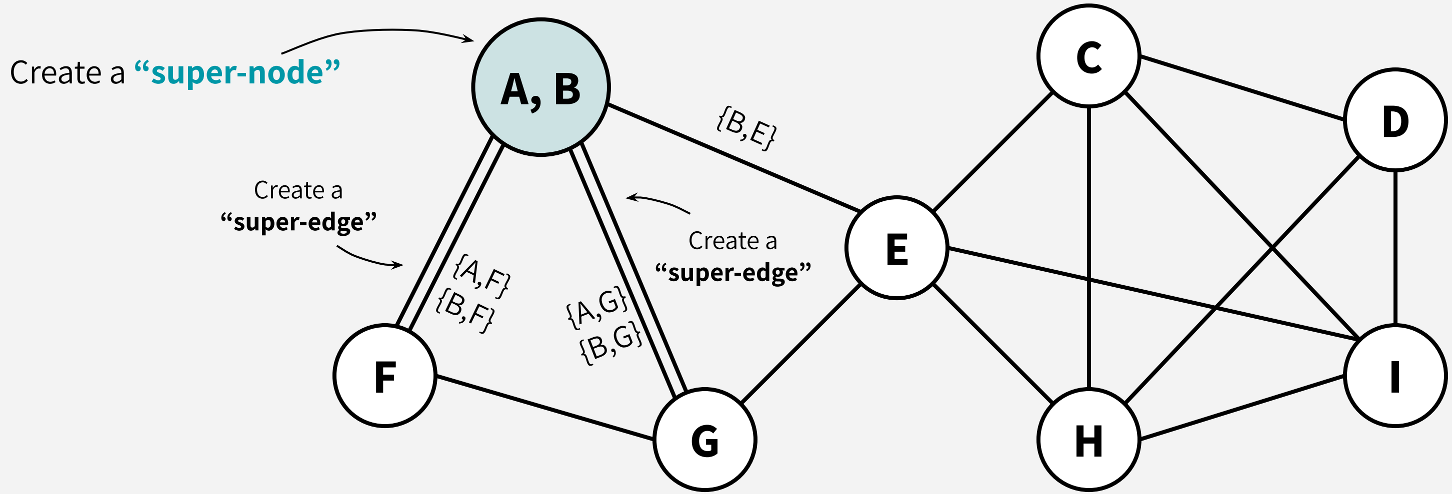

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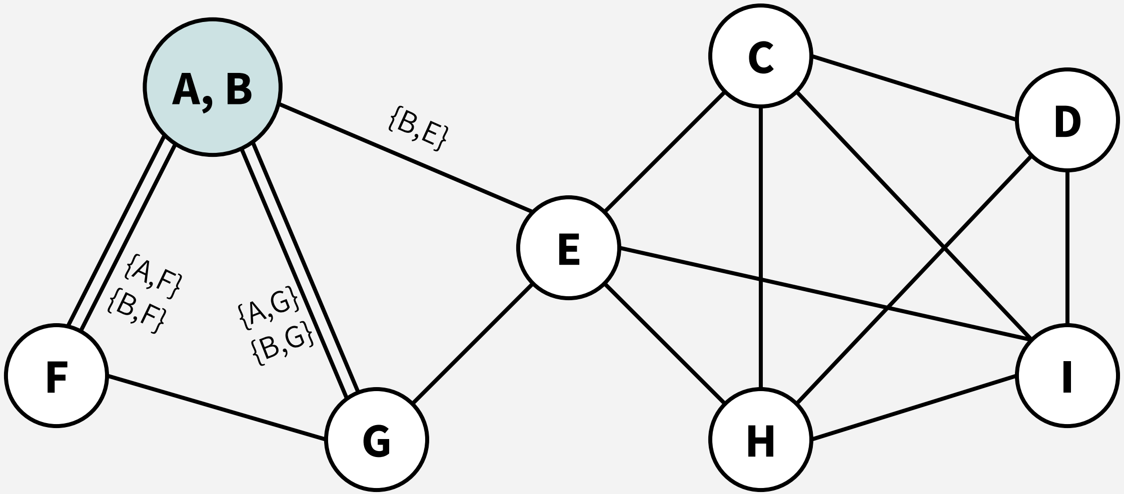
Repeat!



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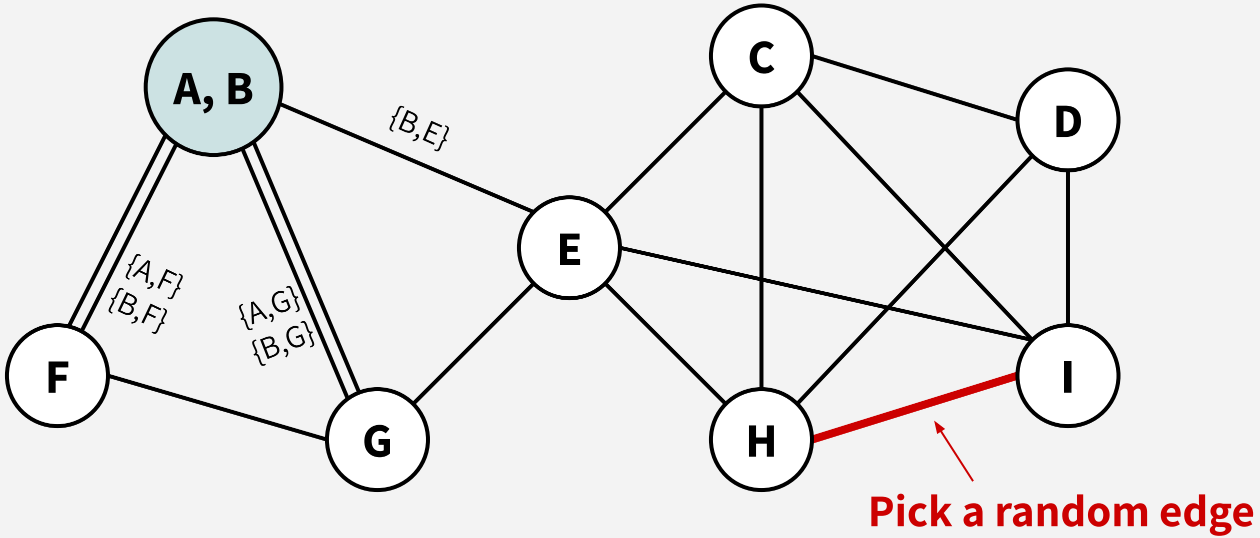

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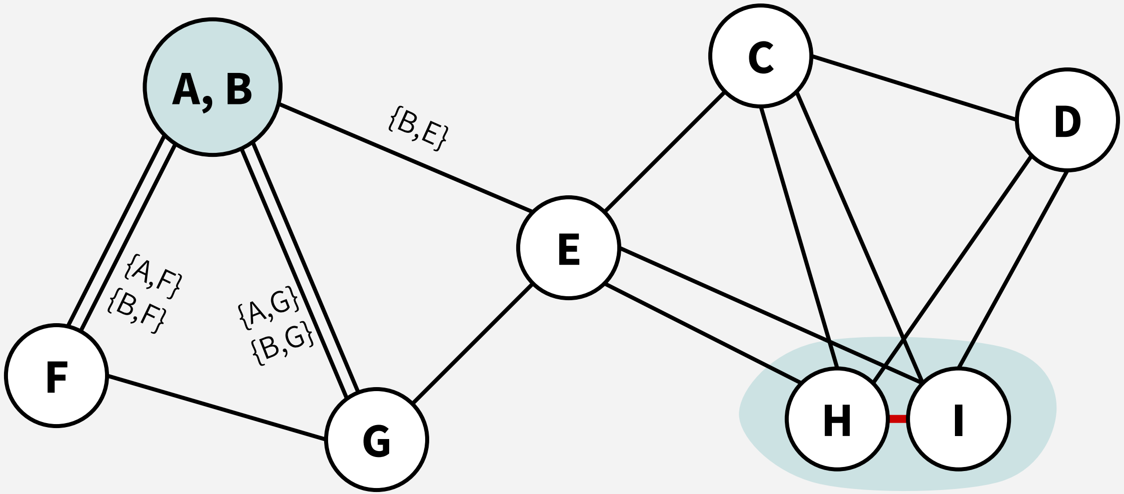

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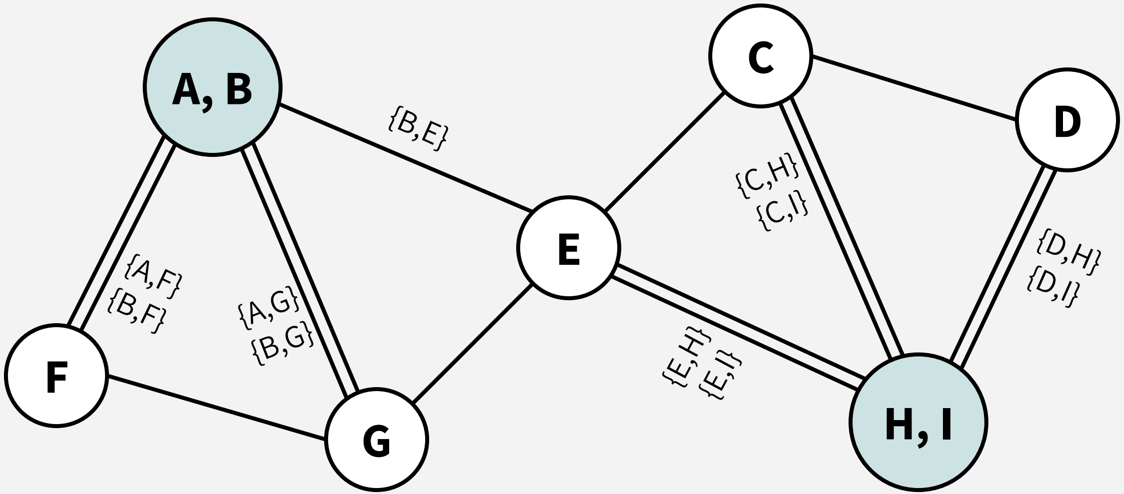

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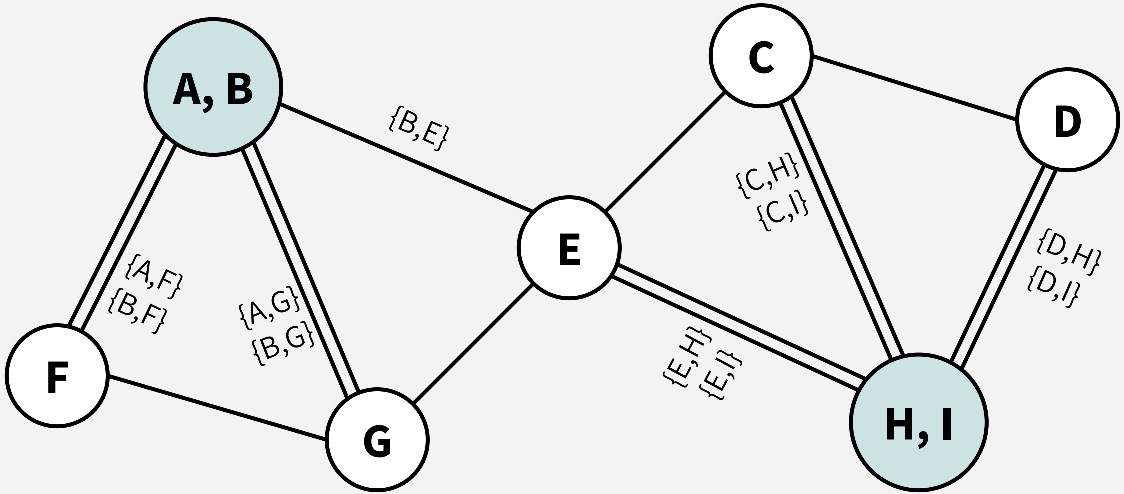

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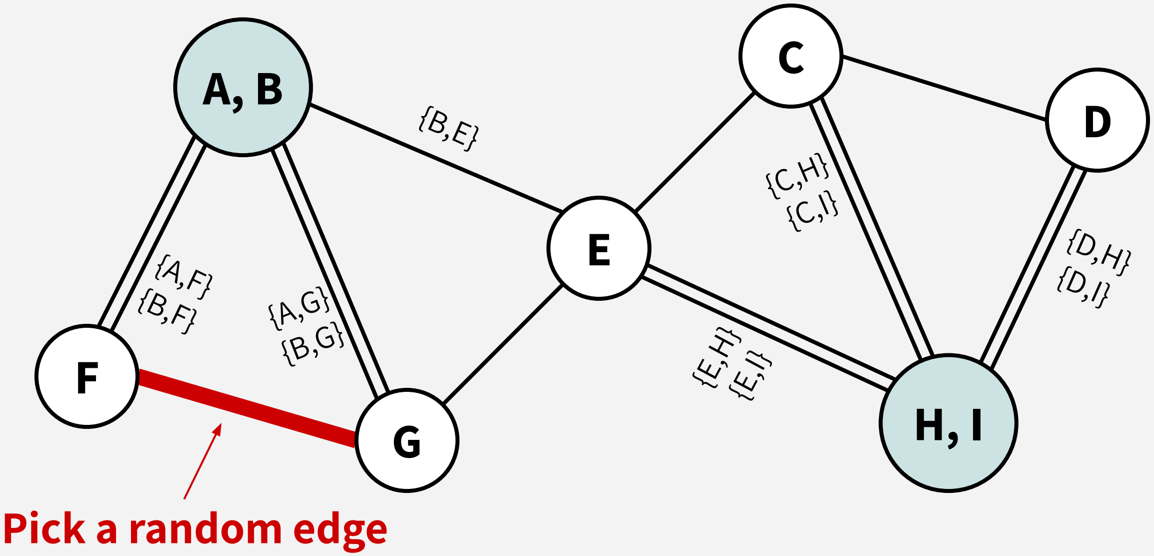

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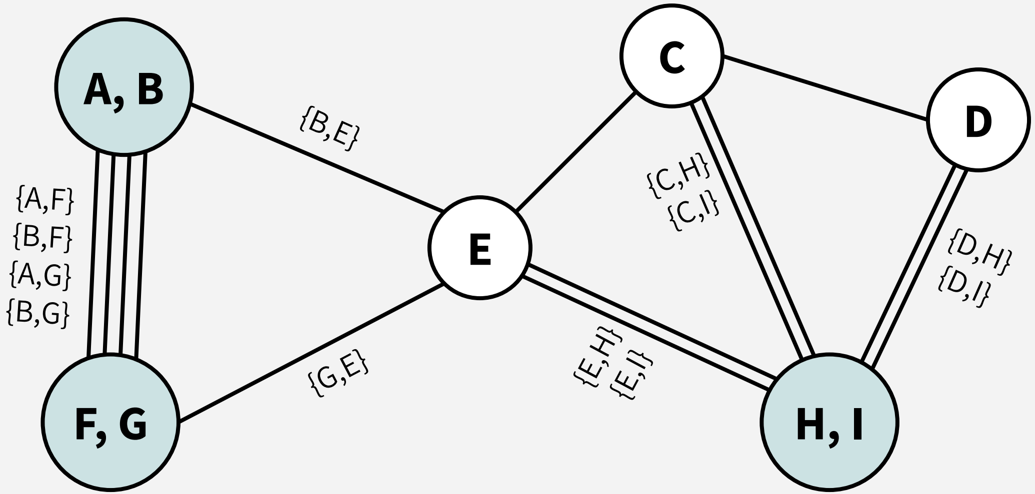

Perform an "edge contraction"

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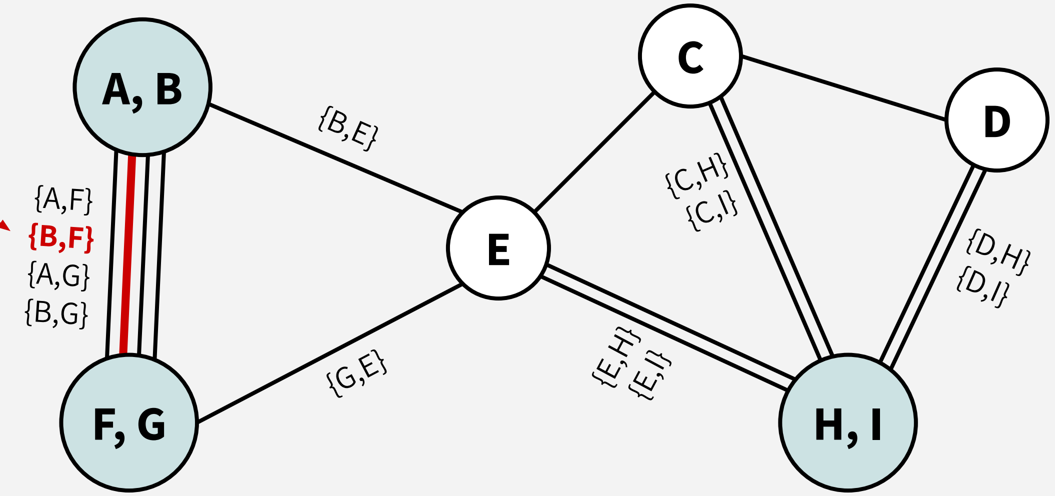

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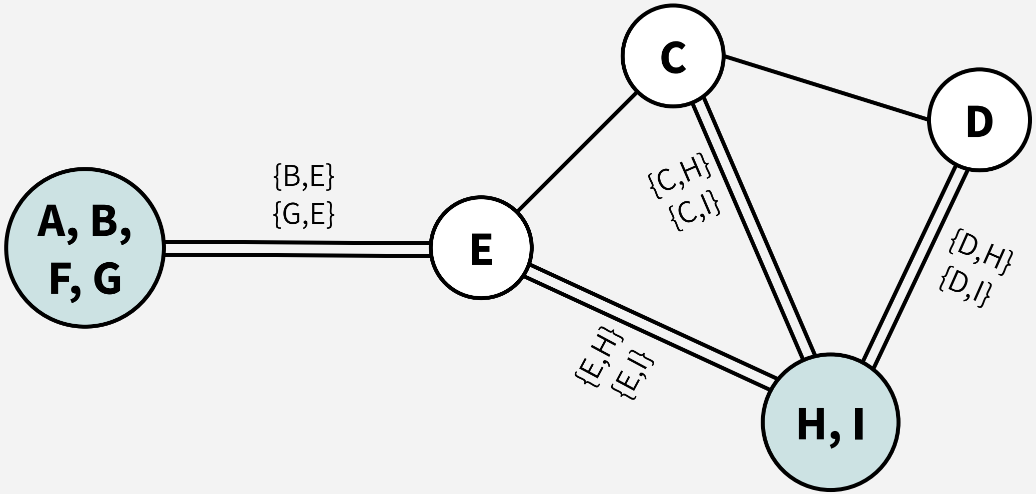

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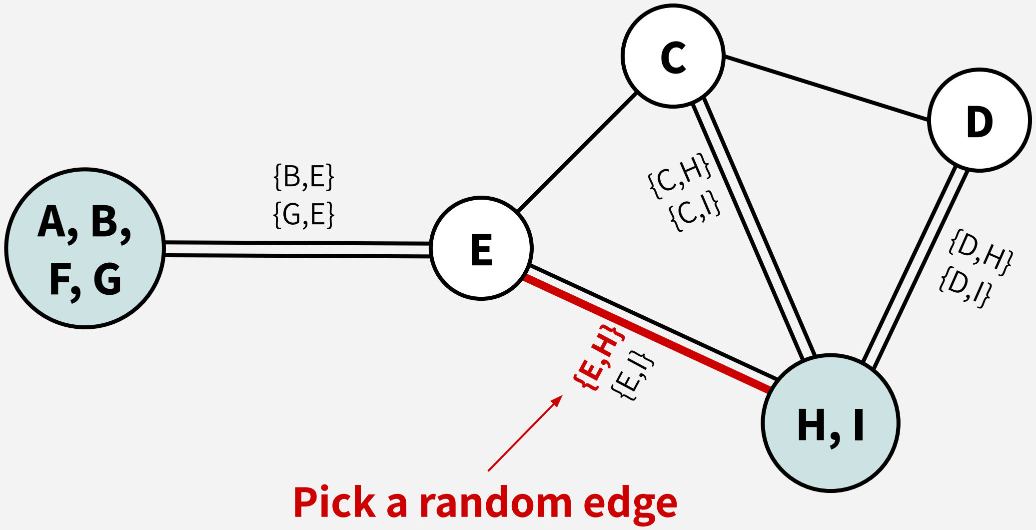

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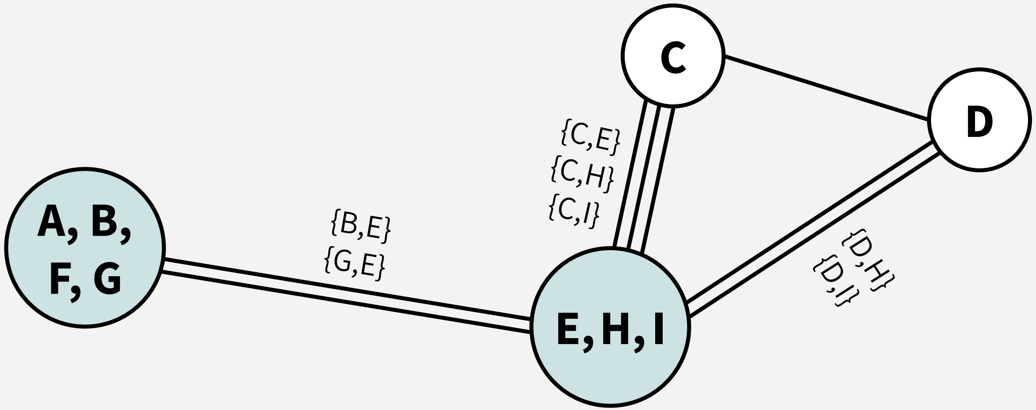

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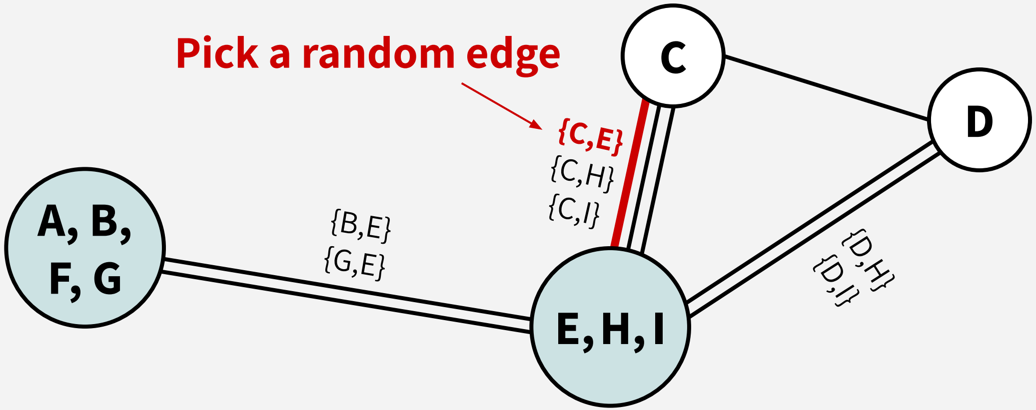

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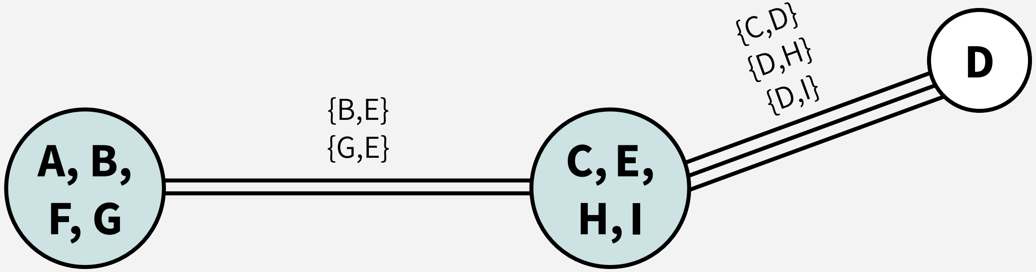

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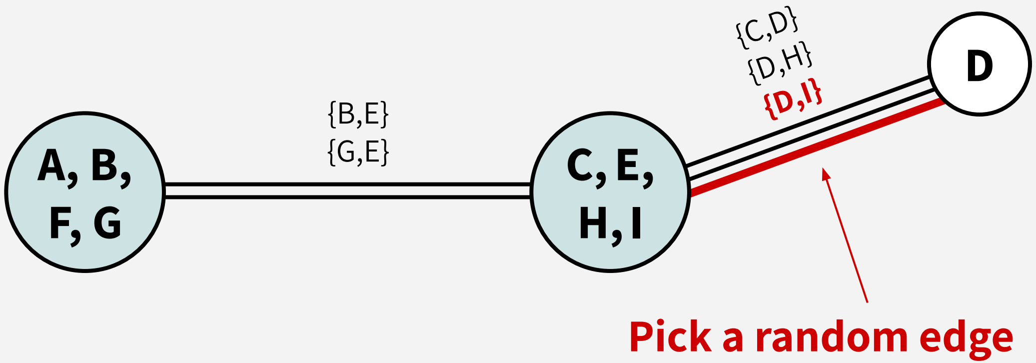

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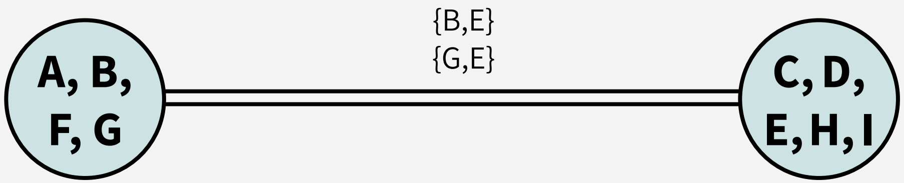

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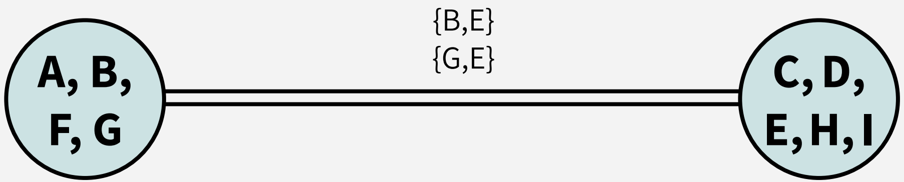
KARGER'S ALGORITHM

Pick a random edge, **contract** it, and repeat until you only have 2 vertices left.

We return the cut given by the remaining 2 super-vertices!



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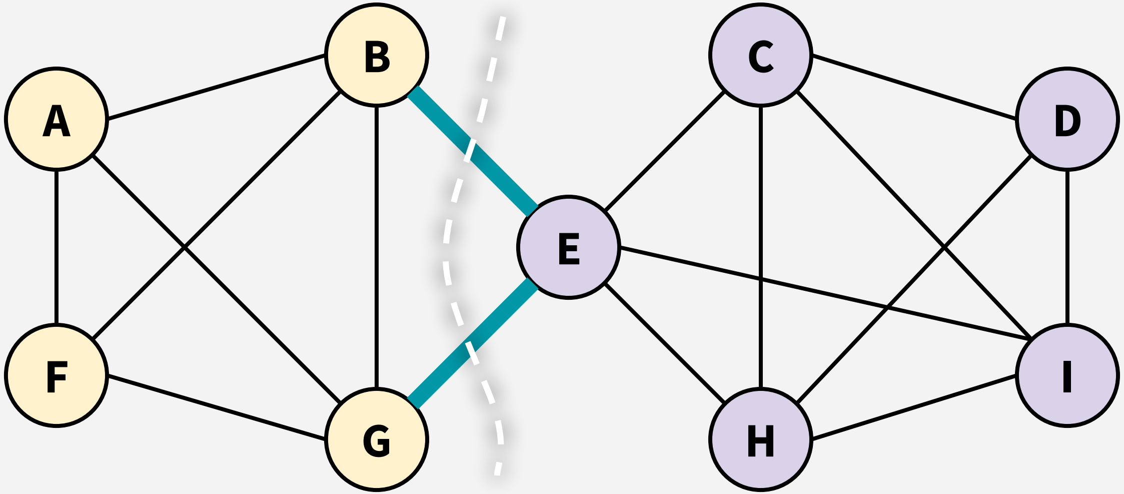
(Just bringing back our original graph to see what this corresponds to)



سوال؟

زمان اجرا و درستی الگوریتم کارگر

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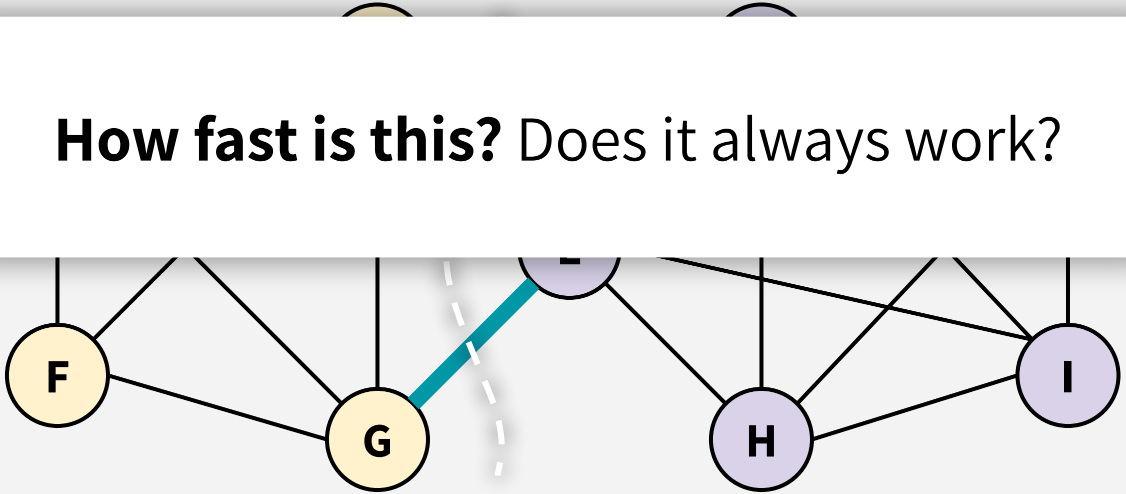

How fast is this? Does it always work?

KARGER'S ALGORITHM: PSEUDOCODE

```
KARGER(G=(V,E)):
   vertices of G' = {supernode(v) for v in V}
   E'u'v'= {(u,v)} for (u,v) in E
   E'u'v'= {} for (u,v) not in E
   F = a copy of E
   while |G'| > 2:
      {(u,v)} = uniformly random edge in F
      MERGE_SUPERNODES(u,v)
      F = F \ Eu'v'
   return the cut given by the remaining two supernodes!

MERGE_SUPERNODES(u,v):
   x' = supernode(nodes in u' ∪ nodes in v')
   for each w' in G' \ {u', v'}:
      Ex'w' = Eu'w' ∪ Ev'w'
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   F = a copy of E                               ← we'll choose randomly from F
   while |G'| > 2:
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one supernode for each vertex

one superedge for each edge

we'll choose randomly from F

now that u' and v' (supernodes of u and v, respectively) are merged, remove edges in the superedge between u' and v' (those edges are no longer "candidate" edges)

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assume that MERGE_SUPERNODES knows about u' and v' (the supernodes u and v belong to)

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x' is our merged result!

KARGER'S ALGORITHM: RUNTIME

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   E'_u'v' = {} for (u,v) not in E
   F = a copy of E
   while |G'| > 2:
      {(u,v)} = uniformly random edge in F
      MERGE_SUPERNODES(u,v)
      F = F \ E_u'v'
   return the cut given by the remaining two supernodes!

MERGE_SUPERNODES(u,v):
   x' = supernode(nodes in u' ∪ nodes in v')
   for each w' in G' \ {u', v'}:
      E_x'w' = E_u'w' ∪ E_v'w'
      remove u' and v' from G' and add x'
```

one supernode for each vertex

one superedge for each edge

we'll choose randomly from F

now that u' and v' (supernodes of u and v, respectively) are merged, remove edges in the superedge between u' and v' (those edges are no longer "candidate" edges)

assume that MERGE_SUPERNODES knows about u' and v' (the supernodes u and v belong to)

x' is our merged result!

MERGE_SUPERNODES takes O(n) time naively

(we can be more clever with a union-find data structure but don't worry about that for today!)

KARGER'S ALGORITHM: PSEUDOCODE

The while loop executes n – 2 times…

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The while loop executes n – 2 times…

MERGE_SUPERNODES takes O(n) time naively
(we can be more clever with a union-find data structure but don't worry about that for today!)

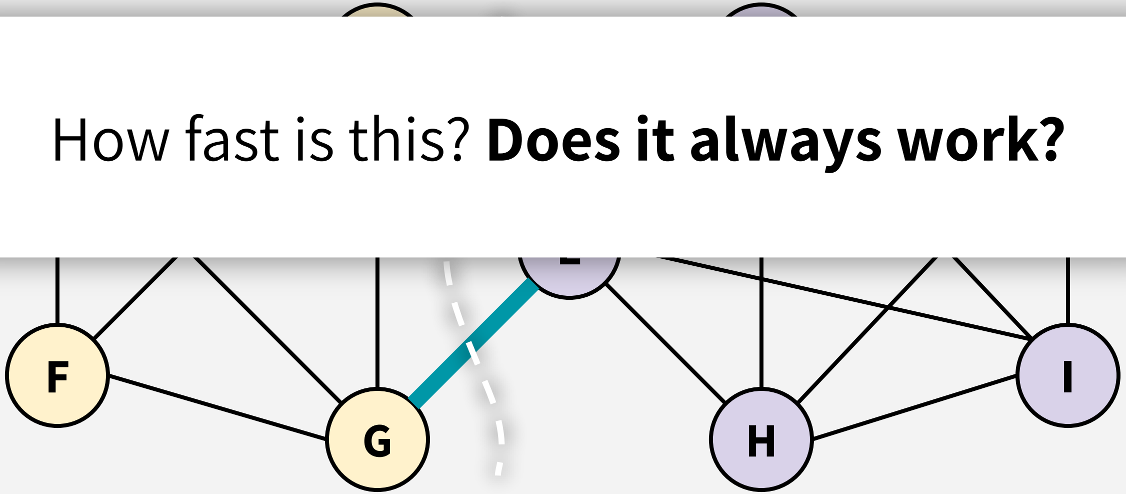
Runtime (w/o fancy data structures): $O(n^2)$

KARGER'S ALGORITHM

Pick a random edge, **contract** it, and repeat until you only have 2 vertices left.

We return the cut given by the remaining 2 super-vertices!

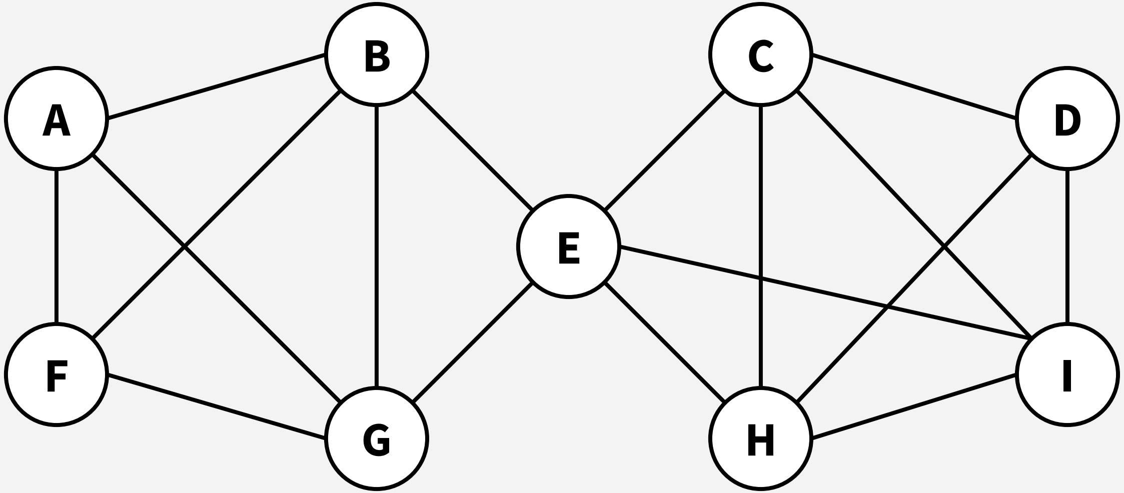
(Just bringing bac
our original grap
to see what this
corresponds to)



How fast is this? **Does it always work?**

WHAT IS AN “UNLUCKY” CHOICE?

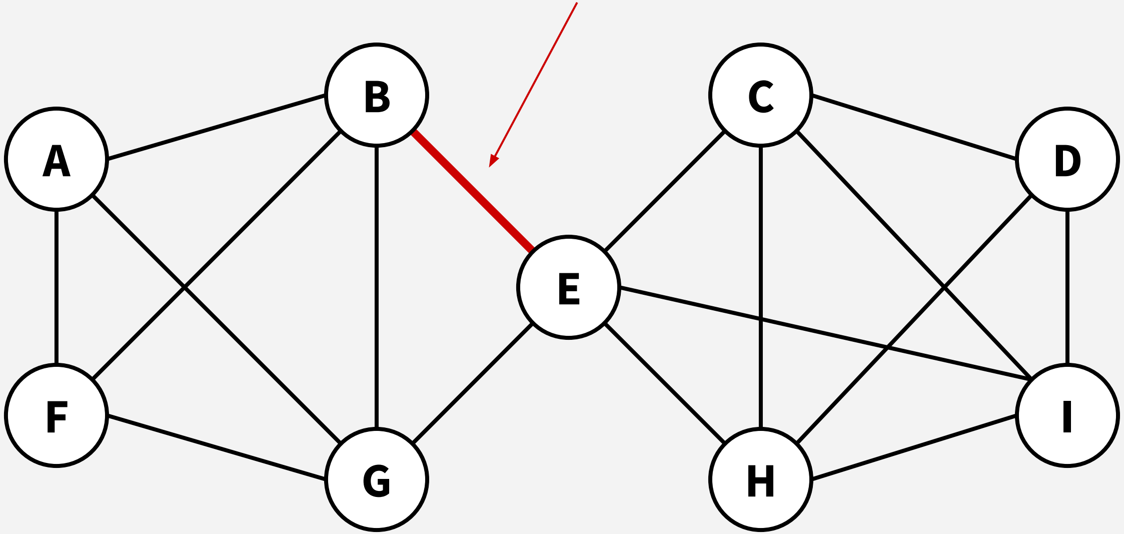
Pick a random edge, **contract** it, and repeat until you only have 2 vertices left.



Let’s see what happens if we’re not so lucky all the time...

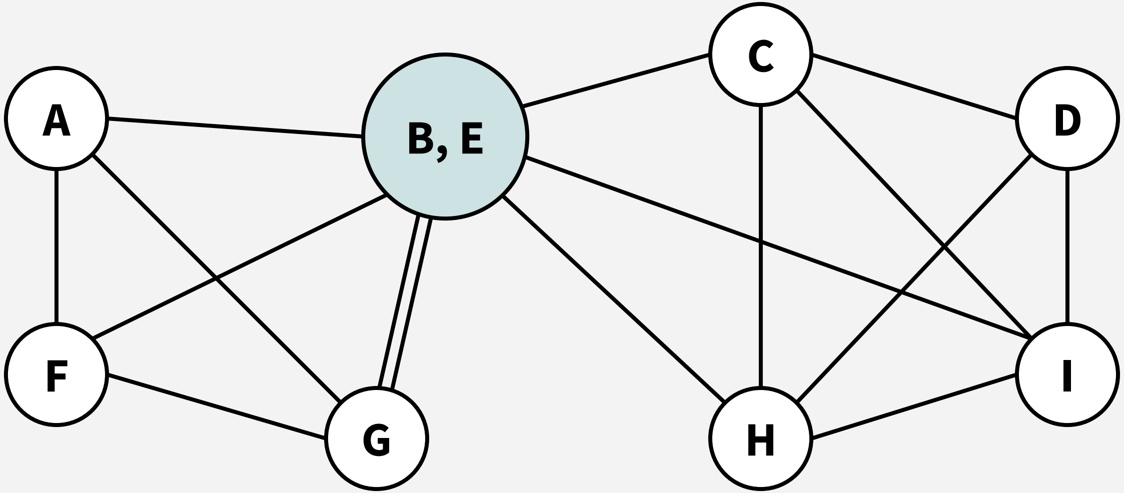
WHAT IS AN “UNLUCKY” CHOICE?

Pick a random edge! (this is a more “unlucky” choice)



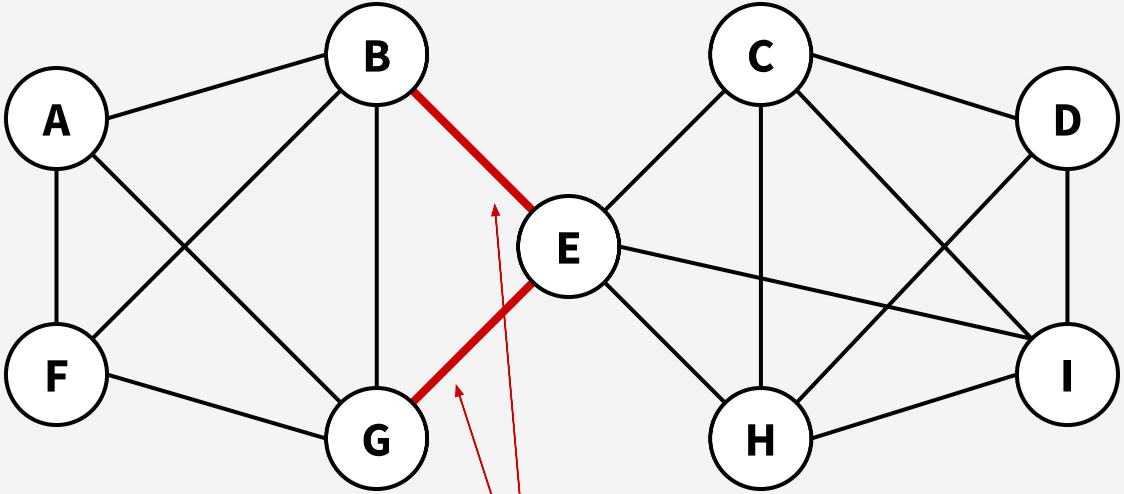
WHAT IS AN "UNLUCKY" CHOICE?

After we do the "edge contraction", *there's no way to return a cut that separates B and E.*



WHAT IS AN "UNLUCKY" CHOICE?

In fact, if Karger's algorithm ever randomly selects **edges in the min-cut**, *then it won't return that min-cut.*



These would be very unlucky edges to pick…

KARGER'S PROBABILITY OF SUCCESS

CLAIM:

The probability that Karger's algorithm returns a minimum cut is ≥ $1/\binom{n}{2}$

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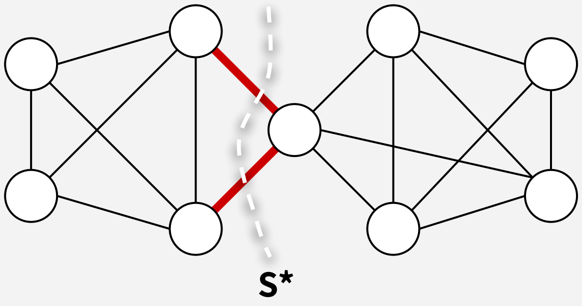

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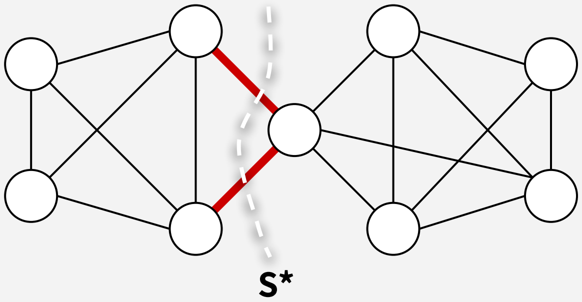

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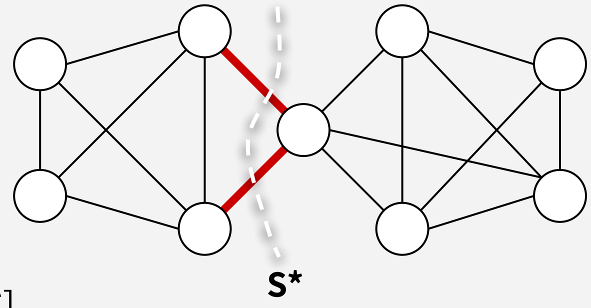

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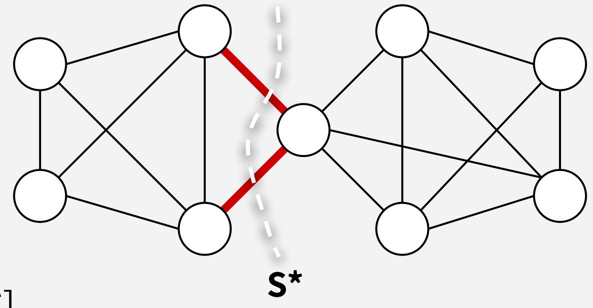
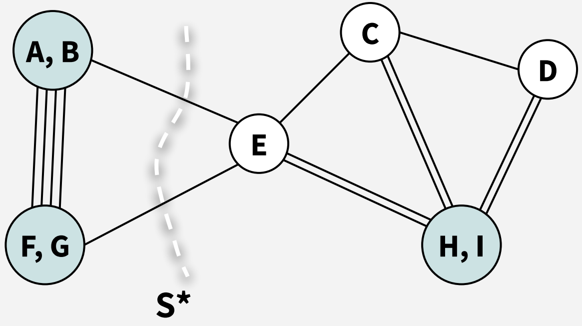


We need to show this! We can find an expression for:
Pr[$\mathbf{e_j}$ doesn't cross S* | $\mathbf{e_1}, \ldots, \mathbf{e_{j-1}}$ don't cross S*]

KARGER'S PROBABILITY OF SUCCESS

PROOF (cont'd): **What is $\Pr[e_j \text{ doesn't cross } S^* \mid e_1, \ldots, e_{j-1} \text{ don't cross } S^*]$?**



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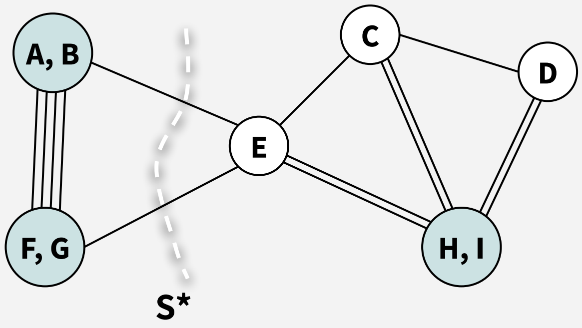
This is equivalent to answering this:

Suppose after j–1 edge choices, Karger hasn't messed up yet!

What's the probability that our j^{th} edge, e_j, also doesn't cross S^*?



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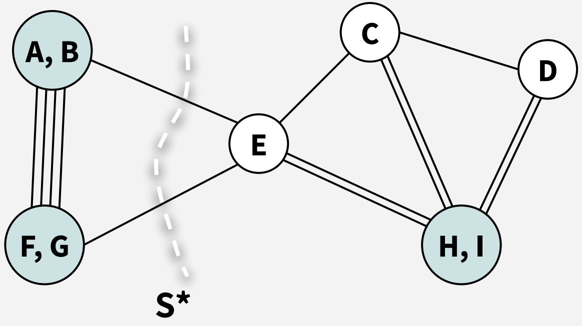
Say there are k edges that cross S*.



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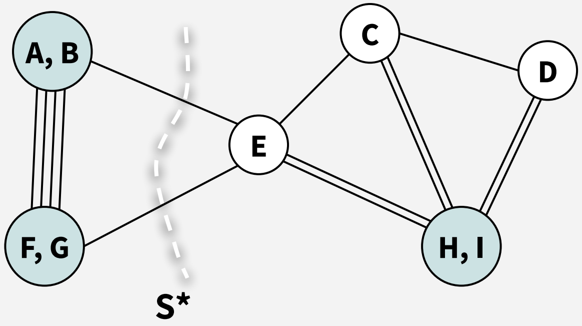

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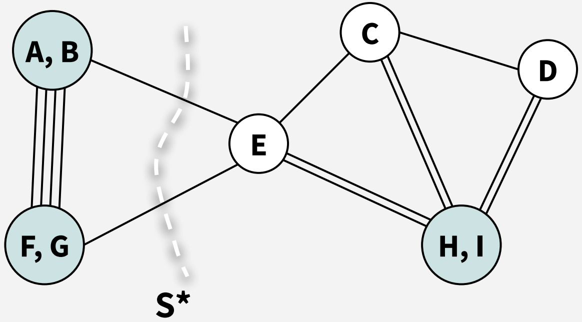

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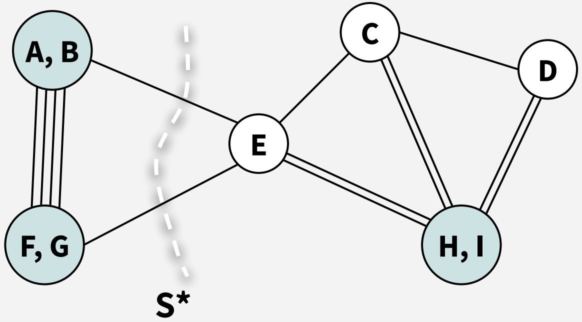

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Thus, there are at least **$k(n-j+1))/2$** remaining edges to choose from!

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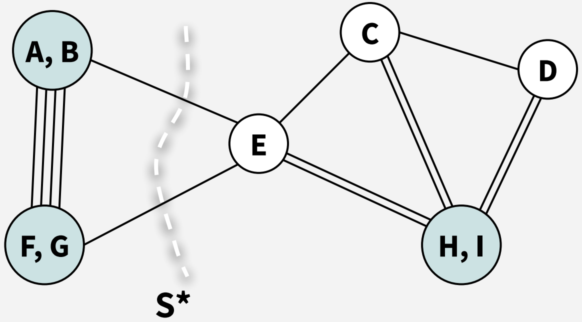

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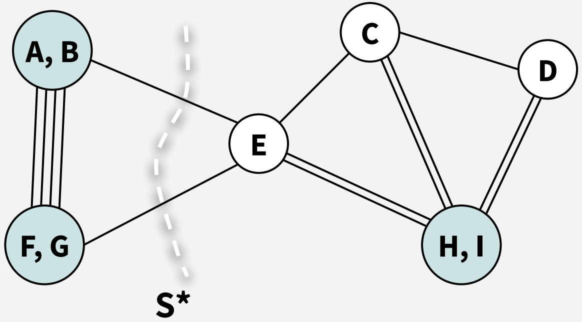

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We basically computed
1 – Pr[failure]
= 1 – (k / total # remaining edges)

So, the probability that our j^{th} edge is also *safe* is: $\geq 1 - \frac{k}{\frac{k(n-j+1)}{2}}$

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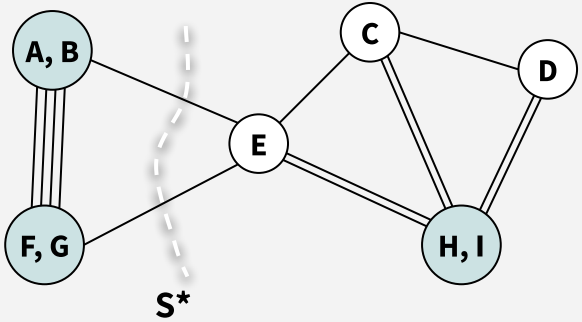

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× Pr[$\mathbf{e_3}$ doesn't cross **S*** | $\mathbf{e_1}$, $\mathbf{e_2}$ don't cross **S***]

. . .

× Pr[$\mathbf{e_{n-2}}$ doesn't cross **S*** | $\mathbf{e_1}, \ldots, \mathbf{e_{n-3}}$ don't cross **S***]

From previous slide:

Pr[e_j doesn't cross S* | $e_1,\ldots,e_{j-1}$ don't cross S*]

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\end{aligned}
$$

$$
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From previous slide:

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\end{aligned}
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$\Pr[e_j \text{ doesn't cross } S^* \mid e_1,\ldots,e_{j-1} \text{ don't cross } S^*]$

$$= \frac{n-j-1}{n-j+1}$$

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KARGER'S PROBABILITY OF SUCCESS

CLAIM:

The probability that Karger's algorithm returns a minimum cut is $\geq 1/\binom{n}{2}$

PROOF (cont'd): Suppose **S*** is a min-cut, and suppose we select edges $\mathbf{e_1}, \mathbf{e_2}, \ldots, \mathbf{e_{n-2}}$. Then:

$$\begin{aligned}
\Pr[\text{return } \mathbf{S^*}] &= \Pr[\text{none of the } \mathbf{e_i} \text{ cross } \mathbf{S^*}] \\
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. . .

× Pr[$\mathbf{e_{n-2}}$ doesn't cross **S*** | $\mathbf{e_1}, \ldots, \mathbf{e_{n-3}}$ don't cross **S***]

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CLAIM:

The probability that Karger's algorithm returns a minimum cut is ≥ $1/\binom{n}{2}$

We proved the claim!

But 1/(n choose 2) doesn't seem like a very high probability.

For example, with our example of n = 9, Karger would return the {A,B,C,D} — {E,F,G,H,I} min cut with probability 1/(9 choose 2) = **0.028**…

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To boost our chances of success, let's employ some repetition!

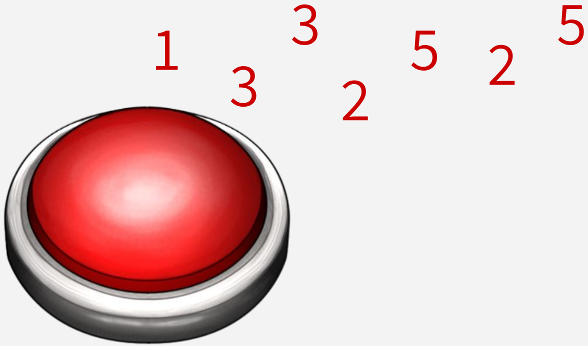
$$\geq \left(\frac{2}{n(n-1)}\right) = 1/\binom{n}{2}$$

A QUICK PROBABILITY EXERCISE

Suppose we have a magic button that produces one of 5 numbers {a,b,c,d,e}, uniformly at random when we push it. (We don’t know what {a,b,c,d,e} are).

Q1: How many times do we need to push the button, in expectation, to see the minimum value?

Q2: What is the probability that you have to push it more than 5 times? 10 times?



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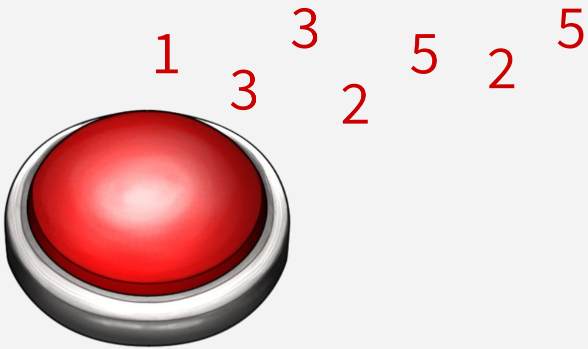

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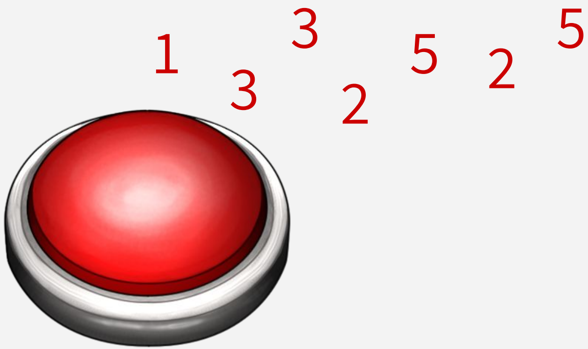

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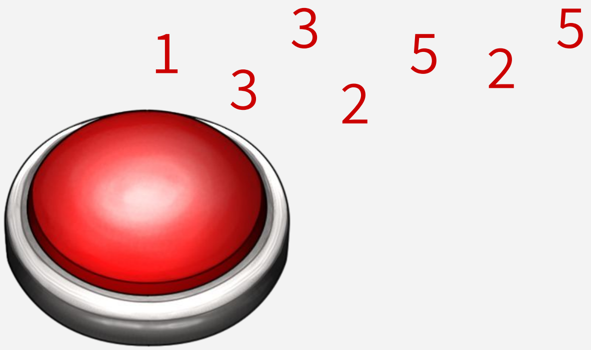

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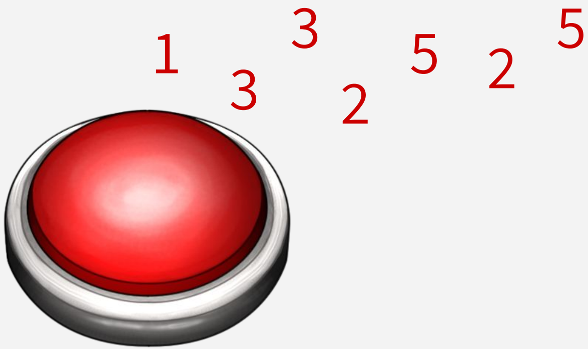

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   Pr[we don't see min after 10 times] = $\mathbf{(1 - 0.2)^{10} \approx 0.1}$

A QUICK PROBABILITY EXERCISE

To boost our chances of success,
let's repeat Karger multiple times and return the smallest cut found!

- Run Karger's! The cut size is 6!
- Run Karger's! The cut size is 3!
- Run Karger's! The cut size is 3!
- Run Karger's! The cut size is 2!
- Run Karger's! The cut size is 5!

$\mathbf{(1 - 0.2)^T}$

is the probability that our repetition fails, i.e. that the minimum value out of the T tries is *not* the true min cut value!

Pr[we don't see min after 10 times] $= \mathbf{(1 - 0.2)^{10}} \approx \mathbf{0.1}$

REPEATING KARGER FOR A BOOST!

THE QUESTION:

How many times should we run Karger's to succeed with probability $\geq 1-\delta$?

Here, δ represents an upper bound on the probability of *failure* we're willing to tolerate

(it's a parameter we can choose, e.g. it can be some small constant like 0.01)

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Pr[we fail to return a min-cut after T trials] = $\mathbf{(1\text{–}p)^T}$, where p = probability a trial succeeds $= 1/\binom{n}{2}$

We want to choose T so that we end up with $\mathbf{(1\text{–}p)^T \leq \delta}$

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- First, we'll make a ~clever~ observation that $\mathbf{1-p \leq e^{-p}}$ (if you plot the lines you'll see why!)
- Thus, we have $\mathbf{(1-p)^T \leq e^{-pT}}$. We want $\mathbf{e^{-pT} \leq \delta}$ $\Rightarrow$ we want $\mathbf{-pT \leq \ln \delta}$ $\Rightarrow$ we want $\mathbf{T \geq (1/p)(-\ln \delta)}$

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- Choose T to be at least (1/p)ln(1/δ), i.e. run Karger's at least $\binom{n}{2}\ln\left(\frac{1}{\delta}\right)$ times!

KARGER'S: PUTTING IT ALL TOGETHER

Runtime of Karger's (without fancy data structures): $\mathbf{O(n^2)}$

To successfully find a minimum cut with probability $\geq 1-\delta$,
repeat Karger's at least $\binom{n}{2} \ln\left(\frac{1}{\delta}\right)$ times.

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Runtime of Karger's (without fancy data structures): **O(n²)**

To successfully find a minimum cut with probability ≥ 1-δ, repeat Karger's at least $\binom{n}{2}\ln\left(\frac{1}{\delta}\right)$ times.

Treating δ as a constant (e.g. 0.01), the total runtime to succeed w/ prob. ≥ 1-δ =

$$O\left(n^2 \cdot \binom{n}{2} \ln\left(\frac{1}{\delta}\right)\right) = \mathbf{O(n^4)}$$

If you want different success probabilities, this overall runtime would be adjusted as well (by adjusting the number of trials needed). Keep in mind that this can be improved by using a union-find data structure as well.

KARGER'S: PUTTING IT ALL TOGETHER

Runtime of Karger's (without fancy data structures): $\mathbf{O(n^2)}$

This is pretty good!

In practice, it's often good enough to use Monte-Carlo algorithms that'll succeed with probability 0.99. Repetition was the key!!!

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GENERAL TAKEAWAY

- If we have a Monte Carlo algorithm with a small success probability
- And we can check how good a solution is (e.g. cost of cut)
- Then we can **boost** our success probability via repetition! Repeat it a bunch of times and take the best solution

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Unfortunately, repeating an $O(n^2)$ time algorithm many times can still be quite expensive...

CAN WE DO BETTER?

Absolutely!!!

The **Karger-Stein** algorithm can achieve a better runtime of $\mathbf{O(n^2 \log^2 n)}$.

THE KEY: We'll do the repetitions in a clever way so that we save on runtime!
Note: this is a trickier algorithm, so we'll just sketch the high-level approach here.

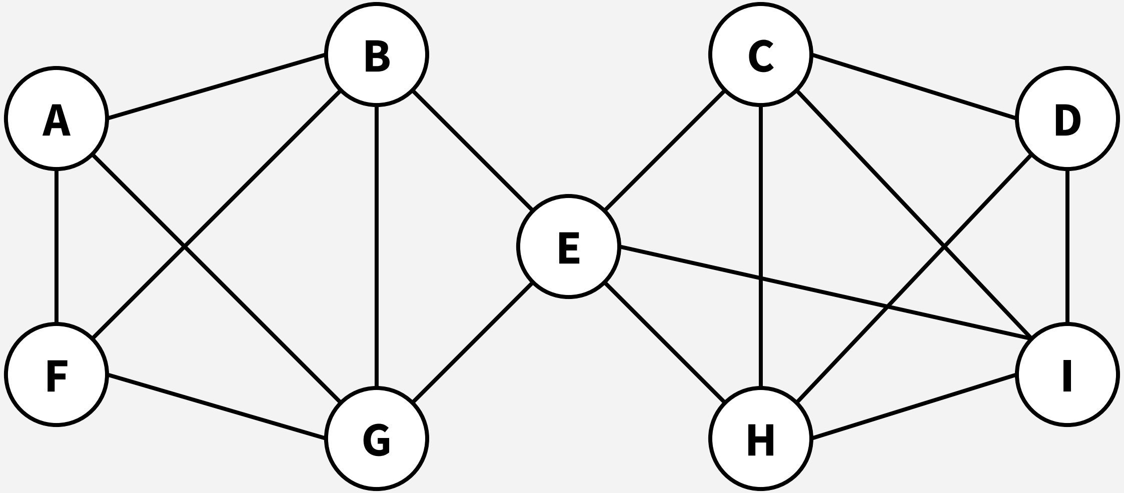
سوال؟

الگوریتم کارگِر-استین

راه بهتری برای تکرار الگوریتم کارگِر

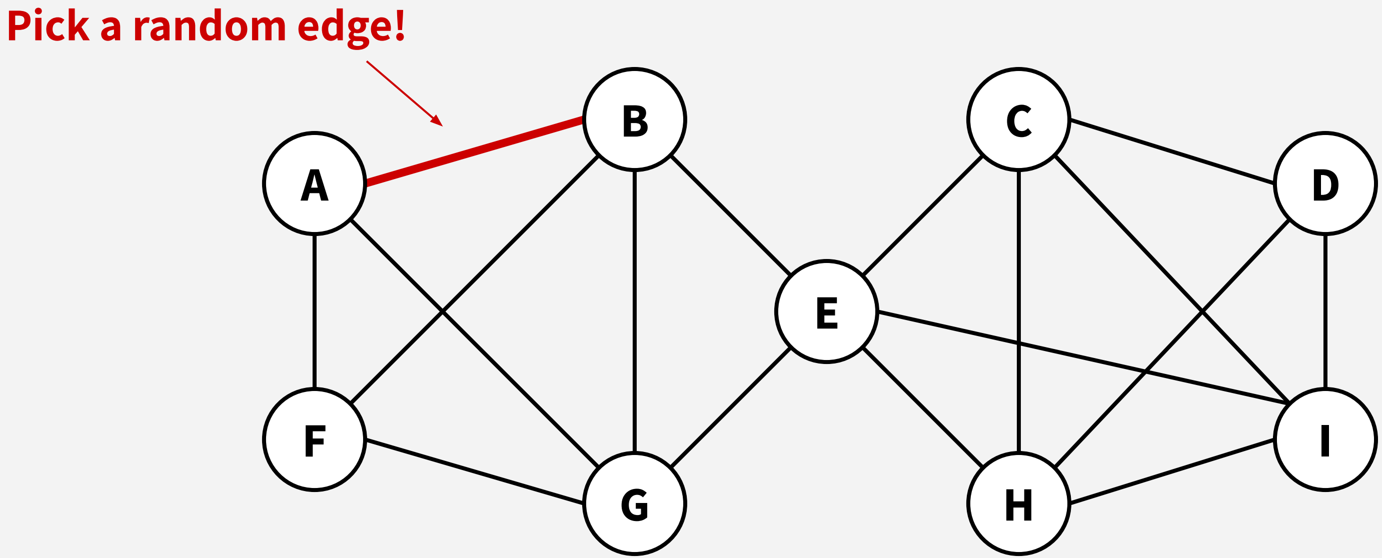
KARGER'S ALGORITHM (AGAIN)

Let's brainstorm how we might save on repetitions...



KARGER'S ALGORITHM (AGAIN)

Pick a random edge, **contract** it, and repeat until you only have 2 vertices left.



There are 17 edges, and 15 are good to contract

Probability that we didn't mess up = 15/17

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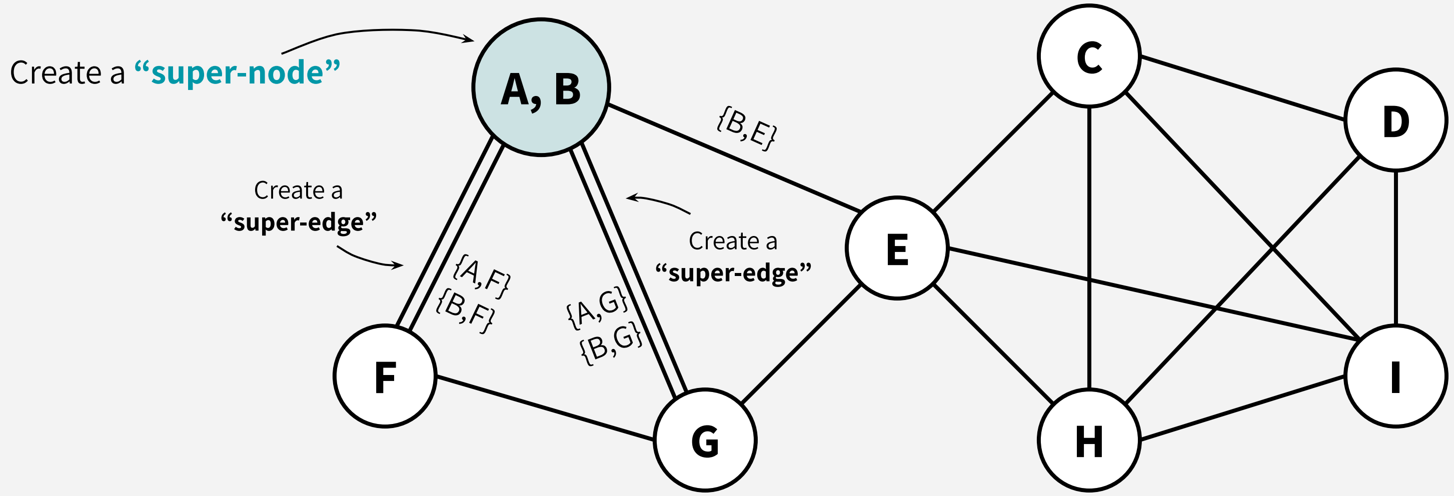
Pick a random edge, **contract** it, and repeat until you only have 2 vertices left.

Now perform an "edge contraction"!



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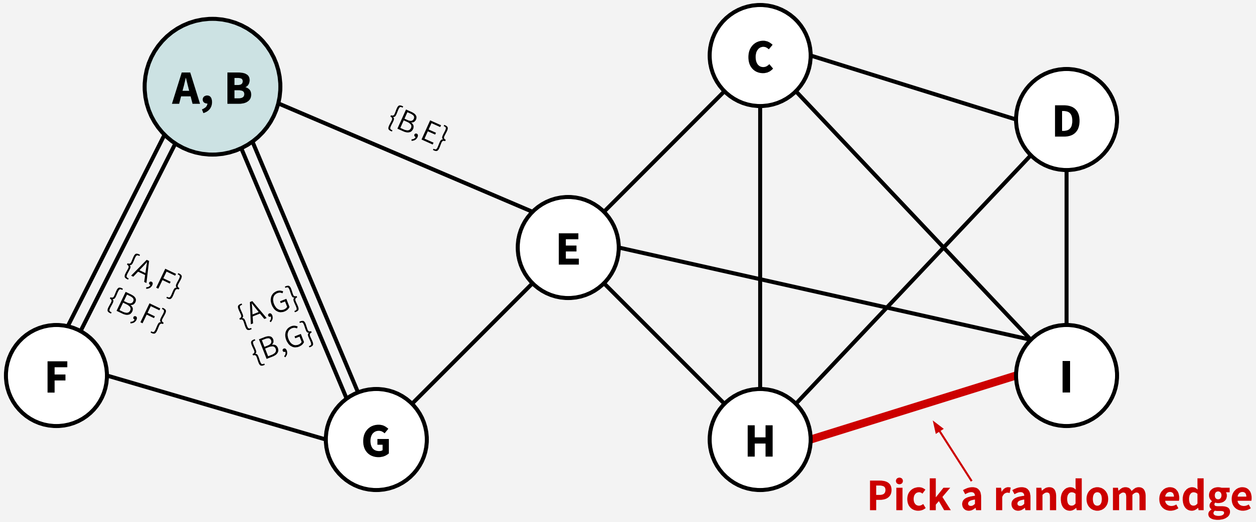

Now there are only 16 edges, since the edge between A and B is gone.

Probability that we didn't mess up = 14/16

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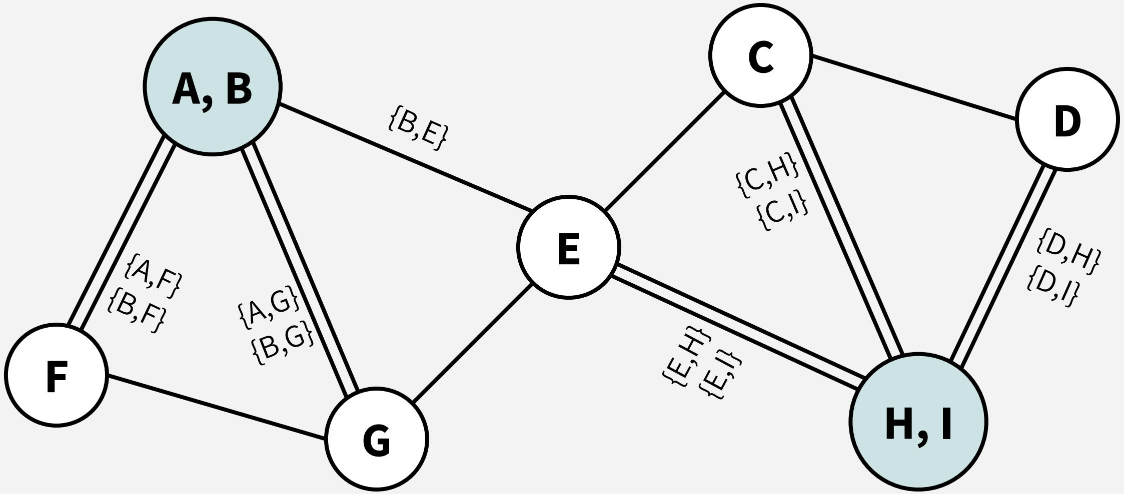

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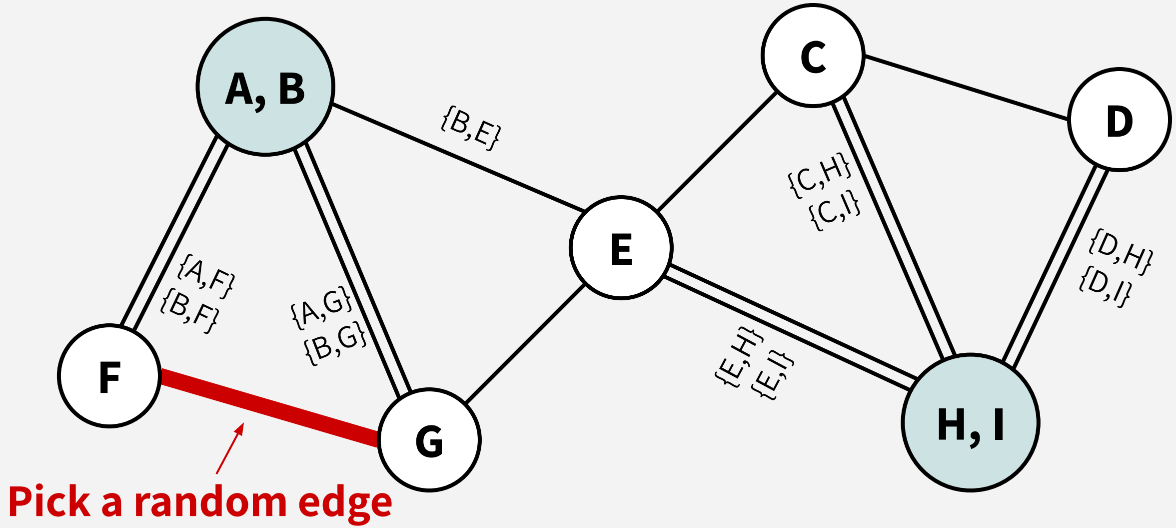

There are 15 edges, since the edge between H and I is gone now.

Again, only 2 edges are bad to pick still.

Probability that we didn't mess up = 13/15

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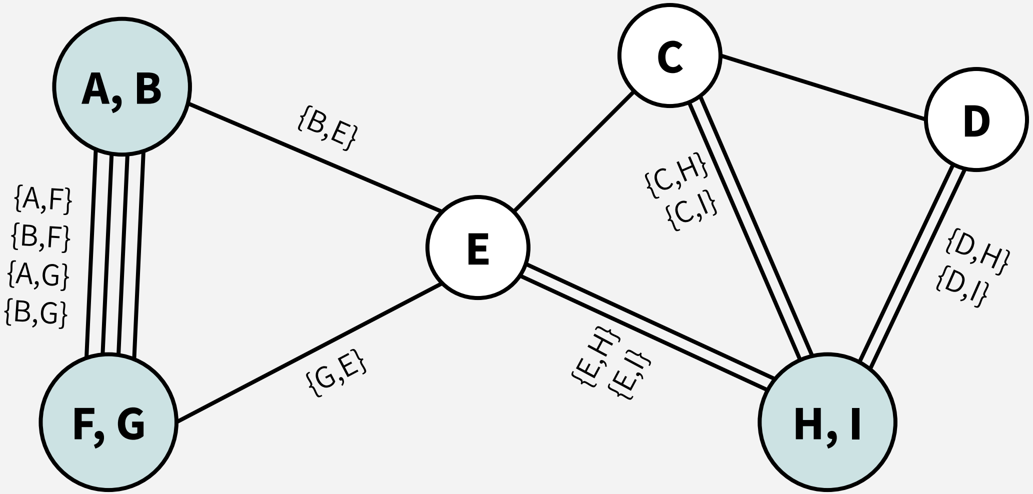
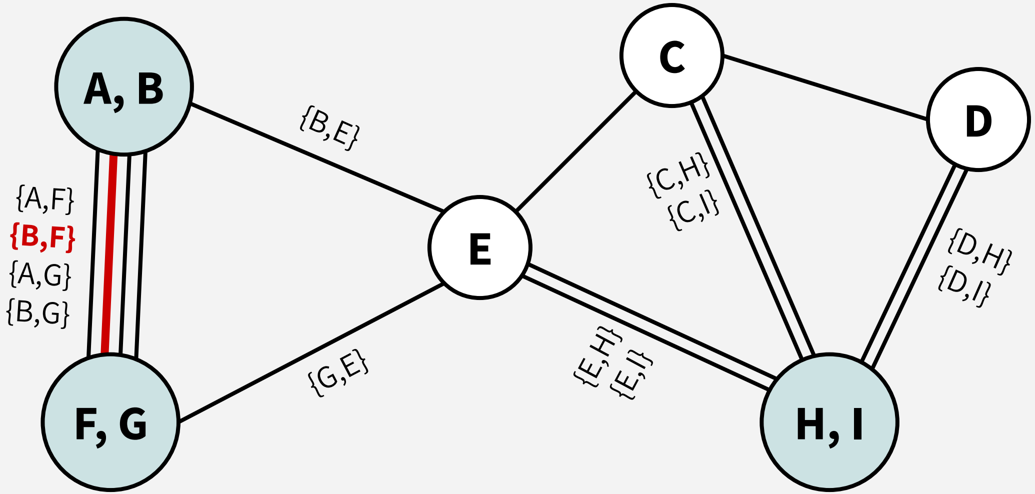


Perform an "edge contraction"

KARGER'S ALGORITHM (AGAIN)

Pick a random edge, **contract** it, and repeat until you only have 2 vertices left.

Pick a random edge



There are now 14 edges, since the edge between F and G is gone now.

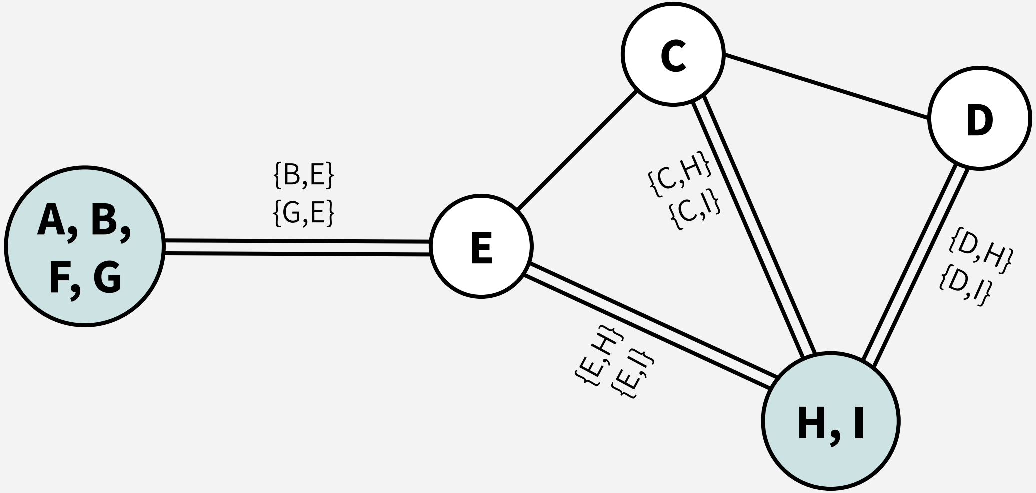
Again, only 2 edges are bad to pick still.

Probability that we didn't mess up = 12/14

KARGER'S ALGORITHM (AGAIN)

Pick a random edge, **contract** it, and repeat until you only have 2 vertices left.

Perform an "edge contraction"



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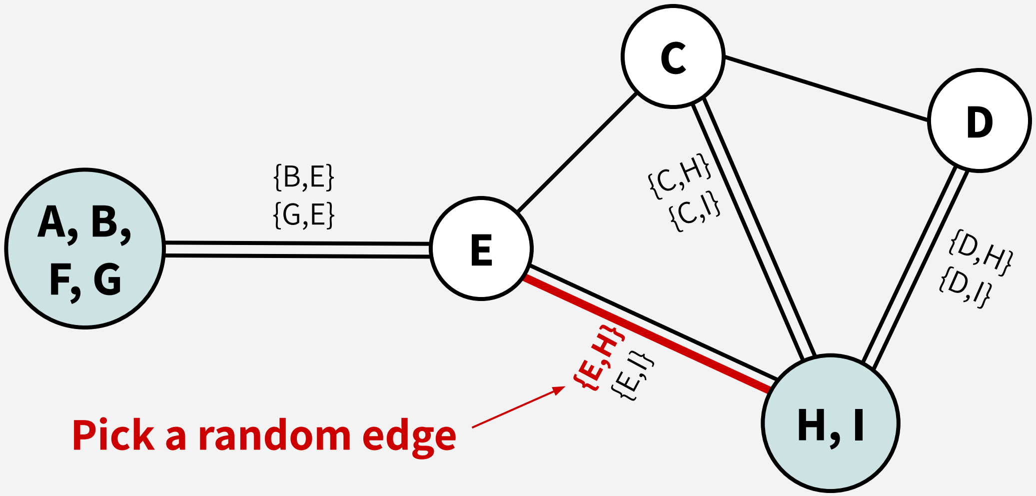


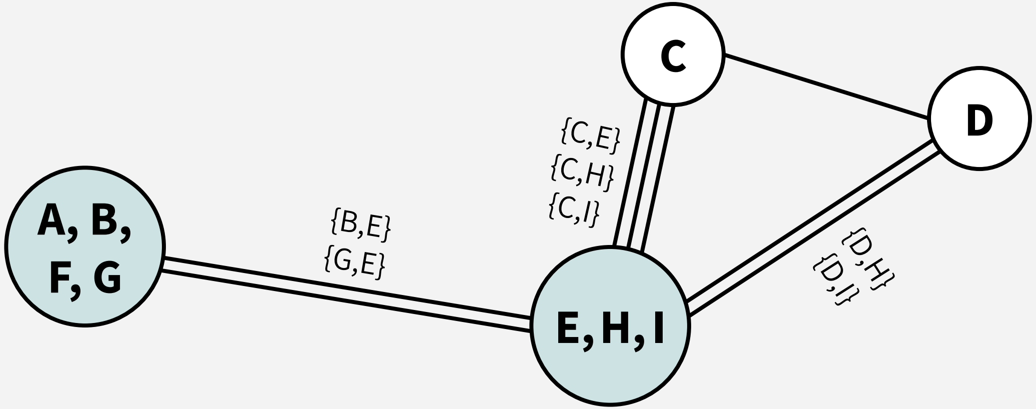
There are now only 10 edges, since the last contraction got rid of 4 edges!

Again, only 2 edges are bad to pick still.

Probability that we didn't mess up = 8/10

KARGER'S ALGORITHM (AGAIN)

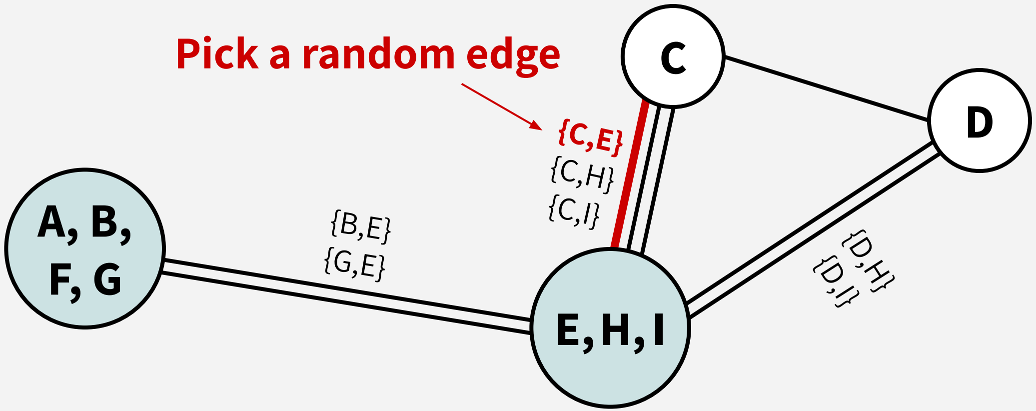
Pick a random edge, **contract** it, and repeat until you only have 2 vertices left.



Perform an "edge contraction"

KARGER'S ALGORITHM (AGAIN)

Pick a random edge, **contract** it, and repeat until you only have 2 vertices left.



There are now only 8 edges, since the last contraction got rid of 2 edges!

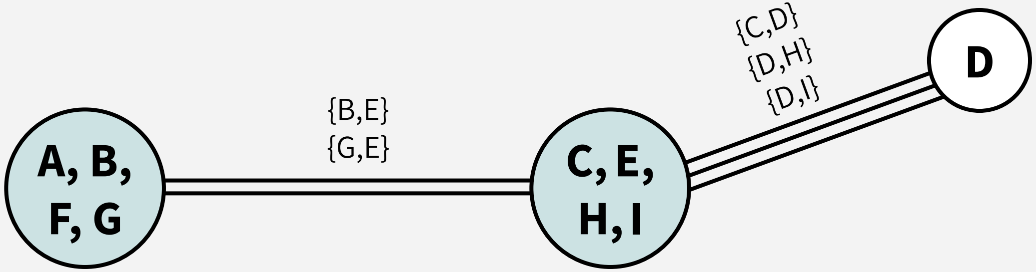
Again, only 2 edges are bad to pick still.

Probability that we didn't mess up = 6/8

KARGER'S ALGORITHM (AGAIN)

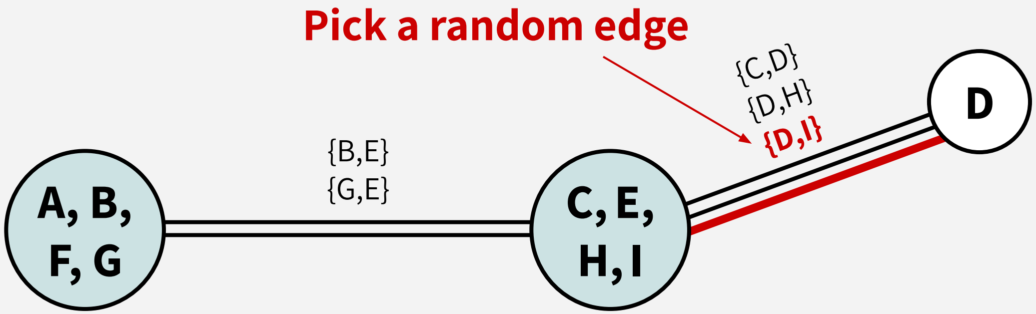
Pick a random edge, **contract** it, and repeat until you only have 2 vertices left.

Perform an "edge contraction"



KARGER'S ALGORITHM (AGAIN)

Pick a random edge, **contract** it, and repeat until you only have 2 vertices left.



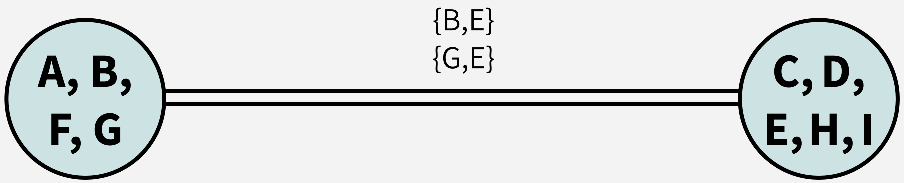
There are now only 5 edges, since the last contraction got rid of 3 edges!

Again, only 2 edges are bad to pick still.

Probability that we didn't mess up = 3/5

KARGER'S ALGORITHM (AGAIN)

Pick a random edge, **contract** it, and repeat until you only have 2 vertices left.

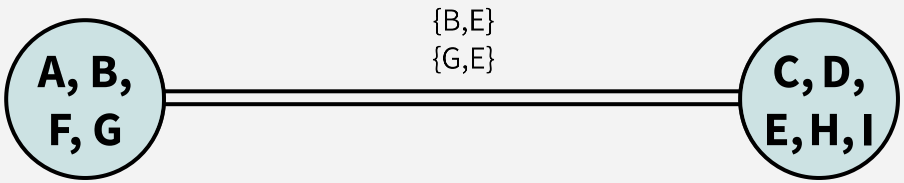


Perform an "edge contraction"

KARGER'S ALGORITHM (AGAIN)

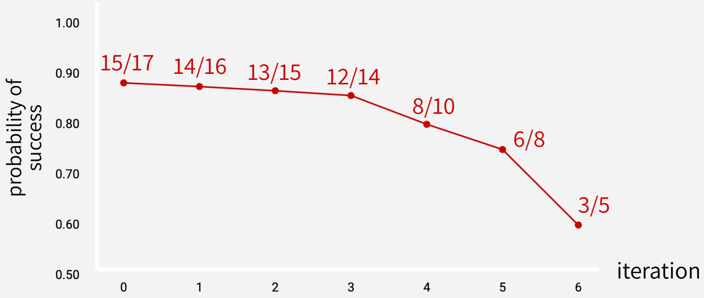
Pick a random edge, **contract** it, and repeat until you only have 2 vertices left.

Here we stop because we only have 2 nodes left!



MORE LIKELY TO MESS UP OVER TIME!

At the beginning of a run of Karger, it's pretty likely our random choice is fine.
The probability that we mess up gets worse and worse over time.



Moral of the walkthrough:

Repeating the beginning of the algorithm is wasteful!!! We weren't really that likely to mess up, so why bother protecting those choices with redundancy?
(i.e. why bother using repetition for our earlier choices?)

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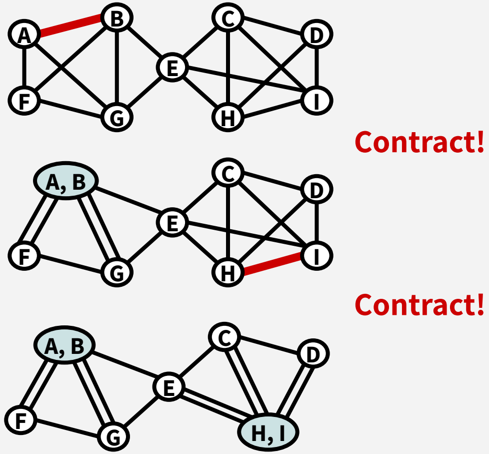
OUR GAMEPLAN (IN PICTURES)

First run Karger's on G for a bit...



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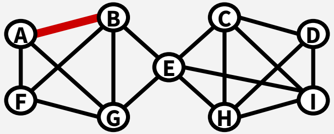
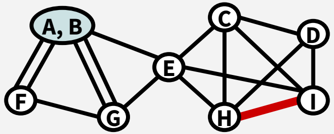


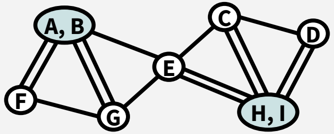
Contract!

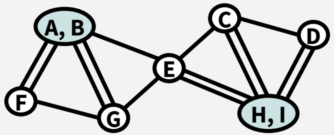


Contract!

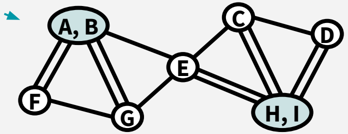
ATTEMPT **A**

ATTEMPT **B**





Now FORK!!!



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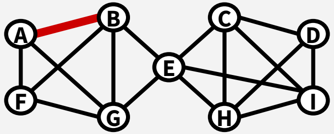
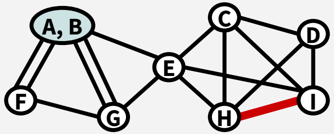


Contract!



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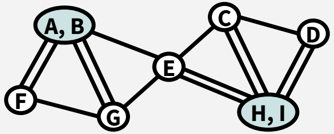
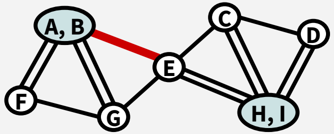
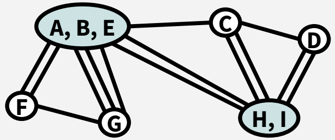


ATTEMPT **A**

This branch made a bad choice 😢



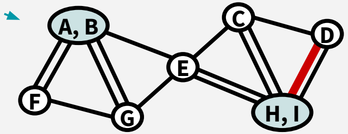
Contract!



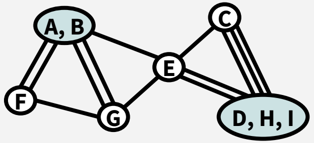
Now FORK!!!

ATTEMPT **B**

But it's okay since branch B made a good choice 😁



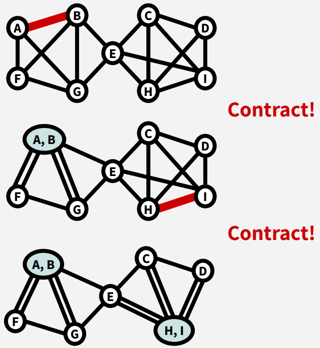
Contract!



OUR GAMEPLAN (IN WORDS)

- First run Karger's on G for a bit until there are $\frac{n}{\sqrt{2}}$ supernodes left.

Why the $\sqrt{2}$ factor? We'll see soon.



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- First run Karger's on G for a bit until there are $\frac{n}{\sqrt{2}}$ supernodes left.
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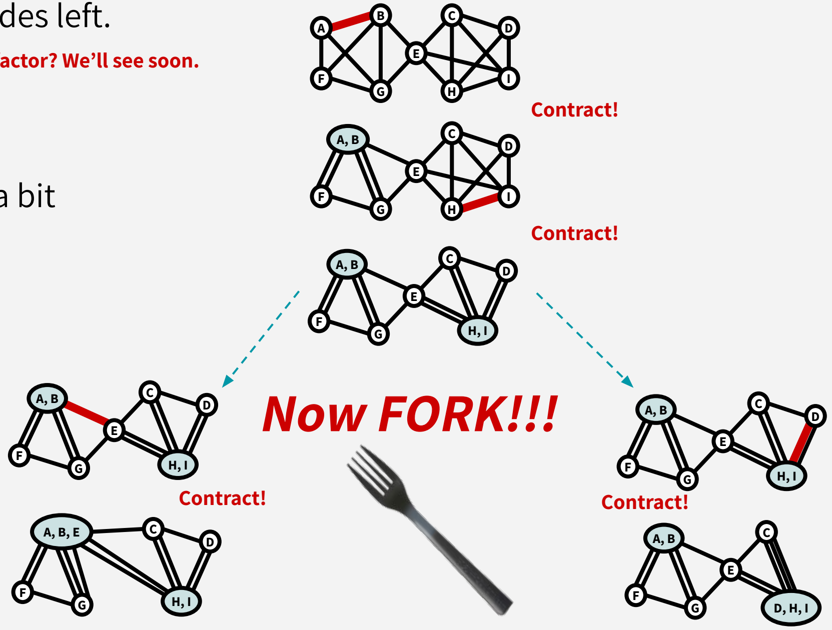
- Then split/fork into two independent copies, $\mathbf{G_1}$ and $\mathbf{G_2}$
- Independently resume Karger's algorithm on $\mathbf{G_1}$ and $\mathbf{G_2}$ for a bit
  (until you reduce # of supernodes by another factor of $\sqrt{2}$)



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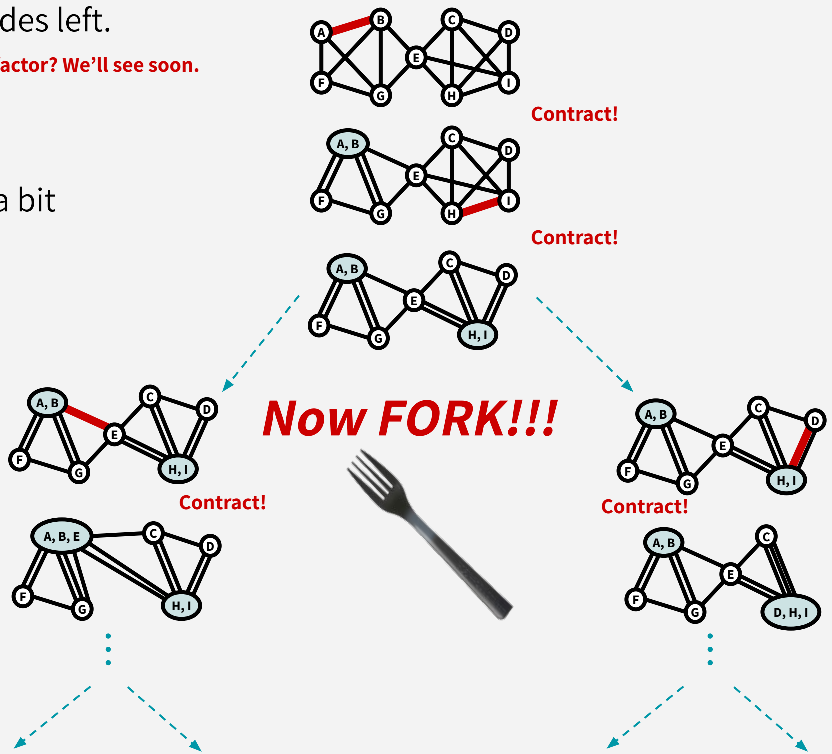
  (until you reduce # of supernodes by another factor of $\sqrt{2}$)
- Then split each of those into two independent copies….
- Etc….



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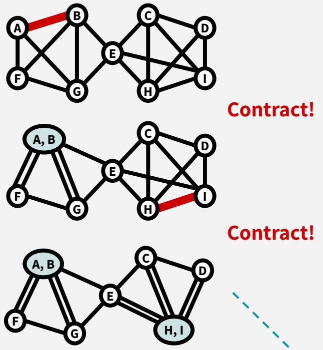
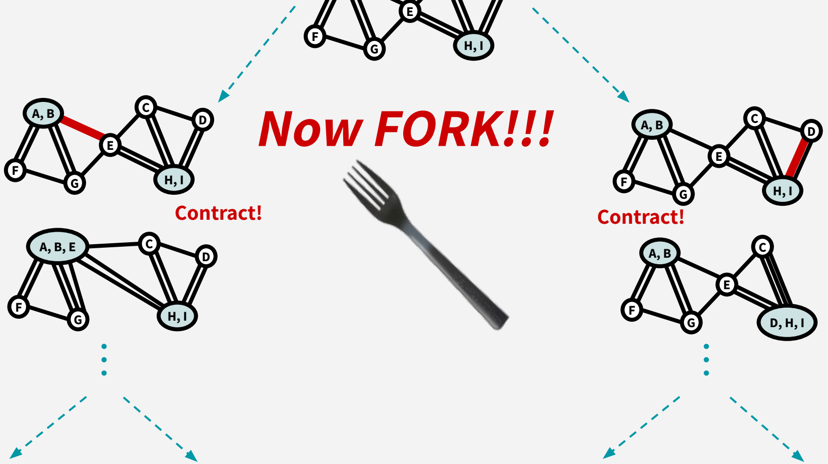




Basically:
Run Kargers for a bit, split into two copies, recurse on each, then return whichever copy's answer was better (i.e. the smaller cut)

KARGER-STEIN PSEUDOCODE

There are some details left out, but here's the skeleton:

```
KARGER-STEIN(G=(V,E)):
  • n ← |V|
  • if n < 4:
        find a min-cut by brute force  ← O(1) time
  • Run Karger's algorithm on G until ⌊n/√2⌋ supernodes
    remain
  • G_1 & G_2 ← copies of what's left in G
  • S_1 = KARGER-STEIN(G_1)
  • S_2 = KARGER-STEIN(G_2)
  • return whichever of S_1, S_2 is the smaller cut
```

KARGER-STEIN (AS RECURSION TREE)

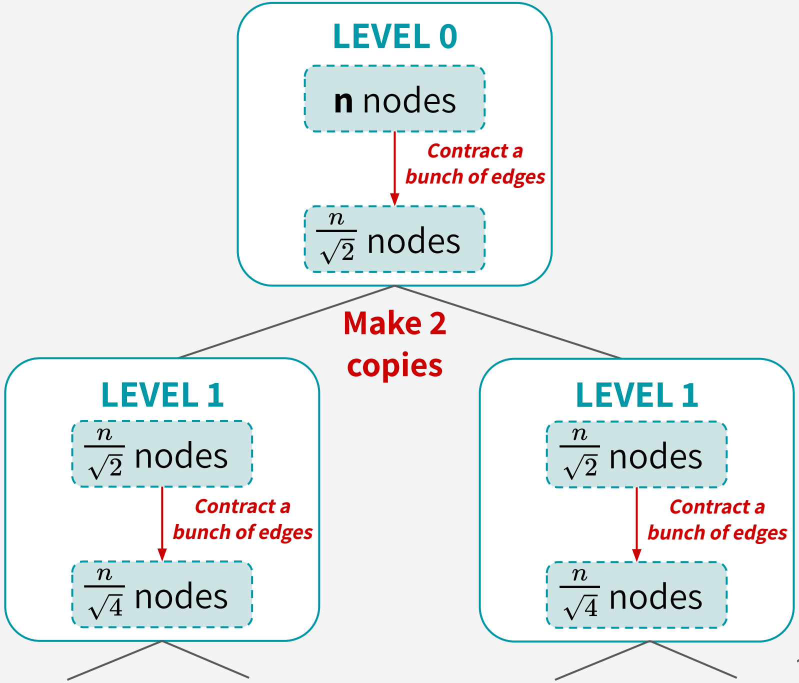
SHRINKING FACTOR: $\sqrt{2}$

DEPTH: $\log_{\sqrt{2}}(n) = \frac{\log n}{\log \sqrt{2}} = 2 \log n$

 # OF LEAVES: $2^{2 \log n} = n^2$

KARGER-STEIN(G=(V,E)):

- n ← |V|
- if n < 4:
  find a min-cut by brute force
- Run Karger's algorithm on G until $\left\lfloor \frac{n}{\sqrt{2}} \right\rfloor$ supernodes remain
- G_1 & G_2 ← copies of what's left in G
- S_1 = KARGER-STEIN(G_1)
- S_2 = KARGER-STEIN(G_2)
- **return** whichever of S_1, S_2 is the smaller cut



KARGER-STEIN RUNTIME

SHRINKING FACTOR: $\sqrt{2}$

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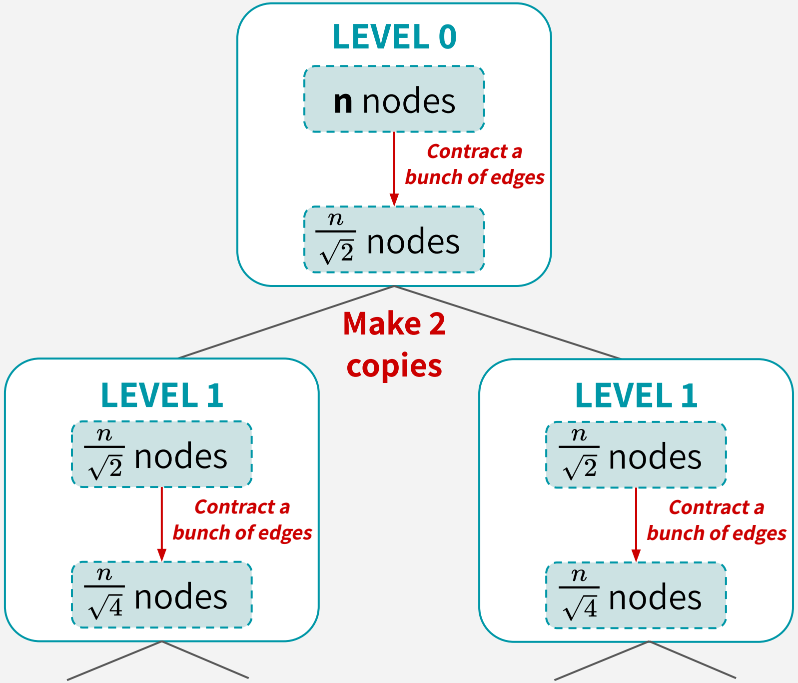

- Amount of work per level = amount of work needed to reduce the number of nodes by a factor of $\sqrt{2}$

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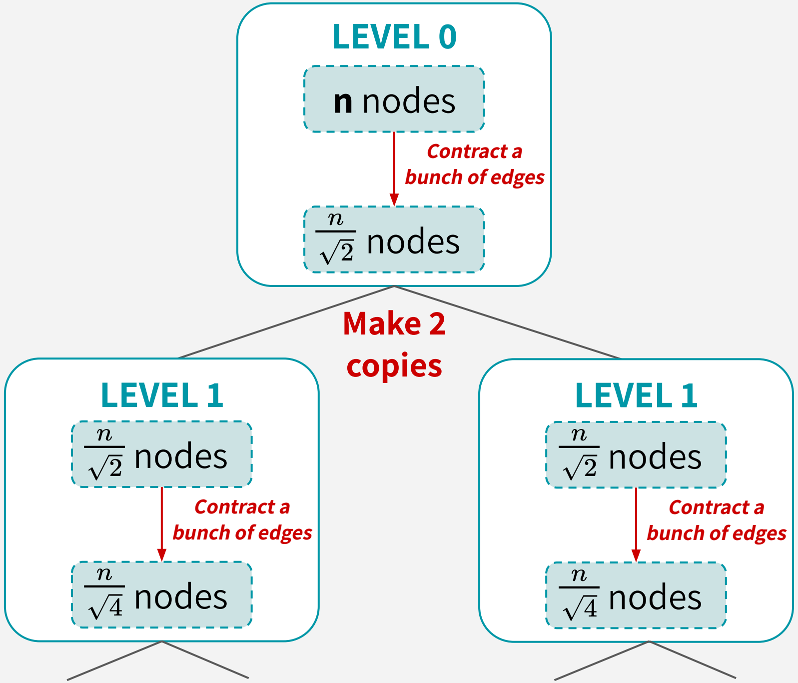

- Amount of work per level = amount of work needed to reduce the number of nodes by a factor of $\sqrt{2}$
- That's at most $O(n^2)$ -- that's the time it takes to run Karger's algorithm once in full!

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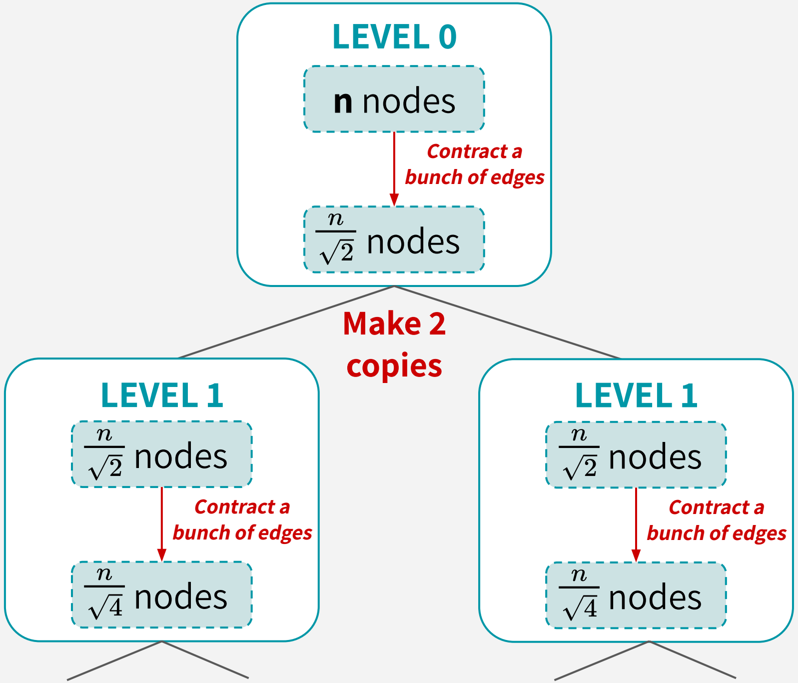

- Amount of work per level = amount of work needed to reduce the number of nodes by a factor of $\sqrt{2}$
- That's at most $O(n^2)$ -- that's the time it takes to run Karger's algorithm once in full!
- Our recurrence relation is

$$T(n) = 2T\left(\frac{n}{\sqrt{2}}\right) + O(n^2)$$

- The Master Theorem says:

$$\boxed{T(n) = O(n^2 \log n)}$$

WHY $n/\sqrt{2}$?

Suppose we contract n–$\frac{n}{\sqrt{2}}$ edges. CLAIM:

Pr[None of those n – $\frac{n}{\sqrt{2}}$ edges cross S*] **≈ 1 / 2** (when n is large)

WHY $n/\sqrt{2}$?

Suppose we contract n– $\frac{n}{\sqrt{2}}$ edges. CLAIM:

Pr[None of those n – $\frac{n}{\sqrt{2}}$ edges cross S*] **≈ 1 / 2** (when n is large)

Suppose **S*** is a min-cut, and suppose we select edges $\mathbf{e_1, e_2, \ldots, e_{n-t}}$. Then:

Pr[None of those n–t edges cross **S***]

= Pr[$\mathbf{e_1}$ doesn't cross **S***]

× Pr[$\mathbf{e_2}$ doesn't cross **S*** | $\mathbf{e_1}$ doesn't cross **S***]

× Pr[$\mathbf{e_3}$ doesn't cross **S*** | $\mathbf{e_1}$, $\mathbf{e_2}$ don't cross **S***]

. . .

× Pr[$\mathbf{e_{n-t}}$ doesn't cross **S*** | $\mathbf{e_1}, \ldots, \mathbf{e_{n-t-1}}$ don't cross **S***]

From earlier:

Pr[e_j doesn't cross S* | $e_1,\ldots,e_{j-1}$ don't cross S*]

$$= \frac{n-j-1}{n-j+1}$$

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Suppose **S*** is a min-cut, and suppose we select edges **e_1, e_2, …, e_{n-t}**. Then:

Pr[None of those n–t edges cross **S***]

= Pr[**e_1** doesn't cross **S***]

× Pr[**e_2** doesn't cross **S*** | **e_1** doesn't cross **S***]

× Pr[**e_3** doesn't cross **S*** | **e_1, e_2** don't cross **S***]

. . .

× Pr[**e_{n-t}** doesn't cross **S*** | **e_1, …,e_{n-t-1}** don't cross **S***]

$$\geq \left(\frac{n-2}{n}\right)\left(\frac{n-3}{n-1}\right)\left(\frac{n-4}{n-2}\right)\left(\frac{n-5}{n-3}\right)\cdots\left(\frac{t+1}{t+3}\right)\left(\frac{t}{t+2}\right)\left(\frac{t-1}{t+1}\right)$$

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$$\geq \left(\frac{n-\cancel{2}}{n}\right)\left(\frac{\cancel{n-3}}{n-1}\right)\left(\frac{\cancel{n-4}}{\cancel{n-2}}\right)\left(\frac{\cancel{n-5}}{\cancel{n-3}}\right)\cdots\left(\frac{\cancel{t+1}}{\cancel{t+3}}\right)\left(\frac{t}{\cancel{t+2}}\right)\left(\frac{t-1}{\cancel{t+1}}\right)$$

$$= \frac{t\cdot(t-1)}{n\cdot(n-1)}$$

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Suppose we contract n– $\frac{n}{\sqrt{2}}$ edges. CLAIM:

Pr[None of those n – $\frac{n}{\sqrt{2}}$ edges cross S*] **≈ 1 / 2** (when n is large)

Suppose **S*** is a min-cut, and suppose we select edges $\mathbf{e_1, e_2, \ldots, e_{n-t}}$. Then:

Pr[None of those n–t edges cross **S***]

$$\geq \left(\frac{\cancel{n-2}}{n}\right)\left(\frac{\cancel{n-3}}{n-1}\right)\left(\frac{\cancel{n-4}}{\cancel{n-2}}\right)\left(\frac{\cancel{n-5}}{\cancel{n-3}}\right)\cdots\left(\frac{\cancel{t+1}}{\cancel{t+3}}\right)\left(\frac{t}{\cancel{t+2}}\right)\left(\frac{t-1}{\cancel{t+1}}\right)$$

$$= \frac{t\cdot(t-1)}{n\cdot(n-1)}$$

(now choose t = $\frac{n}{\sqrt{2}}$)

$$= \frac{\frac{n}{\sqrt{2}}\cdot\left(\frac{n}{\sqrt{2}}-1\right)}{n\cdot(n-1)} \approx \boxed{\frac{1}{2}}$$

(when n is large)

WHY $n/\sqrt{2}$?

Suppose we contract n–$\frac{n}{\sqrt{2}}$ edges. CLAIM:

Pr[None of those n – $\frac{n}{\sqrt{2}}$ edges cross S*] **≈ 1 / 2** (when n is large)

Suppose **S*** is a min-cut, and suppose we select edges $\mathbf{e_1, e_2, \ldots, e_{n-t}}$. Then:

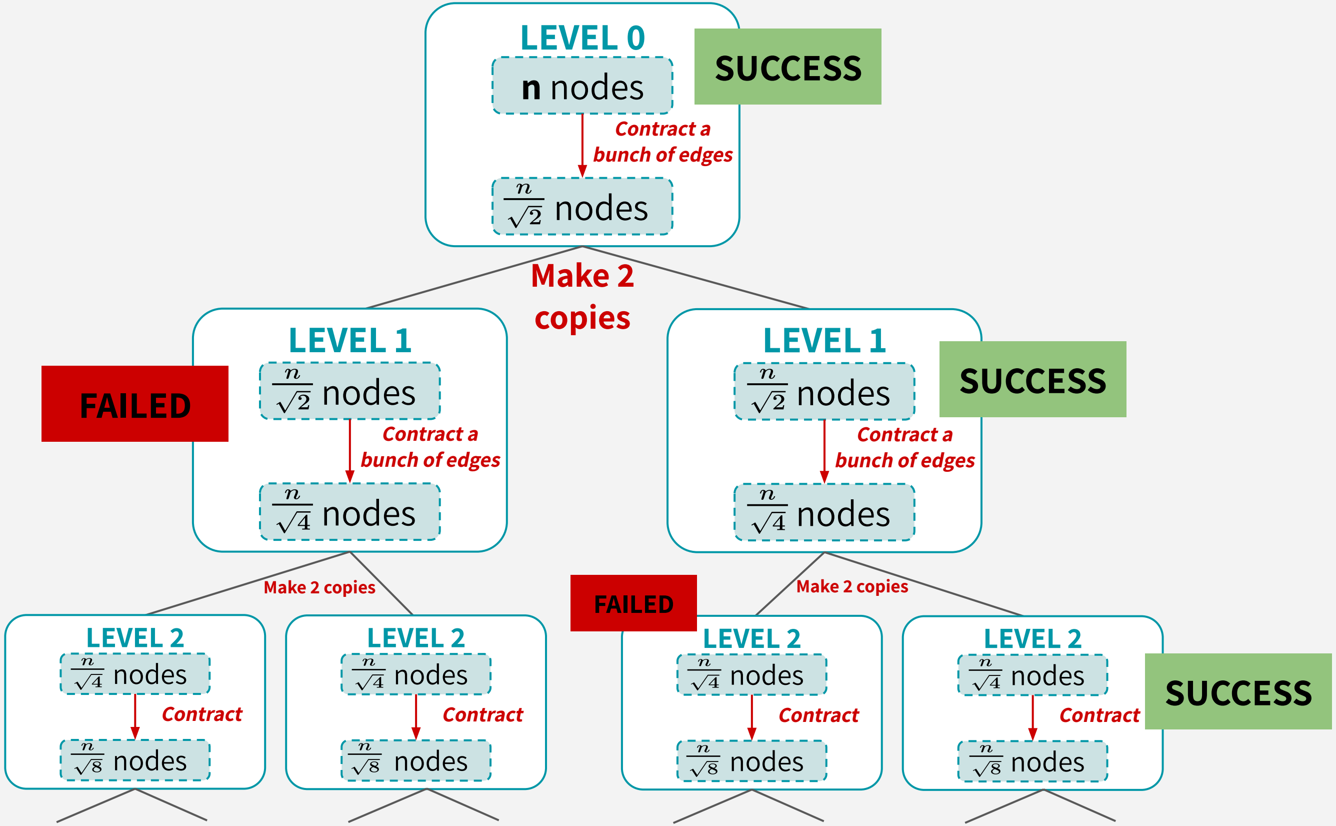
Pr[None of those n–t edges cross **S***]

This means that in every "blob" of running Karger for a bit, the probability of success is ½ !

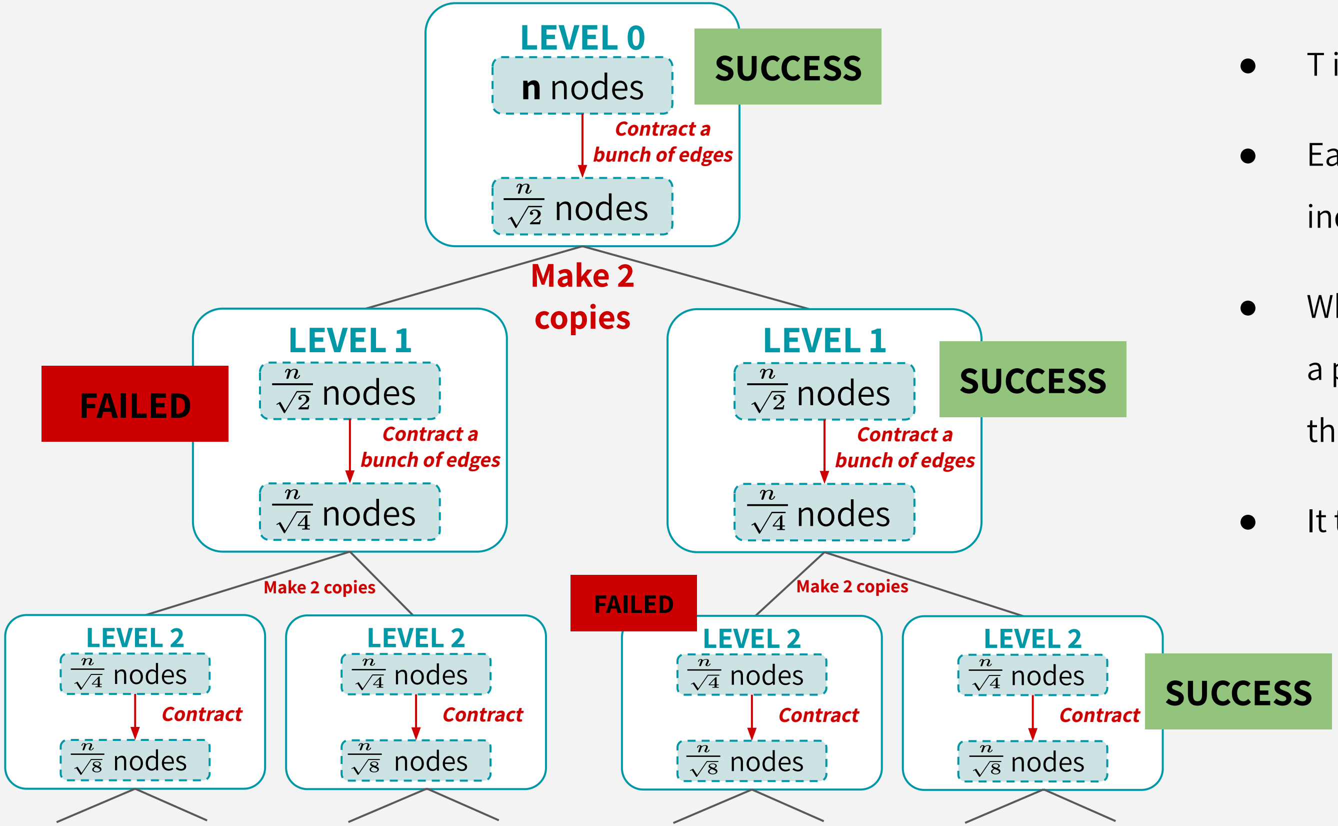
$$= \frac{\frac{n}{\sqrt{2}} \cdot \left(\frac{n}{\sqrt{2}} - 1\right)}{n \cdot (n-1)} \approx \frac{1}{2}$$

(when n is large)

KARGER-STEIN PROB. OF SUCCESS



KARGER-STEIN PROB. OF SUCCESS



- T is a binary tree of depth 2 log n
- Each node of T succeed or fails independently with probability ½
- What is the probability that there's a path from the root to any leaf that's entirely successful?
- It turns out that this is

$$\Omega\left(\frac{1}{\log n}\right)$$

KARGER-STEIN PROB. OF SUCCESS

LEVEL 0

n nodes

Contract a bunch of edges

$\frac{n}{\sqrt{2}}$ nodes

SUCCESS

- T is a binary tree of depth 2 log n
- Each node of T succeed or fails independently with probability ½

The probability of success for one run of KARGER-STEIN is

$$\Omega\left(\frac{1}{\log n}\right)$$

$\frac{n}{\sqrt{4}}$ nodes

Contract $\frac{n}{\sqrt{8}}$ nodes

Contract $\frac{n}{\sqrt{8}}$ nodes

Contract $\frac{n}{\sqrt{8}}$ nodes

Contract $\frac{n}{\sqrt{8}}$ nodes

SUCCESS

ALTOGETHER NOW

- **The probability of success for one run of KARGER-STEIN is** $\Omega\left(\frac{1}{\log n}\right)$
- **We can amplify the success probability with repetition**
  - Run Karger-Stein $O\left((\log n) \cdot \left(\log \frac{1}{\delta}\right)\right)$ times to achieve success probability 1 - δ
- **Each iteration takes time O(n^2 log n)**
  - That's what we proved before.
- **Choosing delta = 0.01 → the total runtime is**

$$O\left(n^2 \log n \cdot \log n\right) = O\left(n^2 \log^2 n\right)$$

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