Department of
Computer Engineering

Data Structure & Algorithms

Growth of functions

Overview

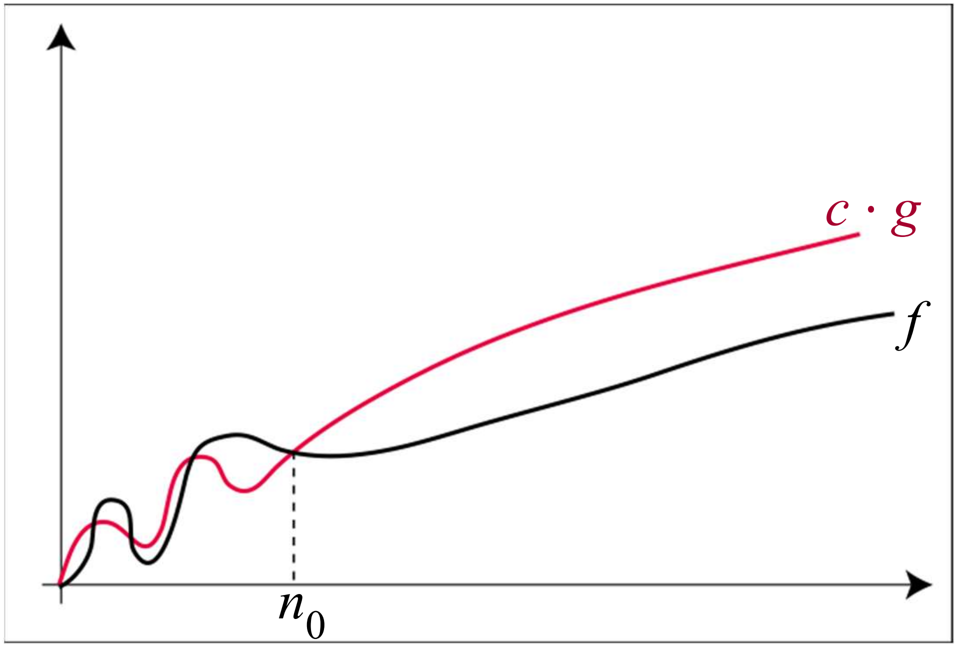
- Order of growth of functions provides a simple characterization of efficiency
- Allows for comparison of relative performance between alternative algorithms
- Concerned with asymptotic efficiency of algorithms
- Best **asymptotic** efficiency usually is best choice except for smaller inputs
- Several standard methods to simplify asymptotic analysis of algorithms

Asymptotic Notation

- Applies to functions whose domains are the set of natural numbers:
$$\boldsymbol{N} = \{0,1,2, \dots\}$$
- If time resource $T(n)$ is being analyzed, the function's range is usually the set of non-negative real numbers:
$$T(n) \in \boldsymbol{R}+$$
- In our textbook, asymptotic categories are expressed in terms of set membership meaning functions belong to a family of functions that exhibit some property.

The O-Notation

$$f(n) \in O(g(n))$$
$$O\big(g(n)\big) = \{\, f(n) \colon \exists c > 0,\ n_0 > 0 \mid \forall n \geq n_0 : f(n) \leq c \cdot g(n) \,\}$$



The O-Notation

Example: $f(n) \in 5n + 10$

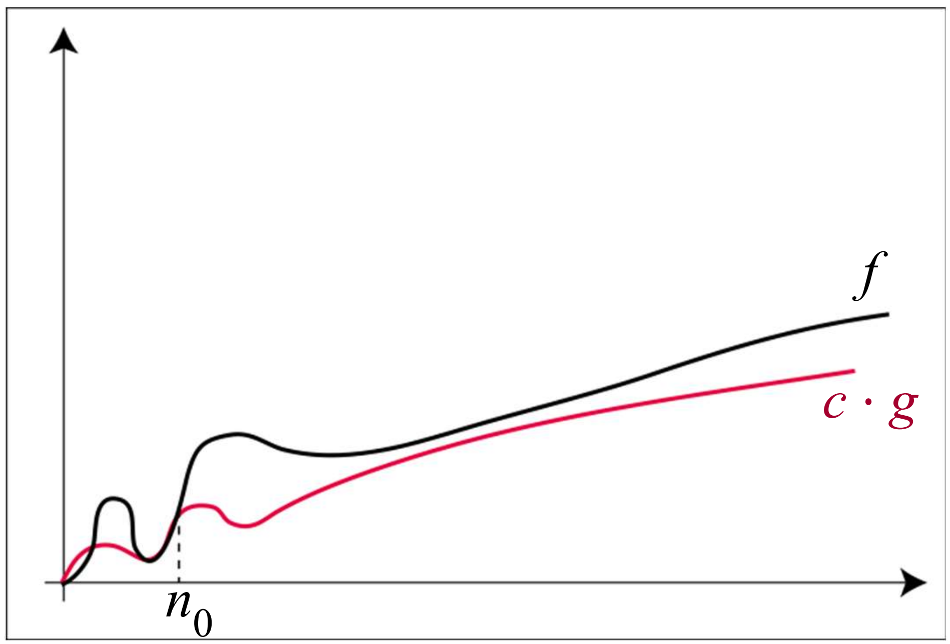
$$f(n) \in O(n)$$

$$f(n) \in O(n^2)$$

$$f(n) \notin O(\sqrt{n})$$

The Ω-Notation

$$f(n) \in \Omega\big(g(n)\big)$$
$$\Omega\big(g(n)\big) = \{\, f(n) \colon \exists c > 0,\, n_0 > 0 \mid \forall n \geq n_0 \colon\ f(n) \geq c \cdot g(n) \,\}$$



The Ω-Notation

Example: $f(n) \in 5n + 10$

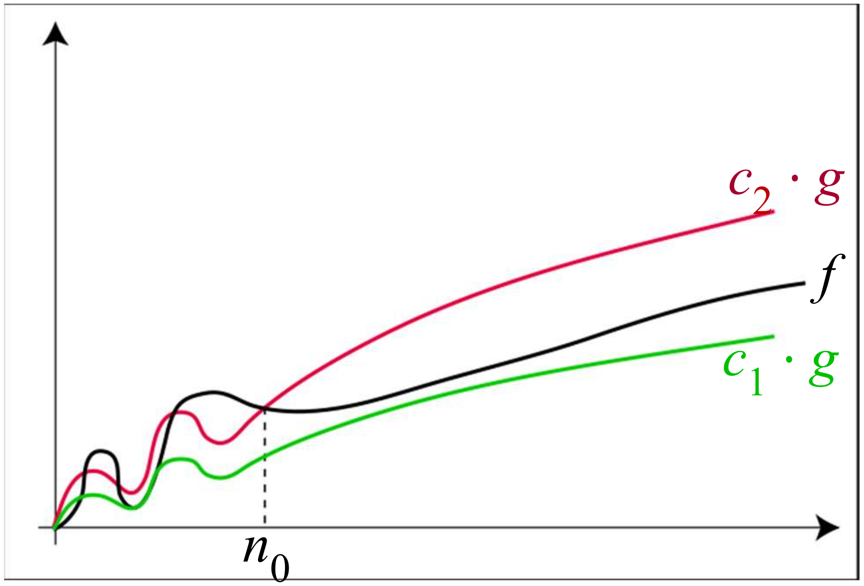
$$f(n) \in \Omega(n)$$

$$f(n) \in \Omega(\sqrt{n})$$

$$f(n) \notin \Omega(n^2)$$

The Θ-Notation

$$f(n) \in \Theta(g(n))$$
$$\Theta\big(g(n)\big) = \{\, f(n)\colon \exists c_1, c_2 > 0, n_0 > 0 \mid \forall n \geq n_0\colon c_1 \cdot g(n) \leq f(n) \leq c_2 \cdot g(n)\}$$



The Θ-Notation

Example: $f(n) \in 5n + 10$

$$f(n) \in \Theta(n)$$

$$f(n) \notin \Theta(\sqrt{n})$$

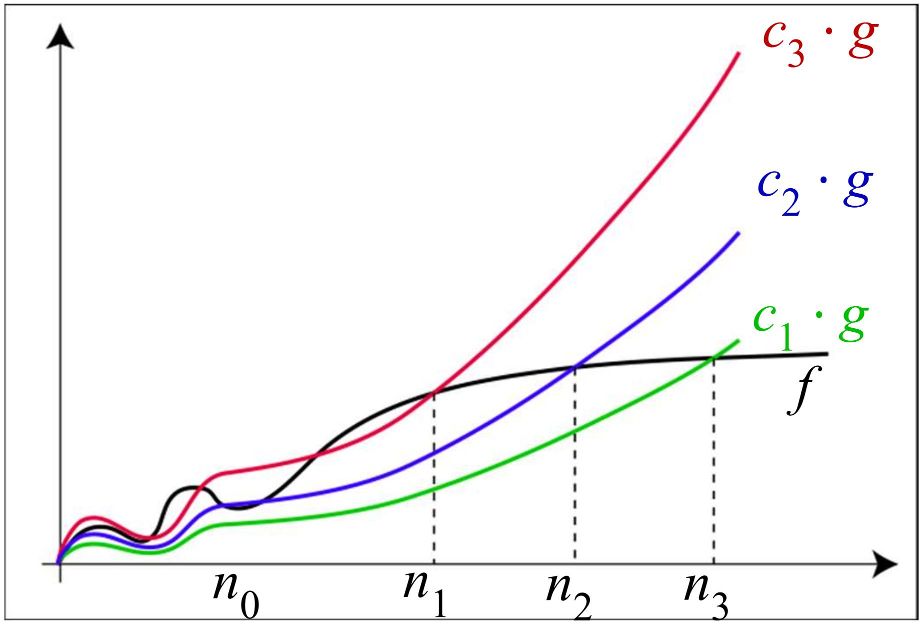
$$f(n) \notin \Theta(n^2)$$

$$If \lim_{n\to\infty} \frac{f(n)}{g(n)} = c,\ for\ some\ constant\ c > 0, then\ f(n) \in \Theta\big(g(n)\big).$$

The o-Notation

$$f(n) \in o(g(n))$$

$$o(g(n)) = \{ f(n) : \forall c > 0\ \exists n_0 > 0 \mid \forall n \geq n_0 : f(n) \leq c \cdot g(n) \}$$



The o-Notation

Example: $f(n) \in 5n + 10$

$$f(n) \notin o(n)$$

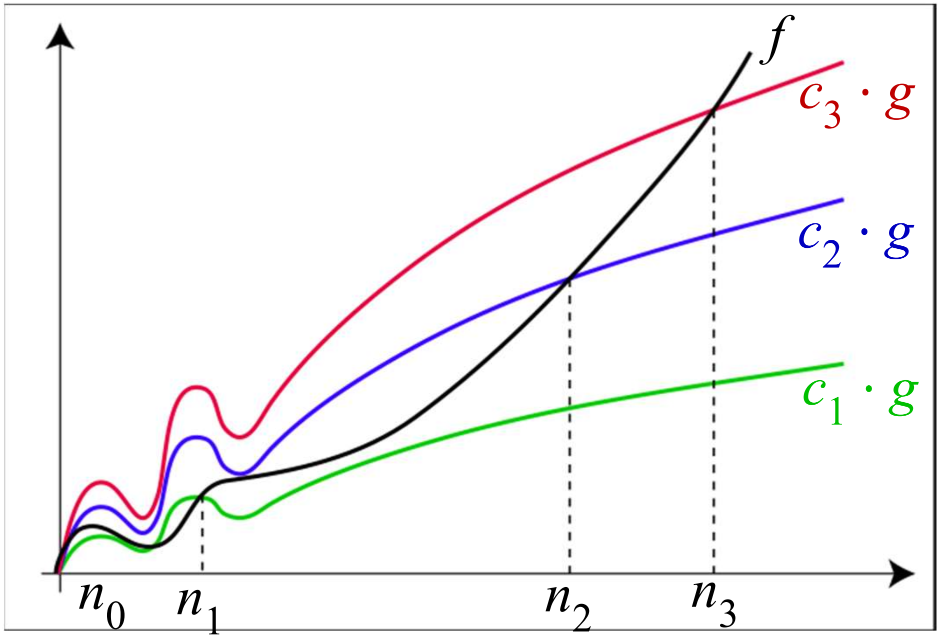
$$f(n) \notin o(\sqrt{n})$$

$$f(n) \in o(n^2)$$

$$If\ f(n) \in o\big(g(n)\big)\ then,\ \lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$$

The ω-Notation

$$f(n) \in \omega(g(n))$$
$$\omega(g(n)) = \{ f(n) : \forall c > 0 \, \exists n_0 > 0 \mid \forall n \geq n_0 : f(n) \geq c \cdot g(n) \}$$



The ω-Notation

Example : $f(n) = 5n + 10$

$$f(n) \notin \omega(n)$$
$$f(n) \in \omega(\sqrt{n})$$
$$f(n) \notin \omega(n^3)$$
$$f(n) \in \omega(logn)$$

$If\ f(n) \in \omega\big(g(n)\big)\ then, \lim_{n\to\infty} \frac{f(n)}{g(n)} = \infty$

Comparison of Functions

Reflexivity:

- $f(n) \in O(f(n))$
- $f(n) \in \Omega(f(n))$
- $f(n) \in \Theta(f(n))$

Transitivity:

- $f(n) \in O(g(n))\ and\ g(n) \in O(h(n)) \Rightarrow f(n) \in O(h(n))$
- $f(n) \in \Omega(g(n))\ and\ g(n) \in \Omega(h(n)) \Rightarrow f(n) \in \Omega(h(n))$
- $f(n) \in \Theta(g(n))\ and\ g(n) \in \Theta(h(n)) \Rightarrow f(n) \in \Theta(h(n))$

Also for o & ω.

Comparison of Functions (cont.)

Symmetry:

- $f(n) \in \Theta(g(n)) \Longleftrightarrow g(n) \in \Theta(f(n))$

Transpose Symmetry:

- $f(n) \in O(g(n)) \Longleftrightarrow g(n) \in \Omega(f(n))$
- $f(n) \in o(g(n)) \Longleftrightarrow g(n) \in \omega(f(n))$

Theorem 3.1:

- $f(n) \in O(g(n))\ and\ f(n) \in \Omega(g(n)) \Rightarrow f(n) \in \Theta(g(n))$

Comparison of Functions (cont.)

Also:

- $f(n) \in O(g(n))\ and\ g(n) \in O(f(n)) \Rightarrow f(n) \in \Theta(g(n))$
- $f(n) \in \Omega(g(n))\ and\ g(n) \in \Omega(f(n)) \Rightarrow f(n) \in \Theta(g(n))$

Comparison of Functions (cont.)

Also:

- $f_1(n) \in O(g_1(n))\ and\ f_2(n) \in O(g_2(n)) \Rightarrow f_1(n) + f_2(n) \in O(g_1(n) + g_2(n))$
- $f(n) \in O(g(n)) \Rightarrow f(n) + g(n) \in O(g(n))$

Correspondence between notations and “<“ , “>”

• $f(n) \in O\big(g(n)\big)$	a $\leq$ b
• $f(n) \in \Omega(g(n))$	a $\geq$ b
• $f(n) \in \Theta\big(g(n)\big)$	a = b
• $f(n) \in o(g(n))$	a < b
• $f(n) \in \omega(g(n))$	a > b

Standard Notations and Common Functions

Floors and ceilings

- For any real number x, the greatest integer less than or equal to x is denoted by $\lfloor x \rfloor$.
- For any real number x, the least integer greater than or equal to x is denoted by $\lceil x \rceil$.
- For all real numbers x,

$$x-1 < \lfloor x \rfloor \le x \le \lceil x \rceil < x+1$$

- Both functions are monotonically increasing (non-decreasing).

- $n \in \mathbb{N}$, $\left\lceil \frac{n}{2} \right\rceil + \left\lfloor \frac{n}{2} \right\rfloor = n$

- $n \in \mathbb{R}$, $n \geq 0$, $a, b > 0$, $\left\lceil \frac{\left\lceil \frac{n}{a} \right\rceil}{b} \right\rceil = \left\lceil \frac{n}{ab} \right\rceil$ & $\left\lfloor \frac{\left\lfloor \frac{n}{a} \right\rfloor}{b} \right\rfloor = \left\lfloor \frac{n}{ab} \right\rfloor$

- a,b $\in \mathbb{R}$, $a > 1$, $\lim_{n \to \infty} \frac{n^b}{a^n} = 0 \rightarrow n^b \in o(a^n)$

- $x \in \mathbb{R}, \quad e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \rightarrow e^x \geq 1 + x$
- $If\ |x| \leq 1\ then\ 1 + x \leq e^x \leq 1 + x + x^2$
- $\lim_{n \to \infty} (1 + \frac{x}{n})^n = e^x$

- $|x| < 1 \rightarrow \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots$

- $x > -1 \rightarrow \frac{x}{x+1} \leq \ln(1+x) \leq x$

Stirling's Approximation:

- $n! = \sqrt{2\pi n}(1 + \theta\left(\frac{1}{n}\right))(\frac{n}{e})^n$

Exercise

- Prove this relations:

  1. $\lg(n!) \in \theta(nlogn)$
  2. $n! \in \omega(2^n)$
  3. $n! \in o(n^n)$

Exponentials

- For all n and $a \geq 1$, the function a^n is the exponential function with base a and is monotonically increasing.

Logarithms

- Textbook adopts the following convention
  - $\log n = \log_2 n$ (binary logarithm)
  - $\ln n = \log_e n$ (natural logarithm)
  - $\log^k n = (\log n)k$ (exponentiation)
  - $\log\log n = \log(\log n)$ (composition)
  - $\log n + k = (\log n) + k$ (precedence of $\log$)

Important relationships

- For all real constants a and b such that a>1,
$$n^b = o(a^n)$$
that is, any exponential function with a base greater than 1, grows faster than any polynomial function.

- For all real constants a and b such that $a > 0$,
$$log^b n = o(n^a)$$
that is, any positive polynomial function grows faster than any polylogarithmic function.

Factorials

- For all n the function $n!$ or "$n\ factorial$" is given by

$$n! = n\times(n-1)\times(n-2)\times(n-3)\times \ldots \times 2\times 1$$

- It can be established that

$$n! = o(n^n)$$
$$n! = \omega(2^n)$$
$$\log(n!) = \Theta(n\log n)$$

Functional iteration

The notation $f^{(i)}(n)$ represents the function $f(n)$ iteratively applied i times to an initial value of n, or, recursively

- $f^{(i)}(n) = n \ if \ n = 0$
- $f^{(i)}(n) = f(f^{(i-1)}(n)) \ if \ n > 0$

Example:

- $If \qquad f(n) = 2n$
- $Then \qquad f^{(2)}(n) = f(2n) = 2(2n) = 2^2 n$
- $Then \qquad f^{(3)}(n) = f(f^{(2)}(n)) = 2(2^2 n) = 2^3 n$
- $Then \qquad f^{(i)}(n) = 2^i n$

Iterated logarithmic function

- The notation log^* n which reads "log $star\ of\ n$" is defined as

$$log^*\, n = \begin{cases} 0 & if\ n \leq 1 \\ 1 + log^*(logn) & if\ n > 1 \end{cases}$$

Example:

- $log^*\, 2 \ = \ 1$
- $log^*\, 4 \ = \ 2$
- $log^*\, 16 \ = \ 3$
- $log^*\, 65536 \ = \ 4$
- $log^*\, 2^{65536} = \ 5$